CHAPTER 17

Introduction to Homological Algebra and Group Cohomology

Let R be a ring with 1. In Section 10.5 we saw that a short exact sequence

$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow 0 \tag{17.1}$$

of R-modules gives rise to an exact sequence of abelian groups

$$0 \longrightarrow \mathrm{Hom}_R(N, D) \xrightarrow{\varphi'} \mathrm{Hom}_R(M, D) \xrightarrow{\psi'} \mathrm{Hom}_R(L, D) \tag{17.2}$$

for any R-module D and that the homomorphism ψ' is in general not surjective so this sequence cannot always be extended to a short exact sequence. Equivalently, homomorphisms from L to D cannot in general be lifted to homomorphisms from M into D. In this chapter we introduce some of the techniques of "homological algebra," which provide a method of extending some exact sequences in a natural way. For the situation above one obtains an infinite exact sequence involving the "cohomology groups" $\mathrm{Ext}_R^n(\_\_, D)$ (cf. Theorem 8), and these groups provide a measure of the set of homomorphisms from L into D that cannot be extended to M. We then consider the analogous questions for the other two functors considered in Section 10.5, namely taking homomorphisms *from* D into the terms of the sequence (1) and tensoring the sequence (1) with D.

In the subsequent sections we concentrate on an important special case of this general type of homological construction—the "cohomology of finite groups." We make explicit the computations in this case and indicate some applications of these techniques to establish some new results in group theory. In this sense, Sections 2–4 may be considered as an explicit "example" illustrating some uses of the general theory in Section 1.

Cohomology and homology groups occur in many areas of mathematics. The formal notions of homology and cohomology groups and the general area of homological algebra arose from algebraic topology around the middle of the 20th century in the study of the relation between the higher homotopy groups and the fundamental group of a topological space, although the study of certain specific cohomology groups, such as Schur's work on group extensions (described in Section 4), predates this by half a century. As with much of algebra, the ideas common to a number of different areas were abstracted into general theories. Much of the language of homology and cohomology reflects its topological origins: homology groups, chains, cycles, boundaries, etc.

17.1 INTRODUCTION TO HOMOLOGICAL ALGEBRA—EXT AND TOR

In this section we describe some general terminology and results in homological algebra leading to the so called Long Exact Sequence in Cohomology. We then define certain (cohomology) groups associated to the sequence (2) and apply the general homological results to obtain a long exact sequence extending this sequence at the right end. We then indicate the corresponding development for sequences obtained by taking homomorphisms from D to the terms in (1) or by tensoring the terms with D.

We begin with a generalization of the notion of an exact sequence, namely a sequence of abelian group homomorphisms where successive maps compose to zero (i.e., the image of one map is contained in the kernel of the next):

Definition. Let C be a sequence of abelian group homomorphisms:

$$0 \longrightarrow C^0 \xrightarrow{d_1} C^1 \longrightarrow \cdots \longrightarrow C^{n-1} \xrightarrow{d_n} C^n \xrightarrow{d_{n+1}} \cdots. \tag{17.3}$$

(1) The sequence C is called a *cochain complex* if the composition of any two successive maps is zero: $d_{n+1} \circ d_n = 0$ for all n.

(2) If C is a cochain complex, its n^{th} *cohomology group* is the quotient group $\ker d_{n+1} / \operatorname{image} d_n$, and is denoted by $H^n(C)$.

There is a completely analogous "dual" version in which the homomorphisms are between groups in *decreasing* order, in which case the sequence corresponding to (3) is written $\cdots \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} \cdots \xrightarrow{d_1} C_0 \to 0$. Then if the composition of any two successive homomorphisms is zero, the complex is called a *chain complex*, and its *homology groups* are defined as $H_n(C) = \ker d_n / \operatorname{image} d_{n+1}$. For chain complexes the notation is often chosen so that the indices appear as subscripts and are decreasing, whereas for cochain complexes the indices are superscripts and are increasing. We shall instead use a uniform notation for the maps on both, since it will be clear from the context whether we are dealing with a chain or a cochain complex.

Chain complexes were the first to arise in topological settings, with cochain complexes soon following. With our applications in Section 2 in mind, we shall concentrate on cochains and cohomology, although all of the general results in this section have similar statements for chains and homology. We shall also be interested in the situation where each C^n is an R-module and the homomorphisms d_n are R-module homomorphisms (referred to simply as a *complex of R-modules*), in which case the groups $H^n(C)$ are also R-modules.

Note that if C is a cochain (respectively, chain) complex then C is an exact sequence if and only if all its cohomology (respectively, homology) groups are zero. Thus the n^{th} cohomology (respectively, homology) group measures the failure of exactness of a complex at the n^{th} stage.

Definition. Let $\mathcal{A} = \{A^n\}$ and $\mathcal{B} = \{B^n\}$ be cochain complexes. A *homomorphism of complexes* $\alpha : \mathcal{A} \to \mathcal{B}$ is a set of homomorphisms $\alpha_n : A^n \to B^n$ such that for every n the following diagram commutes:

$$\begin{tikzcd} \cdots \arrow[r] & A^n \arrow[r] \arrow[d, "\alpha_n"] & A^{n+1} \arrow[r] \arrow[d, "\alpha_{n+1}"] & \cdots \\ \cdots \arrow[r] & B^n \arrow[r] & B^{n+1} \arrow[r] & \cdots \end{tikzcd} \tag{17.4}$$

Proposition 1. A homomorphism $\alpha : \mathcal{A} \to \mathcal{B}$ of cochain complexes induces group homomorphisms from $H^n(\mathcal{A})$ to $H^n(\mathcal{B})$ for $n \geq 0$ on their respective cohomology groups.

Proof: It is an easy exercise to show that the commutativity of (4) implies that the images and kernels at each stage of the maps in the first row are mapped to the corresponding images and kernels for the maps in the second row, thus giving a well defined map on the respective quotient (cohomology) groups.

Definition. Let $\mathcal{A} = \{A^n\}$, $\mathcal{B} = \{B^n\}$ and $\mathcal{C} = \{C^n\}$ be cochain complexes. A *short exact sequence* of complexes $0 \to \mathcal{A} \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\beta} \mathcal{C} \to 0$ is a sequence of homomorphisms of complexes such that $0 \to A^n \xrightarrow{\alpha_n} B^n \xrightarrow{\beta_n} C^n \to 0$ is short exact for every n.

One of the main features of cochain complexes is that they lead to long exact sequences in cohomology, which is our first main result:

Theorem 2. *(The Long Exact Sequence in Cohomology)* Let $0 \to \mathcal{A} \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\beta} \mathcal{C} \to 0$ be a short exact sequence of cochain complexes. Then there is a long exact sequence of cohomology groups:

$$\begin{aligned} 0 \to H^0(\mathcal{A}) \to H^0(\mathcal{B}) \to H^0(\mathcal{C}) &\xrightarrow{\delta_0} H^1(\mathcal{A}) \\ \to H^1(\mathcal{B}) \to H^1(\mathcal{C}) &\xrightarrow{\delta_1} H^2(\mathcal{A}) \to \cdots \end{aligned} \tag{17.5}$$

where the maps between cohomology groups at each level are those in Proposition 1. The maps δ_n are called *connecting homomorphisms*.

Proof: The details of this proof are somewhat lengthy. For each n the verification that the sequence $H^n(\mathcal{A}) \to H^n(\mathcal{B}) \to H^n(\mathcal{C})$ is exact is a straightforward check of the definition of exactness of each map, similar to the proof of Theorem 33 in Section 10.5. The construction of a connecting homomorphism δ_n is outlined in Exercise 2. Some work is then needed to show that δ_n is a homomorphism, and that the sequence is exact at δ_n.

One immediate consequence of the existence of the long exact sequence in Theorem 2 is the fact that if any two of the cochain complexes $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are exact, then so is the third (cf. Exercise 6).

Homomorphisms and the Groups $\mathrm{Ext}^n_R(A,B)$

To apply Theorem 2 to analyze the sequence (2), we try to produce a cochain complex whose first few cohomology groups in the long exact sequence (5) agree with the terms in (2). To do this we introduce the notion of a "resolution" of an R-module:

Definition. Let A be any R-module. A *projective resolution* of A is an exact sequence

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \longrightarrow \cdots \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} A \longrightarrow 0 \tag{17.6}$$

such that each P_i is a projective R-module.

Every R-module has a projective resolution: Let P_0 be any free (hence projective) R-module on a set of generators of A and define an R-module homomorphism ϵ from P_0 onto A by Theorem 6 in Chapter 10. This begins the resolution $\epsilon : P_0 \to A \to 0$. The surjectivity of ϵ ensures that this sequence is exact. Next let $K_0 = \ker \epsilon$ and let P_1 be any free module mapping onto the submodule K_0 of P_0; this gives the second stage $P_1 \to P_0 \to A$ which, by construction, is also exact. We can continue this way, taking at the n^{th} stage a free R-module P_{n+1} that maps surjectively onto the submodule $\ker d_n$ of P_n, obtaining in fact a *free* resolution of A.

One of the reasons that *projective* modules are used in the resolution of A is that this makes it possible to lift various maps (cf. the proof of Proposition 4 following, for instance).

In general a projective resolution is infinite in length, but if A is itself projective, then it has a very simple projective resolution of finite length, namely $0 \to A \xrightarrow{1} A \to 0$ given by the identity map from A to itself.

Given the projective resolution (6), we may form a related sequence by taking homomorphisms of each of the terms into D, keeping in mind that this reverses the direction of the homomorphisms. This yields the sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}_R(A, D) &\xrightarrow{\epsilon} \mathrm{Hom}_R(P_0, D) \xrightarrow{d_1} \mathrm{Hom}_R(P_1, D) \xrightarrow{d_2} \cdots \\ \cdots &\xrightarrow{d_{n-1}} \mathrm{Hom}_R(P_{n-1}, D) \xrightarrow{d_n} \mathrm{Hom}_R(P_n, D) \xrightarrow{d_{n+1}} \cdots \end{aligned} \tag{17.7}$$

where to simplify notation we have denoted the induced maps from $\mathrm{Hom}_R(P_{n-1}, D)$ to $\mathrm{Hom}_R(P_n, D)$ for $n \geq 1$ again by d_n and similarly for the map induced by ϵ (cf. Section 10.5). This sequence is not necessarily exact, however it *is* a cochain complex (this is part of the proof of Theorem 33 in Section 10.5). The corresponding cohomology groups have a special name.

Definition. Let A and D be a R-modules. For any projective resolution of A as in (6) let $d_n : \mathrm{Hom}_R(P_{n-1}, D) \to \mathrm{Hom}_R(P_n, D)$ for all $n \geq 1$ as in (7). Define

$$\mathrm{Ext}^n_R(A, D) = \ker d_{n+1} / \operatorname{image} d_n$$

where $\mathrm{Ext}^0_R(A, D) = \ker d_1$. The group $\mathrm{Ext}^n_R(A, D)$ is called the n^{th} *cohomology group derived from the functor* $\mathrm{Hom}_R(\_\_, D)$. When $R = \mathbb{Z}$ the group $\mathrm{Ext}^n_{\mathbb{Z}}(A, D)$ is also denoted simply $\mathrm{Ext}^n(A, D)$.