where $\nabla^2 \boldsymbol{F}$ is defined by the equation

$$\nabla^2 \boldsymbol{F} = (\nabla^2 P)\boldsymbol{i} + (\nabla^2 Q)\boldsymbol{j} + (\nabla^2 R)\boldsymbol{k}.$$

The identity in (12.38) relates all four operators, gradient, curl, divergence, and Laplacian. The verification of (12.38) is requested in Exercise 7 of Section 12.15.

The curl and divergence have some general properties in common with ordinary derivatives. First, they are ***linear* operators.** That is, if a and b are constants, we have

(12.39) $$\operatorname{div}(a\boldsymbol{F} + b\boldsymbol{G}) = a \operatorname{div} \boldsymbol{F} + b \operatorname{div} \boldsymbol{G},$$

and

(12.40) $$\operatorname{curl}(a\boldsymbol{F} + b\boldsymbol{G}) = a \operatorname{curl} \boldsymbol{F} + b \operatorname{curl} \boldsymbol{G}.$$

They also have a property analogous to the formula for differentiating a product:

(12.41) $$\operatorname{div}(\varphi \boldsymbol{F}) = \varphi \operatorname{div} \boldsymbol{F} + \nabla\varphi \cdot \boldsymbol{F},$$

and

(12.42) $$\operatorname{curl}(\varphi \boldsymbol{F}) = \varphi \operatorname{curl} \boldsymbol{F} + \nabla\varphi \times \boldsymbol{F},$$

where φ is any differentiable scalar field. These properties are immediate consequences of the definitions of curl and divergence; their proofs are requested in Exercise 6 of Section 12.15.

If we use the symbolic vector

$$\nabla = \frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}$$

once more, each of the formulas (12.41) and (12.42) takes a form which resembles more closely the usual rule for differentiating a product:

$$\nabla \cdot (\varphi \boldsymbol{F}) = \varphi \nabla \cdot \boldsymbol{F} + \nabla\varphi \cdot \boldsymbol{F}$$

and

$$\nabla \times (\varphi \boldsymbol{F}) = \varphi \nabla \times \boldsymbol{F} + \nabla\varphi \times \boldsymbol{F}.$$

In Example 3 the Laplacian of a scalar field, $\nabla^2\varphi$, was defined to be $\partial^2\varphi/\partial x^2 + \partial^2\varphi/\partial y^2 + \partial^2\varphi/\partial z^2$. In Example 5 the Laplacian $\nabla^2 \boldsymbol{F}$ of a vector field was defined by components. We get correct formulas for both $\nabla^2\varphi$ and $\nabla^2 \boldsymbol{F}$ if we interpret ∇^2 as the symbolic operator

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

This formula for ∇^2 also arises by dot multiplication of the symbolic vector ∇ with itself.

Thus, we have $\nabla^2 = \nabla \cdot \nabla$ and we can write

$$\nabla^2\varphi = (\nabla \cdot \nabla)\varphi \qquad \text{and} \qquad \nabla^2 \boldsymbol{F} = (\nabla \cdot \nabla)\boldsymbol{F}.$$

Now consider the formula $\nabla \cdot \nabla\varphi$. This can be read as $(\nabla \cdot \nabla)\varphi$, which is $\nabla^2\varphi$; or as $\nabla \cdot (\nabla\varphi)$, which is div $(\nabla\varphi)$. In Example 3 we showed that div $(\nabla\varphi) = \nabla^2\varphi$, so we have

$$(\nabla \cdot \nabla)\varphi = \nabla \cdot (\nabla\varphi)\,;$$

hence we can write $\nabla \cdot \nabla\varphi$ for either of these expressions without danger of ambiguity. This is not true, however, when φ is replaced by a vector field $\boldsymbol{F}$. The expression $(\nabla \cdot \nabla)\boldsymbol{F}$ is $\nabla^2\boldsymbol{F}$, which has been defined. However, $\nabla \cdot (\nabla\boldsymbol{F})$ is meaningless because $\nabla\boldsymbol{F}$ is not defined. Therefore the expression $\nabla \cdot \nabla\boldsymbol{F}$ is meaningful only when it is interpreted as $(\nabla \cdot \nabla)\boldsymbol{F}$. These remarks illustrate that although symbolic formulas sometimes serve as a convenient notation and memory aid, care is needed in manipulating the symbols.

12.15 Exercises

1. For each of the following vector fields determine the Jacobian matrix and compute the curl and divergence.
   (a) $\boldsymbol{F}(x, y, z) = (x^2 + yz)\boldsymbol{i} + (y^2 + xz)\boldsymbol{j} + (z^2 + xy)\boldsymbol{k}$.
   (b) $\boldsymbol{F}(x, y, z) = (2z - 3y)\boldsymbol{i} + (3x - z)\boldsymbol{j} + (y - 2x)\boldsymbol{k}$.
   (c) $\boldsymbol{F}(x, y, z) = (z + \sin y)\boldsymbol{i} - (z - x\cos y)\boldsymbol{j}$.
   (d) $\boldsymbol{F}(x, y, z) = e^{xy}\boldsymbol{i} + \cos xy\,\boldsymbol{j} + \cos xz^2\boldsymbol{k}$.
   (e) $\boldsymbol{F}(x, y, z) = x^2 \sin y\,\boldsymbol{i} + y^2 \sin xz\,\boldsymbol{j} + xy \sin(\cos z)\boldsymbol{k}$.
2. If $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ and $r = \|\boldsymbol{r}\|$, compute curl $[f(r)\boldsymbol{r}]$, where f is a differentiable function.
3. If $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ and $\boldsymbol{A}$ is a constant vector, show that curl $(\boldsymbol{A} \times \boldsymbol{r}) = 2\boldsymbol{A}$.
4. If $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ and $r = \|\boldsymbol{r}\|$, find all integers n for which div $(r^n\boldsymbol{r}) = 0$.
5. Find a vector field whose curl is $x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ or prove that no such vector field exists.
6. Prove the elementary properties of curl and divergence in Equations (12.39) through (12.42).
7. Prove that curl (curl $\boldsymbol{F}$) = grad (div $\boldsymbol{F}$) $- \nabla^2\boldsymbol{F}$ if the components of $\boldsymbol{F}$ have continuous mixed partial derivatives of second order.
8. Prove the identity
$$\nabla \cdot (\boldsymbol{F} \times \boldsymbol{G}) = \boldsymbol{G} \cdot (\nabla \times \boldsymbol{F}) - \boldsymbol{F} \cdot (\nabla \times \boldsymbol{G}),$$
   where $\boldsymbol{F}$ and $\boldsymbol{G}$ are differentiable vector fields.
9. A vector field $\boldsymbol{F}$ will not be the gradient of a potential unless curl $\boldsymbol{F} = \boldsymbol{0}$. However, it may be possible to find a nonzero scalar field μ such that $\mu\boldsymbol{F}$ is a gradient. Prove that if such a μ exists, $\boldsymbol{F}$ is always perpendicular to its curl. When the field is two-dimensional, say $\boldsymbol{F} = P\boldsymbol{i} + Q\boldsymbol{j}$, this exercise gives us a necessary condition for the differential equation $P\,dx + Q\,dy = 0$ to have an integrating factor. (The converse is also true. That is, if $\boldsymbol{F} \cdot \text{curl } \boldsymbol{F} = 0$ in a suitable region, a nonzero μ exists such that $\mu\boldsymbol{F}$ is a gradient. The proof of the converse is not required.)
10. Let $\boldsymbol{F}(x, y, z) = y^2z^2\boldsymbol{i} + z^2x^2\boldsymbol{j} + x^2y^2\boldsymbol{k}$. Show that curl $\boldsymbol{F}$ is not always zero, but that $\boldsymbol{F} \cdot \text{curl } \boldsymbol{F} = 0$. Find a scalar field μ such that $\mu\boldsymbol{F}$ is a gradient.
11. Let $\boldsymbol{V}(x, y) = y^c\boldsymbol{i} + x^c\boldsymbol{j}$, where c is a positive constant, and let $\boldsymbol{r}(x, y) = x\boldsymbol{i} + y\boldsymbol{j}$. Let R be a plane region bounded by a piecewise smooth Jordan curve C. Compute div $(\boldsymbol{V} \times \boldsymbol{r})$ and curl $(\boldsymbol{V} \times \boldsymbol{r})$, and use Green's theorem to show that
$$\oint_C \boldsymbol{V} \times \boldsymbol{r} \cdot d\boldsymbol{\alpha} = 0,$$
   where $\boldsymbol{\alpha}$ describes C.

12. Show that Green's theorem can be expressed as follows:

$$\iint_R (\operatorname{curl} V) \cdot k \, dx \, dy = \oint_C V \cdot T \, ds,$$

where T is the unit tangent to C and s denotes arc length.

13. A plane region R is bounded by a piecewise smooth Jordan curve C. The moments of inertia of R about the x- and y-axes are known to be a and b, respectively. Compute the line integral

$$\oint_C \nabla(r^4) \cdot n \, ds$$

in terms of a and b. Here $r = \|xi + yj\|$, n denotes the unit outward normal of C, and s denotes arc length. The curve is traversed counterclockwise.

14. Let F be a two-dimensional vector field. State a definition for the vector-valued line integral $\int_C F \times d\alpha$. Your definition should be such that the following formula is a consequence of Green's theorem:

$$\int_C F \times d\alpha = \iint_R (\operatorname{div} F) \, dx \, dy,$$

where R is a plane region bounded by a simple closed curve C.

★12.16 Reconstruction of a vector field from its curl

In our study of the gradient we learned how to determine whether or not a given vector field is a gradient. We now ask a similar question concerning the curl. Given a vector field F, is there a G such that curl G = F? Suppose we write $F = Pi + Qj + Rk$ and $G = Li + Mj + Nk$. To solve the equation curl G = F we must solve the system of partial differential equations

$$(12.43) \qquad \frac{\partial N}{\partial y} - \frac{\partial M}{\partial z} = P, \quad \frac{\partial L}{\partial z} - \frac{\partial N}{\partial x} = Q, \quad \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} = R$$

for the three unknown functions L, M, and N when P, Q, and R are given.

It is not always possible to solve such a system. For example, we proved in Section 12.14 that the divergence of a curl is always zero. Therefore, for the system (12.43) to have a solution in some open set S it is necessary to have

$$(12.44) \qquad \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 0$$

everywhere in S. As it turns out, this condition is also sufficient for system (12.43) to have a solution if we suitably restrict the set S in which (12.44) is satisfied. We shall prove now that condition (12.44) suffices when S is a three-dimensional interval.

THEOREM 12.5 *Let F be continuously differentiable on an open interval S in 3-space. Then there exists a vector field G such that* curl $G = F$ *if and only if* div $F = 0$ *everywhere in S.*

Proof. The necessity of the condition div $\boldsymbol{F} = 0$ has already been established, since the divergence of a curl is always zero. To establish the sufficiency we must exhibit three functions L, M, and N that satisfy the three equations in (12.43). Let us try to make a choice with $L = 0$. Then the second and third equations in (12.43) become

$$\frac{\partial N}{\partial x} = -Q \quad \text{and} \quad \frac{\partial M}{\partial x} = R .$$

This means that we must have

$$N(x, y, z) = -\int_{x_0}^{x} Q(t, y, z)\,dt + f(y, z)$$

and

$$M(x, y, z) = \int_{x_0}^{x} R(t, y, z)\,dt + g(y, z),$$

where each integration is along a line segment in S and the "constants of integration" $f(y, z)$ and $g(y, z)$ are independent of x. Let us try to find a solution with $f(y, z) = 0$. The first equation in (12.43) requires

(12.45) $$\frac{\partial N}{\partial y} - \frac{\partial M}{\partial z} = P .$$

For the choice of M and N just described we have

(12.46) $$\frac{\partial N}{\partial y} - \frac{\partial M}{\partial z} = -\frac{\partial}{\partial y}\int_{x_0}^{x} Q(t, y, z)\,dt - \frac{\partial}{\partial z}\int_{x_0}^{x} R(t, y, z)\,dt - \frac{\partial g}{\partial z} .$$

At this stage we interchange the two operations of partial differentiation and integration, using Theorem 10.8. That is, we write

(12.47) $$\frac{\partial}{\partial y}\int_{x_0}^{x} Q(t, y, z)\,dt = \int_{x_0}^{x} D_2Q(t, y, z)\,dt$$

and

(12.48) $$\frac{\partial}{\partial z}\int_{x_0}^{x} R(t, y, z)\,dt = \int_{x_0}^{x} D_3R(t, y, z)\,dt .$$

Then Equation (12.46) becomes

(12.49) $$\frac{\partial N}{\partial y} - \frac{\partial M}{\partial z} = \int_{x_0}^{x} [-D_2Q(t, y, z) - D_3R(t, y, z)]\,dt - \frac{\partial g}{\partial z} .$$

Using condition (12.44) we may replace the integrand in (12.49) by $D_1P(t, y, z)$; Equation (12.49) becomes

$$\frac{\partial N}{\partial y} - \frac{\partial M}{\partial z} = \int_{x_0}^{x} D_1P(t, y, z)\,dt - \frac{\partial g}{\partial z} = P(x, y, z) - P(x_0, y, z) - \frac{\partial g}{\partial z} .$$

Therefore (12.45) will be satisfied if we choose g so that $\partial g/\partial z = -P(x_0, y, z)$. Thus, for example, we may take

$$g(y, z) = -\int_{z_0}^{z} P(x_0, y, u)\, du\,.$$

This argument leads us to consider the vector field $G = L\boldsymbol{i} + M\boldsymbol{j} + N\boldsymbol{k}$, where $L(x, y, z) = 0$ and

$$M(x, y, z) = \int_{x_0}^{x} R(t, y, z)\, dt - \int_{z_0}^{z} P(x_0, y, u)\, du\,, \qquad N(x, y, z) = -\int_{x_0}^{x} Q(t, y, z)\, dt.$$

For this choice of L, M, and N it is easy to verify, with the help of (12.47) and (12.48), that the three equations in (12.43) are satisfied, giving us curl $G = F$, as required.

It should be noted that the foregoing proof not only establishes the existence of a vector field G whose curl is F, but also provides a straightforward method for determining G by integration involving the components of F.

For a given F, the vector field G that we have constructed is not the only solution of the equation curl $G = F$. If we add to this G any continuously differentiable gradient $\nabla\varphi$ we obtain another solution because

$$\text{curl}\,(G + \nabla\varphi) = \text{curl}\, G + \text{curl}\,(\nabla\varphi) = \text{curl}\, G = F,$$

since curl $(\nabla\varphi) = 0$. Moreover, it is easy to show that ***all*** continuously differentiable solutions must be of the form $G + \nabla\varphi$. Indeed, if H is another solution, then curl H = curl G, so curl $(H - G) = 0$. By Theorem 10.9 it follows that $H - G = \nabla\varphi$ for some continuously differentiable gradient $\nabla\varphi$; hence $H = G + \nabla\varphi$, as asserted.

A vector field F for which div $F = 0$ is sometimes called ***solenoidal.*** Theorem 12.5 states that a vector field is solenoidal on an open interval S in 3-space if and only if it is the curl of another vector field on S.

The following example shows that this statement is not true for ***arbitrary*** open sets.

EXAMPLE. ***A solenoidal vector field that is not a curl.*** Let D be the portion of 3-space between two concentric spheres with center at the origin and radii a and b, where $0 < a < b$. Let $\boldsymbol{V} = \boldsymbol{r}/r^3$, where $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ and $r = \|\boldsymbol{r}\|$. It is easy to verify that div $V = 0$ everywhere in D. In fact, we have the general formula

$$\text{div}\,(r^n\boldsymbol{r}) = (n + 3)r^n,$$

and in this example $n = -3$. We shall use Stokes' theorem to prove that this V is ***not*** a curl in D (although it is a curl on every open three-dimensional interval not containing the origin). To do this we assume there is a vector field U such that V = curl U in D and obtain a contradiction. By Stokes' theorem we can write

$$\iint_S (\text{curl}\, \boldsymbol{U}) \cdot \boldsymbol{n}\, dS = \oint_C \boldsymbol{U} \cdot d\boldsymbol{\alpha}, \tag{12.50}$$

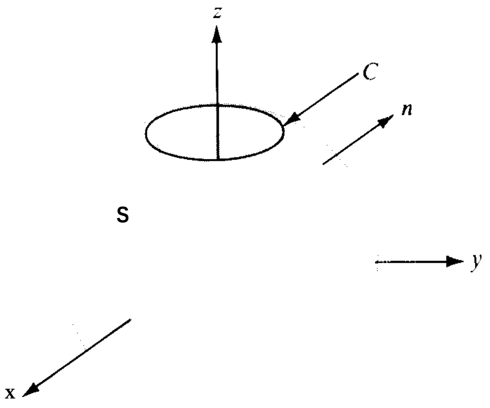

FIGURE 12.13 The surface S and curve C in Equation (12.50).

where S and C are the surface and curve shown in Figure 12.13. To construct S, we take a spherical surface of radius R concentric with the boundaries of D, where $a < R < b$, and we remove a small "polar cap," as indicated in the figure. The portion that remains is the surface S. The curve C is the circular edge shown. Let n denote the unit outer normal of S, so that $n = r/r$. Since curl $U = V = r/r^3$, we have

$$(\text{curl } U) \cdot n = \frac{r}{r^3} \cdot \frac{r}{r} = \frac{1}{r^2},$$

On the surface S this dot product has the constant value $1/R^2$. Therefore we have

$$\iint_S (\text{curl } U) \cdot n \, dS = \frac{1}{R^2} \iint_S dS = \frac{\text{area of } S}{R^2}.$$

When the polar cap shrinks to a point, the area of S approaches $4\pi R^2$ (the area of the whole sphere) and, therefore, the value of the surface integral in (12.50) approaches 4π.

Next we examine the line integral in (12.50). It is easy to prove that for any line integral $\int_C U \cdot d\alpha$ we have the inequality

$$\left| \int_C U \cdot d\alpha \right| \leq M \cdot (\text{length of } C),$$

where M is a constant depending on U. (In fact, M can be taken to be the maximum value of $\|U\|$ on C.) Therefore, as we let the polar cap shrink to a point, the length of C and the value of the line integral both approach zero. Thus we have a contradiction; the surface integral in (12.50) can be made arbitrarily close to 4π, and the corresponding line integral to which it is equal can be made arbitrarily close to 0. Therefore a function U whose curl is V cannot exist in the region D.

The difficulty here is caused by the geometric structure of the region D. Although this region is simply connected (that is, any simple closed curve in D is the edge of a parametric surface lying completely in D) there are closed *surfaces* in D that are not the complete boundaries of solids lying entirely in D. For example, no sphere about the origin is the complete boundary of a solid lying entirely in D. If the region D has the property that *every* closed surface in D is the boundary of a solid lying entirely in D, it can be shown that a vector field U exists such that $V = \operatorname{curl} U$ in D if and only if $\operatorname{div} V = 0$ everywhere in D. The proof of this statement is difficult and will not be given here.

★12.17 Exercises

1. Find a vector field $G(x, y, z)$ whose curl is $2i + j + 3k$ everywhere in 3-space. What is the most general continuously differentiable vector field with this property?
2. Show that the vector field $F(x, y, z) = (y - z)i + (z - x)j + (x - y)k$ is solenoidal, and find a vector field G such that $F = \operatorname{curl} G$ everywhere in 3-space.
3. Let $F(x, y, z) = -zi + xyk$. Find a continuously differentiable vector field G of the form $G(x, y, z) = L(x, y, z)i + M(x, y, z)j$ such that $F = \operatorname{curl} G$ everywhere, in 3-space. What is the most general G of this form?
4. If two vector fields U and V are both irrotational, show that the vector field $U \times V$ is solenoidal.
5. Let $r = xi + yj + zk$ and let $r = \|r\|$. Show that $n = -3$ is the only value of n for which $r^n r$ is solenoidal for $r \neq 0$. For this n, choose a 3-dimensional interval S not containing the origin and express $r^{-3} r$ as a curl in S. Note: Although $r^{-3} r$ is a curl in every such S, it is *not* a curl on the set of all points different from $(0, 0, 0)$.
6. Find the most general continuously differentiable function f of one real variable such that the vector field $f(r)r$ will be solenoidal, where $r = xi + yj + zk$ and $r = \|r\|$.
7. Let V denote a vector field that is continuously differentiable on some open interval S in 3-space. Consider the following two statements about V:
   (i) $\operatorname{curl} V = 0$ and $V = \operatorname{curl} U$ for some continuously differentiable vector field U (everywhere on S).
   (ii) A scalar field φ exists such that $\nabla\varphi$ is continuously differentiable and such that
   $$V = \operatorname{grad} \varphi \quad \text{and} \quad \nabla^2\varphi = 0 \text{ everywhere on } S.$$
   (a) Prove that (i) implies (ii). In other words, a vector field that is both irrotational and solenoidal in S is the gradient of a harmonic function in S.
   (b) Prove that (ii) implies (i), or give a counterexample.
8. Assume continuous differentiability of all vector fields involved, on an open interval S. Suppose $H = F + G$, where F is solenoidal and G is irrotational. Then there exists a vector field U such that $F = \operatorname{curl} U$ and a scalar field φ such that $G = \nabla\varphi$ in S. Show that U and φ satisfy the following partial differential equations in S:
   $$\nabla^2\varphi = \operatorname{div} H, \qquad \operatorname{grad}(\operatorname{div} U) - \nabla^2 U = \operatorname{curl} H.$$
   Note: This exercise has widespread applications, because it can be shown that every continuously differentiable vector field H on S can be expressed in the form $H = F + G$, where F is solenoidal and G is irrotational.
9. Let $H(x, y, z) = x^2yi + y^2zj + z^2xk$. Find vector fields F and G, where F is a curl and G is a gradient, such that $H = F + G$.
10. Let u and v be scalar fields that are continuously differentiable on an open interval R in 3-space.
   (a) Show that a vector field F exists such that $\nabla u \times \nabla v = \operatorname{curl} F$ everywhere in R.

(b) Determine whether or not any of the following three vector fields may be used for F in part (a): (i) $\nabla(uv)$; (ii) $u\nabla v$; (iii) $v\nabla u$.
(c) If $u(x, y, z) = x^3 - y^3 + z^2$ and $v(x, y, z) = x + y + z$, evaluate the surface integral

$$\iint_S \text{Vu} \times \nabla v \cdot \boldsymbol{n}\, dS,$$

where S is the hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$, and $\boldsymbol{n}$ is the unit normal with a non-negative z-component.

12.18 Extensions of Stokes' theorem

Stokes' theorem can be extended to more general simple smooth surfaces. If T is a multiply connected region like that shown in Figure 12.14 (with a finite number of holes), the one-to-one image $S = r(T)$ will contain the same number of holes as T. To extend

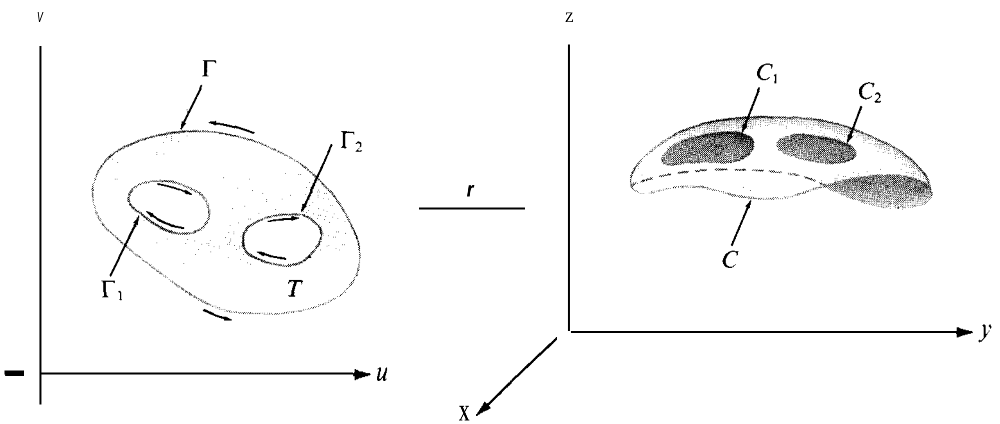

FIGURE 12.14 Extension of Stokes' theorem to surfaces that are one-to-one images of multiply connected regions.

Stokes' theorem to such surfaces we use exactly the same type of argument as in the proof of Stokes' theorem, except that we employ Green's theorem for multiply connected regions (Theorem 11.12). In place of the line integral which appears in Equation (12.27) we need a sum of line integrals, with appropriate signs, taken over the images of the curves forming the boundary of T. For example, if T has two holes, as in Figure 12.14, and if the boundary curves I', Γ_1, and Γ_2 are traversed in the directions shown, the identity in Stokes' theorem takes the form

$$\iint_S (\text{curl}\, \boldsymbol{F}) \cdot \boldsymbol{n}\, dS = \oint_C \boldsymbol{F} \cdot d\boldsymbol{\rho} + \oint_{C_1} \boldsymbol{F} \cdot d\boldsymbol{\rho}_1 + \oint_{C_2} \boldsymbol{F} \cdot d\boldsymbol{\rho}_2,$$

where C, C_1, and C_2 are the images of Γ, Γ_1, and Γ_2, respectively, and $\boldsymbol{\rho}$, $\boldsymbol{\rho}_1$, and $\boldsymbol{\rho}_2$ are the composite functions $\boldsymbol{\rho}(t) = \boldsymbol{r}[\boldsymbol{\gamma}(t)]$, $\boldsymbol{\rho}_1(t) = \boldsymbol{r}[\boldsymbol{\gamma}_1(t)]$, $\boldsymbol{\rho}_2(t) = \boldsymbol{r}[\boldsymbol{\gamma}_2(t)]$. Here y, $\boldsymbol{\gamma}_1$, and $\boldsymbol{\gamma}_2$ are the functions that describe Γ, Γ_1, and Γ_2 in the directions shown. The curves C, C_1, and C_2 will be traversed in the directions inherited from Γ, Γ_1, and Γ_2 through the mapping function r.

Stokes' theorem can also be extended to some (but not all) smooth surfaces that are not simple. We shall illustrate a few of the possibilities with examples.

Consider first the cylinder shown in Figure 12.15. This is the union of two simple smooth parametric surfaces S_1 and S_2, the images of two adjacent rectangles T_1 and T_2, under mappings $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, respectively. If $\boldsymbol{\gamma}_1$ describes the positively oriented boundary Γ_1 of T_1 and $\boldsymbol{\gamma}_2$ describes the positively oriented boundary Γ_2 of T_2, the functions $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ defined by

$$\boldsymbol{\rho}_1(t) = \boldsymbol{r}_1[\boldsymbol{\gamma}_1(t)], \qquad \boldsymbol{\rho}_2(t) = \boldsymbol{r}_2[\boldsymbol{\gamma}_2(t)]$$

describe the images C_1 and C_2 of Γ_1 and Γ_2, respectively. In this example the representations $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ can be chosen so that they agree on the intersection $\Gamma_1 \cap \Gamma_2$. If we apply

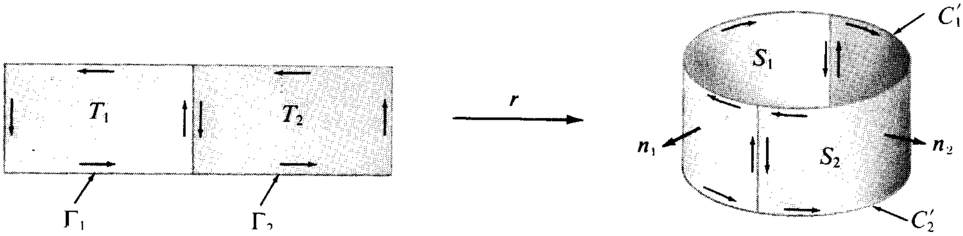

FIGURE 12.15 Extension of Stokes' theorem to a cylinder.

Stokes' theorem to each piece S_1 and S_2 and add the two identities, we obtain

$$\text{(12.51)} \qquad \iint_{S_1} (\operatorname{curl} \boldsymbol{F}) \cdot \boldsymbol{n}_1 \, dS + \iint_{S_2} (\operatorname{curl} \boldsymbol{F}) \cdot \boldsymbol{n}_2 \, dS = \int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{\rho}_1 + \int_{C_2} \boldsymbol{F} \cdot d\boldsymbol{\rho}_2,$$

where $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ are the normals determined by the fundamental vector products of $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, respectively.

Now let $\boldsymbol{r}$ denote the mapping of $T_1 \cup T_2$ which agrees with $\boldsymbol{r}_1$ on T_1 and with $\boldsymbol{r}_2$ on T_2, and let $\boldsymbol{n}$ be the corresponding unit normal determined by the fundamental vector product of r. Since the normals $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ agree in direction on $S_1 \cap S_2$, the unit normal $\boldsymbol{n}$ is the same as $\boldsymbol{n}_1$ on S_1 and the same as $\boldsymbol{n}_2$ on S,. Therefore the sum of the surface integrals on the left of (12.51) is equal to

$$\iint_{S_1 \cup S_2} (\operatorname{curl} \boldsymbol{F}) \cdot \boldsymbol{n} \, dS .$$

For this example, the representations $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ can be chosen so that $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ determine *opposite* directions on each arc of the intersection $C_1 \cap C_2$, as indicated by the arrows in Figure 12.15. The two line integrals on the right of (12.51) can be replaced by a sum of line integrals along the two circles C_1' and C_2' forming the upper and lower edges of $S_1 \cup S_2$, since the line integrals along each arc of the intersection $C_1 \cap C_2$ cancel. Therefore, Equation (12.51) can be written as

$$\text{(12.52)} \qquad \iint_{S_1 \cup S_2} (\operatorname{curl} \boldsymbol{F}) \cdot \boldsymbol{n} \, dS = \int_{C_1'} \boldsymbol{F} \cdot d\boldsymbol{\rho}_1 + \int_{C_2'} \boldsymbol{F} \cdot d\boldsymbol{\rho}_2,$$

where the line integrals are traversed in the directions inherited from Γ_1 and Γ_2. The two circles C_1' and C_2' are said to form the complete boundary of $S_1 \cup S_2$. Equation (12.52) expresses the surface integral of (curl $\boldsymbol{F}$) $\cdot$ $\boldsymbol{n}$ over $S_1 \cup S_2$ as a line integral over the complete boundary of $S_1 \cup S_2$. This equation is the extension of Stokes' theorem for a cylinder.

Suppose now we apply the same concepts to the surface shown in Figure 12.16. This surface is again the union of two smooth simple parametric surfaces S_1 and S_2, the images of two adjacent rectangles T_1 and T_2. This particular surface is called a *Möbius band*;† a

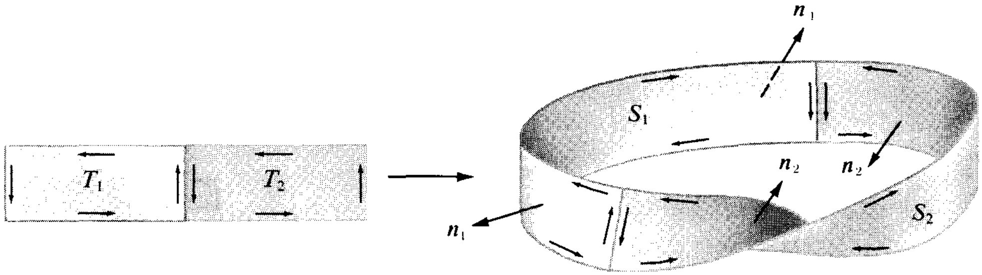

FIGURE 12.16 A Möbius band considered as the union of two simple parametric surfaces. Stokes' theorem does not extend to a Möbius band.

model can easily be constructed from a long rectangular strip of paper by giving one end a half-twist and then fastening the two ends together. We define $\boldsymbol{\rho}_1$, $\boldsymbol{\rho}_2$, C_1, and C_2 for the Möbius band as we defined them for the cylinder above. The edge of $S_1 \cup S_2$ in this case is one simple closed curve C', rather than two. This curve is called the complete boundary of the Mbbius band.

If we apply Stokes' theorem to each piece S_1 and S_2, as we did for the cylinder, we obtain Equation (12.51). But if we try to consolidate the two surface integrals and the two line integrals as we did above, we encounter two difficulties. First, the two normals $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ do not agree in direction everywhere on the intersection $C_1 \cap C_2$. (See Figure 12.16.) Therefore we cannot define a normal n for the whole surface by taking $\boldsymbol{n} = \boldsymbol{n}_1$ on S_1 and $\boldsymbol{n} = \boldsymbol{n}_2$ on S_2, as we did for the cylinder. This is not serious, however, because we can define $\boldsymbol{n}$ to be $\boldsymbol{n}_1$ on S_1 and on $C_1 \cap C_2$, and then define n to be $\boldsymbol{n}_2$ everywhere else. This gives a discontinuous normal, but the discontinuities so introduced form a set of content zero in the uv-plane and do not affect the existence or the value of the surface integral

$$\iint_{S_1 \cup S_2} (\text{curl } \boldsymbol{F}) \cdot \boldsymbol{n} \, dS .$$

A more serious difficulty is encountered when we try to consolidate the line integrals. In this example it is not possible to choose the mappings $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ in such a way that $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ determine opposite directions on each arc of the intersection $C_1 \cap C_2$. This is illustrated by the arrows in Figure 12.16; one of these arcs is traced twice in the same direction. On this arc the corresponding line integrals will not necessarily cancel as they

† After A. F. Möbius (1790–1868), a pupil of Gauss. At the age of 26 he was appointed professor of astronomy at Leipzig, a position he held until his death. He made many contributions to celestial mechanics, but his most important researches were in geometry and in the theory of numbers.

did for the cylinder. Therefore the sum of the line integrals in (12.51) is not necessarily equal to the line integral over the complete boundary of $S_1 \cup S_2$, and Stokes' theorem cannot be extended to the Möbius band.

> ***Note:*** The cylinder and the Mobius band are examples of ***orientable*** and ***nonorientable*** surfaces, respectively. We shall not attempt to define these terms precisely, but shall mention some of their differences. For an orientable surface $S_1 \cup S_2$ formed from two smooth simple parametric surfaces as described above, the mappings $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ can always be chosen so that $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ determine opposite directions on each arc of the intersection $C_1 \cap C_2$. For a nonorientable surface no such choice is possible. For a smooth orientable surface a unit normal vector can be defined in a continuous fashion over the entire surface. For a nonorientable surface no such definition of a normal is possible. A paper model of an orientable surface always has two sides that can be distinguished by painting them with two different colors. Nonorientable surfaces have only one side. For a rigorous discussion of these and other properties of orientable and nonorientable surfaces, see any book on combinatorial topology. Stokes' theorem can be extended to orientable surfaces by a procedure similar to that outlined above for the cylinder.

Another orientable surface is the sphere shown in Figure 12.17. This surface is the union of two simple parametric surfaces (hemispheres) S_1 and S_2, which we may consider

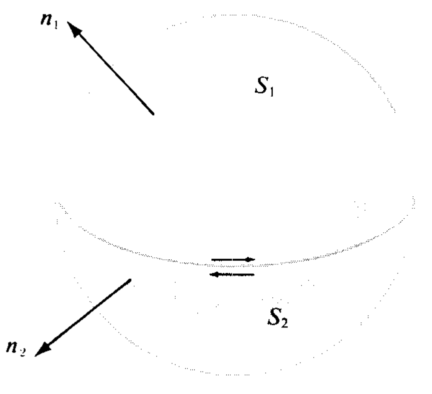

FIGURE 12.17 Extension of Stokes' theorem to a sphere.

images of a circular disk in the xy-plane under mappings $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, respectively. We give $\boldsymbol{r}$, $\boldsymbol{\rho}_1$, $\boldsymbol{\rho}_2$, C_1, C_2 the same meanings as in the above examples. In this case the curves C_1 and C_2 are completely matched by the mapping $\boldsymbol{r}$ (they intersect along the equator), and the surface $S_1 \cup S_2$ is said to be ***closed.*** Moreover, $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ can be chosen so that the directions determined by $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ are opposite on C_1 and C_2, as suggested by the arrows in Figure 12.17. (This is why $S_1 \cup S_2$ is orientable.) If we apply Stokes' theorem to each hemisphere and add the results we obtain Equation (12.51) as before. The normals $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ agree on the intersection $C_1 \cap C_2$, and we can consolidate the integrals over S_1 and S_2 into one integral over the whole sphere. The two line integrals on the right of (12.51) cancel completely, leaving us with the formula

$$\iint_{S_1 \cup S_2} (\operatorname{curl} \boldsymbol{F}) \cdot \boldsymbol{n}\, dS = 0.$$

This holds not only for a sphere, but for any orientable closed surface.