If V is a finite-dimensional (real or complex) vector space and if $(\cdot|\cdot)$ is an inner product on V, there is an associated class of positive linear operators on V. Via (9-10) there is a one-one correspondence between that class of positive operators and the collection of all positive forms on V. We shall use the exercises for this section to emphasize the relationships between positive operators, positive forms, and positive matrices. The following summary may be helpful.

If A is an $n \times n$ matrix over the field of complex numbers, the following are equivalent.

(1) A is positive, i.e., $\sum_j \sum_k A_{kj}x_j\bar{x}_k > 0$ whenever $x_1, \ldots, x_n$ are complex numbers, not all 0.

(2) $(X|Y) = Y^*AX$ is an inner product on the space of $n \times 1$ complex matrices.

(3) Relative to the standard inner product $(X|Y) = Y^*X$ on $n \times 1$ matrices, the linear operator $X \rightarrow AX$ is positive.

(4) $A = P^*P$ for some invertible $n \times n$ matrix P over C.

(5) $A = A^*$, and the principal minors of A are positive.

If each entry of A is real, these are equivalent to:

(6) $A = A^t$, and $\sum_j \sum_k A_{kj}x_jx_k > 0$ whenever $x_1, \ldots, x_n$ are real numbers not all 0.

(7) $(X|Y) = Y^tAX$ is an inner product on the space of $n \times 1$ real matrices.

(8) Relative to the standard inner product $(X|Y) = Y^tX$ on $n \times 1$ real matrices, the linear operator $X \rightarrow AX$ is positive.

(9) There is an invertible $n \times n$ matrix P, with real entries, such that $A = P^tP$.

Exercises

1. Let V be C^2, with the standard inner product. For which vectors α in V is there a positive linear operator T such that $\alpha = T\epsilon_1$?

2. Let V be R^2, with the standard inner product. If θ is a real number, let T be the linear operator 'rotation through θ,'

$$T_\theta(x_1, x_2) = (x_1 \cos\theta - x_2 \sin\theta, x_1 \sin\theta + x_2 \cos\theta).$$

For which values of θ is T_θ a positive operator?

3. Let V be the space of $n \times 1$ matrices over C, with the inner product $(X|Y) = Y^*GX$ (where G is an $n \times n$ matrix such that this is an inner product). Let A be an $n \times n$ matrix and T the linear operator $T(X) = AX$. Find T^*. If Y is a fixed element of V, find the element Z of V which determines the linear functional $X \rightarrow Y^*X$. In other words, find Z such that $Y^*X = (X|Z)$ for all X in V.

4. Let V be a finite-dimensional inner product space. If T and U are positive linear operators on V, prove that $(T + U)$ is positive. Give an example which shows that TU need not be positive.

5. Let

$$A = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix}.$$

(a) Show that A is positive.

(b) Let V be the space of 2×1 real matrices, with the inner product $(X|Y) = Y^tAX$. Find an orthonormal basis for V, by applying the Gram-Schmidt process to the basis $\{X_1, X_2\}$ defined by

$$X_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

(c) Find an invertible 2×2 real matrix P such that $A = P^tP$.

6. Which of the following matrices are positive?

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1+i \\ 1-i & 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 1 \\ 3 & -1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}.$$

7. Give an example of an $n \times n$ matrix which has all its principal minors positive, but which is not a positive matrix.

8. Does $((x_1, x_2)|(y_1, y_2)) = x_1\bar{y}_1 + 2x_2\bar{y}_1 + 2x_1\bar{y}_2 + x_2\bar{y}_2$ define an inner product on C^2?

9. Prove that every entry on the main diagonal of a positive matrix is positive.

10. Let V be a finite-dimensional inner product space. If T and U are linear operators on V, we write $T < U$ if $U - T$ is a positive operator. Prove the following:

(a) $T < U$ and $U < T$ is impossible.
(b) If $T < U$ and $U < S$, then $T < S$.
(c) If $T < U$ and $0 < S$, it need not be that $ST < SU$.

11. Let V be a finite-dimensional inner product space and E the orthogonal projection of V onto some subspace.

(a) Prove that, for any positive number c, the operator $cI + E$ is positive.
(b) Express in terms of E a self-adjoint linear operator T such that $T^2 = I + E$.

12. Let n be a positive integer and A the $n \times n$ matrix

$$A = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+1} \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{2n-1} \end{bmatrix}.$$

Prove that A is positive.

13. Let A be a self-adjoint $n \times n$ matrix. Prove that there is a real number c such that the matrix $cI + A$ is positive.

14. Prove that the product of two positive linear operators is positive if and only if they commute.

15. Let S and T be positive operators. Prove that every characteristic value of ST is positive.

9.4. More on Forms

This section contains two results which give more detailed information about (sesqui-linear) forms.

Theorem 7. *Let* f *be a form on a real or complex vector space* V *and* $\{\alpha_1, \ldots, \alpha_r\}$ *a basis for the finite-dimensional subspace* W *of* V. *Let* M *be the* r $\times$ r *matrix with entries*

$$M_{jk} = f(\alpha_k, \alpha_j)$$

and W′ *the set of all vectors* β *in* V *such that* $f(\alpha, \beta) = 0$ *for all* α *in* W. *Then* W′ *is a subspace of* V, *and* $W \cap W' = \{0\}$ *if and only if* M *is invertible. When this is the case,* $V = W + W'$.

Proof. If β and γ are vectors in W' and c is a scalar, then for every α in W

$$\begin{aligned} f(\alpha, c\beta + \gamma) &= \bar{c}f(\alpha, \beta) + f(\alpha, \gamma) \\ &= 0. \end{aligned}$$

Hence, W' is a subspace of V.

Now suppose $\alpha = \sum_{k=1}^{r} x_k\alpha_k$ and that $\beta = \sum_{j=1}^{r} y_j\alpha_j$. Then

$$\begin{aligned} f(\alpha, \beta) &= \sum_{j,k} \bar{y}_j M_{jk} x_k \\ &= \sum_k \Big(\sum_j \bar{y}_j M_{jk}\Big) x_k. \end{aligned}$$

It follows from this that $W \cap W' \neq \{0\}$ if and only if the homogeneous system

$$\sum_{j=1}^{r} \bar{y}_j M_{jk} = 0, \qquad 1 \leq k \leq r$$

has a non-trivial solution $(y_1, \ldots, y_r)$. Hence $W \cap W' = \{0\}$ if and only if M^* is invertible. But the invertibility of M^* is equivalent to the invertibility of M.

Suppose that M is invertible and let

$$A = (M^*)^{-1} = (M^{-1})^*.$$

Define g_j on V by the equation

$$g_j(\beta) = \sum_{k=1}^{r} A_{jk}\overline{f(\alpha_k, \beta)}.$$

Then

$$\begin{aligned} g_j(c\beta + \gamma) &= \sum_k A_{jk}\overline{f(\alpha_k, c\beta + \gamma)} \\ &= c\sum_k A_{jk}f(\alpha_k, \beta) + \sum_k A_{jk}f(\alpha_k, \gamma) \\ &= cg_j(\beta) + g_j(\gamma). \end{aligned}$$

Hence, each g_j is a linear function on V. Thus we may define a linear operator E on V by setting

$$E\beta = \sum_{j=1}^{r} g_j(\beta)\alpha_j.$$

Since

$$\begin{aligned} g_j(\alpha_n) &= \sum_k A_{jk}\overline{f(\alpha_k, \alpha_n)} \\ &= \sum_k A_{jk}(M^*)_{kn} \\ &= \delta_{jn} \end{aligned}$$

it follows that $E(\alpha_n) = \alpha_n$ for $1 \leq n \leq r$. This implies $E\alpha = \alpha$ for every α in W. Therefore, E maps V onto W and $E^2 = E$. If β is an arbitrary vector in V, then

$$\begin{aligned} f(\alpha_n, E\beta) &= f\left(\alpha_n, \sum_j g_j(\beta)\alpha_j\right) \\ &= \sum_j \overline{g_j(\beta)}f(\alpha_n, \alpha_j) \\ &= \sum_j \left(\sum_k \overline{A}_{jk}f(\alpha_k, \beta)\right)f(\alpha_n, \alpha_j). \end{aligned}$$

Since $A^* = M^{-1}$, it follows that

$$\begin{aligned} f(\alpha_n, E\beta) &= \sum_k \left(\sum_j (M^{-1})_{kj}M_{jn}\right)f(\alpha_k, \beta) \\ &= \sum_k \delta_{kn}f(\alpha_k, \beta) \\ &= f(\alpha_n, \beta). \end{aligned}$$

This implies $f(\alpha, E\beta) = f(\alpha, \beta)$ for every α in W. Hence

$$f(\alpha, \beta - E\beta) = 0$$

for all α in W and β in V. Thus $I - E$ maps V into W'. The equation

$$\beta = E\beta + (I - E)\beta$$

shows that $V = W + W'$. One final point should be mentioned. Since $W \cap W' = \{0\}$, every vector in V is uniquely the sum of a vector in W

and a vector in W'. If β is in W', it follows that $E\beta = 0$. Hence $I - E$ maps V onto W'. ▌

The projection E constructed in the proof may be characterized as follows: $E\beta = \alpha$ if and only if α is in W and $\beta - \alpha$ belongs to W'. Thus E is independent of the basis of W that was used in its construction. Hence we may refer to E as the **projection of** V **on** W that is **determined by** the direct sum decomposition

$$V = W \oplus W'.$$

Note that E is an orthogonal projection if and only if $W' = W^{\perp}$.

Theorem 8. *Let* f *be a form on a real or complex vector space* V *and* A *the matrix of* f *in the ordered basis* $\{\alpha_1, \ldots, \alpha_n\}$ *of* V. *Suppose the principal minors of* A *are all different from* 0. *Then there is a unique upper-triangular matrix* P *with* $P_{kk} = 1$ $(1 \leq k \leq n)$ *such that*

$$P^*AP$$

is upper-triangular.

Proof. Since $\Delta_k(A^*) = \overline{\Delta_k(A)}$ $(1 \leq k \leq n)$, the principal minors of A^* are all different from 0. Hence, by the lemma used in the proof of Theorem 6, there exists an upper-triangular matrix P with $P_{kk} = 1$ such that A^*P is lower-triangular. Therefore, $P^*A = (A^*P)^*$ is upper-triangular. Since the product of two upper-triangular matrices is again upper-triangular, it follows that P^*AP is upper-triangular. This shows the existence but not the uniqueness of P. However, there is another more geometric argument which may be used to prove both the existence and uniqueness of P.

Let W_k be the subspace spanned by $\alpha_1, \ldots, \alpha_k$ and W_k' the set of all β in V such that $f(\alpha, \beta) = 0$ for every α in W_k. Since $\Delta_k(A) \neq 0$, the $k \times k$ matrix M with entries

$$M_{ij} = f(\alpha_j, \alpha_i) = A_{ij}$$

$(1 \leq i, j \leq k)$ is invertible. By Theorem 7

$$V = W_k \oplus W_k'.$$

Let E_k be the projection of V on W_k which is determined by this decomposition, and set $E_0 = 0$. Let

$$\beta_k = \alpha_k - E_{k-1}\alpha_k, \qquad (1 \leq k \leq n).$$

Then $\beta_1 = \alpha_1$, and $E_{k-1}\alpha_k$ belongs to W_{k-1} for $k > 1$. Thus when $k > 1$, there exist unique scalars P_{jk} such that

$$E_{k-1}\alpha_k = -\sum_{j=1}^{k-1} P_{jk}\alpha_j.$$

Setting $P_{kk} = 1$ and $P_{jk} = 0$ for $j > k$, we then have an $n \times n$ upper-triangular matrix P with $P_{kk} = 1$ and

$$\beta_k = \sum_{j=1}^{k} P_{jk}\alpha_j$$

for $k = 1, \ldots, n$. Suppose $1 \leq i < k$. Then β_i is in W_i and $W_i \subset W_{k-1}$. Since β_k belongs to W'_{k-1}, it follows that $f(\beta_i, \beta_k) = 0$. Let B denote the matrix of f in the ordered basis $\{\beta_1, \ldots, \beta_n\}$. Then

$$B_{ki} = f(\beta_i, \beta_k)$$

so $B_{ki} = 0$ when $k > i$. Thus B is upper-triangular. On the other hand,

$$B = P^*AP.$$

Conversely, suppose P is an upper-triangular matrix with $P_{kk} = 1$ such that P^*AP is upper-triangular. Set

$$\beta_k = \sum_j P_{jk}\alpha_j, \qquad (1 \leq k \leq n).$$

Then $\{\beta_1, \ldots, \beta_k\}$ is evidently a basis for W_k. Suppose $k > 1$. Then $\{\beta_1, \ldots, \beta_{k-1}\}$ is a basis for W_{k-1}, and since $f(\beta_i, \beta_k) = 0$ when $i < k$, we see that β_k is a vector in W'_{k-1}. The equation defining β_k implies

$$\alpha_k = -\left(\sum_{j=1}^{k-1} P_{jk}\alpha_j\right) + \beta_k.$$

Now $\sum_{j=1}^{k-1} P_{jk}\alpha_j$ belongs to W_{k-1} and β_k is in W'_{k-1}. Therefore, $P_{1k}, \ldots, P_{k-1k}$ are the unique scalars such that

$$E_{k-1}\alpha_k = -\sum_{j=1}^{k-1} P_{jk}\alpha_j$$

so that P is the matrix constructed earlier. ∎

9.5. Spectral Theory

In this section, we pursue the implications of Theorems 18 and 22 of Chapter 8 concerning the diagonalization of self-adjoint and normal operators.

Theorem 9 (Spectral Theorem). *Let* T *be a normal operator on a finite-dimensional complex inner product space* V *or a self-adjoint operator on a finite-dimensional real inner product space* V. *Let* $c_1, \ldots, c_k$ *be the distinct characteristic values of* T. *Let* W_j *be the characteristic space associated with* c_j *and* E_j *the orthogonal projection of* V *on* W_j. *Then* W_j *is orthogonal to* W_i *when* $i \neq j$, V *is the direct sum of* $W_1, \ldots, W_k$, *and*

$$T = c_1E_1 + \cdots + c_kE_k. \tag{9-11}$$