argues that A has some divisor B, because A is composite. If B is prime we are done; if not, B has a divisor C, and so on. He then claims

> Thus, if the investigation be continued in this way, some prime number will be found which will measure the number before it, which will also measure A.

And his punchline is an appeal to descent:

> For, if it is not found, an infinite series of numbers will measure the number A, each of which is less than the other: which is impossible in numbers.

Euclid also assumes termination of the Euclidean algorithm without comment throughout the *Elements*. It *is* obvious, of course, but hardly more obvious than the existence of a prime divisor. Evidently Euclid was only fleetingly aware of the importance of induction; nevertheless it is to his credit that he noticed it at least once.

Most mathematicians failed to notice descent until around 1640, when Fermat began to announce spectacular new results in number theory and claim they were due to a "method of infinite descent." His most famous proof, and in fact the only one he disclosed, shows that there are no natural numbers a, b, and c such that $a^4 + b^4 = c^2$. He assumed, on the contrary, that there is a solution $a = x_1$, $b = y_1$, $c = z_1$, and showed how to descend to a *smaller* solution $a = x_2$, $b = y_2$, $c = z_2$. By descending indefinitely in this way, one obtains a contradiction that proves the desired result "by infinite descent." The details may be seen in Section 4.7*.

This proof made mathematicians conscious of descent for the first time and hinted at its power. At the same time, unfortunately, the simple logical principle of descent was buried under the technical problem of finding the descent step. Mathematicians continued to use descent until the late 19th century without realizing that an important principle was involved. The Gauss (1801) proof of the prime divisor property (Exercises 1.6.4 and 1.6.5) is a simpler example.

Ascent was likewise used for a long time without the importance of the induction step being noticed. Mathematicians naturally tried to make proofs as simple as possible, so ascent proofs were organized to make the induction step trivial, and hence not worth mentioning. A brilliant example is Euclid's summation of the geometric series (*Elements*, Book IX, Proposition 35). It was also easier to *discover* results in these circumstances. And as long as it was possible to play down the induction step, it was possible to overlook the underlying principle of induction.

The induction step ultimately came to light not in number theory but in combinatorics, where complicated inductions perhaps arise more naturally. The first really precise formulation of induction is by Blaise Pascal (1654), who clearly used the "base step, induction step" format to prove the basic properties of Pascal's triangle.

Understanding did not advance much between 1654 and 1861. Ascending and descending forms of induction were both occasionally used, but without recognition of their importance, or even their equivalence. Certainly, one would not think the time was ripe for a high school teacher to write a textbook using mathematical induction as the *sole basis* of arithmetic! Enter Hermann Grassmann. His *Lehrbuch der Arithmetik für höhere Lehranstalten* (textbook of arithmetic for higher instruction) contains the fundamental idea that everyone else had missed: *the whole of arithmetic follows from the process of succession*. As we explained in Section 1.9*, he did this by using induction to define $+$ and $\times$ from the successor function, and hence prove the ring properties of $\mathbb{Z}$.

But, sadly, Grassmann was a generation ahead of his time. His work fell into obscurity so fast that even like-minded mathematicians of the 1880s and 1890s were unaware of it. Dedekind (1888) rediscovered the inductive definitions of $+$ and $\times$ in terms of the successor function and decided to dig deeper, to explain the nature of *succession* itself. As we asked before Exercise 1.9.5: in the expression $1, 2, 3, \ldots$, what does $\ldots$ mean? It is not enough to say "the remaining values of the successor function $f(n) = n + 1$," because $f(n)$ is also defined on the numbers $n = m + 1/2$ for integers m, and we do not intend these values to be included among the successors of 1. The crux of the problem of defining succession is to exclude such "alien intruders," as Dedekind called them.

As indicated in Section 1.9*, Dedekind's solution makes crucial use of set theory. His discoveries were in fact very influential in the development of logic and set theory in the 20th century. Also, his (and Grassmann's) method of definition by induction led to the theory of recursive functions, and ultimately to computer programming and computer science. This is a surprising twist to a basically philosophical investigation, but mathematics often seems to find its way into the real world, without being asked.

2 CHAPTER Geometry

2.1 Geometric Intuition

Geometry is in many ways opposite or complementary to arithmetic. Arithmetic is discrete, static, computational, and logical; geometry is continuous, fluid, dynamic, and visual. The fundamental geometric quantities (length, area, and volume) are familiar to everyone but hard to define. And some "obvious" geometric facts are not even provable; they can be taken as axioms, but so can their opposites. In geometry, intuition runs ahead of logic. Our imagination leads us to conclusions via steps that "look right" but may not have a purely logical basis. A good example is the Pythagorean theorem, that the square on the hypotenuse of a right-angled triangle equals (in area) the sum of the squares on the other two sides. This theorem has been known since ancient times; was probably first noticed by someone playing with squares and triangles, perhaps as in Figure 2.1.

The picture on the left shows a big square, minus four copies of the triangle, equal in area to the squares on the two sides. The second picture shows that the big square minus four copies of the triangle also equals the square on the hypotenuse. Q.E.D.

This is a wonderful discovery (and it gets better, as we shall see later), but what is it really about? In the physical world, exact