path reaching $y \in T$ extends via $M$ to a vertex of $S$. Hence these edges of $M$ yield a bijection from $T$ to $S - \{u\}$, and we have $|T| = |S - \{u\}|$.

The matching between $T$ and $S - \{u\}$ yields $T \subseteq N(S)$. In fact, $T = N(S)$. Suppose that $y \in Y - T$ has a neighbor $v \in S$. The edge $vy$ cannot be in $M$, since $u$ is unsaturated and the rest of $S$ is matched to $T$ by $M$. Thus adding $vy$ to an $M$-alternating path reaching $v$ yields an $M$-alternating path to $y$. This contradicts $y \notin T$, and hence $vy$ cannot exist.

With $T = N(S)$, we have proved that $|N(S)| = |T| = |S| - 1 < |S|$ for this choice of $S$. This completes the proof of the contrapositive. ∎

One can also prove sufficiency by assuming Hall's Condition, supposing that no matching saturates $X$, and obtaining a contradiction. As we have seen, lack of a matching saturating $X$ yields a violation of Hall's Condition. Contradicting the hypothesis usually means that the contrapositive of the desired implication has been proved. Thus we have stated the proof in that language.

**3.1.12. Remark.** Theorem 3.1.11 implies that whenever an $X, Y$-bigraph has no matching saturating $X$, we can verify this by exhibiting a subset of $X$ with too few neighbors.

Note also that the statement and proof permit multiple edges. ∎

Many proofs of Hall's Theorem have been published; see Mirsky [1971, p38] and Jacobs [1969] for summaries. A proof by M. Hall [1948] leads to a lower bound on the number of matchings that saturate $X$, as a function of the vertex degrees. We consider algorithmic aspects in Section 3.2.

When the sets of the bipartition have the same size, Hall's Theorem is the **Marriage Theorem**, proved originally by Frobenius [1917]. The name arises from the setting of the compatibility relation between a set of $n$ men and a set of $n$ women. If every man is compatible with $k$ women and every woman is compatible with $k$ men, then a perfect matching must exist. Again multiple edges are allowed, which enlarges the scope of applications (see Theorem 3.3.9 and Theorem 7.1.7, for example).

**3.1.13. Corollary.** For $k > 0$, every $k$-regular bipartite graph has a perfect matching.

**Proof:** Let $G$ be a $k$-regular $X, Y$-bigraph. Counting the edges by endpoints in $X$ and by endpoints in $Y$ shows that $k\,|X| = k\,|Y|$, so $|X| = |Y|$. Hence it suffices to verify Hall's Condition; a matching that saturates $X$ will also saturate $Y$ and be a perfect matching.

Consider $S \subseteq X$. Let $m$ be the number of edges from $S$ to $N(S)$. Since $G$ is $k$-regular, $m = k\,|S|$. These $m$ edges are incident to $N(S)$, so $m \le k\,|N(S)|$. Hence $k\,|S| \le k\,|N(S)|$, which yields $|N(S)| \ge |S|$ when $k > 0$. Having chosen $S \subseteq X$ arbitrarily, we have established Hall's condition. ∎

One can also use contradiction here. Assuming that $G$ has no perfect matching yields a set $S \subseteq X$ such that $|N(S)| < |S|$. The argument obtaining a contradiction amounts to a rewording of the direct proof given above.

## MIN-MAX THEOREMS

When a graph $G$ does not have a perfect matching, Theorem 3.1.10 allows us to prove that $M$ is a maximum matching by proving that $G$ has no $M$-augmenting path. Exploring all $M$-alternating paths to eliminate the possibility of augmentation could take a long time.

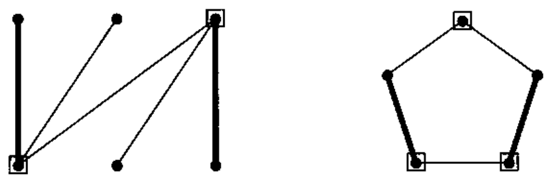
We faced a similar situation when proving that a graph is not bipartite. Instead of checking all possible bipartitions, we can exhibit an odd cycle. Here again, instead of exploring all $M$-alternating paths, we would prefer to exhibit an explicit structure in $G$ that forbids a matching larger than $M$.

**3.1.14. Definition.** A **vertex cover** of a graph $G$ is a set $Q \subseteq V(G)$ that contains at least one endpoint of every edge. The vertices in $Q$ *cover* $E(G)$.

In a graph that represents a road network (with straight roads and no isolated vertices), we can interpret the problem of finding a minimum vertex cover as the problem of placing the minimum number of policemen to guard the entire road network. Thus "cover" means "watch" in this context.

Since no vertex can cover two edges of a matching, the size of every vertex cover is at least the size of every matching. Therefore, obtaining a matching and a vertex cover of the same size PROVES that each is optimal. Such proofs exist for bipartite graphs, but not for all graphs.

**3.1.15. Example.** *Matchings and vertex covers.* In the graph on the left below we mark a vertex cover of size 2 and show a matching of size 2 in bold. The vertex cover of size 2 prohibits matchings with more than 2 edges, and the matching of size 2 prohibits vertex covers with fewer than 2 vertices. As illustrated on the right, the optimal values differ by 1 for an odd cycle. The difference can be arbitrarily large (Exercise 3.3.10). ∎



**3.1.16. Theorem.** (König [1931], Egerváry [1931]) If $G$ is a bipartite graph, then the maximum size of a matching in $G$ equals the minimum size of a vertex cover of $G$.

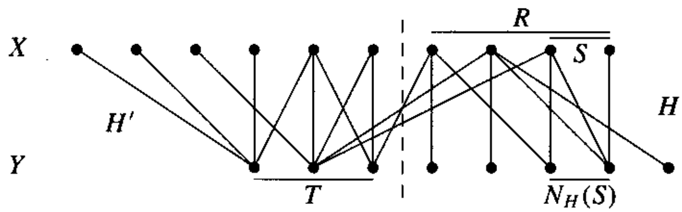
**Proof:** Let $G$ be an $X,Y$-bigraph. Since distinct vertices must be used to cover the edges of a matching, $|Q| \ge |M|$ whenever $Q$ is a vertex cover and $M$ is a matching in $G$. Given a smallest vertex cover $Q$ of $G$, we construct a matching of size $|Q|$ to prove that equality can always be achieved.

Partition $Q$ by letting $R = Q \cap X$ and $T = Q \cap Y$. Let $H$ and $H'$ be the subgraphs of $G$ induced by $R \cup (Y - T)$ and $T \cup (X - R)$, respectively. We use

Hall's Theorem to show that $H$ has a matching that saturates $R$ into $Y - T$ and $H'$ has a matching that saturates $T$. Since $H$ and $H'$ are disjoint, the two matchings together form a matching of size $|Q|$ in $G$.

Since $R \cup T$ is a vertex cover, $G$ has no edge from $Y - T$ to $X - R$. For each $S \subseteq R$, we consider $N_H(S)$, which is contained in $Y - T$. If $|N_H(S)| < |S|$, then we can substitute $N_H(S)$ for $S$ in $Q$ to obtain a smaller vertex cover, since $N_H(S)$ covers all edges incident to $S$ that are not covered by $T$.

The minimality of $Q$ thus yields Hall's Condition in $H$, and hence $H$ has a matching that saturates $R$. Applying the same argument to $H'$ yields the matching that saturates $T$. ■




As graph theory continues to develop, new proofs of fundamental results like the König–Egerváry Theorem appear; see Rizzo [2000].

**3.1.17. Remark.** A **min-max relation** is a theorem stating equality between the answers to a minimization problem and a maximization problem over a class of instances. The König–Egerváry Theorem is such a relation for vertex covering and matching in bipartite graphs.

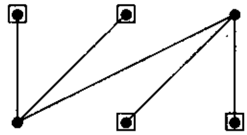
For the discussions in this text, we think of a **dual pair** of optimization problems as a maximization problem **M** and a minimization problem **N**, defined on the same instances (such as graphs), such that for every candidate solution $M$ to **M** and every candidate solution $N$ to **N**, the value of $M$ is less than or equal to the value of $N$. Often the "value" is cardinality, as above where **M** is maximum matching and **N** is minimum vertex cover.

When **M** and **N** are dual problems, obtaining candidate solutions $M$ and $N$ that have the same value PROVES that $M$ and $N$ are optimal solutions for that instance. We will see many pairs of dual problems in this book. A min-max relation states that, on some class of instances, these short proofs of optimality exist. These theorems are desirable because they save work! Our next objective is another such theorem for independent sets in bipartite graphs. ■

## INDEPENDENT SETS AND COVERS

We now turn from matchings to independent sets. The **independence number** of a graph is the maximum size of an independent set of vertices.

**3.1.18. Example.** The independence number of a bipartite graph does *not* always equal the size of a partite set. In the graph below, both partite sets have size 3, but we have marked an independent set of size 4. ■



No vertex covers two edges of a matching. Similarly, no edge contains two vertices of an independent set. This yields another dual covering problem.

**3.1.19. Definition.** An **edge cover** of $G$ is a set $L$ of edges such that every vertex of $G$ is incident to some edge of $L$.

We say that the vertices of $G$ are *covered* by the edges of $L$. In Example 3.1.18, the four edges incident to the marked vertices form an edge cover; the remaining two vertices are covered "for free".

Only graphs without isolated vertices have edge covers. A perfect matching forms an edge cover with $n(G)/2$ edges. In general, we can obtain an edge cover by adding edges to a maximum matching.

**3.1.20. Definition.** For the optimal sizes of the sets in the independence and covering problems we have defined, we use the notation below.

| | |
|---|---|
| maximum size of independent set | $\alpha(G)$ |
| maximum size of matching | $\alpha'(G)$ |
| minimum size of vertex cover | $\beta(G)$ |
| minimum size of edge cover | $\beta'(G)$ |

A graph may have many independent sets of maximum size ($C_5$ has five of them), but the independence number $\alpha(G)$ is a single integer ($\alpha(C_5) = 2$). The notation treats the numbers that answer these optimization problems as graph parameters, like the order, size, maximum degree, diameter, etc. Our use of $\alpha'(G)$ to count the edges in a maximum matching suggests a relationship with the parameter $\alpha(G)$ that counts the vertices in a maximum independent set. We explore this relationship in Section 7.1.

We use $\beta(G)$ for minimum vertex cover due to its interaction with maximum matching. The "prime" goes on $\beta'(G)$ rather than on $\beta(G)$ because $\beta(G)$ counts a set of vertices and $\beta'(G)$ counts a set of edges.

In this notation, the König–Egerváry Theorem states that $\alpha'(G) = \beta(G)$ for every bipartite graph $G$. We will prove that also $\alpha(G) = \beta'(G)$ for bipartite graphs without isolated vertices. Since no edge can cover two vertices of an independent set, the inequality $\beta'(G) \geq \alpha(G)$ is immediate. (When $S \subseteq V(G)$, we often use $\overline{S}$ to denote $V(G) - S$, the remaining vertices).

**3.1.21. Lemma.** In a graph $G$, $S \subseteq V(G)$ is an independent set if and only if $\overline{S}$ is a vertex cover, and hence $\alpha(G) + \beta(G) = n(G)$.

**Proof:** If $S$ is an independent set, then every edge is incident to at least one vertex of $\overline{S}$. Conversely, if $\overline{S}$ covers all the edges, then there are no edges joining vertices of $S$. Hence every maximum independent set is the complement of a minimum vertex cover, and $\alpha(G) + \beta(G) = n(G)$. ■

The relationship between matchings and edge coverings is more subtle. Nevertheless, a similar formula holds.

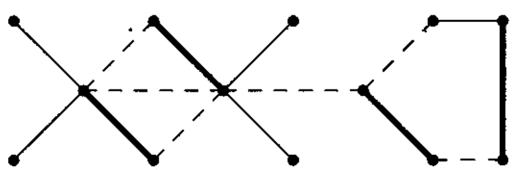
**3.1.22. Theorem.** (Gallai [1959]) If $G$ is a graph without isolated vertices, then $\alpha'(G) + \beta'(G) = n(G)$.

**Proof:** From a maximum matching $M$, we will construct an edge cover of size $n(G) - |M|$. Since a smallest edge cover is no bigger than this cover, this will imply that $\beta'(G) \le n(G) - \alpha'(G)$. Also, from a minimum edge cover $L$, we will construct a matching of size $n(G) - |L|$. Since a largest matching is no smaller than this matching, this will imply that $\alpha'(G) \ge n(G) - \beta'(G)$. These two inequalities complete the proof.

Let $M$ be a maximum matching in $G$. We construct an edge cover of $G$ by adding to $M$ one edge incident to each unsaturated vertex. We have used one edge for each vertex, except that each edge of $M$ takes care of two vertices, so the total size of this edge cover is $n(G) - |M|$, as desired.

Now let $L$ be a minimum edge cover. If both endpoints of an edge $e$ belong to edges in $L$ other than $e$, then $e \notin L$, since $L - \{e\}$ is also an edge cover. Hence each component formed by edges of $L$ has at most one vertex of degree exceeding 1 and is a star (a tree with at most one non-leaf). Let $k$ be the number of these components. Since $L$ has one edge for each non-central vertex in each star, we have $|L| = n(G) - k$. We form a matching $M$ of size $k = n(G) - |L|$ by choosing one edge from each star in $L$. ■

**3.1.23. Example.** The graph below has 13 vertices. A matching of size 4 appears in bold, and adding the solid edges yields an edge cover of size 9. The dashed edges are not needed in the cover. The edge cover consists of four stars; from each we extract one edge (bold) to form the matching. ■



**3.1.24. Corollary.** (König [1916]) If $G$ is a bipartite graph with no isolated vertices, then $\alpha(G) = \beta'(G)$.

**Proof:** By Lemma 3.1.21 and Theorem 3.1.22, $\alpha(G) + \beta(G) = \alpha'(G) + \beta'(G)$. Subtracting the König–Egerváry relation $\alpha'(G) = \beta(G)$ completes the proof. ■

## DOMINATING SETS (optional)

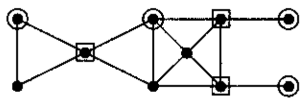
The edges covered by one vertex in a vertex cover are the edges incident to it; they form a star. The vertex cover problem can be described as covering the edge set with the fewest stars. Sometimes we instead want to cover the vertex set with fewest stars. This is equivalent to our next graph parameter.

**3.1.25. Example.** A company wants to establish transmission towers in a remote region. The towers are located at inhabited buildings, and each inhabited building must be reachable. If a transitter at $x$ can reach $y$, then also one at $y$ can reach $x$. Given the pairs that can reach each other, how many transmitters are needed to cover all the buildings?

A similar problem comes from recreational mathematics: How many queens are needed to attack all squares on a chessboard? (Exercise 56). ■

**3.1.26. Definition.** In a graph $G$, a set $S \subseteq V(G)$ is a **dominating set** if every vertex not in $S$ has a neighbor in $S$. The **domination number** $\gamma(G)$ is the minimum size of a dominating set in $G$.

**3.1.27. Example.** The graph $G$ below has a minimal dominating set of size 4 (circles) and a minimum dominating set of size 3 (squares): $\gamma(G) = 3$. ■



Berge [1962] introduced the notion of domination. Ore [1962] coined this terminology, and the notation $\gamma(G)$ appeared in an early survey (Cockayne–Hedetniemi [1977]). An entire book (Haynes–Hedetniemi–Slater [1998]) is devoted to domination and its variations.

**3.1.28. Example.** Covering the vertex set with stars may not require as many stars as covering the edge set. When a graph $G$ has no isolated vertices, every vertex cover *is* a dominating set, so $\gamma(G) \le \beta(G)$. The difference can be large; $\gamma(K_n) = 1$, but $\beta(K_n) = n - 1$. ■

When studying domination as an extremal problem, we try to obtain bounds in terms of other graph parameters, such as the order and the minimum degree. A vertex of degree $k$ dominates itself and $k$ other vertices; thus every dominating set in a $k$-regular graph $G$ has size at least $n(G)/(k+1)$. For every graph with minimum degree $k$, a greedy algorithm produces a dominating set not too much bigger than this.

**3.1.29. Definition.** The **closed neighborhood** $N[v]$ of a vertex $v$ in a graph is $N(v) \cup \{v\}$; it is the set of vertices *dominated by* $v$.

**3.1.30. Theorem.** (Arnautov [1974], Payan [1975]) Every $n$-vertex graph with minimum degree $k$ has a dominating set of size at most $n\frac{1+\ln(k+1)}{k+1}$.

**Proof:** (Alon [1990]) Let $G$ be a graph with minimum degree $k$. Given $S \subseteq V(G)$, let $U$ be the set of vertices not dominated by $S$. We claim that some vertex $y$ outside $S$ dominates at least $|U|(k+1)/n$ vertices of $U$. Each vertex in $U$ has at least $k$ neighbors, so $\sum_{v\in U}|N[v]| \ge |U|(k+1)$. Each vertex of $G$ is counted at most $n$ times by these $|U|$ sets, so some vertex $y$ appears at least $|U|(k+1)/n$ times and satisfies the claim.

We iteratively select a vertex that dominates the most of the remaining undominated vertices. We have proved that when $r$ undominated vertices remain, after the next selection at most $r(1-(k+1)/n)$ undominated vertices remain. Hence after $n\frac{\ln(k+1)}{k+1}$ steps the number of undominated vertices is at most

$$n(1-\tfrac{k+1}{n})^{n\ln(k+1)/(k+1)} < ne^{-\ln(k+1)} = \tfrac{n}{k+1}$$

The selected vertices and these remaining undominated vertices together form a dominating set of size at most $n\frac{1+\ln(k+1)}{k+1}$. ■

**3.1.31. Remark.** This bound is also proved by probabilistic methods in Theorem 8.5.10. Caro–Yuster–West [2000] showed that for large $k$ the total domination number satisfies a bound asymptotic to this. Alon [1990] used probabilistic methods to show that this bound is asymptotically sharp when $k$ is large.

Exact bounds remain of interest for small $k$. Among connected $n$-vertex graphs, $\delta(G) \ge 2$ implies $\gamma(G) \le 2n/5$ (McCuaig–Shepherd [1989], with seven small exceptions), and $\delta(G) \ge 3$ implies $\gamma(G) \le 3n/8$ (Reed [1996]). Exercise 53 requests constructions achieving these bounds. ■

Many variations on the concept of domination are studied. In Example 3.1.25, for example, one might want the transmitters to be able to communicate with each other, which requires that they induce a connected subgraph.

**3.1.32. Definition.** A dominating set $S$ in $G$ is
a **connected dominating set** if $G[S]$ is connected,
an **independent dominating set** if $G[S]$ is independent, and
a **total dominating set** if $G[S]$ has no isolated vertex.

Each variation adds a constraint, so dominating sets of these types are at least as large as $\gamma(G)$. Exercises 54–60 explore these variations. Studying independent dominating sets amounts to studying maximal independent sets. This leads to a nice result about claw-free graphs.

**3.1.33. Lemma.** A set of vertices in a graph is an independent dominating set if and only if it is a maximal independent set.

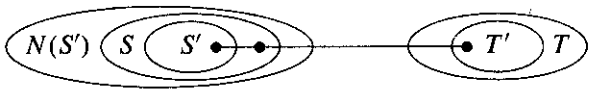
**Proof:** Among independent sets, $S$ is maximal if and only if every vertex outside $S$ has a neighbor in $S$, which is the condition for $S$ to be a dominating set. ■

**3.1.34. Theorem.** (Bollobás–Cockayne [1979]) Every claw-free graph has an independent dominating set of size $\gamma(G)$.

**Proof:** Let $S$ be a minimum dominating set in a claw-free graph $G$. Let $S'$ be a maximal independent subset of $S$. Let $T = V(G) - N(S')$. Let $T'$ be a maximal independent subset of $S$.

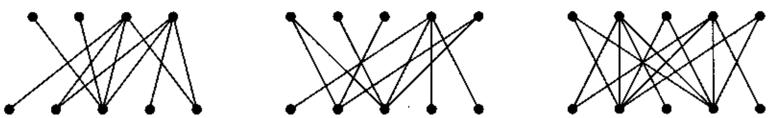
Since $T'$ contains no neighbor of $S'$, $S' \cup T'$ is independent. Since $S'$ is maximal in $S$, we have $S \subseteq N(S')$. Since $T'$ is maximal in $T$, $T'$ dominates $T$. Hence $S' \cup T'$ is a dominating set.

It remains to show that $|S' \cup T'| \le \gamma(G)$. Since $S'$ is maximal in $S$, $T'$ is independent, and $G$ is claw-free, each vertex of $S - S'$ has at most one neighbor in $T'$. Since $S$ is dominating, each vertex of $T'$ has at least one neighbor in $S - S'$. Hence $|T'| \le |S - S'|$, which yields $|S' \cup T'| \le |S| = \gamma(G)$. ■




## EXERCISES

**3.1.1.** (−) Find a maximum matching in each graph below. Prove that it is a maximum matching by exhibiting an optimal solution to the dual problem (minimum vertex cover). Explain why this proves that the matching is optimal.



**3.1.2.** (−) Determine the minimum size of a maximal matching in the cycle $C_n$.

**3.1.3.** (−) Let $S$ be the set of vertices saturated by a matching $M$ in a graph $G$. Prove that some maximum matching also saturates all of $S$. Must the statement be true for every maximum matching?

**3.1.4.** (−) For each of $\alpha$, $\alpha'$, $\beta$, $\beta'$, characterize the simple graphs for which the value of the parameter is 1.

**3.1.5.** (−) Prove that $\alpha(G) \ge \frac{n(G)}{\Delta(G)+1}$ for every graph $G$.

**3.1.6.** (−) Let $T$ be a tree with $n$ vertices, and let $k$ be the maximum size of an independent set in $T$. Determine $\alpha'(T)$ in terms of $n$ and $k$.

**3.1.7.** (−) Use Corollary 3.1.24 to prove that a graph $G$ is bipartite if and only if $\alpha(H) = \beta'(H)$ for every subgraph $H$ of $G$ with no isolated vertices.

• • • • •

**3.1.8.** (!) Prove or disprove: Every tree has at most one perfect matching.

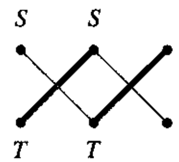
**3.1.9.** (!) Prove that every maximal matching in a graph $G$ has at least $\alpha'(G)/2$ edges.

**3.1.10.** Let $M$ and $N$ be matchings in a graph $G$, with $|M| > |N|$. Prove that there exist matchings $M'$ and $N'$ in $G$ such that $|M'| = |M| - 1$, $|N'| = |N| + 1$, and $M'$, $N'$ have the same union and intersection (as edge sets) as $M, N$.

**3.1.11.** Let $C$ and $C'$ be cycles in a graph $G$. Prove that $C \triangle C'$ decomposes into cycles.

**3.1.12.** Let $C$ and $C'$ be cycles of length $k$ in a graph with girth $k$. Prove that $C \triangle C'$ is a single cycle if and only if $C \cap C'$ is a single path. (Jiang [2001])

**3.1.13.** Let $M$ and $M'$ be matchings in an $X, Y$-bigraph $G$. Suppose that $M$ saturates $S \subseteq X$ and that $M'$ saturates $T \subseteq Y$. Prove that $G$ has a matching that saturates $S \cup T$. For example, below we show $M$ as bold edges and $M'$ as thin edges; we can saturate $S \cup T$ by using one edge from each.




**3.1.14.** Let $G$ be the Petersen graph. In Example 7.1.9, analysis by cases is used to show that if $M$ is a perfect matching in $G$, then $G - M = C_5 + C_5$. Assume this.

a) Prove that every edge of $G$ lies in four 5-cycles, and count the 5-cycles in $G$.

b) Determine the number of perfect matchings in $G$.

**3.1.15.** a) Prove that for every perfect matching $M$ in $Q_k$ and every coordinate $i \in [k]$, there are an even number of edges in $M$ whose endpoints differ in coordinate $i$.

b) Use part (a) to count the perfect matchings in $Q_3$.

**3.1.16.** For $k \geq 2$, prove that $Q_k$ has at least $2^{(2^{k-2})}$ perfect matchings.

**3.1.17.** The *weight* of a vertex in $Q_k$ is the number of 1s in its label. Prove that for every perfect matching in $Q_k$, the number of edges matching words of weight $i$ to words of weight $i+1$ is $\binom{k-1}{i}$, for $0 \leq i \leq k-1$.

**3.1.18.** (!) Two people play a game on a graph $G$, alternately choosing distinct vertices. Player 1 start by choosing any vertex. Each subsequent choice must be adjacent to the preceding choice (of the other player). Thus together they follow a path. The last player able to move wins.

Prove that the second player has a winning strategy if $G$ has a perfect matching, and otherwise the first player has a winning strategy. (Hint: For the second part, the first player should start with a vertex omitted by some maximum matching.)

**3.1.19.** (!) Let $\mathbf{A} = (A_1, \ldots, A_m)$ be a collection of subsets of a set $Y$. A **system of distinct representatives** (SDR) for $\mathbf{A}$ is a set of distinct elements $a_1, \ldots, a_m$ in $Y$ such that $a_i \in A_i$. Prove that $\mathbf{A}$ has an SDR if and only if $|\bigcup_{i \in S} A_i| \geq |S|$ for every $S \subseteq \{1, \ldots, m\}$. (Hint: Transform this to a graph problem.)

**3.1.20.** The people in a club are planning their summer vacations. Trips $t_1, \ldots, t_n$ are available, but trip $t_i$ has capacity $n_i$. Each person likes some of the trips and will travel on at most one. In terms of which people like which trips, derive a necessary and sufficient condition for being able to fill all trips (to capacity) with people who like them.

**3.1.21.** (!) Let $G$ be an $X, Y$-bigraph such that $|N(S)| > |S|$ whenever $\emptyset \neq S \subset X$. Prove that every edge of $G$ belongs to some matching that saturates $X$.

**3.1.22.** Prove that a bipartite graph $G$ has a perfect matching if and only if $|N(S)| \geq |S|$ for all $S \subseteq V(G)$, and present an infinite class of examples to prove that this characterization does not hold for all graphs.

**3.1.23.** (+) *Alternative proof of Hall's Theorem.* Consider a bipartite graph $G$ with bipartition $X, Y$, satisfying $|N(S)| \geq |S|$ for every $S \subseteq X$. Use induction on $|X|$ to prove that $G$ has a matching that saturates $X$. (Hint: First consider the case where $|N(S)| > |S|$ for every proper subset $S$ of $X$. When this does not hold, consider a minimal nonempty $T \subseteq X$ such that $|N(T)| = |T|$.) (M. Hall [1948], Halmos–Vaughan [1950])

**3.1.24.** (!) A **permutation matrix** $P$ is a 0,1-matrix having exactly one 1 in each row and column. Prove that a square matrix of nonnegative integers can be expressed as the sum of $k$ permutation matrices if and only if all row sums and column sums equal $k$.

**3.1.25.** (!) A **doubly stochastic matrix** $Q$ is a nonnegative real matrix in which every row and every column sums to 1. Prove that a doubly stochastic matrix $Q$ can be expressed $Q = c_1P_1 + \cdots + c_mP_m$, where $c_1, \ldots, c_m$ are nonnegative real numbers summing to 1 and $P_1, \ldots, P_m$ are permutation matrices. For example,

$$\begin{pmatrix} 1/2 & 1/3 & 1/6 \\ 0 & 1/6 & 5/6 \\ 1/2 & 1/2 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

(Hint: Use induction on the number of nonzero entries in $Q$.) (Birkhoff [1946], von Neumann [1953])

**3.1.26.** (!) A deck of $mn$ cards with $m$ values and $n$ suits consists of one card of each value in each suit. The cards are dealt into an $n$-by-$m$ array.

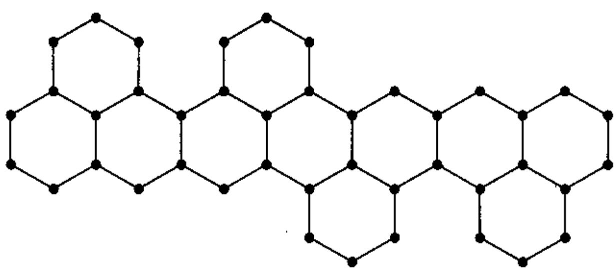
a) Prove that there is a set of $m$ cards, one in each column, having distinct values.

b) Use part (a) to prove that by a sequence of exchanges of cards of the same value, the cards can be rearranged so that each column consists of $n$ cards of distinct suits. (Enchev [1997])

**3.1.27.** (!) *Generalizing Tic-Tac-Toe.* A **positional game** consists of a set $X = x_1, \ldots, x_n$ of positions and a family $W_1, \ldots, W_m$ of winning sets of positions (Tic-Tac-Toe has nine positions and eight winning sets). Two players alternately choose positions; a player wins by collecting a winning set.

Suppose that each winning set has size at least $a$ and each position appears in at most $b$ winning sets (in Tic-Tac-Toe, $a = 3$ and $b = 4$). Prove that Player 2 can force a draw if $a \geq 2b$. (Hint: Form an $X, Y$-bigraph $G$, where $Y = \{w_1, \ldots, w_m\} \cup \{w'_1, \ldots, w'_m\}$, with edges $x_iw_j$ and $x_iw'_j$ whenever $x_i \in W_j$. How can Player 2 use a matching in $G$? Comment: This result implies that Player 2 can force a draw in $d$-dimensional Tic-Tac-Toe when the sides are long enough.)

**3.1.28.** (!) Exhibit a perfect matching in the graph below or give a short proof that it has none. (Lovász–Plummer [1986, p7])



**3.1.29.** (!) Use the König–Egerváry Theorem to prove that every bipartite graph $G$ has a matching of size at least $e(G)/\Delta(G)$. Use this to conclude that every subgraph of $K_{n,n}$ with more than $(k-1)n$ edges has a matching of size at least $k$.

**3.1.30.** (!) Determine the maximum number of edges in a simple bipartite graph that contains no matching with $k$ edges and no star with $l$ edges. (Isaak)

**3.1.31.** Use the König–Egerváry Theorem to prove Hall's Theorem.

**3.1.32.** (!) In an $X,Y$-bigraph $G$, the **deficiency** of a set $S$ is $\mathrm{def}(S) = |S| - |N(S)|$; note that $\mathrm{def}(\varnothing) = 0$. Prove that $\alpha'(G) = |X| - \max_{S\subseteq X} \mathrm{def}(S)$. (Hint: Form a bipartite graph $G'$ such that $G'$ has a matching that saturates $X$ if and only if $G$ has a matching of the desired size, and prove that $G'$ satisfies Hall's Condition.) (Ore [1955])

**3.1.33.** (!) Use Exercise 3.1.32 to prove the König–Egerváry Theorem. (Hint: Obtain a matching and a vertex cover of the same size from a set with maximum deficiency.)

**3.1.34.** (!) Let $G$ be an $X,Y$-bigraph with no isolated vertices, and define *deficiency* as in Exercise 3.1.32. Prove that Hall's Condition holds for a matching saturating $X$ if and only if each subset of $Y$ has deficiency at most $|Y| - |X|$.

**3.1.35.** Let $G$ be an $X,Y$-bigraph. Prove that $G$ is $(k+1)K_2$-free if and only if each $S \subseteq X$ has a subset of size at most $k$ with neighborhood $N(S)$. (Liu–Zhou [1997])

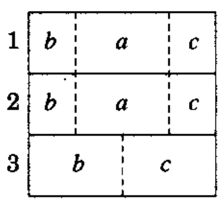
**3.1.36.** Let $G$ be an $X,Y$-bigraph having a matching that saturates $X$. Letting $m = |X|$, prove that $G$ has at most $\binom{m}{2}$ edges belonging to no matching of size $m$. Construct examples to show that this is best possible for every $m$.

**3.1.37.** (+) Let $G$ be an $X,Y$-bigraph having a matching that saturates $X$.

a) Let $S$ and $T$ be subsets of $X$ such that $|N(S)| = |S|$ and $|N(T)| = |T|$. Prove that $|N(S\cap T)| = |S\cap T|$.

b) Prove that $X$ has some vertex $x$ such that every edge incident to $x$ belongs to some maximum matching. (Hint: Consider a minimal nonempty set $S \subseteq X$ such that $|N(S)| = |S|$, if any exists.)

**3.1.38.** (+) An island of area $n$ has $n$ married hunter/farmer couples. The Ministry of Hunting divides the island into $n$ equal-sized hunting regions. The Ministry of Agriculture divides it into $n$ equal-sized farming regions. The Ministry of Marriage requires that each couple receive two overlapping regions. By Exercise 3.1.25, this is always possible. Prove a stronger result: guarantee a pairing where each couple's two regions share area at least $4/(n+1)^2$ when $n$ is odd and $4/[n(n+2)]$ when $n$ is even. Prove also that no larger common area can be guaranteed; the example below achieves equality for $n = 3$. (Marcus–Ree [1959], Floyd [1990])




$$\begin{pmatrix} .5 & .25 & .25 \\ .5 & .25 & .25 \\ 0 & .5 & .5 \end{pmatrix}$$

**3.1.39.** Let $G$ be a nontrivial simple graph. Prove that $\alpha(G) \le n(G) - e(G)/\Delta(G)$. Conclude that $\alpha(G) \le n(G)/2$ when $G$ also is regular. (P. Kwok)

**3.1.40.** Let $G$ be a bipartite graph. Prove that $\alpha(G) = n(G)/2$ if and only if $G$ has a perfect matching.

**3.1.41.** A connected $n$-vertex graph has exactly one cycle if and only if it has exactly $n$ edges (Exercise 2.1.30). Let $C$ be the cycle in such a graph $G$. Assuming the result of Exercise 3.1.40 for trees, prove that $\alpha(G) \ge \lfloor n(G)/2 \rfloor$, with equality if and only if $G - V(C)$ has a perfect matching.

**3.1.42.** (!) An algorithm to greedily build a large independent set iteratively selects a vertex of minimum degree in the remaining graph and deletes it and its neighbors. Prove that this algorithm produces an independent set of size at least $\sum_{v\in V(G)} \frac{1}{d(v)+1}$ in a graph $G$. (Caro [1979], Wei [1981])

**3.1.43.** Let $M$ be a maximal matching and $L$ a minimal edge cover in a graph with no isolated vertices. Prove the statements below. (Norman–Rabin [1959], Gallai [1959])

a) $M$ is a maximum matching if and only if $M$ is contained in a minimum edge cover.

b) $L$ is a minimum edge cover if and only if $L$ contains a maximum matching.

**3.1.44.** (−) Let $G$ be a simple graph in which the sum of the degrees of any $k$ vertices is less than $n - k$. Prove that every maximal independent set in $G$ has more than $k$ vertices. (Meyer [1972])

**3.1.45.** An edge $e$ of a graph $G$ is **$\alpha$-critical** if $\alpha(G - e) > \alpha(G)$. Suppose that $xy$ and $xz$ are $\alpha$-critical edges in $G$. Prove that $G$ has an induced subgraph that is an odd cycle containing $xy$ and $xz$. (Hint: Let $Y, Z$ be maximum independent sets in $G-xy$ and $G-xz$, respectively. Let $H = G[Y \triangle Z]$. Prove that every component of $H$ has the same number of vertices from $Y$ and from $Z$. Use this to prove that $y$ and $z$ belong to the same component of $H$.) (Berge [1970], with a difficult generalization in Markossian–Karapetian [1984])

**3.1.46.** (∗−) Characterize the graphs with domination number 1.

**3.1.47.** (∗−) Find the smallest tree where the domination number and the vertex cover number are not equal.

**3.1.48.** (∗−) Determine $\gamma(C_n)$ and $\gamma(P_n)$.

**3.1.49.** (∗) Let $G$ be a graph without isolated vertices, and let $S$ be a minimal dominating set in $G$. Prove that $\overline{S}$ is a dominating set. Conclude that $\gamma(G) \le n(G)/2$. (Ore [1962])

**3.1.50.** (∗) Prove that $\gamma(G) \le n - \beta'(G) \le n/2$ when $G$ is an $n$-vertex graph without isolated vertices. For $1 \le k \le n/2$, construct a connected $n$-vertex graph $G$ with $\gamma(G) = k$.

**3.1.51.** (∗) Let $G$ be an $n$-vertex graph.

a) Prove that $\lceil n/(1+\Delta(G)\rceil \ge \gamma(G) \le n - \Delta(G)$.

b) Prove that $(1 + \text{diam } G)/3 \le \gamma(G) \le n - \lfloor \text{diam } G/3 \rfloor$.

**3.1.52.** (∗) Prove that if the diameter of $G$ is at least 3, then $\gamma(\overline{G}) \le 2$.

**3.1.53.** (∗) For all $k \in \mathbb{N}$, construct a connected graph with $5k$ vertices and domination number $2k$. Construct a single 3-regular graph $G$ such that $\gamma(G) = 3n(G)/8$.

**3.1.54.** (∗) Determine the domination number of the Petersen graph, and determine the minimum size of a total dominating set in the Petersen graph.

**3.1.55.** (∗) In the hypercube $Q_4$, determine the minimum sizes of a dominating set, an independent dominating set, a connected dominating set, and a total dominating set.

**3.1.56.** (∗) Find a way to place five queens on an eight-by-eight chessboard that attack all other squares. Show that the five queens cannot be placed so that also they do not attack each other. (Comment: Thus the independent domination number of the "queen's graph" exceeds its domination number; it is 7.)

**3.1.57.** (*) For all $n \in \mathbb{N}$, construct an $n$-vertex tree with domination number 2 in which the minimum size of an independent dominating set is $\lfloor n/2 \rfloor$.

**3.1.58.** (*) Prove that a $K_{1,r}$-free graph $G$ has an independent dominating set of size at most $(r-2)\gamma(G) - (r-3)$. (Hint: Generalize the argument of Theorem 3.1.34.) (Bollobás–Cockayne [1979])

**3.1.59.** (*) In a graph $G$ of order $n$, prove that the minimum size of a connected dominating set is $n$ minus the maximum number of leaves in a spanning tree.

**3.1.60.** (*) For $k \le 5$, every graph $G$ with $\delta(G) \le k$ has a connected dominating set of size at most $3n(G)/(k+1)$ (Kleitman–West [1991], Griggs–Wu [1992]). Prove that this is sharp using a graph formed from a cyclic arrangement of $3m$ pairwise-disjoint cliques by making each vertex adjacent to every vertex in the clique before it and the clique after it. Let the clique sizes be $\lceil k/2 \rceil$, $\lfloor k/2 \rfloor$, 1, $\lceil k/2 \rceil$, $\lfloor k/2 \rfloor$, 1, $\cdots$.

# 3.2. Algorithms and Applications

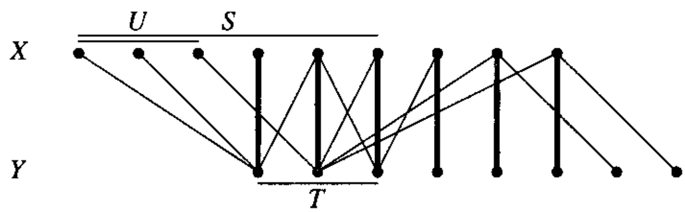
## MAXIMUM BIPARTITE MATCHING

To find a maximum matching, we iteratively seek augmenting paths to enlarge the current matching. In a bipartite graph, if we don't find an augmenting path, we will find a vertex cover with the same size as the current matching, thereby proving that the current matching has maximum size. This yields both an algorithm to solve the maximum matching problem and an algorithmic proof of the König–Egerváry Theorem.

Given a matching $M$ in an $X, Y$-bigraph $G$, we search for $M$-augmenting paths from each $M$-unsaturated vertex in $X$. We need only search from vertices in $X$, because every augmenting path has odd length and thus has ends in both $X$ and $Y$. We will search from the unsaturated vertices in $X$ simultaneously. Starting with a matching of size 0, $\alpha'(G)$ applications of the Augmenting Path Algorithm produce a maximum matching.

**3.2.1. Algorithm.** (Augmenting Path Algorithm).
**Input**: An $X, Y$-bigraph $G$, a matching $M$ in $G$, and the set $U$ of $M$-unsaturated vertices in $X$.
**Idea**: Explore $M$-alternating paths from $U$, letting $S \subseteq X$ and $T \subseteq Y$ be the sets of vertices reached. *Mark* vertices of $S$ that have been explored for path extensions. As a vertex is reached, record the vertex from which it is reached.
**Initialization**: $S = U$ and $T = \varnothing$.
**Iteration**: If $S$ has no unmarked vertex, stop and report $T \cup (X - S)$ as a minimum cover and $M$ as a maximum matching. Otherwise, select an unmarked $x \in S$. To explore $x$, consider each $y \in N(x)$ such that $xy \notin M$. If $y$ is unsaturated, terminate and report an $M$-augmenting path from $U$ to $y$. Otherwise, $y$ is matched to some $w \in X$ by $M$. In this case, include $y$ in $T$ (reached from $x$)

and include $w$ in $S$ (reached from $y$). After exploring all such edges incident to $x$, mark $x$ and iterate. ■




When exploring $x$ in the iterative step, we may reach a vertex $y \in T$ that we have reached previously. Recording $x$ as the previous vertex on the path may change which $M$-augmenting path we report, but it won't change whether such a path exists.

**3.2.2. Theorem.** Repeatedly applying the Augmenting Path Algorithm to a bipartite graph produces a matching and a vertex cover of equal size.

**Proof:** We need only verify that the Augmenting Path Algorithm produces an $M$-augmenting path or a vertex cover of size $|M|$. If the algorithm produces an $M$-augmenting path, we are finished. Otherwise, it terminates by marking all vertices of $S$ and claiming that $R = T \cup (X - S)$ is a vertex cover of size $|M|$. We must prove that $R$ is a vertex cover and has size $|M|$.

To show that $R$ is a vertex cover, it suffices to show that there is no edge joining $S$ to $Y - T$. An $M$-alternating path from $U$ enters $X$ only on an edge of $M$. Hence every vertex $x$ of $S - U$ is matched via $M$ to a vertex of $T$, and there is no edge of $M$ from $S$ to $Y - T$. Also there is no such edge outside $M$. When the path reaches $x \in S$, it can continue along any edge not in $M$, and exploring $x$ puts all other neighbors of $x$ into $T$. Since the algorithm marks all of $S$ before terminating, all edges from $S$ go to $T$.

Now we study the size of $R$. The algorithm puts only saturated vertices in $T$; each $y \in T$ is matched via $M$ to a vertex of $S$. Since $U \subseteq S$, also each vertex of $X - S$ is saturated, and the edges of $M$ incident to $X - S$ cannot involve $T$. Hence they are different from the edges saturating $T$, and we find that $M$ has at least $|T| + |X - S|$ edges. Since there is no matching larger than this vertex cover, we have $|M| = |T| + |X - S| = |R|$. ■

In addition to studying the correctness of algorithms, we are concerned about the time (number of computational steps) they use. We measure this as a function of the size of the input. For graph problems, we usually use the order $n(G)$ and/or size $e(G)$ to measure the input size.

**3.2.3. Definition.** The **running time** of an algorithm is the maximum number of computational steps used, expressed as a function of the size of the input. A **good algorithm** is one that has polynomial running time.

Running time is often expressed as "$O(f)$", where $f$ is a function of the size of the input. Here $O(f)$ denotes the set of functions $g$ such that $|g(x)|$ is bounded by a constant multiple of $|f(x)|$ when $x$ is sufficiently large (that is, there exist $c, a$ such that $|g(x)| \le c\,|f(x)|$ when $|x| \ge a$).

Many problems we study in Chapters 1-4 have good algorithms; other notions of complexity (Appendix B) need not trouble us yet. Since we don't know how long a particular operation may take on a particular computer, constant factors in running time have little meaning. Hence the "Big Oh" notation $O(f)$ is convenient. When $f$ is a quadratic polynomial, we typically abuse notation by writing $O(n^2)$ instead of $O(f)$ to describe functions that grow at most quadratically in terms of $n$.

**3.2.4. Remark.** Let $G$ be an $X, Y$-bigraph with $n$ vertices and $m$ edges. Since $\alpha'(G) \le n/2$, we find a maximum matching in $G$ by applying Algorithm 3.2.1 at most $n/2$ times. Each application explores a vertex of $X$ at most once, just before marking it; thus it considers each edge at most once. If the time for one edge exploration is bounded by a constant, then this algorithm to find a maximum matching runs in time $O(nm)$. Theorem 3.2.22 presents a faster algorithm, with running time $O(\sqrt{n}m)$. Section 3.3 discusses a good algorithm for maximum matching in general graphs. ∎

## WEIGHTED BIPARTITE MATCHING

Our results on maximum matching generalize to weighted $X, Y$-bigraphs, where we seek a matching of maximum total weight. If our graph is not all of $K_{n,n}$, then we insert the missing edges and assign them weight 0. This does not affect the numbers we can obtain as the weight of a matching. Thus we assume that our graph is $K_{n,n}$.

Since we consider only nonnegative edge weights, some maximum weighted matching is a perfect matching; thus we seek a perfect matching. We solve both the maximum weighted matching problem and its dual.

**3.2.5. Example.** *Weighted bipartite matching and its dual.* A farming company owns $n$ farms and $n$ processing plants. Each farm can produce corn to the capacity of one plant. The profit that results from sending the output of farm $i$ to plant $j$ is $w_{i,j}$. Placing weight $w_{i,j}$ on edge $x_i y_j$ gives us a weighted bipartite graph with partite sets $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_n\}$. The company wants to select edges forming a matching to maximize total profit.

The government claims that too much corn is being produced, so it will pay the company not to process corn. The government will pay $u_i$ if the company agrees not to use farm $i$ and $v_j$ if it agrees not to use plant $j$. If $u_i + v_j < w_{i,j}$, then the company makes more by using the edge $x_i y_j$ than by taking the government payments for those vertices. In order to stop all production, the government must offer amounts such that $u_i + v_j \ge w_{i,j}$ for all $i$, $j$. The government wants to find such values to minimize $\sum u_i + \sum v_j$. ∎