If we divide this by $t - x$ and note that $f(t) \to f(x)$ as $t \to x$ (Theorem 5.2), (b) follows. Next, let $h = f/g$. Then

$$\frac{h(t) - h(x)}{t - x} = \frac{1}{g(t)g(x)}\left[g(x)\frac{f(t) - f(x)}{t - x} - f(x)\frac{g(t) - g(x)}{t - x}\right].$$

Letting $t \to x$, and applying Theorems 4.4 and 5.2, we obtain (c).

5.4 Examples The derivative of any constant is clearly zero. If f is defined by $f(x) = x$, then $f'(x) = 1$. Repeated application of (b) and (c) then shows that x^n is differentiable, and that its derivative is nx^{n-1}, for any integer n (if $n < 0$, we have to restrict ourselves to $x \neq 0$). Thus every polynomial is differentiable, and so is every rational function, except at the points where the denominator is zero.

The following theorem is known as the "chain rule" for differentiation. It deals with differentiation of composite functions and is probably the most important theorem about derivatives. We shall meet more general versions of it in Chap. 9.

5.5 Theorem *Suppose f is continuous on $[a, b]$, $f'(x)$ exists at some point $x \in [a, b]$, g is defined on an interval I which contains the range of f, and g is differentiable at the point $f(x)$. If*

$$h(t) = g(f(t)) \qquad (a \le t \le b),$$

then h is differentiable at x, and

$$h'(x) = g'(f(x))f'(x). \tag{3}$$

Proof Let $y = f(x)$. By the definition of the derivative, we have

$$f(t) - f(x) = (t - x)[f'(x) + u(t)], \tag{4}$$

$$g(s) - g(y) = (s - y)[g'(y) + v(s)], \tag{5}$$

where $t \in [a, b]$, $s \in I$, and $u(t) \to 0$ as $t \to x$, $v(s) \to 0$ as $s \to y$. Let $s = f(t)$. Using first (5) and then (4), we obtain

$$\begin{aligned} h(t) - h(x) &= g(f(t)) - g(f(x)) \\ &= [f(t) - f(x)] \cdot [g'(y) + v(s)] \\ &= (t - x) \cdot [f'(x) + u(t)] \cdot [g'(y) + v(s)], \end{aligned}$$

or, if $t \neq x$,

$$\frac{h(t) - h(x)}{t - x} = [g'(y) + v(s)] \cdot [f'(x) + u(t)]. \tag{6}$$

Letting $t \to x$, we see that $s \to y$, by the continuity of f, so that the right side of (6) tends to $g'(y)f'(x)$, which gives (3).

5.6 Examples

(*a*) Let f be defined by

(7)
$$f(x) = \begin{cases} x \sin \dfrac{1}{x} & (x \neq 0), \\ 0 & (x = 0). \end{cases}$$

Taking for granted that the derivative of $\sin x$ is $\cos x$ (we shall discuss the trigonometric functions in Chap. 8), we can apply Theorems 5.3 and 5.5 whenever $x \neq 0$, and obtain

(8)
$$f'(x) = \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x} \qquad (x \neq 0).$$

At $x = 0$, these theorems do not apply any longer, since $1/x$ is not defined there, and we appeal directly to the definition: for $t \neq 0$,

$$\frac{f(t) - f(0)}{t - 0} = \sin \frac{1}{t}.$$

As $t \to 0$, this does not tend to any limit, so that $f'(0)$ does not exist.

(*b*) Let f be defined by

(9)
$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & (x \neq 0), \\ 0 & (x = 0), \end{cases}$$

As above, we obtain

(10)
$$f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x} \qquad (x \neq 0).$$

At $x = 0$, we appeal to the definition, and obtain

$$\left| \frac{f(t) - f(0)}{t - 0} \right| = \left| t \sin \frac{1}{t} \right| \leq |t| \qquad (t \neq 0);$$

letting $t \to 0$, we see that

(11)
$$f'(0) = 0.$$

Thus f is differentiable at all points x, but f' is not a continuous function, since $\cos(1/x)$ in (10) does not tend to a limit as $x \to 0$.

MEAN VALUE THEOREMS

5.7 Definition Let f be a real function defined on a metric space X. We say that f has a *local maximum* at a point $p \in X$ if there exists $\delta > 0$ such that $f(q) \leq f(p)$ for all $q \in X$ with $d(p, q) < \delta$.

Local minima are defined likewise.

Our next theorem is the basis of many applications of differentiation.

5.8 Theorem *Let f be defined on $[a, b]$; if f has a local maximum at a point $x \in (a, b)$, and if $f'(x)$ exists, then $f'(x) = 0$.*

The analogous statement for local minima is of course also true.

Proof Choose δ in accordance with Definition 5.7, so that

$$a < x - \delta < x < x + \delta < b.$$

If $x - \delta < t < x$, then

$$\frac{f(t) - f(x)}{t - x} \geq 0.$$

Letting $t \to x$, we see that $f'(x) \geq 0$.

If $x < t < x + \delta$, then

$$\frac{f(t) - f(x)}{t - x} \leq 0,$$

which shows that $f'(x) \leq 0$. Hence $f'(x) = 0$.

5.9 Theorem *If f and g are continuous real functions on $[a, b]$ which are differentiable in (a, b), then there is a point $x \in (a, b)$ at which*

$$[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x).$$

Note that differentiability is not required at the endpoints.

Proof Put

$$h(t) = [f(b) - f(a)]g(t) - [g(b) - g(a)]f(t) \qquad (a \leq t \leq b).$$

Then h is continuous on $[a, b]$, h is differentiable in (a, b), and

$$h(a) = f(b)g(a) - f(a)g(b) = h(b). \tag{12}$$

To prove the theorem, we have to show that $h'(x) = 0$ for some $x \in (a, b)$.

If h is constant, this holds for every $x \in (a, b)$. If $h(t) > h(a)$ for some $t \in (a, b)$, let x be a point on $[a, b]$ at which h attains its maximum

(Theorem 4.16). By (12), $x \in (a, b)$, and Theorem 5.8 shows that $h'(x) = 0$. If $h(t) < h(a)$ for some $t \in (a, b)$, the same argument applies if we choose for x a point on $[a, b]$ where h attains its minimum.

This theorem is often called a *generalized mean value theorem*; the following special case is usually referred to as "the" mean value theorem:

5.10 Theorem *If f is a real continuous function on $[a, b]$ which is differentiable in (a, b), then there is a point $x \in (a, b)$ at which*

$$f(b) - f(a) = (b - a)f'(x).$$

Proof Take $g(x) = x$ in Theorem 5.9.

5.11 Theorem *Suppose f is differentiable in (a, b).*

(*a*) *If $f'(x) \geq 0$ for all $x \in (a, b)$, then f is monotonically increasing.*

(*b*) *If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant.*

(*c*) *If $f'(x) \leq 0$ for all $x \in (a, b)$, then f is monotonically decreasing.*

Proof All conclusions can be read off from the equation

$$f(x_2) - f(x_1) = (x_2 - x_1)f'(x),$$

which is valid, for each pair of numbers x_1, x_2 in (a, b), for *some* x between x_1 and x_2.

THE CONTINUITY OF DERIVATIVES

We have already seen [Example 5.6(*b*)] that a function f may have a derivative f' which exists at every point, but is discontinuous at some point. However, not every function is a derivative. In particular, derivatives which exist at every point of an interval have one important property in common with functions which are continuous on an interval: Intermediate values are assumed (compare Theorem 4.23). The precise statement follows.

5.12 Theorem *Suppose f is a real differentiable function on $[a, b]$ and suppose $f'(a) < \lambda < f'(b)$. Then there is a point $x \in (a, b)$ such that $f'(x) = \lambda$.*

A similar result holds of course if $f'(a) > f'(b)$.

Proof Put $g(t) = f(t) - \lambda t$. Then $g'(a) < 0$, so that $g(t_1) < g(a)$ for some $t_1 \in (a, b)$, and $g'(b) > 0$, so that $g(t_2) < g(b)$ for some $t_2 \in (a, b)$. Hence g attains its minimum on $[a, b]$ (Theorem 4.16) at some point x such that $a < x < b$. By Theorem 5.8, $g'(x) = 0$. Hence $f'(x) = \lambda$.

Corollary *If f is differentiable on $[a, b]$, then f' cannot have any simple discontinuities on $[a, b]$.*

But f' may very well have discontinuities of the second kind.

L'HOSPITAL'S RULE

The following theorem is frequently useful in the evaluation of limits.

5.13 Theorem *Suppose f and g are real and differentiable in (a, b), and $g'(x) \neq 0$ for all $x \in (a, b)$, where $-\infty \le a < b \le +\infty$. Suppose*

$$\frac{f'(x)}{g'(x)} \to A \text{ as } x \to a. \tag{13}$$

If

$$f(x) \to 0 \text{ and } g(x) \to 0 \text{ as } x \to a, \tag{14}$$

or if

$$g(x) \to +\infty \text{ as } x \to a, \tag{15}$$

then

$$\frac{f(x)}{g(x)} \to A \text{ as } x \to a. \tag{16}$$

The analogous statement is of course also true if $x \to b$, or if $g(x) \to -\infty$ in (15). Let us note that we now use the limit concept in the extended sense of Definition 4.33.

Proof We first consider the case in which $-\infty \le A < +\infty$. Choose a real number q such that $A < q$, and then choose r such that $A < r < q$. By (13) there is a point $c \in (a, b)$ such that $a < x < c$ implies

$$\frac{f'(x)}{g'(x)} < r. \tag{17}$$

If $a < x < y < c$, then Theorem 5.9 shows that there is a point $t \in (x, y)$ such that

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(t)}{g'(t)} < r. \tag{18}$$

Suppose (14) holds. Letting $x \to a$ in (18), we see that

$$\frac{f(y)}{g(y)} \le r < q \qquad (a < y < c). \tag{19}$$

Next, suppose (15) holds. Keeping y fixed in (18), we can choose a point $c_1 \in (a, y)$ such that $g(x) > g(y)$ and $g(x) > 0$ if $a < x < c_1$. Multiplying (18) by $[g(x) - g(y)]/g(x)$, we obtain

$$\frac{f(x)}{g(x)} < r - r\frac{g(y)}{g(x)} + \frac{f(y)}{g(x)} \qquad (a < x < c_1). \tag{20}$$

If we let $x \to a$ in (20), (15) shows that there is a point $c_2 \in (a, c_1)$ such that

$$\frac{f(x)}{g(x)} < q \qquad (a < x < c_2). \tag{21}$$

Summing up, (19) and (21) show that for any q, subject only to the condition $A < q$, there is a point c_2 such that $f(x)/g(x) < q$ if $a < x < c_2$.

In the same manner, if $-\infty < A \leq +\infty$, and p is chosen so that $p < A$, we can find a point c_3 such that

$$p < \frac{f(x)}{g(x)} \qquad (a < x < c_3), \tag{22}$$

and (16) follows from these two statements.

DERIVATIVES OF HIGHER ORDER

5.14 Definition If f has a derivative f' on an interval, and if f' is itself differentiable, we denote the derivative of f' by f'' and call f'' the second derivative of f. Continuing in this manner, we obtain functions

$$f, f', f'', f^{(3)}, \ldots, f^{(n)},$$

each of which is the derivative of the preceding one. $f^{(n)}$ is called the nth derivative, or the derivative of order n, of f.

In order for $f^{(n)}(x)$ to exist at a point x, $f^{(n-1)}(t)$ must exist in a neighborhood of x (or in a one-sided neighborhood, if x is an endpoint of the interval on which f is defined), and $f^{(n-1)}$ must be differentiable at x. Since $f^{(n-1)}$ must exist in a neighborhood of x, $f^{(n-2)}$ must be differentiable in that neighborhood.

TAYLOR'S THEOREM

5.15 Theorem *Suppose f is a real function on $[a, b]$, n is a positive integer, $f^{(n-1)}$ is continuous on $[a, b]$, $f^{(n)}(t)$ exists for every $t \in (a, b)$. Let α, β be distinct points of $[a, b]$, and define*

$$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t - \alpha)^k. \tag{23}$$

Then there exists a point x between α and β such that

(24) $$f(\beta) = P(\beta) + \frac{f^{(n)}(x)}{n!}(\beta - \alpha)^n.$$

For $n = 1$, this is just the mean value theorem. In general, the theorem shows that f can be approximated by a polynomial of degree $n - 1$, and that (24) allows us to estimate the error, if we know bounds on $|f^{(n)}(x)|$.

Proof Let M be the number defined by

(25) $$f(\beta) = P(\beta) + M(\beta - \alpha)^n$$

and put

(26) $$g(t) = f(t) - P(t) - M(t - \alpha)^n \qquad (a \le t \le b).$$

We have to show that $n!M = f^{(n)}(x)$ for some x between α and β. By (23) and (26),

(27) $$g^{(n)}(t) = f^{(n)}(t) - n!M \qquad (a < t < b).$$

Hence the proof will be complete if we can show that $g^{(n)}(x) = 0$ for some x between α and β.

Since $P^{(k)}(\alpha) = f^{(k)}(\alpha)$ for $k = 0, \ldots, n - 1$, we have

(28) $$g(\alpha) = g'(\alpha) = \cdots = g^{(n-1)}(\alpha) = 0.$$

Our choice of M shows that $g(\beta) = 0$, so that $g'(x_1) = 0$ for some x_1 between α and β, by the mean value theorem. Since $g'(\alpha) = 0$, we conclude similarly that $g''(x_2) = 0$ for some x_2 between α and x_1. After n steps we arrive at the conclusion that $g^{(n)}(x_n) = 0$ for some x_n between α and x_{n-1}, that is, between α and β.

DIFFERENTIATION OF VECTOR-VALUED FUNCTIONS

5.16 Remarks Definition 5.1 applies without any change to complex functions f defined on $[a, b]$, and Theorems 5.2 and 5.3, as well as their proofs, remain valid. If f_1 and f_2 are the real and imaginary parts of f, that is, if

$$f(t) = f_1(t) + if_2(t)$$

for $a \le t \le b$, where $f_1(t)$ and $f_2(t)$ are real, then we clearly have

(29) $$f'(x) = f_1'(x) + if_2'(x);$$

also, f is differentiable at x if and only if both f_1 and f_2 are differentiable at x.

Passing to vector-valued functions in general, i.e., to functions $\mathbf{f}$ which map $[a, b]$ into some R^k, we may still apply Definition 5.1 to define $\mathbf{f}'(x)$. The term $\phi(t)$ in (1) is now, for each t, a point in R^k, and the limit in (2) is taken with respect to the norm of R^k. In other words, $\mathbf{f}'(x)$ is that point of R^k (if there is one) for which

$$\lim_{t \to x} \left| \frac{\mathbf{f}(t) - \mathbf{f}(x)}{t - x} - \mathbf{f}'(x) \right| = 0, \tag{30}$$

and $\mathbf{f}'$ is again a function with values in R^k.

If $f_1, \ldots, f_k$ are the components of $\mathbf{f}$, as defined in Theorem 4.10, then

$$\mathbf{f}' = (f_1', \ldots, f_k'), \tag{31}$$

and $\mathbf{f}$ is differentiable at a point x if and only if each of the functions $f_1, \ldots, f_k$ is differentiable at x.

Theorem 5.2 is true in this context as well, and so is Theorem 5.3(a) and (b), if fg is replaced by the inner product $\mathbf{f} \cdot \mathbf{g}$ (see Definition 4.3).

When we turn to the mean value theorem, however, and to one of its consequences, namely, L'Hospital's rule, the situation changes. The next two examples will show that each of these results fails to be true for complex-valued functions.

5.17 Example Define, for real x,

$$f(x) = e^{ix} = \cos x + i \sin x. \tag{32}$$

(The last expression may be taken as the definition of the complex exponential e^{ix}; see Chap. 8 for a full discussion of these functions.) Then

$$f(2\pi) - f(0) = 1 - 1 = 0, \tag{33}$$

but

$$f'(x) = ie^{ix}, \tag{34}$$

so that $|f'(x)| = 1$ for all real x.

Thus Theorem 5.10 fails to hold in this case.

5.18 Example On the segment $(0, 1)$, define $f(x) = x$ and

$$g(x) = x + x^2 e^{i/x^2}. \tag{35}$$

Since $|e^{it}| = 1$ for all real t, we see that

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = 1. \tag{36}$$