(*b*) Let S be the set of all $(x, y) \in R^2$ at which $f(x, y) = 0$. Find those points of S that have no neighborhoods in which the equation $f(x, y) = 0$ can be solved for y in terms of x (or for x in terms of y). Describe S as precisely as you can.

22. Give a similar discussion for

$$f(x, y) = 2x^3 + 6xy^2 - 3x^2 + 3y^2.$$

23. Define f in R^3 by

$$f(x, y_1, y_2) = x^2y_1 + e^x + y_2.$$

Show that $f(0, 1, -1) = 0$, $(D_1f)(0, 1, -1) \neq 0$, and that there exists therefore a differentiable function g in some neighborhood of $(1, -1)$ in R^2, such that $g(1, -1) = 0$ and

$$f(g(y_1, y_2), y_1, y_2) = 0.$$

Find $(D_1g)(1, -1)$ and $(D_2g)(1, -1)$.

24. For $(x, y) \neq (0, 0)$, define $\mathbf{f} = (f_1, f_2)$ by

$$f_1(x, y) = \frac{x^2 - y^2}{x^2 + y^2}, \qquad f_2(x, y) = \frac{xy}{x^2 + y^2}.$$

Compute the rank of $\mathbf{f}'(x, y)$, and find the range of $\mathbf{f}$.

25. Suppose $A \in L(R^n, R^m)$, let r be the rank of A.

(*a*) Define S as in the proof of Theorem 9.32. Show that SA is a projection in R^n whose null space is $\mathscr{N}(A)$ and whose range is $\mathscr{R}(S)$. *Hint:* By (68), $SASA = SA$.

(*b*) Use (*a*) to show that

$$\dim \mathscr{N}(A) + \dim \mathscr{R}(A) = n.$$

26. Show that the existence (and even the continuity) of $D_{12}f$ does not imply the existence of D_1f. For example, let $f(x, y) = g(x)$, where g is nowhere differentiable.

27. Put $f(0, 0) = 0$, and

$$f(x, y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}$$

if $(x, y) \neq (0, 0)$. Prove that

(*a*) f, D_1f, D_2f are continuous in R^2;

(*b*) $D_{12}f$ and $D_{21}f$ exist at every point of R^2, and are continuous except at $(0, 0)$;

(*c*) $(D_{12}f)(0, 0) = 1$, and $(D_{21}f)(0, 0) = -1$.

28. For $t \geq 0$, put

$$\varphi(x, t) = \begin{cases} x & (0 \leq x \leq \sqrt{t}) \\ -x + 2\sqrt{t} & (\sqrt{t} \leq x \leq 2\sqrt{t}) \\ 0 & (\text{otherwise}), \end{cases}$$

and put $\varphi(x, t) = -\varphi(x, |t|)$ if $t < 0$.

Show that φ is continuous on R^2, and

$$(D_2\varphi)(x, 0) = 0$$

for all x. Define

$$f(t) = \int_{-1}^{1} \varphi(x, t)\, dx.$$

Show that $f(t) = t$ if $|t| < \frac{1}{4}$. Hence

$$f'(0) \neq \int_{-1}^{1} (D_2\varphi)(x, 0)\, dx.$$

29. Let E be an open set in R^n. The classes $\mathscr{C}'(E)$ and $\mathscr{C}''(E)$ are defined in the text. By induction, $\mathscr{C}^{(k)}(E)$ can be defined as follows, for all positive integers k: To say that $f \in \mathscr{C}^{(k)}(E)$ means that the partial derivatives $D_1 f, \ldots, D_n f$ belong to $\mathscr{C}^{(k-1)}(E)$.

Assume $f \in \mathscr{C}^{(k)}(E)$. and show (by repeated application of Theorem 9.41) that the kth-order derivative

$$D_{i_1 i_2 \ldots i_k} f = D_{i_1} D_{i_2} \ldots D_{i_k} f$$

is unchanged if the subscripts $i_1, \ldots, i_k$ are permuted.

For instance, if $n \geq 3$, then

$$D_{1213} f = D_{3112} f$$

for every $f \in \mathscr{C}^{(4)}$.

30. Let $f \in \mathscr{C}^{(m)}(E)$, where E is an open subset of R^n. Fix $\mathbf{a} \in E$, and suppose $\mathbf{x} \in R^n$ is so close to $\mathbf{0}$ that the points

$$\mathbf{p}(t) = \mathbf{a} + t\mathbf{x}$$

lie in E whenever $0 \leq t \leq 1$. Define

$$h(t) = f(\mathbf{p}(t))$$

for all $t \in R^1$ for which $\mathbf{p}(t) \in E$.

(*a*) For $1 \leq k \leq m$, show (by repeated application of the chain rule) that

$$h^{(k)}(t) = \sum (D_{i_1 \ldots i_k} f)(\mathbf{p}(t))\, x_{i_1} \ldots x_{i_k}.$$

The sum extends over all ordered k-tuples $(i_1, \ldots, i_k)$ in which each i_j is one of the integers $1, \ldots, n$.

(*b*) By Taylor's theorem (5.15),

$$h(1) = \sum_{k=0}^{m-1} \frac{h^{(k)}(0)}{k!} + \frac{h^{(m)}(t)}{m!}$$

for some $t \in (0, 1)$. Use this to prove Taylor's theorem in n variables by showing that the formula

$$f(\mathbf{a}+\mathbf{x})=\sum_{k=0}^{m-1}\frac{1}{k!}\sum(D_{i_1\ldots i_k}f)(\mathbf{a})x_{i_1}\cdots x_{i_k}+r(\mathbf{x})$$

represents $f(\mathbf{a}+\mathbf{x})$ as the sum of its so-called "Taylor polynomial of degree $m-1$," plus a remainder that satisfies

$$\lim_{\mathbf{x}\to 0}\frac{r(\mathbf{x})}{|\mathbf{x}|^{m-1}}=0.$$

Each of the inner sums extends over all ordered k-tuples $(i_1, \ldots, i_k)$, as in part (a); as usual, the zero-order derivative of f is simply f, so that the constant term of the Taylor polynomial of f at $\mathbf{a}$ is $f(\mathbf{a})$.

(c) Exercise 29 shows that repetition occurs in the Taylor polynomial as written in part (b). For instance, D_{113} occurs three times, as D_{113}, D_{131}, D_{311}. The sum of the corresponding three terms can be written in the form

$$3(D_1^2 D_3 f)(\mathbf{a})x_1^2 x_3.$$

Prove (by calculating how often each derivative occurs) that the Taylor polynomial in (b) can be written in the form

$$\sum\frac{(D_1^{s_1}\cdots D_n^{s_n}f)(\mathbf{a})}{s_1!\cdots s_n!}x_1^{s_1}\cdots x_n^{s_n}.$$

Here the summation extends over all ordered n-tuples $(s_1, \ldots, s_n)$ such that each s_i is a nonnegative integer, and $s_1+\cdots+s_n \le m-1$.

31. Suppose $f \in \mathscr{C}^{(3)}$ in some neighborhood of a point $\mathbf{a} \in R^2$, the gradient of f is $\mathbf{0}$ at $\mathbf{a}$, but not all second-order derivatives of f are 0 at $\mathbf{a}$. Show how one can then determine from the Taylor polynomial of f at $\mathbf{a}$ (of degree 2) whether f has a local maximum, or a local minimum, or neither, at the point $\mathbf{a}$.

Extend this to R^n in place of R^2.

10

INTEGRATION OF DIFFERENTIAL FORMS

Integration can be studied on many levels. In Chap. 6, the theory was developed for reasonably well-behaved functions on subintervals of the real line. In Chap. 11 we shall encounter a very highly developed theory of integration that can be applied to much larger classes of functions, whose domains are more or less arbitrary sets, not necessarily subsets of R^n. The present chapter is devoted to those aspects of integration theory that are closely related to the geometry of euclidean spaces, such as the change of variables formula, line integrals, and the machinery of differential forms that is used in the statement and proof of the n-dimensional analogue of the fundamental theorem of calculus, namely Stokes' theorem.

INTEGRATION

10.1 Definition Suppose I^k is a k-cell in R^k, consisting of all

$$\mathbf{x} = (x_1, \ldots, x_k)$$

such that

$$a_i \le x_i \le b_i \qquad (i = 1, \ldots, k), \tag{1}$$

I^j is the j-cell in R^j defined by the first j inequalities (1), and f is a real continuous function on I^k.

Put $f = f_k$, and define f_{k-1} on I^{k-1} by

$$f_{k-1}(x_1, \ldots, x_{k-1}) = \int_{a_k}^{b_k} f_k(x_1, \ldots, x_{k-1}, x_k)\, dx_k .$$

The uniform continuity of f_k on I^k shows that f_{k-1} is continuous on I^{k-1}. Hence we can repeat this process and obtain functions f_j, continuous on I^j, such that f_{j-1} is the integral of f_j, with respect to x_j, over $[a_j, b_j]$. After k steps we arrive at a *number* f_0, which we call the *integral of f over I^k*; we write it in the form

$$(2) \qquad \int_{I^k} f(\mathbf{x})\, d\mathbf{x} \quad \text{or} \quad \int_{I^k} f.$$

A priori, this definition of the integral depends on the order in which the k integrations are carried out. However, this dependence is only apparent. To prove this, let us introduce the temporary notation $L(f)$ for the integral (2) and $L'(f)$ for the result obtained by carrying out the k integrations in some other order.

10.2 Theorem *For every $f \in \mathscr{C}(I^k)$, $L(f) = L'(f)$.*

Proof If $h(\mathbf{x}) = h_1(x_1) \cdots h_k(x_k)$, where $h_j \in \mathscr{C}([a_j, b_j])$, then

$$L(h) = \prod_{i=1}^{k} \int_{a_i}^{b_i} h_i(x_i)\, dx_i = L'(h).$$

If $\mathscr{A}$ is the set of all finite sums of such functions h, it follows that $L(g) = L'(g)$ for all $g \in \mathscr{A}$. Also, $\mathscr{A}$ is an algebra of functions on I^k to which the Stone-Weierstrass theorem applies.

Put $V = \prod_1^k (b_i - a_i)$. If $f \in \mathscr{C}(I^k)$ and $\varepsilon > 0$, there exists $g \in \mathscr{A}$ such that $\|f - g\| < \varepsilon / V$, where $\|f\|$ is defined as $\max |f(\mathbf{x})|$ ($\mathbf{x} \in I^k$). Then $|L(f-g)| < \varepsilon$, $|L'(f-g)| < \varepsilon$, and since

$$L(f) - L'(f) = L(f-g) + L'(g-f),$$

we conclude that $|L(f) - L'(f)| < 2\varepsilon$.

In this connection, Exercise 2 is relevant.

10.3 Definition The *support* of a (real or complex) function f on R^k is the closure of the set of all points $\mathbf{x} \in R^k$ at which $f(\mathbf{x}) \neq 0$. If f is a continuous function with compact support, let I^k be any k-cell which contains the support of f, and define

$$(3) \qquad \int_{R^k} f = \int_{I^k} f.$$

The integral so defined is evidently independent of the choice of I^k, provided only that I^k contains the support of f.

It is now tempting to extend the definition of the integral over R^k to functions which are limits (in some sense) of continuous functions with compact support. We do not want to discuss the conditions under which this can be done; the proper setting for this question is the Lebesgue integral. We shall merely describe one very simple example which will be used in the proof of Stokes' theorem.

10.4 Example Let Q^k be the k-simplex which consists of all points $\mathbf{x} = (x_1, \ldots, x_k)$ in R^k for which $x_1 + \cdots + x_k \leq 1$ and $x_i \geq 0$ foi $i = 1, \ldots, k$. If $k = 3$, for example, Q^k is a tetrahedron, with vertices at $\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. If $f \in \mathscr{C}(Q^k)$, extend f to a function on I^k by setting $f(\mathbf{x}) = 0$ off Q^k, and define

$$(4) \qquad \int_{Q^k} f = \int_{I^k} f.$$

Here I^k is the "unit cube" defined by

$$0 \leq x_i \leq 1 \qquad (1 \leq i \leq k).$$

Since f may be discontinuous on I^k, the existence of the integral on the right of (4) needs proof. We also wish to show that this integral is independent of the order in which the k single integrations are carried out.

To do this, suppose $0 < \delta < 1$, put

$$(5) \qquad \varphi(t) = \begin{cases} 1 & (t \leq 1 - \delta) \\ \dfrac{(1-t)}{\delta} & (1 - \delta < t \leq 1) \\ 0 & (1 < t), \end{cases}$$

and define

$$(6) \qquad F(\mathbf{x}) = \varphi(x_1 + \cdots + x_k) f(\mathbf{x}) \qquad (\mathbf{x} \in I^k).$$

Then $F \in \mathscr{C}(I^k)$.

Put $\mathbf{y} = (x_1, \ldots, x_{k-1})$, $\mathbf{x} = (\mathbf{y}, x_k)$. For each $\mathbf{y} \in I^{k-1}$, the set of all x_k such that $F(\mathbf{y}, x_k) \neq f(\mathbf{y}; x_k)$ is either empty or is a segment whose length does not exceed δ. Since $0 \leq \varphi \leq 1$, it follows that

$$(7) \qquad |F_{k-1}(\mathbf{y}) - f_{k-1}(\mathbf{y})| \leq \delta \|f\| \qquad (\mathbf{y} \in I^{k-1}),$$

where $\|f\|$ has the same meaning as in the proof of Theorem 10.2, and F_{k-1}, f_{k-1} are as in Definition 10.1.

As $\delta \to 0$, (7) exhibits f_{k-1} as a uniform limit of a sequence of continuous functions. Thus $f_{k-1} \in \mathscr{C}(I^{k-1})$, and the further integrations present no problem.

This proves the existence of the integral (4). Moreover, (7) shows that

$$\left| \int_{I^k} F(\mathbf{x})\, d\mathbf{x} - \int_{I^k} f(\mathbf{x})\, d\mathbf{x} \right| \le \delta \|f\|. \tag{8}$$

Note that (8) is true, regardless of the order in which the k single integrations are carried out. Since $F \in \mathscr{C}(I^k)$, $\int F$ is unaffected by any change in this order. Hence (8) shows that the same is true of $\int f$.

This completes the proof.

Our next goal is the change of variables formula stated in Theorem 10.9. To facilitate its proof, we first discuss so-called primitive mappings, and partitions of unity. Primitive mappings will enable us to get a clearer picture of the local action of a $\mathscr{C}'$-mapping with invertible derivative, and partitions of unity are a very useful device that makes it possible to use local information in a global setting.

PRIMITIVE MAPPINGS

10.5 Definition If $\mathbf{G}$ maps an open set $E \subset R^n$ into R^n, and if there is an integer m and a real function g with domain E such that

$$\mathbf{G}(\mathbf{x}) = \sum_{i \ne m} x_i \mathbf{e}_i + g(\mathbf{x})\mathbf{e}_m \qquad (\mathbf{x} \in E), \tag{9}$$

then we call $\mathbf{G}$ *primitive*. A primitive mapping is thus one that changes at most one coordinate. Note that (9) can also be written in the form

$$\mathbf{G}(\mathbf{x}) = \mathbf{x} + [g(\mathbf{x}) - x_m]\mathbf{e}_m. \tag{10}$$

If g is differentiable at some point $\mathbf{a} \in E$, so is $\mathbf{G}$. The matrix $[\alpha_{ij}]$ of the operator $\mathbf{G}'(\mathbf{a})$ has

$$(D_1 g)(\mathbf{a}), \ldots, (D_m g)(\mathbf{a}), \ldots, (D_n g)(\mathbf{a}) \tag{11}$$

as its mth row. For $j \ne m$, we have $\alpha_{jj} = 1$ and $\alpha_{ij} = 0$ if $i \ne j$. The Jacobian of $\mathbf{G}$ at $\mathbf{a}$ is thus given by

$$J_{\mathbf{G}}(\mathbf{a}) = \det[\mathbf{G}'(\mathbf{a})] = (D_m g)(\mathbf{a}), \tag{12}$$

and we see (by Theorem 9.36) that $\mathbf{G}'(\mathbf{a})$ *is invertible if and only if* $(D_m g)(\mathbf{a}) \ne 0$.

10.6 Definition A linear operator B on R^n that interchanges some pair of members of the standard basis and leaves the others fixed will be called a *flip*.

For example, the flip B on R^4 that interchanges $\mathbf{e}_2$ and $\mathbf{e}_4$ has the form

$$B(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 + x_4\mathbf{e}_4) = x_1\mathbf{e}_1 + x_2\mathbf{e}_4 + x_3\mathbf{e}_3 + x_4\mathbf{e}_2 \tag{13}$$

or, equivalently,

$$B(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 + x_4\mathbf{e}_4) = x_1\mathbf{e}_1 + x_4\mathbf{e}_2 + x_3\mathbf{e}_3 + x_2\mathbf{e}_4. \tag{14}$$

Hence B can also be thought of as interchanging two of the coordinates, rather than two basis vectors.

In the proof that follows, we shall use the projections $P_0, \ldots, P_n$ in R^n, defined by $P_0\mathbf{x} = \mathbf{0}$ and

$$P_m\mathbf{x} = x_1\mathbf{e}_1 + \cdots + x_m\mathbf{e}_m \tag{15}$$

for $1 \le m \le n$. Thus P_m is the projection whose range and null space are spanned by $\{\mathbf{e}_1, \ldots, \mathbf{e}_m\}$ and $\{\mathbf{e}_{m+1}, \ldots, \mathbf{e}_n\}$, respectively.

10.7 Theorem *Suppose* $\mathbf{F}$ *is a* $\mathscr{C}'$*-mapping of an open set* $E \subset R^n$ *into* R^n, $\mathbf{0} \in E$, $\mathbf{F}(\mathbf{0}) = \mathbf{0}$, *and* $\mathbf{F}'(\mathbf{0})$ *is invertible.*

Then there is a neighborhood of $\mathbf{0}$ *in* R^n *in which a representation*

$$\mathbf{F}(\mathbf{x}) = B_1 \cdots B_{n-1}\mathbf{G}_n \circ \cdots \circ \mathbf{G}_1(\mathbf{x}) \tag{16}$$

is valid.

In (16), *each* $\mathbf{G}_i$ *is a primitive* $\mathscr{C}'$*-mapping in some neighborhood of* $\mathbf{0}$; $\mathbf{G}_i(\mathbf{0}) = \mathbf{0}$, $\mathbf{G}_i'(\mathbf{0})$ *is invertible, and each* B_i *is either a flip or the identity operator.*

Briefly, (16) represents $\mathbf{F}$ locally as a composition of primitive mappings and flips.

Proof Put $\mathbf{F} = \mathbf{F}_1$. Assume $1 \le m \le n-1$, and make the following induction hypothesis (which evidently holds for $m = 1$):

V_m is a neighborhood of $\mathbf{0}$, $\mathbf{F}_m \in \mathscr{C}'(V_m)$, $\mathbf{F}_m(\mathbf{0}) = \mathbf{0}$, $\mathbf{F}_m'(\mathbf{0})$ *is invertible, and*

$$P_{m-1}\mathbf{F}_m(\mathbf{x}) = P_{m-1}\mathbf{x} \qquad (\mathbf{x} \in V_m). \tag{17}$$

By (17), we have

$$\mathbf{F}_m(\mathbf{x}) = P_{m-1}\mathbf{x} + \sum_{i=m}^{n} \alpha_i(\mathbf{x})\mathbf{e}_i, \tag{18}$$

where $\alpha_m, \ldots, \alpha_n$ are real $\mathscr{C}'$-functions in V_m. Hence

$$\mathbf{F}_m'(\mathbf{0})\mathbf{e}_m = \sum_{i=m}^{n} (D_m\alpha_i)(\mathbf{0})\mathbf{e}_i. \tag{19}$$