Proof Let $\int$ denote the integral over $[a, b]$, Σ the sum from 1 to n. Then

$$\int f\bar{t}_n = \int f \sum \bar{\gamma}_m \bar{\phi}_m = \sum c_m \bar{\gamma}_m$$

by the definition of $\{c_m\}$,

$$\int |t_n|^2 = \int t_n \bar{t}_n = \int \sum \gamma_m \phi_m \sum \bar{\gamma}_k \bar{\phi}_k = \sum |\gamma_m|^2$$

since $\{\phi_m\}$ is orthonormal, and so

$$\begin{aligned}\int |f - t_n|^2 &= \int |f|^2 - \int f\bar{t}_n - \int \bar{f} t_n + \int |t_n|^2 \\ &= \int |f|^2 - \sum c_m \bar{\gamma}_m - \sum \bar{c}_m \gamma_m + \sum \gamma_m \bar{\gamma}_m \\ &= \int |f|^2 - \sum |c_m|^2 + \sum |\gamma_m - c_m|^2,\end{aligned}$$

which is evidently minimized if and only if $\gamma_m = c_m$.

Putting $\gamma_m = c_m$ in this calculation, we obtain

$$\int_a^b |s_n(x)|^2 \, dx = \sum_1^n |c_m|^2 \le \int_a^b |f(x)|^2 \, dx, \tag{72}$$

since $\int |f - t_n|^2 \ge 0$.

8.12 Theorem *If $\{\phi_n\}$ is orthonormal on $[a, b]$, and if*

$$f(x) \sim \sum_{n=1}^{\infty} c_n \phi_n(x),$$

then

$$\sum_{n=1}^{\infty} |c_n|^2 \le \int_a^b |f(x)|^2 \, dx. \tag{73}$$

In particular,

$$\lim_{n \to \infty} c_n = 0. \tag{74}$$

Proof Letting $n \to \infty$ in (72), we obtain (73), the so-called "Bessel inequality."

8.13 Trigonometric series From now on we shall deal only with the trigonometric system. We shall consider functions f that have period 2π and that are Riemann-integrable on $[-\pi, \pi]$ (and hence on every bounded interval). The Fourier series of f is then the series (63) whose coefficients c_n are given by the integrals (62), and

$$s_N(x) = s_N(f; x) = \sum_{-N}^{N} c_n e^{inx} \tag{75}$$

is the Nth partial sum of the Fourier series of f. The inequality (72) now takes the form

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|s_N(x)|^2\,dx=\sum_{-N}^{N}|c_n|^2\le\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2\,dx. \tag{76}$$

In order to obtain an expression for s_N that is more manageable than (75) we introduce the *Dirichlet kernel*

$$D_N(x)=\sum_{n=-N}^{N}e^{inx}=\frac{\sin(N+\frac{1}{2})x}{\sin(x/2)}. \tag{77}$$

The first of these equalities is the definition of $D_N(x)$. The second follows if both sides of the identity

$$(e^{ix}-1)D_N(x)=e^{i(N+1)x}-e^{-iNx}$$

are multiplied by $e^{-ix/2}$.

By (62) and (75), we have

$$\begin{aligned}s_N(f;x)&=\sum_{-N}^{N}\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)e^{-int}\,dt\,e^{inx}\\&=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)\sum_{-N}^{N}e^{in(x-t)}\,dt,\end{aligned}$$

so that

$$s_N(f;x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)D_N(x-t)\,dt=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x-t)D_N(t)\,dt. \tag{78}$$

The periodicity of all functions involved shows that it is immaterial over which interval we integrate, as long as its length is 2π. This shows that the two integrals in (78) are equal.

We shall prove just one theorem about the pointwise convergence of Fourier series.

8.14 Theorem *If, for some x, there are constants $\delta>0$ and $M<\infty$ such that*

$$|f(x+t)-f(x)|\le M|t| \tag{79}$$

for all $t\in(-\delta,\delta)$, then

$$\lim_{N\to\infty}s_N(f;x)=f(x). \tag{80}$$

Proof Define

$$g(t)=\frac{f(x-t)-f(x)}{\sin(t/2)} \tag{81}$$

for $0 < |t| \le \pi$, and put $g(0) = 0$. By the definition (77),

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(x)\,dx = 1.$$

Hence (78) shows that

$$\begin{aligned} s_N(f;x) - f(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)\sin\left(N+\frac{1}{2}\right)t\,dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[g(t)\cos\frac{t}{2}\right]\sin Nt\,dt + \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[g(t)\sin\frac{t}{2}\right]\cos Nt\,dt. \end{aligned}$$

By (79) and (81), $g(t)\cos(t/2)$ and $g(t)\sin(t/2)$ are bounded. The last two integrals thus tend to 0 as $N \to \infty$, by (74). This proves (80).

Corollary *If $f(x) = 0$ for all x in some segment J, then* $\lim s_N(f;x) = 0$ *for every $x \in J$.*

Here is another formulation of this corollary:

If $f(t) = g(t)$ for all t in some neighborhood of x, then

$$s_N(f;x) - s_N(g;x) = s_N(f-g;x) \to 0 \text{ as } N \to \infty.$$

This is usually called the *localization theorem.* It shows that the behavior of the sequence $\{s_N(f;x)\}$, as far as convergence is concerned, depends only on the values of f in some (arbitrarily small) neighborhood of x. Two Fourier series may thus have the same behavior in one interval, but may behave in entirely different ways in some other interval. We have here a very striking contrast between Fourier series and power series (Theorem 8.5).

We conclude with two other approximation theorems.

8.15 Theorem *If f is continuous (with period 2π) and if $\varepsilon > 0$, then there is a trigonometric polynomial P such that*

$$|P(x) - f(x)| < \varepsilon$$

for all real x.

Proof If we identify x and $x + 2\pi$, we may regard the 2π-periodic functions on R^1 as functions on the unit circle T, by means of the mapping $x \to e^{ix}$. The trigonometric polynomials, i.e., the functions of the form (60), form a self-adjoint algebra $\mathscr{A}$, which separates points on T, and which vanishes at no point of T. Since T is compact, Theorem 7.33 tells us that $\mathscr{A}$ is dense in $\mathscr{C}(T)$. This is exactly what the theorem asserts.

A more precise form of this theorem appears in Exercise 15.

8.16 Parseval's theorem *Suppose f and g are Riemann-integrable functions with period 2π, and*

(82) $$f(x) \sim \sum_{-\infty}^{\infty} c_n e^{inx}, \qquad g(x) \sim \sum_{-\infty}^{\infty} \gamma_n e^{inx}.$$

Then

(83) $$\lim_{N\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x) - s_N(f;x)|^2\,dx = 0,$$

(84) $$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx = \sum_{-\infty}^{\infty} c_n \bar{\gamma}_n,$$

(85) $$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx = \sum_{-\infty}^{\infty} |c_n|^2.$$

Proof Let us use the notation

(86) $$\|h\|_2 = \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} |h(x)|^2\,dx\right\}^{1/2}.$$

Let $\varepsilon > 0$ be given. Since $f \in \mathscr{R}$ and $f(\pi) = f(-\pi)$, the construction described in Exercise 12 of Chap. 6 yields a continuous 2π-periodic function h with

(87) $$\|f - h\|_2 < \varepsilon.$$

By Theorem 8.15, there is a trigonometric polynomial P such that $|h(x) - P(x)| < \varepsilon$ for all x. Hence $\|h - P\|_2 < \varepsilon$. If P has degree N_0, Theorem 8.11 shows that

(88) $$\|h - s_N(h)\|_2 \le \|h - P\|_2 < \varepsilon$$

for all $N \ge N_0$. By (72), with $h - f$ in place of f,

(89) $$\|s_N(h) - s_N(f)\|_2 = \|s_N(h - f)\|_2 \le \|h - f\|_2 < \varepsilon.$$

Now the triangle inequality (Exercise 11, Chap. 6), combined with (87), (88), and (89), shows that

(90) $$\|f - s_N(f)\|_2 < 3\varepsilon \qquad (N \ge N_0).$$

This proves (83). Next,

(91) $$\frac{1}{2\pi}\int_{-\pi}^{\pi} s_N(f)\bar{g}\,dx = \sum_{-N}^{N} c_n \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}\,\overline{g(x)}\,dx = \sum_{-N}^{N} c_n \bar{\gamma}_n,$$

and the Schwarz inequality shows that

(92) $$\left|\int f\bar{g} - \int s_N(f)\bar{g}\right| \le \int |f - s_N(f)||g| \le \left\{\int |f - s_N|^2 \int |g|^2\right\}^{1/2},$$

which tends to 0, as $N \to \infty$, by (83). Comparison of (91) and (92) gives (84). Finally, (85) is the special case $g = f$ of (84).

A more general version of Theorem 8.16 appears in Chap. 11.

THE GAMMA FUNCTION

This function is closely related to factorials and crops up in many unexpected places in analysis. Its origin, history, and development are very well described in an interesting article by P. J. Davis (*Amer. Math. Monthly*, vol. 66, 1959, pp. 849–869). Artin's book (cited in the Bibliography) is another good elementary introduction.

Our presentation will be very condensed, with only a few comments after each theorem. This section may thus be regarded as a large exercise, and as an opportunity to apply some of the material that has been presented so far.

8.17 Definition For $0 < x < \infty$,

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt. \tag{93}$$

The integral converges for these x. (When $x < 1$, both 0 and ∞ have to be looked at.)

8.18 Theorem

(*a*) *The functional equation*

$$\Gamma(x+1) = x\Gamma(x)$$

holds if $0 < x < \infty$.

(*b*) $\Gamma(n+1) = n!$ *for* $n = 1, 2, 3, \ldots$.

(*c*) $\log \Gamma$ *is convex on* $(0, \infty)$.

Proof An integration by parts proves (*a*). Since $\Gamma(1) = 1$, (*a*) implies (*b*), by induction. If $1 < p < \infty$ and $(1/p) + (1/q) = 1$, apply Hölder's inequality (Exercise 10, Chap. 6) to (93), and obtain

$$\Gamma\left(\frac{x}{p} + \frac{y}{q}\right) \le \Gamma(x)^{1/p}\Gamma(y)^{1/q}.$$

This is equivalent to (*c*).

It is a rather surprising fact, discovered by Bohr and Mollerup, that these three properties characterize Γ completely.

8.19 Theorem *If f is a positive function on* $(0, \infty)$ *such that*

(*a*) $f(x+1) = xf(x)$,

(*b*) $f(1) = 1$,

(*c*) $\log f$ *is convex,*

then $f(x) = \Gamma(x)$.

Proof Since Γ satisfies (*a*), (*b*), and (*c*), it is enough to prove that $f(x)$ is uniquely determined by (*a*), (*b*), (*c*), for all $x > 0$. By (*a*), it is enough to do this for $x \in (0, 1)$.

Put $\varphi = \log f$. Then

$$\varphi(x+1) = \varphi(x) + \log x \qquad (0 < x < \infty), \tag{94}$$

$\varphi(1) = 0$, and φ is convex. Suppose $0 < x < 1$, and n is a positive integer. By (94), $\varphi(n+1) = \log(n!)$. Consider the difference quotients of φ on the intervals $[n, n+1]$, $[n+1, n+1+x]$, $[n+1, n+2]$. Since φ is convex

$$\log n \le \frac{\varphi(n+1+x) - \varphi(n+1)}{x} \le \log(n+1).$$

Repeated application of (94) gives

$$\varphi(n+1+x) = \varphi(x) + \log[x(x+1)\cdots(x+n)].$$

Thus

$$0 \le \varphi(x) - \log\left[\frac{n!n^x}{x(x+1)\cdots(x+n)}\right] \le x \log\left(1 + \frac{1}{n}\right).$$

The last expression tends to 0 as $n \to \infty$. Hence $\varphi(x)$ is determined, and the proof is complete.

As a by-product we obtain the relation

$$\Gamma(x) = \lim_{n\to\infty} \frac{n!n^x}{x(x+1)\cdots(x+n)} \tag{95}$$

at least when $0 < x < 1$; from this one can deduce that (95) holds for all $x > 0$, since $\Gamma(x+1) = x\Gamma(x)$.

8.20 Theorem *If* $x > 0$ *and* $y > 0$, *then*

$$\int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{96}$$

This integral is the so-called *beta function* $B(x, y)$.

Proof Note that $B(1, y) = 1/y$, that $\log B(x, y)$ is a convex function of x, for each fixed y, by Hölder's inequality, as in Theorem 8.18, and that

$$B(x+1, y) = \frac{x}{x+y} B(x, y). \tag{97}$$

To prove (97), perform an integration by parts on

$$B(x+1, y) = \int_0^1 \left(\frac{t}{1-t}\right)^x (1-t)^{x+y-1}\, dt.$$

These three properties of $B(x, y)$ show, for each y, that Theorem 8.19 applies to the function f defined by

$$f(x) = \frac{\Gamma(x+y)}{\Gamma(y)} B(x, y).$$

Hence $f(x) = \Gamma(x)$.

8.21 Some consequences The substitution $t = \sin^2 \theta$ turns (96) into

$$2\int_0^{\pi/2} (\sin\theta)^{2x-1} (\cos\theta)^{2y-1}\, d\theta = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{98}$$

The special case $x = y = \frac{1}{2}$ gives

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}. \tag{99}$$

The substitution $t = s^2$ turns (93) into

$$\Gamma(x) = 2\int_0^{\infty} s^{2x-1} e^{-s^2}\, ds \qquad (0 < x < \infty). \tag{100}$$

The special case $x = \frac{1}{2}$ gives

$$\int_{-\infty}^{\infty} e^{-s^2}\, ds = \sqrt{\pi}. \tag{101}$$

By (99), the identity

$$\Gamma(x) = \frac{2^{x-1}}{\sqrt{\pi}} \Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{x+1}{2}\right) \tag{102}$$

follows directly from Theorem 8.19.

8.22 Stirling's formula This provides a simple approximate expression for $\Gamma(x+1)$ when x is large (hence for $n!$ when n is large). The formula is

$$\lim_{x\to\infty} \frac{\Gamma(x+1)}{(x/e)^x \sqrt{2\pi x}} = 1. \tag{103}$$

Here is a proof. Put $t = x(1+u)$ in (93). This gives

$$\Gamma(x+1) = x^{x+1}\, e^{-x} \int_{-1}^{\infty} [(1+u)e^{-u}]^x \, du. \tag{104}$$

Determine $h(u)$ so that $h(0) = 1$ and

$$(1+u)e^{-u} = \exp\left[-\frac{u^2}{2} h(u)\right] \tag{105}$$

if $-1 < u < \infty$, $u \neq 0$. Then

$$h(u) = \frac{2}{u^2}[u - \log(1+u)]. \tag{106}$$

It follows that h is continuous, and that $h(u)$ decreases monotonically from ∞ to 0 as u increases from -1 to ∞.

The substitution $u = s\sqrt{2/x}$ turns (104) into

$$\Gamma(x+1) = x^x e^{-x} \sqrt{2x} \int_{-\infty}^{\infty} \psi_x(s)\, ds \tag{107}$$

where

$$\psi_x(s) = \begin{cases} \exp[-s^2 h(s\sqrt{2/x})] & (-\sqrt{x/2} < s < \infty), \\ 0 & (s \le -\sqrt{x/2}). \end{cases}$$

Note the following facts about $\psi_x(s)$:

(*a*) For every s, $\psi_x(s) \to e^{-s^2}$ as $x \to \infty$.
(*b*) The convergence in (*a*) is uniform on $[-A, A]$, for every $A < \infty$.
(*c*) When $s < 0$, then $0 < \psi_x(s) < e^{-s^2}$.
(*d*) When $s > 0$ and $x > 1$, then $0 < \psi_x(s) < \psi_1(s)$.
(*e*) $\int_0^\infty \psi_1(s)\, ds < \infty$.

The convergence theorem stated in Exercise 12 of Chap. 7 can therefore be applied to the integral (107), and shows that this integral converges to $\sqrt{\pi}$ as $x \to \infty$, by (101). This proves (103).

A more detailed version of this proof may be found in R. C. Buck's "Advanced Calculus," pp. 216–218. For two other, entirely different, proofs, see W. Feller's article in *Amer. Math. Monthly*, vol. 74, 1967, pp. 1223–1225 (with a correction in vol. 75, 1968, p. 518) and pp. 20–24 of Artin's book.

Exercise 20 gives a simpler proof of a less precise result.

EXERCISES

1. Define

$$f(x)=\begin{cases} e^{-1/x^2} & (x\neq 0), \\ 0 & (x=0). \end{cases}$$

Prove that f has derivatives of all orders at $x=0$, and that $f^{(n)}(0)=0$ for $n=1,2,3,\ldots$.

2. Let a_{ij} be the number in the ith row and jth column of the array

$$\begin{array}{ccccc} -1 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & -1 & 0 & 0 & \cdots \\ \frac{1}{4} & \frac{1}{2} & -1 & 0 & \cdots \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & -1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

so that

$$a_{ij}=\begin{cases} 0 & (i<j), \\ -1 & (i=j), \\ 2^{j-i} & (i>j). \end{cases}$$

Prove that

$$\sum_i \sum_j a_{ij}=-2, \qquad \sum_j \sum_i a_{ij}=0.$$

3. Prove that

$$\sum_i \sum_j a_{ij}=\sum_j \sum_i a_{ij}$$

if $a_{ij}\geq 0$ for all i and j (the case $+\infty=+\infty$ may occur).

4. Prove the following limit relations:

(*a*) $\displaystyle\lim_{x\to 0}\frac{b^x-1}{x}=\log b \qquad (b>0).$

(*b*) $\displaystyle\lim_{x\to 0}\frac{\log(1+x)}{x}=1.$

(*c*) $\displaystyle\lim_{x\to 0}(1+x)^{1/x}=e.$

(*d*) $\displaystyle\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n=e^x.$

5. Find the following limits

(*a*) $\lim\limits_{x \to 0} \dfrac{e - (1+x)^{1/x}}{x}$.

(*b*) $\lim\limits_{n \to \infty} \dfrac{n}{\log n} [n^{1/n} - 1]$.

(*c*) $\lim\limits_{x \to 0} \dfrac{\tan x - x}{x(1 - \cos x)}$.

(*d*) $\lim\limits_{x \to 0} \dfrac{x - \sin x}{\tan x - x}$.

6. Suppose $f(x)f(y) = f(x+y)$ for all real x and y.

(*a*) Assuming that f is differentiable and not zero, prove that

$$f(x) = e^{cx}$$

where c is a constant.

(*b*) Prove the same thing, assuming only that f is continuous.

7. If $0 < x < \dfrac{\pi}{2}$, prove that

$$\frac{2}{\pi} < \frac{\sin x}{x} < 1.$$

8. For $n = 0, 1, 2, \dots,$ and x real, prove that

$$|\sin nx| \le n|\sin x|.$$

Note that this inequality may be false for other values of n. For instance,

$$|\sin \tfrac{1}{2}\pi| > \tfrac{1}{2}|\sin \pi|.$$

9. (*a*) Put $s_N = 1 + (\frac{1}{2}) + \cdots + (1/N)$. Prove that

$$\lim_{N \to \infty} (s_N - \log N)$$

exists. (The limit, often denoted by γ, is called Euler's constant. Its numerical value is $0.5772\dots$. It is not known whether γ is rational or not.)

(*b*) Roughly how large must m be so that $N = 10^m$ satisfies $s_N > 100$?

10. Prove that $\sum 1/p$ diverges; the sum extends over all primes.

(This shows that the primes form a fairly substantial subset of the positive integers.)