$$(9\text{-}18)\qquad \begin{aligned} T^*\alpha_j &= a\alpha_j - b\beta_j, \\ T^*\beta_j &= b\alpha_j + a\beta_j \end{aligned} \qquad 1 \le j \le s.$$

Let $W = V_1 + \cdots + V_s$. Then W is the orthogonal direct sum of $V_1, \ldots, V_s$. We shall show that $W = V$. Suppose that this is not the case. Then $W^\perp \neq \{0\}$. Moreover, since (iii) and (9-18) imply that W is invariant under T and T^*, it follows that $W^\perp$ is invariant under T^* and $T = T^{**}$. Let $S = b^{-1}(T - aI)$. Then $S^* = b^{-1}(T^* - aI)$, $S^*S = SS^*$, and $W^\perp$ is invariant under S and S^*. Since $(T - aI)^2 + b^2I = 0$, it follows that $S^2 + I = 0$. Let α be any vector of norm 1 in $W^\perp$ and set $\beta = S\alpha$. Then β is in $W^\perp$ and $S\beta = -\alpha$. Since $T = aI + bS$, this implies

$$\begin{aligned} T\alpha &= a\alpha + b\beta \\ T\beta &= -b\alpha + a\beta. \end{aligned}$$

By the lemma, $S^*\alpha = -\beta$, $S^*\beta = \alpha$, $(\alpha|\beta) = 0$, and $||\beta|| = 1$. Because $T^* = aI + bS^*$, it follows that

$$\begin{aligned} T^*\alpha &= a\alpha - b\beta \\ T^*\beta &= b\alpha + a\beta. \end{aligned}$$

But this contradicts the fact that $V_1, \ldots, V_s$ is a maximal collection of subspaces satisfying (i), (iii), and (9-18). Therefore, $W = V$, and since

$$\det \begin{bmatrix} x - a & b \\ -b & x - a \end{bmatrix} = (x - a)^2 + b^2$$

it follows from (i), (ii) and (iii) that

$$\det(xI - T) = [(x - a)^2 + b^2]^s. \quad \blacksquare$$

Corollary. *Under the conditions of the theorem,* T *is invertible, and*

$$T^* = (a^2 + b^2)T^{-1}.$$

Proof. Since

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} a & b \\ -b & a \end{bmatrix} = \begin{bmatrix} a^2 + b^2 & 0 \\ 0 & a^2 + b^2 \end{bmatrix}$$

it follows from (iii) and (9-18) that $TT^* = (a^2 + b^2)I$. Hence T is invertible and $T^* = (a^2 + b^2)T^{-1}$.

Theorem 19. *Let* T *be a normal operator on a finite-dimensional inner product space* V. *Then any linear operator that commutes with* T *also commutes with* T*. *Moreover, every subspace invariant under* T *is also invariant under* T*.

Proof. Suppose U is a linear operator on V that commutes with T. Let E_j be the orthogonal projection of V on the primary component

W_j $(1 \leq j \leq k)$ of V under T. Then E_j is a polynomial in T and hence commutes with U. Thus

$$E_jUE_j = UE_j^2 = UE_j.$$

Thus $U(W_j)$ is a subset of W_j. Let T_j and U_j denote the restrictions of T and U to W_j. Suppose I_j is the identity operator on W_j. Then U_j commutes with T_j, and if $T_j = c_jI_j$, it is clear that U_j also commutes with $T_j^* = \bar{c}_jI_j$. On the other hand, if T_j is not a scalar multiple of I_j, then T_j is invertible and there exist real numbers a_j and b_j such that

$$T_j^* = (a_j^2 + b_j^2)T_j^{-1}.$$

Since $U_jT_j = T_jU_j$, it follows that $T_j^{-1}U_j = U_jT_j^{-1}$. Therefore U_j commutes with T_j^* in both cases. Now T^* also commutes with E_j, and hence W_j is invariant under T^*. Moreover for every α and β in W_j

$$(T_j\alpha|\beta) = (T\alpha|\beta) = (\alpha|T^*\beta) = (\alpha|T_j^*\beta).$$

Since $T^*(W_j)$ is contained in W_j, this implies T_j^* is the restriction of T^* to W_j. Thus

$$UT^*\alpha_j = T^*U\alpha_j$$

for every α_j in W_j. Since V is the sum of $W_1, \ldots, W_k$, it follows that

$$UT^*\alpha = T^*U\alpha$$

for every α in V and hence that U commutes with T^*.

Now suppose W is a subspace of V that is invariant under T, and let $Z_j = W \cap W_j$. By the corollary to Theorem 17, $W = \sum_j Z_j$. Thus it suffices to show that each Z_j is invariant under T_j^*. This is clear if $T_j = c_jI$. When this is not the case, T_j is invertible and maps Z_j into and hence onto Z_j. Thus $T_j^{-1}(Z_j) = Z_j$, and since

$$T_j^* = (a_j^2 + b_j^2)T_j^{-1}$$

it follows that $T^*(Z_j)$ is contained in Z_j, for every j. ∎

Suppose T is a normal operator on a finite-dimensional inner product space V. Let W be a subspace invariant under T. Then the preceding corollary shows that W is invariant under T^*. From this it follows that $W^\perp$ is invariant under $T^{**} = T$ (and hence under T^* as well). Using this fact one can easily prove the following strengthened version of the cyclic decomposition theorem given in Chapter 7.

Theorem 20. *Let* T *be a normal linear operator on a finite-dimensional inner product space* V *(dim* $\mathrm{V} \geq 1$*). Then there exist* r *non-zero vectors* $\alpha_1, \ldots, \alpha_r$ *in* V *with respective* T*-annihilators* $e_1, \ldots, e_r$ *such that*

(i) $\mathrm{V} = \mathrm{Z}(\alpha_1; \mathrm{T}) \oplus \cdots \oplus \mathrm{Z}(\alpha_r; \mathrm{T})$;
(ii) *if* $1 \leq k \leq r - 1$, *then* e_{k+1} *divides* e_k;

(iii) $Z(\alpha_j; T)$ *is orthogonal to* $Z(\alpha_k; T)$ *when* $j \neq k$. *Furthermore, the integer* r *and the annihilators* $e_1, \ldots, e_r$ *are uniquely determined by conditions* (i) *and* (ii) *and the fact that no* α_k *is* 0.

Corollary. *If* A *is a normal matrix with real (complex) entries, then there is a real orthogonal (unitary) matrix* P *such that* $P^{-1}AP$ *is in rational canonical form.*

It follows that two normal matrices A and B are unitarily equivalent if and only if they have the same rational form; A and B are orthogonally equivalent if they have real entries and the same rational form.

On the other hand, there is a simpler criterion for the unitary equivalence of normal matrices and normal operators.

Definitions. *Let* V *and* V' *be inner product spaces over the same field. A linear transformation*

$$U: V \to V'$$

is called a **unitary transformation** *if it maps* V *onto* V' *and preserves inner products. If* T *is a linear operator on* V *and* T' *a linear operator on* V', *then* T *is* **unitarily equivalent** *to* T' *if there exists a unitary transformation* U *of* V *onto* V' *such that*

$$UTU^{-1} = T'.$$

Lemma. *Let* V *and* V' *be finite-dimensional inner product spaces over the same field. Suppose* T *is a linear operator on* V *and that* T' *is a linear operator on* V'. *Then* T *is unitarily equivalent to* T' *if and only if there is an orthonormal basis* $\mathcal{B}$ *of* V *and an orthonormal basis* $\mathcal{B}'$ *of* V' *such that*

$$[T]_{\mathcal{B}} = [T']_{\mathcal{B}'}.$$

Proof. Suppose there is a unitary transformation U of V onto V' such that $UTU^{-1} = T'$. Let $\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}$ be any (ordered) orthonormal basis for V. Let $\alpha_j' = U\alpha_j$ $(1 \leq j \leq n)$. Then $\mathcal{B}' = \{\alpha_1', \ldots, \alpha_n'\}$ is an orthonormal basis for V' and setting

$$T\alpha_j = \sum_{k=1}^{n} A_{kj}\alpha_k$$

we see that

$$\begin{aligned} T'\alpha_j' &= UT\alpha_j \\ &= \sum_k A_{kj}U\alpha_k \\ &= \sum_k A_{kj}\alpha_k' \end{aligned}$$

Hence $[T]_{\mathcal{B}} = A = [T']_{\mathcal{B}'}$.

Conversely, suppose there is an orthonormal basis $\mathcal{B}$ of V and an orthonormal basis $\mathcal{B}'$ of V' such that

$$[T]_{\mathcal{B}} = [T']_{\mathcal{B}'}$$

and let $A = [T]_{\mathcal{B}}$. Suppose $\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}$ and that $\mathcal{B}' = \{\alpha'_1, \ldots, \alpha'_n\}$. Let U be the linear transformation of V into V' such that $U\alpha_j = \alpha'_j$ $(1 \leq j \leq n)$. Then U is a unitary transformation of V onto V', and

$$\begin{aligned} UTU^{-1}\alpha'_j &= UT\alpha_j \\ &= U \sum_k A_{kj}\alpha_k \\ &= \sum_k A_{kj}\alpha'_k. \end{aligned}$$

Therefore, $UTU^{-1}\alpha'_j = T'\alpha'_j$ $(1 \leq j \leq n)$, and this implies $UTU^{-1} = T'$. ▌

It follows immediately from the lemma that unitarily equivalent operators on finite-dimensional spaces have the same characteristic polynomial. For normal operators the converse is valid.

Theorem 21. *Let* V *and* V′ *be finite-dimensional inner product spaces over the same field. Suppose* T *is a normal operator on* V *and that* T′ *is a normal operator on* V′. *Then* T *is unitarily equivalent to* T′ *if and only if* T *and* T′ *have the same characteristic polynomial.*

Proof. Suppose T and T' have the same characteristic polynomial f. Let W_j $(1 \leq j \leq k)$ be the primary components of V under T and T_j the restriction of T to W_j. Suppose I_j is the identity operator on W_j. Then

$$f = \prod_{j=1}^{k} \det (xI_j - T_j).$$

Let p_j be the minimal polynomial for T_j. If $p_j = x - c_j$ it is clear that

$$\det (xI_j - T_j) = (x - c_j)^{s_j}$$

where s_j is the dimension of W_j. On the other hand, if $p_j = (x - a_j)^2 + b_j^2$ with a_j, b_j real and $b_j \neq 0$, then it follows from Theorem 18 that

$$\det (xI_j - T_j) = p_j^{s_j}$$

where in this case $2s_j$ is the dimension of W_j. Therefore $f = \prod_j p_j^{s_j}$. Now we can also compute f by the same method using the primary components of V' under T'. Since $p_1, \ldots, p_k$ are distinct primes, it follows from the uniqueness of the prime factorization of f that there are exactly k primary components W'_j $(1 \leq j \leq k)$ of V' under T' and that these may be indexed in such a way that p_j is the minimal polynomial for the restriction T'_j of T' to W'_j. If $p_j = x - c_j$, then $T_j = c_jI_j$ and $T'_j = c_jI'_j$ where I'_j is the identity operator on W_j'. In this case it is evident that T_j is unitarily equivalent to T_j'. If $p_j = (x - a_j)^2 + b_j^2$, as above, then using the lemma and Theorem 20, we again see that T_j is unitarily equivalent to T_j'. Thus for each j there are orthonormal bases $\mathcal{B}_j$ and $\mathcal{B}_j'$ of W_j and W_j', respectively, such that

$$[T_j]_{\mathcal{B}_j} = [T_j']_{\mathcal{B}_j'}.$$

Now let U be the linear transformation of V into V' that maps each $\mathcal{B}_j$ onto $\mathcal{B}_j'$. Then U is a unitary transformation of V onto V' such that $UTU^{-1} = T'$. ▮

10. Bilinear Forms

10.1. Bilinear Forms

In this chapter, we treat bilinear forms on finite-dimensional vector spaces. The reader will probably observe a similarity between some of the material and the discussion of determinants in Chapter 5 and of inner products and forms in Chapter 8 and in Chapter 9. The relation between bilinear forms and inner products is particularly strong; however, this chapter does not presuppose any of the material in Chapter 8 or Chapter 9. The reader who is not familiar with inner products would probably profit by reading the first part of Chapter 8 as he reads the discussion of bilinear forms.

This first section treats the space of bilinear forms on a vector space of dimension n. The matrix of a bilinear form in an ordered basis is introduced, and the isomorphism between the space of forms and the space of $n \times n$ matrices is established. The rank of a bilinear form is defined, and non-degenerate bilinear forms are introduced. The second section discusses symmetric bilinear forms and their diagonalization. The third section treats skew-symmetric bilinear forms. The fourth section discusses the group preserving a non-degenerate bilinear form, with special attention given to the orthogonal groups, the pseudo-orthogonal groups, and a particular pseudo-orthogonal group—the Lorentz group.

Definition. *Let* V *be a vector space over the field* F. *A* **bilinear form** *on* V *is a function* f, *which assigns to each ordered pair of vectors* α, β *in* V *a scalar* $f(\alpha, \beta)$ *in* F, *and which satisfies*