The *Jade Mirror of Four Unknowns* does not go beyond four equations in four unknowns (hence the name). The idea is quite general, but it becomes hard to implement on the counting board when there are more than four unknowns. An amusing problem in three unknowns from the *Jade Mirror*, which does not require the full strength of the elimination method, is given in the exercises below.

6.2.4 Problem 2 in the *Jade Mirror* [see Hoe (1977), p. 135] is to find the side a of a right-angled triangle (a,b,c) such that

$$a^2 - (b+c-a) = ab,$$
$$b^2 + (a+c-b) = bc.$$

The *Jade Mirror* suggests choosing the unknowns $x = a$ and $y = b + c$. Using $a^2 = c^2 - b^2$, show that this implies

$$b = (y - x^2/y)/2,$$
$$c = (y + x^2/y)/2.$$

6.2.5 Deduce that the first two equations in Exercise 6.2.4 are equivalent, respectively, to

$$(-2-x)y^2 + (2x+2x^2)y + x^3 = 0,$$
$$(2-x)y^2 + 2xy + x^3 = 0.$$

6.2.6 By subtracting one equation in Exercise 6.2.5 from the other, deduce that $y = x^2/2$. Substitute this back to obtain a quadratic equation for x, with solution $x = a = 4$. What are the values of b and c?

6.3 Quadratic Equations

As early as 2000 BCE, the Babylonians could solve pairs of simultaneous equations of the form

$$x + y = p,$$
$$xy = q,$$

which are equivalent to the quadratic equations

$$x^2 + q = px.$$

The original pair was solved by a method that gave the two roots of the quadratic:

$$x, y = \frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$$

when both were positive (the Babylonians did not admit negative numbers). The steps in the method were as follows:

(i) Form $\frac{x+y}{2}$.

(ii) Form $\left(\frac{x+y}{2}\right)^2$.

(iii) Form $\left(\frac{x+y}{2}\right)^2 - xy$.

(iv) Form $\sqrt{\left(\frac{x+y}{2}\right)^2 - xy} = \frac{x-y}{2}$.

(v) Find x, y by inspection of the values in (i), (iv).

[See Boyer (1968), p. 34, for an actual example.] Of course, these steps were not expressed in symbols but only applied to specific numbers. Nevertheless, a general method is implicit in the many specific cases solved.

An explicit general method, expressed as a formula in words, was given by Brahmagupta (628):

> To the absolute number multiplied by four times the [coefficient of the] square, add the square of the [coefficient of the] middle term; the square root of the same, less the [coefficient of the] middle term, being divided by twice the [coefficient of the] square is the value. [Colebrooke (1817), p. 346]

This is the solution

$$x = \frac{\sqrt{4ac + b^2} - b}{2a}$$

of the equation

$$ax^2 + bx = c,$$

yet one wonders whether Brahmagupta understood it quite this way when, a few lines later, he gives another rule that is trivially equivalent to the first when expressed in our notation:

$$x = \frac{\sqrt{ac + (b/2)^2} - (b/2)}{a}.$$

The methods of the Babylonians and Brahmagupta clearly give correct solutions, but their basis is not clear. The meaning of square roots, for example, was not questioned as it was by Greeks. A rigorous basis for the solution of quadratic equations can be found in Euclid's *Elements*, Book VI.

Proposition 28 can be interpreted as a solution of the general quadratic equation in the case where there is a positive root, as Heath (1925), Vol. 2, p. 263 explains. However, the algebraic interpretation is far from obvious even when one specializes the proposition, which is about parallelograms, to one about rectangles. It seems unlikely that Euclid was aware of the algebra, or he would have expressed it by much simpler geometry.

The transition from geometry to algebra can be seen in al-Khwārizmī's solution of a quadratic equation (Figure 6.1). The solution is still expressed in geometric language, but now the geometry is a direct embodiment of the algebra. It is really the standard algebraic solution, but with "squares" and "products" understood literally as geometric squares and rectangles. To solve $x^2 + 10x = 39$, represent x^2 by a square of side x, and $10x$ by two $5 \times x$ rectangles as in Figure 5.1. The extra square of area 25 "completes the square" of side $x+5$ to one of area $25+39$, since 39 is the given value of $x^2 + 10x$. Thus the big square has area 64, hence its side $x+5 = 8$. This gives the solution $x = 3$.

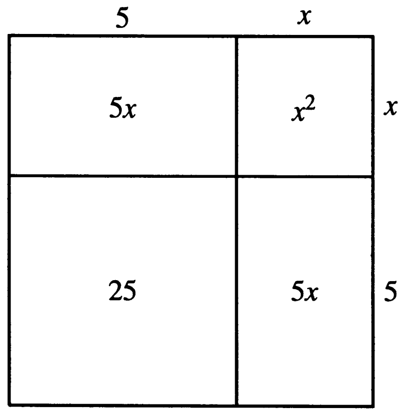

Figure 6.1: Solving a quadratic equation

Euclid and al-Khwārizmī did not admit negative lengths, so the solution $x = -13$ to $x^2 + 10x = 39$ does not appear. This is quite natural, since geometry admits only one square with area 64. Avoidance of negative coefficients, however, causes some unnatural algebraic complications. There is not one general quadratic equation, but three, corresponding to the different ways of distributing positive terms between the two sides: $x^2 + ax = b$, $x^2 = ax + b$, $x^2 + b = ax$.

EXERCISES

Quadratic equations arise frequently in geometry because distance is governed by a quadratic equation (ultimately, by the Pythagorean theorem). In fact, the points created from rational points by any ruler and compass construction can be found by solving a series of linear or quadratic equations, which is why they can be expressed by rational operations and square roots. This result, which was claimed in Section 2.3, can be proved as follows.

6.3.1 Show that the line through two rational points has an equation with rational coefficients.

6.3.2 Show that a circle whose center is a rational point and whose radius is rational has an equation with rational coefficients.

Your proof should show, more generally, that a line or circle constructed from *any* points has an equation with coefficients obtainable from the coordinates of the given points by rational operations. It then suffices to show that intersections of lines and circles can be obtained from the coefficients of their equations by rational operations and square roots.

6.3.3 Show that the intersection of two lines can be computed by rational operations.

6.3.4 Show that the intersection of a line and a circle can be computed by rational operations and a square root (because it depends on solving a quadratic equation).

The last, and hardest, case is finding the intersection of two circles. Fortunately, it is easy to reduce these two quadratic equations to the case just handled in Exercise 6.3.4.

6.3.5 The equations of any two circles can be written in the form

$$(x-a)^2+(y-b)^2=r^2,$$
$$(x-c)^2+(y-d)^2=s^2.$$

Explain why. Now subtract one of these equations from the other, and hence show that their common solutions can be found by rational operations and square roots.

When a sequence of quadratic equations is solved, the solution may involve *nested* square roots, such as $\sqrt{(5+\sqrt{5})/2}$. This very number, in fact, occurs in the icosahedron, as one sees from Pacioli's construction in Section 2.2.

6.3.6 Show that the diagonal of a golden rectangle (which is also the diameter of an icosahedron of edge length 1) is $\sqrt{(5+\sqrt{5})/2}$.

6.4 Quadratic Irrationals

The roots of quadratic equations with rational coefficients are numbers of the form $a+\sqrt{b}$, where a,b are rational. Euclid took the theory of irrationals further in Book X of the *Elements* with a very detailed study of numbers of the form $\sqrt{\sqrt{a}\pm\sqrt{b}}$, where a, b are rational. Book X is the longest book in the *Elements* and it is not clear why Euclid devoted so much space to this topic: perhaps because some of it is needed for the study of regular polyhedra in Book XIII (see Section 2.2 and Exercise 6.3.6), perhaps simply because it was Euclid's favorite topic, or perhaps it was one in which he had some original contributions to show off. It is said that Apollonius took the theory of irrationals further, but unfortunately his work on the subject is lost.

After this, there seems to have been no progress in the theory of irrationals until the Renaissance, except for a remarkable isolated result by Fibonacci (1225). Fibonacci showed that the roots of $x^3+2x^2+10x=20$ are not any of Euclid's irrationals. This is *not* a proof, as some historians have thought, that the roots cannot be constructed by ruler and compasses. Fibonacci did not rule out *all* expressions built from rationals and square roots; nevertheless, it was the first step into the world of irrationals beyond Euclid.

At this point it is worth asking how difficult it is to show that a specific number, say, $\sqrt[3]{2}$, cannot be constructed from rational numbers by square roots. The answer will depend on how well the reader manages the following exercises. The manipulation required would certainly not have been beyond the sixteenth-century algebraists. The subtle part is finding a suitable classification of expressions according to complexity—extending Euclid's classification to expressions in which radical signs are nested to arbitrary depth—and using induction on the level of complexity. This type of thinking did not emerge until the 1820s, hence the relatively late proof that $\sqrt[3]{2}$ is not constructible by ruler and compass [Wantzel (1837)].

EXERCISES

An elementary proof that $\sqrt[3]{2}$ is not constructible was found by the number theorist Edmund Landau (1877–1938) when he was still a student. It is broken down to easy steps below. But first we should check that $\sqrt[3]{2}$ is actually irrational.

6.4.1 Show that the assumption $\sqrt[3]{2}=m/n$, where m and n are integers, leads to a contradiction.

Landau's proof now proceeds by organizing all constructible numbers into sets $F_0, F_1, F_2, \ldots$, according to the "depth of nesting" of square roots.

6.4.2 Let

$$F_0 = \{\text{rational numbers}\}, F_{k+1} = \{a + b\sqrt{c_k} : a, b \in F_k\} \text{ for some } c_k \in F_k.$$

Show that each F_k is a *field*, that is, if x, y are in F_k, then so are $x+y$, $x-y$, xy, and x/y (for $y \neq 0$).

We know from Exercise 6.4.1 that $\sqrt[3]{2}$ is not in F_0, but if it is constructible it will occur in some F_{k+1}. A contradiction now ensues by considering (hypothetically) the first such F_{k+1}.

6.4.3 Show that if $a, b, c \in F_k$ but $\sqrt{c} \notin F_k$, then $a + b\sqrt{c} = 0 \Leftrightarrow a = b = 0$. (For $k = 0$ this is in the *Elements*, Book X, Prop. 79.)

6.4.4 Suppose $\sqrt[3]{2} = a + b\sqrt{c}$ where $a, b, c \in F_k$ but that $\sqrt[3]{2} \notin F_k$. (We know $\sqrt[3]{2} \notin F_0$ from Exercise 6.4.1.) Cube both sides and deduce from Exercise 6.4.3 that

$$2 = a^3 + 3ab^2c \quad \text{and} \quad 0 = 3a^2b + b^3c.$$

6.4.5 Deduce from Exercise 6.4.4 that $\sqrt[3]{2} = a - b\sqrt{c}$ also, and explain why this is a contradiction.

6.5 The Solution of the Cubic

> In our own days Scipione del Ferro of Bologna has solved the case of the cube and first power equal to a constant, a very elegant and admirable accomplishment. Since this art surpasses all human subtlety and the perspicuity of mortal talent and is a truly celestial gift and a very clear test of the capacity of men's minds, whoever applies himself to it will believe that there is nothing that he cannot understand. In emulation of him, my friend Niccolò Tartaglia of Brescia, wanting not to be outdone, solved the same case when he got into a contest with his [Scipione's] pupil, Antonio Maria Fior, and, moved by my many entreaties, gave it to me . . . having received Tartaglia's solution and seeking a proof of it, I came to understand that there were a great many other things that could also be had. Pursuing this thought and with increased confidence, I discovered these others, partly by myself and partly through Lodovico Ferrari, formerly my pupil. [Cardano (1545), p. 8]

The solution of cubic equations in the early sixteenth century was the first clear advance in mathematics since the time of the Greeks. It revealed the power of algebra which the Greeks had not been able to harness, power that was soon to clear a new path to geometry, which was virtually a royal road (analytic geometry and calculus). Cardano's elation at the discovery is completely understandable. Even in the twentieth century, personally discovering the solution of the cubic equation has been the inspiration for at least one distinguished mathematical career [see Kac (1984)].

As far as the history of the original discovery goes, we do not know much more than Cardano tells us. Scipione del Ferro died in 1526, so the first solution was known before then. Tartaglia discovered his solution on February 12, 1535, probably independently, because he solved all problems in the contest with del Ferro's pupil Fior, whereas Fior did not. Cardano has been accused by almost everyone, from Tartaglia on, of stealing Tartaglia's solution, but his own account seems to distribute credit quite fairly. For more background, see the introduction and preface to Cardano (1545) and Crossley (1987).

Cardano presents algebra in the geometric style of al-Khwārizmī (whom he describes as the originator of algebra at the beginning of the book), with the case distinctions that result from avoidance of negative coefficients. By ignoring these complications, his solution can be described as follows. The cubic equation $x^3+ax^2+bx+c=0$ is first transformed into one with no quadratic term by a linear change of variable, namely, $x=y-a/3$. One then has, say,

$$y^3 = py+q.$$

By setting $y=u+v$, the left-hand side becomes

$$(u^3+v^3)+3uv(u+v) = 3uvy+(u^3+v^3),$$

which will equal the previous right-hand side if

$$3uv = p,$$
$$u^3+v^3 = q.$$

Eliminating v gives a quadratic in u^3,

$$u^3+\left(\frac{p}{3u}\right)^3 = q$$

with roots

$$\frac{q}{2}\pm\sqrt{\left(\frac{q}{2}\right)^2-\left(\frac{p}{3}\right)^3}.$$

By symmetry, we obtain the same values for v^3. And since $u^3+v^3=q$, if one of the roots is taken to be u^3, the other is v^3. Without loss of generality we can take

$$u^3=\frac{q}{2}+\sqrt{\left(\frac{q}{2}\right)^2-\left(\frac{p}{3}\right)^3},$$
$$v^3=\frac{q}{2}-\sqrt{\left(\frac{q}{2}\right)^2-\left(\frac{p}{3}\right)^3},$$

and hence

$$y=u+v=\sqrt[3]{\frac{q}{2}+\sqrt{\left(\frac{q}{2}\right)^2-\left(\frac{p}{3}\right)^3}}+\sqrt[3]{\frac{q}{2}-\sqrt{\left(\frac{q}{2}\right)^2-\left(\frac{p}{3}\right)^3}}.$$

EXERCISES

The two equations $3uv=p$, $u^3+v^3=q$ provide another instance of the phenomenon noted in Exercise 6.2.2: when a variable is eliminated between two equations, the degrees of the equations are multiplied.

6.5.1 The equation $3uv=p$ is of degree 2 in u and v, and $u^3+v^3=q$ is of degree 3. What about the equation obtained by eliminating v?

The Cardano formula produces some surprising results, which we look at again in Section 14.3. But first let us test it on a really simple cubic equation.

6.5.2 Use Cardano's formula to solve $y^3=2$. Do you get the obvious solution?

Now try one where the solution is less obvious.

6.5.3 Use Cardano's formula to solve $y^3=6y+6$, and check your answer by substitution.

6.6 Angle Division

Another important contributor to algebra in the sixteenth century was Viète (1540–1603). He helped emancipate algebra from the geometric style of proof by introducing letters for unknowns and using plus and minus signs to facilitate manipulation. Yet at the same time he strengthened its ties with geometry at a higher level by relating algebra to trigonometry. A case in point is his solution of the cubic by circular functions [Viète (1591), Ch. VI, Theorem 3], which shows that solving the cubic is equivalent to trisecting an arbitrary angle.