Proof: By Corollary 2 we may write $V = U_1 \oplus U_2 \oplus \cdots \oplus U_m$, where U_i is an irreducible FG-submodule of V. Let $\mathcal{B}_i$ be a basis of U_i and let $\mathcal{B}$ be the union of the $\mathcal{B}_i$'s. For each $g \in G$, the matrix of $\varphi(g)$ with respect to the basis $\mathcal{B}$ is of the form in the corollary, where $\varphi_i(g)$ is the matrix of $\varphi(g)|_{U_i}$ with respect to the basis $\mathcal{B}_i$.

The converse of Maschke's Theorem is also true. Namely, if the characteristic of F does divide $|G|$, then G possesses (finitely generated) FG-modules which are not completely reducible. Specifically, the regular representation (i.e., the module FG itself) is not completely reducible.

In Section 18.2 we shall discuss the question of uniqueness of the constituents in direct sum decompositions of FG-modules into irreducible submodules.

EXERCISES

Let F be a field, let G be a finite group and let $n \in \mathbb{Z}^+$.

1. Prove that if $\varphi : G \to GL(V)$ is any representation, then φ gives a faithful representation of $G/\ker\varphi$.
2. Let $\varphi : G \to GL_n(F)$ be a matrix representation. Prove that the map $g \mapsto \det(\varphi(g))$ is a degree 1 representation.
3. Prove that the degree 1 representations of G are in bijective correspondence with the degree 1 representations of the abelian group G/G' (where G' is the commutator subgroup of G).
4. Let V be a (possibly infinite dimensional) FG-module (G is a finite group). Prove that for each $v \in V$ there is an FG-submodule containing v of dimension $\leq |G|$.
5. Prove that if $|G| > 1$ then every irreducible FG-module has dimension $< |G|$.
6. Write out the matrices $\varphi(g)$ for every $g \in G$ for each of the following representations that were described in the second set of examples:
   (a) the representation of S_3 described in Example 3 (let $n = 3$ in that example)
   (b) the representation of D_8 described in Example 6 (i.e., let $n = 4$ in that example and write out the values of all the sines and cosines, for all group elements)
   (c) the representation of Q_8 described in Example 7
   (d) the representation of Q_8 described in Example 8.
7. Let V be the 4-dimensional permutation module for S_4 described in Example 3 of the second set of examples. Let $\pi : D_8 \to S_4$ be the permutation representation of D_8 obtained from the action of D_8 by left multiplication on the set of left cosets of its subgroup $\langle s \rangle$. Make V into an FD_8-module via π as described in Example 4 and write out the 4×4 matrices for r and s given by this representation with respect to the basis $e_1, \dots, e_4$.
8. Let V be the FS_n-module described in Examples 3 and 10 in the second set of examples.
   (a) Prove that if v is any element of V such that $\sigma \cdot v = v$ for all $\sigma \in S_n$ then v is an F-multiple of $e_1 + e_2 + \cdots + e_n$.
   (b) Prove that if $n \geq 3$, then V has a unique 1-dimensional submodule, namely the submodule N consisting of all F-multiples of $e_1 + e_2 + \cdots + e_n$.
9. Prove that the 4-dimensional representation of Q_8 on $\mathbb{H}$ described in Example 8 in the second set of examples is irreducible. [Show that any Q_8-stable subspace is a left ideal.]
10. Prove that $GL_2(\mathbb{R})$ has no subgroup isomorphic to Q_8. [This may be done by direct computation using generators and relations for Q_8. Simplify these calculations by putting one generator in rational canonical form.]

11. Let $\varphi : S_n \to GL_n(F)$ be the matrix representation given by the permutation module described in Example 3 in the second set of examples, where the matrices are computed with respect to the basis $e_1, \dots, e_n$. Prove that $\det \varphi(\sigma) = \epsilon(\sigma)$ for all $\sigma \in S_n$, where $\epsilon(\sigma)$ is the sign of the permutation σ. [Check this on transpositions.]

12. Assume the characteristic of F is not 2. Let H be the set of $T \in M_n(F)$ such that T has exactly one nonzero entry in each row and each column and zeros elsewhere, and the nonzero entries are ± 1. Prove that H is a subgroup of $GL_n(F)$ and that H is isomorphic to $E_{2^n} \rtimes S_n$ (semidirect product), where E_{2^n} is the elementary abelian group of order 2^n.

The next few exercises explore an important result known as Schur's Lemma and some of its consequences.

13. Let R be a ring and let M and N be simple (i.e., irreducible) R-modules.
 (a) Prove that every nonzero R-module homomorphism from M to N is an isomorphism. [Consider its kernel and image.]
 (b) Prove Schur's Lemma: if M is a simple R-module then $\mathrm{Hom}_R(M, M)$ is a division ring (recall that $\mathrm{Hom}_R(M, M)$ is the ring of all R-module homomorphisms from M to M, where multiplication in this ring is composition).

14. Let $\varphi : G \to GL(V)$ be a representation of G. The *centralizer* of φ is defined to be the set of all linear transformations, A, from V to itself such that $A\varphi(g) = \varphi(g)A$ for all $g \in G$ (i.e., the linear transformations of V which commute with all $\varphi(g)$'s).
 (a) Prove that a linear transformation A from V to V is in the centralizer of φ if and only if it is an FG-module homomorphism from V to itself (so the centralizer of φ is the same as the *ring* $\mathrm{Hom}_{FG}(V, V)$).
 (b) Show that if z is in the center of G then $\varphi(z)$ is in the centralizer of φ.
 (c) Assume φ is an irreducible representation (so V is a simple FG-module). Prove that if H is any finite *abelian* subgroup of $GL(V)$ such that $A\varphi(g) = \varphi(g)A$ for all $A \in H$ then H is cyclic (in other words, any finite abelian subgroup of the multiplicative group of units in the ring $\mathrm{Hom}_{FG}(V, V)$ is cyclic). [By the preceding exercise, $\mathrm{Hom}_{FG}(V, V)$ is a division ring, so this reduces to proving that a finite abelian subgroup of the multiplicative group of nonzero elements in a division ring is cyclic. Show that the division subring generated by an abelian subgroup of any division ring is a field and use Proposition 18, Section 9.5.]
 (d) Show that if φ is a faithful irreducible representation then the center of G is cyclic.
 (e) Deduce from (d) that if G is abelian and φ is any irreducible representation then $G/\ker\varphi$ is cyclic.

15. Exhibit all 1-dimensional complex representations of a finite cyclic group; make sure to decide which are inequivalent.

16. Exhibit all 1-dimensional complex representations of a finite abelian group. Deduce that the number of inequivalent degree 1 complex representations of a finite abelian group equals the order of the group. [First decompose the abelian group into a direct product of cyclic groups, then use the preceding exercise.]

17. Prove the following variant of Schur's Lemma for complex representations of abelian groups: if G is abelian, any irreducible complex representation, φ, of G is of degree 1 and $G/\ker\varphi$ is cyclic. [This can be done without recourse to Exercise 14 by using the observation that for any $g \in G$ the eigenspaces of $\varphi(g)$ are G-stable. Your proof that φ has degree 1 should also work for infinite abelian groups.]

18. Prove the following general form of Schur's Lemma for complex representations: if $\varphi : G \to GL_n(\mathbb{C})$ is an irreducible matrix representation and A is an $n \times n$ matrix commuting with $\varphi(g)$ for all $g \in G$, then A is a scalar matrix. Deduce that if φ is a faithful, irreducible, complex representation then the center of G is cyclic and $\varphi(z)$ is a scalar matrix for all elements z in the center of G. [As in the preceding exercise, the eigenspaces of A are G-stable.]

19. Prove that if G is an abelian group then any finite dimensional complex representation of G is equivalent to a representation into diagonal matrices (i.e., any finite group of commuting matrices over $\mathbb{C}$ can be simultaneously diagonalized). [This can be done without recourse to Maschke's Theorem by looking at eigenspaces.]

20. Prove that the number of degree 1 complex representations of any finite group G equals $|G : G'|$, where G' is the commutator subgroup of G. [Use Exercises 3 and 16.]

21. Let G be a noncyclic abelian group acting by conjugation on an elementary abelian p-group V, where p is a prime not dividing the order of G.

(a) Prove that if W is an irreducible $\mathbb{F}_pG$-submodule of V then there is some nonidentity element $g \in G$ such that $W \leq C_V(g)$ (here $C_V(g)$ is the subgroup of elements of V that are fixed by g under conjugation).

(b) Prove that V is generated by the subgroups $C_V(g)$ as g runs over all nonidentity elements of G.

22. Let p be a prime, let P be a p-group and let F be a field of characteristic p. Prove that the only irreducible representation of P over F is the trivial representation. [Do this for a group of order p first using the fact that F contains all p^{th} roots of 1 (namely 1 itself). If P is not of order p, let z be an element of order p in the center of P, prove that z is in the kernel of the irreducible representation and apply induction to $P/\langle z \rangle$.]

23. Let p be a prime, let P be a nontrivial p-group and let F be a field of characteristic p. Prove that the regular representation is not completely reducible. [Use the preceding exercise.]

24. Let p be a prime, let P be a nontrivial p-group and let F be a field of characteristic p. Prove that the regular representation is indecomposable.

18.2 WEDDERBURN'S THEOREM AND SOME CONSEQUENCES

In this section we give a famous classification theorem due to Wedderburn which describes, in particular, the structure of the group algebra FG when the characteristic of F does not divide the order of G. From this classification theorem we shall derive various consequences, including the fact that for each finite group G there are only a finite number of nonisomorphic irreducible FG-modules. This result, together with Maschke's Theorem, in some sense completes the Hölder Program for representation theory of finite groups over such fields. The remainder of the book is concerned with developing techniques for determining and working with the irreducible representations as well as applying this knowledge to obtain group-theoretic information.

Theorem 4. *(Wedderburn's Theorem)* Let R be a nonzero ring with 1 (not necessarily commutative). Then the following are equivalent:

(1) every R-module is projective
(2) every R-module is injective
(3) every R-module is completely reducible
(4) the ring R considered as a left R-module is a direct sum:

$$R = L_1 \oplus L_2 \oplus \cdots \oplus L_n,$$