This partially ordered set A has a (unique) maximal element, X.

In many instances the set A consists of some (but not necessarily all) subsets of a set X (i.e., A is a subset of the power set of X) and with the ordering on A again being inclusion. The existence of upper bounds and maximal elements depends on the nature of A.

(2) Let A be the collection of all *proper* subsets of $\mathbb{Z}^+$ ordered under $\subseteq$. In this situation, chains need not have maximal elements, e.g. the chain

$$\{1\} \subseteq \{1,2\} \subseteq \{1,2,3\} \subseteq \cdots$$

does not have an upper bound. The set A does have maximal elements: for example $\mathbb{Z}^+ - \{n\}$ is a maximal element of A for any $n \in \mathbb{Z}^+$.

(3) Let $A = \mathbb{R}$ under the usual $\leq$ relation. In this example every subset of A is a chain (including A itself). The notion of a subset of A having an upper bound is the same as the usual notion of a subset of $\mathbb{R}$ being bounded above by some real number (so some sets, such as intervals of finite length, have upper bounds and others, such as the set of positive reals, do not). The set A does not have a maximal element.

Zorn's Lemma If A is a nonempty partially ordered set in which every chain has an upper bound then A has a maximal element.

It is a nontrivial result that *Zorn's Lemma is independent of the usual (Zermelo–Fraenkel) axioms of set theory*[1] in the sense that if the axioms of set theory are consistent,[2] then so are these axioms together with Zorn's Lemma; and if the axioms of set theory are consistent, then so are these axioms together with the *negation* of Zorn's Lemma. The use of the term "lemma" in Zorn's Lemma is historical.

For the sake of completeness (and to relate Zorn's Lemma to formulations found in other courses) we include two other equivalent formulations of Zorn's Lemma.

The Axiom of Choice The Cartesian product of any nonempty collection of nonempty sets is nonempty. In other words, if I is any nonempty (indexing) set and A_i is a nonempty set for all $i \in I$, then there exists a choice function from I to $\cup_{i \in I} A_i$.

Definition. Let A be a nonempty set. A *well ordering* on A is a total ordering on A such that every nonempty subset of A has a minimum (or smallest) element, i.e., for each nonempty $B \subseteq A$ there is some $s \in B$ such that $s \leq b$, for all $b \in B$.

The Well Ordering Principle Every nonempty set A has a well ordering.

Theorem 2. Assuming the usual (Zermelo–Fraenkel) axioms of set theory, the following are equivalent:

(1) Zorn's Lemma
(2) the Axiom of Choice
(3) the Well Ordering Principle.

Proof: This follows from elementary set theory. We refer the reader to *Real and Abstract Analysis* by Hewitt and Stromberg, Springer-Verlag, 1965, Section 3 for these equivalences and some others.

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[1] See P.J. Cohen's papers in: Proc. Nat. Acad. Sci., 50(1963), and 51(1964).

[2] This is not known to be the case!

EXERCISES

1. Let A be the collection of all finite subsets of $\mathbb{R}$ ordered by inclusion. Discuss the existence (or nonexistence) of upper bounds, minimal and maximal elements (where minimal elements are defined analogously to maximal elements). Explain why this is not a well ordering.
2. Let A be the collection of all infinite subsets of $\mathbb{R}$ ordered by inclusion. Discuss the existence (or nonexistence) of upper bounds, minimal and maximal elements. Explain why this is not a well ordering.
3. Show that the following partial orderings on the given sets are not well orderings:
   (a) $\mathbb{R}$ under the usual relation $\leq$.
   (b) $\mathbb{R}^+$ under the usual relation $\leq$.
   (c) $\mathbb{R}^+ \cup \{0\}$ under the usual relation $\leq$.
   (d) $\mathbb{Z}$ under the usual relation $\leq$.
4. Show that $\mathbb{Z}^+$ is well ordered under the usual relation $\leq$.

APPENDIX II

Category Theory

Category theory provides the language and the mathematical foundations for discussing properties of large classes of mathematical objects such as the class of "all sets" or "all groups" while circumventing problems such as Russell's Paradox. In this framework one may explore the commonality across classes of concepts and methods used in the study of each class: homomorphisms, isomorphisms, etc., and one may introduce tools for studying relations between classes: functors, equivalence of categories, etc. One may then formulate precise notions of a "natural" transformation and "natural" isomorphism, both within a given class or between two classes. (In the text we described "natural" as being "coordinate free.") A prototypical example of natural isomorphisms within a class is the isomorphism of an arbitrary finite dimensional vector space with its double dual in Section 11.3. In fact one of the primary motivations for the introduction of categories and functors by S. Eilenberg and S. MacLane in 1945 was to give a precise meaning to the notions of "natural" in cases such as this. Category theory has also played a foundational role for formalizing new concepts such as schemes (cf. Section 15.5) that are fundamental to major areas of contemporary research (e.g., algebraic geometry). Pioneering work of this nature was done by A. Grothendieck, K. Morita and others.

Our treatment of category theory should be viewed more as an introduction to some of the basic language. Since we have not discussed the Zermelo–Fraenkel axioms of set theory or the Gödel–Bernays axioms of classes we make no mention of the foundations of category theory. To remain consistent with the set theory axioms, however, we implicitly assume that there is a *universe* set **U** which contains all the sets, groups, rings, etc. that one would encounter in "ordinary" mathematics (so that the category of "all sets" implicitly means "all sets in **U**"). The reader is referred to books on set theory, logic, or category theory such as *Categories for the Working Mathematician* by S. MacLane, Springer–Verlag, 1971 for further study.

We have organized this appendix so that wherever possible the examples of each new concept use terminology and structures in the order that these appear in the body of the text. For instance, the first example of a functor involves sets and groups, the second example uses rings, etc. In this way the appendix may be read early on in one's study, and a greater appreciation may be gained through rereading the examples as one becomes conversant with a wider variety of mathematical structures.

1. CATEGORIES AND FUNCTORS

We begin with the basic concept of this appendix.

Definition. A *category* **C** consists of a class of *objects* and sets of *morphisms* between those objects. For every ordered pair A, B of objects there is a set $\mathrm{Hom}_{\mathbf{C}}(A, B)$ of morphisms from A to B, and for every ordered triple A, B, C of objects there is a *law of composition* of morphisms, i.e., a map

$$\mathrm{Hom}_{\mathbf{C}}(A, B) \times \mathrm{Hom}_{\mathbf{C}}(B, C) \longrightarrow \mathrm{Hom}_{\mathbf{C}}(A, C)$$

where $(f, g) \mapsto gf$, and gf is called the composition of g with f. The objects and morphism satisfy the following axioms: for objects A, B, C and D

(i) if $A \neq B$ or $C \neq D$, then $\mathrm{Hom}_{\mathbf{C}}(A, B)$ and $\mathrm{Hom}_{\mathbf{C}}(C, D)$ are disjoint sets,

(ii) composition of morphisms is associative, i.e., $h(gf) = (hg)f$ for every f in $\mathrm{Hom}_{\mathbf{C}}(A, B)$, g in $\mathrm{Hom}_{\mathbf{C}}(B, C)$ and h in $\mathrm{Hom}_{\mathbf{C}}(C, D)$,

(iii) each object has an identity morphism, i.e., for every object A there is a morphism $1_A \in \mathrm{Hom}_{\mathbf{C}}(A, A)$ such that $f1_A = f$ for every $f \in \mathrm{Hom}_{\mathbf{C}}(A, B)$ and $1_A g = g$ for every $g \in \mathrm{Hom}_{\mathbf{C}}(B, A)$.

Morphisms are also called *arrows*. It is an exercise to see that the identity morphism for each object is unique (by the same argument that the identity of a group is unique). We shall write $\mathrm{Hom}(A, B)$ for $\mathrm{Hom}_{\mathbf{C}}(A, B)$ when the category is clear from the context.

The terminology we use throughout the text is common to all categories: a morphism from A to B will be denoted by $f : A \to B$ or $A \xrightarrow{f} B$. The object A is the *domain* of f and B is the *codomain* of f. A morphism from A to A is an endomorphism of A. A morphism $f : A \to B$ is an isomorphism if there is a morphism $g : B \to A$ such that $gf = 1_A$ and $fg = 1_B$.

There is a natural notion of a *subcategory* category $\mathbf{C}$ of $\mathbf{D}$, i.e., when every object of $\mathbf{C}$ is also an object in $\mathbf{D}$, and for objects A, B in $\mathbf{C}$ we have the containment $\mathrm{Hom}_{\mathbf{C}}(A, B) \subseteq \mathrm{Hom}_{\mathbf{D}}(A, B)$.

Examples

In each of the following examples we leave the details of the verification of the axioms for a category as exercises.

(1) **Set** is the category of all sets. For any two sets A and B, $\mathrm{Hom}(A, B)$ is the set of all functions from A to B. Composition of morphisms is the familiar composition of functions: $gf = g \circ f$. The identity in $\mathrm{Hom}(A, A)$ is the map $1_A(a) = a$, for all $a \in A$. This category contains the category of all finite sets as a subcategory.

(2) **Grp** is the category of all groups, where morphisms are group homomorphisms. Note that the composition of group homomorphisms is again a group homomorphism. A subcategory of **Grp** is **Ab**, the category of all abelian groups. Similarly, **Ring** is the category of all nonzero rings with 1, where morphisms are ring homomorphisms that send 1 to 1. The category **CRing** of all commutative rings with 1 is a subcategory of **Ring**.

(3) For a fixed ring R, the category R–**mod** consists of all left R-modules with morphisms being R-module homomorphisms.

(4) **Top** is the category whose objects are topological spaces and morphisms are continuous maps between topological spaces (cf. Section 15.2). Note that the identity (set) map from a space to itself is continuous in every topology, so $\mathrm{Hom}(A, A)$ always has an identity.

(5) Let **0** be the empty category, with no objects and no morphisms. Let **1** denote the category with one object, A, and one morphism: $\mathrm{Hom}(A, A) = \{1_A\}$. Let **2** be the category with two objects, A_1 and A_2, and only one nonidentity morphism: