Applying Theorem 11.5 we find

$$\iint_Q (x \sin y - ye^x)\, dx\, dy = \int_0^{\pi/2} A(y)\, dy = \int_0^{\pi/2} (-ey + y/e)\, dy$$

$$= (1/e - e) \int_0^{\pi/2} y\, dy = (1/e - e)\pi^2/8 .$$

As a check on the calculations we may integrate first with respect to y:

$$\iint_Q (x \sin y - ye^x)\, dx\, dy = \int_{-1}^{1} \left[\int_0^{\pi/2} (x \sin y - ye^x)\, dy \right] dx$$

$$= \int_{-1}^{1} \left(-x \cos y - \tfrac{1}{2}y^2 e^x\right) \Big|_{y=0}^{y=\pi/2} dx$$

$$= \int_{-1}^{1} \left(-\pi^2 e^x/8 + x\right) dx = (1/e - e)\pi^2/8 .$$

EXAMPLE 2. If $Q = [-1, 1] \times [0, 2]$, evaluate the double integral $\iint_Q \sqrt{|y - x^2|}\, dx\, dy$, given that it exists.

Solution. If we integrate first with respect to y and call the result H(x), we have $H(x) = \int_0^2 \sqrt{|y - x^2|}\, dy$. The region of integration is the rectangle shown in Figure 11.6. The parabola $y = x^2$ is also shown because of the presence of $|y - x^2|$ in the integrand. Above this parabola we have $y > x^2$ and below it we have $y < x^2$. This suggests that we split the integral for H(x) as follows:

$$H(x) = \int_0^2 \sqrt{|y - x^2|}\, dy = \int_0^{x^2} \sqrt{x^2 - y}\, dy + \int_{x^2}^2 \sqrt{y - x^2}\, dy .$$

We remember that x is treated as a constant in each of these integrals. In the first integral we make the change of variable $t = x^2 - y$ and in the second we put $t = y - x^2$. This gives us

$$H(x) = \int_0^2 \sqrt{|y - x^2|}\, dy = -\int_{x^2}^{0} \sqrt{t}\, dt + \int_0^{2-x^2} \sqrt{t}\, dt = \tfrac{2}{3}x^3 + \tfrac{2}{3}(2 - x^2)^{3/2}.$$

Applying Theorem 11.5 we find

$$\iint_Q \sqrt{|y - x^2|}\, dx\, dy = \int_{-1}^{1} \left[\tfrac{2}{3}x^3 + \tfrac{2}{3}(2 - x^2)^{3/2}\right] dx = \tfrac{4}{3} \int_0^1 (2 - x^2)^{3/2}\, dx$$

$$= \frac{1}{3}\left[x(2 - x^2)^{3/2} + 3x\sqrt{2 - x^2} + 6 \arcsin\left(\frac{x}{\sqrt{2}}\right)\right] \Bigg|_0^1 = \frac{4}{3} + \frac{\pi}{2} .$$

The same result may be obtained by integrating first with respect to x, but the calculations are more complicated.

11.9 Exercises

Evaluate the double integrals in Exercises 1 through 8 by repeated integration, given that each integral exists.

1. $\iint_Q xy(x+y)\,dx\,dy$, where $Q = [0, 1] \times [0, 1]$.

2. $\iint_Q (x^3 + 3x^2y + y^3)\,dx\,dy$, where $Q = [0, 1] \times [0, 1]$.

3. $\iint_Q (\sqrt{y} + x - 3xy^2)\,dx\,dy$, where $Q = [0, 1] \times [1, 3]$.

4. $\iint_Q \sin^2 x \sin^2 y\,dx\,dy$, where $Q = [0, \pi] \times [0, \pi]$.

5. $\iint_Q \sin(x+y)\,dx\,dy$, where $Q = [0, \pi/2] \times [0, \pi/2]$.

6. $\iint_Q |\cos(x+y)|\,dx\,dy$, where $Q = [0, \pi] \times [0, \pi]$.

7. $\iint_Q f(x+y)\,dx\,dy$, where $Q = [0, 2] \times [0, 2]$, and $f(t)$ denotes the greatest integer $\leq t$.

8. $\iint_Q y^{-3} e^{tx/y}\,dx\,dy$, where $Q = [0, t] \times [1, t]$, $t > 0$.

9. If Q is a rectangle, show that a double integral of the form $\iint_Q f(x)g(y)\,dx\,dy$ is equal to the product of two one-dimensional integrals. State the assumptions about existence that you are using.

10. Let f be defined on the rectangle $Q = [0, 1] \times [0, 1]$ as follows:

$$f(x, y) = \begin{cases} 1 - x - y & \text{if } x + y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Make a sketch of the ordinate set of f over Q and compute the volume of this ordinate set by double integration. (Assume the integral exists.)

11. Solve Exercise 10 when

$$f(x, y) = \begin{cases} x + y & \text{if } x^2 \leq y \leq 2x^2, \\ 0 & \text{otherwise.} \end{cases}$$

12. Solve Exercise 10 when $Q = [-1, 1] \times [-1, 1]$ and

$$f(x, y) = \begin{cases} x^2 + y^2 & \text{if } x^2 + y^2 \leq 1, \\ 0 & \text{otherwise} \end{cases}$$

13. Let f be defined on the rectangle $Q = [1, 2] \times [1, 4]$ as follows:

$$f(x, y) = \begin{cases} (x + y)^{-2} & \text{if } x \leq y \leq 2x, \\ 0 & \text{otherwise.} \end{cases}$$

Indicate, by means of a sketch, the portion of Q in which f is nonzero and compute the value of the double integral $\iint_Q f$, given that the integral exists.

14. Let f be defined on the rectangle $Q = [0, 1] \times [0, 1]$ as follows:

$$f(x, y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y \end{cases}$$

Prove that the double integral $\iint_Q f$ exists and equals 0.

11.10 Integrability of continuous functions

The small-span theorem (Theorem 9.10) can be used to prove integrability of a function which is continuous on a rectangle.

THEOREM 11.6. INTEGRABILITY OF CONTINUOUS FUNCTIONS. *If a function f is continuous on a rectangle $Q = [a, b] \times [c, d]$, then f is integrable on Q. Moreover, the value of the integral can be obtained by iterated integration,*

$$\iint_Q f = \int_c^d \left[\int_a^b f(x, y)\, dx\right] dy = \int_a^b \left[\int_c^d f(x, y)\, dy\right] dx. \tag{11.8}$$

Proof. Theorem 9.8 shows that f is bounded on Q, so f has an upper integral and a lower integral. We shall prove that $\underline{I}(f) = \bar{I}(f)$. Choose $\epsilon > 0$. By the small-span theorem, for this choice of ϵ there is a partition P of Q into a finite number (say n) of subrectangles $Q_1, \ldots, Q_n$ such that the span of f in every subrectangle is less than ϵ. Denote by $M_k(f)$ and $m_k(f)$, respectively, the absolute maximum and minimum values of f in Q_k. Then we have

$$M_k(f) - m_k(f) < \epsilon$$

for each $k = 1, 2, \ldots, n$. Now let s and t be two step functions defined on the interior of each Q_k as follows:

$$s(x) = m_k(f), \qquad t(x) = M_k(f) \qquad \text{if } x \in \operatorname{int} Q_k.$$

At the boundary points we define $s(x) = m$ and $t(x) = M$, where m and M are, respectively, the absolute minimum and maximum values of f on Q. Then we have $s \leq f \leq t$ for all x in Q. Also, we have

$$\iint_Q s = \sum_{k=1}^{n} m_k(f)a(Q_k) \qquad \text{and} \qquad \iint_Q t = \sum_{k=1}^{n} M_k(f)a(Q_k),$$

where $a(Q_k)$ is the area of rectangle Q_k. The difference of these two integrals is

$$\iint_Q t - \iint_Q s = \sum_{k=1}^{n} \{M_k(f) - m_k(f)\} a(Q_k) < \epsilon \sum_{k=1}^{n} a(Q_k) = \epsilon a(Q),$$

where $a(Q)$ is the area of Q. Since $\iint_Q s \le \underline{I}(f) \le \bar{I}(f) \le \iint_Q t$, we obtain the inequality

$$0 \le \bar{I}(f) - \underline{I}(f) \le \epsilon a(Q).$$

Letting $\epsilon \to 0$ we see that $\underline{I}(f) = \bar{I}(f)$, so f is integrable on Q.

Next we prove that the double integral is equal to the first iterated integral in (11.8). For each fixed y in $[c, d]$ the one-dimensional integral $\int_a^b f(x, y)\,dx$ exists since the integrand is continuous on Q. Let $A(y) = \int_a^b f(x, y)\,dx$. We shall prove that A is continuous on $[c, d]$. If y and y_1 are any two points in $[c, d]$ we have

$$A(y) - A(y_1) = \int_a^b \{f(x, y) - f(x, y_1)\}\,dx$$

from which we find

$$|A(y) - A(y_1)| \le (b - a) \max_{a \le x \le b} |f(x, y) - f(x, y_1)| = (b - a)\,|f(x_1, y) - f(x_1, y_1)|$$

where x_1 is a point in $[a, b]$ where $|f(x, y) - f(x, y_1)|$ attains its maximum. This inequality shows that $A(y) \to A(y_1)$ as $y \to y_1$, so A is continuous at y_1. Therefore the integral $\int_c^d A(y)\,dy$ exists and, by Theorem 11.5, it is equal to $\iint_Q f$. A similar argument works when the iteration is taken in the reverse order.

11.11 Integrability of bounded functions with discontinuities

Let f be defined and bounded on a rectangle Q. In Theorem 11.6 we proved that the double integral of f over Q exists if f is continuous everywhere on Q. In this section we prove that the integral also exists if f has discontinuities in Q, provided the set of discontinuities is not too large. To measure the size of the set of discontinuities we introduce the following concept.

DEFINITION OF A BOUNDED SET OF CONTENT ZERO. *Let A be a bounded subset of the plane. The set A is said to have content zero if for every $\epsilon > 0$ there is a finite set of rectangles whose union contains A and the sum of whose areas does not exceed ϵ.*

In other words, a bounded plane set of content zero can be enclosed in a union of rectangles whose total area is arbitrarily small.

The following statements about bounded sets of content zero are easy consequences of this definition. Proofs are left as exercises for the reader.

(a) Any finite set of points in the plane has content zero.

(b) The union of a finite number of bounded sets of content zero is also of content zero.

(c) Every subset of a set of content zero has content zero.

(d) Every line segment has content zero.

THEOREM 11.7. ***Let f be defined and bounded on a rectangle $Q = [a, b] \times [c, d]$. If the set of discontinuities of f in Q is a set of content zero then the double integral $\iint_Q f$ exists.***

Proof. Let $M > 0$ be such that $|f| \leq M$ on Q. Let D denote the set of discontinuities of f in Q. Given $\delta > 0$, let P be a partition of Q such that the sum of the areas of all the subrectangles of P which contain points of D is less than δ. (This is possible since D has content zero.) On these subrectangles define step functions s and t as follows:

$$s(x) = -M, \qquad t(x) = M.$$

On the remaining subrectangles of P define s and t as was done in the proof of Theorem 11.6. Then we have $s \leq f \leq t$ throughout Q. By arguing as in the proof of Theorem 11.6 we obtain the inequality

$$\iint_Q t - \iint_Q s \leq \epsilon a(Q) + 2M\delta. \tag{11.9}$$

The first term, $\epsilon a(Q)$, comes from estimating the integral of $t - s$ over the subrectangles containing only points of continuity of f; the second term, $2M\delta$, comes from estimating the integral of $t - s$ over the subrectangles which contain points of D. From (11.9) we obtain the inequality

$$0 \leq \bar{I}(f) - \underline{I}(f) \leq \epsilon a(Q) + 2M\delta.$$

Letting $\epsilon \to 0$ we find $0 \leq \bar{I}(f) - \underline{I}(f) \leq 2M\delta$. Since δ is arbitrary this implies $\bar{I}(f) = \underline{I}(f)$, so f is integrable on Q.

11.12 Double integrals extended over more general regions

Up to this point the double integral has been defined only for rectangular regions of integration. However, it is not difficult to extend the concept to more general regions.

Let S be a bounded region, and enclose S in a rectangle Q. Let f be defined and bounded on S. Define a new function $\tilde{f}$ on Q as follows:

$$\tilde{f}(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \in S, \\ 0 & \text{if } (x, y) \in Q - S. \end{cases} \tag{11.10}$$

In other words, extend the definition of f to the whole rectangle Q by making the function values equal to 0 outside S. Now ask whether or not the extended function $\tilde{f}$ is integrable on Q. If so, we say that f is integrable on S and that, *by definition,*

$$\iint_S f = \iint_Q \tilde{f}.$$

First we consider sets of points S in the xy-plane described as follows:

$$S = \{(x, y) \mid a \leq x \leq b \quad \text{and} \quad \varphi_1(x) \leq y \leq \varphi_2(x)\},$$

where φ_1 and φ_2 are functions continuous on a closed interval $[a, b]$ and satisfying $\varphi_1 \leq \varphi_2$. An example of such a region, which we call a region of Type I, is shown in Figure 11.7. In a region of Type I, for each point t in $[a, b]$ the vertical line $x = t$ intersects S in a line segment joining the curve $y = \varphi_1(x)$ to $y = \varphi_2(x)$. Such a region is bounded because φ_1 and φ_2 are continuous and hence bounded on $[a, h]$.

Another type of region T (Type II) can be described as follows:

$$T = \{(x, y) \mid c \leq y \leq d \text{ and } \psi_1(y) \leq x \leq \psi_2(y)\},$$

where ψ_1 and ψ_2 are continuous on an interval $[c, d]$ with $\psi_1 \leq \psi_2$. An example is shown in Figure 11.8. In this case horizontal lines intersect T in line segments. Regions of Type II are also bounded. All the regions we shall consider are either of one of these two types or can be split into a finite number of pieces, each of which is of one of these two types.

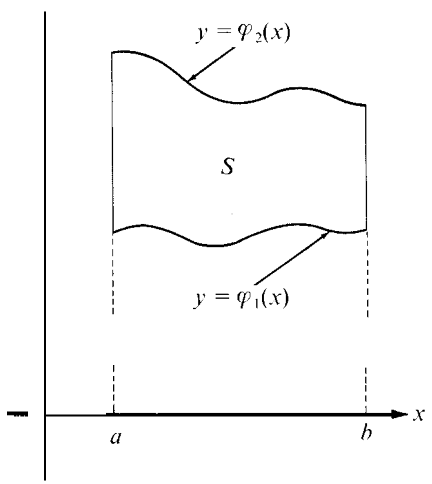

FIGURE 11.7 A region S of Type I

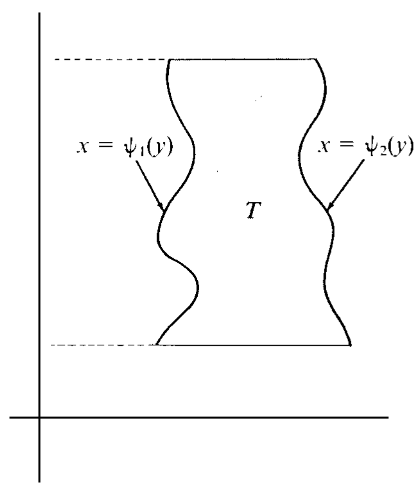

FIGURE 11.8 A region T of Type 11.

Let f be defined and bounded on a region S of Type I. Enclose S in a rectangle Q and definefon Q as indicated in Equation (11.10). The discontinuities of $\tilde{f}$ in Q will consist of the discontinuities off in S plus those points on the boundary of S at which f is nonzero. The boundary of S consists of the graph of φ_1, the graph of φ_2, and two vertical line segments joining these graphs. Each of the line segments has content zero. The next theorem shows that each of the graphs also has content zero.

THEOREM 11.8. *Let φ be a real-valued function that is continuous on an interval [a, b]. Then the graph of φ has content zero.*

Proof. Let A denote the graph of φ, that is, let

$$A = \{(x, y) \mid y = \varphi(x) \text{ and } a \leq x \leq b\}.$$

Choose any $\epsilon > 0$. We apply the small-span theorem (Theorem 3.13 of Volume I) to obtain a partition P of $[a, b]$ into a finite number of subintervals such that the span of φ in every subinterval of P is less than $\epsilon/(b - a)$. Therefore, above each subinterval of P the graph of φ lies inside a rectangle of height $\epsilon/(b - a)$. Hence the entire of graph φ lies within a finite union of rectangles, the sum of whose areas is ϵ. (An example is shown in Figure 11.9.) This proves that the graph of φ has content zero.

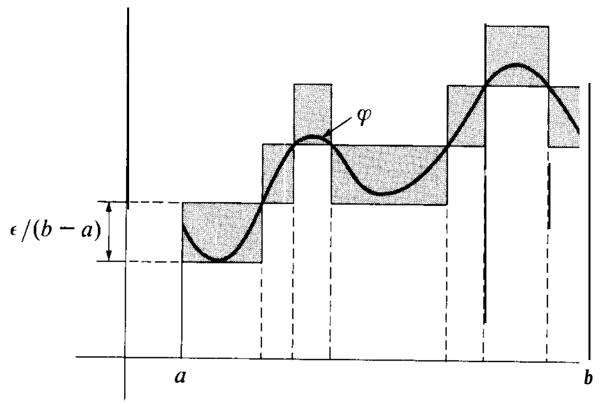

FIGURE 11.9 Proof that the graph of a continuous function φ has content zero.

The next theorem shows that the double integral $\iint_S f$ exists if f is continuous on int S, the interior of S. This is the set

$$\text{int } S = \{(x, y) \mid a < x < b \quad \text{and} \quad \varphi_1(x) < y < \varphi_2(x)\}.$$

THEOREM 11.9. ***Let S be a region of Type I, between the graphs of φ_1 and φ_2. Assume that f is defined and bounded on S and that f is continuous on the interior of S. Then the double integral $\iint_S f$ exists and can be evaluated by repeated one-dimensional integration,***

$$\iint_S f(x, y)\, dx\, dy = \int_a^b \left[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\, dy\right] dx. \tag{11.11}$$

Proof. Let $Q = [a, b] \times [c, d]$ be a rectangle which contains S, and let $\tilde{f}$ be defined by (11.10). The only possible points of discontinuity of $\tilde{f}$ are the boundary points of S. Since the boundary of S has content zero, $\tilde{f}$ is integrable on Q. For each fixed x in (a, b) the one-dimensional integral $\int_c^d \tilde{f}(x, y)\, dy$ exists since the integrand has at most two discontinuities on $[c, d]$. Applying the version of Theorem 11.5 given by Equation (11.7) we have

$$\iint_Q \tilde{f} = \int_a^b \left[\int_c^d \tilde{f}(x, y)\, dy\right] dx. \tag{11.12}$$

Now $\tilde{f}(x, y) = f(x, y)$ if $\varphi_1(x) \le y \le \varphi_2(x)$, and $\tilde{f}(x, y) = 0$ for the remaining values of y in [c, d]. Therefore

$$\int_c^d \tilde{f}(x, y)\, dy = \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\, dy$$

so Equation (11.12) implies (11.11).

There is, of course, an analogous theorem for a region T of Type II. If f is defined and bounded on T and continuous on the interior of T, then f is integrable on T and the formula for repeated integration becomes

$$(11.13) \qquad \iint_T f(x, y)\, dx\, dy = \int_c^d \left[\int_{\psi_1(y)}^{\psi_2(y)} f(x, y)\, dx\right] dy\,.$$

Some regions are of both Type I and Type II. (Regions bounded by circles and ellipses are examples.) In this case the order of integration is immaterial and we may write

$$\int_a^b \left[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\, dy\right] dx = \int_c^d \left[\int_{\psi_1(y)}^{\psi_2(y)} f(x, y)\, dx\right] dy\,.$$

In some cases one of these integrals may be much easier to compute than the other; it is usually worthwhile to examine both before attempting the actual evaluation of a double integral.

11.13 Applications to area and volume

Let S be a region of Type I given by

$$S = \{(x, y) \mid a \le x \le b \quad \text{and} \quad \varphi_1(x) \le y \le \varphi_2(x)\}\,.$$

Applying Theorem 11.9 with $f(x, y) = 1$ for all (x, y) in S we obtain

$$\iint_S dx\, dy = \int_a^b [\varphi_2(x) - \varphi_1(x)]\, dx\,.$$

By Theorem 2.1 of Volume I, the integral on the right is equal to the area of the region S. Thus, double integrals can be used to compute areas.

If f is nonnegative, the set of points (x, y, z) in 3-space such that $(x, y) \in S$ and $0 \le z \le f(x, y)$ is called the *ordinate set of f* over S. An example is shown in Figure 11.10. If f is nonnegative and continuous on S, the integral

$$\int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\, dy$$

represents the area of a cross-section of the ordinate set cut by a plane parallel to the yz-plane (the shaded region in Figure 11.10). Formula (11.11) of Theorem 11.9 shows that the double integral of f over S is equal to the integral of the cross-sectional area. Hence

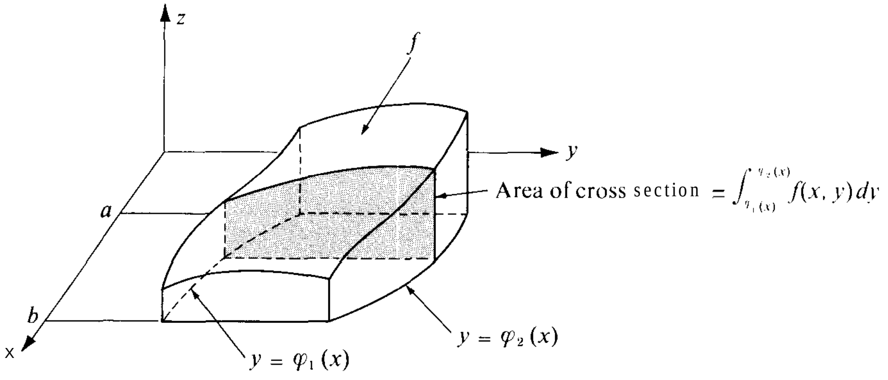

FIGURE 11.10 The integral $\int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\,dy$ is the area of a cross section of the ordinate set. The iterated integral $\int_a^b \left[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\,dy\right] dx$ is the volume of the ordinate set.

the double integral $\iint_S f$ is equal to the volume of the ordinate set of f over S. (See Theorem 2.7 of Volume I, p. 113.)

More generally, if f and g are both continuous on S with $f \le g$, then the double integral $\iint_S (g - f)$ is equal to the volume of the solid lying between the graphs of the functions f and g. Similar remarks apply, of course, to regions of Type II.

11.14 Worked examples

EXAMPLE 1. Compute the volume of the solid enclosed by the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 .$$

Solution. The solid in question lies between the graphs of two functions f and g, where

$$g(x, y) = c\sqrt{1 - x^2/a^2 - y^2/b^2} \qquad \text{and} \qquad f(x, y) = -g(x, y).$$

Here $(x, y) \in S$, where S is the elliptical region given by

$$S = \left\{(x, y) \;\middle|\; \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1\right\}.$$

Applying Theorem 11.9 and using the symmetry we find that the volume V of the ellipsoidal solid is given by

$$V = \iint_S (g - f) = 2 \iint_S g = 8c \int_0^a \left[\int_0^{b\sqrt{1-x^2/a^2}} \sqrt{1 - x^2/a^2 - y^2/b^2}\, dy\right] dx .$$

Let $A = \sqrt{1 - x^2/a^2}$. Then the inner integral is

$$\int_0^{bA} \sqrt{A^2 - y^2/b^2}\, dy = A \int_0^{bA} \sqrt{1 - y^2/(Ab)^2}\, dy .$$

Using the change of variable $y = Ab \sin t$, $dy = Ab \cos t\, dt$, we find that the last integral is equal to

$$A^2 b \int_0^{\pi/2} \cos^2 t\, dt = \frac{\pi}{4} A^2 b = \frac{\pi b}{4}\left(1 - \frac{x^2}{a^2}\right).$$

Therefore

$$V = 8c \int_0^a \frac{\pi b}{4}\left(1 - \frac{x^2}{a^2}\right) dx = \frac{4}{3}\pi abc .$$

In the special case $a = b = c$ the solid is a sphere of radius a and the volume is $\frac{4}{3}\pi a^3$.

EXAMPLE 2. The double integral of a positive function f, $\iint_S f(x, y)\, dx\, dy$, reduces to the repeated integral

$$\int_0^1 \left[\int_{x^2}^{x} f(x, y)\, dy\right] dx .$$

Determine the region S and interchange the order of integration.

Solution. For each fixed x between 0 and 1, the integration with respect to y is over the interval from x^2 to x. This means that the region is of Type I and lies between the two curves $y = x^2$ and $y = x$. The region S is the set of points between these two curves and above the interval [0, 1]. (See Figure 11.11.) Since S is also of Type II we may interchange

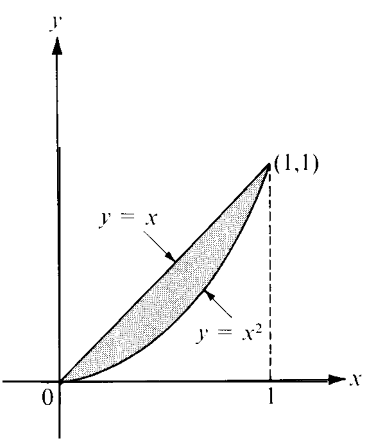

FIGURE 11.11 Example 2.

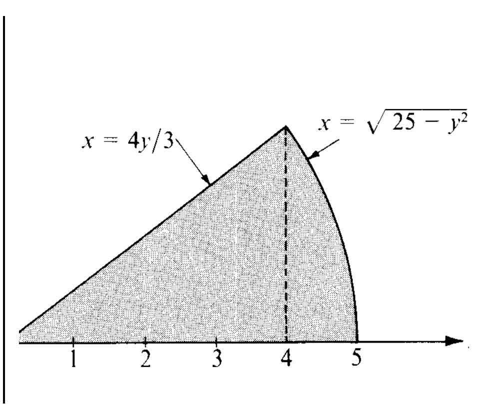

FIGURE 11.12 Example 3.