conic sections to give a very simple solution of the problem of duplication of the cube. In analytic notation, this can be described as finding the intersection of the parabola $y = \frac{1}{2}x^2$ with the hyperbola $xy = 1$. This yields

$$x\frac{1}{2}x^2 = 1 \qquad \text{or} \qquad x^3 = 2.$$

Although the Greeks accepted this as a "construction" for duplicating the cube, they apparently never discussed instruments for actually drawing conic sections. This is very puzzling since a natural generalization of the compass immediately suggests itself (Figure 2.8). The arm A is set at a fixed position relative to a plane P, while the other arm rotates about it at a fixed angle θ, generating a cone with A as its axis of symmetry. The pencil, which is free to slide in a sleeve on this second arm, traces the section of the cone lying in the plane P. According to Coolidge (1945), p. 149, this instrument for drawing conic sections was first described as late as 1000 CE by the Arab mathematician al-Kuji. Yet nearly all the *theoretical* facts one could wish to know about conic sections had already been worked out by Apollonius (around 250–200 BCE)!

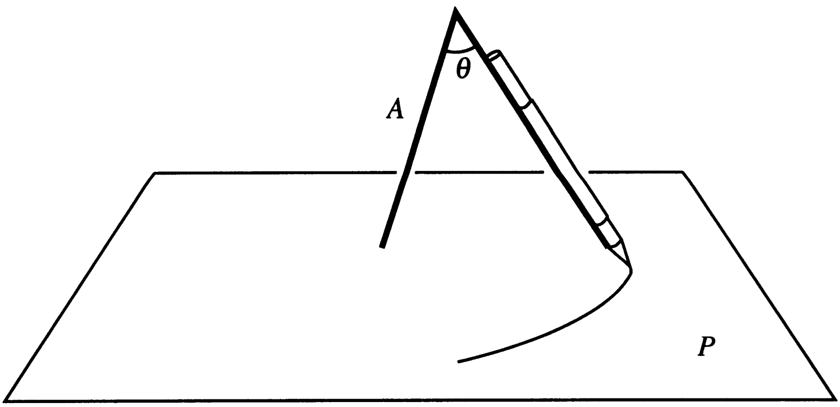

Figure 2.8: Generalized compass

The theory and practice of conic sections finally met when Kepler (1609) discovered the orbits of the planets to be ellipses, and Newton (1687) explained this fact by his law of gravitation. This wonderful vindication of the theory of conic sections has often been described in terms of basic research receiving its long overdue reward, but perhaps one can also see it as a rebuke to Greek disdain for applications. (Kepler would not have been sure which it was. To the end of his days he was proudest of his theory explaining the distances of the planets in terms of the five regular polyhedra (Section 2.2). The fascinating and paradoxical character of Kepler has been warmly described in two excellent books, Koestler (1959) and Banville (1981).

Exercises

A key feature of the ellipse for both geometry and astronomy is a point called the *focus*. The term is the Latin word for fireplace, and it was introduced by Kepler. The ellipse actually has two foci, and they have the geometric property that the sum of the distances from the foci F_1, F_2 to any point P on the ellipse is constant.

2.4.1 This property gives a way to draw an ellipse using two pins and piece of string. Explain how.

2.4.2 By introducing suitable coordinate axes, show that a curve with the above "constant sum" property indeed has an equation of the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

(It is a good idea to start with the two square root terms, representing the distances F_1P and F_2P, on opposite sides of the equation.) Show also that *any* equation of this form is obtainable by suitable choice of F_1, F_2 and F_1P+F_2P.

Another interesting property of the lines from the foci to a point P on the ellipse is that they make equal angles with the tangent at P. It follows that a light ray from F_1 to P is reflected through F_2. A simple proof of this can be based on the *shortest-path property of reflection*, shown in Figure 2.9 and discovered by the Greek scientist Heron around 100 CE.

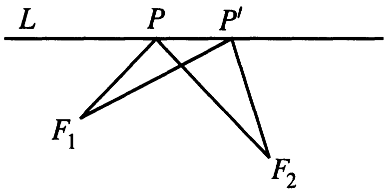

Figure 2.9: The shortest-path property

Shortest-path property. The path F_1PF_2 of reflection in the line L from F_1 to F_2 is shorter than any other path $F_1P'F_2$ from F_1 to L to F_2.

2.4.3 Prove the shortest-path property, by considering the two paths $F_1P\overline{F_2}$ and $F_1P'\overline{F_2}$, where $\overline{F_2}$ is the reflection of the point F_2 in the line L.

Thus to prove that the lines F_1P and F_2P make equal angles with the tangent, it is enough to show that F_1PF_2 is shorter than $F_1P'F_2$ for any other point P' on the tangent at P.

2.4.4 Prove this, using the fact that F_1PF_2 has the same length for all points P on the ellipse.

Kepler's great discovery was that the focus is also significant in astronomy. It is the point occupied by the sun as the planet moves along its ellipse.

2.5 Higher-Degree Curves

The Greeks lacked a systematic theory of higher-degree curves, because they lacked a systematic algebra. They could find what amounted to cartesian equations of individual curves ["symptoms," as they called them; see van der Waerden (1954), p. 241], but they did not consider equations in general or notice any of their properties relevant to the study of curves, for example, the degree. Nevertheless, they studied many interesting special curves, which Descartes and his followers cut their teeth on when algebraic geometry finally emerged in the seventeenth century. An excellent and well-illustrated account of these early investigations may be found in Brieskorn and Knörrer (1981), Chapter 1.

In this section we must confine ourselves to brief remarks on a few examples.

The Cissoid of Diocles (around 100 BCE)

This curve is defined using an auxiliary circle, which for convenience we take to be the unit circle, and vertical lines through x and $-x$. It is the set of all points P seen in Figure 2.10. The portion shown results from varying x between 0 and 1. It is a cubic curve with cartesian equation

$$y^2(1+x) = (1-x)^3.$$

This equation shows that if (x,y) is a point on the curve, then so is $(x,-y)$. Hence one gets the complete picture of it by reflecting the portion shown in Figure 2.10 in the x-axis. The result is a sharp point at R, a *cusp*, a phenomenon that first arises with cubic curves. Diocles showed that the cissoid could be used to duplicate the cube, which is plausible (though still not obvious!) once one knows this curve is cubic.

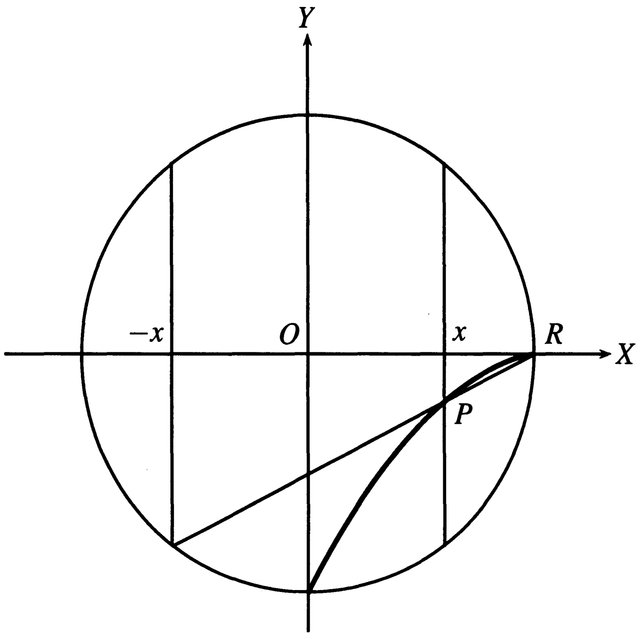

Figure 2.10: Construction of the cissoid

The Spiric Sections of Perseus (around 150 BCE)

Apart from the sphere, cylinder, and cone—whose sections are all conic sections—one of the few surfaces studied by the Greeks was the *torus*. This surface, generated by rotating a circle about an axis outside the circle, but in the same plane, was called a *spira* by the Greeks—hence the name spiric sections for the sections by planes parallel to the axis. These sections, which were first studied by Perseus, have four qualitatively distinct forms [see Figure 2.11, which is adapted from Brieskorn and Knörrer (1981), p. 20].

These forms—convex ovals, "squeezed" ovals, the figure 8, and pairs of ovals—were rediscovered in the seventeenth century when analytic geometers looked at curves of degree 4, of which the spiric sections are examples. For suitable choice of torus, the figure 8 curve becomes the *lemniscate of Bernoulli* and the convex ovals become *Cassini ovals*. Cassini

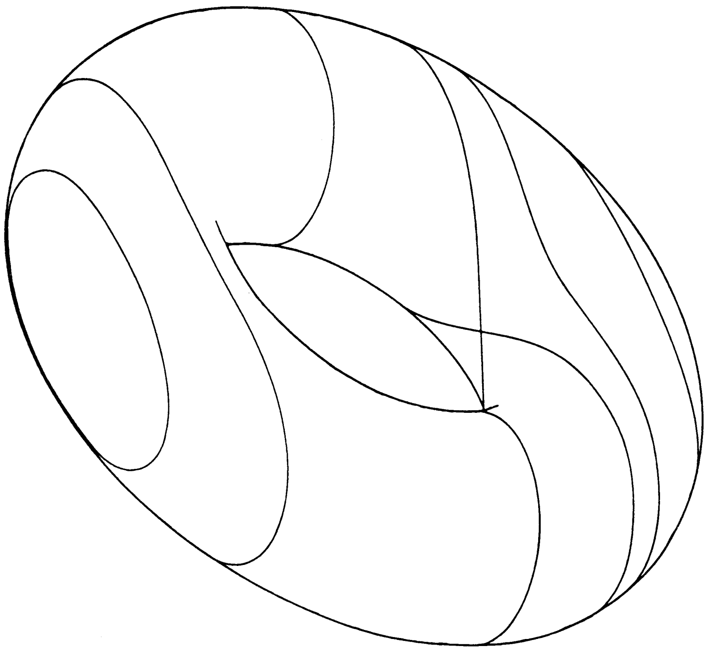

Figure 2.11: Spiric sections

(1625–1712) was a distinguished astronomer but an opponent of Newton's theory of gravitation. He rejected Kepler's ellipses and instead proposed Cassini ovals as orbits for the planets.

The Epicycles of Ptolemy (140 CE)

These curves are known from a famous astronomical work, the *Almagest* of Claudius Ptolemy. Ptolemy himself attributes the idea to Apollonius. It seems almost certain that this is the Apollonius who mastered conic sections, which is ironic, because epicycles were his candidates for the planetary orbits, destined to be defeated by those very same conic sections.

An epicycle, in its simplest form, is the path traced by a point on a circle that rolls on another circle (Figure 2.12). More complicated epicycles can be defined by having a third circle roll on the second, and so on. The Greeks introduced these curves to try to reconcile the complicated movements of the planets, relative to the fixed stars, with a geometry based on the circle. In principle, this is possible! Lagrange (1772) showed that *any* motion along the celestial equator can be approximated arbitrarily closely by epicylic motion, and a more modern version of the result may be found in Sternberg (1969). But Ptolemy's mistake was to accept the apparent complexity of the motions of the planets in the first place. As we now know, the motion becomes simple when one considers motion relative to the sun rather than to the earth and allows orbits to be ellipses.

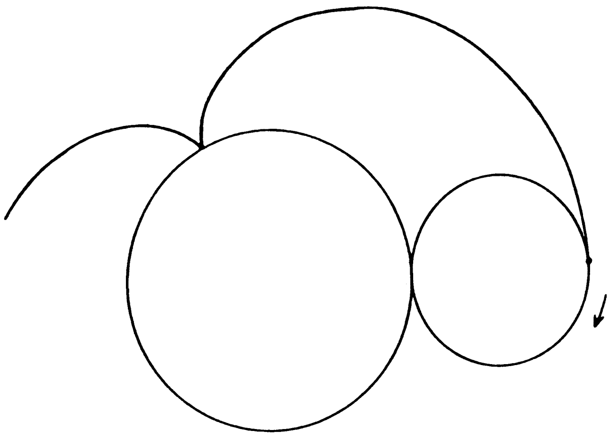

Figure 2.12: Generating an epicycle

Epicycles still have a role to play in engineering, and their mathematical properties are interesting. Some of them are closed curves and turn out to be algebraic, that is, of the form $p(x,y) = 0$ for a polynomial p. Others, such as those that result from rolling circles whose radii have an irrational ratio, lie densely in a certain region of the plane and hence cannot be algebraic; an algebraic curve $p(x,y) = 0$ can meet a straight line $y = mx + c$ in only a finite number of points, corresponding to roots of the polynomial equation $p(x, mx + c) = 0$, and the dense epicycles meet some lines infinitely often.

EXERCISES

The equation of the cissoid is derivable as follows.

2.5.1 Using X and Y for the horizontal and vertical coordinates, show that the straight line RP in Figure 2.10 has equation

$$Y = \frac{\sqrt{1-x^2}}{1+x}(X-1).$$

2.5.2 Deduce the equation of the cissoid from Exercise 2.5.1.

The simplest epicyclic curve is the *cardioid* ("heart-shape"), which results from a circle rolling on a fixed circle of the same size.

2.5.3 Sketch a picture of the cardioid, confirming that it is heart-shaped (sort of).

2.5.4 Show that if both circles have radius 1, and we follow the point on the rolling circle initially at $(1,0)$, then the cardioid it traces out has parametric equations

$$x = 2\cos\theta - \cos 2\theta,$$
$$y = 2\sin\theta - \sin 2\theta.$$

The cardioid is an algebraic curve. Its cartesian equation may be hard to discover, but it is easy to verify, especially if one has a computer algebra system.

2.5.5 Check that the point (x,y) on the cardioid satisfies

$$(x^2+y^2-1)^2 = 4((x-1)^2+y^2).$$

2.6 Biographical Notes: Euclid

Even less is known about Euclid than about Pythagoras. We know only that he flourished around 300 BCE and taught in Alexandria, the Greek city in Egypt founded by Alexander the Great in 322 BCE. Two stories are told about him. The first—the same that is told about Menaechmus and Alexander—has Euclid telling King Ptolemy I "there is no royal road to geometry." The second concerns a student who asked the perennial question: "What shall I gain from learning mathematics?" Euclid called his slave and said: "Give him a coin if he must profit from what he learns."

The most important fact of Euclid's life was undoubtedly his writing of the *Elements*, though we do not know how much of the mathematics in it was actually his own work. Certainly the elementary geometry of triangles and circles was known before Euclid's time. Some of the most sophisticated parts of the *Elements*, too, are due to earlier mathematicians. The theory of irrationals in Book V is due to Eudoxus (around 400–347 BCE), as is the "method of exhaustion" of Book XII (see Chapter 4). The theory of regular polyhedra of Book XIII is due, at least partly, to Theaetetus (around 415–369 BCE).

But whatever Euclid's "research" contribution may have been, it was dwarfed by his contribution to the organization and dissemination of mathematical knowledge. For 2000 years the *Elements* was not only the core of mathematical education but at the heart of Western culture. The most glowing tributes to the *Elements* do not, in fact, come from mathematicians but from philosophers, politicians, and others. We saw Hobbes' response to Euclid in Section 2.1. Here are some others:

> He studied and nearly mastered the six books of Euclid since he was a member of Congress. He regrets his want of education, and does what he can to supply the want.
>
> Abraham Lincoln (writing of himself), *Short Autobiography*
>
> ...he studied Euclid until he could demonstrate with ease all the propositions in the six books.
>
> Herndon's *Life of Lincoln*
>
> At the age of eleven, I began Euclid. ...This was one of the great events of my life, as dazzling as first love. I had not imagined there was anything so delicious in the world.
>
> Bertrand Russell, *Autobiography*, vol. 1

Perhaps the low cultural status of mathematics today, not to mention the mathematical ignorance of politicians and philosophers, reflects the lack of an *Elements* suitable for the modern world.