(a) Prove that if the number of zero rows of A' is strictly larger than the number of zero rows of $(A' \mid C')$ then there are no solutions to $AX = C$.

By (a) we may assume that A' and $(A' \mid C')$ have the same number, r, of nonzero rows (so $n \geq r$).

(b) Prove that if $r = n$ then there is precisely one solution to the system of equations $AX = C$.

(c) Prove that if $r < n$ then there are infinitely many solutions to the system of equations $AX = C$. Prove in fact that the values of the $n - r$ variables corresponding to the nonpivotal columns of $(A' \mid C')$ can be chosen arbitrarily and that the remaining r variables corresponding to the pivotal columns of $(A' \mid C')$ are then determined uniquely.

21. Determine the solutions of the following systems of equations:

(a)
$$\begin{aligned} -3x + 3y + z &= 5 \\ x - y \quad &= 0 \\ 2x - 2y \quad &= -3 \end{aligned}$$

(b)
$$\begin{aligned} x - 2y + z &= 5 \\ x - 4y \quad 6z &= 10 \\ 4x - 11y + 11z &= 12 \end{aligned}$$

(c)
$$\begin{aligned} x - 2y + z &= 5 \\ y - 2z &= 17 \\ 2x - 3y \quad &= 27 \end{aligned}$$

(d)
$$\begin{aligned} x + y - 3z + 2u &= 2 \\ 3x - 2y + 5z + u &= 1 \\ 6x + y - 4z + 3u &= 7 \\ 2x + 2y - 6z \quad &= 4 \end{aligned}$$

(e)
$$\begin{aligned} x + y + 4z + 8u \quad - w &= -1 \\ x + 2y + 3z + 9u \quad - 5w &= -2 \\ -2y + 2z - 2u + v + 14w &= 3 \\ x + 4y + z + 11u \quad - 13w &= -4 \end{aligned}$$

22. Suppose A and B are two row equivalent $m \times n$ matrices.

(a) Prove that the set
$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$
of solutions to the homogeneous linear equations $AX = 0$ as in equation (4) above are the same as the set of solutions to the homogeneous linear equations $BX = 0$. [It suffices to prove this for two matrices differing by an elementary row operation.]

(b) Prove that any linear dependence relation satisfied by the columns of A viewed as vectors in F^m is also satisfied by the columns of B.

(c) Conclude from (b) that the number of linearly independent columns of A is the same as the number of linearly independent columns of B.

23. Let A' be a matrix in reduced row echelon form.

(a) Prove that the nonzero rows of A' are linearly independent. Prove that the pivotal columns of A' are linearly independent and that the nonpivotal columns of A' are linearly dependent on the pivotal columns. (Note the role the pivotal elements play.)

(b) Prove that the number of linearly independent columns of a matrix in reduced row echelon form is the same as the number of linearly independent rows, i.e., the row rank and the column rank of such a matrix are the same.

24. Use the previous two exercises and Exercise 15 above to prove in general that the row rank and the column rank of a matrix are the same.

25. (*Computing Inverses of Matrices*) Let A be an $n \times n$ matrix.

(a) Show that A has an inverse matrix B with columns $B_1, B_2, \dots, B_n$ if and only if the systems of equations:

$$AB_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad AB_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \dots, \quad AB_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

have solutions.

(b) Prove that A has an inverse if and only if A is row equivalent to the $n \times n$ **identity** matrix.

(c) Prove that A has an inverse B if and only if the augmented matrix $(A \mid I)$ can be row reduced to the augmented matrix $(I \mid B)$ where I is the $n \times n$ identity matrix.

26. Determine the inverses of the following matrices using row reduction:

$$A = \begin{pmatrix} -7 & -1 & -4 \\ 7 & 1 & 3 \\ 1 & 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 & 0 & 2 \\ 0 & 2 & 1 & -1 \\ 0 & 2 & 0 & 0 \\ -1 & 1 & 1 & 0 \end{pmatrix}.$$

27. (*Computing Spans, Linear Independence and Linear Dependencies in Vector Spaces*) Let V be an m-dimensional vector space with basis $e_1, e_2, \dots, e_m$ and let $v_1, v_2, \dots, v_n$ be vectors in V. Let A be the $m \times n$ matrix whose columns are the coordinates of the vectors v_i (with respect to the basis $e_1, e_2, \dots, e_m$) and let A' be the reduced row echelon form of A.

(a) Let B be any matrix row equivalent to A. Let $w_1, w_2, \dots, w_n$ be the vectors whose coordinates (with respect to the basis $e_1, e_2, \dots, e_m$) are the columns of B. Prove that any linear relation

$$x_1 v_1 + x_2 v_2 + \dots + x_n v_n = 0 \tag{11.5}$$

satisfied by $v_1, v_2, \dots, v_n$ is also satisfied when v_i is replaced by w_i, $i = 1, 2, \dots, n$.

(b) Prove that the vectors whose coordinates are given by the pivotal columns of A' are linearly independent and that the vectors whose coordinates are given by the nonpivotal columns of A' are linearly dependent on these.

(c) (*Determining Linear Independence of Vectors*) Prove that the vectors $v_1, v_2, \dots, v_n$ are linearly independent if and only if A' has n nonzero rows (i.e., has rank n).

(d) (*Determining Linear Dependencies of Vectors*) By (c), the vectors $v_1, v_2, \dots, v_n$ are linearly dependent if and only if A' has nonpivotal columns. The solutions to (5) defining linear dependence relations among $v_1, v_2, \ldots, v_n$ are given by the linear equations defined by A'. Show that each of the variables $x_1, x_2, \ldots, x_n$ in (5) corresponding to the nonpivotal columns of A' can be prescribed arbitrarily and the values of the remaining variables are then uniquely determined to give a linear dependence relation among $v_1, v_2, \ldots, v_n$ as in (5).

(e) (*Determining the Span of a Set of Vectors*) Prove that the subspace W spanned by $v_1, v_2, \ldots, v_n$ has dimension r where r is the number of nonzero rows of A' and that a basis for W is given by the original vectors v_{j_i} $(i = 1, 2, \ldots, r)$ corresponding to the pivotal columns of A'.

28. Let $V = \mathbb{R}^5$ with the standard basis and consider the vectors

$$v_1 = (1, 1, 3, -2, 3)\ ,\ v_2 = (0, 1, 0, -1, 0)\ ,\ v_3 = (2, 3, 6, -5, 6)$$

$$v_4 = (0, 3, 1, -3, 1)\ ,\ v_5 = (2, -1, -1, -1, -1).$$

(a) Show that the reduced row echelon form of the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 & 0 & 2 \\ 1 & 1 & 3 & 3 & -1 \\ 3 & 0 & 6 & 1 & -1 \\ -2 & -1 & -5 & -3 & -1 \\ 3 & 0 & 6 & 1 & -1 \end{pmatrix}$$

whose columns are the coordinates of v_1, v_2, v_3, v_4, v_5 is the matrix

$$A' = \begin{pmatrix} 1 & 0 & 2 & 0 & 2 \\ 0 & 1 & 1 & 0 & 18 \\ 0 & 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where the 1st, 2nd and 4th columns are pivotal and the remaining two are nonpivotal.

(b) Conclude that these vectors are linearly dependent, that the subspace W spanned by v_1, v_2, v_3, v_4, v_5 is 3-dimensional and that the vectors

$$v_1 = (1, 1, 3, -2, 3), \quad v_2 = (0, 1, 0, -1, 0) \quad \text{and} \quad v_4 = (0, 3, 1, -3, 1)$$

are a basis for W.

(c) Conclude from (a) that the coefficients x_1, x_2, x_3, x_4, x_5 of any linear relation

$$x_1v_1 + x_2v_2 + x_3v_3 + x_4v_4 + x_5v_5 = 0$$

satisfied by v_1, v_2, v_3, v_4, v_5 are given by the equations

$$\begin{aligned} x_1 \quad + 2x_3 \quad + \ 2x_5 &= 0 \\ x_2 + \ x_3 \quad + 18x_5 &= 0 \\ x_4 - \ 7x_5 &= 0. \end{aligned}$$

Deduce that the 3rd and 5th variables, namely x_3 and x_5, corresponding to the nonpivotal columns of A', can be prescribed arbitrarily and the remaining variables are then uniquely determined as:

$$\begin{aligned} x_1 &= -2x_3 - 2x_5 \\ x_2 &= -x_3 - 18x_5 \\ x_4 &= 7x_5 \end{aligned}$$

to give all the linear dependence relations satisfied by v_1, v_2, v_3, v_4, v_5. In particular show that

$$-2v_1 - v_2 + v_3 = 0$$

and

$$-2v_1 - 18v_2 + 7v_4 + v_5 = 0$$

corresponding to $(x_3 = 1, x_5 = 0)$ and $(x_3 = 0, x_5 = 1)$, respectively.

29. For each exercise below, determine whether the given vectors in $\mathbb{R}^4$ are linearly independent. If they are linearly dependent, determine an explicit linear dependence among them.

(a) $(1, -4, 3, 0), (0, -1, 4, -3), (1, -1, 1, -1), (2, 2, -1, -3)$.

(b) $(1, -2, 4, 1), (2, -3, 9, -1), (1, 0, 6, -5), (2, -5, 7, 5)$.

(c) $(1, -2, 0, 1), (2, -2, 0, 0), (-1, 3, 0, -2), (-2, 1, 0, 1)$.

(d) $(0, 1, 1, 0), (1, 0, 1, 1), (2, 2, 2, 0), (0, -1, 1, 1)$.

30. For each exercise below, determine the subspace spanned in $\mathbb{R}^4$ by the given vectors and give a basis for this subspace.

(a) $(1, -2, 5, 3), (2, 3, 1, -4), (3, 8, -3, -5)$.

(b) $(2, -5, 3, 0), (0, -2, 5, -3), (1, -1, 1, -1), (-3, 2, -1, 2)$.

(c) $(1, -2, 0, 1), (2, -2, 0, 0), (-1, 3, 0, -2), (-2, 1, 0, 1)$.

(d) $(1, 1, 0, -1), (1, 2, 3, 0), (2, 3, 3, -1), (1, 2, 2, -2), (2, 3, 2, -3), (1, 3, 4, -3)$.

31. (*Computing the Image and Kernel of a Linear Transformation*) Let V be an n-dimensional vector space with basis $e_1, e_2, \ldots, e_n$ and let W be an m-dimensional vector space with basis $f_1, f_2, \ldots, f_m$. Let φ be a linear transformation from V to W and let A be the corresponding $m \times n$ matrix with respect to these bases: $A = (a_{ij})$ where

$$\varphi(e_j) = \sum_{i=1}^{m} a_{ij} f_i \,, \qquad j = 1, 2, \ldots, n,$$

i.e., the columns of A are the coordinates of the vectors $\varphi(e_1), \varphi(e_2), \ldots, \varphi(e_n)$ with respect to the basis $f_1, f_2, \ldots, f_m$ of W. Let A' be the reduced row echelon form of A.

(a) (*Determining the Image of a Linear Transformation*) Prove that the image $\varphi(V)$ of V under φ has dimension r where r is the number of nonzero rows of A' and that a basis for $\varphi(V)$ is given by the vectors $\varphi(e_{j_i})$ $(i = 1, 2, \ldots, r)$, i.e., the columns of A corresponding to the pivotal columns of A' give the coordinates of a basis for the image of φ.

(b) (*Determining the Kernel of a Linear Transformation*) The elements in the kernel of φ are the vectors in V whose coordinates $(x_1, x_2, \ldots, x_n)$ with respect to the basis $e_1, e_2, \ldots, e_n$ satisfy the equation

$$A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = 0,$$

and the solutions $x_1, x_2, \ldots, x_n$ to this system of linear equations are determined by the matrix A'.

(i) Prove that φ is injective if and only if A' has n nonzero rows (i.e., has rank n).

(ii) By (i), the kernel of φ is nontrivial if and only if A' has nonpivotal columns. Show that each of the variables $x_1, x_2, \ldots, x_n$ above corresponding to the nonpivotal columns of A' can be prescribed arbitrarily and the values of the remaining variables are then uniquely determined to give an element $x_1e_1 + x_2e_2 + \ldots + x_ne_n$ in the kernel of φ. In particular, show that the coordinates of a basis for the kernel are obtained by successively setting one nonpivotal variable equal to 1 and all other nonpivotal variables to 0 and solving for the remaining pivotal variables. Conclude that the kernel of φ has dimension $n - r$ where r is the rank of A.

32. Let $V = \mathbb{R}^5$ and $W = \mathbb{R}^4$ with the standard bases. Let φ be the linear transformation $\varphi : V \to W$ defined by

$$\varphi(x, y, z, u, v) = (x+2y+3z+4u+4v, -2x-4y+2v, x+2y+u-2v, x+2y-v).$$

(a) Prove that the matrix A corresponding to φ and these bases is

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 4 \\ -2 & -4 & 0 & 0 & 2 \\ 1 & 2 & 0 & 1 & -2 \\ 1 & 2 & 0 & 0 & -1 \end{pmatrix}$$

and that the reduced row echelon matrix A' row equivalent to A is

$$A' = \begin{pmatrix} 1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where the 1^{st}, 3^{rd} and 4^{th} columns are pivotal and the remaining two are nonpivotal.

(b) Conclude that the image of φ is 3-dimensional and that the image of the 1^{st}, 3^{rd} and 4^{th} basis elements of V, namely, $(1, -2, 1, 1)$, $(3, 0, 0, 0)$ and $(4, 0, 1, 0)$ give a basis for the image $\varphi(V)$ of V.

(c) Conclude from (a) that the elements in the kernel of φ are the vectors (x, y, z, u, v) satisfying the equations

$$\begin{aligned} x + 2y \qquad\quad - \ v &= 0 \\ z \quad + 3v &= 0 \\ u - \ v &= 0. \end{aligned}$$

Deduce that the 2^{nd} and 5^{th} variables, namely y and v, corresponding to the nonpivotal columns of A' can be prescribed arbitrarily and the remaining variables are then uniquely determined as

$$\begin{aligned} x &= -2y + v \\ z &= -3v \\ u &= v. \end{aligned}$$

Show that $(-2, 1, 0, 0, 0)$ and $(1, 0, -3, 1, 1)$ give a basis for the 2-dimensional kernel of φ, corresponding to $(y = 1, v = 0)$ and $(y = 0, v = 1)$, respectively.

33. Let φ be the linear transformation from $\mathbb{R}^4$ to itself defined by the matrix

$$A = \begin{pmatrix} 1 & -1 & 0 & 3 \\ -1 & 2 & 1 & -1 \\ -1 & 1 & 0 & -3 \\ 1 & -2 & -1 & 1 \end{pmatrix}$$

with respect to the standard basis for $\mathbb{R}^4$. Determine a basis for the image and for the kernel of φ.