Another question is whether every convergent sequence contains a uniformly convergent subsequence. Our next example will show that this need not be so, even if the sequence is uniformly bounded on a compact set. (Example 7.6 shows that a sequence of bounded functions may converge without being uniformly bounded; but it is trivial to see that uniform convergence of a sequence of bounded functions implies uniform boundedness.)

7.21 Example Let

$$f_n(x) = \frac{x^2}{x^2 + (1 - nx)^2} \qquad (0 \le x \le 1, n = 1, 2, 3, \dots).$$

Then $|f_n(x)| \le 1$, so that $\{f_n\}$ is uniformly bounded on $[0, 1]$. Also

$$\lim_{n\to\infty} f_n(x) = 0 \qquad (0 \le x \le 1),$$

but

$$f_n\left(\frac{1}{n}\right) = 1 \qquad (n = 1, 2, 3, \dots),$$

so that no subsequence can converge uniformly on $[0, 1]$.

The concept which is needed in this connection is that of equicontinuity; it is given in the following definition.

7.22 Definition A family $\mathscr{F}$ of complex functions f defined on a set E in a metric space X is said to be *equicontinuous* on E if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon$$

whenever $d(x, y) < \delta$, $x \in E$, $y \in E$, and $f \in \mathscr{F}$. Here d denotes the metric of X.

It is clear that every member of an equicontinuous family is uniformly continuous.

The sequence of Example 7.21 is not equicontinuous.

Theorems 7.24 and 7.25 will show that there is a very close relation between equicontinuity, on the one hand, and uniform convergence of sequences of continuous functions, on the other. But first we describe a selection process which has nothing to do with continuity.

7.23 Theorem *If $\{f_n\}$ is a pointwise bounded sequence of complex functions on a countable set E, then $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}(x)\}$ converges for every $x \in E$.*

Proof Let $\{x_i\}$, $i = 1, 2, 3, \ldots$, be the points of E, arranged in a sequence. Since $\{f_n(x_1)\}$ is bounded, there exists a subsequence, which we shall denote by $\{f_{1,k}\}$, such that $\{f_{1,k}(x_1)\}$ converges as $k \to \infty$.

Let us now consider sequences $S_1, S_2, S_3, \ldots$, which we represent by the array

$$\begin{array}{llllll} S_1: & f_{1,1} & f_{1,2} & f_{1,3} & f_{1,4} & \cdots \\ S_2: & f_{2,1} & f_{2,2} & f_{2,3} & f_{2,4} & \cdots \\ S_3: & f_{3,1} & f_{3,2} & f_{3,3} & f_{3,4} & \cdots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{array}$$

and which have the following properties:

(*a*) S_n is a subsequence of S_{n-1}, for $n = 2, 3, 4, \ldots$.
(*b*) $\{f_{n,k}(x_n)\}$ converges, as $k \to \infty$ (the boundedness of $\{f_n(x_n)\}$ makes it possible to choose S_n in this way);
(*c*) The order in which the functions appear is the same in each sequence; i.e., if one function precedes another in S_1, they are in the same relation in every S_n, until one or the other is deleted. Hence, when going from one row in the above array to the next below, functions may move to the left but never to the right.

We now go down the diagonal of the array; i.e., we consider the sequence

$$S: \; f_{1,1} \;\; f_{2,2} \;\; f_{3,3} \;\; f_{4,4} \cdots.$$

By (*c*), the sequence S (except possibly its first $n - 1$ terms) is a subsequence of S_n, for $n = 1, 2, 3, \ldots$. Hence (*b*) implies that $\{f_{n,n}(x_i)\}$ converges, as $n \to \infty$, for every $x_i \in E$.

7.24 Theorem *If K is a compact metric space, if $f_n \in \mathscr{C}(K)$ for $n = 1, 2, 3, \ldots$, and if $\{f_n\}$ converges uniformly on K, then $\{f_n\}$ is equicontinuous on K.*

Proof Let $\varepsilon > 0$ be given. Since $\{f_n\}$ converges uniformly, there is an integer N such that

$$\|f_n - f_N\| < \varepsilon \qquad (n > N). \tag{42}$$

(See Definition 7.14.) Since continuous functions are uniformly continuous on compact sets, there is a $\delta > 0$ such that

$$|f_i(x) - f_i(y)| < \varepsilon \tag{43}$$

if $1 \le i \le N$ and $d(x, y) < \delta$.

If $n > N$ and $d(x, y) < \delta$, it follows that

$$|f_n(x) - f_n(y)| \le |f_n(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f_n(y)| < 3\varepsilon.$$

In conjunction with (43), this proves the theorem.

7.25 Theorem *If K is compact, if $f_n \in \mathscr{C}(K)$ for $n = 1, 2, 3, \ldots$, and if $\{f_n\}$ is pointwise bounded and equicontinuous on K, then*

(*a*) *$\{f_n\}$ is uniformly bounded on K,*
(*b*) *$\{f_n\}$ contains a uniformly convergent subsequence.*

Proof

(*a*) Let $\varepsilon > 0$ be given and choose $\delta > 0$, in accordance with Definition 7.22, so that

(44) $$|f_n(x) - f_n(y)| < \varepsilon$$

for all n, provided that $d(x, y) < \delta$.

Since K is compact, there are finitely many points $p_1, \ldots, p_r$ in K such that to every $x \in K$ corresponds at least one p_i with $d(x, p_i) < \delta$. Since $\{f_n\}$ is pointwise bounded, there exist $M_i < \infty$ such that $|f_n(p_i)| < M_i$ for all n. If $M = \max(M_1, \ldots, M_r)$, then $|f_n(x)| < M + \varepsilon$ for every $x \in K$. This proves (*a*).

(*b*) Let E be a countable dense subset of K. (For the existence of such a set E, see Exercise 25, Chap. 2.) Theorem 7.23 shows that $\{f_n\}$ has a subsequence $\{f_{n_i}\}$ such that $\{f_{n_i}(x)\}$ converges for every $x \in E$.

Put $f_{n_i} = g_i$, to simplify the notation. We shall prove that $\{g_i\}$ converges uniformly on K.

Let $\varepsilon > 0$, and pick $\delta > 0$ as in the beginning of this proof. Let $V(x, \delta)$ be the set of all $y \in K$ with $d(x, y) < \delta$. Since E is dense in K, and K is compact, there are finitely many points $x_1, \ldots, x_m$ in E such that

(45) $$K \subset V(x_1, \delta) \cup \cdots \cup V(x_m, \delta).$$

Since $\{g_i(x)\}$ converges for every $x \in E$, there is an integer N such that

(46) $$|g_i(x_s) - g_j(x_s)| < \varepsilon$$

whenever $i \geq N, j \geq N, 1 \leq s \leq m$.

If $x \in K$, (45) shows that $x \in V(x_s, \delta)$ for some s, so that

$$|g_i(x) - g_i(x_s)| < \varepsilon$$

for every i. If $i \geq N$ and $j \geq N$, it follows from (46) that

$$\begin{aligned} |g_i(x) - g_j(x)| &\leq |g_i(x) - g_i(x_s)| + |g_i(x_s) - g_j(x_s)| + |g_j(x_s) - g_j(x)| \\ &< 3\varepsilon. \end{aligned}$$

This completes the proof.

THE STONE-WEIERSTRASS THEOREM

7.26 Theorem *If f is a continuous complex function on $[a, b]$, there exists a sequence of polynomials P_n such that*

$$\lim_{n\to\infty} P_n(x) = f(x)$$

uniformly on $[a, b]$. If f is real, the P_n may be taken real.

This is the form in which the theorem was originally discovered by Weierstrass.

Proof We may assume, without loss of generality, that $[a, b] = [0, 1]$. We may also assume that $f(0) = f(1) = 0$. For if the theorem is proved for this case, consider

$$g(x) = f(x) - f(0) - x[f(1) - f(0)] \qquad (0 \le x \le 1).$$

Here $g(0) = g(1) = 0$, and if g can be obtained as the limit of a uniformly convergent sequence of polynomials, it is clear that the same is true for f, since $f - g$ is a polynomial.

Furthermore, we define $f(x)$ to be zero for x outside $[0, 1]$. Then f is uniformly continuous on the whole line.

We put

(47)
$$Q_n(x) = c_n(1 - x^2)^n \qquad (n = 1, 2, 3, \ldots),$$

where c_n is chosen so that

(48)
$$\int_{-1}^{1} Q_n(x)\, dx = 1 \qquad (n = 1, 2, 3, \ldots).$$

We need some information about the order of magnitude of c_n. Since

$$\begin{aligned}\int_{-1}^{1} (1 - x^2)^n\, dx = 2\int_0^1 (1 - x^2)^n\, dx &\ge 2\int_0^{1/\sqrt{n}} (1 - x^2)^n\, dx \\ &\ge 2\int_0^{1/\sqrt{n}} (1 - nx^2)\, dx \\ &= \frac{4}{3\sqrt{n}} \\ &> \frac{1}{\sqrt{n}},\end{aligned}$$

it follows from (48) that

(49)
$$c_n < \sqrt{n}.$$

The inequality $(1 - x^2)^n \geq 1 - nx^2$ which we used above is easily shown to be true by considering the function

$$(1 - x^2)^n - 1 + nx^2$$

which is zero at $x = 0$ and whose derivative is positive in $(0, 1)$.

For any $\delta > 0$, (49) implies

(50) $$Q_n(x) \leq \sqrt{n}\,(1 - \delta^2)^n \qquad (\delta \leq |x| \leq 1),$$

so that $Q_n \to 0$ uniformly in $\delta \leq |x| \leq 1$.

Now set

(51) $$P_n(x) = \int_{-1}^{1} f(x + t)Q_n(t)\,dt \qquad (0 \leq x \leq 1).$$

Our assumptions about f show, by a simple change of variable, that

$$P_n(x) = \int_{-x}^{1-x} f(x + t)Q_n(t)\,dt = \int_0^1 f(t)Q_n(t - x)\,dt,$$

and the last integral is clearly a polynomial in x. Thus $\{P_n\}$ is a sequence of polynomials, which are real if f is real.

Given $\varepsilon > 0$, we choose $\delta > 0$ such that $|y - x| < \delta$ implies

$$|f(y) - f(x)| < \frac{\varepsilon}{2}.$$

Let $M = \sup |f(x)|$. Using (48), (50), and the fact that $Q_n(x) \geq 0$, we see that for $0 \leq x \leq 1$,

$$\begin{aligned}
|P_n(x) - f(x)| &= \left| \int_{-1}^{1} [f(x + t) - f(x)]Q_n(t)\,dt \right| \\
&\leq \int_{-1}^{1} |f(x + t) - f(x)|\,Q_n(t)\,dt \\
&\leq 2M \int_{-1}^{-\delta} Q_n(t)\,dt + \frac{\varepsilon}{2}\int_{-\delta}^{\delta} Q_n(t)\,dt + 2M\int_{\delta}^{1} Q_n(t)\,dt \\
&\leq 4M\sqrt{n}\,(1 - \delta^2)^n + \frac{\varepsilon}{2} \\
&< \varepsilon
\end{aligned}$$

for all large enough n, which proves the theorem.

It is instructive to sketch the graphs of Q_n for a few values of n; also, note that we needed uniform continuity of f to deduce uniform convergence of $\{P_n\}$.

In the proof of Theorem 7.32 we shall not need the full strength of Theorem 7.26, but only the following special case, which we state as a corollary.

7.27 Corollary *For every interval $[-a, a]$ there is a sequence of real polynomials P_n such that $P_n(0) = 0$ and such that*

$$\lim_{n \to \infty} P_n(x) = |x|$$

uniformly on $[-a, a]$.

Proof By Theorem 7.26, there exists a sequence $\{P_n^*\}$ of real polynomials which converges to $|x|$ uniformly on $[-a, a]$. In particular, $P_n^*(0) \to 0$ as $n \to \infty$. The polynomials

$$P_n(x) = P_n^*(x) - P_n^*(0) \qquad (n = 1, 2, 3, \ldots)$$

have desired properties.

We shall now isolate those properties of the polynomials which make the Weierstrass theorem possible.

7.28 Definition A family $\mathscr{A}$ of complex functions defined on a set E is said to be an *algebra* if (i) $f + g \in \mathscr{A}$, (ii) $fg \in \mathscr{A}$, and (iii) $cf \in \mathscr{A}$ for all $f \in \mathscr{A}$, $g \in \mathscr{A}$ and for all complex constants c, that is, if $\mathscr{A}$ is closed under addition, multiplication, and scalar multiplication. We shall also have to consider algebras of real functions; in this case, (iii) is of course only required to hold for all real c.

If $\mathscr{A}$ has the property that $f \in \mathscr{A}$ whenever $f_n \in \mathscr{A}$ $(n = 1, 2, 3, \ldots)$ and $f_n \to f$ uniformly on E, then $\mathscr{A}$ is said to be *uniformly closed.*

Let $\mathscr{B}$ be the set of all functions which are limits of uniformly convergent sequences of members of $\mathscr{A}$. Then $\mathscr{B}$ is called the *uniform closure* of $\mathscr{A}$. (See Definition 7.14.)

For example, the set of all polynomials is an algebra, and the Weierstrass theorem may be stated by saying that the set of continuous functions on $[a, b]$ is the uniform closure of the set of polynomials on $[a, b]$.

7.29 Theorem *Let $\mathscr{B}$ be the uniform closure of an algebra $\mathscr{A}$ of bounded functions. Then $\mathscr{B}$ is a uniformly closed algebra.*

Proof If $f \in \mathscr{B}$ and $g \in \mathscr{B}$, there exist uniformly convergent sequences $\{f_n\}, \{g_n\}$ such that $f_n \to f$, $g_n \to g$ and $f_n \in \mathscr{A}$, $g_n \in \mathscr{A}$. Since we are dealing with bounded functions, it is easy to show that

$$f_n + g_n \to f + g, \qquad f_n g_n \to fg, \qquad cf_n \to cf,$$

where c is any constant, the convergence being uniform in each case.

Hence $f + g \in \mathscr{B}$, $fg \in \mathscr{B}$, and $cf \in \mathscr{B}$, so that $\mathscr{B}$ is an algebra.

By Theorem 2.27, $\mathscr{B}$ is (uniformly) closed.

7.30 Definition Let $\mathscr{A}$ be a family of functions on a set E. Then $\mathscr{A}$ is said to *separate points* on E if to every pair of distinct points $x_1, x_2 \in E$ there corresponds a function $f \in \mathscr{A}$ such that $f(x_1) \neq f(x_2)$.

If to each $x \in E$ there corresponds a function $g \in \mathscr{A}$ such that $g(x) \neq 0$, we say that $\mathscr{A}$ *vanishes at no point of* E.

The algebra of all polynomials in one variable clearly has these properties on R^1. An example of an algebra which does not separate points is the set of all even polynomials, say on $[-1, 1]$, since $f(-x) = f(x)$ for every even function f.

The following theorem will illustrate these concepts further.

7.31 Theorem *Suppose $\mathscr{A}$ is an algebra of functions on a set E, $\mathscr{A}$ separates points on E, and $\mathscr{A}$ vanishes at no point of E. Suppose x_1, x_2 are distinct points of E, and c_1, c_2 are constants (real if $\mathscr{A}$ is a real algebra). Then $\mathscr{A}$ contains a function f such that*

$$f(x_1) = c_1, \qquad f(x_2) = c_2.$$

Proof The assumptions show that $\mathscr{A}$ contains functions g, h, and k such that

$$g(x_1) \neq g(x_2), \qquad h(x_1) \neq 0, \qquad k(x_2) \neq 0.$$

Put

$$u = gk - g(x_1)k, \qquad v = gh - g(x_2)h.$$

Then $u \in \mathscr{A}$, $v \in \mathscr{A}$, $u(x_1) = v(x_2) = 0$, $u(x_2) \neq 0$, and $v(x_1) \neq 0$. Therefore

$$f = \frac{c_1 v}{v(x_1)} + \frac{c_2 u}{u(x_2)}$$

has the desired properties.

We now have all the material needed for Stone's generalization of the Weierstrass theorem.

7.32 Theorem *Let $\mathscr{A}$ be an algebra of real continuous functions on a compact set K. If $\mathscr{A}$ separates points on K and if $\mathscr{A}$ vanishes at no point of K, then the uniform closure $\mathscr{B}$ of $\mathscr{A}$ consists of all real continuous functions on K.*

We shall divide the proof into four steps.

STEP 1 *If $f \in \mathscr{B}$, then $|f| \in \mathscr{B}$.*

Proof Let

$$a = \sup |f(x)| \qquad (x \in K) \tag{52}$$