We want to say a bit more about the special alternating r-linear forms D_J, which we associated with a basis $\{f_1, \ldots, f_n\}$ for V^* in (5-39). It is important to understand that $D_J(\alpha_1, \ldots, \alpha_r)$ is the determinant of a certain $r \times r$ matrix. If

$$A_{ij} = f_j(\alpha_i), \qquad 1 \leq i \leq r, 1 \leq j \leq n,$$

that is, if

$$\alpha_i = A_{i1}\beta_1 + \cdots + A_{in}\beta_n, \qquad 1 \leq i \leq r$$

and J is the r-shuffle $(j_1, \ldots, j_r)$, then

$$D_J(\alpha_1, \ldots, \alpha_r) = \sum_\sigma (\operatorname{sgn} \sigma) A(1, j_{\sigma 1}) \cdots A(n, j_{\sigma n}) \tag{5-42}$$

$$= \det \begin{bmatrix} A(1, j_1) & \cdots & A(1, j_r) \\ \vdots & & \vdots \\ A(r, j_1) & \cdots & A(r, j_r) \end{bmatrix}.$$

Thus $D_J(\alpha_1, \ldots, \alpha_r)$ is the determinant of the $r \times r$ matrix formed from columns $j_1, \ldots, j_r$ of the $r \times n$ matrix which has (the coordinate n-tuples of) $\alpha_1, \ldots, \alpha_r$ as its rows. Another notation which is sometimes used for this determinant is

$$D_J(\alpha_1, \ldots, \alpha_r) = \frac{\partial(\alpha_1, \ldots, \alpha_r)}{\partial(\beta_{j_1}, \ldots, \beta_{j_r})}. \tag{5-43}$$

In this notation, the proof of Theorem 7 shows that every alternating r-linear form L can be expressed relative to a basis $\{\beta_1, \ldots, \beta_n\}$ by the equation

$$L(\alpha_1, \ldots, \alpha_r) = \sum_{j_1 < \cdots < j_r} \frac{\partial(\alpha_1, \ldots, \alpha_r)}{\partial(\beta_{j_1}, \ldots, \beta_{j_r})} L(\beta_{j_1}, \ldots, \beta_{j_r}). \tag{5-44}$$

5.7. The Grassman Ring

Many of the important properties of determinants and alternating multilinear forms are best described in terms of a multiplication operation on forms, called the exterior product. If L and M are, respectively, alternating r and s-linear forms on the module V, we have an associated product of L and M, the tensor product $L \otimes M$. This is not an alternating form unless $L = 0$ or $M = 0$; however, we have a natural way of projecting it into $\Lambda^{r+s}(V)$. It appears that

$$L \cdot M = \pi_{r+s}(L \otimes M) \tag{5-45}$$

should be the 'natural' multiplication of alternating forms. But, is it?

Let us take a specific example. Suppose that V is the module K^n and $f_1, \ldots, f_n$ are the standard coordinate functions on K^n. If $i \neq j$, then

$$f_i \cdot f_j = \pi_2(f_i \otimes f_j)$$

is the (determinant) function

$$D_{ij} = f_i \otimes f_j - f_j \otimes f_i$$

given by (5-39). Now suppose k is an index different from i and j. Then

$$\begin{aligned} D_{ij} \cdot f_k &= \pi_3[(f_i \otimes f_j - f_j \otimes f_i) \otimes f_k] \\ &= \pi_3(f_i \otimes f_j \otimes f_k) - \pi_3(f_j \otimes f_i \otimes f_k). \end{aligned}$$

The proof of the lemma following equation (5-36) shows that for any r-linear form L and any permutation σ of $\{1, \ldots, r\}$

$$\pi_r(L_\sigma) = \operatorname{sgn} \sigma \; \pi_r(L)$$

Hence, $D_{ij} \cdot f_k = 2\pi_3(f_i \otimes f_j \otimes f_k)$. By a similar computation, $f_i \cdot D_{jk} = 2\pi_3(f_i \otimes f_j \otimes f_k)$. Thus we have

$$(f_i \cdot f_j) \cdot f_k = f_i \cdot (f_j \cdot f_k)$$

and all of this looks very promising. But there is a catch. Despite the computation that we have just completed, the putative multiplication in (5-45) is not associative. In fact, if l is an index different from i, j, k, then one can calculate that

$$D_{ij} \cdot D_{kl} = 4\pi_4(f_i \otimes f_j \otimes f_k \otimes f_l)$$

and that

$$(D_{ij} \cdot f_k) \cdot f_l = 6\pi_4(f_i \otimes f_j \otimes f_k \otimes f_l).$$

Thus, in general

$$(f_i \cdot f_j) \cdot (f_k \cdot f_l) \neq [(f_i \cdot f_j) \cdot f_k] \cdot f_l$$

and we see that our first attempt to find a multiplication has produced a non-associative operation.

The reader should not be surprised if he finds it rather tedious to give a direct verification of the two equations showing non-associativity. This is typical of the subject, and it is also typical that there is a general fact which considerably simplifies the work.

Suppose L is an r-linear form and that M is an s-linear form on the module V. Then

$$\begin{aligned} \pi_{r+s}((\pi_r L) \otimes (\pi_s M)) &= \pi_{r+s}\Big(\sum_{\sigma,\tau} (\operatorname{sgn} \sigma)(\operatorname{sgn} \tau) L_\sigma \otimes M_\tau\Big) \\ &= \sum_{\sigma,\tau} (\operatorname{sgn} \sigma)(\operatorname{sgn} \tau) \pi_{r+s}(L_\sigma \otimes M_\tau) \end{aligned}$$

where σ varies over the symmetric group, S_r, of all permutations of $\{1, \ldots, r\}$, and τ varies over S_s. Each pair σ, τ defines an element (σ, τ) of S_{r+s} which permutes the first r elements of $\{1, \ldots, r + s\}$ according to σ and the last s elements according to τ. It is clear that

$$\operatorname{sgn}(\sigma, \tau) = (\operatorname{sgn} \sigma)(\operatorname{sgn} \tau)$$

and that

$$(L \otimes M)_{(\sigma,\tau)} = L_\sigma \otimes L_\tau.$$

Therefore

$$\pi_{r+s}[(\pi_r L) \otimes (\pi_s M)] = \sum_{\sigma,\tau} \operatorname{sgn}(\sigma, \tau)\ \pi_{r+s}\ [(L \otimes M)_{(\sigma,\tau)}].$$

Now we have already observed that

$$\operatorname{sgn}(\sigma, \tau)\pi_{r+s}[(L \otimes M)_{(\sigma,\tau)}] = \pi_{r+s}(L \otimes M).$$

Thus, it follows that

$$\pi_{r+s}[(\pi_r L) \otimes (\pi_s M)] = r!s!\ \pi_{r+s}(L \otimes M). \tag{5-46}$$

This formula simplifies a number of computations. For example, suppose we have an r-shuffle $I = (i_1, \ldots, i_r)$ and s-shuffle $J = (j_1, \ldots, j_s)$. To make things simple, assume, in addition, that

$$i_1 < \cdots < i_r < j_1 < \cdots < j_s.$$

Then we have the associated determinant functions

$$\begin{aligned} D_I &= \pi_r(E_I) \\ D_J &= \pi_s(E_J) \end{aligned}$$

where E_I and E_J are given by (5-30). Using (5-46), we see immediately that

$$\begin{aligned} D_I \cdot D_J &= \pi_{r+s}[\pi_r(E_I) \otimes \pi_s(E_J)] \\ &= r!s!\pi_{r+s}(E_I \otimes E_J). \end{aligned}$$

Since $E_I \otimes E_J = E_{I \cup J}$, it follows that

$$D_I \cdot D_J = r!s!\ D_{I \cup J}.$$

This suggests that the lack of associativity for the multiplication (5-45) results from the fact that $D_I \cdot D_J \neq D_{I \cup J}$. After all, the product of D_I and D_J ought to be $D_{I \cup J}$. To repair the situation, we should define a new product, the **exterior product** (or **wedge product**) of an alternating r-linear form L and an alternating s-linear form M by

$$L \wedge M = \frac{1}{r!s!}\ \pi_{r+s}(L \otimes M). \tag{5-47}$$

We then have

$$D_I \wedge D_J = D_{I \cup J}$$

for the determinant functions on K^n, and, if there is any justice at all, we must have found the proper multiplication of alternating multilinear forms. Unfortunately, (5-47) fails to make sense for the most general case under consideration, since we may not be able to divide by $r!s!$ in the ring K. If K is a field of characteristic zero, then (5-47) is meaningful, and one can proceed quite rapidly to show that the wedge product is associative.

Theorem 8. *Let* K *be a field of characteristic zero and* V *a vector space over* K. *Then the exterior product is an associative operation on the alternating multilinear forms on* V. *In other words, if* L, M, *and* N *are alternating multilinear forms on* V *of degrees* r, s, *and* t, *respectively, then*

$$(L \wedge M) \wedge N = L \wedge (M \wedge N).$$

Proof. It follows from (5-47) that $cd(L \wedge M) = cL \wedge dM$ for any scalars c and d. Hence

$$r!s!t![(L \wedge M) \wedge N] = r!s!(L \wedge M) \wedge t!N$$

and since $\pi_t(N) = t!N$, it results that

$$\begin{aligned} r!s!t![(L \wedge M) \wedge N] &= \pi_{r+s}(L \otimes M) \wedge \pi_t(N) \\ &= \frac{1}{(r+s)!}\frac{1}{t!}\pi_{r+s+t}[\pi_{r+s}(L \otimes M) \otimes \pi_t(N)]. \end{aligned}$$

From (5-46) we now see that

$$r!s!t[(L \wedge M) \wedge N] = \pi_{r+s+t}(L \otimes M \otimes N).$$

By a similar computation

$$r!s!t![L \wedge (M \wedge N)] = \pi_{r+s+t}(L \otimes M \otimes N)$$

and therefore, $(L \wedge M) \wedge N = L \wedge (M \wedge N)$. ∎

Now we return to the general case, in which it is only assumed that K is a commutative ring with identity. Our first problem is to replace (5-47) by an equivalent definition which works in general. If L and M are alternating multilinear forms of degrees r and s respectively, we shall construct a canonical alternating multilinear form $L \wedge M$ of degree $r + s$ such that

$$r!s!(L \wedge M) = \pi_{r+s}(L \otimes M).$$

Let us recall how we define $\pi_{r+s}(L \otimes M)$. With each permutation σ of $\{1, \ldots, r+s\}$ we associate the multilinear function

$$(\operatorname{sgn} \sigma)(L \otimes M)_\sigma \tag{5-48}$$

where

$$(L \otimes M)_\sigma(\alpha_1, \ldots, \alpha_{r+s}) = (L \otimes M)(\alpha_{\sigma 1}, \ldots, \alpha_{\sigma(r+s)})$$

and we sum the functions (5-48) over all permutations σ. There are $(r+s)!$ permutations; however, since L and M are alternating, many of the functions (5-48) are the same. In fact there are at most

$$\frac{(r+s)!}{r!s!}$$

distinct functions (5-48). Let us see why. Let S_{r+s} be the set of permutations of $\{1, \ldots, r+s\}$, i.e., let S_{r+s} be the symmetric group of degree $r + s$. As in the proof of (5-46), we distinguish the subset G that consists of the permutations σ which permute the sets $\{1, \ldots, r\}$ and $\{r+1, \ldots, r+s\}$ within themselves. In other words, σ is in G if $1 \leq \sigma i \leq r$ for each i between 1 and r. (It necessarily follows that $r + 1 \leq \sigma j \leq r + s$ for each j between $r + 1$ and $r + s$.) Now G is a subgroup of S_{r+s}, that is, if σ and τ are in G then $\sigma\tau^{-1}$ is in G. Evidently G has $r!s!$ members.

We have a map

$$S_{r+s} \xrightarrow{\psi} M^{r+s}(V)$$

defined by

$$\psi(\sigma) = (\operatorname{sgn} \sigma)(L \otimes M)_\sigma.$$

Since L and M are alternating,

$$\psi(\gamma) = L \otimes M$$

for every γ in G. Therefore, since $(N\sigma)\tau = N\tau\sigma$ for any $(r + s)$-linear form N on V, we have

$$\psi(\tau\gamma) = \psi(\tau), \qquad \tau \text{ in } S_{r+s}, \ \gamma \text{ in } G.$$

This says that the map ψ is constant on each (left) **coset** τG of the subgroup G. If τ_1 and τ_2 are in S_{r+s}, the cosets $\tau_1 G$ and $\tau_2 G$ are either identical or disjoint, according as $\tau_2^{-1}\tau_1$ is in G or is not in G. Each coset contains $r!s!$ elements; hence, there are

$$\frac{(r+s)!}{r!s!}$$

distinct cosets. If S_{r+s}/G denotes the collection of cosets then ψ defines a function on S_{r+s}/G, i.e., by what we have shown, there is a function $\tilde{\psi}$ on that set so that

$$\psi(\tau) = \tilde{\psi}(\tau G)$$

for every τ in S_{r+s}. If H is a left coset of G, then $\tilde{\psi}(H) = \psi(\tau)$ for every τ in H.

We now define the **exterior product** of the alternating multilinear forms L and M of degrees r and s by setting

$$L \wedge M = \sum_H \tilde{\psi}(H) \tag{5-49}$$

where H varies over S_{r+s}/G. Another way to phrase the definition of $L \wedge M$ is the following. Let S be any set of permutations of $\{1, \ldots, r+s\}$ which contains exactly one element from each left coset of G. Then

$$L \wedge M = \sum_\sigma (\operatorname{sgn} \sigma)(L \otimes M)_\sigma \tag{5-50}$$

where σ varies over S. Clearly

$$r!s!\, L \wedge M = \pi_{r+s}(L \otimes M)$$

so that the new definition is equivalent to (5-47) when K is a field of characteristic zero.

Theorem 9. *Let* K *be a commutative ring with identity and let* V *be a module over* K. *Then the exterior product is an associative operation on the alternating multilinear forms on* V. *In other words, if* L, M, *and* N *are alternating multilinear forms on* V *of degrees* r, s, *and* t, *respectively, then*

$$(\mathrm{L} \wedge \mathrm{M}) \wedge \mathrm{N} = \mathrm{L} \wedge (\mathrm{M} \wedge \mathrm{N}).$$

Proof. Although the proof of Theorem 8 does not apply here, it does suggest how to handle the general case. Let $G(r, s, t)$ be the subgroup of S_{r+s+t} that consists of the permutations which permute the sets

$$\{1, \ldots, r\}, \{r+1, \ldots, r+s\}, \{r+s+1, \ldots, r+s+t\}$$

within themselves. Then $(\operatorname{sgn} \mu)(L \otimes M \otimes N)_\mu$ is the same multilinear function for all μ in a given left coset of $G(r, s, t)$. Choose one element from each left coset of $G(r, s, t)$, and let E be the sum of the corresponding terms $(\operatorname{sgn} \mu)(L \otimes M \otimes N)_\mu$. Then E is independent of the way in which the representatives μ are chosen, and

$$r!s!t!\, E = \pi_{r+s+t}(L \otimes M \otimes N).$$

We shall show that $(L \wedge M) \wedge N$ and $L \wedge (M \wedge N)$ are both equal to E.

Let $G(r+s, t)$ be the subgroup of S_{r+s+t} that permutes the sets

$$\{1, \ldots, r+s\}, \{r+s+1, \ldots, r+s+t\}$$

within themselves. Let T be any set of permutations of $\{1, \ldots, r+s+t\}$ which contains exactly one element from each left coset of $G(r+s, t)$. By (5-50)

$$(L \wedge M) \wedge N = \sum_{\tau} (\operatorname{sgn} \tau)[(L \wedge M) \otimes N]_\tau$$

where the sum is extended over the permutations τ in T. Now let $G(r, s)$ be the subgroup of S_{r+s} that permutes the sets

$$\{1, \ldots, r\}, \{r+1, \ldots, r+s\}$$

within themselves. Let S be any set of permutations of $\{1, \ldots, r+s\}$ which contains exactly one element from each left coset of $G(r, s)$. From (5-50) and what we have shown above, it follows that

$$(L \wedge M) \wedge N = \sum_{\sigma, \tau} (\operatorname{sgn} \sigma)(\operatorname{sgn} \tau)[(L \otimes M)_\sigma \otimes N]_\tau$$

where the sum is extended over all pairs σ, τ in $S \times T$. If we agree to identify each σ in S_{r+s} with the element of S_{r+s+t} which agrees with σ on $\{1, \ldots, r+s\}$ and is the identity on $\{r+s+1, \ldots, r+s+t\}$, then we may write

$$(L \wedge M) \wedge N = \sum_{\sigma, \tau} \operatorname{sgn} (\sigma \tau)[(L \otimes M \otimes N)_\sigma]_\tau.$$

But,

$$[(L \otimes M \otimes N)_\sigma]_\tau = (L \otimes M \otimes N)_{\tau\sigma}.$$

Therefore

$$(L \wedge M) \wedge N = \sum_{\sigma, \tau} \operatorname{sgn} (\tau \sigma)(L \otimes M \otimes N)_{\tau\sigma}.$$

Now suppose we have

$$\tau_1\sigma_1 = \tau_2\sigma_2\, \gamma$$

with σ_i in S, τ_i in T, and γ in $G(r, s, t)$. Then $\tau_2^{-1}\, \tau_1 = \sigma_2\gamma\sigma_1^{-1}$, and since