A, B, C, D such thatg satisfies $\partial^2 g/(\partial u\, \partial v) = 0$. Solve this equation forg and thereby determine f. (Assume equality of the mixed partials.)

6. A function u is defined by an equation of the form

$$u(x, y) = xy f\left(\frac{x+y}{xy}\right).$$

Show that u satisfies a partial differential equation of the form

$$x^2 \frac{\partial u}{\partial x} - y^2 \frac{\partial u}{\partial y} = G(x, y)u,$$

and find $G(x, y)$.

7. The substitution $x = e^s$, $y = e^t$ convertsf(x, y) into $g(s, t)$, where $g(s, t) = f(e^s, e^t)$. If f is known to satisfy the partial differential equation

$$x^2 \frac{\partial^2 f}{\partial x^2} + y^2 \frac{\partial^2 f}{\partial y^2} + x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = 0,$$

show that g satisfies the partial-differential equation

$$\frac{\partial^2 g}{\partial s^2} + \frac{\partial^2 g}{\partial t^2} = 0.$$

8. Let f be a scalar field that is differentiable on an open set Sin $\mathbf{R}^n$. We say that f is *homogeneous of degree p* over S if

$$f(t\boldsymbol{x}) = t^p f(\boldsymbol{x})$$

for every $t > 0$ and every $\boldsymbol{x}$ in S for which $t\boldsymbol{x} \in S$. For a homogeneous scalar field of degreep show that we have

$$\boldsymbol{x} \cdot \nabla f(\boldsymbol{x}) = p f(x) \qquad \text{for each } \boldsymbol{x} \text{ in } S.$$

This is known as *Euler's theorem for homogeneous functions.* If $\boldsymbol{x} = (x_1, \ldots, x_n)$ it can be expressed as

$$x_1 \frac{\partial f}{\partial x_1} + \cdots + x_n \frac{\partial f}{\partial x_n} = p f(x_1, \ldots, x_n).$$

[Hint: For fixed x, define $g(t) = f(t\boldsymbol{x})$ and compute $g'(1)$.]

9. Prove the converse of Euler's theorem. That is, if f satisfies $\boldsymbol{x} \cdot \nabla f(x) = p f(x)$ for all $\boldsymbol{x}$ in an open set S, then f must be homogeneous of degree p over S. *[Hint:* For fixed $\boldsymbol{x}$, define $g(t) = f(t\boldsymbol{x}) - t^p f(\boldsymbol{x})$ and compute $g'(t)$.]

10. Prove the following extension of Euler's theorem for homogeneous functions of degree p in the 2-dimensional case. (Assume equality of the mixed partials.)

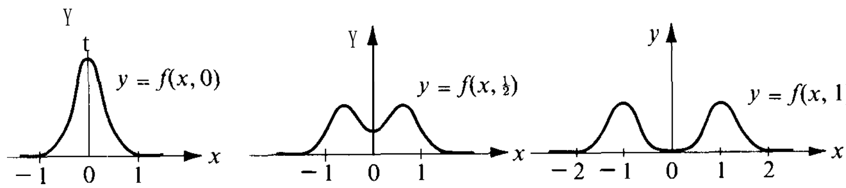
$$x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x\, \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = p(p-1)f.$$

9.4 The one-dimensional wave equation

Imagine a string of infinite length stretched along the x-axis and allowed to vibrate in the xy-plane. We denote by $y = f(x, t)$ the vertical displacement of the string at the point x at time t. We assume that, at time $t = 0$, the string is displaced along a prescribed curve, $y =$ F(x). An example is shown in Figure 9.1(a). Figures 9.1(b) and (c) show possible displacement curves for later values oft. We regard the displacement $f(x, t)$ as an unknown function of x and t to be determined. A mathematical model for this problem (suggested by physical considerations which we shall not discuss here) is the partial differential equation

$$\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2},$$

where c is a positive constant depending on the physical characteristics of the string. This equation is called the *one-dimensional wave equation.* We will solve this equation subject to certain auxiliary conditions.



(a) $t = 0$ (b) $t = \frac{1}{2}$ (c) $t = 1$

FIGURE 9.1 The displacement curve $y = f(x, t)$ shown for various values of t.

Since the initial displacement is the prescribed curve $y =$ P(x), we seek a solution satisfying the condition

$$f(x, 0) = F(x).$$

We also assume that $\partial y/\partial t$, the velocity of the vertical displacement, is prescribed at time $t = 0$, say

$$D_2 f(x, 0) = G(x),$$

where G is a given function. It seems reasonable to expect that this information should suffice to determine the subsequent motion of the string. We will show that, indeed, this is true by determining the functionfin terms of F and G. The solution is expressed in a form given by Jean d'Alembert (1717–1783), a French mathematician and philosopher.

THEOREM 9.2. D'ALEMBERT'S SOLUTION OF THE WAVE EQUATION. *Let Fand G be given functions such that G is differentiable and F is twice differentiable on* $\mathbf{R}^1$. *Then the function* f *given by the formula*

$$f(x, t) = \frac{F(x + ct) + F(x - ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)\, ds \tag{9.11}$$

satisfies the one-dimensional wave equation

(9.12) $$\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2}$$

and the initial conditions

(9.13) $$f(x, 0) = F(x), \qquad D_2 f(x, 0) = G(x).$$

Conversely, any function f with equal mixed partials which satisfies (9.12) and (9.13) necessarily has the form (9.11).

Proof. It is a straightforward exercise to verify that the function f given by (9.11) satisfies the wave equation and the given initial conditions. This verification is left to the reader. We shall prove the converse.

One way to proceed is to assume that f is a solution of the wave equation, introduce a linear change of variables,

$$x = Au + Bv, \qquad t = Cu + Dv,$$

which transforms $f(x, t)$ into a function of u and v, say

$$g(u, v) = f(Au + Bv, Cu + Dv),$$

and choose the constants A, B, C, D so that g satisfies the simpler equation

$$\frac{\partial^2 g}{\partial u \, \partial v} = 0.$$

Solving this equation for g we find that $g(u, v) = \varphi_1(u) + \varphi_2(v)$, where $\varphi_1(u)$ is a function of u alone and $\varphi_2(v)$ is a function of v alone. The constants A, B, C, D can be chosen so that $u = x + ct$, $v = x - ct$, from which we obtain

(9.14) $$f(x, t) = \varphi_1(x + ct) + \varphi_2(x - ct).$$

Then we use the initial conditions (9.13) to determine the functions φ_1 and φ_2 in terms of the given functions F and G.

We will obtain (9.14) by another method which makes use of Theorem 9.1 and avoids the change of variables. First we rewrite the wave equation in the form

(9.15) $$L_1(L_2 f) = 0,$$

where L_1 and L_2 are the first-order linear differential operators given by

$$L_1 = \frac{\partial}{\partial t} - c\frac{\partial}{\partial x}, \qquad L_2 = \frac{\partial}{\partial t} + c\frac{\partial}{\partial x}.$$

Let f be a solution of (9.15) and let

$$u(x, t) = L_2 f(x, t).$$

Equation (9.15) states that u satisfies the first-order equation $L_1(u) = 0$. Hence, by Theorem 9.1 we have

$$u(x, t) = \varphi(x + \text{ct})$$

for some function φ. Let Φ be any primitive of φ, say $\Phi(y) = \int_0^y \varphi(s)\, ds$, and let

$$v(x, t) = \frac{1}{2c}\Phi(x + ct) .$$

We will show that $L_2(v) = L,(f)$. We have

$$\frac{\partial v}{ax} = \frac{1}{2c}\Phi'(x + ct) \qquad \text{and} \qquad \frac{\partial v}{\partial t} = \frac{1}{2}\Phi'(x + \text{ct}),$$

so

$$L_2 v = \frac{\partial v}{\partial t} + c\,\frac{\partial v}{\partial x} = \Phi'(x + ct) = \varphi(x + ct) = u(x, t) = L_2 f.$$

In other words, the difference $f - \text{v}$ satisfies the first-order equation

$$L_2(f - v) = 0.$$

By Theorem 9.1 we must have $f(x, t) - v(x, t) = \psi(x - ct)$ for some function y. Therefore

$$f(x, t) = \text{v}(x, t) + \psi(x - \text{ct}) = \frac{1}{2c}\Phi(x + ct) + \psi(x - ct) .$$

This proves (9.14) with $\varphi_1 = \dfrac{1}{2c}\Phi$ and $\varphi_2 = \text{y}$.

Now we use the initial conditions (9.13) to determine the functions φ_1 and φ_2 in terms of the given functions F and G. The relation $f(x, 0) = F(x)$ implies

(9.16) $$\varphi_1(x) + \varphi_2(x) = F(x).$$

The other initial condition, $D_2 f(x, 0) = G(x)$, implies

(9.17) $$c\varphi_1'(x) - c\varphi_2'(x) = G(x).$$

Differentiating (9.16) we obtain

(9.18) $$\varphi_1'(x) + \varphi_2'(x) = F'(x) .$$

Solving (9.17) and (9.18) for $\varphi_1'(x)$ and $\varphi_2'(x)$ we find

$$\varphi_1'(x) = \frac{1}{2}F'(x) + \frac{1}{2c}G(x), \qquad \varphi_2'(x) = \frac{1}{2}F'(x) - \frac{1}{2c}G(x).$$

Integrating these relations we get

$$\varphi_1(x) - \varphi_1(0) = \frac{F(x) - F(0)}{2} + \frac{1}{2c}\int_0^x G(s)\, ds,$$

$$\varphi_2(x) - \varphi_2(0) = \frac{F(x) - F(0)}{2} - \frac{1}{2c}\int_0^x G(s)\, ds.$$

In the first equation we replace x by $x + ct$; in the second equation we replace x by $x - ct$. Then we add the two resulting equations and use the fact that $\varphi_1(0) + \varphi_2(0) = F(0)$ to obtain

$$f(x, t) = \varphi_1(x + ct) + \varphi_2(x - ct) = \frac{F(x + ct) + F(x - ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)\, ds.$$

This completes the proof.

EXAMPLE. Assume the initial displacement is given by the formula

$$F(x) = \begin{cases} 1 + \cos \pi x & \text{for } -1 \le x \le 1, \\ 0 & \text{for } |x| \ge 1. \end{cases}$$

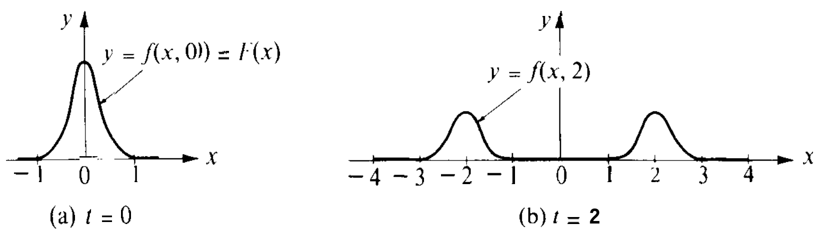


FIGURE 9.2 A solution of the wave equation shown for $t = 0$ and $t = 2$.

The graph of F is shown in Figures 9.1(a) and 9.2(a). Suppose that the initial velocity $G(x) = 0$ for all x. Then the resulting solution of the wave equation is given by the formula

$$f(x, t) = \frac{F(x + ct) + F(x - ct)}{2}.$$

Figures 9.1 and 9.2 show the curve $y = f(x, t)$ for various values of t. The figures illustrate that the solution of the wave equation is a combination of two standing waves, one traveling to the right, the other to the left, each with speed c.

Further examples illustrating the use of the chain rule in the study of partial differential equations are given in the next set of exercises.

9.5 Exercises

In this set of exercises you may assume differentiability of all functions under consideration.

1. If k is a positive constant and $g(x, t) = \frac{1}{2}x/\sqrt{kt}$, let

$$f(x, t) = \int_0^{g(x,t)} e^{-u^2}\, du.$$

(a) Show that $\dfrac{\partial f}{\partial x} = e^{-g^2}\dfrac{\partial g}{\partial x}$ and $\dfrac{\partial f}{\partial t} = e^{-g^2}\dfrac{\partial g}{\partial t}$,

(b) Show that f satisfies the partial differential equation

$$k\frac{\partial^2 f}{\partial x^2} = \frac{\partial f}{\partial t} \quad \text{(the } \textit{heat equation}\text{).}$$

2. Consider a scalar field f defined in $\mathbf{R}^2$ such that $f(x, y)$ depends only on the distance r of (x, y) from the origin, say $f(x, y) = g(r)$, where $r = (x^2 + y^2)^{1/2}$.

(a) Prove that for $(x, y) \neq (0, 0)$ we have

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{1}{r}g'(r) + g''(r).$$

(b) Now assume further that f satisfies ***Laplace's equation,***

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0,$$

for all $(x, y) \neq (0, 0)$. Use part (a) to prove that $f(x, y) = a \log (x^2 + y^2) + b$ for $(x, y) \neq (0, 0)$, where a and b are constants.

3. Repeat Exercise 2 for the n-dimensional case, where $n \geq 3$. That is, assume that $f(x) = f(x_1, \ldots, x_n) = g(r)$, where $r = \|x\|$. Show that

$$\frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} = \frac{n-1}{r}g'(r) + g''(r)$$

for $x \neq 0$. If f satisfies the ***n-dimensional Laplace equation,***

$$\frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} = 0,$$

for all $x \neq 0$, deduce that $f(x) = a\,\|x\|^{2-n} + b$ for $x \neq 0$, where a, b are constants.

Note: The linear operator ∇^2 defined by the equation

$$\nabla^2 f = \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2}$$

is called the ***n-dimensional Laplacian.***

4. ***Two-dimensional Laplacian in polar coordinates.*** The introduction of polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, converts $f(x, y)$ into $g(r, \theta)$. Verify the following formulas:

(a) $\|\nabla f(r\cos\theta, r\sin\theta)\|^2 = \left(\frac{\partial g}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial g}{\partial \theta}\right)^2.$

(b) $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 g}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2} + \frac{1}{r}\frac{\partial g}{\partial r}.$

5. ***Three-dimensional Laplacian in spherical coordinates.*** The introduction of spherical coordinates

$$x = \rho\cos\theta\sin\varphi, \qquad y = \rho\sin\theta\sin\varphi, \qquad z = \rho\cos\varphi,$$

transforms $f(x, y, z)$ to $F(\rho, \theta, \varphi)$. This exercise shows how to express the Laplacian $\nabla^2 f$ in terms of partial derivatives of F.

(a) First introduce polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ to transform $f(x, y, z)$ to $g(r, \theta, z)$. Use Exercise 4 to show that

$$\nabla^2 f = \frac{\partial^2 g}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2} + \frac{1}{r}\frac{\partial g}{\partial r} + \frac{\partial^2 g}{\partial z^2}.$$

(b) Now transform $g(r, \theta, z)$ to $F(\rho, \theta, \varphi)$ by taking $z = \rho\cos\varphi$, $r = \rho\sin\varphi$. Note that, except for a change in notation, this transformation is the same as that used in part (a). Deduce that

$$\nabla^2 f = \frac{\partial^2 F}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial F}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 F}{\partial \varphi^2} + \frac{\cos\varphi}{\rho^2\sin\varphi}\frac{\partial F}{\partial \varphi} + \frac{1}{\rho^2\sin\varphi}\frac{\partial^2 F}{\partial \theta^2}.$$

6. This exercise shows how Legendre's differential equation arises when we seek solutions of Laplace's equation having a special form. Let f be a scalar field satisfying the three-dimensional Laplace equation, $\nabla^2 f = 0$. Introduce spherical coordinates as in Exercise 5 and let $F(\rho, \theta, \varphi) = f(x, y, z)$.

(a) Suppose we seek solutions f of Laplace's equation such that $F(\rho, \theta, \varphi)$ is independent of θ and has the special form $F(\rho, \theta, \varphi) = \rho^n G(\varphi)$. Show that f satisfies Laplace's equation if G satisfies the second-order equation

$$\frac{d^2G}{d\varphi^2} + \cot\varphi\,\frac{dG}{d\varphi} + n(n+1)G = 0.$$

(b) The change of variable $x = \cos\varphi$ ($\varphi = \arccos x$, $-1 \le x \le 1$) transforms $G(\varphi)$ to $g(x)$. Show that g satisfies the Legendre equation

$$(1 - x^2)\frac{d^2g}{dx^2} - 2x\frac{dg}{dx} + n(n+1)g = 0.$$

7. ***Two-dimensional wave equation.*** A thin flexible membrane is stretched over the xy-plane and allowed to vibrate. Let $z = f(x, y, t)$ denote the vertical displacement of the membrane at the point (x, y) at time t. Physical considerations suggest that f satisfies the two-dimensional wave equation,

$$\frac{\partial^2 f}{\partial t^2} = c^2\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right),$$

where c is a positive constant depending on the physical characteristics of the membrane. This exercise reveals a connection between this equation and Bessel's differential equation.
(a) Introduce polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, and let $F(r, \theta, t) = f(r\cos\theta, r\sin\theta, t)$. If f satisfies the wave equation show that F satisfies the equation

$$\frac{\partial^2 F}{\partial t^2} = c^2\left(\frac{\partial^2 F}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2} + \frac{1}{r}\frac{\partial F}{\partial r}\right).$$

(b) If $F(r, \theta, t)$ is independent of θ, say $F(r, \theta, t) = \varphi(r, t)$ the equation in (a) simplifies to

$$\frac{\partial^2 \varphi}{\partial t^2} = c^2\left(\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r}\right).$$

Now let φ be a solution such that $\varphi(r, t)$ factors into a function of r times a function of t, say $\varphi(r, t) = R(r)T(t)$. Show that each of the functions R and T satisfies an ordinary linear differential equation of second order.
(c) If the function T in part (b) is periodic with period $2\pi/c$, show that R satisfies the Bessel equation $r^2R'' + rR' + r^2R = 0$.

9.6 Derivatives of functions defined implicitly

Some surfaces in 3-space are described by Cartesian equations of the form

$$F(x, y, z) = 0.$$

An equation like this is said to provide an *implicit representation* of the surface. For example, the equation $x^2 + y^2 + z^2 - 1 = 0$ represents the surface of a unit sphere with center at the origin. Sometimes it is possible to solve the equation $F(x, y, z) = 0$ for one of the variables in terms of the other two, say for z in terms of x and y. This leads to one or more equations of the form

$$z = f(x, y).$$

For the sphere we have two solutions,

$$z = \sqrt{1 - x^2 - y^2} \qquad \text{and} \qquad z = -\sqrt{1 - x^2 - y^2},$$

one representing the upper hemisphere, the other the lower hemisphere.

In the general case it may not be an easy matter to obtain an explicit formula for z in terms of x and y. For example, there is no easy method for solving for z in the equation $y^2 + xz + z^2 - e^z - 4 = 0$. Nevertheless, a judicious use of the chain rule makes it possible to deduce various properties of the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ without an explicit knowledge of $f(x, y)$. The procedure is described in this section.

We assume that there is a function $f(x, y)$ such that

(9.19) $$F[x, y, f(x, y)] = 0$$

for all (x, y) in some open set S, although we may not have explicit formulas for calculating $f(x, y)$. We describe this by saying that the equation $F(x, y, z) = 0$ defines z *implicitly* as a function of x and y, and we write

$$z = f(x, y).$$

Now we introduce an auxiliary function g defined on S as follows:

$$g(x, y) = F[x, y, f(x, y)].$$

Equation (9.19) states that $g(x, y) = 0$ on S; hence the partial derivatives $\partial g/\partial x$ and $\partial g/\partial y$ are also 0 on S. But we can also compute these partial derivatives by the chain rule. To do this we write

$$g(x, y) = F[u_1(x, y), u_2(x, y), u_3(x, y)],$$

where $u_1(x, y) = x$, $u_2(x, y) = y$, and $u_3(x, y) = f(x, y)$. The chain rule gives us the formulas

$$\frac{\partial g}{\partial x} = D_1F\frac{\partial u_1}{\partial x} + D_2F\frac{\partial u_2}{\partial x} + D_3F\frac{\partial u_3}{\partial x} \quad \text{and} \quad \frac{\partial g}{\partial y} = D_1F\frac{\partial u_1}{\partial y} + D_2F\frac{\partial u_2}{\partial y} + D_3F\frac{\partial u_3}{\partial y},$$

where each partial derivative D_kF is to be evaluated at $(x, y, f(x, y))$. Since we have

$$\frac{\partial u_1}{\partial x} = 1, \quad \frac{\partial u_2}{\partial x} = 0, \quad \frac{\partial u_3}{\partial x} = \frac{\partial f}{\partial x}, \quad \text{and} \quad \frac{\partial g}{\partial x} = 0,$$

the first of the foregoing equations becomes

$$D_1F + D_3F\frac{\partial f}{\partial x} = 0.$$

Solving this for $\partial f/\partial x$ we obtain

$$\frac{\partial f}{\partial x} = -\frac{D_1F[x, y, f(x, y)]}{D_3F[x, y, f(x, y)]} \tag{9.20}$$

at those points at which $D_3F[x, y, f(x, y)] \neq 0$. By a similar argument we obtain a corresponding formula for $\partial f/\partial y$:

$$\frac{\partial f}{\partial y} = -\frac{D_2F[x, y, f(x, y)]}{D_3F[x, y, f(x, y)]} \tag{9.21}$$

at those points at which $D_3F[x, y, f(x, y)] \neq 0$. These formulas are usually written more briefly as follows:

$$\frac{\partial f}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z}, \quad \frac{\partial f}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z}.$$

EXAMPLE. Assume that the equation $y^2 + xz + z^2 - e^z - c = 0$ defines z as a function of x and y, say $z = f(x, y)$. Find a value of the constant c such that $f(0, e) = 2$, and compute the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ at the point $(x, y) = (0, e)$.

Solution. When $x = 0$, $y = e$, and $z = 2$, the equation becomes $e^2 + 4 - e^2 - c = 0$, and this is satisfied by $c = 4$. Let $F(x, y, z) = y^2 + xz + z^2 - e^z - 4$. From (9.20) and (9.21) we have

$$\frac{\partial f}{\partial x} = -\frac{z}{x + 2z - e^z}, \qquad \frac{\partial f}{\partial y} = -\frac{2y}{x + 2z - e^z}.$$

When $x = 0$, $y = e$, and $z = 2$ we find $\partial f/\partial x = 2/(e^2 - 4)$ and $\partial f/\partial y = 2e/(e^2 - 4)$. Note that we were able to compute the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ using only the value of $f(x, y)$ at the single point $(0, e)$.

The foregoing discussion can be extended to functions of more than two variables.

THEOREM 9.3. *Let F be a scalar field differentiable on an open set T in $\mathbf{R}^n$. Assume that the equation*

$$F(x_1, \ldots, x_n) = 0$$

dejines x_n implicitly as a differentiable function of $x_1, \ldots, x_{n-1}$, say

$$x_n = f(x_1, \ldots, x_{n-1}),$$

for allpoints $(x_1, \ldots, x_{n-1})$ in some open set S in $\mathbf{R}^{n-1}$. Then for each $k = 1, 2, \ldots, n - 1$, the partial derivative $D_k f$ is given by the formula

$$D_k f = -\frac{D_k F}{D_n F} \tag{9.22}$$

at those points at which $D_n F \neq 0$. The partial derivatives $D_k F$ and $D_n F$ which appear in (9.22) are to be evaluated at the point $(x_1, x_2, \ldots, x_{n-1}, f(x_1, \ldots, x_{n-1}))$.

The proof is a direct extension of the argument used to derive Equations (9.20) and (9.21) and is left to the reader.

The discussion can be generalized in another way. Suppose we have two surfaces with the following implicit representations :

$$F(x, y, z) = 0, \qquad G(x, y, z) = 0. \tag{9.23}$$

If these surfaces intersect along a curve C, it may be possible to obtain a parametric representation of C by solving the two equations in (9.23) simultaneously for two of the variables in terms of the third, say for x and y in terms of z. Let us suppose that it is possible to solve for x and y and that solutions are given by the equations

$$x = X(z), \qquad y = Y(z)$$

for all z in some open interval *(a, b)*. Then when x and y are replaced by X(z) and Y(z), respectively, the two equations in (9.23) are identically satisfied. That is, we can write

$F[X(z), Y(z), z] = 0$ and $G[X(z), Y(z), z] = 0$ for all z in (a, b). Again, by using the chain rule, we can compute the derivatives $X'(z)$ and $Y'(z)$ without an explicit knowledge of $X(z)$ and $Y(z)$. To do this we introduce new functions f and g by means of the equations

$$f(z) = F[X(z), Y(z), z] \quad \text{and} \quad g(z) = G[X(z), Y(z), z].$$

Then $f(z) = g(z) = 0$ for every z in (a, b) and hence the derivatives $f'(z)$ and $g'(z)$ are also zero on (a, b). By the chain rule these derivatives are given by the formula

$$f'(z) = \frac{\partial F}{\partial x} X'(z) + \frac{\partial F}{\partial y} Y'(z) + \frac{\partial F}{\partial z}, \qquad g'(z) = \frac{\partial G}{\partial x} X'(z) + \frac{\partial G}{\partial y} Y'(z) + \frac{\partial G}{\partial z}.$$

Since $f'(z)$ and $g'(z)$ are both zero we can determine $X'(z)$ and $Y'(z)$ by solving the following pair of simultaneous *linear* equations:

$$\frac{\partial F}{\partial x} X'(z) + \frac{\partial F}{\partial y} Y'(z) = -\frac{\partial F}{\partial z},$$

$$\frac{\partial G}{\partial x} X'(z) + \frac{\partial G}{\partial y} Y'(z) = -\frac{\partial G}{\partial z}.$$

At those points at which the determinant of the system is not zero, these equations have a unique solution which can be expressed as follows, using Cramer's rule:

(9.24)
$$X'(z) = -\frac{\begin{vmatrix} \dfrac{\partial F}{\partial z} & \dfrac{\partial F}{\partial y} \\ \dfrac{\partial G}{\partial z} & \dfrac{\partial G}{\partial y} \end{vmatrix}}{\begin{vmatrix} \dfrac{\partial F}{\partial x} & \dfrac{\partial F}{\partial y} \\ \dfrac{\partial G}{\partial x} & \dfrac{\partial G}{\partial y} \end{vmatrix}}, \qquad Y'(z) = -\frac{\begin{vmatrix} \dfrac{\partial F}{\partial x} & \dfrac{\partial F}{\partial z} \\ \dfrac{\partial G}{\partial x} & \dfrac{\partial G}{\partial z} \end{vmatrix}}{\begin{vmatrix} \dfrac{\partial F}{\partial x} & \dfrac{\partial F}{\partial y} \\ \dfrac{\partial G}{\partial x} & \dfrac{\partial G}{\partial y} \end{vmatrix}}.$$

The determinants which appear in (9.24) are determinants of Jacobian matrices and are called ***Jacobian determinants.*** A special notation is often used to denote Jacobian determinants. We write

$$\frac{\partial(f_1, \ldots, f_n)}{\partial(x_1, \ldots, x_n)} = \det \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & & & \vdots \\ \dfrac{\partial f_n}{\partial x_1} & \dfrac{\partial f_n}{\partial x_2} & \cdots & \dfrac{\partial f_n}{\partial x_n} \end{bmatrix}.$$

In this notation, the formulas in (9.24) can be expressed more briefly in the form

$$(9.25)\qquad X'(z) = \frac{\partial(F, G)/\partial(y, z)}{\partial(F, G)/\partial(x, y)}, \qquad Y'(z) = \frac{\partial(F, G)/\partial(z, x)}{\partial(F, G)/\partial(x, y)}.$$

(The minus sign has been incorporated into the numerators by interchanging the columns.)

The method can be extended to treat more general situations in which m equations in n variables are given, where $n > m$ and we solve for m of the variables in terms of the remaining $n - m$ variables. The partial derivatives of the new functions so defined can be expressed as quotients of Jacobian determinants, generalizing (9.25). An example with $m = 2$ and $n = 4$ is described in Exercise 3 of Section 9.8.

9.7 Worked examples

In this section we illustrate some of the concepts of the foregoing section by solving various types of problems dealing with functions defined implicitly.

EXAMPLE 1. Assume that the equation $g(x, y) = 0$ determines y as a differentiable function of x, say $y = Y(x)$ for all x in some open interval (a, b). Express the derivative $Y'(x)$ in terms of the partial derivatives of g.

Solution. Let $G(x) = g[x, Y(x)]$ for x in (a, b). Then the equation $g(x, y) = 0$ implies $G(x) = 0$ in (a, b). By the chain rule we have

$$G'(x) = \frac{\partial g}{\partial x}\cdot 1 + \frac{\partial g}{\partial y} Y'(x),$$

from which we obtain

$$(9.26)\qquad Y'(x) = -\frac{\partial g/\partial x}{\partial g/\partial y}$$

at those points x in (a, b) at which $\partial g/\partial y \neq 0$. The partial derivatives $\partial g/\partial x$ and $\partial g/\partial y$ are given by the formulas $\partial g/\partial x = D_1 g[x, Y(x)]$ and $\partial g/\partial y = D_2 g[x, Y(x)]$.

EXAMPLE 2. When y is eliminated from the two equations $z = f(x, y)$ and $g(x, y) = 0$, the result can be expressed in the form $z = h(x)$. Express the derivative $h'(x)$ in terms of the partial derivatives off and g.

Solution. Let us assume that the equation $g(x, y) = 0$ may be solved for y in terms of x and that a solution is given by $y = Y(x)$ for all x in some open interval (a, b). Then the function h is given by the formula

$$h(x) = f[x, Y(x)] \qquad \text{if} \quad x \in (a, b).$$

Applying the chain rule we have

$$h'(x) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} Y'(x).$$