Exercises

1. Assuming the usual theory of similar triangles (which Euclid based on the Axiom of Archimedes), prove the parallel case of Desargues's Theorem, and its converse. (The converse is: if $AB, A'B'$ and $AC, A'C'$ and $BC, B'C'$ are three pairs of parallels then AA', BB' and CC' are concurrent.)

2. Descartes showed that, roughly speaking, a curve is a conic if and only if its equation has the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

   Give examples of equations of this form which represent the following 'curves':

   (a) a circle,
   (b) two parallel straight lines,
   (c) a point,
   (d) the whole plane.

3. Why cannot an integer of the form $4n - 1$ be the sum of two integer squares?

4. What is the smallest positive integer which quadruples when its final (base ten) digit is shifted to the front? (For example, it is not 125 since 512 is not 4 times 125.)

5. Prove that, if $x^p + y^p = z^p$ has one solution in positive integers, then it has infinitely many such solutions.

6. Show that, if $x^p + y^p = z^p$, with p an odd prime, then p is a factor of $x + y - z$.

7. Give a proof of the Binomial Theorem.

8. Find two distinct proofs to show that the sum of the entries in any row in Pascal's triangle is a power of 2.

9. Show that the alternating (adding and subtracting) sum of the entries in any row of Pascal's triangle is 0.

10. Using mathematical induction, show that the $(n + 1)$th Fibonacci number is

$$\binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots.$$

11. Project: Prove Pascal's mystic hexagram theorem for the case of the circle. Note that some of the points of intersection may be outside the circle. (Hint: use the Theorem of Menelaus.)

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The Seventeenth Century Continued

Mathematics in the 17th century was by no means confined to France. Elsewhere in Europe, some of the many great mathematicians were the following:

- Bonaventura Cavalieri (1598–1647) in Italy,
- John Wallis (1616–1703) in England,
- Nicolaus Mercator (1620–1687) in Germany and England,
- Christian Huygens (1629–1695) in Holland,
- Isaac Barrow (1630–1677) in England,
- James Gregory (1638–1675) in Scotland,
- Isaac Newton (1642–1727) in England,
- Gottfried Wilhelm Leibniz (1646–1716) in Germany.

Cavalieri rediscovered Archimedes's *method of indivisibles* for calculating areas and volumes. As an example of Cavalieri's approach, let us consider the following derivation of the area of the ellipse $y = (b/a)(a^2 - x^2)^{\frac{1}{2}}$. First we plot the circle $y = (a^2 - x^2)^{\frac{1}{2}}$ on the same rectangular coordinate frame of reference. Second, we note that a vertical chord of the ellipse is just b/a of the corresponding vertical chord of the circle (i.e. the chord lying in the same straight line). Third, we think of the areas of the ellipse and the circle as somehow being the 'sums' of their chords. The ratio of ellipse chord to corresponding circle chord is always b/a, so it follows that the ratio of the area of the ellipse to the area of the circle is also b/a. Hence the area of the ellipse is $(b/a)\pi a^2 = \pi ab$.

Cavalieri's most important result is perhaps the theorem which, in our notation, reads as follows:

$$\int_0^a x^n dx = \frac{a^{n+1}}{n+1}.$$

Wallis was a professor of geometry at Oxford. He was a Royalist, and ended up as chaplain to Charles II. He also invented a method for teaching deaf mutes.

In algebra, he used negative and fractional exponents: $x^{-n} = 1/x^n$ and $x^{p/q} = \sqrt[q]{x^p}$. Using Cavalieri's methods, he calculated the area under the curve

$$y = a_0x^0 + a_1x^1 + \cdots + a_nx^n.$$

He also discovered, but could not give a rigorous proof for, the following expression of π as an infinite product:

$$2 \cdot \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{8}{7} \cdot \frac{8}{9} \cdot \cdots.$$

Mercator calculated the area under the curve

$$y = \frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots,$$

obtaining

$$\log_e(1+x) = x - x^2/2 + x^3/3 - x^4/4 + \cdots$$

(converging for $-1 < x < 1$). (It was Gregory of St.Vincent (1584–1667) who first showed that the integral of $1/x$ is the natural log of x.) The cartographer's Mercator projection is due, not to Nicolaus Mercator, but to Gerhardus Mercator (1512–1592).

Like Pascal, Huygens made a study of the cycloid. This is the curve described by a point fixed to the rim of a wheel as it rolls along a flat surface. Huygens showed that the 'evolute' of a cycloid is again a cycloid. He also discovered (Galileo notwithstanding) that a pendulum had to swing in a cycloidal arc, and not in a circular arc, if the period of oscillation was to be strictly independent of the amplitude of the swing.

Huygen's most profound contribution to physics was the wave theory of light. Assuming that light is propagated in waves in an oscillating 'ether', he was able to explain the laws of reflection and refraction. His theory was in one respect better than Newton's corpuscular (particle) theory, which failed to explain interference phenomena. Nonetheless, Huygen's theory was