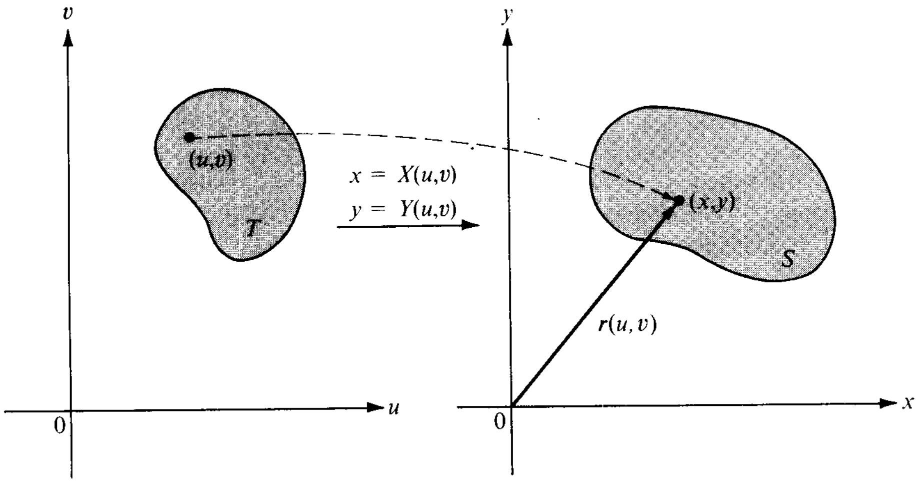

FIGURE 11.23 The mapping defined by the vector equation $\boldsymbol{r}(u, v) = X(u, v)\boldsymbol{i} + Y(u, v)\boldsymbol{j}$.

substitution for double integrals is more elaborate than in the one-dimensional case because there are two formal substitutions to be made, one for x and another for y. This means that instead of the one function g which appears in Equation (11.29), we now have two functions, say X and Y, which connect x, y with u, v as follows:

(11.30) $$x = X(u, v), \qquad y = Y(u, v).$$

The two equations in (11.30) define a mapping which carries a point (u, v) in the uv-plane into an image point (x, y) in the xy-plane. A set T of points in the uv-plane is mapped onto another set S in the xy-plane, as suggested by Figure 11.23. The mapping can also be described by means of a vector-valued function. From the origin in the xy-plane we draw the radius vector $\boldsymbol{r}$ to a general point (x, y) of S, as shown in Figure 11.23. The vector $\boldsymbol{r}$ depends on both u and v and can be considered a vector-valued function of two variables defined by the equation

(11.31) $$\boldsymbol{r}(u, v) = X(u, v)\boldsymbol{i} + Y(u, v)\boldsymbol{j} \quad \text{if} \quad (u, v) \in T.$$

This equation is called a *vector equation* of the mapping. As (u, v) runs through the points of T the endpoint of $\boldsymbol{r}(u, v)$ traces out the points of S.

Sometimes the two equations in (11.30) can be solved for u and v in terms of x and y. When this is possible we may express the result in the form

$$u = U(x, y), \qquad v = V(x, y).$$

These equations define a mapping from the xy-plane to the uv-plane, called the *inverse mapping* of the one defined by (11.30), since it carries points of S back to T. The so-called *one-to-one mappings* are of special importance. These carry *distinct* points of T onto *distinct* points of S; in other words, no two distinct points of T are mapped onto the same point of S by a one-to-one mapping. Each such mapping establishes a one-to-one correspondence between the points in T and those in S and enables us (at least in theory) to go back from S to T by the inverse mapping (which, of course, is also one-to-one).

We shall consider mappings for which the functions X and Y are continuous and have continuous partial derivatives $\partial X/\partial u$, $\partial X/\partial v$, $\partial Y/\partial u$, and $\partial Y/\partial v$ on S. Similar assumptions are made for the functions U and V. These are not serious restrictions since they are satisfied by most functions encountered in practice.

The formula for transforming double integrals may be written as

(11.32) $$\iint_S f(x, y)\, dx\, dy = \iint_T f[X(u, v), Y(u, v)]\, |J(u, v)|\, du\, dv .$$

The factor $J(u, v)$ which appears in the integrand on the right plays the role of the factor $g'(t)$ which appears in the one-dimensional Formula (11.29). This factor is called the *Jacobian determinant* of the mapping defined by (11.30); it is equal to

$$J(u, v) = \begin{vmatrix} \dfrac{\partial X}{\partial u} & \dfrac{\partial Y}{\partial u} \\[2ex] \dfrac{\partial X}{\partial v} & \dfrac{\partial Y}{\partial v} \end{vmatrix} .$$

Sometimes the symbol $\partial(X, Y)/\partial(u, v)$ is used instead of $J(u, v)$ to represent the Jacobian determinant.

We shall not discuss the most general conditions under which the transformation formula (11.32) is valid. It can be shown† that (11.32) holds if, in addition to the continuity assumptions on X, Y, U, and V mentioned above, we assume that the mapping from T to S is one-to-one and that the Jacobian $J(u, v)$ is never zero. The formula is also valid if the mapping fails to be one-to-one on a subset of T of content zero, or if the Jacobian vanishes on a subset of content zero.

In Section 11.30 we show how the transformation formula (11.32) can be derived as a consequence of one of its special cases, namely, the case in which S is a rectangle and the function f has the constant value 1 at each point of S. In this special case (11.32) becomes

(11.33) $$\iint_S dx\, dy = \iint_T |J(u, v)|\, du\, dv .$$

Even for this case a proof is not simple. In Section 11.29 a proof of (11.33) is given with the aid of Green's theorem. The remainder of this section will present a simple geometric argument which explains why a formula like (11.33) should hold.

Geometric motivation for Equation (11.33): Take a region T in the uv-plane, as shown in Figure 11.23, and let S denote the set of points in the xy-plane onto which T is mapped by the vector function $\boldsymbol{r}$ given by (11.31). Now introduce two new vector-valued functions $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ which are obtained by taking the partial derivatives of the components of $\boldsymbol{r}$

† See Theorem 10-30 of the author's *Mathematical Analysis.*

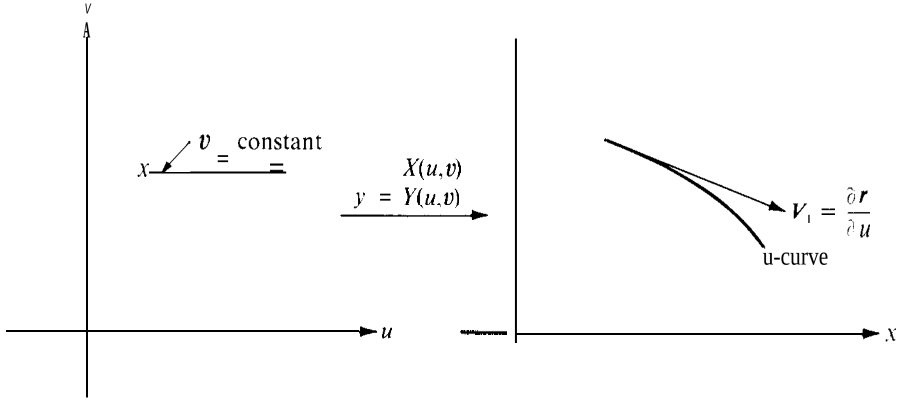

FIGURE 11.24 A u-curve and a corresponding velocity vector.

with respect to u and v, respectively. That is, define

$$V_1 = \frac{\partial r}{\partial u} = \frac{\partial X}{\partial u}\,i + \frac{\partial Y}{\partial u}\,j \qquad \text{and} \qquad V_2 = \frac{\partial r}{\partial v} = \frac{\partial X}{\partial v}\,i + \frac{\partial Y}{\partial v}\,j.$$

These vectors may be interpreted geometrically as follows: Consider a horizontal line segment in the uv-plane (v is constant on such a segment). The vector function r maps this segment onto a curve (called a u-curve) in the xy-plane, as suggested in Figure 11.24. If we think of u as a parameter representing time, the vector V_1 represents the velocity of the position r and is therefore tangent to the curve traced out by the tip of r. In the same way, each vector V_2 represents the velocity vector of a v-curve obtained by setting u = constant. A u-curve and a v-curve pass through each point of the region S.

Consider now a small rectangle with dimensions Au and Au, as shown in Figure 11.25. If Au is the length of a small time interval, then in time Au a point of a u-curve moves along the curve a distance approximately equal to the product $\|V_1\|$ Au (since $\|V_1\|$ represents the speed and Au the time). Similarly, in time Δv a point on a v-curve moves a

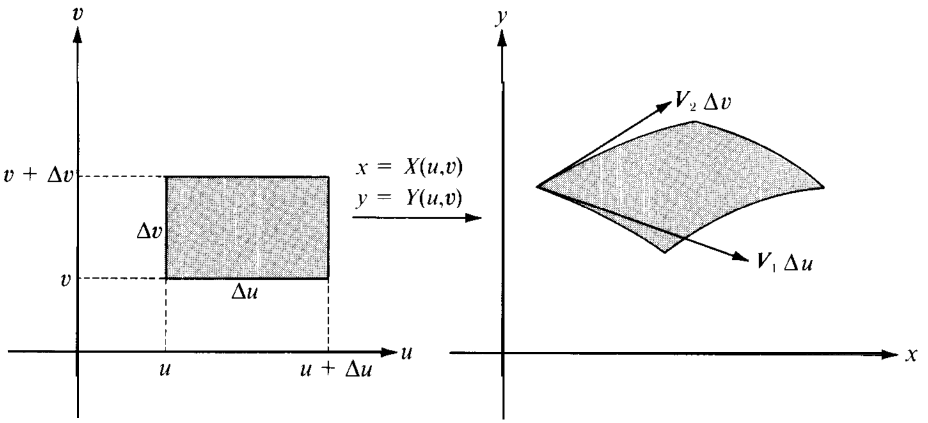

FIGURE 11.25 The image of a rectangular region in the uv-plane is a curvilinear parallelogram in the xy-plane.

distance nearly equal to $\|V_2\| \Delta u$. Hence the rectangular region with dimensions Δu and Δv in the uv-plane is traced onto a portion of the xy-plane that is nearly a parallelogram, whose sides are the vectors $V_1 \Delta u$ and $V_2 \Delta v$, as suggested by Figure 11.25. The area of this parallelogram is the magnitude of the cross product of the two vectors $V_1 \Delta u$ and $V_2 \Delta v$; this is equal to

$$\|(V_1 \Delta u) \times (V_2 \Delta v)\| = \|V_1 \times V_2\| \Delta u \Delta v.$$

If we compute the cross product $V_1 \times V_2$ in terms of the components of V_1 and V_2 we find

$$V_1 \times V_2 = \begin{vmatrix} i & j & k \\ \frac{\partial X}{\partial u} & \frac{\partial Y}{\partial u} & 0 \\ \frac{\partial X}{\partial v} & \frac{\partial Y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial X}{\partial u} & \frac{\partial Y}{\partial u} \\ \frac{\partial X}{\partial v} & \frac{\partial Y}{\partial v} \end{vmatrix} k = J(u, v)\boldsymbol{k}.$$

Therefore the magnitude of $V_1 \times V_2$ is exactly $|J(u, v)|$ and the area of the curvilinear parallelogram in Figure 11.25 is nearly equal to $|J(u, v)| \Delta u \Delta v$.

If $J(u, v) = 1$ for all points in T, then the "parallelogram" has the same area as the rectangle and the mapping preserves areas. Otherwise, to obtain the area of the parallelogram we must multiply the area of the rectangle by $|J(u, v)|$. This suggests that the Jacobian may be thought of as a "magnification factor" for areas.

Now let P be a partition of a large rectangle R enclosing the entire region T and consider a typical subrectangle of P of, say, dimensions Δu and Δv. If Δu and Δv are small, the Jacobian function J is nearly constant on the subrectangle and hence J acts somewhat like a step function on R. (We define J to be zero outside T.) If we think of J as an actual step function, then the double integral of $|J|$ over R (and hence over T) is a sum of products of the form $|J(u, v)| \Delta u \Delta v$ and the above remarks suggest that this sum is nearly equal to the area of S, which we know to be the double integral $\iint_S dx\, dy$.

This geometric discussion, which merely suggests why we might expect an equation like (11.33) to hold, can be made the basis of a rigorous proof, but the details are lengthy and rather intricate. As mentioned above, a proof of (11.33), based on an entirely different approach, will be given in a later section.

If $J(u, v) = 0$ at a particular point (u, v), the two vectors V_1 and V_2 are parallel (since their cross product is the zero vector) and the parallelogram degenerates into a line segment. Such points are called ***singular points*** of the mapping. As we have already mentioned, transformation formula (11.32) is also valid whenever there are only a finite number of such singular points or, more generally, when the singular points form a set of content zero. This is the case for all the mappings we shall use. In the next section we illustrate the use of formula (11.32) in two important examples.

11.27 Special cases of the transformation formula

EXAMPLE 1. ***Polar coordinates.*** In this case we write r and θ instead of u and v and describe the mapping by the two equations:

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

That is, $X(r, \theta) = r\cos\theta$ and $Y(r, \theta) = r\sin\theta$. To obtain a one-to-one mapping we keep $r > 0$ and restrict θ to lie in an interval of the form $\theta_0 \le \theta < \theta_0 + 2\pi$. For example, the mapping is one-to-one on any subset of the rectangle $(0, a] \times [0, 2\pi)$ in the $r\theta$-plane. The Jacobian determinant of this mapping is

$$J(r, \theta) = \begin{vmatrix} \dfrac{\partial X}{\partial r} & \dfrac{\partial Y}{\partial r} \\ \dfrac{\partial X}{\partial \theta} & \dfrac{\partial Y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r\,.$$

Hence the transformation formula in (11.32) becomes

$$\iint_S f(x, y)\,dx\,dy = \iint_T f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta\,.$$

The r-curves are straight lines through the origin and the θ-curves are circles centered at the origin. The image of a rectangle in the $r\theta$-plane is a "parallelogram" in the xy-plane

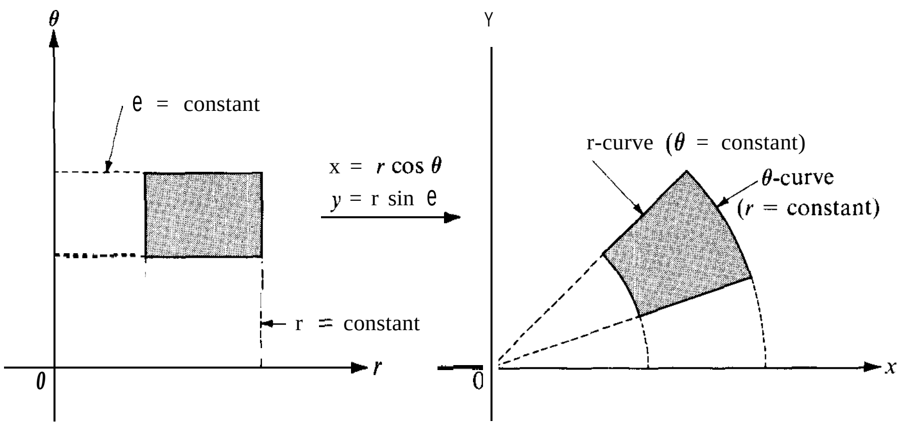

FIGURE 11.26 Transformation by polar coordinates.

bounded by two radial lines and two circular arcs, as shown in Figure 11.26. The Jacobian determinant vanishes when $r = 0$, but this does not affect the validity of the transformation formula because the set of points with $r = 0$ has content zero.

Since $V_1 = \cos\theta\,\boldsymbol{i} + \sin\theta\,\boldsymbol{j}$, we have $\|V_1\| = 1$, so there is no distortion of distances along the r-curves. On the other hand, we have

$$V_2 = -r\sin\theta\,\boldsymbol{i} + r\cos\theta\,\boldsymbol{j}, \qquad \|V_2\| = r,$$

so distances along the θ-curves are multiplied by the factor r.

Polar coordinates are particularly suitable when the region of integration has boundaries along which r or θ is constant. For example, consider the integral for the volume of one

octant of a sphere of radius a,

$$\iint_S \sqrt{a^2 - x^2 - y^2}\, dx\, dy,$$

where the region S is the first quadrant of the circular disk $x^2 + y^2 \le a^2$. In polar coordinates the integral becomes

$$\iint_T \sqrt{a^2 - r^2}\, r\, dr\, d\theta,$$

where the region of integration T is now a rectangle $[0, a] \times [0, \frac{1}{2}\pi]$. Integrating first with respect to θ and then with respect to r we obtain

$$\iint_T \sqrt{a^2 - r^2}\, r\, dr\, d\theta = \frac{\pi}{2}\int_0^a r\sqrt{a^2 - r^2}\, dr = \frac{\pi}{2}\cdot\frac{(a^2 - r^2)^{3/2}}{-3}\bigg|_0^a = \frac{\pi a^3}{6}.$$

The same result can be obtained by integrating in rectangular coordinates but the calculation is more complicated.

EXAMPLE 2. ***Linear transformations.*** Consider a linear transformation defined by a pair of equations of the form

$$x = Au + Bv, \qquad y = Cu + Dv, \tag{11.34}$$

where A, B, C, D are given constants. The Jacobian determinant is

$$J(u, v) = AD - BC,$$

and in order to have an inverse we assume that $AD - BC \neq 0$. This assures us that the two linear equations in (11.34) can be solved for u and v in terms of x and y.

Linear transformations carry parallel lines into parallel lines. Therefore the image of a rectangle in the uv-plane is a parallelogram in the xy-plane, and its area is that of the rectangle multiplied by the factor $|J(u, v)| = |AD - BC|$. Transformation formula (11.32) becomes

$$\iint_S f(x, y)\, dx\, dy = |AD - BC| \iint_T f(Au + Bv, Cu + Dv)\, du\, dv.$$

To illustrate an example in which a linear change of variables is useful, let us consider the integral

$$\iint_S e^{(y-x)/(y+x)}\, dx\, dy,$$

where S is the triangle bounded by the line $x + y = 2$ and the two coordinate axes. (See Figure 11.27.) The presence of $y - x$ and $y + x$ in the integrand suggests the change of variables

$$u = y - x, \qquad v = y + x.$$