(b) If $s > 1$, show that $C(s) = \zeta(s)$, where ζ is the Riemann zeta function defined for $s > 1$ by the series

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}.$$

The series for $\zeta(s)$ diverges for $s \leq 1$. However, since the formula for $C(s)$ in part (a) is meaningful for $0 < s < 1$, it can be used to extend the definition of $\zeta(s)$ to the open interval $0 < s < 1$. Thus, for $s > 0$ and $s \neq 1$ we have the formula

$$\zeta(s) = 1 + \frac{1}{s-1} - s\int_1^{\infty} \frac{t-[t]}{t^{s+1}}\,dt.$$

This is a *theorem* if $s > 1$, and a *definition* if $0 < s < 1$.

In Exercises 4 through 6, φ_2 is the function introduced in Section 15.22.

4. (a) Use Euler's summation formula to prove that

$$\sum_{k=1}^{n} \log^2 k = (n + \tfrac{1}{2})\log^2 n - 2n\log n + 2n - 2 + 2\int_1^n \varphi_2(x)\,\frac{\log x - 1}{x^2}\,dx.$$

(b) Use part (a) to deduce that for $n > e$ we have

$$\sum_{k=1}^{n} \log^2 k = (n + \tfrac{1}{2})\log^2 n - 2n\log n + 2n + A - E(n),$$

where A is a constant and $0 < E(n) < \dfrac{\log n}{4n}$.

5. (a) Use Euler's summation formula to prove that

$$\sum_{k=1}^{n} \frac{\log k}{k} = \frac{1}{2}\log^2 n + \frac{1}{2}\frac{\log n}{n} - \int_1^n \frac{2\log x - 3}{x^3}\,\varphi_2(x)\,dx.$$

(b) Use part (a) to deduce that for $n > e^{3/2}$ we have

$$\sum_{k=1}^{n} \frac{\log k}{k} = \frac{1}{2}\log^2 n + \frac{1}{2}\frac{\log n}{n} + A - E(n),$$

where A is a constant and $0 < E(n) < \dfrac{\log n}{8n^2}$.

6. (a) If $n > 2$ use Euler's summation formula to prove that

$$\sum_{k=2}^{n} \frac{1}{k\log k} = \log(\log n) + \frac{1}{2n\log n} + \frac{1}{4\log 2} - \log(\log 2) - \int_2^n \varphi_2(x)\,\frac{2 + 3\log x + 2\log^2 x}{(x\log x)^3}\,dx.$$

(b) Use part (a) to deduce that for $n > 2$ we have

$$\sum_{k=2}^{n} \frac{1}{k \log k} = \log(\log n) + A + \frac{1}{2n \log n} - E(n),$$

where A is a constant and $0 < E(n) < \dfrac{1}{4n^2 \log n}$.

7. (a) If $a > 0$ and $p > 0$, use Euler's summation formula to prove that

$$\sum_{k=0}^{\infty} e^{-ak^p} = \frac{\Gamma\left(1 + \frac{1}{p}\right)}{a^{1/p}} + \frac{1}{2} - ap \int_0^{\infty} \varphi_1(x) x^{p-1} e^{-ax^p}\, dx,$$

where Γ is the gamma function.

(b) Use part (a) to deduce that

$$\sum_{k=0}^{\infty} e^{-ak^p} = \frac{\Gamma\left(1 + \frac{1}{p}\right)}{a^{1/p}} + \theta, \qquad \text{where } 0 < \theta < 1.$$

8. Deduce the following limit relations with the aid of Stirling's formula and/or Wallis' inequality.

(a) $\displaystyle\lim_{n\to\infty} \frac{n}{(n!)^{1/n}} = e.$

(b) $\displaystyle\lim_{n\to\infty} \frac{(n!)^2 2^{2n}}{(2n)!\,\sqrt{n}} = \sqrt{\pi}.$

(c) $\displaystyle\lim_{n\to\infty} (-1)^n \binom{-\frac{1}{2}}{n} n = \frac{1}{\sqrt{\pi}}.$

9. Let $I_n = \int_0^{\pi/2} \sin^n t\, dt$, where n is a nonnegative integer. In Section 15.22 it was shown that the sequence $\{I_n\}$ satisfies the recursion formula

$$I_{n+2} = \frac{n+1}{n+2} I_n.$$

Let $f(n) = \frac{1}{2}\sqrt{\pi}\,\Gamma\left(\frac{n+1}{2}\right) \Big/ \Gamma\left(\frac{n}{2} + 1\right)$, where Γ is the gamma function.

(a) Use the functional equation $\Gamma(s + 1) = s\Gamma(s)$ to show that

$$f(n + 2) = \frac{n+1}{n+2} f(n).$$

(b) Use part (a) to deduce that

$$\int_0^{\pi/2} \sin^n t\, dt = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2} + 1\right)}.$$

SUGGESTED REFERENCES

This small list contains only a few books suggested for further reading on the general principles of numerical analysis. All these books contain further references to works of a more special nature. The list of tables given in Todd's survey (Reference 9 below) is especially recommended.

Books

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2. P. J. Davis, ***Interpolation and*** Approximation, Blaisdell, Waltham, Mass., 1963.
3. D. R. Hartree, ***Numerical Analysis,*** Oxford Univ. Press (Clarendon), London and New York, 1958.
4. F. B. Hildebrand, ***Introduction to Numerical Analysis,*** McGraw-Hill, New York, 1956.
5. A. S. Householder, ***Principles of Numerical Analysis,*** McGraw-Hill, New York, 1953.
6. W. E. Milne, ***Numerical Calculus,*** Princeton Univ. Press, Princeton, N.J., 1950.
7. J. B. Scarborough, ***Numerical Mathematical Analysis,*** Johns Hopkins Press, Baltimore, Md., 1958, 6th ed., 1966.
8. J. Todd, ***Introduction to the Constructive Theory of Functions,*** Academic Press, New York, 1963.
9. J. Todd (ed.), ***Survey of Numerical Analysis,*** McGraw-Hill, New York, 1962.

Tables

10. L. J. Comrie (ed.), ***Chambers' Six-figure Mathematical Tables,*** W. & R. Chambers, London and Edinburgh, 1949.
11. L. J. Comrie, "Interpolation and Allied Tables," 2nd rev. reprint from ***Nautical Almanac*** for 1937, H.M. Stationery Office, London, 1948.
12. A. J. Fletcher, J. C. P. Miller, and L. Rosenhead, ***Index of Mathematical Tables,*** McGraw-Hill, New York, 1946.
13. ***Tables of Lagrangian Interpolation Coefficients,*** Natl. Bur. Standards Columbia Press Series, Vol. 4., Columbia Univ. Press, New York, 1944.

ANSWERS TO EXERCISES

Chapter 1

1.5 Exercises (page 7)

1. Yes
2. Yes
3. Yes
4. Yes
5. No
6. Yes
7. Yes
8. Yes
9. Yes
10. Yes
11. No
12. Yes
13. Yes
14. No
15. Yes
16. Yes
17. Yes
18. Yes
19. Yes
20. Yes
21. Yes
22. Yes
23. No
24. Yes
25. No
26. Yes
27, Yes
28. Yes

31. (a) No (b) No (c) No (d) No

1.10 Exercises (page 13)

1. Yes; 2
2. Yes; 2
3. Yes; 2
4. Yes; 2
5. Yes; 1
6. No
7. No
8. No
9. Yes; 1
10. Yes; 1
11. Yes; n
12. Yes; n
13. Yes; n
14. Yes; n
15. Yes; n
16. Yes; n

17. Yes; dim $= 1 + \frac{1}{2}n$ if n is even, $\frac{1}{2}(n + 1)$ if n is odd
18. Yes; dim $= \frac{1}{2}n$ if n is even, $\frac{1}{2}(n + 1)$ if n is odd
19. Yes; $k + 1$
20. No
21. (a) dim = 3 (b) dim = 3 (c) dim = 2 (d) dim = 2
23. (a) If $a \neq 0$ and $b \neq 0$, set is independent, dim = 3 ; if one of a or b is zero, set is dependent, dim = 2 (b) Independent, dim = 2 (c) If $a \neq 0$, independent, dim = 3 ; if $a = 0$, dependent, dim = 2 (d) Independent; dim = 3 (e) Dependent; dim = 2 (f) Independent; dim = 2 (g) Independent; dim = 2 (h) Dependent; dim = 2 (i) Independent; dim = 2 (j) Independent; dim = 2

1.13 Exercises (page 20)

1. (a) No (b) No (c) No (d) No (e) Yes

8. (a) $\frac{1}{2}\sqrt{e^2 + 1}$ (b) $g(x) = b\left(x - \dfrac{e^2 + 1}{4}\right)$, b arbitrary

10. (b) $\dfrac{(n + 1)(2n + 1)}{6n}a + \dfrac{n + 1}{2}b$ (c) $g(t) = a\left(t - \dfrac{2n + 1}{3n}\right)$, a arbitrary

11. (c) 43 (d) $g(t) = a(1 - \frac{2}{3}t)$, a arbitrary
12. (a) No (b) No (c) No (d) No
13. (c) 1 (d) $e^2 - 1$
14. (c) $n!/2^{n+1}$

1.17 Exercises (page 30)

1. (a) and (b) $\frac{1}{3}\sqrt{3}$ (1, 1, 1), $\frac{1}{6}\sqrt{6}$ (1, -2, 1)
2. (a) $\frac{1}{2}\sqrt{2}\,(1, 1, 0, 0)$, $\frac{1}{6}\sqrt{6}\,(-1, 1, 2, 0)$, $\frac{1}{6}\sqrt{3}\,(1, -1, 1, 3)$

   (b) $\frac{1}{3}\sqrt{3}\,(1, 1, 0, 1)$, $\dfrac{1}{\sqrt{42}}(1, -2, 6, 1)$

6. $8 - \frac{1}{2}\log^2 3$
7. $e^2 - 1$
8. $\frac{1}{2}(e - e^{-1}) + \dfrac{3}{e}x$; $1 - 7e^{-2}$
9. $\pi - 2\sin x$
10. $\frac{3}{4} - \frac{1}{4}x$

Chapter 2

2.4 **Exercises (page 35)**

1. Linear; nullity 0, rank 2
2. Linear; nullity 0, rank 2
3. Linear; nullity 1, rank 1
4. Linear; nullity 1, rank 1
5. Nonlinear
6. Nonlinear
7. Nonlinear
8. Nonlinear
9. Linear; nullity 0, rank 2
10. Linear ; nullity 0, rank 2
11. Linear ; nullity 0, rank 2
12. Linear ; nullity 0, rank 2
13. Nonlinear
14. Linear; nullity 0, rank 2
15. Nonlinear
16. Linear; nullity 0, rank 3
17. Linear; nullity 1, rank 2
18. Linear; nullity 0, rank 3
19. Nonlinear
20. Nonlinear
21. Nonlinear
22. Nonlinear
23. Linear ; nullity 1, rank 2
24. Linear; nullity 0, rank $n + 1$
25. Linear; nullity 1, rank infinite
26. Linear; nullity infinite, rank 2
27. Linear; nullity 2, rank infinite
28. $N(T)$ is the set of constant sequences; $T(V)$ is the set of sequences with limit 0
29. (d) {1, cos x, sin x} is a basis for T(V); dim $T(V) = 3$ (e) $N(T) = S$ (f) If $T(f) = cf$ with $c \neq 0$, then $c \in T(V)$ so we have $f(x) = c_1 + c_2\cos x + c_3 \sin x$; if $c_1 = 0$, then $c = \pi$ and $f(x) = c_1 \cos x + c_2 \sin x$, where c_1, c_2 are not both zero but otherwise arbitrary; if $c_1 \neq 0$, then $c = 2\pi$ and $f(x) = c_1$, where c_1 is nonzero but otherwise arbitrary.

2.8 Exercises (page 42)

3. Yes; $x = v$, $y = u$
4. Yes; $x = u$, $y = -v$
5. No
6. No
7. No
8. Yes; $x = \log u$, $y = \log v$
9. No
10. Yes; $x = u - 1$, $y = v - 1$
11. Yes; $x = \frac{1}{2}(v + u)$, $y = \frac{1}{2}(v - u)$
12. Yes; $x = \frac{1}{3}(v + u)$, $y = \frac{1}{3}(2v - u)$
13. Yes; $x = w$, $y = v$, $z = u$
14. No
15. Yes; $x = u$, $y = \frac{1}{2}v$, $z = \frac{1}{3}w$
16. Yes; $x = u$, $y = v$, $z = w - u - v$
17. Yes; $x = u - 1$, $y = v - 1$, $z = w + 1$
18. Yes; $x = u - 1$, $y = v - 2$, $z = w - 3$
19. Yes; $x = u$, $y = v - u$, $z = w - v$
20. Yes; $x = \frac{1}{2}(u - v + w)$, $y = \frac{1}{2}(v - w + u)$, $z = \frac{1}{2}(w - u + v)$
25. $(S + T)^2 = S^2 + ST + TS + T^2$;
    $(S + T)^3 = S^3 + TS^2 + STS + S^2T + ST^2 + TST + T^2S + T^3$

26. (a) $(ST)(x, y, z) = (x + y + z, x + y, x)$; $(TS)(x, y, z) = (z, z + y, z + y + x)$;
(ST − TWX, y, $z) = (x + y, x - z, -y - z)$; $S^2(x, y, z) = (x, y, z)$;
$T^2(x, y, z) = (x, 2x + y, 3x + 2y + z)$;
$(ST)^2(x, y, z) = (3x + 2y + z, 2x + 2y + z, x + y + z)$;
$(TS)^2(x, y, z) = (x + y + z, x + 2y + 2z, x + 2y + 3z)$;
$(ST - TS)^2 = (2x + y - z, x + 2y + z, -x + y + 2z)$
(b) $S^{-1}(u, v, w) = (w, v, u)$; $T^{-1}(u, v, w) = (u, v - u, w - v)$;
$(ST)^{-1}(u, v, w) = (w, v - w, u - v)$; $(TS)^{-1}(u, v, w) = (w - v, v - u, u)$
(c) $(T - I)(x, y, z) = (0, x, x + y)$; $(T - I)^2(x, y, z) = (0, 0, x)$;
$(T - I)^n(x, y, z) = (0, 0, 0)$ if $n \geq 3$

28. (a) $Dp(x) = 3 - 2x + 12x^2$; $Tp(x) = 3x - 2x^2 + 12x^3$; $(DT)p(x) = 3 - 4x + 36x^2$;
$(TD)p(x) = -2x + 24x^2$; $(DT - TD)p(x) = 3 - 2x +$ 12x2;
$(T^2D^2 - D^2T^2)p(x) = 8 - 192x$ (b) p(x) = ax, a an arbitrary scalar
(c) $p(x) = ax^2 + b$, a and b arbitrary scalars (d) All p in V

31. (a) $Rp(x) = 2$; $Sp(x) = 3 - x + x^2$; $Tp(x) = 2x + 3x^2 - x^3 + x^4$;
$(ST)p(x) = 2 + 3x - x^2 + x^3$; $(TS)p(x) = 3x - x^2 + x^3$; $(TS)^2p(x) = 3x - x^2 + x^3$;
$(T^2S^2)p(x) = -x^2 + x^3$; $(S^2T^2)p(x) = 2 + 3x - x^2 + x^3$; $(TRS)p(x) = 3x$;
$(RST)p(x) = 2$ (b) $N(R) = \{p \mid p(0) = 0\}$; $R(V) = \{p \mid p$ is constant$\}$; N(S) =
$\{p \mid p$ is constant$\}$; $S(V) = V$; $N(T) = \{O\}$; $T(V) = \{p \mid p(0) = 0\}$ (c) $T^{-1} = s$
(d) $(TS)^n = Z - R$; $S^nT^n = Z$

32. T is not one-to-one on V because it maps all constant sequences onto the same sequence

2.12 Exercises (page 50)

1. (a) The identity matrix Z = (δ_{jk}), where $\delta_{jk} = 1$ ifj = k, and $\delta_{jk} = 0$ ifj $\neq k$
(b) The zero matrix $0 = (a_{jk})$ where each entry $a_{jk} = 0$
(c) The matrix $(c\delta_{jk})$, where (δ_{jk}) is the identity matrix of part (a)

2. (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

3. (a) $-5i + 7j$, $9i - 12j$

(b) $\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$, $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} -\frac{7}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{7}{4} \end{bmatrix}$, $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

4. $\begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$, $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

5. (a) $3i + 4j + 4k$; nullity 0, rank 3 (b) $\begin{bmatrix} -1 & -1 & 2 \\ 1 & -3 & 3 \\ -1 & -5 & 5 \end{bmatrix}$

6. $\begin{bmatrix} 2 & 0 & -2 \\ 1 & -1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$

7. (a) $T(4\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}) = (0, -2)$; nullity 1, rank 2 (b) $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & -2 \end{bmatrix}$ (d) $e_1 = \boldsymbol{j},\ e_2 = \boldsymbol{k},\ e_3 = \boldsymbol{i},\ w_1 = (1, 1),\ w_2 = (1, -1)$

8. (a) $(5, 0, -1)$; nullity 0, rank 2 (b) $\begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$

(c) $e_1 = \boldsymbol{i},\ e_2 = \boldsymbol{i} + \boldsymbol{j},\ w_1 = (1, 0, 1),\ w_2 = (0, 0, 2),\ w_3 = (0, 1, 0)$

9. (a) $(-1, -3, -1)$; nullity 0, rank 2 (b) $\begin{bmatrix} 1 & 1 \\ ) & 1 \\ 1 & 1 \end{bmatrix}$

(c) $e_1 = \boldsymbol{i},\ e_2 = \boldsymbol{j} - \boldsymbol{i},\ w_1 = (1, 0, 1),\ w_2 = (0, 1, 0),\ w_3 = (0, 0, 1)$

10. (a) $e_1 - e_2$; nullity 0, rank 2 (b) $\begin{bmatrix} 1 & 2 \\ & 4 \\ ; & \mathrm{I} \end{bmatrix}$ (c) $a = 5,\ b = 4$

11. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

12. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

13. $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

15. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

16. $\begin{bmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 & -2 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

17. $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$

18. $\begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}, \begin{bmatrix} -5 & -12 \\ 12 & -5 \end{bmatrix}$

19. (a) $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 & [illegible] & [illegible] \\ 0 & 0 & [illegible] & [illegible] \\ 0 & 0 & [illegible] & [illegible] \\ 0 & 0 & [illegible] & [illegible] \end{bmatrix}$ (c) $\begin{bmatrix} [illegible] & [illegible] & 0 & 0 \\ [illegible] & [illegible] & 2 & 0 \\ [illegible] & [illegible] & [illegible] & [illegible] \\ [illegible] & [illegible] & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 0 & [illegible] & [illegible] & [illegible] \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}$ (f) $\begin{bmatrix} [illegible] & [illegible] & [illegible] & [illegible] \\ 0 & 0 & 0 & -48 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -8 & 0 \end{bmatrix}$

20. Choose $(x^3, x^2, x, 1)$ as a basis for V, and (x^2, x) as a basis for W. Then the matrix of TD is

$$\begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}$$

2.16 Exercises (page 57)

1. $B + C = \begin{bmatrix} 3 & 4 \\ 0 & 2 \\ 6 & -5 \end{bmatrix}$, $AB = \begin{bmatrix} 15 & -14 \\ -15 & 14 \end{bmatrix}$, $BA = \begin{bmatrix} -1 & 4 & -2 \\ -4 & 16 & -8 \\ 7 & -28 & 14 \end{bmatrix}$,

$AC = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $CA = \begin{bmatrix} 0 & 0 & 0 \\ 2 & -8 & 4 \\ 4 & -16 & 8 \end{bmatrix}$, $A(2B - 3C) = \begin{bmatrix} 30 & -28 \\ -30 & 28 \end{bmatrix}$

2. (a) $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$, a and b arbitrary (b) $\begin{bmatrix} -2a & a \\ -2b & b \end{bmatrix}$, a and b arbitrary

3. (a) $a = 9$, $b = 6$, $c = 1$, $d = 5$ (b) $a = 1$, $b = 6$, $c = 0$, $d = -2$

4. (a) $\begin{bmatrix} -9 & -2 & -10 \\ 6 & 14 & 8 \\ -7 & 5 & -5 \end{bmatrix}$ (b) $\begin{bmatrix} -3 & 5 & -4 \\ 0 & 3 & 24 \\ 12 & -27 & 0 \end{bmatrix}$

6. $A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$

7. $A^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}$

8. $A^n = \begin{bmatrix} 1 & n & \frac{n(n+1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix}$

9. $\begin{bmatrix} 1 & 0 \\ -100 & 1 \end{bmatrix}$

10. $\begin{bmatrix} a & b \\ c & -a \end{bmatrix}$, where b and c are arbitrary, and a is any solution of the equation $a^2 = -bc$

11. (b) $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$, where a is arbitrary

12. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, and $\begin{bmatrix} a & b \\ c & -a \end{bmatrix}$, where b and c are arbitrary and a is any solution of the equation $a^2 = 1 - bc$

13. $C = \begin{bmatrix} \frac{15}{2} & \frac{13}{2} \\ 8 & 7 \end{bmatrix}$, $D = \begin{bmatrix} \frac{33}{4} & \frac{19}{4} \\ \frac{43}{4} & \frac{25}{4} \end{bmatrix}$

14. (b) $(A + B)^2 = A^2 + AB + BA + B^2$; $(A + B)(A - B) = A^2 + BA - AB - B^2$
(c) For those which commute

2.20 Exercises (page 67)

1. $(x, y, z) = (\frac{8}{5}, -\frac{7}{5}, \frac{8}{5})$
2. No solution
3. $(x, y, z) = (1, -1, 0) + t(-3, 4, 1)$
4. $(x, y, z) = (1, -1, 0) + t(-3, 4, 1)$
5. $(x, y, z, u) = (1, 1, 0, 0) + t(1, 14, 5, 0)$
6. $(x, y, z, u) = (1, 8, 0, -4) + t(2, 7, 3, 0)$
7. $(x, y, z, u, v) = t_1(-1, 1, 0, 0, 0) + t_2(-1, 0, 3, -3, 1)$
8. $(x, y, z, u) = (1, 1, 1, -1) + t_1(-1, 3, 7, 0) + t_2(4, 9, 0, 7)$
9. $(x, y, z) = (\frac{4}{3}, \frac{2}{3}, 0) + t(5, 1, -3)$
10. (a) $(x, y, z, u) = (1, 6, 3, 0) + t_1(4, 11, 7, 0) + t_2(0, 0, 0, 1)$
(b) $(x, y, z, u) = (\frac{3}{11}, 4, \frac{19}{11}, 0) + t(4, -11, 7, 22)$

12. $\begin{bmatrix} -1 & 2 & 1 \\ 5 & -8 & -6 \\ -3 & 5 & 4 \end{bmatrix}$

13. $\begin{bmatrix} -\frac{5}{3} & \frac{2}{3} & \frac{4}{3} \\ -\frac{7}{3} & -\frac{1}{3} & -\frac{5}{3} \end{bmatrix}$

14. $\begin{bmatrix} 14 & 8 & 3 \\ 8 & 5 & 2 \\ 3 & 2 & 1 \end{bmatrix}$

15. $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

16. $\begin{bmatrix} 0 & \frac{1}{2} & 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ -3 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 9 & 0 & -3 & 0 & 1 & 0 \end{bmatrix}$

2.21 Miscellaneous exercises on matrices (page 68)

3. $P = \begin{bmatrix} 2 & 1 \\ 5 & -1 \end{bmatrix}$

4. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and $\begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$, where b and c are arbitrary and a is any solution of the quadratic equation $a^2 - a + bc = 0$

10. (a) $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$, [I: $\begin{matrix} -1 \\ 1 \end{matrix}\Big]$, $\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$, $\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$.

Chapter 3

3.6 Exercises (page 79)

1. (a) 6 (b) 76 (c) $a^3 - 4a$
2. (a) 1 (b) 1 (c) 1
3. (b) $(b-a)(c-a)(c-b)(a+b+c)$ and $(b-a)(c-a)(c-b)(ab+ac+bc)$
4. (a) 8 (b) $(b-a)(c-a)(d-a)(c-b)(d-b)(d-c)$
   (c) $(b-a)(c-a)(d-a)(c-b)(d-b)(d-c)(a+b+c+d)$
   (d) $a(a^2-4)(a^2-16)$ (e) -160

7. $F' = \begin{vmatrix} f_1' & f_2' & f_3' \\ g_1 & g_2 & g_3 \\ h_1 & h_2 & h_3 \end{vmatrix} + \begin{vmatrix} f_1 & f_2 & f_3 \\ g_1' & g_2' & g_3' \\ h_1 & h_2 & h_3 \end{vmatrix} + \begin{vmatrix} f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \\ h_1' & h_2' & h_3' \end{vmatrix}$

8. (b) If $F = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix}$ then $F' = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1''' & f_2''' & f_3''' \end{vmatrix}$

10 $\det A = 16$, $\det(A^{-1}) = \dfrac{1}{16}$, $A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{3}{4} & \frac{1}{8} & \frac{1}{16} \\ 0 & \frac{1}{2} & -\frac{3}{4} & \frac{1}{8} \\ 0 & 0 & \frac{1}{2} & -\frac{3}{4} \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$

3.11 Exercises (page 85)

6. $\det A = (\det B)(\det D)$
7. (a) Independent (b) Independent (c) Dependent

3.17 Exercises (page 94)

1. (a) $\begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & -1 & 1 \\ -6 & 3 & 5 \\ -4 & -2 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 109 & 113 & -41 & -13 \\ -40 & -92 & 74 & 16 \\ -41 & -79 & 7 & 47 \\ -50 & 38 & 16 & 20 \end{bmatrix}$

2. (a) $-\frac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$ (b) $\frac{1}{8}\begin{bmatrix} 2 & -6 & -4 \\ -1 & 3 & -2 \\ 1 & 5 & 2 \end{bmatrix}$ (c) $\frac{1}{306}\begin{bmatrix} 109 & -40 & -41 & -50 \\ 113 & -92 & -79 & 38 \\ -41 & 74 & 7 & 16 \\ -13 & 16 & 47 & 20 \end{bmatrix}$

3. (a) $\lambda = 2$, $\lambda = -3$ (b) $\lambda = 0$, $\lambda = \pm 3$ (c) $\lambda = 3$, $\lambda = \pm i$

5. (a) $x = 0$, $y = 1$, $z = 2$ (b) $x = 1$, $y = 1$, $z = -1$

6. (b) $\det\begin{bmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{bmatrix} = 0$; $\det\begin{bmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{bmatrix} = 0$

(c) $\det\begin{bmatrix} (x - x_1)^2 + (y - y_1)^2 & (x - x_1) & (y - y_1) \\ (x_2 - x_1)^2 + (y_2 - y_1)^2 & (x_2 - x_1) & (y_2 - y_1) \\ (x_3 - x_1)^2 + (y_3 - y_1)^2 & (x_3 - x_1) & (y_3 - y_1) \end{bmatrix} = 0$;

$$\det\begin{bmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{bmatrix} = 0$$

Chapter 4

4.4 Exercises (page 101)

5. Eigenfunctions: $f(t) = Ct^n$, where $C \neq 0$
6. The nonzero constant polynomials
7. Eigenfunctions : $f(t) = Ce^{t/\lambda}$, where $C \neq 0$
8. Eigenfunctions: $f(t) = Ce^{\frac{1}{2}t^2/\lambda}$, where $C \neq 0$
10. Eigenvectors belonging to $\lambda = 0$ are all constant sequences with limit $a \neq 0$. Eigenvectors belonging to $\lambda = -1$ are all nonconstant sequences with limit $a = 0$

4.8 Exercises (page 107)

	Eigenvalue	*Eigenvectors*	dim $E(\lambda)$
1.	(a) 1,1	$(a, b) \neq (0, 0)$	2
	(b) 1, 1	$t(1, 0)$, $t \neq 0$	1
	(c) 1, 1	$t(0, 1)$, $t \neq 0$	1
	(d) 2	$t(1, 1)$ $t \neq 0$	1
	0	$t(1, -1)$, $t \neq 0$	1
2.	$1 + \sqrt{ab}$	$t(\sqrt{a}, \sqrt{b})$, $t \neq 0$	1
	$1 - \sqrt{ab}$	$t(\sqrt{a}, -\sqrt{b})$, $t \neq 0$	1

3. If the field of scalars is the set of real numbers **R**, then real eigenvalues exist only when $\sin \theta = 0$, in which case there are two equal eigenvalues, I, = I, = $\cos \theta$, where $\cos \theta = 1$ or -1. In this case every nonzero vector is an eigenvector, so $\dim E(\lambda_1) = \dim E(\lambda_2) = 2$.
   If the field of scalars is the set of complex numbers C, then the eigenvalues are $\lambda_1 = \cos \theta + i \sin \theta$, I, = $\cos \theta - i \sin \theta$. If $\sin \theta = 0$ these are real and equal. If $\sin \theta \neq 0$ they are distinct complex conjugates; the eigenvectors belonging to I, are $t(i, 1)$, $t \neq 0$; those belonging to λ_2 are $t(1, i)$, $t \neq 0$; $\dim E(\lambda_1) = \dim E(\lambda_2) = 1$.

4. $\begin{bmatrix} a & b \\ c & -a \end{bmatrix}$, where b and c are arbitrary and a is any solution of the equation $a^2 = 1 - bc$.

5. Let A = $\begin{bmatrix} a & 1 \\ c & 1 \end{bmatrix}$, and let A = $(a - d)^2 + 4bc$. The eigenvalues are real and distinct if A > 0, real and equal if A = 0, complex conjugates if A < 0.

6. $a = b = c = d = e = f = 1$.

	Eigenvalue	Eigenvectors	dim *E(I)*
7. (a)	1, 1, 1	$t(0, 0, 1)$, $t \neq 0$	1
(b)	1	$t(1, -1, 0)$, $t \neq 0$	1
	2	$t(3, 3, -1)$, $t \neq 0$	1
	21	$t(1, 1, 6)$, $t \neq 0$	1
(c)	1	$t(3, -1, 3)$, $t \neq 0$	1
	2, 2	$t(2, 2, -1)$, $t \neq 0$	1

8. 1, 1, -1, -1 for each matrix

4.10 Exercises (page 112)

2. (a) Eigenvalues 1, 3; C = $\begin{bmatrix} -2c & 0 \\ c & d \end{bmatrix}$, where $cd \neq 0$
   (b) Eigenvalues 6, -1; C = $\begin{bmatrix} 2a & b \\ 5a & -b \end{bmatrix}$, where $ab \neq 0$
   (c) Eigenvalues 3, 3; if a nonsingular C exists then $C^{-1}AC = 3I$, so $AC = 3C$, $A = 3I$
   (d) Eigenvalues 1, 1; if a nonsingular C exists then $C^{-1}AC = I$, so $AC = C$, $A = Z$
3. $C = A^{-1}B$.
4. (a) Eigenvalues 1, 1, -1; eigenvectors (1, 0, 1), (0, 1, 0), (1, 0, -1);
   $C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$
   (b) Eigenvalues 2, 2, 1; eigenvectors $(1, 0, -1)$, $(0, 1, -1)$, $(1, -1, 1)$;
   $C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$