A_n which is not the identity coset and this is the set of all odd permutations. The signs of permutations obey the usual $\mathbb{Z}/2\mathbb{Z}$ laws:

$$(even)(even) = (odd)(odd) = even$$
$$(even)(odd) = (odd)(even) = odd.$$

Moreover, since ϵ is a homomorphism and every $\sigma \in S_n$ is a product of transpositions, say $\sigma = \tau_1\tau_2\cdots\tau_k$, then $\epsilon(\sigma) = \epsilon(\tau_1)\cdots\epsilon(\tau_k)$; since $\epsilon(\tau_i) = -1$, for $i = 1, 2, \dots, k$, $\epsilon(\sigma) = (-1)^k$. Thus the class of k (mod 2), i.e., the parity of the number of transpositions in the product, is the same no matter how we write σ as a product of transpositions:

$$\epsilon(\sigma) = \begin{cases} +1, & \text{if } \sigma \text{ is a product of an even number of transpositions} \\ -1, & \text{if } \sigma \text{ is a product of an odd number of transpositions.} \end{cases}$$

Finally we give a quick way of computing $\epsilon(\sigma)$ from the cycle decomposition of σ. Recall that an m-cycle may be written as a product of $m - 1$ transpositions. Thus

an m-cycle is an odd permutation if and only if m is even.

For any permutation σ let $\alpha_1\alpha_2\cdots\alpha_k$ be its cycle decomposition. Then $\epsilon(\sigma)$ is given by $\epsilon(\alpha_1)\cdots\epsilon(\alpha_k)$ and $\epsilon(\alpha_i) = -1$ if and only if the length of α_i is even. It follows that for $\epsilon(\sigma)$ to be -1 the product of the $\epsilon(\alpha_i)$'s must contain an odd number of factors of (-1). We summarize this in the following proposition:

Proposition 25. The permutation σ is odd if and only if the number of cycles of even length in its cycle decomposition is odd.

For example, $\sigma = (1\,2\,3\,4\,5\,6)(7\,8\,9)(10\,11)(12\,13\,14\,15)(16\,17\,18)$ has 3 cycles of even length, so $\epsilon(\sigma) = -1$. On the other hand, $\tau = (1\ 12\ 8\ 10\ 4)(2\ 13)(5\ 11\ 7)(6\ 9)$ has exactly 2 cycles of even length, hence $\epsilon(\tau) = 1$.

Be careful not to confuse the terms "odd" and "even" for a permutation σ with the parity of the order of σ. In fact, if σ is of odd order, all cycles in the cycle decomposition of σ have odd length so σ has an even (in this case 0) number of cycles of even length and hence is an even permutation. If $|\sigma|$ is even, σ may be either an even or an odd permutation; e.g., $(1\ 2)$ is odd, $(1\ 2)(3\ 4)$ is even but both have order 2.

As we mentioned in the preceding section, the alternating groups A_n will be important in the study of solvability of polynomials. In the next chapter we shall prove:

A_n is a non-abelian simple group for all $n \geq 5$.

For small values of n, A_n is already familiar to us: A_1 and A_2 are both the trivial group and $|A_3| = 3$ (so $A_3 = \langle (1\,2\,3) \rangle \cong Z_3$). The group A_4 has order 12. Exercise 7 shows A_4 is isomorphic to the group of symmetries of a regular tetrahedron. The lattice of subgroups of A_4 appears in Figure 8 (Exercise 8 asserts that this is its complete lattice of subgroups). One of the nicer aspects of this lattice is that (unlike "virtually all groups") it is a planar graph (there are no crossing lines except at the vertices; see the lattice of D_{16} for a nonplanar lattice).

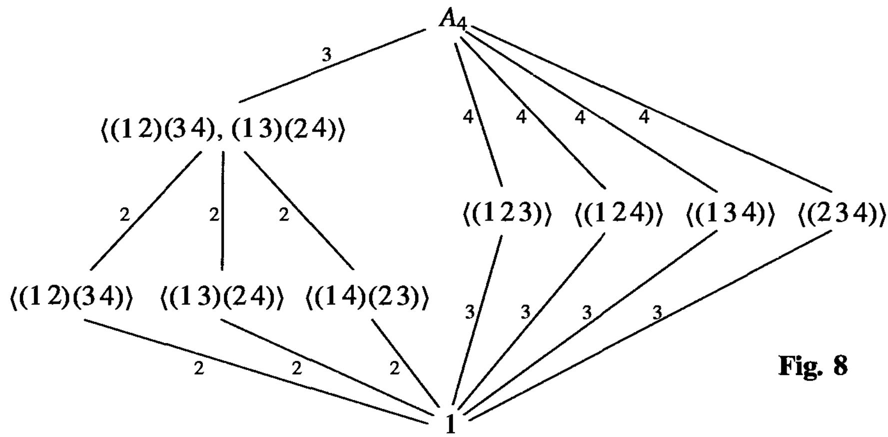

Fig. 8

EXERCISES

1. In Exercises 1 and 2 of Section 1.3 you were asked to find the cycle decomposition of some permutations. Write each of these permutations as a product of transpositions. Determine which of these is an even permutation and which is an odd permutation.
2. Prove that σ^2 is an even permutation for every permutation σ.
3. Prove that S_n is generated by $\{(i \;\; i+1) \mid 1 \le i \le n-1\}$. [Consider conjugates, viz. $(2\,3)(1\,2)(2\,3)^{-1}$.]
4. Show that $S_n = \langle (1\,2),\ (1\,2\,3\ldots n) \rangle$ for all $n \ge 2$.
5. Show that if p is prime, $S_p = \langle \sigma, \tau \rangle$ where σ is any transposition and τ is any p-cycle.
6. Show that $\langle (1\,3), (1\,2\,3\,4) \rangle$ is a proper subgroup of S_4. What is the isomorphism type of this subgroup?
7. Prove that the group of rigid motions of a tetrahedron is isomorphic to A_4. [Recall Exercise 20 in Section 1.7.]
8. Prove the lattice of subgroups of A_4 given in the text is correct. [By the preceding exercise and the comments following Lagrange's Theorem, A_4 has no subgroup of order 6.]
9. Prove that the (unique) subgroup of order 4 in A_4 is normal and is isomorphic to V_4.
10. Find a composition series for A_4. Deduce that A_4 is solvable.
11. Prove that S_4 has no subgroup isomorphic to Q_8.
12. Prove that A_n contains a subgroup isomorphic to S_{n-2} for each $n \ge 3$.
13. Prove that every element of order 2 in A_n is the square of an element of order 4 in S_n. [An element of order 2 in A_n is a product of $2k$ commuting transpositions.]
14. Prove that the subgroup of A_4 generated by any element of order 2 and any element of order 3 is all of A_4.
15. Prove that if x and y are distinct 3-cycles in S_4 with $x \ne y^{-1}$, then the subgroup of S_4 generated by x and y is A_4.
16. Let x and y be distinct 3-cycles in S_5 with $x \ne y^{-1}$.
    (a) Prove that if x and y fix a common element of $\{1, \ldots, 5\}$, then $\langle x, y \rangle \cong A_4$.
    (b) Prove that if x and y do not fix a common element of $\{1, \ldots, 5\}$, then $\langle x, y \rangle = A_5$.
17. If x and y are 3-cycles in S_n, prove that $\langle x, y \rangle$ is isomorphic to Z_3, A_4, A_5 or $Z_3 \times Z_3$.

CHAPTER 4

Group Actions

In this chapter we consider some of the consequences of a group acting on a set. It is an important and recurring idea in mathematics that when one object acts on another then much information can be obtained on both. As more structure is added to the set on which the group acts (for example, groups acting on groups or groups acting on vector spaces (considered in Chapter 18)), more information on the structure of the group becomes available. This study of group actions culminates here in the proof of Sylow's Theorem and the examples and classifications which accrue from it.

The concept of an action will recur as we study modules, vector spaces, canonical forms for matrices and Galois Theory, and is one of the fundamental unifying themes in the text.

4.1 GROUP ACTIONS AND PERMUTATION REPRESENTATIONS

In this section we give the basic theory of group actions and then apply this theory to subgroups of S_n acting on $\{1, 2, \dots, n\}$ to prove that every element of S_n has a unique cycle decomposition. In Sections 2 and 3 we apply the general theory to two other specific group actions to derive some important results.

Let G be a group acting on a nonempty set A. Recall from Section 1.7 that for each $g \in G$ the map

$$\sigma_g : A \to A \qquad \text{defined by} \qquad \sigma_g : a \mapsto g \cdot a$$

is a permutation of A. We also saw in Section 1.7 that there is a homomorphism associated to an action of G on A:

$$\varphi : G \to S_A \qquad \text{defined by} \qquad \varphi(g) = \sigma_g,$$

called the *permutation representation* associated to the given action. Recall some additional terminology associated to group actions introduced in Sections 1.7 and 2.2.

Definition.

(1) The *kernel* of the action is the set of elements of G that act trivially on every element of A: $\{g \in G \mid g \cdot a = a \text{ for all } a \in A\}$.

(2) For each $a \in A$ the *stabilizer* of a in G is the set of elements of G that fix the element a: $\{g \in G \mid g \cdot a = a\}$ and is denoted by G_a.

(3) An action is *faithful* if its kernel is the identity.

Note that the kernel of an action is precisely the same as the kernel of the associated permutation representation; in particular, the kernel is a normal subgroup of G. Two group elements induce the same permutation on A if and only if they are in the same coset of the kernel (if and only if they are in the same fiber of the permutation representation φ). In particular an action of G on A may also be viewed as a faithful action of the quotient group $G/\ker\varphi$ on A. Recall from Section 2.2 that the stabilizer in G of an element a of A is a subgroup of G. If a is a fixed element of A, then the kernel of the action is contained in the stabilizer G_a since the kernel of the action is the set of elements of G that stabilize every point, namely $\bigcap_{a\in A} G_a$.

Examples

(1) Let n be a positive integer. The group $G = S_n$ acts on the set $A = \{1, 2, \dots, n\}$ by $\sigma \cdot i = \sigma(i)$ for all $i \in \{1, \dots, n\}$. The permutation representation associated to this action is the identity map $\varphi : S_n \to S_n$. This action is faithful and for each $i \in \{1, \dots, n\}$ the stabilizer G_i (the subgroup of all permutations fixing i) is isomorphic to S_{n-1} (cf. Exercise 15, Section 3.2).

(2) Let $G = D_8$ act on the set A consisting of the four vertices of a square. Label these vertices 1,2,3,4 in a clockwise fashion as in Figure 2 of Section 1.2. Let r be the rotation of the square clockwise by $\pi/2$ radians and let s be the reflection in the line which passes through vertices 1 and 3. Then the permutations of the vertices given by r and s are

$$\sigma_r = (1\ 2\ 3\ 4) \qquad \text{and} \qquad \sigma_s = (2\ 4).$$

Note that since the permutation representation is a homomorphism, the permutation of the four vertices corresponding to sr is $\sigma_{sr} = \sigma_s\sigma_r = (1\ 4)(2\ 3)$. The action of D_8 on the four vertices of a square is faithful since only the identity symmetry fixes all four vertices. The stabilizer of any vertex a is the subgroup of D_8 of order 2 generated by the reflection about the line passing through a and the center of the square (so, for example, the stabilizer of vertex 1 is $\langle s \rangle$).

(3) Label the four vertices of a square as in the preceding example and now let A be the set whose elements consist of unordered pairs of opposite vertices: $A = \{\, \{1, 3\}\,,\ \{2, 4\}\, \}$. Then D_8 also acts on this set A since each symmetry of the square sends a pair of opposite vertices to a pair of opposite vertices. The rotation r interchanges the pairs $\{1, 3\}$ and $\{2, 4\}$; the reflection s fixes both unordered pairs of opposite vertices. Thus if we label the pairs $\{1, 3\}$ and $\{2, 4\}$ as **1** and **2**, respectively, then the permutations of A given by r and s are

$$\sigma_r = (\mathbf{1}\ \mathbf{2}) \qquad \text{and} \qquad \sigma_s = \text{the identity permutation.}$$

This action of D_8 is not faithful: its kernel is $\langle s, r^2 \rangle$. Moreover, for each $a \in A$ the stabilizer in D_8 of a is the same as the kernel of the action.

(4) Label the four vertices of a square as in Example 2 and now let A be the following set of unordered pairs of vertices: $\{\, \{1, 2\}, \{3, 4\}\, \}$. The group D_8 does *not* act on this set A because $\{1, 2\} \in A$ but $r \cdot \{1, 2\} = \{2, 3\} \notin A$.

The relation between actions and homomorphisms into symmetric groups may be reversed. Namely, given any nonempty set A and any homomorphism φ of the group G into S_A we obtain an action of G on A by defining

$$g \cdot a = \varphi(g)(a)$$