15. Find an element of S_8 which conjugates the subgroup of S_8 obtained in part (a) of Exercise 3, Section 2 to the subgroup of S_8 obtained in part (b) of that same exercise (both of these subgroups are isomorphic to D_8).

16. Find an element of S_4 which conjugates the subgroup of S_4 obtained in part (a) of Exercise 5, Section 2 to the subgroup of S_4 obtained in part (b) of that same exercise (both of these subgroups are isomorphic to D_8).

17. Let A be a nonempty set and let X be any subset of S_A. Let

$$F(X) = \{a \in A \mid \sigma(a) = a \text{ for all } \sigma \in X\} \qquad \text{— the } \textit{fixed set} \text{ of } X.$$

Let $M(X) = A - F(X)$ be the elements which are *moved* by some element of X. Let $D = \{\sigma \in S_A \mid |M(\sigma)| < \infty\}$. Prove that D is a normal subgroup of S_A.

18. Let A be a set, let H be a subgroup of S_A and let $F(H)$ be the fixed points of H on A as defined in the preceding exercise. Prove that if $\tau \in N_{S_A}(H)$ then τ stabilizes the set $F(H)$ and its complement $A - F(H)$.

19. Assume H is a normal subgroup of G, $\mathcal{K}$ is a conjugacy class of G contained in H and $x \in \mathcal{K}$. Prove that $\mathcal{K}$ is a union of k conjugacy classes of equal size in H, where $k = |G : HC_G(x)|$. Deduce that a conjugacy class in S_n which consists of even permutations is either a single conjugacy class under the action of A_n or is a union of two classes of the same size in A_n. [Let $A = C_G(x)$ and $B = H$ so $A \cap B = C_H(x)$. Draw the lattice diagram associated to the Second Isomorphism Theorem and interpret the appropriate indices. See also Exercise 9, Section 1.]

20. Let $\sigma \in A_n$. Show that all elements in the conjugacy class of σ in S_n (i.e., all elements of the same cycle type as σ) are conjugate in A_n if and only if σ commutes with an odd permutation. [Use the preceding exercise.]

21. Let $\mathcal{K}$ be a conjugacy class in S_n and assume that $\mathcal{K} \subseteq A_n$. Show $\sigma \in S_n$ does *not* commute with any odd permutation if and only if the cycle type of σ consists of distinct odd integers. Deduce that $\mathcal{K}$ consists of two conjugacy classes in A_n if and only if the cycle type of an element of $\mathcal{K}$ consists of distinct odd integers. [Assume first that $\sigma \in \mathcal{K}$ does not commute with any odd permutation. Observe that σ commutes with each individual cycle in its cycle decomposition — use this to show that all its cycles must be of odd length. If two cycles have the same odd length, k, find a product of k transpositions which interchanges them and commutes with σ. Conversely, if the cycle type of σ consists of distinct integers, prove that σ commutes *only* with the group generated by the cycles in its cycle decomposition.]

22. Show that if n is odd then the set of all n-cycles consists of two conjugacy classes of equal size in A_n.

23. Recall (cf. Exercise 16, Section 2.4) that a proper subgroup M of G is called *maximal* if whenever $M \le H \le G$, either $H = M$ or $H = G$. Prove that if M is a maximal subgroup of G then either $N_G(M) = M$ or $N_G(M) = G$. Deduce that if M is a maximal subgroup of G that is not normal in G then the number of nonidentity elements of G that are contained in conjugates of M is at most $(|M| - 1)|G : M|$.

24. Assume H is a proper subgroup of the finite group G. Prove $G \ne \cup_{g \in G} gHg^{-1}$, i.e., G is not the union of the conjugates of any proper subgroup. [Put H in some maximal subgroup and use the preceding exercise.]

25. Let $G = GL_2(\mathbb{C})$ and let $H = \{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{C},\ ac \ne 0\}$. Prove that every element of G is conjugate to some element of the subgroup H and deduce that G is the union of conjugates of H. [Show that every element of $GL_2(\mathbb{C})$ has an eigenvector.]

26. Let G be a transitive permutation group on the finite set A with $|A| > 1$. Show that there is some $\sigma \in G$ such that $\sigma(a) \neq a$ for all $a \in A$ (such an element σ is called *fixed point free*).

27. Let $g_1, g_2, \dots, g_r$ be representatives of the conjugacy classes of the finite group G and assume these elements pairwise commute. Prove that G is abelian.

28. Let p and q be primes with $p < q$. Prove that a non-abelian group G of order pq has a nonnormal subgroup of index q, so that there exists an injective homomorphism into S_q. Deduce that G is isomorphic to a subgroup of the normalizer in S_q of the cyclic group generated by the q-cycle $(1\,2\dots q)$.

29. Let p be a prime and let G be a group of order p^α. Prove that G has a subgroup of order p^β, for every β with $0 \leq \beta \leq \alpha$. [Use Theorem 8 and induction on α.]

30. If G is a group of odd order, prove for any nonidentity element $x \in G$ that x and x^{-1} are not conjugate in G.

31. Using the usual generators and relations for the dihedral group D_{2n} (cf. Section 1.2) show that for $n = 2k$ an even integer the conjugacy classes in D_{2n} are the following: $\{1\}$, $\{r^k\}$, $\{r^{\pm 1}\}$, $\{r^{\pm 2}\}$, $\dots$, $\{r^{\pm(k-1)}\}$, $\{sr^{2b} \mid b = 1, \dots, k\}$ and $\{sr^{2b-1} \mid b = 1, \dots, k\}$. Give the class equation for D_{2n}.

32. For $n = 2k + 1$ an odd integer show that the conjugacy classes in D_{2n} are $\{1\}$, $\{r^{\pm 1}\}$, $\{r^{\pm 2}\}$, $\dots$, $\{r^{\pm k}\}$, $\{sr^b \mid b = 1, \dots, n\}$. Give the class equation for D_{2n}.

33. This exercise gives a formula for the size of each conjugacy class in S_n. Let σ be a permutation in S_n and let $m_1, m_2, \dots, m_s$ be the *distinct* integers which appear in the cycle type of σ (including 1-cycles). For each $i \in \{1, 2, \dots, s\}$ assume σ has k_i cycles of length m_i (so that $\Sigma_{i=1}^s k_i m_i = n$). Prove that the number of conjugates of σ is

$$\frac{n!}{(k_1!m_1^{k_1})(k_2!m_2^{k_2})\dots(k_s!m_s^{k_s})}.$$

[See Exercises 6 and 7 in Section 1.3 where this formula was given in some special cases.]

34. Prove that if p is a prime and P is a subgroup of S_p of order p, then $|N_{S_p}(P)| = p(p-1)$. [Argue that every conjugate of P contains exactly $p-1$ p-cycles and use the formula for the number of p-cycles to compute the index of $N_{S_p}(P)$ in S_p.]

35. Let p be a prime. Find a formula for the number of conjugacy classes of elements of order p in S_n (using the greatest integer function).

36. Let $\pi : G \to S_G$ be the left regular representation afforded by the action of G on itself by left multiplication. For each $g \in G$ denote the permutation $\pi(g)$ by σ_g, so that $\sigma_g(x) = gx$ for all $x \in G$. Let $\lambda : G \to S_G$ be the permutation representation afforded by the corresponding right action of G on itself, and for each $h \in G$ denote the permutation $\lambda(h)$ by τ_h. Thus $\tau_h(x) = xh^{-1}$ for all $x \in G$ (λ is called the *right regular representation* of G).

(a) Prove that σ_g and τ_h commute for all $g, h \in G$. (Thus the centralizer in S_G of $\pi(G)$ contains the subgroup $\lambda(G)$, which is isomorphic to G).

(b) Prove that $\sigma_g = \tau_g$ if and only if g is an element of order 1 or 2 in the center of G.

(c) Prove that $\sigma_g = \tau_h$ if and only if g and h lie in the center of G. Deduce that $\pi(G) \cap \lambda(G) = \pi(Z(G)) = \lambda(Z(G))$.

4.4 AUTOMORPHISMS

Definition. Let G be a group. An isomorphism from G onto itself is called an *automorphism* of G. The set of all automorphisms of G is denoted by $\mathrm{Aut}(G)$.

We leave as an exercise the simple verification that $\mathrm{Aut}(G)$ is a group under composition of automorphisms, the *automorphism group* of G (composition of automorphisms is defined since the domain and range of each automorphism is the same). Notice that automorphisms of a group G are, in particular, permutations of the set G so $\mathrm{Aut}(G)$ is a subgroup of S_G.

One of the most important examples of an automorphism of a group G is provided by conjugation by a fixed element in G. The next result discusses this in a slightly more general context.

Proposition 13. Let H be a normal subgroup of the group G. Then G acts by conjugation on H as automorphisms of H. More specifically, the action of G on H by conjugation is defined for each $g \in G$ by

$$h \mapsto ghg^{-1} \qquad \text{for each } h \in H.$$

For each $g \in G$, conjugation by g is an automorphism of H. The permutation representation afforded by this action is a homomorphism of G into $\mathrm{Aut}(H)$ with kernel $C_G(H)$. In particular, $G/C_G(H)$ is isomorphic to a subgroup of $\mathrm{Aut}(H)$.

Proof: (cf. Exercise 17, Section 1.7) Let φ_g be conjugation by g. Note that because g normalizes H, φ_g maps H to itself. Since we have already seen that conjugation defines an action, it follows that $\varphi_1 = 1$ (the identity map on H) and $\varphi_a \circ \varphi_b = \varphi_{ab}$ for all $a, b \in G$. Thus each φ_g gives a bijection from H to itself since it has a 2-sided inverse $\varphi_{g^{-1}}$. Each φ_g is a homomorphism from H to H because

$$\varphi_g(hk) = g(hk)g^{-1} = gh(gg^{-1})kg^{-1} = (ghg^{-1})(gkg^{-1}) = \varphi_g(h)\varphi_g(k)$$

for all $h, k \in H$. This proves that conjugation by any fixed element of G defines an automorphism of H.

By the preceding remark, the permutation representation $\psi : G \to S_H$ defined by $\psi(g) = \varphi_g$ (which we have already proved is a homomorphism) has image contained in the subgroup $\mathrm{Aut}(H)$ of S_H. Finally,

$$\begin{aligned} \ker \psi &= \{g \in G \mid \varphi_g = \mathrm{id}\} \\ &= \{g \in G \mid ghg^{-1} = h \text{ for all } h \in H\} \\ &= C_G(H). \end{aligned}$$

The First Isomorphism Theorem implies the final statement of the proposition.

Proposition 13 shows that a group acts by conjugation on a normal subgroup as *structure preserving* permutations, i.e., as automorphisms. In particular, this action must send subgroups to subgroups, elements of order n to elements of order n, etc. Two specific applications of this proposition are described in the next two corollaries.