$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} =_{def} \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

For a fixed matrix, this function from one vector to another vector is called a *linear transformation*, meaning that it preserves sums and constant multiples of vectors. Using this notation, we can view any set of simultaneous equations of the form $ax + by = e, \quad cx + dy = f$ as equivalent to a single matrix equation $AX = B$, where A denotes the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

X denotes the vector of unknowns $\binom{x}{y}$, and B denotes the vector of constants $\binom{e}{f}$. Stated in words, the simultaneous equations can thus be interpreted as asking to find a vector which when "multiplied" by a certain known matrix gives a certain known vector. Thus, it is analogous to the simple equation $ax = b$, which is solved by multiplying both sides by a^{-1} (assuming $a \neq 0$). Similarly, one way to solve the matrix equation $AX = B$ is to find the inverse of the matrix A, and then apply A^{-1} to both sides to obtain the unique vector solution $X = A^{-1}B$.

By the inverse of the matrix A we mean the matrix which multiplies by it to give the identity matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(the matrix which, when applied to any vector, keeps that vector the same). But not all matrices have inverses. It is not hard to prove that a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has an inverse if and only if its *determinant* $D =_{def} ad - bc$ is nonzero, and that its inverse in that case is

$$\frac{1}{D} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} D^{-1}d & -D^{-1}b \\ -D^{-1}c & D^{-1}a \end{pmatrix}.$$

There are three possibilities for the solutions of the system of simultaneous equations $AX = B$. First, if the determinant D is nonzero, then there is precisely one solution $X = \binom{x}{y}$. If $D = 0$, then either there are no solutions or there are infinitely many. The three possibilities have a simple geometric interpretation. The two equations give straight lines in the xy-plane. If $D \neq 0$, then they intersect in exactly one point (x, y). Otherwise, they are parallel lines, which means either that they don't meet at all (the simultaneous equations have no common solution) or else that they are really the same line (the equations have infinitely many common solutions).