§ **I.3.**

1. (a) $x = 6 + 7n$, $n$ any integer; (b) no solution; (c) same as (a); (d) $219 + 256n$; (e) $36 + 100n$; (f) $636 + 676n$.
2. 0, 1, 4, 9.
3. 3, B.
4. The difference between $n = 10^{k-1}d_{k-1} + \cdots + 10d_1 + d_0$ and the sum of the digits $d_{k-1} + \cdots + d_1 + d_0$ is a sum of multiples of numbers of the form $10^j - 1$, which is divisible by 9.
5. Prove separately that it is divisible by 2, 3 and 5.
6. Let $x$ and $y$ be the two digits. Then 72 — and hence both 8 and 9 — divide the cost $1000x + 60 + y$ cents. Thus, $8|60 + y$, which means that $y = 4$, and then $9|1000x + 64$, which is $\equiv x + 1$ *mod* 9. So $x = 8$. Thus each tile cost $1.12.
7. (a) For example, suppose that $m = 2p^\alpha$. Since $m|(x^2-1) = (x+1)(x-1)$, we must have $\alpha$ powers of $p$ appearing in both $x+1$ and $x-1$ together. But since $p \geq 3$, it follows that $p$ cannot divide both $x + 1$ and $x - 1$ (which are only 2 apart from one another), and so all of the $p$'s must divide one of them. If $p^\alpha|x + 1$, this means that $x \equiv -1$ *mod* $p^\alpha$; if $p^\alpha|x-1$, then $x \equiv 1$ *mod* $p^\alpha$. Finally, since $2|x^2-1$ it follows that $x$ must be odd, i.e., $x \equiv 1 \equiv -1$ *mod* 2. Thus, by Property 5 of congruences, either $x \equiv 1$ *mod* $2p^\alpha$ or $x \equiv -1$ *mod* $2p^\alpha$. (b) First, if $m \geq 8$ is a power of 2, it's easy to show that $x = m/2 + 1$ gives a contradiction to part (a). Next, suppose that $m$ is not a prime power (or twice a prime power), and $p^\alpha||m$. Set $m' = m/p^\alpha$. Use the Chinese Remainder Theorem to find an $x$ which is $\equiv 1$ *mod* $p^\alpha$ and $\equiv -1$ *mod* $m'$. Show that this $x$ contradicts part (a).
8. Pair every integer from 1 to $p - 1$ with its multiplicative inverse. According to Exercise 7(a), only 1 and $-1$ are their own inverses. Thus, when the $p - 1$ numbers are multiplied, each pair containing two numbers which are each other's inverses must cancel, leaving just 1 and $-1$.
9. Of course, 4 has the desired property, but it is not a 3-digit number. By the last part of the Chinese Remainder Theorem, any other number which leaves the right remainders must differ from 4 by a multiple of $7 \cdot 9 \cdot 11 = 693$. The only 3-digit possibility is $4 + 693 = 697$.
10. One can apply the Chinese Remainder Theorem to the congruences $x \equiv 1$ *mod* 11, $x \equiv 2$ *mod* 12, $x \equiv 3$ *mod* 13. Alternately, one can observe that obviously $-10$ leaves the right remainders, and then proceed as in Exercise 9 to get $-10 + 11 \cdot 12 \cdot 13 = 1706$.
11. (a) 1973; (b) 63841; (c) 58837.
12. The quotient leaves remainders of 5, 1, 4 when divided by 9, 10, 11, and so (by the Chinese Remainder Theorem) is of the form $851 + 990m$. Similarly, the divisor is of the form $817 + 990n$. Since the divisor has 3 digits, $n = 0$. Since the product has 6 digits, also $m = 0$. Thus, the answer is 851.