edged this. Two years later, he developed his own version of the Calculus, introducing the notation we still use today. In particular, he obtained the rule for the derivative of a product that is named after him.

Leibniz's work on the Calculus appeared in 1684, before Newton had got around to publishing his results. Not surprisingly, Leibniz was accused of plagiarism by some British mathematicians, not without Newton's acquiescence, and a bitter priority battle ensued. Today, we ascribe the invention of the Calculus to both Newton and Leibniz.

Leibniz had been working as librarian for the Duke of Hannover. When the latter became king of England as George I, he left two people behind: his wife, whom he divorced and shut up in a cloister, and Leibniz, because he did not want to antagonize the British academic establishment.

Leibniz thought of dy and dx in dy/dx as 'infinitesimals'. Thus dx was an infinitely small increment in x which was yet different from 0, and dy, defined as

$$dy = f(x + dx) - f(x)$$

for a given function $y = f(x)$, was also different from 0 (unless f happened to be constant near x). For example, if $f(x) = x^2$, then $dy = (x+dx)^2 - x^2 = 2x(dx) + (dx)^2$. This represented the 'rise' of the function corresponding to the 'run' dx. Hence, the slope of the tangent was rise/run $= dy/dx = 2x + dx$, so that, at x, the tangent has slope $2x$.

The concept of the infinitesimal — also implicit in Newton's fluxions — was criticized by many, including the philosopher and bishop George Berkeley (1685–1753). How, he asked, can we divide by dx if it is 0? How can we get the slope of the tangent right, in our example $2x$, if it is not 0?

Karl Weierstrass (1815–1897) agreed that there were problems, and he responded by putting calculus on the firm footing it has today. At the moment, dy/dx is not seen as a quotient but as a limit of quotients:

$$\frac{dy}{dx} = \lim_{h \to \infty} \frac{f(x+h) - f(x)}{h}.$$

If we insist on the separate existence of dx and dy, we may put $dx = h$ and $dy = \frac{dy}{dx}h$. Weierstrass also gave a perfectly rigorous definition of a limit (the *epsilon-delta* definition) which does not depend on vague notions like 'small', 'approaches', etc.

There is also at the moment a rigorous version of the infinitesimal itself. In 1966 Abraham Robinson introduced a *nonstandard model* for real numbers in which there is an entity ξ such that

$$0 < \xi < 1, \quad 0 < \xi < 1/2, \quad 0 < \xi < 1/3, \quad \ldots \qquad (*)$$

Can this be done consistently? Yes, and the reason, roughly speaking, is as follows. We know from mathematical logic that, if a contradiction were deducible from $(*)$, then this contradiction would be deducible in a finite

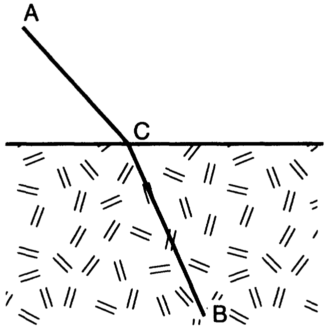

FIGURE 29.1. Refraction of light

number of steps, hence from a finite subset of (∗). Since this is not possible (assuming that the ordinary interpretation of the rational numbers is consistent), there is no contradiction. (See Section 11.)

Leibniz's infinitesimals are simple and intuitive but not rigorous; Robinson's infinitesimals are rigorous but neither simple nor intuitive. Anyone studying nonstandard analysis will discover a complex and bizarre system. On the other hand, it should be stated that nonstandard analysis has helped some mathematicians obtain new results in ordinary analysis.

Leibniz also made contributions to philosophy. He is famous for his view that this is the best of all possible universes (Could God have failed to create the best?). This view was ridiculed by Voltaire in his novel *Candide*. However, it is unlikely that Leibniz intended 'best' to mean 'happiest'. What he had in mind may have been something like the discovery of Willebrord Snell, who was able to explain the refraction of light by assuming that a ray of light minimizes the time in going from point A in one medium to point B in another. (Think of a person trying to walk from point A on a paved surface to a point B in a rough field. Since he finds it harder to walk in the field, he will not go straight from A to B, but in a broken line ACB, as shown in Figure 29.1.) Leibniz makes explicit reference to Snell in Section XXII of the *Discourse on Metaphysics*, where Leibniz discusses the 'easiest way' in which a ray of light might travel. Thus it is quite possible that by 'best' universe, Leibniz meant, among other things, 'easiest', or 'most energy-efficient' universe.

The behaviour of light rays is only one instance of the *Principle of Least Action*, which was put forward by Pierre Maupertuis (1698–1759) in 1751. As now formulated, this principle asserts that any physical process happens in such a way as to minimize the action $\int_a^b E\,dt$, where E is the difference between kinetic and potential energy and t is the time. This principle was formally established by Lagrange.

Some utilitarian philosophers, such as J.S. Mill, have claimed that even human psychology is, or ought to be, determined by something like the