EXAMPLE 22. If V is an n-dimensional inner product space, then each ordered orthonormal basis $\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}$ determines an isomorphism of V onto F^n with the standard inner product. The isomorphism is simply

$$T(x_1\alpha_1 + \cdots + x_n\alpha_n) = (x_1, \ldots, x_n).$$

There is the superficially different isomorphism which $\mathcal{B}$ determines of V onto the space $F^{n\times 1}$ with $(X|Y) = Y^*X$ as inner product. The isomorphism is

$$\alpha \rightarrow [\alpha]_{\mathcal{B}}$$

i.e., the transformation sending α into its coordinate matrix in the ordered basis $\mathcal{B}$. For any ordered basis $\mathcal{B}$, this is a vector space isomorphism; however, it is an isomorphism of the two inner product spaces if and only if $\mathcal{B}$ is orthonormal.

EXAMPLE 23. Here is a slightly less superficial isomorphism. Let W be the space of all 3×3 matrices A over R which are skew-symmetric, i.e., $A^t = -A$. We equip W with the inner product $(A|B) = \frac{1}{2}\operatorname{tr}(AB^t)$, the $\frac{1}{2}$ being put in as a matter of convenience. Let V be the space R^3 with the standard inner product. Let T be the linear transformation from V into W defined by

$$T(x_1, x_2, x_3) = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}.$$

Then T maps V onto W, and putting

$$A = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -y_3 & y_2 \\ y_3 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{bmatrix}$$

we have

$$\begin{aligned} \operatorname{tr}(AB^t) &= x_3y_3 + x_2y_2 + x_3y_3 + x_2y_2 + x_1y_1 \\ &= 2(x_1y_1 + x_2y_2 + x_3y_3). \end{aligned}$$

Thus $(\alpha|\beta) = (T\alpha|T\beta)$ and T is a vector space isomorphism. Note that T carries the standard basis $\{\epsilon_1, \epsilon_2, \epsilon_3\}$ onto the orthonormal basis consisting of the three matrices

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

EXAMPLE 24. It is not always particularly convenient to describe an isomorphism in terms of orthonormal bases. For example, suppose $G = P^*P$ where P is an invertible $n \times n$ matrix with complex entries. Let V be the space of complex $n \times 1$ matrices, with the inner product $[X|Y] = Y^*GX$.

Let W be the same vector space, with the standard inner product $(X|Y) = Y^*X$. We know that V and W are isomorphic inner product spaces. It would seem that the most convenient way to describe an isomorphism between V and W is the following: Let T be the linear transformation from V into W defined by $T(X) = PX$. Then

$$\begin{aligned}(TX|TY) &= (PX|PY)\\ &= (PY)^*(PX)\\ &= Y^*P^*PX\\ &= Y^*GX\\ &= [X|Y].\end{aligned}$$

Hence T is an isomorphism.

EXAMPLE 25. Let V be the space of all continuous real-valued functions on the unit interval, $0 \leq t \leq 1$, with the inner product

$$[f|g] = \int_0^1 f(t)g(t)t^2\,dt.$$

Let W be the same vector space with the inner product

$$(f|g) = \int_0^1 f(t)g(t)\,dt.$$

Let T be the linear transformation from V into W given by

$$(Tf)(t) = tf(t).$$

Then $(Tf|Tg) = [f|g]$, and so T preserves inner products; however, T is *not* an isomorphism of V onto W, because the range of T is not all of W. Of course, this happens because the underlying vector space is not finite-dimensional.

Theorem 11. *Let* V *and* W *be inner product spaces over the same field, and let* T *be a linear transformation from* V *into* W. *Then* T *preserves inner products if and only if* $||T\alpha|| = ||\alpha||$ *for every* α *in* V.

Proof. If T preserves inner products, T 'preserves norms.' Suppose $||T\alpha|| = ||\alpha||$ for every α in V. Then $||T\alpha||^2 = ||\alpha||^2$. Now using the appropriate polarization identity, (8-3) or (8-4), and the fact that T is linear, one easily obtains $(\alpha|\beta) = (T\alpha|T\beta)$ for all α, β in V. ∎

Definition. *A* **unitary operator** *on an inner product space is an isomorphism of the space onto itself.*

The product of two unitary operators is unitary. For, if U_1 and U_2 are unitary, then U_2U_1 is invertible and $||U_2U_1\alpha|| = ||U_1\alpha|| = ||\alpha||$ for each α. Also, the inverse of a unitary operator is unitary, since $||U\alpha|| = ||\alpha||$ says $||U^{-1}\beta|| = ||\beta||$, where $\beta = U\alpha$. Since the identity operator is clearly unitary, we see that the set of all unitary operators on an inner product space is a group, under the operation of composition.

If V is a finite-dimensional inner product space and U is a linear operator on V, Theorem 10 tells us that U is unitary if and only if $(U\alpha|U\beta) = (\alpha|\beta)$ for each α, β in V; or, if and only if for some (every) orthonormal basis $\{\alpha_1, \ldots, \alpha_n\}$ it is true that $\{U\alpha_1, \ldots, U\alpha_n\}$ is an orthonormal basis.

Theorem 12. *Let* U *be a linear operator on an inner product space* V. *Then* U *is unitary if and only if the adjoint* U* *of* U *exists and* $UU^* = U^*U = I$.

Proof. Suppose U is unitary. Then U is invertible and

$$(U\alpha|\beta) = (U\alpha|UU^{-1}\beta) = (\alpha|U^{-1}\beta)$$

for all α, β. Hence U^{-1} is the adjoint of U.

Conversely, suppose U^* exists and $UU^* = U^*U = I$. Then U is invertible, with $U^{-1} = U^*$. So, we need only show that U preserves inner products. We have

$$\begin{aligned}(U\alpha|U\beta) &= (\alpha|U^*U\beta)\\ &= (\alpha|I\beta)\\ &= (\alpha|\beta)\end{aligned}$$

for all α, β. ▌

EXAMPLE 26. Consider $C^{n\times 1}$ with the inner product $(X|Y) = Y^*X$. Let A be an $n \times n$ matrix over C, and let U be the linear operator defined by $U(X) = AX$. Then

$$(UX|UY) = (AX|AY) = Y^*A^*AX$$

for all X, Y. Hence, U is unitary if and only if $A^*A = I$.

Definition. *A complex* $n \times n$ *matrix* A *is called* **unitary,** *if* $A^*A = I$.

Theorem 13. *Let* V *be a finite-dimensional inner product space and let* U *be a linear operator on* V. *Then* U *is unitary if and only if the matrix of* U *in some (or every) ordered orthonormal basis is a unitary matrix.*

Proof. At this point, this is not much of a theorem, and we state it largely for emphasis. If $\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}$ is an ordered orthonormal basis for V and A is the matrix of U relative to $\mathcal{B}$, then $A^*A = I$ if and only if $U^*U = I$. The result now follows from Theorem 12. ▌

Let A be an $n \times n$ matrix. The statement that A is unitary simply means

$$(A^*A)_{jk} = \delta_{jk}$$

or

$$\sum_{r=1}^{n} \overline{A_{rj}} A_{rk} = \delta_{jk}.$$

In other words, it means that the columns of A form an orthonormal set of column matrices, with respect to the standard inner product $(X|Y) = Y^*X$. Since $A^*A = I$ if and only if $AA^* = I$, we see that A is unitary exactly when the rows of A comprise an orthonormal set of n-tuples in C_n (with the standard inner product). So, using standard inner products, A is unitary if and only if the rows and columns of A are orthonormal sets. One sees here an example of the power of the theorem which states that a one-sided inverse for a matrix is a two-sided inverse. Applying this theorem as we did above, say to real matrices, we have the following: Suppose we have a square array of real numbers such that the sum of the squares of the entries in each row is 1 and distinct rows are orthogonal. Then the sum of the squares of the entries in each column is 1 and distinct columns are orthogonal. Write down the proof of this for a 3×3 array, without using any knowledge of matrices, and you should be reasonably impressed.

Definition. *A real or complex* $n \times n$ *matrix* A *is said to be* **orthogonal,** *if* $A^tA = I$.

A real orthogonal matrix is unitary; and, a unitary matrix is orthogonal if and only if each of its entries is real.

EXAMPLE 27. We give some examples of unitary and orthogonal matrices.

(a) A 1×1 matrix $[c]$ is orthogonal if and only if $c = \pm 1$, and unitary if and only if $\bar{c}c = 1$. The latter condition means (of course) that $|c| = 1$, or $c = e^{i\theta}$, where θ is real.

(b) Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then A is orthogonal if and only if

$$A^t = A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

The determinant of any orthogonal matrix is easily seen to be ± 1. Thus A is orthogonal if and only if

$$A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

or

$$A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$$

where $a^2 + b^2 = 1$. The two cases are distinguished by the value of det A.

(c) The well-known relations between the trigonometric functions show that the matrix

$$A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

is orthogonal. If θ is a real number, then A_θ is the matrix in the standard ordered basis for R^2 of the linear operator U_θ, rotation through the angle θ. The statement that A_θ is a real orthogonal matrix (hence unitary) simply means that U_θ is a unitary operator, i.e., preserves dot products.

(d) Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then A is unitary if and only if

$$\begin{bmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

The determinant of a unitary matrix has absolute value 1, and is thus a complex number of the form $e^{i\theta}$, θ real. Thus A is unitary if and only if

$$A = \begin{bmatrix} a & b \\ -e^{i\theta}\bar{b} & e^{i\theta}\bar{a} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix} \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$$

where θ is a real number, and a, b are complex numbers such that $|a|^2 + |b|^2 = 1$.

As noted earlier, the unitary operators on an inner product space form a group. From this and Theorem 13 it follows that the set $U(n)$ of all $n \times n$ unitary matrices is also a group. Thus the inverse of a unitary matrix and the product of two unitary matrices are again unitary. Of course this is easy to see directly. An $n \times n$ matrix A with complex entries is unitary if and only if $A^{-1} = A^*$. Thus, if A is unitary, we have $(A^{-1})^{-1} = A = (A^*)^{-1} = (A^{-1})^*$. If A and B are $n \times n$ unitary matrices, then $(AB)^{-1} = B^{-1}A^{-1} = B^*A^* = (AB)^*$.

The Gram-Schmidt process in C^n has an interesting corollary for matrices that involves the group $U(n)$.

Theorem 14. *For every invertible complex* n $\times$ n *matrix* B *there exists a unique lower-triangular matrix* M *with positive entries on the main diagonal such that* MB *is unitary.*

Proof. The rows $\beta_1, \ldots, \beta_n$ of B form a basis for C^n. Let $\alpha_1, \ldots, \alpha_n$ be the vectors obtained from $\beta_1, \ldots, \beta_n$ by the Gram-Schmidt process. Then, for $1 \leq k \leq n$, $\{\alpha_1, \ldots, \alpha_k\}$ is an orthogonal basis for the subspace spanned by $\{\beta_1, \ldots, \beta_k\}$, and

$$\alpha_k = \beta_k - \sum_{j<k} \frac{(\beta_k|\alpha_j)}{||\alpha_j||^2} \alpha_j.$$

Hence, for each k there exist unique scalars C_{kj} such that

$$\alpha_k = \beta_k - \sum_{j<k} C_{kj}\beta_j.$$

Let U be the unitary matrix with rows

$$\frac{\alpha_1}{||\alpha_1||}, \ldots, \frac{\alpha_n}{||\alpha_n||}$$

and M the matrix defined by

$$M_{kj} = \begin{cases} -\dfrac{1}{||\alpha_k||} \cdot C_{kj}, & \text{if } j < k \\ \dfrac{1}{||\alpha_k||}, & \text{if } j = k \\ 0, & \text{if } j > k. \end{cases}$$

Then M is lower-triangular, in the sense that its entries above the main diagonal are 0. The entries M_{kk} of M on the main diagonal are all > 0, and

$$\frac{\alpha_k}{||\alpha_k||} = \sum_{j=1}^{n} M_{kj}\beta_j, \qquad 1 \leq k \leq n.$$

Now these equations simply say that

$$U = MB.$$

To prove the uniqueness of M, let $T^+(n)$ denote the set of all complex $n \times n$ lower-triangular matrices with positive entries on the main diagonal. Suppose M_1 and M_2 are elements of $T^+(n)$ such that M_iB is in $U(n)$ for $i = 1, 2$. Then because $U(n)$ is a group

$$(M_1B)(M_2B)^{-1} = M_1M_2^{-1}$$

lies in $U(n)$. On the other hand, although it is not entirely obvious, $T^+(n)$ is also a group under matrix multiplication. One way to see this is to consider the geometric properties of the linear transformations

$$X \to MX, \qquad (M \text{ in } T^+(n))$$

on the space of column matrices. Thus M_2^{-1}, $M_1M_2^{-1}$, and $(M_1M_2^{-1})^{-1}$ are all in $T^+(n)$. But, since $M_1M_2^{-1}$ is in $U(n)$, $(M_1M_2^{-1})^{-1} = (M_1M_2^{-1})^*$. The transpose or conjugate transpose of any lower-triangular matrix is an upper-triangular matrix. Therefore, $M_1M_2^{-1}$ is simultaneously upper- and lower-triangular, i.e., diagonal. A diagonal matrix is unitary if and only if each of its entries on the main diagonal has absolute value 1; if the diagonal entries are all positive, they must equal 1. Hence $M_1M_2^{-1} = I$ and $M_1 = M_2$. ∎