4.4.3 The condition for $\mathbf{w}$ to be equidistant from $\mathbf{u}$ and $\mathbf{v}$ is

$$(\mathbf{w}-\mathbf{u})\cdot(\mathbf{w}-\mathbf{u}) = (\mathbf{w}-\mathbf{v})\cdot(\mathbf{w}-\mathbf{v}).$$

Explain why, and show that this condition is equivalent to

$$|\mathbf{u}|^2 - 2\mathbf{w}\cdot\mathbf{u} = |\mathbf{v}|^2 - 2\mathbf{w}\cdot\mathbf{v}.$$

4.4.4 Show that the condition found in Exercise 4.4.3 is equivalent to

$$\left(\mathbf{w}-\frac{\mathbf{u}+\mathbf{v}}{2}\right)\cdot(\mathbf{u}-\mathbf{v}) = 0,$$

and explain why this says that $\mathbf{w}$ is on the perpendicular bisector of the line from $\mathbf{u}$ to $\mathbf{v}$.

Having established that the line equidistant from $\mathbf{u}$ and $\mathbf{v}$ is the perpendicular bisector, we conclude that the perpendicular bisectors of the sides of a triangle are concurrent—because this is obviously true of the equidistant lines of its vertices.

4.5 Inner product and cosine

The inner product of vectors $\mathbf{u}$ and $\mathbf{v}$ depends not only on their lengths $|\mathbf{u}|$ and $|\mathbf{v}|$ but also on the angle θ between them. The simplest way to express its dependence on angle is with the help of the *cosine* function. We write the cosine as a function of angle θ, $\cos\theta$. But, as usual, we avoid measuring angles and instead define $\cos\theta$ as the ratio of sides of a right-angled triangle. For simplicity, we assume that the triangle has vertices $\mathbf{0}$, $\mathbf{u}$, and $\mathbf{v}$ as shown in Figure 4.9.

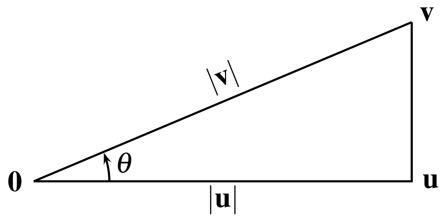

Figure 4.9: Cosine as a ratio of lengths

Then the side $\mathbf{v}$ is the hypotenuse, θ is the angle between the side $\mathbf{u}$ and the hypotenuse, and its cosine is defined by

$$\cos\theta = \frac{|\mathbf{u}|}{|\mathbf{v}|}.$$

We can now use the inner product criterion for perpendicularity to derive the following formula for inner product.

Inner product formula. *If θ is the angle between vectors* $\mathbf{u}$ *and* $\mathbf{v}$*, then*

$$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta.$$

This formula follows because the side $\mathbf{v}-\mathbf{u}$ of the triangle is perpendicular to side $\mathbf{u}$; hence,

$$0 = \mathbf{u}\cdot(\mathbf{v}-\mathbf{u}) = \mathbf{u}\cdot\mathbf{v}-\mathbf{u}\cdot\mathbf{u}.$$

Therefore, $\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{u} = |\mathbf{u}|^{\mathbf{2}} = |\mathbf{u}||\mathbf{v}|\frac{|\mathbf{u}|}{|\mathbf{v}|} = |\mathbf{u}||\mathbf{v}|\cos\theta.$ □

This formula gives a convenient way to calculate the angle (or at least its cosine) between any two lines, because we know from Section 4.4 how to calculate $|\mathbf{u}|$ and $|\mathbf{v}|$. It also gives us the "cosine rule" of trigonometry directly from the calculation of $(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})$.

Cosine rule. *In any triangle, with sides* $\mathbf{u}$*,* $\mathbf{v}$*, and* $\mathbf{u}-\mathbf{v}$*, and angle θ opposite to the side* $\mathbf{u}-\mathbf{v}$*,*

$$|\mathbf{u}-\mathbf{v}|^2 = |\mathbf{u}|^{\mathbf{2}} + |\mathbf{v}|^{\mathbf{2}} - \mathbf{2}|\mathbf{u}||\mathbf{v}|\cos\theta.$$

Figure 4.10 shows the triangle and the relevant sides and angle, but the proof is a purely algebraic consequence of the inner product formula.

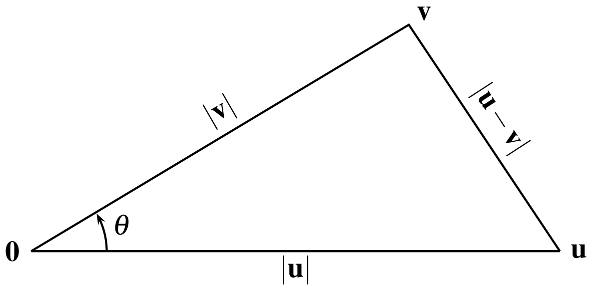

Figure 4.10: Quantities mentioned in the cosine rule

The algebra is simply the following:

$$\begin{aligned} |\mathbf{u}-\mathbf{v}|^2 &= (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) \\ &= \mathbf{u}\cdot\mathbf{u} - 2\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v} \\ &= |\mathbf{u}|^{\mathbf{2}} + |\mathbf{v}|^{\mathbf{2}} - \mathbf{2u}\cdot\mathbf{v} \\ &= |\mathbf{u}|^{\mathbf{2}} + |\mathbf{v}|^{\mathbf{2}} - \mathbf{2}|\mathbf{u}||\mathbf{v}|\cos\theta. \end{aligned}$$ □

A nice way to close this circle of ideas is to consider the special case in which $\mathbf{u}$ and $\mathbf{v}$ are the sides of a right-angled triangle and $\mathbf{u}-\mathbf{v}$ is the hypotenuse. In this case, $\mathbf{u}$ is perpendicular to $\mathbf{v}$, so $\mathbf{u}\cdot\mathbf{v}=0$, and the cosine rule becomes

$$\text{hypotenuse}^2 = |\mathbf{u}-\mathbf{v}|^2 = |\mathbf{u}|^{\mathbf{2}} + |\mathbf{v}|^{\mathbf{2}}$$

—which is the Pythagorean theorem. This result should not be a surprise, however, because we have already seen how the Pythagorean theorem is built into the definition of distance in $\mathbb{R}^2$ and hence into the inner product.

Exercises

The Pythagorean theorem can also be proved directly, by choosing $\mathbf{0}$ at the right angle of a right-angled triangle whose other two vertices are $\mathbf{u}$ and $\mathbf{v}$.

4.5.1 Show that $|\mathbf{v}-\mathbf{u}|^2 = |\mathbf{u}|^{\mathbf{2}} + |\mathbf{v}|^{\mathbf{2}}$ under these conditions, and explain why this is the Pythagorean theorem.

While on the subject of right-angled triangles, we mention a useful formula for studying them.

4.5.2 Show that $(\mathbf{v}+\mathbf{u})\cdot(\mathbf{v}-\mathbf{u}) = |\mathbf{v}|^{\mathbf{2}} - |\mathbf{u}|^{\mathbf{2}}$.

This formula gives a neat proof of the theorem from Section 2.7 about the angle in a semicircle. Take a circle with center $\mathbf{0}$ and a diameter with ends $\mathbf{u}$ and $-\mathbf{u}$ as shown in Figure 4.11. Also, let $\mathbf{v}$ be any other point on the circle.

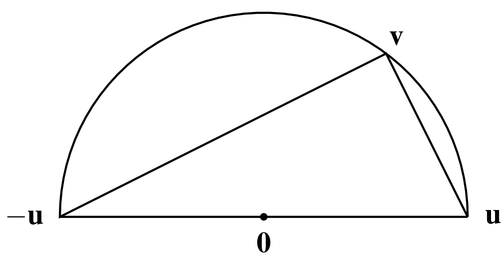

Figure 4.11: Points on a semicircle

4.5.3 Show that the sides of the triangle meeting at $\mathbf{v}$ have directions $\mathbf{v}+\mathbf{u}$ and $\mathbf{v}-\mathbf{u}$ and hence show that they are perpendicular.

4.6 The triangle inequality

In vector geometry, the triangle inequality $|\mathbf{u}+\mathbf{v}| \leq |\mathbf{u}|+|\mathbf{v}|$ of Exercises 3.3.1 to 3.3.3 is usually derived from the fact that

$$|\mathbf{u}\cdot\mathbf{v}| \leq |\mathbf{u}||\mathbf{v}|.$$

This result, known as the *Cauchy–Schwarz inequality*, follows easily from the formula in the previous section. The inner product formula says

$$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta,$$

and therefore,

$$\begin{aligned}|\mathbf{u}\cdot\mathbf{v}| &\leq |\mathbf{u}||\mathbf{v}||\cos\theta| \\ &\leq |\mathbf{u}||\mathbf{v}| \quad \text{because} \quad |\cos\theta| \leq \mathbf{1}.\end{aligned}$$

Now, to get the triangle inequality, it suffices to show that $|\mathbf{u}+\mathbf{v}|^2 \leq (|\mathbf{u}|+|\mathbf{v}|)^{\mathbf{2}}$, which we do as follows:

$$\begin{aligned}|\mathbf{u}+\mathbf{v}|^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) \\ &= |\mathbf{u}|^{\mathbf{2}} + \mathbf{2u}\cdot\mathbf{v} + |\mathbf{v}|^{\mathbf{2}} \quad \text{because } \mathbf{u}\cdot\mathbf{u} = |\mathbf{u}|^{\mathbf{2}} \text{ and } \mathbf{v}\cdot\mathbf{v} = |\mathbf{v}|^{\mathbf{2}} \\ &\leq |\mathbf{u}|^{\mathbf{2}} + \mathbf{2}|\mathbf{u}||\mathbf{v}| + |\mathbf{v}|^{\mathbf{2}} \quad \text{by Cauchy–Schwarz} \\ &= (|\mathbf{u}|+|\mathbf{v}|)^{\mathbf{2}} \qquad \square\end{aligned}$$

The reason for the fuss about the Cauchy–Schwarz inequality is that it holds in spaces more complicated than $\mathbb{R}^2$, with more complicated inner products. Because the triangle inequality follows from Cauchy–Schwarz, it too holds in these complicated spaces. We are mainly concerned with the geometry of the plane, so we do not need complicated spaces. However, it is worth saying a few words about $\mathbb{R}^n$, because linear algebra works just as well there as it does in $\mathbb{R}^2$.

Higher dimensional Euclidean spaces

$\mathbb{R}^n$ is the set of ordered n-tuples $(x_1, x_2, \ldots, x_n)$ of real numbers $x_1, x_2, \ldots, x_n$. These ordered n-tuples are called *n-dimensional vectors*. If $\mathbf{u}$ and $\mathbf{v}$ are in $\mathbb{R}^n$, then we define the *vector sum* $\mathbf{u}+\mathbf{v}$ by

$$\mathbf{u}+\mathbf{v} = (u_1+v_1, u_2+v_2, \ldots, u_n+v_n),$$

and the *scalar multiple* $a\mathbf{u}$ for a real number a by

$$a\mathbf{u} = (au_1, au_2, \ldots, au_n).$$

It is easy to check that $\mathbb{R}^n$ has the properties enumerated at the beginning of Section 4.1. Hence, $\mathbb{R}^n$ is a real vector space under the vector sum and scalar multiplication operations just described.

$\mathbb{R}^n$ becomes a *Euclidean space* when we give it the extra structure of an inner product with the properties enumerated in Section 4.4. These properties hold if we define the inner product $\mathbf{u} \cdot \mathbf{v}$ by

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n,$$

as is easy to check. This inner product enables us to define *distance* in $\mathbb{R}^n$ by the formula

$$|\mathbf{u}|^{\mathbf{2}} = \mathbf{u} \cdot \mathbf{u}$$

which gives the distance $|\mathbf{u}|$ of $\mathbf{u}$ from the origin. This result is compatible with the concept of distance in $\mathbb{R}^2$ or $\mathbb{R}^3$ given by the Pythagorean theorem. For example, the distance of (u_1, u_2, u_3) from $\mathbf{0}$ in $\mathbb{R}^3$ is

$$|\mathbf{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2},$$

as Figure 4.12 shows.

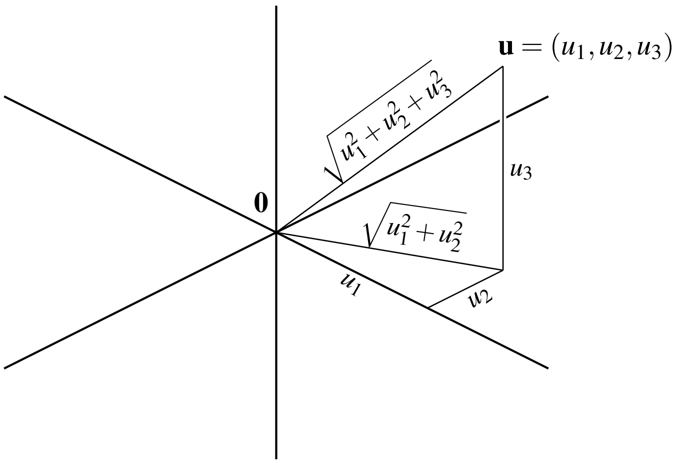

Figure 4.12: Distance in $\mathbb{R}^3$

- $\sqrt{u_1^2+u_2^2}$ is the distance from $\mathbf{0}$ of $(u_1,u_2,0)$ (the hypotenuse of a right-angled triangle with sides u_1 and u_2),

- $\sqrt{u_1^2+u_2^2+u_3^2}$ is the distance from $\mathbf{0}$ of (u_1,u_2,u_3) (the hypotenuse of a right-angled triangle with sides $\sqrt{u_1^2+u_2^2}$ and u_3).

All theorems proved in this chapter for vectors in the plane $\mathbb{R}^2$ hold in $\mathbb{R}^n$. This fact is clear if we take the plane in $\mathbb{R}^n$ to consist of vectors of the form $(x_1,x_2,0,\ldots,0)$, because such vectors behave exactly the same as vectors (x_1,x_2) in $\mathbb{R}^2$. But in fact *any* given plane in $\mathbb{R}^n$ behaves the same as the special plane of vectors $(x_1,x_2,0,\ldots,0)$. We skip the details, but it can be proved by constructing an isometry of $\mathbb{R}^n$ mapping the given plane onto the special plane. As in $\mathbb{R}^2$, any isometry is a product of reflections. In $\mathbb{R}^n$, at most $n+1$ reflections are required, and the proof is similar to the one given in Section 3.7.

Exercises

A proof of Cauchy–Schwarz using only general properties of the inner product can be obtained by an algebraic trick with quadratic equations. The general properties involved are the four listed at the beginning of Section 4.4 and the assumption that $\mathbf{w}\cdot\mathbf{w}=|\mathbf{w}|^2\geq 0$ for any vector $\mathbf{w}$ (an inner product with the latter property is called *positive definite*).

4.6.1 The Euclidean inner product for $\mathbb{R}^n$ defined above is positive definite. Why?

4.6.2 For any real number x, and any vectors $\mathbf{u}$ and $\mathbf{v}$, show that

$$(\mathbf{u}+x\mathbf{v})\cdot(\mathbf{u}+x\mathbf{v})=|\mathbf{u}|^2+2x(\mathbf{u}\cdot\mathbf{v})+x^2|\mathbf{v}|^2,$$

and hence that $|\mathbf{u}|^2+2x(\mathbf{u}\cdot\mathbf{v})+x^2|\mathbf{v}|^2\geq 0$ for any real number x.

4.6.3 If A, B, and C are real numbers and $A+Bx+Cx^2\geq 0$ for any real number x, explain why $B^2-4AC\leq 0$.

4.6.4 By applying Exercise 4.6.3 to the inequality $|\mathbf{u}|^2+2x(\mathbf{u}\cdot\mathbf{v})+x^2|\mathbf{v}|^2\geq 0$, show that

$$(\mathbf{u}\cdot\mathbf{v})^2\leq|\mathbf{u}|^2|\mathbf{v}|^2,\quad\text{and hence}\quad|\mathbf{u}\cdot\mathbf{v}|\leq|\mathbf{u}||\mathbf{v}|.$$

4.7 Rotations, matrices, and complex numbers

Rotation matrices

In Section 3.6, we defined a rotation of $\mathbb{R}^2$ as a function $r_{c,s}$, where c and s are two real numbers such that $c^2+s^2=1$. We described $r_{c,s}$ as the function that sends (x,y) to $(cx-sy, sx+cy)$, but it is also described by the *matrix of coefficients* of x and y, namely

$$\begin{pmatrix} c & -s \\ s & c \end{pmatrix}, \quad \text{where } c=\cos\theta \text{ and } s=\sin\theta.$$

Because most readers will already have seen matrices, it may be useful to translate some previous statements about functions into matrix language, where they may be more familiar. (Readers not yet familiar with matrices will find an introduction in Section 7.2.)

Matrix notation allows us to rewrite $(x,y)\mapsto(cx-sy, sx+cy)$ as

$$\begin{pmatrix} c & -s \\ s & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} cx-sy \\ sx+cy \end{pmatrix}$$

Thus, the function $r_{c,s}$ is applied to the variables x and y by multiplying the column vector $\begin{pmatrix} x \\ y \end{pmatrix}$ on the left by the matrix $\begin{pmatrix} c & -s \\ s & c \end{pmatrix}$. Functions are thereby separated from their variables, so they can be composed without the variables becoming involved—simply by multiplying matrices.

This idea gives proofs of the formulas for $\cos(\theta_1+\theta_2)$ and $\sin(\theta_1+\theta_2)$, similar to Exercises 3.5.3 and 3.5.4, but with the variables x and y filtered out:

- Rotation through angle θ_1 is given by the matrix $\begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix}$.

- Rotation through angle θ_2 is given by the matrix $\begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix}$.

- Hence, rotation through $\theta_1+\theta_2$ is given by the product of these two matrices. That is,

$$\begin{aligned}
&\begin{pmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 & -\cos\theta_1\sin\theta_2 - \sin\theta_1\cos\theta_2 \\ \cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2 & \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 \end{pmatrix}
\end{aligned}$$

by matrix multiplication.

- Finally, equating corresponding entries in the first and last matrices,

$$\cos(\theta_1+\theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2,$$
$$\sin(\theta_1+\theta_2) = \cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2.$$

Complex numbers

One advantage of matrices, which we do not pursue here, is that they can be used to generalize the idea of rotation to any number of dimensions. But, for rotations of $\mathbb{R}^2$, there is a notation even more efficient than the rotation matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

It is the *complex number* $\cos\theta + i\sin\theta$, where $i = \sqrt{-1}$.

We represent the point $(x,y) \in \mathbb{R}^2$ by the complex number $z = x + iy$, and we rotate it through angle θ about O by *multiplying it* by $\cos\theta + i\sin\theta$. This procedure works because $i^2 = -1$, and therefore,

$$(\cos\theta + i\sin\theta)(x+iy) = x\cos\theta - y\sin\theta + i(x\sin\theta + y\cos\theta).$$

Thus, multiplication by $\cos\theta + i\sin\theta$ sends each point (x,y) to the point $(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$, which is the result of rotating (x,y) about O through angle θ. Multiplying all points at once by $\cos\theta + i\sin\theta$, therefore, rotates the *whole plane* about O through angle θ.

It follows that multiplication by $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$ rotates the plane through $\theta_1 + \theta_2$—the first factor rotates it through θ_1 and the second rotates it through θ_2—so it is the same as multiplication by

$\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)$. Equating these two multipliers gives perhaps the ultimate proof of the formulas for $\cos(\theta_1+\theta_2)$ and $\sin(\theta_1+\theta_2)$:

$$\begin{aligned}\cos(\theta_1+\theta_2)+&i\sin(\theta_1+\theta_2)\\ &=(\cos\theta_1+i\sin\theta_1)(\cos\theta_2+i\sin\theta_2)\\ &=\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2+i(\cos\theta_1\sin\theta_2+\sin\theta_1\cos\theta_2)\\ &\quad\text{since } i^2=-1.\end{aligned}$$

Hence, equating real and imaginary parts,

$$\begin{aligned}\cos(\theta_1+\theta_2)&=\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2,\\ \sin(\theta_1+\theta_2)&=\cos\theta_1\sin\theta_2+\sin\theta_1\cos\theta_2.\end{aligned}$$

Exercises

The calculations above show that multiplication by $\cos\theta+i\sin\theta$ is rotation about O through angle θ because of the (seemingly accidental) property $i^2=-1$. In fact, any algebra of points in $\mathbb{R}^2$ that satisfies the same laws as the algebra of $\mathbb{R}$ automatically satisfies the condition $i^2=-1$, where i is the point $(0,1)$.

The following exercises show why. In particular, they reveal geometric consequences of the following algebraic laws:

$$|uv|=|u||v| \qquad \text{(multiplicative absolute value)}$$

$$u(v+w)=uv+uw \qquad \text{(distributive law)}$$

4.7.1 Given that $|x+iy|=\sqrt{x^2+y^2}$, explain why $|v-w|$ equals the distance between the complex numbers v and w.

4.7.2 Assuming the multiplicative absolute value and the distributive law (and, if necessary, any other algebraic laws satisfied by $\mathbb{R}$), show that

$$\text{distance between } uv \text{ and } uw = |u| \times \text{distance between } v \text{ and } w.$$

In other words, multiplying the plane $\mathbb{C}$ of complex numbers by a constant complex number u multiplies all distances by $|u|$.

4.7.3 Deduce from Exercises 4.7.1 and 4.7.2 that multiplication of $\mathbb{C}$ by a number u with $|u|=1$ is an isometry leaving O fixed.

4.7.4 Assuming that $u\neq 1$, and hence that $uz\neq z$ when $z\neq 0$, deduce from Exercise 4.7.3 that multiplication by $u\neq 1$ is a rotation.

These results explain why multiplication by u with $|u|=1$ is a rotation. To find the angle of rotation we assume that the point $(1,0)$ is the 1 of the algebra and observe where the rotation sends 1.

4.7.5 Explain why any u with $|u| = 1$ can be written in the form $\cos\theta + i\sin\theta$ for some angle θ, and conclude that multiplication by u rotates the point 1 (hence the whole plane) through angle θ.

It follows, in particular, that multiplication by $i = (0,1)$ sends $(1,0)$ to $(0,1)$ and hence rotates the plane through $\pi/2$. This result in turn implies $i^2 = -1$, because multiplication by i^2 then rotates the plane through π, which is also the effect of multiplication by -1.

4.8 Discussion

Because the geometric content of a vector space with an inner product is much the same as Euclidean geometry, it is interesting to see how many axioms it takes to describe a vector space. Remember from Section 2.9 that it takes 17 Hilbert axioms to describe the Euclidean plane, or 16 if we are willing to drop completeness of the line.

To define a vector space, we began in Section 4.1 with eight axioms for vector addition and scalar multiplication:

$$
\begin{aligned}
\mathbf{u}+\mathbf{v} &= \mathbf{v}+\mathbf{u} & 1\mathbf{u} &= \mathbf{u}\\
\mathbf{u}+(\mathbf{v}+\mathbf{w}) &= (\mathbf{u}+\mathbf{v})+\mathbf{w} & a(\mathbf{u}+\mathbf{v}) &= a\mathbf{u}+a\mathbf{v}\\
\mathbf{u}+\mathbf{0} &= \mathbf{u} & (a+b)\mathbf{u} &= a\mathbf{u}+b\mathbf{u}\\
\mathbf{u}+(-\mathbf{u}) &= \mathbf{0} & a(b\mathbf{u}) &= (ab)\mathbf{u}.
\end{aligned}
$$

Then, in Section 4.4, we added three (or four, depending on how you count) axioms for the inner product:

$$
\begin{aligned}
\mathbf{u}\cdot\mathbf{v} &= \mathbf{v}\cdot\mathbf{u},\\
\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) &= \mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w},\\
(a\mathbf{u})\cdot\mathbf{v} &= \mathbf{u}\cdot(a\mathbf{v}) = a(\mathbf{u}\cdot\mathbf{v}),
\end{aligned}
$$

We also need relations among inner product, length, and angle—at a minimum the cosine formula,

$$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta,$$

so this is 12 or 13 axioms so far.

But we have also assumed that the scalars $a, b, \ldots$ are real numbers, so there remains the problem of writing down axioms for them. At the very least, one needs axioms saying that the scalars satisfy the ordinary rules of calculation, the so-called *field axioms* (this is usual when defining a vector space):

$$
\begin{array}{rcr}
a+b=b+a, & ab=ba & \text{(commutative laws)} \\
a+(b+c)=(a+b)+c, & a(bc)=(ab)c & \text{(associative laws)} \\
a+0=a, & a1=a & \text{(identity laws)} \\
a+(-a)=0, & aa^{-1}=1 & \text{(inverse laws)} \\
\multicolumn{2}{c}{a(b+c)=ab+ac} & \text{(distributive law)}
\end{array}
$$

Thus, the usual definition of a vector space, with an inner product suitable for Euclidean geometry, takes more than 20 axioms! Admittedly, the field axioms and the vector space axioms are useful in many other parts of mathematics, whereas most of the Hilbert axioms seem meaningful only in geometry. And, by varying the inner product slightly, one can change the geometry of the vector space in interesting ways. For example, one can obtain the geometry of *Minkowski space* used in Einstein's special theory of relativity. To learn more about the vector space approach to geometry, see *Linear Algebra and Geometry, a Second Course* by I. Kaplansky and *Metric Affine Geometry* by E. Snapper and R. J. Troyer.

Still, one can dream of building geometry on a much simpler set of axioms. In Chapter 6, we will realize this dream with *projective geometry*, which we begin studying in Chapter 5.

5

Perspective

PREVIEW

Euclid's geometry concerns figures that can be drawn with straightedge and compass, even though many of its theorems are about straight lines alone. Are there any interesting figures that can be drawn with straightedge alone? Remember, the straightedge has no marks on it, so it is impossible to copy a length. Thus, with a straightedge alone, we cannot draw a square, an equilateral triangle, or any figure involving equal line segments. Yet there is something interesting we *can* draw: a *perspective view of a tiled floor*, such as the one shown in Figure 5.1.

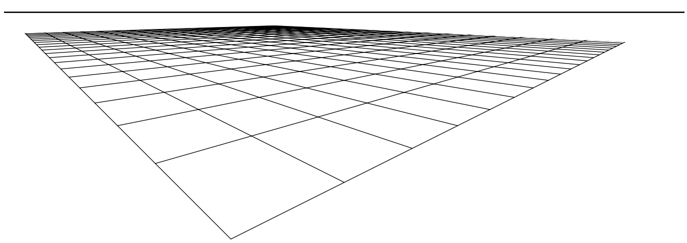

Figure 5.1: Perspective view of a tiled floor

This picture is interesting because it seems clear that all tiles in the view are of equal size. Thus, even though we cannot draw tiles that are actually equal, we can draw tiles that *look* equal.

We will explain how to solve the problem of drawing perspective views in Section 5.2. The solution takes us into a new form of geometry—a geometry of vision—called *projective geometry*.