Since q is prime, either $q \mid p-1$ or $q \mid p+1$. The former is impossible since $q > p$ so the latter holds. Since $q > p$ but $q \mid p+1$, we must have $q = p+1$. This forces $p = 2$, $q = 3$ and $|G| = 12$. The result now follows from the preceding example.

Groups of Order 60

We illustrate how Sylow's Theorems can be used to unravel the structure of groups of a given order even if some groups of that order may be simple. Note the technique of changing from one prime to another and the inductive process where we use results on groups of order < 60 to study groups of order 60.

Proposition 21. If $|G| = 60$ and G has more than one Sylow 5-subgroup, then G is simple.

Proof: Suppose by way of contradiction that $|G| = 60$ and $n_5 > 1$ but that there exists H a normal subgroup of G with $H \neq 1$ or G. By Sylow's Theorem the only possibility for n_5 is 6. Let $P \in Syl_5(G)$, so that $|N_G(P)| = 10$ since its index is n_5.

If $5 \mid |H|$ then H contains a Sylow 5-subgroup of G and since H is normal, it contains all 6 conjugates of this subgroup. In particular, $|H| \geq 1 + 6 \cdot 4 = 25$, and the only possibility is $|H| = 30$. This leads to a contradiction since a previous example proved that any group of order 30 has a normal (hence unique) Sylow 5-subgroup. This argument shows 5 does not divide $|H|$ for any proper normal subgroup H of G.

If $|H| = 6$ or 12, H has a normal, hence characteristic, Sylow subgroup, which is therefore also normal in G. Replacing H by this subgroup if necessary, we may assume $|H| = 2, 3$ or 4. Let $\overline{G} = G/H$, so $|\overline{G}| = 30$, 20 or 15. In each case, $\overline{G}$ has a normal subgroup $\overline{P}$ of order 5 by previous results. If we let H_1 be the complete preimage of $\overline{P}$ in G, then $H_1 \trianglelefteq G$, $H_1 \neq G$ and $5 \mid |H_1|$. This contradicts the preceding paragraph and so completes the proof.

Corollary 22. A_5 is simple.

Proof: The subgroups $\langle (1\,2\,3\,4\,5) \rangle$ and $\langle (1\,3\,2\,4\,5) \rangle$ are distinct Sylow 5-subgroups of A_5 so the result follows immediately from the proposition.

The next proposition shows that there is a unique simple group of order 60.

Proposition 23. If G is a simple group of order 60, then $G \cong A_5$.

Proof: Let G be a simple group of order 60, so $n_2 = 3, 5$ or 15. Let $P \in Syl_2(G)$ and let $N = N_G(P)$, so $|G : N| = n_2$.

First observe that G has no proper subgroup H of index less that 5, as follows: if H were a subgroup of G of index 4, 3 or 2, then, by Theorem 3, G would have a normal subgroup K contained in H with G/K isomorphic to a subgroup of S_4, S_3 or S_2. Since $K \neq G$, simplicity forces $K = 1$. This is impossible since 60 $(= |G|)$ does not divide 4!. This argument shows, in particular, that $n_2 \neq 3$.

If $n_2 = 5$, then N has index 5 in G so the action of G by left multiplication on the set of left cosets of N gives a permutation representation of G into S_5. Since (as above) the kernel of this representation is a proper normal subgroup and G is simple, the kernel is 1 and G is isomorphic to a subgroup of S_5. Identify G with this isomorphic copy so that we may assume $G \leq S_5$. If G is not contained in A_5, then $S_5 = GA_5$ and, by the Second Isomorphism Theorem, $A_5 \cap G$ is of index 2 in G. Since G has no (normal) subgroup of index 2, this is a contradiction. This argument proves $G \leq A_5$. Since $|G| = |A_5|$, the isomorphic copy of G in S_5 coincides with A_5, as desired.

Finally, assume $n_2 = 15$. If for every pair of distinct Sylow 2-subgroups P and Q of G, $P \cap Q = 1$, then the number of nonidentity elements in Sylow 2-subgroups of G would be $(4-1) \cdot 15 = 45$. But $n_5 = 6$ so the number of elements of order 5 in G is $(5-1) \cdot 6 = 24$, accounting for 69 elements. This contradiction proves that there exist distinct Sylow 2-subgroups P and Q with $|P \cap Q| = 2$. Let $M = N_G(P \cap Q)$. Since P and Q are abelian (being groups of order 4), P and Q are subgroups of M and since G is simple, $M \neq G$. Thus 4 divides $|M|$ and $|M| > 4$ (otherwise, $P = M = Q$). The only possibility is $|M| = 12$, i.e., M has index 5 in G (recall M cannot have index 3 or 1). But now the argument of the preceding paragraph applied to M in place of N gives $G \cong A_5$. This leads to a contradiction in this case because $n_2(A_5) = 5$ (cf. the exercises). The proof is complete.

EXERCISES

Let G be a finite group and let p be a prime.

1. Prove that if $P \in Syl_p(G)$ and H is a subgroup of G containing P then $P \in Syl_p(H)$. Give an example to show that, in general, a Sylow p-subgroup of a subgroup of G need not be a Sylow p-subgroup of G.

2. Prove that if H is a subgroup of G and $Q \in Syl_p(H)$ then $gQg^{-1} \in Syl_p(gHg^{-1})$ for all $g \in G$.

3. Use Sylow's Theorem to prove Cauchy's Theorem. (Note that we only used Cauchy's Theorem for abelian groups — Proposition 3.21 — in the proof of Sylow's Theorem so this line of reasoning is not circular.)

4. Exhibit all Sylow 2-subgroups and Sylow 3-subgroups of D_{12} and $S_3 \times S_3$.

5. Show that a Sylow p-subgroup of D_{2n} is cyclic and normal for every odd prime p.

6. Exhibit all Sylow 3-subgroups of A_4 and all Sylow 3-subgroups of S_4.

7. Exhibit all Sylow 2-subgroups of S_4 and find elements of S_4 which conjugate one of these into each of the others.

8. Exhibit two distinct Sylow 2-subgroups of S_5 and an element of S_5 that conjugates one into the other.

9. Exhibit all Sylow 3-subgroups of $SL_2(\mathbb{F}_3)$ (cf. Exercise 9, Section 2.1).

10. Prove that the subgroup of $SL_2(\mathbb{F}_3)$ generated by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ is the unique Sylow 2-subgroup of $SL_2(\mathbb{F}_3)$ (cf. Exercise 10, Section 2.4).

11. Show that the center of $SL_2(\mathbb{F}_3)$ is the group of order 2 consisting of $\pm I$, where I is the identity matrix. Prove that $SL_2(\mathbb{F}_3)/Z(SL_2(\mathbb{F}_3)) \cong A_4$. [Use facts about groups of order 12.]

12. Let $2n = 2^a k$ where k is odd. Prove that the number of Sylow 2-subgroups of D_{2n} is k. [Prove that if $P \in Syl_2(D_{2n})$ then $N_{D_{2n}}(P) = P$.]

13. Prove that a group of order 56 has a normal Sylow p-subgroup for some prime p dividing its order.

14. Prove that a group of order 312 has a normal Sylow p-subgroup for some prime p dividing its order.

15. Prove that a group of order 351 has a normal Sylow p-subgroup for some prime p dividing its order.

16. Let $|G| = pqr$, where p, q and r are primes with $p < q < r$. Prove that G has a normal Sylow subgroup for either p, q or r.

17. Prove that if $|G| = 105$ then G has a normal Sylow 5-subgroup and a normal Sylow 7-subgroup.

18. Prove that a group of order 200 has a normal Sylow 5-subgroup.

19. Prove that if $|G| = 6545$ then G is not simple.

20. Prove that if $|G| = 1365$ then G is not simple.

21. Prove that if $|G| = 2907$ then G is not simple.

22. Prove that if $|G| = 132$ then G is not simple.

23. Prove that if $|G| = 462$ then G is not simple.

24. Prove that if G is a group of order 231 then $Z(G)$ contains a Sylow 11-subgroup of G and a Sylow 7-subgroup is normal in G.

25. Prove that if G is a group of order 385 then $Z(G)$ contains a Sylow 7-subgroup of G and a Sylow 11-subgroup is normal in G.

26. Let G be a group of order 105. Prove that if a Sylow 3-subgroup of G is normal then G is abelian.

27. Let G be a group of order 315 which has a normal Sylow 3-subgroup. Prove that $Z(G)$ contains a Sylow 3-subgroup of G and deduce that G is abelian.

28. Let G be a group of order 1575. Prove that if a Sylow 3-subgroup of G is normal then a Sylow 5-subgroup and a Sylow 7-subgroup are normal. In this situation prove that G is abelian.

29. If G is a non-abelian simple group of order < 100, prove that $G \cong A_5$. [Eliminate all orders but 60.]

30. How many elements of order 7 must there be in a simple group of order 168?

31. For $p = 2$, 3 and 5 find $n_p(A_5)$ and $n_p(S_5)$. [Note that $A_4 \leq A_5$.]

32. Let P be a Sylow p-subgroup of H and let H be a subgroup of K. If $P \trianglelefteq H$ and $H \trianglelefteq K$, prove that P is normal in K. Deduce that if $P \in Syl_p(G)$ and $H = N_G(P)$, then $N_G(H) = H$ (in words: *normalizers of Sylow p-subgroups are self-normalizing*).

33. Let P be a normal Sylow p-subgroup of G and let H be any subgroup of G. Prove that $P \cap H$ is the unique Sylow p-subgroup of H.

34. Let $P \in Syl_p(G)$ and assume $N \trianglelefteq G$. Use the conjugacy part of Sylow's Theorem to prove that $P \cap N$ is a Sylow p-subgroup of N. Deduce that PN/N is a Sylow p-subgroup of G/N (note that this may also be done by the Second Isomorphism Theorem — cf. Exercise 9, Section 3.3).

35. Let $P \in Syl_p(G)$ and let $H \leq G$. Prove that $gPg^{-1} \cap H$ is a Sylow p-subgroup of H for some $g \in G$. Give an explicit example showing that $hPh^{-1} \cap H$ is not necessarily a Sylow p-subgroup of H for any $h \in H$ (in particular, we cannot always take $g = 1$ in the first part of this problem, as we could when H was normal in G).