a real number (since that would be circular). To do this we use a similar set of definitions to those used to define a Cauchy sequence in the first place.

Definition 5.2.1 (ε-close sequences). Let $(a_n)_{n=0}^\infty$ and $(b_n)_{n=0}^\infty$ be two sequences, and let $\varepsilon > 0$. We say that the sequence $(a_n)_{n=0}^\infty$ is *ε-close* to $(b_n)_{n=0}^\infty$ iff a_n is ε-close to b_n for each $n \in \mathbf{N}$. In other words, the sequence $a_0, a_1, a_2, \ldots$ is ε-close to the sequence $b_0, b_1, b_2, \ldots$ iff $|a_n - b_n| \leq \varepsilon$ for all $n = 0, 1, 2, \ldots$.

Example 5.2.2. The two sequences

$$1, -1, 1, -1, 1, \ldots$$

and

$$1.1, -1.1, 1.1, -1.1, 1.1, \ldots$$

are 0.1-close to each other. (Note however that neither of them are 0.1-steady).

Definition 5.2.3 (Eventually ε-close sequences). Let $(a_n)_{n=0}^\infty$ and $(b_n)_{n=0}^\infty$ be two sequences, and let $\varepsilon > 0$. We say that the sequence $(a_n)_{n=0}^\infty$ is *eventually ε-close* to $(b_n)_{n=0}^\infty$ iff there exists an $N \geq 0$ such that the sequences $(a_n)_{n=N}^\infty$ and $(b_n)_{n=N}^\infty$ are ε-close. In other words, $a_0, a_1, a_2, \ldots$ is eventually ε-close to $b_0, b_1, b_2, \ldots$ iff there exists an $N \geq 0$ such that $|a_n - b_n| \leq \varepsilon$ for all $n \geq N$.

Remark 5.2.4. Again, the notations for ε-close sequences and eventually ε-close sequences are not standard in the literature, and we will not use them outside of this section.

Example 5.2.5. The two sequences

$$1.1, 1.01, 1.001, 1.0001, \ldots$$

and

$$0.9, 0.99, 0.999, 0.9999, \ldots$$

are not 0.1-close (because the first elements of both sequences are not 0.1-close to each other). However, the sequences are still eventually 0.1-close, because if we start from the second elements onwards in the sequence, these sequences are 0.1-close. A similar argument shows that the two sequences are eventually 0.01-close (by starting from the third element onwards), and so forth.

Definition 5.2.6 (Equivalent sequences). Two sequences $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ are *equivalent* iff for each rational $\varepsilon > 0$, the sequences $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ are eventually ε-close. In other words, $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$ are equivalent iff for every rational $\varepsilon > 0$, there exists an $N \geq 0$ such that $|a_n - b_n| \leq \varepsilon$ for all $n \geq N$.

Remark 5.2.7. As with Definition 5.1.8, the quantity $\varepsilon > 0$ is currently restricted to be a positive rational, rather than a positive real. However, we shall eventually see that it makes no difference whether ε ranges over the positive rationals or positive reals; see Exercise 6.1.10.

From Definition 5.2.6 it seems that the two sequences given in Example 5.2.5 appear to be equivalent. We now prove this rigourously.

Proposition 5.2.8. *Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be the sequences $a_n = 1 + 10^{-n}$ and $b_n = 1 - 10^{-n}$. Then the sequences a_n, b_n are equivalent.*

Remark 5.2.9. This Proposition, in decimal notation, asserts that $1.0000\ldots = 0.9999\ldots$; see Proposition B.2.3.

Proof. We need to prove that for every $\varepsilon > 0$, the two sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are eventually ε-close to each other. So we fix an $\varepsilon > 0$. We need to find an $N > 0$ such that $(a_n)_{n=N}^{\infty}$ and $(b_n)_{n=N}^{\infty}$ are ε-close; in other words, we need to find an $N > 0$ such that

$$|a_n - b_n| \leq \varepsilon \text{ for all } n \geq N.$$

However, we have

$$|a_n - b_n| = |(1 + 10^{-n}) - (1 - 10^{-n})| = 2 \times 10^{-n}.$$

Since 10^{-n} is a decreasing function of n (i.e., $10^{-m} < 10^{-n}$ whenever $m > n$; this is easily proven by induction), and $n \geq N$, we have $2 \times 10^{-n} \leq 2 \times 10^{-N}$. Thus we have

$$|a_n - b_n| \leq 2 \times 10^{-N} \text{ for all } n \geq N.$$

Thus in order to obtain $|a_n - b_n| \leq \varepsilon$ for all $n \geq N$, it will be sufficient to choose N so that $2 \times 10^{-N} \leq \varepsilon$. This is easy to do using logarithms, but we have not yet developed logarithms yet, so we will use a cruder method. First, we observe 10^N is always greater than N for any $N \geq 1$ (see Exercise 4.3.5). Thus $10^{-N} \leq 1/N$, and so $2 \times 10^{-N} \leq 2/N$. Thus to get $2 \times 10^{-N} \leq \varepsilon$, it will suffice to choose N so that $2/N \leq \varepsilon$, or equivalently that $N \geq 2/\varepsilon$. But by Proposition 4.4.1 we can always choose such an N, and the claim follows. □

Exercise 5.2.1. Show that if $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent sequences of rationals, then $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence if and only if $(b_n)_{n=1}^{\infty}$ is a Cauchy sequence.

Exercise 5.2.2. Let $\varepsilon > 0$. Show that if $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are eventually ε-close, then $(a_n)_{n=1}^{\infty}$ is bounded if and only if $(b_n)_{n=1}^{\infty}$ is bounded.

5.3 The construction of the real numbers

We are now ready to construct the real numbers. We shall introduce a new formal symbol LIM, similar to the formal notations — and // defined earlier; as the notation suggests, this will eventually match the familiar operation of lim, at which point the formal limit symbol can be discarded.

Definition 5.3.1 (Real numbers). A *real number* is defined to be an object of the form $\mathrm{LIM}_{n\to\infty} a_n$, where $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence of rational numbers. Two real numbers $\mathrm{LIM}_{n\to\infty} a_n$ and $\mathrm{LIM}_{n\to\infty} b_n$ are said to be equal iff $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent Cauchy sequences. The set of all real numbers is denoted $\mathbf{R}$.

Example 5.3.2. (Informal) Let $a_1, a_2, a_3, \ldots$ denote the sequence

$$1.4, 1.41, 1.414, 1.4142, 1.41421, \ldots$$

and let $b_1, b_2, b_3, \ldots$ denote the sequence

$$1.5, 1.42, 1.415, 1.4143, 1.41422, \ldots$$

then $\mathrm{LIM}_{n\to\infty} a_n$ is a real number, and is the same real number as $\mathrm{LIM}_{n\to\infty} b_n$, because $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ are equivalent Cauchy sequences: $\mathrm{LIM}_{n\to\infty} a_n = \mathrm{LIM}_{n\to\infty} b_n$.

We will refer to $\mathrm{LIM}_{n\to\infty} a_n$ as the *formal limit* of the sequence $(a_n)_{n=1}^\infty$. Later on we will define a genuine notion of limit, and show that the formal limit of a Cauchy sequence is the same as the limit of that sequence; after that, we will not need formal limits ever again. (The situation is much like what we did with formal subtraction — and formal division //.)

In order to ensure that this definition is valid, we need to check that the notion of equality in the definition obeys the first three laws of equality:

Proposition 5.3.3 (Formal limits are well-defined). *Let $x = \mathrm{LIM}_{n\to\infty} a_n$, $y = \mathrm{LIM}_{n\to\infty} b_n$, and $z = \mathrm{LIM}_{n\to\infty} c_n$ be real numbers. Then, with the above definition of equality for real numbers, we have $x = x$. Also, if $x = y$, then $y = x$. Finally, if $x = y$ and $y = z$, then $x = z$.*

Proof. See Exercise 5.3.1. □

Because of this proposition, we know that our definition of equality between two real numbers is legitimate. Of course, when we define other operations on the reals, we have to check that they obey the law of substitution: two real number inputs which are equal should give equal outputs when applied to any operation on the real numbers.

Now we want to define on the real numbers all the usual arithmetic operations, such as addition and multiplication. We begin with addition.

Definition 5.3.4 (Addition of reals). Let $x = \mathrm{LIM}_{n\to\infty} a_n$ and $y = \mathrm{LIM}_{n\to\infty} b_n$ be real numbers. Then we define the sum $x+y$ to be $x + y := \mathrm{LIM}_{n\to\infty}(a_n + b_n)$.

Example 5.3.5. The sum of $\mathrm{LIM}_{n\to\infty} 1+1/n$ and $\mathrm{LIM}_{n\to\infty} 2+3/n$ is $\mathrm{LIM}_{n\to\infty} 3 + 4/n$.

We now check that this definition is valid. The first thing we need to do is to confirm that the sum of two real numbers is in fact a real number:

Lemma 5.3.6 (Sum of Cauchy sequences is Cauchy). *Let $x = \mathrm{LIM}_{n\to\infty} a_n$ and $y = \mathrm{LIM}_{n\to\infty} b_n$ be real numbers. Then $x + y$ is also a real number (i.e., $(a_n + b_n)_{n=1}^\infty$ is a Cauchy sequence of rationals).*

Proof. We need to show that for every $\varepsilon > 0$, the sequence $(a_n + b_n)_{n=1}^\infty$ is eventually ε-steady. Now from hypothesis we know that $(a_n)_{n=1}^\infty$ is eventually ε-steady, and $(b_n)_{n=1}^\infty$ is eventually ε-steady, but it turns out that this is not quite enough (this can be used to imply that $(a_n + b_n)_{n=1}^\infty$ is eventually 2ε-steady, but that's not what we want). So we need to do a little trick, which is to play with the value of ε.

We know that $(a_n)_{n=1}^\infty$ is eventually δ-steady for every value of δ. This implies not only that $(a_n)_{n=1}^\infty$ is eventually ε-steady, but it is also eventually $\varepsilon/2$-steady. Similarly, the sequence $(b_n)_{n=1}^\infty$ is also eventually $\varepsilon/2$-steady. This will turn out to be enough to conclude that $(a_n + b_n)_{n=1}^\infty$ is eventually ε-steady.

Since $(a_n)_{n=1}^\infty$ is eventually $\varepsilon/2$-steady, we know that there exists an $N \geq 1$ such that $(a_n)_{n=N}^\infty$ is $\varepsilon/2$-steady, i.e., a_n and a_m are $\varepsilon/2$-close for every $n, m \geq N$. Similarly there exists an $M \geq 1$ such that $(b_n)_{n=M}^\infty$ is $\varepsilon/2$-steady, i.e., b_n and b_m are $\varepsilon/2$-close for every $n, m \geq M$.

Let $\max(N, M)$ be the larger of N and M (we know from Proposition 2.2.13 that one has to be greater than or equal to the other). If $n, m \geq \max(N, M)$, then we know that a_n and a_m are $\varepsilon/2$-close, and b_n and b_m are $\varepsilon/2$-close, and so by Proposition 4.3.7 we see that $a_n + b_n$ and $a_m + b_m$ are ε-close for every

$n, m \geq \max(N, M)$. This implies that the sequence $(a_n + b_n)_{n=1}^{\infty}$ is eventually ε-close, as desired. $\square$

The other thing we need to check is the axiom of substitution (see Section A.7): if we replace a real number x by another number equal to x, this should not change the sum $x + y$ (and similarly if we substitute y by another number equal to y).

Lemma 5.3.7 (Sums of equivalent Cauchy sequences are equivalent). *Let $x = \mathrm{LIM}_{n\to\infty} a_n$, $y = \mathrm{LIM}_{n\to\infty} b_n$, and $x' = \mathrm{LIM}_{n\to\infty} a'_n$ be real numbers. Suppose that $x = x'$. Then we have $x+y = x'+y$.*

Proof. Since x and x' are equal, we know that the Cauchy sequences $(a_n)_{n=1}^{\infty}$ and $(a'_n)_{n=1}^{\infty}$ are equivalent, so in other words they are eventually ε-close for each $\varepsilon > 0$. We need to show that the sequences $(a_n + b_n)_{n=1}^{\infty}$ and $(a'_n + b_n)_{n=1}^{\infty}$ are eventually ε-close for each $\varepsilon > 0$. But we already know that there is an $N \geq 1$ such that $(a_n)_{n=N}^{\infty}$ and $(a'_n)_{n=N}^{\infty}$ are ε-close, i.e., that a_n and a'_n are ε-close for each $n \geq N$. Since b_n is of course 0-close to b_n, we thus see from Proposition 4.3.7 that $a_n + b_n$ and $a'_n + b_n$ are ε-close for each $n \geq N$. This implies that $(a_n + b_n)_{n=1}^{\infty}$ and $(a'_n + b_n)_{n=1}^{\infty}$ are eventually ε-close for each $\varepsilon > 0$, and we are done. $\square$

Remark 5.3.8. The above lemma verifies the axiom of substitution for the "x" variable in $x + y$, but one can similarly prove the axiom of substitution for the "y" variable. (A quick way is to observe from the definition of $x + y$ that we certainly have $x + y = y + x$, since $a_n + b_n = b_n + a_n$.)

We can define multiplication of real numbers in a manner similar to that of addition:

Definition 5.3.9 (Multiplication of reals). Let $x = \mathrm{LIM}_{n\to\infty} a_n$ and $y = \mathrm{LIM}_{n\to\infty} b_n$ be real numbers. Then we define the product xy to be $xy := \mathrm{LIM}_{n\to\infty} a_n b_n$.

The following Proposition ensures that this definition is valid, and that the product of two real numbers is in fact a real number:

Proposition 5.3.10 (Multiplication is well defined). *Let $x = \mathrm{LIM}_{n\to\infty}a_n$, $y = \mathrm{LIM}_{n\to\infty}b_n$, and $x' = \mathrm{LIM}_{n\to\infty}a'_n$ be real numbers. Then xy is also a real number. Furthermore, if $x = x'$, then $xy = x'y$.*

Proof. See Exercise 5.3.2. □

Of course we can prove a similar substitution rule when y is replaced by a real number y' which is equal to y.

At this point we embed the rationals back into the reals, by equating every rational number q with the real number $\mathrm{LIM}_{n\to\infty}q$. For instance, if $a_1, a_2, a_3, \ldots$ is the sequence

$$0.5, 0.5, 0.5, 0.5, 0.5, \ldots$$

then we set $\lim_{n\to\infty} a_n$ equal to 0.5. This embedding is consistent with our definitions of addition and multiplication, since for any rational numbers a, b we have

$$(\mathrm{LIM}_{n\to\infty}a) + (\mathrm{LIM}_{n\to\infty}b) = \mathrm{LIM}_{n\to\infty}(a+b) \text{ and}$$
$$(\mathrm{LIM}_{n\to\infty}a) \times (\mathrm{LIM}_{n\to\infty}b) = \mathrm{LIM}_{n\to\infty}(ab);$$

this means that when one wants to add or multiply two rational numbers a, b it does not matter whether one thinks of these numbers as rationals or as the real numbers $\mathrm{LIM}_{n\to\infty}a$, $\mathrm{LIM}_{n\to\infty}b$. Also, this identification of rational numbers and real numbers is consistent with our definitions of equality (Exercise 5.3.3).

We can now easily define negation $-x$ for real numbers x by the formula

$$-x := (-1) \times x,$$

since -1 is a rational number and is hence real. Note that this is clearly consistent with our negation for rational numbers since we have $-q = (-1) \times q$ for all rational numbers q. Also, from our definitions it is clear that

$$-\mathrm{LIM}_{n\to\infty}a_n = \mathrm{LIM}_{n\to\infty}(-a_n)$$

(why?). Once we have addition and negation, we can define subtraction as usual by

$$x - y := x + (-y),$$

note that this implies

$$\mathrm{LIM}_{n\to\infty} a_n - \mathrm{LIM}_{n\to\infty} b_n = \mathrm{LIM}_{n\to\infty}(a_n - b_n).$$

We can now easily show that the real numbers obey all the usual rules of algebra (except perhaps for the laws involving division, which we shall address shortly):

Proposition 5.3.11. *All the laws of algebra from Proposition 4.1.6 hold not only for the integers, but for the reals as well.*

Proof. We illustrate this with one such rule: $x(y+z) = xy + xz$. Let $x = \mathrm{LIM}_{n\to\infty} a_n$, $y = \mathrm{LIM}_{n\to\infty} b_n$, and $z = \mathrm{LIM}_{n\to\infty} c_n$ be real numbers. Then by definition, $xy = \mathrm{LIM}_{n\to\infty} a_n b_n$ and $xz = \mathrm{LIM}_{n\to\infty} a_n c_n$, and so $xy + xz = \mathrm{LIM}_{n\to\infty}(a_n b_n + a_n c_n)$. A similar line of reasoning shows that $x(y+z) = \mathrm{LIM}_{n\to\infty} a_n(b_n + c_n)$. But we already know that $a_n(b_n + c_n)$ is equal to $a_n b_n + a_n c_n$ for the rational numbers a_n, b_n, c_n, and the claim follows. The other laws of algebra are proven similarly. □

The last basic arithmetic operation we need to define is reciprocation: $x \to x^{-1}$. This one is a little more subtle. On obvious first guess for how to proceed would be define

$$(\mathrm{LIM}_{n\to\infty} a_n)^{-1} := \mathrm{LIM}_{n\to\infty} a_n^{-1},$$

but there are a few problems with this. For instance, let a_1, a_2, $a_3, \ldots$ be the Cauchy sequence

$$0.1, 0.01, 0.001, 0.0001, \ldots,$$

and let $x := \mathrm{LIM}_{n\to\infty} a_n$. Then by this definition, x^{-1} would be $\mathrm{LIM}_{n\to\infty} b_n$, where $b_1, b_2, b_3, \ldots$ is the sequence

$$10, 100, 1000, 10000, \ldots$$

but this is not a Cauchy sequence (it isn't even bounded). Of course, the problem here is that our original Cauchy sequence $(a_n)_{n=1}^\infty$ was equivalent to the zero sequence $(0)_{n=1}^\infty$ (why?), and hence that our real number x was in fact equal to 0. So we should only allow the operation of reciprocal when x is non-zero.

However, even when we restrict ourselves to non-zero real numbers, we have a slight problem, because a non-zero real number might be the formal limit of a Cauchy sequence which contains zero elements. For instance, the number 1, which is rational and hence real, is the formal limit $1 = \mathrm{LIM}_{n\to\infty} a_n$ of the Cauchy sequence

$$0, 0.9, 0.99, 0.999, 0.9999, \ldots$$

but using our naive definition of reciprocal, we cannot invert the real number 1, because we can't invert the first element 0 of this Cauchy sequence!

To get around these problems we need to keep our Cauchy sequence away from zero. To do this we first need a definition.

Definition 5.3.12 (Sequences bounded away from zero). A sequence $(a_n)_{n=1}^\infty$ of rational numbers is said to be *bounded away from zero* iff there exists a rational number $c > 0$ such that $|a_n| \geq c$ for all $n \geq 1$.

Examples 5.3.13. The sequence $1, -1, 1, -1, 1, -1, 1, \ldots$ is bounded away from zero (all the coefficients have absolute value at least 1). But the sequence $0.1, 0.01, 0.001, \ldots$ is not bounded away from zero, and neither is $0, 0.9, 0.99, 0.999, 0.9999, \ldots$. The sequence $10, 100, 1000, \ldots$ is bounded away from zero, but is not bounded.

We now show that every non-zero real number is the formal limit of a Cauchy sequence bounded away from zero:

Lemma 5.3.14. *Let x be a non-zero real number. Then $x = \mathrm{LIM}_{n\to\infty} a_n$ for some Cauchy sequence $(a_n)_{n=1}^\infty$ which is bounded away from zero.*

Proof. Since x is real, we know that $x = \mathrm{LIM}_{n\to\infty} b_n$ for some Cauchy sequence $(b_n)_{n=1}^\infty$. But we are not yet done, because we do not know that b_n is bounded away from zero. On the other hand, we are given that $x \neq 0 = \mathrm{LIM}_{n\to\infty} 0$, which means that the sequence $(b_n)_{n=1}^\infty$ is **not** equivalent to $(0)_{n=1}^\infty$. Thus the sequence $(b_n)_{n=1}^\infty$ cannot be eventually ε-close to $(0)_{n=1}^\infty$ for *every* $\varepsilon > 0$. Therefore we can find an $\varepsilon > 0$ such that $(b_n)_{n=1}^\infty$ is **not** eventually ε-close to $(0)_{n=1}^\infty$.

Let us fix this ε. We know that $(b_n)_{n=1}^\infty$ is a Cauchy sequence, so it is eventually ε-steady. Moreover, it is eventually $\varepsilon/2$-steady, since $\varepsilon/2 > 0$. Thus there is an $N \geq 1$ such that $|b_n - b_m| \leq \varepsilon/2$ for all $n, m \geq N$.

On the other hand, we cannot have $|b_n| \leq \varepsilon$ for all $n \geq N$, since this would imply that $(b_n)_{n=1}^\infty$ is eventually ε-close to $(0)_{n=1}^\infty$. Thus there must be some $n_0 \geq N$ for which $|b_{n_0}| > \varepsilon$. Since we already know that $|b_{n_0} - b_n| \leq \varepsilon/2$ for all $n \geq N$, we thus conclude from the triangle inequality (how?) that $|b_n| \geq \varepsilon/2$ for all $n \geq N$.

This almost proves that $(b_n)_{n=1}^\infty$ is bounded away from zero. Actually, what it does is show that $(b_n)_{n=1}^\infty$ is *eventually* bounded away from zero. But this is easily fixed, by defining a new sequence a_n, by setting $a_n := \varepsilon/2$ if $n < N$ and $a_n := b_n$ if $n \geq N$. Since b_n is a Cauchy sequence, it is not hard to verify that a_n is also a Cauchy sequence which is equivalent to b_n (because the two sequences are eventually the same), and so $x = \mathrm{LIM}_{n\to\infty} a_n$. And since $|b_n| \geq \varepsilon/2$ for all $n \geq N$, we know that $|a_n| \geq \varepsilon/2$ for all $n \geq 1$ (splitting into the two cases $n \geq N$ and $n < N$ separately). Thus we have a Cauchy sequence which is bounded away from zero (by $\varepsilon/2$ instead of ε, but that's still OK since $\varepsilon/2 > 0$), and which has x as a formal limit, and so we are done. □

Once a sequence is bounded away from zero, we can take its reciprocal without any difficulty:

Lemma 5.3.15. *Suppose that $(a_n)_{n=1}^\infty$ is a Cauchy sequence which is bounded away from zero. Then the sequence $(a_n^{-1})_{n=1}^\infty$ is also a Cauchy sequence.*

Proof. Since $(a_n)_{n=1}^\infty$ is bounded away from zero, we know that there is a $c > 0$ such that $|a_n| \geq c$ for all $n \geq 1$. Now we need to show that $(a_n^{-1})_{n=1}^\infty$ is eventually ε-steady for each $\varepsilon > 0$. Thus let us fix an $\varepsilon > 0$; our task is now to find an $N \geq 1$ such that $|a_n^{-1} - a_m^{-1}| \leq \varepsilon$ for all $n, m \geq N$. But

$$|a_n^{-1} - a_m^{-1}| = |\frac{a_m - a_n}{a_m a_n}| \leq \frac{|a_m - a_n|}{c^2}$$

(since $|a_m|, |a_n| \geq c$), and so to make $|a_n^{-1} - a_m^{-1}|$ less than or equal to ε, it will suffice to make $|a_m - a_n|$ less than or equal to $c^2\varepsilon$. But since $(a_n)_{n=1}^\infty$ is a Cauchy sequence, and $c^2\varepsilon > 0$, we can certainly find an N such that the sequence $(a_n)_{n=N}^\infty$ is $c^2\varepsilon$-steady, i.e., $|a_m - a_n| \leq c^2\varepsilon$ for all $n \geq N$. By what we have said above, this shows that $|a_n - a_m| \leq \varepsilon$ for all $m, n \geq N$, and hence the sequence $(a_n^{-1})_{n=1}^\infty$ is eventually ε-steady. Since we have proven this for every ε, we have that $(a_n^{-1})_{n=1}^\infty$ is a Cauchy sequence, as desired. □

We are now ready to define reciprocation:

Definition 5.3.16 (Reciprocals of real numbers). Let x be a non-zero real number. Let $(a_n)_{n=1}^\infty$ be a Cauchy sequence bounded away from zero such that $x = \mathrm{LIM}_{n\to\infty} a_n$ (such a sequence exists by Lemma 5.3.14). Then we define the reciprocal x^{-1} by the formula $x^{-1} := \mathrm{LIM}_{n\to\infty} a_n^{-1}$. (From Lemma 5.3.15 we know that x^{-1} is a real number.)

We need to check one thing before we are sure this definition makes sense: what if there are two different Cauchy sequences $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ which have x as their formal limit, $x = \mathrm{LIM}_{n\to\infty} a_n = \mathrm{LIM}_{n\to\infty} b_n$. The above definition might conceivably give *two* different reciprocals x^{-1}, namely $\mathrm{LIM}_{n\to\infty} a_n^{-1}$ and $\mathrm{LIM}_{n\to\infty} b_n^{-1}$. Fortunately, this never happens:

Lemma 5.3.17 (Reciprocation is well defined). *Let $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ be two Cauchy sequences bounded away from zero such that $\mathrm{LIM}_{n\to\infty} a_n = \mathrm{LIM}_{n\to\infty} b_n$ (i.e., the two sequences are equivalent). Then $\mathrm{LIM}_{n\to\infty} a_n^{-1} = \mathrm{LIM}_{n\to\infty} b_n^{-1}$.*

Proof. Consider the following product P of three real numbers:

$$P := (\mathrm{LIM}_{n\to\infty}a_n^{-1}) \times (\mathrm{LIM}_{n\to\infty}a_n) \times (\mathrm{LIM}_{n\to\infty}b_n^{-1}).$$

If we multiply this out, we obtain

$$P = \mathrm{LIM}_{n\to\infty}a_n^{-1}a_nb_n^{-1} = \mathrm{LIM}_{n\to\infty}b_n^{-1}.$$

On the other hand, since $\mathrm{LIM}_{n\to\infty}a_n = \mathrm{LIM}_{n\to\infty}b_n$, we can write P in another way as

$$P = (\mathrm{LIM}_{n\to\infty}a_n^{-1}) \times (\mathrm{LIM}_{n\to\infty}b_n) \times (\mathrm{LIM}_{n\to\infty}b_n^{-1})$$

(cf. Proposition 5.3.10). Multiplying things out again, we get

$$P = \mathrm{LIM}_{n\to\infty}a_n^{-1}b_nb_n^{-1} = \mathrm{LIM}_{n\to\infty}a_n^{-1}.$$

Comparing our different formulae for P we see that $\mathrm{LIM}_{n\to\infty}a_n^{-1} = \mathrm{LIM}_{n\to\infty}b_n^{-1}$, as desired. □

Thus reciprocal is well-defined (for each non-zero real number x, we have exactly one definition of the reciprocal x^{-1}). Note it is clear from the definition that $xx^{-1} = x^{-1}x = 1$ (why?); thus all the field axioms (Proposition 4.2.4) apply to the reals as well as to the rationals. We of course cannot give 0 a reciprocal, since 0 multiplied by anything gives 0, not 1. Also note that if q is a non-zero rational, and hence equal to the real number $\mathrm{LIM}_{n\to\infty}q$, then the reciprocal of $\mathrm{LIM}_{n\to\infty}q$ is $\mathrm{LIM}_{n\to\infty}q^{-1} = q^{-1}$; thus the operation of reciprocal on real numbers is consistent with the operation of reciprocal on rational numbers.

Once one has reciprocal, one can define division x/y of two real numbers x, y, provided y is non-zero, by the formula

$$x/y := x \times y^{-1},$$

just as we did with the rationals. In particular, we have the *cancellation law*: if x, y, z are real numbers such that $xz = yz$, and z is non-zero, then by dividing by z we conclude that $x = y$. Note that this cancellation law does not work when z is zero.

We now have all four of the basic arithmetic operations on the reals: addition, subtraction, multiplication, and division, with all the usual rules of algebra. Next we turn to the notion of order on the reals.

Exercise 5.3.1. Prove Proposition 5.3.3. (Hint: you may find Proposition 4.3.7 to be useful.)

Exercise 5.3.2. Prove Proposition 5.3.10. (Hint: again, Proposition 4.3.7 may be useful.)

Exercise 5.3.3. Let a, b be rational numbers. Show that $a = b$ if and only if $\mathrm{LIM}_{n\to\infty} a = \mathrm{LIM}_{n\to\infty} b$ (i.e., the Cauchy sequences $a, a, a, a, \ldots$ and $b, b, b, b\ldots$ equivalent if and only if $a = b$). This allows us to embed the rational numbers inside the real numbers in a well-defined manner.

Exercise 5.3.4. Let $(a_n)_{n=0}^\infty$ be a sequence of rational numbers which is bounded. Let $(b_n)_{n=0}^\infty$ be another sequence of rational numbers which is equivalent to $(a_n)_{n=0}^\infty$. Show that $(b_n)_{n=0}^\infty$ is also bounded.

Exercise 5.3.5. Show that $\mathrm{LIM}_{n\to\infty} 1/n = 0$.

5.4 Ordering the reals

We know that every rational number is positive, negative, or zero. We now want to say the same thing for the reals: each real number should be positive, negative, or zero. Since a real number x is just a formal limit of rationals a_n, it is tempting to make the following definition: a real number $x = \mathrm{LIM}_{n\to\infty} a_n$ is positive if all of the a_n are positive, and negative if all of the a_n are negative (and zero if all of the a_n are zero). However, one soon realizes some problems with this definition. For instance, the sequence $(a_n)_{n=1}^\infty$ defined by $a_n := 10^{-n}$, thus

$$0.1, 0.01, 0.001, 0.0001, \ldots$$

consists entirely of positive numbers, but this sequence is equivalent to the zero sequence $0, 0, 0, 0, \ldots$ and thus $\mathrm{LIM}_{n\to\infty} a_n = 0$. Thus even though all the rationals were positive, the real formal limit of these rationals was zero rather than positive. Another example is

$$0.1, -0.01, 0.001, -0.0001, \ldots;$$

this sequence is a hybrid of positive and negative numbers, but again the formal limit is zero.

The trick, as with the reciprocals in the previous section, is to limit one's attention to sequences which are bounded away from zero.

Definition 5.4.1. Let $(a_n)_{n=1}^{\infty}$ be a sequence of rationals. We say that this sequence is *positively bounded away from zero* iff we have a positive rational $c > 0$ such that $a_n \geq c$ for all $n \geq 1$ (in particular, the sequence is entirely positive). The sequence is *negatively bounded away from zero* iff we have a negative rational $-c < 0$ such that $a_n \leq -c$ for all $n \geq 1$ (in particular, the sequence is entirely negative).

Examples 5.4.2. The sequence $1.1, 1.01, 1.001, 1.0001, \ldots$ is positively bounded away from zero (all terms are greater than or equal to 1). The sequence $-1.1, -1.01, -1.001, -1.0001, \ldots$ is negatively bounded away from zero. The sequence $1, -1, 1, -1, 1, -1, \ldots$ is bounded away from zero, but is neither positively bounded away from zero nor negatively bounded away from zero.

It is clear that any sequence which is positively or negatively bounded away from zero, is bounded away from zero. Also, a sequence cannot be both positively bounded away from zero and negatively bounded away from zero at the same time.

Definition 5.4.3. A real number x is said to be *positive* iff it can be written as $x = \mathrm{LIM}_{n\to\infty} a_n$ for some Cauchy sequence $(a_n)_{n=1}^{\infty}$ which is positively bounded away from zero. x is said to be *negative* iff it can be written as $x = \mathrm{LIM}_{n\to\infty} a_n$ for some sequence $(a_n)_{n=1}^{\infty}$ which is negatively bounded away from zero.

Proposition 5.4.4 (Basic properties of positive reals). *For every real number x, exactly one of the following three statements is true: (a) x is zero; (b) x is positive; (c) x is negative. A real number x is negative if and only if $-x$ is positive. If x and y are positive, then so are $x + y$ and xy.*

Proof. See Exercise 5.4.1. □