Tensor Products and the Groups $Tor_n^R(A,B)$

The cohomology groups $\text{Ext}_R^n(A, B)$ determine what happens to short exact sequences on the right after applying the left exact functors $\text{Hom}_R(D, \_\_)$ and $\text{Hom}_R(\_\_, D)$. One may similarly ask for the behavior of short exact sequences on the left after applying the right exact functor $D \otimes_R \_\_$ or the right exact functor $\_\_ \otimes_R D$. This leads to the Tor (homology) groups (whose name derives from their relation to torsion submodules), and we now briefly outline the development of these left derived functors. In some respects this theory is "dual" to the theory for Ext_R. We concentrate on the situation for $D \otimes_R \_\_$ when D is a right R-module. When D is a left R-module there is a completely symmetric theory for $\_\_ \otimes_R D$; when R is commutative and all R-modules have the same left and right R action the homology groups resulting from both developments are isomorphic.

Suppose then that D is a right R-module. Then for every left R-module B the tensor product $D \otimes_R B$ is an abelian group and the functor $D \otimes \_\_$ is covariant and right exact, i.e., for any short exact sequence (1) of left R-modules,

$$D \otimes L \longrightarrow D \otimes M \longrightarrow D \otimes N \longrightarrow 0$$

is an exact sequence of abelian groups. This sequence may be extended at the left end to a long exact sequence as follows. Let

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \longrightarrow \cdots \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} B \longrightarrow 0$$

be a projective resolution of B, and take tensor products with D to obtain

$$\cdots \longrightarrow D \otimes P_n \xrightarrow{1 \otimes d_n} D \otimes P_{n-1} \longrightarrow \cdots \xrightarrow{1 \otimes d_1} D \otimes P_0 \xrightarrow{1 \otimes \epsilon} D \otimes B \longrightarrow 0. \quad (17.15)$$

It follows from the argument in Theorem 39 of Section 10.5 that (15) is a chain complex — the composition of any two successive maps is zero — so we may form its homology groups.

Definition. Let D be a right R-module and let B be a left R-module. For any projective resolution of B by left R-modules as above let $1 \otimes d_n : D \otimes P_n \to D \otimes P_{n-1}$ for all $n \geq 1$ as in (15). Then

$$\text{Tor}_n^R(D, B) = \ker(1 \otimes d_n) / \text{image}(1 \otimes d_{n+1})$$

where $\text{Tor}_0^R(D, B) = (D \otimes P_0) / \text{image}(1 \otimes d_1)$. The group $\text{Tor}_n^R(D, B)$ is called the n^{th} *homology group derived from the functor* $D \otimes \_\_$. When $R = \mathbb{Z}$ the group $\text{Tor}_n^{\mathbb{Z}}(D, B)$ is also denoted simply $\text{Tor}_n(D, B)$.

Thus $\text{Tor}_n^R(D, B)$ is the n^{th} homology group of the chain complex obtained from (15) by removing the term $D \otimes B$.

A completely analogous proof to Proposition 3 (but relying on Theorem 39 in Section 10.5) implies the following:

Proposition 13. For any left R-module B we have $\operatorname{Tor}_0^R(D, B) \cong D \otimes B$.

Example

Let $R = \mathbb{Z}$ and let $B = \mathbb{Z}/m\mathbb{Z}$ for some $m \geq 2$. By the proposition, $\operatorname{Tor}_0^{\mathbb{Z}}(D, \mathbb{Z}/m\mathbb{Z})$ is isomorphic to $D \otimes \mathbb{Z}/m\mathbb{Z}$, so we have $\operatorname{Tor}_0^{\mathbb{Z}}(D, \mathbb{Z}/m\mathbb{Z}) \cong D/mD$ (Example 8 following Corollary 12 in Section 10.4). For the higher groups we apply $D \otimes \_\_$ to the projective resolution

$$0 \longrightarrow \mathbb{Z} \xrightarrow{m} \mathbb{Z} \longrightarrow \mathbb{Z}/m\mathbb{Z} \longrightarrow 0$$

of B and use the isomorphisms $D \otimes \mathbb{Z} \cong D$ and $D \otimes \mathbb{Z}/m\mathbb{Z} \cong D/mD$. This gives the chain complex

$$\cdots \longrightarrow 0 \longrightarrow D \xrightarrow{m} D \longrightarrow D/mD \longrightarrow 0.$$

It follows that $\operatorname{Tor}_1^{\mathbb{Z}}(D, \mathbb{Z}/m\mathbb{Z}) \cong {}_mD$ is the subgroup of D annihilated by m and that $\operatorname{Tor}_n^{\mathbb{Z}}(D, \mathbb{Z}/m\mathbb{Z}) = 0$ for all $n \geq 2$, which we summarize as

$$\begin{aligned} &\operatorname{Tor}_0(D, \mathbb{Z}/m\mathbb{Z}) \cong D/mD, \\ &\operatorname{Tor}_1(D, \mathbb{Z}/m\mathbb{Z}) \cong {}_mD, \\ &\operatorname{Tor}_n(D, \mathbb{Z}/m\mathbb{Z}) = 0, \quad \text{for all } n \geq 2. \end{aligned}$$

As for Ext, the Tor groups depend on the ring R (cf. Exercise 20).

Following a similar development to that for Ext_R, one shows:

Proposition 14.

(1) The homology groups $\operatorname{Tor}_n^R(D, B)$ are independent of the choice of projective resolution of B, and

(2) for every R-module homomorphism $f : B \to B'$ there are induced maps $\psi_n : \operatorname{Tor}_n^R(D, B) \to \operatorname{Tor}_n^R(D, B')$ on homology groups (depending only on f).

There is a Long Exact Sequence in Homology analogous to Theorem 2, except that all the arrows are reversed, whose proof follows mutatis mutandis from the argument for cohomology. This together with Simultaneous Resolution gives:

Theorem 15. Let $0 \to L \to M \to N \to 0$ be a short exact sequence of left R-modules. Then there is a long exact sequence of abelian groups

$$\begin{aligned} \cdots \to \operatorname{Tor}_2^R(D, N) \xrightarrow{\delta_1} \operatorname{Tor}_1^R(D, L) \to \operatorname{Tor}_1^R(D, M) \to \\ \operatorname{Tor}_1^R(D, N) \xrightarrow{\delta_0} D \otimes L \to D \otimes M \to D \otimes N \to 0 \end{aligned}$$

where the maps between groups at the same level n are as in Proposition 14 (and the maps δ_n are called connecting homomorphisms).

There is a characterization of flat modules corresponding to Propositions 9 and 11 whose proof is very similar and is left as an exercise.

Proposition 16. For a right R-module D the following are equivalent:

(1) D is a flat R-module,

(2) $\mathrm{Tor}_1^R(D, B) = 0$ for all left R-modules B, and

(3) $\mathrm{Tor}_n^R(D, B) = 0$ for all left R-modules B and all $n \geq 1$.

We have defined $\mathrm{Tor}_n^R(A, B)$ as the homology of the chain complex obtained by tensoring a projective resolution of B on the left with A. The same groups are obtained by taking the homology of the chain complex obtained by tensoring a projective resolution of A on the right by B. Put another way, the $\mathrm{Tor}_n^R(A, B)$ groups define the (covariant) left derived functors for both of the right exact functors $A \otimes_R \_\_$ and $\_\_ \otimes_R B$: if D is a left R-module, then the short exact sequence $0 \to L \to M \to N \to 0$ of right R-modules gives rise to the long exact sequence

$$\cdots \to \mathrm{Tor}_2^R(N, D) \xrightarrow{\gamma_1} \mathrm{Tor}_1^R(L, D) \to \mathrm{Tor}_1^R(M, D) \to$$
$$\mathrm{Tor}_1^R(N, D) \xrightarrow{\gamma_0} L \otimes_R D \to M \otimes_R D \to N \otimes_R D \to 0$$

of abelian groups. In particular, the left R-module D is flat if and only if $\mathrm{Tor}_1^R(A, D) = 0$ for all right R-modules A.

When R is commutative, $A \otimes_R B \cong B \otimes_R A$ (Proposition 20 in Section 10.4) for any two R-modules A and B with the standard R-module structures, and it follows that $\mathrm{Tor}_n^R(A, B) \cong \mathrm{Tor}_n^R(B, A)$ as R-modules. When R is commutative the Tor long exact sequences are exact sequences of R-modules.

Examples

(1) If $R = \mathbb{Z}$, then since $\mathbb{Z}^m$ is free, hence flat (Corollary 42, Section 10.5), we have $\mathrm{Tor}_n(A, \mathbb{Z}^m) = 0$ for all $n \geq 1$ and all abelian groups A.

(2) Since $\mathrm{Tor}_n^R(A, B_1 \oplus B_2) \cong \mathrm{Tor}_n^R(A. B_1) \oplus \mathrm{Tor}_n^R(A, B_2)$ (cf. Exercise 10), the previous two examples together determine $\mathrm{Tor}_n^R(A, B)$ for all abelian groups A and all finitely generated abelian groups B.

(3) As a particular case of the previous example, $\mathrm{Tor}_1(A, B)$ is a torsion group and $\mathrm{Tor}_n(A, B) = 0$ for every abelian group A, every finitely generated abelian group B, and all $n \geq 2$. In fact these results hold without the condition that B be finitely generated.

(4) The exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ gives the long exact sequence

$$\cdots \to \mathrm{Tor}_1(D, \mathbb{Q}) \to \mathrm{Tor}_1(D, \mathbb{Q}/\mathbb{Z}) \to D \otimes \mathbb{Z} \to D \otimes \mathbb{Q} \to D \otimes \mathbb{Q}/\mathbb{Z} \to 0.$$

Since $\mathbb{Q}$ is a flat $\mathbb{Z}$-module (Example 2 following Corollary 42 in Section 10.5), the proposition shows that we have an exact sequence

$$0 \longrightarrow \mathrm{Tor}_1(D, \mathbb{Q}/\mathbb{Z}) \longrightarrow D \longrightarrow D \otimes \mathbb{Q}$$

and so $\mathrm{Tor}_1(D, \mathbb{Q}/\mathbb{Z})$ is isomorphic to the kernel of the natural map from D into $D \otimes \mathbb{Q}$, which is the torsion subgroup of D (cf. Exercise 9 in Section 10.4).

The following results show that, for $R = \mathbb{Z}$, the Tor groups are closely related to torsion subgroups. The Tor groups first arose in applications of torsion abelian groups in topological settings, which helps explain the terminology.

Proposition 17. Let A and B be $\mathbb{Z}$-modules and let $t(A)$ and $t(B)$ denote their respective torsion submodules. Then $\mathrm{Tor}_1(A, B) \cong \mathrm{Tor}_1(t(A), t(B))$.

Proof: In the case where A and B are finitely generated abelian groups this follows by Examples 3 and 4 above. For the general case, cf. Exercise 16.

Corollary 18. If A is an abelian group then A is torsion free if and only if $\mathrm{Tor}_1(A, B) = 0$ for every abelian group B (in which case A is flat as a $\mathbb{Z}$-module).

Proof: By the proposition, if A has no elements of finite order then we have $\mathrm{Tor}_1(A, B) = \mathrm{Tor}_1(t(A), B) = \mathrm{Tor}_1(0, B) = 0$ for every abelian group B. Conversely, if $\mathrm{Tor}_1(A, B) = 0$ for all B, then in particular $\mathrm{Tor}_1(A, \mathbb{Q}/\mathbb{Z}) = 0$, and this group is isomorphic to the torsion subgroup of A by the example above.

The results of Proposition 17 and Corollary 18 hold for any P.I.D. R in place of $\mathbb{Z}$ (cf. Exercise 26 in Section 10.5 and Exercise 16).

Finally, we mention that the cohomology and homology theories we have described may be developed in a vastly more general setting by axiomatizing the essential properties of R-modules and the Hom_R and tensor product functors. This leads to the general notions of *abelian categories* and *additive functors*. In the case of the abelian category of R-modules, any additive functor $\mathcal{F}$ to the category of abelian groups gives rise to a set of *derived functors*, $\mathcal{F}_n$, also from R-modules to abelian groups, for all $n \geq 0$. Then for each short exact sequence $0 \to L \to M \to N \to 0$ of R-modules there is a long exact sequence of (cohomology or homology) groups whose terms are $\mathcal{F}_n(L)$, $\mathcal{F}_n(M)$ and $\mathcal{F}_n(N)$, and these long exact sequences reflect the exactness properties of the functor $\mathcal{F}$. If $\mathcal{F}$ is left or right exact then the 0^{th} derived functor $\mathcal{F}_0$ is naturally equivalent to $\mathcal{F}$ (hence the 0^{th} degree groups $\mathcal{F}_0(X)$ are isomorphic to $\mathcal{F}(X)$), and if $\mathcal{F}$ is an exact functor then $\mathcal{F}_n(X) = 0$ for all $n \geq 1$ and all R-modules X.

EXERCISES

1. Give the details of the proof of Proposition 1.
2. This exercise defines the connecting map δ_n in the Long Exact Sequence of Theorem 2 and proves it is a homomorphism. In the notation of Theorem 2 let $0 \to \mathcal{A} \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\beta} \mathcal{C} \to 0$ be a short exact sequence of cochain complexes, where for simplicity the cochain maps for $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are all denoted by the same d.
   (a) If $c \in C^n$ represents the class $x \in H^n(\mathcal{C})$ show that there is some $b \in B^n$ with $\beta_n(b) = c$.
   (b) Show that $d_{n+1}(b) \in \ker \beta_{n+1}$ and conclude that there is a unique $a \in A^{n+1}$ such that $\alpha_{n+1}(a) = d_{n+1}(b)$. [Use $c \in \ker d_{n+1}$ and the commutativity of the diagram.]
   (c) Show that $d_{n+2}(a) = 0$ and conclude that a defines a class $\overline{a}$ in the quotient group $H^{n+1}(\mathcal{A})$. [Use the fact that α_{n+2} is injective.]
   (d) Prove that $\overline{a}$ is independent of the choice of b, i.e., if b' is another choice and a' is its unique preimage in A^{n+1} then $\overline{a} = \overline{a'}$, and that $\overline{a}$ is also independent of the choice of c representing the class x.
   (e) Define $\delta_n(x) = \overline{a}$ and prove that δ_n is a group homomorphism from $H^n(\mathcal{C})$ to $H^{n+1}(\mathcal{A})$. [Use the fact that $\delta_n(x)$ is independent of the choices of c and b to compute $\delta_n(x_1 + x_2)$.]