and that, in general,

$$x^n = \sum_{k=1}^{n} \frac{\Delta^k f(0)}{k!} x^{(k)},$$

where $f(x) = x^n$ and $\Delta f(x) = f(x+1) - f(x)$. The numbers $T_{k,n} = \Delta^k f(0)/k!$ are called ***Stirling numbers of the second kind.***

(b) Prove that

$$\Delta^k x^{n+1} = (x + k)\, \Delta^k x^n + k\, \Delta^{k-1} x^n$$

and use this to deduce that $T_{k,n+1} = T_{k-1,n} + kT_{k,n}$.

(c) Use the recursion formula in part (b) to verify the entries in Table 15.3, a table of Stirling numbers of the second kind, and construct the next three rows of the table.

TABLE 15.3

n	$T_{1,n}$	$T_{2,n}$	$T_{3,n}$	$T_{4,n}$	$T_{5,n}$	$T_{6,n}$	$T_{7,n}$
1	1						
2	1	1					
3	1	3	1				
4	1	7	6	1			
5	1	15	25	10	1		
6	1	31	90	65	15	1	
7	1	63	301	350	140	21	1

(d) Express the polynomial $x^4 + 3x^3 + 2x - 1$ as a linear combination of factorial polynomials.

6. (a) If p is a positive integer and if a and b are integers with $a < b$, prove that

* différences finies

$$\sum_{k=a}^{b-1} k^{(p)} = \frac{b^{(p+1)} - a^{(p+1)}}{p+1}.$$

This formula is **analogous** to the integration formula for $\int_a^b x^p\, dx$. It should be noted, however, that the upper limit in the sum is $b - 1$, not b.

(b) Verify that $k(k+3) = 4k^{(1)} + k^{(2)}$. The use part (a) to show that

$$\sum_{k=1}^{n} k(k+3) = 4\frac{(n+1)^{(2)}}{2} + \frac{(n+1)^{(3)}}{3} = \frac{n(n+1)(n+5)}{3}.$$

(c) If $f(k)$ is a polynomial in k of degree r, prove that

$$\sum_{k=1}^{n} f(k)$$

is a polynomial in n of degree $r + 1$.

7. Use the method suggested in Exercise 6 to express each of the following sums as a polynomial in n.

(a) $\sum_{k=1}^{n} (4k^2 + 7k + 6)$.

(b) $\sum_{k=1}^{n} k^2(k + 1)$.

(c) $\sum_{k=1}^{n} k(k + 1)(k + 2)$.

(d) $\sum_{k=1}^{n} k^4$.

8. Let A denote the linear operator defined by the equation

$$A(f) = a_0\, \Delta^n f + a_1\, \Delta^{n-1} f + \cdots + a_{n-1}\, \Delta f + a_n f,$$

where $a_0, a_1, \ldots, a_n$ are constants. This is called a *constant-coefficient* ***difference operator.*** It is analogous to the constant-coefficient derivative operator described in Section 6.7. With each such A we can associate the characteristic polynomial p_A defined by

$$p_A(r) = a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n.$$

Conversely, with every polynomial p we can associate an operator A having this polynomial as its characteristic polynomial. If A and B are constant-coefficient difference operators and if λ is a real number, define $A + B$, AB, and λA by the same formulas used in Section 6.7 for derivative operators. Then prove that Theorem 6.6 is valid for constant-coefficient difference operators.

15.14 A minimum problem relative to the max norm

We consider a problem that arises naturally from the theory of polynomial interpolation. In Theorem 15.3 we derived the error formula

$$f(x) - P(x) = \frac{A(x)}{(n+1)!} f^{(n+1)}(c), \tag{15.27}$$

where

$$A(x) = (x - x_0)(x - x_1) \cdots (x - x_n).$$

Here P is the unique polynomial of degree $\le n$ that agrees with f at $n + 1$ distinct points $x_0, x_1, \ldots, x_n$ in $[a, b]$. The function f is assumed to have a derivative of order $n + 1$ on $[a, b]$, and c is an unknown point lying somewhere in $[a, b]$. To estimate the error in (15.27) we need bounds for the $(n + 1)$st derivative $f^{(n+1)}$ and for the product $A(x)$. Since A is a polynomial, its absolute value has a maximum somewhere in the interval $[a, b]$. This maximum will depend on the choice of the points $x_0, x_1, \ldots, x_n$, and it is natural to try to choose these points so the maximum will be as small as possible.

We can denote this maximum by $\|A\|$, where $\|A\|$ is the max norm, given by

$$\|A\| = \max_{a \le x \le b} |A(x)|.$$

The problem is to find a polynomial of specified degree that minimizes $\|A\|$. This problem was first solved by Chebyshev; its solution leads to an interesting class of polynomials that also occur in other connections. First we give a brief account of these polynomials and then return to the minimum problem in question.

15.15 Chebyshev polynomials

Let $x + iy$ be a complex number of absolute value 1. By the binomial theorem we have

$$(x + iy)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}(iy)^k$$

for every integer $n \geq 0$. In this formula we write $x = \cos \theta$, $y = \sin \theta$, and consider the real part of each member. Since

$$(x + iy)^n = (\cos \theta + i \sin \theta)^n = e^{in\theta} = \cos n\theta + i \sin n\theta,$$

the real part of the left member is $\cos n\theta$. The real part of the right member is the sum over even values of k. Hence we have

$$\cos n\theta = x^n - \binom{n}{2} x^{n-2}y^2 + \binom{n}{4} x^{n-4}y^4 - + \cdots. \tag{15.28}$$

Since $y^2 = \sin^2 \theta = 1 - \cos^2 \theta = 1 - x^2$, the right-hand member of (15.28) is a polynomial in x of degree n. This polynomial is called the Chebyshev polynomial of the first kind and is denoted by $T_n(x)$.

DEFINITION. *The Chebyshev polynomial $T_n(x)$ is defined for all real x by the equation*

$$T_n(x) = \sum_{k=0}^{[n/2]} \binom{n}{2k} x^{n-2k}(x^2 - 1)^k.$$

From Equation (15.28) we obtain the following theorem.

THEOREM 15.7. *If $-1 \leq x \leq 1$ we have*

$$T_n(x) = \cos (n \arccos x).$$

Proof. If $\theta = \arccos x$ then $x = \cos \theta$ and $T_n(x) = \cos n\theta$.

The Chebyshev polynomials can be readily computed by taking the real part of $(x + iy)^n$ with $y^2 = 1 - x^2$, or by using the following recursion formula.

THEOREM 15.8. *The Chebyshev polynomials satisfy the recursion formula*

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \quad \text{for } n \geq 1,$$

with $T_0(x) = 1$ and $T_1(x) = x$.

Proof. First assume $-1 \leq x \leq 1$ and put $x = \cos \theta$ in the trigonometric identity

$$\cos (n + 1)\theta + \cos (n - 1)\theta = 2 \cos \theta \cos n\theta .$$

This proves that $T_{n+1}(x) + T_{n-1}(x) = 2xT_n(x)$ for x in the interval $-1 \leq x \leq 1$. But since both members are polynomials, this relation must hold for all x.

The next five polynomials are

$$T_2(x) = 2x^2 - 1, \qquad T_3(x) = 4x^3 - 3x, \qquad T_4(x) = 8x^4 - 8x^2 + 1,$$
$$T_5(x) = 16x^5 - 20x^3 + 5x, \qquad T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1.$$

The recursion formula shows that all the coefficients of $T_n(x)$ are integers; moreover, the coefficient of x^n is 2^{n-1}.

The next theorem shows that $T_n(x)$ has exactly n first order zeros and that they all lie in the interval $[-1, 1]$.

THEOREM 15.9. *If $n \geq 1$ the polynomial $T_n(x)$ has zeros at the n points*

$$x_k = \cos \frac{(2k+1)\pi}{2n}, \qquad k = 0, 1, 2, \ldots, n - 1.$$

Hence $T_n(x)$ has the factorization

$$T_n(x) = 2^{n-1}(x - x_0)(x - x_1) \cdots (x - x_{n-1}) = 2^{n-1} \prod_{k=0}^{n-1} \left(x - \cos \frac{(2k+1)\pi}{2n}\right).$$

Proof. We use the formula $T_n(x) = \cos n\theta$. Since $\cos n\theta = 0$ only if $n\theta$ is an odd multiple of $\pi/2$, we have $T_n(x) = 0$ for x in $[-1, 1]$ only if $n \arccos x = (2k+1)\pi/2$ for some integer k. Therefore the zeros of T_n in the interval $[-1, 1]$ are to be found among the numbers

(15.29) $$x_k = \cos \frac{2k+1}{n} \frac{\pi}{2}, \quad k = 0, \pm 1, \pm 2, \ldots .$$

The values $k = 0, 1, 2, \ldots, n - 1$ give n distinct zeros $x_0, x_1, \ldots, x_{n-1}$, all lying in the open interval $(-1, 1)$. Since a polynomial of degree n cannot have more than n zeros, these must be *all* the zeros of T_n. The remaining x_k in (15.29) are repetitions of these n.

THEOREM 15.10. *In the interval $[-1, 1]$ the extreme values of $T_n(x)$ are $+1$ and -1, taken alternately at the $n + 1$ points*

(15.30) $$t_k = \cos \frac{k\pi}{n}, \qquad \textit{for} \quad k = 0, 1, 2, \ldots, n.$$

Proof. By Rolle's theorem, the relative maxima and minima of T_n must occur between successive zeros; there are $n - 1$ such points in the open interval $(-1, 1)$. From the cosine formula for T_n we see that the extreme values, ± 1, are taken at the $n - 1$ interior points

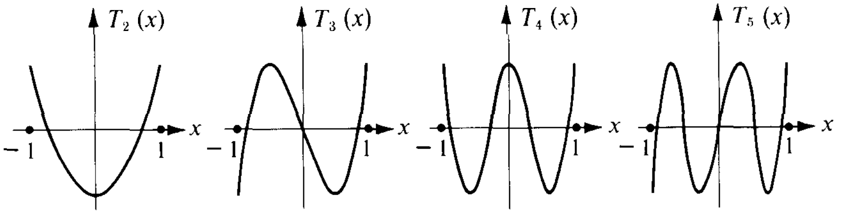

FIGURE 15.2 Graphs of Chebyshev polynomials over the interval $[-1, 1]$.

$\cos(k\pi/n)$, $k = 1, 2, \ldots, n-1$, and also at the two endpoints $x = 1$ and $x = -1$. Therefore in the closed interval $[-1, 1]$ the extreme values $+1$ and -1 are taken *alternately* at the $n + 1$ points $t_0, t_1, \ldots, t_n$ given by $t_k = \cos(k\pi/n)$ for $k = 0, 1, 2, \ldots, n$.

Figure 15.2 shows the graphs of $T_2, \ldots, T_5$ over the interval $[-1, 1]$.

15.16 A minimal property of Chebyshev polynomials

We return now to the problem of finding a polynomial of a specified degree for which the max norm is as small as possible. The problem is solved by the following theorem.

THEOREM 15.11. *Let $p_n(x) = x^n + \cdots$ be any polynomial of degree $n \geq 1$ with leading coefficient 1, and let*

$$\|p_n\| = \max_{-1 \leq x \leq 1} |p_n(x)| .$$

Then we have the inequality

$$\|p_n\| \geq \|\tilde{T}_n\| , \tag{15.31}$$

where $\tilde{T}_n(x) = T_n(x)/2^{n-1}$. Moreover, equality holds in (15.31) *if $p_n = \tilde{T}_n$.*

Proof. In the interval $[-1, 1]$ the polynomial $\tilde{T}_n$ takes its extreme values, $1/2^{n-1}$ and $-1/2^{n-1}$, alternately at the $n + 1$ distinct points t_k in Equation (15.30). Therefore $\|\tilde{T}_n\| = 1/2^{n-1}$.

We show next that the inequality

$$\|p_n\| < \frac{1}{2^{n-1}} \tag{15.32}$$

leads to a contradiction. Assume, then, that p_n satisfies (15.32) and consider the difference

$$r(x) = \tilde{T}_n(x) - p_n(x) .$$

At the points t_k given by (15.30) we have

$$r(t_k) = \frac{(-1)^k}{2^{n-1}} - p_n(t_k) = (-1)^k\left[\frac{1}{2^{n-1}} - (-1)^k p_n(t_k)\right] .$$

Because of (15.32) the factor in square brackets is positive. Therefore $r(t_k)$ has alternating signs at the $n + 1$ points $t_0, t_1, \ldots, t_n$. Since r is continuous it must vanish at least once between consecutive sign changes. Therefore r has at least n distinct zeros. But since r is a polynomial of degree $\leq n - 1$, this means that r is identically zero. Therefore $P_n = \tilde{T}_n$, so $\|P_n\| = \|\tilde{T}_n\| = 1/2^{n-1}$, contradicting (15.32). This proves that we must have $\|p_n\| \geq 1/2^{n-1} = \|\tilde{T}_n\|$.

Although Theorem 15.11 refers to the interval $[-1, 1]$ and to a polynomial with leading coefficient 1, it can be used to deduce a corresponding result for an arbitrary interval $[a, b]$ and an arbitrary polynomial.

THEOREM 15.12. *Let $q_n(x) = c_n x^n + \cdots$ be any polynomial of degree $n \geq 1$, and let*

$$\|q_n\| = \max_{a \leq x \leq b} |q_n(x)|.$$

Then we have the inequality

$$\|q_n\| \geq |c_n| \frac{(b-a)^n}{2^{2n-1}}. \tag{15.33}$$

Moreover, equality holds in (15.33) *if*

$$q_n(x) = c_n \frac{(b-a)^n}{2^{2n-1}} T_n\left(\frac{2x - a - b}{b - a}\right).$$

Proof. Consider the transformation

$$t = \frac{2x - a - b}{b - a}.$$

This maps the interval $a \leq x \leq b$ in a one-to-one fashion onto the interval $-1 \leq t \leq 1$. Since

$$x = \frac{b-a}{2} t + \frac{b+a}{2}$$

we have

$$x^n = \left(\frac{b-a}{2}\right)^n t^n + \text{terms of lower degree,}$$

hence

$$q_n(x) = c_n \left(\frac{b-a}{2}\right)^n p_n(t),$$

where $p_n(t)$ is a polynomial in t of degree n with leading coefficient 1. Applying Theorem 15.11 to p_n we obtain Theorem 15.12.

15.17 Application to the error formula for interpolation

We return now to the error formula (15.27) for polynomial interpolation. If we choose the interpolation points $x_0, x_1, \ldots, x_n$ to be the $n + 1$ zeros of the Chebyshev polynomial

T_{n-1} we can write (15.27) in the form

$$f(x) - P(x) = \frac{T_{n+1}(x)}{2^n(n+1)!} f^{(n+1)}(c).$$

The points $x_0, x_1, \ldots, x_n$ all lie in the open interval $(-1, 1)$ and are given by

$$x_k = \cos\left(\frac{2k+1}{n+1}\frac{\pi}{2}\right) \qquad \text{for} \quad k = 0, 1, 2, \ldots, n.$$

If x is in the interval $[-1, 1]$ we have $|T_{n+1}(x)| \le 1$ and the error is estimated by the inequality

$$|f(x) - P(x)| \le \frac{1}{2^n(n+1)!}|f^{(n+1)}(c)|.$$

If the interpolation takes place in an interval $[a, b]$ with the points

$$y_k = \frac{b-a}{2}x_k + \frac{b+a}{2}$$

as interpolation points, the product

$$A(x) = (x - y_0)(x - y_1)\cdots(x - y_n)$$

satisfies the inequality $|A(x)| \le (b-a)^{n+1}/2^{2n+1}$ for all x in $[a, b]$. The corresponding estimate for $f(x) - P(x)$ is

$$|f(x) - P(x)| \le \frac{(b-a)^{n+1}}{2^{2n+1}(n+1)!}|f^{(n+1)}(c)|.$$

15.18 Exercises

In this set of exercises T_n denotes the Chebyshev polynomial of degree n.

1. Prove that $T_n(-x) = (-1)^n T_n(x)$. This shows that T_n is an even function when n is even and an odd function when n is odd.
2. (a) Prove that in the open interval $(-1, 1)$ the derivative T_n' is given by the formula

$$T_n'(x) = \frac{n \sin n\theta}{\sin \theta}, \qquad \text{where } \theta = \arccos x.$$

   (b) Compute $T_n'(1)$ and $T_n'(-1)$.

3. Prove that $\displaystyle\int_0^x T_n(u)\,du = \frac{1}{2}\left\{\frac{T_{n+1}(x) - T_{n+1}(0)}{n+1} - \frac{T_{n-1}(x) - T_{n-1}(0)}{n-1}\right\}$ if $n \ge 2$.

4. (a) Prove that $2T_m(x)T_n(x) = T_{m+n}(x) + T_{m-n}(x)$.
   (b) Prove that $T_{mn}(x) = T_m[T_n(x)] = T_n[T_m(x)]$.
5. If $x = \cos\theta$, prove that $\sin\theta \sin n\theta$ is a polynomial in x, and determine its degree.
6. The Chebyshev polynomial T_n satisfies the differential equation

$$(1 - x^2)y'' - xy' + n^2y = 0$$

over the entire real axis. Prove this by each of the following methods:

(a) Differentiate the relation $T_n'(x) \sin \theta = n \sin n\theta$ obtained in Exercise 2(a).

(b) Introduce the change of variable $x = \cos \theta$ in the differential equation

$$\frac{d^2 (\cos n\theta)}{d\theta^2} = -n^2 \cos n\theta .$$

7. Determine, in terms of Chebyshev polynomials, a polynomial $Q(x)$ of degree $\leq n$ which best approximates x^{n+1} on the interval $[-1, 1]$ relative to the max norm.
8. Find a polynomial of degree ≤ 4 that best approximates the function $f(x) = x^5$ in the interval $[0, 1]$, relative to the max norm.
9. A polynomial P is called ***primary*** if the coefficient of the term of highest degree is 1. For a given interval $[a, b]$ let $\|P\|$ denote the maximum of $|P|$ on $[u, b]$. Prove each of the following statements :

(a) If $b - a < 4$, for every $\epsilon > 0$ there exists a primary polynomial P with $\|P\| < \epsilon$.

(b) If for every $\epsilon > 0$ there exists a primary polynomial P with $\|P\| < \epsilon$, then $b - a < 4$.

In other words, primary polynomials with arbitrarily small norm exist if, and only if, the interval $[a, b]$ has length less than 4.

10. The Chebyshev polynomials satisfy the following orthogonality relations:

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}\,dx = \begin{cases} 0 & \text{if } n \neq m, \\ \pi & \text{if } n = m = 0, \\ \dfrac{\pi}{2} & \text{if } n = m > 0. \end{cases}$$

Prove this by each of the following methods:

(a) From the differential equation in Exercise 6 deduce that

$$T_m(x)\,\frac{d}{dx}\left(\sqrt{1-x^2}\;T_n'(x)\right) + n^2\,\frac{T_n(x)T_m(x)}{\sqrt{1-x^2}} = 0.$$

Write a corresponding formula with n and m interchanged, subtract the two equations, and integrate from -1 to 1.

(b) Use the orthogonality relations

$$\int_0^{\pi} \cos m\theta \cos n\theta \, d\theta = \begin{cases} 0 & \text{if } n \neq m,\ n > 0,\ m > 0, \\ \pi & \text{if } n = m = 0, \\ \dfrac{\pi}{2} & \text{if } n = m > 0, \end{cases}$$

and introduce the change of variable $x = \cos \theta$.

11. Prove that for $-1 < x < 1$ we have

$$\frac{T_n(x)}{\sqrt{1-x^2}} = (-1)^n \frac{2^n n!}{(2n)!} \frac{d^n}{dx^n} (1 - x^2)^{n-1/2}.$$

12. Let $y_1, y_2, \ldots, y_n$ be n real numbers, and let

$$x_k = \cos \frac{(2k-1)\pi}{2n} \qquad \text{for} \quad k = 1, 2, \ldots, n.$$

Let P be the polynomial of degree $\leq n - 1$ that takes the value y_k at x_k for $1 \leq k \leq n$. If x is not one of the x_k show that

$$P(x) = \frac{1}{n} \sum_{k=1}^{n} (-1)^{k-1} y_k \sqrt{1 - x_k^2} \frac{T_n(x)}{x - x_k}.$$

13. Let P be a polynomial of degree $\leq n - 1$ such that

$$\sqrt{1 - x^2}\, |P(x)| \leq 1$$

for $-1 \leq x \leq 1$. Prove that $\|P\| \leq n$, where $\|P\|$ is the maximum of $|P|$ on the interval $[-1, 1]$.

[*Hint:* Use Exercise 12. Consider three cases: $x_1 \leq x \leq 1$; $-1 \leq x \leq x_n$; $x_n \leq x \leq x_1$; in the first two use Exercise 15(a) of Section 15.9. In the third case note that $\sqrt{1 - x^2} \geq \sin(\pi/2n) > 1/n$.]

In Exercises 14 through 18, $U_n(x) = T'_{n+1}(x)/(n + 1)$ for $n = 0, 1, 2, \ldots$.

14. (a) Prove that $U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x)$ for $n \geq 2$.
(b) Determine the explicit form of the polynomials $U_0, U_1, \ldots, U_5$.
(c) Prove that $|U_n(x)| \leq n + 1$ if $-1 \leq x \leq 1$.

15. Show that U_n satisfies the differential equation

$$(1 - x^2)y'' - 3xy' + n(n + 2)y = 0.$$

16. Derive the orthogonality relations

$$\int_{-1}^{1} \sqrt{1 - x^2}\, U_m(x)U_n(x)\, dx = \begin{cases} 0 & \text{if } m \neq n, \\ \dfrac{\pi}{2} & \text{if } m = n. \end{cases}$$

17. Prove that

$$\sqrt{1 - x^2}\, U_n(x) = (-1)^n \frac{2^n(n + 1)!}{(2n + 1)!} \frac{d^n}{dx^n} (1 - x^2)^{n+1/2}.$$

18. Let $y_1, y_2, \ldots, y_n$ be n real numbers and let

$$x_k = \cos \frac{k\pi}{n + 1} \qquad \text{for } k = 1, 2, \ldots, n.$$

Let P be the polynomial of degree $\leq n - 1$ that takes the value y_k at x_k for $1 \leq k \leq n$. If x is not one of the x_k show that

$$P(x) = \frac{1}{n + 1} \sum_{k=1}^{n} (-1)^{k-1}(1 - x_k^2) y_k \frac{U_n(x)}{x - x_k}.$$

15.19 Approximate integration. The trapezoidal rule

Many problems in both pure and applied mathematics lead to new functions whose properties have not been studied or whose values have not been tabulated. To satisfy certain practical needs of applied science it often becomes necessary to obtain quantitative

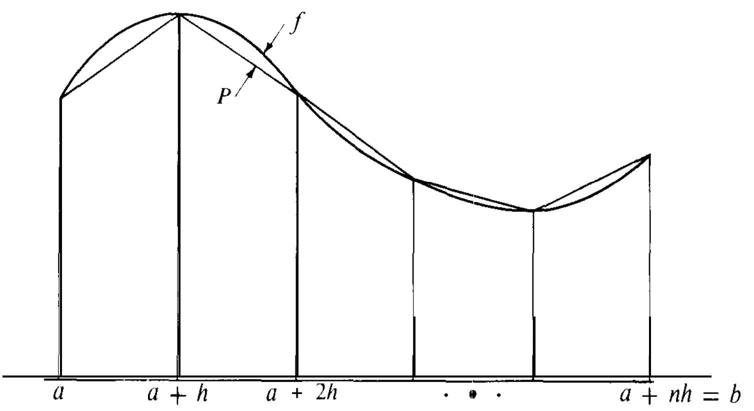

FIGURE 15.3 The trapezoidal rule obtained by piecewise linear interpolation.

information about such functions, either in graphical or numerical form. Many of these functions occur as integrals of the type

$$F(x) = \int_a^x f(t)\,dt\,,$$

where the integrand f is given by an explicit analytic formula or is known in part by tabular data. The remainder of this chapter describes some of the most elementary methods for finding numerical approximations to such integrals. The basic idea is very simple. We approximate the integrand f by another function P whose integral is easily computed, and then we use the integral of P as an approximation to the integral of f.

If f is nonnegative the integral $\int_a^b f(x)\,dx$ represents the area of the ordinate set of f over $[a, b]$. This geometric interpretation of the integral immediately suggests certain procedures for approximate integration. Figure 15.3 shows an example of a function f with known values at $n + 1$ equally spaced points $a, a + h, a + 2h, \ldots, a + nh = b$, where $h = (b - a)/n$. Let $x_k = a + kh$. For each $k = 0, 1, 2, \ldots, n - 1$ the graph of f over the interval $[x_k, x_{k+1}]$ has been approximated by a linear function that agrees with f at the endpoints x_k and x_{k+1}. Let P denote the corresponding piecewise linear interpolating function defined over the full interval $[a, b]$. Then we have

$$\text{(15.34)} \qquad P(x) = \frac{x_{k+1} - x}{h} f(x_k) + \frac{x - x_k}{h} f(x_{k+1}) \qquad \text{if} \quad x_k \le x \le x_{k+1}\,.$$

Integrating over the interval $[x_k, x_{k+1}]$ we find that

$$\int_{x_k}^{x_{k+1}} P(x)\,dx = h\,\frac{f(x_k) + f(x_{k+1})}{2}\,.$$

When f is positive this is the area of the trapezoid determined by the graph of P over $[x_k, x_{k+1}]$. The formula holds, of course, even if f is not positive everywhere. Adding the integrals over all subintervals $[x_k, x_{k+1}]$ we obtain

$$\int_a^b P(x)\,dx = \frac{h}{2}\sum_{k=0}^{n-1}[f(x_k) + f(x_{k+1})] \tag{15.35}$$

$$= \frac{h}{2}\left(f(a) + 2\sum_{k=1}^{n-1} f(a + kh) + f(b)\right).$$

To use this sum as an approximation to the integral $\int_a^b f(x)\,dx$ we need an estimate for the error, $\int_a^b f(x)\,dx - \int_a^b P(x)\,dx$. If f has a continuous second derivative on $[a, b]$ this error is given by the following the&em.

THEOREM 15.13. TRAPEZOIDAL RULE. *Assume f has a continuous second derivative f'' on $[a, b]$. Zf n is a positive integer, let $h = (b - a)/n$. Then we have*

$$\int_a^b f(x)\,dx = \frac{b-a}{2n}\left(f(a) + 2\sum_{k=1}^{n-1} f(a + kh) + f(b)\right) - \frac{(b-a)^3}{12n^2}f''(c) \tag{15.36}$$

for some c in $[a, b]$.

Note: Equation (15.36) is known as the *trapezoidal rule.* The term $-f''(c)(b - a)^3/12n^2$ represents the error in approximating $\int_a^b f(x)\,dx$ by $\int_a^b P(x)\,dx$. Once the maximum value off" on $[a, b]$ is known we can approximate the integral off to any desired degree of accuracy by taking n sufficiently large. Note that no knowledge of the interpolating function *Pis* required to use this formula. It is only necessary to know the values off at the points a, $a + h, \ldots, a + nh$, and to have an estimate for $|f''(c)|$.

Proof. Let P be the interpolating function given by (15.34), where $x_k = a + kh$. In each subinterval $[x_k, x_{k+1}]$ we apply the error estimate for linear interpolation given by Theorem 15.3 and we find

$$f(x) - P(x) = (x - x_k)(x - x_{k+1})\frac{f''(c_k)}{2!} \tag{15.37}$$

for some c_k in (x_k, x_{k+1}). Let M_2 and m_2 denote the maximum and minimum, respectively, off" on $[a, b]$, and let

$$B(x) = (x - x_k)(x_{k+1} - x)/2.$$

Then $B(x) \geq 0$ in the interval $[x_k, x_{k+1}]$, and from (15.37) we obtain the inequalities

$$m_2B(x) \leq P(x) - f(x) \leq M_2B(x)$$

in this interval. Integrating, we have

$$m_2\int_{x_k}^{x_k+h} B(x)\,dx \leq \int_{x_k}^{x_k+h}[P(x) - f(x)]\,dx \leq M_2\int_{x_k}^{x_k+h} B(x)\,dx. \tag{15.38}$$

The integral of B is given by

$$\int_{x_k}^{x_k+h} B(x)\,dx = \frac{1}{2}\int_{x_k}^{x_k+h}(x - x_k)(x_{k+1} - x)\,dx = \frac{1}{2}\int_0^h t(h - t)\,dt = \frac{h^3}{12}.$$

Therefore the inequalities (15.38) give us

$$m_2 \le \frac{12}{h^3}\int_{x_k}^{x_k+h} [P(x) - f(x)]\,dx \le M_2 .$$

Adding these inequalities for $k = 0, 1, 2, \ldots, n - 1$ and dividing by n, we obtain

$$m_2 \le \frac{12}{nh^3}\int_a^b [P(x) - f(x)]\,dx \le M_2 .$$

Since the function f'' is continuous on [a, b], it assumes every value between its minimum m_2 and its maximum M_2 somewhere in [a, b]. In particular, we have

$$f''(c) = \frac{12}{nh^3}\int_a^b [P(x) - f(x)]\,dx$$

for some c in [a, b]. In other words,

$$\int_a^b f(x)\,dx = \int_a^b P(x)\,dx - \frac{nh^3}{12} f''(c) .$$

Using (15.35) and the relation $h = (b - a)/n$ we obtain (15.36).

To derive the trapezoidal rule we used a linear polynomial to interpolate between each adjacent pair of values off. More accurate formulas can be obtained by interpolating with polynomials of higher degree. In the next section we consider an important special case that is remarkable for its simplicity and accuracy.

15.20 Simpson's rule

The solid curve in Figure 15.4 is the graph of a function f over an interval [a, b]. The mid-point of the interval, $(a + b)/2$, is denoted by m. The dotted curve is the graph of a quadratic polynomial P that agrees with f at the three points a, m, and b. If we use the integral $\int_a^b P(x)\,dx$ as an approximation to $\int_a^b f(x)\,dx$ we are led to an approximate integration formula known as ***Simpson's rule.***

Instead of determining P explicitly, we introduce a linear transformation that carries the interval [a, b] onto the interval $[0, 2]$. If we write

$$t = \frac{x - a}{m - u}, \qquad \text{or} \qquad x = a + (m - a)t ,$$

we see that t takes the values 0, 1, 2 when x takes the values a, m, b. Now let

$$\varphi(t) = P[a + (m - a)t] .$$

Then $\varphi(t)$ is a quadratic polynomial in t that takes the values $P(u)$, $P(m)$, $P(b)$ at the points $t = 0, 1, 2$, respectively. Also, we have

$$\int_0^2 \varphi(t)\,dt = \int_0^2 P[a + (m - a)t]\,dt = \frac{1}{m - a}\int_a^b P(x)\,dx ;$$

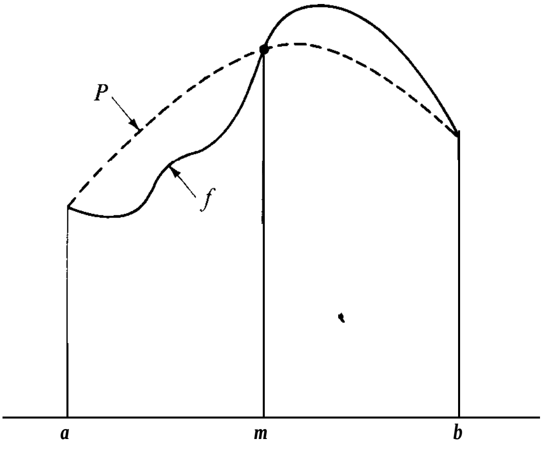

FIGURE 15.4 Interpolation by a quadratic polynomial *P*.

hence

$$\int_a^b P(x)\,dx = (m - a)\int_0^2 \varphi(t)\,dt = \frac{b-a}{2}\int_0^2 \varphi(t)\,dt. \tag{15.39}$$

Now we use Newton's interpolation formula to construct φ. We have

$$\varphi(t) = \varphi(0) + t\,\Delta\varphi(0) + t(t-1)\frac{\Delta^2\varphi(0)}{2!},$$

where $\Delta\varphi(t) = \varphi(t+1) - \varphi(t)$. Integrating from 0 to 2 we obtain

$$\int_0^2 \varphi(t)\,dt = 2\varphi(0) + 2\,\Delta\varphi(0) + \tfrac{1}{3}\,\Delta^2\varphi(0).$$

Since $\Delta\varphi(0) = \varphi(1) - \varphi(0)$ and $\Delta^2\varphi(0) = \varphi(2) - 2\varphi(1) + \varphi(0)$, the integral is equal to

$$\int_0^2 \varphi(t)\,dt = \tfrac{1}{3}[\varphi(0) + 4\varphi(1) + \varphi(2)] = \tfrac{1}{3}[P(a) + 4P(m) + P(b)].$$

Using (15.39) and the fact that P agrees with f at $a, m, b,$ we obtain

$$\int_a^b P(x)\,dx = \frac{b-a}{6}[f(a) + 4f(m) + f(b)]. \tag{15.40}$$

Therefore, we may write

$$\int_a^b f(x)\,dx = \frac{b-a}{6}[f(a) + 4f(m) + f(b)] + R,$$

where $R = \int_a^b f(x)\,dx - \int_a^b P(x)\,dx.$