Definition. If V is a variety, then the field of fractions of the integral domain $k[V]$ is called the field of *rational functions* on V and is denoted by $k(V)$. The *dimension* of a variety V, denoted $\dim V$, is defined to be the transcendence degree of $k(V)$ over k.

Examples

(1) Single points in $\mathbb{A}^n$ are affine varieties since their corresponding ideals in $k[\mathbb{A}^n]$ are maximal ideals. The coordinate ring of a point is isomorphic to k, which is also the field of rational functions. The dimension of a single point is 0. Any finite set is the union of its single point subsets, and this is its unique decomposition into affine subvarieties.

(2) The x-axis in $\mathbb{R}^2$ is irreducible since it has coordinate ring $\mathbb{R}[x, y]/(y) \cong \mathbb{R}[x]$, which is an integral domain. Similarly, the y-axis and, more generally, lines in $\mathbb{R}^2$ are also irreducible (cf. Exercise 23 in Section 1). Linear sets in $\mathbb{R}^n$ are affine varieties. The field of rational functions on the x-axis is the quotient field $\mathbb{R}(x)$ of $\mathbb{R}[x]$, which is why $\mathbb{R}(x)$ is called a rational function field. The dimension of the x-axis (or, more generally, any line) is 1.

(3) The union of the x and y axes in $\mathbb{R}^2$, namely $\mathcal{Z}(xy)$, is not a variety: $\mathcal{Z}(xy) = \mathcal{Z}(x) \cup \mathcal{Z}(y)$ is its unique decomposition into subvarieties. The corresponding coordinate ring $\mathbb{R}[x, y]/(xy)$ contains zero divisors.

(4) The hyperbola $xy = 1$ in $\mathbb{R}^2$ is a variety since we saw in Section 1 that its coordinate ring is the integral domain $\mathbb{R}[x, 1/x]$. Note that the two disjoint branches of the hyperbola (defined by $x > 0$ and $x < 0$) are not subvarieties (cf. also Exercises 12–13).

(5) If $V = \mathcal{Z}(l_1, l_2, \dots, l_m)$ is the zero set of *linear* polynomials $l_1, \dots, l_m$ in $k[x_1, \dots, x_m]$ and $V \neq \emptyset$, then V is an affine variety (called a *linear variety*). Note that determining whether $V \neq \emptyset$ is a linear algebra problem.

We end this section with some general ring-theoretic results that were originally motivated by their connection with decomposition questions in geometry.

Primary Decomposition of Ideals in Noetherian Rings

The second statement in Proposition 17 shows that any ideal of the form $\mathcal{I}(V)$ in $k[\mathbb{A}^n]$ may be written uniquely as a finite intersection of prime ideals, and by Hilbert's Nullstellensatz this applies in particular to all radical ideals when k is algebraically closed. In a large class of commutative rings (including all Noetherian rings) every ideal has a *primary decomposition*, which is a similar decomposition but allows ideals that are analogous to "prime powers" (but see the examples below). This decomposition can be considered as a generalization of the factorization of an integer $n \in \mathbb{Z}$ into the product of prime powers. We shall be primarily concerned with the case of Noetherian rings.

Definition. A proper ideal Q in the commutative ring R is called *primary* if whenever $ab \in Q$ and $a \notin Q$, then $b^n \in Q$ for some positive integer n. Equivalently, if $ab \in Q$ and $a \notin Q$, then $b \in \operatorname{rad} Q$.

Some of the basic properties of primary ideals are given in the following proposition.

Proposition 19. Let R be a commutative ring with 1.

(1) Prime ideals are primary.

(2) The ideal Q is primary if and only if every zero divisor in R/Q is nilpotent.

(3) If Q is primary then rad Q is a prime ideal, and is the unique smallest prime ideal containing Q.

(4) If Q is an ideal whose radical is a maximal ideal, then Q is a primary ideal.

(5) Suppose M is a maximal ideal and Q is an ideal with $M^n \subseteq Q \subseteq M$ for some $n \geq 1$. Then Q is a primary ideal with rad $Q = M$.

Proof: The first two statements are immediate from the definition of a primary ideal. For (3), suppose $ab \in \text{rad } Q$. Then $a^m b^m = (ab)^m \in Q$, and since Q is primary, either $a^m \in Q$, in which case $a \in \text{rad } Q$, or $(b^m)^n \in Q$ for some positive integer n, in which case $b \in \text{rad } Q$. This proves that rad Q is a prime ideal, and it follows that rad Q is the smallest prime ideal containing Q (Proposition 12).

To prove (4) we pass to the quotient ring R/Q; by (2), it suffices to show that every zero divisor in this quotient ring is nilpotent. We are reduced to the situation where $Q = (0)$ and $M = \text{rad } Q = \text{rad}(0)$, which is the nilradical, is a maximal ideal. Since the nilradical is contained in every prime ideal (Proposition 12), it follows that M is the unique prime ideal, so also the unique maximal ideal. If d were a zero divisor, then the ideal (d) would be a proper ideal, hence contained in a maximal ideal. This implies that $d \in M$, hence every zero divisor is indeed nilpotent.

Finally, suppose $M^n \subseteq Q \subseteq M$ for some $n \geq 1$ where M is a maximal ideal. Then $Q \subseteq M$ so rad $Q \subseteq \text{rad } M = M$. Conversely, $M^n \subseteq Q$ shows that $M \subseteq \text{rad } Q$, so rad $Q = M$ is a maximal ideal, and Q is primary by (4).

Definition. If Q is a primary ideal, then the prime ideal $P = \text{rad } Q$ is called the *associated prime* to Q, and Q is said to *belong* to P (or to be P-*primary*).

It is easy to check that a finite intersection of P-primary ideals is again a P-primary ideal (cf. the exercises).

Examples

(1) The primary ideals in $\mathbb{Z}$ are 0 and the ideals (p^m) for p a prime and $m \geq 1$.

(2) For any field k, the ideal (x) in $k[x, y]$ is primary since it is a prime ideal. For any $n \geq 1$, the ideal $(x, y)^n$ is primary since it is a power of the maximal ideal (x, y).

(3) The ideal $Q = (x^2, y)$ in the polynomial ring $k[x, y]$ is primary since we have $(x, y)^2 \subseteq (x^2, y) \subseteq (x, y)$. Similarly, $Q' = (4, x)$ in $\mathbb{Z}[x]$ is a $(2, x)$-primary ideal.

(4) Primary ideals need not be powers of prime ideals. For example, the primary ideal Q in the previous example is not the power of a prime ideal, as follows. If $(x^2, y) = P^k$ for some prime ideal P and some $k \geq 1$, then $x^2, y \in P^k \subseteq P$ so $x, y \in P$. Then $P = (x, y)$, and since $y \notin (x, y)^2$, it would follow that $k = 1$ and $Q = (x, y)$. Since $x \notin (x^2, y)$, this is impossible.

(5) If R is Noetherian, and Q is a primary ideal belonging to the prime ideal P, then

$$P^m \subseteq Q \subseteq P$$

for some $m \geq 1$ by Proposition 14. If P is a maximal ideal, then the last statement in Proposition 19 shows that the converse also holds. This is not necessarily true if P

is a prime ideal that is *not maximal*. For example, consider the ideal $I = (x^2, xy)$ in $k[x, y]$. Then $(x^2) \subset I \subset (x)$, and (x) is a prime ideal, but I is not primary: $xy \in I$ and $x \notin I$, but no positive power of y is an element of I. This example also shows that an ideal whose radical is prime (but not maximal as in (4) of the proposition) is not necessarily primary.

(6) Powers of prime ideals need not be primary. For example, consider the quotient ring $R = \mathbb{R}[x, y, z]/(xy - z^2)$, the coordinate ring of the cone $z^2 = xy$ in $\mathbb{R}^3$, and let $P = (\bar{x}, \bar{z})$ be the ideal generated by $\bar{x}$ and $\bar{z}$ in R. This is a prime ideal in R since the quotient is $R/(\bar{x}, \bar{z}) \cong \mathbb{R}[x, y, z]/(x, z) \cong \mathbb{R}[y]$ (because $(xy - z^2) \subset (x, z)$). The ideal

$$P^2 = (\bar{x}^2, \bar{x}\bar{z}, \bar{z}^2) = (\bar{x}^2, \bar{x}\bar{z}, \bar{x}\bar{y}) = \bar{x}(\bar{x}, \bar{y}, \bar{z}),$$

however, is not primary: $\bar{x}\bar{y} = \bar{z}^2 \in P^2$, but $\bar{x} \notin P^2$, and no power of $\bar{y}$ is in P^2. Note that P^2 is another example of an ideal that is not primary whose radical is prime.

(7) Suppose R is a U.F.D. If π is an irreducible element of R then it is easy to see that the powers (π^n) for $n = 1, 2, \ldots$ are (π)-primary ideals. Conversely, suppose Q is a (π)-primary ideal, and let n be the largest integer with $Q \subseteq (\pi^n)$ (such an integer exists since, for example, $\pi^k \in Q$ for some $k \geq 1$, so $n \leq k$). If q is an element of Q not contained in (π^{n+1}), then $q = r\pi^n$ for some $r \in R$ and $r \notin (\pi)$. Since $r \notin (\pi)$ and Q is (π)-primary, it follows that $\pi^n \in Q$. This shows that $Q = (\pi^n)$.

In the examples above, the ideal (x^2, xy) in $k[x, y]$ is not a primary ideal, but it can be written as the intersection of primary ideals: $(x^2, xy) = (x) \cap (x, y)^2$.

Definition.

(1) An ideal I in R has a *primary decomposition* if it may be written as a finite intersection of primary ideals:

$$I = \bigcap_{i=1}^{m} Q_i \qquad Q_i \text{ a primary ideal.}$$

(2) The primary decomposition above is *minimal* and the Q_i are called the *primary components of I* if
  (a) no primary ideal contains the intersection of the remaining primary ideals, i.e., $Q_i \not\supseteq \cap_{j \neq i} Q_j$ for all i, and
  (b) the associated prime ideals are all distinct: $\operatorname{rad} Q_i \neq \operatorname{rad} Q_j$ for $i \neq j$.

We now prove that in a Noetherian ring every proper ideal has a minimal primary decomposition. This result is often called the Lasker–Noether Decomposition Theorem, since it was first proved for polynomial rings by the chess master Emanuel Lasker and the proof was later greatly simplified and generalized by Emmy Noether.

Definition. A proper ideal I in the commutative ring R is said to be *irreducible* if I cannot be written nontrivially as the intersection of two other ideals, i.e., if $I = J \cap K$ with ideals J, K implies that $I = J$ or $I = K$.

It is easy to see that a prime ideal is irreducible (see Exercise 11 in Section 7.4). The ideal $(x, y)^2$ in $k[x, y]$ in Example 2 earlier shows that primary ideals need not