2. The substitution $u = (x - y)/2$, $v = (x + y)/2$ changes $f(u, v)$ into $F(x, y)$. Use an appropriate form of the chain rule to express the partial derivatives $\partial F/\partial x$ and $\partial F/\partial y$ in terms of the partial derivatives $\partial f/\partial u$ and $\partial f/\partial v$.
3. The equations $u = f(x, y)$, $x = X(s, t)$, $y = Y(s, t)$ define u as a function of s and t, say $u = F(s, t)$.
   (a) Use an appropriate form of the chain rule to express the partial derivatives $\partial F/\partial s$ and $\partial F/\partial t$ in terms of $\partial f/\partial x$, $\partial f/\partial y$, $\partial X/\partial s$, $\partial X/\partial t$, $\partial Y/\partial s$, $\partial Y/\partial t$.
   (b) If $\partial^2 f/(\partial x\, \partial y) = \partial^2 f/(\partial y\, \partial x)$, show that

$$\frac{\partial^2 F}{\partial s^2} = \frac{\partial f}{\partial x}\frac{\partial^2 X}{\partial s^2} + \frac{\partial^2 f}{\partial x^2}\left(\frac{\partial X}{\partial s}\right)^2 + 2\frac{\partial X}{\partial s}\frac{\partial Y}{\partial s}\frac{\partial^2 f}{\partial x\, \partial y} + \frac{\partial f}{\partial y}\frac{\partial^2 Y}{\partial s^2} + \frac{\partial^2 f}{\partial y^2}\left(\frac{\partial Y}{\partial s}\right)^2.$$

   (c) Find similar formulas for the partial derivatives $\partial^2 F/(\partial s\, \partial t)$ and $\partial^2 F/\partial t^2$.
4. Solve Exercise 3 in each of the following special cases:
   (a) $X(s, t) = s + t$, $\quad Y(s, t) = st$.
   (b) $X(s, t) = st$, $\quad Y(s, t) = s/t$.
   (c) $X(s, t) = (s - t)/2$, $\quad Y(s, t) = (s + t)/2$.
5. The introduction of polar coordinates changes $f(x, y)$ into $\varphi(r, \theta)$, where $x = r\cos\theta$ and $y = r\sin\theta$. Express the second-order partial derivatives $\partial^2\varphi/\partial r^2$, $\partial^2\varphi/(\partial r\, \partial\theta)$, and $\partial^2\varphi/(\partial\theta\, \partial r)$ in terms of the partial derivatives of f. You may use the formulas derived in Example 2 of Section 8.21.
6. The equations $u = f(x, y, z)$, $x = X(r, s, t)$, $y = Y(r, s, t)$, and $z = Z(r, s, t)$ define u as a function of r, s, and t, say $u = F(r, s, t)$. Use an appropriate form of the chain rule to express the partial derivatives $\partial F/\partial r$, $\partial F/\partial s$, and $\partial F/\partial t$ in terms of partial derivatives of f, X, Y, and Z.
7. Solve Exercise 6 in each of the following special cases:
   (a) $X(r, s, t) = r + s + t$, $\quad Y(r, s, t) = r - 2s + 3t$, $\quad Z(r, s, t) = 2r + s - t$.
   (b) $X(r, s, t) = r^2 + s^2 + t^2$, $\quad Y(r, s, t) = r^2 - s^2 - t^2$, $\quad Z(r, s, t) = r^2 - s^2 + t^2$.
8. The equations $u = f(x, y, z)$, $x = X(s, t)$, $y = Y(s, t)$, $z = Z(s, t)$ define u as a function of s and t, say $u = F(s, t)$. Use an appropriate form of the chain rule to express the partial derivatives $\partial F/\partial s$ and $\partial F/\partial t$ in terms of partial derivatives of f, X, Y, and Z.
9. Solve Exercise 8 in each of the following special cases:
   (a) $X(s, t) = s^2 + t^2$, $\quad Y(s, t) = s^2 - t^2$, $\quad Z(s, t) = 2st$.
   (b) $X(s, t) = s + t$, $\quad Y(s, t) = s - t$, $\quad Z(s, t) = st$.
10. The equations $u = f(x, y)$, $x = X(r, s, t)$, $y = Y(r, s, t)$ define u as a function of r, s, and t, say $u = F(r, s, t)$. Use an appropriate form of the chain rule to express the partial derivatives $\partial F/\partial r$, $\partial F/\partial s$, and $\partial F/\partial t$ in terms of partial derivatives of f, X, and Y.
11. Solve Exercise 10 in each of the following special cases:
   (a) $X(r, s, t) = r + s$, $\quad Y(r, s, t) = t$.
   (b) $X(r, s, t) = r + s + t$, $\quad Y(r, s, t) = r^2 + s^2 + t^2$.
   (c) $X(r, s, t) = r/s$, $\quad Y(r, s, t) = s/t$.
12. Let $h(x) = f[g(x)]$, where $g = (g_1, \ldots, g_n)$ is a vector field differentiable at a, and f is a scalar field differentiable at $b = g(a)$. Use the chain rule to show that the gradient of h can be expressed as a linear combination of the gradient vectors of the components of g, as follows:

$$\nabla h(a) = \sum_{k=1}^{n} D_k f(b) \nabla g_k(a).$$

13. (a) If $f(x, y, z) = xi + yj + zk$, prove that the Jacobian matrix $Df(x, y, z)$ is the identity matrix of order 3.

(b) Find all differentiable vector fieldsf: $\mathbf{R}^3 \to \mathbf{R}^3$ for which the Jacobian matrix $Df(x, y, z)$ is the identity matrix of order 3.
(c) Find all differentiable vector fieldsf: $\mathbf{R}^3 \to \mathbf{R}^3$ for which the Jacobian matrix is a diagonal matrix of the form diag (p(x), $q(y)$, $r(z)$), wherep, q, and r are given continuous functions.

14. Let f: $\mathbf{R}^2 \to \mathbf{R}^2$ and g: $\mathbf{R}^3 \to \mathbf{R}^2$ be two vector fields defined as follows:

$$f(x, y) = e^{x+2y}\boldsymbol{i} + \sin(y + 2x)\boldsymbol{j},$$

$$g(u, v, w) = (u + 2v^2 + 3w^3)\boldsymbol{i} + (2v - u^2)\boldsymbol{j}.$$

(a) Compute each of the Jacobian matrices $Df(x, y)$ and $Dg(u, v, w)$.
(b) Compute the composition $h(u, v, w) = f[g(u, v, w)]$.
(c) Compute the Jacobian matrix $Dh(1, -1, 1)$.

15. Let f: $\mathbf{R}^3 \to \mathbf{R}^2$ and g: $\mathbf{R}^3 \to \mathbf{R}^3$ be two vector fields defined as follows:

$$f(x, y, z) = (x^2 + y + z)\boldsymbol{i} + (2x + y + z^2)\boldsymbol{j},$$

$$g(u, v, w) = uv^2w^2\boldsymbol{i} + w^2 \sin v\, \boldsymbol{j} + u^2e^v\boldsymbol{k}.$$

(a) Compute each of the Jacobian matrices $Df(x, y, z)$ and $Dg(u, v, w)$.
(b) Compute the composition $h(u, u, w) = f[g(u, v, w)]$.
(c) Compute the Jacobian matrix $Dh(u, 0, w)$.

k8.23 Sufficient conditions for the equality of mixed partial derivatives

Iffis a real-valued function of two variables, the two mixed partial derivatives $D_{1,2}f$ and $D_{2,1}f$ are not necessarily equal. By $D_{1,2}f$ we mean $D_1(D_2f) = \partial^2 f/(\partial x\, \partial y)$, and by $D_{2,1}f$ we mean $D_2(D_1 f) = \partial^2 f/(\partial y\, \partial x)$. For example, if f is defined by the equations

$$f(x, y) = xy\,\frac{x^2 - y^2}{x^2 + y^2} \quad \text{for } (x, y) \neq (0, 0), \qquad f(0, 0) = 0\,,$$

it is easy to prove that $D_{2,1}f(0, 0) = -1$ and $D_{1,2}f(0, 0) = 1$. This may be seen as follows: The definition of $D_{2,1}f(0, 0)$ states that

$$D_{2,1}f(0, 0) = \lim_{k\to 0} \frac{D_1f(0, k) - D_1f(0, 0)}{k}. \tag{8.30}$$

Now we have

$$D_1f(0, 0) = \lim_{h\to 0} \frac{f(h, 0) - f(0, 0)}{h} = 0$$

and, if $(x, y) \neq (0, 0)$, we find

$$D_1f(x, y) = \frac{y(x^4 + 4x^2y^2 - y^4)}{(x^2 + y^2)^2}.$$

Therefore, if $k \neq 0$ we have $D_1f(0, k) = -k^5/k^4 = -k$ and hence

$$\frac{D_1f(0, k) - D_1f(0, 0)}{k} = -1$$

Using this in (8.30) we find that $D_{2,1}f(0, 0) = -1$. A similar argument shows that $D_{1,2}f(0, 0) = 1$, and hence $D_{2,1}f(0, 0) \neq D_{1,2}f(0, 0)$.

In the example just treated the two mixed partials $D_{1,2}f$ and $D_{2,1}f$ are not both continuous at the origin. It can be shown that the two mixed partials ***are*** equal at a point (a, b) if at least one of them is continuous in a neighborhood of the point. We shall prove first that they are equal if ***both*** are continuous. More precisely, we have the following theorem.

THEOREM 8.12. A SUFFICIENT CONDITION FOR EQUALITY OF MIXED PARTIAL DERIVATIVES. ***Assumef is a scalar field such that the partial derivatives D_1f, D_2f, $D_{1,2}f$, and $D_{2,1}f$ exist on an open set S. If (a, b) is a point in S at which both $D_{1,2}f$ and $D_{2,1}f$ are continuous, we have***

$$D_{1,2}f(a, b) = D_{2,1}f(a, b) \tag{8.31}$$

Proof. Choose nonzero ***h*** and ***k*** such that the rectangle $R(h, k)$ with vertices (a, b), $(a + h, b)$, $(a + h, b + k)$, and $(a, b + k)$ lies in S. (An example is shown in Figure 8.9.)

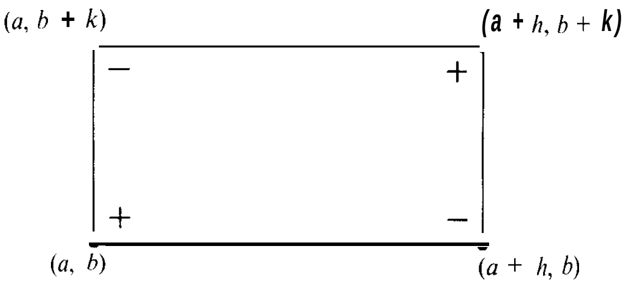

FIGURE 8.9 $\Delta(h, k)$ is a combination of values off at the vertices.

Consider the expression

$$\Delta(h, k) = f(a + h, b + k) - f(a + h, b) - f(a, b + k) + f(a, b).$$

This is a combination of the values off at the vertices of $R(h, k)$, taken with the algebraic signs indicated in Figure 8.9. We shall express $\Delta(h, k)$ in terms of $D_{2,1}f$ and also in terms of $D_{1,2}f$.

We consider a new function G of one variable defined by the equation

$$G(x) = f(x, b + k) - f(x, b)$$

for all x between a and $a + h$. (Geometrically, we are considering the values off at those points at which an arbitrary vertical line cuts the horizontal edges of $R(h, k)$.) Then we have

$$A(h, k) = G(a + h) - G(a). \tag{8.32}$$

Applying the one-dimensional mean-value theorem to the right-hand member of (8.32) we obtain $G(a + h) - G(a) = hG'(x_1)$, where x_1 lies between a and $a + h$. Since $G'(x) = D_1f(x, b + k) - D_1f(x, b)$, Equation (8.32) becomes

$$(8.33) \qquad \Delta(h, k) = h[D_1f(x_1, b + k) - D_1f(x_1, b)].$$

Applying the mean-value theorem to the right-hand member of (8.33) we obtain

$$(8.34) \qquad \Delta(h, k) = hkD_{2,1}f(x_1, y_1),$$

where y_1 lies between b and $b + k$. The point (x_1, y_1) lies somewhere in the rectangle $R(h, k)$.

Applying the same procedure to the function $H(y) = f(a + h, y) - f(a, y)$ we find a second expression for $\Delta(h, k)$, namely,

$$(8.35) \qquad \Delta(h, k) = hkD_{1,2}f(x_2, y_2),$$

where (x_2, y_2) also lies in $R(h, k)$. Equating the two expressions for $\Delta(h, k)$ and cancelling hk we obtain

$$D_{1,2}f(x_1, y_1) = D_{2,1}f(x_2, y_2).$$

Now we let $(h, k) \to (0, 0)$ and use the continuity of $D_{1,2}f$ and $D_{2,1}f$ to obtain (8.31).

The foregoing argument can be modified to prove a stronger version of Theorem 8.12.

THEOREM 8.13. *Let f be a scalar field such that the partial derivatives D_1f, D_2f, and $D_{2,1}f$ exist on an open set S containing (a, b). Assume further that $D_{2,1}f$ is continuous on S. Then the derivative $D_{1,2}f(a, b)$ exists and we have*

$$D_{1,2}f(a, b) = D_{2,1}f(a, b).$$

Proof. We define $\Delta(h, k)$ as in the proof of Theorem 8.12. The part of the proof leading to Equation (8.34) is still valid, giving us

$$(8.36) \qquad \frac{\Delta(h, k)}{hk} = D_{2,1}f(x_1, y_1)$$

for some (x_1, y_1) in the rectangle $R(h, k)$. The rest of the proof is not applicable since it requires the existence of the derivative $D_{1,2}f(a, b)$, which we now wish to prove.

The definition of $D_{1,2}f(a, b)$ states that

$$(8.37) \qquad D_{1,2}f(a, b) = \lim_{h \to 0} \frac{D_2f(a + h, b) - D_2f(a, b)}{h}.$$

We are to prove that this limit exists and has the value $D_{2,1}f\,(a, b)$. From the definition of $D_2 f$ we have

$$D_2 f(a,b) = \lim_{k\to 0}\frac{f(a, b + k) - f(a, b)}{k}$$

and

$$D_2 f(a + h, b) = \lim_{k\to 0}\frac{f(a + h, b + k) - f(a + h, b)}{k}$$

Therefore the difference quotient in (8.37) can be written as

$$\frac{D_2 f(a + h, b) - D_2 f(a, b)}{h} = \lim_{k\to 0}\frac{\Delta(h, k)}{hk}.$$

Using (8.36) we can rewrite this as

$$\text{(8.38)}\qquad \frac{D_2 f(a + h, b) - D_2 f(a, b)}{h} = \lim_{k\to 0} D_{2,1}f(x_1, y_1).$$

To complete the proof we must show that

$$\text{(8.39)}\qquad \lim_{h\to 0}\left[\lim_{k\to 0} D_{2,1}f(x_1, y_1)\right] = D_{2,1}f(a, b).$$

When $k \to 0$, the point $y_1 \to b$, but the behavior of x_1 as a function of k is unknown. If we knew that x_1 approached some limit, say $\bar{x}$, as $k \to 0$, we could use the continuity of $D_{2,1}f$ to deduce that

$$\lim_{k\to 0} D_{2,1}f(x_1, y_1) = D_{2,1}f(\bar{x}, b).$$

Since the limit $\bar{x}$ would have to lie in the interval $a \le \bar{x} \le a + h$, we could then let $h \to 0$ and deduce (8.39). However, the fact that 3 depends on k in an unknown fashion makes a slightly more involved argument necessary.

Because of Equation (8.38) we know that the following limit exists:

$$\lim_{k\to 0} D_{2,1}f(x_1, y_1).$$

Let us denote this limit by F(h). To complete the proof we must show that

$$\lim_{h\to 0} F(h) = D_{2,1}f(a, b).$$

For this purpose we appeal to the definition of continuity of $D_{2,1}f$ at (a, b).

Let ϵ be a given positive number. Continuity of $D_{2,1}f$ at (a, b) means that there is an open disk N with center (a, b) and radius δ, say, such that

$$\text{(8.40)}\qquad |D_{2,1}f(x,y) - D_{2,1}f(a, b)| < \frac{\epsilon}{2} \qquad \text{whenever} \qquad (x, y) \in N.$$

If we choose h and k so that $|h| < \delta/2$ and $|k| < \delta/2$, the entire rectangle shown in Figure 8.9 will lie in the neighborhood N and, specifically, the point (x_1, y_1) will be in N. Therefore (8.40) is valid when $(x, y) = (x_1, y_1)$ and we can write

$$0 \le |D_{2,1}f(x_1, y_1) - D_{2,1}f(a, b)| < \frac{\epsilon}{2}. \tag{8.41}$$

Now keep h fixed and let $k \to 0$. The term $D_{2,1}f(x_1, y_1)$ approaches $F(h)$ and the other terms in (8.41) are independent of k. Therefore we have

$$0 \le |F(h) - D_{2,1}f(a, b)| \le \frac{\epsilon}{2} < \epsilon,$$

provided that $0 < |h| < \delta/2$. But this is precisely the meaning of the statement

$$\lim_{h\to 0} F(h) = D_{2,1}f(a, b)$$

and, as we have already remarked, this completes the proof.

Note: It should be observed that the theorem is also valid if the roles of the two derivatives $D_{1,2}f$ and $D_{2,1}f$ are interchanged.

8.24 Miscellaneous exercises

1. Find a scalar field f satisfying both the following conditions:
(a) The partial derivatives $D_1f(0, 0)$ and $D_2f(0, 0)$ exist and are zero.
(b) The directional derivative at the origin in the direction of the vector $i + j$ exists and has the value 3. Explain why such an f cannot be differentiable at $(0, 0)$.
2. Let f be defined as follows :

$$f(x, y) = y\frac{x^2 - y^2}{x^2 + y^2} \quad \text{if } (x, y) \ne (0, 0), \qquad f(0, 0) = 0.$$

Compute the following partial derivatives, when they exist: $D_1f(0, 0)$, $D_2f(0, 0)$, $D_{2,1}f(0, 0)$, $D_{1,2}f(0, 0)$.
3. Let $f(x, y) = \dfrac{xy^3}{x^3 + y^6}$ if $(x, y) \ne (0, 0)$, and define $f(0, 0) = 0$.
(a) Prove that the derivative $f'(0; a)$ exists for every vector a and compute its value in terms of the components of a.
(b) Determine whether or not f is continuous at the origin.
4. Define $f(x, y) = \int_0^{\sqrt{xy}} e^{-t^2}\, dt$ for $x > 0$, $y > 0$. Compute $\partial f/\partial x$ in terms of x and y.
5. Assume that the equations $u = f(x, y)$, $x = X(t)$, $y = Y(t)$ define u as a function of t, say $u = F(t)$. Compute the third derivative $F'''(t)$ in terms of derivatives of f, X, and Y.
6. The change of variables $x = u + v$, $y = uv^2$ transforms $f(x, y)$ into $g(u, v)$. Compute the value of $\partial^2 g/(\partial v\, \partial u)$ at the point at which $u = 1$, $v = 1$, given that

$$\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial^2 f}{\partial y\, \partial x} = 1$$

at that point.

7. The change of variables $x = uv$, $y = \frac{1}{2}(u^2 - v^2)$ transforms $f(x, y)$ to $g(u, v)$.
(a) Calculate $\partial g/\partial u$, $\partial g/\partial v$, and $\partial^2 g/(\partial u\, \partial v)$ in terms of partial derivatives of f. (You may assume equality of mixed partials.)
(b) If $\|\nabla f(x, y)\|^2 = 2$ for all x and y, determine constants a and b such that

$$a\left(\frac{\partial g}{\partial u}\right)^2 - b\left(\frac{\partial g}{\partial v}\right)^2 = u^2 + v^2.$$

8. Two functions F and G of one variable and a function z of two variables are related by the equation

$$[F(x) + G(y)]^2 e^{z(x,y)} = 2F'(x)G'(y)$$

whenever $F(x) + G(y) \neq 0$. Show that the mixed partial derivative $D_{2,1}z(x, y)$ is never zero. (You may assume the existence and continuity of all derivatives encountered.)

9. A scalar field f is bounded and continuous on a rectangle $R = [a, b] \times [c, d]$. A new scalar field g is defined on R as follows:

$$g(u, v) = \int_c^v \left[\int_a^u f(x, y)\, dx\right] dy.$$

(a) It can be shown that for each fixed u in $[a, b]$ the function A defined on $[c, d]$ by the equation $A(y) = \int_a^u f(x, y)\, dx$ is continuous on $[c, d]$. Use this fact to prove that $\partial g/\partial v$ exists and is continuous on the open rectangle $S = (a, b) \times (c, d)$ (the interior of R).
(b) Assume that

$$\int_c^v \left[\int_a^u f(x, y)\, dx\right] dy = \int_a^u \left[\int_c^v f(x, y)\, dy\right] dx$$

for all (u, v) in R. Prove that g is differentiable on S and that the mixed partial derivatives $D_{1,2}g(u, v)$ and $D_{2,1}g(u, v)$ exist and are equal to $f(u, v)$ at each point of S.

10. Refer to Exercise 9. Suppose u and v are expressed parametrically as follows: $u = A(t)$, $v = B(t)$; and let $\varphi(t) = g[A(t), B(t)]$.
(a) Determine $\varphi'(t)$ in terms of f, A', and B'.
(b) Compute $\varphi'(t)$ in terms of t when $f(x, y) = e^{x+y}$ and $A(t) = B(t) = t^2$. (Assume R lies in the first quadrant.)

11. If $f(x, y, z) = (\boldsymbol{r} \times \boldsymbol{A}) \cdot (\boldsymbol{r} \times \boldsymbol{B})$, where $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ and $\boldsymbol{A}$ and $\boldsymbol{B}$ are constant vectors, show that $\nabla f(x, y, z) = \boldsymbol{B} \times (\boldsymbol{r} \times \boldsymbol{A}) + \boldsymbol{A} \times (\boldsymbol{r} \times \boldsymbol{B})$.

12. Let $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ and let $r = \|\boldsymbol{r}\|$. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are constant vectors, show that:

(a) $\displaystyle \boldsymbol{A} \cdot \nabla \frac{1}{r} = -\frac{\boldsymbol{A} \cdot \boldsymbol{r}}{r^3}.$

(b) $\displaystyle \boldsymbol{B} \cdot \nabla\left(\boldsymbol{A} \cdot \nabla\left(\frac{1}{r}\right)\right) = \frac{3\boldsymbol{A} \cdot \boldsymbol{r}\, \boldsymbol{B} \cdot \boldsymbol{r}}{r^5} - \frac{\boldsymbol{A} \cdot \boldsymbol{B}}{r^3}.$

13. Find the set of all points (a, b, c) in 3-space for which the two spheres $(x - a)^2 + (y - b)^2 + (z - c)^2 = 1$ and $x^2 + y^2 + z^2 = 1$ intersect orthogonally. (Their tangent planes should be perpendicular at each point of intersection.)

14. A cylinder whose equation is $y = f(x)$ is tangent to the surface $z^2 + 2xz + y = 0$ at all points common to the two surfaces. Find $f(x)$.

9

APPLICATIONS OF DIFFERENTIAL CALCULUS

9.1 Partial differential equations

The theorems of differential calculus developed in Chapter 8 have a wide variety of applications. This chapter illustrates their use in some examples related to partial differential equations, implicit functions, and extremum problems. We begin with some elementary remarks concerning partial differential equations.

An equation involving a scalar field f and its partial derivatives is called a *partial differential equation.* Two simple examples in which f is a function of two variables are the first-order equation

$$\frac{\partial f(x, y)}{\partial x} = 0, \tag{9.1}$$

and the second-order equation

$$\frac{\partial^2 f(x, y)}{\partial x^2} + \frac{\partial^2 f(x, y)}{\partial y^2} = 0. \tag{9.2}$$

Each of these is a homogeneous *linear* partial differential equation. That is, each has the form $L(f) = 0$, where L is a linear differential operator involving one or more partial derivatives. Equation (9.2) is called the two-dimensional *Laplace equation.*

Some of the theory of linear ordinary differential equations can be extended to partial differential equations. For example, it is easy to verify that for each of Equations (9.1) and (9.2) the set of solutions is a linear space. However, there is an important difference between ordinary and partial linear differential equations that should be realized at the outset. We illustrate this difference by comparing the partial differential equation (9.1) with the ordinary differential equation

$$f'(x) = 0. \tag{9.3}$$

The most general function satisfying (9.3) is $f(x) = C$, where C is an arbitrary constant: In other words, the solution-space of (9.3) is one-dimensional. But the most general function satisfying (9.1) is

$$f(x, y) = g(y),$$

where g is any function of y. Since g is arbitrary we can easily obtain an infinite set of independent solutions. For example, we can take $g(y) = e^{cy}$ and let c vary over all real numbers. Thus, the solution-space of (9.1) is inznite-dimensional.

In some respects this example is typical of what happens in general. Somewhere in the process of solving a first-order partial differential equation, an integration is required to remove each partial derivative. At this step an arbitrary function is introduced in the solution. This results in an infinite-dimensional solution space.

In many problems involving partial differential equations it is necessary to select from the wealth of solutions a particular solution satisfying one or more auxiliary conditions. As might be expected, the nature of these conditions has a profound effect on the existence or uniqueness of solutions. A systematic study of such problems will not be attempted in this book. Instead, we will treat some special cases to illustrate the ideas introduced in Chapter 8.

9.2 A first-order partial differential equation with constant coefficients

Consider the first-order partial differential equation

$$3\frac{\partial f(x, y)}{\partial x} + 2\frac{\partial f(x, y)}{\partial y} = 0 \tag{9.4}$$

All the solutions of this equation can be found by geometric considerations. We express the left member as a dot product, and write the equation in the form

$$(3\boldsymbol{i} + 2\boldsymbol{j}) \cdot \nabla f(x, y) = 0 .$$

This tells us that the gradient vector $\nabla f(x, y)$ is orthogonal to the vector $3\boldsymbol{i} + 2\boldsymbol{j}$ at each point (x, y). But we also know that $\nabla f(x, y)$ is orthogonal to the level curves of f. Hence these level curves must be straight lines parallel to $3\boldsymbol{i} + 2\boldsymbol{j}$. In other words, the level curves off are the lines

$$2x - 3y = c.$$

Therefore $f(x, y)$ is constant when $2x - 3y$ is constant. This suggests that

$$f(x, y) = g(2x - 3y) \tag{9.5}$$

for some function g.

Now we verify that, for each differentiable function g, the scalar fieldf defined by (9.5) does, indeed, satisfy (9.4). Using the chain rule to compute the partial derivatives off we find

$$\frac{\partial f}{\partial x} = 2g'(2x - 3y), \qquad \frac{\partial f}{\partial y} = -3g'(2x - 3y),$$

$$3\frac{\partial f}{\partial x} + 2\frac{\partial f}{\partial y} = 6g'(2x - 3y) - 6g'(2x - 3y) = 0.$$

Therefore, f satisfies (9.4).

Conversely, we can show that every differentiable f which satisfies (9.4) must necessarily have the form (9.5) for some g. To do this, we introduce a linear change of variables,

$$x = Au + Bv, \qquad y = Cu + Dv. \tag{9.6}$$

This transforms $f(x, y)$ into a function of u and v, say

$$h(u, v) = f(Au + Bv,\ Cu + Dv)\ .$$

We shall choose the constants A, B, C, D so that h satisfies the simpler equation

$$\frac{\partial h(u,\ v)}{\partial u} = 0\,. \tag{9.7}$$

Then we shall solve this equation and show that f has the required form.

Using the chain rule we find

$$\frac{\partial h}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} = \frac{\partial f}{\partial x}A + \frac{\partial f}{\partial y}C\,.$$

Since f satisfies (9.4) we have $\partial f/\partial y = -(3/2)(\partial f/\partial x)$, so the equation for $\partial h/\partial u$ becomes

$$\frac{\partial h}{\partial u} = \frac{\partial f}{\partial x}\left(A - \frac{3}{2}C\right).$$

Therefore, h will satisfy (9.7) if we choose $A = \frac{3}{2}C$. Taking $A = 3$ and C = 2 we find

$$x = 3u + Bv, \qquad y = 2u + Dv. \tag{9.8}$$

For this choice of A and C, the function h satisfies (9.7), so $h(u, v)$ is a function of v alone, say

$$h(u, v) = g(v)$$

for some function g. To express v in terms of x and y we eliminate u from (9.8) and obtain $2x - 3y = (2B - 3D)v$. Now we choose B and D to make $2B - 3D = 1$, say $B = 2$, $D = 1$. For this choice the transformation (9.6) is nonsingular; we have $v = 2x - 3y$, and hence

$$f(x, y) = h(u, v) = g(v) = g(2x - 3y)\,.$$

This shows that every differentiable solution f of (9.4) has the form (9.5).

Exactly the same type of argument proves the following theorem for first-order equations with constant coefficients.

THEOREM 9.1. *Let g he differentiable on $\mathbf{R}^1$, and let f be the scalar field dejined on $\mathbf{R}^2$ by the equation*

$$f(x,\ y) = g(bx - ay), \tag{9.9}$$

where a and b are constants, not both zero. Then f satisfies thejrst-order partial differential equation

$$a\frac{\partial f(x, y)}{\partial x} + b\frac{\partial f(x, y)}{\partial y} = 0 \tag{9.10}$$

everywhere in $\mathbf{R}^2$. Conversely, every differentiable solution of (9.10) necessarily has the form (9.9) for some g.

9.3 Exercises

In this set of exercises you may assume differentiability of all functions under consideration.

1. Determine that solution of the partial differential equation

$$4\frac{\partial f(x, y)}{\partial x} + 3\frac{\partial f(x, y)}{\partial y} = 0$$

which satisfies the condition $f(x, 0) = \sin x$ for all x.

2. Determine that solution of the partial differential equation

$$5\frac{\partial f(x, y)}{\partial x} - 2\frac{\partial f(x, y)}{\partial y} = 0$$

which satisfies the conditions $f(0, 0) = 0$ and $D_1 f(x, 0) = e^x$ for all x.

3. (a) If $u(x, y) = f(xy)$, prove that u satisfies the partial differential equation

$$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = 0.$$

Find a solution such that $u(x, x) = x^4 e^{x^2}$ for all x.

(b) If $v(x, y) = f(x/y)$ for $y \neq 0$, prove that v satisfies the partial-differential equation

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 0.$$

Find a solution such that $v(1, 1) = 2$ and $D_1 v(x, 1/x) = 1/x$ for all $x \neq 0$.

4. If $g(u, v)$ satisfies the partial differential equation

$$\frac{\partial^2 g(u, v)}{\partial u\, \partial v} = 0,$$

prove that $g(u, v) = \varphi_1(u) + \varphi_2(v)$, where $\varphi_1(u)$ is a function of u alone and $\varphi_2(v)$ is a function of v alone.

5. Assume f satisfies the partial differential equation

$$\frac{\partial^2 f}{\partial x^2} - 2\frac{\partial^2 f}{\partial x\, \partial y} - 3\frac{\partial^2 f}{\partial y^2} = 0.$$

Introduce the linear change of variables, $x = Au + Bv$, $y = Cu + Dv$, where A, B, C, D are constant, and let $g(u, v) = f(Au + Bv, Cu + Dv)$. Compute nonzero integer values of