propositions may be left in abeyance if it is, for the moment, inaccessible to human beings.

Aristotle also had something to contribute to the old problem that had divided the Ionic philosophers from those based in Italy: whether the universe is made up of substances or atoms, whether things should be measured or counted. He pointed out that when we talk about a loaf of bread or a glass of wine, bread and wine are measured, but loaves and glasses are counted. He asserted that we measure *matter* by counting its *forms.*

Exercise

How might Aristotle answer Zeno's arguments against motion?

14
Constructions with Ruler and Compass

Ancient Greek mathematicians were haunted by three problems:
I doubling the cube, that is, finding the cube root of 2;
II trisecting any given angle, say an angle of 60°
(of course some angles are easily trisected, for example, one of 90°);
III squaring the circle,
that is, constructing a square equal in area to that of a given circle.

Assorted solutions to these problems were proposed at various times, but these did not conform to the rules of the game, presumably laid down in Plato's Academy, that *only constructions with ruler and compass* be admitted. (Actually, we only know that Pappus attributed these rules to Plato more than 600 years later.) Moreover, the ruler could only be used for joining two points and the compass could only be used for drawing a circle with a given point as center and a given segment as radius. The reader will have no difficulty in carrying out the following constructions with ruler and compass:

(a) to bisect a given angle;

(b) to find the right bisector of a given segment;

(c) to draw a line through a given point parallel to a given line;

(d) to construct an equilateral triangle.

If we adopt a given segment as our unit of length, we can represent any positive real number by a segment, actually by the ratio of this segment to

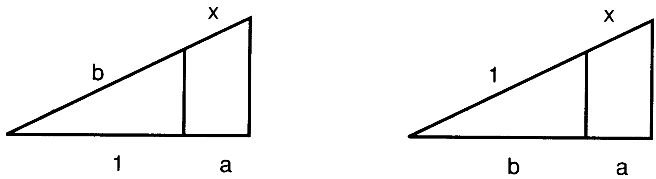

FIGURE 14.1. Finding ab and a/b

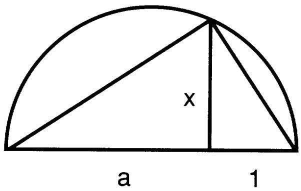

FIGURE 14.2. Constructing root of a

the unit segment. With the help of ruler and compass, the Greeks were able to perform the following arithmetical operations on positive real numbers: adding, subtracting (the smaller from the larger), multiplying, dividing and extracting square roots. The first four of these are called *rational* operations.

Indeed, for addition and subtraction this is obvious. To find $x = ab$ one considers the proportion

$$x : b = a : 1$$

and to find $x = a/b$ one considers the proportion

$$x : 1 = a : b.$$

In both cases, the problem is that of finding the fourth proportional to three given lengths, which can easily be done, following Thales, with the aid of similar triangles, using only ruler and compass constructions. See Figure 14.1.

To find $x = \sqrt{a}$, one looks at the proportion

$$1 : x = x : a.$$

Here the problem is that of finding the *mean* proportional of two given lengths, which the Greeks solved ingeniously by ruler and compass constructions, as illustrated by Figure 14.2, which exhibits the semicircle on a segment of length $a + 1$.

To attack problem **II**, the trisection problem, we would nowadays use trigonometry. First note that we can construct an angle if and only if we can construct its cosine. If $\theta = 60°/3 = 20°$, then $\cos 3\theta = \cos 60° = 1/2$, as is seen from Figure 14.3.

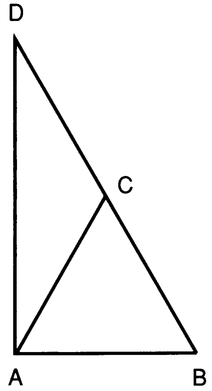

FIGURE 14.3. The cosine of 60°

In Figure 14.3, ABC is an equilateral triangle and $CD = AC$. It easily follows that the angle at B is 60°, the angle at D is 30° and the angle BAD is 90°, hence $\cos 60° = AB/BD = 1/2$.

Now

$$\begin{aligned}\cos 3\theta &= \cos(2\theta + \theta) = \cos 2\theta \cos\theta - \sin 2\theta \sin\theta \\ &= (\cos^2\theta - \sin^2\theta)\cos\theta - 2\sin\theta\cos\theta\sin\theta \\ &= \cos^3\theta - 3\sin^2\theta\cos\theta = \cos^3\theta - 3(1-\cos^2\theta)\cos\theta \\ &= 4\cos^3\theta - 3\cos\theta.\end{aligned}$$

Thus we want to solve the equation $8\cos^3\theta - 6\cos\theta = 1$. Putting $2\cos\theta = u$, we obtain the cubic equation

$$u^3 - 3u - 1 = 0\,.$$

The question is therefore whether a solution of this cubic equation can be expressed in terms of rational operations and square roots. We shall return to this problem in the next chapter. First let us look at a problem which the Greeks were able to solve by their methods.

One of the highlights in Euclid's *Elements* is the construction of a regular pentagon, equivalently, that of an angle of $360°/5 = 72°$. Today we would attack this problem too with the help of trigonometry; for an elegant argument we may even invoke complex numbers. Let $\theta = 72°$, then $5\theta = 360°$, hence, by de Moivre's Theorem,

$$\begin{aligned}(\cos\theta + i\sin\theta)^5 &= \cos 5\theta + i\sin 5\theta \\ &= \cos 360° + i\sin 360° \\ &= 1 + i0 \\ &= 1.\end{aligned}$$