Each polyhedron is bounded by a number of congruent polygonal faces, the same number of faces meet at each vertex, and in each face all the sides and angles are equal, hence the term *regular polyhedron*. A regular polyhedron is a spatial figure analogous to a regular polygon in the plane. But whereas there are regular polygons with any number $n \geq 3$ of sides, there are only five regular polyhedra.

This fact is easily proved and may go back to the Pythagoreans [see, for example Heath (1921), p. 159]. One considers the possible polygons that can occur as faces, their angles, and the numbers of them that can occur at a vertex. For a 3-gon (triangle) the angle is $\pi/3$, so three, four, or five can occur at a vertex, but six cannot, as this would give a total angle 2π and the vertex would be flat. For a 4-gon the angle is $\pi/2$, so three can occur at a vertex, but not four. For a 5-gon the angle is $3\pi/5$, so three can occur at a vertex, but not four. For a 6-gon the angle is $2\pi/3$, so not even three can occur at a vertex. But at least three faces must meet at each vertex of a polyhedron, so 6-gons (and, similarly, 7-gons, 8-gons, $\ldots$, n-gons) cannot occur as faces of a regular polyhedron. This leaves only the five possibilities just listed, which correspond to the five known regular polyhedra.

But do we really know that these five exist? There is no difficulty with the tetrahedron, cube, or octahedron, but it is not clear that, say, 20 equilateral triangles will fit together to form a closed surface. Euclid found this problem difficult enough to be placed near the end of the *Elements*, and few of his readers ever mastered his solution. A beautiful direct construction was given by Luca Pacioli, a friend of Leonardo da Vinci's, in his book *De divina proportione* (1509). Pacioli's construction uses three copies of the *golden rectangle*, with sides 1 and $(1+\sqrt{5})/2$, interlocking as in Figure 2.2. The 12 vertices define 20 triangles such as ABC, and it suffices to show that these are equilateral, that is, $AB = 1$. This is a straightforward exercise in Pythagoras' theorem (Exercise 2.2.2).

The regular polyhedra will make another important appearance in connection with yet another nineteenth-century development, the theory of finite groups and Galois theory. Before the regular polyhedra made this triumphant comeback, they also took part in a famous fiasco: Kepler's theory of planetary distances [Kepler (1596)]. Kepler's theory is summarized by his famous diagram (Figure 2.3) of the five polyhedra, nested in such a way as to produce six spheres of radii proportional to the distances of the six planets then known. Unfortunately, although mathematics could not

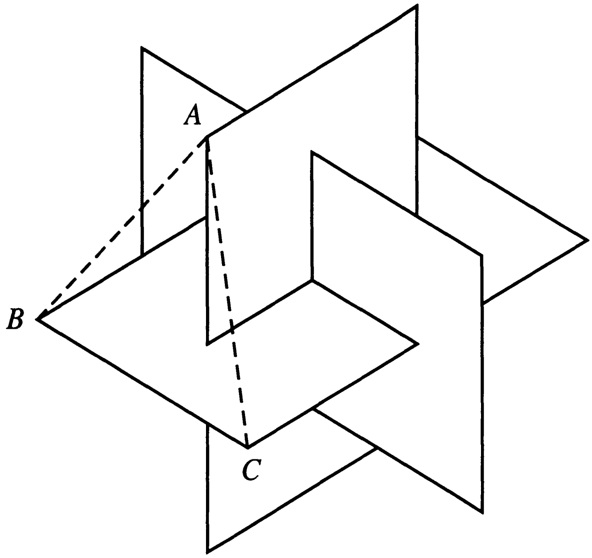

Figure 2.2: Pacioli's construction of the icosahedron

permit any more regular polyhedra, nature could permit more planets, and Kepler's theory was ruined when Uranus was discovered in 1781.

EXERCISES

The ratios between successive radii in Kepler's construction depend on what may be called the *inradius* and *circumradius* of each polyhedron—the radii of the spheres which touch it on the inside and the outside. It happens that the ratio

$$\frac{\text{circumradius}}{\text{inradius}}$$

is the same for the cube and the octahedron, and it is also the same for the dodecahedron and the icosahedron. This implies that the cube and octahedron can be exchanged in Kepler's construction, as can the dodecahedron and the icosahedron. Thus there are at least four different arrangements of the regular polyhedra that yield the same sequence of radii.

It is easy to see why the cube and the octahedron are interchangeable.

2.2.1 Show that $\frac{\text{circumradius}}{\text{inradius}} = \sqrt{3}$ for both the cube and the octahedron.

To compute circumradius/inradius for the icosahedron and the dodecahedron, we pursue Pacioli's construction a little further, with the help of vector addition.

2.2.2 First check Pacioli's construction: use the Pythagorean theorem to show that $AB = BC = CA$ in Figure 2.2. (It may help to use the additional fact

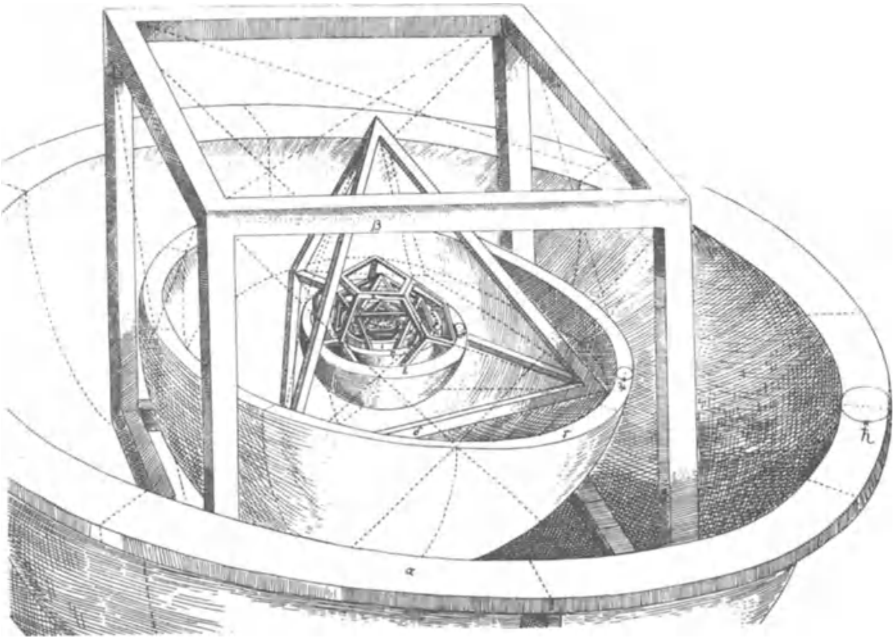

Figure 2.3: Kepler's diagram of the polyhedra

that $\tau = (1+\sqrt{5})/2$ satisfies $\tau^2 = \tau + 1$. This is also useful in the exercises below.)

Now, to simplify coordinates, we take golden rectangles that are twice the normal size—length 2τ and width 2—and place them in the three coordinate planes in the relative positions shown in Figure 2.2, so $O = (0,0,0)$ is at the center of each rectangle.

2.2.3 Show that the coordinates of the vertices of the icosahedron are $(\pm 1, 0, \pm\tau)$, $(\pm\tau, \pm 1, 0)$, and $(0, \pm\tau, \pm 1)$, for all possible combinations of $+$ and $-$ signs.

2.2.4 In particular, show that suitably chosen axes give $A = (1,0,\tau)$, $B = (\tau,-1,0)$, and $C = (\tau,1,0)$ in Figure 2.2. Deduce that

$$\text{circumradius} = \sqrt{\tau+2} \quad \text{for this icosahedron.}$$

To find the inradius, we find the center of the triangle ABC, then compute its distance from O.

2.2.5 Show that the center of the triangle ABC is $\frac{1}{3}(2\tau+1, 0, \tau)$, and hence that

$$\text{inradius} = \frac{1}{3}\sqrt{9\tau+6} \quad \text{for this icosahedron.}$$

It follows that

$$\frac{\text{circumradius}}{\text{inradius}} = \frac{3\sqrt{\tau+2}}{\sqrt{9\tau+6}} \quad \text{for any icosahedron,}$$

but it will be helpful to have this number in a simpler form.

2.2.6 Show that $\frac{3\sqrt{\tau+2}}{\sqrt{9\tau+6}} = \sqrt{3(7-4\tau)} = \sqrt{\frac{15}{4\tau+3}}$.

Now to compute the ratio circumradius/inradius for the dodecahedron, we use the *dual dodecahedron*, whose vertices are the face centers, such as $\frac{1}{3}(A+B+C)$, of the icosahedron above. This immediately gives

$$\text{circumradius of dual dodecahedron} = \text{inradius of icosahedron} = \frac{1}{3}\sqrt{9\tau+6}.$$

Thus it remains to find the inradius of the dual dodecahedron, which is the distance from O to one of *its* face centers. A face of the dual dodecahedron is a pentagon, with vertices, for example,

$$\frac{1}{3}(A+B+C), \quad \frac{1}{3}(A+C+D), \quad \frac{1}{3}(A+D+E), \quad \frac{1}{3}(A+E+F), \quad \frac{1}{3}(A+F+B),$$

where B,C,D,E,F are the five vertices of the icosahedron equidistant from A.

2.2.7 Using $A=(1,0,\tau), B=(\tau,-1,0), C=(\tau,1,0), D=(0,\tau,1), E=(-1,0,\tau)$, and $F=(0,-\tau,1)$, show that the face center of the pentagon with the above vertices is

$$\frac{1}{15}(5A+2B+2C+2D+2E+2F) = \frac{1}{15}(4\tau+3,0,7\tau+4) = \frac{4\tau+3}{15}(1,0,\tau),$$

and hence that

$$\text{inradius of the dual dodecahedron} = \frac{4\tau+3}{15}\sqrt{\tau+2}.$$

2.2.8 Deduce from Exercises 2.2.7 and 2.2.6 that

$$\frac{\text{circumradius}}{\text{inradius}} \text{ for dodecahedron} = \sqrt{\frac{15}{4\tau+3}} = \frac{\text{circumradius}}{\text{inradius}} \text{ for icosahedron.}$$

Another remarkable result follows from this, using the fact that the volume of a pyramid $= 1/3$ base area $\times$ height. The result is attributed to Apollonius.

2.2.9 By dividing the polyhedra into pyramids with bases equal to the faces, and height equal to the inradius, establish the following relationship between the dodecahedron D and the icosahedron I of the same circumradius:

$$\frac{\text{surface area } D}{\text{surface area } I} = \frac{\text{volume } D}{\text{volume } I}.$$

2.3 Ruler and Compass Constructions

Greek geometers prided themselves on their logical purity; nevertheless, they were guided by intuition about physical space. One aspect of Greek geometry that was peculiarly influenced by physical considerations was the theory of constructions. Much of the elementary geometry of straight lines and circles can be viewed as the theory of constructions by ruler and compass. The very subject matter, lines and circles, reflects the instruments used to draw them. And many of the elementary problems of geometry—for example, to bisect a line segment or angle, construct a perpendicular, or draw a circle through three given points—can be solved by ruler and compass constructions.

When coordinates are introduced, it is not hard to show that the points constructible from points $P_1, \ldots, P_n$ have coordinates in the set of numbers generated from the coordinates of $P_1, \ldots, P_n$ by the operations $+$, $-$, $\times$, $\div$ and $\sqrt{\ }$ [see Moise (1963) or the exercises to Section 6.3]. Square roots arise, of course, because of Pythagoras' theorem: if the points (a,b) and (c,d) have been constructed, then so has the distance $\sqrt{(c-a)^2+(d-b)^2}$ between them (Section 1.6 and Figure 2.4). Conversely, it is possible to construct $\sqrt{l}$ for any given length l (Exercise 2.3.2).

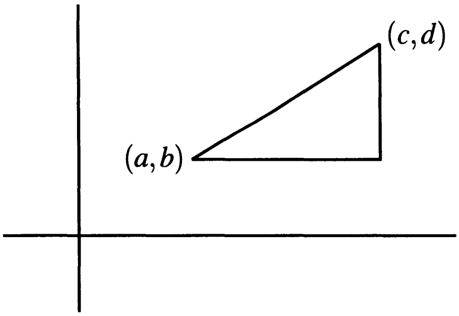

Figure 2.4: Construction of a distance

Looked at from this point of view, ruler and compass constructions look very special and unlikely to yield numbers such as $\sqrt[3]{2}$, for example. However, the Greeks tried very hard to solve just this problem, which was known as *duplication of the cube* (so-called because to double the volume of a cube one must multiply the side by $\sqrt[3]{2}$). Other notorious problems were *trisection of the angle* and *squaring the circle*. The latter problem was to construct a square equal in area to a given circle or to construct the number π, which amounts to the same thing. They never seem to have given up these goals, though the possibility of a negative solution was admitted and solutions by less elementary means were tolerated. We shall see some of these in the next sections.

The impossibility of solving these problems by ruler and compass constructions was not proved until the nineteenth century. For the duplication of the cube and trisection of the angle, impossibility was shown by Wantzel (1837). Wantzel seldom receives credit for settling these problems, which had baffled the best mathematicians for 2000 years, perhaps because his methods were superseded by the more powerful theory of Galois.

The impossibility of squaring the circle was proved by Lindemann (1882), in a very strong way. Not only is π undefinable by rational operations and square roots; it is also *transcendental*, that is, not the root of any polynomial equation with rational coefficients. Like Wantzel's work, this was a rare example of a major result being proved by a minor mathematician. In Lindemann's case the explanation is perhaps that a major step had already been taken when Hermite (1873) proved the transcendence of e. Accessible proofs of both these results can be found in Klein (1924). Lindemann's subsequent career was mathematically undistinguished, even embarrassing. In response to skeptics who thought his success with π had been a fluke, he took aim at the most famous unsolved problem in mathematics, "Fermat's last theorem" (see Chapter 11 for the origin of this problem). His efforts fizzled out in a series of inconclusive papers, each one correcting an error in the one before. Fritsch (1984) has written an interesting biographical article on Lindemann.

One ruler and compass problem is still open: which regular n-gons are constructible? Gauss discovered in 1796 that the 17-gon is constructible and then showed that a regular n-gon is constructible if and only if $n = 2^m p_1 p_2 \cdots p_k$, where each p_i is a prime of the form $2^{2^h} + 1$. (This problem is also known as *circle division*, because it is equivalent to dividing the circumference of a circle, or the angle 2π, into n equal parts.) The proof of necessity was actually completed by Wantzel (1837). However, it is still not explicitly known what these primes are, or even whether there are infinitely many of them. The only ones known are for $h = 0, 1, 2, 3$.

Exercises

Many of the constructions made by the Greeks can be simplified by translating them into algebra, where it turns out that constructible lengths are those that can be built from known lengths by the operations of $+$, $-$, $\times$, $\div$, and $\sqrt{\ }$. It is therefore enough to know constructions for these five basic operations. Addition and subtraction are obvious, and the other operations are covered in the following exercises, together with an example where algebra is a distinct advantage.

2.3.1 Show, using similar triangles, that if lengths l_1 and l_2 are constructible, then so are $l_1 l_2$ and l_1/l_2.

2.3.2 Use similar triangles to explain why $\sqrt{l}$ is the length shown in Figure 2.5, and hence show that $\sqrt{l}$ is constructible from l.

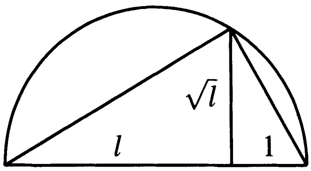

Figure 2.5: Square root construction

One of the finest ruler and compass constructions from ancient times is the regular pentagon, which includes yet another occurrence of the golden ratio $\tau = (1+\sqrt{5})/2$. Knowing (from the questions above) that this number is constructible, it becomes easy for us to construct the pentagon itself.

2.3.3 By finding some parallels and similar triangles in Figure 2.6, show that the diagonal x of the regular pentagon of side 1 satisfies $x/1 = 1/(x-1)$.

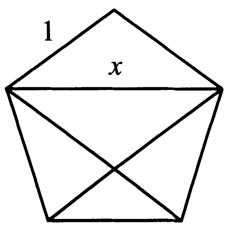

Figure 2.6: The regular pentagon

2.3.4 Deduce from Exercise 2.3.3 that the diagonal of the pentagon is $(1+\sqrt{5})/2$ and hence that the regular pentagon is constructible.

2.4 Conic Sections

Conic sections are the curves obtained by intersecting a circular cone by a plane: hyperbolas, ellipses (including circles), and parabolas (Figure 2.7, left to right). Today we know the conic sections better in terms of their equations in cartesian coordinates:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \qquad \text{(hyperbola)}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad \text{(ellipse)}$$

$$y = ax^2. \qquad \text{(parabola)}$$

More generally, any second-degree equation represents a conic section or a pair of straight lines, a result that was proved by Descartes (1637).

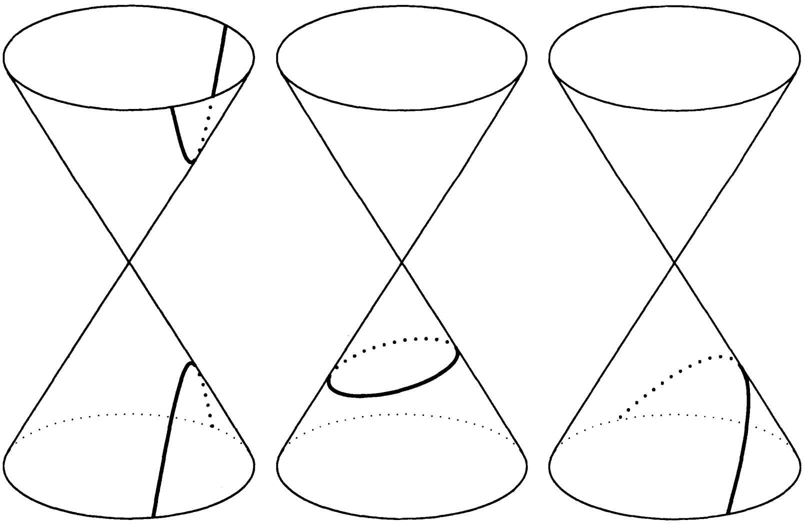

Figure 2.7: The conic sections

The invention of conic sections is attributed to Menaechmus (fourth century BCE), a contemporary of Alexander the Great. Alexander is said to have asked Menaechmus for a crash course in geometry, but Menaechmus refused, saying "There is no royal road to geometry." Menaechmus used