we see that the equation

$$f(x) = \lim_{k \to \infty} f_{n_k}(x)$$

defines $f(x)$ for almost all $x \in X$, and it does not matter how we define $f(x)$ at the remaining points of X.

We shall now show that this function f has the desired properties. Let $\varepsilon > 0$ be given, and choose N as indicated in Definition 11.41. If $n_k > N$, Fatou's theorem shows that

$$\|f - f_{n_k}\| \leq \liminf_{i \to \infty} \|f_{n_i} - f_{n_k}\| \leq \varepsilon.$$

Thus $f - f_{n_k} \in \mathscr{L}^2(\mu)$, and since $f = (f - f_{n_k}) + f_{n_k}$, we see that $f \in \mathscr{L}^2(\mu)$. Also, since ε is arbitrary,

$$\lim_{k \to \infty} \|f - f_{n_k}\| = 0.$$

Finally, the inequality

$$\|f - f_n\| \leq \|f - f_{n_k}\| + \|f_{n_k} - f_n\| \tag{105}$$

shows that $\{f_n\}$ converges to f in $\mathscr{L}^2(\mu)$; for if we take n and n_k large enough, each of the two terms on the right of (105) can be made arbitrarily small.

11.43 The Riesz-Fischer theorem *Let $\{\phi_n\}$ be orthonormal on X. Suppose $\Sigma |c_n|^2$ converges, and put $s_n = c_1\phi_1 + \cdots + c_n\phi_n$. Then there exists a function $f \in \mathscr{L}^2(\mu)$ such that $\{s_n\}$ converges to f in $\mathscr{L}^2(\mu)$, and such that*

$$f \sim \sum_{n=1}^{\infty} c_n \phi_n.$$

Proof For $n > m$,

$$\|s_n - s_m\|^2 = |c_{m+1}|^2 + \cdots + |c_n|^2,$$

so that $\{s_n\}$ is a Cauchy sequence in $\mathscr{L}^2(\mu)$. By Theorem 11.42, there is a function $f \in \mathscr{L}^2(\mu)$ such that

$$\lim_{n \to \infty} \|f - s_n\| = 0.$$

Now, for $n > k$,

$$\int_X f\bar{\phi}_k \, d\mu - c_k = \int_X f\bar{\phi}_k \, d\mu - \int_X s_n \bar{\phi}_k \, d\mu,$$

so that

$$\left|\int_X f\bar{\phi}_k \, d\mu - c_k\right| \le \|f - s_n\| \cdot \|\phi_k\| + \|f - s_n\|.$$

Letting $n \to \infty$, we see that

$$c_k = \int_X f\bar{\phi}_k \, d\mu \qquad (k = 1, 2, 3, \ldots),$$

and the proof is complete.

11.44 Definition An orthonormal set $\{\phi_n\}$ is said to be *complete* if, for $f \in \mathscr{L}^2(\mu)$, the equations

$$\int_X f\bar{\phi}_n \, d\mu = 0 \qquad (n = 1, 2, 3, \ldots)$$

imply that $\|f\| = 0$.

In the Corollary to Theorem 11.40 we deduced the completeness of the trigonometric system from the Parseval equation (101). Conversely, the Parseval equation holds for every complete orthonormal set:

11.45 Theorem *Let $\{\phi_n\}$ be a complete orthonormal set. If $f \in \mathscr{L}^2(\mu)$ and if*

$$f \sim \sum_{n=1}^{\infty} c_n \phi_n, \tag{106}$$

then

$$\int_X |f|^2 \, d\mu = \sum_{n=1}^{\infty} |c_n|^2. \tag{107}$$

Proof By the Bessel inequality, $\Sigma |c_n|^2$ converges. Putting

$$s_n = c_1\phi_1 + \cdots + c_n\phi_n,$$

the Riesz-Fischer theorem shows that there is a function $g \in \mathscr{L}^2(\mu)$ such that

$$g \sim \sum_{n=1}^{\infty} c_n \phi_n, \tag{108}$$

and such that $\|g - s_n\| \to 0$. Hence $\|s_n\| \to \|g\|$. Since

$$\|s_n\|^2 = |c_1|^2 + \cdots + |c_n|^2,$$

we have

$$\int_X |g|^2 \, d\mu = \sum_{n=1}^{\infty} |c_n|^2. \tag{109}$$

Now (106), (108), and the completeness of $\{\phi_n\}$ show that $\|f - g\| = 0$, so that (109) implies (107).

Combining Theorems 11.43 and 11.45, we arrive at the very interesting conclusion that every complete orthonormal set induces a 1-1 correspondence between the functions $f \in \mathscr{L}^2(\mu)$ (identifying those which are equal almost everywhere) on the one hand and the sequences $\{c_n\}$ for which $\Sigma |c_n|^2$ converges, on the other. The representation

$$f \sim \sum_{n=1}^{\infty} c_n \phi_n,$$

together with the Parseval equation, shows that $\mathscr{L}^2(\mu)$ may be regarded as an infinite-dimensional euclidean space (the so-called "Hilbert space"), in which the point f has coordinates c_n, and the functions ϕ_n are the coordinate vectors.

EXERCISES

1. If $f \geq 0$ and $\int_E f\,d\mu = 0$, prove that $f(x) = 0$ almost everywhere on E. *Hint:* Let E_n be the subset of E on which $f(x) > 1/n$. Write $A = \bigcup E_n$. Then $\mu(A) = 0$ if and only if $\mu(E_n) = 0$ for every n.
2. If $\int_A f\,d\mu = 0$ for every measurable subset A of a measurable set E, then $f(x) = 0$ almost everywhere on E.
3. If $\{f_n\}$ is a sequence of measurable functions, prove that the set of points x at which $\{f_n(x)\}$ converges is measurable.
4. If $f \in \mathscr{L}(\mu)$ on E and g is bounded and measurable on E, then $fg \in \mathscr{L}(\mu)$ on E.
5. Put

$$g(x) = \begin{cases} 0 & (0 \leq x \leq \tfrac{1}{2}), \\ 1 & (\tfrac{1}{2} < x \leq 1), \end{cases}$$
$$f_{2k}(x) = g(x) \qquad (0 \leq x \leq 1),$$
$$f_{2k+1}(x) = g(1-x) \qquad (0 \leq x \leq 1).$$

Show that

$$\liminf_{n \to \infty} f_n(x) = 0 \qquad (0 \leq x \leq 1),$$

but

$$\int_0^1 f_n(x)\,dx = \tfrac{1}{2}.$$

[Compare with (77).]

6. Let

$$f_n(x) = \begin{cases} \dfrac{1}{n} & (|x| \le n), \\ 0 & (|x| > n). \end{cases}$$

Then $f_n(x) \to 0$ uniformly on R^1, but

$$\int_{-\infty}^{\infty} f_n \, dx = 2 \qquad (n = 1, 2, 3, \ldots).$$

(We write $\int_{-\infty}^{\infty}$ in place of $\int_{R^1}$.) Thus uniform convergence does not imply dominated convergence in the sense of Theorem 11.32. However, on sets of finite measure, uniformly convergent sequences of bounded functions do satisfy Theorem 11.32.

7. Find a necessary and sufficient condition that $f \in \mathscr{R}(\alpha)$ on $[a, b]$. *Hint:* Consider Example 11.6(b) and Theorem 11.33.

8. If $f \in \mathscr{R}$ on $[a, b]$ and if $F(x) = \int_a^x f(t)\, dt$, prove that $F'(x) = f(x)$ almost everywhere on $[a, b]$.

9. Prove that the function F given by (96) is continuous on $[a, b]$.

10. If $\mu(X) < +\infty$ and $f \in \mathscr{L}^2(\mu)$ on X, prove that $f \in \mathscr{L}(\mu)$ on X. If

$$\mu(X) = +\infty,$$

this is false. For instance, if

$$f(x) = \frac{1}{1 + |x|},$$

then $f \in \mathscr{L}^2$ on R^1, but $f \notin \mathscr{L}$ on R^1.

11. If $f, g \in \mathscr{L}(\mu)$ on X, define the distance between f and g by

$$\int_X |f - g| \, d\mu.$$

Prove that $\mathscr{L}(\mu)$ is a complete metric space.

12. Suppose

(a) $|f(x, y)| \le 1$ if $0 \le x \le 1, 0 \le y \le 1$,

(b) for fixed x, $f(x, y)$ is a continuous function of y,

(c) for fixed y, $f(x, y)$ is a continuous function of x.

Put

$$g(x) = \int_0^1 f(x, y) \, dy \qquad (0 \le x \le 1).$$

Is g continuous?

13. Consider the functions

$$f_n(x) = \sin nx \qquad (n = 1, 2, 3, \ldots, -\pi \le x \le \pi)$$

as points of $\mathscr{L}^2$. Prove that the set of these points is closed and bounded, but not compact.

14. Prove that a complex function f is measurable if and only if $f^{-1}(V)$ is measurable for every open set V in the plane.

15. Let $\mathscr{R}$ be the ring of all elementary subsets of $(0, 1]$. If $0 < a \le b \le 1$, define

$$\phi([a, b]) = \phi([a, b)) = \phi((a, b]) = \phi((a, b)) = b - a,$$

but define

$$\phi((0, b)) = \phi((0, b]) = 1 + b$$

if $0 < b \le 1$. Show that this gives an additive set function ϕ on $\mathscr{R}$, which is not regular and which cannot be extended to a countably additive set function on a σ-ring.

16. Suppose $\{n_k\}$ is an increasing sequence of positive integers and E is the set of all $x \in (-\pi, \pi)$ at which $\{\sin n_k x\}$ converges. Prove that $m(E) = 0$. *Hint:* For every $A \subset E$,

$$\int_A \sin n_k x \, dx \to 0,$$

and

$$2 \int_A (\sin n_k x)^2 \, dx = \int_A (1 - \cos 2n_k x) \, dx \to m(A) \qquad \text{as } k \to \infty.$$

17. Suppose $E \subset (-\pi, \pi)$, $m(E) > 0$, $\delta > 0$. Use the Bessel inequality to prove that there are at most finitely many integers n such that $\sin nx \ge \delta$ for all $x \in E$.

18. Suppose $f \in \mathscr{L}^2(\mu)$, $g \in \mathscr{L}^2(\mu)$. Prove that

$$\left| \int f\bar{g} \, d\mu \right|^2 = \int |f|^2 \, d\mu \int |g|^2 \, d\mu$$

if and only if there is a constant c such that $g(x) = cf(x)$ almost everywhere. (Compare Theorem 11.35.)

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LIST OF SPECIAL SYMBOLS

The symbols listed below are followed by a brief statement of their meaning and by the number of the page on which they are defined.

INDEX