The notion of equivalence can be phrased more succinctly using our language of limits:

Lemma 9.9.7. *Let $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ be sequences of real numbers (not necessarily bounded or convergent). Then $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ are equivalent if and only if* $\lim_{n\to\infty}(a_n - b_n) = 0$.

Proof. See Exercise 9.9.1. □

Meanwhile, the notion of uniform continuity can be phrased using equivalent sequences:

Proposition 9.9.8. *Let X be a subset of $\mathbf{R}$, and let $f : X \to \mathbf{R}$ be a function. Then the following two statements are logically equivalent:*

(a) *f is uniformly continuous on X.*

(b) *Whenever $(x_n)_{n=0}^\infty$ and $(y_n)_{n=0}^\infty$ are two equivalent sequences consisting of elements of X, the sequences $(f(x_n))_{n=0}^\infty$ and $(f(y_n))_{n=0}^\infty$ are also equivalent.*

Proof. See Exercise 9.9.2. □

Remark 9.9.9. The reader should compare this with Proposition 9.3.9. Proposition 9.3.9 asserted that if f was continuous, then f maps convergent sequences to convergent sequences. In contrast, Proposition 9.9.8 asserts that if f is *uniformly* continuous, then f maps *equivalent* pairs of sequences to equivalent pairs of sequences. To see how the two Propositions are connected, observe from Lemma 9.9.7 that $(x_n)_{n=0}^\infty$ will converge to x_* if and only if the sequences $(x_n)_{n=0}^\infty$ and $(x_*)_{n=0}^\infty$ are equivalent.

Example 9.9.10. Consider the function $f : (0,2) \to \mathbf{R}$ defined by $f(x) := 1/x$ considered earlier. From Lemma 9.9.7 we see that the sequence $(1/n)_{n=1}^\infty$ and $(1/2n)_{n=1}^\infty$ are equivalent sequences in $(0,2)$. However, the sequences $(f(1/n))_{n=1}^\infty$ and $(f(1/2n))_{n=1}^\infty$ are not equivalent (why? Use Lemma 9.9.7 again). So by Proposition 9.9.8, f is not uniformly continuous. (These sequences start at 1 instead of 0, but the reader can easily see that this makes no difference to the above discussion.)

Example 9.9.11. Consider the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) := x^2$. This is a continuous function on $\mathbf{R}$, but it turns out not to be uniformly continuous; in some sense the continuity gets "worse and worse" as one approaches infinity. One way to quantify this is via Proposition 9.9.8. Consider the sequences $(n)_{n=1}^{\infty}$ and $(n+\frac{1}{n})_{n=1}^{\infty}$. By Lemma 9.9.7, these sequences are equivalent. But the sequences $(f(n))_{n=1}^{\infty}$ and $(f(n+\frac{1}{n}))_{n=1}^{\infty}$ are not equivalent, since $f(n+\frac{1}{n}) = n^2 + 2 + \frac{1}{n^2} = f(n) + 2 + \frac{1}{n^2}$ does not become eventually 2-close to $f(n)$. By Proposition 9.9.8 we can thus conclude that f is not uniformly continuous.

Another property of uniformly continuous functions is that they map Cauchy sequences to Cauchy sequences.

Proposition 9.9.12. *Let X be a subset of $\mathbf{R}$, and let $f : X \to \mathbf{R}$ be a uniformly continuous function. Let $(x_n)_{n=0}^{\infty}$ be a Cauchy sequence consisting entirely of elements in X. Then $(f(x_n))_{n=0}^{\infty}$ is also a Cauchy sequence.*

Proof. See Exercise 9.9.3. ☐

Example 9.9.13. Once again, we demonstrate that the function $f : (0, 2) \to \mathbf{R}$ defined by $f(x) := 1/x$ is not uniformly continuous. The sequence $(1/n)_{n=1}^{\infty}$ is a Cauchy sequence in $(0, 2)$, but the sequence $(f(1/n))_{n=1}^{\infty}$ is not a Cauchy sequence (why?). Thus by Proposition 9.9.12, f is not uniformly continuous.

Corollary 9.9.14. *Let X be a subset of $\mathbf{R}$, let $f : X \to \mathbf{R}$ be a uniformly continuous function, and let x_0 be an adherent point of X. Then the limit $\lim_{x\to x_0;x\in X} f(x)$ exists (in particular, it is a real number).*

Proof. See Exercise 9.9.4. ☐

We now show that a uniformly continuous function will map bounded sets to bounded sets.

Proposition 9.9.15. *Let X be a subset of $\mathbf{R}$, and let $f : X \to \mathbf{R}$ be a uniformly continuous function. Suppose that E is a bounded subset of X. Then $f(E)$ is also bounded.*

Proof. See Exercise 9.9.5. $\square$

As we have just seen repeatedly, not all continuous functions are uniformly continuous. However, if the domain of the function is a closed interval, then continuous functions are in fact uniformly continuous:

Theorem 9.9.16. *Let $a < b$ be real numbers, and let $f : [a, b] \to \mathbf{R}$ be a function which is continuous on $[a, b]$. Then f is also uniformly continuous.*

Proof. Suppose for sake of contradiction that f is not uniformly continuous. By Proposition 9.9.8, there must therefore exist two equivalent sequences $(x_n)_{n=0}^\infty$ and $(y_n)_{n=0}^\infty$ in $[a, b]$ such that the sequences $(f(x_n))_{n=0}^\infty$ and $(f(y_n))_{n=0}^\infty$ are not equivalent. In particular, we can find an $\varepsilon > 0$ such that $(f(x_n))_{n=0}^\infty$ and $(f(y_n))_{n=0}^\infty$ are not eventually ε-close.

Fix this value of ε, and let E be the set

$$E := \{n \in N : f(x_n) \text{ and } f(y_n) \text{ are not } \varepsilon\text{-close}\}.$$

We must have E infinite, since if E were finite then $(f(x_n))_{n=0}^\infty$ and $(f(y_n))_{n=0}^\infty$ would be eventually ε-close (why?). By Proposition 8.1.5, E is countable; in fact from the proof of that proposition we see that we can find an infinite sequence

$$n_0 < n_1 < n_2 < \dots$$

consisting entirely of elements in E. In particular, we have

$$|f(x_{n_j}) - f(y_{n_j})| > \varepsilon \text{ for all } j \in \mathbf{N}. \tag{9.3}$$

On the other hand, the sequence $(x_{n_j})_{j=0}^\infty$ is a sequence in $[a, b]$, and so by the Heine-Borel theorem (Theorem 9.1.24) there must be a subsequence $(x_{n_{j_k}})_{k=0}^\infty$ which converges to some limit L in $[a, b]$. In particular, f is continuous at L, and so by Proposition 9.4.7,

$$\lim_{k\to\infty} f(x_{n_{j_k}}) = f(L). \tag{9.4}$$

Note that $(x_{n_{j_k}})_{k=0}^\infty$ is a subsequence of $(x_n)_{n=0}^\infty$, and $(y_{n_{j_k}})_{k=0}^\infty$ is a subsequence of $(y_n)_{n=0}^\infty$, by Lemma 6.6.4. On the other hand, from Lemma 9.9.7 we have

$$\lim_{n\to\infty}(x_n - y_n) = 0.$$

By Proposition 6.6.5, we thus have

$$\lim_{k\to\infty}(x_{n_{j_k}} - y_{n_{j_k}}) = 0.$$

Since $x_{n_{j_k}}$ converges to L as $k \to \infty$, we thus have by limit laws

$$\lim_{k\to\infty} y_{n_{j_k}} = L$$

and hence by continuity of f at L

$$\lim_{k\to\infty} f(y_{n_{j_k}}) = f(L).$$

Subtracting this from (9.4) using limit laws, we obtain

$$\lim_{k\to\infty}(f(x_{n_{j_k}}) - f(y_{n_{j_k}})) = 0.$$

But this contradicts (9.3) (why?). From this contradiction we conclude that f is in fact uniformly continuous. □

Remark 9.9.17. One should compare Lemma 9.6.3, Proposition 9.9.15, and Theorem 9.9.16 with each other. No two of these results imply the third, but they are all consistent with each other.

Exercise 9.9.1. Prove Lemma 9.9.7.

Exercise 9.9.2. Prove Proposition 9.9.8. (Hint: you should avoid Lemma 9.9.7, and instead go back to the definition of equivalent sequences in Definition 9.9.5.)

Exercise 9.9.3. Prove Proposition 9.9.12. (Hint: use Definition 9.9.2 directly.)

Exercise 9.9.4. Use Proposition 9.9.12 to prove Corollary 9.9.14. Use this corollary to give an alternate demonstration of the results in Example 9.9.10.

Exercise 9.9.5. Prove Proposition 9.9.15. (Hint: mimic the proof of Lemma 9.6.3. At some point you will need either Proposition 9.9.12 or Corollary 9.9.14.)

Exercise 9.9.6. Let X, Y, Z be subsets of $\mathbf{R}$. Let $f : X \to Y$ be a function which is uniformly continuous on X, and let $g : Y \to Z$ be a function which is uniformly continuous on Y. Show that the function $g \circ f : X \to Z$ is uniformly continuous on X.

9.10 Limits at infinity

Until now, we have discussed what it means for a function $f : X \to \mathbf{R}$ to have a limit as $x \to x_0$ as long as x_0 is a *real* number. We now briefly discuss what it would mean to take limits when x_0 is equal to $+\infty$ or $-\infty$. (This is part of a more general theory of continuous functions on a topological space; see Section 13.5.)

First, we need a notion of what it means for $+\infty$ or $-\infty$ to be adherent to a set.

Definition 9.10.1 (Infinite adherent points). Let X be a subset of $\mathbf{R}$. We say that $+\infty$ is *adherent* to X iff for every $M \in \mathbf{R}$ there exists an $x \in X$ such that $x > M$; we say that $-\infty$ is *adherent* to X iff for every $M \in \mathbf{R}$ there exists an $x \in X$ such that $x < M$.

In other words, $+\infty$ is adherent to X iff X has no upper bound, or equivalently iff $\sup(X) = +\infty$. Similarly $-\infty$ is adherent to X iff X has no lower bound, or iff $\inf(X) = -\infty$. Thus a set is bounded if and only if $+\infty$ and $-\infty$ are not adherent points.

Remark 9.10.2. This definition may seem rather different from Definition 9.1.8, but can be unified using the topological structure of the extended real line $\mathbf{R}^*$, which we will not discuss here.

Definition 9.10.3 (Limits at infinity). Let X be a subset of $\mathbf{R}$ with $+\infty$ as an adherent point, and let $f : X \to \mathbf{R}$ be a function. We say that $f(x)$ *converges to* L as $x \to +\infty$ in X, and write $\lim_{x\to+\infty; x\in X} f(x) = L$, iff for every $\varepsilon > 0$ there exists an M such that f is ε-close to L on $X \cap (M, +\infty)$ (i.e., $|f(x) - L| \le \varepsilon$ for all $x \in X$ such that $x > M$). Similarly we say that $f(x)$ *converges to*

L as $x \to -\infty$ iff for every $\varepsilon > 0$ there exists an M such that f is ε-close to L on $X \cap (-\infty, M)$.

Example 9.10.4. Let $f : (0, \infty) \to \mathbf{R}$ be the function $f(x) := 1/x$. Then we have $\lim_{x \to +\infty; x \in (0,\infty)} 1/x = 0$. (Can you see why, from the definition?)

One can do many of the same things with these limits at infinity as we have been doing with limits at other points x_0; for instance, it turns out that all of the limit laws continue to hold. However, as we will not be using these limits much in this text, we will not devote much attention to these matters. We will note though that this definition is consistent with the notion of a limit $\lim_{n\to\infty} a_n$ of a sequence (Exercise 9.10.1).

Exercise 9.10.1. Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers, then a_n can also be thought of as a function from $\mathbf{N}$ to $\mathbf{R}$, which takes each natural number n to a real number a_n. Show that

$$\lim_{n\to+\infty; n\in\mathbf{N}} a_n = \lim_{n\to\infty} a_n$$

where the left-hand limit is defined by Definition 9.10.3 and the right-hand limit is defined by Definition 6.1.8. More precisely, show that if one of the above two limits exists then so does the other, and then they both have the same value. Thus the two notions of limit here are compatible.

Chapter 10

Differentiation of functions

10.1 Basic definitions

We can now begin the rigourous treatment of calculus in earnest, starting with the notion of a derivative. We can now define derivatives analytically, using limits, in contrast to the geometric definition of derivatives, which uses tangents. The advantage of working analytically is that (a) we do not need to know the axioms of geometry, and (b) these definitions can be modified to handle functions of several variables, or functions whose values are vectors instead of scalar. Furthermore, one's geometric intuition becomes difficult to rely on once one has more than three dimensions in play. (Conversely, one can use one's experience in analytic rigour to extend one's geometric intuition to such abstract settings; as mentioned earlier, the two viewpoints complement rather than oppose each other.)

Definition 10.1.1 (Differentiability at a point). Let X be a subset of $\mathbf{R}$, and let $x_0 \in X$ be an element of X which is also a limit point of X. Let $f : X \to \mathbf{R}$ be a function. If the limit

$$\lim_{x \to x_0; x \in X - \{x_0\}} \frac{f(x) - f(x_0)}{x - x_0}$$

converges to some real number L, then we say that f is *differentiable at x_0 on X with derivative L*, and write $f'(x_0) := L$. If the limit does not exist, or if x_0 is not an element of X or not a limit point of X, we leave $f'(x_0)$ undefined, and say that f is *not differentiable at x_0 on X*.

Remark 10.1.2. Note that we need x_0 to be a limit point in order for x_0 to be adherent to $X - \{x_0\}$, otherwise the limit

$$\lim_{x \to x_0; x \in X - \{x_0\}} \frac{f(x) - f(x_0)}{x - x_0}$$

would automatically be undefined. In particular, we do not define the derivative of a function at an isolated point; for instance, if one restricts the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) := x^2$ to the domain $X := [1, 2] \cup \{3\}$, then the restriction of the function ceases to be differentiable at 3. (See however Exercise 10.1.1 below.) In practice, the domain X will almost always be an interval, and so by Lemma 9.1.21 all elements x_0 of X will automatically be limit points and we will not have to care much about these issues.

Example 10.1.3. Let $f : \mathbf{R} \to \mathbf{R}$ be the function $f(x) := x^2$, and let x_0 be any real number. To see whether f is differentiable at x_0 on $\mathbf{R}$, we compute the limit

$$\lim_{x \to x_0; x \in \mathbf{R} - \{x_0\}} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0; x \in \mathbf{R} - \{x_0\}} \frac{x^2 - x_0^2}{x - x_0}.$$

We can factor the numerator as $(x^2 - x_0^2) = (x - x_0)(x + x_0)$. Since $x \in \mathbf{R} - \{x_0\}$, we may legitimately cancel the factors of $x - x_0$ and write the above limit as

$$\lim_{x \to x_0; x \in \mathbf{R} - \{x_0\}} x + x_0$$

which by limit laws is equal to $2x_0$. Thus the function $f(x)$ is differentiable at x_0 and its derivative there is $2x_0$.

Remark 10.1.4. This point is trivial, but it is worth mentioning: if $f : X \to \mathbf{R}$ is differentiable at x_0, and $g : X \to \mathbf{R}$ is equal to f (i.e., $g(x) = f(x)$ for all $x \in X$), then g is also differentiable at x_0 and $g'(x_0) = f'(x_0)$ (why?). However, if two functions f and

g merely have the same *value* at x_0, i.e., $g(x_0) = f(x_0)$, this does not imply that $g'(x_0) = f'(x_0)$. (Can you see a counterexample?) Thus there is a big difference between two functions being equal on their whole domain, and merely being equal at one point.

Remark 10.1.5. One sometimes writes $\frac{df}{dx}$ instead of f'. This notation is of course very familiar and convenient, but one has to be a little careful, because it is only safe to use as long as x is the only variable used to represent the input for f; otherwise one can get into all sorts of trouble. For instance, the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) := x^2$ has derivative $\frac{df}{dx} = 2x$, but the function $g : \mathbf{R} \to \mathbf{R}$ defined by $g(y) := y^2$ would seem to have derivative $\frac{dg}{dx} = 0$ if y and x are independent variables, despite the fact that g and f are exactly the same function. Because of this possible source of confusion, we will refrain from using the notation $\frac{df}{dx}$ whenever it could possibly lead to confusion. (This confusion becomes even worse in the calculus of several variables, and the standard notation of $\frac{\partial f}{\partial x}$ can lead to some serious ambiguities. There are ways to resolve these ambiguities, most notably by introducing the notion of differentiation along vector fields, but this is beyond the scope of this text.)

Example 10.1.6. Let $f : \mathbf{R} \to \mathbf{R}$ be the function $f(x) := |x|$, and let $x_0 = 0$. To see whether f is differentiable at 0 on $\mathbf{R}$, we compute the limit

$$\lim_{x \to 0; x \in \mathbf{R} - \{0\}} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0; x \in \mathbf{R} - \{0\}} \frac{|x|}{x}.$$

Now we take left limits and right limits. The right limit is

$$\lim_{x \to 0; x \in (0,\infty)} \frac{|x|}{x} = \lim_{x \to 0; x \in (0,\infty)} \frac{x}{x} = \lim_{x \to 0; x \in (0,\infty)} 1 = 1,$$

while the left limit is

$$\lim_{x \to 0; x \in (-\infty,0)} \frac{|x|}{x} = \lim_{x \to 0; x \in (0,\infty)} \frac{-x}{x} = \lim_{x \to 0; x \in (0,\infty)} -1 = -1,$$

and these limits do not match. Thus $\lim_{x\to 0; x\in \mathbf{R}-\{0\}} \frac{|x|}{x}$ does not exist, and f is not differentiable at 0 on $\mathbf{R}$. However, if one restricts f to $[0, \infty)$, then the restricted function $f|_{[0,\infty)}$ *is* differentiable at 0 on $[0, \infty)$, with derivative 1:

$$\lim_{x\to 0; x\in[0,\infty)-\{0\}} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0; x\in(0,\infty)} \frac{|x|}{x} = 1.$$

Similarly, when one restricts f to $(-\infty, 0]$, the restricted function $f|_{(-\infty,0]}$ is differentiable at 0 on $(-\infty, 0]$, with derivative -1. Thus even when a function is not differentiable, it is sometimes possible to restore the differentiability by restricting the domain of the function.

If a function is differentiable at x_0, then it is approximately linear near x_0:

Proposition 10.1.7 (Newton's approximation). *Let X be a subset of $\mathbf{R}$, let x_0 be a limit point of X, let $f : X \to \mathbf{R}$ be a function, and let L be a real number. Then the following statements are logically equivalent:*

(a) *f is differentiable at x_0 on X with derivative L.*

(b) *For every $\varepsilon > 0$, there exists a $\delta > 0$ such that $f(x)$ is $\varepsilon|x - x_0|$-close to $f(x_0) + L(x - x_0)$ whenever $x \in X$ is δ-close to x_0, i.e., we have*

$$|f(x) - (f(x_0) + L(x - x_0))| \le \varepsilon|x - x_0|$$

whenever $x \in X$ and $|x - x_0| \le \delta$.

Remark 10.1.8. Newton's approximation is of course named after the great scientist and mathematician Isaac Newton (1642–1727), one of the founders of differential and integral calculus.

Proof. See Exercise 10.1.2. $\square$

Remark 10.1.9. We can phrase Proposition 10.1.7 in a more informal way: if f is differentiable at x_0, then one has the approximation $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$, and conversely.

As the example of the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) := |x|$ shows, a function can be continuous at a point without being differentiable at that point. However, the converse is true:

Proposition 10.1.10 (Differentiability implies continuity). *Let X be a subset of $\mathbf{R}$, let x_0 be a limit point of X, and let $f : X \to \mathbf{R}$ be a function. If f is differentiable at x_0, then f is also continuous at x_0.*

Proof. See Exercise 10.1.3. □

Definition 10.1.11 (Differentiability on a domain). Let X be a subset of $\mathbf{R}$, and let $f : X \to \mathbf{R}$ be a function. We say that f is *differentiable on X* if, for every $x_0 \in X$, the function f is differentiable at x_0 on X.

From Proposition 10.1.10 and the above definition we have an immediate corollary:

Corollary 10.1.12. *Let X be a subset of $\mathbf{R}$, and let $f : X \to \mathbf{R}$ be a function which is differentiable on X. Then f is also continuous on X.*

Now we state the basic properties of derivatives which you are all familiar with.

Theorem 10.1.13 (Differential calculus). *Let X be a subset of $\mathbf{R}$, let x_0 be a limit point of X, and let $f : X \to \mathbf{R}$ and $g : X \to \mathbf{R}$ be functions.*

(a) *If f is a constant function, i.e., there exists a real number c such that $f(x) = c$ for all $x \in X$, then f is differentiable at x_0 and $f'(x_0) = 0$.*

(b) *If f is the identity function, i.e., $f(x) = x$ for all $x \in X$, then f is differentiable at x_0 and $f'(x_0) = 1$.*

(c) *(Sum rule) If f and g are differentiable at x_0, then $f + g$ is also differentiable at x_0, and $(f + g)'(x_0) = f'(x_0) + g'(x_0)$.*

(*d*) (*Product rule*) *If f and g are differentiable at x_0, then fg is also differentiable at x_0, and $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$.*

(*e*) *If f is differentiable at x_0 and c is a real number, then cf is also differentiable at x_0, and $(cf)'(x_0) = cf'(x_0)$.*

(*f*) (*Difference rule*) *If f and g are differentiable at x_0, then $f-g$ is also differentiable at x_0, and $(f-g)'(x_0) = f'(x_0) - g'(x_0)$.*

(*g*) *If g is differentiable at x_0, and g is non-zero on X (i.e., $g(x) \neq 0$ for all $x \in X$), then $1/g$ is also differentiable at x_0, and $(\frac{1}{g})'(x_0) = -\frac{g'(x_0)}{g(x_0)^2}$.*

(*h*) (*Quotient rule*) *If f and g are differentiable at x_0, and g is non-zero on X, then f/g is also differentiable at x_0, and*

$$(\frac{f}{g})'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}.$$

Remark 10.1.14. The product rule is also known as the *Leibnitz rule*, after Gottfried Leibnitz (1646–1716), who was the other founder of differential and integral calculus besides Newton.

Proof. See Exercise 10.1.4. □

As you are well aware, the above rules allow one to compute many derivatives easily. For instance, if $f : \mathbf{R} - \{1\} \to \mathbf{R}$ is the function $f(x) := \frac{x-2}{x-1}$, then it is easy to use the above rules to show that $f'(x_0) = \frac{1}{(x_0-1)^2}$ for all $x_0 \in \mathbf{R} - \{1\}$. (Why? Note that every point x_0 in $\mathbf{R} - \{1\}$ is a limit point of $\mathbf{R} - \{1\}$.)

Another fundamental property of differentiable functions is the following:

Theorem 10.1.15 (Chain rule). *Let X, Y be subsets of $\mathbf{R}$, let $x_0 \in X$ be a limit point of X, and let $y_0 \in Y$ be a limit point of Y. Let $f : X \to Y$ be a function such that $f(x_0) = y_0$, and such that f is differentiable at x_0. Suppose that $g : Y \to \mathbf{R}$ is a function* *which is differentiable at y_0. Then the function $g \circ f : X \to \mathbf{R}$ is differentiable at x_0, and*

$$(g \circ f)'(x_0) = g'(y_0) f'(x_0).$$

Proof. See Exercise 10.1.7. □

Example 10.1.16. If $f : \mathbf{R} - \{1\} \to \mathbf{R}$ is the function $f(x) := \frac{x-2}{x-1}$, and $g : \mathbf{R} \to \mathbf{R}$ is the function $g(y) := y^2$, then $g \circ f(x) = (\frac{x-2}{x-1})^2$, and the chain rule gives

$$(g \circ f)'(x_0) = 2\left(\frac{x_0 - 2}{x_0 - 1}\right) \frac{1}{(x_0 - 1)^2}.$$

Remark 10.1.17. If one writes y for $f(x)$, and z for $g(y)$, then the chain rule can be written in the more visually appealing manner $\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}$. However, this notation can be misleading (for instance it blurs the distinction between dependent variable and independent variable, especially for y), and leads one to believe that the quantities dz, dy, dx can be manipulated like real numbers. However, these quantities are not real numbers (in fact, we have not assigned any meaning to them at all), and treating them as such can lead to problems in the future. For instance, if f depends on x_1 and x_2, which depend on t, then chain rule for several variables asserts that $\frac{df}{dt} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dt}$, but this rule might seem suspect if one treated df, dt, etc. as real numbers. It is possible to think of dy, dx, etc. as "infinitesimal real numbers" if one knows what one is doing, but for those just starting out in analysis, I would not recommend this approach, especially if one wishes to work rigourously. (There is a way to make all of this rigourous, even for the calculus of several variables, but it requires the notion of a tangent vector, and the derivative map, both of which are beyond the scope of this text.)

Exercise 10.1.1. Suppose that X is a subset of $\mathbf{R}$, x_0 is a limit point of X, and $f : X \to \mathbf{R}$ is a function which is differentiable at x_0. Let $Y \subset X$ be such that x_0 is also limit point of Y. Prove that the restricted function

$f|_Y : Y \to \mathbf{R}$ is also differentiable at x_0, and has the same derivative as f at x_0. Explain why this does not contradict the discussion in Remark 10.1.2.

Exercise 10.1.2. Prove Proposition 10.1.7. (Hint: the cases $x = x_0$ and $x \neq x_0$ have to be treated separately.)

Exercise 10.1.3. Prove Proposition 10.1.10. (Hint: either use the limit laws (Proposition 9.3.14), or use Proposition 10.1.7.)

Exercise 10.1.4. Prove Theorem 10.1.13. (Hint: use the limit laws in Proposition 9.3.14. Use earlier parts of this theorem to prove the latter. For the product rule, use the identity

$$\begin{aligned} & f(x)g(x) - f(x_0)g(x_0) \\ & \quad = f(x)g(x) - f(x)g(x_0) + f(x)g(x_0) - f(x_0)g(x_0) \\ & \quad = f(x)(g(x) - g(x_0)) + (f(x) - f(x_0))g(x_0); \end{aligned}$$

this trick of adding and subtracting an intermediate term is sometimes known as the "middle-man trick" and is very useful in analysis.)

Exercise 10.1.5. Let n be a natural number, and let $f : \mathbf{R} \to \mathbf{R}$ be the function $f(x) := x^n$. Show that f is differentiable on $\mathbf{R}$ and $f'(x) = nx^{n-1}$ for all $x \in \mathbf{R}$. (Hint: use Theorem 10.1.13 and induction.)

Exercise 10.1.6. Let n be a *negative* integer, and let $f : \mathbf{R} - \{0\} \to \mathbf{R}$ be the function $f(x) := x^n$. Show that f is differentiable on $\mathbf{R}$ and $f'(x) = nx^{n-1}$ for all $x \in \mathbf{R} - \{0\}$. (Hint: use Theorem 10.1.13 and Exercise 10.1.5.)

Exercise 10.1.7. Prove Theorem 10.1.15. (Hint: one way to do this is via Newton's approximation, Proposition 10.1.7. Another way is to use Proposition 9.3.9 and Proposition 10.1.10 to convert this problem into one involving limits of sequences, however with the latter strategy one has to treat the case $f'(x_0) = 0$ separately, as some division-by-zero subtleties can occur in that case.)

10.2 Local maxima, local minima, and derivatives

As you learnt in your basic calculus courses, one very common application of using derivatives is to locate maxima and minima. We now present this material again, but this time in a rigourous manner.