35

A Natural Transformation between Vector Spaces

We begin with a review of vector spaces over the field of real numbers, although any other field may be substituted for $\mathbf{R}$.

A *vector space* V over $\mathbf{R}$ is an Abelian group together with a mapping from $V \times \mathbf{R}$ to V sending (x, r) to xr, such that, if r and s are real numbers and x, y are in V, then

$$(x + y)r = xr + yr,$$

$$x(r + s) = xr + xs,$$

$$x(rs) = (xr)s,$$

$$x1 = x.$$

Note that $\mathbf{R}$ is a vector space over itself.

A *linear transformation* from a vector space V to a vector space V' is a mapping $f : V \to V'$ such that $f(x+y) = f(x)+f(y)$ and $f(xr) = (f(x))r$ for any $x, y \in V$ and $r \in \mathbf{R}$.

Taking vector spaces as objects and linear transformations as arrows, it is easy to show that the vector spaces (over the reals) form a concrete category which we shall call *Vect.*

A linear transformation with codomain $\mathbf{R}$ is called a *linear functional.* If f and g are linear functionals on V, we define $(f + g)(x) = f(x) + g(x)$ and $(fr)x = f(xr)$. Now the set of linear functionals on V forms a vector space over $\mathbf{R}$, called the *dual* space V^* of V.

The above procedure may be repeated to obtain the *double dual* of V, namely, $V^{**} = (V^*)^*$. This double dual is closely related to V.

Let $\tilde{}\ : V \to V^{**}$ so that, if $x \in V$, then $\tilde{x}$ is the transformation from V^* to $\mathbf{R}$ that maps any linear functional f to $f(x)$, that is, $\tilde{x}(f) = f(x)$. Two things follow immediately:

I. $\tilde{x}$ is a linear transformation from V^* to $\mathbf{R}$, that is, a linear functional on V^*;

II. $\tilde{}$ is a linear transformation from V to V^{**}.

In the case that V has finite dimension, $\tilde{}$ is an isomorphism.

If h is a linear transformation from a vector space V to a vector space V', we define h^{**} as the function from V^{**} to V'^{**} such that, if $p \in V^{**}$, $h^{**}(p)$ is the member of V'^{**} that maps f' in V'^* to $p(f' \circ h)$. Note that

III. $f' \circ h : V \to \mathbf{R}$ and thus $f' \circ h \in V^*$, which is the domain of p;

IV. h^{**} is a linear transformation from V^{**} to V'^{**}.

In proving **IV**, we note that, if $f' \in V'^*$, then

$$h^{**}(pr)(f') = (pr)(f' \circ h) = p((f' \circ h)r)$$

$$= p((f'r) \circ h) = h^{**}(p)(f'r) = (h^{**}(p)r)(f').$$

Now suppose F maps each object V in the category *Vect* to V^{**}, and each arrow $h : V \to V'$ in *Vect* to h^{**}. If h is the identity function on V, then h^{**} is the identity function on V^{**}, since then $h^{**}(p)(f') = p(f' \circ h) = p(f')$.

Moreover, if $h : V \to V'$ and $g : V' \to V''$ are linear transformations, then so is $g \circ h : V \to V''$. $F(g \circ h) = (g \circ h)^{**}$ maps $p \in V^{**}$ to the member of V''^{**} that maps f'' in V''^* to $p(f'' \circ (g \circ h))$. That is, if $f'' : V'' \to \mathbf{R}$,

$$(g \circ h)^{**}(p)(f'') = p(f'' \circ (g \circ h)).$$

(Note that $f'' \circ (g \circ h) : V \to \mathbf{R}$, so that $f'' \circ (g \circ h) \in V^*$, which is the domain of $p \in V^{**}$.)

Since

$$(F(g) \circ F(h))(p)(f'') = g^{**}(h^{**}(p))(f'') = h^{**}(p)(f'' \circ g)$$

$$= p((f'' \circ g) \circ h) = p(f'' \circ (g \circ h)),$$

it follows that F is a functor from *Vect* to *Vect.*

Another functor from *Vect* to *Vect* is the identity functor I.

Let t assign to every vector space V the linear transformation from V to $F(V) = V^{**}$ which we called $\tilde{}$. That is, let $t(V)(x) = \tilde{x}$. Suppose $h : V \to V'$ and let x be any element of V. Then $(F(h) \circ t(V))(x) = h^{**}(\tilde{x})$.

Also, $(t(V') \circ I(h))(x) = (h(x))\tilde{}$. These two elements of V'^{**} are in fact equal. For let $f' : V' \to \mathbf{R}$, so that $f' \in V'^{*}$. Then

$$\begin{aligned} h^{**}(\tilde{x})(f') &= \tilde{x}(f' \circ h) \\ &= (f' \circ h)(x) \\ &= f'(h(x)) \\ &= (h(x))\tilde{}\ (f'). \end{aligned}$$

We may conclude that $F(h) \circ t(V) = t(V') \circ I(h)$, and hence t is a natural transformation from the functor I to the functor F.

Examples such as this have led to the slogan *that many objects of interest in mathematics are functors and that the arrows between them are natural transformations.* This and the slogans mentioned earlier were first proposed by F. W. Lawvere.

Exercises

1. Show that *Vect* is a concrete category.
2. Show that the sum of two linear transformations (from V to V') is a linear transformation.
3. Show that V^* is a vector space.
4. Verify **I**, **II**, **III** and **IV** from the text.
5. Generalize the results of this chapter from vector spaces to M-sets. (Things become a little easier if it is assumed that multiplication in M is commutative.)

References

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