Exercise 14.1.6. State and prove an analogue of the limit laws in Proposition 9.3.14 when X is now a metric space rather than a subset of $\mathbf{R}$. (Hint: use Corollary 13.2.3.)

14.2 Pointwise and uniform convergence

The most obvious notion of convergence of functions is *pointwise convergence*, or convergence at each point of the domain:

Definition 14.2.1 (Pointwise convergence). Let $(f^{(n)})_{n=1}^{\infty}$ be a sequence of functions from one metric space (X, d_X) to another (Y, d_Y), and let $f : X \to Y$ be another function. We say that $(f^{(n)})_{n=1}^{\infty}$ *converges pointwise to f on X* if we have

$$\lim_{n \to \infty} f^{(n)}(x) = f(x)$$

for all $x \in X$, i.e.

$$\lim_{n \to \infty} d_Y(f^{(n)}(x), f(x)) = 0.$$

Or in other words, for every x and every $\varepsilon > 0$ there exists $N > 0$ such that $d_Y(f^{(n)}(x), f(x)) < \varepsilon$ for every $n > N$. We call the function f the *pointwise limit* of the functions $f^{(n)}$.

Remark 14.2.2. Note that $f^{(n)}(x)$ and $f(x)$ are points in Y, rather than functions, so we are using our prior notion of convergence in metric spaces to determine convergence of functions. Also note that we are not really using the fact that (X, d_X) is a metric space (i.e., we are not using the metric d_X); for this definition it would suffice for X to just be a plain old set with no metric structure. However, later on we shall want to restrict our attention to *continuous* functions from X to Y, and in order to do so we need a metric on X (and on Y), or at least a topological structure. Also when we introduce the concept of *uniform convergence*, then we will definitely need a metric structure on X and Y; there is no comparable notion for topological spaces.

Example 14.2.3. Consider the functions $f^{(n)} : \mathbf{R} \to \mathbf{R}$ defined by $f^{(n)}(x) := x/n$, while $f : \mathbf{R} \to \mathbf{R}$ is the zero function $f(x) := 0$. Then $f^{(n)}$ converges pointwise to f, since for each fixed real number x we have $\lim_{n\to\infty} f^{(n)}(x) = \lim_{n\to\infty} x/n = 0 = f(x)$.

From Proposition 12.1.20 we see that a sequence $(f^{(n)})_{n=1}^{\infty}$ of functions from one metric space (X, d_X) to another (Y, d_Y) can have at most one pointwise limit f (this explains why we can refer to f as *the* pointwise limit). However, it is of course possible for a sequence of functions to have no pointwise limit (can you think of an example?), just as a sequence of points in a metric space do not necessarily have a limit.

Pointwise convergence is a very natural concept, but it has a number of disadvantages: it does not preserve continuity, derivatives, limits, or integrals, as the following three examples show.

Example 14.2.4. Consider the functions $f^{(n)} : [0,1] \to \mathbf{R}$ defined by $f^{(n)}(x) := x^n$, and let $f : [0,1] \to \mathbf{R}$ be the function defined by setting $f(x) := 1$ when $x = 1$ and $f(x) := 0$ when $0 \leq x < 1$. Then the functions $f^{(n)}$ are continuous, and converge pointwise to f on $[0,1]$ (why? treat the cases $x = 1$ and $0 \leq x < 1$ separately), however the limiting function f is not continuous. Note that the same example shows that pointwise convergence does not preserve differentiability either.

Example 14.2.5. If $\lim_{x\to x_0; x\in E} f^{(n)}(x) = L$ for every n, and $f^{(n)}$ converges pointwise to f, we cannot always take limits conclude that $\lim_{x\to x_0; x\in E} f(x) = L$. The previous example is also a counterexample here: observe that $\lim_{x\to 1; x\in[0,1)} x^n = 1$ for every n, but x^n converges pointwise to the function f defined in the previous paragraph, and $\lim_{x\to 1; x\in[0,1)} f(x) = 0$. In particular, we see that

$$\lim_{n\to\infty} \lim_{x\to x_0; x\in X} f^{(n)}(x) \neq \lim_{x\to x_0; x\in X} \lim_{n\to\infty} f^{(n)}(x).$$

(cf. Example 1.2.8). Thus pointwise convergence does not preserve limits.

Example 14.2.6. Suppose that $f^{(n)} : [a,b] \to \mathbf{R}$ a sequence of Riemann-integrable functions on the interval $[a,b]$. If $\int_{[a,b]} f^{(n)} = L$ for every n, and $f^{(n)}$ converges pointwise to some new function f, this does not mean that $\int_{[a,b]} f = L$. An example comes by setting $[a,b] := [0,1]$, and letting $f^{(n)}$ be the function $f^{(n)}(x) := 2n$ when $x \in [1/2n, 1/n]$, and $f^{(n)}(x) := 0$ for all other values of x. Then $f^{(n)}$ converges pointwise to the zero function $f(x) := 0$ (why?). On the other hand, $\int_{[0,1]} f^{(n)} = 1$ for every n, while $\int_{[0,1]} f = 0$. In particular, we have an example where

$$\lim_{n\to\infty} \int_{[a,b]} f^{(n)} \neq \int_{[a,b]} \lim_{n\to\infty} f^{(n)}.$$

One may think that this counterexample has something to do with the $f^{(n)}$ being discontinuous, but one can easily modify this counterexample to make the $f^{(n)}$ continuous (can you see how?).

Another example in the same spirit is the "moving bump" example. Let $f^{(n)} : \mathbf{R} \to \mathbf{R}$ be the function defined by $f^{(n)}(x) := 1$ if $x \in [n, n+1]$ and $f^{(n)}(x) := 0$ otherwise. Then $\int_{\mathbf{R}} f^{(n)} = 1$ for every n (where $\int_{\mathbf{R}} f$ is defined as the limit of $\int_{[-N,N]} f$ as N goes to infinity). On the other hand, $f^{(n)}$ converges pointwise to the zero function 0 (why?), and $\int_{\mathbf{R}} 0 = 0$. In both of these examples, functions of area 1 have somehow "disappeared" to produce functions of area 0 in the limit. See also Example 1.2.9.

These examples show that pointwise convergence is too weak a concept to be of much use. The problem is that while $f^{(n)}(x)$ converges to $f(x)$ for each x, the *rate* of that convergence varies substantially with x. For instance, consider the first example where $f^{(n)} : [0,1] \to \mathbf{R}$ was the function $f^{(n)}(x) := x^n$, and $f : [0,1] \to \mathbf{R}$ was the function such that $f(x) := 1$ when $x = 1$, and $f(x) := 0$ otherwise. Then for each x, $f^{(n)}(x)$ converges to $f(x)$ as $n \to \infty$; this is the same as saying that $\lim_{n\to\infty} x^n = 0$ when $0 \le x < 1$, and that $\lim_{n\to\infty} x^n = 1$ when $x = 1$. But the convergence is much slower near 1 than far away from 1. For instance, consider the statement that $\lim_{n\to\infty} x^n = 0$ for all $0 \le x < 1$. This means, for every $0 \leq x < 1$, that for every ε, there exists an $N \geq 1$ such that $|x^n| < \varepsilon$ for all $n \geq N$ - or in other words, the sequence $1, x, x^2, x^3, \ldots$ will eventually get less than ε, after passing some finite number N of elements in this sequence. But the number of elements N one needs to go out to depends very much on the location of x. For instance, take $\varepsilon := 0.1$. If $x = 0.1$, then we have $|x^n| < \varepsilon$ for all $n \geq 2$ - the sequence gets underneath ε after the second element. But if $x = 0.5$, then we only get $|x^n| < \varepsilon$ for $n \geq 4$ - you have to wait until the fourth element to get within ε of the limit. And if $x = 0.9$, then one only has $|x^n| < \varepsilon$ when $n \geq 22$. Clearly, the closer x gets to 1, the longer one has to wait until $f^{(n)}(x)$ will get within ε of $f(x)$, although it still will get there eventually. (Curiously, however, while the convergence gets worse and worse as x approaches 1, the convergence suddenly becomes perfect when $x = 1$.)

To put things another way, the convergence of $f^{(n)}$ to f is not *uniform* in x - the N that one needs to get $f^{(n)}(x)$ within ε of f depends on x as well as on ε. This motivates a stronger notion of convergence.

Definition 14.2.7 (Uniform convergence). Let $(f^{(n)})_{n=1}^{\infty}$ be a sequence of functions from one metric space (X, d_X) to another (Y, d_Y), and let $f : X \to Y$ be another function. We say that $(f^{(n)})_{n=1}^{\infty}$ *converges uniformly to f on X* if for every $\varepsilon > 0$ there exists $N > 0$ such that $d_Y(f^{(n)}(x), f(x)) < \varepsilon$ for every $n > N$ and $x \in X$. We call the function f the *uniform limit* of the functions $f^{(n)}$.

Remark 14.2.8. Note that this definition is subtly different from the definition for pointwise convergence in Definition 14.2.1. In the definition of pointwise convergence, N was allowed to depend on x; now it is not. The reader should compare this distinction to the distinction between continuity and uniform continuity (i.e., between Definition 13.1.1 and Definition 13.3.4). A more precise formulation of this analogy is given in Exercise 14.2.1.

It is easy to see that if $f^{(n)}$ converges uniformly to f on X, then it also converges pointwise to the same function f (see Exercise

14.2.2); thus when the uniform limit and pointwise limit both exist, then they have to be equal. However, the converse is not true; for instance the functions $f^{(n)} : [0,1] \to \mathbf{R}$ defined earlier by $f^{(n)}(x) := x^n$ converge pointwise, but do not converge uniformly (see Exercise 14.2.2).

Example 14.2.9. Let $f^{(n)} : [0,1] \to \mathbf{R}$ be the functions $f^{(n)}(x) := x/n$, and let $f : [0,1] \to \mathbf{R}$ be the zero function $f(x) := 0$. Then it is clear that $f^{(n)}$ converges to f pointwise. Now we show that in fact $f^{(n)}$ converges to f uniformly. We have to show that for every $\varepsilon > 0$, there exists an N such that $|f^{(n)}(x) - f(x)| < \varepsilon$ for every $x \in [0,1]$ and every $n \geq N$. To show this, let us fix an $\varepsilon > 0$. Then for any $x \in [0,1]$ and $n \geq N$, we have

$$|f^{(n)}(x) - f(x)| = |x/n - 0| = x/n \leq 1/n \leq 1/N.$$

Thus if we choose N such that $N > 1/\varepsilon$ (note that this choice of N does not depend on what x is), then we have $|f^{(n)}(x) - f(x)| < \varepsilon$ for all $n \geq N$ and $x \in [0,1]$, as desired.

We make one trivial remark here: if a sequence $f^{(n)} : X \to Y$ of functions converges pointwise (or uniformly) to a function $f : X \to Y$, then the restrictions $f^{(n)}|_E : E \to Y$ of $f^{(n)}$ to some subset E of X will also converge pointwise (or uniformly) to $f|_Y$. (Why?)

Exercise 14.2.1. The purpose of this exercise is to demonstrate a concrete relationship between continuity and pointwise convergence, and between uniform continuity and uniform convergence. Let $f : \mathbf{R} \to \mathbf{R}$ be a function. For any $a \in \mathbf{R}$, let $f_a : \mathbf{R} \to \mathbf{R}$ be the shifted function $f_a(x) := f(x - a)$.

(a) Show that f is continuous if and only if, whenever $(a_n)_{n=0}^\infty$ is a sequence of real numbers which converges to zero, the shifted functions f_{a_n} converge pointwise to f.

(b) Show that f is uniformly continuous if and only if, whenever $(a_n)_{n=0}^\infty$ is a sequence of real numbers which converges to zero, the shifted functions f_{a_n} converge uniformly to f.

Exercise 14.2.2. (a) Let $(f^{(n)})_{n=1}^{\infty}$ be a sequence of functions from one metric space (X, d_X) to another (Y, d_Y), and let $f : X \to Y$ be another function from X to Y. Show that if $f^{(n)}$ converges uniformly to f, then $f^{(n)}$ also converges pointwise to f.

(b) For each integer $n \geq 1$, let $f^{(n)} : (-1, 1) \to \mathbf{R}$ be the function $f^{(n)}(x) := x^n$. Prove that $f^{(n)}$ converges pointwise to the zero function 0, but does not converge uniformly to any function $f : (-1, 1) \to \mathbf{R}$.

(c) Let $g : (-1, 1) \to \mathbf{R}$ be the function $g(x) := x/(1 - x)$. With the notation as in (b), show that the partial sums $\sum_{n=1}^{N} f^{(n)}$ converges pointwise as $N \to \infty$ to g, but does not converge uniformly to g, on the open interval $(-1, 1)$. (Hint: use Lemma 7.3.3.) What would happen if we replaced the open interval $(-1, 1)$ with the closed interval $[-1, 1]$?

Exercise 14.2.3. Let (X, d_X) a metric space, and for every integer $n \geq 1$, let $f_n : X \to \mathbf{R}$ be a real-valued function. Suppose that f_n converges pointwise to another function $f : X \to \mathbf{R}$ on X (in this question we give $\mathbf{R}$ the standard metric $d(x, y) = |x - y|$). Let $h : \mathbf{R} \to \mathbf{R}$ be a continuous function. Show that the functions $h \circ f_n$ converge pointwise to $h \circ f$ on X, where $h \circ f_n : X \to \mathbf{R}$ is the function $h \circ f_n(x) := h(f_n(x))$, and similarly for $h \circ f$.

Exercise 14.2.4. Let $f_n : X \to Y$ be a sequence of bounded functions from one metric space (X, d_X) to another metric space (Y, d_Y). Suppose that f_n converges uniformly to another function $f : X \to Y$. Suppose that f is a bounded function; i.e., there exists a ball $B_{(Y,d_Y)}(y_0, R)$ in Y such that $f(x) \in B_{(Y,d_Y)}(y_0, R)$ for all $x \in X$. Show that the sequence f_n is *uniformly bounded*; i.e. there exists a ball $B_{(Y,d_Y)}(y_0, R)$ in Y such that $f_n(x) \in B_{(Y,d_Y)}(y_0, R)$ for all $x \in X$ and all positive integers n.

14.3 Uniform convergence and continuity

We now give the first demonstration that uniform convergence is significantly better than pointwise convergence. Specifically, we show that the uniform limit of continuous functions is continuous.

Theorem 14.3.1 (Uniform limits preserve continuity I). *Suppose $(f^{(n)})_{n=1}^{\infty}$ is a sequence of functions from one metric space (X, d_X) to another (Y, d_Y), and suppose that this sequence converges uniformly to another function $f : X \to Y$. Let x_0 be a point in X.*

If the functions $f^{(n)}$ are continuous at x_0 for each n, then the limiting function f is also continuous at x_0.

Proof. See Exercise 14.3.1. □

This has an immediate corollary:

Corollary 14.3.2 (Uniform limits preserve continuity II). *Let $(f^{(n)})_{n=1}^{\infty}$ be a sequence of functions from one metric space (X, d_X) to another (Y, d_Y), and suppose that this sequence converges uniformly to another function $f : X \to Y$. If the functions $f^{(n)}$ are continuous on X for each n, then the limiting function f is also continuous on X.*

This should be contrasted with Example 14.2.4. There is a slight variant of Theorem 14.3.1 which is also useful:

Proposition 14.3.3 (Interchange of limits and uniform limits). *Let (X, d_X) and (Y, d_Y) be metric spaces, with Y complete, and let E be a subset of X. Let $(f^{(n)})_{n=1}^{\infty}$ be a sequence of functions from E to Y, and suppose that this sequence converges uniformly in E to some function $f : E \to Y$. Let $x_0 \in X$ be an adherent point of E, and suppose that for each n the limit $\lim_{x\to x_0; x\in E} f^{(n)}(x)$ exists. Then the limit $\lim_{x\to x_0; x\in E} f(x)$ also exists, and is equal to the limit of the sequence $(\lim_{x\to x_0; x\in E} f^{(n)}(x))_{n=1}^{\infty}$; in other words we have the interchange of limits*

$$\lim_{n\to\infty} \lim_{x\to x_0; x\in E} f^{(n)}(x) = \lim_{x\to x_0; x\in E} \lim_{n\to\infty} f^{(n)}(x).$$

Proof. See Exercise 14.3.2. □

This should be constrasted with Example 14.2.5. Finally, we have a version of these theorems for sequences:

Proposition 14.3.4. *Let $(f^{(n)})_{n=1}^{\infty}$ be a sequence of continuous functions from one metric space (X, d_X) to another (Y, d_Y), and suppose that this sequence converges uniformly to another function $f : X \to Y$. Let $x^{(n)}$ be a sequence of points in X which converge to some limit x. Then $f^{(n)}(x^{(n)})$ converges (in Y) to $f(x)$.*

Proof. See Exercise 14.3.4. □

A similar result holds for bounded functions:

Definition 14.3.5 (Bounded functions). A function $f : X \to Y$ from one metric space (X, d_X) to another (Y, d_Y) is *bounded* if $f(X)$ is a bounded set, i.e., there exists a ball $B_{(Y,d_Y)}(y_0, R)$ in Y such that $f(x) \in B_{(Y,d_Y)}(y_0, R)$ for all $x \in X$.

Proposition 14.3.6 (Uniform limits preserve boundedness). *Let $(f^{(n)})_{n=1}^{\infty}$ be a sequence of functions from one metric space (X, d_X) to another (Y, d_Y), and suppose that this sequence converges uniformly to another function $f : X \to Y$. If the functions $f^{(n)}$ are bounded on X for each n, then the limiting function f is also bounded on X.*

Proof. See Exercise 14.3.6. □

Remark 14.3.7. The above propositions sound very reasonable, but one should caution that it only works if one assumes uniform convergence; pointwise convergence is not enough. (See Exercises 14.3.3, 14.3.5, 14.3.7.)

Exercise 14.3.1. Prove Theorem 14.3.1. Explain briefly why your proof requires uniform convergence, and why pointwise convergence would not suffice. (Hints: it is easiest to use the "epsilon-delta" definition of continuity from Definition 13.1.1. You may find the triangle inequality

$$\begin{aligned} d_Y(f(x), f(x_0)) \leq & d_Y(f(x), f^{(n)}(x)) + d_Y(f^{(n)}(x), f^{(n)}(x_0)) \\ & + d_Y(f^{(n)}(x_0), f(x_0)) \end{aligned}$$

useful. Also, you may need to divide ε as $\varepsilon = \varepsilon/3 + \varepsilon/3 + \varepsilon/3$. Finally, it is possible to prove Theorem 14.3.1 from Proposition 14.3.3, but you may find it easier conceptually to prove Theorem 14.3.1 first.)

Exercise 14.3.2. Prove Proposition 14.3.3. (Hint: this is very similar to Theorem 14.3.1. Theorem 14.3.1 cannot be used to prove Proposition 14.3.3, however it is possible to use Proposition 14.3.3 to prove Theorem 14.3.1.)

Exercise 14.3.3. Compare Proposition 14.3.3 with Example 1.2.8. Can you now explain why the interchange of limits in Example 1.2.8 led to a false statement, whereas the interchange of limits in Proposition 14.3.3 is justified?

Exercise 14.3.4. Prove Proposition 14.3.4. (Hint: again, this is similar to Theorem 14.3.1 and Proposition 14.3.3, although the statements are slightly different, and one cannot deduce this directly from the other two results.)

Exercise 14.3.5. Give an example to show that Proposition 14.3.4 fails if the phrase "converges uniformly" is replaced by "converges pointwise". (Hint: some of the examples already given earlier will already work here.)

Exercise 14.3.6. Prove Propositoin 14.3.6.

Exercise 14.3.7. Give an example to show that Proposition 14.3.6 fails if the phrase "converges uniformly" is replaced by "converges pointwise". (Hint: some of the examples already given earlier will already work here.)

Exercise 14.3.8. Let (X, d) be a metric space, and for every positive integer n, let $f_n : X \to \mathbf{R}$ and $g_n : X \to \mathbf{R}$ be functions. Suppose that $(f_n)_{n=1}^{\infty}$ converges uniformly to another function $f : X \to \mathbf{R}$, and that $(g_n)_{n=1}^{\infty}$ converges uniformly to another function $g : X \to \mathbf{R}$. Suppose also that the functions $(f_n)_{n=1}^{\infty}$ and $(g_n)_{n=1}^{\infty}$ are uniformly bounded, i.e., there exists an $M > 0$ such that $|f_n(x)| \leq M$ and $|g_n(x)| \leq M$ for all $n \geq 1$ and $x \in \mathbf{R}$. Prove that the functions $f_n g_n : X \to \mathbf{R}$ converge uniformly to $fg : X \to \mathbf{R}$.

14.4 The metric of uniform convergence

We have now developed at least four, apparently separate, notions of limit in this text:

(a) limits $\lim_{n\to\infty} x^{(n)}$ of sequences of points in a metric space (Definition 12.1.14; see also Definition 13.5.4);

(b) limiting values $\lim_{x\to x_0; x\in E} f(x)$ of functions at a point (Definition 14.1.1);

(c) pointwise limits f of functions $f^{(n)}$ (Definition 14.2.1; and

(d) uniform limits f of functions $f^{(n)}$ (Definition 14.2.7).

This proliferation of limits may seem rather complicated. However, we can reduce the complexity slightly by observing that (d) can be viewed as a special case of (a), though in doing so it should be cautioned that because we are now dealing with functions instead of points, the convergence is not in X or in Y, but rather in a new space, the space of functions from X to Y.

Remark 14.4.1. If one is willing to work in topological spaces instead of metric spaces, we can also view (b) as a special case of (a), see Exercise 14.1.4, and (c) is also a special case of (a), see Exercise 14.4.4. Thus the notion of convergence in a topological space can be used to unify all the notions of limits we have encountered so far.

Definition 14.4.2 (Metric space of bounded functions)**.** Suppose (X, d_X) and (Y, d_Y) are metric spaces. We let $B(X \to Y)$ denote the space[1] of bounded functions from X to Y:

$$B(X \to Y) := \{f \big| f : X \to Y \text{ is a bounded function}\}.$$

We define a metric $d_\infty : B(X \to Y) \times B(X \to Y) \to \mathbf{R}^+$ by defining

$$d_\infty(f, g) := \sup_{x \in X} d_Y(f(x), g(x)) = \sup\{d_Y(f(x), g(x)) : x \in X\}$$

for all $f, g \in B(X \to Y)$. This metric is sometimes known as the *sup norm metric* or the L^∞ *metric.* We will also use $d_{B(X \to Y)}$ as a synonym for d_∞.

Notice that the distance $d_\infty(f, g)$ is always finite because f and g are assumed to be bounded on X.

Example 14.4.3. Let $X := [0, 1]$ and $Y = \mathbf{R}$. Let $f : [0, 1] \to \mathbf{R}$ and $g : [0, 1] \to \mathbf{R}$ be the functions $f(x) := 2x$ and $g(x) :=$

[1] Note that this is a set, thanks to the power set axiom (Axiom 3.10) and the axiom of specification (Axiom 3.5).

$3x$. Then f and g are both bounded functions and thus live in $B([0,1] \to \mathbf{R})$. The distance between them is

$$d_\infty(f,g) = \sup_{x\in[0,1]} |2x - 3x| = \sup_{x\in[0,1]} |x| = 1.$$

This space turns out to be a metric space (Exercise 14.4.1). Convergence in this metric turns out to be identical to uniform convergence:

Proposition 14.4.4. *Let (X, d_X) and (Y, d_Y) be metric spaces. Let $(f^{(n)})_{n=1}^\infty$ be a sequence of functions in $B(X \to Y)$, and let f be another function in $B(X \to Y)$. Then $(f^{(n)})_{n=1}^\infty$ converges to f in the metric $d_{B(X\to Y)}$ if and only if $(f^{(n)})_{n=1}^\infty$ converges uniformly to f.*

Proof. See Exercise 14.4.2. □

Now let $C(X \to Y)$ be the space of bounded continuous functions from X to Y:

$$C(X \to Y) := \{f \in B(X \to Y) \big| f \text{ is continuous}\}.$$

This set $C(X \to Y)$ is clearly a subset of $B(X \to Y)$. Corollary 14.3.2 asserts that this space $C(X \to Y)$ is closed in $B(X \to Y)$ (why?). Actually, we can say a lot more:

Theorem 14.4.5 (The space of continuous functions is complete). *Let (X, d_X) be a metric space, and let (Y, d_Y) be a complete metric space. The space $(C(X \to Y), d_{B(X\to Y)}|_{C(X\to Y)\times C(X\to Y)})$ is a complete subspace of $(B(X \to Y), d_{B(X\to Y)})$. In other words, every Cauchy sequence of functions in $C(X \to Y)$ converges to a function in $C(X \to Y)$.*

Proof. See Exercise 14.4.3. □

Exercise 14.4.1. Let (X, d_X) and (Y, d_Y) be metric spaces. Show that the space $B(X \to Y)$ defined in Definition 14.4.2, with the metric $d_{B(X\to Y)}$, is indeed a metric space.

Exercise 14.4.2. Prove Proposition 14.4.4.

Exercise 14.4.3. Prove Theorem 14.4.5. (Hint: this is similar, but not identical, to the proof of Theorem 14.3.1).

Exercise 14.4.4. Let (X, d_X) and (Y, d_Y) be metric spaces, and let $Y^X := \{f \big| f : X \to Y\}$ be the space of all functions from X to Y (cf. Axiom 3.10). If $x_0 \in X$ and V is an open set in Y, let $V^{(x_0)} \subseteq Y^X$ be the set

$$V^{(x_0)} := \{f \in Y^X : f(x_0) \in V\}.$$

If E is a subset of Y^X, we say that E is *open* if for every $f \in E$, there exists a finite number of points $x_1, \ldots, x_n \in X$ and open sets $V_1, \ldots, V_n \subseteq Y$ such that

$$f \in V_1^{(x_1)} \cap \ldots \cap V_n^{(x_n)} \subseteq E.$$

- Show that if $\mathcal{F}$ is the collection of open sets in Y^X, then $(Y^X, \mathcal{F})$ is a topological space.
- For each natural number n, let $f^{(n)} : X \to Y$ be a function from X to Y, and let $f : X \to Y$ be another function from X to Y. Show that $f^{(n)}$ converges to f in the topology $\mathcal{F}$ (in the sense of Definition 13.5.4) if and only if $f^{(n)}$ converges to f pointwise (in the sense of Definition 14.2.1).

The topology $\mathcal{F}$ is known as the *topology of pointwise convergence*, for obvious reasons; it is also known as the *product topology*. It shows that the concept of pointwise convergence can be viewed as a special case of the more general concept of convergence in a topological space.

14.5 Series of functions; the Weierstrass M-test

Having discussed sequences of functions, we now discuss infinite series $\sum_{n=1}^\infty f_n$ of functions. Now we shall restrict our attention to functions $f : X \to \mathbf{R}$ from a metric space (X, d_X) to the real line $\mathbf{R}$ (which we of course give the standard metric); this is because we know how to add two real numbers, but don't necessarily know how to add two points in a general metric space Y. Functions whose range is $\mathbf{R}$ are sometimes called *real-valued* functions.

Finite summation is, of course, easy: given any finite collection $f^{(1)}, \ldots, f^{(N)}$ of functions from X to $\mathbf{R}$, we can define the finite