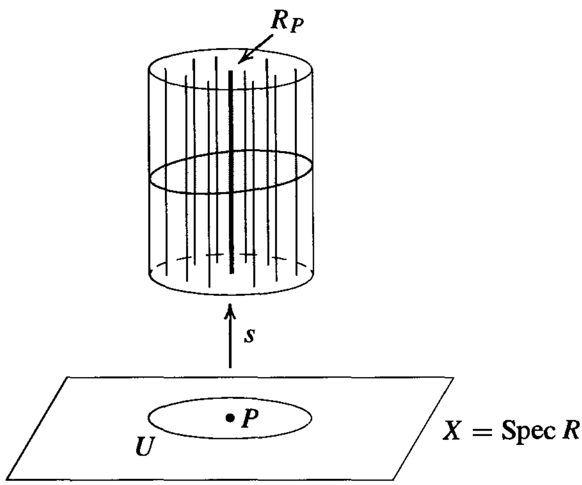
This is an algebraic definition that generalizes the definition of the tangent space $\mathbb{T}_{v,V}$ to a variety V at a point v (by Proposition 52). This can now be used to define what it means for a point in Spec R to be nonsingular: the point $P \in \operatorname{Spec} R$ is *nonsingular* or *smooth* if the local ring R_P is what is called a "regular local ring" (cf. Section 16.2).

Proposition 58 also suggests a nice geometric view of the structure sheaf on Spec R. If we view each point $P \in \operatorname{Spec} R$ as having the local ring R_P above it, then above the open set U in $X = \operatorname{Spec} R$ is a "sheaf" (in the sense of a "bundle") of these "stalks" (in the sense of a "stalk of wheat"), which helps explain some of the terminology. A section s in the structure sheaf $\mathcal{O}(U)$ is a map from U to this bundle of stalks. The image of U under such a section s is indicated by the shaded region in the following figure.



Definition. Let R be a commutative ring with 1. The pair (Spec R, $\mathcal{O}_{\operatorname{Spec} R}$), consisting of the space Spec R with the Zariski topology together with the structure sheaf $\mathcal{O}_{\operatorname{Spec} R}$, is called an *affine scheme*.

The notion of an affine scheme gives a completely algebraic generalization of the geometry of affine algebraic sets valid for arbitrary commutative rings, and is the starting point for modern algebraic geometry.

Examples

(1) If F is any field then $X = \operatorname{Spec} F = \{(0)\}$. In this case there are only two open sets X and $\emptyset$, both of which are principal open sets: $X = X_1$ and $\emptyset = X_0$. The global sections are $\mathcal{O}(X) = F$. There is only one stalk: $\mathcal{O}_{(0)} = F_0 = F$.

(2) If $R = \mathbb{Z}$ then because R is a P.I.D. every open set in $X = \operatorname{Spec} \mathbb{Z}$ is principal open:

$$X_n = \{(p) \mid p \nmid n\} \qquad \text{and}$$

$$\mathcal{O}(X_n) = \mathbb{Z}_n = \mathbb{Z}[1/n] = \{a/b \in \mathbb{Q} \mid \text{if the prime } p \mid b \text{ then } p \mid n\}.$$

For nonzero p the stalk at (p) is the local ring $\mathbb{Z}_{(p)}$, and the stalk at (0) is $\mathbb{Q}$. All the restriction maps as well as the maps from sections to stalks are the natural inclusions.

(3) For a general integral domain R with quotient field F the stalks and sections are

$$\mathcal{O}(U) = \{a/b \in F \mid b \notin P \text{ for all } P \in U\}$$
$$\mathcal{O}_P = R_P = \{a/b \in F \mid b \notin P\}$$

where the stalk at (0) is F, i.e., $\mathcal{O}_{(0)} = F$. Again, the restriction maps and the maps to the stalks are all inclusions.

(4) For the local ring $R = \mathbb{Z}_{(2)} = \{a/b \in \mathbb{Q} \mid b \text{ odd}\}$ we have $\operatorname{Spec} R = \{(0), (2)\}$ with (2) the only closed point and $\{(0)\} = X_2$ a principal open set. The sections $\mathcal{O}(\{(0)\})$ are $R_2 = \mathbb{Q}$, and the stalks are $\mathcal{O}_{(0)} = R_{(0)} = \mathbb{Q}$ and $\mathcal{O}_{(2)} = R_{(2)} = R$.

We next consider the relationship of the affine schemes corresponding to rings R and S with respect to a ring homomorphism from R to S.

Suppose that $\varphi : R \to S$ is a ring homomorphism. We have already seen in Proposition 56(7) that there is an induced continuous map φ^* from $Y = \operatorname{Spec} S$ to $X = \operatorname{Spec} R$ and that under this map the full preimage of the principal open set X_g for $g \in R$ is the principal open set $Y_{\varphi(g)}$. It follows that φ also induces a map on corresponding sections, as follows. Let $Q' \in Y$ be any element in $\operatorname{Spec} S$ and let $Q = \varphi^*(Q') = \varphi^{-1}(Q') \in X$ be the corresponding element in $\operatorname{Spec} R$. If U is a Zariski open set in X containing Q, then $U' = (\varphi^*)^{-1}(U)$ is a Zariski open set in Y containing Q'. Note that φ induces a natural ring homomorphism, φ_Q say, from the localization R_Q to the localization $S_{Q'}$ defined by $\varphi_Q(a/f) = \varphi(a)/\varphi(f) \in S_{Q'}$ for $f \notin Q$. Let $s \in \mathcal{O}_X(U)$ be a section of the structure sheaf of X given locally in the neighborhood X_g of $P \in X$ by a/g^n. It is easy to check that the composite

$$s' : U' \xrightarrow{\varphi^*} U \xrightarrow{s} \bigsqcup_{Q \in U} R_Q \xrightarrow{\varphi} \bigsqcup_{Q' \in U} S_{Q'}$$

defines a map given locally in the neighborhood $Y_{\varphi(g)}$ by the element $\varphi(a)/\varphi(g)^n$, so that $s' \in \mathcal{O}_Y(U')$ is a section of the structure sheaf of Y. It is then straightforward to check that the resulting map $\varphi^\# : \mathcal{O}_X(U) \to \mathcal{O}_Y(U')$ is a ring homomorphism (mapping $1 \in \mathcal{O}_X(U)$ to $1 \in \mathcal{O}_Y(U')$) that is compatible with the restriction maps on $\mathcal{O}_X$ and $\mathcal{O}_Y$ (cf. Exercise 20). It also follows that there is an induced ring homomorphism on the stalks: $\varphi^\# : \mathcal{O}_{X,P} \to \mathcal{O}_{Y,P'}$ for any point $P' \in \operatorname{Spec} S$ and corresponding point $P = \varphi^*(P') \in \operatorname{Spec} R$. Under the isomorphism in Proposition 58, the homomorphism $\varphi^\#$ from $R_P \cong \mathcal{O}_{X,P}$ to $S_{P'} \cong \mathcal{O}_{Y,P'}$ is just the natural ring homomorphism φ_P on the localizations induced by the homomorphism φ. In particular, the inverse image under $\varphi^\#$ of the maximal ideal in the local ring $\mathcal{O}_{Y,P'}$ is the maximal ideal in the local ring $\mathcal{O}_{X,P}$.

Definition. Suppose $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ and $(\operatorname{Spec} S, \mathcal{O}_{\operatorname{Spec} S})$ are two affine schemes. A *morphism of affine schemes* from $(\operatorname{Spec} S, \mathcal{O}_{\operatorname{Spec} S})$ to $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ is a pair $(\varphi^*, \varphi^\#)$ such that

(1) $\varphi^* : \operatorname{Spec} S \to \operatorname{Spec} R$ is Zariski continuous,

(2) there are ring homomorphisms $\varphi^\# : \mathcal{O}(U) \to \mathcal{O}(\varphi^{*-1}(U))$ for every Zariski open subset U in $\operatorname{Spec} R$ that commute with the restriction maps, and

(3) if $P' \in \operatorname{Spec} S$ with corresponding point $P = \varphi^*(P) \in \operatorname{Spec} R$, then under the induced homomorphism on stalks $\varphi^\# : \mathcal{O}_{\operatorname{Spec} R, P} \to \mathcal{O}_{\operatorname{Spec} S, P'}$ the preimage of the maximal ideal of $\mathcal{O}_{\operatorname{Spec} S, P'}$ is the maximal ideal of $\mathcal{O}_{\operatorname{Spec} R, P}$.

A homomorphism $\psi : A \to B$ from the local ring A to the local ring B with the property that the preimage of the maximal ideal of B is the maximal ideal of A is called a *local homomorphism* of local rings. The third condition in the definition is then the statement that the induced homomorphism on stalks is required to be a local homomorphism.

With this terminology, the discussion preceding the definition shows that a ring homomorphism $\varphi : R \to S$ induces a morphism of affine schemes from $(\operatorname{Spec} S, \mathcal{O}_{\operatorname{Spec} S})$ to $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$.

Conversely, suppose $(\varphi^*, \varphi^\#)$ is a morphism of affine schemes from $(\operatorname{Spec} S, \mathcal{O}_{\operatorname{Spec} S})$ to $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$. Then in particular, for $U = \operatorname{Spec} R$, $(\varphi^*)^{-1}(U) = \operatorname{Spec} S$, so by assumption there is a ring homomorphism $\varphi^\# : \mathcal{O}_{\operatorname{Spec} R}(\operatorname{Spec} R) \to \mathcal{O}_{\operatorname{Spec} S}(\operatorname{Spec} S)$ defined on the global sections. By Proposition 57, we have $\mathcal{O}_{\operatorname{Spec} R}(\operatorname{Spec} R) \cong R$ and $\mathcal{O}_{\operatorname{Spec} S}(\operatorname{Spec} S) \cong S$ as rings. Composing with these isomorphisms shows that $\varphi^\#$ gives a ring homomorphism $\varphi : R \to S$. By Proposition 58 we have a local homomorphism $\varphi^\# : R_P \to S_{P'}$, and by the compatibility with the restriction homomorphisms it follows that the diagram

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & S \\ \downarrow & & \downarrow \\ R_P & \xrightarrow{\varphi^\#} & S_{P'} \end{array}$$

commutes, where the two vertical maps are the natural localization homomorphisms. Since $\varphi^\#$ is assumed to be a local homomorphism, $(\varphi^\#)^{-1}(P'S_{P'}) = PR_P$, from which it follows that $\varphi^{-1}(P') = P$. Hence the continuous map from $\operatorname{Spec} S$ to $\operatorname{Spec} R$ induced by φ is the same as φ^*, and it follows easily that φ also induces the homomorphism $\varphi^\#$. This shows that there is a ring homomorphism $\varphi : R \to S$ inducing both φ^* and $\varphi^\#$ as before.

We summarize this in the following proposition:

Theorem 59. Every ring homomorphism $\varphi : R \to S$ induces a morphism

$$(\varphi^*, \varphi^\#) : (\operatorname{Spec} S, \mathcal{O}_{\operatorname{Spec} S}) \to (\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$$

of affine schemes. Conversely, every morphism of affine schemes arises from such a ring homomorphism φ.

Theorem 59 is the analogue for $\operatorname{Spec} R$ of Theorem 6, which converted geometric questions relating to affine algebraic sets to algebraic questions for their coordinate rings.

The condition that the homomorphism on stalks be a local homomorphism in the definition of a morphism of affine schemes is necessary: a continuous map on the spectra together with a set of compatible ring homomorphisms on sections (hence also on stalks) is not sufficient to force these maps to come from a ring homomorphism.