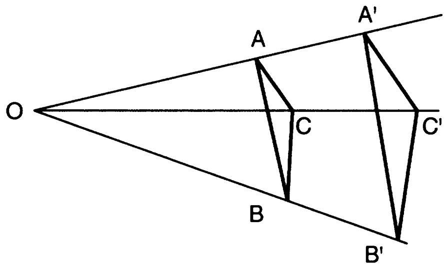

FIGURE 27.1. Desargues's Theorem

> only at infinity), then the points of intersection of pairs of corresponding sides are collinear (if only on the 'line at infinity'), and conversely.

The 'parallel case' for two triangles in the same plane is pictured in Figure 27.1. According to the theorem, if the lines joining corresponding vertices meet in a single point O and if AB is parallel to $A'B'$ and AC is parallel to $A'C'$, then BC is parallel to $B'C'$.

The theorem of Desargues can be proved, in Euclidean plane geometry, using only postulates I to V of Chapter 16. This is shown by David Hilbert (1862–1943) in his *Foundations of Geometry.*

Hilbert also shows that, in the absence of the Axiom of Archimedes (mentioned in Chapters 17 and 19), the Theorem of Desargues can be used for defining multiplication within Euclidean geometry, and for proving, within that geometry, that multiplication is associative. (Hilbert also shows that the Theorem of Pappus does not depend on the Axiom of Archimedes. The Theorem of Pappus is the key ingredient in any proof of the commutativity of multiplication which does not rest on the Axiom of Archimedes.)

For a simple proof of the 'parallel case' of Desargues' Theorem, the reader may wish to consult, e.g., Ewald [1971].

Descartes was educated by the Jesuits, who kindly permitted him to spend the mornings in bed, because of his delicate health. Throughout his life, he did much of his intellectual work in bed. Being a man of 'good' family, he was supposed to choose the church or the army for his career, and he chose the latter. He first served in the army of Maurice of Orange, but transferred to that of the Duke of Bavaria when the Thirty Years War broke out. Even on military campaigns, he spent a good part of his time in bed, thinking about mathematics and philosophy. He resigned his commission in 1621, and travelled for five years, studying mathematics. In the end, he settled in Holland, where he spent twenty years in full time intellectual pursuits. In 1649 he went to Sweden at the invitation of Queen Christina. The vigorous young queen insisted on having mathematics lessons at five in the morning, and within two months the aging Descartes, who would have preferred to sleep in, caught pneumonia and died.

Descartes's first book, *Le Monde*, advanced the Copernican model of the solar system. Just as he was completing it, the Inquisition condemned the Copernican views expressed by Galileo. Descartes decided not to publish his book, writing sadly to Mersenne:

> This has so strongly affected me that I have almost resolved to burn all my manuscript, or at least show it to no one. But on no account will I publish anything that contains a word that might displease the Church.

In 1637 Descartes published his famous *Discours de la Méthode pour bien conduire sa Rasion et chercher la Verité dans les Sciences.* This 'discours' has three important appendices. Appendix I, *La Dioptrique* treats optics and the laws of refraction. Appendix II, *Les Météores*, deals with atmospheric phenomena; in particular, it offers an explanation of the shape of the rainbow. For us, the most important is Appendix III, *La Géometrie.* This is divided into three books and sets forth the principles of analytic geometry. In it we find the usual formulas for the conics, and also Descartes's 'rule of signs' (stated without proof). Descartes was thus the creator of analytic geometry, although Fermat must share equal credit.

The revolutionary idea of analytic geometry was this: points in the Euclidean plane could be represented by pairs of real numbers and, consequently, straight lines and conic sections could be described by sets of pairs (x, y) satisfying equations of the form

$$Ax + By + C = 0$$

and

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

respectively. Thus geometry was reduced to algebra! It was now possible to solve some of the problems which the Greeks had left open, for instance, the problem of doubling the cube, although the actual solution had to wait a couple of centuries more.

Descartes's rule of signs asserts that the number of positive real roots of a polynomial equation $f(x) = 0$ with real coefficients is $v - 2k$, where v is the number of *variations in sign* and k is some natural number. To calculate v one writes $f(x)$ in descending powers of x, omitting all terms with zero coefficients. Then v is the number of times we change sign as we go from left to right. For example, the equation

$$x^6 - 3x^5 - x^4 + 2x - 5 = 0 \tag{1}$$

has three variations in sign, so we can tell that it has either three positive roots or only one. Sometimes we can be certain; for example, putting $x = -y$ in equation (1), we get

$$y^6 + 3y^5 - y^4 - 2y - 5 = 0, \tag{2}$$