

means there is no point trying to prove any sentence which implies $\text{Con}(\Sigma)$. If one wants to use such a sentence, it should be taken as a new axiom.

Sentences of mathematical interest actually arise in this way, most simply in set theory, where consistency is implied by the existence of a “large set.” The usual axioms of set theory (called ZF) say roughly that

(i) \mathbb{N} is a set.

(ii) Further sets result from certain operations, the most important of which are *power* (taking all subsets of a set) and *replacement* (taking the range of a function whose domain is a set).

Because of this, the axioms of ZF can be modelled by any set that contains \mathbb{N} and is closed under power and replacement. Such a set has to be very large—larger than any set whose existence can be proved in ZF—but if it exists then ZF must be consistent, since two contradictory sentences cannot be true of an actually existing object. Thus the existence of a set that is large in the above sense implies $\text{Con}(\text{ZF})$.

If ZF is consistent, then $\text{ZF} + \text{Con}(\text{ZF})$ is also consistent, but an even larger set is required to satisfy the enlarged axiom system. These large-set existence axioms are called *axioms of infinity*. Since they imply $\text{Con}(\text{ZF})$, they cannot be proved in ZF. In particular, one cannot prove the existence of a nontrivial measure on all subsets of \mathbb{R} since, as mentioned in Section 23.4, this implies the existence of a large set. In fact, the existence of a nontrivial measure on \mathbb{R} is an axiom of infinity far stronger than those previously mentioned. Gödel (1946) made the interesting speculation that any true but unprovable proposition is a consequence of some axiom of infinity.

More recently, some “largeness” properties in number theory have been found to imply $\text{Con}(\text{PA})$. The first of these was found by Paris and Harrington (1977), using a modification of a combinatorial theorem of Ramsey (1929). Paris and Harrington found a sentence σ that says that for each $n \in \mathbb{N}$ there is an m such that sets of size $\geq m$ have a certain combinatorial property $C(n)$. They showed that σ follows from Ramsey’s theorem on infinite sets, but that the function

$$f(n) = \text{least } m \text{ such that sets of size } m \text{ have property } C(n)$$

grows faster than any computable function whose existence can be proved in PA. Thus σ in some sense asserts the existence of a “large” function. The property $C(n)$ is such that one can decide whether a finite set has it

or not, hence σ implies (very simply, and certainly in PA) that f is computable. This shows immediately that σ cannot be proved in PA, but Paris and Harrington in fact showed the stronger result that σ implies $\text{Con}(\text{PA})$.

Gödel's theorem shows that something is missing in the purely formal view of mathematics, and the axioms of infinity show that the missing elements may be mathematically interesting and important. Despite this, the official view still seems to be that mathematics consists in the formal deduction of theorems from fixed axioms. As early as 1941 Post protested against this view:

It is to the writer's continuing amazement that ten years after Gödel's remarkable achievement current views on the nature of mathematics are thereby affected only to the point of seeing the need of many formal systems, instead of a universal one. Rather has it seemed to us to be inevitable that these developments will result in a reversal of the entire axiomatic trend of the late nineteenth and early twentieth centuries, with a return to meaning and truth. [Post (1941), p. 345]

I believe that what Post was saying was this. Before Gödel, the goal of mathematical logic had been to distill all mathematics into a set of axioms. It was expected that all of number theory, for example, could be recovered by formal deduction from PA, that is, *by forgetting that the axioms of PA had any meaning*. Gödel showed that this was not so, and in particular that the sentence $\text{Con}(\text{PA})$, which expresses consistency, could not be so recovered. But it is precisely by knowing the *meaning* of the PA axioms that one knows they are consistent: contradictory sentences cannot hold in the actual structure of \mathbb{N} with $+$ and \times . Thus it is the ability to see meaning in PA that enables us to see the truth of $\text{Con}(\text{PA})$ and hence to transcend the power of formal proof.

23.9 Biographical Notes: Gödel

Kurt Gödel (Figure 23.2) was born in 1906 in Brünn, Moravia (now Brno, Czechoslovakia) and died in Princeton in 1978. He was the second son of Rudolf Gödel, the manager of a textile firm, and Marianne Handschuh. Both his parents were members of the substantial German-speaking minority of the region, and his mother had received some of her education

at the French school in Brünn. Her influence seems to have been dominant in Kurt's upbringing, at least in the matter of church and school. He attended Lutheran institutions and was unsympathetic to the Catholic church, to which his father nominally belonged.

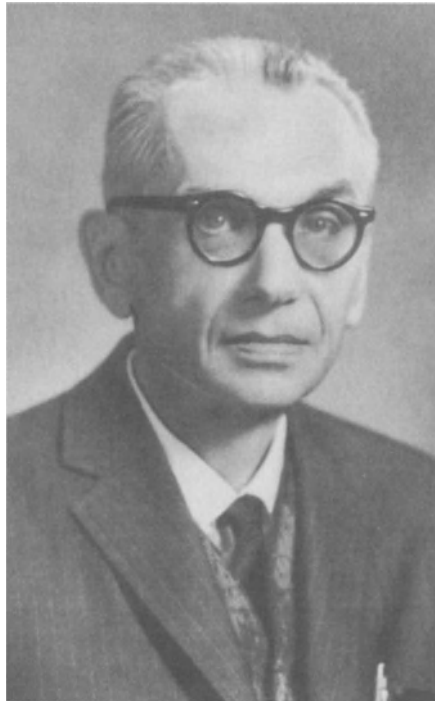


Figure 23.2: Kurt Gödel

Gödel had a generally happy childhood and was noted for his curiosity, being known to his family as *Herr Warum* (Mr. Why). The family was fortunate that Brünn was relatively untouched by World War I, and even after the war the absorption of Moravia into the new nation of Czechoslovakia had little effect on the Gödel family. The most disturbing event of Gödel's childhood was an attack of rheumatic fever at the age of six or seven, followed by his learning, at the age of eight, that rheumatic fever can damage the heart. To the end of his life he was convinced that he had a weak heart and, when doctors found no evidence of this, he developed a distrust of the medical profession as well. This led to a brush with death in the 1940s,

when he left a duodenal ulcer untreated, and he became obsessively cautious and prone to depression,

After completing secondary school, Gödel moved to Vienna (his father's birthplace) to enter university. He was at first undecided between mathematics and physics but opted for mathematics after hearing a brilliant cycle of lectures by the number theorist Fürthwangler. He was introduced to logic and set theory by Hans Hahn, who was interested in point set problems in the theory of real functions. Hahn got Gödel involved in the famous Vienna Circle of philosophers in 1926–1928 and later became his thesis supervisor. The Vienna Circle aimed to put science and philosophy on a rigorous basis by means of formal logic and no doubt had a strong influence on Gödel's work. However, his incompleteness theorem was obviously a blow to the Vienna Circle, just as it was to formalists in mathematics. In fact Gödel began to drift away from the Vienna Circle long before he discovered his theorem, as his philosophical position tended toward the diametric opposite of theirs. The Vienna Circle based its philosophy on strictly material data, whereas Gödel was metaphysical to the point of being interested in ghosts and demons [see for example Kreisel (1980), p. 155].

In 1927 Gödel met his future wife, Adele Nimbusky, a dancer at a nightclub in Vienna. His parents objected to her, on the grounds that she was six years older than Gödel and had been married before, and the couple did not marry until 1938. The marriage endured, and friends noted how much warmer Gödel became in her company. They had no children, and Adele was probably the only person in Gödel's life who could bring him down to earth occasionally.

Gödel became an Austrian citizen in 1929 and rapidly rose to fame after the publication of the incompleteness theorem in 1931. He was invited to the United States and made three visits to the Institute of Advanced Study in Princeton. In between, however, he suffered bouts of depression and spent some time in mental hospitals. In 1938 Hitler annexed Austria and the atmosphere became increasingly oppressive, though Gödel does not seem to have been perceptive about the menace of Nazism. He blamed the situation on Austrian "sloppiness" and decided to leave only when he was judged fit for military service—an obviously incompetent judgment in his opinion.

During this tense period of his life (1937–1940), Gödel tackled the main problems of set theory and proved the consistency of the axiom of

choice and the continuum hypothesis. Thus he arrived at Princeton in 1940 on a second wave of fame. He settled into a position at the Institute of Advanced Study, where he was to stay for the rest of his life. In the early 1940s he continued to work hard on set theory. In 1942 he found a proof of the independence of the axiom of choice but left his work unpublished when he found he was unable to do the same for the continuum hypothesis (namely, to show that if set theory is consistent, one can consistently assume that the axiom of choice is true but the continuum hypothesis is false). These are the results, of course, that were eventually obtained by Cohen (1963).

From 1943 onwards Gödel devoted himself mainly to philosophy. Indeed, Kreisel (1980), p. 150, argues that *all* of Gödel's discoveries stemmed from his philosophical acuteness—allied with the appropriate, but generally elementary, mathematical techniques. The incompleteness theorem, for example, comes from observing the difference between provability and truth. Gödel (1949) made an unexpected foray into another area of mathematics of philosophical interest, the theory of relativity. He showed that there are solutions of Einstein's equations that contain closed timelike lines, theoretically allowing the possibility of time travel. Gödel later calculated that the amount of energy required to travel into one's own past was prohibitively large, but the feasibility of signals to and from the past remained open. Indeed, he seems to have believed that this was a possible basis for the existence of ghosts [Kreisel (1980), p. 155].

Gödel was understandably reticent about expressing such opinions publicly. Even in the case of the incompleteness theorem, whose implications for the question of minds versus machines were widely debated, he did not publish his opinions. His private view, that the mind is more powerful than a machine, may, however, have been important in enabling him to foresee the incompleteness theorem in the first place. Indeed, it may not be too much to say that Gödel's receptiveness to scientifically unconventional ideas paved the way for his unconventional theorems.

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