12. If $H$ is a subgroup of $G$ and $A$ is an abelian group let $M_{G/H}(A)$ denote the abelian group of all maps from the left cosets $gH$ of $H$ in $G$ to $A$.
   (a) Prove that $M_1^G(A) \cong M_1^H(M_{G/H}(A))$ as $H$-modules. [If $\{g_i\}_{i\in\mathcal{I}}$ is a choice of left coset representatives of $H$ in $G$ define the correspondence between $f \in M_1^G(A)$ and $F : H \to M_{G/H}(A)$ by $F(h)(g_i H) = f(g_i h)$, and check that this is an isomorphism of $H$-modules.]
   (b) A $G$-module $A$ such that $H^n(H, A) = 0$ for all $n \geq 1$ and all subgroups $H$ of $G$ is called *cohomologically trivial.* Prove that $M_1^G(A)$ is a cohomologically trivial for any abelian group $A$.
   (c) If $G$ is finite, prove that $\mathbb{Z}G \otimes_{\mathbb{Z}} A$ is cohomologically trivial for all abelian groups $A$.

13. Suppose $A$ is a $G$-module and $H$ is a subgroup of $G$. Prove that the group homomorphism from $H^n(G, A)$ to $H^n(G, M_H^G(A))$ for all $n \geq 0$ induced from the $G$-module homomorphism from $A$ to $M_H^G(A)$ in Example 3 following Corollary 22 composed with the isomorphism $H^n(G, M_H^G(A)) \cong H^n(H, A)$ of Shapiro's Lemma is the restriction homomorphism from $H^n(G, A)$ to $H^n(H, A)$.

14. Suppose $\varphi : H \to G$ is the inclusion map of the subgroup $H$ of $G$ into $G$. If $A$ is an $H$-module and $M_H^G(A)$ the associated induced $G$-module, define the group homomorphism $\psi : M_H^G(A) \to A$ by mapping $f$ to its value at 1: $\psi(f) = f(1)$.
   (a) Prove that $\varphi$ and $\psi$ are compatible homomorphisms.
   (b) Prove that the induced group homomorphism from $H^n(G, M_H^G(A))$ to $H^n(H, A)$ for $n \geq 0$ is the isomorphism in Shapiro's Lemma.

15. Suppose $H$ is a normal subgroup of $G$ and $A$ is a $G$-module. For fixed $g \in G$, let $\psi(a) = ga$ and $\varphi(h) = g^{-1}hg$ for $h \in H$.
   (a) Prove that $\varphi$ and $\psi$ are compatible homomorphisms.
   (b) For each $n \geq 0$, prove that the homomorphism $\theta_g$ from $H^n(H, A)$ to $H^n(H, A)$ induced by the compatible homomorphisms $\varphi$ and $\psi$ is an automorphism of $H^n(H, A)$. [Observe that both $\varphi$ and $\psi$ have inverses.]
   (c) Show that $\theta_g$ acting on $H^0(H, A)$ is the automorphism in Exercise 4.

16. Let $A$ be a $G$-module and for $g \in G$ let $\theta_g$ denote the automorphism of $H^n(G, A)$ defined in the previous exercise.
   (a) Prove that $\theta_g$ acting on $H^0(G, A) = A^G$ is the identity map.
   (b) Prove that $\theta_g$ acting on $H^n(G, A)$ is the identity map for $n \geq 1$. [By induction on $n$ and dimension shifting. For $n = 1$, use the exact sequence in Corollary 22, together with (a) applied to $\theta_g$ on $C^G$. For $n \geq 2$ use the isomorphism $H^{n+1}(G, A) \cong H^n(G, C)$ in Corollary 22.]

17. Suppose that $H$ is a normal subgroup of $G$ and $A$ is a $G$-module. For $n \geq 0$ prove that $H^n(H, A)$ is a $G/H$-module where $gH$ acts by the automorphism $\theta_g$ induced by conjugation by $g$ on $H$ and the natural action of $g$ on $A$ as in Exercise 15. [Use the previous exercise to show this action of a coset is well defined.]

18. Suppose that $G$ is cyclic of order $m$, that $H$ is a subgroup of $G$ of index $d$, and that $\mathbb{Z}$ is a trivial $G$-module. Use the projective $G$-module resolution in Exercise 8 to prove
   (a) that Cor : $H^n(H, \mathbb{Z}) \to H^n(G, \mathbb{Z})$ is multiplication by $d$ from $\mathbb{Z}$ to $\mathbb{Z}$ for $n = 0$, from $\mathbb{Z}/(m/d)\mathbb{Z}$ to $\mathbb{Z}/m\mathbb{Z}$ if $n$ is odd, and from 0 to 0 if $n$ is even, $n \geq 2$, and
   (b) that Res : $H^n(G, \mathbb{Z}) \to H^n(H, \mathbb{Z})$ is the identity map from $\mathbb{Z}$ to $\mathbb{Z}$ for $n = 0$, and is the natural projection map from $\mathbb{Z}/m\mathbb{Z}$ to $\mathbb{Z}/(m/d)\mathbb{Z}$ or from 0 to 0, depending on the parity of $n \geq 1$.

19. Let $p$ be a prime and let $P$ be a Sylow $p$-subgroup of the finite group $G$. Show that for

any $G$-module $A$ and all $n \geq 0$ the map $\text{Res} : H^n(G, A) \to H^n(P, A)$ is injective on the $p$-primary component of $H^1(G, A)$. Deduce that if $|A| = p^a$ then the restriction map is injective on $H^n(G, A)$. [Use Proposition 26.]

**20.** Let $p$ be a prime, let $G = \langle \sigma \rangle$ be cyclic of order $p^m$ and let $W$ be a vector space of dimension $d > 0$ over $\mathbb{F}_p$ on which $\sigma$ acts as a linear transformation. Assume $W$ has a basis such that the matrix of $\sigma$ is a $d \times d$ elementary Jordan block with eigenvalue 1.
- **(a)** Prove that $d \leq p^m$. [Use facts about the minimal polynomial of an elementary Jordan block.]
- **(b)** Prove that $\dim_{\mathbb{F}_p} W^G = 1$.
- **(c)** Prove that $\dim_{\mathbb{F}_p} (\sigma - 1)W = d - 1$.
- **(d)** If $N = 1 + \sigma + \cdots + \sigma^{p^m - 1}$ is the usual norm element, prove that $NW$ is of dimension 1 if $d = p^m$ (respectively, of dimension 0 if $d < p^m$) and that the dimension of ${}_N W$ is $d - 1$ (respectively, $d$). [Let $R$ be the group ring $\mathbb{F}_p G$, and show that every nonzero $R$-submodule of $R$ contains $N$. Note that $W$ is a cyclic $R$-module and let $\varphi : R \to W$ be a surjective homomorphism. Conclude that if $\varphi$ is not an isomorphism then $N \in \ker \varphi$.]
- **(e)** Deduce that if $d = p^m$ then $H^n(G, W) = 0$, and if $d < p^m$ then $H^n(G, W)$ has order $p$, for all $n \geq 1$ (i.e., these cohomology groups are zero if and only if $W$ is a free $\mathbb{F}_p G$-module).

**21.** Let $p$ be a prime, let $G = \langle \sigma \rangle$ be cyclic of order $p^m$ and let $V$ be a $G$-module of exponent $p$. Let $V = V_1 \oplus V_2 \oplus \cdots \oplus V_k$ be a decomposition of $V$ giving the Jordan Canonical Form of $\sigma$, where each $V_i$ is $\sigma$-invariant and a matrix of $\sigma$ on $V_i$ is an $d_i \times d_i$ elementary Jordan block with eigenvalue 1, $d_i \geq 1$ (cf. Section 12.3). Prove that $|V^G| = p^k$ and $|H^n(G, V)| = p^s$ where $s$ is the number of $V_i$ of dimension less than $p^m$ over $\mathbb{F}_p$, for all $n \geq 1$. [Use the preceding exercise and Exercise 5.]

**22.** Suppose $G$ is a topological group, i.e., there is a topology on $G$ such that the maps $G \times G \to G$ defined by $(g_1, g_2) \mapsto g_1 g_2$ and $G \to G$ defined by $g \mapsto g^{-1}$ are continuous.
- **(a)** If $H$ is an open subgroup of $G$ and $g \in G$, prove that the cosets $gH$ and $Hg$ and the subgroup $g^{-1}Hg$ are also open.
- **(b)** Prove that any open subgroup is also closed. [The complement is the union of cosets as in (a).]
- **(c)** Prove that a closed subgroup of finite index is open.
- **(d)** If $G$ is compact prove that every open subgroup $H$ is of finite index.

**23.** Suppose $G$ is a compact topological group. Prove the following are equivalent:
- (i) $G$ is profinite, i.e., $G = \varprojlim G_i$ is the inverse limit of finite groups $G_i$.
- (ii) There exists a family $\{N_i\}$ $(i \in \mathcal{I})$ of open normal subgroups $N_i$ in $G$ such that $\cap_i N_i = 1$ and in this case $G \cong \varprojlim(G/N_i)$.
- (iii) There exists a family $\{H_j\}$ $(j \in \mathcal{J})$ of open subgroups $H_j$ in $G$ such that $\cap_j H_j = 1$.

[To show (iii) implies (ii), let $H$ be open in $G$ and use (d) of the previous exercise to show that $N = \cap_{g \in G} g^{-1}Hg$ is a finite intersection and conclude that $N \subseteq H \subseteq G$ and $N$ is open and normal in $G$.]

**24.** Suppose $N$ and $N'$ are open normal subgroups of the profinite group $G$ and $N' \subseteq N$. Prove that the projection homomorphism $\varphi : G/N' \to G/N$ and the injection $\psi : A^N \to A^{N'}$ are compatible homomorphisms and deduce there is an induced homomorphism from $H^n(G/N, A^N)$ to $H^n(G/N', A^{N'})$.

**25.** If $G$ is an infinite profinite group show that $G$ does not act continuously on $A = \mathbb{Z}G$. [Show that the stabilizer of $a \in A$ is not always of finite index in $G$.]

## 17.3 CROSSED HOMOMORPHISMS AND $H^1(G,A)$

In this section we consider in greater detail the cohomology group $H^1(G, A)$ where $G$ is a group and $A$ is a $G$-module. From the definition of the coboundary map $d_1$ in equation (18), if $f \in C^1(G, A)$ then

$$d_1(f)(g_1, g_2) = g_1 \cdot f(g_2) - f(g_1 g_2) + f(g_1).$$

Thus any function $f : G \to A$ is a 1-cocycle if and only if it satisfies the identity

$$f(gh) = f(g) + gf(h) \qquad \text{for all } g, h \in G. \tag{17.20}$$

Equivalently, a 1-cocycle is determined by a collection $\{a_g\}_{g \in G}$ of elements in $A$ satisfying $a_{gh} = a_g + ga_h$ for $g, h \in G$ (and then the 1-cocycle $f$ is the function sending $g$ to $a_g$). Note that if 1 denotes the identity of $G$, then $f(1) = f(1^2) = f(1) + 1 \cdot f(1) = 2f(1)$, so $f(1) = 0$ is the identity in $A$. Thus 1-cocycles are necessarily "normalized" at the identity. It then follows from the cocycle condition that $f(g^{-1}) = -g^{-1} f(g)$ for all $g \in G$.

If $A$ is a $G$-module on which $G$ acts trivially, then the cocycle condition (20) is simply $f(gh) = f(g) + f(h)$, i.e., $f$ is simply a *homomorphism* from the multiplicative group $G$ to the additive group $A$. Because of this the functions from $G$ to $A$ satisfying (20) are called *crossed homomorphisms*.

A 1-cochain $f$ is a 1-coboundary if there is some $a \in A$ such that

$$f(g) = g \cdot a - a \qquad \text{for all } g \in G, \tag{17.21}$$

(equivalently, $a_g = ga - a$ in the notation above). Note that since $-a \in A$, the coboundary condition in (21) can also be phrased as $f(g) = a - g \cdot a$ for some fixed $a \in A$ and all $g \in G$. The 1-coboundaries are called *principal crossed homomorphisms*. With this terminology the cohomology group $H^1(G, A)$ is the group of crossed homomorphisms modulo the subgroup of principal crossed homomorphisms.

**Example: (Hilbert's Theorem 90)**

Suppose $G = \mathrm{Gal}(K/F)$ is the Galois group of a finite Galois extension $K/F$ of fields. Then the multiplicative group $K^\times$ is a $G$-module and $H^1(G, K^\times) = 0$. To see this, let $\{\alpha_\sigma\}$ be the values $f(\sigma)$ of a 1-cocycle $f$, so that $\alpha_\sigma \in K^\times$ and $\alpha_{\sigma\tau} = \alpha_\sigma \sigma(\alpha_\tau)$ (the cocycle condition written multiplicatively for the group $K^\times$). By the linear independence of automorphisms (Corollary 8 in Section 14.2), there is an element $\gamma \in K$ such that

$$\beta = \sum_{\tau \in G} \alpha_\tau \tau(\gamma)$$

is nonzero, i.e., $\beta \in K^\times$. Then for any $\sigma \in G$ we have

$$\sigma(\beta) = \sum_{\tau \in G} \sigma(\alpha_\tau)\, \sigma\tau(\gamma) = \alpha_\sigma^{-1} \sum_{\tau \in G} \alpha_{\sigma\tau}\, \sigma\tau(\gamma) = \alpha_\sigma^{-1} \beta$$

where the second equality comes from the cocycle condition. Hence $\alpha_\sigma = \beta/\sigma(\beta)$, which is the multiplicative form of the coboundary condition (21) (for the element $a = \beta^{-1}$). Since every 1-cocycle is a 1-coboundary, we have $H^1(G, K^\times) = 0$. The same result holds for infinite Galois extensions by equation (19) in the previous section since $H^1(G, K^\times)$ is the direct limit of trivial groups.