and that this is possible if x and y are "similar plane numbers". This means, in our language, that $x = u^2w$ and $y = v^2w$ for some natural numbers u, v, and w. (The reason for the name is that if a rectangle of area w is magnified by u or v, its area becomes u^2w or v^2w, respectively.) Substituting these "similar plane numbers" for x and y gives the identity

$$(uvw)^2 + \left(\frac{u^2 - v^2}{2}w\right)^2 = \left(\frac{u^2 + v^2}{2}w\right)^2,$$

hence Euclid has solved his problem. The numbers uvw, $\frac{u^2-v^2}{2}w$ and $\frac{u^2+v^2}{2}w$ he has found will be integers if u, v, and w are natural numbers and u, v are both odd or both even (the latter condition ensuring that 2 divides $u^2 - v^2$ and $u^2 + v^2$). Hence he has also found a formula to produce Pythagorean triples.

Euclid then made a throwaway remark that is even more interesting: *xy is a square only if x and y are similar plane numbers*. This is the key to finding all Pythagorean triples, because his numbers $\frac{x-y}{2} = b$, $\frac{x+y}{2} = c$ can equal *any* natural numbers $b < c$ by choosing $x = b + c$, $y = c - b$, and in this case $xy = c^2 - b^2$. If b and c belong to a Pythagorean triple, $c^2 - b^2 = xy$ must therefore be a square, and Euclid's remark is that this happens only if $x = u^2w$, $y = v^2w$ for some natural numbers u, v, w. Thus he is implicitly claiming the following result.

Parameterization of Pythagorean triples *Any Pythagorean triple is of the form*

$$a = uvw, \quad b = \frac{u^2 - v^2}{2}w, \quad c = \frac{u^2 + v^2}{2}w$$

for some natural numbers u, v, and w.

Proof It remains to prove that xy is a square only if $x = u^2w$ and $y = v^2w$ for some natural numbers u, v, and w. In fact, Euclid did this in Proposition 2 of Book IX, which is based on his theory of divisibility. As mentioned in Section 1.6, it is equivalent, and often easier, to use unique prime factorization, and a proof along the latter lines goes as follows.

Suppose x and y are natural numbers and xy is a square. By removing $w = \gcd(x, y)$ from both x and y we obtain natural numbers

$x' = x/w$ and $y' = y/w$ for which $x'y'$ is also a square, but with $\gcd(x', y') = 1$. It follows that the unique prime factorizations of x' and y',

$$x' = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r},$$
$$y' = q_1^{f_1} q_2^{f_2} \cdots q_s^{f_s},$$

have no prime in common. But then, because

$$x'y' = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r} q_1^{f_1} q_2^{f_2} \cdots q_s^{f_s}$$

is a square, unique prime factorization implies that each of the exponents $e_1, e_2, \ldots, e_r, f_1, f_2, \ldots, f_s$ is even. That is, each exponent in the prime factorizations of x' and y' is even, and hence x' and y' are squares themselves. If $x' = u^2$ and $y' = v^2$ this gives $x = u^2w$ and $y = v^2w$, as required. □

It is convenient to call numbers x', y' *relatively prime* when $\gcd(x', y') = 1$. The result that a product of relatively prime x', y' is a square only if x' and y' are squares is one of the most useful consequences of unique prime factorization. Similarly, one finds that the product of relatively prime numbers is a cube only if the numbers themselves are cubes, and so on.

Exercises

There are some variations on Euclid's formula for Pythagorean triples that are worth knowing. One is

$$a = 2uvw, \quad b = (u^2 - v^2)w, \quad c = (u^2 + v^2)w.$$

This is, of course, the double of Euclid's formula, and it does not look completely general, because $a = 2uvw$ is necessarily even. However, it is impossible for a and b both to be odd. If they were, a^2 and b^2 would both leave remainder 1 on division by 4, and hence $c^2 = a^2 + b^2$ would leave remainder 2. But c^2 is an even square and hence leaves remainder 0 on division by 4 (compare with the exercises to Section 1.2).

4.2.1. Deduce from these remarks that in any Pythagorean triple (a, b, c), if the sides are suitably ordered, a is even and b and c are either both even or both odd.

4.2.2. Now use the identity $4xy + (x - y)^2 = (x + y)^2$ to show that any pair $(b, c) = (x - y, x + y)$ of numbers that are both even or both odd extends to a Pythagorean triple (a, b, c) just in case $4xy$ is a square.

4.2.3. Use unique prime factorization to show that $4xy$ is a square if and only if $x = u^2w$ and $y = v^2w$ for some natural numbers u, v, w.

4.2.4. Deduce from the preceding exercises that any Pythagorean triple, if the sides are suitably ordered, is of the form

$$a = 2uvw, \quad b = (u^2 - v^2)w, \quad c = (u^2 + v^2)w$$

for some natural numbers u, v, w.

Pythagorean triples for which a, b, and c have no common divisor except 1 are called *primitive.*

4.2.5. Deduce from Exercise 4.2.4 that each primitive Pythagorean triple, suitably ordered, is of the form

$$a = 2uv, \quad b = u^2 - v^2, \quad c = u^2 + v^2,$$

where u and v are natural numbers with $\gcd(u, v) = 1$, one of them even and the other odd.

An interesting interpretation of the parameters u and v was given in the *Nine Chapters of Mathematical Art,* a Chinese work from the period between 200 B.C. and 200 A.D.

Suppose that one person walks along the sides a, b of a right-angled triangle at speed u, while another walks along the hypotenuse c at speed v, and that both cover the distance in the same time.

4.2.6. Show that, with a suitable choice of unit length,

$$a = 2uv, \quad b = u^2 - v^2, \quad c = u^2 + v^2.$$

(*Hint*: Use the speed condition to find an expression for $b + c$, and substitute it in $a^2 = c^2 - b^2 = (c - b)(c + b)$ to find an expression for $c - b$.)

4.3 Pythagorean Triples in Diophantus

Pythagorean triples may be grouped into classes in which each member of a class is an integer multiple of the smallest member. The smallest member (a, b, c) of each class is one for which a, b, and c have no common divisor > 1, or what we called a *primitive* Pythagorean triple in the exercises. From this viewpoint, we see that the main problem in finding Pythagorean triples is to find the primitive triples. Once we know $(3, 4, 5)$ is a Pythagorean triple, for example, it is trivial to list its multiples $(6, 8, 10)$, $(9, 12, 15)$ .... In fact, they may all be regarded as the "same" triangle, with different choices of the unit of length.

The many integer triples (a, b, c) that are really the "same" may be condensed to a single *rational triple* $(a/c, b/c, 1)$, because if (a, b, c) and (a', b', c') are multiples of the same triple then $a/c = a'/c'$ and $b/c = b'/c'$. Rational numbers really simplify the story here, because we can find a formula for all rational Pythagorean triples without using unique prime factorization (or Euclid's equivalent theory of divisibility). This was discovered around 250 A.D. by the Greek mathematician Diophantus and presented in his book the *Arithmetica*.

Parameterization of rational Pythagorean triples *The nonzero rationals x and y such that $x^2 + y^2 = 1$ are the pairs of the form*

$$x = \frac{1-t^2}{1+t^2}, \qquad y = \frac{2t}{1+t^2},$$

for rational numbers $t \neq 0, \pm 1$.

Proof The problem is to find points on the unit circle $x^2 + y^2 = 1$ with rational coordinates x and y, the so-called *rational points*. Some rational points are obvious, for example, $(-1, 0)$. We also notice that if (x_0, y_0) is any rational point, the line between it and $(-1, 0)$ has rational slope, namely, $t = y_0/(x_0 + 1)$.

Conversely, if $y = t(x+1)$ is any line through $(-1, 0)$ with rational slope t, then its second intersection with the unit circle is a rational point, as the following calculation shows. The intersection of $y = t(x + 1)$ with $x^2 + y^2 = 1$ occurs where

$$x^2 + t^2(x+1)^2 = 1 \quad \text{(substituting } t(x+1) \text{ for } y\text{)},$$

and hence

$$x^2(1+t^2) + 2t^2x + t^2 - 1 = 0,$$

which has solutions

$$\begin{aligned} x &= \frac{-2t^2 \pm \sqrt{4t^4 - 4(1+t^2)(t^2-1)}}{2(1+t^2)} \quad \text{by the quadratic formula} \\ &= \frac{-t^2 \pm 1}{1+t^2} \\ &= -1, \frac{1-t^2}{1+t^2}. \end{aligned}$$

The solution $x = -1$ gives the point $(-1, 0)$ we already know. The solution $x = (1-t^2)/(1+t^2)$ gives the second intersection (Figure 4.2), where

$$y = t(x+1) = t\left(\frac{1-t^2}{1+t^2} + 1\right) = t\left(\frac{1-t^2+1+t^2}{1+t^2}\right) = \frac{2t}{1+t^2}.$$

Thus the coordinates of the second intersection are $x = \frac{1-t^2}{1+t^2}$, $y = \frac{2t}{1+t^2}$, and these are rational because they are built from the rational number t by rational operations.

Hence we have found all rational points on the circle, and all except $(-1, 0)$ have the form $\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$. By taking $t \neq 0, \pm 1$ we also exclude the points $(1, 0)$, $(0, 1)$ and $(0, -1)$. The formula $x = \frac{1-t^2}{1+t^2}$, $y = \frac{2t}{1+t^2}$ then covers exactly the rational Pythagorean triples, because the latter have x and y nonzero. □

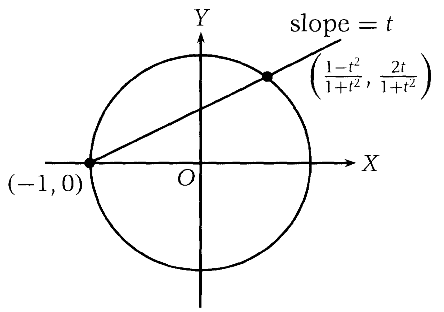

FIGURE 4.2 Constructing rational points on the circle.

Exercises

The formula for rational Pythagorean triples is appealing, because it involves only the single parameter t instead of the u, v, and w required for integer triples. Nevertheless, it is essentially the same formula.

4.3.1. By putting $t = u/v$ for integers u and v, deduce a formula for integer Pythagorean triples from the formula for rational Pythagorean triples.

The formula can in fact be simplified even further by using complex numbers, which we shall say more about later. Those already familiar with $\sqrt{-1} = i$ may enjoy the following exercise.

4.3.2. If points (x, y) of the plane are represented by complex numbers $x + iy$, show that the rational points on the unit circle, other than $(-1, 0)$, are the points of the form $\frac{t-i}{t+i}$, where t is rational.

The formulas

$$x = \frac{1-t^2}{1+t^2}, \qquad y = \frac{2t}{1+t^2}$$

are also useful when t runs through all real values. The point (x, y) then runs through all points of the unit circle (except for $(-1, 0)$), hence the formulas may also be viewed as *parametric equations for the circle.* They are related to the more familiar parametric equations $x = \cos\theta$, $y = \sin\theta$, as we shall see in the next chapter.

We call the functions $x(t) = \frac{1-t^2}{1+t^2}$, $y(t) = \frac{2t}{1+t^2}$ *rational functions* of t because they are built from the variable t and constants by rational operations.

4.3.3. Find rational functions $x(t)$ and $y(t)$ such that $(x(t), y(t))$ runs through all points of the circle except $(1, 0)$.

Rational functions are simpler than the functions $\cos\theta$, $\sin\theta$, and they have certain advantages. Because of this, in algebra and calculus we often want to *rationalize* irrational functions $f(x)$ such as $\sqrt{1-x^2}$. That is, we want to substitute a new function $x(t)$ for x so that $f(x)$ becomes a rational function of the new variable t.

4.3.4. Show that the function $\sqrt{1-x^2}$ is rationalized by the substitution $x = \frac{1-t^2}{1+t^2}$, and also by the substitution $x = \frac{2t}{1+t^2}$.

In contrast to this result, the functions $\sqrt{1-x^3}$ and $\sqrt{1-x^4}$ can *not* be rationalized by substituting a rational function for x. This discovery marks an important boundary between quadratic and higher-degree polynomials and hence leads beyond the scope of this book. Nevertheless, properties of quadratic equations can be used to explain some of the difficulties that arise with higher degree, and we shall show how this comes about in Section 4.7*.

4.4 Rational Triangles

After the discovery of rational right-angled triangles and their complete description by Euclid (Section 4.2), one might expect questions to arise about rational triangles in general. Of course, any three rational numbers can be the sides of a triangle, provided the sum of any two of them is greater than the third. Thus a "rational triangle" should be one that is rational not only in its side lengths, but also in some other quantity, such as altitude or area. Because area $= \frac{1}{2}$base $\times$ height, a triangle with rational sides has rational area if and only all its altitudes are rational, so it is reasonable to define a *rational triangle* to be one with rational sides and rational area.

Many questions can be raised about rational triangles, but they rarely occur in Greek mathematics. As far as we know, the first to treat them thoroughly was the Indian mathematician Brahmagupta, in his *Brâhma-sphuṭa-siddhânta* of 628 A.D. In particular, he found the following complete description of rational triangles.

Parameterization of rational triangles *A triangle with rational sides a, b, c and rational area is of the form*

$$a=\frac{u^2}{v}+v,\quad b=\frac{u^2}{w}+w,\quad c=\frac{u^2}{v}-v+\frac{u^2}{w}-w$$

for some rational numbers u, v, and w.

Brahmagupta (see Colebrooke (1817), p. 306) actually has a factor $1/2$ in each of a, b, and c, but this is superfluous because, for example,

$$\frac{1}{2}\left(\frac{u^2}{v}+v\right)=\frac{(u/2)^2}{v/2}+v/2=\frac{{u_1}^2}{v_1}+v_1,$$