

hyperbola (*continued*)
 area of sector, 295, 297
 area under, 287
 asymptotes, 258
 equation of, 250
 and exponential function, 283
 integer points, 264
 rational points, 126
 relative to horizon, 276
 as section of cone, 248
 transcendental area function,
 287
 two points at infinity, 279
 hyperbolic
 cosine, *see* cosh
 functions, 295
 and Pell equation, 298
 sine, *see* sinh

I

i , 215
 multiplication by, 221
 and Pythagorean triples, 158
 ideal, 244
 nonprincipal
 in $\mathbb{Z}[\sqrt{-5}]$, 245
 principal, 244
 and unique prime
 factorization, 244
 identity
 function, 97
 property of 0, 10
 property of 1, 11
 identity element
 in abelian group, 210
 induction, 1
 ascent form, 24
 in combinatorics, 34

 definition by, 27
 in Dedekind, 35
 descent form, 23
 step, 24
 infinite
 decimal, 81
 ultimately periodic, 82
 descent, 2
 in Fermat, 33, 131
 sets, 28
 infinitesimals, 55
 don't exist in \mathbb{R} , 84
 infinity, 1
 axiom of, 28
 focus at, 255
 line at, 278
 and natural numbers, 1
 point at, 278
 integer, 10
 as a set of pairs, 304
 negative, 10
 positive, 10
 integer points
 on conics, 263
 on ellipse, 263
 on hyperbola, 264
 on line, 126
 on parabola, 264
 on $x^2 + xy - y^2 = 1$, 301
 and Fibonacci numbers,
 301
 on $x^2 - 2y^2 = 1$
 product of, 264
 on $x^2 - dy^2 = 1$, 126, 300
 form infinite cyclic group,
 301
 generation of, 301
 product of, 265
 on $x^2 - y^2 = 1$, 126, 266

- integers
 - Gaussian, 32
 - intersections, 72
 - axioms about, 73
 - of circles, 74
 - of conics, 262
 - of lines, 73
 - inverse
 - in abelian group, 210
 - additive, 10
 - cosine, 152
 - function, 97
 - of logarithm function, 290
 - mod n , 186
 - criterion for existence, 187
 - and division mod n , 187
 - by Euclidean algorithm, 189
 - and the gcd, 187
 - multiplicative, 11
 - tan
 - infinite series, 172
 - irrationality
 - by Cantor's argument, 109
 - clarified by Dedekind, 55
 - Dedekind definition, 79
 - of e , 294
 - from unique prime factorization, 18
 - in music theory, 64
 - "not a ratio", 2
 - of $\log_{10} 2$, 19
 - of $\sqrt[3]{2}$, 19
 - of $\sqrt{2}$, 2, 19
 - and Pythagoreans, 54
 - of $\sqrt{3}$, 19
 - unknown for Euler's constant, 291
 - isometry, 87
 - basic property, 89
 - is composite of reflections, 91
 - of Euclidean plane, 89
 - as complex function, 223
 - of Euclidean space, 97
 - of non-Euclidean plane, 100
 - as complex function, 239
 - of non-Euclidean space, 241
 - as complex function, 241
 - orientation-preserving, 98
 - and complex numbers, 226
 - orientation-reversing
 - and complex numbers, 226
 - of sphere, 97
 - as complex function, 239
 - isomorphic rings, 196
- J**
- Jyesthadeva, 172
- K**
- Kepler, 64
 - and points at infinity, 279
 - introduced term focus, 253
- Klein
 - and motion in geometry, 48
 - definition of geometry, 48, 95

L

Lagrange
 - four squares theorem, 139, 179
 - integer solutions
 - quadratic equations, 309
 - polynomial theorem, 188
 - and Euler's criterion, 198

- Lagrange (*continued*)
 proved Wilson's theorem, 193, 209
 solved Pell equation, 271
 theory of quadratic forms, 242
- Landau, 77
- Laski, 2
- lattice, 244
 shape, 245
- lcm, 20
 how to compute, 21
 in terms of prime factors, 20
- least common multiple, *see* lcm
- least upper bounds, 84
 and area, 284
 and completeness, 84
 exist in \mathbb{R} , 84
 and length of curves, 95
- Legendre
 symbol, 199
 computation, 207
 is multiplicative, 207
 three squares theorem, 179
- Leibniz, 171, 172
 found Wilson's theorem, 208
- length
 of arc
 and angle measure, 144
 of curve, 95
- Lindemann, 295
- line, 82
 absence of gaps, 82
 "continuity", 82
 has degree 1, 247
 equation of, 86
 equidistant, 87, 253
 and reflection, 90
 as equidistant set, 86
 at infinity, 278
 integer points, 126
 non-Euclidean, 100
 order of points, 82
 in $\mathbb{R} \times \mathbb{R}$, 86
 rational points, 126
 reflection in, 87, 90
- linear
 congruence, 197
 equation, 13
 integer solutions, 13
 reason for name, 70
 test for integer solvability, 15
 representation of gcd, 14, 31
- logarithm
 additive property, 287
 consequences, 289
 inverse of, 290
 is transcendental, 293
 natural, 287
 to base a , 294
 to base 2, 305
- lune, 285
- M**
- Madhava, 172
- mathematical induction, *see* induction
- Matijasevič, 309
- matrices, 259
- measure
 of angle, 143
 via arc length, 144
 area, 51
 length, 50
- Menaechmus, 260
 construction of $\sqrt[3]{2}$, 260

Mersenne, 7
 primes, 7
 and perfect numbers, 22
 method of finding 1
 and Chinese remainder
 theorem, 196
 and gcd, 196
 mod notation
 for remainder, 180
 modulus, 180
 Mordell, 142
 motion
 and congruence, 47
 in Klein's definition of
 geometry, 48
 multiplicative inverse, 11
 property, 11

N

\mathbb{N} , 10
 countability, 107
 n -gon
 angle sum, 47
 natural
 logarithm, 287
 numbers, 1
 and infinity, 1
 Neugebauer and Sachs, 112
 Newton, 64
 chord construction, 142
 theory of gravitation
 and conics, 252, 279
 and focus, 252
 Nīlakaṇṭha, 171
Nine Chapters, 116
 Pythagorean triples in, 116
 nonconstructibility of $\sqrt[3]{2}$, 75,
 77

non-Euclidean
 angle, 101
 circle, 101
 construction, 103
 constructible point, 101
 distance, 100
 geometry, 71
 isometry, 100
 line, 100
 line trisection
 sometimes impossible, 104
 plane, 86, 100
 ruler and compass
 constructions, 100
 similarities
 are isometries, 109
 size
 follows from shape, 109
 translation, 102
 non-Euclidean plane, 100
 complex coordinate, 239
 disk model, 239
 isometries, 241
 half turn about i , 239
 and lattice shapes, 245
 in non-Euclidean space, 241
 orientation-preserving
 isometries
 as complex functions, 239
 and parallel axiom, 100
 reflection in imaginary axis,
 239
 tessellation, 239
 generated by reflections,
 239
 non-Euclidean space, 238, 241
 Euclidean planes in, 241
 half space model, 241
 non-Euclidean planes in, 241

- non-Euclidean (*continued*)
 - orientation-preserving
 - isometries
 - as complex functions, 241
 - spheres in, 241
 - nonmodularity, 141
 - norm, 228
 - multiplicative property, 228
 - on $\mathbb{Z}[\sqrt{-2}]$, 230
 - on $\mathbb{Z}[\sqrt{2}]$, 267
 - on $\mathbb{Z}[\sqrt{d}]$, 267, 273
 - and Brahmagupta identity, 267
 - multiplicative property, 273
 - on $\mathbb{Z}[i]$, 228
 - numbers
 - algebraic
 - form countable set, 304
 - ideal, 209
 - complex, 215
 - absolute value, 217
 - addition, 216
 - additivity of argument, 220
 - argument, 220
 - conjugate, 225
 - Hamilton definition, 216
 - imaginary part, 216
 - multiplication, 216
 - as ordered pairs, 216
 - in polar coordinates, 220
 - real part, 216
 - constructed by conics, 247
 - irrational, 79
 - natural, 1
 - division property, 5
 - von Neumann definition, 28
 - perfect, 21
 - prime, 6
 - rational, 11
 - and periodic decimals, 82
 - real, 55, 77
 - Dedekind definition, 79
 - definition of $+$ and \times , 80
 - relatively prime, 115
 - side and diagonal, 266
 - and Euclidean algorithm, 270
 - transcendental, 295
 - Cantor construction, 304
- O**
- octahedron, 59
 - volume of, 59
 - orientation, 98
 - origin, 69
- P**
- Pappus, 47
 - parabola
 - algebraic area function, 286
 - equation of, 250
 - as exceptional case, 248
 - focal property, 253, 254
 - has one focus at infinity, 255
 - integer points, 264
 - one point at infinity, 279
 - relative to horizon, 276
 - as section of cone, 248
 - shape, 251
 - parallel
 - axiom, 44
 - of Euclid, 44
 - fails in non-Euclidean plane, 100

- of Playfair, 44
- construction, 42
- parallelepiped, 56
 - volume of, 56
- parameterization
 - of circle
 - by cos and sin, 119
 - by rational functions, 119
 - of cubic curve
 - by elliptic functions, 141
 - of Pythagorean triples, 114
 - of quadratic curve
 - by rational functions, 127
 - of rational Pythagorean triples, 117
 - of rational triangles, 120
- Pascal, 34
 - triangle, 34
- Pell, 271
- Pell equation, 271
 - and hyperbolic functions, 298
 - nontrivial solution, 273
- pentagon construction, 76
- perfect numbers, 21
 - Euclid's formula, 21
 - Euclid's theorem, 21
 - Euler's theorem, 23
 - and Mersenne primes, 22
- periodic
 - continued fraction, 269
 - decimal, 82
 - function, 150
- periodicity
 - of Euclidean algorithm, 268
- phi function, 189, 197
 - from prime factorization, 197
 - multiplicative property, 197
 - on prime power, 189
- π
 - Archimedes approximation, 146, 171
 - biblical approximation, 145
 - Brouncker continued fraction, 172
 - equals two right angles, 47
 - even powers of
 - Euler series, 173
 - infinite series, 171
 - is transcendental, 295
 - rational approximations, 275
 - and squaring the circle, 143
 - standard continued fraction, 173
 - Viète product, 155
 - Wallis product, 172
 - Zǔ Chōngzhī approximation, 171
 - and continued fraction, 173
- pigeonhole principle, 272
- plane
 - Euclidean, 86
 - non-Euclidean, 100
 - projective, 278
 - reflection in, 97
- Plato
 - and divisors of 5040, 22
 - Epinomis*, 66
 - view of geometry, 65
- Playfair axiom, 44
- Plimpton 322, 111
 - Pythagorean triples in, 111
- Poincaré, 102
 - and linear fractional functions, 241
 - non-Euclidean plane, 100
 - complex coordinate, 239

- Poincaré (*continued*)
 non-Euclidean space
 half space model, 241
 and quadratic forms, 243
- points
 as ordered pairs, 85
 at infinity, 278
- polar
 coordinate grid, 306
 coordinates, 220
- polygon
 area of, 52
 convex, 52
 cut into triangles, 52
- polygonal path, 95
- polynomial, 135
 degree, 135
 division property, 135
 equation
 number of roots, 150
 Euclidean algorithm, 135
 irreducible, 135
 unique factorization, 135
- prime divisor
 Gaussian, 228
 property, 17
 by descent, 19
 for integers, 17
 Gauss remark on, 20
 implies unique prime factorization, 17
 in $\mathbb{Z}[i]$, 232
 known to Euclid, 17
- prime numbers, *see* primes
- primes, 6
 Dirichlet theorem, 9
 equal to sum of two squares, 235
 Euclid's theorem, 7
 in Euler product formula, 173
 Fermat, 7
 and constructible n -gons, 44
 of form $4n + 1$, 9
 of form $4n + 3$, 9
 are Gaussian primes, 229, 235
 infinitely many, 9
 of form $n^2 + 1$, 9, 233
 of form $x^2 + 2y^2$, 242
 of form $x^2 + 3y^2$, 242
 of form $x^2 + y^2$, 242
 infinitely many, 7
 Mersenne, 7
 and perfect numbers, 22
 twin, 233
 in $\mathbb{Z}[\sqrt{-2}]$, 230
 in $\mathbb{Z}[i]$, 228
- primitive
 Pythagorean triples, 116
 root, 200
 existence, 200
- prism, 56
 volume of, 56
- product
 cartesian, 85
 direct, 195
 Euler, 173
 of Pythagorean triples, 158
 tensor, 163
- projection
 circle to conic, 275
 line to circle, 127
 stereographic, 127
- projective
 geometry, 275
 plane, 278
 transformations, 278

Pythagorean theorem, 37
 and Euclidean distance, 85
 and tiled floor, 39
 converse, 40
 and Pythagorean triples, 111
 construction of right angle, 66
 Euclid proof, 53
 true by definition in $\mathbb{R} \times \mathbb{R}$, 86
 Pythagorean triples, 111
 and converse Pythagorean theorem, 111
 in Diophantus, 116
 in Euclid, 113
 Euclid parameterization, 114
 factorization, 159
 and half angle formulas, 156
 and i , 158
 in Plimpton 322, 111
 primitive, 116
 product, 158
 rational, 117
 and tan addition formula, 157
 Pythagoreans, 54
 and $\sqrt{2}$, 54
 music theory, 64
 leads to irrationals, 64
 philosophy, 64

Q

\mathbb{Q} , 11
 basis over, 165
 countability, 107
 quadrant, 69
 quadratic
 character, 199
 of -1 , 201
 of 2 , 202

congruence, 197
 completing the square, 198
 curve
 degenerate, 256
 parameterization, 127
 rational points, 124
 without rational points, 125
 curves
 identified by Fermat and Descartes, 256
 equation
 integer solutions, 309
 solved by Euclid, 56
 forms, 242
 binary, 242
 equivalent, 242
 as matrix products, 259
 studied geometrically, 243
 integers, 230, 243
 nonresidue, 199
 reciprocity, 203
 Rousseau proof, 204
 supplement, 208
 residue, 199
 quadrilateral
 angle sum, 45
 cyclic
 area formula, 123
 quartic equation, 262
 Ferrari solution, 262

R

\mathbb{R} , 80
 basis over \mathbb{Q} for, 175
 uncountability of, 175
 and axiom of choice, 175
 $\mathbb{R}/2\pi\mathbb{Z}$, 164
 $\mathbb{R}/\pi\mathbb{Z}$, 164

- rational
 - addition formula, 157
 - approximation to π , 275
 - combination, 165
 - functions, 119, 134
 - independence, 166
 - theorem, 167
 - points
 - chord construction, 142
 - circle without, 125, 178
 - on circle, 117, 140, 239
 - on cubic curve, 142
 - dividing circle, 170
 - on hyperbola, 126
 - on line, 126
 - on n -sphere, 130
 - on quadratic curve, 124
 - on sphere, 127, 129
 - sphere without, 130
 - tangent construction, 142
 - Pythagorean triples, 117
 - parameterization, 117
 - right triangle, 131
 - area, 131
 - Fermat theorem, 131
 - triangles, 120
 - in Brahmagupta, 120
 - parameterization, 120
 - split, 121
 - values of cosine, 169
- rational numbers, *see* rationals
- rationalization, 119
 - impossible for $\sqrt{1+x^4}$, 134
 - impossible for $\sqrt{1-x^4}$, 120
 - of $\sqrt{1-x^2}$, 119
- rationals, 11
 - and periodic decimals, 82
 - have gaps, 82
 - sum of, 185
- real numbers, *see* reals
- reals, 79
 - completeness, 83
 - and least upper bounds, 84
 - definition of $+$ and \times , 80
 - have least upper bounds, 84
 - have no gaps, 83
 - no infinitesimals property, 84
 - and Archimedean axiom, 84
- rectangle
 - area of, 51
 - existence, 46
- recursive functions, 35
- reflection, 87
 - glide, 93
 - in circle, 100
 - construction, 102
 - in great circle, 97
 - in a line, 87, 90
 - in plane, 97
 - in sphere, 241
- relatively prime numbers, 115
- remainder
 - mod notation for, 180
- Ribet, 141
- Riemann surface, 209
- ring, 12
 - commutative
 - with unit, 12
 - properties, 12, 209
 - of $\mathbb{Z}/n\mathbb{Z}$, 184
 - properties of \mathbb{Z}
 - proved by induction, 27
 - of quadratic integers, 230
- root of 1, 221
- rotation, 93
 - and multiplication by i , 221
 - as complex function, 223

is composite of two
 reflections, 93
 limit, 102
 non-Euclidean, 102
 of sphere, 239
 Rousseau, 204
 RSA, 212

S

series
 for cosine, 293
 for e , 294
 for even powers of π , 173
 for exponential function, 293
 for inverse tan, 172
 for π , 171
 for sine, 293
 Serre, 141
 set
 concept
 used by Dedekind, 27
 countable, 107
 infinite, 28
 theory
 influenced by Dedekind,
 35
 uncountable, 107
 side and diagonal numbers, 266
 recurrence relations, 266
 similarity, 109
 and Euclidean geometry, 109
 group, 109
 non-Euclidean
 is isometry, 109
 preserves angles, 109
 sine, 147
 addition formula, 152
 double angle formula, 156

function
 not algebraic, 150
 is transcendental, 150
 half angle formula, 156
 infinite series, 293
 not algebraic, 282
 wave, 149
 tangent to, 154
 sinh, 293, 295
 addition formula, 297
 space
 Euclidean, 97
 non-Euclidean, 238
 sphere
 complex coordinate, 239
 geometry of, 97
 in non-Euclidean space, 241
 isometries of, 97
 n -dimensional, 130
 rational points, 127, 129
 without rational points, 130
 square root
 and Euclidean algorithm, 267
 construction, 75
 squares
 mod p , 193, 197
 Euler criterion, 198
 splitting one into two, 139
 sum of four
 Bachet question, 138
 Fermat claim, 139
 Lagrange theorem, 139, 179
 sum of three, 179
 Legendre theorem, 179
 sum of two, 139, 178, 179
 Diophantus identity, 216
 divisors, 237
 equal to prime, 235
 factorization, 237

squares (*continued*)
 Fermat theorem, 139, 233
 stereographic projection, 128
 and complex coordinate on
 the sphere, 239
 formulas, 128
 line to circle, 127
 plane to sphere, 128
 subgroup, 97
 infinite cyclic, 301
 orientation-preserving, 98
 successor function, 27
 and Dedekind, 27
 nature of, 34
 properties of, 27
 synthetic geometry, 38
 in Euclid, 38
 in Hilbert, 38

T

tangent
 construction
 of rational points, 142
 to sine wave, 154
 Taylor, 141
 tensor, 164
 tensor product, 163
 tetrahedron, 57
 Dehn invariant, 168
 dihedral angle, 160
 and Hilbert's third problem,
 159
 volume of, 57
 by Euclid, 58
 Thales semicircle theorem, 40,
 49
 three reflections theorem, 91
 transcendental, 282

 functions, 150, 281
 in geometry, 282
 numbers, 295
 Cantor construction, 304
 translation, 88
 as complex function, 223
 is composite of two
 reflections, 92
 non-Euclidean, 102
 triangle
 angle sum, 45
 area, 52, 60
 Hero area formula, 123
 Heronian, 123
 inequality, 93
 in Euclid, 94
 without parallel axiom, 94
 isosceles, 47
 rational, 120
 right, 131
 triangles
 congruent, 47
 rational
 parameterization of, 120
 similar, 75
 trigonometric functions, *see*
 circular functions
 trigonometry, 143
 in Hilbert's third problem,
 159

U

uncountability
 of \mathbb{R} , 107, 175
 and axiom of choice, 175
 proved by Cantor, 108
 unimodular transformations,
 242

as non-Euclidean isometries,
 243
 form nonabelian group, 243
 unique prime factorization, 17
 depends on lattice shape, 243
 and Euler product formula,
 174
 fails in $\mathbb{Z}[\sqrt{-5}]$, 244
 for integers, 17
 and irrationality proofs, 18
 for polynomials, 135
 and principal ideals, 244
 in $\mathbb{Z}[\sqrt{-2}]$, 233
 in $\mathbb{Z}[i]$, 230
 unit
 commutative ring with, 12
 fractions, 3
 of $\mathbb{Z}[i]$, 228

V

Viète
 infinite product for π , 155
 polynomials for cos and sin,
 155
 product of triangles, 219
 and $e^{i\theta}$, 306
 Vitruvius, 66
 volume, 56
 of cuboid, 56
 of octahedron, 59
 of parallelepiped, 56
 of prism, 56
 of tetrahedron, 57
 by Euclid, 58
 von Neumann, 28
 definition of natural numbers,
 28

W

Wallis product for π , 172
 Wantzel, 44
 Waring, 193
 Wiles, 141
 Wilson
 primality criterion, 192
 theorem
 found by al-Haytham, 209
 found by Leibniz, 208
 Gauss proof, 191
 proved by Lagrange, 193,
 209
 published by Waring, 193

Z

\mathbb{Z} , 10
 countability, 107
 $\mathbb{Z}/n\mathbb{Z}$, 183
 ring properties, 184
 $\mathbb{Z}/p\mathbb{Z}$, 187
 field properties, 187
 $\mathbb{Z}[\sqrt{-2}]$, 229
 division property, 233
 and equation $y^3 = x^2 + 2$, 236
 norm, 230
 primes, 230
 unique prime factorization,
 233
 $\mathbb{Z}[\sqrt{-5}]$, 244
 $\mathbb{Z}[\sqrt{2}]$, 267
 $\mathbb{Z}[\sqrt{d}]$, 267
 $\mathbb{Z}[i]$, 227
 Zermelo, 25
 Zù Chōngzhi, 171
 approximation to π , 171
 and continued fraction,
 173

Undergraduate Texts in Mathematics

Anglin: Mathematics: A Concise History and Philosophy.

Readings in Mathematics.

Anglin/Lambek: The Heritage of Thales.

Readings in Mathematics.

Apostol: Introduction to Analytic Number Theory. Second edition.

Armstrong: Basic Topology.

Armstrong: Groups and Symmetry.

Axler: Linear Algebra Done Right. Second edition.

Beardon: Limits: A New Approach to Real Analysis.

Bak/Newman: Complex Analysis. Second edition.

Banchoff/Wermer: Linear Algebra Through Geometry. Second edition.

Berberian: A First Course in Real Analysis.

Bix: Conics and Cubics: A Concrete Introduction to Algebraic Curves.

Brémaud: An Introduction to Probabilistic Modeling.

Bressoud: Factorization and Primality Testing.

Bressoud: Second Year Calculus. *Readings in Mathematics.*

Brickman: Mathematical Introduction to Linear Programming and Game Theory.

Browder: Mathematical Analysis: An Introduction.

Buskes/van Rooij: Topological Spaces: From Distance to Neighborhood.

Callahan: The Geometry of Spacetime: An Introduction to Special and General Relativity.

Carter/van Brunt: The Lebesgue-Stieltjes: A Practical Introduction

Cederberg: A Course in Modern Geometries.

Childs: A Concrete Introduction to Higher Algebra. Second edition.

Chung: Elementary Probability Theory with Stochastic Processes. Third edition.

Cox/Little/O'Shea: Ideals, Varieties, and Algorithms. Second edition.

Croom: Basic Concepts of Algebraic Topology.

Curtis: Linear Algebra: An Introductory Approach. Fourth edition.

Devlin: The Joy of Sets: Fundamentals of Contemporary Set Theory. Second edition.

Dixmier: General Topology.

Driver: Why Math?

Ebbinghaus/Flum/Thomas: Mathematical Logic. Second edition.

Edgar: Measure, Topology, and Fractal Geometry.

Elaydi: An Introduction to Difference Equations. Second edition.

Exner: An Accompaniment to Higher Mathematics.

Exner: Inside Calculus.

Fine/Rosenberger: The Fundamental Theory of Algebra.

Fischer: Intermediate Real Analysis.

Flanigan/Kazdan: Calculus Two: Linear and Nonlinear Functions. Second edition.

Fleming: Functions of Several Variables. Second edition.

Foulds: Combinatorial Optimization for Undergraduates.

Foulds: Optimization Techniques: An Introduction.

Franklin: Methods of Mathematical Economics.

Frazier: An Introduction to Wavelets Through Linear Algebra.

Gordon: Discrete Probability.

Hairer/Wanner: Analysis by Its History. *Readings in Mathematics.*

Halmos: Finite-Dimensional Vector Spaces. Second edition.

Halmos: Naive Set Theory.

Hämmerlin/Hoffmann: Numerical Mathematics.

Readings in Mathematics.

Harris/Hirst/Mossinghoff: Combinatorics and Graph Theory.

Hartshorne: Geometry: Euclid and Beyond.

Hijab: Introduction to Calculus and Classical Analysis.

(continued after index)