day to day and seems likely in this scientific age to develop to enormous proportions) if they continue to be presented in a rigorous form according to the manner of the ancients.

The progress in geometry when Huygens wrote was indeed impressive, considering the very simple system of calculus then available. Virtually all that was known was the differentiation and integration of powers of x (possibly fractional) and implicit differentiation of polynomials in x, y. However, when allied with algebra and analytic geometry, this was sufficient to find tangents, maxima, and minima for all algebraic curves. And when allied with Newton's calculus of infinite series, discovered in the 1660s, the rules for powers of x formed a complete system for differentiation and integration of all functions expressible in power series.

The subsequent development of calculus is a puzzling exception to the normal process of simplification in mathematics. Nowadays we have a much less elegant system, which downplays the use of infinite series and complicates the system of rules for differentiation and integration. The rules for differentiation are still complete, given a sensible set of operations for constructing functions, but the rules for integration are pathetically incomplete. They do not suffice to integrate simple algebraic functions like $\sqrt{1+x^3}$, or even rational functions with undetermined constants like $1/(x^5-x-A)$. Moreover, it is only in recent decades that we have been able to tell *which* algebraic functions are integrable by our rules. [This little-known result is expounded by Davenport (1981).]

The conclusion seems to be that, apart from streamlining the language slightly, we cannot make calculus any simpler than it was in the seventeenth century! It is certainly easier to present the history of the subject if we refrain from imposing modern ideas. This approach also has the advantage of emphasizing the highly combinatorial nature of calculus—it is about *calculation*, after all. In view of the current controversy over the relative merits of calculus and combinatorics, it may be useful to remember that most classical combinatorics was part of the algebra of series, and hence a part of calculus. We develop this theme at greater length in the chapter on infinite series that follows.

Much has been written on the history of calculus, and some particularly useful books are Boyer (1959), Baron (1969), and Edwards (1979). However, historians tend to harp on the question of logical justification and to spend a disproportionate amount of time on the way it was handled in the nineteenth century. This not only obscures the boldness and vigor of early calculus, but it is overly dogmatic about the way in which calculus should be justified. Apart from the justification already available in the seventeenth century (the method of exhaustion), there is also a twentieth-century justification [the theory of infinitesimals of Robinson (1966)], and the sheer diversity of foundations for calculus suggests that we have not yet got to the bottom of it.

9.2 Early Results on Areas and Volumes

The idea of integration is often introduced by approximating the area under curves $y = x^k$ by rectangles (Figure 9.1), say, from 0 to 1. If the base of the region is divided into n equal parts, then the heights of the rectangles are $(1/n)^k, (2/n)^k, \ldots, (n/n)^k$, and finding the area occupied by the rectangles depends on summing the series $1^k + 2^k + \cdots + n^k$. If the curve is revolved around the x-axis, then the rectangles sweep out cylinders of cross-sectional area πr^2, where $r = (1/n)^k, (2/n)^k, \ldots, (n/n)^k$, which necessitates summing the series $1^{2k} + 2^{2k} + \cdots + n^{2k}$.

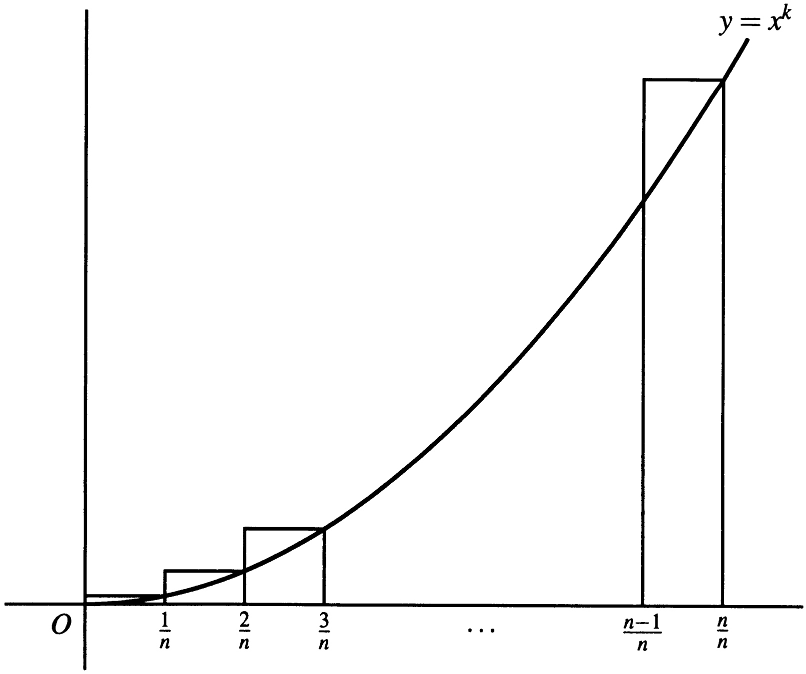


Figure 9.1: Approximating an area by rectangles

After the time of Archimedes, the first new results on areas and volumes were in fact based on summing these series. The Arab mathematician al-Haytham (around 965–1039) summed the series $1^k + 2^k + \cdots + n^k$ for $k = 1,2,3,4$, and used the result to find the volume of the solid obtained by rotating the parabola about its base. [See Baron (1969), p. 70, or Edwards (1979), p. 84, for al-Haytham's method of summing the series, and the exercises for another method.]

Cavalieri (1635) extended these results up to $k = 9$, using them to obtain the equivalent of

$$\int_0^a x^k\, dx = \frac{a^{k+1}}{k+1}$$

and conjecturing this formula for all positive integers k. This result was proved in the 1630s by Fermat, Descartes, and Roberval. Fermat even obtained the result for fractional k [see Baron (1969), pp. 129, 185, and Edwards (1979), p. 116]. Cavalieri is best known for his "method of indivisibles," an early method of discovery which considered areas divided into infinitely thin strips and volumes divided into infinitely thin slices. Archimedes' *Method* used similar ideas but, as mentioned in Section 4.1, this was not known until the twentieth century. Remarkably, Cavalieri's contemporary Torricelli (the inventor of the barometer) speculated that such a method may have been used by the Greeks. Torricelli himself obtained many results using indivisibles, one being almost identical with an area determination for the parabola given by Archimedes in the *Method* [Torricelli (1644)]. Another of his discoveries, which caused astonishment at the time, was that the infinite solid obtained by revolving $y = 1/x$ about the x axis from 1 to ∞ had finite volume [Torricelli (1643) and Exercise 9.2.3]. The philosopher Hobbes (1672) wrote of Torricelli's result that "to understand this for sense, it is not required that a man should be a geometrician or logician, but that he should be mad."

Exercises

9.2.1 Find $1 + 2 + \cdots + n$ by summing the identity $(m+1)^2 - m^2 = 2m + 1$ from $m = 1$ to n. Similarly find $1^2 + 2^2 + \cdots + n^2$ using the identity

$$(m+1)^3 - m^3 = 3m^2 + 3m + 1$$

together with the previous result. Likewise, find $1^3 + 2^3 + \cdots + n^3$ using the identity

$$(m+1)^4 - m^4 = 4m^3 + 6m^2 + 4m + 1$$

and so on.

9.2.2 Show that the approximation to the area under $y=x^2$ by rectangles in Figure 9.1 has value $(2n+1)n(n+1)/6n^3$, and deduce that the area under the curve is $1/3$.

9.2.3 Show that the volume of the solid obtained by rotating the portion of $y=1/x$ from $x=1$ to ∞ about the x axis is finite. Show, on the other hand, that its surface area is infinite.

Cavalieri's most elegant application of his method of indivisibles was to prove Archimedes' formula for the volume of a sphere. His argument is simpler than that of Archimedes, and it goes as follows.

9.2.4 Show that the slice $z=c$ of the sphere $x^2+y^2+z^2=1$ has the same area as the slice $z=c$ of the cylinder $x^2+y^2=1$ *outside* the cone $x^2+y^2=z^2$.

9.2.5 Deduce from Exercise 9.2.4, and the known volume of the cone, that the volume of the sphere is 2/3 the volume of the circumscribing cylinder.

9.3 Maxima, Minima, and Tangents

The idea of differentiation is now considered to be simpler than integration, but historically it developed later. Apart from the construction of the tangent to the spiral $r=a\theta$ by Archimedes, no examples of the characteristic limiting process

$$\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

appeared until it was introduced by Fermat in 1629 for polynomials f and used to find maxima, minima, and tangents. Fermat's work, like his discovery of analytic geometry, was not published until 1679, but it became known to other mathematicians through correspondence after a more complicated tangent method was published by Descartes (1637).

Fermat's calculations involve a sleight of hand also used by Newton and others: introduction of a "small" or "infinitesimal" element E at the beginning, dividing by E to simplify, then omitting E at the end as if it were zero. For example, to find the slope of the tangent to $y=x^2$ at any value x, consider the chord between the points (x,x^2) and $(x+E,(x+E)^2)$ on it.

$$\begin{aligned}\text{slope} &= \frac{(x+E)^2-x^2}{E}\\ &= \frac{2xE+E^2}{E}=2x+E\end{aligned}$$

and we now get the slope of the tangent by neglecting E. This procedure enraged the philosophers, who thought it was being claimed that $2x+E=2x$ and at the same time $E \neq 0$. Of course, it is only necessary to claim that $\lim_{E\to 0}(2x+E)=2x$, but seventeenth-century mathematicians did not know how to say this. In any case, they were too carried away with the power of the method to worry about such criticisms (and it was difficult to take philosophers seriously when they were as obstinate as Hobbes; see Section 9.2). Fermat's method applies to all polynomials $p(x)$, since the highest-degree term in $p(x+E)$ is always canceled by the highest-degree term in $p(x)$, leaving terms divisible by E. Fermat was also able to extend it to curves given by polynomial equations $p(x,y)=0$. He did this in 1638 when Descartes, hoping to stump him, proposed finding the tangent to the folium.

The generality of Fermat's method entitles him to be regarded as one of the founders of calculus. He could certainly find tangents to all curves given by polynomial equations $y=p(x)$ and probably to all algebraic curves $p(x,y)=0$. A completely explicit rule for the latter problem was found by Sluse about 1655 [but not published until Sluse (1673)] and by Hudde in 1657 [published in the 1659 edition of Descartes' *La Géométrie*, Schooten (1659)]. In our notation, if

$$p(x,y)=\sum a_{ij}x^iy^j=0,$$

then

$$\frac{dy}{dx}=-\frac{\sum ia_{ij}x^{i-1}y^j}{\sum ja_{ij}x^iy^{j-1}}.$$

Nowadays, this result is easily obtained by implicit differentiation (see exercises), but it can also be obtained by direct manipulation of polynomials.

Exercises

For evidence that tangents to algebraic curves may be found without calculus, it is enough to look more closely at what we called Diophantus' tangent method in Section 3.5. In his *Arithmetica*, Problem 18, Book VI (previously mentioned in Exercise 3.4.1), Diophantus finds the tangent $y=\frac{3x}{2}+1$ to $y^2=x^3-3x^2+3x+1$ at the point $(0,1)$, apparently by inspection. Without mentioning its geometric interpretation, he simply substitutes $\frac{3x}{2}+1$ for y in $y^2=x^3-3x^2+3x+1$.

9.3.1 Check that this substitution gives the equation

$$x^3-\frac{21}{4}x^2=0.$$

What is the geometric interpretation of the double root $x = 0$?

9.3.2 What would you substitute for y to find the tangent at $(0,1)$ to the curve $y^2 = x^3 - 3x^2 + 5x + 1$?

These examples show how tangents can be found by looking for double roots, though it requires some foresight to make the right substitution. With calculus, the process is more mechanical.

9.3.3 Derive the formula of Hudde and Sluse by differentiating $\sum a_{ij}x^i y^j = 0$ with respect to x.

9.3.4 Use differentiation to find the tangent to the folium $x^3 + y^3 = 3axy$ at point (b,c).

9.4 The *Arithmetica Infinitorum* of Wallis

Wallis's efforts to arithmetize geometry were noted in Section 7.6. In his *Arithmetica Infinitorum* [Wallis (1655a)] he made a similar attempt to arithmetize the theory of areas and volumes of curved figures. Some of his results were, understandably, equivalent to results already known. For example, he gave a proof that

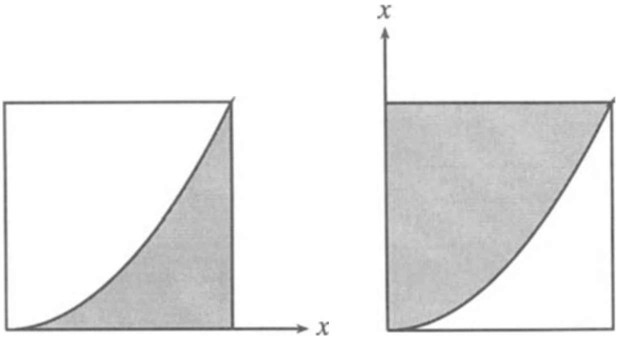
$$\int_0^1 x^p\,dx = \frac{1}{p+1}$$

for positive integers p by showing that

$$\frac{0^p + 1^p + 2^p + \cdots + n^p}{n^p + n^p + n^p + \cdots + n^p} \to \frac{1}{p+1} \quad \text{as} \quad n \to \infty.$$

However, he made a new approach to fractional powers, finding $\int_0^1 x^{m/n}\,dx$ directly rather than by consideration of the curve $y^n = x^m$, as Fermat had done. He first found $\int_0^1 x^{1/2}\,dx$, $\int_0^1 x^{1/3}\,dx, \ldots$ by considering the areas complementary to those under $y = x^2, y = x^3, \ldots$ (Figure 9.2), then guessed the results for other fractional powers by analogy with those already obtained.

Like other early contributors to calculus, Wallis was ambivalent about quantities that tended to zero, treating them as nonzero one minute and zero the next. For this he received a ferocious blast from his arch enemy Thomas Hobbes: "Your scurvy book of *Arithmetica infinitorum*; where your indivisibles have nothing to do, but as they are supposed to have quantity, that is to say, to be *divisibles*" [Hobbes (1656), p. 301]. Quite apart from this



$\int_0^1 x^2\,dx = \frac{1}{3} \qquad \Longrightarrow \qquad \int_0^1 x^{1/2}\,dx = \frac{2}{3}$

Figure 9.2: Areas used by Wallis

fault, which is easily remedied by limit arguments, the reasoning of Wallis is extremely incomplete by today's standards. Observing a pattern in formulas for $p = 1, 2, 3$, for example, he will immediately claim a formula for all positive integers p "by induction" and for fractional p "by interpolation." His boldness reached new heights toward the end of the *Arithmetica infinitorum* in deriving his famous infinite product formula,

$$\frac{\pi}{4} = \frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots.$$

An exposition of his reasoning may be found in Edwards (1979), pp. 171–176, where it is described as "one of the more audacious investigations by analogy and intuition that has ever yielded a correct result."

However, we must bear in mind that Wallis was offering primarily a method of discovery, and what a discovery he made! His infinite product for π was not the first ever given, as Viète (1593) had discovered

$$\begin{aligned}\frac{2}{\pi} &= \cos\frac{\pi}{4}\cos\frac{\pi}{8}\cos\frac{\pi}{16}\cdots\\ &= \sqrt{\frac{1}{2}}\cdot\sqrt{\frac{1}{2}\left(1+\sqrt{\frac{1}{2}}\right)}\cdot\sqrt{\frac{1}{2}\left[1+\sqrt{\frac{1}{2}\left(1+\sqrt{\frac{1}{2}}\right)}\right]}\cdots.\end{aligned}$$

However, the formula of Viète is based on a clever but simple trick (see exercises), whereas that of Wallis is of deeper significance. By relating π to the integers through a sequence of rational operations, Wallis uncovered a sequence of fractions (obtained by terminating the product at the nth factor) he called "hypergeometric." Similar sequences were later found to occur as coefficients in series expansions of many functions, which led to a broad class of functions being called "hypergeometric" by Gauss. Also, Wallis' product was closely related to two other beautiful formulas for π based on sequences of rational operations:

$$\frac{4}{\pi} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \cdots}}}}$$

and

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots .$$

The continued fraction was obtained by Brouncker from Wallis' product and also published in Wallis (1655b). The series is a special case of the series

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{3} - \frac{x^7}{7} + \cdots$$

discovered by Indian mathematicians in the fifteenth century (see Section 10.1) and later rediscovered by Newton, Gregory, and Leibniz. Euler (1748a), p. 311, gave a direct transformation of the series for $\pi/4$ into Brouncker's continued fraction. In addition to setting off this spectacular chain reaction, Wallis' method of "interpolation" had important consequences in the work of Newton, who used it to discover the general binomial theorem (Section 10.2).

EXERCISES

9.4.1 Use the identity $\sin x = 2\sin(x/2)\cos(x/2)$ to show

$$\frac{\sin x}{2^n \sin(x/2^n)} = \cos\frac{x}{2}\cos\frac{x}{2^2}\cdots\cos\frac{x}{2^n}$$

whence

$$\frac{\sin x}{x} = \cos\frac{x}{2}\cos\frac{x}{2^2}\cos\frac{x}{2^3}\cdots$$

9.4.2 Deduce Viète's product by substituting $x = \pi/2$.