$(g_1, \ldots, g_n)$ then P also contains f, i.e., f is contained in the intersection of all the prime ideals P containing $(g_1, \ldots, g_n)$. Since this intersection is $\text{rad}(g_1, \ldots, g_n)$ by Proposition 12, this proves (3).

If $U = X - \mathcal{Z}(I)$ is a Zariski open subset of X, then U is the union of the sets X_f with $f \in I$, which proves (4).

The natural ring homomorphism from R to the localization R_f establishes a bijection between the prime ideals in R_f and the prime ideals in R not containing (f) (Proposition 38). The corresponding Zariski continuous map from $\text{Spec}\, R_f$ to $\text{Spec}\, R$ is therefore continuous and bijective. Since every ideal of R_f is the extension of some ideal of R (cf. Proposition 38(1)), it follows that the inverse map is also continuous, which proves (5).

In (6), every open set is the union of principal open sets by (4), so it suffices to prove that if X is covered by principal open sets X_{g_i} (for i in some index set $\mathcal{J}$) then X is a finite union of some of the X_{g_i}. If the ideal I generated by the g_i were a proper ideal in R, then I would be contained in some maximal ideal P. But in this case the element P in $X = \text{Spec}\, R$ would not be contained in any principal open set X_{g_i}, contradicting the assumption that X is covered by the X_{g_i}. Hence $I = R$ and so $1 \in R$ can be written as a finite sum $1 = a_1 g_{i_1} + \cdots + a_n g_{i_n}$ with $i_1, \ldots, i_n \in \mathcal{J}$. Consider the finite union $X_{g_1} \cup \cdots \cup X_{g_n}$. Any point P in X not contained in this union would be a prime in R that contains $g_{i_1}, \ldots, g_{i_n}$, hence would contain 1, a contradiction. It follows that $X = X_{g_1} \cup \cdots \cup X_{g_n}$ as needed. The second part of (6) follows from (5).

We now define an analogue for $X = \text{Spec}\, R$ of the rational functions on a variety V. As we observed, for the variety V a rational function $\alpha \in k(V)$ is a regular function on some open set U. At each point $v \in U$ there is a representative a/f for α with $f(v) \neq 0$, and this representative is an element in the localization $\mathcal{O}_{v,V} = k[V]_{\mathcal{I}(v)}$. In this way the regular function α on U can be considered as a function from U to the disjoint union of these localizations: the point $v \in U$ is mapped to the representative $a/f \in k[V]_{\mathcal{I}(v)}$. Furthermore the same representative can be used simultaneously not only at v but on the whole Zariski neighborhood V_f of v (so, "locally near v," α is given by a single quotient of elements from $k[V]$). Note that a/f is an element in the localization $k[V]_f$, which is contained in each of the localizations $k[V]_{\mathcal{I}(w)}$ for $w \in V_f$.

We now generalize this to $\text{Spec}\, R$ by considering the collection of functions s from the Zariski open subset U of $\text{Spec}\, R$ to the disjoint union of the localizations R_P for $P \in U$ such that $s(P) \in R_P$ and such that s is given locally by quotients of elements of R. More precisely:

Definition. Suppose U is a Zariski open subset of $\text{Spec}\, R$. If $U = \emptyset$, define $\mathcal{O}(U) = 0$. Otherwise, define $\mathcal{O}(U)$ to be the set of functions $s : U \to \bigsqcup_{Q \in U} R_Q$ from U to the disjoint union of the localizations R_Q for $Q \in U$ with the following two properties:

(1) $s(Q) \in R_Q$ for every $Q \in U$, and

(2) for every $P \in U$ there is an open neighborhood $X_f \subseteq U$ of P in U and an element a/f^n in the localization R_f defining s on X_f, i.e., $s(Q) = a/f^n \in R_Q$ for every $Q \in X_f$.

If s, t are elements in $\mathcal{O}(U)$ then $s + t$ and st are also elements in $\mathcal{O}(U)$ (cf. Exercise 18), so each $\mathcal{O}(U)$ is a ring. Also, every $a \in R$ gives an element in $\mathcal{O}(U)$

defined by $s(Q) = a \in R_Q$, and in particular $1 \in R$ gives an identity for the ring $\mathcal{O}(U)$. If U' is an open subset of U, then there is a natural restriction map from $\mathcal{O}(U)$ to $\mathcal{O}(U')$ which is a homomorphism of rings (cf. Exercise 19).

Definition. Let R be a commutative ring with 1, and let $X = \operatorname{Spec} R$.

(1) The collection of rings $\mathcal{O}(U)$ for the Zariski open sets of X together with the restriction maps $\mathcal{O}(U) \to \mathcal{O}(U')$ for $U' \subseteq U$ is called the *structure sheaf* on X, and is denoted simply by $\mathcal{O}$ (or $\mathcal{O}_X$).

(2) The elements s of $\mathcal{O}(U)$ are called the *sections of* $\mathcal{O}$ over U. The elements of $\mathcal{O}(X)$ are called the *global sections* of $\mathcal{O}$.

The next proposition generalizes the result of Proposition 51 that the only rational functions on a variety V that are regular everywhere are the elements of the coordinate ring $k[V]$.

Proposition 57. Let $X = \operatorname{Spec} R$ and let $\mathcal{O} = \mathcal{O}_X$ be its structure sheaf. The global sections of $\mathcal{O}$ are the elements of R, i.e., $\mathcal{O}(X) \cong R$. More generally, if X_f is a principal open set in X for some $f \in R$, then $\mathcal{O}(X_f)$ is isomorphic to the localization R_f.

Proof: Suppose that a/f^n is an element of the localization R_f. Then the map defined by $s(Q) = a/f^n \in R_Q$ for $Q \in X_f$ gives an element in $\mathcal{O}(X_f)$, and it is immediate that the resulting map ψ from R_f to $\mathcal{O}(X_f)$ is a ring homomorphism. Suppose that $a/f^n = b/f^m$ in R_Q for every $Q \in X_f$, i.e., $g(af^m - bf^n) = 0$ in R for some $g \notin Q$. If I is the ideal in R of elements $r \in R$ with $r(af^m - bf^n) = 0$, it follows from $g \in I$ that I is not contained in Q for any $Q \in X_f$. Put another way, every prime ideal of R containing I also contains f. Hence f is contained in the intersection of all the prime ideals of R containing I, which is to say that $f \in \operatorname{rad} I$. Then $f^N \in I$ for some integer $N \geq 0$, and so $f^N(af^m - bf^n) = 0$ in R. But this shows that $a/f^n = b/f^m$ in R_f and so the map ψ is injective. Suppose now that $s \in \mathcal{O}(X_f)$. Then by definition X_f can be covered by principal open sets X_{g_i} on which $s(Q) = a_i/g_i^{n_i} \in R_Q$ for every $Q \in X_{g_i}$. By (6) of Proposition 56, we may take a finite number of the g_i and then by taking different a_i we may assume all the n_i are equal (since $a_i/g_i^{n_i} = (a_i g_i^{n-n_i})/g_i^n$ if n is the maximum of the n_i). Since $s(Q) = a_i/g_i^n = a_j/g_j^n$ in R_Q for all $Q \in X_{g_i g_j} = X_{g_i} \cap X_{g_j}$, the injectivity of ψ (applied to $R_{g_i g_j}$) shows that $a_i/g_i^n = a_j/g_j^n$ in $R_{g_i g_j}$. This means that $g_i g_j{}^N(a_i g_j^n - a_j g_i^n) = 0$, i.e.,

$$a_i g_i^N g_j{}^{n+N} = a_j g_i^{n+N} g_j{}^N$$

in R for some $N \geq 0$, and we may assume N sufficiently large that this holds for every i and j. Since X_f is the union of the $X_{g_i} = X_{g_i^{n+N}}$, f is contained in the radical of the ideal generated by the g_i^n by (3) of Proposition 56, say

$$f^M = \sum_i b_i g_i^{n+N}$$

for some $M \geq 1$ and $b_i \in R$. Define $a = \sum b_i a_i g_i^N \in R$. Then

$$g_j^N a_j f^M = \sum_i b_i (a_j g_i^{n+N} g_j{}^N) = \sum_i b_i (a_i g_i^N g_j{}^{n+N}) = g_j{}^{n+N} a.$$

It follows that $a/f^M = a_j/g_j^n$ in R_{g_j}, and so the element in $\mathcal{O}(X_f)$ defined by a/f^M in R_f agrees with s on every X_{g_j}, and so on all of X_f since these open sets cover X_f. Hence the map ψ gives an isomorphism $R_f \cong \mathcal{O}(X_f)$. Taking $f = 1$ gives $R \cong \mathcal{O}(X)$, completing the proof.

In the case of affine varieties V the local ring $\mathcal{O}_{v,V}$ at the point $v \in V$ is the collection of all the rational functions in $k(V)$ that are defined at v. Put another way, $\mathcal{O}_{v,V}$ is the union of the rings of regular functions on U for the open sets U containing P, where this union takes place in the function field $k(V)$ of V. In the more general case of $X = \operatorname{Spec} R$, the rings $\mathcal{O}(U)$ for the open sets containing $P \in \operatorname{Spec} R$ are not contained in such an obvious common ring. In this case we proceed by considering the collection of pairs (s, U) with U an open set of X containing P and $s \in \mathcal{O}(U)$. We identify two pairs (s, U) and (s', U') if there is an open set $U'' \subseteq U \cap U'$ containing P on which s and s' restrict to the same element of $\mathcal{O}(U'')$. In the situation of affine varieties, this says that two functions defined in Zariski neighborhoods of the point v define the same regular function at v if they agree in some common neighborhood of v. The collection of equivalence classes of pairs (s, U) defines the *direct limit* of the rings $\mathcal{O}(U)$, and is denoted $\varinjlim \mathcal{O}(U)$ (cf. Exercise 8 in Section 7.6).

Definition. If $P \in X = \operatorname{Spec} R$, then the direct limit, $\varinjlim \mathcal{O}(U)$, of the rings $\mathcal{O}(U)$ for the open sets U of X containing P is called the *stalk* of the structure sheaf at P, and is denoted $\mathcal{O}_P$.

Proposition 58. Let $X = \operatorname{Spec} R$ and let $\mathcal{O} = \mathcal{O}_X$ be its structure sheaf. The stalk of $\mathcal{O}$ at the point $P \in X$ is isomorphic to the localization R_P of R at P: $\mathcal{O}_P \cong R_P$. In particular, the stalk $\mathcal{O}_P$ is a local ring.

Proof: If (s, U) represents an element in the stalk $\mathcal{O}_P$, then $s(P)$ is an element of the localization R_P. By the definition of the direct limit, this element does not depend on the choice of representative (s, U), and so gives a well defined ring homomorphism φ from $\mathcal{O}_P$ to R_P. If $a, f \in R$ with $f \notin P$, then the map $s(Q) = a/f \in R_Q$ defines an element in $\mathcal{O}(X_f)$. Then the class of (s, X_f) in the stalk $\mathcal{O}_P$ is mapped to a/f in R_P by φ, so φ is a surjective map. To see that φ is also injective, suppose that the classes of (s, U) and (s', U') in $\mathcal{O}_P$ satisfy $s(P) = s'(P)$ in R_P. By definition of $\mathcal{O}(U)$, $s = a/g^n$ on X_g for some $g \notin P$. Similarly, $s' = b/(g')^m$ on $X_{g'}$ for some $g' \notin P$. Since $a/g^n = b/(g')^m$ in R_P, there is some $h \notin P$ with $h(a(g')^m - bg^n) = 0$ in R. If $Q \in X_{gg'h} = X_g \cap X_{g'} \cap X_h$ this last equality shows that $a/g^n = b/(g')^m$ in R_Q, so that s and s' agree when restricted to $X_{gg'h}$. By definition of the direct limit, (s, U) and (s', U') define the same element in the stalk $\mathcal{O}_P$, which proves that φ is injective and establishes the proposition.

Proposition 58 shows that the algebraically defined localization R_P for $P \in \operatorname{Spec} R$ plays the role of the local ring $\mathcal{O}_{v,V}$ of regular functions at v for the affine variety V. If $\mathfrak{m}_P$ denotes the maximal ideal PR_P in R_P and $k(P) = R_P/\mathfrak{m}_P$ denotes the corresponding quotient field (which by Proposition 46(1) is also the fraction field of R/P), then the *tangent space* at P is defined to be the $k(P)$-vector space dual of $\mathfrak{m}_P/\mathfrak{m}_P^2$.