

$\psi : H \rtimes_{\varphi_1} K \rightarrow H \rtimes_{\varphi_2} K$ defined by $\psi((h, k)) = (\sigma(h), k^a)$ is a homomorphism. Show ψ is bijective by constructing a 2-sided inverse.]

7. This exercise describes thirteen isomorphism types of groups of order 56. (It is not too difficult to show that every group of order 56 is isomorphic to one of these.)

(a) Prove that there are three abelian groups of order 56.

(b) Prove that every group of order 56 has either a normal Sylow 2-subgroup or a normal Sylow 7-subgroup.

(c) Construct the following non-abelian groups of order 56 which have a normal Sylow 7-subgroup and whose Sylow 2-subgroup S is as specified:

one group when $S \cong Z_2 \times Z_2 \times Z_2$

two nonisomorphic groups when $S \cong Z_4 \times Z_2$

one group when $S \cong Z_8$

two nonisomorphic groups when $S \cong Q_8$

three nonisomorphic groups when $S \cong D_8$.

[For a particular S , two groups are not isomorphic if the kernels of the maps from S into $\text{Aut}(Z_7)$ are not isomorphic.]

(d) Let G be a group of order 56 with a nonnormal Sylow 7-subgroup. Prove that if S is the Sylow 2-subgroup of G then $S \cong Z_2 \times Z_2 \times Z_2$. [Let an element of order 7 act by conjugation on the seven nonidentity elements of S and deduce that they all have the same order.]

(e) Prove that there is a unique group of order 56 with a nonnormal Sylow 7-subgroup. [For existence use the fact that $|GL_3(\mathbb{F}_2)| = 168$; for uniqueness use Exercise 6.]

8. Construct a non-abelian group of order 75. Classify all groups of order 75 (there are three of them). [Use Exercise 6 to show that the non-abelian group is unique.] (The classification of groups of order pq^2 , where p and q are primes with $p < q$ and p not dividing $q - 1$, is quite similar.)

9. Show that the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 4 \end{pmatrix}$ is an element of order 5 in $GL_2(\mathbb{F}_{19})$. Use this matrix to construct a non-abelian group of order 1805 and give a presentation of this group. Classify groups of order 1805 (there are three isomorphism types). [Use Exercise 6 to prove uniqueness of the non-abelian group.] (A general method for finding elements of prime order in $GL_n(\mathbb{F}_p)$ is described in the exercises in Section 12.2; this particular matrix of order 5 in $GL_2(\mathbb{F}_{19})$ appears in Exercise 16 of that section as an illustration of the method.)

10. This exercise classifies the groups of order 147 (there are six isomorphism types).

(a) Prove that there are two abelian groups of order 147.

(b) Prove that every group of order 147 has a normal Sylow 7-subgroup.

(c) Prove that there is a unique non-abelian group whose Sylow 7-subgroup is cyclic.

(d) Let $t_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and $t_2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ be elements of $GL_2(\mathbb{F}_7)$. Prove $P = \langle t_1, t_2 \rangle$ is a Sylow 3-subgroup of $GL_2(\mathbb{F}_7)$ and that $P \cong Z_3 \times Z_3$. Deduce that every subgroup of $GL_2(\mathbb{F}_7)$ of order 3 is conjugate in $GL_2(\mathbb{F}_7)$ to a subgroup of P .

(e) By Example 3 in Section 1 the group P has four subgroups of order 3 and these are: $P_1 = \langle t_1 \rangle$, $P_2 = \langle t_2 \rangle$, $P_3 = \langle t_1 t_2 \rangle$, and $P_4 = \langle t_1 t_2^2 \rangle$. For $i = 1, 2, 3, 4$ let $G_i = (Z_7 \times Z_7) \rtimes_{\varphi_i} Z_3$, where φ_i is an isomorphism of Z_3 with the subgroup P_i of $\text{Aut}(Z_7 \times Z_7)$. For each i describe G_i in terms of generators and relations. Deduce that $G_1 \cong G_2$.

(f) Prove that G_1 is not isomorphic to either G_3 or G_4 . [Show that the center of G_1 has

order 7 whereas the centers of G_3 and G_4 are trivial.]

- (g) Prove that G_3 is not isomorphic to G_4 . [Show that every subgroup of order 7 in G_3 is normal in G_3 but that G_4 has subgroups of order 7 that are not normal.]
(h) Classify the groups of order 147 by showing that the six nonisomorphic groups described above (two from part (a), one from part (c) and G_1 , G_3 , and G_4) are all the groups of order 147. [Use Exercise 6 and part (d).] (The classification of groups of order pq^2 , where p and q are primes with $p < q$ and $p \mid q - 1$, is quite similar.)

11. Classify groups of order 28 (there are four isomorphism types).
12. Classify the groups of order 20 (there are five isomorphism types).
13. Classify groups of order $4p$, where p is a prime greater than 3. [There are four isomorphism types when $p \equiv 3 \pmod{4}$ and five isomorphism types when $p \equiv 1 \pmod{4}$.]
14. This exercise classifies the groups of order 60 (there are thirteen isomorphism types). Let G be a group of order 60, let P be a Sylow 5-subgroup of G and let Q be a Sylow 3-subgroup of G .
 - (a) Prove that if P is not normal in G then $G \cong A_5$. [See Section 4.5.]
 - (b) Prove that if $P \trianglelefteq G$ but Q is not normal in G then $G \cong A_4 \times Z_5$. [Show in this case that $P \leq Z(G)$, $G/P \cong A_4$, a Sylow 2-subgroup T of G is normal and $TQ \cong A_4$.]
 - (c) Prove that if both P and Q are normal in G then $G \cong Z_{15} \rtimes T$ where $T \cong Z_4$ or $Z_2 \times Z_2$. Show in this case that there are six isomorphism types when T is cyclic (one abelian) and there are five isomorphism types when T is the Klein 4-group (one abelian). [Use the same ideas as in the classifications of groups of orders 30 and 20.]
15. Let p be an odd prime. Prove that every element of order 2 in $GL_2(\mathbb{F}_p)$ is conjugate to a diagonal matrix with ± 1 's on the diagonal. Classify the groups of order $2p^2$. [If A is a 2×2 matrix with $A^2 = I$ and v_1, v_2 is a basis for the underlying vector space, look at A acting on the vectors $w_1 = v_1 + v_2$ and $w_2 = v_1 - v_2$.]
16. Show that there are exactly 4 distinct homomorphisms from Z_2 into $\text{Aut}(Z_8)$. Prove that the resulting semidirect products are the groups: $Z_8 \times Z_2$, D_{16} , the quasidihedral group QD_{16} and the modular group M (cf. the exercises in Section 2.5).
17. Show that for any $n \geq 3$ there are exactly 4 distinct homomorphisms from Z_2 into $\text{Aut}(Z_{2^n})$. Prove that the resulting semidirect products give 4 nonisomorphic groups of order 2^{n+1} . [Recall Exercises 21 to 23 in Section 2.3.] (These four groups together with the cyclic group and the generalized quaternion group, $Q_{2^{n+1}}$, are all the groups of order 2^{n+1} which possess a cyclic subgroup of index 2.)
18. Show that if H is any group then there is a group G that contains H as a normal subgroup with the property that for every automorphism σ of H there is an element $g \in G$ such that conjugation by g when restricted to H is the given automorphism σ , i.e., every automorphism of H is obtained as an inner automorphism of G restricted to H .
19. Let H be a group of order n , let $K = \text{Aut}(H)$ and form $G = \text{Hol}(H) = H \rtimes K$ (where φ is the identity homomorphism). Let G act by left multiplication on the left cosets of K in G and let π be the associated permutation representation $\pi : G \rightarrow S_n$.
 - (a) Prove the elements of H are coset representatives for the left cosets of K in G and with this choice of coset representatives π restricted to H is the regular representation of H .
 - (b) Prove $\pi(G)$ is the normalizer in S_n of $\pi(H)$. Deduce that under the regular representation of any finite group H of order n , the normalizer in S_n of the image of H is isomorphic to $\text{Hol}(H)$. [Show $|G| = |N_{S_n}(\pi(H))|$ using Exercises 1 and 2 above.]
 - (c) Deduce that the normalizer of the group generated by an n -cycle in S_n is isomorphic to $\text{Hol}(Z_n)$ and has order $n\varphi(n)$.

20. Let p be an odd prime. Prove that if P is a non-cyclic p -group then P contains a normal subgroup U with $U \cong Z_p \times Z_p$. Deduce that for odd primes p a p -group that contains a unique subgroup of order p is cyclic. (For $p = 2$ it is a theorem that the generalized quaternion groups Q_{2^n} are the only non-cyclic 2-groups which contain a unique subgroup of order 2). [Proceed by induction on $|P|$. Let Z be a subgroup of order p in $Z(P)$ and let $\bar{P} = P/Z$. If \bar{P} is cyclic then P is abelian by Exercise 36 in Section 3.1 — show the result is true for abelian groups. When \bar{P} is not cyclic use induction to produce a normal subgroup \bar{H} of \bar{P} with $\bar{H} \cong Z_p \times Z_p$. Let H be the complete preimage of \bar{H} in P , so $|H| = p^3$. Let $H_0 = \{x \in H \mid x^p = 1\}$ so that H_0 is a characteristic subgroup of H of order p^2 or p^3 by Exercise 9 in Section 4. Show that a suitable subgroup of H_0 gives the desired normal subgroup U .]
21. Let p be an odd prime and let P be a p -group. Prove that if every subgroup of P is normal then P is abelian. (Note that Q_8 is a non-abelian 2-group with this property, so the result is false for $p = 2$.) [Use the preceding exercises and Exercise 15 of Section 4.]
22. Let F be a field let n be a positive integer and let G be the group of upper triangular matrices in $GL_n(F)$ (cf. Exercise 16, Section 2.1)
- (a) Prove that G is the semidirect product $U \rtimes D$ where U is the set of upper triangular matrices with 1's down the diagonal (cf. Exercise 17, Section 2.1) and D is the set of diagonal matrices in $GL_n(F)$.
 - (b) Let $n=2$. Recall that $U \cong F$ and $D \cong F^\times \times F^\times$ (cf. Exercise 11 in Section 3.1). Describe the homomorphism from D into $\text{Aut}(U)$ explicitly in terms of these isomorphisms (i.e., show how each element of $F^\times \times F^\times$ acts as an automorphism on F).
23. Let K and L be groups, let n be a positive integer, let $\rho : K \rightarrow S_n$ be a homomorphism and let H be the direct product of n copies of L . In Exercise 8 of Section 1 an injective homomorphism ψ from S_n into $\text{Aut}(H)$ was constructed by letting the elements of S_n permute the n factors of H . The composition $\psi \circ \rho$ is a homomorphism from G into $\text{Aut}(H)$. The *wreath product* of L by K is the semidirect product $H \rtimes K$ with respect to this homomorphism and is denoted by $L \wr K$ (this wreath product depends on the choice of permutation representation ρ of K — if none is given explicitly, ρ is assumed to be the left regular representation of K).
- (a) Assume K and L are finite groups and ρ is the left regular representation of K . Find $|L \wr K|$ in terms of $|K|$ and $|L|$.
 - (b) Let p be a prime, let $K = L = Z_p$ and let ρ be the left regular representation of K . Prove that $Z_p \wr Z_p$ is a non-abelian group of order p^{p+1} and is isomorphic to a Sylow p -subgroup of S_{p^2} . [The p copies of Z_p whose direct product makes up H may be represented by p disjoint p -cycles; these are cyclically permuted by K .]
24. Let n be an integer > 1 . Prove the following classification: every group of order n is abelian if and only if $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$, where p_1, \dots, p_r are distinct primes, $\alpha_i = 1$ or 2 for all $i \in \{1, \dots, r\}$ and p_i does not divide $p_j^{\alpha_j} - 1$ for all i and j . [See Exercise 56 in Section 4.5.]
25. Let $H(\mathbb{F}_p)$ be the Heisenberg group over the finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ (cf. Exercise 20 in Section 4). Prove that $H(\mathbb{F}_2) \cong D_8$, and that $H(\mathbb{F}_p)$ has exponent p and is isomorphic to the first non-abelian group in Example 7.