Definition. For any $N \leq G$ and any $g \in G$ let

$$gN = \{gn \mid n \in N\} \quad \text{and} \quad Ng = \{ng \mid n \in N\}$$

called respectively a *left coset* and a *right coset* of N in G. Any element of a coset is called a *representative* for the coset.

We have already seen in Proposition 2 that if N is the kernel of a homomorphism and g_1 is any representative for the coset gN then $g_1N = gN$ (and if $g_1 \in Ng$ then $Ng_1 = Ng$). We shall see that this fact is valid for arbitrary subgroups N in Proposition 4 below, which explains the terminology of a *representative*.

If G is an additive group we shall write $g + N$ and $N + g$ for the left and right cosets of N in G with representative g, respectively. In general we can think of the left coset, gN, of N in G as the left translate of N by g. (The reader may wish to review Exercise 18 of Section 1.7 which proves that the right cosets of N in G are precisely the orbits of N acting on G by left multiplication.)

In terms of this definition, Proposition 2 shows that the fibers of a homomorphism are the left cosets of the kernel (and also the right cosets of the kernel), i.e., the elements of the quotient G/K are the left cosets gK, $g \in G$. In the example of $\mathbb{Z}/n\mathbb{Z}$ the multiplication in the quotient group could also be defined in terms of representatives for the cosets. The following result shows the same result is true for G/K in general (provided we know that K is the kernel of some homomorphism), namely that the product of two left cosets X and Y in G/K is computed by choosing any representative u of X, any representative v of Y, multiplying u and v in G and forming the coset $(uv)K$.

Theorem 3. Let G be a group and let K be the kernel of some homomorphism from G to another group. Then the set whose elements are the left cosets of K in G with operation defined by

$$uK \circ vK = (uv)K$$

forms a group, G/K. In particular, this operation is well defined in the sense that if u_1 is any element in uK and v_1 is any element in vK, then $u_1v_1 \in uvK$, i.e., $u_1v_1K = uvK$ so that the multiplication does not depend on the choice of representatives for the cosets. The same statement is true with "right coset" in place of "left coset."

Proof: Let $X, Y \in G/K$ and let $Z = XY$ in G/K, so that by Proposition 2(1) X, Y and Z are (left) cosets of K. By assumption, K is the kernel of some homomorphism $\varphi : G \to H$ so $X = \varphi^{-1}(a)$ and $Y = \varphi^{-1}(b)$ for some $a, b \in H$. By definition of the operation in G/K, $Z = \varphi^{-1}(ab)$. Let u and v be arbitrary representatives of X, Y, respectively, so that $\varphi(u) = a$, $\varphi(v) = b$ and $X = uK$, $Y = vK$. We must show $uv \in Z$. Now

$$\begin{aligned} uv \in Z \quad &\Leftrightarrow \quad uv \in \varphi^{-1}(ab) \\ &\Leftrightarrow \quad \varphi(uv) = ab \\ &\Leftrightarrow \quad \varphi(u)\varphi(v) = ab. \end{aligned}$$

Since the latter equality does hold, $uv \in Z$ hence Z is the (left) coset uvK. (Exercise 2 below shows conversely that every $z \in Z$ can be written as uv, for some $u \in X$ and $v \in Y$.) This proves that the product of X with Y is the coset uvK for any choice of representatives $u \in X$, $v \in Y$ completing the proof of the first statements of the theorem. The last statement in the theorem follows immediately since, by Proposition 2, $uK = Ku$ and $vK = Kv$ for all u and v in G.

In terms of Figure 1, the multiplication in G/K via representatives can be pictured as in the following Figure 3.

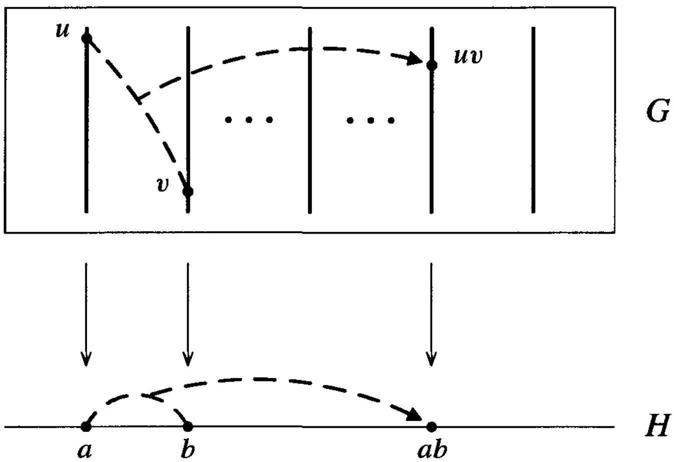

Fig. 3

We emphasize the fact that *the multiplication is independent of the particular representatives chosen.* Namely, the product (or sum, if the group is written additively) of two cosets X and Y is the coset uvK containing the product uv where u and v are *any* representatives for the cosets X and Y, respectively. This process of considering only the coset containing an element, or "reducing mod K" is the same as what we have been doing, in particular, in $\mathbb{Z}/n\mathbb{Z}$. A useful notation for denoting the coset uK containing a representative u is $\bar{u}$. With this notation (which we introduced in the Preliminaries in dealing with $\mathbb{Z}/n\mathbb{Z}$), the quotient group G/K is denoted $\overline{G}$ and the product of elements $\bar{u}$ and $\bar{v}$ is simply the coset containing uv, i.e., $\overline{uv}$. This notation also reinforces the fact that the cosets uK in G/K are *elements* $\bar{u}$ in G/K.

Examples

(1) The first example in this chapter of the homomorphism φ from $\mathbb{Z}$ to Z_n has fibers the left (and also the right) cosets $a + n\mathbb{Z}$ of the kernel $n\mathbb{Z}$. Theorem 3 proves that these cosets form a group under addition of representatives, namely $\mathbb{Z}/n\mathbb{Z}$, which explains the notation for this group. The group is naturally isomorphic to its image under φ, so we recover the isomorphism $\mathbb{Z}/n\mathbb{Z} \cong Z_n$ of Chapter 2.

(2) If $\varphi : G \to H$ is an *isomorphism*, then $K = 1$, the fibers of φ are the singleton subsets of G and so $G/1 \cong G$.

(3) Let G be any group, let $H = 1$ be the group of order 1 and define $\varphi : G \to H$ by $\varphi(g) = 1$, for all $g \in G$. It is immediate that φ is a homomorphism. This map is called the *trivial homomorphism*. Note that in this case $\ker \varphi = G$ and G/G is a group with the single element, G, i.e., $G/G \cong Z_1 = \{1\}$.

(4) Let $G = \mathbb{R}^2$ (operation vector addition), let $H = \mathbb{R}$ (operation addition) and define $\varphi : \mathbb{R}^2 \to \mathbb{R}$ by $\varphi((x, y)) = x$. Thus φ is projection onto the x-axis. We show φ is a homomorphism:

$$\begin{aligned}\varphi((x_1, y_1) + (x_2, y_2)) &= \varphi((x_1 + x_2, y_1 + y_2)) \\ &= x_1 + x_2 = \varphi((x_1, y_1)) + \varphi((x_2, y_2)).\end{aligned}$$

Now

$$\begin{aligned}\ker \varphi &= \{(x, y) \mid \varphi((x, y)) = 0\} \\ &= \{(x, y) \mid x = 0\} = \text{the } y\text{-axis}.\end{aligned}$$

Note that $\ker \varphi$ is indeed a subgroup of $\mathbb{R}^2$ and that the fiber of φ over $a \in \mathbb{R}$ is the translate of the y-axis by a, i.e., the line $x = a$. This is also the left (and the right) coset of the kernel with representative $(a, 0)$ (or any other representative point projecting to a):

$$\overline{(a, 0)} = (a, 0) + y\text{-axis}.$$

Hence Figure 1 in this example becomes

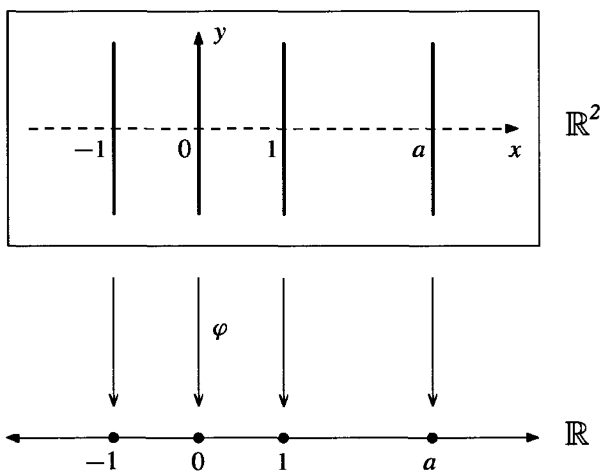

Fig. 4

The group operation (written additively here) can be described either by using the map φ: the sum of the line $(x = a)$ and the line $(x = b)$ is the line $(x = a + b)$; or directly in terms of coset representatives: the sum of the vertical line containing the point (a, y_1) and the vertical line containing the point (b, y_2) is the vertical line containing the point $(a + b, y_1 + y_2)$. Note in particular that the choice of representatives of these vertical lines is not important (i.e., the y-coordinates are not important).

(5) (An example where the group G is non-abelian.) Let $G = Q_8$ and let $H = V_4$ be the Klein 4-group (Section 2.5, Example 2). Define $\varphi : Q_8 \to V_4$ by

$$\varphi(\pm 1) = 1, \qquad \varphi(\pm i) = a, \qquad \varphi(\pm j) = b, \qquad \varphi(\pm k) = c.$$

The check that φ is a homomorphism is left as an exercise — relying on symmetry minimizes the work in showing $\varphi(xy) = \varphi(x)\varphi(y)$ for all x and y in Q_8. It is clear that φ is surjective and that $\ker\varphi = \{\pm 1\}$. One might think of φ as an "absolute value" function on Q_8 so the fibers of φ are the sets $E = \{\pm 1\}$, $A = \{\pm i\}$, $B = \{\pm j\}$ and $C = \{\pm k\}$, which are collapsed to 1, a, b, and c respectively in $Q_8/\langle \pm 1 \rangle$ and these are the left (and also the right) cosets of $\ker\varphi$ (for example, $A = i \cdot \ker\varphi = \{i, -i\} = \ker\varphi \cdot i$).

By Theorem 3, if we are given a subgroup K of a group G which we know is the kernel of some homomorphism, we may define the quotient G/K without recourse to the homomorphism by the multiplication $uKvK = uvK$. This raises the question of whether it is possible to define the quotient group G/N similarly for *any* subgroup N of G. The answer is no in general since this multiplication is not in general well defined (cf. Proposition 5 later). In fact we shall see that it is possible to define the structure of a group on the cosets of N *if and only if* N is the kernel of some homomorphism (Proposition 7). We shall also give a criterion to determine when a subgroup N is such a kernel — this is the notion of a *normal* subgroup and we shall consider non-normal subgroups in subsequent sections.

We first show that the cosets of an arbitrary subgroup of G partition G (i.e., their union is all of G and distinct cosets have trivial intersection).

Proposition 4. Let N be any subgroup of the group G. The set of left cosets of N in G form a partition of G. Furthermore, for all $u, v \in G$, $uN = vN$ if and only if $v^{-1}u \in N$ and in particular, $uN = vN$ if and only if u and v are representatives of the same coset.

Proof: First of all note that since N is a subgroup of G, $1 \in N$. Thus $g = g \cdot 1 \in gN$ for all $g \in G$, i.e.,

$$G = \bigcup_{g \in G} gN.$$

To show that distinct left cosets have empty intersection, suppose $uN \cap vN \neq \emptyset$. We show $uN = vN$. Let $x \in uN \cap vN$. Write

$$x = un = vm, \qquad \text{for some } n, m \in N.$$

In the latter equality multiply both sides on the right by n^{-1} to get

$$u = vmn^{-1} = vm_1, \qquad \text{where } m_1 = mn^{-1} \in N.$$

Now for any element ut of uN ($t \in N$),

$$ut = (vm_1)t = v(m_1 t) \in vN.$$

This proves $uN \subseteq vN$. By interchanging the roles of u and v one obtains similarly that $vN \subseteq uN$. Thus two cosets with nonempty intersection coincide.

By the first part of the proposition, $uN = vN$ if and only if $u \in vN$ if and only if $u = vn$, for some $n \in N$ if and only if $v^{-1}u \in N$, as claimed. Finally, $v \in uN$ is equivalent to saying v is a representative for uN, hence $uN = vN$ if and only if u and v are representatives for the same coset (namely the coset $uN = vN$).