6.15 Theorem *If $a < s < b$, f is bounded on $[a, b]$, f is continuous at s, and $\alpha(x) = I(x - s)$, then*

$$\int_a^b f \, d\alpha = f(s).$$

Proof Consider partitions $P = \{x_0, x_1, x_2, x_3\}$, where $x_0 = a$, and $x_1 = s < x_2 < x_3 = b$. Then

$$U(P, f, \alpha) = M_2, \qquad L(P, f, \alpha) = m_2.$$

Since f is continuous at s, we see that M_2 and m_2 converge to $f(s)$ as $x_2 \to s$.

6.16 Theorem *Suppose $c_n \geq 0$ for 1, 2, 3, ..., Σc_n converges, $\{s_n\}$ is a sequence of distinct points in (a, b), and*

$$\alpha(x) = \sum_{n=1}^{\infty} c_n I(x - s_n). \tag{22}$$

Let f be continuous on $[a, b]$. Then

$$\int_a^b f \, d\alpha = \sum_{n=1}^{\infty} c_n f(s_n). \tag{23}$$

Proof The comparison test shows that the series (22) converges for every x. Its sum $\alpha(x)$ is evidently monotonic, and $\alpha(a) = 0$, $\alpha(b) = \Sigma c_n$. (This is the type of function that occurred in Remark 4.31.)

Let $\varepsilon > 0$ be given, and choose N so that

$$\sum_{N+1}^{\infty} c_n < \varepsilon.$$

Put

$$\alpha_1(x) = \sum_{n=1}^{N} c_n I(x - s_n), \qquad \alpha_2(x) = \sum_{N+1}^{\infty} c_n I(x - s_n).$$

By Theorems 6.12 and 6.15,

$$\int_a^b f \, d\alpha_1 = \sum_{i=1}^{N} c_n f(s_n). \tag{24}$$

Since $\alpha_2(b) - \alpha_2(a) < \varepsilon$,

$$\left| \int_a^b f \, d\alpha_2 \right| \leq M\varepsilon, \tag{25}$$

where $M = \sup|f(x)|$. Since $\alpha = \alpha_1 + \alpha_2$, it follows from (24) and (25) that

(26) $$\left|\int_a^b f\,d\alpha - \sum_{i=1}^{N} c_n f(s_n)\right| \le M\varepsilon.$$

If we let $N \to \infty$, we obtain (23).

6.17 Theorem *Assume α increases monotonically and $\alpha' \in \mathscr{R}$ on $[a, b]$. Let f be a bounded real function on $[a, b]$.*

Then $f \in \mathscr{R}(\alpha)$ if and only if $f\alpha' \in \mathscr{R}$. In that case

(27) $$\int_a^b f\,d\alpha = \int_a^b f(x)\alpha'(x)\,dx.$$

Proof Let $\varepsilon > 0$ be given and apply Theorem 6.6 to α': There is a partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$ such that

(28) $$U(P, \alpha') - L(P, \alpha') < \varepsilon.$$

The mean value theorem furnishes points $t_i \in [x_{i-1}, x_i]$ such that

$$\Delta\alpha_i = \alpha'(t_i)\,\Delta x_i$$

for $i = 1, \ldots, n$. If $s_i \in [x_{i-1}, x_i]$, then

(29) $$\sum_{i=1}^{n} |\alpha'(s_i) - \alpha'(t_i)|\,\Delta x_i < \varepsilon,$$

by (28) and Theorem 6.7(*b*). Put $M = \sup|f(x)|$. Since

$$\sum_{i=1}^{n} f(s_i)\,\Delta\alpha_i = \sum_{i=1}^{n} f(s_i)\alpha'(t_i)\,\Delta x_i$$

it follows from (29) that

(30) $$\left|\sum_{i=1}^{n} f(s_i)\,\Delta\alpha_i - \sum_{i=1}^{n} f(s_i)\alpha'(s_i)\,\Delta x_i\right| \le M\varepsilon.$$

In particular,

$$\sum_{i=1}^{n} f(s_i)\,\Delta\alpha_i \le U(P, f\alpha') + M\varepsilon,$$

for all choices of $s_i \in [x_{i-1}, x_i]$, so that

$$U(P, f, \alpha) \le U(P, f\alpha') + M\varepsilon.$$

The same argument leads from (30) to

$$U(P, f\alpha') \le U(P, f, \alpha) + M\varepsilon.$$

Thus

(31) $$|U(P, f, \alpha) - U(P, f\alpha')| \le M\varepsilon.$$

Now note that (28) remains true if P is replaced by any refinement. Hence (31) also remains true. We conclude that

$$\left| \overline{\int_a^b} f\,d\alpha - \overline{\int_a^b} f(x)\alpha'(x)\,dx \right| \le M\varepsilon.$$

But ε is arbitrary. Hence

$$\overline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f(x)\alpha'(x)\,dx, \tag{32}$$

for *any* bounded f. The equality of the lower integrals follows from (30) in exactly the same way. The theorem follows.

6.18 Remark The two preceding theorems illustrate the generality and flexibility which are inherent in the Stieltjes process of integration. If α is a pure step function [this is the name often given to functions of the form (22)], the integral reduces to a finite or infinite series. If α has an integrable derivative, the integral reduces to an ordinary Riemann integral. This makes it possible in many cases to study series and integrals simultaneously, rather than separately.

To illustrate this point, consider a physical example. The moment of inertia of a straight wire of unit length, about an axis through an endpoint, at right angles to the wire, is

$$\int_0^1 x^2\,dm \tag{33}$$

where $m(x)$ is the mass contained in the interval $[0, x]$. If the wire is regarded as having a continuous density ρ, that is, if $m'(x) = \rho(x)$, then (33) turns into

$$\int_0^1 x^2\,\rho(x)\,dx. \tag{34}$$

On the other hand, if the wire is composed of masses m_i concentrated at points x_i, (33) becomes

$$\sum_i x_i^2\,m_i. \tag{35}$$

Thus (33) contains (34) and (35) as special cases, but it contains much more; for instance, the case in which m is continuous but not everywhere differentiable.

6.19 Theorem (change of variable) *Suppose φ is a strictly increasing continuous function that maps an interval $[A, B]$ onto $[a, b]$. Suppose α is monotonically increasing on $[a, b]$ and $f \in \mathscr{R}(\alpha)$ on $[a, b]$. Define β and g on $[A, B]$ by*

$$\beta(y) = \alpha(\varphi(y)), \qquad g(y) = f(\varphi(y)). \tag{36}$$

Then $g \in \mathscr{R}(\beta)$ *and*

$$\int_A^B g\, d\beta = \int_a^b f\, d\alpha. \tag{37}$$

Proof To each partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$ corresponds a partition $Q = \{y_0, \ldots, y_n\}$ of $[A, B]$, so that $x_i = \varphi(y_i)$. All partitions of $[A, B]$ are obtained in this way. Since the values taken by f on $[x_{i-1}, x_i]$ are exactly the same as those taken by g on $[y_{i-1}, y_i]$, we see that

$$U(Q, g, \beta) = U(P, f, \alpha), \qquad L(Q, g, \beta) = L(P, f, \alpha). \tag{38}$$

Since $f \in \mathscr{R}(\alpha)$, P can be chosen so that both $U(P, f, \alpha)$ and $L(P, f, \alpha)$ are close to $\int f\, d\alpha$. Hence (38), combined with Theorem 6.6, shows that $g \in \mathscr{R}(\beta)$ and that (37) holds. This completes the proof.

Let us note the following special case:

Take $\alpha(x) = x$. Then $\beta = \varphi$. Assume $\varphi' \in \mathscr{R}$ on $[A, B]$. If Theorem 6.17 is applied to the left side of (37), we obtain

$$\int_a^b f(x)\, dx = \int_A^B f(\varphi(y))\varphi'(y)\, dy. \tag{39}$$

INTEGRATION AND DIFFERENTIATION

We still confine ourselves to real functions in this section. We shall show that integration and differentiation are, in a certain sense, inverse operations.

6.20 Theorem *Let* $f \in \mathscr{R}$ *on* $[a, b]$. *For* $a \le x \le b$, *put*

$$F(x) = \int_a^x f(t)\, dt.$$

Then F *is continuous on* $[a, b]$; *furthermore, if* f *is continuous at a point* x_0 *of* $[a, b]$, *then* F *is differentiable at* x_0, *and*

$$F'(x_0) = f(x_0).$$

Proof Since $f \in \mathscr{R}$, f is bounded. Suppose $|f(t)| \le M$ for $a \le t \le b$. If $a \le x < y \le b$, then

$$|F(y) - F(x)| = \left| \int_x^y f(t)\, dt \right| \le M(y - x),$$

by Theorem 6.12(c) and (d). Given $\varepsilon > 0$, we see that

$$|F(y) - F(x)| < \varepsilon,$$

provided that $|y - x| < \varepsilon/M$. This proves continuity (and, in fact, uniform continuity) of F.

Now suppose f is continuous at x_0. Given $\varepsilon > 0$, choose $\delta > 0$ such that

$$|f(t) - f(x_0)| < \varepsilon$$

if $|t - x_0| < \delta$, and $a \le t \le b$. Hence, if

$$x_0 - \delta < s \le x_0 \le t < x_0 + \delta \qquad \text{and} \qquad a \le s < t \le b,$$

we have, by Theorem 6.12(*d*),

$$\left|\frac{F(t) - F(s)}{t - s} - f(x_0)\right| = \left|\frac{1}{t - s}\int_s^t [f(u) - f(x_0)]\, du\right| < \varepsilon.$$

It follows that $F'(x_0) = f(x_0)$.

6.21 The fundamental theorem of calculus *If $f \in \mathscr{R}$ on $[a, b]$ and if there is a differentiable function F on $[a, b]$ such that $F' = f$, then*

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

Proof Let $\varepsilon > 0$ be given. Choose a partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$ so that $U(P,f) - L(P,f) < \varepsilon$. The mean value theorem furnishes points $t_i \in [x_{i-1}, x_i]$ such that

$$F(x_i) - F(x_{i-1}) = f(t_i)\, \Delta x_i$$

for $i = 1, \ldots, n$. Thus

$$\sum_{i=1}^n f(t_i)\, \Delta x_i = F(b) - F(a).$$

It now follows from Theorem 6.7(*c*) that

$$\left|F(b) - F(a) - \int_a^b f(x)\, dx\right| < \varepsilon.$$

Since this holds for every $\varepsilon > 0$, the proof is complete.

6.22 Theorem (integration by parts) *Suppose F and G are differentiable functions on $[a, b]$, $F' = f \in \mathscr{R}$, and $G' = g \in \mathscr{R}$. Then*

$$\int_a^b F(x)g(x)\, dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)\, dx.$$

Proof Put $H(x) = F(x)G(x)$ and apply Theorem 6.21 to H and its derivative. Note that $H' \in \mathscr{R}$, by Theorem 6.13.

INTEGRATION OF VECTOR-VALUED FUNCTIONS

6.23 Definition Let $f_1, \ldots, f_k$ be real functions on $[a, b]$, and let $\mathbf{f} = (f_1, \ldots, f_k)$ be the corresponding mapping of $[a, b]$ into R^k. If α increases monotonically on $[a, b]$, to say that $\mathbf{f} \in \mathscr{R}(\alpha)$ means that $f_j \in \mathscr{R}(\alpha)$ for $j = 1, \ldots, k$. If this is the case, we define

$$\int_a^b \mathbf{f}\, d\alpha = \left(\int_a^b f_1\, d\alpha, \ldots, \int_a^b f_k\, d\alpha\right).$$

In other words, $\int \mathbf{f}\, d\alpha$ is the point in R^k whose jth coordinate is $\int f_j\, d\alpha$.

It is clear that parts (*a*), (*c*), and (*e*) of Theorem 6.12 are valid for these vector-valued integrals; we simply apply the earlier results to each coordinate. The same is true of Theorems 6.17, 6.20, and 6.21. To illustrate, we state the analogue of Theorem 6.21.

6.24 Theorem *If* $\mathbf{f}$ *and* $\mathbf{F}$ *map* $[a, b]$ *into* R^k, *if* $\mathbf{f} \in \mathscr{R}$ *on* $[a, b]$, *and if* $\mathbf{F}' = \mathbf{f}$, *then*

$$\int_a^b \mathbf{f}(t)\, dt = \mathbf{F}(b) - \mathbf{F}(a).$$

The analogue of Theorem 6.13(*b*) offers some new features, however, at least in its proof.

6.25 Theorem *If* $\mathbf{f}$ *maps* $[a, b]$ *into* R^k *and if* $\mathbf{f} \in \mathscr{R}(\alpha)$ *for some monotonically increasing function* α *on* $[a, b]$, *then* $|\mathbf{f}| \in \mathscr{R}(\alpha)$, *and*

$$\left|\int_a^b \mathbf{f}\, d\alpha\right| \le \int_a^b |\mathbf{f}|\, d\alpha. \tag{40}$$

Proof If $f_1, \ldots, f_k$ are the components of $\mathbf{f}$, then

$$|\mathbf{f}| = (f_1^2 + \cdots + f_k^2)^{1/2}. \tag{41}$$

By Theorem 6.11, each of the functions f_i^2 belongs to $\mathscr{R}(\alpha)$; hence so does their sum. Since x^2 is a continuous function of x, Theorem 4.17 shows that the square-root function is continuous on $[0, M]$, for every real M. If we apply Theorem 6.11 once more, (41) shows that $|\mathbf{f}| \in \mathscr{R}(\alpha)$.

To prove (40), put $\mathbf{y} = (y_1, \ldots, y_k)$, where $y_j = \int f_j\, d\alpha$. Then we have $\mathbf{y} = \int \mathbf{f}\, d\alpha$, and

$$|\mathbf{y}|^2 = \sum y_i^2 = \sum y_j \int f_j\, d\alpha = \int \left(\sum y_j f_j\right) d\alpha.$$

By the Schwarz inequality,

$$\sum y_j f_j(t) \le |\mathbf{y}|\, |\mathbf{f}(t)| \qquad (a \le t \le b); \tag{42}$$

hence Theorem 6.12(*b*) implies

$$|\mathbf{y}|^2 \le |\mathbf{y}| \int |\mathbf{f}| \, d\alpha. \tag{43}$$

If $\mathbf{y} = \mathbf{0}$, (40) is trivial. If $\mathbf{y} \neq \mathbf{0}$, division of (43) by $|\mathbf{y}|$ gives (40).

RECTIFIABLE CURVES

We conclude this chapter with a topic of geometric interest which provides an application of some of the preceding theory. The case $k = 2$ (i.e., the case of plane curves) is of considerable importance in the study of analytic functions of a complex variable.

6.26 Definition A continuous mapping γ of an interval $[a, b]$ into R^k is called a *curve* in R^k. To emphasize the parameter interval $[a, b]$, we may also say that γ is a curve on $[a, b]$.

If γ is one-to-one, γ is called an *arc*.
If $\gamma(a) = \gamma(b)$, γ is said to be a *closed curve*.

It should be noted that we define a curve to be a *mapping*, not a point set. Of course, with each curve γ in R^k there is associated a subset of R^k, namely the range of γ, but different curves may have the same range.

We associate to each partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$ and to each curve γ on $[a, b]$ the number

$$\Lambda(P, \gamma) = \sum_{i=1}^{n} |\gamma(x_i) - \gamma(x_{i-1})|.$$

The ith term in this sum is the distance (in R^k) between the points $\gamma(x_{i-1})$ and $\gamma(x_i)$. Hence $\Lambda(P, \gamma)$ is the length of a polygonal path with vertices at $\gamma(x_0)$, $\gamma(x_1), \ldots, \gamma(x_n)$, in this order. As our partition becomes finer and finer, this polygon approaches the range of γ more and more closely. This makes it seem reasonable to define the *length* of γ as

$$\Lambda(\gamma) = \sup \Lambda(P, \gamma),$$

where the supremum is taken over all partitions of $[a, b]$.

If $\Lambda(\gamma) < \infty$, we say that γ is *rectifiable*.

In certain cases, $\Lambda(\gamma)$ is given by a Riemann integral. We shall prove this for *continuously differentiable* curves, i.e., for curves γ whose derivative γ' is continuous.

6.27 Theorem *If γ' is continuous on $[a, b]$, then γ is rectifiable, and*

$$\Lambda(\gamma) = \int_a^b |\gamma'(t)|\, dt.$$

Proof If $a \le x_{i-1} < x_i \le b$, then

$$|\gamma(x_i) - \gamma(x_{i-1})| = \left|\int_{x_{i-1}}^{x_i} \gamma'(t)\, dt\right| \le \int_{x_{i-1}}^{x_i} |\gamma'(t)|\, dt.$$

Hence

$$\Lambda(P, \gamma) \le \int_a^b |\gamma'(t)|\, dt$$

for every partition P of $[a, b]$. Consequently,

$$\Lambda(\gamma) \le \int_a^b |\gamma'(t)|\, dt.$$

To prove the opposite inequality, let $\varepsilon > 0$ be given. Since γ' is uniformly continuous on $[a, b]$, there exists $\delta > 0$ such that

$$|\gamma'(s) - \gamma'(t)| < \varepsilon \qquad \text{if } |s - t| < \delta.$$

Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$, with $\Delta x_i < \delta$ for all i. If $x_{i-1} \le t \le x_i$, it follows that

$$|\gamma'(t)| \le |\gamma'(x_i)| + \varepsilon.$$

Hence

$$\begin{aligned}
\int_{x_{i-1}}^{x_i} |\gamma'(t)|\, dt &\le |\gamma'(x_i)|\, \Delta x_i + \varepsilon\, \Delta x_i \\
&= \left|\int_{x_{i-1}}^{x_i} [\gamma'(t) + \gamma'(x_i) - \gamma'(t)]\, dt\right| + \varepsilon\, \Delta x_i \\
&\le \left|\int_{x_{i-1}}^{x_i} \gamma'(t)\, dt\right| + \left|\int_{x_{i-1}}^{x_i} [\gamma'(x_i) - \gamma'(t)]\, dt\right| + \varepsilon\, \Delta x_i \\
&\le |\gamma(x_i) - \gamma(x_{i-1})| + 2\varepsilon\, \Delta x_i.
\end{aligned}$$

If we add these inequalities, we obtain

$$\begin{aligned}
\int_a^b |\gamma'(t)|\, dt &\le \Lambda(P, \gamma) + 2\varepsilon(b - a) \\
&\le \Lambda(\gamma) + 2\varepsilon(b - a).
\end{aligned}$$

Since ε was arbitrary,

$$\int_a^b |\gamma'(t)|\, dt \le \Lambda(\gamma).$$

This completes the proof.