To see this, one makes use of the identity

$$x^b + 1 = (x+1)(x^{b-1} - x^{b-2} + \cdots + 1)$$

for odd b. If $n = ab$ and b is odd, put $x = 2^a$ and infer that $2^a + 1$ is a factor of $2^n + 1$. Unless $b = 1$, it follows that $2^a + 1$ is an odd proper divisor of $2^n + 1$ different from 1, so $2^n + 1$ cannot be prime. Thus, if $2^n + 1$ is prime, n cannot have any odd proper divisors other than 1, hence n must be a power of 2.

Fermat was under the impression that $2^{2^k} + 1$ is prime for all natural numbers k. Euler later found that $2^{2^5} + 1$, a ten digit number, is divisible by 641. An easy, though tricky, way of seeing this is as follows:

$$\begin{aligned} 2^{2^5} &= 16 \times 2^{28} = (641 - 5^4)2^{28} \\ &= 641m - (5 \times 2^7)^4 \\ &= 641m - (641 - 1)^4 \\ &= 641m - (641n + 1) \\ &= 641(m - n) - 1, \end{aligned}$$

where m and n are integers, hence $2^{2^5} + 1$ is a multiple of 641.

At the moment, the only known Fermat primes are 3, 5, 17, 257, and 65,537, corresponding to $k = 0, 1, 2, 3$ and 4, respectively. Not surprisingly, two people in the 19th century tried to break records by actually constructing regular polygons with 257 and 65,537 sides.

For $k = 5, 6, \ldots, 22$, it is known that $2^{2^k} + 1$ is composite.

It follows from the work of Gauss and Wantzel that a regular polygon with m sides can be constructed by ruler and compass if and only if

$$m = 2^k p_1 \cdots p_l,$$

where $k \geq 0$ and $p_1, \cdots, p_l$ are distinct Fermat primes.

Exercises

1. If $a, b \in F$ but $\sqrt{c} \notin F$, show that $a + b\sqrt{c}$ is either 0 or possesses an inverse in $F[\sqrt{c}]$.
2. If $d \in F[\sqrt{c}]$, show that any element of $F[\sqrt{c}][\sqrt{d}]$ satisfies an equation of degree 4 with coefficients in F.
3. Explain how the Greeks could construct regular polygons with 15 and 60 sides.
4. If n is a positive integer, show that an angle of n degrees can be constructed with ruler and compass if and only if n is a multiple of 3.

16
Euclid

The city of Alexandria, on the mediterranean coast of Egypt, was founded by Alexander the Great in 332 B.C., who brought Greeks, Egyptians and Jews to settle there. One of his generals, Ptolemy I, made Alexandria the capital of his kingdom and founded a dynasty consisting of a long line of rulers, also named 'Ptolemy' and ending with the reign of the famous queen Cleopatra, who picked the wrong side in a Roman civil war.

Ptolemy established a university in Alexandria, called the 'Museum', which was soon to acquire a library holding more than 600,000 papyrus scrolls. For well over 600 years, Alexandria was to be the mathematical and scientific center of the world, with only some schools of philosophy surviving in Athens, although, after the extinction of the Ptolemaic line with Cleopatra, Alexandria was ruled by Rome. It was ultimately conquered by the Arabs in 641 AD.

The first chair of mathematics at the Museum was occupied by Euclid (330 to 275 BC), said to have been a student of a student of Plato. Apart from a couple of anecdotes, we know little about his life, and some ancient authors even thought he was a committee, like the 20th century Nicolas Bourbaki. According to one anecdote, Euclid told the impatient king that 'there is no royal road to learning'. According to another, he gave a small coin to a student who demanded to know the practical value of the lectures he had been attending.

Euclid wrote a number of books, on optics, music, astronomy etc., but his fame rests on the *Elements*, a collection of 13 so-called books (which we would now call chapters), which presented the foundations of all the mathematics known in his day. Nothing like this was to be published again, until