# 8 CHAPTER

# Conic Sections

## 8.1 Too Much, Too Little, and Just Right

Conic sections, as their name suggests, are curves obtained by cutting a cone by a plane. They have been studied since ancient times, originally because of their affinity with the circle, and with revived interest since the 17th century when it was found that they model the paths of projectiles, comets, and planets. Another motive for studying them is their ability to "construct" numbers not constructible by ruler and compass, such as $\sqrt[3]{2}$. Perhaps the best way to explain why the same curves arise in these apparently unrelated situations is to say that conic sections are the *simplest* curves, apart from straight lines. Therefore, of all the curves that can turn up in the world of mathematics, the conic sections will turn up most often.

Their simplicity is measured by the degree of their equations, something that was unknown to the ancients, but independently discovered by Fermat and Descartes when they invented analytic geometry. As we know, straight lines have equations of degree 1, $ax + by = c$. The conic sections (or *conics*, as they are often called) are the curves with equations of degree 2.



J. Stillwell, *Numbers and Geometry*

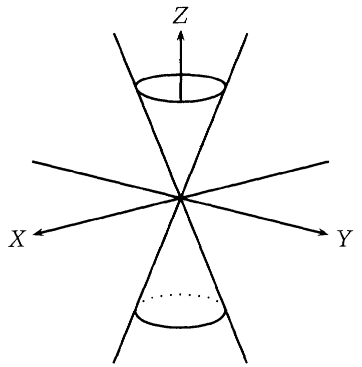


**FIGURE 8.1** The cone.

To see why conics have degree 2, consider the cone $x^2+y^2 = k^2z^2$ shown in Figure 8.1. First of all, why is this the equation of a cone? One sees that the horizontal sections $z =$ constant are the circles $x^2+y^2 = (k \times \text{constant})^2$. It follows that the surface $x^2+y^2 = k^2z^2$ is symmetric about the $z$-axis, and hence all sections by vertical planes through the $z$-axis must look the same. But the section through the plane $x = 0$ is $y^2 = k^2z^2$, which is the pair of lines $y = \pm kz$, so the surface is in fact the cone obtained by rotating these lines about the $z$-axis.

The conic sections proper are the intersections of the cone with planes not passing through the origin. Such a plane can meet the cone in three different ways, and the corresponding curves of intersection (Figure 8.2) are called the *hyperbola*, *ellipse*, and *parabola*, from the Greek meaning roughly "too much," "too little," and "just right." Other English words with the same origin are "hyperbole" (something exaggerated or excessive), "ellipsis" (something cut short), and "parable" (something that runs alongside).

The same broad classification "too much," "too little," and "just right" occurs elsewhere in mathematics. For example, geometries and differential equations are both divided into hyperbolic, elliptic, and parabolic types. The parabolic case is always the exceptional, transitional case between hyperbolic and elliptic. Among the conic sections, the parabola is the exceptional case where the cutting plane is parallel to one of the lines in the cone. This happens when the cutting plane has slope $\pm k$. Hyperbolas occur when the cutting plane