

Figure 1.10: Perpendicular axes

We have already said that the perpendicular distances from P to the axes are the numbers x , y . The distance between points on the same perpendicular to an axis should therefore be defined as the difference between the appropriate coordinates. In Figure 1.11 this is $x_2 - x_1$ for RQ and $y_2 - y_1$ for PQ . But then Pythagoras' theorem tells us that the distance PR is given

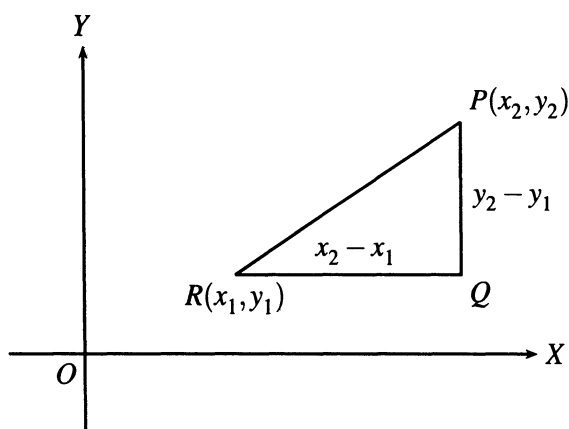


Figure 1.11: Defining distance

by

$$\begin{aligned} PR^2 &= RQ^2 + PQ^2 \\ &= (x_2 - x_1)^2 + (y_2 - y_1)^2. \end{aligned}$$

That is,

$$PR = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \quad (1)$$

Since this construction applies to arbitrary points P, R in the plane, we now have a general formula for the distance between two points.

We derived this formula as a consequence of geometric assumptions, in particular Pythagoras' theorem. Although this makes geometry amenable to arithmetical calculation—a very useful situation, to be sure—it does not say that geometry *is* arithmetic. In the early days of analytic geometry, the latter was a very heretical view (see Section 7.6). Eventually, however, Hilbert (1899) realized it could be made a fact by taking (1) as a *definition* of distance. Of course, all other geometric concepts have to be defined in terms of numbers, too, but this boils down to defining a point, which is simply an *ordered pair* (x, y) of numbers. Equation (1) then gives the distance between the points (x_1, y_1) and (x_2, y_2) .

When geometry is reconstructed in this way, all geometric facts become facts about numbers (though they do not necessarily become easier to see). In particular, Pythagoras' theorem becomes true by definition since it has been built into the definition of distance. This is not to say that Pythagoras' theorem ultimately is trivial. Rather, it shows that Pythagoras' theorem is precisely what is needed to interpret arithmetical facts as geometry.

I mention these more recent developments only to bring Pythagoras' theorem up to date and to give a precise statement of its power to transform arithmetic into geometry. In ancient Greek times, geometry was based much more on seeing than on calculation. We shall see how the Greeks managed to build geometry on the basis of visually evident facts in the next chapter.

EXERCISES

Most mathematicians today are more familiar with coordinates than traditional geometry, yet certain theorems of analytic geometry are seldom proved, because they seem visually obvious. A good example is what Hilbert (1899) calls *additivity of segments*: if A, B, C are points in that order on a line, then $AB + BC = AC$.

1.6.1 By suitably naming the coordinates for A , B , and C , show that the equation $AB + BC = AC$ is equivalent to

$$\sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2 + y_2^2} = \sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2}, \quad (*)$$

where $x_1y_2 = y_1x_2$. *Hint:* It is convenient to let B be the origin.

1.6.2 Prove (*) by proving an equivalent rational equation obtained by squaring twice and using $x_1y_2 = y_1x_2$.

It should be stressed that Hilbert (1899) is not only about defining geometric concepts in terms of coordinates. He is also concerned with the reverse process: setting up geometric assumptions from which coordinates may be rigorously derived. There is more about this in Section 2.1.

1.7 Biographical Notes: Pythagoras

Very little is known for certain about Pythagoras, although he figures in many legends. No documents have survived from the period in which he lived, so we have to rely on stories that were passed down for several centuries before being recorded. It appears that he was born on Samos, a Greek island near the coast of what is now Turkey, around 580 BCE. He traveled to the nearby mainland town of Miletus, where he learned mathematics from Thales (624–547 BCE), traditionally regarded as the founder of Greek mathematics. Pythagoras also traveled to Egypt and Babylon, where he presumably picked up additional mathematical ideas. Around 540 BCE he settled in Croton, a Greek colony in what is now southern Italy.

There he founded a school whose members later became known as the Pythagoreans. The school's motto was "All is number," and the Pythagoreans tried to bring the realms of science, religion, and philosophy all under the rule of number. The very word *mathematics* ("that which is learned") is said to be a Pythagorean invention. The school imposed a strict code of conduct on its members, which included secrecy, vegetarianism, and a curious taboo on the eating of beans. The code of secrecy meant that mathematical results were considered to be the property of the school, and their individual discoverers were not identified to outsiders. Because of this, we do not know who discovered Pythagoras' theorem, the irrationality of $\sqrt{2}$, or other arithmetical results that will be mentioned in Chapter 3.

As mentioned in Section 1.5, the most notable scientific success of the Pythagorean school was the explanation of musical harmony in terms of

whole-number ratios. This success inspired the search for a numerical law governing the motions of planets, a “harmony of the spheres.” Such a law probably cannot be expressed in terms that the Pythagoreans would have accepted; nevertheless, it seems reasonable to view the expansion of the number concept to meet the needs of geometry (and hence mechanics) as a natural extension of the Pythagorean program. In this sense, Newton’s law of gravitation (Section 13.2) expresses the harmony that the Pythagoreans were looking for. Even in the strictest sense, Pythagoreanism is very much alive today. With the digital computer, digital watches, digital audio and video coding everything, at least approximately, into sequences of whole numbers, we are closer than ever to a world in which “all is number.”

Whether the complete rule of number is wise remains to be seen. It is said that when the Pythagoreans tried to extend their influence into politics they met with popular resistance. Pythagoras fled, but he was murdered in nearby Metapontum in 497 BCE.