from the definition of log as an area, and when $n = -1$ we have

$$0 = \log 1 = \log aa^{-1} = \log a + \log a^{-1},$$

which implies

$$\log a^{-1} = -\log a.$$

Finally,

$$\log a^{-2} = \log a^{-1} + \log a^{-1} = -2\log a,$$

and so on, by the additive property again.

2. If $a > 1$ then $\log a > 0$ from the definition of log as an area, and because $\log a^n = n\log a$ we can get arbitrarily large values of the log function by choosing sufficiently large values of the integer n. We can also get *arbitrarily closely spaced* values of the log function, by first choosing a near 1 so as to make $\log a$ as small as we please, then taking the equally spaced values $\log a^2$, $\log a^3$, ... It follows, by an argument like that for the completeness of the real numbers in Section 3.4, that for any real number ρ we can separate the values of $\log t$ into a lower set whose least upper bound is ρ and an upper set whose greatest lower bound is ρ.

   Now it is clear from its definition that $\log t$ increases with t, hence the values of t are separated into a lower set L, for which $\log t$ has least upper bound ρ and an upper set U for which $\log t$ has greatest lower bound ρ. But the only number between the two sets of values of $\log t$ is ρ, hence it follows from the strict increase of the log function that *if τ is the least upper bound of L then $\log \tau$ must be ρ.*

3. It follows that there is a number e such that $\log e = 1$.

4. More generally, any real number $r\log a$ is a value of $\log t$. We define a^r to be the number whose log is $r\log a$, because this is consistent with the meaning of a^n for integers n by part 1. It then follows that e^y is the number x whose log is $y\log e = y$. That is, $x = e^y$ if and only if $y = \log x$.   □

   The fourth property is also described by saying that the *exponential function* e^y of y is the *inverse* of the log function. Thus the graph of $t = e^y$ (Figure 9.7) results from the graph of $y = \log t$ (Figure 9.4) by swapping the t- and y-axes. A consequence of the inverse relationship is that $t = e^{\log t}$, as claimed earlier.
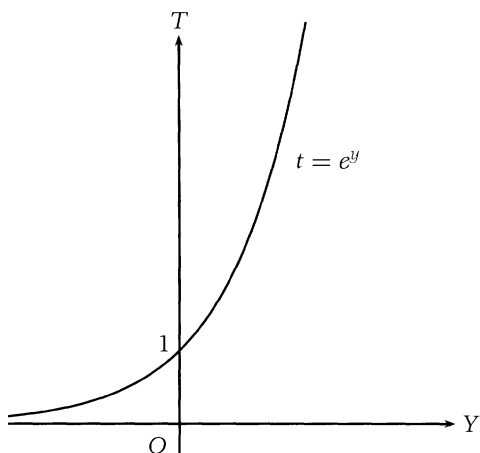
FIGURE 9.7 Graph of the exponential function.

Exercises

In the previous exercise set we saw that the area under the parabola is related to $(1^2+2^2+3^2+\cdots+n^2)/n^3$, which we were able to evaluate exactly. The area under the hyperbola is similarly related to $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, which we do not understand as well. At first it is not even clear whether this sum has a limit as $n \to \infty$, but a quick way to find out is to use what we now know about the area under the hyperbola.

9.3.1. Representing $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ by a suitable step polygon, show that it is greater than $\log n$, and hence that $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \to \infty$ as $n \to \infty$.

In fact, the "rate of growth" of $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ is amazingly close to that of $\log n$. Euler discovered that $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n$ tends to a constant of value approximately 0.5772 as $n \to \infty$. The constant is known as *Euler's constant*, and it has been computed to many decimal places, but it is not known whether it is rational or irrational.

9.3.2. By using upper and lower step polygons, show that

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} > \log n > \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n},$$

and hence deduce that $0 < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n < 1$.

9.3.3. Show by area considerations that $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} - \log n$ is increasing and hence has a limit as $n \to \infty$ by the completeness of the real numbers. Show that $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n$ has the same limit.

9.4   The Exponential Function

The additive property of the log function translates into the following property of the exponential function, which one recognizes as the characteristic property of exponents in general.

Addition formula for the exponential function.   *For real a and b,*

$$e^{a+b} = e^a e^b.$$

Proof   By definition of the exponential function, e^{a+b} is the number whose log is $a + b$. By the additive property of log, it follows that e^{a+b} is the product of the numbers whose logs are a and b. That is, $e^{a+b} = e^a e^b$.   □

Thus e^x behaves like the xth power of e, and we are justified in using a notation that suggests this. But why use powers of the mysterious number e instead of powers of something familiar, like 2 or 10? Most of the reasons for the convenience of e can be traced back to the fact that $\log e = 1$. The properties of general exponential functions a^x often involve a factor $\log a$ (see the exercises), and hence they are simplest when $\log a = 1$.

Other formulas showing e^x to be the simplest exponential function are

$$e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n,$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

We shall not need these formulas, but the second one, in particular, is connected with a spectacular discovery of Euler (1748) we cannot

fail to mention:

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

This can be proved by comparing

$$e^{i\theta} = 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots$$

with known infinite series for $\cos\theta$ and $\sin\theta$:

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots,$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots.$$

Euler's formula is surely the most conclusive argument for the naturalness of e, but if anyone is not yet satisfied, consider the special case $\theta = \pi$:

$$e^{i\pi} = -1.$$

Only e can singlehandedly bring i and π down to earth!

Euler's formula incidentally proves that e^x, and hence its inverse $\log x$, is a transcendental function, as claimed earlier. If e^x were algebraic, then the equation $e^x = 1$ would have only finitely many solutions, real or imaginary. But in fact this equation has infinitely many solutions: $x = 2in\pi$ for all integers n.

Another thing we learn from Euler's formula is that the exponential function e^x is best regarded as a function of a *complex* variable x. It then embraces both the functions cos and sin, and their properties follow from properties of the exponential function. For example, the somewhat complicated addition formulas for cos and sin become consequences of the simple addition formula for e^x (as was verified for $\operatorname{cis}\theta = e^{i\theta}$ in the exercises to Section 5.3). When cos and sin are subordinated to the exponential function in this way, one also understands their uncanny similarity to the so-called *hyperbolic cosine* and *hyperbolic sine*, cosh and sinh, which will be introduced in the next section.

Exercises

Because a^x is the number whose log is $x \log a$, namely, $e^{x \log a}$, powers of any positive real number a can be expressed as powers of e. The inverse of the power function a^x is called the *logarithm to base a* (so the ordinary log is logarithm to base e). If $y = a^x$ we write $x = \log_a y$.

9.4.1. Deduce from these definitions that $(e^a)^b = e^{ab}$, $e^x = a^{x/\log a}$, and $\log_a x = \log x / \log a$.

When we allow complex values of x, the exponential function takes the same value for many values of x, as noticed earlier. Hence its inverse, the log, is no longer a function. The single real value of $\log x$ for each real x is joined by infinitely many complex values. The happier side of this situation is that we can now find values of $\log x$ for *negative* real values of x, because any real $x \neq 0$ occurs as a value of the exponential function.

9.4.2. Find a value of $\log(-1)$.

To find all the values of $\log x$, it is necessary to know that

$$e^{a+ib} = e^a(\cos b + i \sin b) \quad \text{for any real } a \text{ and } b,$$

which follows from the addition formula and Euler's formula.

9.4.3. Assuming $e^{a+ib} = e^a(\cos b + i \sin b)$, find all values of $\log(-1)$.

We can also define a^b as $e^{b \log a}$ for complex numbers a and b, though the expression acquires infinitely many values from the infinitely many values of the log. An interesting example is i^i, which was first evaluated by Euler.

9.4.4. Use Euler's formula to show that $i = e^{i\pi/2}$.

9.4.5. Deduce from Exercises 9.4.1 and 9.4.4 that $e^{-\pi/2}$ is the real value of i^i. What are its other values?

Assuming the infinite series for e^x given earlier, it follows that

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots .$$

The terms of this series tend to 0 very rapidly, which makes it easy to find approximate values of e, such as 2.718. The series also shows, however, that e is an irrational number.

9.4.6. Suppose on the contrary that $e = m/n$ for some integers m and $n \neq 0$. Show the following in turn:

- If $e = m/n$ then $n!e$ is an integer.
- On the other hand,
  $$n!e = \text{an integer} + \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots .$$
- $\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots < \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots = 1.$
- This is a contradiction.

In fact, Charles Hermite showed in 1873 that e is a transcendental number. That is, e is not the root of any polynomial equation with integer coefficients. It was the first "known" number found to be transcendental. Building on Hermite's proof, Ferdinand Lindemann showed in 1882 that π is also transcendental. The proofs use calculus heavily and are a lot more difficult than the proofs that the exponential and circular functions are transcendental. Apparently numbers are harder to understand than functions, at least as far as transcendance goes.

9.5 The Hyperbolic Functions

Just as the circle $x^2 + y^2 = 1$ can be defined by the pair of functions $x = \cos\theta$ and $y = \sin\theta$, the hyperbola $x^2 - y^2 = 1$ can be defined by a pair of functions $x = \cosh\theta$ and $y = \sinh\theta$. It is even possible to interpret the parameter θ as (twice) the area of a "sector" of the hyperbola, and to define the functions cosh and sinh thereby, but to save time we define them so that the equation $\cosh^2\theta - \sinh^2\theta = 1$ is obvious.

The *hyperbolic cosine* $\cosh\theta$ and the *hyperbolic sine* $\sinh\theta$ are defined by

$$\cosh\theta = \frac{e^\theta + e^{-\theta}}{2},$$

$$\sinh\theta = \frac{e^\theta - e^{-\theta}}{2}.$$

It follows easily that

$$\cosh^2\theta - \sinh^2\theta = 1,$$

and hence $(\cosh\theta, \sinh\theta)$ is a point on the hyperbola $x^2 - y^2 = 1$ for each real value of θ.

By investigating $\cosh\theta$ and $\sinh\theta$ a little more closely, we find that each point on the positive branch ($x > 0$) of $x^2 - y^2 = 1$ occurs as $(\cosh\theta, \sinh\theta)$ for exactly one real value of θ. Some of the relevant properties of the hyperbolic cosine and sine can be seen at a glance from their graphs, shown along with those of $\frac{1}{2}e^\theta$ and $\frac{1}{2}e^{-\theta}$ in Figure 9.8. (Proofs are easily constructed from the fact that e^θ takes each positive value exactly once.)

- $\cosh(-\theta) = \cosh\theta$,
- $\cosh\theta$ takes all real values ≥ 1,
- $\sinh(-\theta) = -\sinh\theta$,
- $\sinh\theta$ takes each real value once.

Thus $\sinh\theta$ takes each real value y exactly once, and for each value $y = \sinh\theta$ the value x of $\cosh\theta$ gives a point (x, y) on the positive branch of $x^2 - y^2 = 1$, because $\cosh^2\theta - \sinh^2\theta = 1$. Because there is exactly one point (x, y) on the positive branch for each y, we therefore get each point on the positive branch exactly once as $(\cosh\theta, \sinh\theta)$.

This one-to-one correspondence between points $P_\theta = (\cosh\theta, \sinh\theta)$ on the positive branch and real numbers θ enables us to "add points" by adding their parameter values. We simply define the *sum of points* $P_\theta = (\cosh\theta, \sinh\theta)$ and $P_\phi = (\cosh\phi, \sinh\phi)$ on $x^2 - y^2 = 1$
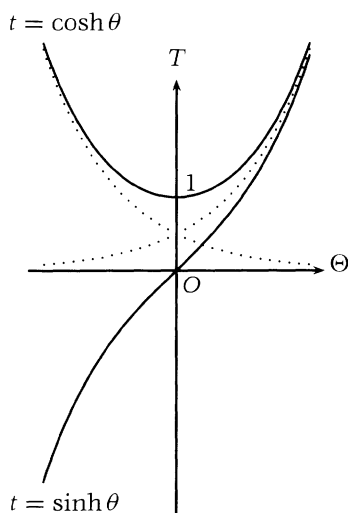

FIGURE 9.8   The graphs of cosh and sinh.

by

$$P_\theta + P_\phi = P_{\theta+\phi} = (\cosh(\theta + \phi), \sinh(\theta + \phi)).$$

Adding points like this may seem an idle and useless thing to do, but we have done it before, with interesting results. The process used in Section 8.5 to generate integer points on $x^2 - 2y^2 = 1$ can be interpreted as repeated "addition" of the point $(3, 2)$ to itself. In the next section we shall see why this is so and how much clearer the process becomes when interpreted as addition of parameter values.

Exercises

The identity $\cosh^2 \theta - \sinh^2 \theta = 1$ is just one of many where the hyperbolic sine and cosine behave almost the same as the ordinary sine and cosine. The similarity is best explained by allowing θ to be complex, so that they all become relatives of the exponential function.

9.5.1. Use Euler's formula for $e^{i\theta}$ and $e^{-i\theta}$ to show that

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \cosh i\theta$$

$$\text{and} \quad \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{1}{i} \sinh i\theta.$$

This explains why cosh and sinh satisfy addition formulas, and other identities, similar to those satisfied by cos and sin. Of course, it is not necessary to use complex numbers to *prove* these identities—they follow from properties of the real exponential function—but complex numbers allow us to predict their existence in the first place and to anticipate their form.

9.5.2. Prove the *addition formulas* for cosh and sinh:

$$\cosh(\theta + \phi) = \cosh \theta \cosh \phi + \sinh \theta \sinh \phi,$$

$$\text{and} \quad \sinh(\theta + \phi) = \sinh \theta \cosh \phi + \cosh \theta \sinh \phi.$$

As mentioned earlier, the parameter θ in $x = \cosh \theta$ and $y = \sinh \theta$ can be interpreted as twice the area of a sector of the hyperbola $x^2 - y^2 = 1$. A geometric proof of this fact can be put together as follows. We first