Next,

(37) $$g'(x) = 1 + \left\{2x - \frac{2i}{x}\right\} e^{i/x^2} \qquad (0 < x < 1),$$

so that

(38) $$|g'(x)| \geq \left|2x - \frac{2i}{x}\right| - 1 \geq \frac{2}{x} - 1.$$

Hence

(39) $$\left|\frac{f'(x)}{g'(x)}\right| = \frac{1}{|g'(x)|} \leq \frac{x}{2 - x}$$

and so

(40) $$\lim_{x \to 0} \frac{f'(x)}{g'(x)} = 0.$$

By (36) and (40), L'Hospital's rule fails in this case. Note also that $g'(x) \neq 0$ on $(0, 1)$, by (38).

However, there *is* a consequence of the mean value theorem which, for purposes of applications, is almost as useful as Theorem 5.10, and which remains true for vector-valued functions: From Theorem 5.10 it follows that

(41) $$|f(b) - f(a)| \leq (b - a) \sup_{a < x < b} |f'(x)|.$$

5.19 Theorem *Suppose* $\mathbf{f}$ *is a continuous mapping of* $[a, b]$ *into* R^k *and* $\mathbf{f}$ *is differentiable in* (a, b). *Then there exists* $x \in (a, b)$ *such that*

$$|\mathbf{f}(b) - \mathbf{f}(a)| \leq (b - a)|\mathbf{f}'(x)|.$$

Proof[1] Put $\mathbf{z} = \mathbf{f}(b) - \mathbf{f}(a)$, and define

$$\varphi(t) = \mathbf{z} \cdot \mathbf{f}(t) \qquad (a \leq t \leq b).$$

Then φ is a real-valued continuous function on $[a, b]$ which is differentiable in (a, b). The mean value theorem shows therefore that

$$\varphi(b) - \varphi(a) = (b - a)\varphi'(x) = (b - a)\mathbf{z} \cdot \mathbf{f}'(x)$$

for some $x \in (a, b)$. On the other hand,

$$\varphi(b) - \varphi(a) = \mathbf{z} \cdot \mathbf{f}(b) - \mathbf{z} \cdot \mathbf{f}(a) = \mathbf{z} \cdot \mathbf{z} = |\mathbf{z}|^2.$$

The Schwarz inequality now gives

$$|\mathbf{z}|^2 = (b - a)|\mathbf{z} \cdot \mathbf{f}'(x)| \leq (b - a)|\mathbf{z}|\,|\mathbf{f}'(x)|.$$

Hence $|\mathbf{z}| \leq (b - a)|\mathbf{f}'(x)|$, which is the desired conclusion.

[1] V. P. Havin translated the second edition of this book into Russian and added this proof to the original one.

EXERCISES

1. Let f be defined for all real x, and suppose that
$$|f(x) - f(y)| \le (x - y)^2$$
for all real x and y. Prove that f is constant.
2. Suppose $f'(x) > 0$ in (a, b). Prove that f is strictly increasing in (a, b), and let g be its inverse function. Prove that g is differentiable, and that
$$g'(f(x)) = \frac{1}{f'(x)} \qquad (a < x < b).$$
3. Suppose g is a real function on R^1, with bounded derivative (say $|g'| \le M$). Fix $\varepsilon > 0$, and define $f(x) = x + \varepsilon g(x)$. Prove that f is one-to-one if ε is small enough. (A set of admissible values of ε can be determined which depends only on M.)
4. If
$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0,$$
where $C_0, \ldots, C_n$ are real constants, prove that the equation
$$C_0 + C_1 x + \cdots + C_{n-1} x^{n-1} + C_n x^n = 0$$
has at least one real root between 0 and 1.
5. Suppose f is defined and differentiable for every $x > 0$, and $f'(x) \to 0$ as $x \to +\infty$. Put $g(x) = f(x+1) - f(x)$. Prove that $g(x) \to 0$ as $x \to +\infty$.
6. Suppose
(*a*) f is continuous for $x \ge 0$,
(*b*) $f'(x)$ exists for $x > 0$,
(*c*) $f(0) = 0$,
(*d*) f' is monotonically increasing.
Put
$$g(x) = \frac{f(x)}{x} \qquad (x > 0)$$
and prove that g is monotonically increasing.
7. Suppose $f'(x)$, $g'(x)$ exist, $g'(x) \ne 0$, and $f(x) = g(x) = 0$. Prove that
$$\lim_{t \to x} \frac{f(t)}{g(t)} = \frac{f'(x)}{g'(x)}.$$
(This holds also for complex functions.)
8. Suppose f' is continuous on $[a, b]$ and $\varepsilon > 0$. Prove that there exists $\delta > 0$ such that
$$\left|\frac{f(t) - f(x)}{t - x} - f'(x)\right| < \varepsilon$$

whenever $0 < |t - x| < \delta$, $a \le x \le b$, $a \le t \le b$. (This could be expressed by saying that f is *uniformly differentiable* on $[a, b]$ if f' is continuous on $[a, b]$.) Does this hold for vector-valued functions too?

9. Let f be a continuous real function on R^1, of which it is known that $f'(x)$ exists for all $x \ne 0$ and that $f'(x) \to 3$ as $x \to 0$. Does it follow that $f'(0)$ exists?
10. Suppose f and g are complex differentiable functions on $(0, 1)$, $f(x) \to 0$, $g(x) \to 0$, $f'(x) \to A$, $g'(x) \to B$ as $x \to 0$, where A and B are complex numbers, $B \ne 0$. Prove that
$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \frac{A}{B}.$$
Compare with Example 5.18. *Hint:*
$$\frac{f(x)}{g(x)} = \left\{\frac{f(x)}{x} - A\right\} \cdot \frac{x}{g(x)} + A \cdot \frac{x}{g(x)}.$$
Apply Theorem 5.13 to the real and imaginary parts of $f(x)/x$ and $g(x)/x$.
11. Suppose f is defined in a neighborhood of x, and suppose $f''(x)$ exists. Show that
$$\lim_{h \to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x).$$
Show by an example that the limit may exist even if $f''(x)$ does not.
Hint: Use Theorem 5.13.
12. If $f(x) = |x|^3$, compute $f'(x)$, $f''(x)$ for all real x, and show that $f^{(3)}(0)$ does not exist.
13. Suppose a and c are real numbers, $c > 0$, and f is defined on $[-1, 1]$ by
$$f(x) = \begin{cases} x^a \sin(|x|^{-c}) & (\text{if } x \ne 0), \\ 0 & (\text{if } x = 0). \end{cases}$$
Prove the following statements:
(*a*) f is continuous if and only if $a > 0$.
(*b*) $f'(0)$ exists if and only if $a > 1$.
(*c*) f' is bounded if and only if $a \ge 1 + c$.
(*d*) f' is continuous if and only if $a > 1 + c$.
(*e*) $f''(0)$ exists if and only if $a > 2 + c$.
(*f*) f'' is bounded if and only if $a \ge 2 + 2c$.
(*g*) f'' is continuous if and only if $a > 2 + 2c$.
14. Let f be a differentiable real function defined in (a, b). Prove that f is convex if and only if f' is monotonically increasing. Assume next that $f''(x)$ exists for every $x \in (a, b)$, and prove that f is convex if and only if $f''(x) \ge 0$ for all $x \in (a, b)$.
15. Suppose $a \in R^1$, f is a twice-differentiable real function on (a, ∞), and M_0, M_1, M_2 are the least upper bounds of $|f(x)|$, $|f'(x)|$, $|f''(x)|$, respectively, on (a, ∞). Prove that
$$M_1^2 \le 4M_0 M_2.$$

Hint: If $h > 0$, Taylor's theorem shows that

$$f'(x) = \frac{1}{2h}[f(x+2h) - f(x)] - hf''(\xi)$$

for some $\xi \in (x, x+2h)$. Hence

$$|f'(x)| \leq hM_2 + \frac{M_0}{h}.$$

To show that $M_1^2 = 4M_0M_2$ can actually happen, take $a = -1$, define

$$f(x) = \begin{cases} 2x^2 - 1 & (-1 < x < 0), \\ \dfrac{x^2 - 1}{x^2 + 1} & (0 \leq x < \infty), \end{cases}$$

and show that $M_0 = 1$, $M_1 = 4$, $M_2 = 4$.

Does $M_1^2 \leq 4M_0M_2$ hold for vector-valued functions too?

16. Suppose f is twice-differentiable on $(0, \infty)$, f'' is bounded on $(0, \infty)$, and $f(x) \to 0$ as $x \to \infty$. Prove that $f'(x) \to 0$ as $x \to \infty$.

Hint: Let $a \to \infty$ in Exercise 15.

17. Suppose f is a real, three times differentiable function on $[-1, 1]$, such that

$$f(-1) = 0, \qquad f(0) = 0, \qquad f(1) = 1, \qquad f'(0) = 0.$$

Prove that $f^{(3)}(x) \geq 3$ for some $x \in (-1, 1)$.

Note that equality holds for $\frac{1}{2}(x^3 + x^2)$.

Hint: Use Theorem 5.15, with $\alpha = 0$ and $\beta = \pm 1$, to show that there exist $s \in (0, 1)$ and $t \in (-1, 0)$ such that

$$f^{(3)}(s) + f^{(3)}(t) = 6.$$

18. Suppose f is a real function on $[a, b]$, n is a positive integer, and $f^{(n-1)}$ exists for every $t \in [a, b]$. Let α, β, and P be as in Taylor's theorem (5.15). Define

$$Q(t) = \frac{f(t) - f(\beta)}{t - \beta}$$

for $t \in [a, b]$, $t \neq \beta$, differentiate

$$f(t) - f(\beta) = (t - \beta)Q(t)$$

$n - 1$ times at $t = \alpha$, and derive the following version of Taylor's theorem:

$$f(\beta) = P(\beta) + \frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta - \alpha)^n.$$

19. Suppose f is defined in $(-1, 1)$ and $f'(0)$ exists. Suppose $-1 < \alpha_n < \beta_n < 1$, $\alpha_n \to 0$, and $\beta_n \to 0$ as $n \to \infty$. Define the difference quotients

$$D_n = \frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n}.$$

Prove the following statements:

(*a*) If $\alpha_n < 0 < \beta_n$, then $\lim D_n = f'(0)$.

(*b*) If $0 < \alpha_n < \beta_n$ and $\{\beta_n/(\beta_n - \alpha_n)\}$ is bounded, then $\lim D_n = f'(0)$.

(*c*) If f' is continuous in $(-1, 1)$, then $\lim D_n = f'(0)$.

Give an example in which f is differentiable in $(-1, 1)$ (but f' is not continuous at 0) and in which α_n, β_n tend to 0 in such a way that $\lim D_n$ exists but is different from $f'(0)$.

20. Formulate and prove an inequality which follows from Taylor's theorem and which remains valid for vector-valued functions.

21. Let E be a closed subset of R^1. We saw in Exercise 22, Chap. 4, that there is a real continuous function f on R^1 whose zero set is E. Is it possible, for each closed set E, to find such an f which is differentiable on R^1, or one which is n times differentiable, or even one which has derivatives of all orders on R^1?

22. Suppose f is a real function on $(-\infty, \infty)$. Call x a *fixed point* of f if $f(x) = x$.

(*a*) If f is differentiable and $f'(t) \neq 1$ for every real t, prove that f has at most one fixed point.

(*b*) Show that the function f defined by

$$f(t) = t + (1 + e^t)^{-1}$$

has no fixed point, although $0 < f'(t) < 1$ for all real t.

(*c*) However, if there is a constant $A < 1$ such that $|f'(t)| \leq A$ for all real t, prove that a fixed point x of f exists, and that $x = \lim x_n$, where x_1 is an arbitrary real number and

$$x_{n+1} = f(x_n)$$

for $n = 1, 2, 3, \ldots$.

(*d*) Show that the process described in (*c*) can be visualized by the zig-zag path

$$(x_1, x_2) \to (x_2, x_2) \to (x_2, x_3) \to (x_3, x_3) \to (x_3, x_4) \to \cdots.$$

23. The function f defined by

$$f(x) = \frac{x^3 + 1}{3}$$

has three fixed points, say α, β, γ, where

$$-2 < \alpha < -1, \qquad 0 < \beta < 1, \qquad 1 < \gamma < 2.$$

For arbitrarily chosen x_1, define $\{x_n\}$ by setting $x_{n+1} = f(x_n)$.

(*a*) If $x_1 < \alpha$, prove that $x_n \to -\infty$ as $n \to \infty$.

(*b*) If $\alpha < x_1 < \gamma$, prove that $x_n \to \beta$ as $n \to \infty$.

(*c*) If $\gamma < x_1$, prove that $x_n \to +\infty$ as $n \to \infty$.

Thus β can be located by this method, but α and γ cannot.

24. The process described in part (*c*) of Exercise 22 can of course also be applied to functions that map $(0, \infty)$ to $(0, \infty)$.

Fix some $\alpha > 1$, and put

$$f(x) = \frac{1}{2}\left(x + \frac{\alpha}{x}\right), \qquad g(x) = \frac{\alpha + x}{1 + x}.$$

Both f and g have $\sqrt{\alpha}$ as their only fixed point in $(0, \infty)$. Try to explain, on the basis of properties of f and g, why the convergence in Exercise 16, Chap. 3, is so much more rapid than it is in Exercise 17. (Compare f' and g', draw the zig-zags suggested in Exercise 22.)

Do the same when $0 < \alpha < 1$.

25. Suppose f is twice differentiable on $[a, b]$, $f(a) < 0$, $f(b) > 0$, $f'(x) \geq \delta > 0$, and $0 \leq f''(x) \leq M$ for all $x \in [a, b]$. Let ξ be the unique point in (a, b) at which $f(\xi) = 0$.

Complete the details in the following outline of *Newton's method* for computing ξ.

(*a*) Choose $x_1 \in (\xi, b)$, and define $\{x_n\}$ by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Interpret this geometrically, in terms of a tangent to the graph of f.

(*b*) Prove that $x_{n+1} < x_n$ and that

$$\lim_{n\to\infty} x_n = \xi.$$

(*c*) Use Taylor's theorem to show that

$$x_{n+1} - \xi = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2$$

for some $t_n \in (\xi, x_n)$.

(*d*) If $A = M/2\delta$, deduce that

$$0 \leq x_{n+1} - \xi \leq \frac{1}{A}[A(x_1 - \xi)]^{2^n}.$$

(Compare with Exercises 16 and 18, Chap. 3.)

(*e*) Show that Newton's method amounts to finding a fixed point of the function g defined by

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

How does $g'(x)$ behave for x near ξ?

(*f*) Put $f(x) = x^{1/3}$ on $(-\infty, \infty)$ and try Newton's method. What happens?

26. Suppose f is differentiable on $[a, b]$, $f(a) = 0$, and there is a real number A such that $|f'(x)| \leq A|f(x)|$ on $[a, b]$. Prove that $f(x) = 0$ for all $x \in [a, b]$. *Hint:* Fix $x_0 \in [a, b]$, let

$$M_0 = \sup|f(x)|, \qquad M_1 = \sup|f'(x)|$$

for $a \leq x \leq x_0$. For any such x,

$$|f(x)| \leq M_1(x_0 - a) \leq A(x_0 - a)M_0.$$

Hence $M_0 = 0$ if $A(x_0 - a) < 1$. That is, $f = 0$ on $[a, x_0]$. Proceed.

27. Let ϕ be a real function defined on a rectangle R in the plane, given by $a \leq x \leq b$, $\alpha \leq y \leq \beta$. A *solution* of the initial-value problem

$$y' = \phi(x, y), \qquad y(a) = c \qquad (\alpha \leq c \leq \beta)$$

is, by definition, a differentiable function f on $[a, b]$ such that $f(a) = c$, $\alpha \leq f(x) \leq \beta$, and

$$f'(x) = \phi(x, f(x)) \qquad (a \leq x \leq b).$$

Prove that such a problem has at most one solution if there is a constant A such that

$$|\phi(x, y_2) - \phi(x, y_1)| \leq A|y_2 - y_1|$$

whenever $(x, y_1) \in R$ and $(x, y_2) \in R$.

Hint: Apply Exercise 26 to the difference of two solutions. Note that this uniqueness theorem does not hold for the initial-value problem

$$y' = y^{1/2}, \qquad y(0) = 0,$$

which has two solutions: $f(x) = 0$ and $f(x) = x^2/4$. Find all other solutions.

28. Formulate and prove an analogous uniqueness theorem for systems of differential equations of the form

$$y_j' = \phi_j(x, y_1, \ldots, y_k), \qquad y_j(a) = c_j \qquad (j = 1, \ldots, k).$$

Note that this can be rewritten in the form

$$\mathbf{y}' = \boldsymbol{\Phi}(x, \mathbf{y}), \qquad \mathbf{y}(a) = \mathbf{c}$$

where $\mathbf{y} = (y_1, \ldots, y_k)$ ranges over a k-cell, $\boldsymbol{\Phi}$ is the mapping of a $(k + 1)$-cell into the Euclidean k-space whose components are the functions $\phi_1, \ldots, \phi_k$, and $\mathbf{c}$ is the vector $(c_1, \ldots, c_k)$. Use Exercise 26, for vector-valued functions.

29. Specialize Exercise 28 by considering the system

$$y_j' = y_{j+1} \qquad (j = 1, \ldots, k - 1),$$

$$y_k' = f(x) - \sum_{j=1}^{k} g_j(x)y_j,$$

where $f, g_1, \ldots, g_k$ are continuous real functions on $[a, b]$, and derive a uniqueness theorem for solutions of the equation

$$y^{(k)} + g_k(x)y^{(k-1)} + \cdots + g_2(x)y' + g_1(x)y = f(x),$$

subject to initial conditions

$$y(a) = c_1, \qquad y'(a) = c_2, \qquad \ldots, \qquad y^{(k-1)}(a) = c_k.$$

6

THE RIEMANN-STIELTJES INTEGRAL

The present chapter is based on a definition of the Riemann integral which depends very explicitly on the order structure of the real line. Accordingly, we begin by discussing integration of real-valued functions on intervals. Extensions to complex- and vector-valued functions on intervals follow in later sections. Integration over sets other than intervals is discussed in Chaps. 10 and 11.

DEFINITION AND EXISTENCE OF THE INTEGRAL

6.1 Definition Let $[a, b]$ be a given interval. By a *partition P* of $[a, b]$ we mean a finite set of points $x_0, x_1, \ldots, x_n$, where

$$a = x_0 \leq x_1 \leq \cdots \leq x_{n-1} \leq x_n = b.$$

We write

$$\Delta x_i = x_i - x_{i-1} \qquad (i = 1, \ldots, n).$$