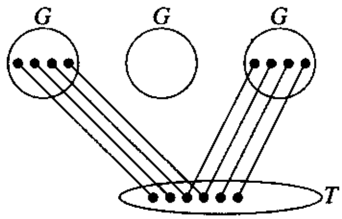
5.2.13. (+) Let G be a k-chromatic graph with girth 6 and order n. Construct G' as follows. Let T be an independent set of kn new vertices. Take $\binom{kn}{n}$ pairwise disjoint copies of G, one for each way to choose an n-set $S \subset T$. Add a matching between each copy of G and its corresponding n-set S. Prove that the resulting graph has chromatic number $k+1$ and girth 6. (Comment: Since C_6 is 2-chromatic with girth 6, the process can start and these graphs exist.) (Blanche Descartes [1947, 1954])



5.2.14. *Chromatic number and cycle lengths.*

a) Let v be a vertex in a graph G. Among all spanning trees of G, let T be one that maximizes $\sum_{u \in V(G)} d_T(u, v)$. Prove that every edge of G joins vertices belonging to a path in T starting at v.

b) Prove that if $\chi(G) > k$, then G has a cycle whose length is one more than a multiple of k. (Hint: Use the tree T of part (a) to define a k-coloring of G.) (Tuza)

5.2.15. (!) Prove that a triangle-free graph with n vertices is colorable with $2\sqrt{n}$ colors. (Comment: Thus every k-chromatic triangle-free graph has at least $k^2/4$ vertices.)

5.2.16. (!) Prove that every n-vertex simple graph with no $r+1$-clique has at most $(1-1/r)n^2/2$ edges. (Hint: This can be proved using Turán's Theorem or by induction on r without Turán's Theorem.)

5.2.17. (!) Let G be a simple n-vertex graph with m edges.

a) Prove that $\omega(G) \ge \lceil n^2/(n^2-2m) \rceil$ and that this bound is sharp. (Hint: Use Exercise 5.2.16. Comment: This also yields $\chi(G) \ge \lceil n^2/(n^2-2m) \rceil$.) (Myers–Liu [1972])

b) Prove that $\alpha(G) \ge \lceil n/(d+1) \rceil$, where d is the average vertex degree of G. (Hint: Use part (a).) (Erdős–Gallai [1961])

5.2.18. The Turán graph $T_{n,r}$ (Example 5.2.7) is the complete r-partite graph with b partite sets of size $a+1$ and $r-b$ partite sets of size a, where $a = \lfloor n/r \rfloor$ and $b = n - ra$.

a) Prove that $e(T_{n,r}) = (1-1/r)n^2/2 - b(r-b)/(2r)$.

b) Since $e(G)$ must be an integer, part (a) implies $e(T_{n,r}) \le \lfloor (1-1/r)n^2/2 \rfloor$. Determine the smallest r such that strict inequality occurs for some n. For this value of r, determine all n such that $e(T_{n,r}) < \lfloor (1-1/r)n^2/2 \rfloor$.

5.2.19. (+) Let $a = \lfloor n/r \rfloor$. Compare the Turán graph $T_{n,r}$ with the graph $\overline{K_a} + K_{n-a}$ to prove directly that $e(T_{n,r}) = \binom{n-a}{2} + (r-1)\binom{a+1}{2}$.

5.2.20. Given positive integers n and k, let $q = \lfloor n/k \rfloor$, $r = n - qk$, $s = \lfloor n/(k+1) \rfloor$, and $t = n - s(k+1)$. Prove that $\binom{q}{2}k + rq \ge \binom{s}{2}(k+1) + ts$. (Hint: Consider the complement of the Turán graph.) (Richter [1993])

5.2.21. Prove that among the n-vertex simple graphs with no $r+1$-clique, the Turán graph $T_{n,r}$ is the *unique* graph having the maximum number of edges. (Hint: Examine the proof of Theorem 5.2.9 more carefully.)

5.2.22. A circular city with diameter four miles will get 18 cellular-phone power stations. Each station has a transmission range of six miles. Prove that no matter where in the city the stations are placed, at least two will each be able to transmit to at least five others. (Adapted from Bondy–Murty [1976, p115])

5.2.23. (!) *Turán's proof of Turán's Theorem*, including uniqueness (Turán [1941]).

a) Prove that a maximal simple graph with no $r+1$-clique has an r-clique.

b) Prove that $e(T_{n,r}) = \binom{r}{2} + (n-r)(r-1) + e(T_{n-r,r})$.

c) Use parts (a) and (b) to prove Turán's Theorem by induction on n, including the characterization of graphs acheiving the bound.

5.2.24. (+) Let $t_r(n) = e(T_{n,r})$. Let G be a graph with n vertices that has $t_r(n) - k$ edges and at least one $r+1$-clique, where $k \ge 0$. Prove that G has at least $f_r(n) + 1 - k$ cliques of order $r+1$, where $f_r(n) = n - \lceil n/r \rceil - r$. (Hint: Prove that a graph with exactly one $r+1$-clique has at most $t_r(n) - f_r(n)$ edges.) (Erdős [1964], Moon [1965c])

5.2.25. *Partial analogue of Turán's Theorem for* $K_{2,m}$.

a) Prove that if G is simple and $\sum_{v\in V(G)} \binom{d(v)}{2} > (m-1)\binom{n}{2}$, then G contains $K_{2,m}$. (Hint: View $K_{2,m}$ as two vertices with m common neighbors.)

b) Prove that $\sum_{v\in V(G)} \binom{d(v)}{2} \ge e(2e/n - 1)$, where G has e edges.

c) Use parts (a) and (b) to prove that a graph with more than $\frac{1}{2}(m-1)^{1/2}n^{3/2} + n/4$ edges contains $K_{2,m}$.

d) Application: Given n points in the plane, prove that the distance is exactly 1 for at most $\frac{1}{\sqrt{2}}n^{3/2} + n/4$ pairs. (Bondy–Murty [1976, p111–112])

5.2.26. For $n \ge 4$, prove that every n-vertex graph with more than $\frac{1}{2}n\sqrt{n-1}$ edges has girth at most 4. (Hint: Use the methods of Exercise 5.2.25

5.2.27. (+) For $n \ge 6$, prove that the maximum number of edges in a simple m-vertex graph not having two edge-disjoint cycles is $n+3$. (Pósa)

5.2.28. (+) For $n \ge 6$, prove that the maximum number of edges in a simple n-vertex graph not having two disjoint cycles is $3n-6$. (Pósa)

5.2.29. (!) Let G be a claw-free graph (no induced $K_{1,3}$).

a) Prove that the subgraph induced by the union of any two color classes in a proper coloring of G consists of paths and even cycles.

b) Prove that if G has a proper coloring using exactly k colors, then G has a proper k-coloring where the color classes differ in size by at most one. (Niessen–Kind [2000])

5.2.30. (+) Prove that if G has a proper coloring g in which every color class has at least two vertices, then G has an optimal coloring f in which every color class has at least two vertices. (Hint: If f has a color class with only one vertex, use g to make an alteration in f. The proof can be given algorithmically or by induction on $\chi(G)$.) (Gallai [1963c])

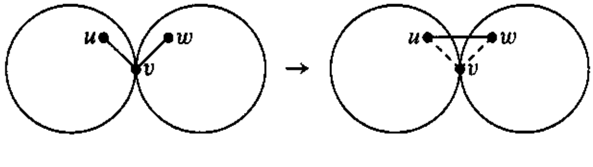
5.2.31. Let G be a connected k-chromatic graph that is not a complete graph or a cycle of length congruent to 3 modulo 6. Prove that every proper k-coloring of G has two vertices of the same color with a common neighbor. (Tomescu)

5.2.32. (!) *The Hajós construction* (Hajós [1961]).

a) Let G and H be k-critical graphs sharing only vertex v, with $vu \in E(G)$ and $vw \in E(H)$. Prove that $(G - vu) \cup (H - vw) \cup uw$ is k-critical.

b) For all $k \ge 3$, use part (a) to obtain a k-critical graph other than K_k.

c) For all $n \ge 4$ except $n = 5$, construct a 4-critical graph with n vertices.



5.2.33. Let G be a k-critical graph having a separating set $S = \{x, y\}$. By Proposition 5.2.18, $x \not\leftrightarrow y$. Prove that G has exactly two S-lobes and that they can be named G_1, G_2 such that $G_1 + xy$ is k-critical and $G_2 \cdot xy$ is k-critical (here $G_2 \cdot xy$ denotes the graph obtained from G_2 by adding xy and then contracting it).

5.2.34. (!) Let G be a 4-critical graph having a separating set S of size 4. Prove that $G[S]$ has at most four edges. (Pritikin)

5.2.35. (+) *Alternative proof that k-critical graphs are k − 1-edge-connected.*

a) Let G be a k-critical graph, with $k \geq 3$. Prove that for every $e, f \in E(G)$ there is a $k-1$-critical subgraph of G containing e but not f. (Toft [1974])

b) Use part (a) and induction on k to prove Dirac's Theorem that every k-critical graph is $k-1$-edge-connected. (Toft [1974])

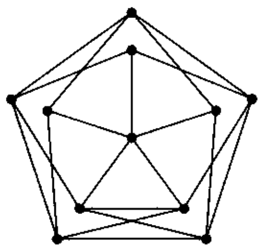
5.2.36. (+) Prove that if G is k-critical and every $k-1$-critical subgraph of G is isomorphic to K_{k-1}, then $G = K_k$ (if $k \geq 4$) (Hint: Use Toft's critical graph lemma—Exercise 5.2.35a.) (Stiebitz [1985])

5.2.37. A graph G is **vertex-color-critical** if $\chi(G - v) < \chi(G)$ for all $v \in V(G)$.

a) Prove that every color-critical graph is vertex-color-critical.

b) Prove that every 3-chromatic vertex-color-critical graph is color-critical.

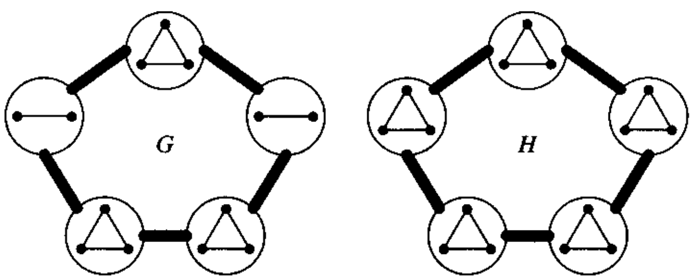
c) Prove that the graph below is vertex-color-critical but not color-critical. (Comment: This is *not* the Grötzsch graph.)



5.2.38. (!) Prove that every simple graph with minimum degree at least 3 contains a K_4-subdivision. (Hint: Prove a stronger result—every nontrivial simple graph with at most one vertex of degree less than 3 contains a K_4-subdivision. The proof of Theorem 5.2.20 already shows that every 3-connected graph contains a K_4-subdivision.) (Dirac [1952a])

5.2.39. (!) Given that $\delta(G) \geq 3$ forces a K_4-subdivision in G, prove that the maximum number of edges in a simple n-vertex graph with no K_4-subdivision is $2n - 3$.

5.2.40. Thick edges below indicate that every vertex in one circle is adjacent to every vertex in the other. Prove that $\chi(G) = 7$ but G has no K_7-subdivision. Prove that $\chi(H) = 8$ but H has no K_8-subdivision. (Catlin [1979])



5.2.41. Let $m = k(k+1)/2$. Prove that $K_{m,m-1}$ has no K_{2k}-subdivision.

5.2.42. (+) Let F be a forest with m edges. Let G be a simple graph such that $\delta(G) \geq m$ and $n(G) \geq n(F)$. Prove that G contains F as a subgraph. (Hint: Delete one leaf from each nontrivial component of F to obtain F'. Let R be the set of neighbors of the deleted vertices. Map R onto an m-set $X \subseteq V(G)$ that minimizes $e(G[X])$. Extend X to a copy of F'. Use Hall's Theorem to show that X can be matched into the remaining vertices to complete a copy of F.) (Brandt [1994])

5.2.43. (+) Let G be a k-chromatic graph. It follows from Lemma 5.1.18 and Proposition 2.1.8 that G contains every k-vertex tree as a subgraph. Strengthen this to a labeled analogue: if f is a proper k-coloring of G and T is a tree with vertex set $\{w_1, \ldots, w_k\}$, then there is an adjacency-preserving map ϕ: $V(T) \to V(G)$ such that $f(\phi(w_i)) = i$ for all i. (Gyárfás–Szemerédi–Tuza [1980], Sumner [1981])

5.2.44. (+) Let G be a k-chromatic graph of girth at least 5. Prove that G contains every k-vertex tree as an induced subgraph. (Gyárfás–Szemerédi–Tuza [1980])

5.3. Enumerative Aspects

Sometimes we can shed light on a hard problem by considering a more general problem. No good algorithm to test existence of a proper k-coloring is known (see Appendix B), but still we can study the number of proper k-colorings (here we fix a particular set of k colors). The chromatic number $\chi(G)$ is the minimum k such that the count is positive; knowing the count for all k would tell us the chromatic number. Birkhoff [1912] introduced this counting problem as a possible way to attack the Four Color Problem (Section 6.3).

In this section, we will discuss properties of the counting function, classes where it is easy to compute, and further related topics.

COUNTING PROPER COLORINGS

We start by defining the counting problem as a function of k.

5.3.1. Definition. Given $k \in \mathbb{N}$ and a graph G, the value $\chi(G; k)$ is the number of proper colorings f: $V(G) \to [k]$. The set of available colors is $[k] = \{1, \ldots, k\}$; the k colors need not all be used in a coloring f. Changing the names of the colors that are used produces a different coloring.

5.3.2. Example. $\chi(\overline{K}_n; k)) = k^n$ *and* $\chi(K_n; k)) = k(k-1)\cdots(k-n+1)$.

When coloring the vertices of $\overline{K}_n$, we can use any of the k colors at each vertex no matter what colors we have used at other vertices. Each of the k^n functions from the vertex set to $[k]$ is a proper coloring, and hence $\chi(\overline{K}_n; k) = k^n$.

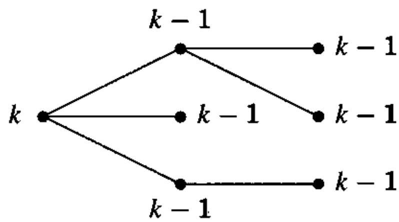
When we color the vertices of K_n, the colors chosen earlier cannot be used on the ith vertex. There remain $k-i+1$ choices for the color of the ith vertex no matter how the earlier colors were chosen. Hence $\chi(K_n; k) = k(k-1)\cdots(k-n+1)$.

We can also count this as $\binom{k}{n}n!$ by first choosing n distinct colors and then multiplying by $n!$ to count the ways to assign the chosen colors to the vertices. For example, $\chi(K_3; 3) = 6$ and $\chi(K_3; 4) = 24$.

The value of the product is 0 when $k < n$. This makes sense, since K_n has no proper k-colorings when $k < n$. ■

5.3.3. Proposition. If T is a tree with n vertices, then $\chi(T; k) = k(k-1)^{n-1}$.

Proof: Choose some vertex v of T as a root. We can color v in k ways. If we extend a proper coloring to new vertices as we grow the tree from v, at each step only the color of the parent is forbidden, and we have $k-1$ choices for the color of the new vertex. Furthermore, deleting a leaf shows inductively that every proper k-coloring arises in this way. Hence $\chi(T; k) = k(k-1)^{n-1}$. ■



Another way to count the colorings is to observe that the color classes of each proper coloring of G partition $V(G)$ into independent sets. Grouping the colorings according to this partition leads to a formula for $\chi(G; k)$ that is a polynomial in k of degree $n(G)$. Note that this holds for the answers in Example 5.3.2 and Proposition 5.3.3. Since every graph has this property, $\chi(G; k)$ as a function of k is called the **chromatic polynomial** of G.

5.3.4. Proposition. Let $x_{(r)} = x(x-1)\cdots(x-r+1)$. If $p_r(G)$ denotes the number of partitions of $V(G)$ into r nonempty independent sets, then $\chi(G; k) = \sum_{r=1}^{n(G)} p_r(G)k_{(r)}$, which is a polynomial in k of degree $n(G)$.

Proof: When r colors are actually used in a proper coloring, the color classes partition $V(G)$ into exactly r independent sets, which can happen in $p_r(G)$ ways. When k colors are available, there are exactly $k_{(r)}$ ways to choose colors and assign them to the classes. All the proper colorings arise in this way, so the formula for $\chi(G; k)$ is correct.

Since $k_{(r)}$ is a polynomial in k and $p_r(G)$ is a constant for each r, this formula implies that $\chi(G; k)$ is a polynomial function of k. When G has n vertices, there is exactly one partition of G into n independent sets and no partition using more sets, so the leading term is k^n. ■

5.3.5. Example. Always $p_n(G) = 1$, using independent sets of size 1. Also $p_1(G) = 0$ unless G has no edges, since only for $\overline{K}_n$ is the entire vertex set an independent set.

Consider $G = C_4$. There is exactly one partition into two independent sets: opposite vertices must be in the same independent set. When $r = 3$, we put two opposite vertices together and leave the other two in sets by themselves; we can do this in two ways. Thus $p_2 = 1$, $p_3 = 2$, $p_4 = 1$.

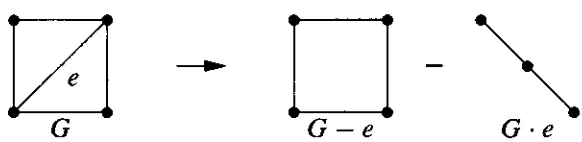
$$\begin{aligned}\chi(C_4; k) &= 1 \cdot k(k-1) + 2 \cdot k(k-1)(k-2) + 1 \cdot k(k-1)(k-2)(k-3) \\ &= k(k-1)(k^2 - 3k + 3).\end{aligned}$$ ∎

Computing the chromatic polynomial in this way is not generally feasible, since there are too many partitions to consider. There is a recursive computation much like that used in Proposition 2.2.8 to count spanning trees. Again $G \cdot e$ denotes the graph obtained by contracting the edge e in G (Definition 2.2.7). Since the number of proper k-colorings is unaffected by multiple edges, **we discard multiple copies of edges that arise from the contraction**, keeping only one copy of each to form a simple graph.

5.3.6. Theorem. (Chromatic recurrence) If G is a simple graph and $e \in E(G)$, then $\chi(G; k) = \chi(G - e; k) - \chi(G \cdot e; k)$.

Proof: Every proper k-coloring of G is a proper k-coloring of $G - e$. A proper k-coloring of $G - e$ is a proper k-coloring of G if and only if it gives distinct colors to the endpoints u, v of e. Hence we can count the proper k-colorings of G by subtracting from $\chi(G - e; k)$ the number of proper k-colorings of $G - e$ that give u and v the same color.

Colorings of $G - e$ in which u and v have the same color correspond directly to proper k-colorings of $G \cdot e$, in which the color of the contracted vertex is the common color of u and v. Such a coloring properly colors all the edges of $G \cdot e$ if and only if it properly colors all the edges of G other than e. ∎



5.3.7. Example. *Proper k-colorings of C_4.* Deleting an edge of C_4 produces P_4, while contracting an edge produces K_3. Since P_4 is a tree and K_3 is a complete graph, we have $\chi(P_4; k) = k(k-1)^3$ and $\chi(K_3; k) = k(k-1)(k-2)$. Using the chromatic recurrence, we obtain

$$\chi(C_4; k) = \chi(P_4; k) - \chi(K_3; k) = k(k-1)(k^2 - 3k + 3).$$ ∎

Because both $G - e$ and $G \cdot e$ have fewer edges than G, we can use the chromatic recurrence inductively to compute $\chi(G; k)$. We need initial conditions for graphs with no edges, which we have already computed: $\chi(\overline{K}_n; k) = k^n$.

5.3.8. Theorem. (Whitney [1933c]) The chromatic polynomial $\chi(G;k)$ has degree $n(G)$, with integer coefficients alternating in sign and beginning $1, -e(G), \cdots$.

Proof: We use induction on $e(G)$. The claims hold trivially when $e(G) = 0$, where $\chi(\overline{K}_n;k) = k^n$. For the induction step, let G be an n-vertex graph with $e(G) \ge 1$. Each of $G-e$ and $G\cdot e$ has fewer edges than G, and $G\cdot e$ has $n-1$ vertices. By the induction hypothesis, there are nonnegative integers $\{a_i\}$ and $\{b_i\}$ such that $\chi(G-e;k) = \sum_{i=0}^{n}(-1)^i a_i k^{n-i}$ and $\chi(G\cdot e;k) = \sum_{i=0}^{n-1}(-1)^i b_i k^{n-1-i}$. By the chromatic recurrence,

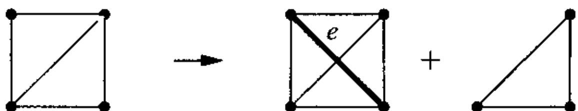
$$
\begin{array}{rlllll}
\chi(G-e;k): & k^n & -\ [e(G)-1]k^{n-1} & +\ a_2k^{n-2} & - \cdots + & (-1)^i \quad a_ik^{n-i}\cdots \\
-\ \chi(G\cdot e;k): & -(& & k^{n-1} - b_1k^{n-2} & + \cdots + & (-1)^{i-1} \quad b_{i-1}k^{n-i}\cdots) \\
\hline
=\quad \chi(G;k): & & k^n - e(G)k^{n-1} & + (a_2+b_1)k^{n-2} & - \cdots + & (-1)^i(a_i+b_{i-1})k^{n-i}\cdots
\end{array}
$$

Hence $\chi(G;k)$ is a polynomial with leading coefficient $a_0 = 1$ and next coefficient $-(a_1+b_0) = -e(G)$, and its coefficients alternate in sign. ∎

5.3.9. Example. When adding an edge yields a graph whose chromatic polynomial is easy to compute, we can use the chromatic recurrence in a different way. Instead of $\chi(G;k) = \chi(G-e;k) - \chi(G\cdot e;k)$, we can write $\chi(G-e;k) = \chi(G;k) + \chi(G\cdot e;k)$. Thus we may be able to compute $\chi(G-e;k)$ using $\chi(G;k)$.

To compute $\chi(K_n - e;k)$, for example, we let G be K_n in this alternative formula and obtain

$$\chi(K_n - e;k) = \chi(K_n;k) + \chi(K_{n-1};k) = (k-n+2)^2\prod_{i=0}^{n-3}(k-i). \qquad ∎$$



$G-e$ $\quad$ G $\quad$ $G\cdot e$

We close our general discussion of $\chi(G;k)$ with an explicit formula. It has exponentially many terms, so its uses are primarily theoretical. The formula summarizes what happens if we iterate the chromatic recurrence until we dispose of all the edges.

5.3.10. Theorem. (Whitney [1932b]) Let $c(G)$ denote the number of components of a graph G. Given a set $S \subseteq E(G)$ of edges in G, let $G(S)$ denote the spanning subgraph of G with edge set S. Then the number $\chi(G;k)$ of proper k-colorings of G is given by

$$\chi(G;k) = \sum_{S\subseteq E(G)}(-1)^{|S|}k^{c(G(S))}$$

Proof: In applying the chromatic recurrence, contraction may produce multiple edges. We have observed that dropping these does not affect $\chi(G;k)$. We claim that deleting extra copies of edges also does not change the claimed formula.

Let e and e' be edges in G with the same endpoints. When $e' \in S$ and $e \notin S$, we have $c(G(S \cup \{e\})) = c(G(S))$, since both endpoints of e are in the same component of $G(S)$. However, $|S \cup \{e\}| = |S| + 1$. Thus the terms for S and $S \cup \{e\}$ in the sum cancel. Therefore, omitting all terms for sets of edges containing e' does not change the sum. This implies that we can keep or drop e' from the graph without changing the formula.

When computing the chromatic recurrence, we therefore obtain the same result if we do not discard multiple edges or loops and instead retain all edges for contraction or deletion. Iterating the recurrence now yields $2^{e(G)}$ terms as we dispose of all the edges; each in turn is deleted or contracted.

When all edges have been deleted or contracted, the graph that remains consists of isolated vertices. Let S be the set of edges that were contracted. The remaining vertices correspond to the components of $G(S)$; each such component becomes one vertex when the edges of S are contracted and the other edges are deleted. The $c(G(S))$ isolated vertices at the end yield a term with $k^{c(G(S))}$ colorings. Furthermore, the sign of the contribution changes for each contracted edge, so the contribution is positive if and only if $|S|$ is even.

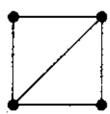
Thus the contribution when S is the set of contracted edges is $(-1)^{|S|}k^{c(G(S))}$, and this accounts for all terms in the sum. ■

5.3.11. Example. *A chromatic polynomial.* When G is a simple graph with n vertices, every spanning subgraph with 0, 1, or 2 edges has n, $n-1$, or $n-2$ components, respectively. When $|S| = 3$, the number of components is $n-2$ if and only if the three edges form a triangle; otherwise it is $n-3$.

For example, when G is a kite (four vertices, five edges) there are ten sets of three edges. For two of these, $G(S)$ consists of a triangle plus one isolated vertex. The other eight sets of three edges yield spanning subgraphs with one component. Both types of triples are counting negatively, since $|S| = 3$. All spanning subgraphs with four or five edges have only one component. Hence Theorem 5.3.10 yields

$$\chi(G;k) = k^4 - 5k^3 + 10k^2 - (2k^2 + 8k^1) + 5k - k = k^4 - 5k^3 + 8k^2 - 4k.$$

This agrees with $\chi(G;k) = k(k-1)(k-2)(k-2)$, computed by counting colorings directly or by using $\chi(G;k) = \chi(C_4;k) - \chi(P_3;k)$. ■



Whitney proved Theorem 5.3.10 using the inclusion-exclusion principle of elementary counting. Among the universe of k-colorings, the proper colorings are those not assigning the same color to the endpoints of any edge. Letting A_i be the set of k-colorings assigning the same color to the endpoints of edge e_i, we want to count the colorings that lie in none of $A_1, \ldots, A_m$ (see Exercise 17).

CHORDAL GRAPHS

Counting colorings is easy for cliques and trees (and the kite) because each such graph arises from K_1 by successively adding a vertex joined to a clique. The chromatic polynomial of such a graph is a product of linear factors.

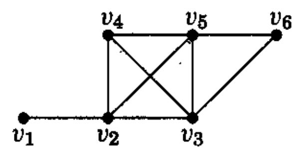
5.3.12. Definition. A vertex of G is **simplicial** if its neighborhood in G induces a clique. A **simplicial elimination ordering** is an ordering $v_n, \ldots, v_1$ for deletion of vertices so that each vertex v_i is a simplicial vertex of the remaining graph induced by $\{v_1, \ldots, v_i\}$. (These orderings are also called **perfect elimination orderings.**)

5.3.13. Example. *Chromatic polynomials from simplicial elimination orderings.* In a tree, a simplicial elimination ordering is a successive deletion of leaves. We have observed that $\chi(G; k) = k(k-1)^{n-1}$ when G is an n-vertex tree.

When $v_n, \ldots, v_1$ is a simplicial elimination ordering for G, the product rule of elementary combinatorics (Appendix A) allows us to count proper k-colorings of G. If we have colored $v_1, \ldots, v_i$, then when we add v_i there are $k - d(i)$ ways to color it, where $d(i) = |N(v_i) \cap \{v_1, \ldots, v_{i-1}\}|$. The factor $k - d(i)$ is independent of how previous color choices were made, because the neighbors of v_i that have been colored form a clique of size $d(i)$ and have distinct colors.

Deleting a simplicial vertex that starts a simplicial elimination ordering yields inductively that every proper k-coloring of G arises in this way. Thus we have expressed the chromatic polynomial as a product of linear factors.

In the graph below, $v_6, \ldots, v_1$ is a simplicial elimination ordering. When we form the graph in the order $v_1, \ldots, v_6$, the values $d(1), \ldots, d(6)$ are 0, 1, 1, 2, 3, 2, and the chromatic polynomial is $k(k-1)(k-1)(k-2)(k-3)(k-2)$. ■



5.3.14. Remark. It is important to note that some graphs without simplicial elimination orderings also have chromatic polynomials that can be expressed as a product of linear factors of the form $k - r_i$ with r_i a nonnegative integer. Exercise 19 presents an example. Thus the existence of a simplicial elimination ordering is a sufficient but not necessary condition for the chromatic polynomial to have this nice factorization property. ■

Trees, cliques, near-complete graphs ($K_n - e$), and interval graphs (Exercise 28) all have simplicial elimination orderings. When $n \geq 3$, the cycle C_n has no simplicial elimination ordering, because a cycle has no simplicial vertex to start the elimination. The existence of simplicial elimination orderings is equivalent to the absence of such cycles as induced subgraphs.

5.3.15. Definition. A **chord** of a cycle C is an edge not in C whose endpoints lie in C. A **chordless cycle** in G is a cycle of length at least 4 in G that has no chord (that is, the cycle is an induced subgraph). A graph G is **chordal** if it is simple and has no chordless cycle.

The motivation for the term "chord" is geometric. If a cycle is drawn with its vertices in order on a circle and its chords are drawn as line segments, then the chords of the cycle are chords of the circle.

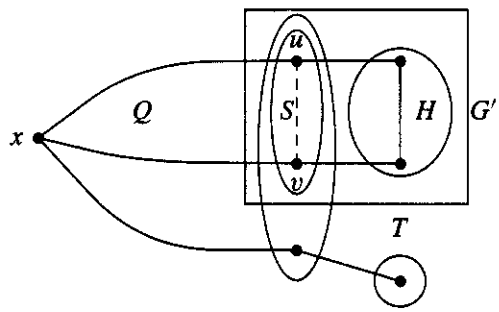
It is fairly easy to show that a graph with a simplicial elimination ordering cannot have a chordless cycle. Thus our characterization of these graphs is another TONCAS theorem. We separate the substantive part of the proof of sufficiency as a lemma that is useful on its own (see also Laskar–Shier [1983]).

5.3.16. Lemma. (Voloshin [1982], Farber–Jamison [1986]) For every vertex x in a chordal graph G, there is a simplicial vertex of G among the vertices farthest from x in G

Proof: We use induction on $n(G)$. Basis step ($n(G) = 1$): The one vertex in K_1 is simplicial.

Induction step ($n(G) \geq 2$): If x is adjacent to all other vertices, then we apply the induction hypothesis to the chordal graph $G - x$. Each simplicial vertex y of $G - x$ is also simplicial in G, since x is adjacent to all of $N(y) \cup \{y\}$.

Otherwise, let T be the set of vertices in G with maximum distance from x, and let H be a component of $G[T]$. Let S be the set of vertices in $G - T$ having neighbors in $V(H)$, and let Q be the component of $G - S$ containing x.

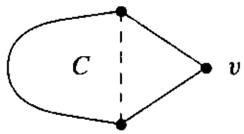


We claim that S is a clique. Each vertex of S has a neighbor in $V(H)$ and a neighbor in Q. For distinct vertices $u, v \in S$, the union of shortest u, v-paths through H and through Q is a cycle of length at least 4. Since there are no edges from $V(H)$ to $V(Q)$, this cycle has no chord other than uv. Since G has no chordless cycle, $u \leftrightarrow v$. Since $u, v \in S$ were chosen arbitrarily, S is a clique.

Now let $G' = G[S \cup V(H)]$; this omits x and thus is smaller than G. We apply the induction hypothesis to G' and a vertex $u \in S$. Since S is a clique, $S - \{u\} \subseteq N(u)$. Whether G' is a clique or not, it thus has a simplicial vertex z within $V(H)$. Since $N_G(z) \subseteq V(G')$, the vertex z is also simplicial in G, and z is a vertex with maximum distance from x, as desired. ■

5.3.17. Theorem. (Dirac [1961]) A simple graph has a simplicial elimination ordering if and only if it is a chordal graph.

Proof: *Necessity.* Let G be a graph with a simplicial elimination ordering. Let C be a cycle in G of length at least 4. At the point when the elimination ordering first deletes a vertex of C, say v, the remaining neighbors of v form a clique. The clique includes the neighbors of v on C; the resulting edge joining them is a chord of C. Hence G has no chordless cycle.



Sufficiency. By Lemma 5.3.16, every chordal graph has a simplicial vertex. This yields a simplicial elimination ordering by induction on $n(G)$, since every induced subgraph of a chordal graph is a chordal graph. ■

Other properties of chordal graphs appear in Exercises 20–27.

A HINT OF PERFECT GRAPHS

In Proposition 5.1.16, we proved that $\chi(G) = \omega(G)$ when G is an interval graph. Furthermore, every induced subgraph of an interval graph is also an interval graph, since we can delete the interval representing v in an interval representation of G to obtain an interval representation of $G - v$. Thus $\chi(H) = \omega(H)$ holds for every induced subgraph H of an interval graph.

5.3.18. Definition. A graph G is **perfect** if $\chi(H) = \omega(H)$ for every induced subgraph $H \subseteq G$. Equivalently, $\chi(G[A]) = \omega(G[A])$ for all $A \subseteq V(G)$.

The **clique cover number** $\theta(G)$ of a graph G is the minimum number of cliques in G needed to cover $V(G)$; note that $\theta(G) = \chi(\overline{G})$.

Since cliques and independent sets exchange roles under complementation, the statement of perfection for $\overline{G}$ is "$\alpha(H) = \theta(H)$ for every induced subgraph H of G". Lovász [1972a, 1972b] proved the **Perfect Graph Theorem** (PGT): G is perfect if and only if its complement $\overline{G}$ is perfect. We prove this in Theorem 8.1.6; here we merely illustrate perfect graphs.

5.3.19. Definition. A family of graphs **G** is **hereditary** if every induced subgraph of a graph in **G** is also a graph in **G**.

5.3.20. Remark. In order to prove that every graph in a hereditary class **G** is perfect, it suffices to verify that $\chi(G) = \omega(G)$ for every $G \in$ **G**. Doing so includes the proof of equality for the induced subgraphs of G. ■

5.3.21. Example. *Bipartite graphs and their line graphs.* Bipartite graphs form a hereditary class, and $\chi(G) = \omega(G)$ for every bipartite graph; hence bipartite graphs are perfect. When H is bipartite, the statement of perfection for $\overline{H}$ is Exercise 5.1.38 and follows from $\alpha(H) = \beta'(H)$ (Corollary 3.1.24). For bipartite graphs, the nontrivial $\alpha(G) = \theta(G) = \beta'(G)$ follows at once from the trivial $\chi(G) = \omega(G)$ by the PGT.

We briefly introduced line graphs in Definition 4.2.18 to prove the edge versions of Menger's Theorem; recall that the line graph $L(G)$ has a vertex for each edge of G, with $e, f \in V(L(G))$ adjacent in $L(G)$ if they have a common endpoint in G. Line graphs of bipartite graphs form a hereditary family, since deleting a vertex in the line graph represents deleting the corresponding edge in the original graph.

Therefore, proving that $\alpha(L(G)) = \theta(L(G))$ when G is bipartite will show that complements of line graphs are perfect. A clique in $L(G)$ (when G is bipartite) consists of edges in G with a common endpoint. Thus covering the vertices of $L(G)$ with cliques corresponds to selecting vertices in G to form a vertex cover. Independent sets in $L(G)$ are matchings in G. Thus perfection for complements of line graphs of bipartite graphs amounts to the König–Egerváry Theorem ($\alpha'(G) = \beta(G)$) for matchings and vertex covers in bipartite graphs.

From this the PGT yields also $\chi(L(G)) = \omega(L(G))$. A proper coloring of $L(G)$ is a partition of $E(G)$ into matchings, and $\omega(L(G)) = \Delta(G)$ (for bipartite G). Hence $\chi(L(G)) = \omega(L(G))$ means that the edges of a bipartite graph G can be partitioned into $\Delta(G)$ matchings. In Theorem 7.1.7, we prove directly this additional result of König [1916]. ■

Since every interval graph is a chordal graph (Exercise 28), proving that all chordal graphs are perfect strengthens Proposition 5.1.16. We explore other characterizations of interval graphs and chordal graphs in Section 8.1.

5.3.22. Theorem. (Berge [1960]) Chordal graphs are perfect.

Proof: Deleting vertices cannot create chordless cycles, so the family is hereditary. By Remark 5.3.20, we need only prove $\chi(G) = \omega(G)$ when G is chordal.

In Theorem 5.3.17, we proved that G has a simplicial elimination ordering. Let $v_1, \ldots, v_n$ be the reverse of such an ordering. For each i, the neighbors of v_i among $\{v_1, \ldots, v_{i-1}\}$ form a clique.

We apply greedy coloring with this ordering. If v_i receives color k, then colors $1, \ldots, k-1$ appear on earlier neighbors of v_i. Since they form a clique, with v_i we have a clique of size k. Thus we obtain a clique whose size equals the number of colors used. ■

The argument of Theorem 5.3.22 shows that greedy coloring relative to the reverse of a simplicial elimination ordering produces an optimal coloring. This generalizes Proposition 5.1.16 about interval graphs.

We present one more fundamental class of perfect graphs; it includes all bipartite graphs.

5.3.23.* Definition. A **transitive orientation** of a graph G is an orientation D such that whenever xy and yz are edges in D, also there is an edge xz in G that is oriented from x to z in D. A simple graph G is a **comparability graph** if it has a transitive orientation.

5.3.24.* Example. If G is an X, Y-bigraph, then directing every edge from X to Y yields a transitive orientation. Thus every bipartite graph is a comparability graph. Transitive orientations arise from order relations; $x \to y$ could mean "x contains y", which is a transitive relation. ■

5.3.25.* Proposition. (Berge [1960]) Comparability graphs are perfect.

Proof: Every induced subdigraph of a transitive digraph is transitive, so the class of comparability graphs is hereditary. Thus we need only show that each comparability graph G is $\omega(G)$-colorable.

Let F be a transitive orientation of G; note that F has no cycle. As shown in proving Theorem 5.1.21, the coloring of G that assigns to each vertex v the number of vertices in the longest path of F ending at v is a proper coloring. By transitivity, the vertices of a path in F form a clique in G. Thus we have $\chi(G) \le \omega(G)$. ■

COUNTING ACYCLIC ORIENTATIONS (optional)

Surprisingly, $\chi(G; k)$ has meaning when k is a negative integer. An **acyclic orientation** of a graph is an orientation having no cycle. Setting $k = -1$ in $\chi(G; k)$ enables us to count the acyclic orientations of G.

5.3.26. Example. Since C_4 has 4 edges, it has 16 orientations. Of these, 14 are acyclic. In Example 5.3.7, we proved that $\chi(C_4; k) = k(k-1)(k^2 - 3k + 3)$. Evaluated at $k = -1$, this equals $(-1)(-2)(7) = 14$. ■

5.3.27. Theorem. (Stanley [1973]) The value of $\chi(G; k)$ at $k = -1$ is $(-1)^{n(G)}$ times the number of acyclic orientations of G.

Proof: We use induction on $e(G)$. Let $a(G)$ be the number of acyclic orientations of G. When G has no edges, $a(G) = 1$ and $\chi(G; -1) = (-1)^{n(G)}$, so the claim holds. We will prove that $a(G) = a(G - e) + a(G \cdot e)$ for $e \in E(G)$. If so, then we apply the recurrence for a, the induction hypothesis for $a(G)$ in terms of $\chi(G; k)$, and the chromatic recurrence to compute

$$a(G) = (-1)^{n(G)}\chi(G - e; -1) + (-1)^{n(G)-1}\chi(G \cdot e; -1) = (-1)^{n(G)}\chi(G; -1).$$

Now we prove the recurrence for a. Every acyclic orientation of G contains an acyclic orientation of $G - e$. An acyclic orientation D of $G - e$ may extend to 0, 1, or 2 acyclic orientations of G by orienting the edge $e = uv$. When D has no u, v-path, we can choose $v \to u$. When D has no v, u-path, we can choose $u \to v$. Since D is acyclic, D cannot have both a u, v-path and a v, u-path, so the two choices for e cannot both be forbidden.

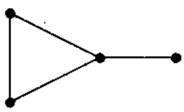
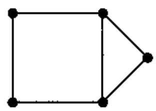
Hence every D extends in at least one way, and $a(G)$ equals $a(G-e)$ plus the number of orientations that extend in both ways. Those extending in both ways are the acyclic orientations of $G - e$ with no u, v-path *and* no v, u-path. There are exactly $a(G \cdot e)$ of these, since a u, v-path or a v, u-path in an orientation of $G - e$ becomes a cycle in $G \cdot e$. ∎

The interpretation of $\chi(G; k)$ for general negative k (Exercise 32) is an instance of the phenomenon of "combinatorial reciprocity" (Stanley [1974]).

EXERCISES

Keep in mind that the notation $\chi(G; k)$ may be viewed as a polynomial or as the number of proper k-colorings of G.

5.3.1. (−) Compute the chromatic polynomials of the graphs below.

5.3.2. (−) Use the chromatic recurrence to obtain the chromatic polynomial of every tree with n vertices.

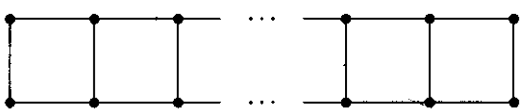
5.3.3. (−) Prove that $k^4 - 4k^3 + 3k^2$ is not a chromatic polynomial.

• • • • •

5.3.4. a) Prove that $\chi(C_n; k) = (k-1)^n + (-1)^n(k-1)$.

b) For $H = G \vee K_1$, prove that $\chi(F; k) = k\chi(G; k-1)$. From this and part (a), find the chromatic polynomial of the wheel $C_n \vee K_1$.

5.3.5. For $n \geq 1$, let $G_n = P_n \,\square\, K_2$; this is the graph with $2n$ vertices and $3n-2$ edges shown below. Prove that $\chi(G_n; k) = (k^2 - 3k + 3)^{n-1}k(k-1)$.



5.3.6. (!) Let G be a graph with n vertices. Use Proposition 5.3.4 to give a non-inductive proof that the coefficient of k^{n-1} in $\chi(G; k)$ is $-e(G)$.

5.3.7. Prove that the chromatic polynomial of an n-vertex graph has no real root larger than $n-1$. (Hint: Use Proposition 5.3.4.)

5.3.8. (!) Prove that the number of proper k-colorings of a connected graph G is less than $k(k-1)^{n-1}$ if $k \geq 3$ and G is not a tree. What happens when $k = 2$?

5.3.9. (!) Prove that $\chi(G; x + y) = \sum_{U \subseteq V(G)} \chi(G[U]; x)\chi(G[\overline{U}]; y)$. (Hint: Since both sides are polynomials, it suffices to prove equality when x and y are positive integers; do this by counting proper $x + y$-colorings in a different way.)