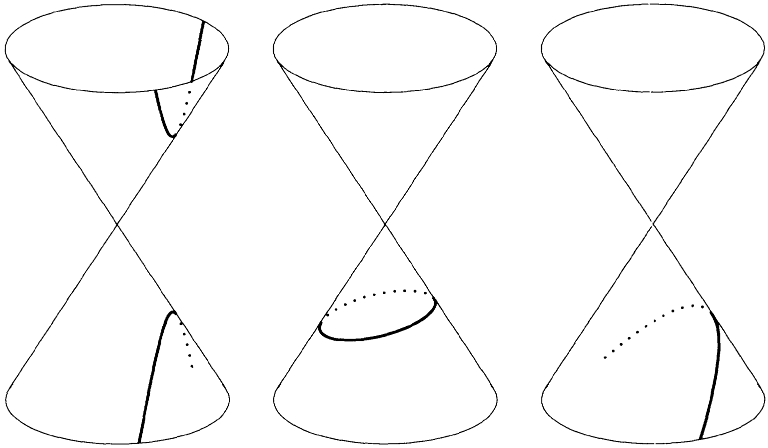

FIGURE 8.2 Hyperbola, ellipse, and parabola as sections of a cone.

slopes "too much" and cuts both halves of the cone, ellipses occur when the cutting plane slopes "too little," and cuts the cone in only a finite curve.

The geometric difference between hyperbola, ellipse, and parabola can also be recognized by algebra. Because the cone is symmetrical about the z-axis, there is no loss of generality in assuming the cutting plane to be perpendicular to the (y, z)-plane, so its equation is of the form $cy+dz = e$. In fact, dividing by the coefficient of z (or by the coefficient of y if the coefficient of z is zero), we get the equation of the cutting plane to be either

$$cy + z = e \qquad \text{or} \qquad y = e.$$

If the latter, we substitute $y = e$ in the equation to the cone and get

$$k^2z^2 - x^2 = e^2$$

as the equation of the resulting hyperbola. If the former, we substitute $z = e - cy$ in the equation $x^2 + y^2 = k^2z^2$ of the cone and get

$$x^2 + y^2(1 - k^2c^2) + 2k^2cey = k^2e^2.$$

Thus the coefficient of y^2 depends on the slope $-c$ of the cutting plane. It is

- Less than 0 if $c^2 > k^2$, which happens if the conic section is a hyperbola.
- Greater than 0 if $c^2 < k^2$, which happens if the conic section is an ellipse.
- Equal to 0 if $c^2 = k^2$, which happens if the conic section is a parabola.

The equation $x^2+y^2(1-k^2c^2)+2k^2cey = k^2e^2$ is the relation between x and y on the conic section, so it is really the equation of the *projection* of the curve in the (x, y)-plane. However, if we introduce coordinates in the cutting plane itself, the only change is to multiply the y-coordinate by a constant factor. It remains true that the coefficients of x^2 and y^2 have opposite signs for a hyperbola, they have the same sign for an ellipse, and there is no y^2 term for a parabola.

We can rewrite the equation

$$x^2 + y^2(1 - k^2c^2) + 2k^2cey = k^2e^2$$

in the form

$$x^2 = Dy + C^2 \qquad \text{or} \qquad x^2 + A(y^2 + 2By) = C^2$$

according as $1 - k^2c^2 = 0$ or not. Both these equations can be simplified by replacing y by y plus a suitable constant, that is, by a change of origin. The first becomes

$$x^2 = Dy \qquad \text{when } y \text{ is replaced by } y - C^2/D,$$

and the second becomes ("completing the square")

$$x^2 + Ay^2 = C^2 - AB^2 \qquad \text{when } y \text{ is replaced by } y - B.$$

With a little further tidying—dividing through to make the constant term 1, writing positive coefficients as squares and negative coefficients as negatives of squares—we finally obtain the simplest possible equations for the conic sections:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{(hyperbola)}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{(ellipse)}$$

$$y = ax^2 \qquad \text{(parabola)},$$

where a and b are nonzero constants.

Exercises

It is possible to check that all nonzero values of a and b actually arise from sections of cones, though this is a little tedious to do directly. In the case of the parabola, it is better to do the following.

8.1.1. Show that all parabolas have the same shape. In particular, if x, y are replaced by cx, cy for a suitable constant c, show that the equation $y = ax^2$ becomes $y = x^2$.

The shape of a hyperbola or ellipse is determined by the ratio of the coefficients of x^2 and y^2.

8.1.2. Show that hyperbolas of arbitrary shape occur as vertical sections of the cone $x^2 + y^2 = k^2z^2$ as k varies.

8.1.3. Show that ellipses of arbitrary shape come from cutting the cone $x^2 + y^2 = z^2$ by suitable planes $y - dz = 1$.

Ellipses can also be obtained as sections of a circular cylinder.

8.1.4. Write down the equation of a circular cylinder symmetric about the z-axis, and find the equation of its intersection with the plane $y = mz$, using suitable coordinates in the latter.

8.2 Properties of Conic Sections

The conic sections have many interesting properties, and in this book we can mention only a few, as we wish to concentrate on our main theme, the relations between numbers, geometry, and functions. However, it is impossible to resist a brief look at some of the properties that make conic sections physically significant. These properties are not closely related to the cone; they come to light when the conic sections are seen from a different geometric viewpoint.

A conic section C can be defined in terms of a point F called its *focus* and a line $\mathcal{D}$ called its *directrix*. C is simply the set of points whose distance from F is a constant multiple of its perpendicular distance from $\mathcal{D}$ (Figure 8.3). The multiple is called the *eccentricity*, e.

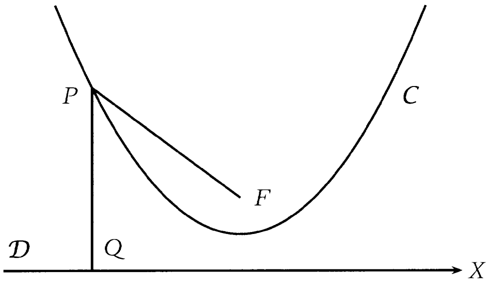

FIGURE 8.3 Focus and directrix of a conic section.

We take $\mathcal{D}$ to be the x-axis and F to be the point $(0, 1)$ for convenience. Then if $P = (x, y)$ we have

$$FP = \sqrt{x^2 + (y-1)^2} \quad \text{and} \quad PQ = y,$$

and therefore

$$\sqrt{x^2 + (y-1)^2} = ey.$$

Squaring both sides gives

$$x^2 + (y-1)^2 = e^2y^2,$$

and therefore

$$x^2 + (1-e^2)y^2 - 2y + 1 = 0.$$

For $e = 1$ this is the parabola $2y = x^2 + 1$. For $e < 1$ it is an ellipse, and for $e > 1$ a hyperbola, as may be checked by completing the square and shifting the origin.

The focus is physically significant in Newton's theory of gravitation, because planets and comets travel on conic sections with the sun at their focus. (Of course, this is an idealization of the real situation. The *mathematical* situation, which closely approximates the real one, assumes two point masses with an inverse square law of attraction. Then if one mass is taken as the origin of coordinates, the other moves along a conic section with the origin as its focus.) This is a very famous result, but it would be a big detour for us to prove it. Instead, we shall prove another important property of the focus, which is the reason for its name, the Latin word for *fireplace*.[1] Kepler

[1]This meaning is also evident in the word for focus used in German, "Brennpunkt," meaning burning point.

gave it this name, knowing that if rays from a distant source fall directly onto a parabolic mirror, they are all reflected to the focus, and hence heat is concentrated there.

Focal property of the parabola. *Lines parallel to the axis of symmetry of a parabola are reflected through the focus*

Proof Given any point P on the parabola, consider the perpendicular from P to Q on the directrix (Figure 8.4). Then if F is the focus, the focus-directrix property gives

$$FP = PQ.$$

It follows that the equidistant line $\mathcal{T}$ of F and Q meets the parabola at P. We wish to show that $\mathcal{T}$ does not meet the parabola at any other point, so that $\mathcal{T}$ is the tangent at P.

If, on the contrary, $\mathcal{T}$ meets the parabola at a second point P', then F and Q are also equidistant from P', so $FP' = P'Q$. But the focus-directrix property of P' says that $FP' = P'Q'$, where $Q' \neq Q$ is the perpendicular projection of P' onto the directrix. This implies $P'Q = P'Q'$, which is contrary to Pythagoras' theorem.

Thus $\mathcal{T}$ is indeed the tangent at P. Because $\mathcal{T}$ is also the equidistant line of F and Q, the angles marked ϕ in Figure 8.4 are equal, and hence so are the angles marked θ, where $\mathcal{N}$ is the normal at P. In particular, this means that the vertical line striking the inside of the parabola at P is reflected through the focus F. □

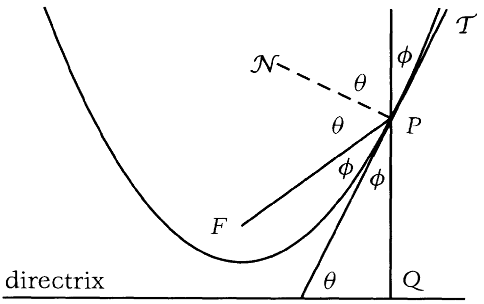

FIGURE 8.4 Tangent, normal, focus, and directrix of the parabola.

Exercises

The focal property of the parabola was first proved by the Greek mathematician Diocles, in a book *On Burning Mirrors* written around 200 B.C. As the title suggests, Diocles was aware of the potential applications of the theorem, and there is indeed a story that Archimedes used burning mirrors against Roman ships. It is probably only a legend, however; the Greeks did not care much for practical applications of geometry, and this one is of doubtful practicality in any case.

8.2.1. Comment on the feasibility of building a parabolic mirror to burn ships.

The most comprehensive ancient book on conic sections is the *Conics* of Apollonius. It was also written around 200 B.C., but it does not mention the focus of the parabola, although it includes a proof of a more difficult focal property of the ellipse.

To give a modern proof of this theorem, it helps to know that the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ has eccentricity $e = \sqrt{1 - b^2/a^2}$, foci at $(\pm ae, 0)$, and directrices $x = \pm a/e$. (There are two of each because of the obvious symmetry of the ellipse.) Once this is known, it is relatively easy to check the focus-directrix property.

8.2.2. Check that the distance from $(ae, 0)$ to any point $P = (x, y)$ on the ellipse is e times the distance of P from the line $x = a/e$.

The focus-directrix property has a practical consequence known as the "thread construction of an ellipse."

8.2.3. Deduce from Exercise 8.2.2 that the sum of distances from the foci to any point on the ellipse is constant.

Thus if a length of thread is tied to a pair of nails at the foci F_1 and F_2 (Figure 8.5) and pulled tight by a pencil at P, then the pencil will draw an ellipse.

The focal property of the ellipse found by Apollonius states that the lines F_1P and F_2P make equal angles with the tangent at P. To prove this, let $\mathcal{T}$ be the line through P that *does* make equal angles with F_2P and F_1P. The problem then is to show that $\mathcal{T}$ does not meet the ellipse at a second point P', so that $\mathcal{T}$ is the tangent.

The latter problem is solved by a classic argument, which shows that the line F_1PF_2 reflected off the line $\mathcal{T}$ is the *shortest* path from F_1 to $\mathcal{T}$ to F_2.

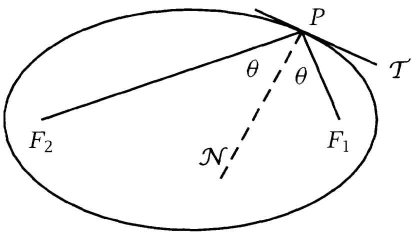

FIGURE 8.5 Foci, tangent, and normal of an ellipse.

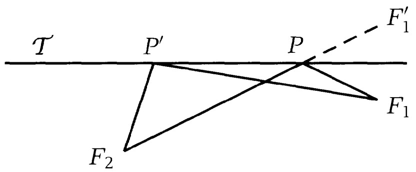

FIGURE 8.6 Minimality of the path of reflection.

8.2.4. By considering the reflection F_1' of F_1 in $\mathcal{T}$ (Figure 8.6) and the triangle inequality, show that F_1PF_2 is shorter than any other path $F_1P'F_2$ from F_1 to $\mathcal{T}$ to F_2.

8.2.5. Deduce from Exercise 8.2.4 and the constancy of the sum of focal distances in the ellipse that $\mathcal{T}$ is the tangent to the ellipse at P.

This proves the focal property of the ellipse. If we fix one focus of the ellipse and let the other tend to infinity, we obtain the parabola as an "ellipse with one focus at infinity." The lines through the focus at infinity become parallel, and thus the focal property of the parabola is a limiting case of the focal property of the ellipse.

8.2.6. Investigate whether there is an analogous focal property of the hyperbola.

8.3 Quadratic Curves

The calculations of Section 8.1 show that all conic sections are quadratic curves, because their equations take the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{(hyperbola)}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{(ellipse)}$$

$$y = ax^2 \qquad \text{(parabola)}$$

when axes are suitably chosen. Thus in each case the equation can be written in the form $q(x, y) = 0$ where q is a quadratic polynomial.

However, it is not yet clear that all quadratic curves are conic sections. There is, in fact, a trivial exception—pairs of straight lines. The straight lines $x = 3y$ and $x = 4y$, for example, can be combined into the quadratic equation $(x - 3y)(x - 4y) = 0$, so the pair is technically a quadratic curve. We call this pair of straight lines a *degenerate* quadratic curve because it results from a genuine curve $(x - 3y)(x - 4y) = c$ as c shrivels to 0. Another degenerate quadratic curve is represented by the equation $x^2 + y^2 = 0$ for the single point $(0, 0)$. This curve results from degeneration of the circle $x^2 + y^2 = c^2$.

Degenerate quadratic curves $q(x, y) = 0$ can be spotted when $q(x, y)$ splits into linear factors (possibly with complex coefficients), so they are not a problem. But what about the genuine curve $xy = 1$? Can we transform its equation into one of the three by suitable choice of axes?

The answer is that all nondegenerate quadratic curves have equations of one of the three types, relative to suitable axes. Hence the genuine quadratic curves are all conic sections. This discovery of Fermat and Descartes was very important in the development of geometry, because it showed for the first time that the algebraic concept of degree is geometrically significant. Their result can be obtained in two steps: rotation of axes and shift of origin.

Suppose that we are given the most general quadratic curve

$$ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

1. We first make the substitution

$$x = x' \cos\theta + y' \sin\theta$$
$$y = -x' \sin\theta + y' \cos\theta,$$

which amounts to choosing x'- and y'-axes at angle θ to the x- and y- axes, by the sin and cos addition formulas of Section 5.3. By suitable choice of θ, $ax^2 + bxy + cy^2$ takes the form $a'x'^2 + c'y'^2$, as