

teristic and minimal polynomials, respectively. What all this tells us is the following. If we consider the relation of similarity on the set of 3×3 matrices over F , the equivalence classes are in one-one correspondence with ordered pairs (f, p) of monic polynomials over F which satisfy (a) and (b).

A.6. The Axiom of Choice

Loosely speaking, the Axiom of Choice is a rule (or principle) of thinking which says that, given a family of non-empty sets, we can choose one element out of each set. To be more precise, suppose that we have an index set A and for each α in A we have an associated set S_α , which is non-empty. To 'choose' one member of each S_α means to give a rule f which associates with each α some element $f(\alpha)$ in the set S_α . The axiom of choice says that this is possible, i.e., given the family of sets $\{S_\alpha\}$, there exists a function f from A into

$$\bigcup_{\alpha} S_{\alpha}$$

such that $f(\alpha)$ is in S_α for each α . This principle is accepted by most mathematicians, although many situations arise in which it is far from clear how any explicit function f can be found.

The Axiom of Choice has some startling consequences. Most of them have little or no bearing on the subject matter of this book; however, one consequence is worth mentioning: Every vector space has a basis. For example, the field of real numbers has a basis, as a vector space over the field of rational numbers. In other words, there is a subset S of R which is linearly independent over the field of rationals and has the property that each real number is a rational linear combination of some finite number of elements of S . We shall not stop to derive this vector space result from the Axiom of Choice. For a proof, we refer the reader to the book by Kelley in the bibliography.

Bibliography

Halmos, P., *Finite-Dimensional Vector Spaces*, D. Van Nostrand Co., Princeton, 1958.

Jacobson, N., *Lectures in Abstract Algebra*, II, D. Van Nostrand Co., Princeton, 1953.

Kelley, John L., *General Topology*, D. Van Nostrand Co., Princeton, 1955.

MacLane, S. and Birkhoff, G., *Algebra*, The Macmillan Co., New York, 1967.

Schreier, O. and Sperner, E., *Introduction to Modern Algebra and Matrix Theory*, 2nd Ed., Chelsea Publishing Co., New York, 1955.

* van der Waerden, B. L., *Modern Algebra* (two volumes), Rev. Ed., Frederick Ungar Publishing Co., New York, 1969.

Index

A

Adjoint:
 classical, 148, 159
 of transformation, 295
Admissible subspace, 232
Algebra, 117
 of formal power series, 119
 self-adjoint, 345
Algebraically closed field, 138
Alternating n -linear function, 144, 169
Annihilator:
 of subset, 101
 of sum and intersection, 106(Ex. 11)
 of vector (T -annihilator), 201, 202, 228
Approximation, 283
Associativity, 1
 of matrix multiplication, 19, 90
 of vector addition, 28
Augmented matrix, 14
Axiom of choice, 400

B

Basis, 41
 change of, 92
 dual, 99, 165
 for module, 164
 ordered, 50
 orthonormal, 281
 standard basis of F^n , 41

Bessel's inequality, 287
Bilinear form, 166, 320, 359
 diagonalization of, 370
 group preserving, 379
 matrix of, 362
 non-degenerate (non-singular), 365
 positive definite, 368
 rank of, 365
 signature of, 372
 skew-symmetric, 375
 symmetric, 367

C

Cauchy-Schwarz inequality, 278
Cayley-Hamilton theorem, 194, 237
Cayley transform, 309(Ex. 7)
Characteristic:
 of a field, 3
 polynomial, 183
 space, 182
 value, 182, 183
 vector, 182
Classical adjoint, 148, 159
Coefficients of polynomial, 120
Cofactor, 158
Column:
 equivalence, 256
 operations, 26, 256
 rank, 72, 114

Commutative:
 algebra, 117
 group, 83
 ring, 140
 Companion matrix, 230
 Complementary subspace, 231
 orthogonal, 286
 Composition, 390
 Conductor, 201, 202, 232
 Congruence, 139, 393, 396
 Conjugate, 271
 transpose, 272
 Conjugation, 276(Ex. 13)
 Coordinates, 50
 coordinate matrix, 51
 Coset, 177, 396
 Cramer's rule, 161
 Cyclic:
 decomposition theorem, 233
 subspace, 227
 vector, 227

D

Degree:
 of multilinear form, 166
 of polynomial, 119
 Dependence, linear, 40, 47
 Derivative of polynomial, 129, 266
 Determinant function, 144
 existence of, 147
 for linear transformations, 172
 uniqueness of, 152
 Determinant rank, 163(Ex. 9)
 Diagonalizable:
 operator, 185
 part of linear operator, 222
 simultaneously, 207
 Diagonalization, 204, 207, 216
 of Hermitian form, 323
 of normal matrix (operator), 317
 of self-adjoint matrix (operator), 314
 of symmetric bilinear form, 370
 unitary, 317
 Differential equations, 223(Ex. 14),
 249(Ex. 8)
 Dimension, 44
 formula, 46
 Direct sum, 210
 invariant, 214
 of matrices, 214
 of operators, 214

Disjoint subspaces (*see* Independent: subspaces)
 Distance, 289(Ex. 4)
 Division with remainder, 128
 Dual:
 basis, 99, 165
 module, 165
 space, 98

E

Eigenvalue (*see* Characteristic: value)
 Elementary:
 column operation, 26, 256
 Jordan matrix, 245
 matrix, 20, 253
 row operation, 6, 252
 Empty set, 388
 Entries of a matrix, 6
 Equivalence relation, 393
 Equivalent systems of equations, 4
 Euclidean space, 277
 Exterior (wedge) product, 175, 177

F

$F^m \times^n$, 29
 F^n , 29
 Factorization of polynomial, 136
 Factors, invariant, 239, 261
 Field, 2
 algebraically closed, 138
 subfield, 2
 Finite-dimensional, 41
 Finitely generated module, 165
 Form:
 alternating, 169
 bilinear, 166, 320, 359
 Hermitian, 323
 matrix of, 322
 multilinear, 166
 non-degenerate, 324(Ex. 6)
 non-negative, 325
 normal, 257, 261
 positive, 325, 328
 quadratic, 273, 368
 r -linear, 166
 rational, 238
 sesqui-linear, 320
 Formal power series, 119
 Free module, 164

Function, 389
 determinant, 144
 identity, 390
 inverse of, 391
 invertible, 390
 linear, 67, 97, 291
 multilinear, 166
 n -linear, 142
 polynomial function, 30
 range of, 389
 restriction of, 391
 Fundamental theorem of algebra, 138

G

Gram-Schmidt process, 280, 287
 Grassman ring, 180
 Greatest common divisor, 133
 Group, 82
 commutative, 83
 general linear, 307
 Lorentz, 382
 orthogonal, 380
 preserving a form, 379
 pseudo-orthogonal, 381
 symmetric, 153

H

Hermitian (*see* Self-adjoint)
 Hermitian form, 323
 Homogeneous system of linear equations, 4
 Hyperspace, 101, 109

I

Ideal, 131
 principal ideal, 131
 Idempotent transformation (*see* Projection)
 Identity:
 element, 117, 140
 function, 390
 matrix, 9
 resolution of, 337, 344
 Independence, linear, 40, 47
 Independent:
 linearly, 40, 47
 subspaces, 209
 Inner product, 271
 matrix of, 274

Inner product (*cont.*):
 quadratic form of, 273
 space, 277
 standard, 271, 272
 Integers, 2
 positive, 2
 Interpolation, 124
 Intersection, 388
 of subspaces, 36
 Invariant:
 direct sum, 214
 factors of a matrix, 239, 261
 subset, 392
 subspace, 199, 206, 314
 Inverse:
 of function, 391
 left, 22
 of matrix, 22, 160
 right, 22
 two-sided, 22
 Invertible:
 function, 390
 linear transformation, 79
 matrix, 22, 160
 Irreducible polynomial, 135
 Isomorphism:
 of inner product spaces, 299
 of vector spaces, 84

J

Jordan form of matrix, 247

K

Kronecker delta, 9

L

Lagrange interpolation formula, 124
 Laplace expansions, 179
 Left inverse, 22
 Linear algebra, 117
 Linear combination:
 of equations, 4
 of vectors, 31
 Linear equations (*see* System of linear equations)
 Linear functional, 97
 Linearly dependent (independent), 40, 47

Linear transformation (operator), 67, 76

- adjoint of, 295
- cyclic decomposition of, 233
- determinant of, 172
- diagonalizable, 185
- diagonalizable part of, 222
- invertible, 79
- matrix in orthonormal basis, 293
- matrix of, 87, 88
- minimal polynomial of, 191
- nilpotent, 222
- non-negative, 329, 341
- non-singular, 79
- normal, 312
- nullity of, 71
- orthogonal, 303
- polar decomposition of, 343
- positive, 329
- product of, 76
- quotient, 397
- range of, 71
- rank of, 71
- self-adjoint, 298, 314
- semi-simple, 263
- trace of, 106(Ex. 15)
- transpose of, 112
- triangular, 202
- unitary, 302

Lorentz:

- group, 382
- transformation, 311(Ex. 15), 382

M

Matrix, 6

- augmented, 14
- of bilinear form, 362
- classical adjoint of, 148, 159
- coefficient, 6
- cofactors, 158
- companion, 230
- conjugate transpose, 272
- coordinate, 51
- elementary, 20, 253
- elementary, Jordan, 245
- of form, 322
- identity, 9
- of inner product, 274
- invariant factors of, 239, 261
- inverse of, 22, 160
- invertible, 22, 160
- Jordan form of, 247.

Matrix (*cont.*):

- of linear transformation, 87, 88
- minimal polynomial of, 191
- nilpotent, 244
- normal, 315
- orthogonal, 162(Ex. 4), 380
- positive, 329
- principal minors of, 326
- product, 17, 90
- rank of, 114
- rational form of, 238
- row rank of, 56, 72, 114
- row-reduced, 9
- row-reduced echelon, 11, 56
- self-adjoint (Hermitian), 35, 314
- similarity of, 94
- skew-symmetric, 162(Ex. 3), 210
- symmetric, 35, 210
- trace of, 98
- transpose of, 114
- triangular, 155(Ex. 7)
- unitary, 163(Ex. 5), 303
- upper-triangular, 27
- Vandermonde, 125
- zero, 12

Minimal polynomial, 191

Module, 164

- basis for, 164
- dual, 165
- finitely generated, 165
- free, 164
- rank of, 165

Monic polynomial, 120

Multilinear function (form), 166

- degree of, 166

Multiplicity, 130

N

 n -linear function, 142

- alternating, 144, 169

 n -tuple, 29

Nilpotent:

- matrix, 244
- operator, 222

Non-degenerate:

- bilinear form, 365
- form, 324(Ex. 6)

Non-negative:

- form, 325
- operator, 329, 341

Non-singular:
 form (*see* Non-degenerate)
 linear transformation, 79

Norm, 273

Normal:
 form, 257, 261
 matrix, 315
 operator, 312

Nullity of linear transformation, 71

Null space, 71

Numbers:
 complex, 2
 rational, 3
 real, 2

O

Onto, 389

Operator, linear, 76

Ordered basis, 50

Orthogonal:
 complement, 285
 equivalence of matrices, 308
 group, 380
 linear transformation, 304
 matrix, 162(Ex. 4), 380
 projection, 285
 set, 278
 vectors, 278, 368

Orthogonalization, 280

Orthonormal:
 basis, 281
 set, 278

P

Parallelogram law, 276(Ex. 9)

Permutation, 151
 even, odd, 152
 product of, 153
 sign of, 152

Polar decomposition, 343

Polarization identities, 274, 368

Polynomial, 119
 characteristic, 183
 coefficients of, 120
 degree of, 119
 derivative of, 129, 266
 function, 30
 irreducible (prime), 135
 minimal, 191

Polynomial (*cont.*):
 monic, 120
 primary decomposition of, 137
 prime (irreducible), 135
 prime factorization of, 136
 reducible, 135
 root of, 129
 scalar, 120
 zero of, 129

Positive:
 form, 325, 328
 integers, 2
 matrix, 329
 operator, 329

Positive definite, 368

Power series, 119

Primary components, 351

Primary decomposition:
 of polynomial, 137
 theorem, 220

Prime:
 factorization of polynomial, 136
 polynomial, 135

Principal:
 access theorem, 323
 ideal, 131
 minors, 326

Product:
 exterior (wedge), 175, 177
 of linear transformations, 76
 of matrices, 14, 90
 of permutations, 153
 tensor, 168

Projection, 211

Proper subset, 388

Pseudo-orthogonal group, 381

Q

Quadratic form, 273, 368

Quotient:
 space, 397
 transformation, 397

R

Range, 71

Rank:
 of bilinear form, 365
 column, 72, 114
 determinant, 163(Ex. 9)