15. In each case, find a linear differential equation with constant coefficients satisfied by all the given functions.
(a) $u_1(x) = e^x, \quad u_2(x) = e^{-x}, \quad u_3(x) = e^{2x}, \quad u_4(x) = e^{-2x}$.
(b) $u_1(x) = e^{-2x}, \quad u_2(x) = xe^{-2x}, \quad u_3(x) = x^2e^{-2x}$.
(c) $u_1(x) = 1, \quad u_2(x) = x, \quad u_3(x) = e^x, \quad u_4(x) = xe^x$.
(d) $u_1(x) = x, \quad u_2(x) = e^x, \quad u_3(x) = xe^x$.
(e) $u_1(x) = x^2, \quad u_2(x) = e^x, \quad u_3(x) = xe^x$.
(f) $u_1(x) = e^{-2x}\cos 3x, \quad u_2(x) = e^{-2x}\sin 3x, \quad u_3(x) = e^{-2x}, \quad u_4(x) = xe^{-2x}$.
(g) $u_1(x) = \cosh x, \quad u_2(x) = \sinh x, \quad u_3(x) = x\cosh x, \quad u_4(x) = x\sinh x$.
(h) $u_1(x) = \cosh x \sin x, \quad u_2(x) = \sinh x \cos x, \quad u_3(x) = x$.

16. Let $r_1, \ldots, r_n$ be n distinct real numbers, and let $Q_1, \ldots, Q_n$ be n polynomials, none of which is the zero polynomial. Prove that the n functions

$$u_1(x) = Q_1(x)e^{r_1x}, \ldots, u_n(x) = Q_n(x)e^{r_nx}$$

are independent.

***Outline** of proof.* Use induction on n. For $n = 1$ and $n = 2$ the result is easily verified. Assume the statement is true for $n = p$ and let $c_1, \ldots, c_{p+1}$ be $p + 1$ real scalars such that

$$\sum_{k=1}^{p+1} c_kQ_k(x)e^{r_kx} = 0.$$

Multiply both sides by $e^{-r_{p+1}x}$ and differentiate the resulting equation. Then use the induction hypothesis to show that all the scalars c_k are 0. An alternate proof can be given based on order of magnitude as $x \to +\infty$, as was done in Example 7 of Section 1.7 (p. 10).

17. Let $m_1, m_2, \ldots, m_k$ be k positive integers, let $r_1, r_2, \ldots, r_k$ be k distinct real numbers, and let $n = m_1 + \cdots + m_k$. For each pair of integers p, q satisfying $1 \le p \le k$, $1 \le q \le m_p$, let

$$u_{q,p}(x) = x^{q-1}e^{r_px}.$$

For example, when $p = 1$ the corresponding functions are

$$u_{1,1}(x) = e^{r_1x}, \quad u_{2,1}(x) = xe^{r_1x}, \ldots, u_{m_1,1}(x) = x^{m_1-1}e^{r_1x}.$$

Prove that the n functions $u_{q,p}$ so defined are independent. *[Hint:* Use Exercise 16.]

18. Let L be a constant-coefficient linear differential operator of order n with characteristic polynomial $p(r)$. Let L' be the constant-coefficient operator whose characteristic polynomial is the derivative $p'(r)$. For example, if $L = 2D^2 - 3D + 1$ then $L' = 4D - 3$. More generally, define the mth derivative $L^{(m)}$ to be the operator whose characteristic polynomial is the mth derivative $p^{(m)}(r)$. (The operator $L^{(m)}$ should not be confused with the mth power L^m.)
(a) If u has n derivatives, prove that

$$L(u) = \sum_{k=0}^{n} \frac{p^{(k)}(0)}{k!} u^{(k)}.$$

(b) If u has $n - m$ derivatives, prove that

$$L^{(m)}(u) = \sum_{k=0}^{n-m} \frac{p^{(k+m)}(0)}{k!} u^{(k)} \qquad \text{for} \qquad m = 0, 1, 2, \ldots, n,$$

where $L^{(0)} = L$.

19. Refer to the notation of Exercise 18. If u and v have n derivatives, prove that

$$L(uv) = \sum_{k=O}^{n} \frac{L^{(k)}(u)}{k!} v^{(k)}.$$

[Hint: Use Exercise 18 along with Leibniz's formula for the kth derivative of a product:

$$(uv)^{(k)} = \sum_{r=0}^{k} \binom{k}{r} u^{(k-r)} v^{(r)} .]$$

20. (a) Let $p(t) = q(t)?(t)$, where q and r are polynomials and m is a positive integer. Prove that $p'(t) = q(t)^{m-1}s(t)$, where s is a polynomial.
(b) Let L be a constant-coefficient operator which annihilates u, where u is a given function of x. Let $M = L^m$, the mth power of L, where $m > 1$. Prove that each of the derivatives M', $M'', \ldots, M^{(m-1)}$ also annihilates u.
(c) Use part (b) and Exercise 19 to prove that M annihilates each of the functions u, xu, . . . , $x^{m-1}u$.
(d) Use part (c) to show that the operator $(D^2 - 2\alpha D + \alpha^2 + \beta^2)^m$ annihilates each of the functions $x^q e^{\alpha x} \sin \beta x$ and $x^q e^{\alpha x} \cos \beta x$ for $q = 1, 2, \ldots, m-1$.
21. Let L be a constant-coefficient operator of order n with characteristic polynomial $p(r)$. If α is constant and if u has n derivatives, prove that

$$L(e^{\alpha x}u(x)) = e^{\alpha x} \sum_{k=O}^{n} \frac{p^{(k)}(\alpha)}{k!} u^{(k)}(x).$$

6.10 The relation between the homogeneous and nonhomogeneous equations

We return now to the general linear differential equation of order n with coefficients that are not necessarily constant. The next theorem describes the relation between solutions of a homogeneous equation $L(y) = 0$ and those of a nonhomogeneous equation $L(y) = R(x)$.

THEOREM 6.10. *Let $L : \mathscr{C}_n(J) \to \mathscr{C}(J)$ be a linear differential operator of order n. Let $u_1, \ldots, u_n$ be n independent solutions of the homogeneous equation $L(y) = 0$, and let y_1 be a particular solution of the nonhomogeneous equation $L(y) = R$, where $R \in V(J)$. Then every solution $y = f(x)$ of the nonhomogeneous equation has the form*

$$f(x) = y_1(x) + \sum_{k=1}^{n} c_k u_k(x), \tag{6.19}$$

where $c_1, \ldots, c_n$ are constants.

Proof. By linearity we have $L(f - y_1) = L(f) - L(y_1) = R - R = 0$. Therefore $f - y_1$ is in the solution space of the homogeneous equation $L(y) = 0$, so $f - y_1$ is a linear combination of $u_1, \ldots, u_n$, say $f - y_1 = c_1u_1 + \cdots + c_nu_n$. This proves (6.19).

Since all solutions of $L(y) = R$ are found in (6.19), the sum on the right of (6.19) (with arbitrary constants $c_1, c_2, \ldots, c_n$) is called the general solution of the nonhomogeneous equation. Theorem 6.10 states that the general solution of the nonhomogeneous equation is obtained by adding toy, the general solution of the homogeneous equation.

Note: Theorem 6.10 has a simple geometric analogy which helps give an insight into its meaning. To determine all points on a plane we find a particular point on the plane and add to it all points on the parallel plane through the origin. To find all solutions of $L(y) = R$ we find a particular solution and add to it all solutions of the homogeneous equation $L(y) = 0$. The set of solutions of the nonhomogeneous equation is analogous to a plane through a particular point. The solution space of the homogeneous equation is analogous to a parallel plane through the origin.

To use Theorem 6.10 in practice we must solve two problems: (1) Find the general solution of the homogeneous equation $L(y) = 0$, and (2) find a particular solution of the nonhomogeneous equation $L(y) = R$. In the next section we show that we can always solve problem (2) if we can solve problem (1).

6.11 Determination of a particular solution of the nonhomogeneous equation. The method of variation of parameters

We turn now to the problem of determining one particular solution y_1 of the nonhomogeneous equation $L(y) = R$. We shall describe a method known as ***variation of parameters*** which tells us how to determine y_1 if we know n independent solutions $u_1, \ldots, u_n$ of the homogeneous equation $L(y) = 0$. The method provides a particular solution of the form

$$y_1 = v_1u_1 + \cdots + v_nu_n, \tag{6.20}$$

where $v_1, \ldots, v_n$ are functions that can be calculated in terms of $u_1, \ldots, u_n$ and the right-hand member R. The method leads to a system of n linear algebraic equations satisfied by the derivatives $v_1', \ldots, v_n'$. This system can always be solved because it has a nonsingular coefficient matrix. Integration of the derivatives then gives the required functions $v_1, \ldots, v_n$. The method was first used by Johann Bernoulli to solve linear equations of first order, and then by Lagrange in 1774 to solve linear equations of second order.

For the nth order case the details can be simplified by using vector and matrix notation. The right-hand member of (6.20) can be written as an inner product,

$$y_1 = (v, u), \tag{6.21}$$

where v and u are n-dimensional vector functions given by

$$v = (v_1, \ldots, v_n), \qquad u = (u_1, \ldots, u_n).$$

We try to choose v so that the inner product defining y_1 will satisfy the nonhomogeneous equation $L(y) = R$, given that $L(u) = 0$, where $L(u) = (L(u_1), \ldots, L(u_n))$.

We begin by calculating the first derivative of y_1. We find

$$y_1' = (v, u') + (v', u). \tag{6.22}$$

We have n functions $v_1, \ldots, v_n$ to determine, so we should be able to put n conditions on them. If we impose the condition that the second term on the right of (6.22) should vanish, the formula for y_1' simplifies to

$$y_1' = (v, u'), \qquad \text{provided that } (v', u) = 0.$$

Differentiating the relation for y_1' we find

$$y_1'' = (v, u'') + (v', u').$$

If we can choose v so that $(v', u') = 0$ then the formula for y_1'' also simplifies and we get

$$y_1'' = (v, u'') \qquad \text{provided that also } (v', u') = 0.$$

If we continue in this manner for the first $n - 1$ derivatives of y_1 we find

$$y_1^{(n-1)} = (v, u^{(n-1)}), \qquad \text{provided that also } (v', u^{(n-2)}) = 0.$$

So far we have put $n - 1$ conditions on v. Differentiating once more we get

$$y_1^{(n)} = (v, u^{(n)}) + (v', u^{(n-1)}).$$

This time we impose the condition $(v', u^{(n-1)}) = R(x)$, and the last equation becomes

$$y_1^{(n)} = (v, u^{(n)}) + R(x), \qquad \text{provided that also } (v', u^{(n-1)}) = R(x).$$

Suppose, for the moment, that we can satisfy the n conditions imposed on v. Let $L = D^n + P_1(x)D^{n-1} + \cdots + P_n(x)$. When we apply L to y_1 we find

$$\begin{aligned} L(y_1) &= y_1^{(n)} + P_1(x)y_1^{(n-1)} + \cdots + P_n(x)y_1 \\ &= \{(v, u^{(n)}) + R(x)\} + P_1(x)(v, u^{(n-1)}) + \cdots + P_n(x)(v, u) \\ &= (v, L(u)) + R(x) = (v, O) + R(x) = R(x). \end{aligned}$$

Thus $L(y_1) = R(x)$, so y_1 is a solution of the nonhomogeneous equation.

The method will succeed if we can satisfy the n conditions we have imposed on v. These conditions state that $(v', u^{(k)}) = 0$ for $k = 0, 1, \ldots, n - 2$, and that $(v', u^{(n-1)}) = R(x)$. We can write these n equations as a single matrix equation,

$$W(x)v'(x) = R(x)\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \tag{6.23}$$

where $v'(x)$ is regarded as an $n \times 1$ column matrix, and where W is the $n \times n$ matrix function whose rows consist of the components of u and its successive derivatives:

$$W = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \\ u_1' & u_2' & \cdots & u_n' \\ \vdots & & & \vdots \\ u_1^{(n-1)} & u_2^{(n-1)} & \cdots & u_n^{(n-1)} \end{bmatrix}.$$

The matrix W is called the *Wronskian matrix* of $u_1, \ldots, u_n$, after J. M. H. Wronski (1778-1853).

In the next section we shall prove that the Wronskian matrix is nonsingular. Therefore we can multiply both sides of (6.23) by $W(x)^{-1}$ to obtain

$$v'(x) = R(x)W(x)^{-1}\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Choose two points c and x in the interval J under consideration and integrate this vector equation over the interval from c to x to obtain

$$v(x) = v(c) + \int_c^x R(t)W(t)^{-1}\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} dt = v(c) + z(x),$$

where

$$z(x) = \int_c^x R(t)W(t)^{-1}\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} dt.$$

The formula $y_1 = (u, v)$ for the particular solution now becomes

$$y_1 = (u, v) = (u, v(c) + z) = (u, v(c)) + (u, z).$$

The first term $(u, v(c))$ satisfies the homogeneous equation since it is a linear combination of $u_1, \ldots, u_n$. Therefore we can omit this term and use the second term (u, z) as a particular solution of the nonhomogeneous equation. In other words, a particular solution of

$L(y) = R$ is given by the inner product

$$(u(x),\ z(x)) = \left(u(x), \int_c^x R(t)W(t)^{-1}\begin{bmatrix}0\\ \vdots\\ 0\\ 1\end{bmatrix} dt\right).$$

Note that it is not necessary that the function R be continuous on the interval J. All that is required is that R be integrable on $[c, x]$.

We can summarize the results of this section by the following theorem.

THEOREM 6.11. *Let $u_1, \ldots, u_n$ be n independent solutions of the homogeneous nth order linear differential equation $L(y) = 0$ on an interval J. Then a particular solution y_1 of the nonhomogeneous equation $L(y) = R$ is given by the formula*

$$y_1(x) = \sum_{k=1}^{n} u_k(x)v_k(x),$$

where $v_1, \ldots, v_n$ are the entries in the $n \times 1$ column matrix v determined by the equation

$$v(x) = \int_c^x R(t)W(t)^{-1}\begin{bmatrix}0\\ \vdots\\ 0\\ 1\end{bmatrix} dt. \tag{6.24}$$

In this formula, W is the Wronskian matrix of $u_1, \ldots, u_n$, and c is any point in J.

Note: The definite integral in (6.24) can be replaced by any indefinite integral

$$\int R(x)W(x)^{-1}\begin{bmatrix}0\\ \vdots\\ 0\\ 1\end{bmatrix} dx.$$

EXAMPLE 1. Find the general solution of the differential equation

$$y'' - y = \frac{2}{1 + e^x}$$

on the interval $(-\infty, +\infty)$.

Solution. The homogeneous equation, $(D^2 - 1)y = 0$ has the two independent solutions $u_1(x) = e^x$, $u_2(x) = e^{-x}$. The Wronskian matrix of u_1 and u_2 is

$$W(x) = \begin{bmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{bmatrix}.$$

Since det $W(x) = -2$, the matrix is nonsingular and its inverse is given by

$$W(x)^{-1} = -\frac{1}{2}\begin{bmatrix} -e^{-x} & -e^{-x} \\ -e^{x} & e^{x} \end{bmatrix}.$$

Therefore

$$W(x)^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} -e^{-x} \\ e^{x} \end{bmatrix}$$

and we have

$$R(x)W(x)^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\frac{1}{2}\,\frac{2}{1+e^{x}}\begin{bmatrix} -e^{-x} \\ e^{x} \end{bmatrix} = \begin{bmatrix} \dfrac{e^{-x}}{1+e^{x}} \\[2ex] \dfrac{-e^{x}}{1+e^{x}} \end{bmatrix}.$$

Integrating each component of the vector on the right we find

$$v_1(x) = \int \frac{e^{-x}}{1+e^{x}}\,dx = \int \left(e^{-x} - 1 + \frac{e^{x}}{1+e^{x}}\right) dx = -e^{-x} - x + \log(1 + e^{x})$$

and

$$v_2(x) = \int \frac{-e^{x}}{1+e^{x}}\,dx = -\log(1 + e^{x}).$$

Therefore the general solution of the differential equation is

$$\begin{aligned} y &= c_1u_1(x) + c_2u_2(x) + v_1(x)u_1(x) + v_2(x)u_2(x) \\ &= c_1e^{x} + c_2e^{-x} - 1 - xe^{x} + (e^{x} - e^{-x}) \log(1 + e^{x}). \end{aligned}$$

6.12 Nonsingularity of the Wronskian matrix of n independent solutions of a homogeneous linear equation

In this section we prove that the Wronsklan matrix W of n independent solutions $u_1, \ldots, u_n$ of a homogeneous equation $L(y) = 0$ is nonsingular. We do this by proving that the determinant of W is an exponential function which is never zero on the interval J under consideration.

Let $w(x) = \det W(x)$ for each x in J, and assume that the differential equation satisfied by $u_1, \ldots, u_n$ has the form

$$y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_n(x)y = 0. \tag{6.25}$$

Then we have:

THEOREM 6.12. *The Wronskian determinant satisfies the first-order differential equation*

$$w' + P_1(x)w = 0 \tag{6.26}$$

on J. Therefore, if $c \in J$ we have

$$w(x) = w(c) \exp\left[-\int_c^x P_1(t)\,dt\right] \qquad \textit{(Abel's formula)}. \tag{6.27}$$

Moreover, $w(x) \neq 0$ for all x in J.

Proof. Let u be the row-vector $u = (u_1, \ldots, u_n)$. Since each component of u satisfies the differential equation (6.25) the same is true of u. The rows of the Wronskian matrix W are the vectors $u, u', \ldots, u^{(n-1)}$. Hence we can write

$$w = \det W = \det(u, u', \ldots, u^{(n-1)}).$$

The derivative of w is the determinant of the matrix obtained by differentiating the last row of W (see Exercise 8 of Section 3.17). That is

$$w' = \det(u, u', \ldots, u^{(n-2)}, u^{(n)}).$$

Multiplying the last row of w by $P_1(x)$ we also have

$$P_1(x)w = \det(u, u', \ldots, u^{(n-2)}, P_1(x)u^{(n-1)}).$$

Adding these last two equations we find

$$w' + P_1(x)w = \det(u, u', \ldots, u^{(n-2)}, u^{(n)} + P_1(x)u^{(n-1)}).$$

But the rows of this last determinant are dependent since u satisfies the differential equation (6.25). Therefore the determinant is zero, which means that w satisfies (6.26). Solving (6.26) we obtain Abel's formula (6.27).

Next we prove that $w(c) \neq 0$ for some c in J. We do this by a contradiction argument. Suppose that $w(t) = 0$ for all t in J. Choose a fixed t in J, say $t = t_0$, and consider the linear system of algebraic equations

$$W(t_0)X = 0,$$

where X is a column vector. Since $\det W(t_0) = 0$, the matrix $W(t_0)$ is singular so this system has a nonzero solution, say $X = (c_1, \ldots, c_n) \neq (0, \ldots, 0)$. Using the components of this nonzero vector, let f be the linear combination

$$f(t) = c_1u_1(t) + \cdots + c_nu_n(t).$$

The function f so defined satisfies $L(f) = 0$ on J since it is a linear combination of $u_1, \ldots, u_n$. The matrix equation $W(t_0)X = 0$ implies that

$$f(t_0) = f'(t_0) = \cdots = f^{(n-1)}(t_0) = 0.$$

Therefore f has the initial-value vector 0 at $t = t_0$ so, by the uniqueness theorem, f is the zero solution. This means $c_1 = \cdots = c_n = 0$, which is a contradiction. Therefore $w(t) \neq 0$ for some t in J. Taking c to be this t in Abel's formula we see that $w(x) \neq 0$ for all x in J. This completes the proof of Theorem 6.12.

6.13 Special methods for determining a particular solution of the nonhomogeneous equation. Reduction to a system of first-order linear equations

Although variation of parameters provides a general method for determining a particular solution of $L(y) = R$, special methods are available that are often easier to apply when the equation has certain special forms. For example, if the equation has constant coefficients we can reduce the problem to that of solving a succession of linear equations of first order. The general method is best illustrated with a simple example.

EXAMPLE 1. Find a particular solution of the equation

$$(D - 1)(D - 2)y = xe^{x+x^2}. \tag{6.28}$$

Solution. Let $u = (D - 2)y$. Then the equation becomes

$$(D - 1)u = xe^{x+x^2}.$$

This is a first-order linear equation in u which can be solved using Theorem 6.1. A particular solution is

$$u = \tfrac{1}{2}e^{x+x^2}$$

Substituting this in the equation $u = (D - 2)y$ we obtain

$$(D - 2)y = \tfrac{1}{2}e^{x+x^2},$$

a first-order linear equation for y. Solving this by Theorem 6.1 we find that a particular solution (with $y_1(0) = 0$) is given by

$$y_1(x) = \tfrac{1}{2}e^{2x}\int_0^x e^{t^2-t}\,dt.$$

Although the integral cannot be evaluated in terms of elementary functions we consider the equation as having been solved, since the solution is expressed in terms of integrals of familiar functions. The general solution of (6.28) is

$$y = c_1e^x + c_2e^{2x} + \tfrac{1}{2}e^{2x}\int_0^x e^{t^2-t}\,dt.$$

6.14 The annihilator method for determining a particular solution of the nonhomogeneous equation

We describe next a method which can be used if the equation $L(y) = R$ has constant coefficients and if the right-hand member R is itself annihilated by a constant-coefficient operator, say $A(R) = 0$. In principle, the method is very simple. We apply the operator A to both members of the differential equation $L(y) = R$ and obtain a new equation $AL(y) = 0$ which must be satisfied by all solutions of the original equation. Since AL is another constant-coefficient operator we can determine its null space by calculating the roots of the characteristic equation of AL. Then the problem remains of choosing from this null space a particular function y_1 that satisfies $L(y_1) = R$. The following examples illustrate the process.

EXAMPLE 1. Find a particular solution of the equation

$$(D^4 - 16)y = x^4 + x + 1.$$

Solution. The right-hand member, a polynomial of degree 4, is annihilated by the operator D^5. Therefore any solution of the given equation is also a solution of the equation

$$D^5(D^4 - 16)y = 0. \tag{6.29}$$

The roots of the characteristic equation are 0, 0, 0, 0, 0, 2, -2, $2i$, $-2i$, so all the solutions of (6.29) are to be found in the linear combination

$$y = c_1 + c_2x + c_3x^2 + c_4x^3 + c_5x^4 + c_6e^{2x} + c_7e^{-2x} + c_8\cos 2x + c_9 \sin 2x.$$

We want to choose the c_i so that $L(y) = x^4 + x + 1$, where $L = D^4 - 16$. Since the last four terms are annihilated by L, we can take $c_6 = c_7 = c_8 = c_9 = 0$ and try to find $c_1, \ldots, c_5$ so that

$$L(c_1 + c_2x + c_3x^2 + c_4x^3 + c_5x^4) = x^4 + x + 1.$$

In other words, we seek a particular solution y_1 which is a polynomial of degree 4 satisfying $L(y_1) = x^4 + x + 1$. To simplify the algebra we write

$$16y_1 = ax^4 + bx^3 + cx^2 + dx + e.$$

This gives us $16y_1^{(4)} = 24a$, so $y_1^{(4)} = 3a/2$. Substituting in the differential equation $L(y_1) = x^4 + x + 1$, we must determine a, b, c, d, e to satisfy

$$\tfrac{3}{2}a - ax^4 - bx^3 - cx^2 - dx - e = x^4 + x + 1.$$

Equating coefficients of like powers of x we obtain

$$a = -1, \qquad b = c = 0, \qquad d = -1, \qquad e = -\tfrac{5}{2},$$

so the particular solution y_1 is given by

$$y_1 = -\tfrac{1}{16}x^4 - \tfrac{1}{16}x - \tfrac{5}{32}.$$

EXAMPLE 2. Solve the differential equation $y'' - 5y' + 6y = xe^x$.

Solution. The differential equation has the form

$$L(y) = R, \tag{6.30}$$

where $R(x) = xe^x$ and $L = D^2 - 5D + 6$. The corresponding homogeneous equation can be written as

$$(D - 2)(D - 3)y = 0;$$

it has the independent solutions $u_1(x) = e^{2x}$, $u_2(x) = e^{3x}$. Now we seek a particular solution y_1 of the nonhomogeneous equation. We recognize the function $R(x) = xe^x$ as a solution of the homogeneous equation

$$(D - 1)^2 y = 0 .$$

Therefore, if we operate on both sides of (6.30) with the operator $(D - 1)^2$ we find that any function which satisfies (6.30) must also satisfy the equation

$$(D - 1)^2(D - 2)(D - 3)y = 0.$$

This differential equation has the characteristic roots 1, 1, 2, 3, so all its solutions are to be found in the linear combination

$$y = ae^x + bxe^x + ce^{2x} + de^{3x},$$

where a, b, c, d are constants. We want to choose a, b, c, d so that the resulting solution y_1 satisfies $L(y_1) = xe^x$. Since $L(ce^{2x} + de^{3x}) = 0$ for every choice of c and d, we need only choose a and b so that $L(ae^x + bxe")$ = xe^x and take $c = d = 0$. If we put

$$y_1 = ae" + bxe^x,$$

we have

$$D(y_1) = (a + b)e^x + bxe^x, \qquad D^2(y_1) = (a + 2b)e^x + bxe",$$

so the equation $(D^2 - 5D + 6)y_1 = xe^x$ becomes

$$(2a - 3b)e^x + 2bxe^x = xe^x .$$

Canceling e^x and equating coefficients of like powers of x we find $a = \frac{3}{4}$, $b = \frac{1}{2}$. *Therefore* $y_1 = \frac{3}{4}e^x + \frac{1}{2}xe^x$ and the general solution of $L(y) = R$ is given by the formula

$$y = c_1e^{2x} + c_2e^{3x} + \tfrac{3}{4}e^x + \tfrac{1}{2}xe^x .$$

The method used in the foregoing examples is called the ***annihilator method.*** It will always work if we can find a constant coefficient operator A that annihilates R. From our knowledge of homogeneous linear differential equations with constant coefficients, we know that the only real-valued functions annihilated by constant-coefficient operators are linear combinations of terms of the form

$$x^{m-1}e^{\alpha x}, \qquad x^{m-1}e^{\alpha x}\cos\beta x, \qquad x^{m-1}e^{\alpha x}\sin\beta x,$$

where m is a positive integer and α and β are real constants. The function $y = x^{m-1}e^{\alpha x}$ is a solution of a differential equation with a characteristic root α having multiplicity m.

Therefore, this function has the annihilator $(D - \alpha)^m$. Each of the functions $y = x^{m-1}e^{\alpha x}\cos \beta x$ and $y = x^{m-1}e^{\alpha x}\sin \beta x$ is a solution of a differential equation with complex characteristic roots $\alpha \pm i\beta$, each occurring with multiplicity m, so they are annihilated by the operator $[D^2 - 2\alpha D + (\alpha^2 + \beta^2)]^m$. For ease of reference, we list these annihilators in Table 6.1, along with some of their special cases.

TABLE 6.1

Function	*Annihilator*
$y = x^{m-1}$	D^m
$y = e^{\alpha x}$	$D - \alpha$
$y = x^{m-1}e^{\alpha x}$	$(D - \alpha)^m$
$y = \cos \beta x$ or $y = \sin \beta x$	$D^2 + \beta^2$
$y = x^{m-1}\cos \beta x$ or $y = x^{m-1}\sin \beta x$	$(D^2 + \beta^2)^m$
$y = e^{\alpha x}\cos \beta x$ or $y = e^{\alpha x}\sin \beta x$	$D^2 - 2\alpha D + (\alpha^2 + \beta^2)$
$y = x^{m-1}e^{\alpha x}\cos \beta x$ or $y = x^{m-1}e^{\alpha x}\sin \beta x$	$[D^2 - 2\alpha D + (\alpha^2 + \beta^2)]^m$

Although the annihilator method is very efficient when applicable, it is limited to equations whose right members R have a constant-coefficient annihilator. If $R(x)$ has the form e^{x^2}, $\log x$, or $\tan x$, the method will not work; we must then use variation of parameters or some other method to find a particular solution.

6.15 Exercises

In each of Exercises 1 through 10, find the general solution on the interval $(-\infty, +\infty)$.

1. $y'' - y' = x^2$.
2. $y'' - 4y = e^{2x}$.
3. $y'' + 2y' = 3xe^x$.
4. $y'' + 4y = \sin x$.
5. $y'' - 2y' + y = e^x + e^{2x}$.
6. $y''' - y' = e^x$.
7. $y''' - y' = e^x + e^{-x}$.
8. $y''' + 3y'' + 3y' + y = xe^{-x}$.
9. $y'' + y = xe^x \sin 2x$.
10. $y^{(4)} - y = x^2e^{-x}$.
11. If a constant-coefficient operator A annihilates f and if a constant-coefficient operator B annihilates g, show that the product AB annihilates $f + g$.
12. Let A be a constant-coefficient operator with characteristic polynomial p_A.
    (a) Use the annihilator method to prove that the differential equation $A(y) = e^{\alpha x}$ has a particular solution of the form
    $$y_1 = \frac{e^{\alpha x}}{p_A(\alpha)}$$
    if α is not a zero of the polynomial p_A.
    (b) If α is a simple zero of p_A (multiplicity 1), prove that the equation $A(y) = e^{\alpha x}$ has the particular solution
    $$y_1 = \frac{xe^{\alpha x}}{p'_A(\alpha)}.$$
    (c) Generalize the results of (a) and (b) when α is a zero of p_A with multiplicity m.

13. Given two constant-coefficient operators A and B whose characteristic polynomials have no zeros in common. Let $C = AB$.
(a) Prove that every solution of the differential equation $C(y) = 0$ has the form $y = y_1 + y_2$, where $A(y_1) = 0$ and $B(y_2) = 0$.
(b) Prove that the functions y_1 and y_2 in part (a) are uniquely determined. That is, for a given y satisfying $C(y) = 0$ there is only one pair y_1, y_2 with the properties in part (a).
14. If $L(y) = y'' + ay' + by$, where a and b are constants, let f be that particular solution of $L(y) = 0$ satisfying the conditions $f(0) = 0$ and $f'(0) = 1$. Show that a particular solution of $L(y) = R$ is given by the formula

$$y_1(x) = \int_c^x f(x - t)R(t)\,dt$$

for any choice of c. In particular, if the roots of the characteristic equation are equal, say $r_1 = r_2 = m$, show that the formula for $y_1(x)$ becomes

$$y_1(x) = e^{mx}\int_c^x (x - t)e^{-mt}R(t)\,dt.$$

15. Let Q be the operator "multiplication by x." That is, $Q(y)(x) = x \cdot y(x)$ for each y in class $\mathscr{C}^\infty$ and each real x. Let I denote the identity operator, defined by $I(y) = y$ for each y in $\mathscr{C}^\infty$.
(a) Prove that $DQ - QD = I$.
(b) Show that $D^2Q - QD^2$ is a constant-coefficient operator of first order, and determine this operator explicitly as a linear polynomial in D.
(c) Show that $D^3Q - QD^3$ is a constant-coefficient operator of second order, and determine this operator explicitly as a quadratic polynomial in D.
(d) Guess the generalization suggested for the operator $D^nQ - QD^n$, and prove your result by induction.

In each of Exercises 16 through 20, find the general solution of the differential equation in the given interval.

16. $y'' - y = 1/x$, $\quad (0, +\infty)$.
17. $y'' + 4y = \sec 2x$, $\quad \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$.
18. $y'' - y = \sec^3 x - \sec x$, $\quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.
19. $y'' - 2y' + y = e^{e^x}(e^x - 1)^2$, $\quad (-\infty, +\infty)$.
20. $y''' - 7y'' + 14y' - 8y = \log x$, $\quad (0, +\infty)$.

6.16 Miscellaneous exercises on linear differential equations

1. An integral curve $y = u(x)$ of the differential equation $y'' - 3y' - 4y = 0$ intersects an integral curve $y = v(x)$ of the differential equation $y'' + 4y' - 5y = 0$ at the origin. Determine the functions u and v if the two curves have equal slopes at the origin and if

$$\lim_{x\to\infty} \frac{[v(x)]^4}{u(x)} = \frac{5}{6}.$$

2. An integral curve $y = u(x)$ of the differential equation $y'' - 4y' + 29y = 0$ intersects an integral curve $y = v(x)$ of the differential equation $y'' + 4y' + 13y = 0$ at the origin. The two curves have equal slopes at the origin. Determine u and v if $u'(\pi/2) = 1$.

3. Given that the differential equation $y'' + 4xy' + Q(x)y = 0$ has two solutions of the form $y_1 = u(x)$ and $y_2 = xu(x)$, where $u(0) = 1$. Determine both $u(x)$ and $Q(x)$ explicitly in terms of x.
4. Let $L(y) = y'' + P_1y' + P_2y$. To solve the nonhomogeneous equation $L(y) = R$ by variation of parameters, we need to know two linearly independent solutions of the homogeneous equation. This exercise shows that if *one* solution u_1 of $L(y) = 0$ is known, and if u_1 is never zero on an interval J, a second solution u_2 of the homogeneous equation is given by the formula

$$u_2(x) = u_1(x)\int_c^x \frac{Q(t)}{[u_1(t)]^2}\,dt,$$

where $Q(x) = e^{-\int P_1(x)\,dx}$, and c is any point in J. These two solutions are independent on J.
(a) Prove that the function u_2 does, indeed, satisfy $L(y) = 0$.
(b) Prove that u_1 and u_2 are independent on J.
5. Find the general solution of the equation

$$xy'' - 2(x + 1)y' + (x + 2)y = x^3e^{2x}$$

for $x > 0$, given that the homogeneous equation has a solution of the form $y = e^{mx}$.
6. Obtain one nonzero solution by inspection and then find the general solution of the differential equation

$$(y'' - 4y') + x^2(y' - 4y) = 0.$$

7. Find the general solution of the differential equation

$$4x^2y'' + 4xy' - y = 0,$$

given that there is a particular solution of the form $y = x^m$ for $x > 0$.
8. Find a solution of the homogeneous equation by trial, and then find the general solution of the equation

$$x(1 - x)y'' - (1 - 2x)y' + (x^2 - 3x + 1)y = (1 - x)^3.$$

9. Find the general solution of the equation

$$(2x - 3x^3)y'' + 4y' + 6xy = 0,$$

given that it has a solution that is a polynomial in x.
10. Find the general solution of the equation

$$x^2(1 - x)y'' + 2x(2 - x)y' + 2(1 + x)y = x^2,$$

given that the homogeneous equation has a solution of the form $y = x^c$.
11. Let $g(x) = \int_1^x e^t/t\,dt$ if $x > 0$. (Do not attempt to evaluate this integral.) Find all values of the constant a such that the function f defined by

$$f(x) = \frac{1}{x}e^{ag(x)}$$