4.4.2 Deduce from Exercise 4.4.1, by the method of exhaustion, that the areas under $y = 1/x$ from 1 to a and from b to ab are equal.

4.4.3 Deduce from Exercise 4.4.2, and the above definition of log, that

$$\log ab = \log a + \log b.$$

4.5 Biographical Notes: Archimedes

Archimedes is one of the few ancient mathematicians whose life is known in any detail, thanks to the attention he received from classical authors such as Plutarch, Livy, and Cicero and his participation in the historically significant siege of Syracuse in 212 BCE. He was born in Syracuse (a Greek city in what is now Sicily) around 287 BCE and did most of his important work there, though he may have studied for a time in Alexandria. He seems to have been related to the ruler of Syracuse, King Hieron II, or at least on good terms with him. There are many stories of mechanisms invented by Archimedes for the benefit of Hieron: compound pulleys for moving ships, ballistic devices for the defense of Syracuse, and a model planetarium.

The most famous story about Archimedes is the one told by Vitruvius (*De architectura*, Book IX, Ch. 3), which has Archimedes leaping from his bath with a shout of "Eureka!" when he realized that weighing a crown immersed in water would give a means of testing whether it was pure gold. Historians doubt the authenticity of this story, but it does at least recognize Archimedes' understanding of hydrostatics.

In ancient times Archimedes' reputation rested on his mechanical inventions, which no doubt were more understandable to most people than his pure mathematics. However, it can also be argued that his theoretical mechanics (including the law of the lever, centers of mass, equilibrium, and hydrostatic pressure) was his most original contribution to science. Before Archimedes there was no mathematical theory of mechanics at all, only the thoroughly incorrect mechanics of Aristotle. In pure mathematics, Archimedes did not make any comparable *conceptual* advances, except perhaps in his *Method*, which uses his ideas from statics as a means of discovering results on areas and volumes. The concepts that Archimedes needed for proofs in geometry—the theory of proportions and the method of exhaustion—had already been supplied by Eudoxus, and it was Archimedes' phenomenal insight and technique that lifted him head and shoulders above his contemporaries.

The story of Archimedes' death has often been told, though with varying details. He was killed by a Roman soldier when Syracuse fell to the Romans under Marcellus in 212 BCE. Probably he was doing mathematics at the time of his death, but whether he enraged a soldier by saying "Stand away from my diagram!" is conjectural. This story has come down to us from Tzetzes (*Chiliad*, Book II). Other versions of the death of Archimedes are given in Plutarch's *Marcellus* (Ch. XIX). Plutarch also tells us that Archimedes asked that his gravestone be inscribed with a figure and description of his favorite result, the relation between the volumes of the sphere and the cylinder. [He showed that the volume of the sphere is two-thirds that of the enveloping cylinder. See Heath (1897), p. 43, and Exercise 9.2.5.] A century and a half later, Cicero (*Tusculan Disputations*, Book V) reported finding the gravestone when he was a quaestor in Sicily in 75 BCE. The grave had been neglected, but the figure of the sphere and cylinder was still recognizable.

5

Number Theory in Asia

5.1 The Euclidean Algorithm

It is clear from the preceding chapters of this book that ancient Greece had an enormous influence on world mathematics and that most of the fundamental concepts of mathematics can be found there. This does not mean, however, that the Greeks discovered everything first, or that they did everything best. We have already seen that the Pythagorean theorem was known in Babylon earlier than in Greece, and that Pythagorean triples were understood better there than they ever were in Greece, at least until the time of Diophantus.

In fact, the Pythagorean theorem and Pythagorean triples were also known in ancient China and India. As far as we know these were independent discoveries, so it rather seems that the Pythagorean theorem is mathematically universal, likely to arise in any sufficiently advanced civilization. Other such cultural universals are the concept of π—the ratio of radius to circumference in the circle—and the Euclidean algorithm. As we shall see in this chapter, the Euclidean algorithm seems to arise whenever there is an interest in multiples, divisors, or integer solutions of linear and quadratic equations.

For Euclid, there were two quite separate applications of the Euclidean algorithm. In the first, the algorithm was applied to integers and used to draw conclusions about divisibility and primes. In the second, the algorithm was applied to line segments and was used as a *criterion for irrationality*: if the algorithm does not terminate, then the ratio of the segments is irrational. As we saw in Section 3.4, it is possible that the Greeks pushed