On the other hand, consider the tensor product $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$, which is generated as an abelian group by the elements $0 \otimes 0 = 1 \otimes 0 = 0 \otimes 1 = 0$ and $1 \otimes 1$. In this case $1 \otimes 1 \neq 0$ since, for example, the map $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ defined by $(a, b) \mapsto ab$ is clearly nonzero and linear in both a and b. Since $2(1 \otimes 1) = 2 \otimes 1 = 0 \otimes 1 = 0$, the element $1 \otimes 1$ is of order 2. Hence $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$.

(3) In general,

$$\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z},$$

where d is the g.c.d. of the integers m and n. To see this, observe first that

$$a \otimes b = a \otimes (b \cdot 1) = (ab) \otimes 1 = ab(1 \otimes 1),$$

from which it follows that $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ is a cyclic group with $1 \otimes 1$ as generator. Since $m(1 \otimes 1) = m \otimes 1 = 0 \otimes 1 = 0$ and similarly $n(1 \otimes 1) = 1 \otimes n = 0$, we have $d(1 \otimes 1) = 0$, so the cyclic group has order dividing d. The map $\varphi : \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/d\mathbb{Z}$ defined by $\varphi(a \bmod m, b \bmod n) = ab \bmod d$ is well defined since d divides both m and n. It is clearly $\mathbb{Z}$-bilinear. The induced map $\Phi : \mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/d\mathbb{Z}$ from Corollary 12 maps $1 \otimes 1$ to the element $1 \in \mathbb{Z}/d\mathbb{Z}$, which is an element of order d. In particular $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ has order at least d. Hence $1 \otimes 1$ is an element of order d and Φ gives an isomorphism $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z}$.

(4) In $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$ a simple tensor has the form $(a/b \bmod \mathbb{Z}) \otimes (c/d \bmod \mathbb{Z})$ for some rational numbers a/b and c/d. Then

$$\begin{aligned}(\frac{a}{b} \bmod \mathbb{Z}) \otimes (\frac{c}{d} \bmod \mathbb{Z}) &= d(\frac{a}{bd} \bmod \mathbb{Z}) \otimes (\frac{c}{d} \bmod \mathbb{Z}) \\ &= (\frac{a}{bd} \bmod \mathbb{Z}) \otimes d(\frac{c}{d} \bmod \mathbb{Z}) = (\frac{a}{bd} \bmod \mathbb{Z}) \otimes 0 = 0\end{aligned}$$

and so

$$\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0.$$

In a similar way, $A \otimes_{\mathbb{Z}} B = 0$ for any *divisible* abelian group A and *torsion* abelian group B (an abelian group in which every element has finite order). For example

$$\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0.$$

(5) The structure of a tensor product can vary considerably depending on the ring over which the tensors are taken. For example $\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}$ and $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ are isomorphic as left $\mathbb{Q}$-modules (both are one dimensional vector spaces over $\mathbb{Q}$) — cf. the exercises. On the other hand we shall see at the end of this section that $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ and $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ are not isomorphic $\mathbb{C}$-modules (the former is a 1-dimensional vector space over $\mathbb{C}$ and the latter is 2-dimensional over $\mathbb{C}$).

(6) *General extension of scalars or change of base:* Let $f : R \to S$ be a ring homomorphism with $f(1_R) = 1_S$. Then $s \cdot r = sf(r)$ gives S the structure of a right R-module with respect to which S is an (S, R)-bimodule. Then for any left R-module N, the resulting tensor product $S \otimes_R N$ is a left S-module obtained by *changing the base* from R to S. This gives a slight generalization of the notion of extension of scalars (where R was a subring of S).

(7) Let $f : R \to S$ be a ring homomorphism as in the preceding example. Then we have $S \otimes_R R \cong S$ as left S-modules, as follows. The map $\varphi : S \times R \to S$ defined by $(s, r) \mapsto sr$ (where $sr = sf(r)$ by definition of the right R-action on S), is an R-balanced map, as is easily checked. For example,

$$\varphi(s_1 + s_2, r) = (s_1 + s_2)r = s_1 r + s_2 r = \varphi(s_1, r) + \varphi(s_2, r)$$

and

$$\varphi(sr, r') = (sr)r' = s(rr') = \varphi(s, rr').$$

By Theorem 10 we have an associated group homomorphism $\Phi : S \otimes_R R \to S$ with $\Phi(s \otimes r) = sr$. Since $\Phi(s'(s \otimes r)) = \Phi(s's \otimes r) = s'sr = s'\Phi(s \otimes r)$, it follows that Φ is also an S-module homomorphism. The map $\Phi' : S \to S \otimes_R R$ with $s \mapsto s \otimes 1$ is an S-module homomorphism that is inverse to Φ because $\Phi \circ \Phi'(s) = \Phi(s \otimes 1) = s$ gives $\Phi\Phi' = 1$, and

$$\Phi' \circ \Phi(s \otimes r) = \Phi'(sr) = sr \otimes 1 = s \otimes r$$

shows that $\Phi'\Phi$ is the identity on simple tensors, hence $\Phi'\Phi = 1$.

(8) Let R be a ring (not necessarily commutative), let I be a two sided ideal in R, and let N be a left R-module. Then as previously mentioned, R/I is an $(R/I, R)$-bimodule, so the tensor product $R/I \otimes_R N$ is a left R/I-module. This is an example of "extension of scalars" with respect to the natural projection homomorphism $R \to R/I$.

Define

$$IN = \left\{ \sum_{\text{finite}} a_i n_i \mid a_i \in I, n_i \in N \right\},$$

which is easily seen to be a left R-submodule of N (cf. Exercise 5, Section 1). Then

$$(R/I) \otimes_R N \cong N/IN,$$

as left R-modules, as follows. The tensor product is generated as an abelian group by the simple tensors $(r \bmod I) \otimes n = r(1 \otimes n)$ for $r \in R$ and $n \in N$ (viewing the R/I-module tensor product as an R-module on which I acts trivially). Hence the elements $1 \otimes n$ generate $(R/I) \otimes_R N$ as an R/I-module. The map $N \to (R/I) \otimes_R N$ defined by $n \mapsto 1 \otimes n$ is a left R-module homomorphism and, by the previous observation, is surjective. Under this map $a_i n_i$ with $a_i \in I$ and $n_i \in N$ maps to $1 \otimes a_i n_i = a_i \otimes n_i = 0$, and so IN is contained in the kernel. This induces a surjective R-module homomorphism $f : N/IN \to (R/I) \otimes_R N$ with $f(n \bmod I) = 1 \otimes n$. We show f is an isomorphism by exhibiting its inverse. The map $(R/I) \times N \to N/IN$ defined by mapping $(r \bmod I, n)$ to $(rn \bmod IN)$ is well defined and easily checked to be R-balanced. It follows by Theorem 10 that there is an associated group homomorphism $g : (R/I) \otimes N \to N/IN$ with $g((r \bmod I) \otimes n) = rn \bmod IN$. As usual, $fg = 1$ and $gf = 1$, so f is a bijection and $(R/I) \otimes_R N \cong N/IN$, as claimed.

As an example, let $R = \mathbb{Z}$ with ideal $I = m\mathbb{Z}$ and let N be the $\mathbb{Z}$-module $\mathbb{Z}/n\mathbb{Z}$. Then $IN = m(\mathbb{Z}/n\mathbb{Z}) = (m\mathbb{Z} + n\mathbb{Z})/n\mathbb{Z} = d\mathbb{Z}/n\mathbb{Z}$ where d is the g.c.d. of m and n. Then $N/IN \cong \mathbb{Z}/d\mathbb{Z}$ and we recover the isomorphism $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z}$ of Example 3 above.

We now establish some of the basic properties of tensor products. Note the frequent application of Theorem 10 to establish the existence of homomorphisms.

Theorem 13. *(The "Tensor Product" of Two Homomorphisms)* Let M, M' be right R-modules, let N, N' be left R-modules, and suppose $\varphi : M \to M'$ and $\psi : N \to N'$ are R-module homomorphisms.

(1) There is a unique group homomorphism, denoted by $\varphi \otimes \psi$, mapping $M \otimes_R N$ into $M' \otimes_R N'$ such that $(\varphi \otimes \psi)(m \otimes n) = \varphi(m) \otimes \psi(n)$ for all $m \in M$ and $n \in N$.

(2) If M, M' are also (S, R)-bimodules for some ring S and φ is also an S-module homomorphism, then $\varphi \otimes \psi$ is a homomorphism of left S-modules. In particular, if R is commutative then $\varphi \otimes \psi$ is always an R-module homomorphism for the standard R-module structures.

(3) If $\lambda : M' \to M''$ and $\mu : N' \to N''$ are R-module homomorphisms then $(\lambda \otimes \mu) \circ (\varphi \otimes \psi) = (\lambda \circ \varphi) \otimes (\mu \circ \psi)$.

Proof: The map $(m, n) \mapsto \varphi(m) \otimes \psi(n)$ from $M \times N$ to $M' \otimes_R N'$ is clearly R-balanced, so (1) follows immediately from Theorem 10.

In (2) the definition of the (left) action of S on M together with the assumption that φ is an S-module homomorphism imply that on simple tensors

$$(\varphi \otimes \psi)(s(m \otimes n)) = (\varphi \otimes \psi)(sm \otimes n) = \varphi(sm) \otimes \psi(n) = s\varphi(m) \otimes \psi(n).$$

Since $\varphi \otimes \psi$ is additive, this extends to sums of simple tensors to show that $\varphi \otimes \psi$ is an S-module homomorphism. This gives (2).

The uniqueness condition in Theorem 10 implies (3), which completes the proof.

The next result shows that we may write $M \otimes N \otimes L$, or more generally, an n-fold tensor product $M_1 \otimes M_2 \otimes \cdots \otimes M_n$, unambiguously whenever it is defined.

Theorem 14. *(Associativity of the Tensor Product)* Suppose M is a right R-module, N is an (R, T)-bimodule, and L is a left T-module. Then there is a unique isomorphism

$$(M \otimes_R N) \otimes_T L \cong M \otimes_R (N \otimes_T L)$$

of abelian groups such that $(m \otimes n) \otimes l \mapsto m \otimes (n \otimes l)$. If M is an (S, R)-bimodule, then this is an isomorphism of S-modules.

Proof: Note first that the (R, T)-bimodule structure on N makes $M \otimes_R N$ into a right T-module and $N \otimes_T L$ into a left R-module, so both sides of the isomorphism are well defined. For each fixed $l \in L$, the mapping $(m, n) \mapsto m \otimes (n \otimes l)$ is R-balanced, so by Theorem 10 there is a homomorphism $M \otimes_R N \to M \otimes_R (N \otimes_T L)$ with $m \otimes n \mapsto m \otimes (n \otimes l)$. This shows that the map from $(M \otimes_R N) \times L$ to $M \otimes_R (N \otimes_T L)$ given by $(m \otimes n, l) \mapsto m \otimes (n \otimes l)$ is well defined. Since it is easily seen to be T-balanced, another application of Theorem 10 implies that it induces a homomorphism $(M \otimes_R N) \otimes_T L \to M \otimes_R (N \otimes_T L)$ such that $(m \otimes n) \otimes l \mapsto m \otimes (n \otimes l)$. In a similar way we can construct a homomorphism in the opposite direction that is inverse to this one. This proves the group isomorphism.

Assume in addition M is an (S, R)-bimodule. Then for $s \in S$ and $t \in T$ we have

$$s\,((m \otimes n)t) = s(m \otimes nt) = sm \otimes nt = (sm \otimes n)t = (s(m \otimes n))\,t$$

so that $M \otimes_R N$ is an (S, T)-bimodule. Hence $(M \otimes_R N) \otimes_T L$ is a left S-module. Since $N \otimes_T L$ is a left R-module, also $M \otimes_R (N \otimes_T L)$ is a left S-module. The group isomorphism just established is easily seen to be a homomorphism of left S-modules by the same arguments used in previous proofs: it is additive and is S-linear on simple tensors since $s\,((m \otimes n) \otimes l) = s(m \otimes n) \otimes l = (sm \otimes n) \otimes l$ maps to the element $sm \otimes (n \otimes l) = s\,(m \otimes (n \otimes l))$. The proof is complete.