Theorem 14. *Let V be a finite-dimensional inner product space and let T be any linear operator on V. Then there exist a unitary operator U on V and a non-negative operator N on V such that T = UN. The non-negative operator N is unique. If T is invertible, the operator U is also unique.*

Proof. Suppose we have $T = UN$, where U is unitary and N is non-negative. Then $T^* = (UN)^* = N^*U^* = NU^*$. Thus $T^*T = NU^*UN = N^2$. This shows that N is uniquely determined as the non-negative square root of the non-negative operator T^*T.

So, to begin the proof of the existence of U and N, we use Theorem 13 to define N as the unique non-negative square root of T^*T. If T is invertible, then so is N because

$$(N\alpha|N\alpha) = (N^2\alpha|\alpha) = (T^*T\alpha|\alpha) = (T\alpha|T\alpha).$$

In this case, we define $U = TN^{-1}$ and prove that U is unitary. Now $U^* = (TN^{-1})^* = (N^{-1})^*T^* = (N^*)^{-1}T^* = N^{-1}T^*$. Thus

$$\begin{aligned} UU^* &= TN^{-1}N^{-1}T^* \\ &= T(N^{-1})^2T^* \\ &= T(N^2)^{-1}T^* \\ &= T(T^*T)^{-1}T^* \\ &= TT^{-1}(T^*)^{-1}T^* \\ &= I \end{aligned}$$

and U is unitary.

If T is not invertible, we shall have to do a bit more work to define U. We first define U on the range of N. Let α be a vector in the range of N, say $\alpha = N\beta$. We define $U\alpha = T\beta$, motivated by the fact that we want $UN\beta = T\beta$. We must verify that U is well-defined on the range of N; in other words, if $N\beta' = N\beta$ then $T\beta' = T\beta$. We verified above that $||N\gamma||^2 = ||T\gamma||^2$ for every γ in V. Thus, with $\gamma = \beta - \beta'$, we see that $N(\beta - \beta') = 0$ if and only if $T(\beta - \beta') = 0$. So U is well-defined on the range of N and is clearly linear where defined. Now if W is the range of N, we are going to define U on $W^\perp$. To do this, we need the following observation. Since T and N have the same null space, their ranges have the same dimension. Thus $W^\perp$ has the same dimension as the orthogonal complement of the range of T. Therefore, there exists an (inner product space) isomorphism U_0 of $W^\perp$ onto $T(V)^\perp$. Now we have defined U on W, and we define U on $W^\perp$ to be U_0.

Let us repeat the definition of U. Since $V = W \oplus W^\perp$, each α in V is uniquely expressible in the form $\alpha = N\beta + \gamma$, where $N\beta$ is in the range W of N, and γ is in $W^\perp$. We define

$$U\alpha = T\beta + U_0\gamma.$$

This U is clearly linear, and we verified above that it is well-defined. Also

$$\begin{aligned}(U\alpha|U\alpha) &= (T\beta + U_0\gamma|T\beta + U_0\gamma)\\ &= (T\beta|T\beta) + (U_0\gamma|U_0\gamma)\\ &= (N\beta|N\beta) + (\gamma|\gamma)\\ &= (\alpha|\alpha)\end{aligned}$$

and so U is unitary. We also have $UN\beta = T\beta$ for each β. ▌

We call $T = UN$ a **polar decomposition** for T. We certainly cannot call it *the* polar decomposition, since U is not unique. Even when T is invertible, so that U is unique, we have the difficulty that U and N may not commute. Indeed, they commute if and only if T is normal. For example, if $T = UN = NU$, with N non-negative and U unitary, then

$$TT^* = (NU)(NU)^* = NUU^*N = N^2 = T^*T.$$

The general operator T will also have a decomposition $T = N_1U_1$, with N_1 non-negative and U_1 unitary. Here, N_1 will be the non-negative square root of TT^*. We can obtain this result by applying the theorem just proved to the operator T^*, and then taking adjoints.

We turn now to the problem of what can be said about the simultaneous diagonalization of commuting families of normal operators. For this purpose the following terminology is appropriate.

Definitions. *Let $\mathfrak{F}$ be a family of operators on an inner product space* V. *A function* r *on $\mathfrak{F}$ with values in the field* F *of scalars will be called a* **root** *of $\mathfrak{F}$ if there is a non-zero α in* V *such that*

$$\mathrm{T}\alpha = \mathrm{r}(\mathrm{T})\alpha$$

for all T *in $\mathfrak{F}$. For any function* r *from $\mathfrak{F}$ to* F, *let* V(r) *be the set of all α in* V *such that* $\mathrm{T}\alpha = \mathrm{r}(\mathrm{T})\alpha$ *for every* T *in $\mathfrak{F}$.*

Then $V(r)$ is a subspace of V, and r is a root of $\mathfrak{F}$ if and only if $V(r) \neq \{0\}$. Each non-zero α in $V(r)$ is simultaneously a characteristic vector for every T in $\mathfrak{F}$.

Theorem 15. *Let $\mathfrak{F}$ be a commuting family of diagonalizable normal operators on a finite-dimensional inner product space* V. *Then $\mathfrak{F}$ has only a finite number of roots. If* $\mathrm{r}_1, \ldots, \mathrm{r}_k$ *are the distinct roots of $\mathfrak{F}$, then*

(i) $\mathrm{V}(\mathrm{r}_i)$ *is orthogonal to* $\mathrm{V}(\mathrm{r}_j)$ *when* $\mathrm{i} \neq \mathrm{j}$, *and*
(ii) $\mathrm{V} = \mathrm{V}(\mathrm{r}_1) \oplus \cdots \oplus \mathrm{V}(\mathrm{r}_k)$.

Proof. Suppose r and s are distinct roots of F. Then there is an operator T in $\mathfrak{F}$ such that $r(T) \neq s(T)$. Since characteristic vectors belonging to distinct characteristic values of T are necessarily orthogonal, it follows that $V(r)$ is orthogonal to $V(s)$. Because V is finite-dimensional, this implies $\mathfrak{F}$ has at most a finite number of roots. Let $r_1, \ldots, r_k$ be the roots of F. Suppose $\{T_1, \ldots, T_m\}$ is a maximal linearly independent subset of $\mathfrak{F}$, and let

$$\{E_{i1}, E_{i2}, \ldots\}$$

be the resolution of the identity defined by T_i $(1 \leq i \leq m)$. Then the projections E_{ij} form a commutative family. For each E_{ij} is a polynomial in T_i and $T_1, \ldots, T_m$ commute with one another. Since

$$I = (\sum_{j_1} E_{1j_1}) (\sum_{j_2} E_{2j_2}) \cdots (\sum_{j_m} E_{mj_m})$$

each vector α in V may be written in the form

$$\alpha = \sum_{j_1, \ldots, j_m} E_{1j_1}E_{2j_2} \cdots E_{mj_m}\alpha. \tag{9-13}$$

Suppose $j_1, \ldots, j_m$ are indices for which $\beta = E_{1j_1}E_{2j_2} \cdots E_{mj_m}\alpha \neq 0$. Let

$$\beta_i = (\prod_{n \neq i} E_{nj_n})\, \alpha.$$

Then $\beta = E_{ij_i}\beta_i$; hence there is a scalar c_i such that

$$T_i\beta = c_i\beta, \qquad 1 \leq i \leq m.$$

For each T in $\mathfrak{F}$, there exist unique scalars b_i such that

$$T = \sum_{i=1}^{m} b_iT_i.$$

Thus

$$\begin{aligned} T\beta &= \sum_i b_iT_i\beta \\ &= (\sum_i b_ic_i)\, \beta. \end{aligned}$$

The function $T \to \sum_i b_ic_i$ is evidently one of the roots, say r_t of $\mathfrak{F}$, and β lies in $V(r_t)$. Therefore, each non-zero term in (9-13) belongs to one of the spaces $V(r_1), \ldots, V(r_k)$. It follows that V is the orthogonal direct sum of $V(r_1), \ldots, V(r_k)$. ∎

Corollary. *Under the assumptions of the theorem, let* P_j *be the orthogonal projection of* V *on* $V(r_j)$, $(1 \leq j \leq k)$. *Then* $P_iP_j = 0$ *when* $i \neq j$,

$$I = P_1 + \cdots + P_k,$$

and every T *in* $\mathfrak{F}$ *may be written in the form*

$$T = \sum_j r_j(T)P_j. \tag{9-14}$$

Definitions. *The family of orthogonal projections* $\{P_1, \ldots, P_k\}$ *is called the* **resolution of the identity determined by** $\mathfrak{F}$, *and* (9-14) *is the* **spectral resolution of** T **in terms of this family.**

Although the projections $P_1, \ldots, P_k$ in the preceding corollary are canonically associated with the family $\mathfrak{F}$, they are generally not in $\mathfrak{F}$ nor even linear combinations of operators in $\mathcal{F}$; however, we shall show that they may be obtained by forming certain products of polynomials in elements of $\mathcal{F}$.

In the study of any family of linear operators on an inner product space, it is usually profitable to consider the self-adjoint algebra generated by the family.

Definition. *A* **self-adjoint algebra of operators** *on an inner product space* V *is a linear subalgebra of* $L(V, V)$ *which contains the adjoint of each of its members.*

An example of a self-adjoint algebra is $L(V, V)$ itself. Since the intersection of any collection of self-adjoint algebras is again a self-adjoint algebra, the following terminology is meaningful.

Definition. *If* $\mathcal{F}$ *is a family of linear operators on a finite-dimensional inner product space, the* **self-adjoint algebra generated by** $\mathcal{F}$ *is the smallest self-adjoint algebra which contains* $\mathcal{F}$.

Theorem 16. *Let* $\mathcal{F}$ *be a commuting family of diagonalizable normal operators on a finite-dimensional inner product space* V*, and let* $\mathcal{A}$ *be the self-adjoint algebra generated by* $\mathcal{F}$ *and the identity operator. Let* $\{P_1, \ldots, P_k\}$ *be the resolution of the identity defined by* $\mathcal{F}$. *Then* $\mathcal{A}$ *is the set of all operators on* V *of the form*

$$T = \sum_{j=1}^{k} c_j P_j \tag{9-15}$$

where $c_1, \ldots, c_k$ *are arbitrary scalars.*

Proof. Let $\mathcal{C}$ denote the set of all operators on V of the form (9-15). Then $\mathcal{C}$ contains the identity operator and the adjoint

$$T^* = \sum_j \bar{c}_j P_j$$

of each of its members. If $T = \sum_j c_j P_j$ and $U = \sum_j d_j P_j$, then for every scalar a

$$aT + U = \sum_j (ac + d_j) P_j$$

and

$$\begin{aligned} TU &= \sum_{i,j} c_i d_j P_i P_j \\ &= \sum_j c_j d_j P_j \\ &= UT. \end{aligned}$$

Thus $\mathcal{C}$ is a self-adjoint commutative algebra containing $\mathcal{F}$ and the identity operator. Therefore $\mathcal{C}$ contains $\mathcal{A}$.

Now let $r_1, \ldots, r_k$ be all the roots of $\mathfrak{F}$. Then for each pair of indices (i, n) with $i \neq n$, there is an operator T_{in} in $\mathfrak{F}$ such that $r_i(T_{in}) \neq r_n(T_{in})$. Let $a_{in} = r_i(T_{in}) - r_n(T_{in})$ and $b_{in} = r_n(T_{in})$. Then the linear operator

$$Q_i = \prod_{n \neq i} a_{in}^{-1}(T_{in} - b_{in}I)$$

is an element of the algebra $\mathfrak{A}$. We will show that $Q_i = P_i$ $(1 \leq i \leq k)$. For this, suppose $j \neq i$ and that α is an arbitrary vector in $V(r_j)$. Then

$$\begin{aligned} T_{ij}\alpha &= r_j(T_{ij})\alpha \\ &= b_{ij}\alpha \end{aligned}$$

so that $(T_{ij} - b_{ij}I)\alpha = 0$. Since the factors in Q_i all commute, it follows that $Q_i\alpha = 0$. Hence Q_i agrees with P_i on $V(r_j)$ whenever $j \neq i$. Now suppose α is a vector in $V(r_i)$. Then $T_{in}\alpha = r_i(T_{in})\alpha$, and

$$a_{in}^{-1}(T_{in} - b_{in}I)\alpha = a_{in}^{-1}[r_i(T_{in}) - r_n(T_{in})]\alpha = \alpha.$$

Thus $Q_i\alpha = \alpha$ and Q_i agrees with P_i on $V(r_i)$; therefore, $Q_i = P_i$ for $i = 1, \ldots, k$. From this it follows that $\mathfrak{A} = \mathfrak{C}$. ▌

The theorem shows that the algebra $\mathfrak{A}$ is commutative and that each element of $\mathfrak{A}$ is a diagonalizable normal operator. We show next that $\mathfrak{A}$ has a single generator.

Corollary. *Under the assumptions of the theorem, there is an operator* T *in* $\mathfrak{A}$ *such that every member of* $\mathfrak{A}$ *is a polynomial in* T.

Proof. Let $T = \sum_{j=1}^{k} t_jP_j$ where $t_1, \ldots, t_k$ are distinct scalars. Then

$$T^n = \sum_{j=1}^{k} t_j^nP_j$$

for $n = 1, 2, \ldots$. If

$$f = \sum_{n=1}^{s} a_nx^n$$

it follows that

$$\begin{aligned} f(T) &= \sum_{n=1}^{s} a_nT^n = \sum_{n=1}^{s}\sum_{j=1}^{k} a_nt_j^nP_j \\ &= \sum_{j=1}^{k}\left(\sum_{n=1}^{s} a_nt_j^n\right)P_j \\ &= \sum_{j=1}^{k} f(t_j)P_j. \end{aligned}$$

Given an arbitrary

$$U = \sum_{j=1}^{k} c_jP_j$$

in $\mathfrak{A}$, there is a polynomial f such that $f(t_j) = c_j$ $(1 \leq j \leq k)$, and for any such f, $U = f(T)$. ▌

Exercises

1. Give a reasonable definition of a non-negative $n \times n$ matrix, and then prove that such a matrix has a unique non-negative square root.

2. Let A be an $n \times n$ matrix with complex entries such that $A^* = -A$, and let $B = e^A$. Show that

(a) $\det B = e^{\operatorname{tr} A}$;
(b) $B^* = e^{-A}$;
(c) B is unitary.

3. If U and T are normal operators which commute, prove that $U + T$ and UT are normal.

4. Let T be a linear operator on the finite-dimensional complex inner product space V. Prove that the following ten statements about T are equivalent.

(a) T is normal.
(b) $\|T\alpha\| = \|T^*\alpha\|$ for every α in V.
(c) $T = T_1 + iT_2$, where T_1 and T_2 are self-adjoint and $T_1T_2 = T_2T_1$.
(d) If α is a vector and c a scalar such that $T\alpha = c\alpha$, then $T^*\alpha = \bar{c}\alpha$.
(e) There is an orthonormal basis for V consisting of characteristic vectors for T.
(f) There is an orthonormal basis $\mathcal{B}$ such that $[T]_{\mathcal{B}}$ is diagonal.
(g) There is a polynomial g with complex coefficients such that $T^* = g(T)$.
(h) Every subspace which is invariant under T is also invariant under T^*.
(i) $T = NU$, where N is non-negative, U is unitary, and N commutes with U.
(j) $T = c_1E_1 + \cdots + c_kE_k$, where $I = E_1 + \cdots + E_k$, $E_iE_j = 0$ for $i \neq j$, and $E_j^2 = E_j = E_j^*$.

5. Use Exercise 3 to show that any commuting family of normal operators (not necessarily diagonalizable ones) on a finite-dimensional inner product space generates a commutative self-adjoint algebra of normal operators.

6. Let V be a finite-dimensional complex inner product space and U a unitary operator on V such that $U\alpha = \alpha$ implies $\alpha = 0$. Let

$$f(z) = i\frac{(1+z)}{(1-z)}, \qquad z \neq 1$$

and show that

(a) $f(U) = i(I + U)(I - U)^{-1}$;
(b) $f(U)$ is self-adjoint;
(c) for every self-adjoint operator T on V, the operator

$$U = (T - iI)(T + iI)^{-1}$$

is unitary and such that $T = f(U)$.

7. Let V be the space of complex $n \times n$ matrices equipped with the inner product

$$(A|B) = \operatorname{tr}(AB^*).$$