Definition 19.3.2 (Lebesgue integral). Let $f : \Omega \to \mathbf{R}^*$ be an absolutely integrable function. We define the *Lebesgue integral* $\int_\Omega f$ of f to be the quantity

$$\int_\Omega f := \int_\Omega f^+ - \int_\Omega f^-.$$

Note that since f is absolutely integrable, $\int_\Omega f^+$ and $\int_\Omega f^-$ are less than or equal to $\int_\Omega |f|$ and hence are finite. Thus $\int_\Omega f$ is always finite; we are never encountering the indeterminate form $+\infty - (+\infty)$.

Note that this definition is consistent with our previous definition of the Lebesgue integral for non-negative functions, since if f is non-negative then $f^+ = f$ and $f^- = 0$. We also have the useful *triangle inequality*

$$|\int_\Omega f| \le \int_\Omega f^+ + \int_\Omega f^- = \int_\Omega |f| \qquad (19.1)$$

(Exercise 19.3.1).

Some other properties of the Lebesgue integral:

Proposition 19.3.3. *Let Ω be a measurable set, and let $f : \Omega \to \mathbf{R}$ and $g : \Omega \to \mathbf{R}$ be absolutely integrable functions.*

(a) *For any real number c (positive, zero, or negative), we have that cf is absolutely integrable and $\int_\Omega cf = c \int_\Omega f$.*

(b) *The function $f + g$ is absolutely integrable, and $\int_\Omega (f + g) = \int_\Omega f + \int_\Omega g$.*

(c) *If $f(x) \le g(x)$ for all $x \in \Omega$, then we have $\int_\Omega f \le \int_\Omega g$.*

(d) *If $f(x) = g(x)$ for almost every $x \in \Omega$, then $\int_\Omega f = \int_\Omega g$.*

Proof. See Exercise 19.3.2.   □

As mentioned in the previous section, one cannot necessarily interchange limits and integrals, $\lim \int f_n = \int \lim f_n$, as the "moving bump example" showed. However, it is possible to exclude the

moving bump example, and successfully interchange limits and integrals, if we know that the functions f_n are all majorized by a single absolutely integrable function. This important theorem is known as the *Lebesgue dominated convergence theorem*, and is extremely useful:

Theorem 19.3.4 (Lebesgue dominated convergence thm). *Let Ω be a measurable subset of $\mathbf{R}^n$, and let $f_1, f_2, \ldots$ be a sequence of measurable functions from Ω to $\mathbf{R}^*$ which converge pointwise. Suppose also that there is an absolutely integrable function F : $\Omega \to [0, \infty]$ such that $|f_n(x)| \leq F(x)$ for all $x \in \Omega$ and all $n = 1, 2, 3, \ldots$. Then*

$$\int_\Omega \lim_{n \to \infty} f_n = \lim_{n \to \infty} \int_\Omega f_n.$$

Proof. Let $f : \Omega \to \mathbf{R}^*$ be the function $f(x) := \lim_{n \to \infty} f_n(x)$; this function exists by hypothesis. By Lemma 18.5.10, f is measurable. Also, since $|f_n(x)| \leq F(x)$ for all n and all $x \in \Omega$, we see that each f_n is absolutely integrable, and by taking limits we obtain $|f(x)| \leq F(x)$ for all $x \in \Omega$, so f is also absolutely integrable. Our task is to show that $\lim_{n \to \infty} \int_\Omega f_n = \int_\Omega f$.

The functions $F + f_n$ are non-negative and converge pointwise to $F + f$. So by Fatou's lemma (Lemma 19.2.13)

$$\int_\Omega F + f \leq \lim_{n \to \infty} \inf \int_\Omega F + f_n$$

and thus

$$\int_\Omega f \leq \lim_{n \to \infty} \inf \int_\Omega f_n.$$

But the functions $F - f_n$ are also non-negative and converge pointwise to $F - f$. So by Fatou's lemma again

$$\int_\Omega F - f \leq \lim_{n \to \infty} \inf \int_\Omega F - f_n.$$

Since the right-hand side is $\int_\Omega F - \limsup_{n \to \infty} \int_\Omega f_n$ (why did the lim inf become a lim sup?), we thus have

$$\int_\Omega f \geq \lim_{n \to \infty} \sup \int_\Omega f_n.$$

Thus the lim inf and lim sup of $\int_\Omega f_n$ are both equal to $\int_\Omega f$, as desired.   □

Finally, we record a lemma which is not particularly interesting in itself, but will have some useful consequences later in these notes.

Definition 19.3.5 (Upper and lower Lebesgue integral). Let Ω be a measurable subset of $\mathbf{R}^n$, and let $f : \Omega \to \mathbf{R}$ be a function (not necessarily measurable). We define the *upper Lebesgue integral* $\overline{\int}_\Omega f$ to be

$$\overline{\int}_\Omega f := \inf\{\int_\Omega g : g \text{ is an absolutely integrable function}$$

$$\text{from } \Omega \text{ to } \mathbf{R} \text{ that majorizes } f\}$$

and the *lower Lebesgue integral* $\underline{\int}_\Omega f$ to be

$$\underline{\int}_\Omega f := \sup\{\int_\Omega g : g \text{ is an absolutely integrable function}$$

$$\text{from } \Omega \text{ to } \mathbf{R} \text{ that minorizes } f\}.$$

It is easy to see that $\underline{\int}_\Omega f \leq \overline{\int}_\Omega f$ (why? use Proposition 19.3.3(c)). When f is absolutely integrable then equality occurs (why?). The converse is also true:

Lemma 19.3.6. *Let Ω be a measurable subset of $\mathbf{R}^n$, and let $f : \Omega \to \mathbf{R}$ be a function (not necessarily measurable). Let A be a real number, and suppose $\overline{\int}_\Omega f = \underline{\int}_\Omega f = A$. Then f is absolutely integrable, and*

$$\int_\Omega f = \overline{\int}_\Omega f = \underline{\int}_\Omega f = A.$$

Proof. By definition of upper Lebesgue integral, for every integer $n \geq 1$ we may find an absolutely integrable function $f_n^+ : \Omega \to \mathbf{R}$ which majorizes f such that

$$\int_\Omega f_n^+ \leq A + \frac{1}{n}.$$

Similarly we may find an absolutely integrable function $f_n^- : \Omega \to$ **R** which minorizes f such that

$$\int_\Omega f_n^- \leq A - \frac{1}{n}.$$

Let $F^+ := \inf_n f_n^+$ and $F^- := \sup_n f_n^-$. Then F^+ and F^- are measurable (by Lemma 18.5.10) and absolutely integrable (because they are squeezed between the absolutely integrable functions f_1^+ and f_1^-, for instance). Also, F^+ majorizes f and F^- minorizes f. Finally, we have

$$\int_\Omega F^+ \leq \int_\Omega f_n^+ \leq A + \frac{1}{n}$$

for every n, and hence

$$\int_\Omega F^+ \leq A.$$

Similarly we have

$$\int_\Omega F^- \geq A.$$

but F^+ majorizes F^-, and hence $\int_\Omega F^+ \geq \int_\Omega F^-$. Hence we must have

$$\int_\Omega F^+ = \int_\Omega F^- = A.$$

In particular

$$\int_\Omega F^+ - F^- = 0.$$

By Proposition 19.2.6(a), we thus have $F^+(x) = F^-(x)$ for almost every x. But since f is squeezed between F^- and F^+, we thus have $f(x) = F^+(x) = F^-(x)$ for almost every x. In particular, f differs from the absolutely integrable function F^+ only on a set of measure zero and is thus measurable (see Exercise 18.5.5) and absolutely integrable, with

$$\int_\Omega f = \int_\Omega F^+ = \int_\Omega F^- = A$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Exercise 19.3.1. Prove (19.1) whenever Ω is a measurable subset of $\mathbf{R}^n$ and f is an absolutely integrable function.

Exercise 19.3.2. Prove Proposition 19.3.3. (Hint: for (b), break f, g, and $f + g$ up into positive and negative parts, and try to write everything in terms of integrals of non-negative functions only, using Lemma 19.2.10.)

Exercise 19.3.3. Let $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ be absolutely integrable, measurable functions such that $f(x) \leq g(x)$ for all $x \in \mathbf{R}$, and that $\int_{\mathbf{R}} f = \int_R g$. Show that $f(x) = g(x)$ for almost every $x \in \mathbf{R}$ (i.e., that $f(x) = g(x)$ for all $x \in \mathbf{R}$ except possibly for a set of measure zero).

19.4   Comparison with the Riemann integral

We have spent a lot of effort constructing the Lebesgue integral, but have not yet addressed the question of how to actually compute any Lebesgue integrals, and whether Lebesgue integration is any different from the Riemann integral (say for integrals in one dimension). Now we show that the Lebesgue integral is a generalization of the Riemann integral. To clarify the following discussion, we shall temporarily distinguish the Riemann integral from the Lebesgue integral by writing the Riemann integral $\int_I f$ as $R. \int_I f$.

Our objective here is to prove

Proposition 19.4.1. *Let $I \subseteq \mathbf{R}$ be an interval, and let $f : I \to \mathbf{R}$ be a Riemann integrable function. Then f is also absolutely integrable, and $\int_I f = R. \int_I f$.*

Proof. Write $A := R. \int_I f$. Since f is Riemann integrable, we know that the upper and lower Riemann integrals are equal to A. Thus, for every $\varepsilon > 0$, there exists a partition $\mathbf{P}$ of I into smaller intervals J such that

$$A - \varepsilon \leq \sum_{J \in \mathbf{P}} |J| \inf_{x \in J} f(x) \leq A \leq \sum_{J \in \mathbf{P}} |J| \sup_{x \in J} f(x) \leq A + \varepsilon,$$

where $|J|$ denotes the length of J. Note that $|J|$ is the same as $m(J)$, since J is a box.

Let $f_\varepsilon^- : I \to \mathbf{R}$ and $f_\varepsilon^+ : I \to \mathbf{R}$ be the functions

$$f_\varepsilon^-(x) = \sum_{J \in \mathbf{P}} \inf_{x \in J} f(x) \chi_J(x)$$

and

$$f_\varepsilon^+(x) = \sum_{J \in \mathbf{P}} \sup_{x \in J} f(x) \chi_J(x);$$

these are simple functions and hence measurable and absolutely integrable. By Lemma 19.1.9 we have

$$\int_I f_\varepsilon^- = \sum_{J \in \mathbf{P}} |J| \inf_{x \in J} f(x)$$

and

$$\int_I f_\varepsilon^+ = \sum_{J \in \mathbf{P}} |J| \sup_{x \in J} f(x)$$

and hence

$$A - \varepsilon \le \int_I f_\varepsilon^- \le A \le \int_I f_\varepsilon^+ \le A + \varepsilon.$$

Since f_ε^+ majorizes f, and f_ε^- minorizes f, we thus have

$$A - \varepsilon \le \underline{\int}_\Omega f \le \overline{\int}_\Omega f \le A + \varepsilon$$

for every ε, and thus

$$\underline{\int}_\Omega f = \overline{\int}_\Omega f = A$$

and hence by Lemma 19.3.6, f is absolutely integrable with $\int_I f = A$, as desired. □

Thus every Riemann integrable function is also Lebesgue integrable, at least on bounded intervals, and we no longer need the $R. \int_I f$ notation. However, the converse is not true. Take for

instance the function $f : [0,1] \to \mathbf{R}$ defined by $f(x) := 1$ when
x is rational, and $f(x) := 0$ when x is irrational. Then from
Proposition 11.7.1 we know that f is not Riemann integrable.
On the other hand, f is the characteristic function of the set
$\mathbf{Q} \cap [0,1]$, which is countable and hence measure zero. Thus f
is Lebesgue integrable and $\int_{[0,1]} f = 0$. Thus the Lebesgue inte-
gral can handle more functions than the Riemann integral; this is
one of the primary reasons why we use the Lebesgue integral in
analysis. (The other reason is that the Lebesgue integral interacts
well with limits, as the Lebesgue monotone convergence theorem,
Fatou's lemma, and Lebesgue dominated convergence theorem al-
ready attest. There are no comparable theorems for the Riemann
integral).

19.5   Fubini's theorem

In one dimension we have shown that the Lebesgue integral is
connected to the Riemann integral. Now we will try to understand
the connection in higher dimensions. To simplify the discussion we
shall just study two-dimensional integrals, although the arguments
we present here can easily be extended to higher dimensions.

    We shall study integrals of the form $\int_{\mathbf{R}^2} f$. Note that once
we know how to integrate on $\mathbf{R}^2$, we can integrate on measurable
subsets Ω of $\mathbf{R}^2$, since $\int_\Omega f$ can be rewritten as $\int_{\mathbf{R}^2} f\chi_\Omega$.

    Let $f(x,y)$ be a function of two variables. In principle, we
have three different ways to integrate f on $\mathbf{R}^2$. First of all, we
can use the two-dimensional Lebesgue integral, to obtain $\int_{\mathbf{R}^2} f$.
Secondly, we can fix x and compute a one-dimensional integral in
y, and then take that quantity and integrate in x, thus obtaining
$\int_{\mathbf{R}} (\int_{\mathbf{R}} f(x,y)\ dy)\ dx$. Secondly, we could fix y and integrate in
x, and then integrate in y, thus obtaining $\int_{\mathbf{R}} (\int_{\mathbf{R}} f(x,y)\ dx)\ dy$.

    Fortunately, if the function f is absolutely integrable on f,
then all three integrals are equal:

Theorem 19.5.1 (Fubini's theorem). *Let $f : \mathbf{R}^2 \to \mathbf{R}$ be an ab-
solutely integrable function. Then there exists absolutely integrable*

functions $F : \mathbf{R} \to \mathbf{R}$ and $G : \mathbf{R} \to \mathbf{R}$ such that for almost every x, $f(x, y)$ is absolutely integrable in y with

$$F(x) = \int_{\mathbf{R}} f(x, y) \, dy,$$

and for almost every y, $f(x, y)$ is absolutely integrable in x with

$$G(y) = \int_{\mathbf{R}} f(x, y) \, dx.$$

Finally, we have

$$\int_{\mathbf{R}} F(x) \, dx = \int_{\mathbf{R}^2} f = \int_{\mathbf{R}} G(y) \, dy.$$

Remark 19.5.2. Very roughly speaking, Fubini's theorem says that

$$\int_{\mathbf{R}} (\int_{\mathbf{R}} f(x, y) \, dy) \, dx = \int_{\mathbf{R}^2} f = \int_{\mathbf{R}} (\int_{\mathbf{R}} f(x, y) \, dx) \, dy.$$

This allows us to compute two-dimensional integrals by splitting them into two one-dimensional integrals. The reason why we do not write Fubini's theorem this way, though, is that it is possible that the integral $\int_{\mathbf{R}} f(x, y) \, dy$ does not actually exist for every x, and similarly $\int_{\mathbf{R}} f(x, y) \, dx$ does not exist for every y; Fubini's theorem only asserts that these integrals only exist for *almost every* x and y. For instance, if $f(x, y)$ is the function which equals 1 when $x > 0$ and $y = 0$, equals -1 when $x < 0$ and $y = 0$, and is zero otherwise, then f is absolutely integrable on $\mathbf{R}^2$ and $\int_{\mathbf{R}^2} f = 0$ (since f equals zero almost everywhere in $\mathbf{R}^2$), but $\int_{\mathbf{R}} f(x, y) \, dy$ is not absolutely integrable when $x = 0$ (though it is absolutely integrable for every other x).

Proof. The proof of Fubini's theorem is quite complicated and we will only give a sketch here. We begin with a series of reductions. Roughly speaking (ignoring issues relating to sets of measure zero), we have to show that

$$\int_{\mathbf{R}} (\int_{\mathbf{R}} f(x, y) \, dy) \, dx = \int_{\mathbf{R}^2} f$$

together with a similar equality with x and y reversed. We shall just prove the above equality, as the other one is very similar.

First of all, it suffices to prove the theorem for non-negative functions, since the general case then follows by writing a general function f as a difference $f^+ - f^-$ of two non-negative functions, and applying Fubini's theorem to f^+ and f^- separately (and using Proposition 19.3.3(a) and (b)). Thus we will henceforth assume that f is non-negative.

Next, it suffices to prove the theorem for non-negative functions f supported on a bounded set such as $[-N, N] \times [-N, N]$ for some positive integer N. Indeed, once one obtains Fubini's theorem for such functions, one can then write a general function f as the supremum of such compactly supported functions as

$$f = \sup_{N>0} f\chi_{[-N,N]\times[-N,N]},$$

apply Fubini's theorem to each function $f\chi_{[-N,N]\times[-N,N]}$ separately, and then take suprema using the monotone convergence theorem. Thus we will henceforth assume that f is supported on $[-N, N] \times [-N, N]$.

By another similar argument, it suffices to prove the theorem for non-negative simple functions supported on $[-N, N]\times[-N, N]$, since one can use Lemma 19.1.4 to write f as the supremum of simple functions (which must also be supported on $[-N, N]$), apply Fubini's theorem to each simple function, and then take suprema using the monotone convergence theorem. Thus we may assume that f is a non-negative simple function supported on $[-N, N] \times [-N, N]$.

Next, we see that it suffices to prove the theorem for characteristic functions supported in $[-N, N] \times [-N, N]$. This is because every simple function is a linear combination of characteristic functions, and so we can deduce Fubini's theorem for simple functions from Fubini's theorem for characteristic functions. Thus we may take $f = \chi_E$ for some measurable $E \subseteq [-N, N] \times [-N, N]$. Our task is then to show (ignoring sets of measure zero) that

$$\int_{[-N,N]} (\int_{[-N,N]} \chi_E(x,y)\ dy)\ dx = m(E).$$

It will suffice to show the upper Lebesgue integral estimate

$$\overline{\int}_{[-N,N]} \left(\overline{\int}_{[-N,N]} \chi_E(x,y) \, dy \right) dx \le m(E). \qquad (19.2)$$

We will prove this estimate later. Once we show this for every set E, we may substitute E with $[-N,N] \times [-N,N] \backslash E$ and obtain

$$\overline{\int}_{[-N,N]} \left(\overline{\int}_{[-N,N]} (1 - \chi_E(x,y)) \, dy \right) dx \le 4N^2 - m(E).$$

But the left-hand side is equal to

$$\overline{\int}_{[-N,N]} \left(2N - \underline{\int}_{[-N,N]} \chi_E(x,y) \, dy \right) dx$$

which is in turn equal to

$$4N^2 - \underline{\int}_{[-N,N]} \left(\underline{\int}_{[-N,N]} \chi_E(x,y) \, dy \right) dx$$

and thus we have

$$\underline{\int}_{[-N,N]} \left(\underline{\int}_{[-N,N]} \chi_E(x,y) \, dy \right) dx \ge m(E).$$

In particular we have

$$\underline{\int}_{[-N,N]} \left(\overline{\int}_{[-N,N]} \chi_E(x,y) \, dy \right) dx \ge m(E)$$

and hence by Lemma 19.3.6 we see that $\overline{\int}_{[-N,N]} \chi_E(x,y) \, dy$ is absolutely integrable and

$$\int_{[-N,N]} \left(\overline{\int}_{[-N,N]} \chi_E(x,y) \, dy \right) dx = m(E).$$

A similar argument shows that

$$\int_{[-N,N]} \left(\underline{\int}_{[-N,N]} \chi_E(x,y) \, dy \right) dx = m(E)$$

and hence

$$\int_{[-N,N]} (\overline{\int}_{[-N,N]} \chi_E(x,y) \, dy - \underline{\int}_{[-N,N]} \chi_E(x,y)) \, dx = 0.$$

Thus by Proposition 19.2.6(a) we have

$$\underline{\int}_{[-N,N]} \chi_E(x,y) \, dy = \overline{\int}_{[-N,N]} \chi_E(x,y) \, dy$$

for almost every $x \in [-N, N]$. Thus $\chi_E(x,y)$ is absolutely integrable in y for almost every x, and $\int_{[-N,N]} \chi_E(x,y)$ is thus equal (almost everywhere) to a function $F(x)$ such that

$$\int_{[-N,N]} F(x) \, dx = m(E)$$

as desired.

It remains to prove the bound (19.2). Let $\varepsilon > 0$ be arbitrary. Since $m(E)$ is the same as the outer measure $m^*(E)$, we know that there exists an at most countable collection $(B_j)_{j \in J}$ of boxes such that $E \subseteq \bigcup_{j \in J} B_j$ and

$$\sum_{j \in J} m(B_j) \le m(E) + \varepsilon.$$

Each box B_j can be written as $B_j = I_j \times I_j'$ for some intervals I_j and I_j'. Observe that

$$m(B_j) = |I_j||I_j'| = \int_{I_j} |I_j'| \, dx = \int_{I_j} (\int_{I_j'} dy) dx$$

$$= \int_{[-N,N]} (\int_{[-N,N]} \chi_{I_j \times I_j'}(x,y) \, dx) dy = \int_{[-N,N]} (\int_{[-N,N]} \chi_{B_j}(x,y) \, dx) dy.$$

Adding this over all $j \in J$ (using Corollary 19.2.11) we obtain

$$\sum_{j \in J} m(B_j) = \int_{[-N,N]} (\int_{[-N,N]} \sum_{j \in J} \chi_{B_j}(x,y) \, dx) dy.$$

In particular we have

$$\overline{\int}_{[-N,N]} (\overline{\int}_{[-N,N]} \sum_{j \in J} \chi_{B_j}(x,y) \ dx) dy \leq m(E) + \varepsilon.$$

But $\sum_{j \in J} \chi_{B_j}$ majorizes χ_E (why?) and thus

$$\overline{\int}_{[-N,N]} (\overline{\int}_{[-N,N]} \chi_E(\hat{x},y) \ dx) dy \leq m(E) + \varepsilon.$$

But ε is arbitrary, and so we have (19.2) as desired. This completes the proof of Fubini's theorem. $\square$

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