If D is fixed, then given any R-module X we have an associated abelian group $\mathrm{Hom}_R(D, X)$. Further, an R-module homomorphism $\alpha : X \to Y$ induces an abelian group homomorphism $\alpha' : \mathrm{Hom}_R(D, X) \to \mathrm{Hom}_R(D, Y)$, defined by $\alpha'(f) = \alpha \circ f$. Put another way, the map $\mathrm{Hom}_R(D, \_\_)$ is a *covariant functor* from the category of R-modules to the category of abelian groups (cf. Appendix II). Theorem 28 shows that applying this functor to the terms in the exact sequence

$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow 0$$

produces an exact sequence

$$0 \to \mathrm{Hom}_R(D, L) \xrightarrow{\psi'} \mathrm{Hom}_R(D, M) \xrightarrow{\varphi'} \mathrm{Hom}_R(D, N).$$

This is referred to by saying that $\mathrm{Hom}_R(D, \_\_)$ is a *left exact* functor. By Proposition 30, the functor $\mathrm{Hom}_R(D, \_\_)$ is *exact*, i.e., always takes short exact sequences to short exact sequences, if and only if D is projective. We summarize this as

Corollary 32. If D is an R-module, then the functor $\mathrm{Hom}_R(D, \_\_)$ from the category of R-modules to the category of abelian groups is left exact. It is exact if and only if D is a projective R-module.

Note that if $\mathrm{Hom}_R(D, \_\_)$ takes short exact sequences to short exact sequences, then it takes exact sequences of any length to exact sequences since any exact sequence can be broken up into a succession of short exact sequences.

As we have seen, the functor $\mathrm{Hom}_R(D, \_\_)$ is in general not exact on the right. Measuring the extent to which functors such as $\mathrm{Hom}_R(D, \_\_)$ fail to be exact leads to the notions of "homological algebra," considered in Chapter 17.

Examples

(1) We shall see in Section 11.1 that if $R = F$ is a field then every F-module is projective (although we only prove this for finitely generated modules).

(2) By Corollary 31, $\mathbb{Z}$ is a projective $\mathbb{Z}$-module. This can be seen directly as follows: suppose f is a map from $\mathbb{Z}$ to N and $M \xrightarrow{\varphi} N \to 0$ is exact. The homomorphism f is uniquely determined by the value $n = f(1)$. Then f can be lifted to a homomorphism $F : \mathbb{Z} \to M$ by first defining $F(1) = m$, where m is any element in M mapped to n by φ, and then extending F to all of $\mathbb{Z}$ by additivity.

By the first statement in Proposition 30, since $\mathbb{Z}$ is projective, if

$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow 0$$

is an exact sequence of $\mathbb{Z}$-modules, then

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, L) \xrightarrow{\psi'} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, M) \xrightarrow{\varphi'} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, N) \longrightarrow 0$$

is also an exact sequence. This can also be seen directly using the isomorphism $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, M) \cong M$ of abelian groups, which shows that the two exact sequences above are essentially the same.

(3) Free $\mathbb{Z}$-modules have no nonzero elements of finite order so no nonzero finite abelian group can be isomorphic to a submodule of a free module. By Corollary 31 it follows that no nonzero finite abelian group is a projective $\mathbb{Z}$-module.

(4) As a particular case of the preceding example, we see that for $n \geq 2$ the $\mathbb{Z}$-module $\mathbb{Z}/n\mathbb{Z}$ is not projective. By Theorem 28 it must be possible to find a short exact sequence which after applying the functor $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \_\_)$ is no longer exact on the right. One such sequence is the exact sequence of Example 2 following Corollary 23:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0,$$

for $n \geq 2$. Note first that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = 0$ since there are no nonzero $\mathbb{Z}$-module homomorphisms from $\mathbb{Z}/n\mathbb{Z}$ to $\mathbb{Z}$. It is also easy to see that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$, as follows. Every homomorphism f is uniquely determined by $f(1) = a \in \mathbb{Z}/n\mathbb{Z}$, and given any $a \in \mathbb{Z}/n\mathbb{Z}$ there is a unique homomorphism f_a with $f_a(1) = a$; the map $f_a \mapsto a$ is easily checked to be an isomorphism from $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$ to $\mathbb{Z}/n\mathbb{Z}$.

Applying $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \_\_)$ to the short exact sequence above thus gives the sequence

$$0 \longrightarrow 0 \xrightarrow{n'} 0 \xrightarrow{\pi'} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

which is not exact at its only nonzero term.

(5) Since $\mathbb{Q}/\mathbb{Z}$ is a torsion $\mathbb{Z}$-module it is not a submodule of a free $\mathbb{Z}$-module, hence is not projective. Note also that the exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \xrightarrow{\pi} \mathbb{Q}/\mathbb{Z} \to 0$ does not split since $\mathbb{Q}$ contains no submodule isomorphic to $\mathbb{Q}/\mathbb{Z}$.

(6) The $\mathbb{Z}$-module $\mathbb{Q}$ is not projective (cf. the exercises).

(7) We shall see in Chapter 12 that a finitely generated $\mathbb{Z}$-module is projective if and only if it is free.

(8) Let R be the commutative ring $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ under componentwise addition and multiplication. If P_1 and P_2 are the principal ideals generated by $(1, 0)$ and $(0, 1)$ respectively then $R = P_1 \oplus P_2$, hence both P_1 and P_2 are projective R-modules by Proposition 30. Neither P_1 nor P_2 is free, since any free module has order a multiple of four.

(9) The direct sum of two projective modules is again projective (cf. Exercise 3).

(10) We shall see in Part VI that if F is any field and $n \in \mathbb{Z}^+$ then the ring $R = M_n(F)$ of all $n \times n$ matrices with entries from F has the property that every R-module is projective. We shall also see that if G is a finite group of order n and $n \neq 0$ in the field F then the group ring FG also has the property that every module is projective.

Injective Modules and $Hom_R(\_\_, D)$

If $0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow 0$ is a short exact sequence of R-modules then, instead of considering maps *from* an R-module D into L or N and the extent to which these determine maps from D into M, we can consider the "dual" question of maps from L or N *to* D. In this case, it is easy to dispose of the situation of a map from N to D: an R-module map from N to D immediately gives a map from M to D simply by composing with φ. It is easy to check that this defines an injective homomorphism of abelian groups

$$\begin{aligned} \varphi' : \mathrm{Hom}_R(N, D) &\longrightarrow \mathrm{Hom}_R(M, D) \\ f &\longmapsto f' = f \circ \varphi, \end{aligned}$$

or, put another way,

$$\text{if} \quad M \xrightarrow{\varphi} N \to 0 \quad \text{is exact,}$$

$$\text{then} \quad 0 \to \operatorname{Hom}_R(N, D) \xrightarrow{\varphi'} \operatorname{Hom}_R(M, D) \quad \text{is exact.}$$

(Note that the associated maps on the homomorphism groups are in the reverse direction from the original maps.)

On the other hand, given an R-module homomorphism f from L to D it may not be possible to extend f to a map F from M to D, i.e., given f it may not be possible to find a map F making the following diagram commute:

$$\begin{tikzcd} L \arrow[r, "\psi"] \arrow[d, "f"'] & M \arrow[dl, dashed, "F"] \\ D & \end{tikzcd}$$

For example, consider the exact sequence $0 \longrightarrow \mathbb{Z} \xrightarrow{\psi} \mathbb{Z} \xrightarrow{\varphi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$ of $\mathbb{Z}$-modules, where ψ is multiplication by 2 and φ is the natural projection. Take $D = \mathbb{Z}/2\mathbb{Z}$ and let $f : \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ be reduction modulo 2 on the first $\mathbb{Z}$ in the sequence. There is only one nonzero homomorphism F from the second $\mathbb{Z}$ in the sequence to $\mathbb{Z}/2\mathbb{Z}$ (namely, reduction modulo 2), but this F does not lift the map f since $F \circ \psi(\mathbb{Z}) = F(2\mathbb{Z}) = 0$, so $F \circ \psi \neq f$.

Composition with ψ induces an abelian group homomorphism ψ' from$\operatorname{Hom}_R(M, D)$ to $\operatorname{Hom}_R(L, D)$, and in terms of the map ψ', the homomorphism $f \in \operatorname{Hom}_R(L, D)$ can be lifted to a homomorphism from M to D if and only if f is in the image of ψ'. The example above shows that

$$\text{if} \quad 0 \longrightarrow L \xrightarrow{\psi} M \quad \text{is exact,}$$

$$\text{then} \quad \operatorname{Hom}_R(M, D) \xrightarrow{\psi'} \operatorname{Hom}_R(L, D) \longrightarrow 0 \quad \text{is } \textit{not necessarily} \text{ exact.}$$

We can summarize these results in the following dual version of Theorem 28:

Theorem 33. Let D, L, M, and N be R-modules. If

$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow 0 \quad \text{is exact,}$$

then the associated sequence

$$0 \to \operatorname{Hom}_R(N, D) \xrightarrow{\varphi'} \operatorname{Hom}_R(M, D) \xrightarrow{\psi'} \operatorname{Hom}_R(L, D) \quad \text{is exact.} \tag{10.12}$$

A homomorphism $f : L \to D$ lifts to a homomorphism $F : M \to D$ if and only if $f \in \operatorname{Hom}_R(L, D)$ is in the image of ψ'. In general $\psi' : \operatorname{Hom}_R(M, D) \to \operatorname{Hom}_R(L, D)$ need not be surjective; the map ψ' is surjective if and only if every homomorphism from L to D lifts to a homomorphism from M to D, in which case the sequence (12) can be extended to a short exact sequence.

The sequence (12) is exact for *all* R-modules D if and only if the sequence

$$L \xrightarrow{\psi} M \xrightarrow{\varphi} N \to 0 \quad \text{is exact.}$$