12.6 Elliptic Functions

The idea of inverting elliptic integrals to obtain elliptic functions is due to Gauss, Abel, and Jacobi. Gauss had the idea in the late 1790s but did not publish it; Abel had the idea in 1823 and published it in 1827, independently of Gauss. Jacobi's independence is not quite so clear. He seems to have been approaching the idea of inversion in 1827, but he was only stung into action by the appearance of Abel's paper. At any rate, his ideas subsequently developed at an explosive rate, and he published the first book on elliptic functions, the *Fundamenta nova theoriae functionum ellipticarum*, two years later [Jacobi (1829)].

Gauss first considered inverting an elliptic integral in 1796, in the case of $\int dt/\sqrt{1-t^3}$. The following year he inverted the lemniscatic integral and made better progress. Defining the "lemniscatic sine function" $x = sl(u)$ by

$$u = \int_0^x \frac{dt}{\sqrt{1-t^4}},$$

he found that this function was periodic, like the sine, with period

$$2\varpi = 4\int_0^1 \frac{dt}{\sqrt{1-t^4}}.$$

Gauss also noticed that $sl(u)$ invites complex arguments, since it follows from $i^2 = -1$ that

$$\frac{d(it)}{\sqrt{1-(it)^4}} = i\frac{dt}{\sqrt{1-t^4}},$$

hence $sl(iu) = isl(u)$ and the lemniscatic sine has a second period $2i\varpi$. Thus Gauss discovered *double periodicity*, one of the key properties of the elliptic functions, though at first he did not realize its universality. The scope and importance of elliptic functions hit him on May 30, 1799, when he discovered an extraordinary numerical coincidence. His diary entry of that day reads:

> We have established that the arithmetic-geometric mean between 1 and $\sqrt{2}$ is π/ϖ to 11 places; the demonstration of this fact will surely open up an entirely new field of analysis.

Gauss had been fascinated by the arithmetic-geometric mean since 1791, when he was 14. The arithmetic-geometric mean of two positive numbers a and b is the common limit, $\mathrm{agM}(a,b)$ of the two sequences $\{a_n\}$ and $\{b_n\}$ defined by

$$a_0 = a, \qquad b_0 = b,$$
$$a_{n+1} = \frac{a_n+b_n}{2}, \qquad b_{n+1} = \sqrt{a_n b_n}.$$

[For more information on the theory and history of the agM function, see Cox (1984).]

It is indeed true that $\mathrm{agM}(1,\sqrt{2}) = \pi/\varpi$, as Gauss soon proved, and the "entirely new field of analysis" he created from the merger of these ideas was extraordinarily rich. It encompassed elliptic functions in general, the theta functions later rediscovered by Jacobi, and the modular functions later rediscovered by Klein. The theory was not clearly improved until the 1850s, when Riemann showed that double periodicity becomes obvious when elliptic integrals are placed in a suitable geometric setting.

Unfortunately, Gauss released virtually none of his results on elliptic functions. Apart from publishing an expression for $\mathrm{agM}(a,b)$ as an elliptic integral [Gauss (1818)], he did nothing until Abel's results appeared in 1827—then promptly claimed them as his own. He wrote to Bessel [Gauss (1828)]:

> I shall most likely not soon prepare my investigations on the transcendental functions which I have had for many years—since 1798. ...Herr Abel has now, as I see, anticipated me and relieved me of the burden in regard to one third of these matters.

It was disingenuous of Gauss to claim he had more results than Abel, because Abel also had results unknown to Gauss. True, Gauss had priority on the key ideas of inversion and double periodicity, but priority isn't everything, as Gauss himself perhaps knew. His own cherished discovery of the relation between agM and elliptic integrals had not only been found earlier, but even published by Lagrange (1785).

EXERCISES

The following exercises show how the lemniscatic sine and its derivative are quite analogous to the ordinary sine and its derivative, the cosine.

12.6.1 Show that $sl'(u) = \sqrt{1 - sl^4(u)}$.

12.6.2 Deduce from the Euler addition theorem (Section 12.4) that

$$sl(u+v) = \frac{sl(u)sl'(v) + sl(v)sl'(u)}{1 + sl^2(u)sl^2(v)}.$$

12.7 A Postscript on the Lemniscate

The duplication of the arc of the lemniscate had some interesting consequences for the lemniscate itself. Fagnano showed, by similar arguments, that a quadrant of the lemniscate could be divided into two, three, or five equal arcs by ruler and compass [see Ayoub (1984)]. This raised a question: for which n can the lemniscate be divided into n equal parts by ruler and compasses? Recall from Section 2.3 that the corresponding question for the circle had been answered by Gauss (1801), Art. 366. The answer is $n = 2^m p_1 p_2 \cdots p_k$, where each p_i is a prime of the form $2^{2^h} + 1$. In the introduction to his theory, Gauss claims

> The principles of the theory which we are going to explain actually extend much further than we will indicate. For they can be applied not only to circular functions but just as well to other transcendental functions, e.g. to those which depend on the integral $\int (1/\sqrt{1-x^4})\,dx$. (Art. 355)

However, his surviving papers do not include any result on the lemniscate as incisive as his result on the circle. There is only a diary entry of March 21, 1797, stating divisibility of the lemniscate into five equal parts.

The answer to the problem of dividing the lemniscate into n equal parts was found by Abel (1827), transforming Gauss' obscurity into crystal clarity: division by ruler and compasses is possible for *precisely the same n* as for the circle. This wonderful result serves, perhaps better than any other, to underline the unifying role of elliptic functions in geometry, algebra, and number theory. A modern proof of it may be found in Rosen (1981).

12.8 Biographical Notes: Abel and Jacobi

Niels Henrik Abel was born in the small town of Finnøy, on the southwestern coast of Norway, in 1802 and died in Oslo in 1829. In his short life he managed to win the esteem of the best mathematicians in Europe, but he fell victim to official indifference, terrible family burdens, and tuberculosis. His heart-breaking story is not unlike that of his great contemporary in another field, the poet John Keats (1797–1823).

Like several mathematicians before him (Wallis, Gregory, Euler), Abel was the son of a Protestant minister. His father, Søren, distinguished himself in theology and philology at the University of Copenhagen and was a supporter of the new literary and social movements of his time. Søren's liberality, particularly toward the consumption of alcohol, was unfortunately not matched by good judgment, and his marriage to Anne Marie Simonsen in 1799 eventually led to disaster. The beautiful Anne Marie was a talented pianist and singer but completely irresponsible and later openly unfaithful to her husband. The family held together during Abel's early years, when he was educated by his father, but both parents were becoming frequently drunk and unstable by 1815, when Niels and his older brother, Hans Mathias, were sent to the Cathedral School in Oslo.

At first school was not much better than home. Some of its best teachers had gone to the recently opened Oslo University, and discipline had deteriorated to the point where fights between staff and students were common. The mathematics teacher, Bader, was particularly brutal, beating even good students like Abel, and injuring one boy severely enough to cause his death. This led to Bader's dismissal (without his being brought to court, however) and to the appointment of a new mathematics teacher, Bernt Michael Holmboe, in 1818. Although not a creative mathematician, Holmboe knew his subject and was an inspiring teacher. He introduced Abel to Euler's calculus texts, and Abel soon abandoned all other reading for the works of Newton, Lagrange, and Gauss, among others. By 1819 Holmboe was writing in his report book: "With the most excellent genius he combines an insatiable interest and desire for mathematics, so that if he lives he probably will become a great mathematician" [see Ore (1957), p. 33]. Ore informs us that the last three words are a revision, probably of the phrase "the world's foremost mathematician," which Holmboe may have been asked to tone down by the school principal. Why Holmboe chose to balance the phrase with the ominous "if he lives" is a mystery, though uncomfortably close to correct prophecy.

During his last two years at the Cathedral School, around 1820, Abel believed he had discovered the solution of the quintic equation. The mathematicians in Oslo were skeptical but unable to fault Abel's argument, so it was sent to the Danish mathematician Ferdinand Degen. Degen, too, was unable to find an error, but he prudently asked Abel for more details and a numerical illustration. When Abel attempted to compute one he discovered his error. However, Degen also had another suggestion: Abel would do better to apply his energy to "the elliptic transcendentals."

Meanwhile, Abel's family was disintegrating. Hans Mathias, after a promising start at the Cathedral School, slipped to the bottom of the class and was sent home, eventually to become feeble-minded. His father drunk himself to death in 1820, leaving the family penniless. Niels Henrik, now the oldest responsible member of the family, took steps which were to save his sister Elisabeth and younger brother Peder. He found another home for Elisabeth and took Peder with him when he entered the University of Oslo in 1821.

Before long, Abel had read most of the advanced mathematical works in the university library, and his own research began in earnest. By 1823 he had discovered the inversion that was the key to elliptic functions, proved the unsolvability of the quintic, and discovered a wonderful general theorem on integration, now known as Abel's theorem, which implicitly introduces the concept of genus. On a trip to Copenhagen in 1823 to tell Degen of these results, he met and fell in love with Christine ("Crelly") Kemp. Like Abel, she came from an educated but impoverished family; she was making a living for herself by tutoring. The remaining six years of Abel's life were consumed by the struggle for recognition of his mathematics and attempts to gain a position that paid enough to allow him to marry Crelly.

In 1824 he won a government grant to travel and meet other scientists, and he became engaged to Crelly at Christmas. She was now working in Oslo as a governess, a job that Abel had arranged for her. The grant was mainly intended to take him to Paris, but when he finally set off, late in 1825, he impulsively detoured to Berlin to visit friends. There he also met August Crelle, an engineer and amateur mathematician, who was about to found the first German mathematical journal. The meeting was fortuitous, as Crelle was able to give an international circulation to Abel's first important results, while Abel could supply papers of a quality that ensured the success of the new journal. In meeting influential mathematicians Abel was less lucky. He made no effort to visit Gauss while in Germany, being convinced that Gauss was "absolutely unapproachable," and failed to make an impression on Cauchy in Paris, though he presented him with a copy of the memoir on Abel's theorem. During his stay in Paris, Abel discovered his theorem on the lemniscate and sat for his only known portrait (Figure 12.3).

Figure 12.3: Neils Henrik Abel

By the end of 1826 Abel was running out of money and eating only one meal a day. He feared he was losing touch with Crelly, as she had returned to Copenhagen and her letters were infrequent. He left Paris for Berlin on December 29, while he still had money to pay for the journey, and found a letter from Crelly waiting. Some good news at last! Crelly stood by him as ever, and their plans for the future were revived. Abel returned to Oslo in May 1827, via Copenhagen, and arranged another job in Norway for Crelly. Unfortunately, the university was still unwilling to give him more than a temporary appointment, which paid barely enough to meet his family's debts. In September 1827 Abel's first memoir on elliptic functions was published in Crelle's journal. In the same month, Jacobi appeared on the scene with the first announcement of his results. There were results that Abel knew how to prove, and when Jacobi's proofs appeared, some months later, Abel was shocked to see Jacobi using the method of inversion without acknowledging its previous appearance in Abel's paper. Abel was initially bitter over this blow and strove to "knock out" Jacobi with a second memoir. However, he ceased to bear a grudge after he learned how much Jacobi really admired his work. Jacobi in fact admitted that his first announcement had been based on guesswork and that he had realized inversion was the key to the proof only after reading Abel.

In May 1828 Abel finally received a decent job offer from Berlin, only to have it withdrawn two months later. Crelle had been working in support of Abel, but another candidate had pushed in ahead of him. Then a group of French mathematicians petitioned the king of Norway-Sweden to use his influence on Abel's behalf, but still the University of Oslo remained unmoved. By now, time was running out. Abel's health worsened and in January 1829 he began spitting blood. Crelle renewed his efforts in Berlin, but it was too late. Abel died on April 6, 1829, just two days before the arrival of a letter from Crelle informing him of his appointment as professor in Berlin.

Carl Gustav Jacob Jacobi (Figure 12.4) was born in Potsdam in 1804 and died in Berlin in 1851. He was the second of three sons of Simon Jacobi, a banker. The oldest son, Moritz, became a physicist and inventor of a popular pseudoscience called "galvanoplastics," which made him more famous in his time than Carl. The youngest, Eduard, carried on the family business, and there was also a sister, Therese. Jacobi's mother's name has not come down to us, though her side of the family was also important, one of her brothers taking charge of Jacobi's education until he entered secondary school in 1816. He was promoted to the top class after only a few months, but he had to remain there for four years, until he became old enough to enter university. During his school days Jacobi excelled in classics and history as well as mathematics. He studied Euler's *Introductio in analysin infinitorum* [Euler (1748a)] and attempted, like Abel, to solve the quintic equation.

Entering the University of Berlin in 1821, Jacobi continued his broad classical education for two years, before private study of the works of Euler, Lagrange, Laplace, and Gauss convinced him that he had time only for mathematics. He gained his first degree in 1824 and began lecturing (in differential geometry) at the University of Berlin in 1825. Despite a reputation for bluntness and sarcasm, Jacobi made rapid progress in his career. He moved to Königsberg in 1826, becoming associate professor there in 1827 and full professor in 1832. Overriding Jacobi's sometimes abrasive manner were his exceptional energy and enthusiasm for both research and teaching. He managed to combine the two by lecturing up to 10 hours a

Figure 12.4: Carl Gustav Jacob Jacobi

week on elliptic functions, incorporating his latest discoveries. Such high-intensity instruction was unheard of then, as it is now, yet Jacobi built up a school of talented pupils.

In 1831 he married Marie Schwink, the daughter of a formerly wealthy man who had lost his fortune through speculation. Nine years later, with a growing family (eventually five sons and three daughters), Jacobi found himself in a similar predicament. His father's fortune had vanished and he had to support his widowed mother. In 1843 he suffered a breakdown from overwork, and diabetes was diagnosed. His friend Dirichlet managed to secure a grant for Jacobi to travel to Italy for the sake of his health. After eight months there, Jacobi was well enough to return. He was given permission to move to Berlin, because of its milder climate, and an increase in salary to meet the higher living costs in the capital. However, in 1849 the salary bonus was retracted. Jacobi had to move out of his house to an inn, and he sent the rest of his family to the small town of Gotha, where housing was cheaper. Early in 1851 he came down with influenza after visiting them. Before he had quite recovered, he was stricken with smallpox and died within a week.

Jacobi is remembered for his contributions to many fields of mathematics, including differential geometry, mechanics, and number theory as well as elliptic functions. He was a great admirer of Euler and planned the edition of Euler's works that eventually began to appear, on a reduced scale, in 1911. In fact, in many ways Jacobi was a second, if lesser, Euler. He saw elliptic functions not so much as things in themselves, as Abel did, but as a source of dazzling formulas with implications in number theory. An astounding collection of formulas may be found in his major work on elliptic functions, the *Fundamenta nova* [Jacobi (1829)]. At the same time, he was deeply impressed by Abel's ideas and selflessly campaigned to make them better known. He introduced the terms "Abelian integral" and "Abelian function" for the generalizations of elliptic integrals and functions considered by Abel as well as "Abelian theorem" for Abel's theorem, which he described as "the greatest mathematical discovery of our time."