This says that

$$T\left(\sum_{i=k+1}^{n} c_i\alpha_i\right) = 0$$

and accordingly the vector $\alpha = \sum_{i=k+1}^{n} c_i\alpha_i$ is in the null space of T. Since $\alpha_1, \ldots, \alpha_k$ form a basis for N, there must be scalars $b_1, \ldots, b_k$ such that

$$\alpha = \sum_{i=1}^{k} b_i\alpha_i.$$

Thus

$$\sum_{i=1}^{k} b_i\alpha_i - \sum_{j=k+1}^{n} c_j\alpha_j = 0$$

and since $\alpha_1, \ldots, \alpha_n$ are linearly independent we must have

$$b_1 = \cdots = b_k = c_{k+1} = \cdots = c_n = 0.$$

If r is the rank of T, the fact that $T\alpha_{k+1}, \ldots, T\alpha_n$ form a basis for the range of T tells us that $r = n - k$. Since k is the nullity of T and n is the dimension of V, we are done. ▮

Theorem 3. *If* A *is an* m $\times$ n *matrix with entries in the field* F, *then*

$$\text{row rank (A)} = \text{column rank (A)}.$$

Proof. Let T be the linear transformation from $F^{n\times 1}$ into $F^{m\times 1}$ defined by $T(X) = AX$. The null space of T is the solution space for the system $AX = 0$, i.e., the set of all column matrices X such that $AX = 0$. The range of T is the set of all $m \times 1$ column matrices Y such that $AX = Y$ has a solution for X. If $A_1, \ldots, A_n$ are the columns of A, then

$$AX = x_1A_1 + \cdots + x_nA_n$$

so that the range of T is the subspace spanned by the columns of A. In other words, the range of T is the column space of A. Therefore,

$$\text{rank } (T) = \text{column rank } (A).$$

Theorem 2 tells us that if S is the solution space for the system $AX = 0$, then

$$\dim S + \text{column rank } (A) = n.$$

We now refer to Example 15 of Chapter 2. Our deliberations there showed that, if r is the dimension of the row space of A, then the solution space S has a basis consisting of $n - r$ vectors:

$$\dim S = n - \text{row rank } (A).$$

It is now apparent that

$$\text{row rank } (A) = \text{column rank } (A). \quad ▮$$

The proof of Theorem 3 which we have just given depends upon explicit calculations concerning systems of linear equations. There is a more conceptual proof which does not rely on such calculations. We shall give such a proof in Section 3.7.

Exercises

1. Which of the following functions T from R^2 into R^2 are linear transformations?

(a) $T(x_1, x_2) = (1 + x_1, x_2)$;
(b) $T(x_1, x_2) = (x_2, x_1)$;
(c) $T(x_1, x_2) = (x_1^2, x_2)$;
(d) $T(x_1, x_2) = (\sin x_1, x_2)$;
(e) $T(x_1, x_2) = (x_1 - x_2, 0)$.

2. Find the range, rank, null space, and nullity for the zero transformation and the identity transformation on a finite-dimensional space V.

3. Describe the range and the null space for the differentiation transformation of Example 2. Do the same for the integration transformation of Example 5.

4. Is there a linear transformation T from R^3 into R^2 such that $T(1, -1, 1) = (1, 0)$ and $T(1, 1, 1) = (0, 1)$?

5. If

$$\begin{aligned} \alpha_1 &= (1, -1), & \beta_1 &= (1, 0) \\ \alpha_2 &= (2, -1), & \beta_2 &= (0, 1) \\ \alpha_3 &= (-3, 2), & \beta_3 &= (1, 1) \end{aligned}$$

is there a linear transformation T from R^2 into R^2 such that $T\alpha_i = \beta_i$ for $i = 1, 2$ and 3?

6. Describe explicitly (as in Exercises 1 and 2) the linear transformation T from F^2 into F^2 such that $T\epsilon_1 = (a, b)$, $T\epsilon_2 = (c, d)$.

7. Let F be a subfield of the complex numbers and let T be the function from F^3 into F^3 defined by

$$T(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3, 2x_1 + x_2, -x_1 - 2x_2 + 2x_3).$$

(a) Verify that T is a linear transformation.

(b) If (a, b, c) is a vector in F^3, what are the conditions on a, b, and c that the vector be in the range of T? What is the rank of T?

(c) What are the conditions on a, b, and c that (a, b, c) be in the null space of T? What is the nullity of T?

8. Describe explicitly a linear transformation from R^3 into R^3 which has as its range the subspace spanned by $(1, 0, -1)$ and $(1, 2, 2)$.

9. Let V be the vector space of all $n \times n$ matrices over the field F, and let B be a fixed $n \times n$ matrix. If

$$T(A) = AB - BA$$

verify that T is a linear transformation from V into V.

10. Let V be the set of all complex numbers regarded as a vector space over the field of *real* numbers (usual operations). Find a function from V into V which is a linear transformation on the above vector space, but which is not a linear transformation on C^1, i.e., which is not complex linear.

11. Let V be the space of $n \times 1$ matrices over F and let W be the space of $m \times 1$ matrices over F. Let A be a fixed $m \times n$ matrix over F and let T be the linear transformation from V into W defined by $T(X) = AX$. Prove that T is the zero transformation if and only if A is the zero matrix.

12. Let V be an n-dimensional vector space over the field F and let T be a linear transformation from V into V such that the range and null space of T are identical. Prove that n is even. (Can you give an example of such a linear transformation T?)

13. Let V be a vector space and T a linear transformation from V into V. Prove that the following two statements about T are equivalent.

(a) The intersection of the range of T and the null space of T is the zero subspace of V.

(b) If $T(T\alpha) = 0$, then $T\alpha = 0$.

3.2. The Algebra of Linear Transformations

In the study of linear transformations from V into W, it is of fundamental importance that the set of these transformations inherits a natural vector space structure. The set of linear transformations from a space V into itself has even more algebraic structure, because ordinary composition of functions provides a 'multiplication' of such transformations. We shall explore these ideas in this section.

Theorem 4. *Let* V *and* W *be vector spaces over the field* F. *Let* T *and* U *be linear transformations from* V *into* W. *The function* (T + U) *defined by*

$$(\mathrm{T} + \mathrm{U})(\alpha) = \mathrm{T}\alpha + \mathrm{U}\alpha$$

is a linear transformation from V *into* W. *If* c *is any element of* F, *the function* (cT) *defined by*

$$(\mathrm{cT})(\alpha) = \mathrm{c}(\mathrm{T}\alpha)$$

is a linear transformation from V *into* W. *The set of all linear transformations from* V *into* W, *together with the addition and scalar multiplication defined above, is a vector space over the field* F.

Proof. Suppose T and U are linear transformations from V into W and that we define $(T + U)$ as above. Then

$$\begin{aligned}(T + U)(c\alpha + \beta) &= T(c\alpha + \beta) + U(c\alpha + \beta)\\ &= c(T\alpha) + T\beta + c(U\alpha) + U\beta\\ &= c(T\alpha + U\alpha) + (T\beta + U\beta)\\ &= c(T + U)(\alpha) + (T + U)(\beta)\end{aligned}$$

which shows that $(T + U)$ is a linear transformation. Similarly,

$$\begin{aligned}(cT)(d\alpha + \beta) &= c[T(d\alpha + \beta)]\\ &= c[d(T\alpha) + T\beta]\\ &= cd(T\alpha) + c(T\beta)\\ &= d[c(T\alpha)] + c(T\beta)\\ &= d[(cT)\alpha] + (cT)\beta\end{aligned}$$

which shows that (cT) is a linear transformation.

To verify that the set of linear transformations of V into W (together with these operations) is a vector space, one must directly check each of the conditions on the vector addition and scalar multiplication. We leave the bulk of this to the reader, and content ourselves with this comment: The zero vector in this space will be the zero transformation, which sends every vector of V into the zero vector in W; each of the properties of the two operations follows from the corresponding property of the operations in the space W. ▮

We should perhaps mention another way of looking at this theorem. If one defines sum and scalar multiple as we did above, then the set of *all* functions from V into W becomes a vector space over the field F. This has nothing to do with the fact that V is a vector space, only that V is a non-empty set. When V is a vector space we can define a linear transformation from V into W, and Theorem 4 says that the linear transformations are a subspace of the space of all functions from V into W.

We shall denote the space of linear transformations from V into W by $L(V, W)$. We remind the reader that $L(V, W)$ is defined only when V and W are vector spaces over the same field.

Theorem 5. *Let* V *be an* n*-dimensional vector space over the field* F, *and let* W *be an* m*-dimensional vector space over* F. *Then the space* L(V, W) *is finite-dimensional and has dimension* mn.

Proof. Let

$$\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\} \quad \text{and} \quad \mathcal{B}' = \{\beta_1, \ldots, \beta_m\}$$

be ordered bases for V and W, respectively. For each pair of integers (p, q) with $1 \leq p \leq m$ and $1 \leq q \leq n$, we define a linear transformation $E^{p,q}$ from V into W by

$$\begin{aligned}E^{p,q}(\alpha_i) &= \begin{cases}0, & \text{if } \; i \neq q\\ \beta_p, & \text{if } \; i = q\end{cases}\\ &= \delta_{iq}\beta_p.\end{aligned}$$

According to Theorem 1, there is a unique linear transformation from V into W satisfying these conditions. The claim is that the mn transformations $E^{p,q}$ form a basis for $L(V, W)$.

Let T be a linear transformation from V into W. For each j, $1 \leq j \leq n$, let $A_{1j}, \ldots, A_{mj}$ be the coordinates of the vector $T\alpha_j$ in the ordered basis $\mathcal{B}'$, i.e.,

$$T\alpha_j = \sum_{p=1}^{m} A_{pj}\beta_p. \tag{3-1}$$

We wish to show that

$$T = \sum_{p=1}^{m} \sum_{q=1}^{n} A_{pq}E^{p,q}. \tag{3-2}$$

Let U be the linear transformation in the right-hand member of (3-2). Then for each j

$$\begin{aligned} U\alpha_j &= \sum_p \sum_q A_{pq}E^{p,q}(\alpha_j) \\ &= \sum_p \sum_q A_{pq}\delta_{jq}\beta_p \\ &= \sum_{p=1}^{m} A_{pj}\beta_p \\ &= T\alpha_j \end{aligned}$$

and consequently $U = T$. Now (3-2) shows that the $E^{p,q}$ span $L(V, W)$; we must prove that they are independent. But this is clear from what we did above; for, if the transformation

$$U = \sum_p \sum_q A_{pq}E^{p,q}$$

is the zero transformation, then $U\alpha_j = 0$ for each j, so

$$\sum_{p=1}^{m} A_{pj}\beta_p = 0$$

and the independence of the β_p implies that $A_{pj} = 0$ for every p and j. ∎

Theorem 6. *Let* V, W, *and* Z *be vector spaces over the field* F. *Let* T *be a linear transformation from* V *into* W *and* U *a linear transformation from* W *into* Z. *Then the composed function* UT *defined by* $(UT)(\alpha) = U(T(\alpha))$ *is a linear transformation from* V *into* Z.

Proof.

$$\begin{aligned} (UT)(c\alpha + \beta) &= U[T(c\alpha + \beta)] \\ &= U(cT\alpha + T\beta) \\ &= c[U(T\alpha)] + U(T\beta) \\ &= c(UT)(\alpha) + (UT)(\beta). \quad ∎ \end{aligned}$$

In what follows, we shall be primarily concerned with linear transformation of a vector space into itself. Since we would so often have to write 'T is a linear transformation from V into V,' we shall replace this with 'T is a linear operator on V.'

Definition. *If* V *is a vector space over the field* F, *a* **linear operator on** V *is a linear transformation from* V *into* V.

In the case of Theorem 6 when $V = W = Z$, so that U and T are linear operators on the space V, we see that the composition UT is again a linear operator on V. Thus the space $L(V, V)$ has a 'multiplication' defined on it by composition. In this case the operator TU is also defined, and one should note that in general $UT \neq TU$, i.e., $UT - TU \neq 0$. We should take special note of the fact that if T is a linear operator on V then we can compose T with T. We shall use the notation $T^2 = TT$, and in general $T^n = T \cdots T$ (n times) for $n = 1, 2, 3, \ldots$. We define $T^0 = I$ if $T \neq 0$.

Lemma. *Let* V *be a vector space over the field* F; *let* U, T_1 *and* T_2 *be linear operators on* V; *let* c *be an element of* F.

(a) $IU = UI = U$;
(b) $U(T_1 + T_2) = UT_1 + UT_2$; $(T_1 + T_2)U = T_1U + T_2U$;
(c) $c(UT_1) = (cU)T_1 = U(cT_1)$.

Proof. (a) This property of the identity function is obvious. We have stated it here merely for emphasis.

(b)
$$\begin{aligned} [U(T_1 + T_2)](\alpha) &= U[(T_1 + T_2)(\alpha)] \\ &= U(T_1\alpha + T_2\alpha) \\ &= U(T_1\alpha) + U(T_2\alpha) \\ &= (UT_1)(\alpha) + (UT_2)(\alpha) \end{aligned}$$

so that $U(T_1 + T_2) = UT_1 + UT_2$. Also

$$\begin{aligned} [(T_1 + T_2)U](\alpha) &= (T_1 + T_2)(U\alpha) \\ &= T_1(U\alpha) + T_2(U\alpha) \\ &= (T_1U)(\alpha) + (T_2U)(\alpha) \end{aligned}$$

so that $(T_1 + T_2)U = T_1U + T_2U$. (The reader may note that the proofs of these two distributive laws do not use the fact that T_1 and T_2 are linear, and the proof of the second one does not use the fact that U is linear either.)

(c) We leave the proof of part (c) to the reader. ∎

The contents of this lemma and a portion of Theorem 5 tell us that the vector space $L(V, V)$, together with the composition operation, is what is known as a linear algebra with identity. We shall discuss this in Chapter 4.

Example 8. If A is an $m \times n$ matrix with entries in F, we have the linear transformation T defined by $T(X) = AX$, from $F^{n\times 1}$ into $F^{m\times 1}$. If B is a $p \times m$ matrix, we have the linear transformation U from $F^{m\times 1}$ into $F^{p\times 1}$ defined by $U(Y) = BY$. The composition UT is easily described:

$$\begin{aligned} (UT)(X) &= U(T(X)) \\ &= U(AX) \\ &= B(AX) \\ &= (BA)X. \end{aligned}$$

Thus UT is 'left multiplication by the product matrix BA.'