time, that there are exactly six triangular numbers that are products of three consecutive integers (See Tzanakis and de Weger [1989].)

Exercises

1. How might a Pythagorean have derived the fact that the angle at the tip of his star is $36°$?

2. Check that 220 and 284 are amicable. If 12,285 is one member of an amicable pair, find the other.

3. Prove that the sum of the first n cubes is the square of the sum of the first n numbers. (Use mathematical induction or construct a square figure with sides of length $1 + 2 + \cdots + n$ and divide it into figures whose areas are the first n perfect cubes.)

8

Perfect Numbers

The Pythagoreans were interested in perfect numbers, that is, numbers such as 6 and 28, which are equal to the sum of their proper divisors. They may also be described as numbers which are amicable with themselves. Nowadays we usually speak about the sum of all the divisors of a positive integer n, including n itself. If $\sigma(n)$ denotes this sum, then n is *perfect* if and only if $\sigma(n) = 2n$. As the culmination of Book IX of the *Elements* (300 BC), Euclid proved that any positive integer of the form

$$n = 2^{m-1}(2^m - 1)$$

is perfect, whenever $2^m - 1$ is prime. This fact had probably been discovered by the Pythagoreans.

Proof of Proposition IX 36 (*Perfect Number Theorem*):

If $p = 2^m - 1$ is prime, then the divisors of $n = 2^{m-1}p$ are

$$1,\ 2,\ 2^2,\ \ldots,\ 2^{m-1},\ p,\ 2p,\ 2^2p,\ \ldots,\ 2^{m-1}p$$

(thanks to unique factorization). The sum of these divisors is thus

$$\begin{aligned}\sigma(n) &= (1 + 2 + 2^2 + \cdots + 2^{m-1})(1 + p)\\ &= (2^m - 1)(1 + p)\\ &= 2(2^{m-1}(2^m - 1)) = 2n.\end{aligned}$$

Even though $2^m p$ is not square-free, Euclid did have a rigorous proof of the special case of the unique factorization theorem which is used in the

2	107	9689
3	127	9941
5	521	11213
7	607	19937
13	1279	21701
17	2203	23209
19	2281	44497
31	3217	86243
61	4253	132049
89	4423	216091

TABLE 8.1. Values of m making $2^m - 1$ prime

above proof, and he also had a rigorous proof for the formula for the sum of a geometric progression (IX 35).

An integer of the form $2^m - 1$ can only be prime if m is prime. For if $m = ab$ with $a, b > 1$, we have the factorization

$$2^{ab} - 1 = (2^a - 1)((2^a)^{b-1} + (2^a)^{b-2} + \cdots + 2^a + 1)$$

into two factors greater than 1. The converse is not true. Although 11 is prime, $2^{11} - 1$ is not; for $2^{11} - 1 = 2047 = 23 \times 89$.

Primes of the form $2^m - 1$ are called *Mersenne* primes, after Father Marin Mersenne (1588–1648). In the preface of his *Cogitata Physico-Mathematica* (1644), Mersenne correctly stated that the first 8 perfect numbers are given by the values $m = 2, 3, 5, 7, 13, 17, 19$ and 31. He also claimed that $2^{67} - 1$ is prime, and hence $2^{66}(2^{67} - 1)$ is perfect. Here he was wrong. In 1903, Frank Nelson Cole gave a lecture which consisted of two calculations. First Cole calculated $2^{67} - 1$. Then he worked out the product

$$193,707,721 \times 761,838,257,287.$$

He did not say a word as he did this. The two calculations agreed, and Cole received a standing ovation. He had factored $2^{67} - 1$ and proved Mersenne wrong.

Edouard Lucas (1842–1891) found a very efficient way of testing whether $2^m - 1$ is prime. Let $u_1 = 4$ and $u_{n+1} = u_n^2 - 2$. Thus $u_2 = 14, u_3 = 194$ and $u_4 = 37,634$. If $m > 2$ then $2^m - 1$ is prime just in case $2^m - 1$ is a factor of u_{m-1}. For example, since $2^5 - 1$ is a factor of 37,634 it follows that $2^5 - 1$ is prime (and hence $2^4(2^5 - 1) = 496$ is perfect). (For an elementary proof of Lucas's Theorem, see Sierpinski [1964].)

Thanks to Lucas's test — and the modern computer — we know, since about 1985, that $2^m - 1$ is prime when m has the 30 values given in Table