define the Cartesian product $\prod_{\alpha \in I} X_\alpha$ to be the set

$$\prod_{\alpha \in I} X_\alpha = \left\{ (x_\alpha)_{\alpha \in I} \in (\bigcup_{\beta \in I} X_\beta)^I : x_\alpha \in X_\alpha \text{ for all } \alpha \in I \right\},$$

where we recall (from Axiom 3.10) that $(\bigcup_{\alpha \in I} X_\alpha)^I$ is the set of all functions $(x_\alpha)_{\alpha \in I}$ which assign an element $x_\alpha \in \bigcup_{\beta \in I} X_\beta$ to each $\alpha \in I$. Thus $\prod_{\alpha \in I} X_\alpha$ is a subset of that set of functions, consisting instead of those functions $(x_\alpha)_{\alpha \in I}$ which assign an element $x_\alpha \in X_\alpha$ to each $\alpha \in I$.

Example 8.4.2. For any sets I and X, we have $\prod_{\alpha \in I} X = X^I$ (why?). If I is a set of the form $I := \{i \in \mathbf{N} : 1 \leq i \leq n\}$, then $\prod_{\alpha \in I} X_\alpha$ is the same set as the set $\prod_{1 \leq i \leq N} X_i$ defined in Definition 3.5.7 (why?).

Recall from Lemma 3.5.12 that if $X_1, \ldots, X_n$ were any finite collection of non-empty sets, then the finite Cartesian product $\prod_{1 \leq i \leq n} X_i$ was also non-empty. The Axiom of choice asserts that this statement is also true for infinite Cartesian products:

Axiom 8.1 (Choice). *Let I be a set, and for each $\alpha \in I$, let X_α be a non-empty set. Then $\prod_{\alpha \in I} X_\alpha$ is also non-empty. In other words, there exists a function $(x_\alpha)_{\alpha \in I}$ which assigns to each $\alpha \in I$ an element $x_\alpha \in X_\alpha$.*

Remark 8.4.3. The intuition behind this axiom is that given a (possibly infinite) collection of non-empty sets X_α, one should be able to choose a single element x_α from each one, and then form the possibly infinite tuple $(x_\alpha)_{\alpha \in I}$ from all the choices one has made. On one hand, this is a very intuitively appealing axiom; in some sense one is just applying Lemma 3.1.6 over and over again. On the other hand, the fact that one is making an infinite number of arbitrary choices, with no explicit rule as to *how* to make these choices, is a little disconcerting. Indeed, there are many theorems proven using the axiom of choice which assert the abstract existence of some object x with certain properties, without saying at all *what* that object is, or how to construct it. Thus the axiom of choice can lead to proofs which are *non-constructive* - demonstrating existence of an object without actually constructing the object explicitly. This problem is not unique to the axiom of choice - it already appears for instance in Lemma 3.1.6 - but the objects shown to exist using the axiom of choice tend to be rather extreme in their level of non-constructiveness. However, as long as one is aware of the distinction between a non-constructive existence statement, and a constructive existence statement (with the latter being preferable, but not strictly necessary in many cases), there is no difficulty here, except perhaps on a philosophical level.

Remark 8.4.4. There are many equivalent formulations of the axiom of choice; we give some of these in the exercises below.

In analysis one often does not need the full power of the axiom of choice. Instead, one often only needs the *axiom of countable choice*, which is the same as the axiom of choice but with the index set I restricted to be at most countable. We give a typical example of this below.

Lemma 8.4.5. *Let E be a non-empty subset of the real line with $\sup(E) < \infty$ (i.e., E is bounded from above). Then there exists a sequence $(a_n)_{n=1}^{\infty}$ whose elements a_n all lie in E, such that $\lim_{n\to\infty} a_n = \sup(E)$.*

Proof. For each positive natural number n, let X_n denote the set

$$X_n := \{x \in E : \sup(E) - 1/n \leq x \leq \sup(E)\}.$$

Since $\sup(E)$ is the least upper bound for E, then $\sup(E) - 1/n$ cannot be an upper bound for E, and so X_n is non-empty for each n. Using the axiom of choice (or the axiom of countable choice), we can then find a sequence $(a_n)_{n=1}^{\infty}$ such that $a_n \in X_n$ for all $n \geq 1$. In particular $a_n \in E$ for all n, and $\sup(E) - 1/n \leq a_n \leq \sup(E)$ for all n. But then we have $\lim_{n\to\infty} a_n = \sup(E)$ by the squeeze test (Corollary 6.4.14). □

Remark 8.4.6. In many special cases, one can obtain the conclusion of this lemma without using the axiom of choice. For instance, if E is a closed set (Definition 12.2.12) then one can define a_n without choice by the formula $a_n := \inf(X_n)$; the extra hypothesis that E is closed will ensure that a_n lies in E.

Another formulation of the axiom of choice is as follows.

Proposition 8.4.7. *Let X and Y be sets, and let $P(x, y)$ be a property pertaining to an object $x \in X$ and an object $y \in Y$ such that for every $x \in X$ there is at least one $y \in Y$ such that $P(x, y)$ is true. Then there exists a function $f : X \to Y$ such that $P(x, f(x))$ is true for all $x \in X$.*

Proof. See Exercise 8.4.1. □

Exercise 8.4.1. Show that the axiom of choice implies Proposition 8.4.7. (Hint: consider the sets $Y_x := \{y \in y : P(x, y) \text{ is true}\}$ for each $x \in X$.) Conversely, show that if Proposition 8.4.7 is true, then the axiom of choice is also true.

Exercise 8.4.2. Let I be a set, and for each $\alpha \in I$ let X_α be a non-empty set. Suppose that all the sets X_α are disjoint from each other, i.e., $X_\alpha \cap X_\beta = \emptyset$ for all distinct $\alpha, \beta \in I$. Using the axiom of choice, show that there exists a set Y such that $\#(Y \cap X_\alpha) = 1$ for all $\alpha \in I$ (i.e., Y intersects each X_α in exactly one element). Conversely, show that if the above statement was true for an arbitrary choice of sets I and non-empty disjoint sets X_α, then the axiom of choice is true. (Hint: the problem is that in Axiom 8.1 the sets X_α are not assumed to be disjoint. But this can be fixed by the trick by looking at the sets $\{\alpha\} \times X_\alpha = \{(\alpha, x) : x \in X_\alpha\}$ instead.)

Exercise 8.4.3. Let A and B be sets such that there exists a surjection $g : B \to A$. Using the axiom of choice, show that there then exists an injection $f : A \to B$; in other words A has lesser or equal cardinality to B in the sense of Exercise 3.6.7. (Hint: consider the inverse images $g^{-1}(\{a\})$ for each $a \in A$.) Compare this with Exercise 3.6.8. Conversely, show that if the above statement is true for arbitrary sets A, B and surjections $g : B \to A$, then the axiom of choice is true. (Hint: use Exercise 8.4.2.)

8.5 Ordered sets

The axiom of choice is intimately connected to the theory of *ordered sets.* There are actually many types of ordered sets; we will concern ourselves with three such types, the *partially ordered sets,* the *totally ordered sets*, and the *well-ordered sets.*

Definition 8.5.1 (Partially ordered sets)**.** A *partially ordered set* (or *poset*) is a set X, together[1] with a relation $\leq_X$ on X (thus for any two objects $x, y \in X$, the statement $x \leq_X y$ is either a true statement or a false statement). Furthermore, this relation is assumed to obey the following three properties:

- (Reflexivity) For any $x \in X$, we have $x \leq_X x$.
- (Anti-symmetry) If $x, y \in X$ are such that $x \leq_X y$ and $y \leq_X x$, then $x = y$.
- (Transitivity) If $x, y, z \in X$ are such that $x \leq_X y$ and $y \leq_X z$, then $x \leq_X z$.

We refer to $\leq_X$ as the *ordering relation.* In most situations it is understood what the set X is from context, and in those cases we shall simply write $\leq$ instead of $\leq_X$. We write $x <_X y$ (or $x < y$ for short) if $x \leq_X y$ and $x \neq y$.

Examples 8.5.2. The natural numbers $\mathbf{N}$ together with the usual less-than-or-equal-to relation $\leq$ (as defined in Definition 2.2.11) forms a partially ordered set, by Proposition 2.2.12. Similar arguments (using the appropriate definitions and propositions) show that the integers $\mathbf{Z}$, the rationals $\mathbf{Q}$, the reals $\mathbf{R}$, and the extended reals $\mathbf{R}^*$ are also partially ordered sets. Meanwhile, if X is any collection of sets, and one uses the relation of is-a-subset-of $\subseteq$ (as defined in Definition 3.1.15) for the ordering relation $\leq_X$, then X

[1]Strictly speaking, a partially ordered set is not a set X, but rather a pair $(X, \leq_X)$. But in many cases the ordering $\leq_X$ will be clear from context, and so we shall refer to X itself as the partially ordered set even though this is technically incorrect.

is also partially ordered (Proposition 3.1.18). Note that it is certainly possible to give these sets a different partial ordering than the standard one; see for instance Exercise 8.5.3.

Definition 8.5.3 (Totally ordered set). Let X be a partially ordered set with some order relation $\leq_X$. A subset Y of X is said to be *totally ordered* if, given any two $y, y' \in Y$, we either have $y \leq_X y'$ or $y' \leq_X y$ (or both). If X itself is totally ordered, we say that X is a *totally ordered set* (or *chain*) with order relation $\leq_X$.

Examples 8.5.4. The natural numbers $\mathbf{N}$, the integers $\mathbf{Z}$, the rationals $\mathbf{Q}$, reals $\mathbf{R}$, and the extended reals $\mathbf{R}^*$, all with the usual ordering relation $\leq$, are totally ordered (by Proposition 2.2.13, Lemma 4.1.11, Proposition 4.2.9, Proposition 5.4.7, and Proposition 6.2.5 respectively). Also, any subset of a totally ordered set is again totally ordered (why?). On the other hand, a collection of sets with the $\subseteq$ relation is usually not totally ordered. For instance, if X is the set $\{\{1,2\},\{2\},\{2,3\},\{2,3,4\},\{5\}\}$, ordered by the set inclusion relation $\subseteq$, then the elements $\{1,2\}$ and $\{2,3\}$ of X are not comparable to each other (i.e., $\{1,2\} \not\subseteq \{2,3\}$ and $\{2,3\} \not\subseteq \{1,2\}$).

Definition 8.5.5 (Maximal and minimal elements). Let X be a partially ordered set, and let Y be a subset of X. We say that y is a *minimal element of* Y if $y \in Y$ and there is no element $y' \in Y$ such that $y' < y$. We say that y is a *maximal element of* Y if $y \in Y$ and there is no element $y' \in Y$ such that $y < y'$.

Example 8.5.6. Using the set X from the previous example, $\{2\}$ is a minimal element, $\{1,2\}$ and $\{2,3,4\}$ are maximal elements, $\{5\}$ is both a minimal and a maximal element, and $\{2,3\}$ is neither a minimal nor a maximal element. This example shows that a partially ordered set can have multiple maxima and minima; however, a totally ordered set cannot (Exercise 8.5.7).

Example 8.5.7. The natural numbers $\mathbf{N}$ (ordered by $\leq$) has a minimal element, namely 0, but no maximal element. The set of integers $\mathbf{Z}$ has no maximal and no minimal element.

Definition 8.5.8 (Well-ordered sets). Let X be a partially ordered set, and let Y be a totally ordered subset of X. We say that Y is *well-ordered* if every non-empty subset of Y has a minimal element $\min(Y)$.

Examples 8.5.9. The natural numbers $\mathbf{N}$ are well-ordered by Proposition 8.1.4. However, the integers $\mathbf{Z}$, the rationals $\mathbf{Q}$, and the real numbers $\mathbf{R}$ are not (see Exercise 8.1.2). Every finite totally ordered set is well-ordered (Exercise 8.5.8). Every subset of a well-ordered set is again well-ordered (why?).

One advantage of well-ordered sets is that they automatically obey a principle of strong induction (cf. Proposition 2.2.14):

Proposition 8.5.10 (Principle of strong induction). *Let X be a well-ordered set with an ordering relation $\leq$, and let $P(n)$ be a property pertaining to an element $n \in X$ (i.e., for each $n \in X$, $P(n)$ is either a true statement or a false statement). Suppose that for every $n \in X$, we have the following implication: if $P(m)$ is true for all $m \in X$ with $m <_X n$, then $P(n)$ is also true. Prove that $P(n)$ is true for all $n \in X$.*

Remark 8.5.11. It may seem strange that there is no "base" case in strong induction, corresponding to the hypothesis $P(0)$ in Axiom 2.5. However, such a base case is automatically included in the strong induction hypothesis. Indeed, if 0 is the minimal element of X, then by specializing the hypothesis "if $P(m)$ is true for all $m \in X$ with $m <_X n$, then $P(n)$ is also true" to the $n = 0$ case, we automatically obtain that $P(0)$ is true. (Why?)

Proof. See Exercise 8.5.10. □

So far we have not seen the axiom of choice play any rôle. This will come in once we introduce the notion of an upper bound and a strict upper bound.

Definition 8.5.12 (Upper bounds and strict upper bounds). Let X be a partially ordered set with ordering relation $\leq$, and let Y be a subset of X. If $x \in X$, we say that x is an *upper bound* for

Y iff $y \leq x$ for all $y \in Y$. If in addition $x \notin Y$, we say that x is a *strict upper bound* for Y. Equivalently, x is a strict upper bound for Y iff $y < x$ for all $y \in Y$. (Why is this equivalent?)

Example 8.5.13. Let us work in the real number system $\mathbf{R}$ with the usual ordering $\leq$. Then 2 is an upper bound for the set $\{x \in \mathbf{R} : 1 \leq x \leq 2\}$ but is not a strict upper bound. The number 3, on the other hand, is a strict upper bound for this set.

Lemma 8.5.14. *Let X be a partially ordered set with ordering relation $\leq$, and let x_0 be an element of X. Then there is a well-ordered subset Y of X which has x_0 as its minimal element, and which has no strict upper bound.*

Proof. The intuition behind this lemma is that one is trying to perform the following algorithm: we initalize $Y := \{x_0\}$. If Y has no strict upper bound, then we are done; otherwise, we choose a strict upper bound and add it to Y. Then we look again to see if Y has a strict upper bound or not. If not, we are done; otherwise we choose another strict upper bound and add it to Y. We continue this algorithm "infinitely often" until we exhaust all the strict upper bounds; the axiom of choice comes in because infinitely many choices are involved. This is however not a rigourous proof because it is quite difficult to precisely pin down what it means to perform an algorithm "infinitely often". Instead, what we will do is that we will isolate a collection of "partially completed" sets Y, which we shall call *good sets*, and then take the union of all these good sets to obtain a "completed" object Y_∞ which will indeed have no strict upper bound.

We now begin the rigourous proof. Suppose for sake of contradiction that every well-ordered subset Y of X which has x_0 as its minimal element has at least one strict upper bound. Using the axiom of choice (in the form of Proposition 8.4.7), we can thus assign a strict upper bound $s(Y) \in X$ to each well-ordered subset Y of X which has x_0 as its minimal element.

Let us define a special class of subsets Y of X. We say that a subset Y of X is *good* iff it is well-ordered, contains x_0 as its minimal element, and obeys the property that

$$x = s(\{y \in Y : y < x\}) \text{ for all } x \in Y \backslash \{x_0\}.$$

Note that if $x \in Y\backslash\{x_0\}$ then the set $\{y \in Y : y < x\}$ is a subset of X which is well-ordered and contains x_0 as its minimal element. Let $\Omega := \{Y \subseteq X : Y \text{ is good}\}$ be the collection of all good subsets of X. This collection is not empty, since the subset $\{x_0\}$ of X is clearly good (why?).

We make the following important observation: if Y and Y' are two good subsets of X, then every element of $Y'\backslash Y$ is a strict upper bound for Y, and every element of $Y\backslash Y'$ is a strict upper bound for Y'. (Exercise 8.5.13). In particular, given any two good sets Y and Y', at least one of $Y'\backslash Y$ and $Y\backslash Y'$ must be empty (since they are both strict upper bounds of each other). In other words, Ω is totally ordered by set inclusion: given any two good sets Y and Y', either $Y \subseteq Y'$ or $Y' \subseteq Y$.

Let $Y_\infty := \bigcup \Omega$, i.e., Y_∞ is the set of all elements of X which belong to at least one good subset of X. Clearly $x_0 \in Y_\infty$. Also, since each good subset of X has x_0 as its minimal element, the set Y_∞ also has x_0 as its minimal element (why?).

Next, we show that Y_∞ is totally ordered. Let x, x' be two elements of Y_∞. By definition of Y_∞, we know that x lies in some good set Y and x' lies in some good set Y'. But since Ω is totally ordered, one of these good sets contains the other. Thus x, x' are contained in a single good set (either Y or Y'); since good sets are totally ordered, we thus see that either $x \leq x'$ or $x' \leq x$ as desired.

Next, we show that Y_∞ is well-ordered. Let A be any non-empty subset of Y_∞. Then we can pick an element $a \in A$, which then lies in Y_∞. Therefore there is a good set Y such that $a \in Y$. Then $A \cap Y$ is a non-empty subset of Y; since Y is well-ordered, the set $A \cap Y$ thus has a minimal element, call it b. Now recall that for any other good set Y', every element of $Y'\backslash Y$ is a strict upper bound for Y, and in particular is larger than b. Since b is a minimal element of $A \cap Y$, this implies that b is also a minimal element of $A \cap Y'$ for any good set Y' (why?). Since every element of A belongs to Y_∞ and hence belongs to at least one good set Y', we thus see that b is a minimal element of A. Thus Y_∞ is well-ordered as claimed.

Since Y_∞ is well-ordered with x_0 as its minimal element, it has a strict upper bound $s(Y_\infty)$. But then $Y_\infty \cup \{s(Y_\infty)\}$ is well-ordered (why? see Exercise 8.5.11) and has x_0 as its minimal element (why?). Thus this set is good, and must therefore be contained in Y_∞. But this is a contradiction since $s(Y_\infty)$ is a *strict* upper bound for Y_∞. Thus we have constructed a set with no strict upper bound, as desired. □

The above lemma has the following important consequence:

Lemma 8.5.15 (Zorn's lemma). *Let X be a non-empty partially ordered set, with the property that every totally ordered subset Y of X has an upper bound. Then X contains at least one maximal element.*

Proof. See Exercise 8.5.14. □

We give some applications of Zorn's lemma (also called the *principle of transfinite induction*) in the exercises below.

Exercise 8.5.1. Consider the empty set $\emptyset$ with the empty order relation $\leq_\emptyset$ (this relation is vacuous because the empty set has no elements). Is this set partially ordered? totally ordered? well-ordered? Explain.

Exercise 8.5.2. Give examples of a set X and a relation $\leq$ such that

(a) The relation $\leq$ is reflexive and anti-symmetric, but not transitive;

(b) The relation $\leq$ is reflexive and transitive, but not anti-symmetric;

(c) The relation $\leq$ is anti-symmetric and transitive, but not reflexive.

Exercise 8.5.3. Given two positive integers $n, m \in \mathbf{N}\backslash\{0\}$, we say that *$n$ divides m*, and write $n|m$, if there exists a positive integer a such that $m = na$. Show that the set $\mathbf{N}\backslash\{0\}$ with the ordering relation $|$ is a partially ordered set but not a totally ordered one. Note that this is a different ordering relation from the usual $\leq$ ordering of $\mathbf{N}\backslash\{0\}$.

Exercise 8.5.4. Show that the set of positive reals $\mathbf{R}^+ := \{x \in \mathbf{R} : x > 0\}$ have no minimal element.

Exercise 8.5.5. Let $f : X \to Y$ be a function from one set X to another set Y. Suppose that Y is partially ordered with some ordering relation $\leq_Y$. Define a relation $\leq_X$ on X by defining $x \leq_X x'$ if and only if $f(x) \leq_Y f(x')$. Show that this relation $\leq_X$ turns X into a partially ordered set. If we know in addition that the relation $\leq_Y$ makes Y totally ordered, does this mean that the relation $\leq_X$ makes X totally ordered also? If not, what additional assumption needs to be made on f in order to ensure that $\leq_X$ makes X totally ordered?

Exercise 8.5.6. Let X be a partially ordered set. For any x in X, define the *order ideal* $(x) \subset X$ to be the set $(x) := \{y \in X : y \leq x\}$. Let $(X) := \{(x) : x \in X\}$ be the set of all order ideals, and let $f : X \to (X)$ be the map $f(x) := (x)$ that sends every element of x to its order ideal. Show that f is a bijection, and that given any $x, y \in X$, that $x \leq_X y$ if and only if $f(x) \subseteq f(y)$. This exercise shows that any partially ordered set can be *represented* by a collection of sets whose ordering relation is given by set inclusion.

Exercise 8.5.7. Let X be a partially ordered set, and let Y be a totally ordered subset of X. Show that Y can have at most one maximum and at most one minimum.

Exercise 8.5.8. Show that every finite non-empty subset of a totally ordered set has a minimum and a maximum. (Hint: use induction.) Conclude in particular that every finite totally ordered set is well-ordered.

Exercise 8.5.9. Let X be a totally ordered set such that every non-empty subset of X has both a minimum and a maximum. Show that X is finite. (Hint: assume for sake of contradiction that X is infinite. Start with the minimal element x_0 of X and then construct an increasing sequence $x_0 < x_1 < \ldots$ in X.)

Exercise 8.5.10. Prove Proposition 8.5.10, without using the axiom of choice. (Hint: consider the set

$$Y := \{n \in X : P(m) \text{ is false for some } m \in X \text{ with } m \leq_X n\},$$

and show that Y being non-empty would lead to a contradiction.)

Exercise 8.5.11. Let X be a partially ordered set, and let Y and Y' be well-ordered subsets of X. Show that $Y \cup Y'$ is well-ordered if and only if it is totally ordered.

Exercise 8.5.12. Let X and Y be partially ordered sets with ordering relations $\leq_X$ and $\leq_Y$ respectively. Define a relation $\leq_{X \times Y}$ on the Cartesian product $X \times Y$ by defining $(x, y) \leq_{X \times Y} (x', y')$ if $x \leq_X x'$, or if

$x = x'$ and $y \leq_Y y'$. (This is called the *lexicographical ordering* on $X \times Y$, and is similar to the alphabetical ordering of words; a word w appears earlier in a dictionary than another word w' if the first letter of w is earlier in the alphabet than the first letter of w', or if the first letters match and the second letter of w is earlier than the second letter of w', and so forth.) Show that $\leq_{X\times Y}$ defines a partial ordering on $X \times Y$. Furthermore, show that if X and Y are totally ordered, then so is $X \times Y$, and if X and Y are well-ordered, then so is $X \times Y$.

Exercise 8.5.13. Prove the claim in the proof of Lemma 8.5.14, namely that every element of $Y' \backslash Y$ is an upper bound for Y and vice versa. (Hint: Show using Proposition 8.5.10 that

$$\{y \in Y : y \leq a\} = \{y \in Y' : y \leq a\} = \{y \in Y \cap Y' : y \leq a\}$$

for all $a \in Y \cap Y'$. Conclude that $Y \cap Y'$ is good, and hence $s(Y \cap Y')$ exists. Show that $s(Y \cap Y') = \min(Y' \backslash Y)$ if $Y' \backslash Y$ is non-empty, and similarly with Y and Y' interchanged. Since $Y' \backslash Y$ and $Y \backslash Y'$ are disjoint, one can then conclude that one of these sets is empty, at which point the claim becomes easy to establish.)

Exercise 8.5.14. Use Lemma 8.5.14 to prove Lemma 8.5.15. (Hint: first show that if X had no maximal elements, then any subset of X which has an upper bound, also has a strict upper bound.)

Exercise 8.5.15. Let A and B be two non-empty sets such that A does not have lesser or equal cardinality to B. Using the principle of transfinite induction, prove that B has lesser or equal cardinality to A. (Hint: for every subset $X \subseteq B$, let $P(X)$ denote the property that there exists an injective map from X to A.) This exercise (combined with Exercise 8.3.3) shows that the cardinality of any two sets is comparable, as long as one assumes the axiom of choice.

Exercise 8.5.16. Let X be a set, and let P be the set of all partial orderings of X. (For instance, if $X := \mathbf{N} \backslash \{0\}$, then both the usual partial ordering $\leq$, and the partial ordering in Exercise 8.5.3, are elements of P.) We say that one partial ordering $\leq \in P$ is *coarser* than another partial ordering $\leq' \in P$ if for any $x, y \in P$, we have the implication $(x \leq y) \implies (x \leq' y)$. Thus for instance the partial ordering in Exercise 8.5.3 is coarser than the usual ordering $\leq$. Let us write $\leq \preceq \leq'$ if $\leq$ is coarser than $\leq'$. Show that $\preceq$ turns P into a partially ordered set; thus the set of partial orderings on X is itself partially ordered. There is exactly one minimal element of P; what is it? Show that the maximal elements of P are precisely the total orderings of P. Using

Zorn's lemma, show that given any partial ordering $\leq$ of X there exists a total ordering $\leq'$ such that $\leq$ is coarser than $\leq'$.

Exercise 8.5.17. Use Zorn's lemma to give another proof of the claim in Exercise 8.4.2. (Hint: let Ω be the set of all $Y \subseteq \bigcup_{\alpha \in I} X_\alpha$ such that $\#(Y \cap X_\alpha) \leq 1$ for all $\alpha \in I$, i.e., all sets which intersect each X_α in at most one element. Use Zorn's lemma to locate a maximal element of Ω.) Deduce that Zorn's lemma and the axiom of choice are in fact logically equivalent (i.e., they can be deduced from each other).

Exercise 8.5.18. Using Zorn's lemma, prove *Hausdorff's maximality principle*: if X is a partially ordered set, then there exists a totally ordered subset Y of X which is maximal with respect to set inclusion (i.e. there is no other totally ordered subset Y' of X which contains Y. Conversely, show that if Hausdorff's maximality principle is true, then Zorn's lemma is true. Tthus by Exercise 8.5.17, these two statements are logically equivalent to the axiom of choice.

Exercise 8.5.19. Let X be a set, and let Ω be the space of all pairs $(Y, \leq)$, where Y is a subset of X and $\leq$ is a well-ordering of Y. If $(Y, \leq)$ and $(Y', \leq')$ are elements of Ω, we say that $(Y, \leq)$ is an *initial segment* of $(Y', \leq')$ if there exists an $x \in Y'$ such that $Y := \{y \in Y' : y < x\}$ (so in particular $Y \subsetneq Y'$), and for any $y, y' \in Y$, $y \leq y'$ if and only if $y \leq' y'$. Define a relation $\preceq$ on Ω by defining $(Y, \leq) \preceq (Y', \leq')$ if either $(Y, \leq) = (Y', \leq')$, or if $(Y, \leq)$ is an initial segment of $(Y', \leq')$. Show that $\preceq$ is a partial ordering of Ω. There is exactly one minimal element of Ω; what is it? Show that the maximal elements of Ω are precisely the well-orderings $(X, \leq)$ of X. Using Zorn's lemma, conclude the *well ordering principle*: every set X has at least one well-ordering. Conversely, use the well-ordering principle to prove the axiom of choice, Axiom 8.1. (Hint: place a well-ordering $\leq$ on $\bigcup_{\alpha \in I} X_\alpha$, and then consider the minimal elements of each X_α.) We thus see that the axiom of choice, Zorn's lemma, and the well-ordering principle are all logically equivalent to each other.

Exercise 8.5.20. Let X be a set, and let $\Omega \subset 2^X$ be a collection of subsets of X. Using Zorn's lemma, show that there is a subcollection $\Omega' \subseteq \Omega$ such that all the elements of Ω' are disjoint from each other (i.e., $A \cap B = \emptyset$ whenever A, B are distinct elements of Ω'), but that all the elements of Ω intersect at least one element of Ω' (i.e., for all $C \in \Omega$ there exists $A \in \Omega'$ such that $C \cap A \neq \emptyset$). (Hint: consider all the subsets of Ω whose elements are all disjoint from each other, and locate a maximal element of this collection.) Conversely, if the above claim is true, show that it implies the claim in Exercise 8.4.2, and thus this is yet another claim which is logically equivalent to the axiom of choice. (Hint: let Ω be the set of all pair sets of the form $\{(0, \alpha), (1, x_\alpha)\}$, where $\alpha \in I$ and $x_\alpha \in X_\alpha$.)