EXERCISES

1. Suppose α increases on $[a, b]$, $a \leq x_0 \leq b$, α is continuous at x_0, $f(x_0) = 1$, and $f(x) = 0$ if $x \neq x_0$. Prove that $f \in \mathscr{R}(\alpha)$ and that $\int f\,d\alpha = 0$.

2. Suppose $f \geq 0$, f is continuous on $[a, b]$, and $\int_a^b f(x)\,dx = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$. (Compare this with Exercise 1.)

3. Define three functions $\beta_1, \beta_2, \beta_3$ as follows: $\beta_j(x) = 0$ if $x < 0$, $\beta_j(x) = 1$ if $x > 0$ for $j = 1, 2, 3$; and $\beta_1(0) = 0$, $\beta_2(0) = 1$, $\beta_3(0) = \frac{1}{2}$. Let f be a bounded function on $[-1, 1]$.

(*a*) Prove that $f \in \mathscr{R}(\beta_1)$ if and only if $f(0+) = f(0)$ and that then

$$\int f\,d\beta_1 = f(0).$$

(*b*) State and prove a similar result for β_2.

(*c*) Prove that $f \in \mathscr{R}(\beta_3)$ if and only if f is continuous at 0.

(*d*) If f is continuous at 0 prove that

$$\int f\,d\beta_1 = \int f\,d\beta_2 = \int f\,d\beta_3 = f(0).$$

4. If $f(x) = 0$ for all irrational x, $f(x) = 1$ for all rational x, prove that $f \notin \mathscr{R}$ on $[a, b]$ for any $a < b$.

5. Suppose f is a bounded real function on $[a, b]$, and $f^2 \in \mathscr{R}$ on $[a, b]$. Does it follow that $f \in \mathscr{R}$? Does the answer change if we assume that $f^3 \in \mathscr{R}$?

6. Let P be the Cantor set constructed in Sec. 2.44. Let f be a bounded real function on $[0, 1]$ which is continuous at every point outside P. Prove that $f \in \mathscr{R}$ on $[0, 1]$. *Hint:* P can be covered by finitely many segments whose total length can be made as small as desired. Proceed as in Theorem 6.10.

7. Suppose f is a real function on $(0, 1]$ and $f \in \mathscr{R}$ on $[c, 1]$ for every $c > 0$. Define

$$\int_0^1 f(x)\,dx = \lim_{c \to 0} \int_c^1 f(x)\,dx$$

if this limit exists (and is finite).

(*a*) If $f \in \mathscr{R}$ on $[0, 1]$, show that this definition of the integral agrees with the old one.

(*b*) Construct a function f such that the above limit exists, although it fails to exist with $|f|$ in place of f.

8. Suppose $f \in \mathscr{R}$ on $[a, b]$ for every $b > a$ where a is fixed. Define

$$\int_a^\infty f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx$$

if this limit exists (and is finite). In that case, we say that the integral on the left *converges*. If it also converges after f has been replaced by $|f|$, it is said to converge *absolutely*.

Assume that $f(x) \geq 0$ and that f decreases monotonically on $[1, \infty)$. Prove that

$$\int_1^\infty f(x)\,dx$$

converges if and only if

$$\sum_{n=1}^\infty f(n)$$

converges. (This is the so-called "integral test" for convergence of series.)

9. Show that integration by parts can sometimes be applied to the "improper" integrals defined in Exercises 7 and 8. (State appropriate hypotheses, formulate a theorem, and prove it.) For instance show that

$$\int_0^\infty \frac{\cos x}{1+x}\,dx = \int_0^\infty \frac{\sin x}{(1+x)^2}\,dx.$$

Show that one of these integrals converges *absolutely*, but that the other does not.

10. Let p and q be positive real numbers such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Prove the following statements.

(*a*) If $u \geq 0$ and $v \geq 0$, then

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q}.$$

Equality holds if and only if $u^p = v^q$.

(*b*) If $f \in \mathscr{R}(\alpha)$, $g \in \mathscr{R}(\alpha)$, $f \geq 0$, $g \geq 0$, and

$$\int_a^b f^p\,d\alpha = 1 = \int_a^b g^q\,d\alpha,$$

then

$$\int_a^b fg\,d\alpha \leq 1.$$

(*c*) If f and g are complex functions in $\mathscr{R}(\alpha)$, then

$$\left|\int_a^b fg\,d\alpha\right| \leq \left\{\int_a^b |f|^p\,d\alpha\right\}^{1/p}\left\{\int_a^b |g|^q\,d\alpha\right\}^{1/q}.$$

This is *Hölder's inequality*. When $p = q = 2$ it is usually called the Schwarz inequality. (Note that Theorem 1.35 is a very special case of this.)

(*d*) Show that Hölder's inequality is also true for the "improper" integrals described in Exercises 7 and 8.

11. Let α be a fixed increasing function on $[a, b]$. For $u \in \mathscr{R}(\alpha)$, define

$$\|u\|_2 = \left\{ \int_a^b |u|^2 \, d\alpha \right\}^{1/2}.$$

Suppose $f, g, h \in \mathscr{R}(\alpha)$, and prove the triangle inequality

$$\|f - h\|_2 \leq \|f - g\|_2 + \|g - h\|_2$$

as a consequence of the Schwarz inequality, as in the proof of Theorem 1.37.

12. With the notations of Exercise 11, suppose $f \in \mathscr{R}(\alpha)$ and $\varepsilon > 0$. Prove that there exists a continuous function g on $[a, b]$ such that $\|f - g\|_2 < \varepsilon$.

Hint: Let $P = \{x_0, \ldots, x_n\}$ be a suitable partition of $[a, b]$, define

$$g(t) = \frac{x_i - t}{\Delta x_i} f(x_{i-1}) + \frac{t - x_{i-1}}{\Delta x_i} f(x_i)$$

if $x_{i-1} \leq t \leq x_i$.

13. Define

$$f(x) = \int_x^{x+1} \sin(t^2) \, dt.$$

(*a*) Prove that $|f(x)| < 1/x$ if $x > 0$.

Hint: Put $t^2 = u$ and integrate by parts, to show that $f(x)$ is equal to

$$\frac{\cos(x^2)}{2x} - \frac{\cos[(x+1)^2]}{2(x+1)} - \int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{3/2}} \, du.$$

Replace $\cos u$ by -1.

(*b*) Prove that

$$2xf(x) = \cos(x^2) - \cos[(x+1)^2] + r(x)$$

where $|r(x)| < c/x$ and c is a constant.

(*c*) Find the upper and lower limits of $xf(x)$, as $x \to \infty$.

(*d*) Does $\int_0^\infty \sin(t^2) \, dt$ converge?

14. Deal similarly with

$$f(x) = \int_x^{x+1} \sin(e^t) \, dt.$$

Show that

$$e^x |f(x)| < 2$$

and that

$$e^x f(x) = \cos(e^x) - e^{-1} \cos(e^{x+1}) + r(x),$$

where $|r(x)| < Ce^{-x}$, for some constant C.

15. Suppose f is a real, continuously differentiable function on $[a, b]$, $f(a) = f(b) = 0$, and

$$\int_a^b f^2(x)\,dx = 1.$$

Prove that

$$\int_a^b xf(x)f'(x)\,dx = -\tfrac{1}{2}$$

and that

$$\int_a^b [f'(x)]^2\,dx \cdot \int_a^b x^2 f^2(x)\,dx > \tfrac{1}{4}.$$

16. For $1 < s < \infty$, define

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

(This is Riemann's zeta function, of great importance in the study of the distribution of prime numbers.) Prove that

(*a*) $\zeta(s) = s \displaystyle\int_1^{\infty} \frac{[x]}{x^{s+1}}\,dx$

and that

(*b*) $\zeta(s) = \dfrac{s}{s-1} - s \displaystyle\int_1^{\infty} \frac{x - [x]}{x^{s+1}}\,dx,$

where $[x]$ denotes the greatest integer $\leq x$.

Prove that the integral in (*b*) converges for all $s > 0$.

Hint: To prove (*a*), compute the difference between the integral over $[1, N]$ and the Nth partial sum of the series that defines $\zeta(s)$.

17. Suppose α increases monotonically on $[a, b]$, g is continuous, and $g(x) = G'(x)$ for $a \leq x \leq b$. Prove that

$$\int_a^b \alpha(x)g(x)\,dx = G(b)\alpha(b) - G(a)\alpha(a) - \int_a^b G\,d\alpha.$$

Hint: Take g real, without loss of generality. Given $P = \{x_0, x_1, \ldots, x_n\}$, choose $t_i \in (x_{i-1}, x_i)$ so that $g(t_i)\,\Delta x_i = G(x_i) - G(x_{i-1})$. Show that

$$\sum_{i=1}^{n} \alpha(x_i)g(t_i)\,\Delta x_i = G(b)\alpha(b) - G(a)\alpha(a) - \sum_{i=1}^{n} G(x_{i-1})\,\Delta\alpha_i.$$

18. Let $\gamma_1, \gamma_2, \gamma_3$ be curves in the complex plane, defined on $[0, 2\pi]$ by

$$\gamma_1(t) = e^{it}, \qquad \gamma_2(t) = e^{2it}, \qquad \gamma_3(t) = e^{2\pi i t \sin(1/t)}.$$

Show that these three curves have the same range, that γ_1 and γ_2 are rectifiable, that the length of γ_1 is 2π, that the length of γ_2 is 4π, and that γ_3 is not rectifiable.

19. Let γ_1 be a curve in R^k, defined on $[a, b]$; let ϕ be a continuous 1-1 mapping of $[c, d]$ onto $[a, b]$, such that $\phi(c) = a$; and define $\gamma_2(s) = \gamma_1(\phi(s))$. Prove that γ_2 is an arc, a closed curve, or a rectifiable curve if and only if the same is true of γ_1. Prove that γ_2 and γ_1 have the same length.

7

SEQUENCES AND SERIES OF FUNCTIONS

In the present chapter we confine our attention to complex-valued functions (including the real-valued ones, of course), although many of the theorems and proofs which follow extend without difficulty to vector-valued functions, and even to mappings into general metric spaces. We choose to stay within this simple framework in order to focus attention on the most important aspects of the problems that arise when limit processes are interchanged.

DISCUSSION OF MAIN PROBLEM

7.1 Definition Suppose $\{f_n\}$, $n = 1, 2, 3, \ldots$, is a sequence of functions defined on a set E, and suppose that the sequence of numbers $\{f_n(x)\}$ converges for every $x \in E$. We can then define a function f by

$$f(x) = \lim_{n \to \infty} f_n(x) \qquad (x \in E). \tag{1}$$

Under these circumstances we say that $\{f_n\}$ converges on E and that f is the *limit*, or the *limit function*, of $\{f_n\}$. Sometimes we shall use a more descriptive terminology and shall say that "$\{f_n\}$ converges to f *pointwise* on E" if (1) holds. Similarly, if $\Sigma f_n(x)$ converges for every $x \in E$, and if we define

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \qquad (x \in E), \tag{2}$$

the function f is called the *sum* of the series Σf_n.

The main problem which arises is to determine whether important properties of functions are preserved under the limit operations (1) and (2). For instance, if the functions f_n are continuous, or differentiable, or integrable, is the same true of the limit function? What are the relations between f_n' and f', say, or between the integrals of f_n and that of f?

To say that f is continuous at a limit point x means

$$\lim_{t \to x} f(t) = f(x).$$

Hence, to ask whether the limit of a sequence of continuous functions is continuous is the same as to ask whether

$$\lim_{t \to x} \lim_{n \to \infty} f_n(t) = \lim_{n \to \infty} \lim_{t \to x} f_n(t), \tag{3}$$

i.e., whether the order in which limit processes are carried out is immaterial. On the left side of (3), we first let $n \to \infty$, then $t \to x$; on the right side, $t \to x$ first, then $n \to \infty$.

We shall now show by means of several examples that limit processes cannot in general be interchanged without affecting the result. Afterward, we shall prove that under certain conditions the order in which limit operations are carried out is immaterial.

Our first example, and the simplest one, concerns a "double sequence."

7.2 Example For $m = 1, 2, 3, \ldots, n = 1, 2, 3, \ldots$, let

$$s_{m,n} = \frac{m}{m+n}.$$

Then, for every fixed n,

$$\lim_{m \to \infty} s_{m,n} = 1,$$

so that

$$\lim_{n \to \infty} \lim_{m \to \infty} s_{m,n} = 1. \tag{4}$$

On the other hand, for every fixed m,

$$\lim_{n\to\infty} s_{m,n} = 0,$$

so that

(5) $$\lim_{m\to\infty}\lim_{n\to\infty} s_{m,n} = 0.$$

7.3 Example Let

$$f_n(x) = \frac{x^2}{(1+x^2)^n} \qquad (x \text{ real}; n = 0, 1, 2, \ldots),$$

and consider

(6) $$f(x) = \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}.$$

Since $f_n(0) = 0$, we have $f(0) = 0$. For $x \neq 0$, the last series in (6) is a convergent geometric series with sum $1 + x^2$ (Theorem 3.26). Hence

(7) $$f(x) = \begin{cases} 0 & (x = 0), \\ 1 + x^2 & (x \neq 0), \end{cases}$$

so that a convergent series of continuous functions may have a discontinuous sum.

7.4 Example For $m = 1, 2, 3, \ldots$, put

$$f_m(x) = \lim_{n\to\infty} (\cos m!\pi x)^{2n}.$$

When $m!x$ is an integer, $f_m(x) = 1$. For all other values of x, $f_m(x) = 0$. Now let

$$f(x) = \lim_{m\to\infty} f_m(x).$$

For irrational x, $f_m(x) = 0$ for every m; hence $f(x) = 0$. For rational x, say $x = p/q$, where p and q are integers, we see that $m!x$ is an integer if $m \geq q$, so that $f(x) = 1$. Hence

(8) $$\lim_{m\to\infty}\lim_{n\to\infty} (\cos m!\pi x)^{2n} = \begin{cases} 0 & (x \text{ irrational}), \\ 1 & (x \text{ rational}). \end{cases}$$

We have thus obtained an everywhere discontinuous limit function, which is not Riemann-integrable (Exercise 4, Chap. 6).

7.5 Example Let

$$f_n(x) = \frac{\sin nx}{\sqrt{n}} \qquad (x \text{ real}, n = 1, 2, 3, \ldots), \tag{9}$$

and

$$f(x) = \lim_{n\to\infty} f_n(x) = 0.$$

Then $f'(x) = 0$, and

$$f_n'(x) = \sqrt{n} \cos nx,$$

so that $\{f_n'\}$ does not converge to f'. For instance,

$$f_n'(0) = \sqrt{n} \to +\infty$$

as $n \to \infty$, whereas $f'(0) = 0$.

7.6 Example Let

$$f_n(x) = n^2 x(1 - x^2)^n \qquad (0 \le x \le 1, n = 1, 2, 3, \ldots). \tag{10}$$

For $0 < x \le 1$, we have

$$\lim_{n\to\infty} f_n(x) = 0,$$

by Theorem 3.20(d). Since $f_n(0) = 0$, we see that

$$\lim_{n\to\infty} f_n(x) = 0 \qquad (0 \le x \le 1). \tag{11}$$

A simple calculation shows that

$$\int_0^1 x(1 - x^2)^n \, dx = \frac{1}{2n + 2}.$$

Thus, in spite of (11),

$$\int_0^1 f_n(x) \, dx = \frac{n^2}{2n + 2} \to +\infty$$

as $n \to \infty$.

If, in (10), we replace n^2 by n, (11) still holds, but we now have

$$\lim_{n\to\infty} \int_0^1 f_n(x) \, dx = \lim_{n\to\infty} \frac{n}{2n + 2} = \frac{1}{2},$$

whereas

$$\int_0^1 \left[\lim_{n\to\infty} f_n(x)\right] dx = 0.$$

Thus the limit of the integral need not be equal to the integral of the limit, even if both are finite.

After these examples, which show what can go wrong if limit processes are interchanged carelessly, we now define a new mode of convergence, stronger than pointwise convergence as defined in Definition 7.1, which will enable us to arrive at positive results.

UNIFORM CONVERGENCE

7.7 Definition We say that a sequence of functions $\{f_n\}$, $n = 1, 2, 3, \ldots$, converges *uniformly* on E to a function f if for every $\varepsilon > 0$ there is an integer N such that $n \geq N$ implies

$$|f_n(x) - f(x)| \leq \varepsilon \tag{12}$$

for all $x \in E$.

It is clear that every uniformly convergent sequence is pointwise convergent. Quite explicitly, the difference between the two concepts is this: If $\{f_n\}$ converges pointwise on E, then there exists a function f such that, for every $\varepsilon > 0$, and for every $x \in E$, there is an integer N, depending on ε *and* on x, such that (12) holds if $n \geq N$; if $\{f_n\}$ converges uniformly on E, it is possible, for each $\varepsilon > 0$, to find *one* integer N which will do for *all* $x \in E$.

We say that the series $\Sigma f_n(x)$ converges uniformly on E if the sequence $\{s_n\}$ of partial sums defined by

$$\sum_{i=1}^{n} f_i(x) = s_n(x)$$

converges uniformly on E.

The Cauchy criterion for uniform convergence is as follows.

7.8 Theorem *The sequence of functions $\{f_n\}$, defined on E, converges uniformly on E if and only if for every $\varepsilon > 0$ there exists an integer N such that $m \geq N$, $n \geq N$, $x \in E$ implies*

$$|f_n(x) - f_m(x)| \leq \varepsilon. \tag{13}$$

Proof Suppose $\{f_n\}$ converges uniformly on E, and let f be the limit function. Then there is an integer N such that $n \geq N$, $x \in E$ implies

$$|f_n(x) - f(x)| \leq \frac{\varepsilon}{2},$$

so that

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)| \leq \varepsilon$$

if $n \geq N$, $m \geq N$, $x \in E$.