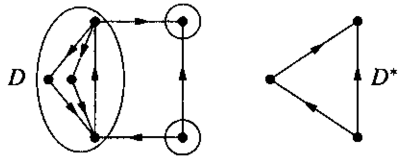
1.4.13. a) Prove that the strong components of a digraph are pairwise disjoint.

b) Let $D_1, \dots, D_k$ be the strong components of a digraph D. Let D^* be the loopless digraph with vertices $v_1, \dots, v_k$ such that $v_i \to v_j$ if and only if $i \neq j$ and D has an edge from D_i to D_j. Prove that D^* has no cycle.



1.4.14. (!) Let G be an n-vertex digraph with no cycles. Prove that the vertices of G can be ordered as $v_1, \dots, v_n$ so that if $v_i v_j \in E(G)$, then $i < j$.

1.4.15. Let G be the simple digraph with vertex set $\{(i, j) \in \mathbb{Z}^2 \colon 0 \le i \le m \text{ and } 0 \le n\}$ and an edge from (i, j) to (i', j') if and only if (i', j') is obtained from (i, j) by adding 1 to one coordinate. Prove that the number of paths from $(0, 0)$ to (m, n) in G is $\binom{m+n}{n}$.

1.4.16. (+) *Fermat's Little Theorem.* Let $\mathbb{Z}_n$ denote the set of congruence classes of integers modulo n (see Appendix A). Let a be a natural number having no common prime factors with n; multiplication by a defines a permutation of $\mathbb{Z}_n$. Let l be the least natural number such that $a^l \equiv a \bmod n$.

a) Let G be the functional digraph with vertex set $\mathbb{Z}_n$ for the permutation defined by multiplication by a. Prove that all cycles in G (except the loop on n) have length $l - 1$.

b) Conclude from part (a) that $a^{n-1} \equiv 1 \bmod n$.

1.4.17. (∗) Prove that a (directed) odd cycle is a digraph with no kernel. Construct a digraph that has an odd cycle as an induced subgraph but does have a kernel.

1.4.18. (∗) Prove that a digraph having no cycle has a unique kernel.

1.4.19. Use Lemma 1.4.23 and induction on the number of edges to prove the characterization of Eulerian digraphs (Theorem 1.4.24). (Hint: Follow Theorem 1.2.26.)

1.4.20. Prove the characterization of Eulerian digraphs (Theorem 1.4.24) using the notion of maximal trails. (Hint: Follow 1.2.32, the second proof of Theorem 1.2.26.)

1.4.21. Theorem 1.4.24 establishes necessary and sufficient conditions for a digraph to have an Eulerian circuit. Determine (with proof), the necessary and sufficient conditions for a digraph to have an Eulerian trail (Definition 1.4.22). (Good [1946])

1.4.22. Let D be a digraph with $d^-(v) = d^+(v)$ for every vertex v, except that $d^+(x) - d^-(x) = k = d^-(y) - d^+(y)$. Use the characterization of Eulerian digraphs to prove that D contains k pairwise edge-disjoint x, y-paths.

1.4.23. Prove that every graph G has an orientation D that is "balanced" at each vertex, meaning that $\left|d_D^+(v) - d_D^-(v)\right| \le 1$ for every $v \in V(G)$.

1.4.24. Prove or disprove: Every graph G has an orientation such that for every $S \subseteq V(G)$, the number of edges entering S and leaving S differ by at most 1.

1.4.25. (!) *Orientations and P_3-decomposition.*

a) Prove that every connected graph has an orientation in which the number of vertices with odd outdegree is at most 1. (Rotman [1991])

b) Use part (a) to conclude that a simple connected graph with an even number of edges can be decomposed into paths with two edges.

1.4.26. Arrange seven 0's and seven 1's cyclically so that the 14 strings of four consecutive bits are all the 4-digit binary strings other than 0101 and 1010.

1.4.27. *DeBruijn sequence for any alphabet and length.* Let A be an alphabet of size k. Prove that there exists a cyclic arrangement of k^l characters chosen from A such that the k^l strings of length l in the sequence are all distinct. (Good [1946], Rees [1946])

1.4.28. Let S be an alphabet of size m. Explain how to produce a cyclic arrangement of $m^4 - m$ letters from S such that all four-letter strings of consecutive letters are different and contain at least two distinct letters.

1.4.29. (!) Suppose that G is a graph and D is an orientation of G that is strongly connected. Prove that if G has an odd cycle, then D has an odd cycle. (Hint: Consider each pair $\{v_i, v_{i+1}\}$ in an odd cycle $(v_1, \ldots, v_k)$ of G.)

1.4.30. (+) Given a strong digraph D, let $f(D)$ be the length of the shortest closed walk visiting every vertex. Prove that the maximum value of $f(D)$ over all strong digraphs with n vertices is $\lfloor (n+1)^2/4 \rfloor$ if $n \geq 2$. (Cull [1980])

1.4.31. Determine the minimum n such that there is a pair of nonisomorphic n-vertex tournaments with the same list of outdegrees.

1.4.32. Let $p = p_1, \ldots, p_m$ and $q = q_1, \ldots, q_n$ be lists of nonnegative integers. The pair (p, q) is **bigraphic** if there is a simple bipartite graph in which $p_1, \ldots, p_m$ are the degrees for one partite set and $q_1, \ldots, q_n$ are the degrees for the other. When p has positive sum, prove that (p, q) is bigraphic if and only if (p', q') is bigraphic, where (p', q') is obtained from (p, q) by deleting the largest element Δ from p and subtracting 1 from each of the Δ largest elements of q. (Hint: Follow the method of Theorem 1.3.31.)

1.4.33. (∗) Let A and B be two m by n matrices with entries in $\{0, 1\}$. An *exchange* operation substitutes a submatrix of the form $\binom{01}{10}$ for a submatrix of the form $\binom{10}{01}$ or vice versa. Prove that if A and B have the same list of row sums and have the same list of column sums, then A can be transformed into B by a sequence of exchange operations. Interpret this conclusion in the context of bipartite graphs. (Ryser [1957])

1.4.34. (!) Let G and H be two tournaments on a vertex set V. Prove that $d_G^+(v) = d_H^+(v)$ for all $v \in V$ if and only if G can be turned into H by a sequence of direction-reversals on cycles of length 3. (Hint: Consider a vertex of maximum outdegree in the subgraph of G consisting of edges oriented oppositely in H.) (Ryser [1964])

1.4.35. (+) Let $p_1, \ldots, p_n$ be nonnegative integers with $p_1 \leq \cdots \leq p_n$. Let $p'_k = \sum_{i=1}^{k} p_i$. Prove that there exists a tournament with outdegrees $p_1, \ldots, p_n$ if and only if $p'_k \geq \binom{k}{2}$ for $1 \leq k < n$ and $p'_n = \binom{n}{2}$. (Hint: Use induction on $\sum_{k=1}^{n} [p'_k - \binom{k}{2}]$.) (Landau [1953])

1.4.36. By Proposition 1.4.30, every tournament has a king. Let T be a tournament having no vertex with indegree 0.

a) Prove that if x is a king in T, then T has another king in $N^-(x)$.

b) Use part (a) to prove that T has at least three kings.

c) For each $n \geq 3$, construct a tournament T with $\delta^-(T) > 0$ and only 3 kings.

(Comment: There exists an n-vertex tournament having exactly k kings whenever $n \geq k \geq 1$ except when $k = 2$ and when $n = k = 4$.) (Maurer [1980])

1.4.37. Consider the following algorithm whose input is a tournament T.

1) Select a vertex x in T.

2) If x has indegree 0, call x a king of T and stop.

3) Otherwise, delete $\{x\} \cup N^+(x)$ from T to form T'.

4) Run the algorithm on T'; call the output a king in T and stop.

Prove that this algorithm terminates and produces a king in T

1.4.38. (+) For $n \in \mathbb{N}$, prove that there is an n-vertex tournament in which every vertex is a king if and only if $n \notin \{2, 4\}$.

1.4.39. (+) Prove that every loopless digraph D has a set S of pairwise nonadjacent vertices such that every vertex outside S is reached from S by a path of length at most 2. (Hint: Use strong induction on $n(D)$. Comment: This generalizes Proposition 1.4.30.) (Chvátal–Lovász [1974])

1.4.40. A directed graph is **unipathic** if for every pair of vertices x, y there is at most one (directed) x, y-path. Let T_n be the tournament on n vertices with the edge between v_i and v_j directed toward the vertex with larger index. What is the maximum number of edges in a unipathic subgraph of T_n? How many unipathic subgraphs are there with the maximum number of edges? (Hint: Show that the underlying graph has no triangles.) (Maurer–Rabinovitch–Trotter [1980])

1.4.41. Let G be a tournament. Let L_0 be a listing of $V(G)$ in some order. If y immediately follows x in L_0 but $y \rightarrow x$ in G, then yx is a **reverse edge**. We can interchange x and y in the order when yx is a reverse edge (this may increase the number of reverse edges). Suppose that a sequence $L_0, L_1, \cdots$ is produced by successively switching one reverse edge in the current order. Prove that this always leads to a list with no reverse edges. Determine the maximum number of steps to termination. (Comment: In the special case where the vertices are numbers and each edge points to the higher number of the pair, the result says that successively switching adjacent numbers that are out of order always eventually sorts the list.) (Locke [1995])

1.4.42. (!) Given an ordering $\sigma = v_1, \ldots, v_n$ of the vertices of a tournament, let $f(\sigma)$ be the sum of the lengths of the feedback edges, meaning the sum of $j - i$ over edges $v_j v_i$ such that $j > i$. Prove that every ordering minimizing $f(\sigma)$ places the vertices in nonincreasing order of outdegree. (Hint: Determine how $f(\sigma)$ changes when consecutive elements of σ are exchanged.) (Kano–Sakamoto [1983], Isaak–Tesman [1991])

Chapter 2

Trees and Distance

2.1. Basic Properties

The word "tree" suggests branching out from a root and never completing a cycle. Trees as graphs have many applications, especially in data storage, searching, and communication.

2.1.1. Definition. A graph with no cycle is **acyclic**. A **forest** is an acyclic graph. A **tree** is a connected acyclic graph. A **leaf** (or **pendant vertex**) is a vertex of degree 1. A **spanning subgraph** of G is a subgraph with vertex set $V(G)$. A **spanning tree** is a spanning subgraph that is a tree.

2.1.2. Example. A tree is a connected forest, and every component of a forest is a tree. A graph with no cycles has no odd cycles; hence trees and forests are bipartite.

Paths are trees. A tree is a path if and only if its maximum degree is 2. A **star** is a tree consisting of one vertex adjacent to all the others. The n-vertex star is the biclique $K_{1,n-1}$.

A graph that is a tree has exactly one spanning tree; the full graph itself. A spanning subgraph of G need not be connected, and a connected subgraph of G need not be a spanning subgraph. For example:

If $n(G) > 1$, then the empty subgraph with vertex set $V(G)$ and edge set $\emptyset$ is spanning but not connected.

If $n(G) > 2$, then a subgraph consisting of one edge and its endpoints is connected but not spanning. ■

PROPERTIES OF TREES

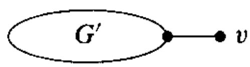
Trees have many equivalent characterizations, any of which could be taken as the definition. Such characterizations are useful because we need only verify that a graph satisfies any one of them to prove that it is a tree, after which we can use all the other properties.

We first prove that deleting a leaf from a tree yields a smaller tree.

2.1.3. Lemma. Every tree with at least two vertices has at least two leaves. Deleting a leaf from an n-vertex tree produces a tree with $n - 1$ vertices.

Proof: A connected graph with at least two vertices has an edge. In an acyclic graph, an endpoint of a maximal nontrivial path has no neighbor other than its neighbor on the path. Hence the endpoints of a such a path are leaves.

Let v be a leaf of a tree G, and let $G' = G - v$. A vertex of degree 1 belongs to no path connecting two other vertices. Therefore, for $u, w \in V(G')$, every u, w-path in G is also in G'. Hence G' is connected. Since deleting a vertex cannot create a cycle, G' also is acyclic. Thus G' is a tree with $n - 1$ vertices. ■



Lemma 2.1.3 implies that every tree with more than one vertex arises from a smaller tree by adding a vertex of degree 1 (all our graphs are finite). This rescues some proofs from the induction trap: growing an $n+1$-vertex tree from an arbitrary n-vertex tree by adding a new neighbor at an arbitrary old vertex generates all trees with $n + 1$ vertices. The word "arbitrary" means that the discussion considers all ways of making the choice.

Our proof of equivalence of characterizations of trees uses induction, prior results, a counting argument, extremality, and contradiction.

2.1.4. Theorem. For an n-vertex graph G (with $n \geq 1$), the following are equivalent (and characterize the trees with n vertices).

A) G is connected and has no cycles.
B) G is connected and has $n - 1$ edges.
C) G has $n - 1$ edges and no cycles.
D) For $u, v \in V(G)$, G has exactly one u, v-path.

Proof: We first demonstrate the equivalence of A, B, and C by proving that any two of {connected, acyclic, $n - 1$ edges} together imply the third.

A $\Rightarrow$ {B, C}. We use induction on n. For $n = 1$, an acyclic 1-vertex graph has no edge. For $n > 1$, we suppose that the implication holds for graphs with fewer than n vertices. Given an acyclic connected graph G, Lemma 2.1.3 provides a leaf v and states that $G' = G - v$ also is acyclic and connected (see figure above). Applying the induction hypothesis to G' yields $e(G') = n - 2$. Since only one edge is incident to v, we have $e(G) = n - 1$.

B $\Rightarrow$ {A, C}. Delete edges from cycles of G one by one until the resulting graph G' is acyclic. Since no edge of a cycle is a cut-edge (Theorem 1.2.14), G' is connected. Now the preceding paragraph implies that $e(G') = n - 1$. Since we are given $e(G) = n - 1$, no edges were deleted. Thus $G' = G$, and G is acyclic.

C $\Rightarrow$ {A, B}. Let $G_1, \ldots, G_k$ be the components of G. Since every vertex appears in one component, $\sum_i n(G_i) = n$. Since G has no cycles, each component satisfies property A. Thus $e(G_i) = n(G_i) - 1$. Summing over i yields $e(G) = \sum_i [n(G_i) - 1] = n - k$. We are given $e(G) = n - 1$, so $k = 1$, and G is connected.

A $\Rightarrow$ D. Since G is connected, each pair of vertices is connected by a path. If some pair is connected by more than one, we choose a shortest (total length) pair P, Q of distinct paths with the same endpoints. By this extremal choice, no internal vertex of P or Q can belong to the other path (see figure below). This implies that $P \cup Q$ is a cycle, which contradicts the hypothesis A.

D $\Rightarrow$ A. If there is a u, v-path for every $u, v \in V(G)$, then G is connected. If G has a cycle C, then G has two u, v-paths for $u, v \in V(C)$; hence G is acyclic (this also forbids loops). ■

2.1.5. Corollary. a) Every edge of a tree is a cut-edge.
b) Adding one edge to a tree forms exactly one cycle.
c) Every connected graph contains a spanning tree.

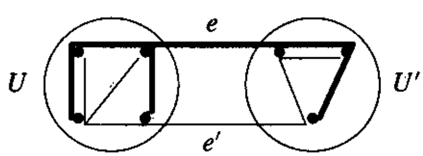
Proof: (a) A tree has no cycles, so Theorem 1.2.14 implies that every edge is a cut-edge. (b) A tree has a unique path linking each pair of vertices (Theorem 2.1.4D), so joining two vertices by an edge creates exactly one cycle. (c) As in the proof of B$\Rightarrow$A,C in Theorem 2.1.4, iteratively deleting edges from cycles in a connected graph yields a connected acyclic subgraph. ■

We apply Corollary 2.1.5 to prove two results about pairs of spanning trees. We use subtraction and addition to indicate deletion and inclusion of edges.

2.1.6. Proposition. If T, T' are spanning trees of a connected graph G and $e \in E(T) - E(T')$, then there is an edge $e' \in E(T') - E(T)$ such that $T - e + e'$ is a spanning tree of G.

Proof: By Corollary 2.1.5a, every edge of T is a cut-edge of T. Let U and U' be the two components of $T - e$. Since T' is connected, T' has an edge e' with endpoints in U and U'. Now $T - e + e'$ is connected, has $n(G) - 1$ edges, and is a spanning tree of G.

(In the figure below, T is bold, T' is solid, and they share two edges.) ■



2.1.7. Proposition. If T, T' are spanning trees of a connected graph G and $e \in E(T) - E(T')$, then there is an edge $e' \in E(T') - E(T)$ such that $T' + e - e'$ is a spanning tree of G.

Proof: By Corollary 2.1.5b, The graph $T' + e$ contains a unique cycle C. Since T is acyclic, there is an edge $e' \in E(C) - E(T)$. Deleting e' breaks the only cycle in $T' + e$. Now $T' + e - e'$ is connected and acyclic and is a spanning tree of G.

(In the figure above, adding e to T creates a cycle C of length five; all four edges of $C - e$ belong to $E(T) - E(T')$ and can serve as e'.) ∎

The edge e' can be chosen to satisfy the conclusions of Propositions 2.1.6–2.1.7 simultaneously, as illustrated in the figure between them (Exercise 37).

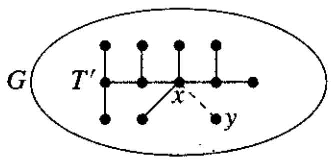
The next result illustrates proof by induction using deletion of a leaf.

2.1.8. Proposition. If T is a tree with k edges and G is a simple graph with $\delta(G) \geq k$, then T is a subgraph of G.

Proof: We use induction on k. Basis step: $k = 0$. Every simple graph contains K_1, which is the only tree with no edges.

Induction step: $k > 0$. We assume that the claim holds for trees with fewer than k edges. Since $k > 0$, Lemma 2.1.3 allows us to choose a leaf v in T; let u be its neighbor. Consider the smaller tree $T' = T - v$. By the induction hypothesis, G contains T' as a subgraph, since $\delta(G) \geq k > k - 1$.

Let x be the vertex in this copy of T' that corresponds to u (see illustration). Because T' has only $k - 1$ vertices other than u and $d_G(x) \geq k$, x has a neighbor y in G that is not in this copy of T'. Adding the edge xy expands this copy of T' into a copy of T in G, with y playing the role of v. ∎



The inequality of Proposition 2.1.8 is sharp; the graph K_k has minimum degree $k - 1$, but it contains no tree with k edges. The proposition implies that every n-vertex simple graph G with more than $n(k - 1)$ edges has T as a subgraph (Exercise 34). Erdős and Sós conjectured the stronger statement that $e(G) > n(k - 1)/2$ forces T as a subgraph (Erdős [1964]). This has been proved for graphs without 4-cycles (Saclé–Woźniak [1997]). Ajtai, Komlós, and Szemerédi proved an asymptotic version, as reported in Soffer [2000].

DISTANCE IN TREES AND GRAPHS

When using graphs to model communication networks, we want vertices to be close together to avoid communication delays. We measure distance using lengths of paths.

2.1.9. Definition. If G has a u, v-path, then the **distance** from u to v, written $d_G(u, v)$ or simply $d(u, v)$, is the least length of a u, v-path. If G has no such path, then $d(u, v) = \infty$. The **diameter** (diam G) is $\max_{u,v \in V(G)} d(u, v)$.

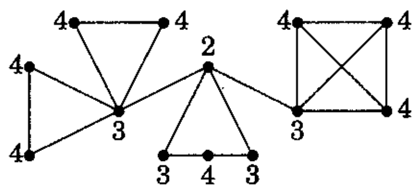
The **eccentricity** of a vertex u, written $\epsilon(u)$, is $\max_{v \in V(G)} d(u, v)$. The **radius** of a graph G, written rad G, is $\min_{u \in V(G)} \epsilon(u)$.

The diameter equals the maximum of the vertex eccentricities. In a disconnected graph, the diameter and radius (and every eccentricity) are infinite, because distance between vertices in different components is infinite. We use the word "diameter" due to its use in geometry, where it is the greatest distance between two elements of a set.

2.1.10. Example. The Petersen graph has diameter 2, since nonadjacent vertices have a common neighbor. The hypercube Q_k has diameter k, since it takes k steps to change all k coordinates. The cycle C_n has diameter $\lfloor n/2 \rfloor$. In each of these, every vertex has the same eccentricity, and diam G = rad G.

For $n \geq 3$, the n-vertex tree of least diameter is the star, with diameter 2 and radius 1. The one of largest diameter is the path, with diameter $n - 1$ and radius $\lceil (n-1)/2 \rceil$. Every path in a tree is the shortest (the only!) path between its endpoints, so the diameter of a tree is the length of its longest path.

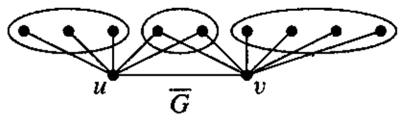
In the graph below, each vertex is labeled with its eccentricity. The radius is 2, the diameter is 4, and the length of the longest path is 7. ■



To have large diameter, many edges must be missing. Thus we expect the complement of a graph with large diameter to have small diameter. We use the simple observation that a graph has diameter at most 2 if and only if nonadjacent vertices always have common neighbors (see also Exercise 15).

2.1.11. Theorem. If G is a simple graph, then diam $G \geq 3 \Rightarrow$ diam $\overline{G} \leq 3$.

Proof: When diam $G > 2$, there exist nonadjacent vertices $u, v \in V(G)$ with no common neighbor. Hence every $x \in V(G) - \{u, v\}$ has at least one of $\{u, v\}$ as a nonneighbor. This makes x adjacent in $\overline{G}$ to at least one of $\{u, v\}$ in $\overline{G}$. Since also $uv \in E(\overline{G})$, for every pair x, y there is an x, y-path of length at most 3 in $\overline{G}$ through $\{u, v\}$. Hence diam $\overline{G} \leq 3$. ■



2.1.12. Definition. The **center** of a graph G is the subgraph induced by the vertices of minimum eccentricity.

The center of a graph is the full graph if and only if the radius and diameter are equal. We next describe the centers of trees. In the induction step, we delete *all* leaves instead of just one.

2.1.13. Theorem. (Jordan [1869]) The center of a tree is a vertex or an edge.

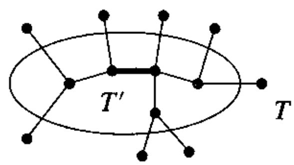
Proof: We use induction on the number of vertices in a tree T.

Basis step: $n(T) \le 2$. With at most two vertices, the center is the entire tree.

Induction step: $n(T) > 2$. Form T' by deleting every leaf of T. By Lemma 2.1.3, T' is a tree. Since the internal vertices on paths between leaves of T remain, T' has at least one vertex.

Every vertex at maximum distance in T from a vertex $u \in V(T)$ is a leaf (otherwise, the path reaching it from u can be extended farther). Since all the leaves have been removed and no path between two other vertices uses a leaf, $\epsilon_{T'}(u) = \epsilon_T(u) - 1$ for every $u \in V(T')$. Also, the eccentricity of a leaf in T is greater than the eccentricity of its neighbor in T. Hence the vertices minimizing $\epsilon_T(u)$ are the same as the vertices minimizing $\epsilon_{T'}(u)$.

We have shown that T and T' have the same center. By the induction hypothesis, the center of T' is a vertex or an edge. ■



In a communication network, large diameter may be acceptable if most pairs can communicate via short paths. This leads us to study the average distance instead of the maximum. Since the average is the sum divided by $\binom{n}{2}$ (the number of vertex pairs), it is equivalent to study $D(G) = \sum_{u,v \in V(G)} d_G(u, v)$.

The sum $D(G)$ has been called the **Wiener index** of G (also written $W(G)$). Wiener used it to study the boiling point of paraffin. Molecules can be modeled by graphs with vertices for atoms and edges for atomic bonds. Many chemical properties of molecules are related to the Wiener index of the corresponding graphs. We study the extreme values of $D(G)$.

2.1.14. Theorem. Among trees with n vertices, the Wiener index $D(T) = \sum_{u,v} d(u, v)$ is minimized by stars and maximized by paths, both uniquely.

Proof: Since a tree has $n - 1$ edges, it has $n - 1$ pairs of vertices at distance 1, and all other pairs have distance at least 2. The star achieves this and hence minimizes $D(T)$. To show that no other tree achieves this, consider a leaf x in T, and let v be its neighbor. If all other vertices have distance 2 from x, then they must be neighbors of v, and T is the star. The value is $D(K_{1,n-1}) = (n - 1) + 2\binom{n-1}{2} = (n - 1)^2$.

For the maximization, consider first $D(P_n)$. This equals the sum of the distances from an endpoint u to the other vertices, plus $D(P_{n-1})$. We have $\sum_{v\in V(P_n)} d(u,v) = \sum_{i=0}^{n-1} i = \binom{n}{2}$. Thus $D(P_n) = D(P_{n-1}) + \binom{n}{2}$. With Pascal's Formula $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$ (see Appendix A), induction yields $D(P_n) = \binom{n+1}{3}$.

We prove by induction on n that among n-vertex tree, P_n is the only tree that maximizes $D(T)$.

Basis step: $n = 1$. The only tree with one vertex is P_1.

Induction step: $n > 1$. Let u be a leaf of an n-vertex tree T. Now $D(T) = D(T-u) + \sum_{v\in V(T)} d(u,v)$. By the induction hypothesis, $D(T-u) \le D(P_{n-1})$, with equality if and only if $T - u$ is a path. Thus it suffices to show that $\sum_{v\in V(T)} d(u,v)$ is maximized only when T is a path and u is an endpoint of T.

Consider the list of distances from u. In P_n, this list is $1, 2, \ldots, n-1$, all distinct. A shortest path from u to a vertex farthest from u contains vertices at all distances from u, so in any tree the set of distances from u to other vertices has no gaps. Thus any repetition makes $\sum_{v\in V(T)} d(u,v)$ smaller than when u is a leaf of a path. When T is not a path, such a repetition occurs. ■

Over all connected n-vertex graphs, $D(G)$ is minimized by K_n. The maximization problem reduces to what we have already done with trees.

2.1.15. Lemma. If H is a subgraph of G, then $d_G(u,v) \le d_H(u,v)$.

Proof: Every u,v-path in H appears also in G, so the shortest u,v-path in G is no longer than the shortest u,v-path in H. ■

2.1.16. Corollary. If G is a connected n-vertex graph, then $D(G) \le D(P_n)$.

Proof: Let T be a spanning tree of G. By Lemma 2.1.15, $D(G) \le D(T)$. By Theorem 2.1.14, $D(T) \le D(P_n)$. ■

DISJOINT SPANNING TREES (optional)

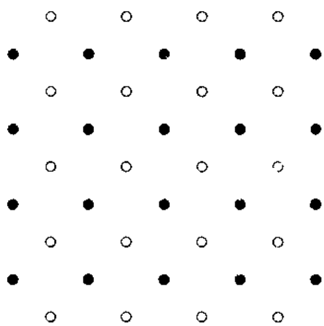
We have seen that every connected graph has a spanning tree. Edge-disjoint spanning trees provide alternate routes when an edge in the primary tree fails. Tutte [1961a] and Nash-Williams [1961] independently characterized graphs having k pairwise edge-disjoint spanning trees (see Exercise 67).

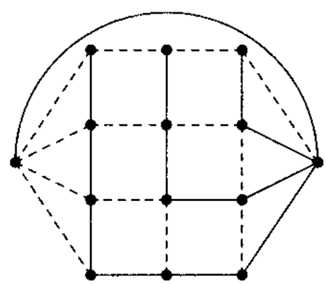
We describe one application of edge-disjoint spanning trees. David Gale devised a game marketed under the name "Bridg-it" (copyright 1960 by Hassenfeld Bros., Inc.—"Hasbro Toys"). Each of two players owns a rectangular grid of posts. The players move alternately, at each move joining two of their own posts by a unit-length bridge. The figure on the left below illustrates the board; Player 1's posts are solid, and Player 2's are hollow. The object of Player 1 is to construct a path of bridges from the left column to the right column; Player 2 wants a path of bridges from the top row to the bottom row.

Bridges cannot cross. Therefore, every bridge that is played eliminates a potential move for the other player. Since every path from left to right crosses every path from top to bottom, the players cannot both win. Note also that the layout of the board is symmetric in the two players.

We argue that Player 2 cannot have a winning strategy. Suppose otherwise. Because the board is symmetric, Player 1 can start with any move and then followthe strategy of Player 2, making an arbitrary move if the strategy of Player 2 ever calls for a bridge that has already been played. Before Player 2 can win, Player 1 wins by using the same strategy.

If the game is played until no further moves are possible, then some player must have won (Exercise 70). Since Player 2 has no winning strategy, this implies that Player 1 has a winning strategy. Here we give an explicit strategythat Player 1 can use to win. (The argument holds more generally in the contextof "matroids"—see Theorem 8.2.46.)





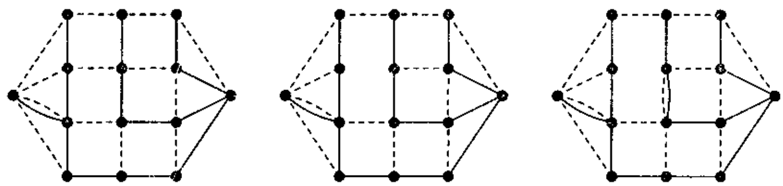
2.1.17. Theorem. Player 1 has a winning strategy in Bridg-it.

Proof: We form a graph of the potential connections for Player 1. Posts on the same end are equivalent, so we collect the (solid) posts from the end columns as single vertices. We add an auxiliary edge between the ends. The figure on the right above illustrates that this graph is the union of two edge-disjoint spanning trees; we omit a technical description of the two trees.

Together, the two trees contain edge-disjoint paths between the goal vertices. Since the auxiliary edge doesn't really exist, we pretend Player 2 moved first and took that edge. A move by Player 2 cuts one edge e in the graph and makes it no longer available. This cuts one of the trees into two components. By Proposition 2.1.6, some edge e' from the other tree reconnects it.

Player 1 chooses such an edge e'. This makes e' uncuttable, in effect putting e' in both spanning trees. After deleting e and making e' a double edge with one copy in each tree, our graph still consists of two edge-disjoint spanning trees. Since Player 2 cannot cut a double edge, Player 2 cannot cut both trees. Thus Player 1 can always defend. The figure below illustrates the strategy.

The process stops when Player 1 has won or when no single edges remain to be cut. In the latter case the remaining edges are double edges and form a spanning tree of bridges built by Player 1. Thus in either case Player 1 has constructed a path connecting the special vertices. ■



Player 1 reconnects Player 2 cuts Player 1 reconnects

EXERCISES

2.1.1. (−) For each k, list the isomorphism classes of trees with maximum degree k and at most six vertices. Do the same for diameter k. (Explain why there are no others.)

2.1.2. (−) Let G be a graph.
a) Prove that G is a tree if and only if G is connected and every edge is a cut-edge.
b) Prove that G is a tree if and only if adding any edge with endpoints in $V(G)$ creates exactly one cycle.

2.1.3. (−) Prove that a graph is a tree if and only if it is loopless and has exactly one spanning tree.

2.1.4. (−) Prove or disprove: Every graph with fewer edges than vertices has a component that is a tree.

2.1.5. (−) Let G be a graph. Prove that a maximal acyclic subgraph of G consists of a spanning tree from each component of G.

2.1.6. (−) Let T be a tree with average degree a. In terms of a, determine $n(T)$.

2.1.7. (−) Prove that every n-vertex graph with m edges has at least $m - n + 1$ cycles.

2.1.8. (−) Prove that each property below characterizes forests.
a) Every induced subgraph has a vertex of degree at most 1.
b) Every connected subgraph is an induced subgraph.
c) The number of components is the number of vertices minus the number of edges.

2.1.9. (−) For $2 \le k \le n-1$, prove that the n-vertex graph formed by adding one vertex adjacent to every vertex of P_{n-1} has a spanning tree with diameter k.

2.1.10. (−) Let u and v be vertices in a connected n-vertex simple graph. Prove that if $d(u,v) > 2$, then $d(u) + d(v) \le n + 1 - d(u,v)$. Construct examples to show that this can fail whenever $n \ge 3$ and $d(u,v) \le 2$.

2.1.11. (−) Let x and y be adjacent vertices in a graph G. For all $z \in V(G)$, prove that $|d_G(x,z) - d_G(y,z)| \le 1$.

2.1.12. (−) Compute the diameter and radius of the biclique $K_{m,n}$.

2.1.13. (−) Prove that every graph with diameter d has an independent set with at least $\lceil (1+d)/2 \rceil$ vertices.

2.1.14. (−) Suppose that the processors in a computer are named by binary k-tuples, and pairs can communicate directly if and only if their names are adjacent in the k-dimensional cube Q_k. A processor with name u wants to send a message to the processor with name v. How can it find the first step on a shortest path to v?

2.1.15. (−) Let G be a simple graph with diameter at least 4. Prove that $\overline{G}$ has diameter at most 2. (Hint: Use Theorem 2.1.11.)

2.1.16. (−) Given a simple graph G, define G' to be the simple graph on the same vertex set such that $xy \in E(G')$ if and only if x and y are adjacent in G or have a common neighbor in G. Prove that $\text{diam}\,(G') = \lceil \text{diam}\,(G)/2 \rceil$.

• • • • •

2.1.17. (!) Prove C $\Rightarrow$ {A, B} in Theorem 2.1.4 by adding edges to connect components.

2.1.18. (!) Prove that every tree with maximum degree $\Delta > 1$ has at least Δ vertices of degree 1. Show that this is best possible by constructing an n-vertex tree with exactly Δ leaves, for each choice of n, Δ with $n > \Delta \geq 2$.

2.1.19. Prove or disprove: If n_i denotes the number of vertices of degree i in a tree T, then $\sum in_i$ depends only on the number of vertices in T.

2.1.20. A *saturated hydrocarbon* is a molecule formed from k carbon atoms and l hydrogen atoms by adding bonds between atoms such that each carbon atom is in four bonds, each hydrogen atom is in one bond, and no sequence of bonds forms a cycle of atoms. Prove that $l = 2k + 2$. (Bondy–Murty [1976, p27])

2.1.21. Let G be an n-vertex simple graph having a decomposition into k spanning trees. Suppose also that $\Delta(G) = \delta(G) + 1$. For $2k \geq n$, show that this is impossible. For $2k < n$, determine the degree sequence of G in terms of n and k.

2.1.22. Let T be an n-vertex tree having one vertex of each degree i with $2 \leq i \leq k$; the remaining $n - k + 1$ vertices are leaves. Determine n in terms of k.

2.1.23. Let T be a tree in which every vertex has degree 1 or degree k. Determine the possible values of $n(T)$.

2.1.24. Prove that every nontrivial tree has at least two maximal independent sets, with equality only for stars. (Note: maximal $\neq$ maximum.)

2.1.25. Prove that among trees with n vertices, the star has the most independent sets.

2.1.26. (!) For $n \geq 3$, let G be an n-vertex graph such that every graph obtained by deleting one vertex is a tree. Determine $e(G)$, and use this to determine G itself.

2.1.27. (!) Let $d_1, \ldots, d_n$ be positive integers, with $n \geq 2$. Prove that there exists a tree with vertex degrees $d_1, \ldots, d_n$ if and only if $\sum d_i = 2n - 2$.

2.1.28. Let $d_1 \geq \cdots \geq d_n$ be nonnegative integers. Prove that there exists a connected graph (loops and multiple edges allowed) with degree sequence $d_1, \ldots, d_n$ if and only if $\sum d_i$ is even, $d_n \geq 1$, and $\sum d_i \geq 2n - 2$. (Hint: Consider a realization with the fewest components.) Is the statement true for simple graphs?

2.1.29. (!) Every tree is bipartite. Prove that every tree has a leaf in its larger partite set (in both if they have equal size).

2.1.30. Let T be a tree in which all vertices adjacent to leaves have degree at least 3. Prove that T has some pair of leaves with a common neighbor.

2.1.31. Prove that a simple connected graph having exactly two vertices that are not cut-vertices is a path.

2.1.32. Prove that an edge e of a connected graph G is a cut-edge if and only if e belongs to every spanning tree. Prove that e is a loop if and only if e belongs to no spanning tree.

2.1.33. (!) Let G be a connected n-vertex graph. Prove that G has exactly one cycle if and only if G has exactly n edges.

2.1.34. (!) Let T be a tree with k edges, and let G be a simple n-vertex graph with more than $n(k-1) - \binom{k}{2}$ edges. Use Proposition 2.1.8 to prove that $T \subseteq G$ if $n > k$.

2.1.35. (!) Let T be a tree. Prove that the vertices of T all have odd degree if and only if for all $e \in E(T)$, both components of $T - e$ have odd order.

2.1.36. (!) Let T be a tree of even order. Prove that T has exactly one spanning subgraph in which every vertex has odd degree.

2.1.37. (!) Let T, T' be two spanning trees of a connected graph G. For $e \in E(T) - E(T')$, prove that there is an edge $e' \in E(T') - E(T)$ such that $T' + e - e'$ and $T - e + e'$ are both spanning trees of G.

2.1.38. Let T, T' be two trees on the same vertex set such that $d_T(v) = d'_T(v)$ for each vertex v. Prove that T' can be obtained from T' using 2-switches (Definition 1.3.32) so that every graph along the way is also a tree. (Kelmans [1998])

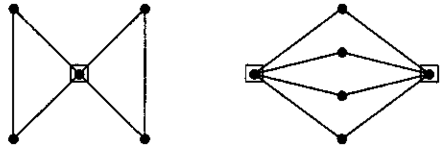
2.1.39. (!) Let G be a tree with $2k$ vertices of odd degree. Prove that G decomposes into k paths. (Hint: Prove the stronger result that the claim holds for all forests.)

2.1.40. (!) Let G be a tree with k leaves. Prove that G is the union of paths $P_1, \ldots, P_{\lceil k/2 \rceil}$ such that $P_i \cap P_j \neq \emptyset$ for all $i \neq j$. (Ando–Kaneko–Gervacio [1996])

2.1.41. For $n \geq 4$, let G be a simple n-vertex graph with $e(G) \geq 2n - 3$. Prove that G has two cycles of equal length. (Chen–Jacobson–Lehel–Shreve [1999] strengthens this.)

2.1.42. Let G be a connected Eulerian graph with at least three vertices. A vertex v in G is *extendible* if every trail beginning at v can be extended to form an Eulerian circuit of G. For example, in the graphs below only the marked vertices are extendible. Prove the following statements about G (adapted from Chartrand–Lesniak [1986, p61]).

a) A vertex $v \in V(G)$ is extendible if and only if $G - v$ is a forest. (Ore [1951])
b) If v is extendible, then $d(v) = \Delta(G)$. (Bäbler [1953])
c) All vertices of G are extendible if and only if G is a cycle.
d) If G is not a cycle, then G has at most two extendible vertices.

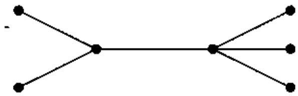


2.1.43. Let u be a vertex in a connected graph G. Prove that it is possible to select shortest paths from u to all other vertices of G so that the union of the paths is a tree.

2.1.44. (!) Prove or disprove: If a simple graph with diameter 2 has a cut-vertex, then its complement has an isolated vertex.

2.1.45. Let G be a graph having spanning trees with diameter 2 and diameter l. For $2 < k < l$, prove that G also has a spanning tree with diameter k. (Galvin)

2.1.46. (!) Prove that the trees with diameter 3 are the **double-stars** (two central vertices plus leaves). Count the isomorphism classes of double-stars with n vertices.



2.1.47. (!) *Diameter and radius.*

a) Prove that the distance function $d(u, v)$ on pairs of vertices of a graph satisfies the triangle inequality: $d(u, v) + d(v, w) \geq d(u, w)$.

b) Use part (a) to prove that diam $G \leq 2$rad G for every graph G.

c) For all positive integers r and d that satisfy $r \leq d \leq 2r$, construct a simple graph with radius r and diameter d. (Hint: Build a suitable graph with one cycle.)

2.1.48. (!) For $n \geq 4$, prove that the minimum number of edges in an n-vertex graph with diameter 2 and maximum degree $n - 2$ is $2n - 4$.

2.1.49. Let G be a simple graph. Prove that rad $G \geq 3 \Rightarrow$ rad $\overline{G} \leq 2$.

2.1.50. *Radius and eccentricity.*

a) Prove that the eccentricities of adjacent vertices differ by at most 1.

b) In terms of the radius r, determine the maximum possible distance from a vertex of eccentricity $r + 1$ to the center of G. (Hint: Use a graph with one cycle.)

2.1.51. Let x and y be distinct neighbors of a vertex v in a graph G.

a) Prove that if G is a tree, then $2\epsilon(v) \leq \epsilon(x) + \epsilon(y)$.

b) Determine the smallest graph where this inequality can fail.

2.1.52. Let x be a vertex in a graph G, and suppose that $\epsilon(x) >$ rad G.

a) Prove that if G is a tree, then x has a neighbor with eccentricity $k - 1$.

b) Show that part (a) does not hold for all graphs by constructing, for each even r that is at least 4, a graph with radius r in which x has eccentricity $r + 2$ and has no neighbor with eccentricity $r + 1$. (Hint: Use a graph with one cycle.)

2.1.53. Prove that the center of a graph can be disconnected and can have components arbitrarily far apart by constructing a graph where the center consists of two vertices and the distance between these two vertices is k.

2.1.54. *Centers of trees.* Let T be a tree.

a) Give a noninductive proof that the center of T is a vertex or an edge.

b) Prove that the center of T is one vertex if and only if diam $T = 2$rad T.

c) Use part (a) to prove that if $n(T)$ is odd, then every automorphism of T maps some vertex to itself.

2.1.55. Given $x \in V(G)$, let $s(x) = \sum_{v \in V(G)} d(x, v)$. The **barycenter** of G is the subgraph induced by the set of vertices minimizing $s(x)$ (the set is also called the **median**).

a) Prove that the barycenter of a tree is a single vertex or an edge. (Hint: Study $s(u) - s(v)$ when u and v are adjacent.) (Jordan [1869])

b) Determine the maximum distance between the center and the barycenter in a tree of diameter d. (Example: in the tree below, the center is the edge xy, the barycenter contains only z, and the distance between them is 1.)