6

Polynomial Equations

6.1 Algebra

The word "algebra" comes from the Arabic word *al-jabr* meaning "restoring." It passed into mathematics through the book *Al-jabr w'al mûqabala* (Science of restoring and opposition) of al-Khwārizmī in 830 CE, a work on the solution of equations. In this context, "restoring" meant adding equal terms to both sides and "opposition" meant setting the two sides equal. For many centuries, al-jabr more commonly meant the resetting of broken bones, and the surgical meaning accompanied the mathematical one when "al-jabr" became "algebra" in Spanish, Italian, and English. Even today the surgical meaning is included in the *Oxford English Dictionary*. Al-Khwārizmī's own name has given us the word "algorithm," so his work has had a lasting impact on mathematics, even though its content was quite elementary.

His algebra went no further than the solution of quadratic equations, which had already been understood by the Babylonians, presented from the geometric viewpoint by Euclid, and reduced to a formula by Brahmagupta (628) (see Section 6.3). Brahmagupta's work, the high point of Indian mathematics to that time, was more advanced than al-Khwārizmī's in several respects—notation, admission of negative numbers, and the treatment of Diophantine equations—even though it predated al-Khwārizmī and was very likely known to him. Indian mathematics had spread to the Arab world with the general promotion of culture by the eighth-century caliphs of Baghdad, and Arab mathematicians acknowledged the Indian origin of cer-

J. Stillwell, *Mathematics and Its History*

tain ideas, for instance, decimal numerals. Why then did al-Khwārizmī's work rather than Brahmagupta's become the definitive "algebra"?

Perhaps this is a case (like "Pell's equation," to mention another pertinent example) where a mathematical term caught on for accidental reasons. However, it may be that the time was ripe for the idea of algebra to be cultivated, and the simple algebra of al-Khwārizmī served this purpose better than those of his more sophisticated predecessors. In Indian mathematics, algebra was inseparable from number theory and elementary arithmetic. In Greek mathematics, algebra was hidden by geometry. Other possible sources of algebra, Babylonia and China, were lost or cut off from the West until it was too late for them to be influential. Arabic mathematics developed at the right time and place to absorb both the geometry of the West and the algebra of the East and to recognize algebra as a separate field with its own methods. The concept of algebra that emerged—the theory of polynomial equations—proved its worth by holding firm for 1000 years. Only in the nineteenth century did algebra grow beyond the bounds of the theory of equations, and this was a time when most fields of mathematics were outgrowing their established habitats.

The early algebraic methods seemed only superficially different from geometric methods, as we shall see in the case of quadratic equations in Section 6.3. Algebraic methods for solving equations became distinct from, and superior to, the geometric only with the advent of new manipulative techniques and efficient notation in the sixteenth century (Section 6.5). Algebra did not break away from geometry, however, but actually gave geometry a new lease on life, thanks to the development of analytic geometry by Fermat and Descartes around 1630. This recombination of algebra and geometry at a higher level is discussed in Chapter 7. It led to the modern field of algebraic geometry.

The story of algebraic geometry unfolds along with the story of polynomial equations, becoming entwined with many other mathematical threads in the process. We shall study several of the decisive early events in this story. One we have already seen is Diophantus' chord and tangent methods for finding rational solutions of equations (Section 3.5). Another relevant event, though not in fact historically connected with Western mathematics, was the method of elimination developed by Chinese mathematicians between the early Christian era and the Middle Ages. Since this method predates any comparable method in the West, and concerns equations of the lowest degree, it is logical to discuss it first.

6.2 Linear Equations and Elimination

The Chinese discovered a method for solving linear equations in any number of unknowns during the Han dynasty (206 BCE–220 CE). It appears in the famous book *Jiuzhang suanshu* [Nine Chapters of Mathematical Art, see Shen *et al.* (1999)], which was written during this period, and survives today in a third-century version with a commentary by Liu Hui. The method was essentially what we call "Gaussian elimination," systematically eliminating terms in a system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

by subtracting a suitable multiple of each equation from the one below it until a triangular system is obtained:

$$\begin{aligned} a'_{11}x_1 + a'_{12}x_2 + \cdots + a'_{1n}x_n &= b'_1 \\ a'_{22}x_2 + \cdots + a'_{2n}x_n &= b'_2 \\ \ddots \qquad &\vdots \\ a'_{nn}x_n &= b'_n \end{aligned}$$

then solving for $x_n, x_{n-1}, \ldots, x_1$ in turn by successive substitutions. This type of calculation was particularly suited to a Chinese device called the counting board, which held the array of coefficients and facilitated manipulations similar to those we perform with matrices. For further details, see Li and Du (1987).

Around the twelfth century Chinese mathematicians discovered that elimination could be adapted to simultaneous polynomial equations in two or more variables. For example, one can eliminate y between a pair of equations

$$a_0(x)y^m + a_1(x)y^{m-1} + \cdots + a_m(x) = 0, \tag{1}$$

$$b_0(x)y^m + b_1(x)y^{m-1} + \cdots + b_m(x) = 0, \tag{2}$$

where the $a_i(x), b_j(x)$ are polynomials in x. The y^m term can be eliminated by forming the equation $b_0(x) \times (1) - a_0(x) \times (2)$, say,

$$c_0(x)y^{m-1} + c_1(x)y^{m-2} + \cdots + c_{m-1} = 0. \tag{3}$$

We can form a second equation of degree $m-1$ in y by multiplying (3) by y, then again eliminating y^m between $(3)\times y$ and (1), giving, say,

$$d_0(x)y^{m-1}+d_1(x)y^{m-2}+\cdots+d_{m-1}=0. \tag{4}$$

The problem is now reduced to eliminating y between the equations (3) and (4), which are of lower degree in y than (1) and (2). Thus one can continue inductively until an equation in x alone is obtained. This method was extended to four variables in the work of Zhū Shijié (1303) entitled *Siyuan yujian* (Jade Mirror of Four Unknowns).

As we shall see in Chapter 7, the two-variable polynomial problem arose in the West in the seventeenth century, in the context of finding intersections of curves. This led first to a rediscovery of the method of elimination for polynomials; only later was this method based on an understanding of linear equations. The well-known Cramer's rule for the solution of linear equations was named after its appearance in a book on algebraic curves [Cramer (1750)].

Exercises

The first interesting case of elimination between two-variable polynomials occurs when the polynomials have degree 2. Geometrically, this amounts to finding the intersections of two conic sections.

6.2.1 Derive an equation that is linear in y from the two equations

$$x^2+xy+y^2=1,$$
$$4x^2+3xy+2y^2=3,$$

and hence show that $y=(1-2x^2)/x$.

6.2.2 Deduce that the intersections of the two curves in Exercise 6.2.1 occur where x satisfies $3x^4-4x^2+1=0$.

This example, where the two equations of degree 2 yield a single equation of degree 4 $(=2\times 2)$, illustrates a general phenomenon where degrees are multiplied. We shall observe other instances, and study it more deeply, as the book progresses.

The present example is not a typical equation of degree 4, since it is quadratic in $x^2=z$. However, this makes it a lot easier to solve.

6.2.3 Solve $3z^2-4z+1=0$ for $z=x^2$ by factorizing the left-hand side, and hence find four solutions for x.

Give geometric reasons why you would expect two curves of degree 2 to have up to four intersections. Could they have more than four?