Evidently $A - 2I$ also has rank 2, so that the space of characteristic vectors associated with the characteristic value 2 has dimension 1. Evidently $T\alpha = 2\alpha$ if and only if α is a scalar multiple of $\alpha_2 = (1, 1, 2)$.

Definition. *Let* T *be a linear operator on the finite-dimensional space* V. *We say that* T *is* **diagonalizable** *if there is a basis for* V *each vector of which is a characteristic vector of* T.

The reason for the name should be apparent; for, if there is an ordered basis $\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}$ for V in which each α_i is a characteristic vector of T, then the matrix of T in the ordered basis $\mathcal{B}$ is diagonal. If $T\alpha_i = c_i\alpha_i$, then

$$[T]_{\mathcal{B}} = \begin{bmatrix} c_1 & 0 & \cdots & 0 \\ 0 & c_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & c_n \end{bmatrix}.$$

We certainly do not require that the scalars $c_1, \ldots, c_n$ be distinct; indeed, they may all be the same scalar (when T is a scalar multiple of the identity operator).

One could also define T to be diagonalizable when the characteristic vectors of T span V. This is only superficially different from our definition, since we can select a basis out of any spanning set of vectors.

For Examples 1 and 2 we purposely chose linear operators T on R^n which are not diagonalizable. In Example 1, we have a linear operator on R^2 which is not diagonalizable, because it has no characteristic values. In Example 2, the operator T has characteristic values; in fact, the characteristic polynomial for T factors completely over the real number field: $f = (x - 1)(x - 2)^2$. Nevertheless T fails to be diagonalizable. There is only a one-dimensional space of characteristic vectors associated with each of the two characteristic values of T. Hence, we cannot possibly form a basis for R^3 which consists of characteristic vectors of T.

Suppose that T is a diagonalizable linear operator. Let $c_1, \ldots, c_k$ be the *distinct* characteristic values of T. Then there is an ordered basis $\mathcal{B}$ in which T is represented by a diagonal matrix which has for its diagonal entries the scalars c_i, each repeated a certain number of times. If c_i is repeated d_i times, then (we may arrange that) the matrix has the block form

$$(6\text{-}3) \qquad [T]_{\mathcal{B}} = \begin{bmatrix} c_1I_1 & 0 & \cdots & 0 \\ 0 & c_2I_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & c_kI_k \end{bmatrix}$$

where I_j is the $d_j \times d_j$ identity matrix. From that matrix we see two things. First, the characteristic polynomial for T is the product of (possibly repeated) linear factors:

$$f = (x - c_1)^{d_1} \cdots (x - c_k)^{d_k}.$$

If the scalar field F is algebraically closed, e.g., the field of complex numbers, every polynomial over F can be so factored (see Section 4.5); however, if F is not algebraically closed, we are citing a special property of T when we say that its characteristic polynomial has such a factorization. The second thing we see from (6-3) is that d_i, the number of times which c_i is repeated as root of f, is equal to the dimension of the space of characteristic vectors associated with the characteristic value c_i. That is because the nullity of a diagonal matrix is equal to the number of zeros which it has on its main diagonal, and the matrix $[T - c_iI]_{\mathcal{B}}$ has d_i zeros on its main diagonal. This relation between the dimension of the characteristic space and the multiplicity of the characteristic value as a root of f does not seem exciting at first; however, it will provide us with a simpler way of determining whether a given operator is diagonalizable.

Lemma. *Suppose that* $T\alpha = c\alpha$. *If* f *is any polynomial, then* $f(T)\alpha = f(c)\alpha$.

Proof. Exercise.

Lemma. *Let* T *be a linear operator on the finite-dimensional space* V. *Let* $c_1, \ldots, c_k$ *be the distinct characteristic values of* T *and let* W_i *be the space of characteristic vectors associated with the characteristic value* c_i. *If* $W = W_1 + \cdots + W_k$, *then*

$$\dim W = \dim W_1 + \cdots + \dim W_k.$$

In fact, if $\mathcal{B}_i$ *is an ordered basis for* W_i, *then* $\mathcal{B} = (\mathcal{B}_1, \ldots, \mathcal{B}_k)$ *is an ordered basis for* W.

Proof. The space $W = W_1 + \cdots + W_k$ is the subspace spanned by all of the characteristic vectors of T. Usually when one forms the sum W of subspaces W_i, one expects that $\dim W < \dim W_1 + \cdots + \dim W_k$ because of linear relations which may exist between vectors in the various spaces. This lemma states that the characteristic spaces associated with different characteristic values are independent of one another.

Suppose that (for each i) we have a vector β_i in W_i, and assume that $\beta_1 + \cdots + \beta_k = 0$. We shall show that $\beta_i = 0$ for each i. Let f be any polynomial. Since $T\beta_i = c_i\beta_i$, the preceding lemma tells us that

$$\begin{aligned} 0 = f(T)0 &= f(T)\beta_1 + \cdots + f(T)\beta_k \\ &= f(c_1)\beta_1 + \cdots + f(c_k)\beta_k. \end{aligned}$$

Choose polynomials $f_1, \ldots, f_k$ such that

$$f_i(c_j) = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

Then

$$0 = f_i(T)0 = \sum_j \delta_{ij}\beta_j$$
$$= \beta_i.$$

Now, let $\mathcal{B}_i$ be an ordered basis for W_i, and let $\mathcal{B}$ be the sequence $\mathcal{B} = (\mathcal{B}_1, \ldots, \mathcal{B}_k)$. Then $\mathcal{B}$ spans the subspace $W = W_1 + \cdots + W_k$. Also, $\mathcal{B}$ is a linearly independent sequence of vectors, for the following reason. Any linear relation between the vectors in $\mathcal{B}$ will have the form $\beta_1 + \cdots + \beta_k = 0$, where β_i is some linear combination of the vectors in $\mathcal{B}_i$. From what we just did, we know that $\beta_i = 0$ for each i. Since each $\mathcal{B}_i$ is linearly independent, we see that we have only the trivial linear relation between the vectors in $\mathcal{B}$. ▌

Theorem 2. *Let* T *be a linear operator on a finite-dimensional space* V. *Let* $c_1, \ldots, c_k$ *be the distinct characteristic values of* T *and let* W_i *be the null space of* $(T - c_iI)$. *The following are equivalent.*

(i) T *is diagonalizable.*
(ii) *The characteristic polynomial for* T *is*

$$f = (x - c_1)^{d_1} \cdots (x - c_k)^{d_k}$$

and dim $W_i = d_i$, $i = 1, \ldots, k$.
(iii) *dim* $W_1 + \cdots +$ *dim* $W_k =$ *dim* V.

Proof. We have observed that (i) implies (ii). If the characteristic polynomial f is the product of linear factors, as in (ii), then $d_1 + \cdots + d_k = \dim V$. For, the sum of the d_i's is the degree of the characteristic polynomial, and that degree is dim V. Therefore (ii) implies (iii). Suppose (iii) holds. By the lemma, we must have $V = W_1 + \cdots + W_k$, i.e., the characteristic vectors of T span V. ▌

The matrix analogue of Theorem 2 may be formulated as follows. Let A be an $n \times n$ matrix with entries in a field F, and let $c_1, \ldots, c_k$ be the distinct characteristic values of A in F. For each i, let W_i be the space of column matrices X (with entries in F) such that

$$(A - c_iI)X = 0,$$

and let $\mathcal{B}_i$ be an ordered basis for W_i. The bases $\mathcal{B}_1, \ldots, \mathcal{B}_k$ collectively string together to form the sequence of columns of a matrix P:

$$P = [P_1, P_2, \ldots] = (\mathcal{B}_1, \ldots, \mathcal{B}_k).$$

The matrix A is similar over F to a diagonal matrix if and only if P is a square matrix. When P is square, P is invertible and $P^{-1}AP$ is diagonal.

Example 3. Let T be the linear operator on R^3 which is represented in the standard ordered basis by the matrix

$$A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}.$$

Let us indicate how one might compute the characteristic polynomial, using various row and column operations:

$$\begin{aligned}
\begin{vmatrix} x-5 & 6 & 6 \\ 1 & x-4 & -2 \\ -3 & 6 & x+4 \end{vmatrix} &= \begin{vmatrix} x-5 & 0 & 6 \\ 1 & x-2 & -2 \\ -3 & 2-x & x+4 \end{vmatrix} \\
&= (x-2)\begin{vmatrix} x-5 & 0 & 6 \\ 1 & 1 & -2 \\ -3 & -1 & x+4 \end{vmatrix} \\
&= (x-2)\begin{vmatrix} x-5 & 0 & 6 \\ 1 & 1 & -2 \\ -2 & 0 & x+2 \end{vmatrix} \\
&= (x-2)\begin{vmatrix} x-5 & 6 \\ -2 & x+2 \end{vmatrix} \\
&= (x-2)(x^2-3x+2) \\
&= (x-2)^2(x-1).
\end{aligned}$$

What are the dimensions of the spaces of characteristic vectors associated with the two characteristic values? We have

$$A - I = \begin{bmatrix} 4 & -6 & -6 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{bmatrix}$$

$$A - 2I = \begin{bmatrix} 3 & -6 & -6 \\ -1 & 2 & 2 \\ 3 & -6 & -6 \end{bmatrix}.$$

We know that $A - I$ is singular and obviously rank $(A - I) \geq 2$. Therefore, rank $(A - I) = 2$. It is evident that rank $(A - 2I) = 1$.

Let W_1, W_2 be the spaces of characteristic vectors associated with the characteristic values 1, 2. We know that dim $W_1 = 1$ and dim $W_2 = 2$. By Theorem 2, T is diagonalizable. It is easy to exhibit a basis for R^3 in which T is represented by a diagonal matrix. The null space of $(T - I)$ is spanned by the vector $\alpha_1 = (3, -1, 3)$ and so $\{\alpha_1\}$ is a basis for W_1. The null space of $T - 2I$ (i.e., the space W_2) consists of the vectors (x_1, x_2, x_3) with $x_1 = 2x_2 + 2x_3$. Thus, one example of a basis for W_2 is

$$\begin{aligned} \alpha_2 &= (2, 1, 0) \\ \alpha_3 &= (2, 0, 1). \end{aligned}$$

If $\mathcal{B} = \{\alpha_1, \alpha_2, \alpha_3\}$, then $[T]_{\mathcal{B}}$ is the diagonal matrix

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

The fact that T is diagonalizable means that the original matrix A is similar (over R) to the diagonal matrix D. The matrix P which enables us to change coordinates from the basis $\mathcal{B}$ to the standard basis is (of course) the matrix which has the transposes of $\alpha_1, \alpha_2, \alpha_3$ as its column vectors:

$$P = \begin{bmatrix} 3 & 2 & 2 \\ -1 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}.$$

Furthermore, $AP = PD$, so that

$$P^{-1}AP = D.$$

Exercises

1. In each of the following cases, let T be the linear operator on R^2 which is represented by the matrix A in the standard ordered basis for R^2, and let U be the linear operator on C^2 represented by A in the standard ordered basis. Find the characteristic polynomial for T and that for U, find the characteristic values of each operator, and for each such characteristic value c find a basis for the corresponding space of characteristic vectors.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

2. Let V be an n-dimensional vector space over F. What is the characteristic polynomial of the identity operator on V? What is the characteristic polynomial for the zero operator?

3. Let A be an $n \times n$ triangular matrix over the field F. Prove that the characteristic values of A are the diagonal entries of A, i.e., the scalars A_{ii}.

4. Let T be the linear operator on R^3 which is represented in the standard ordered basis by the matrix

$$\begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}.$$

Prove that T is diagonalizable by exhibiting a basis for R^3, each vector of which is a characteristic vector of T.

5. Let

$$A = \begin{bmatrix} 6 & -3 & -2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{bmatrix}.$$

Is A similar over the field R to a diagonal matrix? Is A similar over the field C to a diagonal matrix?