the diagram

$$\begin{array}{ccc} \mathcal{O}_X(U) & \xrightarrow{\varphi^\#} & \mathcal{O}_Y(U') \\ \downarrow & & \downarrow \\ \mathcal{O}_X(V) & \xrightarrow{\varphi^\#} & \mathcal{O}_Y(V') \end{array}$$

is commutative, where the vertical maps are the restriction homomorphisms.

21. Suppose D is a multiplicatively closed subset of R. Show that the localization homomorphism $R \to D^{-1}R$ induces a homeomorphism from $\operatorname{Spec}(D^{-1}R)$ to the collection of prime ideals P of R with $P \cap D = \emptyset$.

22. Show that $\operatorname{Spec} k[x, y]/(xy)$ is connected but is the union of two proper closed subsets each homeomorphic to $\operatorname{Spec} k[x]$, hence is not irreducible (cf. Exercise 16).

23. For each of the following rings R exhibit the elements of $\operatorname{Spec} R$, the open sets U in $\operatorname{Spec} R$, the sections $\mathcal{O}(U)$ of the structure sheaf for $\operatorname{Spec} R$ for each open U, and the stalks $\mathcal{O}_P$ at each point $P \in \operatorname{Spec} R$:
(a) $\mathbb{Z}/4\mathbb{Z}$ **(b)** $\mathbb{Z}/6\mathbb{Z}$ **(c)** $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ **(d)** $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

24. **(a)** If every ideal of R is principal, show every open set in $\operatorname{Spec} R$ is a principal open set.
(b) Show that if $R = \mathbb{Z}[x]/(4, x^2)$ then R contains a nonprincipal ideal, but every open set in $\operatorname{Spec} R$ is a principal open set.

25. **(a)** If M is an R-module prove that $\operatorname{Supp}(M)$ is a Zariski closed subset of $\operatorname{Spec} R$. [Use Exercise 33 of Section 4.]
(b) If M is a finitely generated R-module prove that $\operatorname{Supp}(M) = \mathcal{Z}(\operatorname{Ann}(M)) \subseteq \operatorname{Spec} R$. [Use Exercise 34 of Section 4.]

26. Suppose M is a finitely generated module over the Noetherian ring R.
(a) Prove that there are finitely many minimal primes $*P_1, \dots, P_n$ containing $\operatorname{Ann}(M)$. [Use Corollary 22.]
(b) Prove that $\{P_1, \dots, P_n\}$ is also the set of minimal primes in $\operatorname{Ass}_R(M)$ and that $\operatorname{Supp}(M)$ is the union of the Zariski closed sets $\mathcal{Z}(P_1), \dots, \mathcal{Z}(P_n)$ in $\operatorname{Spec} R$. [Use the previous exercise and Exercise 40 in Section 4.]

The previous exercise gives a geometric view of a finitely generated module M over a Noetherian ring R: over each point P in $\operatorname{Spec} R$ is the localization M_P (the *stalk* over P). The stalk is nonzero precisely over the points in the Zariski closed subsets $\mathcal{Z}(P_1), \dots, \mathcal{Z}(P_n)$ where the P_i are the minimal primes in $\operatorname{Ass}_R(M)$. These ideas lead to the notion of the *(coherent) module sheaf on* $\operatorname{Spec} R$ *associated to* M (with a picture similar to that of the structure sheaf following Proposition 58), which is a powerful tool in modern algebraic geometry.

27. Let $R = k[x, y]$ and let M be the ideal (x, y) in R. Prove that $\operatorname{Supp}(M) = \operatorname{Spec} R$ and $\operatorname{Ass}_R(M) = \{0\}$.

The next two exercises show that the associated primes for an ideal I in a Noetherian ring R in the sense of primary decomposition are the associated primes for I in the sense of $\operatorname{Ass}_R(R/I)$.

28. This exercise proves that the ideal Q in a Noetherian ring R is P-primary if and only if $\operatorname{Ass}_R(R/Q) = \{P\}$.
(a) Suppose Q is a P-primary ideal and let M be the R-module R/Q. If $0 \neq m \in M$, show that $Q \subseteq \operatorname{Ann}(m) \subseteq P$ and that $\operatorname{rad}\operatorname{Ann}(m) = P$. Deduce that if $\operatorname{Ann}(m)$ is a prime ideal then it is equal to P and hence that $\operatorname{Ass}_R(R/Q) = \{P\}$. [Use Exercise 33 in Section 1.]

(b) For any ideal Q of R, let $0 \neq M \subseteq R/Q$. Prove that the radical of $\mathrm{Ann}(M)$ is the intersection of the prime ideals in $\mathrm{Supp}(M)$. [Use Proposition 12 and Exercise 25.]

(c) For M as in (b), prove that the radical of $\mathrm{Ann}M$ is also the intersection of the prime ideals in $\mathrm{Ass}_R(M)$. [Use Exercise 26(b).]

(d) If Q is an ideal of R with $\mathrm{Ass}_R(R/Q) = \{P\}$ prove that $\mathrm{rad}\, Q = P$. [Use the fact that $Q = \mathrm{Ann}(R/Q)$ and (c).]

(e) If Q is an ideal of R with $\mathrm{Ass}_R(R/Q) = \{P\}$ prove that Q is P-primary. [If $ab \in Q$ with $a \notin Q$ consider $0 \neq M = (Ra + Q)/Q \subseteq R/Q$ and show that b is contained in $\mathrm{Ann}M \subseteq \mathrm{rad}\,\mathrm{Ann}(M)$. Use Exercises 33–34 in Section 1, to show that $\mathrm{Ass}_R(M) = \{P\}$, then use (c) to show that $\mathrm{rad}\,\mathrm{Ann}(M) = P$, and conclude finally that $b \in P$.]

29. Suppose $I = Q_1 \cap \cdots \cap Q_n$ is a minimal primary decomposition of the ideal I in the Noetherian ring R with $P_i = \mathrm{rad}\, Q_i$, $i = 1, \ldots, n$. This exercise proves that $\mathrm{Ass}_R(R/I) = \{P_1, \ldots, P_n\}$.

(a) Prove that the natural projection homomorphisms induce an injection of R/I into $R/Q_1 \oplus \cdots \oplus R/Q_n$ and deduce that $\mathrm{Ass}_R(R/I) \subseteq \{P_1, \ldots, P_n\}$. [Use Exercise 34 in Section 1 and the previous exercise.]

(b) Let $Q_i' = \cap_{j \neq i} Q_j$. Show that the minimality of the decomposition implies that $0 \neq Q_i'/I = (Q_i' + Q_i)/Q_i \subseteq R/Q_i$. Deduce that $\mathrm{Ass}_R(Q_i'/I) = \{P_i\}$. [Use Exercises 33–34 in Section 1 and the previous exercise.] Deduce that $\{P_i\} \in \mathrm{Ass}_R(R/I)$, so that $\mathrm{Ass}_R(R/I) = \{P_1, \ldots, P_n\}$. [Use $Q_i'/I \subseteq R/I$ and Exercise 34 in Section 1.]

30. Let I be the ideal (x^2, xy, xz, yz) in $R = k[x, y, z]$. Prove that $\mathrm{Ass}_R(R/I)$ consists of the primes $\{(x, y), (x, z), (x, y, z)\}$.

31. (Spec *for Quadratic Integer Rings)* Let R be the ring of integers in the quadratic field $K = \mathbb{Q}(\sqrt{D})$ where D is a squarefree integer and let P be a nonzero prime ideal in R. This exercise shows how the prime ideals in R are determined explicitly from the primes (p) in $\mathbb{Z}$, giving in particular a description of Spec R fibered over Spec $\mathbb{Z}$.

As in the discussion and example following Theorem 29, we have $R = \mathbb{Z}[\omega]$ where $\omega = \sqrt{D}$ if $D \equiv 2, 3 \bmod 4$ (respectively, $\omega = (1+\sqrt{D})/2$ if $D \equiv 1 \bmod 4$), with minimal polynomial $m_\omega(x) = x^2 - D$ (respectively, $m_\omega(x) = x^2 - x + (1-D)/4$), and $P \cap \mathbb{Z} = p\mathbb{Z}$ is a nonzero prime ideal of $\mathbb{Z}$.

(a) For any prime p in $\mathbb{Z}$ show that $R/pR \cong \mathbb{Z}[x]/(p, m_\omega(x)) \cong \mathbb{F}_p[x]/(\overline{m}_\omega(x))$ as rings, where $\overline{m}_\omega(x)$ is the reduction of $m_\omega(x)$ modulo p. Deduce that there is a prime ideal P in R with $P \cap \mathbb{Z} = (p)$ (this gives an alternate proof of Theorem 26(2) in this case).

(b) Use the isomorphism in (a) to prove that P is determined explicitly by the factorization of $m_\omega(x)$ modulo p:

(i) If $\overline{m}_\omega(x) \equiv (x - a)^2 \bmod p$ where $a \in \mathbb{Z}$ then $P = (p, \omega - a)$ and $pR = P^2$. Show that this case occurs only for the finitely many primes p dividing the discriminant of $m_\omega(x)$.

(ii) If $\overline{m}_\omega(x) \equiv (x - a)(x - b) \bmod p$ with integers $a, b \in \mathbb{Z}$ that are distinct modulo p then P is either $P_1 = (p, \omega - a)$ or $P_2 = (p, \omega - b)$ and P_1, P_2 are distinct prime ideals in R with $pR = P_1 P_2$.

(iii) If $\overline{m}_\omega(x)$ is irreducible modulo p then $P = pR$.

(c) Show that the picture for Spec R over Spec $\mathbb{Z}$ for any D is similar to that for the case $R = \mathbb{Z}[i]$ when $D = -1$: there is precisely one nonclosed point $(0) \in$ Spec R over $(0) \in$ Spec $\mathbb{Z}$, precisely one closed point $P \in$ Spec R over each of the primes (p) in Spec $\mathbb{Z}$ in (i) (called *ramified* primes) and over the primes in (iii) (called *inert* primes), and precisely two closed points over the primes in (ii) (called *split* primes).

CHAPTER 16

Artinian Rings, Discrete Valuation Rings, and Dedekind Domains

Throughout this chapter R will denote a commutative ring with $1 \neq 0$.

16.1 ARTINIAN RINGS

In this section we shall study the basic theory of commutative rings that satisfy the descending chain condition (D.C.C.) on ideals, the Artinian rings (named after E. Artin). While one might at first expect that these rings have properties analogous to those for the commutative rings satisfying the ascending chain condition (the Noetherian rings), in fact this is not the case. The structure of Artinian rings is very restricted; for example an Artinian ring is necessarily also Noetherian (Theorem 3). Noncommutative Artinian rings play a central role in Representation Theory (cf. Chapters 18 and 19).

Definition. For any commutative ring R the *Krull dimension* (or simply the *dimension*) of R is the maximum possible length of a chain $P_0 \subset P_1 \subset P_2 \subset \cdots \subset P_n$ of distinct prime ideals in R. The dimension of R is said to be infinite if R has arbitrarily long chains of distinct prime ideals.

A ring with finite dimension must satisfy both the ascending and descending chain conditions on prime ideals (although not necessarily on all ideals). A field has dimension 0 and a Principal Ideal Domain that is not a field has dimension 1.

We shall see shortly that rings with D.C.C. on ideals always have dimension 0 (i.e., primes are maximal). If R is an integral domain that is also a finitely generated k–algebra over a field k, then the dimension of R is equal to the transcendence degree over k of the field of fractions of R (cf. Exercise 11). In particular, the Krull dimension agrees with the definition introduced earlier for the dimension of an affine variety. The advantage of the definition above is that it does not refer to any k–algebra structure and applies to arbitrary commutative rings R.

Definition. The *Jacobson radical* of R is the intersection of all maximal ideals of R and is denoted by Jac R.