From the definition and Lemma 15.6.11, we see that

$$zz^{-1} = z^{-1}z = |z|^{-2}\overline{z}z = |z|^{-2}|z|^2 = 1,$$

and so z^{-1} is indeed the reciprocal of z. We can thus define a notion of quotient z/w for any two complex numbers z, w with $w \neq 0$ in the usual manner by the formula $z/w := zw^{-1}$.

The complex numbers can be given a distance by defining $d(z, w) = |z - w|$.

Lemma 15.6.13. *The complex numbers* $\mathbf{C}$ *with the distance* d *form a metric space. If* $(z_n)_{n=1}^{\infty}$ *is a sequence of complex numbers, and* z *is another complex number, then we have* $\lim_{n\to\infty} z_n = z$ *in this metric space if and only if* $\lim_{n\to\infty} \Re(z_n) = \Re(z)$ *and* $\lim_{n\to\infty} \Im(z_n) = \Im(z)$.

Proof. See Exercise 15.6.9. □

This metric space is in fact complete and connected, but not compact: see Exercises 15.6.10, 15.6.12, 15.6.13. We also have the usual limit laws:

Lemma 15.6.14 (Complex limit laws). *Let* $(z_n)_{n=1}^{\infty}$ *and* $(w_n)_{n=1}^{\infty}$ *be convergent sequences of complex numbers, and let* c *be a complex number. Then the sequences* $(z_n+w_n)_{n=1}^{\infty}$, $(z_n-w_n)_{n=1}^{\infty}$, $(cz_n)_{n=1}^{\infty}$, $(z_n w_n)_{n=1}^{\infty}$, *and* $(\overline{z_n})_{n=1}^{\infty}$ *are also convergent, with*

$$\begin{aligned}
\lim_{n\to\infty} z_n + w_n &= \lim_{n\to\infty} z_n + \lim_{n\to\infty} w_n \\
\lim_{n\to\infty} z_n - w_n &= \lim_{n\to\infty} z_n - \lim_{n\to\infty} w_n \\
\lim_{n\to\infty} cz_n &= c \lim_{n\to\infty} z_n \\
\lim_{n\to\infty} z_n w_n &= (\lim_{n\to\infty} z_n)(\lim_{n\to\infty} w_n) \\
\lim_{n\to\infty} \overline{z_n} &= \overline{\lim_{n\to\infty} z_n}
\end{aligned}$$

Also, if the w_n *are all non-zero and* $\lim_{n\to\infty} w_n$ *is also non-zero, then* $(z_n/w_n)_{n=1}^{\infty}$ *is also a convergent sequence, with*

$$\lim_{n\to\infty} z_n/w_n = (\lim_{n\to\infty} z_n)/(\lim_{n\to\infty} w_n).$$

Proof. See Exercise 15.6.14. □

Observe that the real and complex number systems are in fact quite similar; they both obey similar laws of arithmetic, and they have similar structure as metric spaces. Indeed many of the results in this textbook that were proven for real-valued functions, are also valid for complex-valued functions, simply by replacing "real" with "complex" in the proofs but otherwise leaving all the other details of the proof unchanged. Alternatively, one can always split a complex-valued function f into real and imaginary parts $\Re(f)$, $\Im(f)$, thus $f = \Re(f) + i\Im(f)$, and then deduce results for the complex-valued function f from the corresponding results for the real-valued functions $\Re(f), \Im(f)$. For instance, the theory of pointwise and uniform convergence from Chapter 14, or the theory of power series from this chapter, extends without any difficulty to complex-valued functions. In particular, we can define the complex exponential function in exactly the same manner as for real numbers:

Definition 15.6.15 (Complex exponential). If z is a complex number, we define the function $\exp(z)$ by the formula

$$\exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

One can state and prove the ratio test for complex series and use it to show that $\exp(z)$ converges for every z. It turns out that many of the properties from Theorem 15.5.2 still hold: we have that $\exp(z+w) = \exp(z)\exp(w)$, for instance; see Exercise 15.6.16. (The other properties require complex differentiation and complex integration, but these topics are beyond the scope of this text.) Another useful observation is that $\overline{\exp(z)} = \exp(\overline{z})$; this can be seen by conjugating the partial sums $\sum_{n=0}^{N} \frac{z^n}{n!}$ and taking limits as $N \to \infty$.

The complex logarithm turns out to be somewhat more subtle, mainly because exp is no longer invertible, and also because the various power series for the logarithm only have a finite radius of convergence (unlike exp, which has an infinite radius of convergence). This rather delicate issue is beyond the scope of this text and will not be discussed here.

Exercise 15.6.1. Prove Lemma 15.6.4.

Exercise 15.6.2. Prove Lemma 15.6.6.

Exercise 15.6.3. Prove Lemma 15.6.7.

Exercise 15.6.4. Prove Lemma 15.6.9.

Exercise 15.6.5. If z is a complex number, show that $\Re(z) = \frac{z+\overline{z}}{2}$ and $\Im(z) = \frac{z-\overline{z}}{2i}$.

Exercise 15.6.6. Prove Lemma 15.6.6. (Hint: to prove the triangle inequality, first prove that $\Re(z\overline{w}) \leq |z||w|$, and hence (from Exercise 15.6.5) that $z\overline{w} + \overline{z}w \leq 2|z||w|$. Then add $|z|^2 + |w|^2$ to both sides of this inequality.)

Exercise 15.6.7. Show that if z, w are complex numbers with $w \neq 0$, then $|z/w| = |z|/|w|$.

Exercise 15.6.8. Let z, w be non-zero complex numbers. Show that $|z+w| = |z|+|w|$ if and only if there exists a positive real number $c > 0$ such that $z = cw$.

Exercise 15.6.9. Prove Lemma 15.6.13.

Exercise 15.6.10. Show that the complex numbers $\mathbf{C}$ (with the usual metric d) form a complete metric space.

Exercise 15.6.11. Let $f : \mathbf{R}^2 \to \mathbf{C}$ be the map $f(a,b) := a + bi$. Show that f is a bijection, and that f and f^{-1} are both continuous maps.

Exercise 15.6.12. Show that the complex numbers $\mathbf{C}$ (with the usual metric d) form a connected metric space. (Hint: first show that $\mathbf{C}$ is path connected, as in Exercise 13.4.7.)

Exercise 15.6.13. Let E be a subset of $\mathbf{C}$. Show that E is compact if and only if E is closed and bounded. (Hint: combine Exercise 15.6.11 with the Heine-Borel theorem, Theorem 12.5.7.) In particular, show that $\mathbf{C}$ is not compact.

Exercise 15.6.14. Prove Lemma 15.6.14. (Hint: split z_n and w_n into real and imaginary parts and use the usual limit laws, Lemma 6.1.19, combined with Lemma 15.6.13.)

Exercise 15.6.15. The purpose of this exercise is to explain why we do not try to organize the complex numbers into positive and negative parts. Suppose that there was a notion of a "positive complex number" and a "negative complex number" which obeyed the following reasonable axioms (cf. Proposition 4.2.9):

- (Trichotomy) For every complex number z, exactly one of the following statements is true: z is positive, z is negative, z is zero.
- (Negation) If z is a positive complex number, then $-z$ is negative. If z is a negative complex number, then $-z$ is positive.
- (Additivity) If z and w are positive complex numbers, then $z+w$ is also positive.
- (Multiplicativity) If z and w are positive complex numbers, then zw is also positive.

Show that these four axioms are inconsistent, i.e., one can use these axioms to deduce a contradiction. (Hints: first use the axioms to deduce that 1 is positive, and then conclude that -1 is negative. Then apply the Trichotomy axiom to $z = i$ and obtain a contradiction in any one of the three cases).

Exercise 15.6.16. Prove the ratio test for complex series, and use it to show that the series used to define the complex exponential is absolutely convergent. Then prove that $\exp(z+w) = \exp(z)\exp(w)$ for all complex numbers z, w.

15.7 Trigonometric functions

We now discuss the next most important class of special functions, after the exponential and logarithmic functions, namely the trigonometric functions. (There are several other useful special functions in mathematics, such as the hyperbolic trigonometric functions and hypergeometric functions, the gamma and zeta functions, and elliptic functions, but they occur more rarely and will not be discussed here.)

Trigonometric functions are often defined using geometric concepts, notably those of circles, triangles, and angles. However, it is also possible to define them using more analytic concepts, and in particular the (complex) exponential function.

Definition 15.7.1 (Trigonometric functions). If z is a complex number, then we define

$$\cos(z) := \frac{e^{iz} + e^{-iz}}{2}$$

and

$$\sin(z) := \frac{e^{iz} - e^{-iz}}{2i}.$$

We refer to cos and sin as the *cosine* and *sine* functions respectively.

These formulae were discovered by Leonhard Euler (1707–1783) in 1748, who recognized the link between the complex exponential and the trigonometric functions. Note that since we have defined the sine and cosine for complex numbers z, we automatically have defined them also for real numbers x. In fact in most applications one is only interested in the trigonometric functions when applied to real numbers.

From the power series definition of exp, we have

$$e^{iz} = 1 + iz - \frac{z^2}{2!} - \frac{iz^3}{3!} + \frac{z^4}{4!} + \ldots$$

and

$$e^{-iz} = 1 - iz - \frac{z^2}{2!} + \frac{iz^3}{3!} + \frac{z^4}{4!} - \ldots$$

and so from the above formulae we have

$$\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$$

and

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}.$$

In particular, $\cos(x)$ and $\sin(x)$ are always real whenever x is real. From the ratio test we see that the two power series $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$,

$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ are absolutely convergent for every x, thus $\sin(x)$ and $\cos(x)$ are real analytic at 0 with an infinite radius of convergence. From Exercise 15.2.8 we thus see that the sine and cosine functions are real analytic on all of $\mathbf{R}$. (They are also complex analytic on all of $\mathbf{C}$, but we will not pursue this matter in this text). In particular the sine and cosine functions are continuous and differentiable.

We list some basic properties of the sine and cosine functions below.

Theorem 15.7.2 (Trigonometric identities). *Let x, y be real numbers.*

(a) *We have* $\sin(x)^2 + \cos(x)^2 = 1$. *In particular, we have* $\sin(x) \in [-1, 1]$ *and* $\cos(x) \in [-1, 1]$ *for all* $x \in \mathbf{R}$.

(b) *We have* $\sin'(x) = \cos(x)$ *and* $\cos'(x) = -\sin(x)$.

(c) *We have* $\sin(-x) = -\sin(x)$ *and* $\cos(-x) = \cos(x)$.

(d) *We have* $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ *and* $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$.

(e) *We have* $\sin(0) = 0$ *and* $\cos(0) = 1$.

(f) *We have* $e^{ix} = \cos(x) + i\sin(x)$ *and* $e^{-ix} = \cos(x) - i\sin(x)$. *In particular* $\cos(x) = \Re(e^{ix})$ *and* $\sin(x) = \Im(e^{ix})$.

Proof. See Exercise 15.7.1. □

Now we describe some other properties of sin and cos.

Lemma 15.7.3. *There exists a positive number x such that* $\sin(x)$ *is equal to* 0.

Proof. Suppose for sake of contradiction that $\sin(x) \neq 0$ for all $x \in (0, \infty)$. Observe that this would also imply that $\cos(x) \neq 0$ for all $x \in (0, \infty)$, since if $\cos(x) = 0$ then $\sin(2x) = 0$ by Theorem 15.7.2(d) (why?). Since $\cos(0) = 1$, this implies by the intermediate value theorem (Theorem 9.7.1) that $\cos(x) > 0$ for all

$x > 0$ (why?). Also, since $\sin(0) = 0$ and $\sin'(0) = 1 > 0$, we see that sin increasing near 0, hence is positive to the right of 0. By the intermediate value theorem again we conclude that $\sin(x) > 0$ for all $x > 0$ (otherwise sin would have a zero on $(0, \infty)$).

In particular if we define the cotangent function $\cot(x) := \cos(x)/\sin(x)$, then $\cot(x)$ would be positive and differentiable on all of $(0, \infty)$. From the quotient rule (Theorem 10.1.13(h)) and Theorem 15.7.2 we see that the derivative of $\cot(x)$ is $-1/\sin(x)^2$ (why?) In particular, we have $\cot'(x) \leq -1$ for all $x > 0$. By the fundamental theorem of calculus (Theorem 11.9.1) this implies that $\cot(x+s) \leq \cot(x)-s$ for all $x > 0$ and $s > 0$. But letting $s \to \infty$ we see that this contradicts our assertion that cot is positive on $(0, \infty)$ (why?). □

Let E be the set $E := \{x \in (0, +\infty) : \sin(x) = 0\}$, i.e., E is the set of roots of sin on $(0, +\infty)$. By Lemma 15.7.3, E is non-empty. Since $\sin'(0) > 0$, there exists a $c > 0$ such that $E \subseteq [c, +\infty)$ (see Exercise 15.7.2). Also, since sin is continuous in $[c, +\infty)$, E is closed in $[c, +\infty)$ (why? use Theorem 13.1.5(d)). Since $[c, +\infty)$ is closed in $\mathbf{R}$, we conclude that E is closed in $\mathbf{R}$. Thus E contains all its adherent points, and thus contains $\inf(E)$. Thus if we make the definition

Definition 15.7.4. We define π to be the number

$$\pi := \inf\{x \in (0, \infty) : \sin(x) = 0\}$$

then we have $\pi \in E \subseteq [c, +\infty)$ (so in particular $\pi > 0$) and $\sin(\pi) = 0$. By definition of π, sin cannot have any zeroes in $(0, \pi)$, and so in particular must be positive on $(0, \pi)$, (cf. the arguments in Lemma 15.7.3 using the intermediate value theorem). Since $\cos'(x) = -\sin(x)$, we thus conclude that $\cos(x)$ is strictly decreasing on $(0, \pi)$. Since $\cos(0) = 1$, this implies in particular that $\cos(\pi) < 1$; since $\sin^2(\pi) + \cos^2(\pi) = 1$ and $\sin(\pi) = 0$, we thus conclude that $\cos(\pi) = -1$.

In particular we have Euler's famous formula

$$e^{\pi i} = \cos(\pi) + i \sin(\pi) = -1.$$

We now conclude with some other properties of sine and cosine.

Theorem 15.7.5 (Periodicity of trigonometric functions). *Let x be a real number.*

(a) *We have* $\cos(x+\pi) = -\cos(x)$ *and* $\sin(x+\pi) = -\sin(x)$. *In particular we have* $\cos(x+2\pi) = \cos(x)$ *and* $\sin(x+2\pi) = \sin(x)$, *i.e.,* sin *and* cos *are periodic with period* 2π.

(b) *We have* $\sin(x) = 0$ *if and only if* x/π *is an integer.*

(c) *We have* $\cos(x) = 0$ *if and only if* x/π *is an integer plus* $1/2$.

Proof. See Exercise 15.7.3. □

We can of course define all the other trigonometric functions: tangent, cotangent, secant, and cosecant, and develop all the familiar identities of trigonometry; some examples of this are given in the exercises.

Exercise 15.7.1. Prove Theorem 15.7.2. (Hint: write everything in terms of exponentials whenever possible.)

Exercise 15.7.2. Let $f : \mathbf{R} \to \mathbf{R}$ be a function which is differentiable at x_0, with $f(x_0) = 0$ and $f'(x_0) \neq 0$. Show that there exists a $c > 0$ such that $f(y)$ is non-zero whenever $0 < |x - y| < c$. Conclude in particular that there exists a $c > 0$ such that $\sin(x) \neq 0$ for all $0 < x < c$.

Exercise 15.7.3. Prove Theorem 15.7.5. (Hint: for (c), you may wish to first compute $\sin(\pi/2)$ and $\cos(\pi/2)$, and then link $\cos(x)$ to $\sin(x + \pi/2)$.)

Exercise 15.7.4. Let x, y be real numbers such that $x^2 + y^2 = 1$. Show that there is exactly one real number $\theta \in (-\pi, \pi]$ such that $x = \sin(\theta)$ and $y = \cos(\theta)$. (Hint: you may need to divide into cases depending on whether x, y are positive, negative, or zero.)

Exercise 15.7.5. Show that if $r, s > 0$ are positive real numbers, and θ, α are real numbers such that $re^{i\theta} = se^{i\alpha}$, then $r = s$ and $\theta = \alpha + 2\pi k$ for some integer k.

Exercise 15.7.6. Let z be a non-zero complex real number. Using Exercise 15.7.4, show that there is exactly one pair of real numbers r, θ such that $r > 0$, $\theta \in (-\pi, \pi]$, and $z = re^{i\theta}$. (This is sometimes known as the *standard polar representation* of z.)

Exercise 15.7.7. For any real number θ and integer n, prove the *de Moivre identities*

$$\cos(n\theta) = \Re((\cos\theta + i\sin\theta)^n); \quad \sin(n\theta) = \Im((\cos\theta + i\sin\theta)^n).$$

Exercise 15.7.8. Let $\tan : (-\pi/2, \pi/2) \to \mathbf{R}$ be the tangent function $\tan(x) := \sin(x)/\cos(x)$. Show that tan is differentiable and monotone increasing, with $\frac{d}{dx}\tan(x) = 1 + \tan(x)^2$, and that $\lim_{x\to\pi/2}\tan(x) = +\infty$ and $\lim_{x\to-\pi/2}\tan(x) = -\infty$. Conclude that tan is in fact a bijection from $(-\pi/2, \pi/2) \to \mathbf{R}$, and thus has an inverse function $\tan^{-1} : \mathbf{R} \to (-\pi/2, \pi/2)$ (this function is called the *arctangent function*). Show that $\tan^{-1}$ is differentiable and $\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$.

Exercise 15.7.9. Recall the arctangent function $\tan^{-1}$ from Exercise 15.7.8. By modifying the proof of Theorem 15.5.6(e), establish the identity

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

for all $x \in (-1, 1)$. Using Abel's theorem (Theorem 15.3.1) to extend this identity to the case $x = 1$, conclude in particular the identity

$$\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \ldots = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.$$

(Note that the series converges by the alternating series test, Proposition 7.2.12). Conclude in particular that $4 - \frac{4}{3} < \pi < 4$. (One can of course compute $\pi = 3.1415926\ldots$ to much higher accuracy, though if one wishes to do so it is advisable to use a different formula than the one above, which converges very slowly).

Exercise 15.7.10. Let $f : \mathbf{R} \to \mathbf{R}$ be the function

$$f(x) := \sum_{n=1}^{\infty} 4^{-n}\cos(32^n \pi x).$$

(a) Show that this series is uniformly convergent, and that f is continuous.

(b) Show that for every integer j and every integer $m \geq 1$, we have

$$|f(\frac{j+1}{32^m}) - f(\frac{j}{32^m})| \geq 4^{-m}.$$

(Hint: use the identity

$$\sum_{n=1}^{\infty} a_n = (\sum_{n=1}^{m-1} a_n) + a_m + \sum_{n=m+1}^{\infty} a_n$$

for certain sequences a_n. Also, use the fact that the cosine function is periodic with period 2π, as well as the geometric series formula $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ for any $|r| < 1$. Finally, you will need the inequality $|\cos(x) - \cos(y)| \leq |x - y|$ for any real numbers x and y; this can be proven by using the mean value theorem (Corollary 10.2.9), or the fundamental theorem of calculus (Theorem 11.9.4).)

(c) Using (b), show that for every real number x_0, the function f is not differentiable at x_0. (Hint: for every x_0 and every $m \geq 1$, there exists an integer j such that $j \leq 32^m x_0 \leq j + 1$, thanks to Exercise 5.4.3.)

(d) Explain briefly why the result in (c) does not contradict Corollary 14.7.3.

Chapter 16

Fourier series

In the previous two chapters, we discussed the issue of how certain functions (for instance, compactly supported continuous functions) could be approximated by polynomials. Later, we showed how a different class of functions (real analytic functions) could be written exactly (not approximately) as an infinite polynomial, or more precisely a power series.

Power series are already immensely useful, especially when dealing with special functions such as the exponential and trigonometric functions discussed earlier. However, there are some circumstances where power series are not so useful, because one has to deal with functions (e.g., $\sqrt{x}$) which are not real analytic, and so do not have power series.

Fortunately, there is another type of series expansion, known as *Fourier series*, which is also a very powerful tool in analysis (though used for slightly different purposes). Instead of analyzing compactly supported functions, it instead analyzes *periodic functions*; instead of decomposing into polynomials, it decomposes into *trigonometric polynomials*. Roughly speaking, the theory of Fourier series asserts that just about every periodic function can be decomposed as an (infinite) sum of sines and cosines.

Remark 16.0.6. Jean-Baptiste Fourier (1768-1830) was, among other things, the governor of Egypt during the reign of Napoleon. After the Napoleonic wars, he returned to mathematics. He introduced Fourier series in an important 1807 paper in which he used them to solve what is now known as the heat equation. At the time, the claim that every periodic function could be expressed as a sum of sines and cosines was extremely controversial, even such leading mathematicians as Euler declared that it was impossible. Nevertheless, Fourier managed to show that this was indeed the case, although the proof was not completely rigourous and was not totally accepted for almost another hundred years.

There will be some similarities between the theory of Fourier series and that of power series, but there are also some major differences. For instance, the convergence of Fourier series is usually not uniform (i.e., not in the L^∞ metric), but instead we have convergence in a different metric, the L^2-metric. Also, we will need to use complex numbers heavily in our theory, while they played only a tangential rôle in power series.

The theory of Fourier series (and of related topics such as Fourier integrals and the Laplace transform) is vast, and deserves an entire course in itself. It has many, many applications, most directly to differential equations, signal processing, electrical engineering, physics, and analysis, but also to algebra and number theory. We will only give the barest bones of the theory here, however, and almost no applications.

16.1 Periodic functions

The theory of Fourier series has to do with the analysis of *periodic functions*, which we now define. It turns out to be convenient to work with complex-valued functions rather than real-valued ones.

Definition 16.1.1. Let $L > 0$ be a real number. A function $f : \mathbf{R} \to \mathbf{C}$ is *periodic with period L*, or *L-periodic*, if we have $f(x + L) = f(x)$ for every real number x.

Example 16.1.2. The real-valued functions $f(x) = \sin(x)$ and $f(x) = \cos(x)$ are 2π-periodic, as is the complex-valued function $f(x) = e^{ix}$. These functions are also 4π-periodic, 6π-periodic, etc. (why?). The function $f(x) = x$, however, is not periodic. The constant function $f(x) = 1$ is L-periodic for every L.

Remark 16.1.3. If a function f is L-periodic, then we have $f(x + kL) = f(x)$ for every integer k (why? Use induction for the positive k, and then use a substitution to convert the positive k result to a negative k result. The $k = 0$ case is of course trivial). In particular, if a function f is 1-periodic, then we have $f(x+k) = f(x)$ for every $k \in \mathbf{Z}$. Because of this, 1-periodic functions are sometimes also called $\mathbf{Z}$-*periodic* (and L-periodic functions called $L\mathbf{Z}$-periodic).

Example 16.1.4. For any integer n, the functions $\cos(2\pi nx)$, $\sin(2\pi nx)$, and $e^{2\pi inx}$ are all $\mathbf{Z}$-periodic. (What happens when n is not an integer?). Another example of a $\mathbf{Z}$-periodic function is the function $f : \mathbf{R} \to \mathbf{C}$ defined by $f(x) := 1$ when $x \in [n, n + \frac{1}{2})$ for some integer n, and $f(x) := 0$ when $x \in [n+\frac{1}{2}, n+1)$ for some integer n. This function is an example of a *square wave*.

Henceforth, for simplicity, we shall only deal with functions which are $\mathbf{Z}$-periodic (for the Fourier theory of L-periodic functions, see Exercise 16.5.6). Note that in order to completely specify a $\mathbf{Z}$-periodic function $f : \mathbf{R} \to \mathbf{C}$, one only needs to specify its values on the interval $[0, 1)$, since this will determine the values of f everywhere else. This is because every real number x can be written in the form $x = k + y$ where k is an integer (called the *integer part* of x, and sometimes denoted $[x]$) and $y \in [0, 1)$ (this is called the *fractional part* of x, and sometimes denoted $\{x\}$); see Exercise 16.1.1. Because of this, sometimes when we wish to describe a $\mathbf{Z}$-periodic function f we just describe what it does on the interval $[0, 1)$, and then say that it is *extended periodically* to all of $\mathbf{R}$. This means that we define $f(x)$ for any real number x by setting $f(x) := f(y)$, where we have decomposed $x = k + y$ as discussed above. (One can in fact replace the interval $[0, 1)$ by any other half-open interval of length 1, but we will not do so here.)

The space of complex-valued continuous $\mathbf{Z}$-periodic functions is denoted $C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$. (The notation $\mathbf{R}/\mathbf{Z}$ comes from algebra, and denotes the quotient group of the additive group $\mathbf{R}$ by the additive group $\mathbf{Z}$; more information in this can be found in any algebra text.) By "continuous" we mean continuous at all points on $\mathbf{R}$; merely being continuous on an interval such as $[0,1]$ will not suffice, as there may be a discontinuity between the left and right limits at 1 (or at any other integer). Thus for instance, the functions $\sin(2\pi nx)$, $\cos(2\pi nx)$, and $e^{2\pi inx}$ are all elements of $C(\mathbf{R}/\mathbf{Z};\mathbf{C})$, as are the constant functions, however the square wave function described earlier is not in $C(\mathbf{R}/\mathbf{Z};\mathbf{C})$ because it is not continuous. Also the function $\sin(x)$ would also not qualify to be in $C(\mathbf{R}/\mathbf{Z};\mathbf{C})$ since it is not $\mathbf{Z}$-periodic.

Lemma 16.1.5 (Basic properties of $C(\mathbf{R}/\mathbf{Z};\mathbf{C})$).

(a) *(Boundedness) If $f \in C(\mathbf{R}/\mathbf{Z};\mathbf{C})$, then f is bounded (i.e., there exists a real number $M > 0$ such that $|f(x)| \leq M$ for all $x \in \mathbf{R}$).*

(b) *(Vector space and algebra properties) If $f, g \in C(\mathbf{R}/\mathbf{Z};\mathbf{C})$, then the functions $f+g$, $f-g$, and fg are also in $C(\mathbf{R}/\mathbf{Z};\mathbf{C})$. Also, if c is any complex number, then the function cf is also in $C(\mathbf{R}/\mathbf{Z};\mathbf{C})$.*

(c) *(Closure under uniform limits) If $(f_n)_{n=1}^{\infty}$ is a sequence of functions in $C(\mathbf{R}/\mathbf{Z};\mathbf{C})$ which converges uniformly to another function $f : \mathbf{R} \to \mathbf{C}$, then f is also in $C(\mathbf{R}/\mathbf{Z};\mathbf{C})$.*

Proof. See Exercise 16.1.2. □

One can make $C(\mathbf{R}/\mathbf{Z};\mathbf{C})$ into a metric space by re-introducing the now familiar sup-norm metric

$$d_\infty(f,g) = \sup_{x\in\mathbf{R}} |f(x) - g(x)| = \sup_{x\in[0,1)} |f(x) - g(x)|$$

of uniform convergence. (Why is the first supremum the same as the second?) See Exercise 16.1.3.

Exercise 16.1.1. Show that every real number x can be written in exactly one way in the form $x = k+y$, where k is an integer and $y \in [0,1)$. (Hint: to prove existence of such a representation, set $k := \sup\{l \in \mathbf{Z} : l \leq x\}$.)