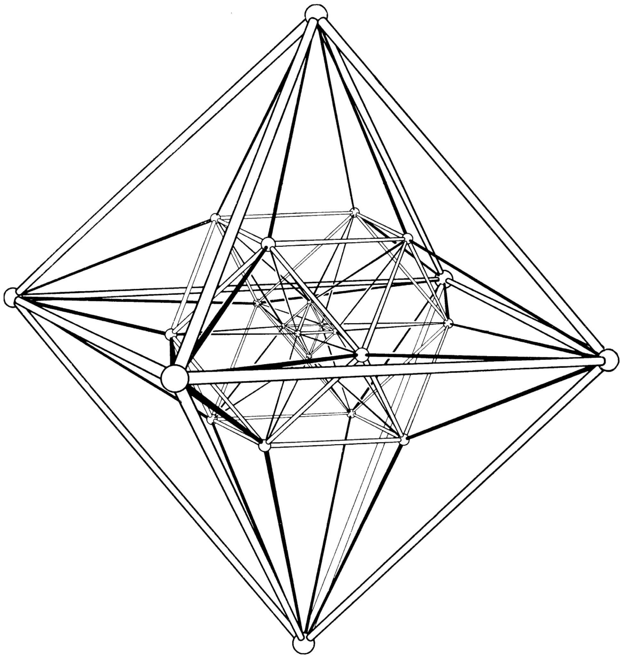

Figure 7.8: The 24-cell

7.8 The groups $\mathbb{S}^3$ and $\mathbb{RP}^3$

The rotations of the tetrahedron, which were discussed in Section 7.7, vividly show that *a group of rotations is itself a geometric object*. This statement is just as true of the group of all rotations of $\mathbb{S}^2$. In fact, this group is closely related to two important geometric objects: the *3-sphere* $\mathbb{S}^3$ and the *three-dimensional real projective space* $\mathbb{RP}^3$.

Just as the 1- and 2-spheres are the sets of points at unit distance from O in $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively, the 3-sphere is the set of points in $\mathbb{R}^4$ at unit distance from O:

$$\mathbb{S}^3 = \{(a,b,c,d) \in \mathbb{R}^4 : a^2+b^2+c^2+d^2 = 1\}.$$

The points (a,b,c,d) on $\mathbb{S}^3$ correspond to quaternions $\mathbf{q} = a\mathbf{1}+b\mathbf{i}+c\mathbf{j}+d\mathbf{k}$ with $|\mathbf{q}| = 1$, because $|\mathbf{q}|^2 = a^2+b^2+c^2+d^2$. Hence, *rotations of* $\mathbb{S}^2$, *which correspond to pairs* $\pm\mathbf{q}$ *of such quaternions, correspond to point pairs* $\pm(a,b,c,d)$ *on* $\mathbb{S}^3$.