We observe that every coset $A$ of the rationals $\mathbf{R}$ has a non-empty intersection with $[0,1]$. Indeed, if $A$ is a coset, then $A = x + \mathbf{Q}$ for some real numbcr $x$. If we then pick a rational number $q$ in $[-x, 1-x]$ then we see that $x+q \in [0,1]$, and thus $A \cap [0,1]$ contains $x+q$.

Let $\mathbf{R}/\mathbf{Q}$ denote the set of all cosets of $\mathbf{Q}$; note that this is a set whose elements are themselves sets (of real numbers). For each coset $A$ in $\mathbf{R}/\mathbf{Q}$, let us pick an element $x_A$ of $A \cap [0,1]$. (This requires us to make an infinite number of choices, and thus requires the axiom of choice, see Section 8.4.) Let $E$ be the set of all such $x_A$, i.e., $E := \{x_A : A \in \mathbf{R}/\mathbf{Q}\}$. Note that $E \subseteq [0,1]$ by constrution.

Now consider the set

$$X = \bigcup_{q \in \mathbf{Q} \cap [-1,1]} (q + E).$$

Clearly this set is contained in $[-1,2]$ (since $q + x \in [-1,2]$ whenever $q \in [-1,1]$ and $x \in E \subseteq [0,1]$). We claim that this set contains the interval $[0,1]$. Indeed, for any $y \in [0,1]$, we know that $y$ must belong to some coset $A$ (for instance, it belongs to the coset $y + \mathbf{Q}$). But we also have $x_A$ belonging to the same coset, and thus $y - x_A$ is equal to some rational $q$. Since $y$ and $x_A$ both live in $[0,1]$, then $q$ lives in $[-1,1]$. Since $y = q + x_A$, we have $y \in q + E$, and hence $y \in X$ as desired.

We claim that

$$m^*(X) \neq \sum_{q \in \mathbf{Q} \cap [-1,1]} m^*(q+E),$$

which would prove the claim. To see why this is true, observe that since $[0,1] \subseteq X \subseteq [-1,2]$, that we have $1 \leq m^*(X) \leq 3$ by monotonicity and Proposition 18.2.6. For the right hand side, observe from translation invariance that

$$\sum_{q \in \mathbf{Q} \cap [-1,1]} m^*(q+E) = \sum_{q \in \mathbf{Q} \cap [-1,1]} m^*(E).$$

The set $\mathbf{Q} \cap [-1,1]$ is countably infinite (why?). Thus the right-hand side is either 0 (if $m^*(E) = 0$) or $+\infty$ (if $m^*(E) > 0$). Either way, it cannot be between 1 and 3, and the claim follows. $\square$

**Remark 18.3.2.** The above proof used the axiom of choice. This turns out to be absolutely necessary; one can prove using some advanced techniques in mathematical logic that if one does not assume the axiom of choice, then it is possible to have a mathematical model where outer measure is countably additive.

One can refine the above argument, and show in fact that $m^*$ is not finitely additive either:

**Proposition 18.3.3** (Failure of finite additivity). *There exists a finite collection $(A_j)_{j\in J}$ of disjoint subsets of $\mathbf{R}$, such that*

$$m^*(\bigcup_{j\in J} A_j) \neq \sum_{j\in J} m^*(A_j).$$

*Proof.* This is accomplished by an indirect argument. Suppose for sake of contradiction that $m^*$ *was* finitely additive. Let $E$ and $X$ be the sets introduced in Proposition 18.3.1. From countable sub-additivity and translation invariance we have

$$m^*(X) \leq \sum_{q\in\mathbf{Q}\cap[-1,1]} m^*(q+E) = \sum_{q\in\mathbf{Q}\cap[-1,1]} m^*(E).$$

Since we know that $1 \leq m^*(X) \leq 3$, we thus have $m^*(E) \neq 0$, since otherwise we would have $m^*(X) \leq 0$, a contradiction.

Since $m^*(E) \neq 0$, there exists a finite integer $n > 0$ such that $m^*(E) > 1/n$. Now let $J$ be a finite subset of $\mathbf{Q} \cap [-1,1]$ of cardinality 3n. If $m^*$ were finitely additive, then we would have

$$m^*(\sum_{q\in J} q+E) = \sum_{q\in J} m^*(q+E) = \sum_{q\in J} m^*(E) > 3n\frac{1}{n} = 3.$$

But we know that $\sum_{q\in J} q+E$ is a subset of $X$, which has outer measure at most 3. This contradicts monotonicity. Hence $m^*$ cannot be finitely additive. $\square$

**Remark 18.3.4.** The examples here are related to the *Banach-Tarski paradox*, which demonstrates (using the axiom of choice) that one can partition the unit ball in $\mathbf{R}^3$ into a finite number of pieces which, when rotated and translated, can be reassembled to form *two* complete unit balls! Of course, this partition involves non-measurable sets. We will not present this paradox here as it requires some group theory which is beyond the scope of this text.

## 18.4 Measurable sets

In the previous section we saw that certain sets were badly behaved with respect to outer measure, in particular they could be used to contradict finite or countable additivity. However, those sets were rather pathological, being constructed using the axiom of choice and looking rather artificial. One would hope to be able to exclude them and then somehow recover finite and countable additivity. Fortunately, this can be done, thanks to a clever definition of Constantin Carathéodory (1873–1950):

**Definition 18.4.1** (Lebesgue measurability)**.** Let $E$ be a subset of $\mathbf{R}^n$. We say that $E$ is *Lebesgue measurable*, or *measurable* for short, iff we have the identity

$$m^*(A) = m^*(A \cap E) + m^*(A \backslash E)$$

for every subset $A$ of $\mathbf{R}^n$. If $E$ is measurable, we define the *Lebesgue measure* of $E$ to be $m(E) = m^*(E)$; if $E$ is not measurable, we leave $m(E)$ undefined.

In other words, $E$ being measurable means that if we use the set $E$ to divide up an arbitrary set $A$ into two parts, we keep the additivity property. Of course, if $m^*$ were finitely additive then every set $E$ would be measurable; but we know from Proposition 18.3.3 that not every set is finitely additive. One can think of the measurable sets as the sets for which finite additivity works. We sometimes subscript $m(E)$ as $m_n(E)$ to emphasize the fact that we are using $n$-dimensional Lebesgue measure.

The above definition is somewhat hard to work with, and in practice one does not verify a set is measurable directly from this definition. Instead, we will use this definition to prove various useful properties of measurable sets (Lemmas 18.4.2-18.4.11), and after that we will rely more or less exclusively on the properties in those lemmas, and no longer need to refer to the above definition.

We begin by showing that a large number of sets are indeed measurable. The empty set $E = \emptyset$ and the whole space $E = \mathbf{R}^n$ are clearly measurable (why?). Here is another example of a measurable set:

**Lemma 18.4.2** (Half-spaces are measurable). *The half-space*

$$\{(x_1, \ldots, x_n) \in \mathbf{R}^n : x_n > 0\}$$

*is measurable.*

*Proof.* See Exercise 18.4.3. □

**Remark 18.4.3.** A similar argument will also show that any half-space of the form $\{(x_1, \ldots, x_n) \in \mathbf{R}^n : x_j > 0\}$ or $\{(x_1, \ldots, x_n) \in \mathbf{R}^n : x_j < 0\}$ for some $1 \leq j \leq n$ is measurable.

Now for some more properties of measurable sets.

**Lemma 18.4.4** (Properties of measurable sets).

(a) *If $E$ is measurable, then $\mathbf{R}^n \backslash E$ is also measurable.*

(b) *(Translation invariance) If $E$ is measurable, and $x \in \mathbf{R}^n$, then $x + E$ is also measurable, and $m(x + E) = m(E)$.*

(c) *If $E_1$ and $E_2$ are measurable, then $E_1 \cap E_2$ and $E_1 \cup E_2$ are measurable.*

(d) *(Boolean algebra property) If $E_1, E_2, \ldots, E_N$ are measurable, then $\bigcup_{j=1}^N E_j$ and $\bigcap_{j=1}^N E_j$ are measurable.*

(e) *Every open box, and every closed box, is measurable.*

(f) *Any set $E$ of outer measure zero (i.e., $m^*(E) = 0$) is measurable.*

*Proof.* See Exercise 18.4.4. □

From Lemma 18.4.4, we have proven properties (ii), (iii), (xiii) on our wish list of measurable sets, and we are making progress towards (i). We also have finite additivity (property (ix) on our wish list):

**Lemma 18.4.5** (Finite additivity). *If $(E_j)_{j\in J}$ are a finite collection of disjoint measurable sets and any set $A$ (not necessarily measurable), we have*

$$m^*(A \cap \bigcup_{j\in J} E_j) = \sum_{j\in J} m^*(A \cap E_j).$$

*Furthermore, we have $m(\bigcup_{j\in J} E_j) = \sum_{j\in J} m(E_j)$.*

*Proof.* See Exercise 18.4.6. □

**Remark 18.4.6.** Lemma 18.4.5 and Proposition 18.3.3, when combined, imply that there exist non-measurable sets: see Exercise 18.4.5.

**Corollary 18.4.7.** *If $A \subseteq B$ are two measurable sets, then $B\backslash A$ is also measurable, and*

$$m(B\backslash A) = m(B) - m(A).$$

*Proof.* See Exercise 18.4.7. □

Now we show countable additivity.

**Lemma 18.4.8** (Countable additivity). *If $(E_j)_{j\in J}$ are a countable collection of disjoint measurable sets, then $\bigcup_{j\in J} E_j$ is measurable, and $m(\bigcup_{j\in J} E_j) = \sum_{j\in J} m(E_j)$.*

*Proof.* Let $E := \bigcup_{j\in J} E_j$. Our first task will be to show that $E$ is measurable. Thus, let $A$ be any other measurable set; we need to show that

$$m^*(A) = m^*(A \cap E) + m^*(A\backslash E).$$

Since $J$ is countable, we may write $J = \{j_1, j_2, j_3, \ldots\}$. Note that

$$A \cap E = \bigcup_{k=1}^{\infty}(A \cap E_{j_k})$$

(why?) and hence by countable sub-additivity

$$m^*(A \cap E) \leq \sum_{k=1}^{\infty} m^*(A \cap E_{j_k}).$$

We rewrite this as

$$m^*(A \cap E) \leq \sup_{N\geq 1} \sum_{k=1}^{N} m^*(A \cap E_{j_k}).$$

Let $F_N$ be the set $F_N := \bigcup_{k=1}^{N} E_{j_k}$. Since the $A \cap E_{j_k}$ are all disjoint, and their union is $A \cap F_N$, we see from Lemma 18.4.5 that

$$\sum_{k=1}^{N} m^*(A \cap E_{j_k}) = m^*(A \cap F_N)$$

and hence

$$m^*(A \cap E) \leq \sup_{N\geq 1} m^*(A \cap F_N).$$

Now we look at $A\backslash E$. Since $F_N \subseteq E$ (why?), we have $A\backslash E \subseteq A\backslash F_N$ (why?). By monotonicity, we thus have

$$m^*(A\backslash E) \leq m^*(A\backslash F_N)$$

for all $N$. In particular, we see that

$$m^*(A \cap E) + m^*(A\backslash E) \leq \sup_{N\geq 1} m^*(A \cap F_N) + m^*(A\backslash E)$$

$$\leq \sup_{N \geq 1} m^*(A \cap F_N) + m^*(A \backslash F_N).$$

But from Lemma 18.4.5 we know that $F_N$ is measurable, and hence

$$m^*(A \cap F_N) + m^*(A \backslash F_N) = m^*(A).$$

Putting this all together we obtain

$$m^*(A \cap E) + m^*(A \backslash E) \leq m^*(A).$$

But from finite sub-additivity we have

$$m^*(A \cap E) + m^*(A \backslash E) \geq m^*(A)$$

and the claim follows. This shows that $E$ is measurable.

To finish the lemma, we need to show that $m(E)$ is equal to $\sum_{j \in J} m(E_j)$. We first observe from countable sub-additivity that

$$m(E) \leq \sum_{j \in J} m(E_j) = \sum_{k=1}^{\infty} m(E_{j_k}).$$

On the other hand, by finite additivity and monotonicity we have

$$m(E) \geq m(F_N) = \sum_{k=1}^{N} m(E_{j_k}).$$

Taking limits as $N \to \infty$ we obtain

$$m(E) \geq \sum_{k=1}^{\infty} m(E_{j_k})$$

and thus we have

$$m(E) = \sum_{k=1}^{\infty} m(E_{j_k}) = \sum_{j \in J} m(E_j)$$

as desired. $\square$

This proves property (xi) on our wish list. Next, we do countable unions and intersections.

**Lemma 18.4.9** ($\sigma$-algebra property). *If $(\Omega_j)_{j\in J}$ are any countable collection of measurable sets (so $J$ is countable), then the union $\bigcup_{j\in J}\Omega_j$ and the intersection $\bigcap_{j\in J}\Omega_j$ are also measurable.*

*Proof.* See Exercise 18.4.8. □

The final property left to verify on our wish list is (a). We first need a preliminary lemma.

**Lemma 18.4.10.** *Every open set can be written as a countable or finite union of open boxes.*

*Proof.* We first need some notation. Call a box $B = \prod_{i=1}^n (a_i, b_i)$ *rational* if all of its components $a_i, b_i$ are rational numbers. Observe that there are only a countable number of rational boxes (this is since a rational box is described by $2n$ rational numbers, and so has the same cardinality as $\mathbf{Q}^{2n}$. But $\mathbf{Q}$ is countable, and the Cartesian product of any finite number of countable sets is countable; see Corollaries 8.1.14, 8.1.15).

We make the following claim: given any open ball $B(x, r)$, there exists a rational box $B$ which is contained in $B(x, r)$ and which contains $x$. To prove this claim, write $x = (x_1, \ldots, x_n)$. For each $1 \le i \le n$, let $a_i$ and $b_i$ be rational numbers such that

$$x_i - \frac{r}{n} < a_i < x_i < b_i < x_i + \frac{r}{n}.$$

Then it is clear that the box $\prod_{i=1}^n (a_i, b_i)$ is rational and contains $x$. A simple computation using Pythagoras' theorem (or the triangle inequality) also shows that this box is contained in $B(x, r)$; we leave this to the reader.

Now let $E$ be an open set, and let $\Sigma$ be the set of all rational boxes $B$ which are subsets of $E$, and consider the union $\bigcup_{B\in\Sigma} B$ of all those boxes. Clearly, this union is contained in $E$, since every box in $\Sigma$ is contained in $E$ by construction. On the other hand, since $E$ is open, we see that for every $x \in E$ there is a ball $B(x, r)$

contained in $E$, and by the previous claim this ball contains a rational box which contains $x$. In particular, $x$ is contained in $\bigcup_{B\in\Sigma} B$. Thus we have

$$E = \bigcup_{B\in\Sigma} B$$

as desired; note that $\Sigma$ is countable or finite because it is a subset of the set of all rational boxes, which is countable. □

**Lemma 18.4.11** (Borel property). *Every open set, and every closed set, is Lebesgue measurable.*

*Proof.* It suffices to do this for open sets, since the claim for closed sets then follows by Lemma 18.4.4(a) (i.e., property (ii)). Let $E$ be an open set. By Lemma 18.4.10, $E$ is the countable union of boxes. Since we already know that boxes are measurable, and that the countable union of measurable sets is measurable, the claim follows. □

The construction of Lebesgue measure and its basic properties are now complete. Now we make the next step in constructing the Lebesgue integral - describing the class of functions we can integrate.

*Exercise* 18.4.1. If $A$ is an open interval in $\mathbf{R}$, show that $m^*(A) = m^*(A \cap (0,\infty)) + m^*(A\backslash(0,\infty))$.

*Exercise* 18.4.2. If $A$ is an open box in $\mathbf{R}^n$, and $E$ is the half-plane $E := \{(x_1,\ldots,x_n) \in \mathbf{R}^n : x_n > 0\}$, show that $m^*(A) = m^*(A \cap E) + m^*(A\backslash E)$. (Hint: use Exercise 18.4.1.)

*Exercise* 18.4.3. Prove Lemma 18.4.2. (Hint: use Exercise 18.4.2.)

*Exercise* 18.4.4. Prove Lemma 18.4.4. (Hints: for (c), first prove that

$$m^*(A) = m^*(A\cap E_1\cap E_2)+m^*(A\cap E_1\backslash E_2)+m^*(A\cap E_2\backslash E_1)+m^*(A\backslash(E_1\cup E_2)).$$

A Venn diagram may be helpful. Also you may need the finite subadditivity property. Use (c) to prove (d), and use (bd) and the various versions of Lemma 18.4.2 to prove (e)).

*Exercise* 18.4.5. Show that the set $E$ used in the proof of Propositions 18.3.1 and 18.3.3 is non-measurable.

*Exercise* 18.4.6. Prove Lemma 18.4.5.

*Exercise* 18.4.7. Use Lemma 18.4.5 to prove Corollary 18.4.7.

*Exercise* 18.4.8. Prove Lemma 18.4.9. (Hint: for the countable union problem, write $J = \{j_1, j_2, \ldots\}$, write $F_N := \bigcup_{k=1}^{N} \Omega_{j_k}$, and write $E_N := F_N \backslash F_{N-1}$, with the understanding that $F_0$ is the empty set. Then apply Lemma 18.4.8. For the countable intersection problem, use what you just did and Lemma 18.4.4(a).)

*Exercise* 18.4.9. Let $A \subseteq \mathbf{R}^2$ be the set $A := [0,1]^2 \backslash \mathbf{Q}^2$; i.e $A$ consists of all the points $(x, y)$ in $[0,1]^2$ such that $x$ and $y$ are not both rational. Show that $A$ is measurable and $m(A) = 1$, but that $A$ has no interior points. (Hint: it's easier to use the properties of outer measure and measure, including those in the exercises above, than to try to do this problem from first principles.)

*Exercise* 18.4.10. Let $A \subseteq B \subseteq \mathbf{R}^n$. Show that if $B$ is Lebesgue measurable with measure zero, then $A$ is also Lebesgue measurable with measure zero.

## 18.5 Measurable functions

In the theory of the Riemann integral, we are only able to integrate a certain class of functions - the Riemann integrable functions. We will now be able to integrate a much larger range of functions - the *measurable functions*. More precisely, we can only integrate those measurable functions which are absolutely integrable - but more on that later.

**Definition 18.5.1** (Measurable functions). Let $\Omega$ be a measurable subset of $\mathbf{R}^n$, and let $f : \Omega \to \mathbf{R}^m$ be a function. A function $f$ is *measurable* iff $f^{-1}(V)$ is measurable for every open set $V \subseteq \mathbf{R}^m$.

As discussed earlier, most sets that we deal with in real life are measurable, so it is only natural to learn that most functions we deal with in real life are also measurable. For instance, continuous functions are automatically measurable:

**Lemma 18.5.2** (Continuous functions are measurable). *Let $\Omega$ be a measurable subset of $\mathbf{R}^n$, and let $f : \Omega \to \mathbf{R}^m$ be continuous. Then $f$ is also measurable.*

*Proof.* Let $V$ be any open subset of $\mathbf{R}^m$. Then since $f$ is continuous, $f^{-1}(V)$ is open relative to $\Omega$ (see Theorem 13.1.5(c)), i.e., $f^{-1}(V) = W \cap \Omega$ for some open set $W \subseteq \mathbf{R}^n$ (see Proposition 12.3.4(a)). Since $W$ is open, it is measurable; since $\Omega$ is measurable, $W \cap \Omega$ is also measurable. $\square$

Because of Lemma 18.4.10, we have an easy criterion to test whether a function is measurable or not:

**Lemma 18.5.3.** *Let $\Omega$ be a measurable subset of $\mathbf{R}^n$, and let $f : \Omega \to \mathbf{R}^m$ be a function. Then $f$ is measurable if and only if $f^{-1}(B)$ is measurable for every open box $B$.*

*Proof.* See Exercise 18.5.1. $\square$

**Corollary 18.5.4.** *Let $\Omega$ be a measurable subset of $\mathbf{R}^n$, and let $f : \Omega \to \mathbf{R}^m$ be a function. Suppose that $f = (f_1, \ldots, f_m)$, where $f_j : \Omega \to \mathbf{R}$ is the $j^{th}$ co-ordinate of $f$. Then $f$ is measurable if and only if all of the $f_j$ are individually measurable.*

*Proof.* See Exercise 18.5.2. $\square$

Unfortunately, it is not true that the composition of two measurable functions is automatically measurable; however we can do the next best thing: a continuous function applied to a measurable function is measurable.

**Lemma 18.5.5.** *Let $\Omega$ be a measurable subset of $\mathbf{R}^n$, and let $W$ be an open subset of $\mathbf{R}^m$. If $f : \Omega \to W$ is measurable, and $g : W \to \mathbf{R}^p$ is continuous, then $g \circ f : \Omega \to \mathbf{R}^p$ is measurable.*

*Proof.* See Exercise 18.5.3. $\square$

This has an immediate corollary:

**Corollary 18.5.6.** *Let $\Omega$ be a measurable subset of $\mathbf{R}^n$. If $f : \Omega \to \mathbf{R}$ is a measurable function, then so is $|f|$, $\max(f, 0)$, and $\min(f, 0)$.*

*Proof.* Apply Lemma 18.5.5 with $g(x) := |x|$, $g(x) := \max(x, 0)$, and $g(x) := \min(x, 0)$. □

A slightly less immediate corollary:

**Corollary 18.5.7.** *Let $\Omega$ be a measurable subset of $\mathbf{R}^n$. If $f : \Omega \to \mathbf{R}$ and $g : \Omega \to \mathbf{R}$ are measurable functions, then so is $f + g$, $f - g$, $fg$, $\max(f, g)$, and $\min(f, g)$. If $g(x) \neq 0$ for all $x \in \Omega$, then $f/g$ is also measurable.*

*Proof.* Consider $f + g$. We can write this as $k \circ h$, where $h : \Omega \to \mathbf{R}^2$ is the function $h(x) = (f(x), g(x))$, and $k : \mathbf{R}^2 \to \mathbf{R}$ is the function $k(a, b) := a + b$. Since $f, g$ are measurable, then $h$ is also measurable by Corollary 18.5.4. Since $k$ is continuous, we thus see from Lemma 18.5.5 that $k \circ h$ is measurable, as desired. A similar argument deals with all the other cases; the only thing concerning the $f/g$ case is that the space $\mathbf{R}^2$ must be replaced with $\{(a, b) \in \mathbf{R}^2 : b \neq 0\}$ in order to keep the map $(a, b) \mapsto a/b$ continuous and well-defined. □

Another characterization of measurable functions is given by

**Lemma 18.5.8.** *Let $\Omega$ be a measurable subset of $\mathbf{R}^n$, and let $f : \Omega \to \mathbf{R}$ be a function. Then $f$ is measurable if and only if $f^{-1}((a, \infty))$ is measurable for every real number $a$.*

*Proof.* See Exercise 18.5.4. □

Inspired by this lemma, we extend the notion of a measurable function to the extended real number system $\mathbf{R}^* := \mathbf{R} \cup \{+\infty\} \cup \{-\infty\}$:

**Definition 18.5.9** (Measurable functions in the extended reals). Let $\Omega$ be a measurable subset of $\mathbf{R}^n$. A function $f : \Omega \to \mathbf{R}^*$ is said to be *measurable* iff $f^{-1}((a, \infty))$ is measurable for every real number $a$.

Note that Lemma 18.5.8 ensures that the notion of measurability for functions taking values in the extended reals $\mathbf{R}^*$ is compatible with that for functions taking values in just the reals $\mathbf{R}$.

Measurability behaves well with respect to limits:

**Lemma 18.5.10** (Limits of measurable functions are measurable). *Let $\Omega$ be a measurable subset of $\mathbf{R}^n$. For each positive integer $n$, let $f_n : \Omega \to \mathbf{R}^*$ be a measurable function. Then the functions $\sup_{n\geq 1} f_n$, $\inf_{n\geq 1} f_n$, $\limsup_{n\to\infty} f_n$, and $\liminf_{n\to\infty} f_n$ are also measurable. In particular, if the $f_n$ converge pointwise to another function $f : \Omega \to \mathbf{R}$, then $f$ is also measurable.*

*Proof.* We first prove the claim about $\sup_{n\geq 1} f_n$. Call this function $g$. We have to prove that $g^{-1}((a,\infty))$ is measurable for every $a$. But by the definition of supremum, we have

$$g^{-1}((a,\infty)) = \bigcup_{n\geq 1} f_n^{-1}((a,\infty))$$

(why?), and the claim follows since the countable union of measurable sets is again measurable.

A similar argument works for $\inf_{n\geq 1} f_n$. The claim for lim sup and lim inf then follow from the identities

$$\limsup_{n\to\infty} f_n = \inf_{N\geq 1} \sup_{n\geq N} f_n$$

and

$$\liminf_{n\to\infty} f_n = \sup_{N\geq 1} \inf_{n\geq N} f_n$$

(see Definition 6.4.6). □

As you can see, just about anything one does to a measurable function will produce another measurable function. This is basically why almost every function one deals with in mathematics is measurable. (Indeed, the only way to construct non-measurable functions is via artificial means such as invoking the axiom of choice.)

*Exercise* 18.5.1. Prove Lemma 18.5.3. (Hint: use Lemma 18.4.10 and the $\sigma$-algebra property.)

*Exercise* 18.5.2. Use Lemma 18.5.3 to deduce Corollary 18.5.4.

*Exercise* 18.5.3. Prove Lemma 18.5.5.

*Exercise* 18.5.4. Prove Lemma 18.5.8. (Hint: use Lemma 18.5.3. As a preliminary step, you may need to show that if $f^{-1}((a,\infty))$ is measurable for all $a$, then $f^{-1}([a,\infty))$ is also measurable for all $a$.)

*Exercise* 18.5.5. Let $f : \mathbf{R}^n \to \mathbf{R}$ be Lebesgue measurable, and let $g : \mathbf{R}^n \to \mathbf{R}$ be a function which agrees with $f$ outside of a set of measure zero, thus there exists a set $A \subseteq \mathbf{R}^n$ of measure zero such that $f(x) = g(x)$ for all $x \in \mathbf{R}^n \backslash A$. Show that $g$ is also Lebesgue measurable. (Hint: use Exercise 18.4.10.)

# Chapter 19

# Lebesgue integration

In Chapter 11, we approached the Riemann integral by first integrating a particularly simple class of functions, namely the *piecewise constant* functions. Among other things, piecewise constant functions only attain a finite number of values (as opposed to most functions in real life, which can take an infinite number of values). Once one learns how to integrate piecewise constant functions, one can then integrate other Riemann integrable functions by a similar procedure.

We shall use a similar philosophy to construct the Lebesgue integral. We shall begin by considering a special subclass of measurable functions - the *simple* functions. Then we will show how to integrate simple functions, and then from there we will integrate all measurable functions (or at least the absolutely integrable ones).

## 19.1 Simple functions

**Definition 19.1.1** (Simple functions). Let $\Omega$ be a measurable subset of $\mathbf{R}^n$, and let $f : \Omega \to \mathbf{R}$ be a measurable function. We say that $f$ is a *simple function* if the image $f(\Omega)$ is finite. In other words, there exists a finite number of real numbers $c_1, c_2, \ldots, c_N$ such that for every $x \in \Omega$, we have $f(x) = c_j$ for some $1 \leq j \leq N$.

**Example 19.1.2.** Let $\Omega$ be a measurable subset of $\mathbf{R}^n$, and let $E$ be a measurable subset of $\Omega$. We define the *characteristic function*