Since $g - f_n \geq 0$, we see similarly that

$$\int_E (g - f)\, d\mu \leq \liminf_{n\to\infty} \int_E (g - f_n)\, d\mu,$$

so that

$$-\int_E f\, d\mu \leq \liminf_{n\to\infty} \left[-\int_E f_n\, d\mu \right],$$

which is the same as

$$\int_E f\, d\mu \geq \limsup_{n\to\infty} \int_E f\, d\mu. \tag{86}$$

The existence of the limit in (84) and the equality asserted by (84) now follow from (85) and (86).

Corollary *If $\mu(E) < +\infty$, $\{f_n\}$ is uniformly bounded on E, and $f_n(x) \to f(x)$ on E, then* (84) *holds.*

A uniformly bounded convergent sequence is often said to be boundedly convergent.

COMPARISON WITH THE RIEMANN INTEGRAL

Our next theorem will show that every function which is Riemann-integrable on an interval is also Lebesgue-integrable, and that Riemann-integrable functions are subject to rather stringent continuity conditions. Quite apart from the fact that the Lebesgue theory therefore enables us to integrate a much larger class of functions, its greatest advantage lies perhaps in the ease with which many limit operations can be handled; from this point of view, Lebesgue's convergence theorems may well be regarded as the core of the Lebesgue theory.

One of the difficulties which is encountered in the Riemann theory is that limits of Riemann-integrable functions (or even continuous functions) may fail to be Riemann-integrable. This difficulty is now almost eliminated, since limits of measurable functions are always measurable.

Let the measure space X be the interval $[a, b]$ of the real line, with $\mu = m$ (the Lebesgue measure), and $\mathfrak{M}$ the family of Lebesgue-measurable subsets of $[a, b]$. Instead of

$$\int_X f\, dm$$

it is customary to use the familiar notation

$$\int_a^b f\, dx$$

for the Lebesgue integral of f over $[a, b]$. To distinguish Riemann integrals from Lebesgue integrals, we shall now denote the former by

$$\mathscr{R}\int_a^b f\,dx.$$

11.33 Theorem

(a) If $f \in \mathscr{R}$ on $[a, b]$, then $f \in \mathscr{L}$ on $[a, b]$, and

$$\int_a^b f\,dx = \mathscr{R}\int_a^b f\,dx. \tag{87}$$

(b) Suppose f is bounded on $[a, b]$. Then $f \in \mathscr{R}$ on $[a, b]$ if and only if f is continuous almost everywhere on $[a, b]$.

Proof Suppose f is bounded. By Definition 6.1 and Theorem 6.4 there is a sequence $\{P_k\}$ of partitions of $[a, b]$, such that P_{k+1} is a refinement of P_k, such that the distance between adjacent points of P_k is less than $1/k$, and such that

$$\lim_{k\to\infty} L(P_k, f) = \mathscr{R}\underline{\int} f\,dx, \qquad \lim_{k\to\infty} U(P_k, f) = \mathscr{R}\overline{\int} f\,dx. \tag{88}$$

(In this proof, all integrals are taken over $[a, b]$.)

If $P_k = \{x_0, x_1, \ldots, x_n\}$, with $x_0 = a$, $x_n = b$, define

$$U_k(a) = L_k(a) = f(a);$$

put $U_k(x) = M_i$ and $L_k(x) = m_i$ for $x_{i-1} < x \le x_i$, $1 \le i \le n$, using the notation introduced in Definition 6.1. Then

$$L(P_k, f) = \int L_k\,dx, \qquad U(P_k, f) = \int U_k\,dx, \tag{89}$$

and

$$L_1(x) \le L_2(x) \le \cdots \le f(x) \le \cdots \le U_2(x) \le U_1(x) \tag{90}$$

for all $x \in [a, b]$, since P_{k+1} refines P_k. By (90), there exist

$$L(x) = \lim_{k\to\infty} L_k(x), \qquad U(x) = \lim_{k\to\infty} U_k(x). \tag{91}$$

Observe that L and U are bounded measurable functions on $[a, b]$, that

$$L(x) \le f(x) \le U(x) \qquad (a \le x \le b), \tag{92}$$

and that

$$\int L\,dx = \mathscr{R}\underline{\int} f\,dx, \qquad \int U\,dx = \mathscr{R}\overline{\int} f\,dx, \tag{93}$$

by (88), (90), and the monotone convergence theorem.

So far, nothing has been assumed about f except that f is a bounded real function on $[a, b]$.

To complete the proof, note that $f \in \mathscr{R}$ if and only if its upper and lower Riemann integrals are equal, hence if and only if

$$\int L\,dx = \int U\,dx; \tag{94}$$

since $L \le U$, (94) happens if and only if $L(x) = U(x)$ for almost all $x \in [a, b]$ (Exercise 1).

In that case, (92) implies that

$$L(x) = f(x) = U(x) \tag{95}$$

almost everywhere on $[a, b]$, so that f is measurable, and (87) follows from (93) and (95).

Furthermore, if x belongs to no P_k, it is quite easy to see that $U(x) = L(x)$ if and only if f is continuous at x. Since the union of the sets P_k is countable, its measure is 0, and we conclude that f is continuous almost everywhere on $[a, b]$ if and only if $L(x) = U(x)$ almost everywhere, hence (as we saw above) if and only if $f \in \mathscr{R}$.

This completes the proof.

The familiar connection between integration and differentiation is to a large degree carried over into the Lebesgue theory. If $f \in \mathscr{L}$ on $[a, b]$, and

$$F(x) = \int_a^x f\,dt \qquad (a \le x \le b), \tag{96}$$

then $F'(x) = f(x)$ almost everywhere on $[a, b]$.

Conversely, if F is differentiable at every point of $[a, b]$ ("almost everywhere" is not good enough here!) and if $F' \in \mathscr{L}$ on $[a, b]$, then

$$F(x) - F(a) = \int_a^x F'(t) \qquad (a \le x \le b).$$

For the proofs of these two theorems, we refer the reader to any of the works on integration cited in the Bibliography.

INTEGRATION OF COMPLEX FUNCTIONS

Suppose f is a complex-valued function defined on a measure space X, and $f = u + iv$, where u and v are real. We say that f is measurable if and only if both u and v are measurable.

It is easy to verify that sums and products of complex measurable functions are again measurable. Since

$$|f| = (u^2 + v^2)^{1/2},$$

Theorem 11.18 shows that $|f|$ is measurable for every complex measurable f.

Suppose μ is a measure on X, E is a measurable subset of X, and f is a complex function on X. We say that $f \in \mathscr{L}(\mu)$ on E provided that f is measurable and

$$\int_E |f| \, d\mu < +\infty, \tag{97}$$

and we define

$$\int_E f \, d\mu = \int_E u \, d\mu + i \int_E v \, d\mu$$

if (97) holds. Since $|u| \leq |f|$, $|v| \leq |f|$, and $|f| \leq |u| + |v|$, it is clear that (97) holds if and only if $u \in \mathscr{L}(\mu)$ and $v \in \mathscr{L}(\mu)$ on E.

Theorems 11.23(*a*), (*d*), (*e*), (*f*), 11.24(*b*), 11.26, 11.27, 11.29, and 11.32 can now be extended to Lebesgue integrals of complex functions. The proofs are quite straightforward. That of Theorem 11.26 is the only one that offers anything of interest:

If $f \in \mathscr{L}(\mu)$ on E, there is a complex number c, $|c| = 1$, such that

$$c \int_E f \, d\mu \geq 0.$$

Put $g = cf = u + iv$, u and v real. Then

$$\left| \int_E f \, d\mu \right| = c \int_E f \, d\mu = \int_E g \, d\mu = \int_E u \, d\mu \leq \int_E |f| \, d\mu.$$

The third of the above equalities holds since the preceding ones show that $\int g \, d\mu$ is real.

FUNCTIONS OF CLASS $\mathscr{L}^2$

As an application of the Lebesgue theory, we shall now extend the Parseval theorem (which we proved only for Riemann-integrable functions in Chap. 8) and prove the Riesz-Fischer theorem for orthonormal sets of functions.

11.34 Definition Let X be a measurable space. We say that a complex function $f \in \mathscr{L}^2(\mu)$ on X if f is measurable and if

$$\int_X |f|^2 \, d\mu < +\infty.$$

If μ is Lebesgue measure, we say $f \in \mathscr{L}^2$. For $f \in \mathscr{L}^2(\mu)$ (we shall omit the phrase "on X" from now on) we define

$$\|f\| = \left\{ \int_X |f|^2 \, d\mu \right\}^{1/2}$$

and call $\|f\|$ the $\mathscr{L}^2(\mu)$ norm of f.

11.35 Theorem *Suppose $f \in \mathscr{L}^2(\mu)$ and $g \in \mathscr{L}^2(\mu)$. Then $fg \in \mathscr{L}(\mu)$, and*

$$\int_X |fg| \, d\mu \le \|f\| \, \|g\|. \tag{98}$$

This is the Schwarz inequality, which we have already encountered for series and for Riemann integrals. It follows from the inequality

$$0 \le \int_X (|f| + \lambda|g|)^2 \, d\mu = \|f\|^2 + 2\lambda \int_X |fg| \, d\mu + \lambda^2 \|g\|^2,$$

which holds for every real λ.

11.36 Theorem *If $f \in \mathscr{L}^2(\mu)$ and $g \in \mathscr{L}^2(\mu)$, then $f + g \in \mathscr{L}^2(\mu)$, and*

$$\|f + g\| \le \|f\| + \|g\|.$$

Proof The Schwarz inequality shows that

$$\begin{aligned} \|f + g\|^2 &= \int |f|^2 + \int f\bar{g} + \int \bar{f}g + \int |g|^2 \\ &\le \|f\|^2 + 2\|f\| \, \|g\| + \|g\|^2 \\ &= (\|f\| + \|g\|)^2. \end{aligned}$$

11.37 Remark If we define the distance between two functions f and g in $\mathscr{L}^2(\mu)$ to be $\|f - g\|$, we see that the conditions of Definition 2.15 are satisfied, except for the fact that $\|f - g\| = 0$ does not imply that $f(x) = g(x)$ for all x, but only for almost all x. Thus, if we identify functions which differ only on a set of measure zero, $\mathscr{L}^2(\mu)$ is a metric space.

We now consider $\mathscr{L}^2$ on an interval of the real line, with respect to Lebesgue measure.

11.38 Theorem *The continuous functions form a dense subset of $\mathscr{L}^2$ on $[a, b]$.*

More explicitly, this means that for any $f \in \mathscr{L}^2$ on $[a, b]$, and any $\varepsilon > 0$, there is a function g, continuous on $[a, b]$, such that

$$\|f-g\| = \left\{\int_a^b |f-g|^2\, dx\right\}^{1/2} < \varepsilon.$$

Proof We shall say that f is approximated in $\mathscr{L}^2$ by a sequence $\{g_n\}$ if $\|f-g_n\| \to 0$ as $n \to \infty$.

Let A be a closed subset of $[a, b]$, and K_A its characteristic function. Put

$$t(x) = \inf |x-y| \qquad (y \in A)$$

and

$$g_n(x) = \frac{1}{1+nt(x)} \qquad (n = 1, 2, 3, \ldots).$$

Then g_n is continuous on $[a, b]$, $g_n(x) = 1$ on A, and $g_n(x) \to 0$ on B, where $B = [a, b] - A$. Hence

$$\|g_n - K_A\| = \left\{\int_B g_n^2\, dx\right\}^{1/2} \to 0$$

by Theorem 11.32. Thus characteristic functions of closed sets can be approximated in $\mathscr{L}^2$ by continuous functions.

By (39) the same is true for the characteristic function of any measurable set, and hence also for simple measurable functions.

If $f \geq 0$ and $f \in \mathscr{L}^2$, let $\{s_n\}$ be a monotonically increasing sequence of simple nonnegative measurable functions such that $s_n(x) \to f(x)$. Since $|f - s_n|^2 \leq f^2$, Theorem 11.32 shows that $\|f - s_n\| \to 0$.

The general case follows.

11.39 Definition We say that a sequence of complex functions $\{\phi_n\}$ is an *orthonormal* set of functions on a measurable space X if

$$\int_X \phi_n \bar{\phi}_m\, d\mu = \begin{cases} 0 & (n \neq m), \\ 1 & (n = m). \end{cases}$$

In particular, we must have $\phi_n \in \mathscr{L}^2(\mu)$. If $f \in \mathscr{L}^2(\mu)$ and if

$$c_n = \int_X f\bar{\phi}_n\, d\mu \qquad (n = 1, 2, 3, \ldots),$$

we write

$$f \sim \sum_{n=1}^{\infty} c_n \phi_n,$$

as in Definition 8.10.

The definition of a trigonometric Fourier series is extended in the same way to $\mathscr{L}^2$ (or even to $\mathscr{L}$) on $[-\pi, \pi]$. Theorems 8.11 and 8.12 (the Bessel inequality) hold for any $f \in \mathscr{L}^2(\mu)$. The proofs are the same, word for word.

We can now prove the Parseval theorem.

11.40 Theorem *Suppose*

$$f(x) \sim \sum_{-\infty}^{\infty} c_n e^{inx}, \tag{99}$$

where $f \in \mathscr{L}^2$ on $[-\pi, \pi]$. Let s_n be the nth partial sum of (99). Then

$$\lim_{n \to \infty} \|f - s_n\| = 0, \tag{100}$$

$$\sum_{-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f|^2 \, dx. \tag{101}$$

Proof Let $\varepsilon > 0$ be given. By Theorem 11.38, there is a continuous function g such that

$$\|f - g\| < \frac{\varepsilon}{2}.$$

Moreover, it is easy to see that we can arrange it so that $g(\pi) = g(-\pi)$. Then g can be extended to a periodic continuous function. By Theorem 8.16, there is a trigonometric polynomial T, of degree N, say, such that

$$\|g - T\| < \frac{\varepsilon}{2}.$$

Hence, by Theorem 8.11 (extended to $\mathscr{L}^2$), $n \geq N$ implies

$$\|s_n - f\| \leq \|T - f\| < \varepsilon,$$

and (100) follows. Equation (101) is deduced from (100) as in the proof of Theorem 8.16.

Corollary *If $f \in \mathscr{L}^2$ on $[-\pi, \pi]$, and if*

$$\int_{-\pi}^{\pi} f(x) e^{-inx} \, dx = 0 \qquad (n = 0, \pm 1, \pm 2, \ldots),$$

then $\|f\| = 0$.

Thus if two functions in $\mathscr{L}^2$ have the same Fourier series, they differ at most on a set of measure zero.

11.41 Definition Let f and $f_n \in \mathscr{L}^2(\mu)$ $(n = 1, 2, 3, \ldots)$. We say that $\{f_n\}$ converges to f in $\mathscr{L}^2(\mu)$ if $\|f_n - f\| \to 0$. We say that $\{f_n\}$ is a Cauchy sequence in $\mathscr{L}^2(\mu)$ if for every $\varepsilon > 0$ there is an integer N such that $n \geq N$, $m \geq N$ implies $\|f_n - f_m\| \leq \varepsilon$.

11.42 Theorem *If $\{f_n\}$ is a Cauchy sequence in $\mathscr{L}^2(\mu)$, then there exists a function $f \in \mathscr{L}^2(\mu)$ such that $\{f_n\}$ converges to f in $\mathscr{L}^2(\mu)$.*

This says, in other words, that $\mathscr{L}^2(\mu)$ is a *complete* metric space.

Proof Since $\{f_n\}$ is a Cauchy sequence, we can find a sequence $\{n_k\}$, $k = 1, 2, 3, \ldots$, such that

$$\|f_{n_k} - f_{n_{k+1}}\| < \frac{1}{2^k} \qquad (k = 1, 2, 3, \ldots).$$

Choose a function $g \in \mathscr{L}^2(\mu)$. By the Schwarz inequality,

$$\int_X |g(f_{n_k} - f_{n_{k+1}})| \, d\mu \leq \frac{\|g\|}{2^k}.$$

Hence

$$\sum_{k=1}^{\infty} \int_X |g(f_{n_k} - f_{n_{k+1}})| \, d\mu \leq \|g\|. \tag{102}$$

By Theorem 11.30, we may interchange the summation and integration in (102). It follows that

$$|g(x)| \sum_{k=1}^{\infty} |f_{n_k}(x) - f_{n_{k+1}}(x)| < +\infty \tag{103}$$

almost everywhere on X. Therefore

$$\sum_{k=1}^{\infty} |f_{n_{k+1}}(x) - f_{n_k}(x)| < +\infty \tag{104}$$

almost everywhere on X. For if the series in (104) were divergent on a set E of positive measure, we could take $g(x)$ to be nonzero on a subset of E of positive measure, thus obtaining a contradiction to (103).

Since the kth partial sum of the series

$$\sum_{k=1}^{\infty} (f_{n_{k+1}}(x) - f_{n_k}(x)),$$

which converges almost everywhere on X, is

$$f_{n_{k+1}}(x) - f_{n_1}(x),$$