That is,

$$|a_n| < |a_N|\beta^{-N} \cdot \beta^n$$

for $n \ge N$, and (*a*) follows from the comparison test, since $\Sigma\beta^n$ converges.

If $|a_{n+1}| \ge |a_n|$ for $n \ge n_0$, it is easily seen that the condition $a_n \to 0$ does not hold, and (*b*) follows.

Note: The knowledge that $\lim a_{n+1}/a_n = 1$ implies nothing about the convergence of Σa_n. The series $\Sigma 1/n$ and $\Sigma 1/n^2$ demonstrate this.

3.35 Examples

(*a*) Consider the series

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots,$$

for which

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0,$$

$$\liminf_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[2n]{\frac{1}{3^n}} = \frac{1}{\sqrt{3}},$$

$$\limsup_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[2n]{\frac{1}{2^n}} = \frac{1}{\sqrt{2}},$$

$$\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{2}\left(\frac{3}{2}\right)^n = +\infty.$$

The root test indicates convergence; the ratio test does not apply.

(*b*) The same is true for the series

$$\frac{1}{2} + 1 + \frac{1}{8} + \frac{1}{4} + \frac{1}{32} + \frac{1}{16} + \frac{1}{128} + \frac{1}{64} + \cdots,$$

where

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{8},$$

$$\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = 2,$$

but

$$\lim \sqrt[n]{a_n} = \tfrac{1}{2}.$$

3.36 Remarks The ratio test is frequently easier to apply than the root test, since it is usually easier to compute ratios than nth roots. However, the root test has wider scope. More precisely: Whenever the ratio test shows convergence, the root test does too; whenever the root test is inconclusive, the ratio test is too. This is a consequence of Theorem 3.37, and is illustrated by the above examples.

Neither of the two tests is subtle with regard to divergence. Both deduce divergence from the fact that a_n does not tend to zero as $n \to \infty$.

3.37 Theorem *For any sequence $\{c_n\}$ of positive numbers,*

$$\liminf_{n\to\infty} \frac{c_{n+1}}{c_n} \le \liminf_{n\to\infty} \sqrt[n]{c_n},$$

$$\limsup_{n\to\infty} \sqrt[n]{c_n} \le \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}.$$

Proof We shall prove the second inequality; the proof of the first is quite similar. Put

$$\alpha = \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}.$$

If $\alpha = +\infty$, there is nothing to prove. If α is finite, choose $\beta > \alpha$. There is an integer N such that

$$\frac{c_{n+1}}{c_n} \le \beta$$

for $n \ge N$. In particular, for any $p > 0$,

$$c_{N+k+1} \le \beta c_{N+k} \qquad (k = 0, 1, \ldots, p-1).$$

Multiplying these inequalities, we obtain

$$c_{N+p} \le \beta^p c_N,$$

or

$$c_n \le c_N \beta^{-N} \cdot \beta^n \qquad (n \ge N).$$

Hence

$$\sqrt[n]{c_n} \le \sqrt[n]{c_N \beta^{-N}} \cdot \beta,$$

so that

$$\limsup_{n\to\infty} \sqrt[n]{c_n} \le \beta, \tag{18}$$

by Theorem 3.20(b). Since (18) is true for every $\beta > \alpha$, we have

$$\limsup_{n\to\infty} \sqrt[n]{c_n} \le \alpha.$$

POWER SERIES

3.38 Definition Given a sequence $\{c_n\}$ of complex numbers, the series

$$\sum_{n=0}^{\infty} c_n z^n \tag{19}$$

is called a *power series*. The numbers c_n are called the *coefficients* of the series; z is a complex number.

In general, the series will converge or diverge, depending on the choice of z. More specifically, with every power series there is associated a circle, the circle of convergence, such that (19) converges if z is in the interior of the circle and diverges if z is in the exterior (to cover all cases, we have to consider the plane as the interior of a circle of infinite radius, and a point as a circle of radius zero). The behavior on the circle of convergence is much more varied and cannot be described so simply.

3.39 Theorem *Given the power series* $\Sigma c_n z^n$, *put*

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{|c_n|}, \qquad R = \frac{1}{\alpha}.$$

(*If* $\alpha = 0$, $R = +\infty$; *if* $\alpha = +\infty$, $R = 0$.) *Then* $\Sigma c_n z^n$ *converges if* $|z| < R$, *and diverges if* $|z| > R$.

Proof Put $a_n = c_n z^n$, and apply the root test:

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = |z| \limsup_{n\to\infty} \sqrt[n]{|c_n|} = \frac{|z|}{R}.$$

Note: R is called the radius of convergence of $\Sigma c_n z^n$.

3.40 Examples

(a) The series $\Sigma n^n z^n$ has $R = 0$.

(b) The series $\sum \frac{z^n}{n!}$ has $R = +\infty$. (In this case the ratio test is easier to apply than the root test.)

(*c*) The series Σz^n has $R = 1$. If $|z| = 1$, the series diverges, since $\{z^n\}$ does not tend to 0 as $n \to \infty$.

(*d*) The series $\sum \frac{z^n}{n}$ has $R = 1$. It diverges if $z = 1$. It converges for all other z with $|z| = 1$. (The last assertion will be proved in Theorem 3.44.)

(*e*) The series $\sum \frac{z^n}{n^2}$ has $R = 1$. It converges for all z with $|z| = 1$, by the comparison test, since $|z^n/n^2| = 1/n^2$.

SUMMATION BY PARTS

3.41 Theorem *Given two sequences $\{a_n\}$, $\{b_n\}$, put*

$$A_n = \sum_{k=0}^{n} a_k$$

if $n \geq 0$; put $A_{-1} = 0$. Then, if $0 \leq p \leq q$, we have

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p. \tag{20}$$

Proof

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q} (A_n - A_{n-1}) b_n = \sum_{n=p}^{q} A_n b_n - \sum_{n=p-1}^{q-1} A_n b_{n+1},$$

and the last expression on the right is clearly equal to the right side of (20).

Formula (20), the so-called "partial summation formula," is useful in the investigation of series of the form $\Sigma a_n b_n$, particularly when $\{b_n\}$ is monotonic. We shall now give applications.

3.42 Theorem *Suppose*

(*a*) *the partial sums A_n of Σa_n form a bounded sequence;*
(*b*) $b_0 \geq b_1 \geq b_2 \geq \cdots$;
(*c*) $\lim_{n \to \infty} b_n = 0$.

Then $\Sigma a_n b_n$ converges.

Proof Choose M such that $|A_n| \le M$ for all n. Given $\varepsilon > 0$, there is an integer N such that $b_N \le (\varepsilon/2M)$. For $N \le p \le q$, we have

$$\begin{aligned}\left|\sum_{n=p}^{q} a_n b_n\right| &= \left|\sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p\right| \\ &\le M \left|\sum_{n=p}^{q-1} (b_n - b_{n+1}) + b_q + b_p\right| \\ &= 2Mb_p \le 2Mb_N \le \varepsilon.\end{aligned}$$

Convergence now follows from the Cauchy criterion. We note that the first inequality in the above chain depends of course on the fact that $b_n - b_{n+1} \ge 0$.

3.43 Theorem *Suppose*

(*a*) $|c_1| \ge |c_2| \ge |c_3| \ge \cdots$;
(*b*) $c_{2m-1} \ge 0$, $c_{2m} \le 0$ $\quad (m = 1, 2, 3, \ldots)$;
(*c*) $\lim_{n\to\infty} c_n = 0$.

Then Σc_n *converges.*

Series for which (*b*) holds are called "alternating series"; the theorem was known to Leibnitz.

Proof Apply Theorem 3.42, with $a_n = (-1)^{n+1}$, $b_n = |c_n|$.

3.44 Theorem *Suppose the radius of convergence of* $\Sigma c_n z^n$ *is* 1, *and suppose* $c_0 \ge c_1 \ge c_2 \ge \cdots$, $\lim_{n\to\infty} c_n = 0$. *Then* $\Sigma c_n z^n$ *converges at every point on the circle* $|z| = 1$, *except possibly at* $z = 1$.

Proof Put $a_n = z^n$, $b_n = c_n$. The hypotheses of Theorem 3.42 are then satisfied, since

$$|A_n| = \left|\sum_{m=0}^{n} z^m\right| = \left|\frac{1 - z^{n+1}}{1 - z}\right| \le \frac{2}{|1 - z|},$$

if $|z| = 1$, $z \ne 1$.

ABSOLUTE CONVERGENCE

The series Σa_n is said to *converge absolutely* if the series $\Sigma |a_n|$ converges.

3.45 Theorem *If* Σa_n *converges absolutely, then* Σa_n *converges.*

Proof The assertion follows from the inequality

$$\left|\sum_{k=n}^{m} a_k\right| \le \sum_{k=n}^{m} |a_k|,$$

plus the Cauchy criterion.

3.46 Remarks For series of positive terms, absolute convergence is the same as convergence.

If Σa_n converges, but $\Sigma|a_n|$ diverges, we say that Σa_n converges *nonabsolutely*. For instance, the series

$$\sum \frac{(-1)^n}{n}$$

converges nonabsolutely (Theorem 3.43).

The comparison test, as well as the root and ratio tests, is really a test for absolute convergence, and therefore cannot give any information about nonabsolutely convergent series. Summation by parts can sometimes be used to handle the latter. In particular, power series converge absolutely in the interior of the circle of convergence.

We shall see that we may operate with absolutely convergent series very much as with finite sums. We may multiply them term by term and we may change the order in which the additions are carried out, without affecting the sum of the series. But for nonabsolutely convergent series this is no longer true, and more care has to be taken when dealing with them.

ADDITION AND MULTIPLICATION OF SERIES

3.47 Theorem *If* $\Sigma a_n = A$, *and* $\Sigma b_n = B$, *then* $\Sigma(a_n + b_n) = A + B$, *and* $\Sigma c a_n = cA$, *for any fixed c.*

Proof Let

$$A_n = \sum_{k=0}^{n} a_k, \qquad B_n = \sum_{k=0}^{n} b_k.$$

Then

$$A_n + B_n = \sum_{k=0}^{n} (a_k + b_k).$$

Since $\lim_{n\to\infty} A_n = A$ and $\lim_{n\to\infty} B_n = B$, we see that

$$\lim_{n\to\infty} (A_n + B_n) = A + B.$$

The proof of the second assertion is even simpler.

Thus two convergent series may be added term by term, and the resulting series converges to the sum of the two series. The situation becomes more complicated when we consider multiplication of two series. To begin with, we have to define the product. This can be done in several ways; we shall consider the so-called "Cauchy product."

3.48 Definition Given Σa_n and Σb_n, we put

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} \qquad (n = 0, 1, 2, \ldots)$$

and call Σc_n the *product* of the two given series.

This definition may be motivated as follows. If we take two power series $\Sigma a_n z^n$ and $\Sigma b_n z^n$, multiply them term by term, and collect terms containing the same power of z, we get

$$\begin{aligned}\sum_{n=0}^{\infty} a_n z^n \cdot \sum_{n=0}^{\infty} b_n z^n &= (a_0 + a_1 z + a_2 z^2 + \cdots)(b_0 + b_1 z + b_2 z^2 + \cdots) \\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0)z + (a_0 b_2 + a_1 b_1 + a_2 b_0)z^2 + \cdots \\ &= c_0 + c_1 z + c_2 z^2 + \cdots.\end{aligned}$$

Setting $z = 1$, we arrive at the above definition.

3.49 Example If

$$A_n = \sum_{k=0}^{n} a_k, \qquad B_n = \sum_{k=0}^{n} b_k, \qquad C_n = \sum_{k=0}^{n} c_k,$$

and $A_n \to A$, $B_n \to B$, then it is not at all clear that $\{C_n\}$ will converge to AB, since we do not have $C_n = A_n B_n$. The dependence of $\{C_n\}$ on $\{A_n\}$ and $\{B_n\}$ is quite a complicated one (see the proof of Theorem 3.50). We shall now show that the product of two convergent series may actually diverge.

The series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}} = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$$

converges (Theorem 3.43). We form the product of this series with itself and obtain

$$\begin{aligned}\sum_{n=0}^{\infty} c_n = 1 - \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{2}\sqrt{2}} + \frac{1}{\sqrt{3}}\right) \\ - \left(\frac{1}{\sqrt{4}} + \frac{1}{\sqrt{3}\sqrt{2}} + \frac{1}{\sqrt{2}\sqrt{3}} + \frac{1}{\sqrt{4}}\right) + \cdots,\end{aligned}$$

so that

$$c_n = (-1)^n \sum_{k=0}^{n} \frac{1}{\sqrt{(n-k+1)(k+1)}}.$$

Since

$$(n-k+1)(k+1) = \left(\frac{n}{2}+1\right)^2 - \left(\frac{n}{2}-k\right)^2 \le \left(\frac{n}{2}+1\right)^2.$$

we have

$$|c_n| \ge \sum_{k=0}^{n} \frac{2}{n+2} = \frac{2(n+1)}{n+2},$$

so that the condition $c_n \to 0$, which is necessary for the convergence of Σc_n, is not satisfied.

In view of the next theorem, due to Mertens, we note that we have here considered the product of two nonabsolutely convergent series.

3.50 Theorem *Suppose*

(*a*) $\sum_{n=0}^{\infty} a_n$ *converges absolutely,*

(*b*) $\sum_{n=0}^{\infty} a_n = A,$

(*c*) $\sum_{n=0}^{\infty} b_n = B,$

(*d*) $c_n = \sum_{k=0}^{n} a_k b_{n-k} \qquad (n = 0, 1, 2, \ldots).$

Then

$$\sum_{n=0}^{\infty} c_n = AB.$$

That is, the product of two convergent series converges, and to the right value, if at least one of the two series converges absolutely.

Proof Put

$$A_n = \sum_{k=0}^{n} a_k, \qquad B_n = \sum_{k=0}^{n} b_k, \qquad C_n = \sum_{k=0}^{n} c_k, \qquad \beta_n = B_n - B.$$

Then

$$\begin{aligned} C_n &= a_0 b_0 + (a_0 b_1 + a_1 b_0) + \cdots + (a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0) \\ &= a_0 B_n + a_1 B_{n-1} + \cdots + a_n B_0 \\ &= a_0(B + \beta_n) + a_1(B + \beta_{n-1}) + \cdots + a_n(B + \beta_0) \\ &= A_n B + a_0 \beta_n + a_1 \beta_{n-1} + \cdots + a_n \beta_0 \end{aligned}$$

Put

$$\gamma_n = a_0 \beta_n + a_1 \beta_{n-1} + \cdots + a_n \beta_0 .$$

We wish to show that $C_n \to AB$. Since $A_n B \to AB$, it suffices to show that

(21) $$\lim_{n \to \infty} \gamma_n = 0.$$

Put

$$\alpha = \sum_{n=0}^{\infty} |a_n|.$$

[It is here that we use (*a*).] Let $\varepsilon > 0$ be given. By (*c*), $\beta_n \to 0$. Hence we can choose N such that $|\beta_n| \le \varepsilon$ for $n \ge N$, in which case

$$\begin{aligned} |\gamma_n| &\le |\beta_0 a_n + \cdots + \beta_N a_{n-N}| + |\beta_{N+1} a_{n-N-1} + \cdots + \beta_n a_0| \\ &\le |\beta_0 a_n + \cdots + \beta_N a_{n-N}| + \varepsilon\alpha. \end{aligned}$$

Keeping N fixed, and letting $n \to \infty$, we get

$$\limsup_{n \to \infty} |\gamma_n| \le \varepsilon\alpha,$$

since $a_k \to 0$ as $k \to \infty$. Since ε is arbitrary, (21) follows.

Another question which may be asked is whether the series Σc_n, if convergent, must have the sum AB. Abel showed that the answer is in the affirmative.

3.51 Theorem *If the series Σa_n, Σb_n, Σc_n converge to A, B, C, and $c_n = a_0 b_n + \cdots + a_n b_0$, then $C = AB$.*

Here no assumption is made concerning absolute convergence. We shall give a simple proof (which depends on the continuity of power series) after Theorem 8.2.

REARRANGEMENTS

3.52 Definition Let $\{k_n\}$, $n = 1, 2, 3, \ldots$, be a sequence in which every positive integer appears once and only once (that is, $\{k_n\}$ is a 1-1 function from J onto J, in the notation of Definition 2.2). Putting

$$a'_n = a_{k_n} \qquad (n = 1, 2, 3, \ldots),$$

we say that $\Sigma a'_n$ is a *rearrangement* of Σa_n.

If $\{s_n\}$, $\{s_n'\}$ are the sequences of partial sums of Σa_n, $\Sigma a_n'$, it is easily seen that, in general, these two sequences consist of entirely different numbers. We are thus led to the problem of determining under what conditions all rearrangements of a convergent series will converge and whether the sums are necessarily the same.

3.53 Example Consider the convergent series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \cdots \tag{22}$$

and one of its rearrangements

$$1 + \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{5} + \tfrac{1}{7} - \tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{11} - \tfrac{1}{6} + \cdots \tag{23}$$

in which two positive terms are always followed by one negative. If s is the sum of (22), then

$$s < 1 - \tfrac{1}{2} + \tfrac{1}{3} = \tfrac{5}{6}.$$

Since

$$\frac{1}{4k-3} + \frac{1}{4k-1} - \frac{1}{2k} > 0$$

for $k \geq 1$, we see that $s_3' < s_6' < s_9' < \cdots$, where s_n' is nth partial sum of (23). Hence

$$\limsup_{n\to\infty} s_n' > s_3' = \tfrac{5}{6},$$

so that (23) certainly does not converge to s [we leave it to the reader to verify that (23) does, however, converge].

This example illustrates the following theorem, due to Riemann.

3.54 Theorem *Let Σa_n be a series of real numbers which converges, but not absolutely. Suppose*

$$-\infty \leq \alpha \leq \beta \leq \infty.$$

Then there exists a rearrangement $\Sigma a_n'$ with partial sums s_n' such that

$$\liminf_{n\to\infty} s_n' = \alpha, \qquad \limsup_{n\to\infty} s_n' = \beta. \tag{24}$$

Proof Let

$$p_n = \frac{|a_n| + a_n}{2}, \qquad q_n = \frac{|a_n| - a_n}{2} \qquad (n = 1, 2, 3, \ldots).$$