5.1 Perspective drawing

Sometime in the 15th century, Italian artists discovered how to draw three-dimensional scenes in correct perspective. Figures 5.2 and 5.3 illustrate the great advance in realism this skill achieved, with pictures drawn before and after the discovery. The "before" picture, Figure 5.2, is a drawing I found in the book *Perspective in Perspective* by L. Wright. It is thought to date from the late 15th century, but it comes from England, where knowledge of perspective had evidently not reached at that time.

Figure 5.2: *The birth of St Edmund*, by an unknown artist

The "after" picture, Figure 5.3, is the 1514 engraving *Saint Jerome in his study*, by the great German artist Albrecht Dürer (1471–1528). Dürer made study tours of Italy in 1494 and 1505 and became a master of all aspects of drawing, including perspective.

The simplest test of perspective drawing is the depiction of a tiled floor. The picture in Figure 5.2 clearly fails this test. All the tiles are drawn as rectangles, which makes the floor look vertical. We know from experience that a horizontal rectangle does not *look* rectangular—its angles are not all right angles because its sides converge to a common point on the horizon, as in the tabletop in Dürer's engraving.

Figure 5.3: *St Jerome in his study*, by Albrecht Dürer

The Italians drew tiles by a method called the *costruzione legittima* (legitimate construction), first published by Leon Battista Alberti in 1436. The bottom edge of the picture coincides with a line of tile edges, and any other horizontal line is chosen as the horizon. Then lines drawn from equally spaced points on the bottom edge to a point on the horizon depict the parallel columns of tiles perpendicular to the bottom edge (Figure 5.4). Another horizontal line, near the bottom, completes the first row of tiles.

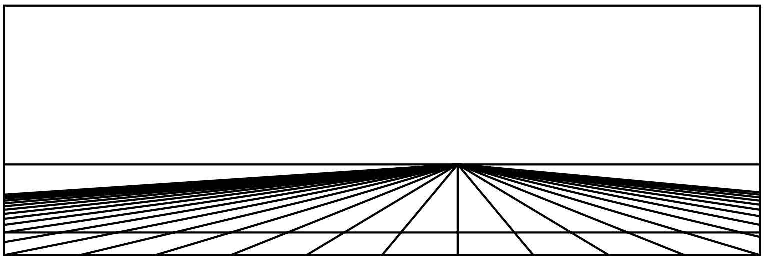

Figure 5.4: Beginning the *costruzione legittima*

The real problem comes next. How do we find the correct horizontal lines to depict the 2nd, 3rd, 4th, ... rows of tiles? The answer is surprisingly simple: Draw the *diagonal* of any tile in the bottom row (shown in gray in Figure 5.5). The diagonal necessarily crosses successive columns at the corners of tiles in the 2nd, 3rd, 4th, ... rows; hence, these rows can be constructed by drawing horizontal lines at the successive crossings. Voilà!

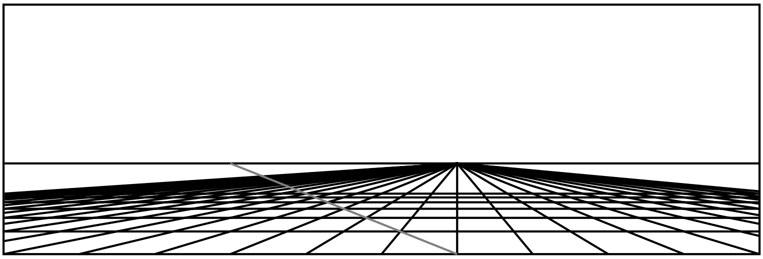

Figure 5.5: Completing the *costruzione legittima*

Exercises

Suppose that the floor has rows of tiles crossing the x-axis at $x = 0, 1, 2, 3, \ldots$, and that the artist copies the view of the floor onto a vertical transparent screen through the y-axis, keeping a fixed eye position at the point $(-1, 1)$. Then the perspective view of the points $x = 0, 1, 2, 3, \ldots$ will be the series of points on the y-axis shown in Figure 5.6.

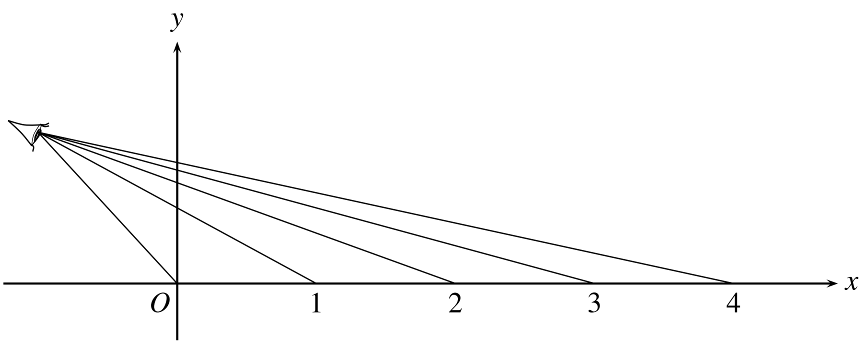

Figure 5.6: Perspective view of equally spaced points

5.1.1 Show that the line from $(-1,1)$ to $(n,0)$ crosses the y-axis at $y = \frac{n}{n+1}$. Hence, the perspective images of the points $x = 0, 1, 2, 3, \ldots$ are the points $y = 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$.

If each of the points $0, 1, 2, 3, \ldots$ is sent to the next, then each of their perspective images $y = 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$ is sent to the next.

5.1.2 Show that the function $f(y) = \frac{1}{2-y}$ effects this move.

5.1.3 Which point on the y-axis is not moved by the function $f(y) = \frac{1}{2-y}$, and what is the geometric significance of this point?

5.2 Drawing with straightedge alone

The *construzione legittima* takes advantage of something that is visually obvious but mathematically mysterious—the fact that parallel lines generally do not look parallel, but appear to meet on the horizon. The point where a family of parallels appear to meet is called their "vanishing point" by artists, and their *point at infinity* by mathematicians. The horizon itself, which consists of all the points at infinity, is called the *line at infinity*.

However, the *costruzione legittima* does not take full advantage of points at infinity. It involves some parallels that are really *drawn* parallel, so we need both straightedge and compass as used in Chapter 1. The construction also needs measurement to lay out the equally spaced points on the bottom line of the picture, and this again requires a compass. Thus, the *costruzione legittima* is a Euclidean construction at heart, requiring both a straightedge and a compass.

Is it possible to draw a perspective view of a tiled floor with a straightedge alone? Absolutely! All one needs to get started is the horizon and a tile placed obliquely. The tile is created by the two pairs of parallel lines, which are simply pairs that meet on the horizon (Figure 5.7).

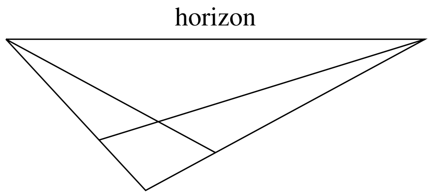

Figure 5.7: The first tile

We then draw the diagonal of this tile and extend it to the horizon, obtaining the point at infinity of all diagonals parallel to this first one. This step allows us to draw two more diagonals, of tiles adjacent to the first one.

These diagonals give us the remaining sides of the adjacent tiles, and we can then repeat the process. The first few steps are shown in Figure 5.8. Figure 5.1 at the beginning of the chapter is the result of carrying out many steps (and deleting the construction lines).

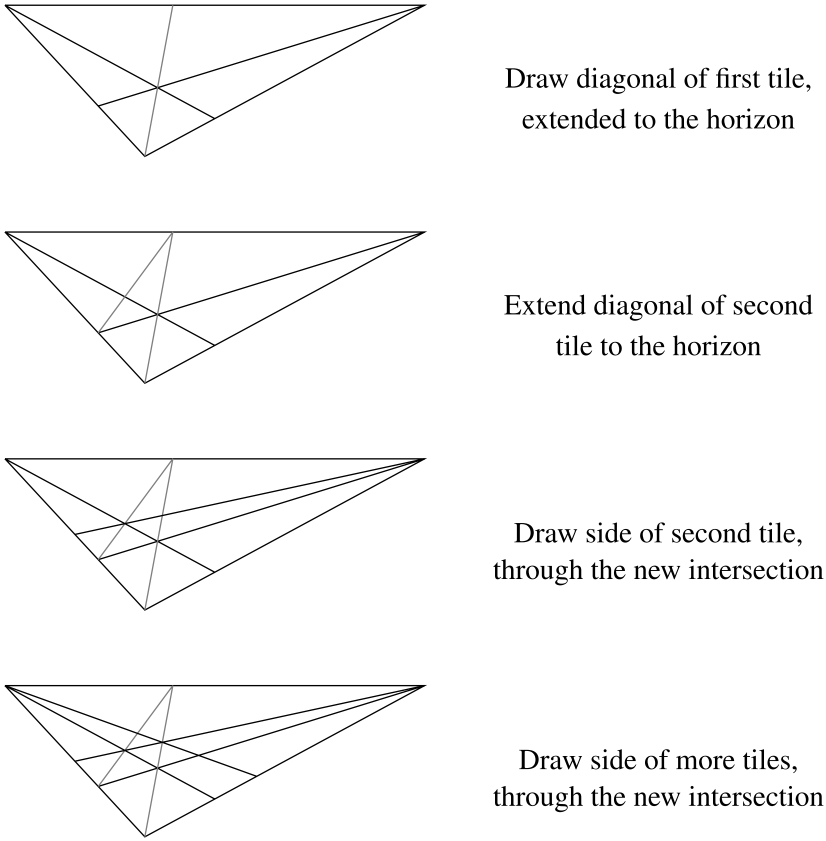

Figure 5.8: Constructing the tiled floor

This construction is easy and fun to do, and we urge the reader to get a straightedge and try it. Also try the constructions suggested in the exercises, which create pictures of floors with differently shaped tiles.

Exercises

Consider the triangular tile shown shaded in Figure 5.9. Notice that this triangle could be half of the quadrangular tile shown in Figure 5.7 (this is a hint).

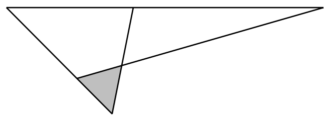

Figure 5.9: A triangular tile

5.2.1 Draw a perspective view of the plane filled with many copies of this tile.

5.2.2 Also, by deleting some lines in your solution to Exercise 5.2.1, create a perspective view of the plane filled with congruent hexagons.

5.3 Projective plane axioms and their models

Drawing a tiled floor with straightedge alone requires a "horizon"—a line at infinity. Apart from this requirement, the construction works because certain things remain the same in any view of the plane:

- straight lines remain straight
- intersections remain intersections
- parallel lines remain parallel or meet on the horizon.

Now parallel lines *always* meet on the horizon if you point yourself in the right direction, so if we could look in all directions at once we would see that any two lines have a point in common. This idea leads us to believe in a structure called a *projective plane*, containing objects called *"points"* and *"lines"* satisfying the following axioms. We write "points" and "lines" in quotes because they may not be the same as ordinary points and lines.

Axioms for a projective plane

1. Any two "points" are contained in a unique "line."
2. Any two "lines" contain a unique "point."
3. There exist four "points", no three of which are in a "line."

Notice that these are axioms about *incidence*: They involve only meetings between "points" and "lines," not things such as length or angle. Some of Euclid's and Hilbert's axioms are of this kind, but not many.

Axiom 1 is essentially Euclid's first axiom for the construction of lines. Axiom 2 says that there are no exceptional pairs of lines that do not meet. We can define "parallels" to be lines that meet on a line called the "horizon," but this does not single out a special class of lines—in a projective plane, the "horizon" behaves the same as any other line. Axiom 3 says that a projective plane has "enough points to be interesting." We can think of the four points as the four vertices of a quadrilateral, from which one may generate the complicated structure seen in the pictures of a tiled floor at the beginning of this chapter.

The real projective plane

If there is such a thing as a projective plane, it should certainly satisfy these axioms. But does *anything* satisfy them? After all, we humans can never see all of the horizon at once, so perhaps it is inconsistent to suppose that all parallels meet. These doubts are dispelled by the following *model*, or *interpretation*, of the axioms for a projective plane. The model is called the *real projective plane* $\mathbb{RP}^2$, and it gives a mathematical meaning to the terms "point," "line," and "plane" that makes all the axioms true.

Take "points" to be lines through O in $\mathbb{R}^3$, "lines" to be planes through O in $\mathbb{R}^3$, and the "plane" to be the set of all lines through O in $\mathbb{R}^3$. Then

1. Any two "points" are contained in a unique "line" because two given lines through O lie in a unique plane through O.

2. Any two "lines" contain a unique "point" because any two planes through O meet in a unique line through O.

3. There are four different "points," no three of which are in a "line": for example, the lines from O to the four points $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, and $(1,1,1)$, because no three of these lines lie in the same plane through O.

The last claim is perhaps a little hard to grasp by visualization, but it can be checked algebraically because any plane through O has an equation of the form

$$ax+by+cz=0 \quad \text{for some real numbers} \quad a,b,c.$$

If, say, $(1,0,0)$ and $(0,1,0)$ are on this plane, then we find by substituting these values of x,y,z in the equation that

$$a = 0 \quad \text{and} \quad b = 0, \quad \text{hence the plane is } z = 0.$$

But $(0,0,1)$ and $(1,1,1)$ do *not* lie on the plane $z = 0$. It can be checked similarly that the plane through any two of the points does not contain the other two.

It is no fluke that lines and planes through O in $\mathbb{R}^3$ behave as we want "points" and "lines" of a projective plane to behave, because they capture the idea of *viewing with an all-seeing eye.* The point O is the position of the eye, and the lines through O connect the eye to points in the plane. Consider how the eye sees the plane $z = -1$, for example (Figure 5.10).

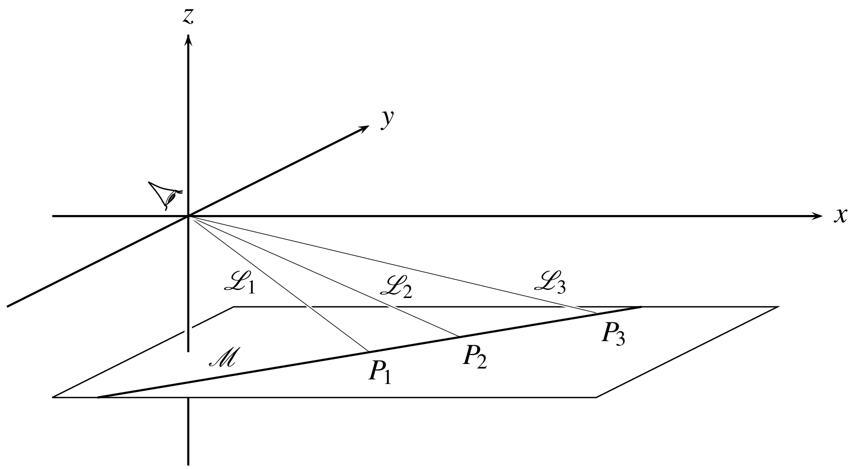

Figure 5.10: Viewing a plane from O

Points $P_1, P_2, P_3, \ldots$ in the plane $z = -1$ are joined to the eye by lines $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3, \ldots$ through O, and as the point P_n tends to infinity, the line $\mathscr{L}_n$ tends toward the horizontal. Therefore, it is natural to call the horizontal lines through O the "points at infinity" of the plane $z = -1$, and to call the plane of all horizontal lines through O the "horizon" or "line at infinity" of the plane $z = -1$.

Unlike the lines $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3, \ldots$, corresponding to points $P_1, P_2, P_3, \ldots$ of the *Euclidean plane* $z = -1$, horizontal lines through O have no counterparts in the Euclidean plane: They *extend* the Euclidean plane to a *projective plane*. However, the extension arises in a natural way. Once we