## 12.2 Parameterization of Cubic Curves

To see how to construct parameterizing functions for a cubic curve, we first reconstruct the parameterizing functions

$$x = \sin u,$$
$$y = \cos u$$

for the circle $x^2 + y^2 = 1$, pretending that we do not know this curve geometrically but only as an algebraic relation between $x$ and $y$.

The sine function can be defined as the inverse $f$ of $f^{-1}(x) = \sin^{-1} x$, which in turn is definable as the integral

$$f^{-1}(x) = \int_0^x \frac{dt}{\sqrt{1-t^2}}.$$

Finally, the integral can be viewed as an outgrowth of the equation $y^2 = 1 - x^2$, because the integrand $1/\sqrt{1-x^2}$ is simply $1/y$. Why do we use this integrand rather than any other to define $u = f^{-1}(x)$ and hence obtain $x$ as a function $f(u)$? The answer is that we then obtain $y$ as $f'(u)$, hence $x$, $y$ are both functions of the parameter $u$. This is confirmed by the following calculation:

$$f'(u) = \frac{dx}{du} = 1 \Big/ \frac{du}{dx}$$

and

$$\frac{du}{dx} = \frac{d}{dx} \int_0^x \frac{dt}{\sqrt{1-t^2}} = \frac{1}{\sqrt{1-x^2}} = \frac{1}{y},$$

hence $y = f'(u)$ (which of course is $\cos u$).

Exactly the same construction can be used to parameterize any relation of the form $y^2 = p(x)$. We put

$$u = g^{-1}(x) = \int_0^x \frac{dt}{\sqrt{p(t)}}$$

to get $x = g(u)$, and then find that $y = g'(u)$ by differentiation of $u$. Thus in a sense it is trivial to parameterize curves of the form $y^2 = p(x)$ (which we know from Section 8.4 to include all cubic curves, up to a projective transformation of $x$ and $y$). As we shall see in the next section, the integrals $\int dt/\sqrt{p(t)}$ had been studied since the 1600s for $p$ a polynomial of degree 3 or 4; however, no one thought to invert them until about 1800.

Jacobi had a deep knowledge of both the integrals *and* inversion when he wrote his cryptic paper [Jacobi (1834)] pointing out the relation between integrals and rational points on curves (cf. Sections 11.6 and 12.5). Thus it seems likely he understood the preceding parameterization, though such a parameterization was first given explicitly by Clebsch (1864).

EXERCISES

It may happen that the integral $\int_0^x dt/\sqrt{p(t)}$ does not converge because of the behavior of $1/\sqrt{p(t)}$ at $t=0$. But in that case one can use the parameter $u=f^{-1}(x)=\int_a^x dt/\sqrt{p(t)}$ for some other value of $a$.

**12.2.1** Check that $y=f'(u)$ remains true with this change of definition.

When the cubic curve is $y^2=x^3$, which has a rational parameterization, the parameterizing functions constructed above indeed turn out to be rational.

**12.2.2** Given $y=x^{3/2}$, find $x=f(u)$ and $y=f'(u)$, where $u=f^{-1}(x)=\int_a^x \frac{dt}{t^{3/2}}$.

## 12.3 Elliptic Integrals

Integrals of the form $\int R(t,\sqrt{p(t)})\,dt$, where $R$ is a rational function and $p$ is a polynomial of degree 3 or 4 without multiple factors, are called *elliptic integrals*, because the first example occurs in the formula for the arc length of the ellipse. (The functions obtained by inverting elliptic integrals are called *elliptic functions*, and the curves that require elliptic functions for their parameterization are called *elliptic curves*. This drift in the meaning of "elliptic" is rather unfortunate because the ellipse, being parameterizable by rational functions, is not an elliptic curve!)

Elliptic integrals arise in many important problems of geometry and mechanics, for example, as arc lengths of the ellipse and hyperbola, period of the simple pendulum, and deflection of a thin elastic bar [see Chapter 13 and, for example, Melzak (1976), pp. 253–269]. When these problems first arose in the late seventeenth century they posed the first obstacle to Leibniz's program of integration in "closed form" or "by elementary functions." As mentioned in Section 9.6, Leibniz considered the proper solution of an integration problem $\int f(x)\,dx$ to be a known function $g(x)$ such that $g'(x)=f(x)$. The functions then "known," and now called "elementary," were those composed from algebraic, circular, and exponential functions and their inverses.

All efforts to express elliptic integrals in these terms failed, and as early as 1694 Jakob Bernoulli conjectured that the task was impossible. The conjecture was eventually confirmed by Liouville (1833), in the course of showing that a large class of integrals is nonelementary. In the meantime, mathematicians had discovered so many properties of elliptic integrals, and the elliptic functions obtained from them by inversion, that they could be considered known even if not elementary.

The key that unlocked many of the secrets of elliptic integrals was the curve known as the *lemniscate of Bernoulli* (Figure 12.1). This curve is mentioned briefly in Section 2.5 as one of the spiric sections of Perseus. It has cartesian equation

$$(x^2+y^2)^2 = x^2 - y^2$$

and polar equation

$$r^2 = \cos 2\theta.$$

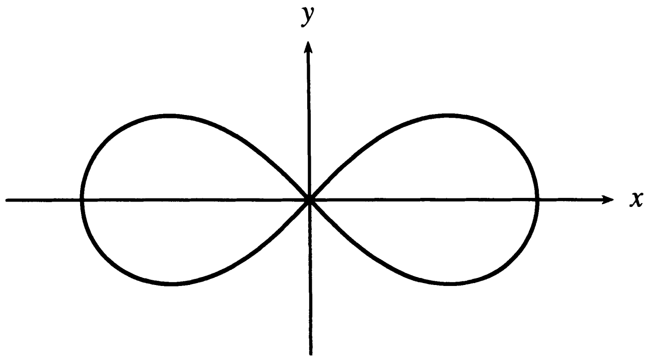


Figure 12.1: The lemniscate of Bernoulli

The first to consider it in its own right was Jakob Bernoulli (1694). He showed that its arc length is expressed by the elliptic integral $\int_0^x dt/\sqrt{1-t^4}$, subsequently known as the *lemniscatic integral*, and thus he gave this formal expression a concrete geometric interpretation. Many later developments in the theory of elliptic integrals and functions grew out of interplay between the lemniscate and the lemniscatic integral. Being the simplest elliptic integral, or at any rate the most analogous to the arcsine integral $\int_0^x dt/\sqrt{1-t^2}$, the lemniscatic integral $\int_0^x dt/\sqrt{1-t^4}$ was the most amenable to manipulation. It was often possible, after some property had been extracted from the lemniscatic integral, to extend the argument to more general elliptic integrals.

The most notable example of this methodology was in the discovery of the addition theorems, which we discuss in the next section.

EXERCISES

The properties of the lemniscate mentioned above are easily proved by some standard analytic geometry and calculus.

**12.3.1** Deduce the cartesian equation of the lemniscate from its polar equation

$$r^2 = \cos 2\theta.$$

**12.3.2** Use the polar equation of the lemniscate and the formula for the element of arc in polar coordinates,

$$ds = \sqrt{(rd\theta)^2 + dr^2}$$

to deduce that arc length of the lemniscate is given by

$$s = \int \frac{d\theta}{r}.$$

**12.3.3** Conclude, by changing the variable of integration to $r$, that the total length of the lemniscate is $4\int_0^1 dr/\sqrt{1-r^4}$.

Unlike the arcsine integrand $1/\sqrt{1-t^2}$, which is rationalized by substituting $2v/(1+v^2)$ for $t$, the lemniscatic integrand $1/\sqrt{1-t^4}$ cannot be rationalized by replacing $t$ by any rational function.

**12.3.4** Explain how this follows from the exercises in Section 11.6.

It was this connection between the lemniscatic integral and Fermat's theorem on the impossibility of $r^4 - s^4 = v^2$ in positive integers that led Jakob Bernoulli to suspect the impossibility of evaluating the lemniscatic integral by known functions.

## 12.4 Doubling the Arc of the Lemniscate

An addition theorem is a formula expressing $f(u_1+u_2)$ in terms of $f(u_1)$ and $f(u_2)$, and perhaps also $f'(u_1)$ and $f'(u_2)$. For example, the addition theorem for the sine function is

$$\sin(u_1+u_2) = \sin u_1 \cos u_2 + \sin u_2 \cos u_1.$$

Since the derivative, $\cos u$, of $\sin u$ equals $\sqrt{1-\sin^2 u}$, we can also write the addition theorem as

$$\sin(u_1+u_2) = \sin u_1 \sqrt{1-\sin^2 u_2} + \sin u_2 \sqrt{1-\sin^2 u_1},$$

showing that $\sin(u_1 + u_2)$ is an algebraic function of $\sin u_1$ and $\sin u_2$.

To simplify the comparison with elliptic functions we consider the following special case of the sine addition theorem:

$$\sin 2u = 2 \sin u \sqrt{1 - \sin^2 u}. \tag{1}$$

If we let

$$u = \sin^{-1} x = \int_0^x \frac{dt}{\sqrt{1-t^2}},$$

then

$$2u = 2\int_0^x \frac{dt}{\sqrt{1-t^2}}.$$

But from (1) we also have

$$2u = \sin^{-1}(2x\sqrt{1-x^2}),$$

so

$$2\int_0^x \frac{dt}{\sqrt{1-t^2}} = \int_0^{2x\sqrt{1-x^2}} \frac{dt}{\sqrt{1-t^2}}. \tag{2}$$

Bearing in mind that $\sin^{-1} x = \int_0^x dt/\sqrt{1-t^2}$ represents the angle $u$ seen in Figure 12.2, equation (2) tells us that the angle (or arc length) $u$ is doubled by going from $x$ to $2x\sqrt{1-x^2}$. The latter number, since it is obtained from $x$ by rational operations and square roots, is constructible from $x$ by ruler and compasses (confirming the geometrically obvious fact that an angle can be duplicated by ruler and compasses).

All this has a remarkable parallel in the properties of the lemniscate and its arc length integral $\int_0^x dt/\sqrt{1-t^4}$. The discovery of a formula for doubling the arc of the lemniscate by Fagnano (1718) showed that geometric information could be extracted from the previously intractable elliptic integrals, and we can also view it as the first step toward the theory of elliptic functions. In our notation, Fagnano's formula was

$$2\int_0^x \frac{dt}{\sqrt{1-t^4}} = \int_0^{2x\sqrt{1-x^4}/(1+x^4)} \frac{dt}{\sqrt{1-t^4}}. \tag{3}$$

Since $2x\sqrt{1-x^4}/(1+x^4)$ is obtained from $x$ by rational operations and square roots, (3) shows, like (2), that the arc can be doubled by ruler and compass construction.

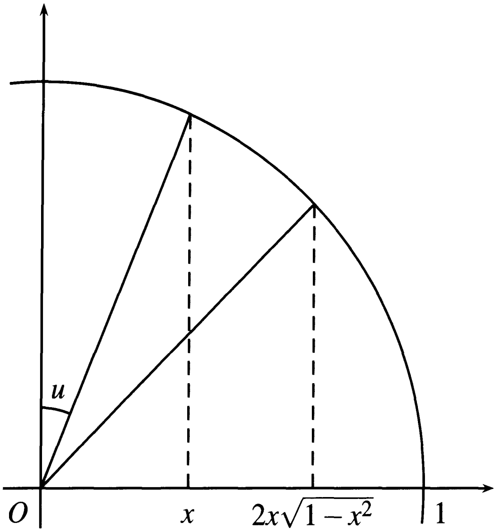


Figure 12.2: Doubling a circular arc

Fagnano derived his formula by two substitutions that, as Siegel (1969), p. 3, points out, are analogous to a natural substitution for the arcsine integral (see exercises below).

EXERCISES

The following exercises compare the effect of the substitution $t = 2v/(1+v^2)$ in $dt/\sqrt{1-t^2}$ with analogous substitutions for $t^2$ in $dt/\sqrt{1-t^4}$.

**12.4.1** Show that substituting $t = 2v/(1+v^2)$ gives $\sqrt{1-t^2} = (1-v^2)/(1+v^2)$ and hence that $dt/\sqrt{1-t^2} = 2dv/(1+v^2)$.

**12.4.2** Show that $t^2 = 2v^2/(1+v^4)$ gives $\sqrt{1-t^4} = (1-v^4)/(1+v^4)$ and hence

$$\frac{dt}{\sqrt{1-t^4}} = \sqrt{2}\frac{dv}{\sqrt{1+v^4}}.$$

It follows that this change of variable corresponds to a certain relation between integrals, which turns out to be "half way" to the Fagnano formula.

**12.4.3** Deduce from Exercise 12.4.2 that

$$\sqrt{2}\int_0^x \frac{dv}{\sqrt{1+v^4}} = \int_0^{\sqrt{2}x/\sqrt{1+x^4}} \frac{dt}{\sqrt{1-t^4}}.$$

To complete the journey to the Fagnano formula we make a second, similar, substitution that recreates the lemniscatic integral.

**12.4.4** Similarly show that the substitution $v^2 = 2w^2/(1-w^4)$ gives

$$\frac{dv}{\sqrt{1+v^4}} = \sqrt{2}\frac{dw}{\sqrt{1-w^4}}.$$

**12.4.5** Check that the result of the substitutions in 12.4.2 and 12.4.4 is

$$t = \frac{2w\sqrt{1-w^4}}{1+w^4}$$

and that the corresponding relation between integrals is the Fagnano duplication formula.

## 12.5 General Addition Theorems

Fagnano's duplication formula remained a little-known curiosity until Euler received a copy of Fagnano's works on December 23, 1751, a date later described by Jacobi as "the birth day of the theory of elliptic functions." Euler was the first to see that Fagnano's substitution trick was not just a curious fluke but a revelation into the behavior of elliptic integrals. With his superb manipulative skill Euler was quickly able to extend it to very general addition theorems. First to the addition theorem for the lemniscatic integral,

$$\int_0^x \frac{dt}{\sqrt{1-t^4}} + \int_0^y \frac{dt}{\sqrt{1-t^4}} = \int_0^{(x\sqrt{1-y^4}+y\sqrt{1-x^4})/(1+x^2y^2)} \frac{dt}{\sqrt{1-t^4}}$$

then to $\int dt/\sqrt{p(t)}$, where $p(t)$ is an arbitrary polynomial of degree 4. An ingenious reconstruction of Euler's train of thought, by analogy with the arcsine addition theorem

$$\int_0^x \frac{dt}{\sqrt{1-t^2}} + \int_0^y \frac{dt}{\sqrt{1-t^2}} = \int_0^{x\sqrt{1-y^2}+y\sqrt{1-x^2}} \frac{dt}{\sqrt{1-t^2}},$$

has been given by Siegel (1969), pp. 1–10. Brilliant as his results were, Euler was dealing only with elliptic integrals, *not* with elliptic functions, their inverses, so one could still quibble with Jacobi's assessment. But one has to remember that Jacobi could see an elliptic function a mile off, probably more easily than we can see that the arcsine addition theorem is really a theorem about sines!

It should be mentioned that Euler's addition theorems do not cover all elliptic integrals. The general form $\int R(t, \sqrt{p(t)})\,dt$ does, however, reduce to just three kinds, of which Euler's are the first and most important. The classical theory of elliptic integrals of the different kinds, with their various addition and transformation theorems, was systematized by Legendre (1825). Ironically, this was just before the appearance of elliptic functions, which made much of Legendre's work obsolete.

These early investigations exploited some of the formal similarities between $\int dt/\sqrt{p(t)}$, where $p$ is a polynomial of degree 4, and $\int dt/\sqrt{q(t)}$, where $q$ is a quadratic. There is no real difference if $p$ is of degree 3, as an easy transformation shows (Exercise 12.5.1). This is why $\int dt/\sqrt{p(t)}$ is also called an elliptic integral when $p$ is of degree 3. In fact, it eventually turned out that the most convenient integral to use as a basis for the theory of elliptic functions is $\int dt/\sqrt{4t^3-g_2t-g_3}$, whose inverse is known as the Weierstrass $\wp$-function.

The addition theorem for this integral is

$$\int_0^{x_1}\frac{dt}{\sqrt{4t^3-g_2t-g_3}}+\int_0^{x_2}\frac{dt}{\sqrt{4t^3-g_2t-g_3}}=\int_0^{x_3}\frac{dt}{\sqrt{4t^3-g_2t-g_3}},$$

where $x_3$ is none other than the $x$-coordinate of the third point on

$$y^2=4x^3-g_2x-g_3$$

of the straight line through $(x_1,y_1)$ and $(x_2,y_2)$ (see Section 11.6). Now that we know, from Section 12.2, that this curve is parameterized by $x=\wp(u)$, $y=\wp'(u)$, defined by inverting the integral, some connection between the geometry of the curve and the addition theorem is understandable. But the stunning simplicity of the relationship seems to demand a deeper explanation. This lies in the realm of complex numbers, which we shall enter briefly in the next section and more thoroughly in Sections 16.4 and 16.5.

EXERCISES

**12.5.1** Show that the substitution $t=1/u$ transforms

$$\frac{dt}{\sqrt{(t-a)(t-b)(t-c)}}\quad\text{into}\quad\frac{-du}{\sqrt{u(1-ua)(1-ub)(1-uc)}}.$$

Conversely, we can transform quartic polynomials under the square root sign to cubics, even in cases where the quartic is not of the form obtained in Exercise 12.5.1.

**12.5.2** Transform

$$\frac{dt}{\sqrt{1-t^4}}\quad\text{into}\quad\frac{du}{\sqrt{\text{cubic polynomial in }u}}$$

by making a suitable substitution for $t$.