For arbitrary digraphs, we prove an analogue of Dirac's theorem (Theorem 7.2.8). Indeed, it yields Dirac's theorem as a special case (Exercise 49). Meyniel [1973] subtantially strengthened the theorem by weakening the hypothesis (Theorem 8.4.42).

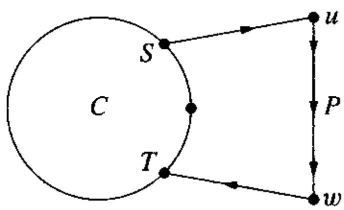
7.2.21. Definition. A digraph is **strict** if it has no loops and has at most one copy of each ordered pair as an edge.

7.2.22. Theorem. (Ghouilà-Houri [1960]) If D is a strict digraph, and $\min\{\delta^+(D), \delta^-(D)\} \geq n(D)/2$, then D is Hamiltonian.

Proof: Again we use contradiction and extremality. In an n-vertex counterexample D, let C be a longest cycle, with length l. As we have observed, $l > \max\{\delta^+, \delta^-\} \geq n/2$. Let P be a longest path in $D - V(C)$, beginning at u, ending at w, and having length $m \geq 0$. Now $l > n/2$ and $n \geq l + m + 1$ imply $m < n/2$.

Let S be the set of predecessors of u on C, and let T be the set of successors of w on C. By the maximality of P, every predecessor of u and successor of w lies in $V(C) \cup V(P)$. Thus S and T each have size at least $\min\{\delta^+, \delta^-\} - m$, which is at least $\geq n/2 - m$ and hence is positive. Thus S and T are nonempty.

The maximality of C guarantees that the distance along C from a vertex $u' \in S$ to a vertex $w' \in T$ must exceed $m + 1$. Otherwise, traveling along P instead of C from u' to w' yields a longer cycle. Hence we may assume that every vertex of S is followed on C by more than m vertices not in T.



If the distance between successive vertices of S along C is always at most $m + 1$, then there is no legal place to put a vertex of T. Since both S and T are nonempty, we may thus assume there is a vertex of S followed on C by at least $m + 1$ vertices not in S. These are forbidden from T, as is the immediate successor on C of all the other vertices of S.

Thus at least $|S| - 1 + m + 1 \geq n/2$ vertices of C are not in T. Together withthe vertices that are in T, this yields $|V(C)| \geq n - m$, which contradicts $l \leq n - m - 1$. The contradiction implies that C must be a spanning cycle. ■

EXERCISES

7.2.1. (−) For which values of r is $K_{r,r}$ Hamiltonian?

7.2.2. (−) Is the Grötzsch graph (Example 5.2.2) Hamiltonian?

7.2.3. (−) For $n > 1$, prove that $K_{n,n}$ has $(n-1)!n!/2$ Hamiltonian cycles.

7.2.4. (−) Prove that G has a Hamiltonian path only if for every $S \subseteq V(G)$, the number of components of $G - S$ is at most $|S| + 1$.

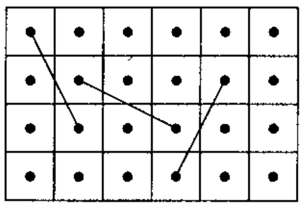
• • • • •

7.2.5. Prove that every 5-vertex path in the dodecahedron lies in a Hamiltonian cycle.

7.2.6. (!) Let G be a Hamiltonian bipartite graph, and choose $x, y \in V(G)$. Prove that $G - x - y$ has a perfect matching if and only if x and y are on opposite sides of the bipartition of G. Apply this to prove that deleting two unit squares from an 8 by 8 chessboard leaves a board that can be partitioned into 1 by 2 rectangles if and only if the two missing squares have opposite colors.

7.2.7. A mouse eats its way through a $3 \times 3 \times 3$ cube of cheese by eating all the $1 \times 1 \times 1$ subcubes. If it starts at a corner subcube and always moves on to an adjacent subcube (sharing a face of area 1), can it do this and eat the center subcube last? Give a method or prove impossible. (Ignore gravity.)

7.2.8. (!) On a chessboard, a **knight** can move from one square to another that differs by 1 in one coordinate and by 2 in the other coordinate, as shown below. Prove that no $4 \times n$ chessboard has a **knight's tour**: a traversal by knight's moves that visits each square once and returns to the start. (Hint: Find an appropriate set of vertices in the corresponding graph to violate the necessary condition.)



7.2.9. Construct an infinite family of non-Hamiltonian graphs satisfying the necessary condition of Proposition 7.2.3.

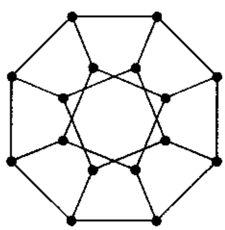
7.2.10. (!) *Hamiltonian vs. Eulerian.*

a) Find a 2-connected non–Eulerian graph whose line graph is Hamiltonian.

b) Prove that $L(G)$ is Hamiltonian if and only if G has a closed trail that contains at least one endpoint of each edge. (Harary and Nash-Williams [1965])

7.2.11. Construct a 3-regular 3-connected graph whose line graph is not Hamiltonian. (Hint: Replace each vertex in the Petersen graph with an appropriate graph and apply Exercise 7.2.10.)

7.2.12. Determine whether the graph below is Hamiltonian.

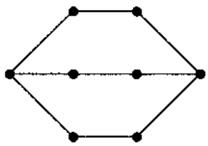


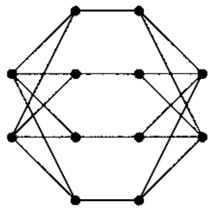
7.2.13. Let G be the 3-regular graph obtained from the Petersen graph by replacing one vertex with a triangle, matching the vertices of the triangle to the former neighbors of the deleted vertex. Prove that G is not Hamiltonian. (Comment: Except for this graph and the Petersen graph, every 2-connected, k-regular graph with at most $3k+3$ vertices is Hamiltonian.) (Hilbig [1986])

7.2.14. A graph G is **uniquely** k-edge-colorable if all proper k-edge-colorings of G induce the same partition of the edges. Prove that every uniquely 3-edge-colorable 3-regular graph is Hamiltonian. (Greenwell–Kronk [1973])

7.2.15. Place n points around a circle. Let G_n be the 4-regular graph obtained by joining each point to the nearest two points in each direction. If $n \geq 5$, prove that G_n is the union of two Hamiltonian cycles.

7.2.16. For $k \geq 3$, let G_k be the graph obtained from two disjoint copies of $K_{k,k-2}$ by adding a matching between the two "partite sets" of size k. Determine all values of k such that G_k is Hamiltonian.





7.2.17. (!) Prove that the cartesian product of two Hamiltonian graphs is Hamiltonian. Conclude that the k-dimensional cube Q_k is Hamiltonian for $k \geq 2$.

7.2.18. Prove that the cartesian product of two graphs with Hamiltonian paths fails to have a Hamiltonian cycle if and only if both graphs are bipartite and have odd order, in which case the product has a Hamiltonian path.

7.2.19. (+) For each odd natural number k, construct a $k-1$-connected k-regular simple bipartite graph that is not Hamiltonian.

7.2.20. (!) The kth **power** of a simple graph G is the simple graph G^k with vertex set $V(G)$ and edge set $\{uv\colon d_G(u, v) \leq k\}$.

a) Suppose that $G - x$ has at least three nontrivial components in each of which x has exactly one neighbor. Prove that G^2 is not Hamiltonian. (Hint: Consider the second graph in Example 7.2.5.)

b) Prove that the cube of each connected graph (with at least three vertices) is Hamiltonian. (Hint: Reduce this to the special case of trees, and prove it for trees by proving the stronger result that if xy is an edge of the tree T, then T^3 has a Hamiltonian cycle using the edge xy. Comment: Fleischner [1974] proved that the square of each 2-connected graph is Hamiltonian.)

7.2.21. Let $n = k(2l+1)$. Construct a non-Hamiltonian complete k-partite graph with n vertices and minimum degree $\frac{n}{2}\frac{k-1}{k}\frac{2l}{2l+1}$. (Snevily)

7.2.22. Let $\mathbf{G}(k, t)$ be the class of connected k-partite graphs in which each partite set has size t and each subgraph induced by two partite sets is a matching of size t. For $k \geq 4$ and $t \geq 4$, construct a graph in $\mathbf{G}(k, t)$ that is not Hamiltonian. (Hint: There is a graph in $\mathbf{G}(4, 4)$ with a 3-set whose deletion leaves four components; generalize this example. Comment: $\mathbf{G}(3, t) = \{C_{3t}\}$, and also every graph in $\mathbf{G}(k, 3)$ is Hamiltonian.) (Ayel [1982])

7.2.23. (∗) Prove that the Petersen graph has toughness 4/3.

7.2.24. (∗) Let $t(G)$ denote the toughness of G.

a) Prove that $t(G) \le \kappa(G)/2$. (Chvátal [1973])

b) Prove that equality holds in part (a) for claw-free graphs. (Hint: Consider a set S such that $|S| = t(G) \cdot c(G - S)$.) (Matthews–Sumner [1984])

7.2.25. (!) Let G be a simple graph that is not a forest and has girth at least 5. Prove that $\overline{G}$ is Hamiltonian. (Hint: Use Ore's condition.) (N. Graham)

7.2.26. (!) Prove that if G fails Chvátal's condition, then $\overline{G}$ has at least $n - 2$ edges. Conclude from this that the maximum number of edges in a simple non-Hamiltonian n-vertex graph is $\binom{n-1}{2} + 1$. (Ore [1961], Bondy [1972b])

7.2.27. Prove directly by induction on n that the maximum number of edges in a simple non-Hamiltonian n-vertex graph is $\binom{n-1}{2} + 1$.

7.2.28. *Generalization of the edge bound.*

a) Let $f(i) = 2i^2 - i + (n-i)(n-i-1)$, and suppose that $n \ge 6k$. Prove that on the interval $k \le i \le n/2$, the maximum value of $f(i)$ is $f(k)$.

b) Let G be a simple graph with minimum degree k. Use part (a) and Chvátal's condition to prove that if G has at least $6k$ vertices and has more than $\binom{n(G)-k}{2} + k^2$ edges, then G is Hamiltonian. (Erdős [1962])

7.2.29. (!) Let G be a simple graph with vertex degrees $d_1 \le \cdots \le d_n$, and let $d'_1 \le \cdots \le d'_n$ be the vertex degrees in $\overline{G}$. Prove that if $d_i \ge d'_i$ for all $i \le n/2$, then G has a Hamiltonian path. Conclude that every simple graph isomorphic to its complement has a Hamiltonian path. (Clapham [1974])

7.2.30. Obtain Lemma 7.2.9 (sufficiency of Ore's condition) from Theorem 7.2.13 (sufficiency of Chvátal's condition). (Bondy [1978])

7.2.31. (!) Prove or disprove: If G is a simple graph with at least three vertices, and G has at least $\alpha(G)$ vertices of degree $n(G) - 1$, then G is Hamiltonian.

7.2.32. (+) Suppose that n is even and G is a simple bipartite graph with partite sets X, Y of size $n/2$. Let the vertex degrees of G be $d_1, \ldots, d_n$. Let G' be the supergraph of G obtained by adding edges so that $G[Y] = K_{n/2}$.

a) Prove that G is Hamiltonian if and only if G' is Hamiltonian, and describe the relationship between the degree sequences of G and G'.

b) Suppose that $d_k > k$ or $d_{n/2} > n/2 - k$ whenever $k \le n/4$. Prove that G is Hamiltonian. (Hint: Assume that the degree sequence of G' fails Chvátal's condition for some $i < n/2$, and obtain a contradiction.) (Chvátal [1972])

7.2.33. (!) A graph is **Hamiltonian-connected** if for every pair of vertices u, v there is a Hamiltonian path from u to v. Prove that a simple graph G is Hamiltonian if $e(G) \ge \binom{n(G)-1}{2} + 2$ and Hamiltonian-connected if $e(G) \ge \binom{n(G)-1}{2} + 3$. (Proving the two together permits a simpler proof.) (Ore [1963])

7.2.34. *Necessary condition for Hamiltonian-connected.* (Moon [1965a])

a) Prove that every Hamiltonian-connected graph G with at least four vertices has at least $\lceil 3n(G)/2 \rceil$ edges.

b) Prove that the bound in part (a) is best possible by showing that $C_m \square K_2$ is Hamiltonian-connected if m is odd.

7.2.35. (!) *Sufficient condition for Hamiltonian-connected.* (Ore [1963])

a) Prove that a simple graph G is Hamiltonian-connected if $x \not\leftrightarrow y$ implies $d(x) +$

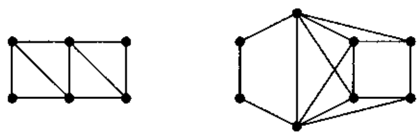
$d(y) > n(G)$. (Hint: Prove that appropriate graphs related to G are Hamiltonian by considering their closures.)

b) Prove that part (a) is sharp by constructing, for each even n greater than 2, a simple n-vertex graph with minimum degree $n/2$ that is not Hamiltonian-connected.

7.2.36. Las Vergnas' condition for a simple n-vertex graph is the existence of a vertex ordering $v_1, \dots, v_n$ such that there is no nonadjacent pair v_i, v_j satisfying $i < j$, $d(v_i) \le i$, $d(v_j) < j$, $d(v_i) + d(v_j) < n$, and $i + j \ge n$. Las Vergnas [1971] proved that this condition is sufficient for the existence of a spanning cycle.

a) Prove that Chvátal's condition (Theorem 7.2.13) implies Las Vergnas' condition, which means that Las Vergnas' theorem strengthens Chvátal's theorem.

b) Prove that each of the graphs below fails Chvátal's condition but has a complete graph as its Hamiltonian closure. Prove that the smaller graph satisfies Las Vergnas' condition but the larger one does not.



7.2.37. For $\emptyset \ne S \subset V(G)$, let $t(S) = |\overline{S} \cap N(S)|/|\overline{S}|$. Let $\theta(G) = \min t(S)$. Lu [1994] proved that if $\theta(G)n(G) \ge \alpha(G)$, then G is Hamiltonian. Prove that $\kappa(G) \ge \alpha(G)$ implies $\theta(G)n(G) \ge \alpha(G)$. (Comment: This shows that Lu's theorem implies the Chvátal–Erdős Theorem and is a stronger result.)

7.2.38. (!) *Long paths and cycles.* Let G be a connected simple graph with $\delta(G) = k \ge 2$ and $n(G) > 2k$.

a) Let P be a maximal path in G (not a subgraph of any longer path). If $n(P) \le 2k$, prove that the induced subgraph $G[V(P)]$ has a spanning cycle (this cycle need not have its vertices in the same order as P).

b) Use part (a) to prove that G has a path with at least $2k + 1$ vertices. Give an example for each odd value of n to show that G need not have a cycle with more than $k + 1$ vertices.

7.2.39. Prove that if a simple graph G has degree sequence $d_1 \le \cdots \le d_n$ and $d_1 + d_2 < n$, then G has a path of length at least $d_1 + d_2 + 1$ unless G is the join of $n - (d_1 + 1)$ isolated vertices with a graph on $d_1 + 1$ vertices or $G = pK_{d_1} \vee K_1$ for some $p \ge 3$. (Ore [1967b])

7.2.40. (!) Dirac [1952b] proved that every 2-connected simple graph G has a cycle of length at least $\min\{n(G), 2\delta(G)\}$. Use this to prove that every $2k$-regular graph with $4k + 1$ vertices is Hamiltonian. (Nash-Williams)

7.2.41. Scott Smith conjectured that any two longest cycles in a k-connected graph have at least k common vertices. The approach below works for small k.

a) Suppose that G is a 4-regular graph with n vertices that is the union of two cycles (multiple edges may arise). Let G' be the 4-regular graph on $n+2$ vertices obtained from G by subdividing two edges and adding a double edge between the two new vertices. Show that G' is also the union of two spanning cycles if $n \le 5$.

b) Use part (a) to conclude that any pair of longest cycles in a k-connected graph intersect in at least k points if $k \le 6$. (Smith, Burr)

7.2.42. (+) Let G be an Eulerian graph. Let V' be the set of Eulerian circuits of G, considering a circuit and its reversal to be the same. Let G' be the graph with vertex set V' such that two circuits are adjacent if and only if one arises from the other by reversing the edge order on a proper closed subcircuit. Prove that G' is Hamiltonian if $\Delta(G) \le 4$. (Hint: Use induction on the number of vertices of degree 4, proving that there is a Hamiltonian cycle through every edge of G'. Comment: The conclusion also holds without restriction on $\Delta(G)$.) (Xia [1982], Zhang–Guo [1986])

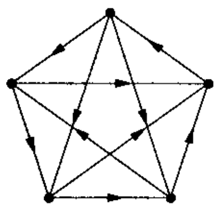
7.2.43. Prove that the Eulerian circuit graph G' of Exercise 7.2.42 is regular, and derive a formula for its vertex degree. Compare $\delta(G')$ and $n(G')$ when $n(G) = 2$ to show that the preceding problem cannot be solved by applying general results on Hamiltonicity of regular graphs with specified degree.

7.2.44. Prove that every tournament has a Hamiltonian path (a spanning directed path). (Hint: Use extremality). (Rédei [1934])

7.2.45. Let T be a strong tournament. For each $u \in V(T)$ and each k such that $3 \le k \le n$, prove that u belongs to a cycle of length k in T. (Hint: Use induction on k.) (Moon [1966])

7.2.46. Let G be a 7-vertex tournament in which every vertex has outdegree 3. Use Exercise 7.2.45 to prove that G has two vertex-disjoint cycles.

7.2.47. (+) Prove that every tournament has a Hamiltonian path that is not contained in a Hamiltonian cycle, except the cyclic tournament on three vertices and the tournament T_5 on five vertices drawn below. (Hint: Induction works, but some care is needed to prove the claim for six vertices. In all cases, find the desired configuration or $G = T_5$.) (Grünbaum, in Harary [1969, p211])



7.2.48. (∗) Prove that Theorem 7.2.22 is best possible by showing that the strictness condition on the digraph cannot be weakened to allow loops. In particular, construct for each even n an n-vertex digraph D that is not Hamiltonian even though at most one copy of each ordered pair is an edge and $\min\{\delta^-(D), \delta^+(D)\} \ge n/2$.

7.2.49. (∗) Obtain Theorem 7.2.8 (sufficiency of Dirac's condition in graphs) from Theorem 7.2.22 (sufficiency of Ghouilà-Houri's condition on digraphs). (Hint: Transform a simple graph G into a strict digraph by replacing each edge with a pair of directed edges in opposite directions.)

7.3. Planarity, Colorings, and Cycles

We return to the Four Color Problem to explore its historical relationship with the problems of edge-coloring and Hamiltonian cycles. We then consider ways in which the problem generalizes.

TAIT'S THEOREM

In 1878, Tait proved a theorem relating face-coloring and edge-coloring of plane graphs, and he used this in an approach to the Four Color Theorem. This stimulated interest in edge-coloring. We first define face-coloring precisely.

7.3.1. Definition. A **proper face-coloring** of a 2-edge-connected plane graph is an assignment of colors to its faces so that faces having a common edge in their boundaries have distinct colors.

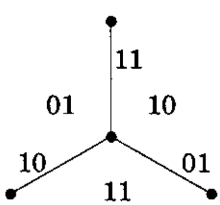
We often think of a face-coloring as a coloring of the dual graph. For this reason, we restrict our attention to face-colorings of 2-edge-connected graphs. When a plane graph has a cut-edge, its dual has a loop. We say that graphs with loops do not have proper colorings. In a plane graph with a cut-edge, a face shares a boundary with itself and is thus uncolorable.

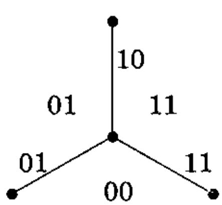
Since adding edges does not make ordinary coloring easier, to prove the Four Color Theorem it suffices to prove that all triangulations are 4-colorable. Equivalently, we could show that all duals of triangulations are 4-face-colorable. The dual G^* of a plane triangulation G is a 3-regular, 2-edge-connected plane graph (Exercise 6.1.11). Tait showed that for such graphs, proper 4-face-colorings are equivalent to proper 3-edge-colorings.

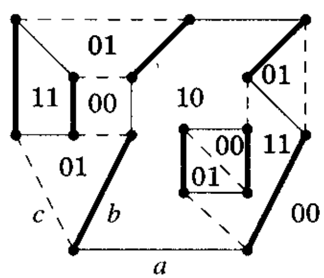
7.3.2. Theorem. (Tait [1878]) A simple 2-edge-connected 3-regular plane graph is 3-edge-colorable if and only if it is 4-face-colorable.

Proof: Let G be such a graph. Suppose first that G is 4-face-colorable; we obtain a 3-edge-coloring. Let the four colors be denoted by binary ordered pairs: $c_0 = 00$, $c_1 = 01$, $c_2 = 10$, $c_3 = 11$. Color $E(G)$ by assigning to the edge between faces with colors c_i and c_j the color obtained by adding c_i and c_j coordinatewise using addition modulo 2. (Thus $c_2 + c_3 = c_1$, for example.) We show that this is a proper 3-edge-coloring.

Because G is 2-edge-connected, each edge bounds two distinct faces. Hence the color 00 never occurs as a sum. We check that the edges at a vertex receive distinct colors. At vertex v the faces bordering the three incident edges must have distinct colors $\{c_i, c_j, c_k\}$, as illustrated below. If color 00 is not in this set, then the sum of any two of these is the third, and hence $\{c_i, c_j, c_k\}$ is also the set of colors on the edges. If $c_k = 00$, then c_i and c_j appear on two of the edges, and the third receives color $c_i + c_j$, which is the color not in $\{c_i, c_j, c_k\}$.





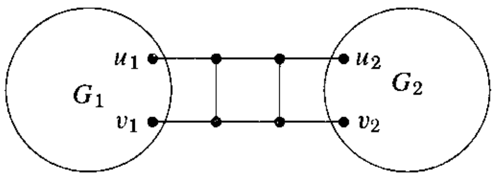


For the converse, suppose that G has a proper 3-edge-coloring using colors a, b, c (shown bold, solid, and dashed). Let E_a, E_b, E_c be the edge sets having the three colors, respectively. We construct a 4-face-coloring using the four colors defined above. Since G is 3-regular, each color appears at every vertex, and the union of any two of E_a, E_b, E_c is 2-regular, which makes it a union of disjoint cycles. Each face of this subgraph is a union of faces of the original graph. Let $H_1 = E_a \cup E_b$ and $H_2 = E_b \cup E_c$. To each face of G, assign the color whose ith coordinate ($i \in \{1, 2\}$) is the parity of the number of cycles in H_i that contain it (0 for even, 1 for odd).

We claim that this is a proper 4-face-coloring, as illustrated above. Faces F, F' sharing an edge e are distinct faces, since G is 2-edge-connected. Edge e belongs to a cycle C in at least one of H_1, H_2 (in both if e has color b). By the Jordan Curve Theorem, one of F, F' is inside C and the other is outside. All other cycles in H_1 and H_2 fail to separate F and F', leaving them on the same side. Hence if e has color a, c, or b, then the parity of the number of cycles containing F and F' is different in H_1, in H_2, or in *both*, respectively. Thus F and F' receive different colors in the face-coloring we have constructed. ■

Due to this theorem, a proper 3-edge-coloring of a 3-regular graph is called a **Tait coloring**. The problem of showing that every 2-edge-connected 3-regular planar graph is 3-edge-colorable reduces to showing that every 3-connected 3-regular planar graph is 3-edge-colorable.

7.3.3.* Lemma. If G is a 3-regular graph with edge-connectivity 2, then G has subgraphs G_1, G_2 and vertices $u_1, v_1 \in V(G_1)$ and $u_2, v_2 \in V(G_2)$ such that $u_1 \not\leftrightarrow v_1$, also $u_2 \not\leftrightarrow v_2$, and G consists of G_1, G_2 and a *ladder* of some length joining G_1, G_2 at u_1, v_1, u_2, v_2 as shown below.



Proof: If G has an edge cut of size 2 in which the two edges are incident, then the third edge incident to their common vertex is a cut-edge, contradicting $\kappa' = 2$. Hence we may assume that the four endpoints in our minimum edge cut xy, uv are distinct. If $x \not\leftrightarrow y$ and $u \not\leftrightarrow v$, then these are the four desired vertices and the ladder has only these two edges.

When $x \leftrightarrow y$, we extend the ladder (a similar argument applies when $u \leftrightarrow v$). Let w be the third neighbor of x and z the third neighbor of y. If $w = z$, then the third edge incident to this vertex is a cut-edge. Hence $w \neq z$ and the ladder extends. If $w \not\leftrightarrow z$, then we are finished in this direction; otherwise, we repeat the argument till we obtain a nonadjacent pair at the base of the ladder. ■

7.3.4.* Theorem. All 2-edge-connected 3-regular simple planar graphs are 3-edge-colorable if and only if all 3-connected 3-regular simple planar graphs are 3-edge-colorable.

Proof: The second family is contained in the first. Hence it suffices to show that 3-edge-colorability for all graphs in the smaller family implies it also for the larger family. We use induction on $n(G)$.

Basis step ($n(G) = 4$): The only 2-edge-connected 3-regular simple planar graph with at most 4 vertices is K_4, which is 3-edge-colorable.

Induction step ($n(G) > 4$): Since $\kappa(G) = \kappa'(G)$ when G is 3-regular (Theorem 4.1.11), we may restrict our attention to 3-regular graphs with edge-connectivity 2. Lemma 7.3.3 gives us a decomposition of G into G_1, G_2, and a ladder joining them. The *length* of the ladder is the distance from G_1 to G_2.

Both $G_1+u_1v_1$ and $G_2+u_2v_2$ are 2-edge-connected and 3-regular. By the induction hypothesis, they are 3-edge-colorable; let f_i be a proper 3-edge-coloring of $G_i + u_iv_i$. Permute names of colors so that $f_1(u_1v_1) = 1$ and so that $f_2(u_2v_2)$ is chosen from $\{1, 2\}$ to have the same parity as the length of the ladder.

Returning to G, color each G_i as in f_i. Beginning from the end of the ladder at G_1, color the rungs of the ladder with 3, and color the paths forming the sides of the ladder alternately with 1 and 2. The edges of the ladder at u_i and v_i now have the color $f_i(u_iv_i)$. Thus we have assembled a proper 3-edge-coloring of G. ∎

Thus the Four Color Theorem reduces to finding Tait colorings of 3-edge-connected 3-regular planar graphs. The statement of their existence was known as **Tait's conjecture** and is equivalent to the Four Color Theorem.

GRINBERG'S THEOREM

Every Hamiltonian 3-regular graph has a Tait coloring (Exercise 1). Tait believed that this completed a proof of the Four Color Theorem, because he assumed that every 3-connected 3-regular planar graph is Hamiltonian. Not until 1946 was an explicit counterexample found, although the gap in the proof was noticed earlier. Later, Grinberg [1968] discovered a simple necessary condition that led to many 3-regular 3-connected non-Hamiltonian planar graphs, including the Grinberg graph of Exercise 16.

7.3.5. Theorem. (Grinberg [1968]) If G is a loopless plane graph having a Hamiltonian cycle C, and G has f_i' faces of length i inside C and f_i'' faces of length i outside C, then $\sum_i (i-2)(f_i' - f_i'') = 0$.

Proof: Considering the faces inside and outside C separately, we want to show that $\sum_i (i-2)f_i' = \sum_i (i-2)f_i''$. No changes on one side affect the sum on the other side. Furthermore, we can switch inside and outside by projecting the embedding onto a sphere and puncturing a face inside C.

Hence we need only show that $\sum_i (i-2)f_i'$ is constant. When there are no inside edges, the sum is $n-2$. With this as the basis step, we prove by induction on the number of inside edges that the sum is always $n-2$.

Suppose that $\sum_i (i-2)f_i' = n-2$ when there are k edges inside C. We can obtain any graph with $k+1$ edges inside C by adding an edge to such a graph.

The added edge cuts a face of some length r into two faces of lengths s and t. We have $s + t = r + 2$, because the new edge contributes to both new faces and each edge on the old face contributes to one new face.

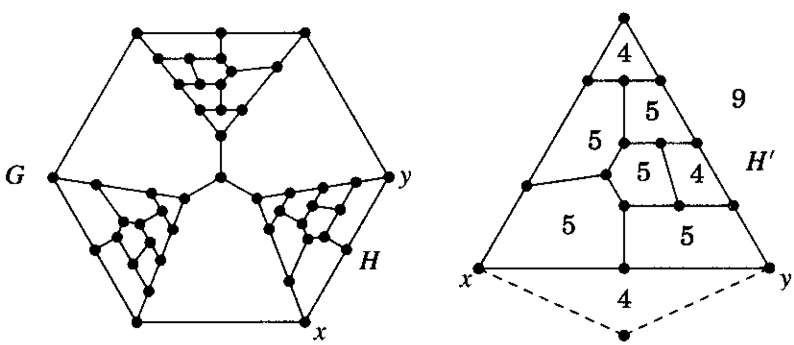
No other contribution to the sum changes. Since $(s-2)+(t-2)=(r-2)$, the contribution from these faces also remains the same. By the induction hypothesis, the sum is $n-2$. ■

Being a necessary condition, Grinberg's condition can be used to show that graphs are *not* Hamiltonian. The arguments can often be simplified using modular arithmetic. Two numbers that are not congruent mod k are not equal.

We apply this to the first known non-Hamiltonian 3-connected 3-regular planar graph (Tutte [1946]). Tutte used an *ad hoc* argument to prove that this graph is not Hamiltonian. For many years it was the only known example (see Exercise 17 for the smallest now known).

7.3.6. Example. *Grinberg's condition and the Tutte graph.* The Tutte graph G appears on the left below. Let H denote each component obtained by deleting the central vertex and the three long edges. Since a Hamiltonian cycle must visit the central vertex of G, it must traverse one copy of H along a Hamiltonian path joining the other entrances to H, which we call x and y.

We therefore study a graph that has a Hamiltonian cycle if and only if H has a Hamiltonian x, y-path. Such a graph H' (on the right below) is obtained by adding an x, y-path of length two through a new vertex.



The plane graph H' has five 5-faces, three 4-faces, and one 9-face. Grinberg's condition becomes $2a_4 + 3a_5 + 7a_9 = 0$, where $a_i = f_i' - f_i''$. Since the unbounded face is always outside, the equation reduces mod 3 to $2a_4 \equiv 7 \bmod 3$. Since $f_4' + f_4'' = 3$, the possibilities for a_4 are $+3, +1, -1, -3$. The only choice satisfying $2a_4 \equiv 7 \bmod 3$ is $a_4 = -1$, which requires that two of the 4-faces lie outside the Hamiltonian cycle. However, the 4-faces having having a vertex of degree 2 cannot lie outside the cycle, since the edges incident to the vertex of degree 2 separate the face from the outside face.

We can reach a contradiction faster by subdividing one edge incident to each vertex of degree 2. This does not change the existence of a spanning cycle. The resulting graph has seven 5-faces, one 4-face, and one 11-face. The required equation becomes $2 \cdot (\pm 1) = 9 - 3a_5$, which has no solution since the left side is not a multiple of 3. ■

We have not presented a systematic procedure for proving the nonexistence of solutions to equations with integer variables. Our arguments involving divisibility are merely tricks to avoid listing cases, but such tricks often work.

High connectivity makes it harder to avoid spanning cycles. Tutte [1956] (extended by Thomassen [1983]) proved that every 4-connected planar graph is Hamiltonian. Barnette [1969] conjectured that every planar 3-connected 3-regular bipartite graph is Hamiltonian.

SNARKS (optional)

Another approach to the Four Color Theorem is to study which 3-regular graphs are 3-edge-colorable. In a discussion focusing on 3-regular graphs and graphs without cut-edges, it is convenient to have simple adjectives to describe these properties.

7.3.7. Definition. A **bridgeless graph** is a graph without cut-edges. A **cubic graph** is a graph that is regular of degree 3.

7.3.8. Conjecture. (3-edge-coloring Conjecture—Tutte [1967]) Every bridgeless cubic non-3-edge-colorable graph contains a subdivision of the Petersen graph.

Conjecture 7.3.8 has been proved! Like the Four Color Theorem, its computer-assisted proof uses discharging methods. The proof will appear in a series of five papers by Robertson, Sanders, Seymour, and Thomas [2001].

Since every subdivision of the Petersen graph is nonplanar, Conjecture 7.3.8 implies Tait's Conjecture and hence the Four Color Theorem. One natural approach to the conjecture, like the idea of reducibility for the Four Color Theorem, is to derive properties that a minimal counterexample must have. In this language, Theorem 7.3.4 says that a minimal counterexample must be 3-edge-connected. In the next lemma, we make this statement precise and obtain several other properties.

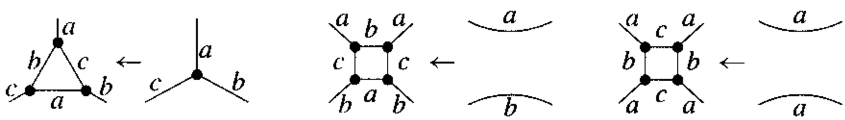
7.3.9. Definition. A **trivial edge cut** is an edge cut whose deletion isolates a single vertex. Other edge cuts are **nontrivial**.

7.3.10. Lemma. If a non-3-edge-colorable cubic graph G has connectivity 2 or girth less than 4 or a nontrivial 3-edge cut, then G contains a subdivision of a smaller non-3-edge-colorable cubic graph.

Proof: Suppose first that G has an edge cut of size 2. As discussed in Lemma 7.3.3, these edges have no common vertices. Deleting the edge cut and adding one edge to each piece yields cubic graphs $G_1 + u_1v_1$ and $G_2 + u_2v_2$. As argued in Theorem 7.3.4, at least one of these graphs is not 3-edge-colorable. Since the added edge can be replaced by a path through the other piece, G contains a subdivision of this smaller non-3-edge-colorable graph.

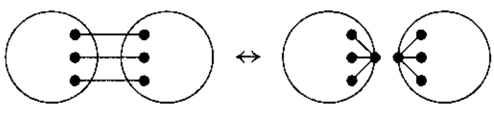
Next suppose that G contains a triangle. Let G' be the graph obtained from G by contracting the triangle to a single vertex. A proper 3-edge-coloring of G' could be expanded into a proper 3-edge-coloring of G as shown below. Also, G contains a subdivision of G', obtained by deleting one edge of the triangle.

Suppose that G contains a 4-cycle but no triangle. Let G' be the cubic graph obtained from G by deleting two opposite edges of the 4-cycle and replacing the resulting paths of length 3 with single edges. Since G has no triangle, the new edges are not loops. A proper 3-edge-coloring of G' yields a proper 3-edge-coloring of G via the two cases shown below. Also G contains a subdivision of G', so G' is the desired smaller graph.



Finally, suppose that G contains a nontrivial 3-edge cut $[S, \overline{S}]$. Since we may assume that G is 3-edge-connected, the three edges of the cut are pairwise disjoint. The two graphs obtained by contracting $G[S]$ or $G[\overline{S}]$ to a single vertex are also 3-regular. If both are 3-edge-colorable, then the colors can be renamed to agree on the edges of the cut, yielding a proper 3-edge-coloring of G. Thus at least one of these graphs is not 3-edge-colorable.

It remains only to show that G contains a subdivision of $G[S]$ (and similarly of $G[\overline{S}]$). Let a, b, c be the endpoints in $\overline{S}$ of the edges in the cut. Since G is 3-edge-connected, the cut is a bond, and $G[\overline{S}]$ is connected (Proposition 4.1.15). Thus $G[\overline{S}]$ contains an a, b-path P and a path from c to P. Adding these paths and the edges of the cut to $G[S]$ completes a subdivision of $G[S]$. ■



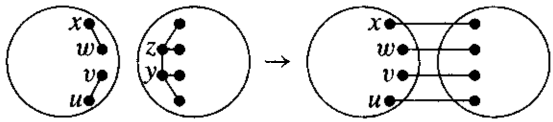
7.3.11. Definition. A **snark** is a 2-edge-connected 3-regular graph that is not 3-colorable, has girth at least 5, and has no non-trivial 3-edge cut. A **prime snark** is one that contains no subdivision of a smaller snark.

In this language, we have reduced Tutte's 3-edge-coloring Conjecture to the statement that the Petersen graph is the only prime snark. Again, we note that the conjecture has been proved (Robertson–Sanders–Seymour–Thomas [2001]).

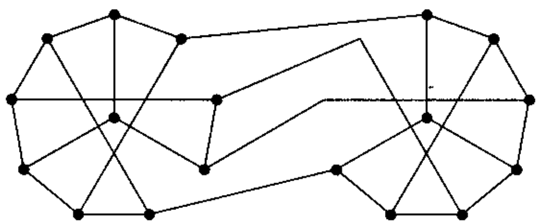
After the Petersen graph in 1898, by 1975 only three more snarks had been found: the 18-vertex Blanuša [1946] snark, the 210-vertex Descartes [1948] snark, and the 50-vertex Szekeres [1973] snark. This prompted Martin Gardner [1976] to invent the term "snark", evoking the rarity of the creature in Lewis Carroll's "The Hunting of the Snark".

Isaacs [1975] then showed that the earlier snarks arise from the Petersen graph via an operation that generates infinite families of snarks.

7.3.12. Definition. The **dot product** of cubic graphs G and H is the cubic graph formed from $G + H$ by deleting disjoint edges uv and wx from G, deleting adjacent vertices y and z from H, and adding edges from u and v to $N_H(y) - \{z\}$ and from w and x to $N_H(z) - \{y\}$.

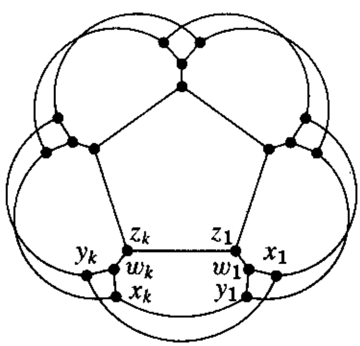


The dot product of two snarks is a snark (Exercise 23). Applying it to two copies of the Petersen graph yields the Blanuša snark shown below. This graph has a non-trivial 4-edge cut. Kochol [1996] introduced a more general operation that yields snarks with large girth and higher connectedness properties.



7.3.13. Example. *The flower snarks.* Isaacs also found an explicit infinite family of snarks (Exercise 21) that don't arise via the dot product. Independently discovered by Grinberg, they have $4k$ vertices, for odd $k \geq 5$.

Begin with three disjoint k-cycles. Let $\{x_i\}$, $\{y_i\}$, $\{z_i\}$ be the three vertex sets, indexed cyclically. For each i add a vertex w_i with $N(w_i) = \{x_i, y_i, z_i\}$. The resulting graph G_k is 3-edge-colorable. Let H_k be the graph obtained by replacing the edges x_kx_1 and y_ky_1 with x_ky_1 and y_kx_1. If k is odd and $k \geq 5$, then H_k is a snark. If k is even, then H_k is 3-edge-colorable. The drawing of H_k in which $\{z_i\}$ is a central cycle suggests the name "flower snark". ∎



FLOWS AND CYCLE COVERS (optional)

Tait's Theorem (Theorem 7.3.2) states that 3-edge-colorability and 4-face-colorability are equivalent for plane triangulations. When extending this beyond planar graphs, we need a concept that makes sense for all graphs and is equivalent to 4-face-coloring on plane graphs. Additional information about this topic (and about snarks) appears in the monograph by Zhang [1997].

7.3.14. Definition. A **flow** on a graph G is a pair (D, f) such that
1) D is an orientation of G,
2) f is a weight function on $E(G)$, and
3) each $v \in V(G)$ satisfies $\sum_{w \in N_D^+(v)} f(vw) = \sum_{u \in N_D^-(v)} f(uv)$.

A k-**flow** is an integer-valued flow such that $|f(e)| \le k-1$ for all $e \in E(G)$. A flow is **nowhere-zero** or **positive** if $f(e)$ is nonzero or positive, respectively, for all $e \in E(G)$.

The usage of "flow" here is somewhat different from that in Chapter 4. In both contexts, the word "flow" suggests the conservation constraints imposed at each vertex. The bound of $k-1$ on flow value evokes the notion of capacity.

We can alter the orientation to make all weights positive.

7.3.15. Proposition. For a graph G, the following are equivalent:
A) G has a positive k-flow.
B) G has a nowhere-zero k-flow.
C) G has a nowhere-zero k-flow for each orientation of G.

Proof: Simultaneously changing the orientation of an edge and the sign of its weight does not affect the conservation constraints. ■

Thus the existence of a nowhere-zero k-flow does not depend on the choice of the orientation. We can also take linear combinations of flows.

7.3.16. Proposition. If $(D, f_1), \ldots, (D, f_r)$ are flows on G, and $g = \sum_{i=1}^{r} \alpha_i f_i$, then (D, g) is a flow on G.

Proof: For each $v \in V(G)$, the net flow out of v under each f_i is zero, and hence it is also zero under g. ■

7.3.17. Proposition. For a flow on G, the net flow out of any set $S \subseteq V(G)$ is zero. Thus a graph with a nowhere-zero flow has no cut-edge.

Proof: We sum the net flows out of vertices of S. Edges leaving S contribute with positive weight, edges entering S contribute with negative weight, and edges within S contribute positively at their tails and negatively at their heads. The net flow out of S is thus the sum of the net flows out of the vertices of S, which is zero.

This implies that the net flow across any edge cut is zero, so it cannot consist of a single edge with nonzero weight. ■

Thus we restrict our attention to graphs without cut-edges (bridgeless graphs). What distinguishes flows here from circulations in Section 4.3 is that we forbid zero as a weight. Nowhere-zero flows enable us to extend Tait's Theorem. We begin by interpreting Eulerian graphs in the context of nowhere-zero flows; connectedness is no longer important.

7.3.18. Definition. A graph is an **even graph** if every vertex has even degree.

7.3.19. Proposition. A graph has a nowhere-zero 2-flow if and only if it is an even graph.

Proof: Given a nowhere-zero 2-flow, we obtain a positive 2-flow. Since this assigns weight 1 to every edge, the orientation must have as many edges entering each vertex as leaving it. Thus each vertex degree is even.

Conversely, when each vertex degree is even, each component has an Eulerian circuit. Orienting the edges to follow such a circuit and assigning weight 1 to each edge yields a positive 2-flow. ∎

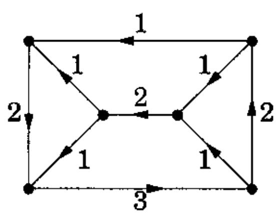
Nowhere-zero 3-flows are more subtle, even for 3-regular graphs.

7.3.20. Proposition. (Tutte [1949]) A cubic graph has a nowhere-zero 3-flow if and only if it is bipartite.

Proof: Let G be a cubic X, Y-bigraph. Every regular bipartite graph has a 1-factor. Orient the edges of a 1-factor from X to Y, and give them weight 2. Orient all other edges from Y to X, and give them weight 1. The flow in and out of every vertex is 2, so this is a nowhere-zero 3-flow.

Conversely, let G be a cubic graph with a nowhere-zero 3-flow. By Proposition 7.3.15, we may assume that the flow is 1 or 2 on each edge. Since the net flow is 0, there must be one edge with flow 2 and two edges with flow 1 at each vertex. Thus the edges with flow 2 form a matching. The X be the set of tails and Y the set of heads of these edges. Since the net flow is 0 at each vertex, each edge with flow 2 points from X to Y, and each edge with flow 1 points from Y to X. Thus X, Y is a bipartition of G. ∎

7.3.21. Example. Since the Petersen graph is cubic and not bipartite, it has no nowhere-zero 3-flow. We will see that it also has no nowhere-zero 4-flow. Below we show a nowhere-zero 4-flow in the 3-regular simple graph $C_3 \square K_2$. ∎



To understand the duality between flows and colorings, we characterize the plane graphs with nowhere-zero k-flows.