5.2.19. Modify $\overline{K}_a + K_{n-a}$ to obtain $T_{n,r}$, counting the changes in the number of edges.

5.2.21. The proof of Theorem 5.2.9 transforms a graph with no $r+1$-clique into an r-partite graph with at least at many edges. The number of edges strictly increases unless equality holds in each step of the computation. What is needed for equality to hold?

5.2.27. For the upper bound, deleting the edges of a cycle from a counterexample leaves a forest, which restricts the girth. Reducing to the case $\delta(G) \geq 3$ now requires $n \leq 8$, which contradicts Exercise 5.2.26.

5.2.28. For the upper bound, reduce to the case $\delta(G) \geq 3$, and consider a shortest cycle C. Deleting $V(C)$ leaves a forest, and its leaves have at least three neighbors on C.

5.2.29. For part (b), if the largest and smallest color classes differ in size by more than 1, use part (a) to alter the coloring appropriately.

5.2.32. Part (a) uses the properties of k-critical graphs for G and H. In part (c), one can give explicit examples with orders 4, 6, 8 and then apply part (a).

5.2.40. To compute the chromatic number, consider independent sets. To forbid subdivisions of complete graphs, consider vertex cuts; a subdivision of K_k must have $k-1$ pairwise internally disjoint paths joining two of its vertices of degree $k-1$.

5.2.43. Use induction on k. In the induction step, discard an appropriate subset of $V(G)$.

5.2.44. Reduce to the case $\delta(G) \geq k-1$ and then use induction on k.

5.3.3. If $\chi(G;k) = k^4 - 4k^3 + 3k^2$, then how many edges and vertices does G have?

5.3.4. For part (a), use the chromatic recurrence or Theorem 5.3.10.

5.3.6. Explain how contributions to the coefficient of $k^{n(G)-1}$ arise in computing $\sum_{r=1}^{n} p_r(G)k_{(r)}$.

5.3.12. For part (a), use the chromatic recurrence. For part (b), consult Exercise 1.3.40 for the maximum number of edges in an n-vertex graph with r components.

5.3.18. Part (a) implies that part (b) needs only one computation. Expressing a chromatic polynomial as the sum of two chromatic polynomials involves addition of an edge as in Example 5.3.9.

5.3.23. Use a simplicial elimination ordering.

5.3.26. For part (a), use a simplicial elimination ordering; the simplicial vertex may or may not be in $G \cap H$. For part (b), $N(x)$ may or may not equal $V(G) - x$.

5.3.28. Build a simplicial elimination ordering of G and a transitive orientation of $\overline{G}$.

6.1.20. Which plane graphs have only one face? When a plane graph has more than one face, what kind of edge can be deleted to reduce the number of faces?

6.1.24. In the induction step, delete an edge of a cycle.

6.1.25. Use Euler's Formula.

6.1.28. This can be proved by applying Euler's Formula to an appropriate planar graph. Also it can be proved using induction.

6.1.30. Mimic the proof of Theorem 6.1.23.

6.2.6. In order to apply the claim and the induction hypothesis, find an appropriate vertex to delete from a larger graph.

6.2.7. Given the graph G to be tested, construct a graph H such that G is outerplanar if and only if H is planar. Kuratowski's Theorem then applies to H.

6.2.8. Several cases need to be considered concerning how a subdivision of K_5 might be arranged in the graph.

6.2.9. For the construction, start with $n = 5$. Reduce the upper bound to considering planar graphs with $2n$ edges that have minimum degree at least three and contain a triangle. In each case, obtain disjoint cycles or a subdivision of $K_{3,3}$.

6.2.11. Let H' be a subgraph contractible to H. Prove that if H is not all of H', then H' has an edge incident to a vertex of degree 2. Use this in an inductive proof.

6.3.5. Use the Four Color Theorem.

6.3.9. When C has length 4, replacing the inside (or outside) with a single edge between opposite vertices of C allows one to obtain a 4-coloring of a C-lobe that uses distinct colors on two opposite vertices of C.

6.3.12. For the construction, use groups of three vertices to build "alcoves" such that no guard can see into more than one alcove.

6.3.20. Show that the lower bound resulting from Proposition 6.3.10 is at least $r/2$ when $s > (r-2)^2/2$, and provide a decomposition into $r/2$ planar subgraphs.

6.3.27. Consider the copies of $K_{m,n}$ in a drawing of $K_{m+1,n}$.

6.3.28. (also next problem). Consider what happens when a vertex moves across an edge. For the second part, consider a drawing where the crossings are easy to count.

7.1.11. For necessity in part (b), show first that the average vertex degree must be 2.

7.1.16. Letting $G = L(H)$, use H to show for $S \subseteq V(G)$ that the number of components of $G - S$ is at most $|S| + 1$.

7.1.17. Extract an embedding of G from an embedding of $L(G)$.

7.1.20. Consider Definition 1.4.20.

7.1.24. It may help to consider first the special case $H = K_2$, though this need not be proved separately.

7.1.26. Every edge incident to the cut-vertex must appear in some color.

7.1.33. Show that an improvement can be made when one color appears too often and another not enough.

7.2.17. Reduce the problem to the case where the two Hamiltonian graphs are cycles.

7.2.23. For each k, determine how large S must be so that $G - S$ has k components.

7.2.29. Write the given condition involving the degrees of G and $\overline{G}$ solely in terms of the degree of G; then show that G satisfies Chvátal's condition for spanning paths.

7.2.31. Use the Chvátal–Erdős Theorem

7.2.32. Be careful about how the transformation modifies the degrees; the condition is stated correctly. (Theorem 7.2.19).

7.3.3. The dual graph is 3-regular.

7.3.4. Consider the faces inside and outside a spanning cycle separately.

7.3.5. Translate this problem into a statement proved earlier.

7.3.17. Reduce this to studying the graph analyzed in Example 7.3.6.

7.3.18. Modify the graph to obtain a situation where Grinberg's Theorem applies.

7.3.21. Given the triple of colors on $\{x_{i-1}x_i, y_{i-1}y_i, z_{i-1}z_i\}$ consider the possibilities for the triple of colors on $\{x_ix_{i+1}, y_iy_{i+1}, z_iz_{i+1}\}$.

Appendix D

Glossary of Terms

In addition to terms used in this book, this glossary also contains other related terms that the reader may encounter in further study. This includes some alternative terminology used by other authors.

The items are informal sketches of definitions. Numbers in brackets are page references for the full definition or the first usage. When used without specification, "G" indicates a graph or possibly also a digraph, "D" indicates a digraph, v or e indicate a vertex or edge, and n indicates the number of vertices.

Absorption property (matroids) [351]: $r(X) = r(X \cup e) = r(X \cup f)$ implies $r(X) = r(X \cup f \cup e)$
Acyclic [67]: without cycles
Acyclic orientation [203, 208]: orientation without cycles
Adjacency matrix A [6]: entry $a_{i,j}$ is number of edges from vertex i to vertex j
Adjacency relation: set of unordered or ordered pairs forming edges in graph or digraph
Adjacency set $N(v)$: the set of vertices adjacent to v
Adjacent [2]: vertices that are endpoints of an edge, sometimes used to describe edges with a common endpoint
Adjoins: is adjacent to
Adjugate: matrix of cofactors
Almost always [430]: having asymptotic probability 1
M-alternating path: a path alternating between edges in M and not in M
Ancestor [100]: in a rooted tree, a vertex along the path to the root
Antichain: family of pairwise incomparable items (under an order relation)
Anticlique: stable set
Antihole: induced subgraph isomorphic to the complement of a cycle
Approximation algorithm [496]: polynomial-time algorithm with bounded performance ratio
Approximation scheme [496]: family of approximation algorithms with arbitrarily good performance ratio
Arborescence: a directed forest in which every vertex has outdegree at most one
Arboricity $\Upsilon(G)$ [372]: minimum number of forests covering the edges
Arc: directed edge (ordered pair of vertices)
k-arc-connected: same as k-edge-connected for digraphs
Articulation point: a vertex whose deletion increases the number of components
Assignment Problem [126]: minimize (or maximize) the sum of the edge weights in a perfect matching of a complete bipartite graph with equal part-sizes
Asteroidal triple [346]: three distinct vertices with each pair connected by a path avoiding the neighborhood of the third

Asymmetric: having no automorphisms other than the identity
Asymptotic [431]: having ratio approaching 1
Augmentation property (matroids) [352]: $I_1, I_2 \in \mathbf{I}$ with $|I_2| > |I_1|$ implies the existence of $e \in I_2 - I_1$ such that $I_1 \cup e \in \mathbf{I}$
Augmenting path [109]: for a matching, an alternating path that can be used to increase the size of the matching; for a flow, increases the flow value
Automorphism [14]: a permutation of the vertices that preserves the adjacency relation
Automorphism group Γ: the group of automorphisms under composition
Average degree: $\sum d(v)/n(G) = 2e(G)/n(G)$.
Azuma's Inequality: a bound on the probability in the tail of a distribution

Backtracking [156]: depth-first-search
Balanced graph [434]: the full graph is the subgraph with the largest average vertex degree
Balanced k-partite: partite sets differ by at most one in size (see equipartite)
Bandwidth: the minimum, over vertex numberings by distinct integers, of the maximum difference between labels of adjacent vertices
Barycenter [78]: vertex minimizing the sum of distances to other vertices
Base (matroids) [349]: maximal independent set
Base exchange property (matroids) [351]: for all $B_1, B_2 \in \mathbf{B}$ and $e \in B_1 - B_2$, there exists an element $f \in B_2 - B_1$ such that $B_1 - e + f$ is a base.
Berge graph [340]: a graph with no odd hole or odd antihole
Best possible: fails to be true when some condition is loosened
Bicentral tree: a tree whose center is an edge
Biclique [9]: complete bipartite graph
Biconnected: 2-connected
Bigraphic [65, 185]: a pair of sequences realizable as the vertex degrees for the partite sets in a simple bipartite graph
X, Y-bigraph [24]: a bipartite graph with bipartition X, Y
Binary matrix (or vector): all entries in $\{0, 1\}$
Binary matroid [357]: representable over the field with two elements
Binary tree [101]: rooted tree in which every non-leaf vertex has at most two children
Binomial coefficient [487] $\binom{n}{k}$: the number of ways to choose a subset of size k from an n-element set, equal to $n!/[k!(n-k)!]$.
Biparticity [422]: number of bipartite subgraphs needed to partition the edges
Bipartite graph [4]: a graph whose vertices can be covered by two independent sets
Bipartite Ramsey number: for a bipartite G, the minimum n such that 2-coloring the edges of $K_{n,n}$ forces a monochromatic G
Bipartition [24]: a partition of the vertex set into two independent sets
Birkhoff diamond [259]: a particular reducible configuration for the Four Color Problem
Block [155]: (1) a maximal subgraph with no cut-vertex; (2) a graph with no cut-vertex; (3) a class in a partition of a set
Block-cutpoint graph 156: simple bipartite graph in which the partite sets are the blocks and the cutvertices of G and the adjacency relation is containment
Block graph: intersection graph of the blocks in G
Blossom [142]: an odd cycle arising in Edmonds' algorithm for general matching
Bond [154]: a minimal edge cut
Bond matroid [362]: dual of the cycle matroid of a graph
Bond space [452]: orthogonal complement to the cycle space; linear combinations of bonds (over field of two elements)
Book embedding: a decomposition of G into outerplanar graphs with a consistent ordering of the vertices (as on the spine of a book)
Bouquet: a graph consisting of one vertex and some number of loops
Branch vertex [249]: a vertex of degree at least 3
Branching: a digraph where each vertex has indegree one except one that has indegree 0
r-branching [404]: branching rooted at r
Breadth-first search [99]: a search exploring vertices in order by distance from root
Breadth-first tree: tree generated by a breadth-first search from a root

Bridge [304]: cut-edge
H-bridge of G: H-fragment (used by other authors)
Bridgeless graph [304]: graph without cut-edges
Brooks' Theorem: $\chi(G) \le \Delta(G)$ for connected graphs, except for cliques and odd cycles

Cactus [160]: a connected graph in which every edge appears in at most one cycle
(k, g)-cage [49]: a $k$regular graph of smallest order among those with girth g
Capacity [176, 178]: a limit on flow (1) through an edge in a network; (2) across a cut
Cartesian product $G_1 \square G_2$ [193]: the graph with vertex set $V(G_1) \times V(G_2)$ and edges given by $(u_1, u_2) \leftrightarrow (v_1, v_2)$ if 1) $u_1 = v_1$ and $u_2 \leftrightarrow v_2$ in G_2 or 2) $u_2 = v_2$ and $u_1 \leftrightarrow v_1$ in G_1
Caterpillar [88]: a tree with a single path containing at least one endpoint of every edge
Cayley's Formula [81]: statement there are n^{n-2} trees with vertex set $[n]$
2-cell [268]: on a surface, a region homeomorphic to a disc, meaning that every closed curve is contractible to a point
2-cell embedding [268]: an embedding in which every region is a 2-cell
Center [72]: subgraph induced by the vertices of minimum eccentricity
Central tree [78]: a tree whose center is one vertex
α, β-chain: a path alternating between colors α and β
Characteristic polynomial $\phi(G; \lambda)$ [453]: characteristic polynomial of the adjacency matrix of the graph, whose roots are the eigenvalues
Children [100]: in a rooted tree, neighbors of the current vertex that are farther from the root
Chinese Postman Problem [99]: problem of finding the cheapest closed walk covering all the edges in an edge-weighted graph
Choice number [408]: choosability
Choosability $\chi_l(G)$ [408]: minimum k such that G is k-choosable
k-choosable [408]: for all lists of size k assigned to vertices of G, there is a proper coloring that selects a color for each vertex from its list
Chord [225]: edge joining two nonconsecutive vertices of a path or cycle
Chordal graph [225]: having no chordless cycle
Chordless cycle [225]: an induced cycle of length at least 4
Chordless path: a path that is an induced subgraph
Chromatic index $\chi'(G)$ [275]: edge-chromatic number
Chromatic number $\chi(G)$ [5, 191]: minimum number of colors in a proper coloring.
Chromatic polynomial $\chi(G; k)$ [220]: a polynomial whose value at k is the number of proper colorings of G using colors from $\{1, \ldots, k\}$.
Chromatic recurrence: recurrence relation for chromatic polynomial
k-chromatic [192]: having chromatic number k
Circle graph [341]: an intersection graph of chords of a circle
Circuit [27, 60]: equivalence class of closed trails without specifying starting vertex (an even graph); (caution—used by some authors to mean *cycle*)
Circulant graph: a graph constructed as equally-spaced vertices on a circle with adjacency depending only on distance
Circular-arc graph [341]: an intersection graph of arcs of a circle
Circulation [187]: a flow in a network with net flow 0 at each vertex
Circumference [293]: the length of the longest cycle
Clause [499]: a collection of literals in a logical (Boolean) formula
Claw [12]: the graph $K_{1,3}$
Claw-free: having no induced $K_{1,3}$
Clique [4]: set of pairwise-adjacent vertices (used by many authors to mean *complete graph*)
Clique cover [226]: a set of cliques covering the vertices (minimum size = $\theta(G)$)
Clique decomposition: a partition of the edge set into complete subgraphs
Clique edge cover: a set of complete subgraphs covering the edges
Clique identification: a perfection-preserving operation that merges cliques in two graphs
Clique number $\omega(G)$: maximum order of a clique in G
Clique partition number: minimum size of a clique decomposition
Clique tree [327]: an intersection representation of a chordal graph, consisting of a host tree with a bijection between its vertices and the maximal cliques of G such that the cliques containing each vertex form a subtree of the host

Clique-vertex incidence matrix [328]: 0,1-matrix in which entry (i, j) is 1 if and only if vertex j belongs to maximal clique i
Closed ear [164]: a path between two (possibly equal) old vertices through new vertices
Closed-ear decomposition [164]: construction of a graph from a cycle by addition of closed ears
Closed neigborhood [116]: a vertex and all its neighbors
Closed set (matroids) [360]: a set whose span is itself
Closed walk [20]: a walk whose last vertex is the same as its first
Closure [289, 360]: (1) the graph $C(G)$ obtained from G by iteratively adding edges joining nonadjacent vertices with degree-sum at least $n(G)$; (2) image under a closure operator
Closure operator [360]: an operator that is expansive, order-preserving, and idempotent
Cobase [360]: a base of the dual matroid
Cocircuit [360]: a circuit of the dual matroid
Cocritical pair: two nonadjacent vertices whose addition as an edge increases the clique number
Cocycle matroid [362]: the dual of a cycle matroid
Cocycle space: bond space
Cograph [202]: P_4-free graph (equivalent to *complement reducible* graph)
Color class [191]: in a coloring, a set of objects having the same color
Color-critical [192]: a graph such that every proper subgraph has smaller chromatic number
k-colorable [191]: having a proper coloring with at most k colors
k-coloring [191, 380]: a partition into k sets
P coloring: a vertex partition into subsets inducing graphs with property **P**
Column matroid $M(A)$ [351]: matroid whose independent sets are the linearly independent subsets of columns of the matrix A
Comma-free code: no code word is a prefix of another
Common system of distinct representatives (CSDR) [171]: given families **A** and **B** of sets, a CSDR is a set of elements that is an SDR of **A** and is an SDR of **B**
Comparability graph [228]: graph having a transitive orientation
Complement $\overline{G}$[3]: simple graph or digraph with the same vertex set as G, defined by $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$
Complement reducible [344]: reducible to the trivial graph by iteratively taking complements of components
Complete graph K_n [9]: simple graph in which each two vertices are adjacent
Complete k-partite graph $K_{n_1,\dots,n_k}$ [207]: k-partite graph in which every pair of vertices not belonging to the same partite set is adjacent (sizes of the partite sets are $n_1, \dots, n_k$)
Completely labeled cell [388]: simplicial region with distinct labels on corners
Complexity [494]: the worst-case number of operations needed, as a function of the input size
Component [22]: maximal connected subgraph
S-component of G: see S-lobe
Composition $G_1[G_2]$ [332]: a graph whose vertex set is the cartesian product $V(G_1) \times V(G_2)$, defined by $(u_1, u_2) \leftrightarrow (v_1, v_2)$ if and only if $u_1 \leftrightarrow v_1$ in G_1, or $u_1 = v_1$ and $u_2 \leftrightarrow v_2$ in G_2
Conflict graph [252]: graph whose vertices are the bridges of a cycle, with bridges adjacent (conflicting) when they have three common endpoints or four alternating endpoints on the cycle
Conflicting chords: two chords whose endpoints alternate on a specified cycle
Conjugate partition: two partitions of n such that one gives the row sizes and the other the column sizes of a Ferrers diagram
Connected [6]: having a u, v-path for every pair of vertices u, v
k-connected [149, 164]: having connectivity at least k
Connection relation [21]: relation satisfied by vertices x, y if there is an x, y-path
Connectivity $\kappa(G)$ [149, 164]: the minimum number of vertices whose deletion disconnects the graph or reduces it to one vertex (sometimes called "vertex connectivity" for clarity)
Consecutive 1s property (for rows) [328]: having a permutation of columns so 1s appear consecutively in each row
Conservation constraint [176]: for a flow, the condition of net flow 0 at a vertex
Consistent rounding [186]: conversion of the data and the row/column sums in a matrix to nearest integers up or down such that row and column sums remain correct
Construction procedure: a procedure for iteratively building members of a class of graphs from a small base graph or graphs
Contraction [84]: replaces edge uv by a vertex w incident to the edges formerly incident to u or v

Converse D^{-1}: obtained from digraph D by switching the head and tail in each edge
Convex embedding [248]: a plane graph in which every bounded face is a convex set and the outer boundary is a convex polygon
Convex function [443]: satisfies the inequality $f(\lambda a + (1-\lambda)b) \leq \lambda f(a) + (1-\lambda)f(b)$ for all a, b and $0 \leq \lambda \leq 1$
Convex quadrilateral: not no corner in the triangle formed by the other three
Cost [125]: name of the objective function for many weighted minimization problems
Cotree: with respect to a graph, the edges not belonging to a given spanning tree
F-covering: covering of edge set by subgraphs in the family **F**
Critical edge [122, 339]: edge whose deletion increases the independence number
Critical graph: used with respect to many graph properties, indicating that the deletion of any vertex (or edge, depending on context) destroys the property
k-critical graph [192]: usually means color-critical with chromatic number k
Critically 2-connected: deletion of an edge destroys 2-connectedness
Crossing [234]: in a drawing of a graph, an internal intersection of two edges
Crossing number $\nu(G)$ [262]: minimum number of crossings when drawing G in the plane
k-cube Q_k [36]: k-dimensional cube
Cubic graph [304]: a regular graph of degree 3
Cut $[S, \overline{S}]$ [166]: the edges from a vertex subset to its complement (especially in networks)
Cut-edge [23]: an edge whose deletion increases the number of components
Cutset: a separating set of vertices
Cut-vertex [23]: vertex whose deletion increases the number of components
Cycle [5, 55]: a simple graph whose vertices can be placed on a circle so that vertices are adjacent if and only if they appear consecutively on the circle (caution—used by some authors to mean *even graph*)
Cycle double cover [312]: a list of cycles such that each edge appears in two items in the list
k-cycle [9]: a cycle of length k, consisting of k vertices and k edges
Cycle matroid $M(G)$ [350]: the matroid whose circuits are the cycles of G
Cycle rank: dimension of cycle space, equal to #edges − #vertices + #components
Cycle space [452]: the nullspace of the incidence matrix; the elements correspond to the even subgraphs
Cyclic edge-connectivity: number of edges that must be deleted to disconnect a component so that every remaining component contains a cycle
Cyclically k-edge-connected: cyclic edge-connectivity at least k

de Bruijn graph [61]: digraph encoding possible transitions between k-ary n-tuples as additional characters are received
Decision problem [494]: a computational problem with a YES/NO answer
Decomposition [11]: an expression of G as a union of edge-disjoint subgraphs
F-decomposition [397]: decomposition using graphs in the family **F**
F-decomposition number of G: minimum number of graphs in an **F**-decomposition of G
Degree $d(v)$ [6, 34]: (1) for a vertex, the number of times it appears in edges (may be modified by "in-" or "out-" in a digraph); (2) for a regular graph, the degree of each vertex
Degree sequence $d_1 \geq \cdots \geq d_n$ [44]: the list of vertex degrees, usually indexed in nonincreasing order regardless of vertex order
Degree set: the set of vertex degrees (appearing once each)
Degree-sum Formula: $\sum d(v) = 2e(G)$
Deletion method [428] a strengthening of the existence argument in the probabilistic method
Demand [184]: sink constraint in transportation network
Density [435]: ratio of number of edges to number of vertices
Dependent edge [231]: an edge in an acyclic orientation whose reversal creates a cycle
Dependent set (matroids) [349]: a set containing a circuit
Depth-first search [156]: backtracking search from a vertex, exploring from the most recently reached vertex and backing up when it has no new neighbors
Descendants of x [100]: in a rooted tree, members of the subtree rooted at x
Diagonal Ramsey number [385]: Ramsey number for an instance where the thresholds (numbers or graphs) are equal

Diameter [70]: the maximum of the distance $d(u, v)$ over vertex pairs u, v
Digraph [53]: directed graph
Dijkstra's Algorithm [97]: algorithm to compute shortest paths from one vertex
Dilworth's Theorem [413]: maximum number of pairwise incomparable elements equals minimum number of totally ordered subsets needed to cover all elements
k-dimensional cube Q_k [36]: simple graph with vertex set $\{0, 1\}^k$ where vertices are adjacent if and only if their names differ in exactly one coordinate
Dinitz Conjecture [410]: each bipartite graph G is $\Delta(G)$-list-edge-colorable
Directed graph [53]: vertex set, edge set, and specification of head and tail for each edge
Directed walk, trail, path, cycle, etc. [57]: same as without the adjective "directed" (the head of an edge is the tail of the next edge)
Disc: a planar region bounded by a simple closed curve
Disconnected [6]: a graph with more than one component
Disconnecting set [152]: a set of edges whose deletion makes some vertex unreachable from some other vertex
Disjoint union $G_1 + G_2$ [39]: the union of two graphs with disjoint vertex sets
Disjointness graph: complement of intersection graph
Distance $d(u, v)$ [70]: the minimum length of a u, v-path
Distance-preserving embedding [400]: mapping $f\colon V(G) \to V(H)$ so that $d_H(f(u), f(v)) = d_G(u, v)$.
Dodecahedron [243]: planar graph with 20 vertices, 30 edges, and 12 faces of length 5
Dominating set [116]: a set $S \subseteq V$ such that every vertex outside S has a neighbor in S
Domination number [116]: the minimum size of a dominating set of vertices
Double jump [437]: the markedly different structure of the random graph in Model A for probability functions of the form c/n with $c < 1$, $c = 1$, and $c > 1$.
Double star [77]: a tree with at most two vertices of degree more than 1
Double torus [266]: the (orientable) surface with two handles
Double triangle [280]: $K_4 - e$
Doubly stochastic matrix [120]: square matrix having sum 1 in each row and column
Dual augmentation property (matroids) [362]: disjoint sets independent in a matroid and its dual can be enlarged to a complementary base and cobase
Dual edge e^* [236]: the edge of the dual graph G^* corresponding to edge e of a plane graph G
Dual graph G^* [236]: for a plane graph G, the graph with a vertex for each region of G, where vertices are adjacent if the boundaries of their regions in G share an edge (extends to 2-cell embeddings on any surface)
Dual hereditary system (or matroid) M [360]: the hereditary system whose bases are the complements of the bases of M
Dual problem [113]: for a problem $\max c^T x$ such that $Ax \le b$ and $x \ge 0$, the dual is $\min y^T b$ such that $yA \ge c$ and $y \ge 0$
Duality gap: strict inequality between optimal values of a pair of dual integer programs
Duplication of vertex x [321]: adding x' with $N(x') = N(x)$

Ear [163]: path whose internal vertices have degree two (or are "new")
Ear decomposition [163]: construction of G from a cycle by addition of ears
Eccentricity $\epsilon_G(v)$ [70]: for a vertex, the maximum distance to other vertices
Edge [2]: (1) in a graph, a pair of vertices ($E(G)$ denotes the edge set); (2) in a hypergraph, a subset of the vertex set
Edge-choosability $\chi_l'(G)$ [409]: minimum k such that G is k-edge-choosable
k-edge-choosable [409]: for all lists of size k assigned to edges of G, there exists a proper edge-coloring that selects a color for each edge from its list
Edge-chromatic number $\chi'(G)$ [275]: the minimum number of colors in a proper edge-coloring
k-edge-colorable [275]: having a proper edge-coloring with at most k colors
Edge-coloring [274]: an assignment of labels to the edges
k-edge-connected [152, 164]: having edge-connectivity at least k
Edge-connectivity $\kappa'(G)$ [152]: the minimum number of edges whose deletion disconnects G
Edge cover [114]: a set of edges incident to all the vertices
Edge cut $[S, \overline{S}]$ [152, 164]: the set of edges joining a vertex in S to a vertex not in S
Edge-reconstructible: a graph that can be determined (up to isomorphism) by knowing the multiset of subgraphs obtained by deleting single edges

Edge-Reconstruction Conjecture: the conjecture that every graph with at least four edges is edge-reconstructible
Edge-transitive [18]: having for each pair $e, f \in E(G)$ a permutation that maps e to f
Eigenvalue [453]: for a graph, an eigenvalue of the adjacency matrix
Eigenvector of A [453]: a vector x such that $A_x = \lambda x$ for some constant λ
Elementary contraction [84]: contraction
Elementary cycle: boundary of a region in a plane graph (caution - some authors who use "cycle" to mean *circuit* use "elementary cycle" to mean *cycle*)
Elementary subdivision [162]: replacement of an edge by a path of two edges connecting the endpoints of the original edge (see *edge subdivision*)
Embedding [234]: a mapping of a graph into a surface, such that (the images of) its edges do not intersect except for shared endpoints
Empty graph [22]: graph having no edges
Endpoint [2]: (1) each member of an edge; (2) the first or last vertex of a path, trail, or walk
End-vertex: a vertex of degree 1
Equipartite [207]: having partite sets differing in size by at most 1
Equitable coloring: having color classes differing in size by at most 1
Equivalence [399]: as a graph, a union of pairwise disjoint complete graphs
Equivalence relation [490]: reflexive, symmetric, and transitive relation
Erdős number: distance from Erdős in the collaboration graph of mathematicians
Euler characteristic: for a surface of genus γ, $2 - 2\gamma$
Euler tour: Eulerian circuit
Eulerian circuit [26, 60]: a closed trail containing every edge
Eulerian (di)graph [26, 60]: a graph or digraph having an Eulerian circuit
Eulerian trail [26, 60]: a trail containing every edge
Euler's Formula [241]: the formula $n - e + f = 2 - 2\gamma$ for 2-cell embeddings of a connected n-vertex graph with e edges and f faces on a surface of genus γ
Even cycle [24]: cycle with an even number of edges (or vertices)
Even graph [26]: graph with all vertex degrees even
Even pair [348]: vertex pair x, y such that every chordless x, y-path has even length
Even triangle [280]: triangle T such that every vertex has an even number of neighbors in T
Even vertex [26]: vertex of even degree
Evolution: the model of generating random graphs by successively adding random edges
(n, k, c)-expander [463]: bipartite graph with partite sets of size n and vertex degrees at most k such that each set S with at most half the vertices of the first partite set has at least $(1 + c(1 - |S|/n)|S|$ neighbors
Expansion: in 3-regular graph, subdivides two edges and adds one edge joining the new vertices
Expansion Lemma [162]: adding a vertex of degree k to a k-connected graph preserves k-connectedness
Expansive property [358]: for a function σ on the subsets of a set, the requirement that $X \subseteq \sigma(X)$ for all X
Expectation [427]: for a discrete random variable, $\sum k \text{Prob}(X = k)$
Exterior region: the unbounded region in a plane graph
Exterior vertex: vertex on the unbounded region

Face 235]: a region of an embedding
Factor [136]: a spanning subgraph
f-factor [140]: a spanning subgraph with $d(v) = f(v)$
k-factor [140]: a spanning k-regular subgraph
k-factorable [276]: having a decomposition into k-factors
Factorization: an expression of G as the edge-disjoint union of spanning subgraphs
k-factorization [276]: a decomposition of a graph into k-factors
x, U-fan [170]: pairwise internally-disjoint paths from x to distinct vertices of U
Fáry's Theorem [246]: a planar graph has a straight-line embedding in the plane
Fat triangle [275]: a 3-vertex graph in which each pair has the same edge multiplicity
Feasible flow [176]: a network flow satisfying edge-constraints and having net flow 0 at each internal vertex

Feasible solution [322]: a choice of values for the variables that satisfies all the constraints in an optimization problem

Ferrers digraph: a digraph (loops allowed) with no x, y, z, w (not necessarily distinct) such that $x \to y$ and $z \to w$ but $z \not\to y$ and $x \not\to w$; equivalently, the successor sets or predecessor sets are ordered by inclusion; equivalently, the adjacency matrix has no 2-by-2 permutation submatrix.

Five Color Theorem [257]: the theorem that planar graphs are 5-colorable

Flat [266]: a closed set in a matroid

Flow [176]: an assignment of weights to edges of a network

k-flow [307]: an assignment of weights in $\{-k+1, \dots, k-1\}$ to edges of a digraph so that net flow out is zero at each vertex

Flower (in Edmonds' Blossom Algorithm) [142]: consists of a stem (alternating path from an unsaturated vertex) and a blossom (odd cycle with a nearly-perfect matching)

Forcibly Hamiltonian: a degree sequence such that every simple graph with that degree sequence is Hamiltonian

Forest [67]: a disjoint union of trees, an acyclic graph

Four Color Theorem [260]: the theorem that planar graphs are 4-colorable

Fraternal orientation [345]: an orientation such that two vertices are adjacent if they have a common successor

H-fragment of G [252]: a component of $G - H$ together with the edges to its vertices of attachment

H-free [41]: having no copy of H as an induced subgraph

Free matroid [357]: matroid in which every set of elements is independent

Friendship Theorem [467]: if every pair of people in a set have exactly one common friend in the set, then someone is everyone's friend

Fundamental cycle [374]: for a spanning tree, a cycle formed by adding an edge to it

Gammoid [377]: a matroid on E arising from vertex sets F, E in a digraph by letting independent sets be those that are saturated by a set of disjoint paths starting in F

Generalized chromatic number: minimum number of classes needed to partition the vertices so that the subgraph induced by each color class has property **P**

Generalized Petersen graph [316]: the graph with vertices $\{u_1, \dots, u_n\}$ and $\{v_1, \dots, v_n\}$ and edges $\{u_i u_{i+1}\}$, $\{u_i v_i\}$, and $\{v_i v_{i+k}\}$, where addition is modulo n

Generalized Ramsey number $r(G_1, \dots, G_k)$ [386]: the minimum n such that k-coloring the edges of K_n forces a copy of G_i in color i for some i

Genus γ [266]: (1) for a surface, the number of handles in its topological description (2) for a graph, the minimum genus surface on which it embeds

Geodesic: a shortest path between its endpoints

Geodetic: having the property that each pair of vertices u, v are the endpoints of a unique path of length $d(u, v)$

Girth [13]: the length of a shortest cycle in G

k-gon: in an embedding, a k-cycle bounding a region

Good algorithm [124]: algorithm running in polynomial time

Good characterization [495]: a characterization that is checkable in polynomial time

Good coloring: often means proper coloring

Gossip problem [406]: minimize the number of calls so that each vertex transmits to every other by an increasing path

Graceful labeling [87]: an assignment of distinct integers to vertices such that 1) the integers are between 0 and $e(G)$, and 2) the differences between the labels at the endpoints of the edges yield the integers $1, \dots, e(G)$

Graceful graph [87]: a graph with a graceful labeling

Graceful tree [87]: a tree with a graceful labeling

Graceful tree conjecture [87]: every tree has a graceful labeling

Graph [2]: a set of vertices, a set of edges, and an assignment of a set at most two vertices as endpoints of each edge

Graphic matroid $M(G)$ [350]: matroid whose independent sets are the acyclic subsets of $E(G)$

Graphic sequence [44]: a list of integers realizable as the degree sequence of a simple graph

Greedy algorithm [95, 354]: a fast algorithm to find a good feasible solution by iteratively making a heuristically good choice

Greedy coloring [194]: with respect to some vertex ordering, color each vertex with the least-indexed color not already appearing among the neighbors of the vertex being colored
Grinberg condition [303]: necessary for Hamiltonian cycles in planar graphs, that summing (length−2) over the inside faces or over the outside faces yields the same total
Grötzsch graph [205]: the smallest triangle-free 4-chromatic graph
Grundy number: the maximum number of colors in an application of the greedy coloring algorithm

Hadwiger conjecture [213]: every k-chromatic graph has a subgraph contractible to K_k (true for "almost all" graphs)
Hajós conjecture [213]: every k-chromatic graph contains a K_k-subdivision (false for $k > 5$)
Hall's condition [110]: for every subset S of a partite set X in a bipartite graph, at least $|S|$ vertices have neighbors in S
Hall's theorem [110]: Hall's condition is necessary and sufficient for the existence of a matching that saturates X
Hamilton tour: Hamiltonian cycle
Hamiltonian [286]: having a Hamiltonian cycle
Hamiltonian closure [289]: graph obtained by successively adding edges joining vertices whose degree-sum is as large as the number of vertices
Hamiltonian-connected [297]: having a Hamiltonian path from each vertex to every other
Hamiltonian cycle [286]: a cycle containing each vertex
Hamiltonian path [291]: a path containing each vertex
Harary graphs [150]: a family of k-connected n-vertex graphs with the fewest edges
Head [53]: the second vertex of an edge in a digraph
Heawood's Formula [268]: the chromatic number of a graph embedded on the oriented surface with γ handles is at most $\lfloor 1/2(7 + \sqrt{1 + 48\gamma}) \rfloor$.
Helly property [80]: the property of the real line (or trees) that pairwise intersecting subsets have a common intersection point
Hereditary class [226]: a class F such that all induced subgraphs of graphs in F are also in F
Hereditary family [349]: a family F of sets such that every subset of a member of F is in F
Hereditary system [349]: a system consisting of a hereditary family and the alternative ways of specifying that family
Hole [340]: a chordless cycle in a graph
Homeomorphic: two graphs obtainable from the same graph by subdivision of edges
Homogeneous [380]: in Ramsey theory, a set whose colored pieces have the same color
Homomorphism: a map $f : V(G) \to V(H)$ that preserves adjacency
Huffman code [103]: prefix-free encoding of data to minimize expected search time
Hungarian Algorithm [126]: an algorithm for solving the assignment problem
Hypercube Q_k [36]: k-dimensional cube
Hypergraph [449]: a generalization of graph in which edges may be any subset of the vertices
Hyperplane (matroids) [360]: a maximal closed proper subset of the ground set
Hypohamiltonian: a non-Hamiltonian graph whose vertex-deleted subgraphs are all Hamiltonian
Hypotraceable: a non-traceable graph whose vertex-deleted subgraphs are all traceable

Icosahedron [243]: planar triangulation with 12 faces, 30 edges, and 20 vertices
Idempotence property (matroids) [359]: $\sigma^2(X) = \sigma(X)$ for all X
Identification: an operation replacing two vertices by a single vertex with the combined incidences (same as contraction if the vertices are adjacent)
Imperfect graph [232]: has $\chi(H) > \omega(H)$ for some induced subgraph H
Incidence matrix [6]: (1) for a graph, the 0,1-matrix in which entry (i, j) is 1 if and only if vertex i and edge j are incident; (2) for a digraph, entry (i, j) is 1 if vertex i is the head of edge j, -1 if it is the tail, 0 otherwise; (2) in general, the matrix of a membership relation
Incident [6]: 1) a vertex v and edge e with $v \in e$; 2) two edges with a common endpoint
Inclusion-exclusion principle [223]: number of objects outside $A_1, \ldots, A_n$ is $\sum_{S \in [n]} (-1)^{|S|} \left| \bigcap_{i \in S} A_i \right|$
Incomparability graph: the complement of a comparability graph
Incorporation property (matroids) [359]: $r(\sigma(X)) = r(X)$
Indegree [58]: for a vertex in a directed graph, the number of edges of which it is the head
Independence number $\alpha(G)$ [113]: maximum size of an independent set of vertices

Independent domination number [117]: minimum size of an independent dominating set
Independent set [3]: a set of pairwise nonadjacent vertices
Indicator variable [427]: a random variable taking values in $\{0, 1\}$
Induced circuit property (matroids) [355]: adding an element to an independent set creates at most one circuit
Induced sub(di)graph $G[A]$ [23]: the sub(di)graph on vertex set $A \subseteq V(G)$ obtained by taking A and all edges of G having both endpoints in A
Integer program [323]: linear program plus requirement that variables be integer-valued
Integrality Theorem [181]: in a network with integer edge capacities, there is an optimal flow expressible as units of flow along source/sink paths
Interlacing Theorem [458]: for each vertex x, the eigenvalues $\{\lambda_i\}$ of G and $\{\mu_i\}$ of $G - x$ satisfy $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \cdots \geq \mu_n \geq \lambda_n$
Internal vertices [20]: (1) for a path, the non-endpoints; (2) for a plane graph, the vertices not on the boundary of the exterior face
Internally disjoint paths [161]: paths intersecting only at endpoints
Intersection graph [324]: for a family of sets, the graph having a vertex for each set and having vertices adjacent when the sets intersect
Intersection number [397]: minimum size of a set U such that G is an intersection graph of subsets of U (equals minimum number of complete subgraphs covering $E(G)$)
Intersection of matroids [366]: the hereditary system whose independent sets are the common independent sets in the matroids
Intersection representation [324]: an assignment of a set S_v to each vertex v such that $u \leftrightarrow v$ if and only if $S_u \cap S_v \neq \emptyset$
Interval graph [195]: a graph having an interval representation
Interval number [451]: minimum t such that G has a t-interval representation
Interval representation of G [195]: a collection of intervals whose intersection graph is G
t-interval [451]: a union of at most t intervals in $\mathbb{R}$
t-interval representation [451]: an intersection representation where each assigned set is a t-interval
In-tree [89]: a directed tree in which each edge is oriented toward the root
Involution [470]: a permutation whose square is the identity
Isolated vertex or edge [22]: incident to no (other) edge
Isometric embedding [400]: a distance-preserving mapping of $V(G)$ into $V(H)$
Isomorphic decomposition: decomposition into isomorphic subgraphs
Isomorphism [7]: a vertex bijection preserving the adjacency relation
Isthmus: a cut-edge
Join $G \vee H$ [138]: the disjoint union $G + H$ plus the edges $\{uv :\ u \in V(G), v \in V(H)\}$
Joined to: adjacent to
Junction: vertex of degree at least three

Kempe chain [258]: a path between two vertices that alternates between two colors (particularly as used in forbidding minimal 5-chromatic planar graphs)
Kernel [57, 410]: in a digraph, an independent in-dominating set
Kernel perfect [410]: having a kernel in each induced subgraph
Kirchhoff's current law: net flow around a closed walk is 0
Kite [12]: simple 4-vertex graph obtained by deleting one edge from K_4
König-Egerváry Theorem [112]: maximum matching and minimum vertex in a bipartite graph have equal size
König's Other Theorem [115]: maximum independent and minimum edge cover in a bipartite graph with no isolated vertices have equal size
Krausz decomposition [285]: edge covering by complete subgraphs using each vertex at most twice (leads to the graph for which this is the line graph)
Kronecker product: tensor product
Kruskal's algorithm [95]: grows a minimum weighted spanning tree by iteratively adding the cheapest edge in the graph that does not complete a cycle
Kuratowski subgraph [247]: subdivision of K_5 or $K_{3,3}$
Kuratowski's Theorem [246]: a graph is planar if and only if it has no subdivision of K_5 or $K_{3,3}$

Labeling: assignment of integers to vertices
Leaf [67]: vertex of degree 1
Leaf block [156]: a block containing only one cut-vertex
Length [20]: the number of steps (or sum of weights) from start to finish
Lexicographic product $G[H]$ [393]: composition
Line: another name for edge
Line graph $L(G)$ [168, 273]: the intersection graph of the edges of G, where vertices correspond to edges of G and are adjacent if the corresponding edges share a vertex
Linear matroid [351]: matroid whose independent sets are the sets of independent columns of some matrix over some field
Linear program [179]: problem of optimizing a linear function with linear constraints
Link: edge
k-linked: a stronger condition than k-connected, in which for every choice of two k-tuples of vertices $(u_1, \ldots, u_k)$ and $(v_1, \ldots, v_k)$, there exists a set of k internally disjoint paths connecting corresponding vertices u_i, v_i.
List chromatic index [409]: edge-choosability
List chromatic number [408]: choosability
List Coloring Conjecture [409]: edge-choosability always equals edge-chromatic number
Literal [500]: a logical (true/false) variable or its negation
S-lobe [211]: a subgraph of G induced by $S \cup V_i$, where V_i is the vertex set of a component of $G - S$
Local search: technique for solving optimization problems by successively making small changes in a feasible solution
Loop [2]: an edge whose endpoints are the same
Loopless [6]: having no loops

(n, k, c)-magnifier [463]: n-vertex graph of maximum degree k in which each set S with at most half the vertices has at least $c\,|S|$ neighbors outside S
Markov chain [54]: discrete system with transition probabilities
Markov's inequality [432]: for a nonnegative random variable, Prob $(X \geq t) \leq E(X)/t$
Martingale [443]: sequence of random variables such that $E(X_i | X_0, \ldots, X_{i-1}) = X_{i-1}$
Matching [107]: a set of edges sharing no endpoints
b-matching: given a constraint vector b, a subgraph H with $d_H(v) \leq b(v)$ for all v
Matrix rounding [186]: problem of converting the data and row/column sums in a matrix to nearest integers up or down such that row and column sums remain correct
Matrix-Tree Theorem [86]: subtracting the adjacency matrix from the diagonal matrix of degrees, deleting a row and column, and taking the determinant yields the number of spanning trees
Matroid [354]: a hereditary system satisfying any one of a list of many equivalent properties
Matroid basis graph [376]: graph whose vertex set is the collection of bases of a matroid, adjacent when their symmetric difference has two elements
Matroid Covering Theorem [372]: the number of independent sets needed to cover the elements of a matroid is $\max_{X \subseteq E} \lceil |X| / r(X) \rceil$
Matroid Intersection Theorem [367]: the maximum size of a common independent set in two matroids on E equals the minimum over $X \subseteq E$ of the rank of X in the first matroid plus the rank of $\overline{X}$ in the second matroid
Matroid Packing Theorem [372]: the maximum number of pairwise disjoint bases in a matroid is $\min_{r(X)<r(E)} \lfloor (|E| - CA(X))/(r(E) - r(X)) \rfloor$
Matroid Union Theorem [370]: the union of matroids $M_1, \ldots, M_k$ is a matroid with rank function $r(X) = \min_{Y \subseteq X}(|X - Y| + \sum r_i(Y))$
Max-flow Min-cut Theorem [180]: maximum flow value equals minimum cut value
Maximal clique [31]: a maximal set of pairwise adjacent vertices
Maximal path or trail [27]: non-extendible path or trail
Maximal planar graph [242]: equivalent to planar triangulation
Maximum Cardinality Search [325]: an algorithm for recognizing chordal graphs
Maximum degree Δ [34]: maximum of the vertex degrees
Maximum flow [176]: a feasible network flow of maximum value, or the value itself
Maximum genus $\gamma_M(G)$: the maximum genus surface on which G has a 2-cell embedding
Maximum (P-object) [31]: for a property P, no larger object of the same type also has property P