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More about Mesopotamian Mathematics

In *Science Awakening I*, B. L. van der Waerden quotes the beginning of 'AO8862', a Babylonian clay tablet going back to about the same time as the Rhind Papyrus:

> Length, width. I have multiplied the length and the width, thus obtaining the area. Then I added to the area, the excess of the length over the width: 183 was the result. Moreover, I have added the length and the width: 27. Required length, width and area.
>
> 27 and 183, the sums; 15 the length; 180 the area; 12 the width;
>
> One follows this method:
>
> $27 + 183 = 210$, $2 + 27 = 29$.
>
> Take one half of 29 (this gives $14\frac{1}{2}$),
>
> $14\frac{1}{2} \times 14\frac{1}{2} = 210\frac{1}{4}$,
> $210\frac{1}{4} - 210 = \frac{1}{4}$.
>
> The square root of $1/4$ is $1/2$.
>
> $14\frac{1}{2} + \frac{1}{2} = 15$, the length;
> $14\frac{1}{2} - \frac{1}{2} = 14$, the width.
>
> Subtract 2, which has been added to 27, from 14, the width. 12 is the actual width. I have multiplied the length 15 by the width 12.
>
> $15 \times 12 = 180$, the area;
> $15 - 12 = 3$;
> $180 + 3 = 183$.

What is going on here? In modern notation, we would write x and y for length and width, respectively. The problem is to find a solution for the simultaneous equations

$$xy + (x - y) = 183 \text{ and } x + y = 27.$$

The answer is given as $x = 15$ and $y = 12$. The scribe's method is this: consider

$$xy + x - y + x + y = x(y + 2) = 210.$$

Putting $y' = y + 2$, we have $xy' = 210$. On the other hand, adding the factors of 210, we get

$$x + y' = x + y + 2 = 29;$$

$$\text{hence } \tfrac{1}{2}(x + y') = \tfrac{1}{2}(29) = 14\tfrac{1}{2};$$

$$\text{hence } \frac{x^2 + 2xy' + y'^2}{4} = (14\tfrac{1}{2})^2 = 210\tfrac{1}{4};$$

$$\text{hence } \frac{x^2 - 2xy' + y'^2}{4} = 210\tfrac{1}{4} - 210 = \tfrac{1}{4} \text{ (the so-called discriminant);}$$

$$\text{hence } \frac{x - y'}{2} = \tfrac{1}{2}.$$

Adding and subtracting $\frac{1}{2}(x + y')$ and $\frac{1}{2}(x - y')$, we get $x = 14\frac{1}{2} + \frac{1}{2} = 15$ and $y' = 14\frac{1}{2} - \frac{1}{2} = 14$. Note that 14 is not really the width; but $y = y' - 2 = 14 - 2 = 12$ is. The scribe then computes the area and checks his work. The scribe did not consider the possibility $x = 14, y + 2 = 15$, which gives the second solution $x = 14$, $y = 13$. He did not know how to take the negative square root of $\frac{1}{4}$.

The Babylonians could solve many kinds of equations, including: $ax = b$, $x^2 \pm ax = b$, $x^3 = a$, $x^2(x + 1) = a$. They could also solve simultaneous equations having the following forms:

$$x \pm y = a, \quad xy = b;$$

$$x \pm y = a, \quad x^2 + y^2 = b.$$

They even managed to solve the following pair of equations:

$$x^3\sqrt{x^2 + y^2} = 3{,}200{,}000; \quad xy = 1200. \qquad (*)$$

As we saw just above, the Babylonians knew that

$$a^2 - b^2 = (a + b)(a - b).$$

They also knew that

$$(a + b)^2 = a^2 + 2ab + b^2.$$

Like the Egyptians, the Babylonians built pyramids, or *ziggurats*. If each story of a ziggurat consists of a square platform measuring $1 \times m \times m$, then the volume of a ziggurat with a base of length n is

$$(1 \times n \times n) + \cdots + (1 \times 2 \times 2) + (1 \times 1 \times 1) = 1^2 + 2^2 + 3^2 + \cdots + n^2.$$

The Babylonians knew that the formula for this sum is

$$n(n+1)(2n+1)/6\ ,$$

a result also known to Pythagoras, but perhaps first proved by Archimedes.

According to the biblical story of the Tower of Babel, there was once an attempt to build a ziggurat 'with its top reaching heaven'. Perhaps the people behind this project thought that there was only a finite distance between heaven and earth, or perhaps they thought that they could calculate the sum of $1^2 + 2^2 + 3^2 + \cdots$, not realizing that the series diverges.

A remarkable fact about ancient Babylonian mathematics is that it included not just the so-called theorem of Pythagoras, but a theory of 'Pythagorean triangles'. (A *Pythagorean triangle* is a triple (x, y, z) of positive integers such that $x^2 + y^2 = z^2$, and thus x, y and z are sides of a right angled triangle.) From a clay tablet called 'Plimpton 322' (dating from 1900–1600 BC), we can deduce that the Babylonians used a result of which the following is a modern version:

> Suppose u and v are *relatively prime positive integers*, that is, integers whose greatest common divisor is 1. Assume that not both are odd and that $u > v$. Then, if $a = 2uv, b = u^2 - v^2$ and $c = u^2 + v^2$, we have $\gcd(a, b, c) = 1$ and $a^2 + b^2 = c^2$.

Included in Plimpton 322 is the triangle (13500, 12709, 18541), which is generated by taking $u = 125$ and $v = 54$.

The converse of the above theorem is also true. That is, if a, b and c are relatively prime positive integers, with a even, such that $a^2 + b^2 = c^2$, then there are relatively prime positive integers u and v, not both odd, such that $a = 2uv, b = u^2 - v^2$ and $c = u^2 + v^2$. It is not impossible that the Babylonians knew this, but the earliest record we have of this result is in the solutions of Problems 8 and 9 of Book II of the *Arithmetica* of Diophantus (250 AD). Indeed, since Diophantus explained his ideas in terms of special cases, it is correct to say that the first explicit, rigorous proof of the converse of the Babylonian theorem was given only in 1738, by C. A. Koerbero (Dickson [1971], Vol. II).

According to a tablet found in 1936 in Susa, an ancient city in what is now Iran, the Babylonians sometimes used the value $3\frac{1}{8}$ for π. At other times, they seem to have been satisfied with $\pi \approx 3$. It has been suggested that this Babylonian usage is behind 1 *Kings* 7:23–24:

> He [Hiram of Tyre] made the basin of cast metal, ten cubits from rim to rim, circular in shape and five cubits high; a cord