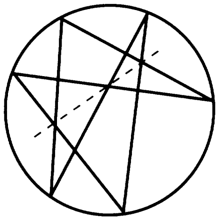

FIGURE 27.2. Pascal's Theorem

Over the centuries, research into Fermat's Last Theorem proved very fruitful. As one example, Kummer's research led to the discovery of 'ideals' and their unique factorization. As another example, Faltings's research led to advances in algebraic geometry — this as recently as 1983. In fact, Faltings came close to proving Fermat's Last Theorem, by showing that, for any $n > 2$, the equation $x^n + y^n = 1$ has at most a finite number of rational solutions.

Pascal was educated at home and forbidden to study mathematics. By the age of 12, he had rediscovered many of Euclid's theorems, so his father relented and gave him a copy of the *Elements.* At 14 Pascal began attending meetings of a group of mathematicians which included Mersenne. At 16 he wrote an essay on conic sections (with an account of the 'mystical hexagram') and, at 18, he constructed a calculating machine, one of the first computers.

At 27, Pascal abandoned mathematics to devote himself wholly to the philosophy of religion and the worship of God. At least one historian of mathematics (E.T. Bell) has taken this as a sure indication of insanity, but it is unlikely that a madman could have produced the elegant French prose or the brilliant philosophical analyses that we find in the *Pensées*, which Pascal wrote during this period. In his later life, Pascal returned to mathematics for a couple of brief periods. Pascal died at age 39. His last feat was the creation of a public transportation system, with the profits going to help the poor.

What did Pascal do as a mathematician? In 1639 (at age 16) he used some ideas of Desargues to obtain 'Pascal's Theorem' (with the 'mystic hexagram'):

Theorem 27.1. *If a hexagon is inscribed in a conic, the points of intersection of opposite sides, assumed to be nonparallel, lie in a straight line* (Figure 27.2).

Actually, this is a theorem in projective geometry, and it suffices to prove it for the case when the conic is a circle, since other conics can be obtained from the circle by projection. If one views a pair of straight lines as a degenerate conic, one obtains the Theorem of Pappus as a special case of the Theorem of Pascal. A simple proof for Pascal's Theorem in the case of the circle is found in Coxeter and Greitzer's *Geometry Revisited.*

In 1653, Pascal rediscovered what we call 'Pascal's triangle':

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \\ & & . & . & . & . & . & & \end{array}$$

As we noted, this goes back to the Chinese, but Pascal was the first to give clear and complete demonstrations of its basic properties (making the first explicit use of mathematical induction). He showed that the kth entry in the nth row is $\binom{n-1}{k-1}$, this being the number of ways of choosing $k-1$ out of $n-1$ things. (Today we prefer to call $\binom{n}{k}$ the number of k-element subsets of an n-element set.) Pascal showed that

$$\binom{n-1}{k-1} = \frac{(n-1)(n-2)...(n-k+1)}{(k-1)!}$$

and he showed that

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

(This latter equation follows from the observation that, to choose k out of n things, we may set one of the n things aside and consider two cases: in the first case, the special thing is included in the choice, and hence there are $\binom{n-1}{k-1}$ possibilities; in the second case the special thing is excluded, leaving $\binom{n-1}{k}$ possibilities.) Finally, Pascal proved that $\binom{n}{k}$ is the coefficient of $x^k y^{n-k}$ in the binomial expansion of $(x+y)^n$.

With Fermat, Pascal developed the theory of probability, including the concept of mathematical expectation. Pascal uses probability in the *Pensées* as part of a proof that it is wiser to believe in God (Pascal, *Oeuvres Complètes*, p. 1212). This argument, called 'Pascal's wager', goes as follows: Even if the probability that a god exists is very small (but positive), if he rewards those who believe in him with eternal happiness, assumed to be of infinite value, a rational human being ought to believe in this god. (Did Pascal consider the possibility that God might punish those who adopt their beliefs in expectation of personal gain?)