Proof. We apply Chebyshev's inequality to $\bar{X}$. For this we need to know the expectation and variance of $\bar{X}$. These are

$$E(\bar{X}) = m \qquad \text{and} \qquad \operatorname{Var}(\bar{X}) = \frac{\sigma^2}{n}.$$

(See Exercise 5 in Section 14.27.) Chebyshev's inequality becomes $P(|\bar{X} - m| > c) \leq \sigma^2/(nc^2)$. Letting $n \to \infty$ and replacing c by ϵ we obtain (14.49) and hence (14.50).

Note: To show that the limit relation in (14.48) is a special case of Theorem 14.12, we assume each X_k has the possible values 0 and 1, with probabilities $P(X_k = 1) = p$ and $P(X_k = 0) = 1 - p$. Then $\bar{X}$ is the relative frequency of success in n independent trials, $E(X) = p$, and (14.49) reduces to (14.48).

Theorem 14.12 is called a *weak* law because there is also a strong law of large numbers which (under the same hypotheses) states that

$$P\left(\lim_{n\to\infty} |\bar{X} - m| = 0\right) = 1. \tag{14.51}$$

The principal difference between (14.51) and (14.50) is that the operations "limit" and "probability" are interchanged. It can be shown that the strong law implies the weak law, but not conversely.

Notice that the strong law in (14.51) seems to be closer to formula (14.45) than (14.50) is. In fact, (14.51) says that we have $\lim_{n\to\infty} \bar{X} = m$ "almost always," that is, with probability 1. When applied to coin tossing, in particular, it says that the failure of Equation (14.45) is no more likely than the chance of tossing a fair coin repeatedly and always getting heads. The strong law really shows why probability theory corresponds to experience and to our intuitive feeling of what probability "should be."

The proof of the strong law is lengthy and will be omitted. Proofs appear in the books listed as References 1, 3, 8, and 10 at the end of this chapter.

14.30 The central limit theorem of the calculus of probabilities

In many applications of probability theory, the random variables of interest are sums of other random variables. For example, the financial outcome after several plays of a game is the sum of the winnings at each play. A surprising thing happens when a large number of independent random variables are added together. Under general conditions (applicable in almost every situation that occurs in practice) the distribution of the sum tends to be normal, regardless of the distributions of the individual random variables that make up the sum. The precise statement of this remarkable fact is known as the *central limit theorem of the calculus of probabilities.* It accounts for the importance of the normal distribution in both theory and practice. A thorough discussion of this theorem belongs to the advanced study of probability theory. This section will merely describe what the theorem asserts.

Suppose we have an infinite sequence of random variables, say $X_1, X_2, \ldots,$ with finite expectations and variances. Let

$$m_k = E(X_k) \qquad \text{and} \qquad \sigma_k^2 = \operatorname{Var}(X_k), \qquad k = 1, 2, \ldots.$$

We form a new random variable S_n by adding the first n differences $X_k - m_k$:

(14.52) $$S_n = \sum_{k=1}^{n} (X_k - m_k).$$

We add the *differences* rather than the X_k alone so that the sum S_n will have expected value 0. The problem here is to determine the limiting form, as $n \to \infty$, of the distribution function of S_n.

If $X_1, X_2, \ldots, X_n$ are ***independent,*** then [by Exercise 4(c) of Section 14.27] we have

$$\text{Var}(S_n) = \sum_{k=1}^{n} \text{Var}(X_k - m_k) = \sum_{k=1}^{n} \text{Var}(X_k) = \sum_{k=1}^{n} \sigma_k^2.$$

Ordinarily, Var (S_n) will be large even though the individual variances σ_k^2 may be small. Random variables with a large variance are not fruitful objects of study because their values tend to be widely dispersed from the expected value. For this reason, a new random variable T_n is introduced by the equation

(14.53) $$T_n = \frac{S_n}{\sqrt{\text{Var}(S_n)}}.$$

This new variable has expectation 0 and variance 1 and is called a ***standardized*** random variable. The standardized variable T_n is meaningful even if the random variables $X_1, X_2, \ldots, X_n$ are not independent.

We now introduce the following definition:

DEFINITION OF THE CENTRAL LIMIT PROPERTY. *Let*

(14.54) $$X_1, X_2, X_3, \ldots$$

be a sequence of random variables (not necessarily independent), where each X_k has a finite expectation m_k and a finite variance σ_k^2. Define S_n and T_n by (14.52) and (14.53). The sequence in (14.54) is said to satisfy the central limit property if, for all a and b with $a \le b$, we have

(14.55) $$\lim_{n\to\infty} P(a \le T_n \le b) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-u^2/2}\, du.$$

In other words, the random variables in (14.54) satisfy the central limit property if the distribution of the standardized variable T_n approaches a standard normal distribution as $n \to \infty$. [Equation (14.55) is to hold also if $a = -\infty$ or $b = +\infty$.]

Laplace was the first to realize that this property is shared by many sequences of random variables, although a special case (random variables describing a Bernoullian sequence of trials) had been known earlier by DeMoivre. (Figure 14.11 shows a binomial distribution and a corresponding normal approximation.) Laplace stated a general central limit theorem which was first completely proved by the Russian mathematician A. Lyapunov in 1901. In 1922, J. W. Lindeberg generalized Laplace's result by showing that the property is satisfied if the random variables are independent and have a common distribution giving them the same expectations and variances, say $E(X_k) = m$ and Var $(X_k) = \sigma^2$ for all k. In this case the standardized variable becomes

$$T_n = \frac{\sum_{k=1}^{n} X_k - nm}{\sigma\sqrt{n}}.$$

Lindeberg realized that independence alone is not sufficient to guarantee the central limit property, but he formulated another condition (now known as the ***Lindeberg condition***) which, along with independence, ***is*** sufficient. In 1935, W. Feller showed that the Lindeberg condition is both necessary and sufficient for independent random variables to satisfy the central limit property. We shall not discuss the Lindeberg condition here except to mention that it implies

$$\text{Var}(S_n) \to \infty \quad \text{as} \quad n \to \infty.$$

Fortunately, many independent random variables that occur in practice automatically satisfy the Lindeberg condition and therefore also have the central limit property. Up to now, the theory for ***dependent*** random variables is incomplete. Only a few special cases have been treated. Much of the contemporary research in probability theory centers about the search for general theorems dealing with dependent variables.

14.31 Exercises

1. Carry out the proof of Chebyshev's inequality in the discrete case.
2. If a is any real number, prove that

$$P(|X - a| > c\lambda) \le \frac{1}{c^2}$$

for every $c > 0$, where $\lambda^2 = \int_{-\infty}^{+\infty} (t - a)^2 f_X(t)\, dt$. Chebyshev's inequality is the special case in which $a = E(X)$.

3. Let X denote the random variable which counts the number of successes in n independent trials of a Bernoullian sequence; the probability of success is p. Show that, for every $\epsilon > 0$,

$$P\left(\left|\frac{X}{n} - p\right| > \epsilon\right) \le \frac{1}{4n\epsilon^2}.$$

4. A fair coin is tossed n times; the number of heads is denoted by X. Find the smallest n for which Chebyshev's inequality implies

$$P\left(0.4 < \frac{X}{n} < 0.6\right) > 0.90.$$

5. In a production line the number X of the defective articles manufactured in any given hour is known to have a Poisson distribution with mean $E(X) = 100$. Use Chebyshev's inequality to compute a lower bound for the probability that in a given hour there will be between 90 and 110 defective articles produced.
6. Assume that a random variable X has a standard normal distribution (mean 0 and variance 1). Let p denote the probability that X differs from its expectation $E(X)$ by more than three

times its standard deviation. Use Chebyshev's inequality to find an upper bound for p. Then use suitable tables of the normal distribution to show that there is an upper bound for p that is approximately one-fiftieth of that obtained by Chebyshev's inequality.

7. Given a sequence of independent random variables $X_1, X_2, \ldots,$ each of which has a normal distribution. Let $m_k = E(X_k)$ and let $\sigma_k^2 = \operatorname{Var}(X_k)$. Show that this sequence has the central limit property. [Hint: Refer to Exercise 7 in Section 14.24.]

8. Let $X_1, X_2, \ldots$ be independent random variables having the same binomial distribution. Assume each X_k takes the possible values 0 and 1 with probabilities $P(X_k = 1) = p$ and $P(X_k = 0) = q$, where $p + q = 1$. Let $Z_n = X_1 + \cdots + X_n$. The random variable Z_n counts the number of successes in n Bernoulli trials.

(a) Show that the central limit property takes the following form:

$$\lim_{n\to\infty} P\left(\frac{Z_n - np}{\sqrt{npq}} \le t\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2}\, du .$$

(b) Use the approximation suggested by part (a) to estimate the probability of obtaining between 45 and 55 heads if a fair coin is tossed 100 times. Refer to Table 14.1, p. 536 for the computation.

9. With the notation of Exercise 8, the central limit theorem for random variables describing a Bernoullian sequence of trials can be written in the form

$$\lim_{n\to\infty} \frac{P\left(t_1 \le \dfrac{Z_n - np}{\sqrt{npq}} \le t_2\right)}{\Phi(t_2) - \Phi(t_1)} = 1 ,$$

where Φ is the standard normal distribution. For this particular case it can be shown that the formula is also valid when t_1 and t_2 are functions of n given by $t_1 = (a - np)/\sqrt{npq}$ and $t_2 = (b - np)/\sqrt{npq}$, where a and b are fixed positive constants, $a < b$.

(a) Show that this relation implies the asymptotic formula

$$\sum_{k=a}^{b} \binom{n}{k} p^k q^{n-k} \sim \Phi\left(\frac{b - np + \frac{1}{2}}{\sqrt{npq}}\right) - \Phi\left(\frac{a - np - \frac{1}{2}}{\sqrt{npq}}\right) \qquad \text{as} \quad n \to \infty .$$

(b) An unbiased die is tossed 180 times. Use the approximation suggested in part (a) to estimate the probability that the upturned face is a six exactly 30 times. Refer to Table 14.1, p. 536 for the computation.

10. An unbiased die is tossed 100 times. Use the approximation suggested in Exercise 9(a) to estimate the probability that the upturned face is a six (a) exactly 25 times, (b) at least 25 times. Refer to Table 14.1, p. 536 for the computation.

Suggested References

1. H. Cramér, *Elements of Probability Theory,* Wiley, New York, 1955.
2. H. Cramér, *Mathematical Methods of Statistics,* Princeton Univ. Press, Princeton, N.J., 1946.
3. W. Feller, *An Introduction to Probability Theory and Its Applications,* 2nd ed., Wiley, New York, 1957.
4. B. V. Gnedenko and A. N. Kolmogorov, *Limit Distributions for Sums of Independent Random Variables,* Addison-Wesley, Reading, Mass., 1954.

5. S. Goldberg, ***Probability, an Introduction,*** Prentice-Hall, Englewood Cliffs, N.J., 1960.
6. H. Levy and L. Roth, ***Elements of Probability,*** Oxford Univ. Press, London and New York, 1936.
7. M. Loève, ***Probability Theory: Foundations, Random Sequences,*** Van Nostrand, New York, 1955.
8. M. E. Munroe, ***Theory of Probability***, McGraw-Hill, New York, 1951.
9. J. Neyman, ***First Course in Probability and Statistics,*** Holt, Rinehart and Winston, New York, 1950.
10. E. Parzen, ***Modern Probability Theory and Its Applications,*** Wiley, New York, 1960.
11. I. Todhunter, ***A History of the Mathematical Theory of Probability from the Time of Pascal to Laplace***, Chelsea, New York, 1949.
12. J. V. Uspensky, ***Introduction to Mathematical Probability,*** McGraw-Hill, New York, 1937.

15

INTRODUCTION TO NUMERICAL ANALYSIS

15.1 Historical introduction

The planet Uranus was discovered in 1781 by a gifted amateur astronomer, William Herschel (1738–1822), with a homemade IO-ft. telescope. With the use of Kepler's laws, the expected orbit of Uranus was quickly calculated from a few widely separated observations. It was found that the mean distance of Uranus from the sun was about twice that of Saturn and that one complete orbit would require 84 years. By 1830 the accumulated empirical data showed deviations from the scheduled orbit that could not be accounted for. Some astronomers felt that Newton's law of universal gravitation might not hold for distances as large as that of Uranus from the sun; others suspected that the perturbations were due to a hitherto undiscovered comet or more distant planet.

An undergraduate student at Cambridge University, John Couch Adams (1819–1892), was intrigued by the possibility of an undiscovered planet. He set himself the difficult task of calculating what the orbit of such a planet must be to account for the observed positions of Uranus, assuming the validity of Newton's law of gravitation. He completed his calculations in 1845 and asked the Royal Observatory at Greenwich to search for the hypothetical planet, but his request was not taken seriously.

A similar calculation was made independently and almost simultaneously by Jean Joseph Leverrier (1811-1877) of Paris, who asked Johann Galle, head of the Berlin Observatory, to confirm his prediction. The same evening that he received Leverrier's letter, Galle found the new planet, Neptune, almost exactly in its calculated position. This was another triumph for Newton's law of gravitation, and one of the first major triumphs of ***numerical analysis,*** the art and science of computation.

The history of numerical analysis goes back to ancient times. As early as 2000 B.C. the Babylonians were compiling mathematical tables. One clay tablet has been found containing the squares of the integers from 1 to 60. The Babylonians worshipped the heavenly bodies and kept elaborate astronomical records. The celebrated Alexandrian astronomer Claudius Ptolemy (circa 1.50 A.D.) possessed a Babylonian record of eclipses dating from 747 B.C.

In 220 B.C., Archimedes used regular polygons as approximations to a circle and deduced the inequalities $3\frac{10}{71} < \pi < 3\frac{1}{7}$. Numerical work from that time until the 17th century was centered principally around the preparation of astronomical tables. The advent of algebra in the 16th century brought about renewed activity in all branches of mathematics, including numerical analysis. In 1614, Napier published the first table of logarithms. In 1620, the logarithms of the sine and tangent functions were tabulated to seven decimal places. By 1628, fourteen-place tables of the logarithms of the numbers from 1 to 100,000 had been computed.

Computations with infinite series began to flourish near the end of the 17th century, along with the development of the calculus. Early in the 18th century Jacob Stirling and Brook Taylor laid the foundations of the ***calculus of finite differences***, which now plays a central role in numerical analysis. With the prediction of the existence and location of the planet Neptune by Adams and Leverrier in 1845, the scientific importance of numerical analysis became established once and for all.

Late in the 19th century the development of automatic calculating machinery further stimulated the growth of numerical analysis. This growth has been explosive since the end of World War II because of the progress in high-speed electronic computing devices. The new machines have made possible a great many outstanding scientific achievements which previously seemed unattainable.

The art of computation (as distinct from the science of computation) lays much stress on the detailed planning required in a particular calculation. It also deals with such matters as precision, accuracy, errors, and checking. This aspect of numerical analysis will not be discussed here; it is best learned by carrying out actual numerical calculations with specific problems. For valuable advice on practical methods and techniques the reader should consult the existing books on numerical analysis, some of which are listed in the bibliography at the end of this chapter. The bibliography also contains some of the standard mathematical tables; many of them also give practical information on how to carry out a specific calculation.

This chapter provides an introduction to the ***science*** of computation. It contains some of the basic mathematical principles that might be required of almost anyone who uses numerical analysis, whether he works with a desk calculator or with a large-scale high-speed computing machine. Aside from its practical value, the material in this chapter is of interest in its own right, and it is hoped that this brief introduction will stimulate the reader to learn more about this important and fascinating branch of mathematics.

15.2 Approximations by polynomials

A basic idea in numerical analysis is that of using simple functions, usually polynomials, to approximate a given function f. One type of polynomial approximation was discussed in Volume I in connection with Taylor's formula (Theorem 7.1). The problem there was to find a polynomial P which agrees with a given function f and some of its derivatives at a given point. We proved that if f is a function with a derivative of order n at a point a, there is one and only one polynomial P of degree $\leq n$ which satisfies the $n + 1$ relations

$$P(a) = f(a), \qquad P'(a) = f'(a), \qquad \ldots, \qquad P^{(n)}(a) = f^{(n)}(a).$$

The solution is given by the ***Taylor polynomial,***

$$P(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x - a)^k .$$

We also discussed the error incurred in approximating $f(x)$ by $P(x)$ at points x other than a. This error is defined to be the difference E,(x) $= f(x) -$ P(x), so we can write

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x - a)^k + E_n(x).$$

To make further statements about the error we need more information about f. For example, if f has a continuous derivative of order $n + 1$ in some interval containing a, then for every x in this interval the error can be expressed as an integral or as an $(n + 1)$st derivative :

$$E,(x) = \frac{1}{n!} \int_{a}^{x} (x - t)^n f^{(n+1)}(t)\, dt = \frac{f^{(n+1)}(c)}{(n + 1)!} (x - a)^{n+1},$$

where c lies between a and x. (See Sections 7.5 and 7.7 in Volume I.)

There are many other ways to approximate a given functionfby polynomials, depending on the use to be made of the approximation. For example, instead of asking for a polynomial that agrees with f and some of its derivatives at a given point, we can ask for a polynomial that takes the same values as f at a number of distinct points. Specifically, if the given distinct points are $x_0, x_1, \ldots, x_n$ we seek a polynomial P satisfying the conditions

$$P(x_0) = f(x_0), \qquad P(x_1) = f(x_1), \qquad \ldots, \qquad P(x_n) = f(x_n). \tag{15.1}$$

Since there are $n + 1$ conditions to be satisfied we try a polynomial of degree $\leq n$, say

$$P(x) = \sum_{k=0}^{n} a_k x^k,$$

with $n + 1$ coefficients $a_0, a_1, \ldots, a_n$ to be determined. The $n + 1$ conditions (15.1) lead to a system of $n + 1$ linear equations for the coefficients. From the theory of linear equations it can be shown that this system has one and only one solution; hence such a polynomial always exists. If the equations are solved by Cramer's rule the coefficients a_0, $a_1, \ldots, a_n$ are expressed as quotients of determinants. In practice, however, the polynomial P is seldom determined in this manner because the calculations are extremely laborious when n is large. Simpler methods have been developed to calculate the polynomial approximation. Some of these will be discussed in later sections. The polynomial which solves the foregoing problem is called an *interpolating polynomial.*

Another common type of polynomial approximation is the so-called *least-square approximation*. Here the given function f is defined and integrable on an interval $[a, b]$ and we seek a polynomial P of degree $\leq n$ such that the mean-square error

$$\int_a^b |f(x) - P(x)|^2\, dx$$

will be as small as possible. In Section 15.4 we shall prove that for a continuous f such a polynomial exists and is uniquely determined. The Legendre polynomials introduced in Section 1.14 play a fundamental role in the solution of this problem.

15.3 Polynomial approximation and normed linear spaces

All the different types of polynomial approximation described in the foregoing section can be related by one central idea which is best described in the language of linear spaces.

Let V be a linear space of functions which contains all polynomials of degree $\leq n$ and which also contains the function f to be approximated. The polynomials form a finite-dimensional subspace S, with dim S = n + 1. When we speak of approximating f by a polynomial P in S, we consider the difference $f - P$, which we call the error of the approximation, and then we decide on a way to measure the size of this error.

If V is a Euclidean space, then it has an inner product (x, y) and a corresponding norm given by $\|x\| = (x, x)^{1/2}$, and we can use the norm $\|f - P\|$ as a measure of the size of the error.

Sometimes norms can be introduced in non-Euclidean linear spaces, that is, in linear spaces which do not have an inner product. These norms were introduced in Section 7.26. For convenience we repeat the definition here.

DEFINITION OF A NORM. *Let V be a linear space. A real-valued function N defined on V is called a norm if it has the following properties:*

(a) $N(f) \geq 0$ *for all* f *in* V.

(b) $N(cf) = |c|\, N(f)$ *for all* f *in* V *and every scalar* c.

(c) $N(f + g) \leq N(f) + N(g)$ *for all* f *and* g *in* V.

(d) $N(f) = 0$ *implies* $f = 0$.

A linear space with a norm assigned to it is called a normed linear space.

The norm off is sometimes written $\|f\|$ instead of $N(f)$. In this notation, the fundamental properties become :

(a) $\|f\| \geq 0$,

(b) $\|cf\| = |c|\, \|f\|$,

(c) $\|f + g\| \leq \|f\| + \|g\|$,

(d) $\|f\| = 0$ implies $f = 0$.

A function N that satisfies properties (a), (b), and (c), but *not* (d), is called a *seminorm.* Some problems in the theory of approximation deal with seminormed linear spaces; others with normed linear spaces. The following examples will be discussed in this chapter.

EXAMPLE 1. *Taylor seminorm.* For. a fixed integer $n \geq 1$, let V denote the linear space of functions having a derivative of order n at a given point a. If $f \in V$, let

$$N(f) = \sum_{k=0}^{n} |f^{(k)}(a)| .$$

It is easy to verify that the function N so defined is a seminorm. It is not a norm because $N(f) = 0$ if and only if

$$f(a) = f'(a) = \cdots = f^{(n)}(a) = 0,$$

and these equations can be satisfied by a nonzero function. For example, $N(f) = 0$ when $f(x) = (x - a)^{n+1}$.

EXAMPLE 2. ***Interpolation seminorm.*** Let V denote the linear space of all real-valued functions defined on an interval $[a, b]$. For a fixed set of $n + 1$ distinct points $x_0, x_1, \ldots, x_n$ in $[a, b]$, let N be defined by the equation

$$N(f) = \sum_{k=0}^{n} |f(x_k)|$$

if $f \in V$. This function N is a seminorm on V. It is not a norm because $N(f) = 0$ if and only if $f(x_0) = f(x_1) = \cdots = f(x_n) = 0$, and it is clear that these equations can be satisfied by a function f that is not zero everywhere on $[a, b]$.

EXAMPLE 3. ***Square norm.*** Let C denote the linear space of functions continuous on an interval $[a, b]$. If $f \in C$ define

$$N(f) = \left(\int_a^b |f(x)|^2 \, dx\right)^{1/2}. \tag{15.2}$$

This is a norm inherited from the inner product

$$(f, g) = \int_a^b f(x)\overline{g(x)} \, dx.$$

Note: Let S denote the set of functions f that are integrable on $[a, b]$. The set S is a linear space, and the function N defined by (15.2) is a seminorm on S. It is not a norm because we can have $N(f) = 0$ without f being identically zero on $[a, b]$.

EXAMPLE 4. ***Max norm.*** Let C denote the linear space of functions continuous on an interval $[a, b]$. If $f \in C$, define

$$N(f) = \max_{a \le x \le b} |f(x)|,$$

where the symbol on the right stands for the absolute maximum value of $|f|$ on $[a, b]$. The verification of all four norm properties is requested in Exercise 4 of Section 15.5.

15.4 Fundamental problems in polynomial approximation

Let C be the space of functions continuous on a given interval $[a, b]$, and let S be the linear subspace consisting of all polynomials of degree $\le n$. Assume also that a norm or seminorm has been defined on C. Choose a function f in C. If there is a polynomial P in S such that

$$\|f - P\| \le \|f - Q\|$$

for all polynomials Q in S, we say that P is a ***best polynomial approximation*** to f with the specified degree. The term "best" is, of course, relative to the given norm (or seminorm). The best polynomial for one choice of norm need not be best for another choice of norm.

Once a norm or seminorm has been chosen, three problems immediately suggest themselves.

1. ***Existence.*** Given f in C, is there a best polynomial approximation to f with the specified degree ?

2. Uniqueness. If a best polynomial approximation to f exists with the specified degree, is it uniquely determined?

3. Construction. If a best polynomial approximation to f exists with the specified degree, how can it be determined ?

There are, of course, many other problems that can be considered. For example, if a unique best polynomial P_n of degree $\leq n$ exists, we may wish to obtain upper bounds for $\|f - P_n\|$ that can be used to satisfy practical requirements. Or we may ask whether $\|f - P_n\| \to 0$ as $n \to \infty$ for the given norm or possibly for some other norm. If so, we say that the polynomial approximations converge to f in this norm. In such a case arbitrarily close approximations exist relative to this norm if n is sufficiently large. These examples illustrate some of the types of problems considered in the general theory of polynomial approximation. In this introductory treatment we restrict our attention primarily to the three problems of existence, uniqueness, and construction, as described above.

For approximation by Taylor polynomials these three problems can be completely solved. If f has a derivative of order n at a point a, it is easy to prove that the best polynomial approximation of degree $\leq n$ relative to the Taylor seminorm for this n is the Taylor polynomial

$$P(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x - a)^k. \tag{15.3}$$

In fact, for this polynomial we have

$$\|f - P\| = \sum_{k=0}^{n} |f^{(k)}(a) - P^{(k)}(a)| = 0,$$

so the inequality $\|f - P\| \leq \|f - Q\|$ is trivially satisfied for all polynomials Q. Therefore P is a best polynomial approximation relative to this seminorm. To establish uniqueness, we consider any polynomial Q of degree $\leq n$ such that $\|f - Q\| = 0$. This equation implies that

$$Q(a) = f(a), \qquad Q'(a) = f'(a), \qquad \ldots, \qquad Q^{(n)}(a) = f^{(n)}(a).$$

From Theorem 7.1 of Volume I we know that the Taylor polynomial in (15.3) is the only polynomial satisfying all these equations. Therefore Q = P. Equation (15.3) also solves the problem of construction.

All three problems can also be solved for any norm derived from an inner product. In this case, Theorem 1.16 tells us that there is a unique polynomial in S for which the norm $\|f - P\|$ is as small as possible. In fact, this P is the projection off on S and is given by an explicit formula,

$$P(x) = \sum_{k=0}^{n} (P, e_i)e_i(x),$$

where $e_0, e_1, \ldots, e_n$ are functions forming an orthonormal basis for S.

For example, if C is the space of real functions continuous on the interval [- 1, 1] and if

$$(f, g) = \int_{-1}^{1} f(x)g(x)\,dx,$$

the normalized Legendre polynomials φ_0 , φ_1, . . . , φ_n form an orthonormal basis for S, and the projection f_n off on S is given by

$$f_n(x) = \sum_{k=0}^{n} (f, \varphi_k)\varphi_k(x), \qquad \text{where} \quad (f, \varphi_k) = \int_{-1}^{1} f(t)\varphi_k(t)\,dt.$$

We recall that the normalized Legendre polynomials are given by

$$\varphi_k(x) = \sqrt{\frac{2k+1}{2}}\, P_k(x), \qquad \text{where} \quad P_k(x) = \frac{1}{2^k k!}\frac{d^k}{dx^k}(x" - 1)“.$$

The first six normalized polynomials are

$$\varphi_0(x) = \sqrt{\tfrac{1}{2}}, \qquad \varphi_1(x) = \sqrt{\tfrac{3}{2}}\,x, \qquad \varphi_2(x) = \tfrac{1}{2}\sqrt{\tfrac{5}{2}}\,(3x2 - 1), \qquad \varphi_3(x) = \tfrac{1}{2}\sqrt{\tfrac{7}{2}}\,(5x^3 - 3x),$$

$$\varphi_4(x) = \tfrac{1}{8}\sqrt{\tfrac{9}{2}}\,(35x4 - 30x^2 + 3)) \qquad \varphi_5(x) = \tfrac{1}{8}\sqrt{\tfrac{11}{2}}\,(63x^5 - 70x^3 + 15x).$$

The corresponding problems for the interpolation seminorm will be treated next in Section 15.6. In later sections we discuss polynomial approximation relative to the max norm.

15.5 Exercises

1. Prove that each of the following collections of functions is a linear space.
   (a) All polynomials.
   (b) All polynomials of degree $\leq n$.
   (c) All functions continuous on an interval I.
   (d) All functions having a derivative at each point of I.
   (e) All functions having a derivative of order n at each point of I.
   (f) All functions having a derivative of order n at a fixed point x_0.
   (g) All functions having power-series expansions in a neighborhood of a given point x_0 .
2. Determine whether or not each of the following collections of real-valued functions is a linear space.
   (a) All polynomials of degree II.
   (b) All functions defined and bounded on an interval [a, b].
   (c) All step functions defined on an interval [a, b].
   (d) All functions monotonic on an interval [a, b].
   (e) All functions integrable on an interval [a, b].
   (f) All functions that are piecewise monotonic on an interval [a, b].
   (g) All functions that can be expressed in the form $f - g$, where f and g are monotonic increasing on an interval [a, b].
3. Let C denote the linear space of real-valued functions continuous on an interval [a, b]. A function N is defined on C by the equation given. In each case, determine which of the four

properties of a norm are satisfied by N, and determine thereby whether N is a norm, a seminorm, or neither.

(a) $N(f) = f(a)$.

(b) $N(f) = |f(a)|$.

(c) $N(f) = |f(b) - f(a)|$.

(d) $N(f) = \int_a^b f(x)\,dx$.

(e) $N(f) = \left|\int_a^b f(x)\,dx\right|$.

(f) $N(f) = \int_a^b |f(x)|\,dx$.

(g) $N(f) = \int_a^b |f(x)|^2\,dx$.

(h) $N(f) = \left|\int_a^b f(x)\,dx\right|^2$.

4. Let C be the linear space of functions continuous on an interval $[a, b]$. If $f \in C$, define

$$N(f) = \max_{a \le x \le b} |f(x)|.$$

Show that N is a norm for C.

5. Let B denote the linear space of all real-valued functions that are defined and bounded on an interval $[a, b]$. If $f \in B$, define

$$N(f) = \sup_{a \le x \le b} |f(x)|,$$

where the symbol on the right stands for the supremum (least upper bound) of the set of all numbers $|f(x)|$ for x in $[a, b]$. Show that N is a norm for B. This is called the ***sup norm.***

6. Refer to Exercise 3. Determine which of the given functions N have the property that $N(fg) \le N(f)N(g)$ for all f and g in C.

7. For a fixed integer $n \ge 1$, let S be the set of all functions having a derivative of order n at a fixed point x_0. If $f \in S$, let

$$N(f) = \sum_{k=0}^{n} \frac{1}{k!} |f^{(k)}(x_0)|.$$

(a) Show that N is a seminorm on S.

(b) Show that $N(fg) \le N(f)N(g)$ for all f, g in S. Prove also that the Taylor seminorm does not have this property.

8. Let f be a real continuous function on the interval $[-1, 1]$.

(a) Prove that the best quadratic polynomial approximation relative to the square norm on $[-1, 1]$ is given by

$$P(x) = \tfrac{1}{2}\int_{-1}^{1} f(t)\,dt + \tfrac{3}{2}x\int_{-1}^{1} tf(t)\,dt + \tfrac{5}{8}(3x^2 - 1)\int_{-1}^{1}(3t^2 - 1)f(t)\,dt.$$

(b) Find a similar formula for the best polynomial approximation of degree ≤ 4.

9. Calculate constants a, b, c so that the integral $\int_{-1}^{1} |e^x - (a + bx + cx^2)|^2\,dx$ will be as small as possible.

10. Let $f(x) = |x|$ for $-1 \le x \le 1$. Determine the polynomial of degree ≤ 4 that best approximates f on $[-1, 1]$ relative to the square norm.

11. Let C denote the linear space of real continuous functions on $[a, b]$ with inner product $(f, g) = \int_a^b f(x)g(x)\,dx$. Let $e_0, \ldots, e_n$ be an orthonormal basis for the subspace S of polynomials of degree $\le n$. Let P be the polynomial in S that best approximates a given f in C relative to the square norm.

(a) Prove that the square of the norm of the error is given by

$$\|f - P\|^2 = \|f\|^2 - \sum_{k=0}^{n} (f, e_k)^2.$$

(b) Calculate this error explicitly when $[a, b] = [-1, 1]$, $n = 2$, and $f(x) = |x|$.

12. Let $f(x) = 1/x$ for $x \neq 0$.
(a) Show that the constant polynomial P that best approximates f over the interval $[1, n]$ relative to the square norm is $P(x) = (\log n)/(n - 1)$. Compute $\|P - f\|^2$ for this P.
(b) Find the linear polynomial P that best approximates f over the interval $[1, n]$ relative to the square norm. Compute $\|P - f\|^2$ for this P when $n = 2$.
13. Let $f(x) = e^x$.
(a) Show that the constant polynomial P that best approximates f over the interval $[0, n]$ relative to the square norm is $P(x) = (e^n - 1)/n$. Compute $\|P - f\|^2$ for this P.
(b) Find the linear polynomial P that best approximates f over the interval $[0, 1]$ relative to the square norm. Compute $\|P - f\|^2$ for this P.
14. Let $P_0, P_1, \ldots, P_n$ be $n + 1$ polynomials orthonormal on $[a, b]$ relative to the inner product in Exercise 11. Assume also that P_k has degree k.
(a) Prove that any three consecutive polynomials in this set are connected by a recurrence relation of the form

$$P_{k+1}(x) = (a_k x + b_k)P_k(x) + c_k P_{k-1}(x)$$

for $1 \leq k \leq n - 1$, where a_k, b_k, c_k are constants.
(b) Determine this recurrence relation explicitly when the polynomials are the orthonormal Legendre polynomials.
15. Refer to Exercise 14, and let p_k denote the coefficient of x^k in $P_k(x)$.
(a) Show that $a_k = p_{k+1}/p_k$.
(b) Use the recurrence relation in Exercise 14 to derive the formula

$$\sum_{k=0}^{m} P_k(x)P_k(y) = \frac{p_m}{p_{m+1}} \frac{P_{m+1}(x)P_m(y) - P_m(x)P_{m+1}(y)}{x - y},$$

valid for $x \neq y$. Discuss also the limiting case $x = y$.

15.6 Interpolating polynomials

We turn now to approximation by interpolation polynomials. The values of a function f are known at $n + 1$ distinct points $x_0, x_1, \ldots, x_n$ and we seek a polynomial P of degree $\leq n$ that satisfies the conditions

$$P(x_0) = f(x_0), \quad P(x_1) = f(x_1), \quad \ldots, \quad P(x_n) = f(x_n). \tag{15.4}$$

First we prove that if such a polynomial exists it is unique. Then we prove it exists by explicit construction. This polynomial minimizes the distance from f to P, measured in the interpolation seminorm for this n,

$$\|f - P\| = \sum_{k=0}^{n} |f(x_k) - P(x_k)|.$$

Since this distance is 0 if P satisfies (15.4), the interpolating polynomial P is the best approximation relative to this seminorm.

THEOREM 15.1. UNIQUENESS THEOREM. *Given $n + 1$ distinct points $x_0, x_1, \ldots, x_n$, let P and Q be two polynomials of degree $\leq n$ such that*

$$P(x_k) = Q(x_k)$$

for each $k = 0, 1, 2, \ldots, n$. Then $P(x) = Q(x)$ for all x.

Proof. Let $R(x) = P(x) - Q(x)$. The function R is a polynomial of degree $\leq n$ which has $n + 1$ distinct zeros at the points $x_0, x_1, \ldots, x_n$. The only polynomial with this property is the zero polynomial. Therefore $R(x) = 0$ for all x, so $P(x) = Q(x)$ for all x.

The interpolating polynomial P can be constructed in many ways. We describe first a method of Lagrange. Let $A(x)$ be the polynomial given by the equation

$$A(x) = (x - x_0)(x - x_1) \cdots (x - x_n) = \prod_{j=0}^{n} (x - x_j). \tag{15.5}$$

This polynomial has a simple zero at each of the points x_j. Let $A_k(x)$ denote the polynomial of degree n obtained from $A(x)$ by deleting the factor $x - x_k$. That is, let

$$A_k(x) = \prod_{\substack{j=0 \\ j \neq k}}^{n} (x - x_j). \tag{15.6}$$

The polynomial $A_k(x)$ has a simple zero at each point $x_j \neq x_k$. At the point x_k itself we have

$$A_k(x_k) = \prod_{\substack{j=0 \\ j \neq k}}^{n} (x_k - x_j). \tag{15.7}$$

This is nonzero since no factor in the product is zero. Therefore the polynomial $A_k(x)/A_k(x_k)$ has the value 1 when $x = x_k$ and the value 0 when $x = x_j$ for $x_j \neq x_k$. Now let

$$P(x) = \sum_{k=0}^{n} \frac{f(x_k)A_k(x)}{A_k(x_k)}.$$

When $x = x_j$, each term in this sum vanishes except the jth term, which has the value $f(x_j)$. Therefore $P(x_j) = f(x_j)$ for each j. Since each term of this sum is a polynomial of degree n, the sum itself is a polynomial of degree $\leq n$. Thus, we have found a polynomial satisfying the required conditions. These results can be summarized by the following theorem:

THEOREM 15.2. ***Given $n + 1$ distinct points $x_0, x_1, \ldots, x_n$ and $n + 1$ real numbers $f(x_0), f(x_1), \ldots, f(x_n)$, not necessarily distinct, there exists one and only one polynomial P of degree $\leq n$ such that $P(x_j) = f(x_j)$ for each $j = 0, 1, 2, \ldots, n$. This polynomial is given by the formula***

$$P(x) = \sum_{k=0}^{n} \frac{f(x_k)A_k(x)}{A_k(x_k)}, \tag{15.8}$$

where $A_k(x)$ is the polynomial defined by (15.6).

Formula (15.8) for $P(x)$ is called *Lagrange's interpolation formula.* We can write it in the form

$$P(x) = \sum_{k=0}^{n} f(x_k)L_k(x),$$

where $L_k(x)$ is a polynomial of degree n given by

$$L_k(x) = \frac{A_k(x)}{A_k(x_k)}. \tag{15.9}$$

Thus, for each fixed x, $P(x)$ is a linear combination of the prescribed values $f(x_0), f(x_1), \ldots, f(x_n)$. The multipliers $L_k(x)$ depend only on the points $x_0, x_1, \ldots, x_n$ and not on the prescribed values. They are called *Lagrange interpolation coefficients.* If we use the formulas in (15.6) and (15.7) we can write Equation (15.9) in the form

$$L_k(x) = \prod_{\substack{j=0 \\ j \neq k}}^{n} \frac{x - x_j}{x_k - x_j}. \tag{15.10}$$

This product formula provides an efficient method for evaluating the number $L_k(x)$ for a given x.

Note: The Lagrange coefficients $L_k(x)$ are often expressed in the form

$$L_k(x) = \frac{A_k(x)}{A'(x_k)},$$

where A' is the derivative of the polynomial in (15.5). To prove this formula it suffices to show that $A'(x_k) = A_k(x_k)$. Differentiating the relation

$$A(x) = (x - x_k)A_k(x)$$

we obtain $A'(x) = (x - x_k)A_k'(x) + A_k(x)$. When $x = x_k$ this gives us $A'(x_k) = A_k(x_k)$.

EXAMPLE. Determine the polynomial of degree ≤ 3 that takes the values y_0, y_1, y_2, y_3 at the points $-2, -1, 1, 2$, respectively.

Solution. We take $x_0 = -2$, $x_1 = -1$, $x_2 = 1$, $x_3 = 2$. The polynomials $L_k(x)$ in (15.10) are given by the formulas

$$L_0(x) = \frac{(x+1)(x-1)(x-2)}{(-2+1)(-2-1)(-2-2)} = -\frac{1}{12}(x+1)(x-1)(x-2),$$

$$L_1(x) = \frac{(x+2)(x-1)(x-2)}{(-1+2)(-1-1)(-1-2)} = \frac{1}{6}(x+2)(x-1)(x-2),$$

$$L_2(x) = \frac{(x+2)(x+1)(x-2)}{(1+2)(1+1)(1-2)} = -\frac{1}{6}(x+2)(x+1)(x-2),$$

$$L_3(x) = \frac{(x+2)(x+1)(x-1)}{(2+2)(2+1)(2-1)} = \frac{1}{12}(x+2)(x+1)(x-1).$$