2.12 Exercises

In all exercises involving the vector space V_n, the usual basis of unit coordinate vectors is to be chosen unless another basis is specifically mentioned. In exercises concerned with the matrix of a linear transformation $T: V \to W$ where $V = W$, we take the same basis in both V and W unless another choice is indicated.

1. Determine the matrix of each of the following linear transformations of V_n into V_n:
   (a) the identity transformation,
   (b) the zero transformation,
   (c) multiplication by a fixed scalar c.
2. Determine the matrix for each of the following projections.
   (a) $T: V_3 \to V_2$, where $T(x_1, x_2, x_3) = (x_1, x_2)$.
   (b) $T: V_3 \to V_2$, where $T(x_1, x_2, x_3) = (x_2, x_3)$.
   (c) $T: V_5 \to V_3$, where $T(x_1, x_2, x_3, x_4, x_5) = (x_2, x_3, x_4)$.
3. A linear transformation $T: V_2 \to V_2$ maps the basis vectors i andj as follows:

$$T(i) = i + j, \qquad T(j) = 2i - j.$$

   (a) Compute $T(3i - 4j)$ and $T^2(3i - 4j)$ in terms of i and j.
   (b) Determine the matrix of T and of T^2.
   (c) Solve part (b) if the basis (i, j) is replaced by (e_1, e_2), where $e_1 = i - j$, $e_2 = 3i + j$.
4. A linear transformation $T: V_2 \to V_2$ is defined as follows: Each vector (x, y) is reflected in the y-axis and then doubled in length to yield $T(x, y)$. Determine the matrix of T and of T^2.
5. Let $T: V_3 \to V_3$ be a linear transformation such that

$$T(k) = 2i + 3j + 5k, \qquad T(j+k) = i, \qquad T(i+j+k) = j - k.$$

   (a) Compute $T(i + 2j + 3k)$ and determine the nullity and rank of T.
   (b) Determine the matrix of T.
6. For the linear transformation in Exercise 5, choose both bases to be (e_1, e_2, e_3), where $e_1 = (2, 3, 5)$, $e_2 = (1, 0, 0)$, $e_3 = (0, 1, -1)$, and determine the matrix of T relative to the new bases.
7. A linear transformation $T: V_3 \to V_2$ maps the basis vectors as follows: $T(i) = (0, 0)$, $T(j) = (1, 1)$, $T(k) = (1, -1)$.
   (a) Compute $T(4i - j + k)$ and determine the nullity and rank of T.
   (b) Determine the matrix of T.
   (c) Use the basis (i, j, k) in V_3 and the basis (w_1, w_2) in V_2, where $w_1 = (1, 1)$, $w_2 = (1, 2)$. Determine the matrix of T relative to these bases.
   (d) Find bases (e_1, e_2, e_3) for V_3 and (w_1, w_2) for V_2 relative to which the matrix of T will be in diagonal form.
8. A linear transformation $T: V_2 \to V_3$ maps the basis vectors as follows: $T(i) = (1, 0, 1)$, $T(j) = (-1, 0, 1)$.
   (a) Compute $T(2i - 3j)$ and determine the nullity and rank of T.
   (b) Determine the matrix of T.
   (c) Find bases (e_1, e_2) for V_2 and (w_1, w_2, w_3) for V_3 relative to which the matrix of T will be in diagonal form.
9. Solve Exercise 8 if $T(i) = (1, 0, 1)$ and $T(j) = (1, 1, 1)$.
10. Let V and W be linear spaces, each with dimension 2 and each with basis (e_1, e_2). Let $T: V \to W$ be a linear transformation such that $T(e_1 + e_2) = 3e_1 + 9e_2$, $T(3e_1 + 2e_2) = 7e_1 + 23e_2$.
    (a) Compute $T(e_2 - e_1)$ and determine the nullity and rank of T.
    (b) Determine the matrix of T relative to the given basis.

(c) Use the basis (e_1, e_2) for V and find a new basis of the form $(e_1 + ae_2, 2e_1 + be_2)$ for W, relative to which the matrix of T will be in diagonal form.

In the linear space of all real-valued functions, each of the following sets is independent and spans a finite-dimensional subspace V. Use the given set as a basis for V and let $D: V \to V$ be the differentiation operator. In each case, find the matrix of D and of D^2 relative to this choice of basis.

11. $(\sin x, \cos x)$.
12. $(1, x, e^x)$.
13. $(1, 1 + x, 1 + x + e^x)$.
14. (e^x, xe^x).
15. $(-\cos x, \sin x)$.
16. $(\sin x, \cos x, x \sin x, x \cos x)$.
17. $(e^x \sin x, e^x \cos x)$.
18. $(e^{2x} \sin 3x, e^{2x} \cos 3x)$.

19. Choose the basis $(1, x, x^2, x^3)$ in the linear space V of all real polynomials of degree ≤ 3. Let D denote the differentiation operator and let $T: V \to V$ be the linear transformation which maps $p(x)$ onto $xp'(x)$. Relative to the given basis, determine the matrix of each of the following transformations: (a) T; (b) DT; (c) TD; (d) $TD - DT$; (e) T^2; (f) $T^2D^2 - D^2T^2$.
20. Refer to Exercise 19. Let W be the image of V under TD. Find bases for V and for W relative to which the matrix of TD is in diagonal form.

2.13 Linear spaces of matrices

We have seen how matrices arise in a natural way as representations of linear transformations. Matrices can also be considered as objects existing in their own right, without necessarily being connected to linear transformations. As such, they form another class of mathematical objects on which algebraic operations can be defined. The connection with linear transformations serves as motivation for these definitions, but this connection will be ignored for the moment.

Let m and n be two positive integers, and let $I_{m,n}$ be the set of all pairs of integers (i, j) such that $1 \leq i \leq m$, $1 \leq j \leq n$. Any function A whose domain is $I_{m,n}$ is called an $m \times n$ *matrix*. The function value $A(i, j)$ is called the *ij-entry* or *ij-element* of the matrix and will be denoted also by a_{ij}. It is customary to display all the function values in a rectangular array consisting of m rows and n columns, as follows:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

The elements a_{ij} may be arbitrary objects of any kind. Usually they will be real or complex numbers, but sometimes it is convenient to consider matrices whose elements are other objects, for example, functions. We also denote matrices by the more compact notation

$$A = (a_{ij})_{i,j=1}^{m,n} \qquad \text{or} \qquad A = (a_{ij}).$$

If $m = n$, the matrix is said to be a *square matrix*. A $1 \times n$ matrix is called a *row matrix*; an $m \times 1$ matrix is called a *column matrix*.

Two functions are equal if and only if they have the same domain and take the same function value at each element in the domain. Since matrices are functions, two matrices $A = (a_{ij})$ and $B = (b_{ij})$ are equal if and only if they have the same number of rows, the same number of columns, and equal entries $a_{ij} = b_{ij}$ for each pair (i, j).

Now we assume the entries are numbers (real or complex) and we define addition of matrices and multiplication by scalars by the same method used for any real- or complex-valued functions.

DEFINITION. *If $A = (a_{ij})$ and $B = (b_{ij})$ are two $m \times n$ matrices and if c is any scalar, we define matrices $A + B$ and cA as follows:*

$$A + B = (a_{ij} + b_{ij}), \qquad cA = (ca_{ij}).$$

The sum is defined only when A and B have the same size.

EXAMPLE. If

$$A = \begin{bmatrix} 1 & 2 & -3 \\ -1 & 0 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5 & 0 & 1 \\ 1 & -2 & 3 \end{bmatrix},$$

then we have

$$A + B = \begin{bmatrix} 6 & 2 & -2 \\ 0 & -2 & 7 \end{bmatrix}, \quad 2A = \begin{bmatrix} 2 & 4 & -6 \\ -2 & 0 & 8 \end{bmatrix}, \quad (-1)B = \begin{bmatrix} -5 & 0 & -1 \\ -1 & 2 & -3 \end{bmatrix}.$$

We define the zero matrix 0 to be the $m \times n$ matrix all of whose elements are 0. With these definitions, it is a straightforward exercise to verify that the collection of all $m \times n$ matrices is a linear space. We denote this linear space by $M_{m,n}$. If the entries are real numbers, the space $M_{m,n}$ is a real linear space. If the entries are complex, $M_{m,n}$ is a complex linear space. It is also easy to prove that this space has dimension mn. In fact, a basis for $M_{m,n}$ consists of the mn matrices having one entry equal to 1 and all others equal to 0. For example, the six matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

form a basis for the set of all 2×3 matrices.

2.14 Isomorphism between linear transformations and matrices

We return now to the connection between matrices and linear transformations. Let V and W be finite-dimensional linear spaces with $\dim V = n$ and $\dim W = m$. Choose a basis $(e_1, \ldots, e_n)$ for V and a basis $(w_1, \ldots, w_m)$ for W. In this discussion, these bases are kept fixed. Let $\mathscr{L}(V, W)$ denote the linear space of all linear transformations of V into W. If $T \in \mathscr{L}(V, W)$, let $m(T)$ denote the matrix of T relative to the given bases. We recall that $m(T)$ is defined as follows.

The image of each basis element e_k is expressed as a linear combination of the basis elements in W:

(2.19)
$$T(e_k) = \sum_{i=1}^{m} t_{ik} w_i \qquad \text{for} \quad k = 1, 2, \ldots, n.$$

The scalar multipliers t_{ik} are the ik-entries of $m(T)$. Thus, we have

(2.20)
$$m(T) = (t_{ik})_{i,k=1}^{m,n}.$$

Equation (2.20) defines a new function m whose domain is $\mathscr{L}(V, W)$ and whose values are matrices in $M_{m,n}$. Since every $m \times n$ matrix is the matrix $m(T)$ for some T in $\mathscr{L}(V, W)$, the range of m is $M_{m,n}$. The next theorem shows that the transformation $m: \mathscr{L}(V, W) \to M_{m,n}$ is linear and one-to-one on $\mathscr{L}(V, W)$.

THEOREM 2.15. ISOMORPHISM THEOREM. *For all S and T in $\mathscr{L}(V, W)$ and all scalars c, we have*

$$m(S + T) = m(S) + m(T) \qquad \textit{and} \qquad m(cT) = cm(T).$$

Moreover,

$$m(S) = m(T) \qquad \textit{implies} \qquad S = T,$$

so m is one-to-one on $\mathscr{L}(V, W)$.

Proof. The matrix $m(T)$ is formed from the multipliers t_{ik} in (2.19). Similarly, the matrix $m(S)$ is formed from the multipliers s_{ik} in the equations

(2.21)
$$S(e_k) = \sum_{i=1}^{m} s_{ik} w_i \qquad \text{for} \quad k = 1, 2, \ldots, n.$$

Since we have

$$(S + T)(e_k) = \sum_{i=1}^{m} (s_{ik} + t_{ik}) w_i \qquad \text{and} \qquad (cT)(e_k) = \sum_{i=1}^{m} (ct_{ik}) w_i,$$

we obtain $m(S + T) = (s_{ik} + t_{ik}) = m(S) + m(T)$ and $m(cT) = (ct_{ik}) = cm(T)$. This proves that m is linear.

To prove that m is one-to-one, suppose that $m(S) = m(T)$, where $S = (s_{ik})$ and $T = (t_{ik})$. Equations (2.19) and (2.21) show that $S(e_k) = T(e_k)$ for each basis element e_k, so $S(x) = T(x)$ for all x in V, and hence $S = T$.

Note: The function m is called an ***isomorphism.*** For a given choice of bases, m establishes a one-to-one correspondence between the set of linear transformations $\mathscr{L}(V, W)$ and the set of $m \times n$ matrices $M_{m,n}$. The operations of addition and multiplication by scalars are preserved under this correspondence. The linear spaces $\mathscr{L}(V, W)$ and $M_{m,n}$ are said to be ***isomorphic.*** Incidentally, Theorem 2.11 shows that the domain of a one-to-one linear transformation has the same dimension as its range. Therefore, $\dim \mathscr{L}(V, W) = \dim M_{m,n} = mn$.

If $V = W$ and if we choose the same basis in both V and W, then the matrix $m(I)$ which corresponds to the identity transformation $I: V \to V$ is an $n \times n$ diagonal matrix with each diagonal entry equal to 1 and all others equal to 0. This is called the *identity* or *unit matrix* and is denoted by I or by I_n.

2.15 Multiplication of matrices

Some linear transformations can be multiplied by means of composition. Now we shall define multiplication of matrices in such a way that the product of two matrices corresponds to the composition of the linear transformations they represent.

We recall that if $T: U \to V$ and $S: V \to W$ are linear transformations, their composition $ST: U \to W$ is a linear transformation given by

$$ST(x) = S[T(x)] \quad \text{for all } x \text{ in } U.$$

Suppose that U, V, and W are finite-dimensional, say

$$\dim U = n, \qquad \dim V = p, \qquad \dim W = m.$$

Choose bases for U, V, and W. Relative to these bases, the matrix $m(S)$ is an $m \times p$ matrix, the matrix T is a $p \times n$ matrix, and the matrix of ST is an $m \times n$ matrix. The following definition of matrix multiplication will enable us to deduce the relation $m(ST) = m(S)m(T)$. This extends the isomorphism property to products.

DEFINITION. *Let A be any $m \times p$ matrix, and let B be any $p \times n$ matrix, say*

$$A = (a_{ij})_{i,j=1}^{m,p} \quad \text{and} \quad B = (b_{ij})_{i,j=1}^{p,n}.$$

Then the product AB is defined to be the $m \times n$ matrix $C = (c_{ij})$ whose ij-entry is given by

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}. \tag{2.22}$$

Note: The product AB is not defined unless the number of columns of A is equal to the number of rows of B.

If we write A_i for the ith row of A, and B^j for the jth column of B, and think of these as p-dimensional vectors, then the sum in (2.22) is simply the dot product $A_i \cdot B^j$. In other words, the ij-entry of AB is the dot product of the ith row of A with the jth column of B:

$$AB = (A_i \cdot B^j)_{i,j=1}^{m,n}.$$

Thus, matrix multiplication can be regarded as a generalization of the dot product.

EXAMPLE 1. Let $A = \begin{bmatrix} 3 & 1 & 2 \\ -1 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 6 \\ 5 & -1 \\ 0 & 2 \end{bmatrix}$. Since A is 2×3 and B is 3×2, the product AB is the 2×2 matrix

$$AB = \begin{bmatrix} A_1 \cdot B^1 & A_1 \cdot B^2 \\ A_2 \cdot B^1 & A_2 \cdot B^2 \end{bmatrix} = \begin{bmatrix} 17 & 21 \\ 1 & -7 \end{bmatrix}.$$

The entries of AB are computed as follows:

$$A_1 \cdot B^1 = 3 \cdot 4 + 1 \cdot 5 + 2 \cdot 0 = 17, \qquad A_1 \cdot B^2 = 3 \cdot 6 + 1 \cdot (-1) + 2 \cdot 2 = 21,$$

$$A_2 \cdot B^1 = (-1) \cdot 4 + 1 \cdot 5 + 0 \cdot 0 = 1, \qquad A_2 \cdot B^2 = (-1) \cdot 6 + 1 \cdot (-1) + 0 \cdot 2 = -7.$$

EXAMPLE 2. Let

$$A = \begin{bmatrix} 2 & 1 & -3 \\ 1 & 2 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}.$$

Here A is 2×3 and B is 3×1, so AB is the 2×1 matrix given by

$$AB = \begin{bmatrix} A_1 \cdot B^1 \\ A_2 \cdot B^1 \end{bmatrix} = \begin{bmatrix} -9 \\ 8 \end{bmatrix},$$

since $A_1 \cdot B^1 = 2 \cdot (-2) + 1 \cdot 1 + (-3) \cdot 2 = -9$ and $A_2 \cdot B^1 = 1 \cdot (-2) + 2 \cdot 1 + 4 \cdot 2 = 8$.

EXAMPLE 3. If A and B are both square matrices of the same size, then both AB and BA are defined. For example, if

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 4 \\ 5 & 2 \end{bmatrix},$$

we find that

$$AB = \begin{bmatrix} 13 & 8 \\ 2 & -2 \end{bmatrix}, \qquad BA = \begin{bmatrix} -1 & 10 \\ 3 & 12 \end{bmatrix}.$$

This example shows that in general $AB \neq BA$. If $AB = BA$, we say A and B *commute.*

EXAMPLE 4. If I_p is the $p \times p$ identity matrix, then $I_pA = A$ for every $p \times n$ matrix A, and $BI_p = B$ for every $m \times p$ matrix B. For example,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix},$$

Now we prove that the matrix of a composition ST is the product of the matrices $m(S)$ and $m(T)$.

THEOREM 2.16. *Let $T: U \to V$ and $S: V \to W$ be linear transformations, where U, V, W are finite-dimensional linear spaces. Then, for a fixed choice of bases, the matrices of S, T, and ST are related by the equation*

$$m(ST) = m(S)m(T).$$

Proof. Assume $\dim U = n$, $\dim V = p$, $\dim W = m$. Let $(u_1, \ldots, u_n)$ be a basis for U, $(v_1, \ldots, v_p)$ a basis for V, and $(w_1, \ldots, w_m)$ a basis for W. Relative to these bases, we have

$$m(S) = (s_{ij})_{i,j=1}^{m,p}, \quad \text{where} \quad S(v_k) = \sum_{i=1}^{m} s_{ik} w_i \quad \text{for} \quad k = 1, 2, \ldots, p,$$

and

$$m(T) = (t_{ij})_{i,j=1}^{p,n}, \quad \text{where} \quad T(u_j) = \sum_{k=1}^{p} t_{kj} v_k \quad \text{for} \quad j = 1, 2, \ldots, n.$$

Therefore, we have

$$ST(u_j) = S[T(u_j)] = \sum_{k=1}^{p} t_{kj} S(v_k) = \sum_{k=1}^{p} t_{kj} \sum_{i=1}^{m} s_{ik} w_i = \sum_{i=1}^{m} \left(\sum_{k=1}^{p} s_{ik} t_{kj} \right) w_i,$$

so we find that

$$m(ST) = \left(\sum_{k=1}^{p} s_{ik} t_{kj} \right)_{i,j=1}^{m,n} = m(S)m(T).$$

We have already noted that matrix multiplication does not always satisfy the commutative law. The next theorem shows that it does satisfy the associative and distributive laws.

THEOREM 2.17. ASSOCIATIVE AND DISTRIBUTIVE LAWS FOR MATRIX MULTIPLICATION. *Given matrices A, B, C.*

(a) *If the products $A(BC)$ and $(AB)C$ are meaningful, we have*

$$A(BC) = (AB)C \qquad \text{(associative law).}$$

(b) *Assume A and B are of the same size. If AC and BC are meaningful, we have*

$$(A + B)C = AC + BC \qquad \text{(right distributive law),}$$

whereas if CA and CB are meaningjiil, we have

$$C(A + B) = CA + CB \qquad \text{(left distributive law).}$$

Proof. These properties can be deduced directly from the definition of matrix multiplication, but we prefer the following type of argument. Introduce finite-dimensional linear spaces U, V, W, X and linear transformations $T: U \to V$, $S: V \to W$, $R: W \to X$ such that, for a fixed choice of bases, we have

$$A = m(R), \qquad B = m(S), \qquad C = m(T).$$

By Theorem 2.16, we have $m(RS) = AB$ and $m(ST) = BC$ From the associative law for composition, we find that $R(ST) = (RS)T$. Applying Theorem 2.16 once more to this equation, we obtain $m(R)m(ST) = m(RS)m(T)$ or $A(BC) = (AB)C$, which proves (a). The proof of (b) can be given by a similar type of argument.

DEFINITION. *If A is a square matrix, we define integral powers of A inductively as follows:*

$$A” = I, \qquad A^n = AA^{n-1} \quad \text{for } n \geq 1.$$

2.16 Exercises

1. If $A = \begin{bmatrix} 1 & -4 & 2 \\ -1 & 4 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 5 & -2 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 2 \\ 1 & -1 \\ 1 & -3 \end{bmatrix}$, compute $B + C$, AB, BA, AC, CA, $A(2B - 3C)$.

2. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$. Find all 2 x 2 matrices B such that (a) $AB = 0$; (b) $BA = 0$.

3. In each case find a, b, c, d to satisfy the given equation.

(a) $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 6 \\ 5 \end{bmatrix}$; (b) $\begin{bmatrix} a & b & c & d \\ 1 & 4 & 9 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 6 & 6 \\ 1 & 9 & 8 & 4 \end{bmatrix}$.

4. Calculate $AB - BA$ in each case.

(a) $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 1 & 1 \\ -4 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix}$;

(b) $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 & -2 \\ 3 & -2 & 4 \\ -3 & 5 & 11 \end{bmatrix}$.

5. If A is a square matrix, prove that $A^n A^m = A^{m+n}$ for all integers $m \geq 0$, $n \geq 0$.

6. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Verify that $A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and compute A^n.

7. Let $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Verify that $A^2 = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$ and compute $A”$.

8. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Verify that $A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$. Compute A^3 and A^4. Guess a general formula for $A”$ and prove it by induction,

9. Let $A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$. Prove that $A^2 = 2A - Z$ and compute A^{100}.

10. Find all 2×2 matrices A such that $A^2 = 0$.

11. (a) Prove that a 2×2 matrix A commutes with every 2×2 matrix if and only if A commutes with each of the four matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

(b) Find all such matrices A.

12. The equation $A^2 = Z$ is satisfied by each of the 2×2 matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ c & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & b \\ 0 & -1 \end{bmatrix},$$

where b and c are arbitrary real numbers. Find all 2×2 matrices A such that $A^2 = I$.

13. If $A = \begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 6 \\ 9 & 8 \end{bmatrix}$, find 2×2 matrices C and D such that $AC = B$ and $DA = B$.

14. (a) Verify that the algebraic identities

$$(A + B)^2 = A^2 + 2AB + B^2 \qquad \text{and} \qquad (A + B)(A - B) = A^2 - B^2$$

do not hold for the 2×2 matrices $A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$.

(b) Amend the right-hand members of these identities to obtain formulas valid for all square matrices A and B.

(c) For which matrices A and B are the identities valid as stated in (a)?

2.17 Systems of linear equations

Let A $= (a_{ij})$ be a given $m \times n$ matrix of numbers, and let $c_1, \ldots, c_m$ be m further numbers. A set of m equations of the form

$$\sum_{k=1}^{n} a_{ik} x_k = c_i \qquad \text{for } i = 1, 2, \ldots, m, \tag{2.23}$$

is called a system of m linear equations in n unknowns. Here $x_1, \ldots, x_n$ are regarded as unknown. A *solution* of the system is any n-tuple of numbers $(x_1, \ldots, x_n)$ for which all the equations are satisfied. The matrix A is called the *coefficient-matrix* of the system.

Linear systems can be studied with the help of linear transformations. Choose the usual bases of unit coordinate vectors in V_n and in V_m. The coefficient-matrix A determines a linear transformation, $T\colon V_n \to V_m$, which maps an arbitrary vector x = $(x_1, \ldots,$ x,) in V_n onto the vector y = $(y_1, \ldots, y_m)$ in V_m given by the m linear equations

$$y_i = \sum_{k=1}^{n} a_{ik} x_k \qquad \text{for} \quad i = 1, 2, \ldots, m.$$

Let c = $(c_1, \ldots,$ c,) be the vector in V_m whose components are the numbers appearing in system (2.23). This system can be written more simply as;

$$T(x) = c.$$

The system has a solution if and only if c is in the range of T. If exactly one x in V_n maps onto c, the system has exactly one solution. If more than one x maps onto c, the system has more than one solution.

EXAMPLE: 1. *A system with no solution.* The system x + y = 1, $x + y = 2$ has no solution. 'The sum of two numbers cannot be both I and 2.

EXAMPLE 2. *A system with exactly one solution.* The system x + y = 1, x − y = 0 has exactly one solution: (x, y) = $(\frac{1}{2}, \frac{1}{2})$.

EXAMPLE: 3. *A system with more than one solution.* The system x + y = 1 , consisting of one equation in two unknowns, has more than one solution. Any two numbers whose sum is 1 gives ii solution.

With each linear system (2.23), we can associate another system

$$\sum_{k=1}^{n} a_{ik} x_k = 0 \qquad \text{for} \quad i = 1, 2, \ldots, m,$$

obtained by replacing each c_i in (2.23) by 0. This is called the *homogeneous system* corresponding to (2.23). If c $\neq O$, system (2.23) is called a *nonhomogeneous system.* A vector x in V_n will satisfy the homogeneous system if and only if

$$T(x) = O,$$

where T is the linear transformation determined by the coefficient-matrix. The homogeneous system always has one solution, namely x = O, but it may have others. The set of solutions of the homogeneous system is the null space of T. The next theorem describes the relation between solutions of the homogeneous system and those of the nonhomogeneous system.

THEOREM 2.18. *Assume the nonhomogeneous system (2.23) has a solution, say b.*

(a) *If a vector x is a solution of the nonhomogeneous system, then the vector v = x − b is a solution of the corresponding homogeneous system.*

(b) *If a vector v is a solution of the homogeneous+ system, then the vector x = v + b is a solution of the nonhomogeneous system.*

Proof. Let $T: V_n \to V_m$ be the linear transformation determined by the coefficient-matrix, as described above. Since b is a solution of the nonhomogeneous system we have $T(b) = c$. Let x and v be two vectors in V_n such that $v = x - b$. Then we have

$$T(v) = T(x - b) = T(x) - T(b) = T(x) - c.$$

Therefore $T(x) = c$ if and only if $T(v) = 0$. This proves both (a) and (b).

This theorem shows that the problem of finding all solutions of a nonhomogeneous system splits naturally into two parts: (1) Finding all solutions v of the homogeneous system, that is, determining the null space of T; and (2) finding one particular solution b of the nonhomogeneous system. By adding b to each vector v in the null space of T, we thereby obtain all solutions $x = v + b$ of the nonhomogeneous system.

Let k denote the dimension of $N(T)$ (the nullity of T). If we can find k *independent* solutions $v_1, \ldots, v_k$ of the homogeneous system, they will form a basis for $N(T)$, and we can obtain every v in $N(T)$ by forming all possible linear combinations

$$v = t_1 v_1 + \cdots + t_k v_k,$$

where $t_1, \ldots, t_k$ are arbitrary scalars. This linear combination is called the *general solution of the homogeneous system.* If b is one particular solution of the nonhomogeneous system, then all solutions x are given by

$$x = b + t_1 v_1 + \cdots + t_k v_k.$$

This linear combination is called the *general solution of the nonhomogeneous system.* Theorem 2.18 can now be restated as follows.

THEOREM 2.19. *Let $T: V_n \to V_m$ be the linear transformation such that $T(x) = y$, where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_m)$ and*

$$y_i = \sum_{k=1}^{n} a_{ik} x_k \qquad \text{for } i = 1, 2, \ldots, m.$$

Let k denote the nullity of T. If $v_1, \ldots, v_k$ are k independent solutions of the homogeneous system $T(x) = 0$, and if b is one particular solution of the nonhomogeneous system $T(x) = c$, then the general solution of the nonhomogeneous system is

$$x = b + t_1 v_1 + \cdots + t_k v_k,$$

where $t_1, \ldots, t_k$ are arbitrary scalars.

This theorem does not tell us how to decide if a nonhomogeneous system has a particular solution b, nor does it tell us how to determine solutions $v_1, \ldots, v_k$ of the homogeneous system. It does tell us what to expect when the nonhomogeneous system has a solution. The following example, although very simple, illustrates the theorem.

EXAMPLE. The system $x + y = 2$ has for its associated homogeneous system the equation $x + y = 0$. Therefore, the null space consists of all vectors in V_2 of the form $(t, -t)$, where t is arbitrary. Since $(t, -t) = t(1, -1)$, this is a one-dimensional subspace of V_2 with basis $(1, -1)$. A particular solution of the nonhomogeneous system is $(0, 2)$. Therefore the general solution of the nonhomogeneous system is given by

$$(x, y) = (0, 2) + t(1, -1) \qquad \text{or} \qquad x = t, \quad y = 2 - t,$$

where t is arbitrary.

2.18 Computation techniques

We turn now to the problem of actually computing the solutions of a nonhomogeneous linear system. Although many methods have been developed for attacking this problem, all of them require considerable computation if the system is large. For example, to solve a system of ten equations in as many unknowns can require several hours of hand computation, even with the aid of a desk calculator.

We shall discuss a widely-used method, known as the *Gauss-Jordan elimination method,* which is relatively simple and can be easily programmed for high-speed electronic computing machines. The method consists of applying three basic types of operations on the equations of a linear system:

(1) Interchanging two equations;
(2) Multiplying all the terms of an equation by a nonzero scalar:
(3) Adding to one equation a multiple of another.

Each time we perform one of these operations on the system we obtain a new system having exactly the same solutions. Two such systems are called *equivalent.* By performing these operations over and over again in a systematic fashion we finally arrive at an equivalent system which can be solved by inspection.

We shall illustrate the method with some specific examples. It will then be clear how the method is to be applied in general.

EXAMPLE 1. *A system with a unique solution.* Consider the system

$$\begin{aligned} 2x - 5y + 4z &= -3 \\ x - 2y + z &= 5 \\ x - 4y + 6z &= 10. \end{aligned}$$

This particular system has a unique solution, $x = 124$, $y = 75$, $z = 31$, which we shall obtain by the Gauss-Jordan elimination process. To save labor we do not bother to copy the letters x, y, z and the equals sign over and over again, but work instead with the *augmented matrix*

$$\left[\begin{array}{ccc|c} 2 & -5 & 4 & -3 \\ 1 & -2 & 1 & 5 \\ 1 & -4 & 6 & 10 \end{array}\right] \tag{2.24}$$

obtained by adjoining the right-hand members of the system to the coefficient matrix. The three basic types of operation mentioned above are performed on the rows of the augmented matrix and are called *row operations*. At any stage of the process we can put the letters x, y, z back again and insert equals signs along the vertical line to obtain equations. Our ultimate goal is to arrive at the augmented matrix

$$\text{(2.25)}\qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 124 \\ 0 & 1 & 0 & 75 \\ 0 & 0 & 1 & 31 \end{array}\right]$$

after a succession of row operations. The corresponding system of equations is x = 124, y = 75, z = 31, which gives the desired solution.

The first step is to obtain a 1 in the upper left-hand corner of the matrix. We can do this by interchanging the first row of the given matrix (2.24) with either the second or third row. Or, we can multiply the first row by $\frac{1}{2}$. Interchanging the first and second rows, we get

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 5 \\ 2 & -5 & 4 & -3 \\ 1 & -4 & 6 & 10 \end{array}\right].$$

The next step is to make all the remaining entries in the first column equal to zero, leaving the first row intact. To do this we multiply the first row by -2 and add the result to the second row. Then we multiply the first row by -1 and add the result to the third row. After these two operations, we obtain

$$\text{(2.26)}\qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 5 \\ 0 & -1 & 2 & -13 \\ 0 & -2 & 5 & 5 \end{array}\right].$$

Now we repeat the process on the smaller matrix $\left[\begin{array}{cc|c} -1 & 2 & -13 \\ -2 & 5 & 5 \end{array}\right]$ which appears adjacent to the two zeros. We can obtain a 1 in *its* upper left-hand corner by multiplying the second row of (2.26) by -1. This gives us the matrix

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 5 \\ 0 & 1 & -2 & 13 \\ 0 & -2 & 5 & 5 \end{array}\right].$$

Multiplying the second row by 2 and adding the result to the third, we get

$$\text{(2.27)}\qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 5 \\ 0 & 1 & -2 & 13 \\ 0 & 0 & 1 & 31 \end{array}\right].$$

At this stage, the corresponding system of equations is given by

$$\begin{aligned} x - 2y + z &= 5 \\ y - 2z &= 13 \\ z &= 31. \end{aligned}$$

These equations can be solved in succession, starting with the third one and working backwards, to give us

$$z = 31, \qquad y = 13 + 2z = 13 + 62 = 75, \qquad x = 5 + 2y - z = 5 + 150 - 31 = 124.$$

Or, we can continue the Gauss-Jordan process by making all the entries zero above the diagonal elements in the second and third columns. Multiplying the second row of (2.27) by 2 and adding the result to the first row, we obtain

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 31 \\ 0 & 1 & -2 & 13 \\ 0 & 0 & 1 & 31 \end{array}\right].$$

Finally, we multiply the third row by 3 and add the result to the first row, and then multiply the third row by 2 and add the result to the second row to get the matrix in (2.25).

EXAMPLE 2. *A system with more than one solution.* Consider the following system of 3 equations in 5 unknowns:

$$\begin{aligned} 2x - 5y + 4z + u - v &= -3 \\ x - 2y + z - u + v &= 5 \\ x - 4y + 6z + 2u - v &= 10. \end{aligned} \tag{2.28}$$

The corresponding augmented matrix is

$$\left[\begin{array}{ccccc|c} 2 & -5 & 4 & 1 & -1 & -3 \\ 1 & -2 & 1 & -1 & 1 & 5 \\ 1 & -4 & 6 & 2 & -1 & 10 \end{array}\right].$$

The coefficients of x, y, z and the right-hand members are the same as those in Example 1. If we perform the same row operations used in Example 1, we finally arrive at the augmented matrix

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & -16 & 19 & 124 \\ 0 & 1 & 0 & -9 & 11 & 75 \\ 0 & 0 & 1 & -3 & 4 & 31 \end{array}\right].$$

The corresponding system of equations can be solved for x, y, and z in terms of u and v, giving us

$$\begin{aligned} x &= 124 + 16u - 19v \\ y &= 75 + 9u - 11v \\ z &= 31 + 3u - 4v. \end{aligned}$$

If we let $u = t_1$ and $v = t_2$, where t_1 and t_2 are arbitrary real numbers, and determine x, y, z by these equations, the vector (x, y, z, u, v) in V_5 given by

$$(x, y, z, u, v) = (124 + 16t_1 - 19t_2,\ 75 + 9t_1 - 11t_2,\ 31 + 3t_1 - 4t_2,\ t_1,\ t_2)$$

is a solution. By separating the parts involving t_1 and t_2, we can rewrite this as follows:

$$(x, y, z, u, v) = (124, 75, 31, 0, 0) + t_1(16, 9, 3, 1, 0) + t_2(-19, -11, -4, 0, 1).$$

This equation gives the general solution of the system. The vector $(124, 75, 31, 0, 0)$ is a particular solution of the nonhomogeneous system (2.28). The two vectors $(16, 9, 3, 1, 0)$ and $(-19, -11, -4, 0, 1)$ are solutions of the corresponding homogeneous system. Since they are independent, they form a basis for the space of all solutions of the homogeneous system.

EXAMPLE 3. *A system with no solution.* Consider the system

$$\begin{aligned} 2x - 5y + 4z &= -3 \\ x - 2y + z &= 5 \\ x - 4y + 5z &= 10. \end{aligned} \tag{2.29}$$

This system is almost identical to that of Example 1 except that the coefficient of z in the third equation has been changed from 6 to 5. The corresponding augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & -5 & 4 & -3 \\ 1 & -2 & 1 & 5 \\ 1 & -4 & 5 & 10 \end{array}\right].$$

Applying the same row operations used in Example 1 to transform (2.24) into (2.27), we arrive at the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 5 \\ 0 & 1 & -2 & 13 \\ 0 & 0 & 0 & 31 \end{array}\right]. \tag{2.30}$$

When the bottom row is expressed as an equation, it states that $0 = 31$. Therefore the original system has no solution since the two systems (2.29) and (2.30) are equivalent.

In each of the foregoing examples, the number of equations did not exceed the number of unknowns. If there are more equations than unknowns, the Gauss-Jordan process is still applicable. For example, suppose we consider the system of Example 1, which has the solution $x = 124$, $y = 75$, $z = 31$. If we adjoin a new equation to this system which is also satisfied by the same triple, for example, the equation $2x - 3y + z = 54$, then the elimination process leads to the agumented matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 124 \\ 0 & 1 & 0 & 75 \\ 0 & 0 & 1 & 31 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

with a row of zeros along the bottom. But if we adjoin a new equation which is not satisfied by the triple (124, 75, 31), for example the equation $x + y + z = 1$, then the elimination process leads to an augmented matrix of the form

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 124 \\ 0 & 1 & 0 & 75 \\ 0 & 0 & 1 & 31 \\ 0 & 0 & 0 & a \end{array}\right],$$

where $a \neq 0$. The last row now gives a contradictory equation $0 = a$ which shows that the system has no solution.

2.19 Inverses of square matrices

Let $A = (a_{ij})$ be a square $n \times n$ matrix. If there is another $n \times n$ matrix B such that $BA = I$, where Z is the $n \times n$ identity matrix, then A is called *nonsingular* and B is called a *left inverse* of A.

Choose the usual basis of unit coordinate vectors in V_n and let $T: V_n \to V_n$ be the linear transformation with matrix $m(T) = A$. Then we have the following.

THEOREM 2.20. *The matrix A is nonsingular if and only if T is invertible. If $BA = I$ then $B = m(T^{-1})$.*

Proof. Assume that A is nonsingular and that $BA = I$. We shall prove that $T(x) = O$ implies $x = O$. Given x such that $T(x) = O$, let X be the $n \times 1$ column matrix formed from the components of x. Since $T(x) = O$, the matrix product AX is an $n \times 1$ column matrix consisting of zeros, so $B(AX)$ is also a column matrix of zeros. But $B(AX) = (BA)X = IX = X$, so every component of x is 0. Therefore, Tis invertible, and the equation $TT^{-1} =$ Z implies that $m(T)m(T^{-1}) =$ Z or $Am(T^{-1}) = I$. Multiplying on the left by B, we find $m(T^{-1}) = B$. Conversely, if T is invertible, then $T^{-1}T$ is the identity transformation so $m(T^{-1})m(T)$ is the identity matrix. Therefore A is nonsingular and $m(T^{-1})A = I$.

All the properties of invertible linear transformations have their counterparts for nonsingular matrices. In particular, left inverses (if they exist) are unique, and every left inverse is also a right inverse. In other words, if A is nonsingular and $BA = I$, then $AB = I$. We call B the *inverse* of A and denote it by A^{-1}. The inverse A^{-1} is also nonsingular and *its* inverse is A.

Now we show that the problem of actually determining the entries of the inverse of a nonsingular matrix is equivalent to solving n separate nonhomogeneous linear systems.

Let $A = (a_{ij})$ be nonsingular and let $A^{-1} = (b_{ij})$ be its inverse. The entries of A and A^{-1} are related by the n^2 equations

$$\sum_{k=1}^{n} a_{ik} b_{kj} = \delta_{ij}, \tag{2.31}$$

where $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ if $i \neq j$. For each fixed choice of j, we can regard this as a nonhomogeneous system of n linear equations in n unknowns $b_{1j}, b_{2j}, \ldots, b_{nj}$. Since A is nonsingular, each of these systems has a unique solution, the jth column of B. All these systems have the same coefficient-matrix A and differ only in their right members. For example, if A is a 3×3 matrix, there are 9 equations in (2.31) which can be expressed as 3 separate linear systems having the following augmented matrices :

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & 1 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \end{array}\right], \quad \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 1 \\ a_{31} & a_{32} & a_{33} & 0 \end{array}\right], \quad \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 1 \end{array}\right].$$

If we apply the Gauss-Jordan process, we arrive at the respective augmented matrices

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & b_{11} \\ 0 & 1 & 0 & b_{21} \\ 0 & 0 & 1 & b_{31} \end{array}\right], \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & b_{12} \\ 0 & 1 & 0 & b_{22} \\ 0 & 0 & 1 & b_{32} \end{array}\right], \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & b_{13} \\ 0 & 1 & 0 & b_{23} \\ 0 & 0 & 1 & b_{33} \end{array}\right].$$

In actual practice we exploit the fact that all three systems have the same coefficient-matrix and solve all three systems at once by working with the enlarged matrix

$$\left[\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right].$$

The elimination process then leads to

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & b_{11} & b_{12} & b_{13} \\ 0 & 1 & 0 & b_{21} & b_{22} & b_{23} \\ 0 & 0 & 1 & b_{31} & b_{32} & b_{33} \end{array}\right].$$