

# 14

## Complex Numbers in Algebra

### 14.1 Impossible Numbers

Over the last few chapters it has often been claimed that certain mysteries—de Moivre’s formula for  $\sin n\theta$  (Section 6.6), factorization of polynomials (Section 6.7), classification of cubic curves (Section 8.4), branch points (Section 10.5), genus (Section 11.3), and behavior of elliptic functions (Sections 11.6 and 12.6)—are clarified by the introduction of complex numbers. That complex numbers do all this and more is one of the miracles of mathematics. At the beginning of their history, complex numbers  $a + b\sqrt{-1}$  were considered to be “impossible numbers,” tolerated only in a limited algebraic domain because they seemed useful in the solution of cubic equations. But their significance turned out to be geometric and ultimately led to the unification of algebraic functions with conformal mapping, potential theory, and another “impossible” field, noneuclidean geometry. This resolution of the paradox of  $\sqrt{-1}$  was so powerful, unexpected, and beautiful that only the word “miracle” seems adequate to describe it.

In the present chapter we shall see how complex numbers emerged from the theory of equations and enabled its fundamental theorem to be proved—at which point it became clear that complex numbers had meaning far beyond algebra. Their impact on curves and function theory, which is where conformal mapping and potential theory come in, is described in Chapters 15 and 16. Noneuclidean geometry had entirely different origins but arrived at the same place as function theory in the 1880s, thanks to complex numbers. This unexpected meeting is described in Chapter 18, after some geometric preparations in Chapter 17.

## 14.2 Quadratic Equations

The usual way to introduce complex numbers in a mathematics course is to point out that they are needed to solve certain quadratic equations, such as the equation  $x^2 + 1 = 0$ . However, this did not happen when quadratic equations first appeared, since at that time there was no *need* for all quadratic equations to have solutions. Many quadratic equations are implicit in Greek geometry, as one would expect when circles, parabolas, and the like are being investigated, but one does not demand that every geometric problem have a solution. If one asks whether a particular circle and line intersect, say, then the answer can be yes or no. If yes, the quadratic equation for the intersection has a solution; if no, it has no solution. An “imaginary solution” is uncalled for in this context.

Even when quadratic equations appeared in algebraic form, with Diophantus and the Arab mathematicians, there was initially no reason to admit complex solutions. One still wanted to know only whether there were real solutions, and if not the answer was simply—no solution. This is plainly the appropriate answer when quadratics are solved by geometrically completing the square (Section 6.3), as was still done up to the time of Cardano. A square of negative area did not exist in geometry. The story might have been different had mathematicians used symbols more and dared to consider the symbol  $\sqrt{-1}$  as an object in its own right, but this did not happen until quadratics had been overtaken by cubics, at which stage complex numbers became unavoidable, as we shall now see.

## 14.3 Cubic Equations

The del Ferro–Tartaglia–Cardano solution of the cubic equation

$$y^3 = py + q$$

is

$$y = \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}} + \sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}}$$

as we saw in Section 6.5. The formula involves complex numbers when  $(q/2)^2 - (p/3)^3 < 0$ . However, it is not possible to dismiss this as a case with no solution, *because a cubic always has at least one real root* (since  $y^3 - py - q$  is positive for sufficiently large positive  $y$  and negative for sufficiently large negative  $y$ ). Thus the Cardano formula raises the problem

of reconciling a real value, found by inspection, say, with an expression of the form

$$\sqrt[3]{a+b\sqrt{-1}} + \sqrt[3]{a-b\sqrt{-1}}$$

Cardano did not face up to this problem in his *Ars magna* [Cardano (1545)]. He did, it is true, once mention complex numbers, but in connection with a quadratic equation and accompanied by the comment that these numbers were “as subtle as they are useless” [Cardano (1545), Ch. 37, Rule II].

The first work to take complex numbers seriously and achieve the necessary reconciliation was Bombelli (1572). Bombelli worked out the formal algebra of complex numbers, with the particular aim of reducing expressions  $\sqrt[3]{a+b\sqrt{-1}}$  to the form  $c+d\sqrt{-1}$ . His method enabled him to show the reality of some expressions resulting from Cardano’s formula. For example, the solution of

$$x^3 = 15x + 4$$

is

$$x = \sqrt[3]{2+11\sqrt{-1}} + \sqrt[3]{2-11\sqrt{-1}}$$

according to the formula. On the other hand, inspection gives the solution  $x = 4$ . Bombelli had the hunch that the two parts of  $x$  in the Cardano formula were of the form  $2+n\sqrt{-1}$ ,  $2-n\sqrt{-1}$ , and he found by cubing these expressions formally [using  $(\sqrt{-1})^2 = -1$ ] that indeed

$$\begin{aligned}\sqrt[3]{2+11\sqrt{-1}} &= 2 + \sqrt{-1}, \\ \sqrt[3]{2-11\sqrt{-1}} &= 2 - \sqrt{-1},\end{aligned}$$

hence the Cardano formula also gives  $x = 4$ .

Figure 14.1 is a facsimile of the manuscript page on which Bombelli stated his result. It is not hard to pick out the preceding expressions when one allows for the notation and the fact that  $11\sqrt{-1}$  is written as  $\sqrt{0-121}$ .

Much later, Hölder (1896) showed that any algebraic formula for the solution of the cubic must involve square roots of quantities that become negative for particular values of the coefficients. A proof of Hölder’s result may be found in van der Waerden (1949), p. 180.

#### EXERCISES

**14.3.1** Check that  $(2 + \sqrt{-1})^3 = 2 + 11\sqrt{-1}$ .