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Non-Euclidean Geometry and Hilbert's Axioms

The parallel postulate V, Euclid's fifth postulate, seems less natural or convincing than the others. Ever since Euclid's time, people have felt that it ought to be deducible from Euclid's other postulates I to IV or from some logically equivalent set of axioms.

Noteworthy attempts to prove Euclid's fifth postulate were made by Proclus (410–85 AD), Saccheri (1667–1733), Thibault (1775–1822), and many others. We now know that these attempts were doomed to fail. Postulate V is independent of I to IV and one of Euclid's contributions to mathematics was his implicit recognition of this fact by presenting V as an axiom.

Given 'absolute geometry' (that is, the geometry based only on postulates I to IV) there are a number of important statements equivalent to the parallel postulate:

- Through a point not on a line there is exactly one line parallel to that line — Playfair.

- Every segment is a side of a square (with four right angles) — Legendre.

- Not every pair of similar triangles is congruent — Wallis.

- Every triangle has a circumcircle — Legendre.

- There is at least one triangle whose angle sum is 180° — Legendre.

(Heath notes that the first of these is due to Proclus.)

The search for a proof of the parallel postulate led to the discovery of many such equivalent statements, but each one was felt to be insufficiently 'self-evident' or 'basic' to count as a proper Euclidean axiom. What was really wanted was a deduction of postulate V from postulates I to IV alone.

Gauss may have been the first person to suspect the truth. In a letter to Franz Taurinus, written in 1824, Gauss says that he is sure that the parallel postulate cannot be proved.

Consider the alternative postulate:

> **(H)** Through any point not on a line, there are at least two lines through that point and parallel to that line.

If we replace Euclid's parallel postulate by (H), we get the axioms of 'hyperbolic geometry'. It seems that Gauss believed hyperbolic geometry to be consistent.

The first person to publish results in hyperbolic geometry was the Russian N. I. Lobachevsky (1793–1856), of the University of Kasan, in 1829. In the same year, essentially the same results were discovered independently by the Hungarian J. Bolyai.

It was not until 1868 that it was proved that postulates I to IV do not imply postulate V. In that year, E. Beltrami (1835–1900) gave a Euclidean model for hyperbolic geometry. This showed that, if hyperbolic geometry contained any logical contradiction (for example, the assertion that both (H) and V are true), then that contradiction could be translated into a contradiction in Euclidean geometry. Since, presumably, there is no inconsistency in Euclidean geometry, there is none in hyperbolic geometry either.

In 1882, in the first article ever published in *Acta Mathematica*, Henri Poincaré (1854–1912) gave a sketch of a second Euclidean model for hyperbolic geometry. This model goes as follows. We interpret 'point' as 'point of the Cartesian plane in the interior of the unit circle $x^2 + y^2 = 1$'. We interpret a 'line' to mean either a 'diameter of the unit circle (minus endpoints)', or else 'a circular arc in the interior of the unit circle and orthogonal to it'. (Two arcs are 'orthogonal' if they intersect at right angles.)

'Betweenness' is defined in the obvious way. Segment equality is defined as follows. Let AB be a 'segment', that is, part of a diameter or orthogonal arc. Let A^e be the endpoint of that diameter or orthogonal arc which is on A's side of the diameter or arc. Let B^e be the other endpoint. Let

$$d(AB) = (AB^e/BB^e)(BA^e/AA^e),$$

where the segments on the right of this equation are ordinary Euclidean segments. Then two Poincaré segments AB and CD are 'equal' if and only if $d(AB) = d(CD)$.

The 'angle' between two Poincaré lines is the Euclidean angle between their tangents through the point where the lines meet. Angle equality is defined in the usual way.