for every ε. Since $\overline{\int}_I \max(f,g) - \underline{\int}_I \max(f,g)$ does not depend on ε, we thus see that

$$\overline{\int}_I \max(f,g) - \underline{\int}_I \max(f,g) = 0$$

and hence that $\max(f,g)$ is Riemann integrable.   □

Corollary 11.4.4 (Absolute values preserve Riemann integrability). *Let I be a bounded interval. If $f : I \to \mathbf{R}$ is a Riemann integrable function, then the positive part $f_+ := \max(f,0)$ and the negative part $f_- := \min(f,0)$ are also Riemann integrable on I. Also, the absolute value $|f| = f_+ - f_-$ is also Riemann integrable on I.*

Theorem 11.4.5 (Products preserve Riemann integrability). *Let I be a bounded interval. If $f : I \to \mathbf{R}$ and $g : I \to \mathbf{R}$ are Riemann integrable, then $fg : I \to \mathbf{R}$ is also Riemann integrable.*

Proof. This one is a little trickier. We split $f = f_+ + f_-$ and $g = g_+ + g_-$ into positive and negative parts; by Corollary 11.4.4, the functions f_+, f_-, g_+, g_- are Riemann integrable. Since

$$fg = f_+ g_+ + f_+ g_- + f_- g_+ + f_- g_-$$

then it suffices to show that the functions $f_+ g_+, f_+ g_-, f_- g_+, f_- g_-$ are individually Riemann integrable. We will just show this for $f_+ g_+$; the other three are similar.

Since f_+ and g_+ are bounded and positive, there are $M_1, M_2 > 0$ such that

$$0 \le f_+(x) \le M_1 \text{ and } 0 \le g_+(x) \le M_2$$

for all $x \in I$. Now let $\varepsilon > 0$ be arbitrary. Then, as in the proof of Theorem 11.4.3, we can find a piecewise constant function $\underline{f_+}$ minorizing f_+ on I, and a piecewise constant function $\overline{f_+}$ majorizing f_+ on I, such that

$$\int_I \overline{f_+} \le \int_I f_+ + \varepsilon$$

and

$$\int_I \underline{f_+} \geq \int_I f_+ - \varepsilon.$$

Note that f_+ may be negative at places, but we can fix this by replacing $\underline{f_+}$ by $\max(\underline{f_+}, 0)$, since this still minorizes f_+ (why?) and still has integral greater than or equal to $\int_I f_+ - \varepsilon$ (why?). So without loss of generality we may assume that $\underline{f_+}(x) \geq 0$ for all $x \in I$. Similarly we may assume that $\overline{f_+}(x) \leq M_1$ for all $x \in I$; thus

$$0 \leq \underline{f_+}(x) \leq f_+(x) \leq \overline{f_+}(x) \leq M_1$$

for all $x \in I$.

Similar reasoning allows us to find piecewise constant $\underline{g_+}$ minorizing g_+, and $\overline{g_+}$ majorizing g_+, such that

$$\int_I \overline{g_+} \leq \int_I g_+ + \varepsilon$$

and

$$\int_I \underline{g_+} \geq \int_I g_+ - \varepsilon,$$

and

$$0 \leq \underline{g_+}(x) \leq g_+(x) \leq \overline{g_+}(x) \leq M_2$$

for all $x \in I$.

Notice that $\underline{f_+}\,\underline{g_+}$ is piecewise constant and minorizes $f_+ g_+$, while $\overline{f_+}\,\overline{g_+}$ is piecewise constant and majorizes $f_+ g_+$. Thus

$$0 \leq \overline{\int_I} f_+ g_+ - \underline{\int_I} f_+ g_+ \leq \int_I \overline{f_+}\,\overline{g_+} - \underline{f_+}\,\underline{g_+}.$$

However, we have

$$\overline{f_+}(x)\overline{g_+}(x) - \underline{f_+}(x)\underline{g_+}(x) = \overline{f_+}(x)(\overline{g_+} - \underline{g_+})(x) + \underline{g_+}(x)(\overline{f_+} - \underline{f_+}(x))$$

$$\leq M_1(\overline{g_+} - \underline{g_+})(x) + M_2(\overline{f_+} - \underline{f_+}(x))$$

for all $x \in I$, and thus

$$0 \leq \overline{\int_I} f_+ g_+ - \underline{\int_I} f_+ g_+ \leq M_1 \int_I (\overline{g_+} - \underline{g_+}) + M_2 \int_I (\overline{f_+} - \underline{f_+})$$

$$\leq M_1(2\varepsilon) + M_2(2\varepsilon).$$

Again, since ε was arbitrary, we can conclude that f_+g_+ is Riemann integrable, as before. Similar argument show that f_+g_-, f_-g_+, f_-g_- are Riemann integrable; combining them we obtain that fg is Riemann integrable.   □

Exercise 11.4.1. Prove Theorem 11.4.1. (Hint: you may find Theorem 11.2.16 to be useful. For part (b): First do the case $c > 0$. Then do the case $c = -1$ and $c = 0$ separately. Using these cases, deduce the case of $c < 0$. You can use earlier parts of the theorem to prove later ones.)

Exercise 11.4.2. Let $a < b$ be real numbers, and let $f : [a, b] \to \mathbf{R}$ be a continuous, non-negative function (so $f(x) \geq 0$ for all $x \in [a, b]$). Suppose that $\int_{[a,b]} f = 0$. Show that $f(x) = 0$ for all $x \in [a, b]$. (Hint: argue by contradiction.)

Exercise 11.4.3. Let I be a bounded interval, let $f : I \to \mathbf{R}$ be a Riemann integrable function, and let $\mathbf{P}$ be a partition of I. Show that

$$\int_I f = \sum_{J \in \mathbf{P}} \int_J f.$$

Exercise 11.4.4. Without repeating all the computations in the above proofs, give a short explanation as to why the remaining cases of Theorem 11.4.3 and Theorem 11.4.5 follow automatically from the cases presented in the text. (Hint: from Theorem 11.4.1 we know that if f is Riemann integrable, then so is $-f$.)

11.5   Riemann integrability of continuous functions

We have already said a lot about Riemann integrable functions so far, but we have not yet actually produced any such functions other than the piecewise constant ones. Now we rectify this by showing that a large class of useful functions are Riemann integrable. We begin with the uniformly continuous functions.

Theorem 11.5.1. *Let I be a bounded interval, and let f be a function which is uniformly continuous on I. Then f is Riemann integrable.*

Proof. From Proposition 9.9.15 we see that f is bounded. Now we have to show that $\underline{\int}_I f = \overline{\int}_I f$.

If I is a point or the empty set then the theorem is trivial, so let us assume that I is one of the four intervals $[a, b]$, (a, b), $(a, b]$, or $[a, b)$ for some real numbers $a < b$.

Let $\varepsilon > 0$ be arbitrary. By uniform continuity, there exists a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in I$ are such that $|x - y| < \delta$. By the Archimedean principle, there exists an integer $N > 0$ such that $(b - a)/N < \delta$.

Note that we can partition I into N intervals $J_1, \ldots, J_N$, each of length $(b-a)/N$. (How? One has to treat each of the cases $[a, b]$, (a, b), $(a, b]$, $[a, b)$ slightly differently.) By Proposition 11.3.12, we thus have

$$\overline{\int}_I f \leq \sum_{k=1}^{N} (\sup_{x \in J_k} f(x))|J_k|$$

and

$$\underline{\int}_I f \geq \sum_{k=1}^{N} (\inf_{x \in J_k} f(x))|J_k|$$

so in particular

$$\overline{\int}_I f - \underline{\int}_I f \leq \sum_{k=1}^{N} (\sup_{x \in J_k} f(x) - \inf_{x \in J_k} f(x))|J_k|.$$

However, we have $|f(x) - f(y)| < \varepsilon$ for all $x, y \in J_k$, since $|J_k| = (b - a)/N < \delta$. In particular we have

$$f(x) < f(y) + \varepsilon \text{ for all } x, y \in J_k.$$

Taking suprema in x, we obtain

$$\sup_{x \in J_k} f(x) \leq f(y) + \varepsilon \text{ for all } y \in J_k,$$

and then taking infima in y we obtain

$$\sup_{x \in J_k} f(x) \leq \inf_{y \in J_k} f(y) + \varepsilon.$$

Inserting this bound into our previous inequality, we obtain

$$\overline{\int}_I f - \underline{\int}_I f \le \sum_{k=1}^{N} \varepsilon |J_k|,$$

but by Theorem 11.1.13 we thus have

$$\overline{\int}_I f - \underline{\int}_I f \le \varepsilon(b-a).$$

But $\varepsilon > 0$ was arbitrary, while $(b - a)$ is fixed. Thus $\overline{\int}_I f - \underline{\int}_I f$ cannot be positive. By Lemma 11.3.3 and the definition of Riemann integrability we thus have that f is Riemann integrable.
□

Combining Theorem 11.5.1 with Theorem 9.9.16, we thus obtain

Corollary 11.5.2. *Let $[a, b]$ be a closed interval, and let $f :$ $[a, b] \to \mathbf{R}$ be continuous. Then f is Riemann integrable.*

Note that this Corollary is not true if $[a, b]$ is replaced by any other sort of interval, since it is not even guaranteed then that continuous functions are bounded. For instance, the function $f : (0, 1) \to \mathbf{R}$ defined by $f(x) := 1/x$ is continuous but not Riemann integrable. However, if we assume that a function is both continuous *and* bounded, we can recover Riemann integrability:

Proposition 11.5.3. *Let I be a bounded interval, and let $f : I \to$ $\mathbf{R}$ be both continuous and bounded. Then f is Riemann integrable on I.*

Proof. If I is a point or an empty set then the claim is trivial; if I is a closed interval the claim follows from Corollary 11.5.2. So let us assume that I is of the form $(a, b]$, (a, b), or $[a, b)$ for some $a < b$.

We have a bound M for f, so that $-M \le f(x) \le M$ for all $x \in I$. Now let $0 < \varepsilon < (b-a)/2$ be a small number. The function

f when restricted to the interval $[a + \varepsilon, b - \varepsilon]$ is continuous, and hence Riemann integrable by Corollary 11.5.2. In particular, we can find a piecewise constant function $h : [a + \varepsilon, b - \varepsilon] \to \mathbf{R}$ which majorizes f on $[a + \varepsilon, b - \varepsilon]$ such that

$$\int_{[a+\varepsilon, b-\varepsilon]} h \leq \int_{[a+\varepsilon, b-\varepsilon]} f + \varepsilon.$$

Define $\tilde{h} : I \to \mathbf{R}$ by

$$\tilde{h}(x) := \begin{cases} h(x) & \text{if } x \in [a + \varepsilon, b - \varepsilon] \\ M & \text{if } x \in I \backslash [a + \varepsilon, b - \varepsilon] \end{cases}$$

Clearly $\tilde{h}$ is piecewise constant on I and majorizes f; by Theorem 11.2.16 we have

$$\int_I \tilde{h} = \varepsilon M + \int_{[a+\varepsilon, b-\varepsilon]} h + \varepsilon M \leq \int_{[a+\varepsilon, b-\varepsilon]} f + (2M + 1)\varepsilon.$$

In particular we have

$$\overline{\int}_I f \leq \int_{[a+\varepsilon, b-\varepsilon]} f + (2M + 1)\varepsilon.$$

A similar argument gives

$$\underline{\int}_I f \geq \int_{[a+\varepsilon, b-\varepsilon]} f - (2M + 1)\varepsilon$$

and hence

$$\overline{\int}_I f - \underline{\int}_I f \leq (4M + 2)\varepsilon.$$

But ε is arbitrary, and so we can argue as in the proof of Theorem 11.5.1 to conclude Riemann integrability. $\square$

This gives a large class of Riemann integrable functions already; the bounded continuous functions. But we can expand this class a little more, to include the bounded *piecewise* continuous functions.

Definition 11.5.4. Let I be a bounded interval, and let $f : I \to$ **R**. We say that f is *piecewise continuous on I* iff there exists a partition **P** of I such that $f|_J$ is continuous on J for all $J \in$ **P**.

Example 11.5.5. The function $f : [1,3] \to$ **R** defined by

$$F(x) := \begin{cases} x^2 & \text{if } 1 \leq x < 2 \\ 7 & \text{if } x = 2 \\ x^3 & \text{if } 2 < x \leq 3 \end{cases}$$

is not continuous on $[1,3]$, but it is piecewise continuous on $[1,3]$ (since it is continuous when restricted to $[1,2)$ or $\{2\}$ or $(2,3]$, and those three intervals partition $[1,3]$).

Proposition 11.5.6. *Let I be a bounded interval, and let $f : I \to$ **R** be both piecewise continuous and bounded. Then f is Riemann integrable.*

Proof. See Exercise 11.5.1.   □

Exercise 11.5.1. Prove Proposition 11.5.6. (Hint: use Theorem 11.4.1(a) and (h).)

11.6   Riemann integrability of monotone functions

In addition to piecewise continuous functions, another wide class of functions is Riemann integrable, namely the monotone functions. We give two instances of this:

Proposition 11.6.1. *Let $[a,b]$ be a closed and bounded interval and let $f : [a,b] \to$ **R** be a monotone function. Then f is Riemann integrable on $[a,b]$.*

Remark 11.6.2. From Exercise 9.8.5 we know that there exist monotone functions which are not piecewise continuous, so this proposition is not subsumed by Proposition 11.5.6.

Proof. Without loss of generality we may take f to be monotone increasing (instead of monotone decreasing). From Exercise 9.8.1 we know that f is bounded. Now let $N > 0$ be an integer, and partition $[a, b]$ into N half-open intervals $\{[a + \frac{b-a}{N}j, a + \frac{b-a}{N}(j+1)) : 0 \leq j \leq N - 1\}$ of length $(b-a)/N$, together with the point $\{b\}$. Then by Proposition 11.3.12 we have

$$\overline{\int}_I f \leq \sum_{j=0}^{N-1} \left(\sup_{x \in [a + \frac{b-a}{N}j, a + \frac{b-a}{N}(j+1))} f(x) \right) \frac{b-a}{N},$$

(the point $\{b\}$ clearly giving only a zero contribution). Since f is monotone increasing, we thus have

$$\overline{\int}_I f \leq \sum_{j=0}^{N-1} f\left(a + \frac{b-a}{N}(j+1)\right) \frac{b-a}{N}.$$

Similarly we have

$$\underline{\int}_I f \geq \sum_{j=0}^{N-1} f\left(a + \frac{b-a}{N}j\right) \frac{b-a}{N}.$$

Thus we have

$$\overline{\int}_I f - \underline{\int}_I f \leq \sum_{j=0}^{N-1} \left(f\left(a + \frac{b-a}{N}(j+1)\right) - f\left(a + \frac{b-a}{N}j\right) \right) \frac{b-a}{N}.$$

Using telescoping series (Lemma 7.2.15) we thus have

$$\overline{\int}_I f - \underline{\int}_I f \leq \left(f\left(a + \frac{b-a}{N}(N)\right) - f\left(a + \frac{b-a}{N}0\right) \right) \frac{b-a}{N}$$

$$= (f(b) - f(a)) \frac{b-a}{N}.$$

But N was arbitrary, so we can conclude as in the proof of Theorem 11.5.1 that f is Riemann integrable. $\square$

Corollary 11.6.3. *Let I be a bounded interval, and let $f : I \to \mathbf{R}$ be both monotone and bounded. Then f is Riemann integrable on I.*

Proof. See Exercise 11.6.1.   □

We now give the famous integral test for determining convergence of monotone decreasing series.

Proposition 11.6.4 (Integral test). *Let* $f : [0, \infty) \to \mathbf{R}$ *be a monotone decreasing function which is non-negative (i.e., $f(x) \geq 0$ for all $x \geq 0$). Then the sum $\sum_{n=0}^{\infty} f(n)$ is convergent if and only if $\sup_{N>0} \int_{[0,N]} f$ is finite.*

Proof. See Exercise 11.6.3.   □

Corollary 11.6.5. *Let p be a real number. Then $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges absolutely when $p > 1$ and diverges when $p \leq 1$.*

Proof. See Exercise 11.6.5.   □

Exercise 11.6.1. Use Proposition 11.6.1 to prove Corollary 11.6.3. (Hint: adapt the proof of Proposition 11.5.3.)

Exercise 11.6.2. Formulate a reasonable notion of a piecewise monotone function, and then show that all bounded piecewise monotone functions are Riemann integrable.

Exercise 11.6.3. Prove Proposition 11.6.4. (Hint: what is the relationship between the sum $\sum_{n=1}^{N} f(n)$, the sum $\sum_{n=0}^{N-1} f(n)$, and the integral $\int_{[0,N]} f$?)

Exercise 11.6.4. Give examples to show that both directions of the integral test break down if f is not assumed to be monotone decreasing.

Exercise 11.6.5. Use Proposition 11.6.4 to prove Corollary 11.6.5.

11.7   A non-Riemann integrable function

We have shown that there are large classes of bounded functions which are Riemann integrable. Unfortunately, there do exist bounded functions which are not Riemann integrable:

Proposition 11.7.1. *Let* $f : [0, 1] \to \mathbf{R}$ *be the discontinuous function*

$$f(x) := \begin{cases} 1 & \text{if } x \in \mathbf{Q} \\ 0 & \text{if } x \notin \mathbf{Q} \end{cases}$$

considered in Example 9.3.21. Then f *is bounded but not Riemann integrable.*

Proof. It is clear that f is bounded, so let us show that it is not Riemann integrable.

Let $\mathbf{P}$ be any partition of $[0, 1]$. For any $J \in \mathbf{P}$, observe that if J is not a point or the empty set, then

$$\sup_{x \in J} f(x) = 1$$

(by Proposition 5.4.14). In particular we have

$$(\sup_{x \in J} f(x))|J| = |J|.$$

(Note this is also true when J is a point, since both sides are zero.) In particular we see that

$$U(f, \mathbf{P}) = \sum_{J \in \mathbf{P}: J \neq \emptyset} |J| = [0, 1] = 1$$

by Theorem 11.1.13; note that the empty set does not contribute anything to the total length. In particular we have $\overline{\int}_{[0,1]} f = 1$, by Proposition 11.3.12.

A similar argument gives that

$$\inf_{x \in J} f(x) = 0$$

for all J (other than points or the empty set), and so

$$L(f, \mathbf{P}) = \sum_{J \in \mathbf{P}: J \neq \emptyset} 0 = 0.$$

In particular we have $\underline{\int}_{[0,1]} f = 0$, by Proposition 11.3.12. Thus the upper and lower Riemann integrals do not match, and so this function is not Riemann integrable. $\square$

Remark 11.7.2. As you can see, it is only rather "artificial" bounded functions which are not Riemann integrable. Because of this, the Riemann integral is good enough for a large majority of cases. There are ways to generalize or improve this integral, though. One of these is the *Lebesgue integral*, which we will define in Chapter 19. Another is the *Riemann-Stieltjes integral* $\int_I f d\alpha$, where $\alpha : I \to \mathbf{R}$ is a monotone increasing function, which we define in the next section.

11.8   The Riemann-Stieltjes integral

Let I be a bounded interval, let $\alpha : I \to \mathbf{R}$ be a monotone increasing function, and let $f : I \to \mathbf{R}$ be a function. Then there is a generalization of the Riemann integral, known as the *Riemann-Stieltjes integral*. This integral is defined just like the Riemann integral, but with one twist: instead of taking the length $|J|$ of intervals J, we take the α-length $\alpha[J]$, defined as follows. If J is a point or the empty set, then $\alpha[J] := 0$. If J is an interval of the form $[a, b]$, (a, b), $(a, b]$, or $[a, b)$, then $\alpha[J] := \alpha(b) - \alpha(a)$. Note that in the special case where α is the identity function $\alpha(x) := x$, then $\alpha[J]$ is just the same as $|J|$. However, for more general monotone functions α, the α-length $\alpha[J]$ is a different quantity from $|J|$. Nevertheless, it turns out one can still do much of the above theory, but replacing $|J|$ by $\alpha[J]$ throughout.

Definition 11.8.1 (α-length). Let I be a bounded interval, and let $\alpha : X \to \mathbf{R}$ be a function defined on some domain X which contains I. Then we define the *α-length* $\alpha[I]$ of I as follows. If I is a point or the empty set, we set $\alpha[I] = 0$. If I is an interval of the form $[a, b]$, $[a, b)$, $(a, b]$, or (a, b) for some $b > a$, then we set $\alpha[I] = \alpha(b) - \alpha(a)$.

Example 11.8.2. Let $\alpha : \mathbf{R} \to \mathbf{R}$ be the function $\alpha(x) := x^2$. Then $\alpha[[2, 3]] = \alpha(3) - \alpha(2) = 9 - 4 = 5$, while $\alpha[(-3, -2)] = -5$. Meanwhile $\alpha[\{2\}] = 0$ and $\alpha[\emptyset] = 0$.

Example 11.8.3. Let $\alpha : \mathbf{R} \to \mathbf{R}$ be the identity function $\alpha(x) :=$

x. Then $\alpha[I] = |I|$ for all bounded intervals I (why?) Thus the notion of length is a special case of the notion of α-length.

We sometimes write $\alpha|_a^b$ or $\alpha(x)|_{x=a}^{x=b}$ instead of $\alpha[[a, b]]$.

One of the key theorems for the theory of the Riemann integral was Theorem 11.1.13, which concerned length and partitions, and in particular showed that $|I| = \sum_{J \in \mathbf{P}} |J|$ whenever $\mathbf{P}$ was a partition of I. We now generalize this slightly.

Lemma 11.8.4. *Let I be a bounded interval, let $\alpha : X \to \mathbf{R}$ be a function defined on some domain X which contains I, and let $\mathbf{P}$ be a partition of I. Then we have*

$$\alpha[I] = \sum_{J \in \mathbf{P}} \alpha[J].$$

Proof. See Exercise 11.8.1.   □

We can now define a generalization of Definition 11.2.9.

Definition 11.8.5 (P.c. Riemann-Stieltjes integral). Let I be a bounded interval, and let $\mathbf{P}$ be a partition of I. Let $\alpha : X \to \mathbf{R}$ be a function defined on some domain X which contains I, and let $f : I \to \mathbf{R}$ be a function which is piecewise constant with respect to $\mathbf{P}$. Then we define

$$\text{p.c.} \int_{[\mathbf{P}]} f \; d\alpha := \sum_{J \in \mathbf{P}} c_J \alpha[J]$$

where c_J is the constant value of f on J.

Example 11.8.6. Let $f : [1, 3] \to \mathbf{R}$ be the function

$$f(x) = \begin{cases} 4 & \text{when } x \in [1, 2) \\ 2 & \text{when } x \in [2, 3], \end{cases}$$

let $\alpha : \mathbf{R} \to \mathbf{R}$ be the function $\alpha(x) := x^2$, and let $\mathbf{P}$ be the partition $\mathbf{P} := \{[1, 2), [2, 3]\}$. Then

$$\text{p.c.} \int_{[\mathbf{P}]} f \; d\alpha = c_{[1,2)} \alpha[[1, 2)] + c_{[2,3]} \alpha[[2, 3]]$$

$$= 4(\alpha(2) - \alpha(1)) + 2(\alpha(3) - \alpha(2)) = 4 \times 3 + 2 \times 5 = 22.$$

Example 11.8.7. Let $\alpha : \mathbf{R} \to \mathbf{R}$ be the identity function $\alpha(x) := x$. Then for any bounded interval I, any partition $\mathbf{P}$ of I, and any function f that is piecewise constant with respect to $\mathbf{P}$, we have $p.c. \int_{[\mathbf{P}]} f \ d\alpha = p.c. \int_{[\mathbf{P}]} f$ (why?).

We can obtain an exact analogue of Proposition 11.2.13 by replacing all the integrals $p.c. \int_{[\mathbf{P}]} f$ in the proposition with $p.c. \int_{[\mathbf{P}]} f$ (Exercise 11.8.2). We can thus define $p.c. \int_I f \ d\alpha$ for any piecewise constant function $f : I \to \mathbf{R}$ and any $\alpha : X \to \mathbf{R}$ defined on a domain containing I, in analogy to before, by the formula

$$p.c. \int_I f \ d\alpha := p.c. \int_{[\mathbf{P}]} f \ d\alpha$$

for any partition $\mathbf{P}$ on I with respect to which f is piecewise constant.

Up until now we have made no assumption on α. Let us now assume that α is *monotone increasing*, i.e., $\alpha(y) \geq \alpha(x)$ whenever $x, y \in X$ are such that $y \geq x$. This implies that $\alpha(I) \geq 0$ for all intervals in X (why?). From this one can easily verify that all the results from Theorem 11.2.16 continue to hold when the integrals $p.c. \int_I f$ are replaced by $p.c. \int_I f \ d\alpha$, and the lengths $|I|$ are replaced by the α-lengths $\alpha(I)$; see Exercise 11.8.3.

We can then define upper and lower Riemann-Stieltjes integrals $\overline{\int}_I f \ d\alpha$ and $\underline{\int}_I f \ d\alpha$ whenever $f : I \to \mathbf{R}$ is bounded and α is defined on a domain containing I, by the usual formulae

$$\overline{\int}_I f \ d\alpha := \inf\{p.c. \int_I g \ d\alpha : g \text{ is p.c. on } I \text{ and majorizes } f\}$$

and

$$\underline{\int}_I f \ d\alpha := \sup\{p.c. \int_I g \ d\alpha : g \text{ is p.c. on } I \text{ and minorizes } f\}.$$

We then say that f is *Riemann-Stieltjes integrable on I with respect to α* if the upper and lower Riemann-Stieltjes integrals match, in which case we set

$$\int_I f \ d\alpha := \overline{\int}_I f \ d\alpha = \underline{\int}_I f \ d\alpha.$$

As before, when α is the identity function $\alpha(x) := x$ then the Riemann-Stieltjes integral is identical to the Riemann integral; thus the Riemann-Stieltjes integral is a generalization of the Riemann integral. (We shall see another comparison between the two integrals a little later, in Corollary 11.10.3.) Because of this, we sometimes write $\int_I f$ as $\int_I f\, dx$ or $\int_I f(x)\, dx$.

Most (but not all) of the remaining theory of the Riemann integral then can be carried over without difficulty, replacing Riemann integrals with Riemann-Stieltjes integrals and lengths with α-lengths. There are a couple results which break down; Theorem 11.4.1(g), Proposition 11.5.3, and Proposition 11.5.6 are not necessarily true when α is discontinuous at key places (e.g., if f and α are both discontinuous at the same point, then $\int_I f\, d\alpha$ is unlikely to be defined. However, Theorem 11.5.1 is still true (Exercise 11.8.4).

Exercise 11.8.1. Prove Lemma 11.8.1. (Hint: modify the proof of Theorem 11.1.13.)

Exercise 11.8.2. State and prove a version of Proposition 11.2.13 for the Riemann-Stieltjes integral.

Exercise 11.8.3. State and prove a version of Theorem 11.2.16 for the Riemann-Stieltjes integral.

Exercise 11.8.4. State and prove a version of Theorem 11.5.1 for the Riemann-Stieltjes integral. (Hint: one has to be careful with the proof; the problem here is that some of the references to the length of $|J_k|$ should remain unchanged, and other references to the length of $|J_k|$ should be changed to the α-length $\alpha(J_k)$ - basically, all of the occurrences of $|J_k|$ which appear inside a summation should be replaced with $\alpha(J_k)$, but the rest should be unchanged.)

Exercise 11.8.5. Let sgn : $\mathbf{R} \to \mathbf{R}$ be the signum function

$$\mathrm{sgn}(x) := \begin{cases} 1 & \text{when } x > 0 \\ 0 & \text{when } x = 0 \\ -1 & \text{when } x < 0. \end{cases}$$

Let $f : [-1, 1] \to \mathbf{R}$ be a continuous function. Show that f is Riemann-Stieltjes integrable with respect to sgn, and that

$$\int_{[-1,1]} f\, d\mathrm{sgn} = f(0).$$

(Hint: for every $\varepsilon > 0$, find piecewise constant functions majorizing and minorizing f whose Riemann-Stieltjes integral is ε-close to $f(0)$.)

11.9   The two fundamental theorems of calculus

We now have enough machinery to connect integration and differentiation via the familiar fundamental theorem of calculus. Actually, there are two such theorems, one involving the derivative of the integral, and the other involving the integral of the derivative.

Theorem 11.9.1 (First Fundamental Theorem of Calculus). *Let $a < b$ be real numbers, and let $f : [a, b] \to \mathbf{R}$ be a Riemann integrable function. Let $F : [a, b] \to \mathbf{R}$ be the function*

$$F(x) := \int_{[a,x]} f.$$

Then F is continuous. Furthermore, if $x_0 \in [a, b]$ and f is continuous at x_0, then F is differentiable at x_0, and $F'(x_0) = f(x_0)$.

Proof. Since f is Riemann integrable, it is bounded (by Definition 11.3.4). Thus we have some real number M such that $-M \leq f(x) \leq M$ for all $x \in [a, b]$.

Now let $x < y$ be two elements of $[a, b]$. Then notice that

$$F(y) - F(x) = \int_{[a,y]} f - \int_{[a,x]} f = \int_{[x,y]} f$$

by Theorem 11.4.1(h). By Theorem 11.4.1(e) we thus have

$$\int_{[x,y]} f \leq \int_{[x,y]} M = p.c. \int_{[x,y]} M = M(y - x)$$

and

$$\int_{[x,y]} f \geq \int_{[x,y]} -M = p.c. \int_{[x,y]} -M = -M(y - x)$$

and thus

$$|F(y) - F(x)| \leq M(y - x).$$