The product of a scalar c and the matrix A is defined by

$$(cA)_{ij} = cA_{ij}. \tag{2-4}$$

Note that $F^{1\times n} = F^n$.

EXAMPLE 3. **The space of functions from a set to a field.** Let F be any field and let S be any non-empty set. Let V be the set of all functions from the set S into F. The sum of two vectors f and g in V is the vector $f + g$, i.e., the function from S into F, defined by

$$(f + g)(s) = f(s) + g(s). \tag{2-5}$$

The product of the scalar c and the function f is the function cf defined by

$$(cf)(s) = cf(s). \tag{2-6}$$

The preceding examples are special cases of this one. For an n-tuple of elements of F may be regarded as a function from the set S of integers $1, \ldots, n$ into F. Similarly, an $m \times n$ matrix over the field F is a function from the set S of pairs of integers, (i, j), $1 \leq i \leq m$, $1 \leq j \leq n$, into the field F. For this third example we shall indicate how one verifies that the operations we have defined satisfy conditions (3) and (4). For vector addition:

(a) Since addition in F is commutative,

$$f(s) + g(s) = g(s) + f(s)$$

for each s in S, so the functions $f + g$ and $g + f$ are identical.

(b) Since addition in F is associative,

$$f(s) + [g(s) + h(s)] = [f(s) + g(s)] + h(s)$$

for each s, so $f + (g + h)$ is the same function as $(f + g) + h$.

(c) The unique zero vector is the zero function which assigns to each element of S the scalar 0 in F.

(d) For each f in V, $(-f)$ is the function which is given by

$$(-f)(s) = -f(s).$$

The reader should find it easy to verify that scalar multiplication satisfies the conditions of (4), by arguing as we did with the vector addition.

EXAMPLE 4. **The space of polynomial functions over a field F.** Let F be a field and let V be the set of all functions f from F into F which have a rule of the form

$$f(x) = c_0 + c_1x + \cdots + c_nx^n \tag{2-7}$$

where $c_0, c_1, \ldots, c_n$ are fixed scalars in F (independent of x). A function of this type is called a **polynomial function on F**. Let addition and scalar multiplication be defined as in Example 3. One must observe here that if f and g are polynomial functions and c is in F, then $f + g$ and cf are again polynomial functions.

EXAMPLE 5. The field C of complex numbers may be regarded as a vector space over the field R of real numbers. More generally, let F be the field of real numbers and let V be the set of n-tuples $\alpha = (x_1, \ldots, x_n)$ where $x_1, \ldots, x_n$ are *complex* numbers. Define addition of vectors and scalar multiplication by (2-1) and (2-2), as in Example 1. In this way we obtain a vector space over the field R which is quite different from the space C^n and the space R^n.

There are a few simple facts which follow almost immediately from the definition of a vector space, and we proceed to derive these. If c is a scalar and 0 is the zero vector, then by 3(c) and 4(c)

$$c0 = c(0 + 0) = c0 + c0.$$

Adding $-(c0)$ and using 3(d), we obtain

$$c0 = 0. \tag{2-8}$$

Similarly, for the scalar 0 and any vector α we find that

$$0\alpha = 0. \tag{2-9}$$

If c is a non-zero scalar and α is a vector such that $c\alpha = 0$, then by (2-8), $c^{-1}(c\alpha) = 0$. But

$$c^{-1}(c\alpha) = (c^{-1}c)\alpha = 1\alpha = \alpha$$

hence, $\alpha = 0$. Thus we see that if c is a scalar and α a vector such that $c\alpha = 0$, then either c is the zero scalar or α is the zero vector.

If α is any vector in V, then

$$0 = 0\alpha = (1 - 1)\alpha = 1\alpha + (-1)\alpha = \alpha + (-1)\alpha$$

from which it follows that

$$(-1)\alpha = -\alpha. \tag{2-10}$$

Finally, the associative and commutative properties of vector addition imply that a sum involving a number of vectors is independent of the way in which these vectors are combined and associated. For example, if $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are vectors in V, then

$$(\alpha_1 + \alpha_2) + (\alpha_3 + \alpha_4) = [\alpha_2 + (\alpha_1 + \alpha_3)] + \alpha_4$$

and such a sum may be written without confusion as

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4.$$

Definition. *A vector β in V is said to be a* **linear combination** *of the vectors $\alpha_1, \ldots, \alpha_n$ in V provided there exist scalars $c_1, \ldots, c_n$ in F such that*

$$\begin{aligned}\beta &= c_1\alpha_1 + \cdots + c_n\alpha_n \\ &= \sum_{i=1}^{n} c_i\alpha_i.\end{aligned}$$

Other extensions of the associative property of vector addition and the distributive properties 4(c) and 4(d) of scalar multiplication apply to linear combinations:

$$\sum_{i=1}^{n} c_i\alpha_i + \sum_{i=1}^{n} d_i\alpha_i = \sum_{i=1}^{n} (c_i + d_i)\alpha_i$$

$$c \sum_{i=1}^{n} c_i\alpha_i = \sum_{i=1}^{n} (cc_i)\alpha_i.$$

Certain parts of linear algebra are intimately related to geometry. The very word 'space' suggests something geometrical, as does the word 'vector' to most people. As we proceed with our study of vector spaces, the reader will observe that much of the terminology has a geometrical connotation. Before concluding this introductory section on vector spaces, we shall consider the relation of vector spaces to geometry to an extent which will at least indicate the origin of the name 'vector space.' This will be a brief intuitive discussion.

Let us consider the vector space R^3. In analytic geometry, one identifies triples (x_1, x_2, x_3) of real numbers with the points in three-dimensional Euclidean space. In that context, a vector is usually defined as a directed line segment PQ, from a point P in the space to another point Q. This amounts to a careful formulation of the idea of the 'arrow' from P to Q. As vectors are used, it is intended that they should be determined by their length and direction. Thus one must identify two directed line segments if they have the same length and the same direction.

The directed line segment PQ, from the point $P = (x_1, x_2, x_3)$ to the point $Q = (y_1, y_2, y_3)$, has the same length and direction as the directed line segment from the origin $O = (0, 0, 0)$ to the point $(y_1 - x_1, y_2 - x_2, y_3 - x_3)$. Furthermore, this is the only segment emanating from the origin which has the same length and direction as PQ. Thus, if one agrees to treat only vectors which emanate from the origin, there is exactly one vector associated with each given length and direction.

The vector OP, from the origin to $P = (x_1, x_2, x_3)$, is completely determined by P, and it is therefore possible to identify this vector with the point P. In our definition of the vector space R^3, the vectors are simply defined to be the triples (x_1, x_2, x_3).

Given points $P = (x_1, x_2, x_3)$ and $Q = (y_1, y_2, y_3)$, the definition of the sum of the vectors OP and OQ can be given geometrically. If the vectors are not parallel, then the segments OP and OQ determine a plane and these segments are two of the edges of a parallelogram in that plane (see Figure 1). One diagonal of this parallelogram extends from O to a point S, and the sum of OP and OQ is defined to be the vector OS. The coordinates of the point S are $(x_1 + y_1, x_2 + y_2, x_3 + y_3)$ and hence this geometrical definition of vector addition is equivalent to the algebraic definition of Example 1.

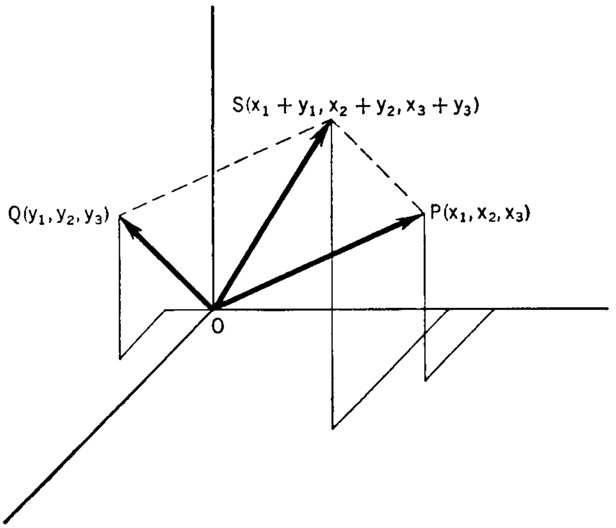

FIGURE 1

Scalar multiplication has a simpler geometric interpretation. If c is a real number, then the product of c and the vector OP is the vector from the origin with length $|c|$ times the length of OP and a direction which agrees with the direction of OP if $c > 0$, and which is opposite to the direction of OP if $c < 0$. This scalar multiplication just yields the vector OT where $T = (cx_1, cx_2, cx_3)$, and is therefore consistent with the algebraic definition given for R^3.

From time to time, the reader will probably find it helpful to 'think geometrically' about vector spaces, that is, to draw pictures for his own benefit to illustrate and motivate some of the ideas. Indeed, he should do this. However, in forming such illustrations he must bear in mind that, because we are dealing with vector spaces as algebraic systems, all proofs we give will be of an algebraic nature.

Exercises

1. If F is a field, verify that F^n (as defined in Example 1) is a vector space over the field F.

2. If V is a vector space over the field F, verify that

$$(\alpha_1 + \alpha_2) + (\alpha_3 + \alpha_4) = [\alpha_2 + (\alpha_3 + \alpha_1)] + \alpha_4$$

for all vectors α_1, α_2, α_3, and α_4 in V.

3. If C is the field of complex numbers, which vectors in C^3 are linear combinations of $(1, 0, -1)$, $(0, 1, 1)$, and $(1, 1, 1)$?

4. Let V be the set of all pairs (x, y) of real numbers, and let F be the field of real numbers. Define

$$\begin{aligned}(x, y) + (x_1, y_1) &= (x + x_1, y + y_1)\\ c(x, y) &= (cx, y).\end{aligned}$$

Is V, with these operations, a vector space over the field of real numbers?

5. On R^n, define two operations

$$\begin{aligned}\alpha \oplus \beta &= \alpha - \beta\\ c \cdot \alpha &= -c\alpha.\end{aligned}$$

The operations on the right are the usual ones. Which of the axioms for a vector space are satisfied by $(R^n, \oplus, \cdot)$?

6. Let V be the set of all complex-valued functions f on the real line such that (for all t in R)

$$f(-t) = \overline{f(t)}.$$

The bar denotes complex conjugation. Show that V, with the operations

$$\begin{aligned}(f + g)(t) &= f(t) + g(t)\\ (cf)(t) &= cf(t)\end{aligned}$$

is a vector space over the field of *real* numbers. Give an example of a function in V which is not real-valued.

7. Let V be the set of pairs (x, y) of real numbers and let F be the field of real numbers. Define

$$\begin{aligned}(x, y) + (x_1, y_1) &= (x + x_1, 0)\\ c(x, y) &= (cx, 0).\end{aligned}$$

Is V, with these operations, a vector space?

2.2. Subspaces

In this section we shall introduce some of the basic concepts in the study of vector spaces.

Definition. *Let* V *be a vector space over the field* F. *A* **subspace** *of* V *is a subset* W *of* V *which is itself a vector space over* F *with the operations of vector addition and scalar multiplication on* V.

A direct check of the axioms for a vector space shows that the subset W of V is a subspace if for each α and β in W the vector $\alpha + \beta$ is again in W; the 0 vector is in W; for each α in W the vector $(-\alpha)$ is in W; for each α in W and each scalar c the vector $c\alpha$ is in W. The commutativity and associativity of vector addition, and the properties (4)(a), (b), (c), and (d) of scalar multiplication do not need to be checked, since these are properties of the operations on V. One can simplify things still further.