(a) *For every $x \in E$ we have $x \leq \sup(E)$ and $x \geq \inf(E)$.*

(b) *Suppose that M is an upper bound for E, i.e., $x \leq M$ for all $x \in E$. Then we have $\sup(E) \leq M$.*

(c) *Suppose that M is a lower bound for E, i.e., $x \geq M$ for all $x \in E$. Then we have $\inf(E) \geq M$.*

Proof. See Exercise 6.2.2. □

Exercise 6.2.1. Prove Proposition 6.2.5. (Hint: you may need Proposition 5.4.7.)

Exercise 6.2.2. Prove Proposition 6.2.11. (Hint: you may need to break into cases depending on whether $+\infty$ or $-\infty$ belongs to E. You can of course use Definition 5.5.10, *provided that E consists only of real numbers.*)

6.3 Suprema and infima of sequences

Having defined the notion of a supremum and infimum of sets of reals, we can now also talk about the supremum and infimum of sequences.

Definition 6.3.1 (Sup and inf of sequences). Let $(a_n)_{n=m}^\infty$ be a sequence of real numbers. Then we define $\sup(a_n)_{n=m}^\infty$ to be the supremum of the set $\{a_n : n \geq m\}$, and $\inf(a_n)_{n=m}^\infty$ to the infimum of the same set $\{a_n : n \geq m\}$.

Remark 6.3.2. The quantities $\sup(a_n)_{n=m}^\infty$ and $\inf(a_n)_{n=m}^\infty$ are sometimes written as $\sup_{n\geq m} a_n$ and $\inf_{n\geq m} a_n$ respectively.

Example 6.3.3. Let $a_n := (-1)^n$; thus $(a_n)_{n=1}^\infty$ is the sequence $-1, 1, -1, 1, \ldots$. Then the set $\{a_n : n \geq 1\}$ is just the two-element set $\{-1, 1\}$, and hence $\sup(a_n)_{n=1}^\infty$ is equal to 1. Similarly $\inf(a_n)_{n=1}^\infty$ is equal to -1.

Example 6.3.4. Let $a_n := 1/n$; thus $(a_n)_{n=1}^\infty$ is the sequence $1, 1/2, 1/3, \ldots$. Then the set $\{a_n : n \geq 1\}$ is the countable set $\{1, 1/2, 1/3, 1/4, \ldots\}$. Thus $\sup(a_n)_{n=1}^\infty = 1$ and $\inf(a_n)_{n=1}^\infty = 0$ (Exercise 6.3.1). Notice here that the infimum of the sequence is not actually a member of the sequence, though it becomes very close to the sequence eventually. (So it is a little inaccurate to think of the supremum and infimum as the "largest element of the sequence" and "smallest element of the sequence" respectively.)

Example 6.3.5. Let $a_n := n$; thus $(a_n)_{n=1}^\infty$ is the sequence 1, 2, 3, 4, ... Then the set $\{a_n : n \geq 1\}$ is just the positive integers $\{1, 2, 3, 4, \ldots\}$. Then $\sup(a_n)_{n=1}^\infty = +\infty$ and $\inf(a_n)_{n=1}^\infty = 1$.

As the last example shows, it is possible for the supremum or infimum of a sequence to be $+\infty$ or $-\infty$. However, if a sequence $(a_n)_{n=m}^\infty$ is bounded, say bounded by M, then all the elements a_n of the sequence lie between $-M$ and M, so that the set $\{a_n : n \geq m\}$ has M as an upper bound and $-M$ as a lower bound. Since this set is clearly non-empty, we can thus conclude that the supremum and infimum of a bounded sequence are real numbers (i.e., not $+\infty$ and $-\infty$).

Proposition 6.3.6 (Least upper bound property). *Let $(a_n)_{n=m}^\infty$ be a sequence of real numbers, and let x be the extended real number $x := \sup(a_n)_{n=m}^\infty$. Then we have $a_n \leq x$ for all $n \geq m$. Also, whenever $M \in \mathbf{R}^*$ is an upper bound for a_n (i.e., $a_n \leq M$ for all $n \geq m$), we have $x \leq M$. Finally, for every extended real number y for which $y < x$, there exists at least one $n \geq m$ for which $y < a_n \leq x$.*

Proof. See Exercise 6.3.2. □

Remark 6.3.7. There is a corresponding Proposition for infima, but with all the references to order reversed, e.g., all upper bounds should now be lower bounds, etc. The proof is exactly the same.

Now we give an application of these concepts of supremum and infimum. In the previous section we saw that all convergent sequences are bounded. It is natural to ask whether the converse is true: are all bounded sequences convergent? The answer is no; for instance, the sequence $1, -1, 1, -1, \dots$ is bounded, but not Cauchy and hence not convergent. However, if we make the sequence both bounded and *monotone* (i.e., increasing or decreasing), then it is true that it must converge:

Proposition 6.3.8 (Monotone bounded sequences converge). *Let $(a_n)_{n=m}^\infty$ be a sequence of real numbers which has some finite upper bound $M \in \mathbf{R}$, and which is also increasing (i.e., $a_{n+1} \geq a_n$ for all $n \geq m$). Then $(a_n)_{n=m}^\infty$ is convergent, and in fact*

$$\lim_{n\to\infty} a_n = \sup(a_n)_{n=m}^\infty \leq M.$$

Proof. See Exercise 6.3.3. □

One can similarly prove that if a sequence $(a_n)_{n=m}^\infty$ is bounded below and decreasing (i.e., $a_{n+1} \leq a_n$), then it is convergent, and that the limit is equal to the infimum.

A sequence is said to be *monotone* if it is either increasing or decreasing. From Proposition 6.3.8 and Corollary 6.1.17 we see that a monotone sequence converges if and only if it is bounded.

Example 6.3.9. The sequence $3, 3.1, 3.14, 3.141, 3.1415, \dots$ is increasing, and is bounded above by 4. Hence by Proposition 6.3.8 it must have a limit, which is a real number less than or equal to 4.

Proposition 6.3.8 asserts that the limit of a monotone sequence exists, but does not directly say what that limit is. Nevertheless, with a little extra work one can often find the limit once one is given that the limit does exist. For instance:

Proposition 6.3.10. *Let $0 < x < 1$. Then we have* $\lim_{n\to\infty} x^n = 0$.

Proof. Since $0 < x < 1$, one can show that the sequence $(x^n)_{n=1}^\infty$ is decreasing (why?). On the other hand, the sequence $(x^n)_{n=1}^\infty$ has a lower bound of 0. Thus by Proposition 6.3.8 (for infima instead of suprema) the sequence $(x^n)_{n=1}^\infty$ converges to some limit L. Since

$x^{n+1} = x \times x^n$, we thus see from the limit laws (Theorem 6.1.19) that $(x^{n+1})_{n=1}^{\infty}$ converges to xL. But the sequence $(x^{n+1})_{n=1}^{\infty}$ is just the sequence $(x^n)_{n=2}^{\infty}$ shifted by one, and so they must have the same limits (why?). So $xL = L$. Since $x \neq 1$, we can solve for L to obtain $L = 0$. Thus $(x^n)_{n=1}^{\infty}$ converges to 0. □

Note that this proof does not work when $x > 1$ (Exercise 6.3.4).

Exercise 6.3.1. Verify the claim in Example 6.3.4.

Exercise 6.3.2. Prove Proposition 6.3.6. (Hint: use Theorem 6.2.11.)

Exercise 6.3.3. Prove Proposition 6.3.8. (Hint: use Proposition 6.3.6, together with the assumption that a_n is increasing, to show that a_n converges to $\sup(a_n)_{n=m}^{\infty}$.)

Exercise 6.3.4. Explain why Proposition 6.3.10 fails when $x > 1$. In fact, show that the sequence $(x^n)_{n=1}^{\infty}$ diverges when $x > 1$. (Hint: prove by contradiction and use the identity $(1/x)^n x^n = 1$ and the limit laws in Theorem 6.1.19.) Compare this with the argument in Example 1.2.3; can you now explain the flaws in the reasoning in that example?

6.4 Limsup, liminf, and limit points

Consider the sequence

$$1.1, -1.01, 1.001, -1.0001, 1.00001, \ldots.$$

If one plots this sequence, then one sees (informally, of course) that this sequence does not converge; half the time the sequence is getting close to 1, and half the time the sequence is getting close to -1, but it is not converging to either of them; for instance, it never gets eventually 1/2-close to 1, and never gets eventually 1/2-close to -1. However, even though -1 and +1 are not quite limits of this sequence, it does seem that in some vague way they "want" to be limits. To make this notion precise we introduce the notion of a *limit point.*

Definition 6.4.1 (Limit points). Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers, let x be a real number, and let $\varepsilon > 0$ be a real number. We say that x is *ε-adherent* to $(a_n)_{n=m}^\infty$ iff there exists an $n \geq m$ such that a_n is ε-close to x. We say that x is *continually ε-adherent* to $(a_n)_{n=m}^\infty$ iff it is ε-adherent to $(a_n)_{n=N}^\infty$ for every $N \geq m$. We say that x is a *limit point* or *adherent point* of $(a_n)_{n=m}^\infty$ iff it is continually ε-adherent to $(a_n)_{n=m}^\infty$ for every $\varepsilon > 0$.

Remark 6.4.2. The verb "to adhere" means much the same as "to stick to"; hence the term "adhesive".

Unwrapping all the definitions, we see that x is a limit point of $(a_n)_{n=m}^\infty$ if, for every $\varepsilon > 0$ and every $N \geq m$, there exists an $n \geq N$ such that $|a_n - x| \leq \varepsilon$. (Why is this the same definition?) Note the difference between a sequence being ε-close to L (which means that *all* the elements of the sequence stay within a distance ε of L) and L being ε-adherent to the sequence (which only needs a *single* element of the sequence to stay within a distance ε of L). Also, for L to be continually ε-adherent to $(a_n)_{n=m}^\infty$, it has to be ε-adherent to $(a_n)_{n=N}^\infty$ for *all* $N \geq m$, whereas for $(a_n)_{n=m}^\infty$ to be eventually ε-close to L, we only need $(a_n)_{n=N}^\infty$ to be ε-close to L for *some* $N \geq m$. Thus there are some subtle differences in quantifiers between limits and limit points.

Note that limit points are only defined for finite real numbers. It is also possible to rigourously define the concept of $+\infty$ or $-\infty$ being a limit point; see Exercise 6.4.8.

Example 6.4.3. Let $(a_n)_{n=1}^\infty$ denote the sequence

$$0.9, 0.99, 0.999, 0.9999, 0.99999, \ldots.$$

The number 0.8 is 0.1-adherent to this sequence, since 0.8 is 0.1-close to 0.9, which is a member of this sequence. However, it is not *continually* 0.1-adherent to this sequence, since once one discards the first element of this sequence there is no member of the sequence to be 0.1-close to. In particular, 0.8 is not a limit point of this sequence. On the other hand, the number 1 is 0.1-adherent to this sequence, and in fact is continually 0.1-adherent to this sequence, since no matter how many initial members of the sequence one discards, there is still something for 1 to be 0.1-close to. In fact, it is continually ε-adherent for every ε, and is hence a limit point of this sequence.

Example 6.4.4. Now consider the sequence

$$1.1, -1.01, 1.001, -1.0001, 1.00001, \ldots.$$

The number 1 is 0.1-adherent to this sequence; in fact it is continually 0.1-adherent to this sequence, because no matter how many elements of the sequence one discards, there are some elements of the sequence that 1 is 0.1-close to. (As discussed earlier, one does not need *all* the elements to be 0.1-close to 1, just some; thus 0.1-adherent is weaker than 0.1-close, and continually 0.1-adherent is a different notion from eventually 0.1-close.) In fact, for every $\varepsilon > 0$, the number 1 is continually ε-adherent to this sequence, and is thus a limit point of this sequence. Similarly -1 is a limit point of this sequence; however 0 (say) is not a limit point of this sequence, since it is not continually 0.1-adherent to it.

Limits are of course a special case of limit points:

Proposition 6.4.5 (Limits are limit points). *Let $(a_n)_{n=m}^{\infty}$ be a sequence which converges to a real number c. Then c is a limit point of $(a_n)_{n=m}^{\infty}$, and in fact it is the only limit point of $(a_n)_{n=m}^{\infty}$.*

Proof. See Exercise 6.4.1. □

Now we will look at two special types of limit points: the limit superior (lim sup) and limit inferior (lim inf).

Definition 6.4.6 (Limit superior and limit inferior). Suppose that $(a_n)_{n=m}^{\infty}$ is a sequence. We define a new sequence $(a_N^+)_{N=m}^{\infty}$ by the formula

$$a_N^+ := \sup(a_n)_{n=N}^{\infty}.$$

More informally, a_N^+ is the supremum of all the elements in the sequence from a_N onwards. We then define the *limit superior* of the sequence $(a_n)_{n=m}^{\infty}$, denoted $\limsup_{n\to\infty} a_n$, by the formula

$$\limsup_{n\to\infty} a_n := \inf(a_N^+)_{N=m}^{\infty}.$$

Similarly, we can define

$$a_N^- := \inf(a_n)_{n=N}^\infty$$

and define the *limit inferior* of the sequence $(a_n)_{n=m}^\infty$, denoted $\liminf_{n\to\infty} a_n$, by the formula

$$\liminf_{n\to\infty} a_n := \sup(a_N^-)_{N=m}^\infty.$$

Example 6.4.7. Let $a_1, a_2, a_3, \ldots$ denote the sequence

$$1.1, -1.01, 1.001, -1.0001, 1.00001, \ldots.$$

Then $a_1^+, a_2^+, a_3^+, \ldots$ is the sequence

$$1.1, 1.001, 1.001, 1.00001, 1.00001, \ldots$$

(why?), and its infimum is 1. Hence the limit superior of this sequence is 1. Similarly, $a_1^-, a_2^-, a_3^-, \ldots$ is the sequence

$$-1.01, -1.01, -1.0001, -1.0001, -1.000001, \ldots$$

(why?), and the supremum of this sequence is -1. Hence the limit inferior of this sequence is -1. One should compare this with the supremum and infimum of the sequence, which are 1.1 and -1.01 respectively.

Example 6.4.8. Let $a_1, a_2, a_3, \ldots$ denote the sequence

$$1, -2, 3, -4, 5, -6, 7, -8, \ldots$$

Then $a_1^+, a_2^+, \ldots$ is the sequence

$$+\infty, +\infty, +\infty, +\infty, \ldots$$

(why?) and so the limit superior is $+\infty$. Similarly, $a_1^-, a_2^-, \ldots$ is the sequence

$$-\infty, -\infty, -\infty, -\infty, \ldots$$

and so the limit inferior is $-\infty$.

Example 6.4.9. Let $a_1, a_2, a_3, \ldots$ denote the sequence

$$1, -1/2, 1/3, -1/4, 1/5, -1/6, \ldots$$

Then $a_1^+, a_2^+, \ldots$ is the sequence

$$1, 1/3, 1/3, 1/5, 1/5, 1/7, \ldots$$

which has an infimum of 0 (why?), so the limit superior is 0. Similarly, $a_1^-, a_2^-, \ldots$ is the sequence

$$-1/2, -1/2, -1/4, -1/4, -1/6, -1/6$$

which has a supremum of 0. So the limit inferior is also 0.

Example 6.4.10. Let $a_1, a_2, a_3, \ldots$ denote the sequence

$$1, 2, 3, 4, 5, 6, \ldots$$

Then $a_1^+, a_2^+, \ldots$ is the sequence

$$+\infty, +\infty, +\infty, \ldots$$

so the limit superior is $+\infty$. Similarly, $a_1^-, a_2^-, \ldots$ is the sequence

$$1, 2, 3, 4, 5, \ldots$$

which has a supremum of $+\infty$. So the limit inferior is also $+\infty$.

Remark 6.4.11. Some authors use the notation $\overline{\lim}_{n\to\infty} a_n$ instead of $\limsup_{n\to\infty} a_n$, and $\underline{\lim}_{n\to\infty} a_n$ instead of $\liminf_{n\to\infty} a_n$. Note that the starting index m of the sequence is irrelevant (see Exercise 6.4.2).

Returning to the piston analogy, imagine a piston at $+\infty$ moving leftward until it is stopped by the presence of the sequence $a_1, a_2, \ldots$. The place it will stop is the supremum of $a_1, a_2, a_3, \ldots$, which in our new notation is a_1^+. Now let us remove the first element a_1 from the sequence; this may cause our piston to slip leftward, to a new point a_2^+ (though in many cases the piston will not move and a_2^+ will just be the same as a_1^+). Then we remove the second element a_2, causing the piston to slip a little more. If we keep doing this the piston will keep slipping, but there will be some point where it cannot go any further, and this is the limit superior of the sequence. A similar analogy can describe the limit inferior of the sequence.

We now describe some basic properties of limit superior and limit inferior.

Proposition 6.4.12. *Let $(a_n)_{n=m}^\infty$ be a sequence of real numbers, let L^+ be the limit superior of this sequence, and let L^- be the limit inferior of this sequence (thus both L^+ and L^- are extended real numbers).*

(a) *For every $x > L^+$, there exists an $N \geq m$ such that $a_n < x$ for all $n \geq N$. (In other words, for every $x > L^+$, the elements of the sequence $(a_n)_{n=m}^\infty$ are eventually less than x.) Similarly, for every $y < L^-$ there exists an $N \geq m$ such that $a_n > y$ for all $n \geq N$.*

(b) *For every $x < L^+$, and every $N \geq m$, there exists an $n \geq N$ such that $a_n > x$. (In other words, for every $x < L^+$, the elements of the sequence $(a_n)_{n=m}^\infty$ exceed x infinitely often.) Similarly, for every $y > L^-$ and every $N \geq m$, there exists an $n \geq N$ such that $a_n < y$.*

(c) *We have $\inf(a_n)_{n=m}^\infty \leq L^- \leq L^+ \leq \sup(a_n)_{n=m}^\infty$.*

(d) *If c is any limit point of $(a_n)_{n=m}^\infty$, then we have $L^- \leq c \leq L^+$.*

(e) *If L^+ is finite, then it is a limit point of $(a_n)_{n=m}^\infty$. Similarly, if L^- is finite, then it is a limit point of $(a_n)_{n=m}^\infty$.*

(f) *Let c be a real number. If $(a_n)_{n=m}^\infty$ converges to c, then we must have $L^+ = L^- = c$. Conversely, if $L^+ = L^- = c$, then $(a_n)_{n=m}^\infty$ converges to c.*

Proof. We shall prove (a) and (b), and leave the remaining parts to the exercises. Suppose first that $x > L^+$. Then by definition of L^+, we have $x > \inf(a_N^+)_{N=m}^\infty$. By Proposition 6.3.6, there must then exist an integer $N \geq m$ such that $x > a_N^+$. By definition of a_N^+, this means that $x > \sup(a_n)_{n=N}^\infty$. Thus by Proposition 6.3.6 again, we have $x > a_n$ for all $n \geq N$, as desired. This proves the first part of (a); the second part of (a) is proven similarly.

Now we prove (b). Suppose that $x < L^+$. Then we have $x < \inf(a_N^+)_{N=m}^\infty$. If we fix any $N \geq m$, then by Proposition 6.3.6, we thus have $x < a_N^+$. By definition of a_N^+, this means that $x < \sup(a_n)_{n=N}^\infty$. By Proposition 6.3.6 again, there must thus exist $n \geq N$ such that $a_n > x$, as desired. This proves the first part of (b), the second part of (b) is proven similarly.

The proofs of (c), (d), (e), (f) are left to Exercise 6.4.3. □

Parts (c) and (d) of Proposition 6.4.12 say, in particular, that L^+ is the largest limit point of $(a_n)_{n=m}^\infty$, and L^- is the smallest limit point (providing that L^+ and L^- are finite. Proposition 6.4.12 (f) then says that if L^+ and L^- coincide (so there is only one limit point), then the sequence in fact converges. This gives a way to test if a sequence converges: compute its limit superior and limit inferior, and see if they are equal.

We now give a basic comparison property of limit superior and limit inferior.

Lemma 6.4.13 (Comparison principle). *Suppose that $(a_n)_{n=m}^\infty$ and $(b_n)_{n=m}^\infty$ are two sequences of real numbers such that $a_n \leq b_n$ for all $n \geq m$. Then we have the inequalities*

$$\sup(a_n)_{n=m}^\infty \leq \sup(b_n)_{n=m}^\infty$$
$$\inf(a_n)_{n=m}^\infty \leq \inf(b_n)_{n=m}^\infty$$
$$\limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} b_n$$
$$\liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} b_n$$

Proof. See Exercise 6.4.4. □

Corollary 6.4.14 (Squeeze test). *Let $(a_n)_{n=m}^{\infty}$, $(b_n)_{n=m}^{\infty}$, and $(c_n)_{n=m}^{\infty}$ be sequences of real numbers such that*

$$a_n \leq b_n \leq c_n$$

for all $n \geq M$. Suppose also that $(a_n)_{n=m}^{\infty}$ and $(c_n)_{n=m}^{\infty}$ both converge to the same limit L. Then $(b_n)_{n=m}^{\infty}$ is also convergent to L.

Proof. See Exercise 6.4.5. □

Example 6.4.15. We already know (see Proposition 6.1.11) that $\lim_{n\to\infty} 1/n = 0$. By the limit laws (Theorem 6.1.19), this also implies that $\lim_{n\to\infty} 2/n = 0$ and $\lim_{n\to\infty} -2/n = 0$. The squeeze test then shows that any sequence $(b_n)_{n=1}^{\infty}$ for which

$$-2/n \leq b_n \leq 2/n \text{ for all } n \geq 1$$

is convergent to 0. For instance, we can use this to show that the sequence $(-1)^n/n + 1/n^2$ converges to zero, or that 2^{-n} converges to zero. Note one can use induction to show that $0 \leq 2^{-n} \leq 1/n$ for all $n \geq 1$.

Remark 6.4.16. The squeeze tęst, combined with the limit laws and the principle that monotone bounded sequences always have limits, allows to compute a large number of limits. We give some examples in the next chapter.

One commonly used consequence of the squeeze test is

Corollary 6.4.17 (Zero test for sequences). *Let $(a_n)_{n=M}^{\infty}$ be a sequence of real numbers. Then the limit $\lim_{n\to\infty} a_n$ exists and is equal to zero if and only if the limit $\lim_{n\to\infty} |a_n|$ exists and is equal to zero.*

Proof. See Exercise 6.4.7. □

We close this section with the following improvement to Proposition 6.1.12.

Theorem 6.4.18 (Completeness of the reals). *A sequence $(a_n)_{n=1}^{\infty}$ of real numbers is a Cauchy sequence if and only if it is convergent.*

Remark 6.4.19. Note that while this is very similar in spirit to Proposition 6.1.15, it is a bit more general, since Proposition 6.1.15 refers to Cauchy sequences of rationals instead of real numbers.

Proof. Proposition 6.1.12 already tells us that every convergent sequence is Cauchy, so it suffices to show that every Cauchy sequence is convergent.

Let $(a_n)_{n=1}^{\infty}$ be a Cauchy sequence. We know from Corollary 6.1.17 that the sequence $(a_n)_{n=1}^{\infty}$ is bounded; by Lemma 6.4.13 (or Proposition 6.4.12(c)) this implies that $L^- := \liminf_{n\to\infty} a_n$ and $L^+ := \limsup_{n\to\infty} a_n$ of the sequence are both finite. To show that the sequence converges, it will suffice by Proposition 6.4.12(f) to show that $L^- = L^+$.

Now let $\varepsilon > 0$ be any real number. Since $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence, it must be eventually ε-steady, so in particular there exists an $N \geq 1$ such that the sequence $(a_n)_{n=N}^{\infty}$ is ε-steady. In particular, we have $a_N - \varepsilon \leq a_n \leq a_N + \varepsilon$ for all $n \geq N$. By Proposition 6.3.6 (or Lemma 6.4.13) this implies that

$$a_N - \varepsilon \leq \inf(a_n)_{n=N}^{\infty} \leq \sup(a_n)_{n=N}^{\infty} \leq a_N + \varepsilon$$

and hence by the definition of L^- and L^+ (and Proposition 6.3.6 again)

$$a_N - \varepsilon \leq L^- \leq L^+ \leq a_N + \varepsilon.$$

Thus we have

$$0 \leq L^+ - L^- \leq 2\varepsilon.$$

But this is true for all $\varepsilon > 0$, and L^+ and L^- do not depend on ε; so we must therefore have $L^+ = L^-$. (If $L^+ > L^-$ then we could set $\varepsilon := (L^+ - L^-)/3$ and obtain a contradiction.) By Proposition 6.4.12(f) we thus see that the sequence converges. □

Remark 6.4.20. In the language of metric spaces (see Chapter 12), Theorem 6.4.18 asserts that the real numbers are a *complete* metric space - that they do not contain "holes" the same way the rationals do. (Certainly the rationals have lots of Cauchy sequences which do not converge to other rationals; take for instance the sequence $1, 1.4, 1.41, 1.414, 1.4142, \ldots$ which converges to the irrational $\sqrt{2}$.) This property is closely related to the least upper bound property (Theorem 5.5.9), and is one of the principal characteristics which make the real numbers superior to the rational numbers for the purposes of doing analysis (taking limits, taking derivatives and integrals, finding zeroes of functions, that kind of thing), as we shall see in later chapters.

Exercise 6.4.1. Prove Proposition 6.4.5.

Exercise 6.4.2. State and prove analogues of Exercises 6.1.3 and 6.1.4 for limit points, limit superior, and limit inferior.

Exercise 6.4.3. Prove parts (c),(d),(e),(f) of Proposition 6.4.12. (Hint: you can use earlier parts of the proposition to prove later ones.)

Exercise 6.4.4. Prove Lemma 6.4.13.

Exercise 6.4.5. Use Lemma 6.4.13 to prove Corollary 6.4.14.

Exercise 6.4.6. Give an example of two bounded sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ such that $a_n < b_n$ for all $n \geq 1$, but that $\sup(a_n)_{n=1}^{\infty} \not< \sup(b_n)_{n=1}^{\infty}$. Explain why this does not contradict Lemma 6.4.13.

Exercise 6.4.7. Prove Corollary 6.4.17. Is the corollary still true if we replace zero in the statement of this Corollary by some other number?

Exercise 6.4.8. Let us say that a sequence $(a_n)_{n=M}^{\infty}$ of real numbers has $+\infty$ as a limit point iff it has no finite upper bound, and that it has $-\infty$ as a limit point iff it has no finite lower bound. With this definition, show that $\limsup_{n\to\infty} a_n$ is a limit point of $(a_n)_{n=M}^{\infty}$, and furthermore that it is larger than all the other limit points of $(a_n)_{n=M}^{\infty}$; in other words, the limit superior is the largest limit point of a sequence. Similarly, show that the limit inferior is the smallest limit point of a sequence. (One can use Proposition 6.4.12 in the course of the proof.)

Exercise 6.4.9. Using the definition in Exercise 6.4.8, construct a sequence $(a_n)_{n=1}^{\infty}$ which has exactly three limit points, at $-\infty$, 0, and $+\infty$.

Exercise 6.4.10. Let $(a_n)_{n=N}^{\infty}$ be a sequence of real numbers, and let $(b_m)_{m=M}^{\infty}$ be another sequence of real numbers such that each b_m is a limit point of $(a_n)_{n=N}^{\infty}$. Let c be a limit point of $(b_m)_{m=M}^{\infty}$. Prove that c is also a limit point of $(a_n)_{n=N}^{\infty}$. (In other words, limit points of limit points are themselves limit points of the original sequence.)

6.5 Some standard limits

Armed now with the limit laws and the squeeze test, we can now compute a large number of limits.

A particularly simple limit is that of the *constant sequence* $c, c, c, c, \ldots$; we clearly have

$$\lim_{n\to\infty} c = c$$

for any constant c (why?).

Also, in Proposition 6.1.11, we proved that $\lim_{n\to\infty} 1/n = 0$. This now implies

Corollary 6.5.1. *We have* $\lim_{n\to\infty} 1/n^{1/k} = 0$ *for every integer* $k \geq 1$.

Proof. From Lemma 5.6.6 we know that $1/n^{1/k}$ is a decreasing function of n, while being bounded below by 0. By Proposition 6.3.8 (for decreasing sequences instead of increasing sequences) we thus know that this sequence converges to some limit $L \geq 0$:

$$L = \lim_{n\to\infty} 1/n^{1/k}.$$

Raising this to the k^{th} power and using the limit laws (or more precisely, Theorem 6.1.19(b) and induction), we obtain

$$L^k = \lim_{n\to\infty} 1/n.$$

By Proposition 6.1.11 we thus have $L^k = 0$; but this means that L cannot be positive (else L^k would be positive), so $L = 0$, and we are done. $\square$

Some other basic limits: