Since $am = 0$ for all $a \in I$ and all $m \in M$ this is well defined and one easily checks that it makes M into an (R/I)-module. In particular, when I is a maximal ideal in the commutative ring R and $IM = 0$, then M is a vector space over the field R/I (cf. the following example).

The next example is of sufficient importance as to be singled out. It will form the basis for our proof of the Fundamental Theorem of Finitely Generated Abelian Groups in Chapter 12.

Example: ($\mathbb{Z}$-modules)

Let R = $\mathbb{Z}$, let A be *any* abelian group (finite or infinite) and write the operation of A as $+$. Make A into a $\mathbb{Z}$-module as follows: for any $n \in \mathbb{Z}$ and $a \in A$ define

$$na = \begin{cases} a + a + \cdots + a \quad (n \text{ times}) & \text{if } n > 0 \\ 0 & \text{if } n = 0 \\ -a - a - \cdots - a \quad (-n \text{ times}) & \text{if } n < 0 \end{cases}$$

(here 0 is the identity of the additive group A). This definition of an action of the integers on A makes A into a $\mathbb{Z}$-module, and the module axioms show that this is the only possible action of $\mathbb{Z}$ on A making it a (unital) $\mathbb{Z}$-module. Thus every abelian group is a $\mathbb{Z}$-module. Conversely, if M is any $\mathbb{Z}$-module, a fortiori M is an abelian group, so

$\mathbb{Z}$-modules are the same as abelian groups.

Furthermore, it is immediate from the definition that

$\mathbb{Z}$-submodules are the same as subgroups.

Note that for the cyclic group $\langle a \rangle$ written multiplicatively the additive notation na becomes a^n, that is, we have all along been using the fact that $\langle a \rangle$ is a right $\mathbb{Z}$-module (checking that this "exponential" notation satisfies the usual laws of exponents is equivalent to checking the $\mathbb{Z}$-module axioms — this was given as an exercise at the end of Section 1.1). Note that since $\mathbb{Z}$ is commutative these definitions of left and right actions by ring elements give the same module structure.

If A is an abelian group containing an element x of finite order n then $nx = 0$. Thus, in contrast to vector spaces, a $\mathbb{Z}$-module may have nonzero elements x such that $nx = 0$ for some nonzero ring element n. In particular, if A has order m, then by Lagrange's Theorem (Corollary 9, Section 3.2) $mx = 0$, for *all* $x \in A$. Note that then A is a module over $\mathbb{Z}/m\mathbb{Z}$.

In particular, if p is a prime and A is an abelian group (written additively) such that $px = 0$, for all $x \in A$, then (as noted in Example 5) A is a $\mathbb{Z}/p\mathbb{Z}$-module, i.e., can be considered as a vector space over the field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. For instance, the Klein 4-group is a (2-dimensional) vector space over $\mathbb{F}_2$. These groups are the *elementary abelian p-groups* discussed in Section 4.4 (see, in particular, Proposition 17(3)).

The next example is also of fundamental importance and will form the basis for our study of canonical forms of matrices in Sections 12.2 and 12.3.

Example: ($F[x]$-modules)

Let F be a field, let x be an indeterminate and let R be the polynomial ring $F[x]$. Let V be a vector space over F and let T be a linear transformation from V to V (we shall review the theory of linear transformations in the next chapter — for the purposes of this example one only needs to know the definition of a linear transformation). We have already seen that V is an F-module; the linear map T will enable us to make V into an $F[x]$-module.

First, for the nonnegative integer n, define

$$T^0 = I,$$

$$\vdots$$

$$T^n = T \circ T \circ \cdots \circ T \qquad (n \text{ times})$$

where I is the identity map from V to V and $\circ$ denotes function composition (which makes sense because the domain and codomain of T are the same). Also, for any two linear transformations A, B from V to V and elements $\alpha, \beta \in F$, let $\alpha A + \beta B$ be defined by

$$(\alpha A + \beta B)(v) = \alpha(A(v)) + \beta(B(v))$$

(i.e., addition and scalar multiplication of linear transformations are defined pointwise). Then $\alpha A + \beta B$ is easily seen to be a linear transformation from V to V, so that linear combinations of linear transformations are again linear transformations.

We now define the action of any polynomial in x on V. Let $p(x)$ be the polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where $a_0, \ldots, a_n \in F$. For each $v \in V$ define an action of the ring element $p(x)$ on the module element v by

$$\begin{aligned} p(x)v &= (a_n T^n + a_{n-1} T^{n-1} + \cdots + a_1 T + a_0)(v) \\ &= a_n T^n(v) + a_{n-1} T^{n-1}(v) + \cdots + a_1 T(v) + a_0 v \end{aligned}$$

(i.e., $p(x)$ acts by substituting the linear transformation T for x in $p(x)$ and applying the resulting linear transformation to v). Put another way, x acts on V as the linear transformation T and we extend this to an action of all of $F[x]$ on V in a natural way. It is easy to check that this definition of an action of $F[x]$ on V satisfies all the module axioms and makes V into an $F[x]$-module.

The field F is naturally a subring of $F[x]$ (the constant polynomials) and the action of these field elements is by definition the same as their action when viewed as constant polynomials. In other words, the definition of the $F[x]$ action on V is consistent with the given action of the field F on the vector space V, i.e., the definition *extends* the action of F to an action of the larger ring $F[x]$.

The way $F[x]$ acts on V depends on the choice of T so that there are in general many different $F[x]$-module structures on the same vector space V. For instance, if $T = 0$, and $p(x), v$ are as above, then $p(x)v = a_0 v$, that is, the polynomial $p(x)$ acts on v simply by multiplying by the constant term of $p(x)$, so that the $F[x]$-module structure is just the F-module structure. If, on the other hand, T is the identity transformation (so $T^n(v) = v$, for all n and v), then $p(x)v = a_n v + a_{n-1} v + \cdots + a_0 v = (a_n + \cdots + a_0)v$, so that now $p(x)$ multiplies v by the sum of the coefficients of $p(x)$.

To give another specific example, let V be affine n-space F^n and let T be the "shift operator"

$$T(x_1, x_2, \ldots, x_n) = (x_2, x_3, \ldots, x_n, 0).$$

Let e_i be the usual i^{th} basis vector $(0, 0, \ldots, 0, 1, 0, \ldots, 0)$ where the 1 is in position i. Then

$$T^k(e_i) = \begin{cases} e_{i-k} & \text{if } i > k \\ 0 & \text{if } i \leq k \end{cases}$$

so for example, if $m < n$,

$$(a_m x^m + a_{m-1} x^{m-1} + \cdots + a_0) e_n = (0, \ldots, 0, a_m, a_{m-1}, \ldots, a_0).$$

From this we can determine the action of any polynomial on any vector.

The construction of an $F[x]$-module from a vector space V over F and a linear transformation T from V to V in fact describes *all* $F[x]$-modules; namely, an $F[x]$-module is a vector space together with a linear transformation which specifies the action of x. This is because if V is any $F[x]$-module, then V is an F-module and the action of the ring element x on V is a linear transformation from V to V. The axioms for a module ensure that the actions of F and x on V uniquely determine the action of any element of $F[x]$ on V. Thus there is a bijection between the collection of $F[x]$-modules and the collection of pairs V, T

$$\left\{ V \text{ an } F[x]\text{-module} \right\} \quad \longleftrightarrow \quad \left\{ \begin{array}{c} V \text{ a vector space over } F \\ \text{and} \\ T : V \to V \text{ a linear transformation} \end{array} \right\}$$

given by

the element x acts on V as the linear transformation T.

Now we consider $F[x]$-submodules of V where, as above, V is any $F[x]$-module and T is the linear transformation from V to V given by the action of x. An $F[x]$-submodule W of V must first be an F-submodule, i.e., W must be a vector subspace of V. Secondly, W must be sent to itself under the action of the ring element x, i.e., we must have $T(w) \in W$, for all $w \in W$. Any vector subspace U of V such that $T(U) \subseteq U$ is called *T-stable* or *T-invariant*. If U is any T-stable subspace of V it follows that $T^n(U) \subseteq U$, for all $n \in \mathbb{Z}^+$ (for example, $T(U) \subseteq U$ implies $T^2(U) = T(T(U)) \subseteq T(U) \subseteq U$). Moreover any linear combination of powers of T then sends U into U so that U is also stable by the action of *any* polynomial in T. Thus U is an $F[x]$-submodule of V. This shows that

the $F[x]$-submodules of V are precisely the T-stable subspaces of V.

In terms of the bijection above,

$$\left\{ W \text{ an } F[x]\text{-submodule} \right\} \quad \longleftrightarrow \quad \left\{ \begin{array}{c} W \text{ a subspace of } V \\ \text{and} \\ W \text{ is } T\text{-stable} \end{array} \right\}$$

which gives a complete dictionary between $F[x]$-modules V and vector spaces V together with a given linear transformation T from V to V.

For instance, if T is the shift operator defined on affine n-space above and k is any integer in the range $0 \leq k \leq n$, then the subspace

$$U_k = \{ (x_1, x_2, \ldots, x_k, 0, \ldots, 0) \mid x_i \in F \}$$

is clearly T-stable so is an $F[x]$-submodule of V.