EXERCISES

Let R be a commutative ring with 1 and let $\mathcal{J}$ be its Jacobson radical.

1. Suppose R is an Artinian ring and I is an ideal in R. Prove that R/I is also Artinian.

2. Show that every finite commutative ring with 1 is Artinian.

3. Prove that an integral domain of Krull dimension 0 is a field.

4. Prove that an Artinian integral domain is a field.

5. Suppose I is a nilpotent ideal in R and $M = IM$ for some R-module M. Prove that $M = 0$.

6. Suppose that $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is an exact sequence of R-modules. Prove that M is an Artinian R-module if and only if M' and M'' are Artinian R-modules.

7. Suppose $R = F$ is a field. Prove that an R-module M is Artinian if and only if it is Noetherian if and only if M is a finite dimensional vector space over F.

8. Let M be a maximal ideal of the ring R and suppose that $M^n = 0$ for some $n \geq 1$. Prove that R is Noetherian if and only if R is Artinian. [Observe the each successive quotient $M^i/M^{i+1}, i = 0, \dots, n-1$ in the filtration $R \supseteq M \supseteq \cdots \supseteq M^{n-1} \supseteq M^n = 0$ is a module over the field $F = R/M$. Then use the previous two exercises and Exercise 6 of Section 15.1.]

9. Let M be a finitely generated R-module. Prove that if $x_1, \dots, x_n$ are elements of M whose images in $M/\mathcal{J}M$ generate $M/\mathcal{J}M$, then they generate M. Deduce that if R is Noetherian and the images of $a_1, \dots, a_n$ in $\mathcal{J}/\mathcal{J}^2$ generate $\mathcal{J}/\mathcal{J}^2$, then $\mathcal{J} = (a_1, \dots, a_n)$. [Let N be the submodule generated by $x_1, \dots, x_n$ and apply Nakayama's Lemma to the module $A = M/N$.]

10. Let $R = \mathbb{Z}_{(2)}$ be the localization of $\mathbb{Z}$ at the prime ideal (2). Prove that Jac $R = (2)$ is the ideal generated by 2. If $M = \mathbb{Q}$, prove that $M/2M$ is a finitely generated R-module but that M is not finitely generated over R. Why doesn't this contradict the previous exercise? [Note the hypotheses in Nakayama's Lemma.]

11. Let V be an affine variety over a field k and let $R = k[V]$ be its coordinate ring. Let $d_t(R)$ denote the transcendence degree of the field of fractions $k(V)$ over k, and let $d_p(R)$ be the Krull dimension of R defined in terms of chains of prime ideals. This exercise shows $d_t(R) = d_p(R)$. By Noether's Normalization Lemma there is a polynomial subring $R_1 = k[y_1, \dots, y_m]$ of R such that R is integral over R_1.

(a) Show that $d_t(R_1) = d_t(R) = m$ and that $d_p(R_1) = d_p(R)$. Deduce that we may assume $R = R_1$. [Use the Going-up and Going-down Theorems (cf. Theorem 26, Section 15.3) to prove the second equality.]

(b) When $R = R_1$ show that $d_p(R) \geq d_t(R)$ by exhibiting an explicit chain of prime ideals of length m.

(c) When $R = R_1$ show that any nonzero prime ideal of R contains an element f such that $R(f)$ is transcendental over R of transcendence degree 1. Use induction to show that $d_p(R) \leq d_t(R)$, and deduce that $d_p(R) = d_t(R)$.

12. Let R be a Noetherian local ring with maximal ideal M.

(a) The quotient M/M^2 is a module (i.e., vector space) over the field R/M. Prove that $d = \dim_{R/M}(M/M^2)$ is finite.

(b) Prove that M can be generated as an ideal in R by d elements and by no fewer. [Use Exercise 9.]

(c) Let $R = k[x_1, \dots, x_n]_{(x_1, \dots, x_n)}$ be the localization of the polynomial ring $k[x_1, \dots, x_n]$ over the field k at the maximal ideal $(x_1, \dots, x_n)$, and let M be the maximal ideal in

R. Prove that $\dim_{R/M}(M/M^2) = n = \dim R$. [Cf. the previous exercise.]

It can be shown that $\dim_{R/M}(M/M^2) \geq \dim R$ for any Noetherian local ring R with maximal ideal M. A Noetherian local ring R is called a *regular local ring* if $\dim_{R/M}(M/M^2) = \dim R$. It is a fact that a regular local ring is necessarily an integral domain and is also integrally closed.

13. If R is a Noetherian ring, prove that the Zariski topology on Spec R is discrete (i.e., every subset is Zariski open and also Zariski closed) if and only if R is Artinian.

14. Suppose I is the ideal $(x_1, x_2^2, x_3^3, \dots)$ in the polynomial ring $k[x_1, x_2, x_3, \dots]$ where k is a field and let R be the quotient ring $k[x_1, x_2, x_3, \dots]/I$. Prove that the image of the ideal $(x_1, x_2, x_3, \dots)$ in R is the unique prime ideal in R but is not finitely generated. Deduce that R is a local ring of Krull dimension 0 but is not Artinian.

16.2 DISCRETE VALUATION RINGS

In the previous section we showed that the Artinian rings are the Noetherian rings having Krull dimension 0. We now consider the easiest Noetherian rings of dimension 1, the Discrete Valuation Rings first introduced in Section 8.1:

Definition.

(1) A *discrete valuation* on a field K is a function $\nu : K^\times \to \mathbb{Z}$ satisfying
- **(i)** ν is surjective,
- **(ii)** $\nu(xy) = \nu(x) + \nu(y)$ for all $x, y \in K^\times$,
- **(iii)** $\nu(x + y) \geq \min\{\nu(x), \nu(y)\}$ for all $x, y \in K^\times$ with $x + y \neq 0$.

The subring $\{x \in K \mid \nu(x) \geq 0\} \cup \{0\}$ is called the *valuation ring* of ν.

(2) An integral domain R is called a *Discrete Valuation Ring* (D.V.R.) if R is the valuation ring of a discrete valuation ν on the field of fractions of R.

The valuation ν is often extended to all of K by defining $\nu(0) = +\infty$, in which case (ii) and (iii) hold for all $a, b \in K$.

Examples

(1) The localization $\mathbb{Z}_{(p)}$ of $\mathbb{Z}$ at any nonzero prime ideal (p) is a D.V.R. with respect to the discrete valuation ν_p on $\mathbb{Q}$ defined as follows (cf. Exercise 27, Section 7.1). Every element $a/b \in \mathbb{Q}^\times$ can be written uniquely in the form $p^n(a_1/b_1)$ where $n \in \mathbb{Z}$, $a_1/b_1 \in \mathbb{Q}^\times$ and both a_1 and b_1 are relatively prime to p. Define

$$\nu_p\left(\frac{a}{b}\right) = \nu_p\left(p^n \frac{a_1}{b_1}\right) = n.$$

One easily checks that the axioms for a D.V.R. are satisfied. We call ν_p the *p-adic valuation* on $\mathbb{Q}$. The corresponding valuation ring is the set of rational numbers with $n \geq 0$ together with 0, i.e., the rational numbers a/b where b is not divisible by p, which is $\mathbb{Z}_{(p)}$.

(2) For any field F, let f be an irreducible polynomial in $F[x]$. Every nonzero element in the field $F(x)$ can be written uniquely in the form $f^n(a/b)$ where $n \in \mathbb{Z}$, $a/b \in F[x]^\times$ and both a and b are relatively prime to f. Then

$$\nu_f\left(f^n \frac{a}{b}\right) = n$$

defines a valuation on $F(x)$ and the corresponding valuation ring is the localization $F[x]_f$ of $F[x]$ at f consisting of the rational functions in $F(x)$ whose denominator is not divisible by f. When $f = x-\alpha$ is a polynomial of degree 1 in $F[x]$, the valuation ν_f gives the *order of the zero* (if $n \geq 0$) or *pole* (if $n < 0$) of the element in $F(x)$ at $x = \alpha$.

(3) The ring of formal Laurent series $F((x))$ with coefficients in the field F has a discrete valuation ν defined by

$$\nu\left(\sum_{i\geq n}^{\infty} a_i x^i\right) = n$$

(cf. Exercise 5, Section 7.2). The corresponding D.V.R. is the ring $F[[x]]$ of power series in x with coefficients in F.

Note that $\nu(1) = \nu(1) + \nu(1)$ implies that $\nu(1) = 0$, so every Discrete Valuation Ring R is a ring with identity $1 \neq 0$. Since R is a subring of a field by definition, R is in particular an integral domain. It is easy to see that a D.V.R. is a Euclidean Domain (cf. Example 4 in Section 8.1), so in particular is also a P.I.D. and a U.F.D. In fact the factorization and ideal structure of a D.V.R. is very simple, as the next proposition shows.

Proposition 5. Suppose R is a Discrete Valuation Ring with respect to the valuation ν, and let t be any element of R with $\nu(t) = 1$. Then

(1) A nonzero element $u \in R$ is a unit if and only if $\nu(u) = 0$.

(2) Every nonzero element $r \in R$ can be written in the form $r = ut^n$ for some unit $u \in R$ and some $n \geq 0$. Every nonzero element x in the field of fractions of R can be written in the form $x = ut^n$ for some unit $u \in R$ and some $n \in \mathbb{Z}$.

(3) Every nonzero ideal of R is a principal ideal of the form (t^n) for some $n \geq 0$. In particular, R is a Noetherian ring.

Proof: If u is a unit, then $uv = 1$ for some $v \in R$ and then $\nu(u)+\nu(v) = \nu(uv) = 1$ with $\nu(u) \geq 0$ and $\nu(v) \geq 0$ shows that $\nu(u) = 0$. Conversely, if u is nonzero and $\nu(u) = 0$ then $u^{-1} \in K$ satisfies $\nu(u^{-1}) + \nu(u) = \nu(1) = 0$. Hence $\nu(u^{-1}) = 0$ and $u^{-1} \in R$, so u is a unit. This proves (1).

For (2), note that if $\nu(x) = n$ then $\nu(xt^{-n}) = 0$, so $xt^{-n} = u$ is a unit in R by (1). Hence $x = ut^n$, where $x \in R$ if and only if $n = \nu(x) \geq 0$.

If I is a nonzero ideal in R, let $r \in I$ be an element with $\nu(r)$ minimal. If $\nu(r) = n$, then r differs from t^n by a unit by (2), so $t^n \in I$ and $(t^n) \subseteq I$. If now a is any nonzero element of I, then $\nu(a) \geq n$ by choice of n. Then $\nu(at^{-n}) \geq 0$ and so $at^{-n} \in R$, which shows that $a \in (t^n)$. Hence $I = (t^n)$, proving the first statement in (3). It is then clear that ascending chains of ideals in R are finite, proving that R is Noetherian and completing the proof.

Definition. If R is a D.V.R. with valuation ν, then an element t of R with $\nu(t) = 1$ is called a *uniformizing* (or *local*) *parameter* for R.