

The next application of the fixed-point theorem establishes an existence theorem for the integral equation

$$(7.83) \quad \varphi(x) = \psi(x) + \lambda \int_a^b K(x, t) \varphi(t) dt.$$

Here ψ is a given function, continuous on $[a, b]$, λ is a given constant, and K is a given function defined and bounded on the square

$$S = \{(x, y) \mid a \leq x \leq b, a \leq y \leq b\}.$$

The function K is called the *kernel* of the integral equation. Let C be the linear space of continuous functions on $[a, b]$. We assume that the kernel K is such that the operator T given by

$$T(\varphi)(x) = \psi(x) + \lambda \int_a^b K(x, t) \varphi(t) dt$$

maps C into C . In other words, we assume that $T(\varphi) \in C$ whenever $\varphi \in C$. A solution of the integral equation is any function φ in C that satisfies (7.83).

THEOREM 7.22. AN EXISTENCE THEOREM FOR INTEGRAL EQUATIONS. *If, under the foregoing conditions, we have*

$$(7.84) \quad |K(x, y)| \leq M$$

for all (x, y) in S , where $M > 0$, then for each λ such that

$$(7.85) \quad |\lambda| < \frac{1}{M(b-a)}$$

there is one and only one function φ in C that satisfies the integral equation (7.83).

Proof. We shall prove that T is a contraction operator. Take any two functions φ_1 and φ_2 in C and consider the difference

$$T(\varphi_1)(x) - T(\varphi_2)(x) = \lambda \int_a^b K(x, t) [\varphi_1(t) - \varphi_2(t)] dt.$$

Using the inequality (7.84) we may write

$$|T(\varphi_1)(x) - T(\varphi_2)(x)| \leq |\lambda| M(b-a) \|\varphi_1 - \varphi_2\| = \alpha \|\varphi_1 - \varphi_2\|,$$

where $\alpha = |\lambda| M(b-a)$. Because of (7.85) we have $0 \leq \alpha < 1$, so T is a contraction operator with contraction constant α . Therefore T has a unique fixed point φ in C . This function φ satisfies (7.83).

PART 2

NONLINEAR ANALYSIS

8

DIFFERENTIAL CALCULUS OF SCALAR AND VECTOR FIELDS

8.1 Functions from \mathbf{R}^n to \mathbf{R}^m . Scalar and vector fields

Part 1 of this volume dealt primarily with linear transformations

$$T: V \rightarrow W$$

from one linear space V into another linear space W . In Part 2 we drop the requirement that T be linear but restrict the spaces V and W to be finite-dimensional. Specifically, we shall consider functions with domain in n -space \mathbf{R}^n and with range in m -space \mathbf{R}^m .

When both n and m are equal to 1, such a function is called a real-valued function of a real variable. When $n = 1$ and $m > 1$ it is called a vector-valued function of a real variable. Examples of such functions were studied extensively in Volume I.

In this chapter we assume that $n > 1$ and $m \geq 1$. When $m = 1$ the function is called a real-valued function of a vector variable or, more briefly, a *scalar field*. When $m > 1$ it is called a vector-valued function of a vector variable, or simply a *vector field*.

This chapter extends the concepts of limit, continuity, and derivative to scalar and vector fields. Chapters 10 and 11 extend the concept of the integral.

Notation: Scalars will be denoted by light-faced type, and vectors by bold-faced type. If f is a scalar field defined at a point $\mathbf{x} = (x_1, \dots, x_n)$ in \mathbf{R}^n , the notations $f(\mathbf{x})$ and $f(x_1, \dots, x_n)$ are both used to denote the value of f at that particular point. If \mathbf{f} is a vector field we write $\mathbf{f}(\mathbf{x})$ or $\mathbf{f}(x_1, \dots, x_n)$ for the function value at \mathbf{x} . We shall use the inner product

$$\mathbf{x} \cdot \mathbf{y} = \sum_{k=1}^n x_k y_k$$

and the corresponding norm $\|\mathbf{x}\| = (\mathbf{x} \cdot \mathbf{x})^{1/2}$, where $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$. Points in \mathbf{R}^2 are usually denoted by (x, y) instead of (x_1, x_2) ; points in \mathbf{R}^3 by (x, y, z) instead of (x_1, x_2, x_3) .

Scalar and vector fields defined on subsets of \mathbf{R}^2 and \mathbf{R}^3 occur frequently in the applications of mathematics to science and engineering. For example, if at each point \mathbf{x} of the atmosphere we assign a real number $f(\mathbf{x})$ which represents the temperature at \mathbf{x} , the function

\mathbf{f} so defined is a scalar field. If we assign a vector which represents the wind velocity at that point, we obtain an example of a vector field.

In physical problems dealing with either scalar or vector fields it is important to know how the field changes as we move from one point to another. In the one-dimensional case the derivative is the mathematical tool used to study such changes. Derivative theory in the one-dimensional case deals with functions defined on open intervals. To extend the theory to \mathbf{R}^n we consider generalizations of open intervals called *open sets*.

8.2 Open balls and open sets

Let \mathbf{a} be a given point in \mathbf{R}^n and let r be a given positive number. The set of all points \mathbf{x} in \mathbf{R}^n such that

$$\|\mathbf{x} - \mathbf{a}\| < r$$

is called an *open n-ball* of radius r and center \mathbf{a} . We denote this set by $B(\mathbf{a})$ or by $B(\mathbf{a}; r)$.

The ball $B(\mathbf{a}; r)$ consists of all points whose distance from \mathbf{a} is less than r . In \mathbf{R}^1 this is simply an open interval with center at \mathbf{a} . In \mathbf{R}^2 it is a circular disk, and in \mathbf{R}^3 it is a spherical solid with center at \mathbf{a} and radius r .

DEFINITION OF AN INTERIOR POINT. *Let S be a subset of \mathbf{R}^n , and assume that $\mathbf{a} \in S$. Then \mathbf{a} is called an interior point of S if there is an open n -ball with center at \mathbf{a} , all of whose points belong to S .*

In other words, every interior point \mathbf{a} of S can be surrounded by an n -ball $B(\mathbf{a})$ such that $B(\mathbf{a}) \subseteq S$. The set of all interior points of S is called the *interior* of S and is denoted by $\text{int } S$. An open set containing a point \mathbf{a} is sometimes called a *neighborhood* of \mathbf{a} .

DEFINITION OF AN OPEN SET. *A set S in \mathbf{R}^n is called open if all its points are interior points. In other words, S is open if and only if $S = \text{int } S$.*

EXAMPLES. In \mathbf{R}^1 the simplest type of open set is an open interval. The union of two or more open intervals is also open. A closed interval $[a, b]$ is not an open set because neither endpoint of the interval can be enclosed in a 1-ball lying within the given interval.

The 2-ball $S = B(\mathbf{0}; 1)$ shown in Figure 8.1 is an example of an open set in \mathbf{R}^2 . Every point \mathbf{a} of S is the center of a disk lying entirely in S . For some points the radius of this disk is very small.

Some open sets in \mathbf{R}^2 can be constructed by taking the Cartesian product of open sets in \mathbf{R}^1 . If A_1 and A_2 are subsets of \mathbf{R}^1 , their Cartesian product $A_1 \times A_2$ is the set in \mathbf{R}^2 defined by

$$A_1 \times A_2 = \{(\mathbf{a}_1, \mathbf{a}_2) \mid \mathbf{a}_1 \in A_1 \text{ and } \mathbf{a}_2 \in A_2\}.$$

An example is shown in Figure 8.2. The sets A_1 and A_2 are intervals, and $A_1 \times A_2$ is a rectangle.

If A_1 and A_2 are open subsets of \mathbf{R}^1 , then $A_1 \times A_2$ will be an open subset of \mathbf{R}^2 . To prove this, choose any point $\mathbf{a} = (\mathbf{a}_1, \mathbf{a}_2)$ in $A_1 \times A_2$. We must show that \mathbf{a} is an interior point



Circular disk

FIGURE 8.1 The disk $B(O; 1)$ is an open set in \mathbf{R}^2 .

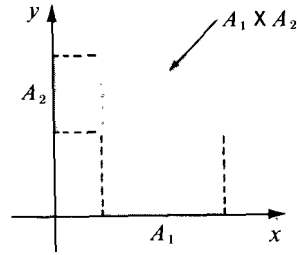


FIGURE 8.2 The Cartesian product of two open intervals is an open rectangle.

of $A_1 \times A_2$. Since A_1 and A_2 are open in \mathbf{R}^1 there is a 1-ball $B(a_1; r_1)$ in A_1 and a 1-ball $B(a_2; r_2)$ in A_2 . Let $r = \min\{r_1, r_2\}$. We can easily show that the 2-ball $B(a; r) \subseteq A_1 \times A_2$. In fact, if $x = (x_1, x_2)$ is any point of $B(a; r)$ then $\|x - a\| < r$, so $|x_1 - a_1| < r_1$ and $|x_2 - a_2| < r_2$. Hence $x_1 \in B(a_1; r_1)$ and $x_2 \in B(a_2; r_2)$. Therefore $x_1 \in A_1$ and $x_2 \in A_2$, so $(x_1, x_2) \in A_1 \times A_2$. This proves that every point of $B(a; r)$ is in $A_1 \times A_2$. Therefore every point of $A_1 \times A_2$ is an interior point, so $A_1 \times A_2$ is open.

The reader should realize that an open subset of \mathbf{R}^1 is no longer an open set when it is considered as a subset of \mathbf{R}^2 , because a subset of \mathbf{R}^1 cannot contain a 2-ball.

DEFINITIONS OF EXTERIOR AND BOUNDARY. A point x is said to be exterior to a set S in \mathbf{R}^n if there is an n -ball $B(x)$ containing no points of S . The set of all points in \mathbf{R}^n exterior to S is called the exterior of S and is denoted by $\text{ext } S$. A point which is neither exterior to S nor an interior point of S is called a boundary point of S . The set of all boundary points of S is called the boundary of S and is denoted by ∂S .

These concepts are illustrated in Figure 8.1. The exterior of S is the set of all x with $\|x\| > 1$. The boundary of S consists of all x with $\|x\| = 1$.

8.3 Exercises

1. Let f be a scalar field defined on a set S and let c be a given real number. The set of all points x in S such that $f(x) = c$ is called a *level set* of f . (Geometric and physical problems dealing with level sets will be discussed later in this chapter.) For each of the following scalar fields, S is the whole space \mathbf{R}^n . Make a sketch to describe the level sets corresponding to the given values of c .

- | | |
|--|---|
| (a) $f(x, y) = x^2 + y^2$, | $c = 0, 1, 4, 9$. |
| (b) $f(x, y) = e^{xy}$, | $c = e^{-2}, e^{-1}, 1, e, e^2, e^3$. |
| (c) $f(x, y) = \cos(x + y)$, | $c = -1, 0, \frac{1}{2}, \frac{1}{2}\sqrt{2}, 1$. |
| (d) $f(x, y, z) = x + y + z$, | $c = -1, 0, 1$. |
| (e) $f(x, y, z) = x^2 + 2y^2 + 3z^2$, | $c = 0, 6, 12$. |
| (f) $f(x, y, z) = \sin(x^2 + y^2 + z^2)$, | $c = -1, -\frac{1}{2}, 0, \frac{1}{2}\sqrt{2}, 1$. |

2. In each of the following cases, let S be the set of all points (x, y) in the plane satisfying the given inequalities. Make a sketch showing the set S and explain, by a geometric argument, whether or not S is open. Indicate the boundary of S on your sketch.
- (a) $x^2 + y^2 < 1$. (h) $1 \leq x \leq 2$ and $3 < y < 4$.
 (b) $3x^2 + 2y^2 < 6$. (i) $1 < x < 2$ and $y > 0$.
 (c) $|x| < 1$ and $|y| < 1$. (j) $x \geq y$.
 (d) $x \geq 0$ and $y > 0$. (k) $x > y$.
 (e) $|x| \leq 1$ and $|y| \leq 1$. (l) $y > x^2$ and $|x| < 2$.
 (f) $x > 0$ and $y < 0$. (m) $(x^2 + y^2 - 1)(4 - x^2 - y^2) > 0$.
 (g) $xy < 1$. (n) $(2x - x^2 - y^2)(x^2 + y^2 - x) > 0$.
3. In each of the following, let S be the set of all points (x, y, z) in 3-space satisfying the given inequalities and determine whether or not S is open.
- (a) $z^2 - x^2 - y^2 - 1 > 0$.
 (b) $|x| < 1$, $|y| < 1$, and $|z| < 1$.
 (c) $x + y + z < 1$.
 (d) $|x| \leq 1$, $|y| < 1$, and $|z| < 1$.
 (e) $x + y + z < 1$ and $x > 0$, $y > 0$, $z > 0$.
 (f) $x^2 + 4y^2 + 4z^2 - 2x + 16y + 40z + 113 < 0$.
4. (a) If A is an open set in n -space and if $\mathbf{x} \in A$, show that the set $A - \{\mathbf{x}\}$, obtained by removing the point \mathbf{x} from A , is open.
 (b) If A is an open interval on the real line and B is a closed subinterval of A , show that $A - B$ is open.†
 (c) If A and B are open intervals on the real line, show that $A \cup B$ and $A \cap B$ are open.
 (d) If A is a closed interval on the real line, show that its complement (relative to the whole real line) is open.
5. Prove the following properties of open sets in \mathbf{R}^n :
- (a) The empty set \emptyset is open.
 (b) \mathbf{R}^n is open.
 (c) The union of any collection of open sets is open.
 (d) The intersection of a finite collection of open sets is open.
 (e) Give an example to show that the intersection of an infinite collection of open sets is not necessarily open.

Closed sets. A set S in \mathbf{R}^n is called **closed** if its complement $\mathbf{R}^n - S$ is open. The next three exercises discuss properties of closed sets.

6. In each of the following cases, let S be the set of all points (x, y) in \mathbf{R}^2 satisfying the given conditions. Make a sketch showing the set S and give a geometric argument to explain whether S is open, closed, both open and closed, or neither open nor closed.
- (a) $x^2 + y^2 \geq 0$. (g) $1 \leq x \leq 2$, $3 \leq y \leq 4$.
 (b) $x^2 + y^2 < 0$. (h) $1 \leq x \leq 2$, $3 \leq y < 4$.
 (c) $x^2 + y^2 \leq 1$. (i) $y = x^2$.
 (d) $1 < x^2 + y^2 < 2$. (j) $y \geq x^2$.
 (e) $1 \leq x^2 + y^2 \leq 2$. (k) $y \geq x^2$ and $|x| < 2$.
 (f) $1 < x^2 + y^2 \leq 2$. (l) $y \geq x^2$ and $|x| \leq 2$.
7. (a) If A is a closed set in n -space and \mathbf{x} is a point not in A , prove that $A \cup \{\mathbf{x}\}$ is also closed.
 (b) Prove that a closed interval $[a, b]$ on the real line is a closed set.
 (c) If A and B are closed intervals on the real line, show that $A \cup B$ and $A \cap B$ are closed.

† If A and B are sets, the difference $A - B$ (called the *complement of B relative to A*) is the set of all elements of A which are not in B .

8. Prove the following properties of closed sets in \mathbf{R}^n . You may use the results of Exercise 5.
 - (a) The empty set \emptyset is closed.
 - (b) \mathbf{R}^n is closed.
 - (c) The intersection of any collection of closed sets is closed.
 - (d) The union of a finite number of closed sets is closed.
 - (e) Give an example to show that the union of an infinite collection of closed sets is not necessarily closed.
9. Let S be a subset of \mathbf{R}^n .
 - (a) Prove that both $\text{int } S$ and $\text{ext } S$ are open sets.
 - (b) Prove that $\mathbf{R}^n = (\text{int } S) \cup (\text{ext } S) \cup \partial S$, a union of disjoint sets, and use this to deduce that the boundary ∂S is always a closed set.
10. Given a set S in \mathbf{R}^n and a point \mathbf{x} with the property that every ball $B(\mathbf{x})$ contains both interior points of S and points exterior to S . Prove that \mathbf{x} is a boundary point of S . Is the converse statement true? That is, does every boundary point of S necessarily have this property?
11. Let S be a subset of \mathbf{R}^n . Prove that $\text{ext } S = \text{int}(\mathbf{R}^n - S)$.
12. Prove that a set S in \mathbf{R}^n is closed if and only if $S = (\text{int } S) \cup \partial S$.

8.4 Limits and continuity

The concepts of limit and continuity are easily extended to scalar and vector fields. We shall formulate the definitions for vector fields; they apply also to scalar fields.

We consider a function $f: S \rightarrow \mathbf{R}^m$, where S is a subset of \mathbf{R}^n . If $\mathbf{a} \in \mathbf{R}^n$ and $\mathbf{b} \in \mathbf{R}^m$ we write

$$(8.1) \quad \lim_{\mathbf{x} \rightarrow \mathbf{a}} f(\mathbf{x}) = \mathbf{b} \quad (\text{or, } f(\mathbf{x}) \rightarrow \mathbf{b} \text{ as } \mathbf{x} \rightarrow \mathbf{a})$$

to mean that

$$(8.2) \quad \lim_{\|\mathbf{x} - \mathbf{a}\| \rightarrow 0} \|f(\mathbf{x}) - \mathbf{b}\| = 0.$$

The limit symbol in equation (8.2) is the usual limit of elementary calculus. In this definition it is not required that \mathbf{b} be defined at the point \mathbf{a} itself.

If we write $\mathbf{h} = \mathbf{x} - \mathbf{a}$, Equation (8.2) becomes

$$\lim_{\|\mathbf{h}\| \rightarrow 0} \|f(\mathbf{a} + \mathbf{h}) - \mathbf{b}\| = 0.$$

For points in \mathbf{R}^2 we write (x, y) for \mathbf{x} and (a, b) for \mathbf{a} and express the limit relation (8.1) as follows :

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y) = \mathbf{b}.$$

For points in \mathbf{R}^3 we put $\mathbf{x} = (x, y, z)$ and $\mathbf{a} = (a, b, c)$ and write

$$\lim_{(x,y,z) \rightarrow (a,b,c)} f(x, y, z) = \mathbf{b}.$$

A function f is said to be *continuous* at \mathbf{a} if f is defined at \mathbf{a} and if

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} f(\mathbf{x}) = f(\mathbf{a}).$$

We say f is *continuous on a set S* iff f is continuous at each point of S .

Since these definitions are straightforward extensions of those in the one-dimensional case, it is not surprising to learn that many familiar properties of limits and continuity can also be extended. For example, the usual theorems concerning limits and continuity of sums, products, and quotients also hold for scalar fields. For vector fields, quotients are not defined but we have the following theorem concerning sums, multiplication by scalars, inner products, and norms.

THEOREM 8.1. *If $\lim_{x \rightarrow a} f(x) = b$ and $\lim_{x \rightarrow a} g(x) = c$, then we also have:*

- (a) $\lim_{x \rightarrow a} [f(x) + g(x)] = b + c$.
- (b) $\lim_{x \rightarrow a} \lambda f(x) = \lambda b$ for every scalar λ .
- (c) $\lim_{x \rightarrow a} f(x) \cdot g(x) = b \cdot c$.
- (d) $\lim_{x \rightarrow a} \|f(x)\| = \|b\|$.

Proof. We prove only parts (c) and (d); proofs of (a) and (b) are left as exercises for the reader.

To prove (c) we write

$$f(x) \cdot g(x) - b \cdot c = [f(x) - b] \cdot [g(x) - c] + b \cdot [g(x) - c] + c \cdot [f(x) - b].$$

Now we use the triangle inequality and the Cauchy-Schwarz inequality to obtain

$$0 \leq \|f(x) \cdot g(x) - b \cdot c\| \leq \|f(x) - b\| \|g(x) - c\| + \|b\| \|g(x) - c\| + \|c\| \|f(x) - b\|.$$

Since $\|f(x) - b\| \rightarrow 0$ and $\|g(x) - c\| \rightarrow 0$ as $x \rightarrow a$, this shows that $\|f(x) \cdot g(x) - b \cdot c\| \rightarrow 0$ as $x \rightarrow a$, which proves (c).

Taking $f(x) = g(x)$ in part (c) we find

$$\lim_{x \rightarrow a} \|f(x)\|^2 = \|b\|^2,$$

from which we obtain (d).

EXAMPLE 1. Continuity and components of a vector field. If a vector field f has values in \mathbb{R}^m , each function value $f(x)$ has m components and we can write

$$f(x) = (f_1(x), \dots, f_m(x)).$$

The m scalar fields f_1, \dots, f_m are called **components** of the vector field f . We shall prove that f is continuous at a point if, and only if, each component f_k is continuous at that point.

Let e_k denote the k th unit coordinate vector (all the components of e_k are 0 except the k th, which is equal to 1). Then $f_k(x)$ is given by the dot product

$$f_k(x) = f(x) \cdot e_k.$$

Therefore, part (c) of Theorem 8.1 shows that each point of continuity of \mathbf{f} is also a point of continuity of f_k . Moreover, since we have

$$\mathbf{f}(\mathbf{x}) = \sum_{k=1}^m f_k(\mathbf{x})\mathbf{e}_k,$$

repeated application of parts (a) and (b) of Theorem 8.1 shows that a point of continuity of all m components f_1, \dots, f_m is also a point of continuity of \mathbf{f} .

EXAMPLE 2. Continuity of the identity function. The identity function, $\mathbf{f}(\mathbf{x}) = \mathbf{x}$, is continuous everywhere in \mathbf{R}^n . Therefore its components are also continuous everywhere in \mathbf{R}^n . These are the n scalar fields given by

$$f_1(\mathbf{x}) = x_1, f_2(\mathbf{x}) = x_2, \dots, f_n(\mathbf{x}) = x_n.$$

EXAMPLE 3. Continuity of linear transformations. Let $\mathbf{f}: \mathbf{R}^n \rightarrow \mathbf{R}^m$ be a linear transformation. We will prove that \mathbf{f} is continuous at each point \mathbf{a} in \mathbf{R}^n . By linearity we have

$$\mathbf{f}(\mathbf{a} + \mathbf{h}) = \mathbf{f}(\mathbf{a}) + \mathbf{f}(\mathbf{h}).$$

Therefore, it suffices to prove that $\mathbf{f}(\mathbf{h}) \rightarrow \mathbf{0}$ as $\mathbf{h} \rightarrow \mathbf{0}$. Writing \mathbf{h} in terms of its components we have $\mathbf{h} = h_1\mathbf{e}_1 + \dots + h_n\mathbf{e}_n$. Using linearity again we find $\mathbf{f}(\mathbf{h}) = h_1\mathbf{f}(\mathbf{e}_1) + \dots + h_n\mathbf{f}(\mathbf{e}_n)$. This shows that $\mathbf{f}(\mathbf{h}) \rightarrow \mathbf{0}$ as $\mathbf{h} \rightarrow \mathbf{0}$.

EXAMPLE 4. Continuity of polynomials in n variables. A scalar field P defined on \mathbf{R}^n by a formula of the form

$$P(\mathbf{x}) = \sum_{k_1=0}^{p_1} \dots \sum_{k_n=0}^{p_n} c_{k_1 \dots k_n} x_1^{k_1} \dots x_n^{k_n}$$

is called a polynomial in n variables x_1, \dots, x_n . A polynomial is continuous everywhere in \mathbf{R}^n because it is a finite sum of products of scalar fields continuous everywhere in \mathbf{R}^n . For example, a polynomial in two variables x and y , given by

$$P(x, y) = \sum_{i=0}^p \sum_{j=0}^q c_{ij} x^i y^j$$

is continuous at every point (x, y) in \mathbf{R}^2 .

EXAMPLE 5. Continuity of rational functions. A scalar field f given by $f(\mathbf{x}) = P(\mathbf{x})/Q(\mathbf{x})$, where P and Q are polynomials in the components of \mathbf{x} , is called a rational function. A rational function is continuous at each point where $Q(\mathbf{x}) \neq 0$.

Further examples of continuous function can be constructed with the help of the next theorem, which is concerned with continuity of composite functions.

THEOREM 8.2. Let \mathbf{f} and g be functions such that the composite function $\mathbf{f} \circ g$ is defined at a , where

$$(\mathbf{f} \circ g)(\mathbf{x}) = \mathbf{f}[g(\mathbf{x})].$$

If g is continuous at a and if \mathbf{f} is continuous at $g(a)$, then the composition $\mathbf{f} \circ g$ is continuous at a .

Proof. Let $\mathbf{y} = \mathbf{f}(\mathbf{x})$ and $\mathbf{b} = g(a)$. Then we have

$$\mathbf{f}[g(\mathbf{x})] - \mathbf{f}[g(a)] = \mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{b}).$$

By hypothesis, $\mathbf{y} \rightarrow \mathbf{b}$ as $\mathbf{x} \rightarrow a$, so we have

$$\lim_{\|\mathbf{x}-a\| \rightarrow 0} \|\mathbf{f}[g(\mathbf{x})] - \mathbf{f}[g(a)]\| = \lim_{\|\mathbf{y}-\mathbf{b}\| \rightarrow 0} \|\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{b})\| = 0.$$

Therefore $\lim_{\mathbf{x} \rightarrow a} \mathbf{f}[g(\mathbf{x})] = \mathbf{f}[g(a)]$, so $\mathbf{f} \circ g$ is continuous at a .

EXAMPLE 6. The foregoing theorem implies continuity of scalar fields h , where $h(x, y)$ is given by formulas such as

$$\sin(x^2y), \quad \log(x^2 + y^2), \quad \frac{x+y}{x+y}, \quad \log[\cos(x^2 + y^2)].$$

These examples are continuous at all points at which the functions are defined. The first is continuous at all points in the plane, and the second at all points except the origin. The third is continuous at all points (x, y) at which $x + y \neq 0$, and the fourth at all points at which $x^2 + y^2$ is not an odd multiple of $n\pi/2$, $n = 1, 3, 5, \dots$, is a family of circles centered at the origin.] These examples show that the discontinuities of a function of two variables may consist of isolated points, entire curves, or families of curves.

EXAMPLE 7. A function of two variables may be continuous in each variable separately and yet be discontinuous as a function of the two variables together. This is illustrated by the following example:

$$f(x, y) = \frac{xy}{x^2 + y^2} \quad \text{if } (x, y) \neq (0, 0), \quad f(0, 0) = 0.$$

For points (x, y) on the x -axis we have $y = 0$ and $f(x, y) = f(x, 0) = 0$, so the function has the constant value 0 everywhere on the x -axis. Therefore, if we put $y = 0$ and think of f as a function of x alone, f is continuous at $x = 0$. Similarly, f has the constant value 0 at all points on the y -axis, so if we put $x = 0$ and think off as a function of y alone, f is continuous at $y = 0$. However, as a function of two variables, f is not continuous at the origin. In fact, at each point of the line $y = x$ (except at the origin) the function has the constant value $\frac{1}{2}$ because $f(x, x) = x^2/(2x^2) = \frac{1}{2}$; since there are points on this line arbitrarily close to the origin and since $f(0, 0) \neq \frac{1}{2}$, the function is not continuous at $(0, 0)$.

8.5 Exercises

The exercises in this section are concerned with limits and continuity of scalar fields defined on subsets of the plane.

1. In each of the following examples a scalar field is defined by the given equation for all points (x, y) in the plane for which the expression on the right is defined. In each example determine the set of points (x, y) at which it is continuous.

$$(a) f(x, y) = x^4 + y^4 - 4x^2y^2.$$

$$(f) f(x, y) = \arcsin \frac{x}{\sqrt{x^2 + y^2}}.$$

$$(b) f(x, y) = \log(x^2 + y^2).$$

$$(g) f(x, y) = \arctan \frac{x+y}{1-xy}.$$

$$(c) f(x, y) = \frac{1}{y} \cos x^2.$$

$$(h) f(x, y) = \frac{x}{\sqrt{x^2 + y^2}}.$$

$$(d) f(x, y) = \tan \frac{x^2}{y}.$$

$$(i) f(x, y) = x^{(y^2)}.$$

$$(e) f(x, y) = \arctan \frac{y}{x}.$$

$$(j) f(x, y) = \arccos \sqrt{x/y}.$$

2. If $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L$, and if the one-dimensional limits

$$\lim_{x \rightarrow a} f(x, y) \quad \text{and} \quad \lim_{y \rightarrow b} f(x, y)$$

both exist, prove that

$$\lim_{x \rightarrow a} [\lim_{y \rightarrow b} f(x, y)] = \lim_{y \rightarrow b} [\lim_{x \rightarrow a} f(x, y)] = L.$$

The two limits in this equation are called *iterated* limits; the exercise shows that the existence of the two-dimensional limit and of the two one-dimensional limits implies the existence and equality of the two iterated limits. (The converse is not always true. A counter example is given in Exercise 4.)

3. Let $f(x, y) = (x - y)/(x + y)$ if $x + y \neq 0$. Show that

$$\lim_{x \rightarrow 0} [\lim_{y \rightarrow 0} f(x, y)] = 1 \quad \text{but that} \quad \lim_{y \rightarrow 0} [\lim_{x \rightarrow 0} f(x, y)] = -1.$$

Use this result with Exercise 2 to deduce that $f(x, y)$ does not tend to a limit as $(x, y) \rightarrow (0, 0)$.

4. Let

$$f(x, y) = \frac{x^2 y^2}{x^2 y^2 + (x - y)^2} \quad \text{whenever } x^2 y^2 + (x - y)^2 \neq 0.$$

Show that

$$\lim_{x \rightarrow 0} [\lim_{y \rightarrow 0} f(x, y)] = \lim_{y \rightarrow 0} [\lim_{x \rightarrow 0} f(x, y)] = 0$$

but that $f(x, y)$ does not tend to a limit as $(x, y) \rightarrow (0, 0)$. [Hint: Examine f on the line $y = x$.] This example shows that the converse of Exercise 2 is not always true.

5. Let

$$f(x, y) = \begin{cases} x \sin \frac{1}{y} & \text{if } y \neq 0, \\ 0 & \text{if } y = 0. \end{cases}$$

Show that $f(x, y) \rightarrow 0$ as $(x, y) \rightarrow (0, 0)$ but that

$$\lim_{y \rightarrow 0} [\lim_{x \rightarrow 0} f(x, y)] \neq \lim_{x \rightarrow 0} [\lim_{y \rightarrow 0} f(x, y)].$$

Explain why this does not contradict Exercise 2.

6. If $(x, y) \neq (0, 0)$, let $f(x, y) = (x^2 - y^2)/(x^2 + y^2)$. Find the limit of $f(x, y)$ as $(x, y) \rightarrow (0, 0)$ along the line $y = mx$. Is it possible to define $f(0, 0)$ so as to make f continuous at $(0, 0)$?
7. Let $f(x, y) = 0$ if $y \leq 0$ or if $y \geq x^2$ and let $f(x, y) = 1$ if $0 < y < x^2$. Show that $f(x, y) \rightarrow 0$ as $(x, y) \rightarrow (0, 0)$ along any straight line through the origin. Find a curve through the origin along which (except at the origin) $f(x, y)$ has the constant-value 1. Is f continuous at the origin?
8. If $f(x, y) = [\sin(x^2 + y^2)]/(x^2 + y^2)$ when $(x, y) \neq (0, 0)$ how must $f(0, 0)$ be defined so as to make f continuous at the origin?
9. Let f be a scalar field continuous at an interior point a of a set S in \mathbf{R}^n . If $f(a) \neq 0$, prove that there is an n -ball $B(a)$ in which f has the same sign as $f(a)$.

8.6 The derivative of a scalar field with respect to a vector

This section introduces derivatives of scalar fields. Derivatives of vector fields are discussed in Section 8.18.

Let f be a scalar field defined on a set S in \mathbf{R}^n , and let a be an interior point of S . We wish to study how the field changes as we move from a to a nearby point. For example, suppose $f(a)$ is the temperature at a point a in a heated room with an open window. If we move toward the window the temperature will decrease, but if we move toward the heater it will increase. In general, the manner in which a field changes will depend on the direction in which we move away from a .

Suppose we specify this direction by another vector y . That is, suppose we move from a toward another point $a + y$ along the line segment joining a and $a + y$. Each point on this segment has the form $a + hy$, where h is a real number. An example is shown in Figure 8.3. The distance from a to $a + hy$ is $\|hy\| = |h| \|y\|$.

Since a is an interior point of S , there is an n -ball $B(a; r)$ lying entirely in S . If h is chosen so that $|h| \|y\| < r$, the segment from a to $a + hy$ will lie in S . (See Figure 8.4.) We keep

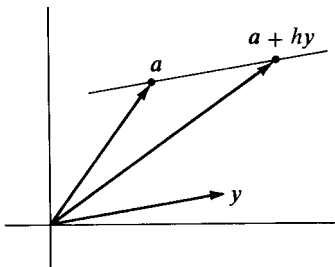


FIGURE 8.3 The point $a + hy$ lies on the line through a parallel to y .

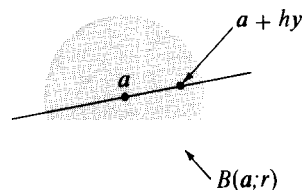


FIGURE 8.4 The point $a + hy$ lies in the n -ball $B(a; r)$ if $\|hy\| < r$.

$h \neq 0$ but small enough to guarantee that $a + hy \in S$ and we form the difference quotient

$$(8.3) \quad \frac{f(a + hy) - f(a)}{h}.$$

The numerator of this quotient tells us how much the function changes when we move from a to $a + hy$. The quotient itself is called the **average rate of change** off over the line segment joining a to $a + hy$. We are interested in the behavior of this quotient as $h \rightarrow 0$. This leads us to the following definition.

DEFINITION OF THE DERIVATIVE OF A SCALAR FIELD WITH RESPECT TO A VECTOR. Given a scalar field $f: S \rightarrow \mathbb{R}$, where $S \subseteq \mathbb{R}^n$. Let a be an interior point of S and let y be an arbitrary point in \mathbb{R}^n . The derivative off at a with respect to y is denoted by the symbol $f'(a; y)$ and is defined by the equation

$$(8.4) \quad f'(a; y) = \lim_{h \rightarrow 0} \frac{f(a + hy) - f(a)}{h}$$

when the limit on the right exists.

EXAMPLE 1. If $y = \mathbf{0}$, the difference quotient (8.3) is 0 for every $h \neq 0$, so $f'(a; \mathbf{0})$ always exists and equals 0.

EXAMPLE 2. Derivative of a linear transformation. If $f: S \rightarrow \mathbb{R}$ is linear, then $f(a + hy) = f(a) + hf(y)$ and the difference quotient (8.3) is equal to $f(y)$ for every $h \neq 0$. In this case, $f'(a; y)$ always exists and is given by

$$f'(a; y) = f(y)$$

for every a in S and every y in \mathbb{R}^n . In other words, the derivative of a linear transformation with respect to y is equal to the value of the function at y .

To study how f behaves on the line passing through a and $a + y$ for $y \neq \mathbf{0}$ we introduce the function

$$g(t) = f(a + ty).$$

The next theorem relates the derivatives $g'(t)$ and $f'(a + ty; y)$.

THEOREM 8.3. Let $g(t) = f(a + ty)$. If one of the derivatives $g'(t)$ or $f'(a + ty; y)$ exists then the other also exists and the two are equal,

$$(8.5) \quad g'(t) = f'(a + ty; y).$$

In particular, when $t = 0$ we have $g'(0) = f'(a; y)$.