3 CHAPTER Coordinates

3.1 Lines and Circles

The most important step in geometry since ancient times was the introduction of coordinates by Descartes in his *Geometry* of 1637. Coordinates are a simple idea, but not much use without algebra and a symbolic notation, which is probably why the idea did not take off earlier. The time was ripe for coordinates in 1637, because algebra and its notation had matured over the preceding century, to a level similar to high school algebra today. In fact, Fermat in 1629 hit on the same idea as Descartes, and he illustrated it with similar results, but they were not published at the time.

The coordinates of a point P of the plane are its distances (x, y) from perpendicular axes OY and OX (Figure 3.1). O is called the *origin* of coordinates, and its own coordinates are $(0, 0)$.

Distances are normally regarded as positive numbers, so this scheme initially applies only to points P above and to the right of O, the so-called *positive quadrant*. In fact, Fermat and Descartes did not look beyond this quadrant, because they did not consider the possibility of negative distances. But if points to the left of O are

J. Stillwell, *Numbers and Geometry*

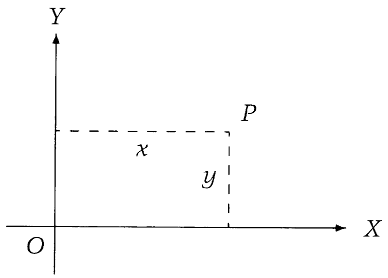

FIGURE 3.1 Coordinates of a point.

given negative x coordinates and points below O are given negative y coordinates, everything works smoothly. (This is one of the benefits of making the rules of arithmetic for negative quantities the same as the rules for positive quantities, as we did in Section 1.4.) In particular, some curves that look "sawn off" in the positive quadrant have natural extensions to the rest of the plane.

For example, a straight line through the origin (other than OY) has the property that the ratio y/x of distances from the axes is a constant, say m. This gives us the equation $y = mx$, and the negative values of x and y satisfying the same equation lie on the same straight line (Figure 3.2).

Generalizing this idea slightly, we arrive at the general equation $ax + by = c$, which gives any straight line by suitable choice of constants a, b, and c. This is why we call this equation *linear*.

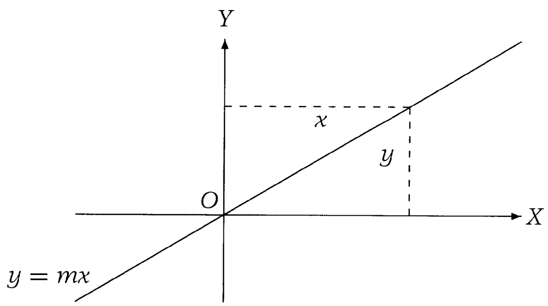

FIGURE 3.2 Straight line through the origin.

The next most important curve is the circle, whose points have the property of being at constant distance from its center. If we take the center to be O, and radius r, then Pythagoras' theorem tells us that

$$x^2 + y^2 = r^2,$$

because r is the hypotenuse of a right-angled triangle with sides x and y. Similarly, the points at distance r from an arbitrary point (a, b) satisfy the equation

$$(x - a)^2 + (y - b)^2 = r^2,$$

hence this is the equation for a general circle.

These equations are no doubt very familiar to most readers, but it is worth reflecting on what we assumed in deriving them. Among other things, we used the existence of rectangles, similar triangles of different sizes, and Pythagoras' theorem. In Chapter 2 we saw that all of these are characteristic of Euclidean geometry, each of them equivalent to the parallel axiom.

The representation of straight lines and circles by equations in fact gives yet another way to define Euclidean geometry. This did not dawn on Fermat and Descartes—they took their geometry straight out of Euclid and saw the coordinates merely as a way of handling it more efficiently—but it became important when non-Euclidean geometry was discovered in the 19th century. One finally had to wonder: what *is* Euclidean geometry? And what is non-Euclidean geometry, for that matter? The beauty of coordinates is that they allow all geometries to be built on a common foundation of numbers, with different geometries distinguished by different equations. When we look at geometry this way, Euclidean geometry turns out to be the one with the simplest equations.

Of course, the improved handling obtained by the use of coordinates is also important. Coordinate geometry is called *analytic* because situations are "analyzed" rather than "synthesized." Typically, points, lines, and circles are found by solving equations, rather than by ruler and compass construction as in Euclid. Thanks to the power of algebra, analysis is a method of much greater scope. As we shall see, it can even show the *impossibility* of certain constructions sought by the Greeks.