When we say that a compound experiment E is determined by two stochastically independent experiments E_1 and E_2, we mean that the probability space $(S, \mathscr{B}, P)$ is defined in the manner just described, "independence" being reflected in the fact that $P(x, y)$ is the product $P_1(x)P_2(y)$. It can be shown that the assignment of probabilities in (13.14) implies the formula

$$(13.16) \qquad P(U \times V) = P_1(U)P_2(V)$$

for every pair of subsets U in $\mathscr{B}_1$ and V in $\mathscr{B}_2$. (See Exercise 12 in Section 13.23 for an outline of the proof.) We shall deduce some important consequences of this formula.

Let A be an event (in the compound experiment E) of the form

$$A = C_1 \times S_2,$$

where $C_1 \in \mathscr{B}_1$. Each outcome in A is an ordered pair (x, y) where x is restricted to be an outcome of C_1 (in the first experiment E_1) but y can be any outcome of S_2 (in the second experiment E_2). If we apply (13.16) we find

$$P(A) = P(C_1 \times S_2) = P_1(C_1)P_2(S_2) = P_1(C_1),$$

since $P_2(S_2) = 1$. Thus the definition of P assigns the same probability to A that P_1 assigns to C_1. For this reason, such an event A is said to be *determined by the first experiment* E_1. Similarly, if B is an event of E of the form

$$B = S_1 \times C_2,$$

where $C_2 \in \mathscr{B}_2$, we have

$$P(B) = P(S_1 \times C_2) = P_1(S_1)P_2(C_2) = P_2(C_2)$$

and B is said to be *determined by the second experiment* E_2. We shall now show, using (13.16), that two such events A and B are *independent*. That is, we have

$$(13.17) \qquad P(A \cap B) = P(A)P(B).$$

First we note that

$$\begin{aligned} A \cap B &= \{(x, y) \mid (x, y) \in C_1 \times S_2 \text{ and } (x, y) \in S_1 \times C_2\} \\ &= \{(x, y) \mid x \in C_1 \text{ and } y \in C_2\} \\ &= C_1 \times C_2. \end{aligned}$$

Hence, by (13.16), we have

$$(13.18) \qquad P(A \cap B) = P(C_1 \times C_2) = P_1(C_1)P_2(C_2).$$

Since $P_1(C_1) = P(A)$ and $P_2(C_2) = P(B)$ we obtain (13.17). Note that Equation (13.18) also shows that we can compute the probability $P(A \cap B)$ as a product of probabilities in the individual sample spaces S_1 and S_2; hence no calculations with probabilities in the compound experiment are needed.

The generalization to compound experiments determined by n experiments E_1, E_2, . . . , E_n is carried out in the same way. The points in the new sample space are n-tuples $(x_1, x_2, \ldots, x_n)$ and the point probabilities are defined as the product of the probabilities of the separate outcomes,

$$P(x_1, x_2, \ldots, x_n) = P_1(x_1)P_2(x_2) \cdots P_n(x_n). \tag{13.19}$$

When this definition of P is used we say that E is determined by *n independent experiments* $E_1, E_2, \ldots, E_n$. In the special case in which all the experiments are associated with the same probability space, the compound experiment E is said to be an example of *independent repeated trials under identical conditions.* Such an example is considered in the next section.

13.16 Bernoulli trials

An important example of a compound experiment was studied extensively by Jakob Bernoulli and is now known as a *Bernoulli sequence of trials.* This is a sequence of repeated trials executed under the same conditions, each result being stochastically independent of all the others. The experiment being repeated has just two possible outcomes, usually called "success" and "failure;" the probability of success is denoted by p and that of failure by q. Of course, $q = 1 - p$. The main result associated with Bernoulli sequences is the following theorem:

THEOREM 13.3. BERNOULLI'S FORMULA. *The probability of exactly k successes in n Bernoulli trials is*

$$\binom{n}{k} p^k q^{n-k}, \tag{13.20}$$

where $\binom{n}{k}$ *denotes the binomial coefficient,* $\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$.

Proof. Denote "success" by S and "failure" by F and consider a particular sequence of n results. This may be represented by an n-tuple

$$(x_1, x_2, \ldots, x_n),$$

where each x_i is either an S or an F. The event A in which we are interested is the collection of all n-tuples that contain exactly k S's and $n - k$ F's. Let us compute the point probability of a particular n-tuple in A. The probability of each S is p, and that of each F is q. Hence, by (13.19), the probability of each particular n-tuple in A is the product of k factors equal to p with $n - k$ factors equal to q. That is,

$$P(x_1, x_2, \ldots, x_n) = p^k q^{n-k} \qquad \text{if } (x_1, x_2, \ldots, x_n) \in A.$$

Therefore, to compute P(A) we need only count the number of elements in A and multiply this number by $p^k q^{n-k}$. But the number of elements in A is simply the number of ways of putting exactly k S's into the n possible positions of the n-tuple. This is the same as the number of subsets of k elements that can be formed from a set consisting of n elements; we have already seen that this number is $\binom{n}{k}$. Therefore, if we add the point probabilities for all points in A we obtain

$$P(A) = \binom{n}{k} p^k q^{n-k}.$$

EXAMPLE 1. An unbiased coin is tossed 50 times. Compute the probability of exactly 25 heads.

Solution. We interpret this experiment as a sequence of 50 Bernoulli trials, in which "success" means "heads" and "failure" means "tails." Since the coin is unbiased we assign the probabilities $p = q = \frac{1}{2}$, and formula (13.20) gives us $\binom{50}{k}(\frac{1}{2})^{50}$ for the probability of exactly k heads in 50 tosses. In particular, when $k = 25$ we obtain

$$\binom{50}{25}\left(\frac{1}{2}\right)^{50} = \frac{50!}{25!\,25!}\left(\frac{1}{2}\right)^{50}.$$

To express this number as a decimal it is best to use logarithms, since tables of logarithms of factorials are readily available. If we denote the number in question by P, a table of common logarithms (base 10) gives us

$$\begin{aligned} \log P &= \log 50! - 2\log 25! - 50\log 2 \\ &= 64.483 - 50.381 - 15.052 = -0.950 = 0.05 - 1.00 \\ &= \log 1.12 - \log 10 = \log 0.112, \end{aligned}$$

so $P = 0.112$.

EXAMPLE 2. What is the probability of at least r successes in n Bernoulli trials?

Solution. Let A_k denote the event "exactly k successes in n trials." Then the event E in which we are interested is the union

$$E = A_r \cup A_{r+1} \cup \cdots \cup A_n.$$

Since the A_k are disjoint, we find

$$P(E) = \sum_{k=r}^{n} P(A_k) = \sum_{k=r}^{n} \binom{n}{k} p^k q^{n-k}.$$

Since

$$\sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} = (p+q)^n = 1,$$

the probability of the complementary event E' can be computed as follows:

$$P(E') = 1 - P(E) = \sum_{k=0}^{r-1} \binom{n}{k} p^k q^{n-k}.$$

This last sum gives us the probability of at most $r - 1$ successes in n trials.

13.17 The most probable number of successes in n Bernoulli trials

A pair of fair dice is rolled 28 times. What is the most probable number of sevens? To solve this problem we let $f(k)$ denote the probability of exactly k sevens in 28 tosses. The probability of tossing a seven is $\frac{1}{6}$. Bernoulli's formula tells us that

$$f(k) = \binom{28}{k} \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{28-k}.$$

We wish to determine what value (or values) of k in the range $k = 0, 1, 2, \ldots, 28$ make $f(k)$ as large as possible. The next theorem answers this question for any sequence of Bernoulli trials.

THEOREM 13.4. *Given an integer $n \geq 1$ and a real p, $0 < p < 1$, consider the set of numbers*

$$f(k) = \binom{n}{k} p^k (1 - p)^{n-k}, \qquad \text{for} \quad k = 0, 1, \ldots, n.$$

(a) *If $(n + 1)p$ is not an integer, the largest value of $f(k)$ occurs for exactly one k:*

$$k = [(n + 1)p], \qquad \text{the greatest integer} < (n + 1)p.$$

(b) *If $(n + 1)p$ is an integer, the largest value of $f(k)$ occurs for exactly two values of k:*

$$k = (n + 1)p \qquad \text{and} \qquad k = (n + 1)p - 1.$$

Proof. To study the behavior of $f(k)$ we consider the ratio

$$r(k) = \frac{f(k)}{f(k + 1)} = \frac{k + 1}{n - k} \frac{1 - p}{p}$$

for $k = 0, 1, \ldots, n - 1$. The function $r(k)$ is strictly increasing so we have

$$0 < r(0) < r(1) < \cdots < r(n - 1).$$

We consider six cases, illustrated in Figure 13.2. In the first three cases we show that $f(k)$ takes its largest value for exactly one k. In the remaining cases $f(k)$ takes its largest value for two consecutive values of k.

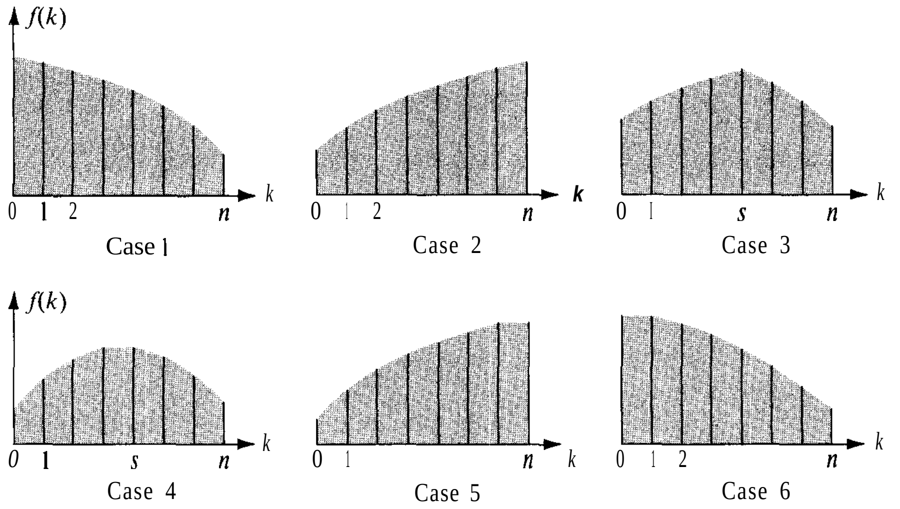

FIGURE 13.2 Calculation of the most probable number of successes in n Bernoulli trials.

CASE 1. $r(0) > 1$. In this case $r(k) > 1$ for every k so we have

$$f(0) > f(1) > \cdots > f(n).$$

Therefore the largest value of $f(k)$ occurs only for $k = 0$. Also, $r(0) = (1 - p)/(np) > 1$, so $1 - p > np$, $(n + 1)p < 1$, hence $[(n + 1)p] = 0$.

CASE 2. $r(n - 1) < 1$. In this case $r(k) < 1$ for every k so $f(0) < f(1) < \cdots < f(n)$ and the largest value of $f(k)$ occurs only for $k = n$. Since $r(n - 1) = n(1 - p)/p < 1$, we have $n - np < p$, hence $n < (n + 1)p < n + 1$, so $[(n + 1)p] = n$.

CASE 3. $r(0) < 1$, $r(n - 1) > 1$, *and* $r(k) \neq 1$ *for all* k. In this case there is a unique integer s, $0 < s < n$, such that $r(s - 1) < 1$ and $r(s) > 1$. The function $f(k)$ increases in the range $0 \le k \le s$ and decreases in the range $s \le k \le n$. Therefore $f(k)$ has a unique maximum at $k = s$. Since $r(s - 1) = s(1 - p)/(np - sp + p) < 1$ we have $s < (n + 1)p$. The inequality $r(s) > 1$ shows that $(n + 1)p < s + 1$, hence $[(n + 1)p] = s$.

Note that in each of the first three cases the maximum value of $f(k)$ occurs when $k = [(n + 1)p]$; also $(n + 1)p$ is not an integer in any of these cases.

CASE 4. $r(0) < 1$, $r(n - 1) > 1$, *and* $r(s - 1) = 1$ *for some* s, $2 \le s < n$. In this case $f(k)$ increases for $0 \le k \le s - 1$ and decreases for $s \le k \le n$. The maximum value of $f(k)$ occurs twice, when $k = s - 1$ and when $k = s$. The equation $r(s - 1) = 1$ implies $(n + 1)p = s$.

CASE 5. $r(n - 1) = 1$. In this case $r(k) < 1$ for $k \le n - 2$, so $f(k)$ increases in the range $0 \le k \le n - 1$, and $f(n - 1) = f(n)$. Hence the maximum of $f(k)$ occurs twice, when $k = n - 1$ and when $k = n$. The equation $r(n - 1) = 1$ implies $(n + 1)p = n$.

CASE 6. $r(0) = 1$. In this case $r(k) > 1$ for $k \geq 1$, so f'(k) decreases in the range $1 \leq k \leq n$. The maximum $f(k)$ occurs twice, when $k = 0$ and when $k = 1$. The equation $r(0) = 1$ implies $(n + 1)p = 1$.

In each of the last three cases the maximum value off(k) occurs for $k = (n + 1)p$ and for $k = (n + 1)p - 1$. This completes the proof.

EXAMPLE 1. A pair of fair dice is rolled 28 times. What is the most probable number of sevens ?

Solution. We apply Theorem 13.4 with $n = 28$, $p = \frac{1}{6}$, and $(n + 1)p = \frac{29}{6}$. This is not an integer so the largest value off(k) occurs for $k = [\frac{29}{6}] = 4$.

Note: If the dice are rolled 29 times there are two solututions, $k = 4$ and $k = 5$.

EXAMPLE 2. Find the smallest n such that if a pair of fair dice is thrown n times the probability of getting exactly four sevens is at least as large as the probability of getting any other number of sevens.

Solution. We take $p = \frac{1}{6}$ in Theorem 13.4. We want the largest value of $f(k)$ to occur when $k = 4$. This requires either $[(n + 1)p] = 4$, $(n + 1)p = 4$, or $(n + 1)p - 1 = 4$. The smallest n satisfying any of these relations is $n = 23$.

13.18 Exercises

1. A coin is tossed twice, the probability of heads on the first toss being p_1 and that on the second toss p_2. Consider this a compound experiment determined by two stochastically independent experiments, and let the sample space be

$$S = \{(H, H), (H, T), (T, H), (T, T)\}.$$

(a) Compute the probability of each element of S.
(b) Can p_1 and p_2 be assigned so that

$$P(H, H) = \tfrac{1}{9}, \qquad P(H, T) = P(T, H) = \tfrac{2}{9}, \qquad P(T, T) = \tfrac{4}{9}?$$

(c) Can p_1 and p_2 be assigned so that

$$P(H, H) = P(T, T) = \tfrac{1}{3}, \qquad P(H, T) = P(T, H) = \tfrac{1}{6}\ ?$$

(d) Consider the following four events (subsets of S):

H_1 : heads on the first toss,
H_2 : heads on the second toss,
T_1 : tails on the first toss,
T_2 : tails on the second toss.

Determine which pairs of these four events are independent.

In each of Exercises 2 through 12 describe your sample space, your assignment of probabilities, and the event whose probability you are computing.

2. A student takes a true – false examination consisting of 10 questions. He is completely unprepared so he plans to guess each answer. The guesses are to be made at random. For example, he may toss a fair coin and use the outcome to determine his guess.
   (a) Compute the probability that he guesses correctly at least five times.
   (b) Compute the probability that he guesses correctly at least nine times.
   (c) What is the smallest n such that the probability of guessing at least n correct answers is less than $\frac{1}{2}$?
3. Ten fair dice are tossed together. What is the probability that exactly three sixes occur?
4. A fair coin is tossed five times. What is the probability of getting (a) exactly three heads? (b) at least three heads? (c) at most one head?
5. A man claims to have a divining rod which locates hidden sources of oil. The Caltech Geology Department conducts the following experiment to test his claim. He is taken into a room in which there are 10 sealed barrels. He is told that five of them contain oil and five contain water. His task is to decide which of the five contain oil and which do not.
   (a) What is the probability that he locates the five oil barrels correctly just by chance?
   (b) What is the probability that he locates at least three of the oil barrels correctly by chance?
6. A little old lady from Pasadena claims that by tasting a cup of tea made with milk she can tell whether the milk or the tea was added first to the cup. The lady's claim is tested by requiring her to taste and classify 10 pairs of cups of tea, each pair containing one cup of tea made by each of the two methods under consideration. Let p denote her "true" probability of classifying a pair of cups correctly. (If she is skillful, p is substantially greater than $\frac{1}{2}$; if not, $p \leq \frac{1}{2}$.) Assume the 10 pairs of cups are classified under independent and identical conditions.
   (a) Compute, in terms of p, the probability that she classifies correctly at least eight of the 10 pairs of cups.
   (b) Evaluate this probability explicitly when $p = \frac{1}{2}$.
7. (Another problem of Chevalier de Mere.) Determine whether or not it is advantageous to bet even money on at least one 6 appearing in four throws of an unbiased die. [Hint: Show that the probability of at least one 6 in n throws is $1 - (\frac{5}{6})^n$.]
8. An urn contains w white balls and b black balls. If $k \leq n$, compute the probability of drawing k white balls in n drawings, if each ball is replaced before the next one is drawn.
9. Two dice are thrown eight times. Compute the probability that the sum is 11 exactly three times.
10. Throw a coin 10 times or 10 coins once and count the number of heads. Find the probability of obtaining at least six heads.
11. After a long series of tests on a certain kind of rocket engine it has been determined that in approximately 5 % of the trials there will be a malfunction that will cause the rocket to misfire. Compute the probability that in 10 trials there will be at least one failure.
12. A coin is tossed repeatedly. Compute the probability that the total number of heads will be at least 6 before the total number of tails reaches 5.
13. Exercise 12 may be generalized as follows: Show that the probability of at least m successes before n failures in a sequence of Bernoullian trials is

$$\sum_{k=m}^{m+n-1} \binom{m+n-1}{k} p^k q^{m+n-k-1}.$$

14. Determine all n with the following property: If a pair of fair dice is thrown n times, the probability of getting exactly ten sevens is at least as large as the probability of getting any other number of sevens.

15. A binary slot machine has three identical and independent wheels. When the machine is played the possible outcomes are ordered triples (x, y, z), where each of x, y, z can be 0 or 1. On each wheel the probability of 0 is p and the probability of 1 is $1 - p$, where $0 < p < 1$. The machine pays \$2 if the outcome is $(1, 1, 1)$ or $(0, 0, 0)$; it pays \$1 for the outcome $(1, 1, 0)$; otherwise it pays nothing. Let $f(p)$ denote the probability that the machine pays \$1 or more when it is played once.
(a) Calculate $f(p)$.
(b) Define the "payoff" to be the sum $\sum_{x\in S} g(x)P(x)$, where S is the sample space, $P(x)$ is the probability of outcome x, and $g(x)$ is the number of dollars paid by outcome x. Calculate the value of p for which the payoff is smallest.

13.19 Countable and uncountable sets

Up to now we have discussed probability theory only for finite sample spaces. We wish now to extend the theory to infinite sample spaces. For this purpose it is necessary to distinguish between two types of infinite sets, *countable* and *uncountable.* This section describes these two concepts.

To count the members of an n-element set we match the set, element by element, with the set of integers $\{1, 2, \ldots, n\}$. Comparing the sizes of two sets by matching them element by element takes the place of counting when we deal with infinite sets. The process of "matching" can be given a neat mathematical formulation by employing the function concept:

DEFINITION. *Two sets A and B are said to be in one-to-one correspondence if a function f exists with the following properties:*
(a) *The domain of f is A and the range of f is B.*
(b) *If x and y are distinct elements of A, then f(x) and f(y) are distinct elements of B. That is, for all x and y in A,*

$$x \neq y \quad \text{implies} \quad f(x) \neq f(y). \tag{13.21}$$

A function satisfying property (13.21) is said to be *one-to-one* on A. Two sets A and B in one-to-one correspondence are also said to be *equivalent*, and we indicate this by writing $A \sim B$. It is clear that every set A is equivalent to itself, since we may let $f(x) = x$ for each x in A.

A set can be equivalent to a proper subset of itself. For example, the set $P = \{1, 2, 3, \ldots\}$, consisting of all the positive integers, is equivalent to the proper subset $Q = \{2, 4, 6, \ldots\}$ consisting of the even integers. In this case a one-to-one function which makes them equivalent is given by $f(x) = 2x$ for x in P.

If $A \sim B$ we can easily show that $B \sim A$. In fact, if f is one-to-one on A and if the range of f is B, then for each b in B there is exactly one a in A such that $f(a) = b$. Therefore we can define an inverse function g on B as follows: If $b \in B$, $g(b) = a$, where a is the unique element of A such that $f(a) = b$. This g is one-to-one on B and its range is A; hence $B \sim A$. This property of equivalence is known as *symmetry:*

$$A \sim B \quad \text{implies} \quad B \sim A. \tag{13.22}$$

It is also easy to show that equivalence has the following property, known as *transitivity:*

$$A \sim B \quad \text{and} \quad B \sim C \quad \text{implies} \quad A \sim C. \tag{13.23}$$

A proof of the transitive property is requested in Exercise 2 of Section 13.20.

A set S is called *finite* and is said to contain n elements if

$$S \sim \{1, 2, \ldots, n\}.$$

The empty set is also considered to be finite. Sets which are not finite are called *infinite sets.* A set S is said to be *countably injinite* if it is equivalent to the set of all positive integers, that is, if

$$S \sim \{1, 2, 3, \ldots\}. \tag{13.24}$$

In this case there is a function f which establishes a one-to-one correspondence between the positive integers and the elements of S; hence the set S can be displayed in roster notation as follows:

$$S = \{f(1), f(2), f(3), \ldots\}.$$

Often we use subscripts and denote $f(k)$ by a_k (or by a similar notation) and we write $S = \{a_1, a_2, a_3, \ldots\}$. The important thing here is that the correspondence in (13.24) enables us to use the positive integers as "labels" for the elements of S.

A set is said to be *countable* if it is finite or countably infinite. A set which is not countable is called *uncountable.*† (Examples will be given presently.) Many set operations when performed on countable sets produce countable sets. For example, we have the following properties :

(a) Every subset of a countable set is countable.
(b) The intersection of any collection of countable sets is countable.
(c) The union of a countable collection of countable sets is countable.
(d) The Cartesian product of a finite number of countable sets is countable.

Since we shall do very little with countably infinite sets in this book, detailed proofs of these properties will not be given.‡ Instead, we shall give a number of examples to show how these properties may be used to construct new countable sets from given ones.

EXAMPLE 1. The set S of all integers (positive, negative, or zero) is countable.

Proof. If $n \in S$, let $f(n) = 2n$ if n is positive, and let $f(n) = 2|n| + 1$ if n is negative or zero. The domain off is S and its range is the set of positive integers. Since f is one-to-one on S, this shows that S is countable.

EXAMPLE 2. The set R of all rational numbers is countable.

Proof. For each fixed integer $n \geq 1$, let S_n denote the set of rational numbers of the form x/n, where x belongs to the set S of Example 1. Each set S_n is equivalent to S [take

† The words *denumerable* and ***nondenumerable*** are sometimes used as synonyms for ***countable*** and ***un-countable,*** respectively.

‡ Proofs are outlined in Exercises 3 through 8 of Section 13.20.

$f(t) = nt$ if $t \in S_n$] and hence each S_n is countable. Since R is the union of all the S_n, property (c) implies that R is countable.

Note. If $\mathscr{F} = \{A_1, A_2, A_3, \ldots\}$ is a countable collection of sets, the union of all sets in the family $\mathscr{F}$ is denoted by the symbols

$$\bigcup_{k=1}^{\infty} A_k \quad \text{or} \quad A_1 \cup A_2 \cup A_3 \cup \cdots .$$

EXAMPLE 3. Let A be a countably infinite set, say $A = \{a_1, a_2, a_3, \ldots\}$. For each integer $n \geq 1$, let $\mathscr{F}_n$ denote the family of n-element subsets of A. That is, let

$$\mathscr{F}_n = \{S \mid S \subseteq A \text{ and } S \text{ has } n \text{ elements}\}.$$

Then each $\mathscr{F}_n$ is countable.

Proof. If S is an n-element subset of A, we may write

$$S = \{a_{k_1}, a_{k_2}, \ldots, a_{k_n}\},$$

where $k_1 < k_2 < \cdots < k_n$. Let $f(S) = (a_{k_1}, a_{k_2}, \ldots, a_{k_n})$. That is, f is the function which associates with S the ordered n-tuple $(a_{k_1}, a_{k_2}, \ldots, a_{k_n})$. The domain of f is $\mathscr{F}_n$ and its range, which we denote by T_n, is a subset of the Cartesian product $C_n = A \times A \times \cdots \times A$ (n factors). Since A is countable, so is C_n [by property (d)] and hence T_n is also [by property (a)]. But $T_n \sim \mathscr{F}_n$ because f is one-to-one. This shows that $\mathscr{F}_n$ is countable.

EXAMPLE 4. The collection of all finite subsets of a countable set is countable.

Proof. The result is obvious if the given set is finite. Assume, then, that the given set (call it A) is countably infinite, and let $\mathscr{F}$ denote the class of all finite subsets of A:

$$\mathscr{F} = \{S \mid S \subseteq A \text{ and } S \text{ is finite}\}.$$

Then $\mathscr{F}$ is the union of all the families $\mathscr{F}_n$ of Example 3; hence, by property (c), $\mathscr{F}$ is countable.

EXAMPLE 5. The collection of *all* subsets of a countably infinite set is uncountable.

Proof. Let A denote the given countable set and let $\mathscr{A}$ denote the family of all subsets of A. We shall assume that $\mathscr{A}$ is countable and arrive at a contradiction. If $\mathscr{A}$ is countable, then $\mathscr{A} \sim A$ and hence there exists a one-to-one function f whose domain is A and whose range is $\mathscr{A}$. Thus for each a in A, the function value $f(a)$ is a subset of A. This subset may or may not contain the element a. We denote by B the set of elements a such that $a \notin f(a)$. Thus,

$$B = \{a \mid a \in A \text{ but } a \notin f(a)\}.$$

This B, being a subset of A, must belong to the family $\mathscr{A}$. This means that $B = f(b)$ for some b in A. Now there are only two possibilities: (i) $b \in B$, or (ii) $b \notin B$. If $b \in B$, then by the definition of B *we* have $b \notin f(b)$, which is a contradiction since $f(b) = B$. Therefore (i) is impossible. In case (ii), $b \notin B$, which means $b \notin f(b)$. This contradicts the definition of B, so case (ii) is also impossible. Therefore the assumption that $\mathscr{A}$ is countable leads to a contradiction and we must conclude that $\mathscr{A}$ is uncountable.

We give next an example of an uncountable set that is easier to visualize than that in Example 5.

EXAMPLE 6. The set of real x satisfying $0 < x < 1$ is uncountable.

Proof. Again, we assume the set is countable and arrive at a contradiction. If the set is countable we may display its elements as follows: {x„ x_2, x_3, . . .}. Now we shall construct a real number y satisfying $0 < y < 1$ which is not in this list. For this purpose we write each element x_n as a decimal:

$$x_n = 0.a_{n,1}\, a_{n,2}\, a_{n,3} \cdots,$$

where each $a_{n,i}$ is one of the integers in the set {O., $1, 2, \ldots, 9$}. Let y be the real number which has the decimal expansion

$$y = 0.y_1 y_2 y_3 \cdots,$$

where

$$\text{Yn} = \begin{cases} 1 & \text{if } a_{n,n} \neq 1, \\ 2 & \text{if } a_{n,n} = 1. \end{cases}$$

Then no element of the set $\{x_1, x_2, x_3, \ldots\}$ can be equal to y, because y differs from x_1 in the first decimal place, differs from x_2 in the second decimal place, and in general, y differs from x_k in the kth decimal place. (A situation like $x_n = 0.249999 \cdots$ and $y = 0.250000 \cdots$ cannot occur here because of the way the y_n are chosen.) Since this y satisfies $0 < y < 1$, we have a contradiction, and hence the set of real numbers in the open interval (0, 1) is uncountable.

13.20 Exercises

1. Let $P = \{1, 2, 3, \ldots\}$ denote the set of positive integers. For each of the following sets, exhibit a one-to-one function f whose domain is P and whose range is the set in question:
   (a) $A = \{2, 4, 6, \ldots\}$, the set of even positive integers.
   (b) $B = (3, 3^2, 3^3, \ldots\}$, the set of powers of 3.
   (c) C = (2, 3, 5, 7, 11, 13, . . .}, the set of primes. *[Note:* Part of the proof consists in showing that C is an infinite set.]
   (d) $P \times P$, the Cartesian product of P with itself.
   (e) The set of integers of the form $2^m 3^n$, where m and n are positive integers.
2. Prove the transitive property of set equivalence:

$$\text{If } A - B \text{ and } B - C, \text{ then } A - C.$$

[Hint: Iffmakes A equivalent to B and ifg makes B equivalent to C, show that the composite function $h = g \circ f$ makes A equivalent to C.]