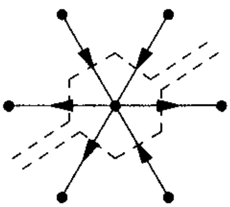
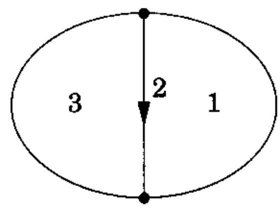
7.3.22. Theorem. (Tutte [1954b]) A plane bridgeless graph is k-face-colorable if and only if it has a nowhere-zero k-flow.

Proof: (Younger [1983], refined by Seymour) Let f be a flow on a plane graph G. We define a function g on the set of faces by letting $g(F)$ be the net flow accumulated by traveling from face F out to the unbounded face. Each time we cross an edge e we count $+f(e)$ if e is directed toward our right, $-f(e)$ if e is directed toward our left. The value assigned to the outside face is 0.

The function g is well-defined; that is, $g(F)$ is independent of our route to the outside face. We can change a route into any other by a succession of changes where we go the "other way" around some vertex v (shown on the left below). The change increases or decreases our accumulation for this portion by the net flow out of v, which is 0. Note that the difference between the values on faces with a common edge e is $\pm f(e)$.





Conversely, given a function g defined on the faces, we can invert the process to obtain a flow (shown on the right above). As we stand on face F and look and face F' across edge e, we let $f(e) = g(F) - g(F')$ if e is directed toward our right, $f(e) = g(F') - g(F)$ if e is directed toward our left.

Thus flows correspond to face-colorings. The face-coloring is proper if and only if the flow is nowhere-zero. If the flow is a nowhere-zero k-flow, then reducing the labels in the coloring to congruence classes in $\{0, \dots, k-1\}$ produces a proper k-coloring. Conversely, a proper k-face-coloring using these colors produces a nowhere-zero k-flow. ■

The correspondence between face-labelings and flows in Theorem 7.3.22 is valid when the labels come from any abelian group. Applied using the group of binary ordered pairs under addition ((0, 0) is the identity), the statement proved by this argument is precisely Tait's Theorem itself.

Since we can study flows on all graphs, we can consider the flow problem as a general dual notion to vertex coloring. "Nowhere-zero" is the analogue of "proper". Since every nowhere-zero k-flow *is* a nowhere-zero $k+1$-flow, the natural problem is to minimize k such that G has a nowhere-zero k-flow. This minimum is the **flow number** of G, by analogy with "chromatic number". Since we say "G is k-colorable" when G has a proper k-coloring, the natural analogue would be to say "G is k-**flowable**" instead of "G has a nowhere-zero k-flow". This language is not yet common, so we will use it sparingly.

By Tait's Theorem, Theorem 7.3.22 states that a cubic bridgeless planar graph is 3-edge-colorable if and only if it has a nowhere-zero 4-flow. We want to extend this correspondence by dropping the condition on planarity. A simple observation about parity will be useful.

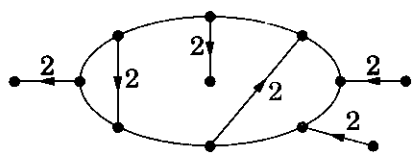
7.3.23. Lemma. In a nowhere-zero k-flow, every vertex is incident to an even number of edges of odd weight.

Proof: Since at each vertex the total weight on entering edges equals the total weight on exiting edges, the sum of the weights is even. ■

7.3.24. Theorem. Let G be a cubic graph. If G has a nowhere-zero 4-flow, then G is 3-edge-colorable.

Proof: By Proposition 7.3.15, we may assume that G has a positive 4-flow (D, f), and thus $f(e) \in \{1, 2, 3\}$ for each edge e. By Lemma 7.3.23, each vertex is incident to exactly one edge of weight 2. Thus the edges of weight 2 form a 1-factor in G, and deleting them leaves a union of disjoint cycles. To complete a 1-factorization, it suffices to show that each of these cycles has even length.

Let C be such a cycle. The edges of weight 2 that are incident to vertices of C are chords or join $V(C)$ with $\overline{V(C)}$. The chords occupy an even size subset of $V(C)$. Thus it suffices to show that the number of edges between $V(C)$ and $\overline{V(C)}$ is even. These edges all have weight 2. Since the net flow out of $V(C)$ must be 0 and all edges between $V(C)$ and $\overline{V(C)}$ have flow 2, the number of edges leaving $V(C)$ must equal the number of edges entering it. ■



Since the Petersen graph is not 3-edge-colorable, Theorem 7.3.24 implies that it is not 4-flowable. Existence of nowhere-zero k-flows is preserved by subdivision: when an edge e of weight j in a nowhere-zero k-flow is subdivided, replacing it with a path of length 2 oriented in the same direction with weight j on both edges yields a nowhere-zero k-flow in the new graph. Thus subdivisions of the Petersen graph also have no nowhere-zero 4-flows.

The converse of Theorem 7.3.24 is true but not trivial, since it may not be possible to treat the color classes as edge sets of fixed weight and orient the graph to make this a 4-flow. In the graph $C_3 \square K_2$ of Example 7.3.21, there is essentially only one proper 3-edge-coloring, and when the color classes are labeled $1, 2, 3$ it is not possible to obtain a 4-flow. In the positive 4-flow in Example 7.3.21, the edges of weight 1 do not form a matching.

Nevertheless, we can apply the next theorem to guarantee nowhere-zero 4-flows in cubic graphs. The characterization is more general, since it does not require regularity.

7.3.25. Theorem. A graph has a nowhere-zero 4-flow if and only if it is the union of two even graphs.

Proof: Let G_1, G_2 be even graphs with $G = G_1 \cup G_2$. Let D be an orientation of G, restricting to D_i on G_i. By Proposition 7.3.19 and Proposition 7.3.15, G_i has a nowhere-zero 2-flow (D_i, f_i). Extend f_i to $E(G)$ by letting $f_i(e) = 0$ for $e \in E(G) - E(G_i)$. Let $f = f_1 + 2f_2$. This weight function is odd on $E(G_1)$ and is ± 2 on $E(G) - E(G_1)$, so it is nowhere-zero. Its magnitude is always at most 3, and by Proposition 7.3.16 (D, f) is a flow; thus it is a nowhere-zero 4-flow.

Conversely, let (D, f) be a nowhere-zero 4-flow on G. Let $E_1 = \{e \in E(G): f(e) \text{ is odd}\}$. By Lemma 7.3.23, E_1 forms an even subgraph of G. Thus there is a nowhere-zero 2-flow (D_1, f_1) on E_1, where D_1 agrees with D. Extend f_1 to $E(G)$ by letting $f_1(e) = 0$ for $e \in E(G) - E_1$; now (D, f_1) is a 2-flow on G.

Define f_2 on $E(G)$ by $f_2 = (f - f_1)/2$. By Proposition 7.3.16, (D, f_2) is a flow on G. It is an integer flow, since $f(e) - f_1(e)$ is always even. By Lemma 7.3.23, the set $E_2 = \{e \in E(G): f_2(e) \text{ is odd}\}$ forms an even subgraph of G. For $e \in E(G) - E_1$, we have $f(e) = \pm 2$ and $f_1(e) = 0$, which yields $f_2(e) = \pm 1$, so $E(G) - E_1 \subseteq E_2$. Now G is the union of two even subgraphs. ∎

7.3.26. Corollary. If G is a cubic graph, then G is 3-edge-colorable if and only if G has a nowhere-zero 4-flow.

Proof: Every 3-edge-colorable cubic graph is the union of two even subgraphs: the edges of colors 1 and 2, and the edges of colors 1 and 3. ∎

In light of Theorem 7.3.22, Corollary 7.3.26 generalizes Tait's Theorem.

We have seen that subdivisions of the Petersen graph are not 4-flowable. Among bridgeless graphs, Tutte conjectured that excluding such subgraphs yields nowhere-zero 4-flows.

7.3.27. Conjecture. (Tutte's 4-flow Conjecture—Tutte [1966b]) Every bridgeless graph containing no subdivision of the Petersen graph is 4-flowable. ∎

Since every graph containing a subdivision of the Petersen graph is nonplanar, Tutte's 4-flow Conjecture implies the Four Color Theorem. Since nowhere-zero 4-flows are equivalent to 3-edge-colorings on cubic graphs, the 4-flow Conjecture also implies the 3-edge-coloring Conjecture (which has been proved). Researchers have hoped for an elegant proof of Tutte's 4-flow Conjecture as a way of obtaining a shorter proof of the Four Color Theorem.

We close this section by describing of several other famous conjectures related to these. Every nowhere-zero k-flow is a nowhere-zero $k+1$-flow, so conditions for nowhere-zero 3-flows or 5-flows should be more or less restrictive, respectively, than conditions for a nowhere-zero 4-flow. Statements of Tutte's 3-flow Conjecture appear in Steinberg [1976] and in Bondy–Murty [1976, Unsolved Problem 48].

7.3.28. Conjecture. (Tutte's 3-flow Conjecture) Every 4-edge-connected graph has a nowhere-zero 3-flow. ∎

7.3.29. Conjecture. (Tutte's 5-flow Conjecture—Tutte [1954b]) Every bridgeless graph has a nowhere-zero 5-flow. ∎

Kilpatrick [1975] and Jaeger [1979] proved that every bridgeless graph is 8-flowable. Seymour [1981] proved that these graphs are 6-flowable. We sketch the ideas of the 8-flow Theorem; details are requested in exercises.

Both proofs reduce to the 3-edge-connected case, by showing that a smallest bridgeless graph without a nowhere-zero k-flow is simple, 2-connected, and 3-edge-connected (Exercise 26). The main step is then to express a 3-edge-connected graph as a union of subgraphs with good flows. A generalization of Theorem 7.3.25 then applies: If G_1 has a nowhere-zero k_1-flow and G_2 has a nowhere-zero k_2-flow, then $G_1 \cup G_2$ has a nowhere-zero $k_1 k_2$-flow (Exercise 24). (The converse also holds but is not needed.)

For the 8-flow Theorem, it then suffices to prove that a 3-edge-connected graph can be expressed as the union of three even subgraphs. First, adding an additional copy of each edge in G yields a 6-edge-connected graph G'. Then, the Tree–Packing Theorem of Nash-Williams (Corollary 8.2.59) yields three pairwise edge-disjoint spanning trees in G'. These correspond to three spanning trees in G. Since we obtained them as edge-disjoint trees in G', each edge of G appears in at most two of them.

Within a spanning tree of G, one can find a **parity subgraph** of G, meaning a spanning subgraph H such that $d_H(v) \equiv d_G(v) \bmod 2$ for all $v \in V(G)$ (Exercise 25). The complement within $E(G)$ of the edge set of a parity subgraph is an even subgraph of G. Since our three spanning trees have no common edge, the complements of their parity subgraphs express G as a union of three even subgraphs. By Proposition 7.3.19, each has a nowhere-zero 2-flow, and hence G has a nowhere-zero 8-flow.

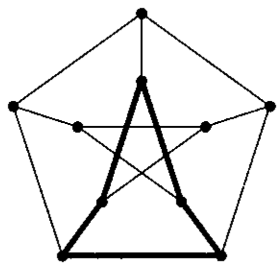
The approach in Seymour [1981] is similar; the task is to express a 3-edge-connected graph as a union of an even graph and a 3-flowable graph. This uses more subtle concepts, including a notion of "modular" flows originally introduced by Tutte [1949]. Seymour's proof was refined by Younger [1983] and Jaeger [1988]. We refer the reader to Zhang [1997] for an exposition.

Celmins [1984] proved that if the 5-flow Conjecture is false, then the smallest counterexample is a snark having girth at least 7 and no nontrivial edge cut with four edges.

We describe one additional conjecture and its relation to earlier topics. In a 2-edge-connected plane graph, all facial boundaries are cycles. Each edge lies in the boundary of two faces, so the facial cycles together cover every edge exactly twice. It is reasonable to ask whether such a covering can be obtained also for graphs that are not planar.

7.3.30. Definition. A **cover** of a graph G is a list of subgraphs whose union is G. A **double cover** is a cover with each edge appearing in exactly two subgraphs in the list. A **cycle double cover (CDC)** is a double cover consisting of cycles.

7.3.31. Example. Together with the outer 5-cycle, the 5 rotations of the 5-cycle illustrated below form a CDC of the Petersen graph. The Petersen graph also has CDCs using cycles of other lengths (Exercise 36). ■



Since cut-edges appear in no cycles, only bridgeless graphs have CDCs.

7.3.32. Conjecture. (Cycle Double Cover Conjecture—Szekeres [1973], Seymour [1979b]) Every bridgeless graph has a cycle double cover. ∎

One might think that the CDC Conjecture follows immediately using embeddings on surfaces with handles, but such embeddings may have facial boundaries that traverse the same edge twice. The **Strong Embedding Conjecture** asserts that every 2-connected graph has an embedding (on some surface) in which the boundary of each face is a single cycle. Applying this to each block of a 2-edge-connected graph would yield the CDC Conjecture.

In discussing the CDC, we must alert the reader to an unfortunate conflict in terminology. Throughout this book, we use the definition of *cycle* that is common in discussing connectivity, girth, circumference, planarity, etc. In this language, a *circuit* is an equivalence class of closed trails (ignoring the starting vertex), and an *even graph* is a graph whose vertex degrees are all even. A circuit traverses a connected even graph.

The literature on cycle covers generally reverses this terminology, using "circuit" to mean what we call a cycle and "cycle" to mean what we call an even graph. Since the term "even graph" strongly evokes its definition, we hope that our usage will be clear.

The alternative usage arises from other contexts. In a matroid (Section 8.2), the circuits are the minimal dependent sets, and in the cycle matroid of a graph these are the edge sets of the cycles. The cycle space of a graph is a vector space (using scalars $\{0, 1\}$) where the coordinates are indexed by the edges and the vectors correspond to the even subgraphs.

The original CDC Conjecture states that every bridgeless graph has a double cover by even subgraphs. That phrasing is equivalent to ours, since every even graph is an edge-disjoint union of cycles.

Thus we might seek a double cover by using a small number of even subgraphs. The cycles in a cycle double cover are even subgraphs; when cycles are pairwise edge-disjoint, they can be combined to form a single even subgraph. This leads to the connection between integer flows and cycle double covers.

7.3.33. Proposition. A graph has a nowhere-zero 4-flow if and only if it has a cycle double cover forming three even subgraphs.

Proof: Theorem 7.3.25 states that a graph has a nowhere-zero 4-flow if and only if it is the union of two even subgraphs E_1, E_2. Let $E_3 = E_1 \triangle E_2$. At each vertex v the degree in E_3 is the sum of the degrees in E_1 and E_2 minus twice the number of common incident edges; hence it is even. Hence E_3 is an even subgraph, and it contains precisely the edges that appear in just one of $\{E_1, E_2\}$. Cycle decompositions of E_1, E_2, E_3 thus combine to yield a CDC.

Conversely, if a CDC forms three even subgraphs, then omitting one of them leaves the graph expressed as the union of two even subgraphs, and hence a nowhere-zero 4-flow exists. ■

Let **P** denote the family of graphs that do not contain a subdivision of the Petersen graph. By Proposition 7.3.33, Tutte's 4-flow Conjecture implies that every graph in **P** has a CDC. Alspach–Goddyn–Zhang [1994] proved a deep result that yields cycle double covers for graphs in **P**. (They proved that a stronger covering property holds for G if and only if $G \in$ **P**.) In light of Proposition 7.3.33, this is a partial result toward Tutte's 4-flow Conjecture.

The CDC Conjecture is also related to snarks. Goddyn [1985] proved that if the CDC Conjecture is false, then the smallest counterexample is a snark with girth at least 8.

EXERCISES

7.3.1. (−) Prove that every Hamiltonian 3-regular graph has a Tait coloring.

7.3.2. (−) Exhibit 3-regular simple graphs with the following properties.
a) Planar but not 3-edge-colorable.
b) 2-connected but not 3-edge-colorable.
c) Planar with connectivity 2, but not Hamiltonian.

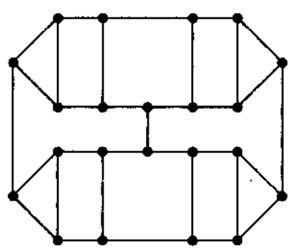
• • • • •

7.3.3. Prove that every maximal plane graph other than K_4 is 3-face-colorable.

7.3.4. Without using the Four Color Theorem, prove that every Hamiltonian plane graph is 4-face-colorable (nothing is assumed about the vertex degrees).

7.3.5. Prove that a 2-edge-connected plane graph is 2-face-colorable if and only if it is Eulerian.

7.3.6. Use Tait's Theorem (Theorem 7.3.2) to prove that $\chi'(G) = 3$ for the graph G below.

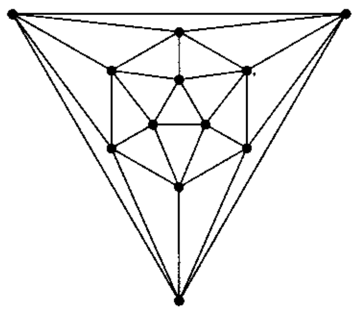


7.3.7. (!) Let G be a plane triangulation.

a) Prove that the dual G^* has a 2-factor.

b) Use part (a) to prove that the vertices of G can be 2-colored so that every face has vertices of both colors. (Hint: Use the idea in the proof of Theorem 7.3.2.) (Buršteĭn [1974], Penaud [1975])

7.3.8. (+) It has been conjectured that every planar triangulation has edge-chromatic number $\Delta(G)$, and this has been proved when $\Delta(G)$ is high enough. Show that $\chi'(G) = \Delta(G)$ for the graph of the icosahedron, illustrated below.



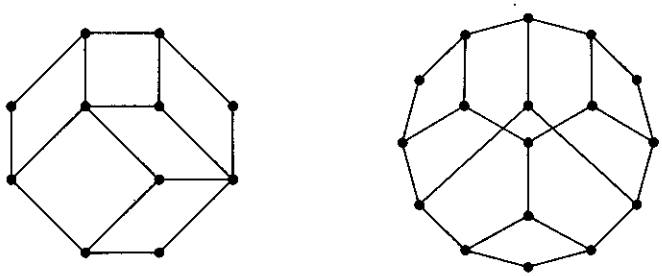
7.3.9. Prove that a proper 4-coloring of the icosahedron uses each color exactly 5 times.

7.3.10. Whitney [1931] proved that every 4-connected planar triangulation is Hamiltonian. Use this to reduce the Four Color Problem to the problem of proving that every Hamiltonian planar graph is 4-colorable.

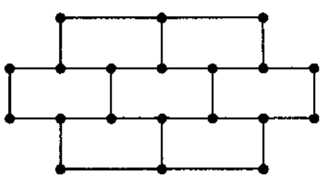
7.3.11. Find a 5-connected planar graph. Does there exist a 6-connected planar graph?

7.3.12. Let G be a planar graph with at least three faces. Prove that G has a vertex partition into two sets whose induced subgraphs are trees if and only if G^* is Hamiltonian.

7.3.13. (!) For each of the planar graphs below, present a Hamiltonian cycle or use planarity (Grinberg's condition) to prove that it is non-Hamiltonian.

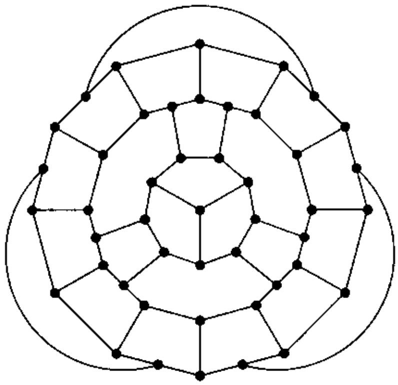


7.3.14. Let G be the graph below. Prove that G has no Hamiltonian cycle. Explain why Grinberg's Theorem cannot be used directly to prove this.

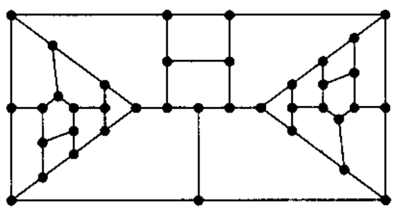


7.3.15. (!) Prove Grinberg's Theorem using Euler's Formula.

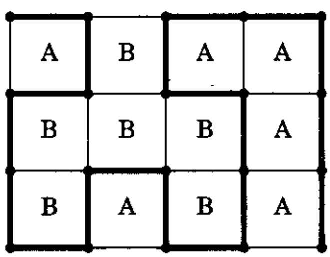
7.3.16. (!) Use Grinberg's condition to prove that the Grinberg graph (below) is not Hamiltonian.



7.3.17. (!) The smallest known 3-regular 3-connected planar graph that is not Hamiltonian has 38 vertices and appears below. Prove that this graph is not Hamiltonian. (Lederberg [1966], Bosák [1966], Barnette)



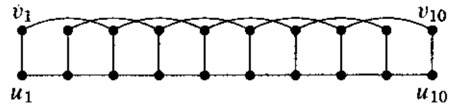
7.3.18. Let G be the grid graph $P_m \square P_n$. Let Q be a Hamiltonian path from the upper left corner vertex to the lower right corner vertex, such as that shown in bold below. Note that Q partitions the grid into regions, of which some open to the left or downward and others open to the right or upward. Prove that the total area of the up-right regions (B) equals the total area of the down-left regions (A). (Fisher–Collins–Krompart [1994])



7.3.19. (!) The **generalized Petersen graph** $P(n, k)$ is the graph with vertices $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_n\}$ and edges $\{u_i u_{i+1}\}$, $\{u_i v_i\}$, and $\{v_i v_{i+k}\}$, where addition is modulo n. The Petersen graph itself is $P(5, 2)$.

a) Prove that the subgraph of $P(n, 2)$ induced by k consecutive pairs $\{u_i, v_i\}$ has a spanning cycle if $k \equiv 1 \bmod 3$ and $k \geq 4$.

b) Use part (a) to prove that $\chi'(P(n, 2)) = 3$ if $n \geq 6$.



7.3.20. (−) Let G be a 3-regular graph. Prove that if G is the union of three cycles, then G is 3-edge-colorable.

7.3.21. (+) *"Flower snarks"*. Let G_k and H_k be as constructed in (Example 7.3.13).

a) Prove that G_k is 3-edge-colorable.

b) Prove that H_k is not 3-edge-colorable when k is odd. (Isaacs [1975])

7.3.22. Prove that every edge cut of $K_k \square C_l$ that does not isolate a vertex has at least $2k$ edges.

7.3.23. (∗) Prove that applying the dot product operation (Definition 7.3.12) to two snarks yields a third snark. (Isaacs [1975])

7.3.24. (∗!) Let G_1 and G_2 be graphs. Prove that if G_1 has a nowhere-zero k_1-flow and G_2 has a nowhere-zero k_2-flow, then $G_1 \cup G_2$ has a nowhere-zero k_1k_2-flow.

7.3.25. (!) A **parity subgraph** of G is a subgraph H such that $d_H(v) \equiv d_G(v) \bmod 2$ for all $v \in V(G)$. Prove that every spanning tree of a connected graph G contains a partity subgraph of G. (Itai–Rodeh [1978])

7.3.26. (∗) For $k \geq 3$, prove that a smallest nontrivial 2-edge-connected graph G having no nowhere-zero k-flow must be simple, 2-connected, and 3-edge-connected. (Hint: First exclude loops and vertices of degree 2 and reduce to consideration of blocks. Then exclude multiple edges and finally edge cuts of size 2. In each case, compare G to a graph obtained from it by deleting or contracting edges.)

7.3.27. (∗) Prove that every Hamiltonian graph has a nowhere-zero 4-flow.

7.3.28. (∗) Prove that every bridgeless graph with a Hamiltonian path has a nowhere-zero 5-flow. (Jaeger [1978])

7.3.29. (∗) Embed K_6 on the torus, and let G be the dual graph. Find a nowhere-zero 5-flow on G.

7.3.30. (∗) Prove that a graph G is the union of r even subgraphs if and only if G has a nowhere-zero 2^r-flow. (Matthews [1978])

7.3.31. (∗) Let G be a graph having a cycle double cover forming 2^r even subgraphs. Prove that G has a nowhere-zero 2^r-flow. (Jaeger [1988])

7.3.32. (∗!) A **modular 3-orientation** of a graph G is an orientation D such that $d_D^+(v) \equiv d_D^-(v) \bmod 3$ for all $v \in V(G)$. Prove that a bridgeless graph has a nowhere-zero 3-flow if and only if it has a modular 3-orientation. (Steinberg–Younger [1989])

7.3.33. (∗) *Characterization of nowhere-zero k-flows.* Let G be a bridgeless graph, let D be an orientation of G, and let a and b be positive integers. Prove that the following statements are equivalent. (Hoffman [1958])

a) $\frac{a}{b} \leq \frac{|[S,\overline{S}]|}{|[\overline{S},S]|} \leq \frac{b}{a}$ for every nonempty proper vertex subset S.

b) G has an integer flow using weights in the interval $[a, b]$.

c) G has a real-valued flow using weights in the interval $[a, b]$.

7.3.34. $(*)$ Find cycle double covers for the graphs $C_m \vee K_1$, $C_m \vee 2K_1$, and $C_m \vee K_2$.

7.3.35. $(*)$ Find the cycle double covers with fewest cycles for every 3-regular simple graph with 6 vertices.

7.3.36. $(*-)$ Let G be the Petersen graph. Find a cycle double cover of G whose elements are not all 5-cycles. Find a double cover of G consisting of 1-factors. (Hint: Consider the drawing of G having a 9-cycle on the "outside". Comment: Fulkerson [1971] conjectured that every bridgeless cubic graph has a double cover consisting of 6 perfect matchings.)

7.3.37. $(*)$ Prove that any two 6-cycles in the Petersen graph must have at least two common edges. Conclude that the Petersen graph has no CDC consisting of five 6-cycles. Use this and Exercise 7.3.20 to conclude that the Petersen graph has no CDC consisting of even cycles. (C.Q. Zhang)

7.3.38. $(*!)$ A cycle double cover is **orientable** if its cycles can be oriented as directed cycles so that for each edge, the two cycles containing it traverse it in opposite directions. A digraph is **even** if $d^-(v) = d^+(v)$ for each vertex v.

a) Suppose that G has a nonnegative k-flow (D, f). Prove that f can be expressed as $\sum_{i=1}^{k-1} f_i$, where each (D, f_i) is a nonnegative 2-flow on G. (Hint: Use induction on k.) (Little–Tutte–Younger [1988])

b) Prove that a graph G has a positive k-flow (D, f) if and only if D is the union of $k-1$ even digraphs such that each edge e in D appears in exactly $f(e)$ of them. (Little–Tutte–Younger [1988])

c) Prove that a graph G has a nowhere-zero 3-flow if and only if it has an orientable cycle double cover forming three even subgraphs. (Tutte [1949])

7.3.39. $(*)$ Let G be a graph having a CDC formed from four even subgraphs. Prove that G also has a CDC formed from three even subgraphs. (Hint: Use symmetric differences.)

7.3.40. $(*)$ In the Petersen graph, prove that the solution to the Chinese Postman Problem has total length 20, but the minimum total length of cycles covering the Petersen graph is 21.

7.3.41. $(*)$ Let M be a perfect matching in the Petersen graph. Prove that there is no list of cycles in the Petersen graph that together cover every edge of M exactly twice and all other edges exactly once. (Itai–Rodeh [1978], Seymour [1979b])

7.3.42. $(*)$ Let G be a graph in which a shortest covering walk (that is, an optimal solution to the Chinese Postman Problem) decomposes into cycles. Prove that G has a cycle cover of total length at most $e(G) + n(G) - 1$. Determine the minimum length of a cycle cover of $K_{3,t}$ in terms of the number of edges and vertices.

Chapter 8

Additional Topics

In this chapter we explore more advanced or specialized material. Each section gives a glimpse of a topic that deserves its own chapter (or book). Several sections treat more difficult material near the end.

8.1. Perfect Graphs

We have discussed the lower bound $\chi(G) \geq \omega(G)$ for chromatic number; the vertices of a clique need different colors. In Section 5.3, we discussed graphs whose induced subgraphs all achieve equality in this bound.

8.1.1. Definition. A graph G is **perfect** if $\chi(H) = \omega(H)$ for every induced subgraph H of G.

When discussing perfect graphs, it is common to use **stable set** to mean an independent set of vertices. As before, a **clique** is a set of pairwise adjacent vertices. As usual, **maximum** means maximum-sized.

Since we focus on vertex coloring, again in this section we restrict our attention to simple graphs. Complementation converts cliques to stable sets and vice versa, so $\omega(\overline{H}) = \alpha(H)$. Properly coloring $\overline{H}$ means expressing $V(H)$ as a union of cliques in H; such a set of cliques in H is a **clique covering** of H. Thus for every graph G we have four optimization parameters of interest.

independence number	$\alpha(G)$	max size of a stable set
clique number	$\omega(G)$	max size of a clique
chromatic number	$\chi(G)$	min size of a coloring
clique covering number	$\theta(G)$	min size of a clique covering

Berge actually defined two types of perfection:

G is **γ-perfect** if $\chi(G[A]) = \omega(G[A])$ for all $A \subseteq V(G)$.
G is **α-perfect** if $\theta(G[A]) = \alpha(G[A])$ for all $A \subseteq V(G)$.

Our definition of perfect is the same as this definition of γ-perfect (Berge used $\gamma(G)$ for chromatic number). Since $\overline{G}[A]$ is the complement of $G[A]$, the definition of α-perfect can be stated in terms of $\overline{G}$ as "$\chi(\overline{G}[A]) = \omega(\overline{G}[A])$ for all $A \subseteq V(G)$". Thus "G is α-perfect" has the same meaning as "$\overline{G}$ is γ-perfect".

We now use only one definition of perfection, because Lovász [1972a] proved "G is γ-perfect if and only if G is α-perfect". In terms of our original definition of perfection, this becomes "G is perfect if and only if $\overline{G}$ is perfect". This statement is the **Perfect Graph Theorem (PGT)**.

Always $\chi(G) \geq \omega(G)$ and $\theta(G) \geq \alpha(G)$, since a clique and a stable set share at most one vertex. A statement of perfection for a class of graphs is thus an integral min-max relation. We observed in Example 5.3.21 that several familiar min-max relations are statements that bipartite graphs, their line graphs, and the complements of such graphs are perfect.

If $k \geq 2$, then $\chi(C_{2k+1}) > \omega(C_{2k+1})$ and $\chi(\overline{C}_{2k+1}) > \omega(\overline{C}_{2k+1})$ (Exercise 1). Thus odd cycles and their complements (except C_3 and $\overline{C}_3$) are imperfect.

8.1.2. Conjecture. (**Strong Perfect Graph Conjecture (SPGC)**—Berge [1960]) A graph G is perfect if and only if both G and $\overline{G}$ have no induced subgraph that is an odd cycle of length at least 5. ■

The SPGC remains open. Since the condition in the conjecture is self-complementary, the SPGC implies the PGT.

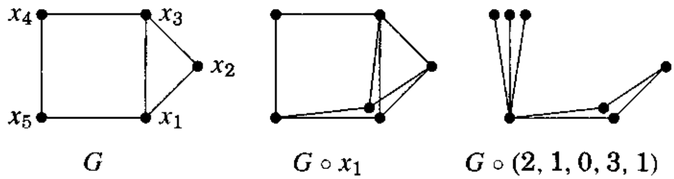
Having presented several classical families of perfect graphs in Section 5.3, our goal now is to prove the Perfect Graph Theorem. Later we also study properties of minimal imperfect graphs and classes of perfect graphs. For further reading, Golumbic [1980] provides a thorough introduction to the subject. Berge–Chvátal [1984] collects and updates many of the classical papers.

THE PERFECT GRAPH THEOREM

In 1960, Berge conjectured that γ-perfection and α-perfection are equivalent (see Berge [1961]). Lovász [1972a] stunned the world of combinatorics by proving this important and well-known conjecture at the age of 22. Fulkerson also studied it, reducing it to a statement he thought was too strong to be true. When Berge told him that Lovász had proved it, within hours he proved the missing lemma (Lemma 8.1.4), thus illustrating that a theorem becomes easier to prove when known to be true (Fulkerson [1971]).

We will prove the Perfect Graph Theorem using an operation that enlarges a graph without affecting the property of perfection.

8.1.3. Definition. **Duplicating a vertex** x of G produces a new graph $G \circ x$ by adding a vertex x' with $N(x') = N(x)$. The **vertex multiplication** of G by the nonnegative integer vector $h = (h_1, \ldots, h_n)$ is the graph $H = G \circ h$ whose vertex set consists of h_i copies of each $x_i \in V(G)$, with copies of x_i and x_j adjacent in H if and only if $x_i \leftrightarrow x_j$ in G.



8.1.4. Lemma. Vertex multiplication preserves γ-perfection and α-perfection.

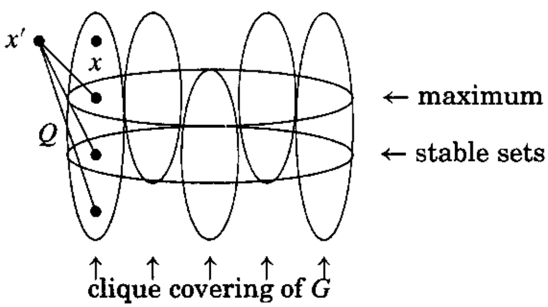
Proof: We first observe that $G \circ h$ can be obtained from an induced subgraph of G by successive vertex duplications. If every h_i is 0 or 1, then $G \circ h = G[A]$, where $A = \{i\colon h_i > 0\}$. Otherwise, start with $G[A]$ and perform duplications until there are h_i copies of x_i (for each i). Each vertex duplication preserves the property that copies of x_i and x_j are adjacent if and only if $x_i x_j \in E(G)$, so the resulting graph is $G \circ h$.

If G is α-perfect but $G \circ h$ is not, then some operation in the creation of $G \circ h$ from $G[A]$ produces a graph that is not α-perfect from an α-perfect graph. It thus suffices to prove that vertex duplication preserves α-perfection. The same reduction holds for γ-perfection. Since every proper induced subgraph of $G \circ x$ is an induced subgraph of G or a vertex duplication of an induced subgraph of G, we further reduce our claim to showing that $\chi(G \circ x) = \omega(G \circ x)$ when G is γ-perfect and that $\alpha(G \circ x) = \theta(G \circ x)$ when G is α-perfect.

When G is γ-perfect, we extend a proper coloring of G to a proper coloring of $G \circ x$ by giving x' the same color as x. No clique contains both x and x', so $\omega(G \circ x) = \omega(G)$. Hence $\chi(G \circ x) = \chi(G) = \omega(G) = \omega(G \circ x)$.

When G is α-perfect, we consider two cases. If x belongs to a maximum stable set in G, then adding x' to it yields $\alpha(G \circ x) = \alpha(G) + 1$. Since $\theta(G) = \alpha(G)$, we can obtain a clique covering of this size by adding x' as a 1-vertex clique to some set of $\theta(G)$ cliques covering G.

If x belongs to no maximum stable set in G, then $\alpha(G \circ x) = \alpha(G)$. Let Q be the clique containing x in a minimum clique cover of G. Since $\theta(G) = \alpha(G)$, Q intersects every maximum stable set in G. Since x belongs to no maximum stable set, $Q' = Q - x$ also intersects every maximum stable set. This yields $\alpha(G - Q') = \alpha(G) - 1$. Applying the α-perfection of G to the induced subgraph $G - Q'$ (which contains x) yields $\theta(G - Q') = \alpha(G - Q')$. Adding $Q' \cup \{x'\}$ to a set of $\alpha(G) - 1$ cliques covering $G - Q'$ yields a set of $\alpha(G)$ cliques covering $G \circ x$. ■



8.1.5. Lemma. In a minimal imperfect graph, no stable set intersects every maximum clique.

Proof: If a stable set S in G intersects every $\omega(G)$-clique, then perfection of $G - S$ yields $\chi(G-S) = \omega(G-S) = \omega(G) - 1$, and S completes a proper $\omega(G)$-coloring of G. This makes G perfect. ■

8.1.6. Theorem. (The Perfect Graph Theorem (PGT) - Lovász [1972a, 1972b]) A graph is perfect if and only if its complement is perfect.

Proof: It suffices to show that α-perfection of G implies γ-perfection of G; applying this to $\overline{G}$ yields the converse. If the claim fails, then we consider a minimal graph G that is α-perfect but not γ-perfect. By Lemma 8.1.5, we may assume that every maximal stable set S in G misses some maximum clique $Q(S)$.

We design a special vertex multiplication of G. Let $\mathbf{S} = \{S_i\}$ be the list of maximal stable sets of G. We weight each vertex by its frequency in $\{Q(S_i)\}$, letting h_j be the number of stable sets $S_i \in \mathbf{S}$ such that $x_j \in Q(S_i)$. By Lemma 8.1.4, $H = G \circ h$ is α-perfect, yielding $\alpha(H) = \theta(H)$. We use counting arguments for $\alpha(H)$ and $\theta(H)$ to obtain a contradiction.

Let A be the 0,1-matrix of the incidence relation between $\{Q(S_i)\}$ and $V(G)$; we have $a_{i,j} = 1$ if and only if $x_j \in Q(S_i)$. By construction, h_j is the number of 1s in column j of A, and $n(H)$ is the total number of 1s in A. Since each row has $\omega(G)$ 1s, also $n(H) = \omega(G)\,|\mathbf{S}|$. Since vertex duplication cannot enlarge cliques, we have $\omega(H) = \omega(G)$. Therefore, $\theta(H) \ge n(H)/\omega(H) = |\mathbf{S}|$.

We obtain a contradiction by proving that $\alpha(H) < |\mathbf{S}|$. Every stable set in H consists of copies of elements in some stable set of G, so a maximum stable set in H consists of all copies of all vertices in some maximal stable set of G. Hence $\alpha(H) = \max_{T \in \mathbf{S}} \sum_{i:\, x_i \in T} h_i$. The sum counts the 1s in A that appear in the columns indexed by T. If we count these 1s instead by rows, we obtain $\alpha(H) = \max_{T \in \mathbf{S}} \sum_{S \in \mathbf{S}} |T \cap Q(S)|$. Since T is a stable set, it has at most one vertex in each chosen clique $Q(S)$. Furthermore, T is disjoint from $Q(T)$. With $|T \cap Q(S)| \le 1$ for all $S \in \mathbf{S}$, and $|T \cap Q(T)| = 0$, we have $\alpha(H) \le |\mathbf{S}| - 1$. ■

$$
\begin{array}{c|ccc|}
 & & V(G) & \\
\cline{2-4}
Q(S_1) & \vdots & \vdots & \vdots \\
Q(T) & 0 & 0 & 0 \\
Q(S_n) & \vdots & \vdots & \vdots \\
\cline{2-4}
\multicolumn{1}{c}{} & \uparrow & \uparrow & \multicolumn{1}{c}{\uparrow} \\
\multicolumn{1}{c}{} & T & T & \multicolumn{1}{c}{T}
\end{array}
$$

8.1.7.* Remark. *Linear optimization and duality.* Clique-vertex incidence matrices also arise in expressing α and θ as integer optimization problems. A linear (maximization) program can be written as "maximize $c \cdot x$ over nonnegative vectors x such that $Ax \le b$", where A is a matrix, b, c are vectors, and each row of $Ax \leq b$ is a linear constraint $a_i \cdot x \leq b_i$ on the vector x of variables. A vector x satisfying all the constraints is a **feasible solution**.

An **integer linear program** requires that each x_j also be an integer. Let A be the incidence matrix between maximal cliques and vertices in a graph G; we have $a_{i,j} = 1$ when $v_j \in Q_i$. By definition, $\alpha(G)$ is the solution to "max $\mathbf{1}_n \cdot x$ such that $Ax \leq \mathbf{1}_m$" when the variables are required to be nonnegative integers. In the solution, x_j is 1 or 0 depending on whether v_j is in the maximum stable set; the constraints prevent choosing adjacent vertices. Similarly, when B is the incidence matrix between maximal stable sets and vertices, $\omega(G)$ is the solution to "max $\mathbf{1}_n \cdot x$ such that $Bx \leq \mathbf{1}_p$" with integer variables.

Every maximization program has a dual minimization program. When the max program is "max $c \cdot x$ such that $Ax \leq b$", the dual is "min $y \cdot b$ such that $y^T A \geq c$". This program has a variable y_i for each original constraint and a constraint for each original variable x_j, and it interchanges $c, \max, \leq$ with $b, \min, \geq$. When stated in this form, the variables in both programs must be nonnegative. The integer programs dual to ω and α seek the minimum number of stable sets that cover the vertices and the minimum number of cliques that cover the vertices, respectively; this describes $\chi(G)$ and $\theta(G)$.

Using the nonnegativity of the variables, the constraints yield

$$c \cdot x \leq y^T Ax \leq y \cdot b.$$

The statement "$c \cdot x \leq y \cdot b$" for feasible solutions x, y is **weak duality**. The (strong) **Duality Theorem of Linear Programming** states that dual programs having feasible solutions have optimal solutions with the same value when integer solutions are not required.

The statements $\chi \geq \omega$ and $\theta \geq \alpha$ are statements of weak duality for dual pairs of linear programs. A guarantee of strong duality using solutions that have only integer values is a combinatorial min-max relation. We have presented many such relations and observed that they guarantee quick proofs of optimality. They also often lead to fast algorithms for finding optimal solutions, which is one motivation for studying families of perfect graphs. ■

8.1.8.* Example. *Fractional solutions for an imperfect graph.* For the 5-cycle, the linear programs for $\omega, \chi, \alpha, \theta$ all have optimal value 5/2. There are five maximal cliques and five maximal stable sets, each of size 2. Setting each $x_j = 1/2$ gives weight 1 to each clique and stable set, thereby satisfying the constraints for either maximization problem. Setting each $y_i = 1/2$ in the dual programs covers each vertex with a total weight of 1, so again the constraints are satisfied. These programs have no optimal solution in integers, and the integer programs have a "duality gap": $\chi = 3 > 2 = \omega$ and $\theta = 3 > 2 = \omega$. ■

CHORDAL GRAPHS REVISITED

Like trees, the more general class of chordal graphs has many characterizations. The definition by forbidding chordless cycles is a **forbidden substructure characterization**. A finite list of forbidden substructures such as induced subgraphs yields a fast algorithm for testing membership in the class, but for chordal graphs the list is infinite and other methods are needed.

A chordal graph can be built from a single vertex by iteratively adding a vertex joined to a clique; this is the reverse of a simplicial elimination ordering, and we have seen that greedy coloring with respect to such a construction ordering yields an optimal coloring. Many classes of perfect graphs have such a **construction procedure** that produces the graphs in the class and no others. A construction procedure or the reverse **decomposition procedure** may lead to fast algorithms for computations on graphs in the class.

Next we consider another type of characterization.

8.1.9. Definition. An **intersection representation** of a graph G is a family of sets $\{S_v\colon v \in V(G)\}$ such that $u \leftrightarrow v$ if and only if $S_u \cap S_v \neq \emptyset$. If $\{S_v\}$ is an intersection representation of G, then G is the **intersection graph** of $\{S_v\}$.

Interval graphs are the graphs having intersection representations where each set in the family is an interval on the real line. Line graphs also form an intersection class; the allowed sets are pairs of natural numbers, corresponding to edges of the graph H such that $G = L(H)$. An intersection characterization for chordal graphs was found independently by Walter [1972, 1978], Gavril [1974], and Buneman [1974].

8.1.10. Lemma. If $T_1, \ldots, T_k$ are pairwise intersecting subtrees of a tree T, then there is a vertex belonging to all of $T_1, \ldots, T_k$.

Proof: (Lehel) We prove the contrapositive. If each vertex v misses some $T(v)$ among $T_1, \ldots, T_k$, we mark the edge that leaves v on the unique path to $T(v)$. If T has n vertices, then we make n marks, so some edge uw has been marked twice. Now $T(u)$ and $T(w)$ have no common vertex. ∎

8.1.11. Theorem. A graph is chordal if and only if it has an intersection representation using subtrees of a tree (a **subtree representation**).

Proof: We prove that the condition is equivalent to the existence of a simplicial elimination ordering. We use induction, with trivial basis K_1.

Let $v_1, \ldots, v_n$ be a simplicial elimination ordering for G. Since $v_2, \ldots, v_n$ is a simplicial elimination ordering for $G - v_1$, the induction hypothesis yields a subtree representation of $G - v_1$ in a host tree T. Since v_1 is simplicial in G, the set $S = N_G(v_1)$ induces a clique in $G - v_1$. Therefore, the subtrees of T assigned to vertices of S are pairwise intersecting.

By Lemma 8.1.10, these subtrees have a common vertex x. We enlarge T to a tree T' by adding a leaf y adjacent to x, and we add the edge xy to the subtrees representing vertices of S. We represent v_1 by the subtree consisting only of y. This completes a subtree representation of G in T'.

Conversely, let T be a smallest host tree for a subtree representation of G, with each $v \in V(G)$ represented by $T(v) \subseteq T$. If $xy \in E(T)$, then G must have a vertex u such that $T(u)$ contains x but not y; otherwise, contracting xy into y would yield a representation in a smaller tree.