(e) *Let X, Y be disjoint finite sets (so $X \cap Y = \emptyset$), and $f : X \cup Y \to \mathbf{R}$ is a function. Then we have*

$$\sum_{z \in X \cup Y} f(z) = \left(\sum_{x \in X} f(x)\right) + \left(\sum_{y \in Y} f(y)\right).$$

(f) *(Linearity, part I) Let X be a finite set, and let $f : X \to \mathbf{R}$ and $g : X \to \mathbf{R}$ be functions. Then*

$$\sum_{x \in X} (f(x) + g(x)) = \sum_{x \in X} f(x) + \sum_{x \in X} g(x).$$

(g) *(Linearity, part II) Let X be a finite set, let $f : X \to \mathbf{R}$ be a function, and let c be a real number. Then*

$$\sum_{x \in X} cf(x) = c \sum_{x \in X} f(x).$$

(h) *(Monotonicity) Let X be a finite set, and let $f : X \to \mathbf{R}$ and $g : X \to \mathbf{R}$ be functions such that $f(x) \le g(x)$ for all $x \in X$. Then we have*

$$\sum_{x \in X} f(x) \le \sum_{x \in X} g(x).$$

(i) *(Triangle inequality) Let X be a finite set, and let $f : X \to \mathbf{R}$ be a function, then*

$$|\sum_{x \in X} f(x)| \le \sum_{x \in X} |f(x)|.$$

Proof. See Exercise 7.1.2. □

Remark 7.1.12. The substitution rule in Proposition 7.1.11(c) can be thought of as making the substitution $x := g(y)$ (hence the name). Note that the assumption that g is a bijection is essential;

can you see why the rule will fail when g is not one-to-one or not onto? From Proposition 7.1.11(c) and (d) we see that

$$\sum_{i=n}^{m} a_i = \sum_{i=n}^{m} a_{f(i)}$$

for any bijection f from the set $\{i \in \mathbf{Z} : n \leq i \leq m\}$ to itself. Informally, this means that we can rearrange the elements of a finite sequence at will and still obtain the same value.

Now we look at double finite series - finite series of finite series - and how they connect with Cartesian products.

Lemma 7.1.13. *Let X, Y be finite sets, and let $f : X \times Y \to \mathbf{R}$ be a function. Then*

$$\sum_{x\in X}(\sum_{y\in Y} f(x,y)) = \sum_{(x,y)\in X\times Y} f(x,y).$$

Proof. Let n be the number of elements in X. We will use induction on n (cf. Proposition 7.1.8); i.e., we let $P(n)$ be the assertion that Lemma 7.1.13 is true for any set X with n elements, and any finite set Y and any function $f : X \times Y \to \mathbf{R}$. We wish to prove $P(n)$ for all natural numbers n.

The base case $P(0)$ is easy, following from Proposition 7.1.11(a) (why?). Now suppose that $P(n)$ is true; we now show that $P(n+1)$ is true. Let X be a set with $n + 1$ elements. In particular, by Lemma 3.6.9, we can write $X = X' \cup \{x_0\}$, where x_0 is an element of X and $X' := X - \{x_0\}$ has n elements. Then by Proposition 7.1.11(e) we have

$$\sum_{x\in X}\left(\sum_{y\in Y} f(x,y)\right) = (\sum_{x\in X'}\left(\sum_{y\in Y} f(x,y))\right) + \left(\sum_{y\in Y} f(x_0,y)\right);$$

by the induction hypothesis this is equal to

$$\sum_{(x,y)\in X'\times Y} f(x,y) + (\sum_{y\in Y} f(x_0,y)).$$

By Proposition 7.1.11(c) this is equal to

$$\sum_{(x,y)\in X'\times Y} f(x,y) + (\sum_{(x,y)\in\{x_0\}\times Y} f(x,y)).$$

By Proposition 7.1.11(e) this is equal to

$$\sum_{(x,y)\in X\times Y} f(x,y)$$

(why?) as desired. □

Corollary 7.1.14 (Fubini's theorem for finite series). *Let X, Y be finite sets, and let $f : X \times Y \to \mathbf{R}$ be a function. Then*

$$\begin{aligned}\sum_{x\in X}(\sum_{y\in Y} f(x,y)) &= \sum_{(x,y)\in X\times Y} f(x,y)\\ &= \sum_{(y,x)\in Y\times X} f(x,y)\\ &= \sum_{y\in Y}(\sum_{x\in X} f(x,y)).\end{aligned}$$

Proof. In light of Lemma 7.1.13, it suffices to show that

$$\sum_{(x,y)\in X\times Y} f(x,y) = \sum_{(y,x)\in Y\times X} f(x,y).$$

But this follows from Proposition 7.1.11(c) by applying the bijection $h : X \times Y \to Y \times X$ defined by $h(x, y) := (y, x)$. (Why is this a bijection, and why does Proposition 7.1.11(c) give us what we want?) □

Remark 7.1.15. This should be contrasted with Example 1.2.5; thus we anticipate something interesting to happen when we move from finite sums to infinite sums. However, see Theorem 8.2.2.

Exercise 7.1.1. Prove Lemma 7.1.4. (Hint: you will need to use induction, but the base case might not necessarily be at 0.)

Exercise 7.1.2. Prove Proposition 7.1.11. (Hint: this is not as lengthy as it may first appear. It is largely a matter of choosing the right bijections to turn these sums over sets into finite series, and then applying Lemma 7.1.4.)

Exercise 7.1.3. Form a definition for the finite products $\prod_{i=1}^n a_i$ and $\prod_{x \in X} f(x)$. Which of the above results for finite series have analogues for finite products? (Note that it is dangerous to apply logarithms because some of the a_i or $f(x)$ could be zero or negative. Besides, we haven't defined logarithms yet.)

Exercise 7.1.4. Define the *factorial function* $n!$ for natural numbers n by the recursive definition $0! := 1$ and $(n+1)! := n! \times (n+1)$. If x and y are real numbers, prove the *binomial formula*

$$(x+y)^n = \sum_{j=0}^n \frac{n!}{j!(n-j)!} x^j y^{n-j}$$

for all natural numbers n. (Hint: induct on n.)

Exercise 7.1.5. Let X be a finite set, let m be an integer, and for each $x \in X$ let $(a_n(x))_{n=m}^\infty$ be a convergent sequence of real numbers. Show that the sequence $(\sum_{x \in X} a_n(x))_{n=m}^\infty$ is convergent, and

$$\lim_{n \to \infty} \sum_{x \in X} a_n(x) = \sum_{x \in X} \lim_{n \to \infty} a_n(x).$$

(Hint: induct on the cardinality of X, and use Theorem 6.1.19(a).) Thus we may always interchange finite sums with convergent limits. Things however get trickier with infinite sums; see Exercise 19.2.11.

7.2 Infinite series

We are now ready to sum infinite series.

Definition 7.2.1 (Formal infinite series). A (formal) infinite series is any expression of the form

$$\sum_{n=m}^\infty a_n,$$

where m is an integer, and a_n is a real number for any integer $n \geq m$. We sometimes write this series as

$$a_m + a_{m+1} + a_{m+2} + \ldots.$$

At present, this series is only defined *formally*; we have not set this sum equal to any real number; the notation $a_m + a_{m+1} + a_{m+2} + \ldots$ is of course designed to look very suggestively like a sum, but is not actually a finite sum because of the "..." symbol. To rigourously define what the series actually sums to, we need another definition.

Definition 7.2.2 (Convergence of series). Let $\sum_{n=m}^{\infty} a_n$ be a formal infinite series. For any integer $N \geq m$, we define the N^{th} *partial sum* S_N of this series to be $S_N := \sum_{n=m}^{N} a_n$; of course, S_N is a real number. If the sequence $(S_N)_{n=m}^{\infty}$ converges to some limit L as $N \to \infty$, then we say that the infinite series $\sum_{n=m}^{\infty} a_n$ is *convergent*, and *converges to* L; we also write $L = \sum_{n=m}^{\infty} a_n$, and say that L is the *sum* of the infinite series $\sum_{n=m}^{\infty} a_n$. If the partial sums S_N diverge, then we say that the infinite series $\sum_{n=m}^{\infty} a_n$ is *divergent*, and we do not assign any real number value to that series.

Remark 7.2.3. Note that Proposition 6.1.7 shows that if a series converges, then it has a unique sum, so it is safe to talk about *the* sum $L = \sum_{n=m}^{\infty} a_n$ of a convergent series.

Examples 7.2.4. Consider the formal infinite series

$$\sum_{n=1}^{\infty} 2^{-n} = 2^{-1} + 2^{-2} + 2^{-3} + \ldots.$$

The partial sums can be verified to equal

$$S_N = \sum_{n=1}^{N} 2^{-n} = 1 - 2^{-N}$$

by an easy induction argument (or by Lemma 7.3.3 below); the sequence $1 - 2^{-N}$ converges to 1 as $N \to \infty$, and hence we have

$$\sum_{n=1}^{\infty} 2^{-n} = 1.$$

In particular, this series is convergent. On the other hand, if we consider the series

$$\sum_{n=1}^{\infty} 2^n = 2^1 + 2^2 + 2^3 + \dots$$

then the partial sums are

$$S_N = \sum_{n=1}^{N} 2^n = 2^{N+1} - 2$$

and this is easily shown to be an unbounded sequence, and hence divergent. Thus the series $\sum_{n=1}^{\infty} 2^n$ is divergent.

Now we address the question of when a series converges. The following proposition shows that a series converges iff the "tail" of the sequence is eventually less than ε for any $\varepsilon > 0$:

Proposition 7.2.5. *Let $\sum_{n=m}^{\infty} a_n$ be a formal series of real numbers. Then $\sum_{n=m}^{\infty} a_n$ converges if and only if, for every real number $\varepsilon > 0$, there exists an integer $N \geq m$ such that*

$$|\sum_{n=p}^{q} a_n| \leq \varepsilon \textit{ for all } p, q \geq N.$$

Proof. See Exercise 7.2.2. □

This Proposition, by itself, is not very handy, because it is not so easy to compute the partial sums $\sum_{n=p}^{q} a_n$ in practice. However, it has a number of useful corollaries. For instance:

Corollary 7.2.6 (Zero test). *Let $\sum_{n=m}^{\infty} a_n$ be a convergent series of real numbers. Then we must have $\lim_{n\to\infty} a_n = 0$. To put this another way, if $\lim_{n\to\infty} a_n$ is non-zero or divergent, then the series $\sum_{n=m}^{\infty} a_n$ is divergent.*

Proof. See Exercise 7.2.3. □

Example 7.2.7. The sequence $a_n := 1$ does not converge to 0 as $n \to \infty$, so we know that $\sum_{n=1}^\infty 1$ is a divergent series. (Note however that $1, 1, 1, 1, \ldots$ is a convergent *sequence*; convergence of series is a different notion from convergence of sequences.) Similarly, the sequence $a_n := (-1)^n$ diverges, and in particular does not converge to zero; thus the series $\sum_{n=1}^\infty (-1)^n$ is also divergent.

If a sequence $(a_n)_{n=m}^\infty$ *does* converge to zero, then the series $\sum_{n=m}^\infty a_n$ may or may not be convergent; it depends on the series. For instance, we will soon see that the series $\sum_{n=1}^\infty 1/n$ is divergent despite the fact that $1/n$ converges to 0 as $n \to \infty$.

Definition 7.2.8 (Absolute convergence)**.** Let $\sum_{n=m}^\infty a_n$ be a formal series of real numbers. We say that this series is *absolutely convergent* iff the series $\sum_{n=m}^\infty |a_n|$ is convergent.

In order to distinguish convergence from absolute convergence, we sometimes refer to the former as *conditional* convergence.

Proposition 7.2.9 (Absolute convergence test)**.** *Let $\sum_{n=m}^\infty a_n$ be a formal series of real numbers. If this series is absolutely convergent, then it is also conditionally convergent. Furthermore, in this case we have the triangle inequality*

$$|\sum_{n=m}^\infty a_n| \leq \sum_{n=m}^\infty |a_n|.$$

Proof. See Exercise 7.2.4. □

Remark 7.2.10. The converse to this proposition is not true; there exist series which are conditionally convergent but not absolutely convergent. See Example 7.2.13.

Remark 7.2.11. We consider the class of conditionally convergent series to include the class of absolutely convergent series as a subclass. Thus when we say a statement such as "$\sum_{n=m}^\infty a_n$ is conditionally convergent", this does not automatically mean that $\sum_{n=m}^\infty a_n$ is not absolutely convergent. If we wish to say that a series is conditionally convergent but not absolutely convergent, then we will instead use a phrasing such as "$\sum_{n=m}^{\infty} a_n$ is *only* conditionally convergent", or "$\sum_{n=m}^{\infty} a_n$ converges conditionally, but not absolutely".

Proposition 7.2.12 (Alternating series test). *Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers which are non-negative and decreasing, thus $a_n \geq 0$ and $a_n \geq a_{n+1}$ for every $n \geq m$. Then the series $\sum_{n=m}^{\infty}(-1)^n a_n$ is convergent if and only if the sequence a_n converges to 0 as $n \to \infty$.*

Proof. From the zero test, we know that if $\sum_{n=m}^{\infty}(-1)^n a_n$ is a convergent series, then the sequence $(-1)^n a_n$ converges to 0, which implies that a_n also converges to 0, since $(-1)^n a_n$ and a_n have the same distance from 0.

Now suppose conversely that a_n converges to 0. For each N, let S_N be the partial sum $S_N := \sum_{n=m}^{N}(-1)^n a_n$; our job is to show that S_N converges. Observe that

$$\begin{aligned} S_{N+2} &= S_N + (-1)^{N+1} a_{N+1} + (-1)^{N+2} a_{N+2} \\ &= S_N + (-1)^{N+1}(a_{N+1} - a_{N+2}). \end{aligned}$$

But by hypothesis, $(a_{N+1} - a_{N+2})$ is non-negative. Thus we have $S_{N+2} \geq S_N$ when N is odd and $S_{N+2} \leq S_N$ if N is even.

Now suppose that N is even. From the above discussion and induction we see that $S_{N+2k} \leq S_N$ for all natural numbers k (why?). Also we have $S_{N+2k+1} \geq S_{N+1} = S_N - a_{N+1}$ (why?). Finally, we have $S_{N+2k+1} = S_{N+2k} - a_{N+2k+1} \leq S_{N+2k}$ (why?). Thus we have

$$S_N - a_{N+1} \leq S_{N+2k+1} \leq S_{N+2k} \leq S_N$$

for all k. In particular, we have

$$S_N - a_{N+1} \leq S_n \leq S_N \text{ for all } n \geq N$$

(why?). In particular, the sequence S_n is eventually a_{N+1}-steady. But the sequence a_N converges to 0 as $N \to \infty$, thus this implies that S_n is eventually ε-steady for every $\varepsilon > 0$ (why?). Thus S_n converges, and so the series $\sum_{n=m}^{\infty}(-1)^n a_n$ is convergent. $\square$

Example 7.2.13. The sequence $(1/n)_{n=1}^{\infty}$ is non-negative, decreasing, and converges to zero. Thus $\sum_{n=1}^{\infty}(-1)^n/n$ is convergent (but it is not absolutely convergent, because $\sum_{n=1}^{\infty} 1/n$ diverges, see Corollary 7.3.7). Thus absolute divergence does not imply conditional divergence, even though absolute convergence implies conditional convergence.

Some basic identities concerning convergent series are collected below.

Proposition 7.2.14 (Series laws).

(a) *If $\sum_{n=m}^{\infty} a_n$ is a series of real numbers converging to x, and $\sum_{n=m}^{\infty} b_n$ is a series of real numbers converging to y, then $\sum_{n=m}^{\infty}(a_n + b_n)$ is also a convergent series, and converges to $x + y$. In particular, we have*

$$\sum_{n=m}^{\infty}(a_n + b_n) = \sum_{n=m}^{\infty} a_n + \sum_{n=m}^{\infty} b_n.$$

(b) *If $\sum_{n=m}^{\infty} a_n$ is a series of real numbers converging to x, and c is a real number, then $\sum_{n=m}^{\infty}(ca_n)$ is also a convergent series, and converges to cx. In particular, we have*

$$\sum_{n=m}^{\infty}(ca_n) = c\sum_{n=m}^{\infty} a_n.$$

(c) *Let $\sum_{n=m}^{\infty} a_n$ be a series of real numbers, and let $k \geq 0$ be an integer. If one of the two series $\sum_{n=m}^{\infty} a_n$ and $\sum_{n=m+k}^{\infty} a_n$ are convergent, then the other one is also, and we have the identity*

$$\sum_{n=m}^{\infty} a_n = \sum_{n=m}^{m+k-1} a_n + \sum_{n=m+k}^{\infty} a_n.$$

(d) *Let $\sum_{n=m}^{\infty} a_n$ be a series of real numbers converging to x, and let k be an integer. Then $\sum_{n=m+k}^{\infty} a_{n-k}$ also converges to x.*

Proof. See Exercise 7.2.5. □

From Proposition 7.2.14(c) we see that the convergence of a series does not depend on the first few elements of the series (though of course those elements do influence which value the series converges to). Because of this, we will usually not pay much attention as to what the initial index m of the series is.

There is one type of series, called *telescoping series*, which are easy to sum:

Lemma 7.2.15 (Telescoping series). *Let $(a_n)_{n=0}^\infty$ be a sequence of real numbers which converge to 0, i.e., $\lim_{n\to\infty} a_n = 0$. Then the series $\sum_{n=0}^\infty (a_n - a_{n+1})$ converges to a_0.*

Proof. See Exercise 7.2.6. □

Exercise 7.2.1. Is the series $\sum_{n=1}^\infty (-1)^n$ convergent or divergent? Justify your answer. Can you now resolve the difficulty in Example 1.2.2?

Exercise 7.2.2. Prove Proposition 7.2.5. (Hint: use Proposition 6.1.12 and Theorem 6.4.18.)

Exercise 7.2.3. Use Proposition 7.2.5 to prove Corollary 7.2.6.

Exercise 7.2.4. Prove Proposition 7.2.9. (Hint: use Proposition 7.2.5 and Proposition 7.1.4(e).)

Exercise 7.2.5. Prove Proposition 7.2.14. (Hint: use Theorem 6.1.19.)

Exercise 7.2.6. Prove Lemma 7.2.15. (Hint: First work out what the partial sums $\sum_{n=0}^N (a_n - a_{n+1})$ should be, and prove your assertion using induction.)

7.3 Sums of non-negative numbers

Now we specialize the preceding discussion in order to consider sums $\sum_{n=m}^\infty a_n$ where all the terms a_n are non-negative. This situation comes up, for instance, from the absolute convergence test, since the absolute value $|a_n|$ of a real number a_n is always non-negative. Note that when all the terms in a series are non-negative, there is no distinction between conditional convergence and absolute convergence.

Suppose $\sum_{n=m}^{\infty} a_n$ is a series of non-negative numbers. Then the partial sums $S_N := \sum_{n=m}^{N} a_n$ are increasing, i.e., $S_{N+1} \geq S_N$ for all $N \geq m$ (why?). From Proposition 6.3.8 and Corollary 6.1.17, we thus see that the sequence $(S_N)_{n=m}^{\infty}$ is convergent if and only if it has an upper bound M. In other words, we have just shown

Proposition 7.3.1. *Let $\sum_{n=m}^{\infty} a_n$ be a formal series of non-negative real numbers. Then this series is convergent if and only if there is a real number M such that*

$$\sum_{n=m}^{N} a_n \leq M \text{ for all integers } N \geq m.$$

A simple corollary of this is

Corollary 7.3.2 (Comparison test). *Let $\sum_{n=m}^{\infty} a_n$ and $\sum_{n=m}^{\infty} b_n$ be two formal series of real numbers, and suppose that $|a_n| \leq b_n$ for all $n \geq m$. Then if $\sum_{n=m}^{\infty} b_n$ is convergent, then $\sum_{n=m}^{\infty} a_n$ is absolutely convergent, and in fact*

$$|\sum_{n=m}^{\infty} a_n| \leq \sum_{n=m}^{\infty} |a_n| \leq \sum_{n=m}^{\infty} b_n.$$

Proof. See Exercise 7.3.1. □

We can also run the comparison test in the contrapositive: if we have $|a_n| \leq b_n$ for all $n \geq m$, and $\sum_{n=m}^{\infty} a_n$ is not absolutely convergent, then $\sum_{n=m}^{\infty} b_n$ is conditionally divergent. (Why does this follow immediately from Corollary 7.3.2?)

A useful series to use in the comparison test is the *geometric series*

$$\sum_{n=0}^{\infty} x^n,$$

where x is some real number:

Lemma 7.3.3 (Geometric series). *Let x be a real number. If $|x| \geq 1$, then the series $\sum_{n=0}^{\infty} x^n$ is divergent. If however $|x| < 1$, then the series is absolutely convergent and*

$$\sum_{n=0}^{\infty} x^n = 1/(1-x).$$

Proof. See Exercise 7.3.2. □

We now give a useful criterion, known as the *Cauchy criterion*, to test whether a series of non-negative but decreasing terms is convergent.

Proposition 7.3.4 (Cauchy criterion). *Let $(a_n)_{n=1}^{\infty}$ be a decreasing sequence of non-negative real numbers (so $a_n \geq 0$ and $a_{n+1} \leq a_n$ for all $n \geq 1$). Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the series*

$$\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \ldots$$

is convergent.

Remark 7.3.5. An interesting feature of this criterion is that it only uses a small number of elements of the sequence a_n (namely, those elements whose index n is a power of 2, $n = 2^k$) in order to determine whether the whole series is convergent or not.

Proof. Let $S_N := \sum_{n=1}^{N} a_n$ be the partial sums of $\sum_{n=1}^{\infty} a_n$, and let $T_K := \sum_{k=0}^{K} 2^k a_{2^k}$ be the partial sums of $\sum_{k=0}^{K} 2^k a_{2^k}$. In light of Proposition 7.3.1, our task is to show that the sequence $(S_N)_{N=1}^{\infty}$ is bounded if and only if the sequence $(T_K)_{K=0}^{\infty}$ is bounded. To do this we need the following claim:

Lemma 7.3.6. *For any natural number K, we have $S_{2^{K+1}-1} \leq T_K \leq 2S_{2^K}$.*

Proof. We use induction on K. First we prove the claim when $K = 0$, i.e.

$$S_1 \leq T_0 \leq 2S_1.$$

This becomes

$$a_1 \le a_1 \le 2a_1$$

which is clearly true, since a_1 is non-negative.

Now suppose the claim has been proven for K, and now we try to prove it for $K+1$:

$$S_{2^{K+2}-1} \le T_{K+1} \le 2S_{2^{K+1}}.$$

Clearly we have

$$T_{K+1} = T_K + 2^{K+1}a_{2^{K+1}}.$$

Also, we have (using Lemma 7.1.4(a) and (f), and the hypothesis that the a_n are decreasing)

$$S_{2^{K+1}} = S_{2^K} + \sum_{n=2^K+1}^{2^{K+1}} a_n \ge S_{2^K} + \sum_{n=2^K+1}^{2^{K+1}} a_{2^{K+1}} = S_{2^K} + 2^K a_{2^{K+1}}$$

and hence

$$2S_{2^{K+1}} \ge 2S_{2^K} + 2^{K+1}a_{2^{K+1}}.$$

Similarly we have

$$\begin{aligned} S_{2^{K+2}-1} &= S_{2^{K+1}-1} + \sum_{n=2^{K+1}}^{2^{K+2}-1} a_n \\ &\le S_{2^{K+1}-1} + \sum_{n=2^{K+1}}^{2^{K+2}-1} a_{2^{K+1}} \\ &= S_{2^{K+1}-1} + 2^{K+1}a_{2^{K+1}}. \end{aligned}$$

Combining these inequalities with the induction hypothesis

$$S_{2^{K+1}-1} \le T_K \le 2S_{2^K}$$

we obtain

$$S_{2^{K+2}-1} \le T_{K+1} \le 2S_{2^{K+1}}$$

as desired. This proves the claim. □

From this claim we see that if $(S_N)_{N=1}^{\infty}$ is bounded, then $(S_{2^K})_{K=0}^{\infty}$ is bounded, and hence $(T_K)_{K=0}^{\infty}$ is bounded. Conversely, if $(T_K)_{K=0}^{\infty}$ is bounded, then the claim implies that $S_{2^{K+1}-1}$ is bounded, i.e., there is an M such that $S_{2^{K+1}-1} \leq M$ for all natural numbers K. But one can easily show (using induction) that $2^{K+1} - 1 \geq K + 1$, and hence that $S_{K+1} \leq M$ for all natural numbers K, hence $(S_N)_{N=1}^{\infty}$ is bounded. □

Corollary 7.3.7. *Let $q > 0$ be a rational number. Then the series $\sum_{n=1}^{\infty} 1/n^q$ is convergent when $q > 1$ and divergent when $q \leq 1$.*

Proof. The sequence $(1/n^q)_{n=1}^{\infty}$ is non-negative and decreasing (by Lemma 5.6.9(d)), and so the Cauchy criterion applies. Thus this series is convergent if and only if

$$\sum_{k=0}^{\infty} 2^k \frac{1}{(2^k)^q}$$

is convergent. But by the laws of exponentiation (Lemma 5.6.9) we can rewrite this as the geometric series

$$\sum_{k=0}^{\infty} (2^{1-q})^k.$$

As mentioned earlier, the geometric series $\sum_{k=0}^{\infty} x^k$ converges if and only if $|x| < 1$. Thus the series $\sum_{n=1}^{\infty} 1/n^q$ will converge if and only if $|2^{1-q}| < 1$, which happens if and only if $q > 1$ (why? Try proving it just using Lemma 5.6.9, and without using logarithms). □

In particular, the series $\sum_{n=1}^{\infty} 1/n$ (also known as the *harmonic series*) is divergent, as claimed earlier. However, the series $\sum_{n=1}^{\infty} 1/n^2$ is convergent.

Remark 7.3.8. The quantity $\sum_{n=1}^{\infty} 1/n^q$, when it converges, is called $\zeta(q)$, the *Riemann-zeta function of* q. This function is very important in number theory, and in particular in the distribution of the primes; there is a very famous unsolved problem regarding