$\chi_E : \Omega \to \mathbf{R}$ by setting $\chi_E(x) := 1$ if $x \in E$, and $\chi_E(x) := 0$ if $x \notin E$. (In some texts, χ_E is also written 1_E, and is referred to as an *indicator function*). Then χ_E is a measurable function (why?), and is a simple function, because the image $\chi_E(\Omega)$ is $\{0, 1\}$ (or $\{0\}$ if E is empty, or $\{1\}$ if $E = \Omega$).

We remark on three basic properties of simple functions: that they form a vector space, that they are linear combinations of characteristic functions, and that they approximate measurable functions. More precisely, we have the following three lemmas:

Lemma 19.1.3. *Let Ω be a measurable subset of $\mathbf{R}^n$, and let $f : \Omega \to \mathbf{R}$ and $g : \Omega \to \mathbf{R}$ be simple functions. Then $f + g$ is also a simple function. Also, for any scalar $c \in \mathbf{R}$, the function cf is also a simple function.*

Proof. See Exercise 19.1.1. $\square$

Lemma 19.1.4. *Let Ω be a measurable subset of $\mathbf{R}^n$, and let $f : \Omega \to \mathbf{R}$ be a simple function. Then there exists a finite number of real numbers $c_1, \ldots, c_N$, and a finite number of disjoint measurable sets $E_1, E_2, \ldots, E_N$ in Ω, such that $f = \sum_{i=1}^N c_i \chi_{E_i}$.*

Proof. See Exercise 19.1.2. $\square$

Lemma 19.1.5. *Let Ω be a measurable subset of $\mathbf{R}^n$, and let $f : \Omega \to \mathbf{R}$ be a measurable function. Suppose that f is always non-negative, i.e., $f(x) \geq 0$ for all $x \in \Omega$. Then there exists a sequence $f_1, f_2, f_3, \ldots$ of simple functions, $f_n : \Omega \to \mathbf{R}$, such that the f_n are non-negative and increasing,*

$$0 \leq f_1(x) \leq f_2(x) \leq f_3(x) \leq \ldots \text{ for all } x \in \Omega$$

and converge pointwise to f:

$$\lim_{n\to\infty} f_n(x) = f(x) \text{ for all } x \in \Omega.$$

Proof. See Exercise 19.1.3. $\square$

We now show how to compute the integral of simple functions.

Definition 19.1.6 (Lebesgue integral of simple functions). Let Ω be a measurable subset of $\mathbf{R}^n$, and let $f : \Omega \to \mathbf{R}$ be a simple function which is non-negative; thus f is measurable and the image $f(\Omega)$ is finite and contained in $[0, \infty)$. We then define the *Lebesgue integral* $\int_\Omega f$ of f on Ω by

$$\int_\Omega f := \sum_{\lambda \in f(\Omega); \lambda > 0} \lambda m(\{x \in \Omega : f(x) = \lambda\}).$$

We will also sometimes write $\int_\Omega f$ as $\int_\Omega f\, dm$ (to emphasize the rôle of Lebesgue measure m) or use a dummy variable such as x, e.g., $\int_\Omega f(x)\, dx$.

Example 19.1.7. Let $f : \mathbf{R} \to \mathbf{R}$ be the function which equals 3 on the interval $[1, 2]$, equals 4 on the integral $(2, 4)$, and is zero everywhere else. Then

$$\int_\Omega f := 3 \times m([1, 2]) + 4 \times m((2, 4)) = 3 \times 1 + 4 \times 2 = 11.$$

Or if $g : \mathbf{R} \to \mathbf{R}$ is the function which equals 1 on $[0, \infty)$ and is zero everywhere else, then

$$\int_\Omega g = 1 \times m([0, \infty)) = 1 \times +\infty = +\infty.$$

Thus the simple integral of a simple function can equal $+\infty$. (The reason why we restrict this integral to non-negative functions is to avoid ever encountering the indefinite form $+\infty + (-\infty)$).

Remark 19.1.8. Note that this definition of integral corresponds to one's intuitive notion of integration (at least of non-negative functions) as the area under the graph of the function (or volume, if one is in higher dimensions).

Another formulation of the integral for non-negative simple functions is as follows.

Lemma 19.1.9. *Let Ω be a measurable subset of $\mathbf{R}^n$, and let $E_1, \ldots, E_N$ are a finite number of disjoint measurable subsets in Ω.*

Let $c_1, \ldots, c_N$ be non-negative numbers (not necessarily distinct). Then we have

$$\int_\Omega \sum_{j=1}^N c_j \chi_{E_j} = \sum_{j=1}^n c_j m(E_j).$$

Proof. We can assume that none of the c_j are zero, since we can just remove them from the sum on both sides of the equation. Let $f := \sum_{j=1}^N c_j \chi_{E_j}$. Then $f(x)$ is either equal to one of the c_j (if $x \in E_j$) or equal to 0 (if $x \notin \bigcup_{j=1}^N E_j$). Thus f is a simple function, and $f(\Omega) \subseteq \{0\} \cup \{c_j : 1 \leq j \leq N\}$. Thus, by the definition,

$$\int_\Omega f = \sum_{\lambda \in \{c_j : 1 \leq j \leq N\}} \lambda m(\{x \in \Omega : f(x) = \lambda\})$$

$$= \sum_{\lambda \in \{c_j : 1 \leq j \leq N\}} \lambda m(\bigcup_{1 \leq j \leq N : c_j = \lambda} E_j).$$

But by the finite additivity property of Lebesgue measure, this is equal to

$$\sum_{\lambda \in \{c_j : 1 \leq j \leq N\}} \lambda \sum_{1 \leq j \leq N : c_j = \lambda} m(E_j)$$

$$\sum_{\lambda \in \{c_j : 1 \leq j \leq N\}} \sum_{1 \leq j \leq N : c_j = \lambda} c_j m(E_j).$$

Each j appears exactly once in this sum, since c_j is only equal to exactly one value of λ. So the above expression is equal to $\sum_{1 \leq j \leq N} c_j m(E_j)$ as desired. □

Some basic properties of Lebesgue integration of non-negative simple functions:

Proposition 19.1.10. *Let Ω be a measurable set, and let $f : \Omega \to \mathbf{R}$ and $g : \Omega \to \mathbf{R}$ be non-negative simple functions.*

(a) *We have $0 \leq \int_\Omega f \leq \infty$. Furthermore, we have $\int_\Omega f = 0$ if and only if $m(\{x \in \Omega : f(x) \neq 0\}) = 0$.*

(b) *We have* $\int_\Omega (f+g) = \int_\Omega f + \int_\Omega g$.

(c) *For any positive number* c, *we have* $\int_\Omega cf = c\int_\Omega f$.

(d) *If* $f(x) \le g(x)$ *for all* $x \in \Omega$, *then we have* $\int_\Omega f \le \int_\Omega g$.

We make a very convenient notational convention: if a property $P(x)$ holds for all points in Ω, except for a set of measure zero, then we say that P holds for *almost every* point in Ω. Thus (a) asserts that $\int_\Omega f = 0$ if and only if f is zero for almost every point in Ω.

Proof. From Lemma 19.1.4 or from the formula

$$f = \sum_{\lambda \in f(\Omega)\backslash\{0\}} \lambda \chi_{\{x \in \Omega: f(x) = \lambda\}}$$

we can write f as a combination of characteristic functions, say

$$f = \sum_{j=1}^{N} c_j \chi_{E_j},$$

where $E_1, \ldots, E_N$ are disjoint subsets of Ω and the c_j are positive. Similarly we can write

$$g = \sum_{k=1}^{M} d_k \chi_{F_k}$$

where $F_1, \ldots, F_M$ are disjoint subsets of Ω and the d_k are positive.

(a) Since $\int_\Omega f = \sum_{j=1}^N c_j m(E_j)$ it is clear that the integral is between 0 and infinity. If f is zero almost everywhere, then all of the E_j must have measure zero (why?) and so $\int_\Omega f = 0$. Conversely, if $\int_\Omega f = 0$, then $\sum_{j=1}^N c_j m(E_j) = 0$, which can only happen when all of the $m(E_j)$ are zero (since all the c_j are positive). But then $\bigcup_{j=1}^N E_j$ has measure zero, and hence f is zero almost everywhere in Ω.

(b) Write $E_0 := \Omega \backslash \bigcup_{j=1}^{N} E_j$ and $c_0 := 0$, then we have $\Omega = E_0 \cup E_1 \cup \ldots \cup E_N$ and

$$f = \sum_{j=0}^{N} c_j \chi_{E_j}.$$

Similarly if we write $F_0 := \Omega \backslash \bigcup_{k=1}^{M} F_k$ and $d_0 := 0$ then

$$g = \sum_{k=0}^{M} d_k \chi_{F_k}.$$

Since $\Omega = E_0 \cup \ldots \cup E_N = F_0 \cup \ldots \cup F_M$, we have

$$f = \sum_{j=0}^{N} \sum_{k=0}^{M} c_j \chi_{E_j \cap F_k}$$

and

$$g = \sum_{k=0}^{M} \sum_{j=0}^{N} d_k \chi_{E_j \cap F_k}$$

and hence

$$f + g = \sum_{0 \leq j \leq N; 0 \leq k \leq M} (c_j + d_k) \chi_{E_j \cap F_k}.$$

By Lemma 19.1.9, we thus have

$$\int_{\Omega} (f + g) = \sum_{0 \leq j \leq N; 0 \leq k \leq M} (c_j + d_k) m(E_j \cap F_k).$$

On the other hand, we have

$$\int_{\Omega} f = \sum_{0 \leq j \leq N} c_j m(E_j) = \sum_{0 \leq j \leq N; 0 \leq k \leq M} c_j m(E_j \cap F_k)$$

and similarly

$$\int_{\Omega} g = \sum_{0 \leq k \leq M} d_k m(F_k) = \sum_{0 \leq j \leq N; 0 \leq k \leq M} d_k m(E_j \cap F_k)$$

and the claim (b) follows.

(c) Since $cf = \sum_{j=1}^N cc_j\chi_{E_j}$, we have $\int_\Omega cf = \sum_{j=1}^N cc_jm(E_j)$. Since $\int_\Omega f = \sum_{j=1}^N c_jm(E_j)$, the claim follows.

(d) Write $h := g - f$. Then h is simple and non-negative and $g = f + h$, hence by (b) we have $\int_\Omega g = \int_\Omega f + \int_\Omega h$. But by (a) we have $\int_\Omega h \geq 0$, and the claim follows.

$\square$

Exercise 19.1.1. Prove Lemma 19.1.3.

Exercise 19.1.2. Prove Lemma 19.1.4.

Exercise 19.1.3. Prove Lemma 19.1.5. (Hint: set

$$f_n(x) := \sup\{\frac{j}{2^n} : j \in \mathbf{Z}, \frac{j}{2^n} \leq \min(f(x), 2^n)\},$$

i.e., $f_n(x)$ is the greatest integer multiple of 2^{-n} which does not exceed either $f(x)$ or 2^n. You may wish to draw a picture to see how f_1, f_2, f_3, etc. works. Then prove that f_n obeys all the required properties.)

19.2 Integration of non-negative measurable functions

We now pass from the integration of non-negative simple functions to the integration of non-negative measurable functions. We will allow our measurable functions to take the value of $+\infty$ sometimes.

Definition 19.2.1 (Majorization). Let $f : \Omega \to \mathbf{R}$ and $g : \Omega \to \mathbf{R}$ be functions. We say that f *majorizes* g, or g *minorizes* f, iff we have $f(x) \geq g(x)$ for all $x \in \Omega$.

We sometimes use the phrase "f dominates g" instead of "f majorizes g".

Definition 19.2.2 (Lebesgue integral for non-negative functions). Let Ω be a measurable subset of $\mathbf{R}^n$, and let $f : \Omega \to [0, \infty]$

be measurable and non-negative. Then we define the *Lebesgue integral* $\int_\Omega f$ of f on Ω to be

$$\int_\Omega f := \sup\{\int_\Omega s : s \text{ is simple and non-negative, and minorizes } f\}$$

Remark 19.2.3. The reader should compare this notion to that of a lower Riemann integral from Definition 11.3.2. Interestingly, we will not need to match this lower integral with an upper integral here.

Remark 19.2.4. Note that if Ω' is any measurable subset of Ω, then we can define $\int_{\Omega'} f$ as well by restricting f to Ω', thus $\int_{\Omega'} f := \int_{\Omega'} f|_{\Omega'}$.

We have to check that this definition is consistent with our previous notion of Lebesgue integral for non-negative simple functions; in other words, if $f : \Omega \to \mathbf{R}$ is a non-negative simple function, then the value of $\int_\Omega f$ given by this definition should be the same as the one given in the previous definition. But this is clear because f certainly minorizes itself, and any other non-negative simple function s which minorizes f will have an integral $\int_\Omega s$ less than or equal to $\int_\Omega f$, thanks to Proposition 19.1.10(d).

Remark 19.2.5. Note that $\int_\Omega f$ is always at least 0, since 0 is simple, non-negative, and minorizes f. Of course, $\int_\Omega f$ could equal $+\infty$.

Some basic properties of the Lebesgue integral on non-negative measurable functions (which supercede Proposition 19.1.10):

Proposition 19.2.6. *Let Ω be a measurable set, and let $f : \Omega \to [0, \infty]$ and $g : \Omega \to [0, \infty]$ be non-negative measurable functions.*

(a) *We have $0 \leq \int_\Omega f \leq \infty$. Furthermore, we have $\int_\Omega f = 0$ if and only if $f(x) = 0$ for almost every $x \in \Omega$.*

(b) *For any positive number c, we have $\int_\Omega cf = c\int_\Omega f$.*

(c) *If $f(x) \leq g(x)$ for all $x \in \Omega$, then we have $\int_\Omega f \leq \int_\Omega g$.*

(d) *If* $f(x) = g(x)$ *for almost every* $x \in \Omega$, *then* $\int_\Omega f = \int_\Omega g$.

(e) *If* $\Omega' \subseteq \Omega$ *is measurable, then* $\int_{\Omega'} f = \int_\Omega f\chi_{\Omega'} \leq \int_\Omega f$.

Proof. See Exercise 19.2.1. □

Remark 19.2.7. Proposition 19.2.6(d) is quite interesting; it says that one can modify the values of a function on any measure zero set (e.g., you can modify a function on every rational number), and not affect its integral at all. It is as if no individual point, or even a measure zero collection of points, has any "vote" in what the integral of a function should be; only the collective set of points has an influence on an integral.

Remark 19.2.8. Note that we do not yet try to interchange sums and integrals. From the definition it is fairly easy to prove that $\int_\Omega (f + g) \geq \int_\Omega f + \int_\Omega g$ (Exercise 19.2.2), but to prove equality requires more work and will be done later.

As we have seen in previous chapters, we cannot always interchange an integral with a limit (or with limit-like concepts such as supremum). However, with the Lebesgue integral it is possible to do so if the functions are increasing:

Theorem 19.2.9 (Lebesgue monotone convergence theorem). *Let* Ω *be a measurable subset of* $\mathbf{R}^n$, *and let* $(f_n)_{n=1}^\infty$ *be a sequence of non-negative measurable functions from* Ω *to* $\mathbf{R}$ *which are increasing in the sense that*

$$0 \leq f_1(x) \leq f_2(x) \leq f_3(x) \leq \ldots \text{ for all } x \in \Omega.$$

(Note we are assuming that $f_n(x)$ *is increasing with respect to* n*; this is a different notion from* $f_n(x)$ *increasing with respect to* x*.) Then we have*

$$0 \leq \int_\Omega f_1 \leq \int_\Omega f_2 \leq \int_\Omega f_3 \leq \ldots$$

and

$$\int_\Omega \sup_n f_n = \sup_n \int_\Omega f_n.$$

Proof. The first conclusion is clear from Proposition 19.2.6(c). Now we prove the second conclusion. From Proposition 19.2.6(c) again we have

$$\int_\Omega \sup_m f_m \geq \int_\Omega f_n$$

for every n; taking suprema in n we obtain

$$\int_\Omega \sup_m f_m \geq \sup_n \int_\Omega f_n$$

which is one half of the desired conclusion. To finish the proof we have to show

$$\int_\Omega \sup_m f_m \leq \sup_n \int_\Omega f_n.$$

From the definition of $\int_\Omega \sup_m f_m$, it will suffice to show that

$$\int_\Omega s \leq \sup_n \int_\Omega f_n$$

for all simple non-negative functions which minorize $\sup_m f_m$.

Fix s. We will show that

$$(1-\varepsilon)\int_\Omega s \leq \sup_n \int_\Omega f_n$$

for every $0 < \varepsilon < 1$; the claim then follows by taking limits as $\varepsilon \to 0$.

Fix ε. By construction of s, we have

$$s(x) \leq \sup_n f_n(x)$$

for every $x \in \Omega$. Hence, for every $x \in \Omega$ there exists an N (depending on x) such that

$$f_N(x) \geq (1-\varepsilon)s(x).$$

Since the f_n are increasing, this will imply that $f_n(x) \geq (1-\varepsilon)s(x)$ for all $n \geq N$. Thus, if we define the sets E_n by

$$E_n := \{x \in \Omega : f_n(x) \geq (1-\varepsilon)s(x)\}$$

then we have $E_1 \subset E_2 \subset E_3 \subset \ldots$ and $\bigcup_{n=1}^{\infty} E_n = \Omega$.

From Proposition 19.2.6(cdf) we have

$$(1-\varepsilon)\int_{E_n} s = \int_{E_n} (1-\varepsilon)s \leq \int_{E_n} f_n \leq \int_{\Omega} f_n$$

so to finish the argument it will suffice to show that

$$\sup_n \int_{E_n} s = \int_{\Omega} s.$$

Since s is a simple function, we may write $s = \sum_{j=1}^{N} c_j \chi_{F_j}$ for some measurable F_j and positive c_j. Since

$$\int_{\Omega} s = \sum_{j=1}^{N} c_j m(F_j)$$

and

$$\int_{E_n} s = \int_{E_n} \sum_{j=1}^{N} c_j \chi_{F_j \cap E_n} = \sum_{j=1}^{N} c_j m(F_j \cap E_n)$$

it thus suffices to show that

$$\sup_n m(F_j \cap E_n) = m(F_j)$$

for each j. But this follows from Exercise 18.2.3(a). □

This theorem is extremely useful. For instance, we can now interchange addition and integration:

Lemma 19.2.10 (Interchange of addition and integration). *Let Ω be a measurable subset of $\mathbf{R}^n$, and let $f : \Omega \to [0, \infty]$ and $g : \Omega \to [0, \infty]$ be measurable functions. Then $\int_{\Omega}(f+g) = \int_{\Omega} f + \int_{\Omega} g$.*

Proof. By Lemma 19.1.5, there exists a sequence $0 \leq s_1 \leq s_2 \leq \ldots \leq f$ of simple functions such that $\sup_n s_n = f$, and similarly a sequence $0 \leq t_1 \leq t_2 \leq \ldots \leq g$ of simple functions such that $\sup_n t_n = g$. Since the s_n are increasing and the t_n are increasing, it is then easy to check that $s_n + t_n$ is also increasing and

$\sup_n(s_n + t_n) = f + g$ (why?). By the monotone convergence theorem (Theorem 19.2.9) we thus have

$$\int_\Omega f = \sup_n \int_\Omega s_n$$

$$\int_\Omega g = \sup_n \int_\Omega t_n$$

$$\int_\Omega (f+g) = \sup_n \int_\Omega (s_n + t_n).$$

But by Proposition 19.1.9(b) we have $\int_\Omega (s_n + t_n) = \int_\Omega s_n + \int_\Omega t_n$. By Proposition 19.1.9(d), $\int_\Omega s_n$ and $\int_\Omega t_n$ are both increasing in n, so

$$\sup_n(\int_\Omega s_n + \int_\Omega t_n) = (\sup_n \int_\Omega s_n) + (\sup_n \int_\Omega t_n)$$

and the claim follows. □

Of course, once one can interchange an integral with a sum of two functions, one can handle an integral and any finite number of functions by induction. More surprisingly, one can handle infinite sums as well of *non-negative* functions:

Corollary 19.2.11. *If Ω is a measurable subset of $\mathbf{R}^n$, and g_1, $g_2, \ldots$ are a sequence of non-negative functions from Ω to $[0, \infty]$, then*

$$\int_\Omega \sum_{n=1}^\infty g_n = \sum_{n=1}^\infty \int_\Omega g_n.$$

Proof. See Exercise 19.2.3. □

Remark 19.2.12. Note that we do not need to assume anything about the convergence of the above sums; it may well happen that both sides are equal to $+\infty$. However, we *do* need to assume non-negativity; see Exercise 19.2.4.

One could similarly ask whether we could interchange limits and integrals; in other words, is it true that

$$\int_\Omega \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int_\Omega f_n.$$

Unfortunately, this is not true, as the following "moving bump" example shows. For each $n = 1, 2, 3 \ldots$, let $f_n : \mathbf{R} \to \mathbf{R}$ be the function $f_n = \chi_{[n,n+1)}$. Then $\lim_{n\to\infty} f_n(x) = 0$ for every x, but $\int_{\mathbf{R}} f_n = 1$ for every n, and hence $\lim_{n\to\infty} \int_{\mathbf{R}} f_n = 1 \neq 0$. In other words, the limiting function $\lim_{n\to\infty} f_n$ can end up having significantly smaller integral than any of the original integrals. However, the following very useful lemma of Fatou shows that the reverse cannot happen - there is no way the limiting function has larger integral than the (limit of the) original integrals:

Lemma 19.2.13 (Fatou's lemma). *Let Ω be a measurable subset of $\mathbf{R}^n$, and let $f_1, f_2, \ldots$ be a sequence of non-negative functions from Ω to $[0, \infty]$. Then*

$$\int_\Omega \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int_\Omega f_n.$$

Proof. Recall that

$$\lim \inf_{n\to\infty} f_n = \sup_n (\inf_{m\geq n} f_m)$$

and hence by the monotone convergence theorem

$$\int_\Omega \lim \inf_{n\to\infty} f_n = \sup_n \int_\Omega (\inf_{m\geq n} f_m).$$

By Proposition 19.2.6(c) we have

$$\int_\Omega (\inf_{m\geq n} f_m) \leq \int_\Omega f_j$$

for every $j \geq n$; taking infima in j we obtain

$$\int_\Omega (\inf_{m\geq n} f_m) \leq \inf_{j\geq n} \int_\Omega f_j.$$

Thus

$$\int_\Omega \lim \inf_{n\to\infty} f_n \leq \sup_n \inf_{j\geq n} \int_\Omega f_j = \lim \inf_{n\to\infty} \int_\Omega f_n$$

as desired. $\square$

Note that we are allowing our functions to take the value $+\infty$ at some points. It is even possible for a function to take the value $+\infty$ but still have a finite integral; for instance, if E is a measure zero set, and $f : \Omega \to \mathbf{R}$ is equal to $+\infty$ on E but equals 0 everywhere else, then $\int_\Omega f = 0$ by Proposition 19.2.6(a). However, if the integral is finite, the function must be finite almost everywhere:

Lemma 19.2.14. *Let Ω be a measurable subset of $\mathbf{R}^n$, and let $f : \Omega \to [0, \infty]$ be a non-negative measurable function such that $\int_\Omega f$ is finite. Then f is finite almost everywhere (i.e., the set $\{x \in \Omega : f(x) = +\infty\}$ has measure zero).*

Proof. See Exercise 19.2.5. □

Form Corollary 19.2.11 and Lemma 19.2.14 one has a useful lemma:

Lemma 19.2.15 (Borel-Cantelli lemma). *Let $\Omega_1, \Omega_2, \ldots$ be measurable subsets of $\mathbf{R}^n$ such that $\sum_{n=1}^\infty m(\Omega_n)$ is finite. Then the set*

$$\{x \in \mathbf{R}^n : x \in \Omega_n \text{ for infinitely many } n\}$$

is a set of measure zero. In other words, almost every point belongs to only finitely many Ω_n.

Proof. See Exercise 19.2.6. □

Exercise 19.2.1. Prove Proposition 19.2.6. (Hint: do not attempt to mimic the proof of Proposition 19.1.10; rather, try to use Proposition 19.1.10 and Definition 19.2.2. For one direction of part (a), start with $\int_\Omega f = 0$ and conclude that $m(\{x \in \Omega : f(x) > 1/n\}) = 0$ for every $n = 1, 2, 3, \ldots$, and then use the countable sub-additivity. To prove (e), first prove it for simple functions.)

Exercise 19.2.2. Let Ω be a measurable subset of $\mathbf{R}^n$, and let $f : \Omega \to [0, +\infty]$ and $g : \Omega \to [0, +\infty]$ be measurable functions. Without using Theorem 19.2.9 or Lemma 19.2.10, prove that $\int_\Omega (f + g) \geq \int_\Omega f + \int_\Omega g$.

Exercise 19.2.3. Prove Corollary 19.2.11. (Hint: use the monotone convergence theorem with $f_N := \sum_{n=1}^N g_n$.)

Exercise 19.2.4. For each $n = 1, 2, 3, \ldots$, let $f_n : \mathbf{R} \to \mathbf{R}$ be the function $f_n = \chi_{[n,n+1)} - \chi_{[n+1,n+2)}$; i.e., let $f_n(x)$ equal $+1$ when $x \in [n, n+1)$, equal -1 when $x \in [n+1, n+2)$, and 0 everywhere else. Show that

$$\int_{\mathbf{R}} \sum_{n=1}^{\infty} f_n \neq \sum_{n=1}^{\infty} \int_{\mathbf{R}} f_n.$$

Explain why this does not contradict Corollary 19.2.11.

Exercise 19.2.5. Prove Lemma 19.2.14.

Exercise 19.2.6. Use Corollary 19.2.11 and Lemma 19.2.14 to prove Lemma 19.2.15. (Hint: use the indicator functions χ_{Ω_n}.)

Exercise 19.2.7. Let $p > 2$ and $c > 0$. Using the Borel-Cantelli lemma, show that the set

$$\{x \in [0,1] : |x - \frac{a}{q}| \leq \frac{c}{q^p} \text{ for infinitely many positive integers } a, q\}$$

has measure zero. (Hint: one only has to consider those integers a in the range $0 \leq a \leq q$ (why?). Use Corollary 11.6.5 to show that the sum $\sum_{q=1}^{\infty} \frac{c(q+1)}{q^p}$ is finite.)

Exercise 19.2.8. Call a real number $x \in \mathbf{R}$ *diophantine* if there exist real numbers $p, C > 0$ such that $|x - \frac{a}{q}| > C/|q|^p$ for all non-zero integers q and all integers a. Using Exercise 19.2.7, show that almost every real number is diophantine. (Hint: first work in the interval $[0,1]$. Show that one can take p and C to be rational and one can also take $p > 2$. Then use the fact that the countable union of measure zero sets has measure zero.)

Exercise 19.2.9. For every positive integer n, let $f_n : \mathbf{R} \to [0, \infty)$ be a non-negative measurable function such that

$$\int_{\mathbf{R}} f_n \leq \frac{1}{4^n}.$$

Show that for every $\varepsilon > 0$, there exists a set E of Lebesgue measure $m(E) \leq \varepsilon$ such that $f_n(x)$ converges pointwise to zero for all $x \in \mathbf{R} \backslash E$. (Hint: first prove that $m(\{x \in \mathbf{R} : f_n(x) > \frac{1}{\varepsilon 2^n}\}) \leq \frac{\varepsilon}{2^n}$ for all $n = 1, 2, 3, \ldots$, and then consider the union of all the sets $\{x \in \mathbf{R} : f_n(x) > \frac{1}{\varepsilon 2^n}\}$.)

Exercise 19.2.10. For every positive integer n, let $f_n : \mathbf{R} \to [0, \infty)$ be a non-negative measurable function such that f_n converges pointwise to zero. Show that for every $\varepsilon > 0$, there exists a set E of Lebesgue measure $m(E) \leq \varepsilon$ such that $f_n(x)$ converges *uniformly* to zero for all $x \in \mathbf{R}\backslash E$. (This is a special case of *Egoroff's theorem.* To prove it, first show that for any positive integer m, we can find an $N > 0$ such that $m(\{x \in \mathbf{R} : f_n(x) > 1/m\}) \leq \varepsilon/2^m$ for all $n \geq N$.)

Exercise 19.2.11. Give an example of a bounded non-negative function $f : \mathbf{N} \times \mathbf{N} \to \mathbf{R}^+$ such that $\sum_{m=1}^\infty f(n,m)$ converges for every n, and such that $\lim_{n\to\infty} f(n,m)$ exists for every m, but such that

$$\lim_{n\to\infty} \sum_{m=1}^\infty f(n,m) \neq \sum_{m=1}^\infty \lim_{n\to\infty} f(n,m).$$

(Hint: modify the moving bump example. It is even possible to use a function f which only takes the values 0 and 1.) This shows that interchanging limits and infinite sums can be dangerous.

19.3 Integration of absolutely integrable functions

We have now completed the theory of the Lebesgue integral for non-negative functions. Now we consider how to integrate functions which can be both positive and negative. However, we do wish to avoid the indefinite expression $+\infty + (-\infty)$, so we will restrict our attention to a subclass of measurable functions - the *absolutely integrable functions.*

Definition 19.3.1 (Absolutely integrable functions). Let Ω be a measurable subset of $\mathbf{R}^n$. A measurable function $f : \Omega \to \mathbf{R}^*$ is said to be *absolutely integrable* if the integral $\int_\Omega |f|$ is finite.

Of course, $|f|$ is always non-negative, so this definition makes sense even if f changes sign. Absolutely integrable functions are also known as $L^1(\Omega)$ functions.

If $f : \Omega \to \mathbf{R}^*$ is a function, we define the *positive part* $f^+ : \Omega \to [0,\infty]$ and *negative part* $f^- : \Omega \to [0,\infty]$ by the formulae

$$f^+ := \max(f, 0); \quad f^- := -\min(f, 0).$$

From Corollary 18.5.6 we know that f^+ and f^- are measurable. Observe also that f^+ and f^- are non-negative, that $f = f^+ - f^-$, and $|f| = f^+ + f^-$. (Why?).