10.40 Theorem *Fix k, $1 \le k \le n$. Let $E \subset R^n$ be an open set in which every closed k-form is exact. Let T be a 1-1 $\mathscr{C}''$-mapping of E onto an open set $U \subset R^n$ whose inverse S is also of class $\mathscr{C}''$.*

Then every closed k-form in U is exact in U.

Note that every convex open set E satisfies the present hypothesis, by Theorem 10.39. The relation between E and U may be expressed by saying that they are *$\mathscr{C}''$-equivalent.*

Thus every closed form is exact in any set which is $\mathscr{C}''$-equivalent to a convex open set.

Proof Let ω be a k-form in U, with $d\omega = 0$. By Theorem 10.22(c), ω_T is a k-form in E for which $d(\omega_T) = 0$. Hence $\omega_T = d\lambda$ for some $(k-1)$-form λ in E. By Theorem 10.23, and another application of Theorem 10.22(c),

$$\omega = (\omega_T)_S = (d\lambda)_S = d(\lambda_S).$$

Since λ_S is a $(k-1)$-form in U, ω is exact in U.

10.41 Remark In applications, cells (see Definition 2.17) are often more convenient parameter domains than simplexes. If our whole development had been based on cells rather than simplexes, the computation that occurs in the proof of Stokes' theorem would be even simpler. (It is done that way in Spivak's book.) The reason for preferring simplexes is that the definition of the boundary of an oriented simplex seems easier and more natural than is the case for a cell. (See Exercise 19.) Also, the partitioning of sets into simplexes (called "triangulation") plays an important role in topology, and there are strong connections between certain aspects of topology, on the one hand, and differential forms, on the other. These are hinted at in Sec. 10.35. The book by Singer and Thorpe contains a good introduction to this topic.

Since every cell can be triangulated, we may regard it as a chain. For dimension 2, this was done in Example 10.32; for dimension 3, see Exercise 18.

Poincaré's lemma (Theorem 10.39) can be proved in several ways. See, for example, page 94 in Spivak's book, or page 280 in Fleming's. Two simple proofs for certain special cases are indicated in Exercises 24 and 27.

VECTOR ANALYSIS

We conclude this chapter with a few applications of the preceding material to theorems concerning vector analysis in R^3. These are special cases of theorems about differential forms, but are usually stated in different terminology. We are thus faced with the job of translating from one language to another.

10.42 Vector fields Let $\mathbf{F} = F_1 \mathbf{e}_1 + F_2 \mathbf{e}_2 + F_3 \mathbf{e}_3$ be a continuous mapping of an open set $E \subset R^3$ into R^3. Since $\mathbf{F}$ associates a vector to each point of E, $\mathbf{F}$ is sometimes called a vector field, especially in physics. With every such $\mathbf{F}$ is associated a 1-form

$$\lambda_{\mathbf{F}} = F_1\, dx + F_2\, dy + F_3\, dz \tag{124}$$

and a 2-form

$$\omega_{\mathbf{F}} = F_1\, dy \wedge dz + F_2\, dz \wedge dx + F_3\, dx \wedge dy. \tag{125}$$

Here, and in the rest of this chapter, we use the customary notation (x, y, z) in place of (x_1, x_2, x_3).

It is clear, conversely, that every 1-form λ in E is $\lambda_{\mathbf{F}}$ for some vector field $\mathbf{F}$ in E, and that every 2-form ω is $\omega_{\mathbf{F}}$ for some $\mathbf{F}$. In R^3, the study of 1-forms and 2-forms is thus coextensive with the study of vector fields.

If $u \in \mathscr{C}'(E)$ is a real function, then its *gradient*

$$\nabla u = (D_1 u)\mathbf{e}_1 + (D_2 u)\mathbf{e}_2 + (D_3 u)\mathbf{e}_3$$

is an example of a vector field in E.

Suppose now that $\mathbf{F}$ is a vector field in E, of class $\mathscr{C}'$. Its *curl* $\nabla \times \mathbf{F}$ is the vector field defined in E by

$$\nabla \times \mathbf{F} = (D_2 F_3 - D_3 F_2)\mathbf{e}_1 + (D_3 F_1 - D_1 F_3)\mathbf{e}_2 + (D_1 F_2 - D_2 F_1)\mathbf{e}_3$$

and its *divergence* is the real function $\nabla \cdot \mathbf{F}$ defined in E by

$$\nabla \cdot \mathbf{F} = D_1 F_1 + D_2 F_2 + D_3 F_3 .$$

These quantities have various physical interpretations. We refer to the book by O. D. Kellogg for more details.

Here are some relations between gradients, curls, and divergences.

10.43 Theorem *Suppose E is an open set in R^3, $u \in \mathscr{C}''(E)$, and $\mathbf{G}$ is a vector field in E, of class C''.*

(*a*) *If $\mathbf{F} = \nabla u$, then $\nabla \times \mathbf{F} = \mathbf{0}$.*
(*b*) *If $\mathbf{F} = \nabla \times \mathbf{G}$, then $\nabla \cdot \mathbf{F} = 0$.*

Furthermore, if E is $\mathscr{C}''$-equivalent to a convex set, then (*a*) *and* (*b*) *have converses, in which we assume that $\mathbf{F}$ is a vector field in E, of class $\mathscr{C}'$:*

(*a'*) *If $\nabla \times \mathbf{F} = \mathbf{0}$, then $\mathbf{F} = \nabla u$ for some $u \in \mathscr{C}''(E)$.*
(*b'*) *If $\nabla \cdot \mathbf{F} = 0$, then $\mathbf{F} = \nabla \times \mathbf{G}$ for some vector field $\mathbf{G}$ in E, of class $\mathscr{C}''$*

Proof If we compare the definitions of ∇u, $\nabla \times \mathbf{F}$, and $\nabla \cdot \mathbf{F}$ with the differential forms $\lambda_{\mathbf{F}}$ and $\omega_{\mathbf{F}}$ given by (124) and (125), we obtain the following four statements:

$$\begin{aligned} \mathbf{F} &= \nabla u && \text{if and only if} \quad \lambda_{\mathbf{F}} = du. \\ \nabla \times \mathbf{F} &= \mathbf{0} && \text{if and only if} \quad d\lambda_{\mathbf{F}} = 0. \\ \mathbf{F} &= \nabla \times \mathbf{G} && \text{if and only if} \quad \omega_{\mathbf{F}} = d\lambda_{\mathbf{G}}. \\ \nabla \cdot \mathbf{F} &= 0 && \text{if and only if} \quad d\omega_{\mathbf{F}} = 0. \end{aligned}$$

Now if $\mathbf{F} = \nabla u$, then $\lambda_{\mathbf{F}} = du$, hence $d\lambda_{\mathbf{F}} = d^2 u = 0$ (Theorem 10.20), which means that $\nabla \times \mathbf{F} = \mathbf{0}$. Thus (*a*) is proved.

As regards (*a'*), the hypothesis amounts to saying that $d\lambda_{\mathbf{F}} = 0$ in E. By Theorem 10.40, $\lambda_{\mathbf{F}} = du$ for some 0-form u. Hence $\mathbf{F} = \nabla u$.

The proofs of (*b*) and (*b'*) follow exactly the same pattern.

10.44 Volume elements The k-form

$$dx_1 \wedge \cdots \wedge dx_k$$

is called the volume element in R^k. It is often denoted by dV (or by dV_k if it seems desirable to indicate the dimension explicitly), and the notation

$$\int_{\Phi} f(\mathbf{x})\, dx_1 \wedge \cdots \wedge dx_k = \int_{\Phi} f\, dV \tag{126}$$

is used when Φ is a positively oriented k-surface in R^k and f is a continuous function on the range of Φ.

The reason for using this terminology is very simple: If D is a parameter domain in R^k, and if Φ is a 1-1 $\mathscr{C}'$-mapping of D into R^k, with positive Jacobian J_{Φ}, then the left side of (126) is

$$\int_D f(\Phi(\mathbf{u}))J_{\Phi}(\mathbf{u})\, d\mathbf{u} = \int_{\Phi(D)} f(\mathbf{x})\, d\mathbf{x},$$

by (35) and Theorem 10.9.

In particular, when $f = 1$, (126) defines the *volume* of Φ. We already saw a special case of this in (36).

The usual notation for dV_2 is dA.

10.45 Green's theorem *Suppose E is an open set in R^2, $\alpha \in \mathscr{C}'(E)$, $\beta \in \mathscr{C}'(E)$, and Ω is a closed subset of E, with positively oriented boundary $\partial\Omega$, as described in Sec.* 10.31. *Then*

$$\int_{\partial\Omega} (\alpha\, dx + \beta\, dy) = \int_{\Omega} \left(\frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y}\right) dA. \tag{127}$$

Proof Put $\lambda = \alpha\, dx + \beta\, dy$. Then

$$d\lambda = (D_2\alpha)\, dy \wedge dx + (D_1\beta)\, dx \wedge dy$$
$$= (D_1\beta - D_2\alpha)\, dA,$$

and (127) is the same as

$$\int_{\partial\Omega} \lambda = \int_{\Omega} d\lambda,$$

which is true by Theorem 10.33.

With $\alpha(x, y) = -y$ and $\beta(x, y) = x$, (127) becomes

$$\tfrac{1}{2}\int_{\partial\Omega} (x\, dy - y\, dx) = A(\Omega), \tag{128}$$

the area of Ω.

With $\alpha = 0$, $\beta = x$, a similar formula is obtained. Example 10.12(*b*) contains a special case of this.

10.46 Area elements in R^3 Let Φ be a 2-surface in R^3, of class $\mathscr{C}'$, with parameter domain $D \subset R^2$. Associate with each point $(u, v) \in D$ the vector

$$\mathbf{N}(u, v) = \frac{\partial(y, z)}{\partial(u, v)}\mathbf{e}_1 + \frac{\partial(z, x)}{\partial(u, v)}\mathbf{e}_2 + \frac{\partial(x, y)}{\partial(u, v)}\mathbf{e}_3 . \tag{129}$$

The Jacobians in (129) correspond to the equation

$$(x, y, z) = \Phi(u, v). \tag{130}$$

If f is a continuous function on $\Phi(D)$, the *area integral* of f over Φ is defined to be

$$\int_{\Phi} f\, dA = \int_D f(\Phi(u, v))\,|\mathbf{N}(u, v)|\; du\, dv. \tag{131}$$

In particular, when $f = 1$ we obtain the *area* of Φ, namely,

$$A(\Phi) = \int_D |\mathbf{N}(u, v)|\; du\, dv. \tag{132}$$

The following discussion will show that (131) and its special case (132) are reasonable definitions. It will also describe the geometric features of the vector $\mathbf{N}$.

Write $\Phi = \varphi_1\mathbf{e}_1 + \varphi_2\mathbf{e}_2 + \varphi_3\mathbf{e}_3$, fix a point $\mathbf{p}_0 = (u_0, v_0) \in D$, put $\mathbf{N} = \mathbf{N}(\mathbf{p}_0)$, put

$$\alpha_i = (D_1\varphi_i)(\mathbf{p}_0), \qquad \beta_i = (D_2\varphi_i)(\mathbf{p}_0) \qquad (i = 1, 2, 3) \tag{133}$$

and let $T \in L(R^2, R^3)$ be the linear transformation given by

$$T(u, v) = \sum_{i=1}^{3} (\alpha_i u + \beta_i v)\mathbf{e}_i. \tag{134}$$

Note that $T = \Phi'(\mathbf{p}_0)$, in accordance with Definition 9.11.

Let us now assume that the rank of T is 2. (If it is 1 or 0, then $\mathbf{N} = \mathbf{0}$, and the tangent plane mentioned below degenerates to a line or to a point.) The range of the affine mapping

$$(u, v) \to \Phi(\mathbf{p}_0) + T(u, v)$$

is then a plane Π, called the *tangent plane* to Φ at $\mathbf{p}_0$. [One would like to call Π the tangent plane at $\Phi(\mathbf{p}_0)$, rather than at $\mathbf{p}_0$; if Φ is not one-to-one, this runs into difficulties.]

If we use (133) in (129), we obtain

$$\mathbf{N} = (\alpha_2 \beta_3 - \alpha_3 \beta_2)\mathbf{e}_1 + (\alpha_3 \beta_1 - \alpha_1 \beta_3)\mathbf{e}_2 + (\alpha_1 \beta_2 - \alpha_2 \beta_1)\mathbf{e}_3, \tag{135}$$

and (134) shows that

$$T\mathbf{e}_1 = \sum_{i=1}^{3} \alpha_i \mathbf{e}_i, \qquad T\mathbf{e}_2 = \sum_{i=1}^{3} \beta_i \mathbf{e}_i. \tag{136}$$

A straightforward computation now leads to

$$\mathbf{N} \cdot (T\mathbf{e}_1) = 0 = \mathbf{N} \cdot (T\mathbf{e}_2). \tag{137}$$

Hence $\mathbf{N}$ is perpendicular to Π. It is therefore called the *normal to* Φ *at* $\mathbf{p}_0$.

A second property of $\mathbf{N}$, also verified by a direct computation based on (135) and (136), is that the determinant of the linear transformation of R^3 that takes $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to $\{T\mathbf{e}_1, T\mathbf{e}_2, \mathbf{N}\}$ is $|\mathbf{N}|^2 > 0$ (Exercise 30). The 3-simplex

$$[\mathbf{0}, T\mathbf{e}_1, T\mathbf{e}_2, \mathbf{N}] \tag{138}$$

is thus *positively oriented.*

The third property of $\mathbf{N}$ that we shall use is a consequence of the first two: The above-mentioned determinant, whose value is $|\mathbf{N}|^2$, is the volume of the parallelepiped with edges $[\mathbf{0}, T\mathbf{e}_1]$, $[\mathbf{0}, T\mathbf{e}_2]$, $[\mathbf{0}, \mathbf{N}]$. By (137), $[\mathbf{0}, \mathbf{N}]$ is perpendicular to the other two edges. *The area of the parallelogram with vertices*

$$\mathbf{0}, T\mathbf{e}_1, T\mathbf{e}_2, T(\mathbf{e}_1 + \mathbf{e}_2) \tag{139}$$

is therefore $|\mathbf{N}|$.

This parallelogram is the image under T of the unit square in R^2. If E is any rectangle in R^2, it follows (by the linearity of T) that the area of the parallelogram $T(E)$ is

$$A(T(E)) = |\mathbf{N}| A(E) = \int_E |\mathbf{N}(u_0, v_0)| \, du \, dv. \tag{140}$$

We conclude that (132) is correct when Φ is affine. To justify the definition (132) in the general case, divide D into small rectangles, pick a point (u_0, v_0) in each, and replace Φ in each rectangle by the corresponding tangent plane. The sum of the areas of the resulting parallelograms, obtained via (140), is then an approximation to $A(\Phi)$. Finally, one can justify (131) from (132) by approximating f by step functions.

10.47 Example Let $0 < a < b$ be fixed. Let K be the 3-cell determined by

$$0 \le t \le a, \qquad 0 \le u \le 2\pi, \qquad 0 \le v \le 2\pi.$$

The equations

$$\begin{aligned} x &= t \cos u \\ y &= (b + t \sin u) \cos v \\ z &= (b + t \sin u) \sin v \end{aligned} \tag{141}$$

describe a mapping Ψ of R^3 into R^3 which is 1-1 in the interior of K, such that $\Psi(K)$ is a solid torus. Its Jacobian is

$$J_\Psi = \frac{\partial(x, y, z)}{\partial(t, u, v)} = t(b + t \sin u)$$

which is positive on K, except on the face $t = 0$. If we integrate J_Ψ over K, we obtain

$$\text{vol}\,(\Psi(K)) = 2\pi^2 a^2 b$$

as the volume of our solid torus.

Now consider the 2-chain $\Phi = \partial\Psi$. (See Exercise 19.) Ψ maps the faces $u = 0$ and $u = 2\pi$ of K onto the same cylindrical strip, but with opposite orientations. Ψ maps the faces $v = 0$ and $v = 2\pi$ onto the same circular disc, but with opposite orientations. Ψ maps the face $t = 0$ onto a circle, which contributes 0 to the 2-chain $\partial\Psi$. (The relevant Jacobians are 0.) Thus Φ is simply the 2-surface obtained by setting $t = a$ in (141), with parameter domain D the square defined by $0 \le u \le 2\pi$, $0 \le v \le 2\pi$.

According to (129) and (141), the normal to Φ at $(u, v) \in D$ is thus the vector

$$\mathbf{N}(u, v) = a(b + a \sin u)\mathbf{n}(u, v)$$

where

$$\mathbf{n}(u, v) = (\cos u)\mathbf{e}_1 + (\sin u \cos v)\mathbf{e}_2 + (\sin u \sin v)\mathbf{e}_3\,.$$

Since $|\mathbf{n}(u, v)| = 1$, we have $|\mathbf{N}(u, v)| = a(b + a \sin u)$, and if we integrate this over D, (131) gives

$$A(\Phi) = 4\pi^2 ab$$

as the surface area of our torus.

If we think of $\mathbf{N} = \mathbf{N}(u, v)$ as a directed line segment, pointing from $\Phi(u, v)$ to $\Phi(u, v) + \mathbf{N}(u, v)$, then $\mathbf{N}$ points *outward*, that is to say, away from $\Psi(K)$. This is so because $\mathbf{J}_\Psi > 0$ when $t = a$.

For example, take $u = v = \pi/2$, $t = a$. This gives the largest value of z on $\Psi(K)$, and $\mathbf{N} = a(b + a)\mathbf{e}_3$ points "upward" for this choice of (u, v).

10.48 Integrals of 1-forms in R^3 Let γ be a $\mathscr{C}'$-curve in an open set $E \subset R^3$, with parameter interval $[0, 1]$, let $\mathbf{F}$ be a vector field in E, as in Sec. 10.42, and define $\lambda_\mathbf{F}$ by (124). The integral of $\lambda_\mathbf{F}$ over γ can be rewritten in a certain way which we now describe.

For any $u \in [0, 1]$,

$$\gamma'(u) = \gamma_1'(u)\mathbf{e}_1 + \gamma_2'(u)\mathbf{e}_2 + \gamma_3'(u)\mathbf{e}_3$$

is called the *tangent vector* to γ at u. We define $\mathbf{t} = \mathbf{t}(u)$ to be the unit vector in the direction of $\gamma'(u)$. Thus

$$\gamma'(u) = |\gamma'(u)|\,\mathbf{t}(u).$$

[If $\gamma'(u) = \mathbf{0}$ for some u, put $\mathbf{t}(u) = \mathbf{e}_1$; any other choice would do just as well.] By (35),

$$\begin{aligned} \int_\gamma \lambda_\mathbf{F} &= \sum_{i=1}^{3} \int_0^1 F_i(\gamma(u))\gamma_i'(u)\,du \\ &= \int_0^1 \mathbf{F}(\gamma(u)) \cdot \gamma'(u)\,du \qquad (142) \\ &= \int_0^1 \mathbf{F}(\gamma(u)) \cdot \mathbf{t}(u)\,|\gamma'(u)|\,du. \end{aligned}$$

Theorem 6.27 makes it reasonable to call $|\gamma'(u)|\,du$ the *element of arc length along* γ. A customary notation for it is ds, and (142) is rewritten in the form

$$\int_\gamma \lambda_\mathbf{F} = \int_\gamma (\mathbf{F} \cdot \mathbf{t})\,ds. \qquad (143)$$

Since $\mathbf{t}$ is a unit tangent vector to γ, $\mathbf{F} \cdot \mathbf{t}$ is called the *tangential component* of $\mathbf{F}$ along γ.