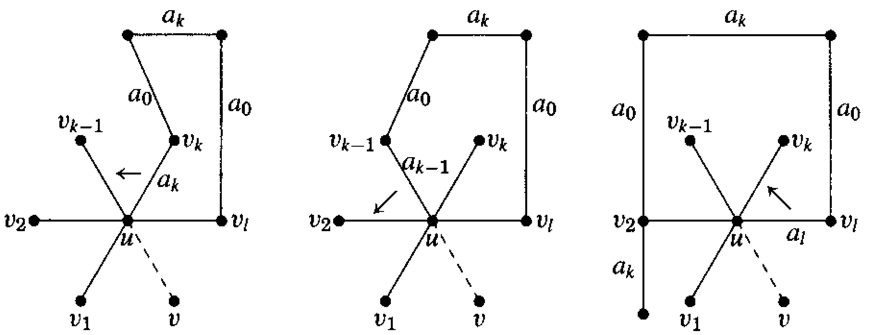
Let P be the maximal alternating path of edges colored a_0 and a_k that begins at v_l along color a_0. There is only one such path, because each vertex has at most one incident edge in each color (we ignore edges not yet colored). To complete the augmentation, we will interchange colors a_0 and a_k on P and downshift from an appropriate neighbor of u, depending on where P goes.

If P reaches v_k, then it arrives at v_k along an edge with color a_0, follows $v_k u$ in color a_k, and stops at u, which lacks color a_0. In this case, we downshift from v_k and switch colors on P (left picture below).

If P reaches v_{k-1}, then it reaches v_{k-1} on color a_0 and stops there, because a_k does not appear at v_{k-1}. In this case, we downshift from v_{k-1}, give color a_0 to uv_{k-1}, and switch colors on P (middle picture).

If P does not reach v_k or v_{k-1}, then it ends at some vertex outside $\{u, v_l, v_k, v_{k-1}\}$. In this case, we downshift from v_l, give color a_0 to uv_l, and switch colors on P (rightmost picture).

In each case, the changes described yield a proper $\Delta(G) + 1$-edge-coloring of $G' + uv$, so we have completed the desired augmentation. ■



P reaches v_k $\qquad$ P reaches v_{k-1} $\qquad$ otherwise

For simple graphs, we now have only two possibilities for χ'.

7.1.11. Definition. A simple graph G is **Class 1** if $\chi'(G) = \Delta(G)$. It is **Class 2** if $\chi'(G) = \Delta(G) + 1$.

Determining whether a graph is Class 1 or Class 2 is generally hard (Holyer [1981]; see Appendix B). Thus we seek conditions that forbid or guarantee $\Delta(G)$-edge-colorability. Examples of such conditions include Exercises 24–27.

7.1.12.* Remark. There is an obvious necessary condition for a graph to be Class 1 that is conjectured to be sufficient when $\Delta(G) > \frac{3}{10}n(G)$. Part (a) of Exercise 27 observes that a subgraph of G with odd order is an obstruction to $\Delta(G)$-edge-colorability if it has too many edges. A subgraph H of a simple graph G is an **overfull subgraph** if $n(H)$ is odd and $2e(H)/(n(H) - 1) > \Delta(G)$.

The **Overfull Conjecture** (Chetwynd–Hilton [1986]—see also Hilton [1989]) states that if $\Delta(G) > n(G)/3$, then a simple graph G is Class 1 if and only if G has no overfull subgraph. The Petersen graph with a vertex deleted shows that the condition is not sufficient when $\Delta(G) = n(G)/3$ (Exercise 28).

The Overfull Conjecture implies the **1-factorization Conjecture**: If $r \ge m$ (or $r \ge m - 1$ if m is even), then every r-regular simple graph of order $2m$ is Class 1. This also is sharp (Exercise 29).

The conclusions of the two conjectures hold when $\Delta(G)$ is large enough (Chetwynd–Hilton [1989], Niessen–Volkmann [1990], Perkovic–Reed [1997], Plantholt [2001]. ∎

When G has multiple edges, $\chi'(G) \le \lfloor 3\Delta(G)/2 \rfloor$ (Shannon [1949]) and $\chi'(G) \le \Delta(G) + \mu(G)$ (Vizing [1964, 1965], Gupta [1966]) These bounds follow (Exercise 35) from that of Andersen [1977] and Goldberg [1977, 1984]:

$$\chi'(G) \le \max\{\Delta(G), \max_{\mathbf{P}} \lfloor \tfrac{1}{2}(d(x) + \mu(xy) + \mu(yz) + d(z)) \rfloor\}$$

where $\mathbf{P} = \{x, y, z \in V(G)\colon y \in N(x) \cap N(z)\}$. Proving this bound uses the methods of Theorem 7.1.10 plus counting arguments. To illustrate the use of counting arguments, we prove Shannon's Theorem from that of Vizing and Gupta.

7.1.13.* Theorem. (Shannon [1949]) If G is a graph, then $\chi'(G) \le \frac{3}{2}\Delta(G)$.

Proof: Let $k = \chi'(G)$, and assume $k \ge (3/2)\Delta(G)$. Let G' be a minimal subgraph of G with $\chi'(G') = k$. Since $k \le \Delta(G') + \mu(G')$ (Vizing–Gupta), we obtain $\mu(G') \ge \Delta(G)/2$. Let e with endpoints x, y be an edge with multiplicity $\mu(G')$.

Let f be a proper $k - 1$-edge-coloring of $G' - e$. In $G' - e$, both x and y have degree at most $\Delta(G) - 1$, so under f at least $(k - 1) - (\Delta(G) - 1)$ colors are missing at x, and similarly at y. No color is missing at both, since G' is not $k - 1$-edge-colorable. Accounting for the $\mu(G') - 1$ colors used on edges with endpoints x, y yields

$$2(k - \Delta(G)) + (\Delta(G)/2) - 1 \le 2(k - \Delta(G)) + \mu(G') - 1 \le k - 1,$$

and hence $k \le (3/2)\Delta(G)$. ∎

Finally, there is a general conjecture analogous to the Overfull Conjecture.

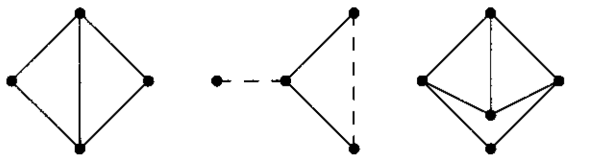
7.1.14.* Conjecture. (Goldberg [1973, 1984], Seymour [1979a])
If $\chi'(G) \ge \Delta(G) + 2$, then $\chi'(G) = \max_{H \subseteq G} \left\lceil \frac{e(H)}{\lfloor n(H)/2 \rfloor} \right\rceil$. ∎

CHARACTERIZATION OF LINE GRAPHS (optional)

Characterizations of line graphs can lead to good algorithms to test whether a graph G is a line graph and, if so, to obtain H such that $L(H) = G$.

7.1.15. Example. To illustrate the ideas, we prove that the rightmost graph below is not the line graph of a simple graph. The kite G (two triangles with a common edge) is the line graph of the paw H (a claw plus an edge). By case analysis, we find that H is the only simple graph whose line graph is G, and the edges becoming the vertices of degree 2 in G must be the dashed edges.

The rightmost graph adds a vertex to G having only the vertices of degree 2 as neighbors. The result is not a line graph, because there is no way to add an edge to H that shares an endpoint with each dashed edge without sharing an endpoint with a solid edge. ■



$G = L(H)$ $\quad$ H $\quad$ not a line graph

Our first characterization encodes the process of taking the line graph. If $G = L(H)$ and H is simple, then each $v \in V(H)$ with $d(v) \geq 2$ generates a clique $Q(v)$ in G corresponding to edges incident to v. These cliques partition $E(G)$. Furthermore, each vertex $e \in V(G)$ belongs only to the cliques generated by the two endpoints of $e \in E(H)$.

For example, when G is the kite, we can partition $E(G)$ into three cliques (a triangle plus two edges), each vertex covered at most twice. These three cliques correspond to the vertices of degree at least 2 in the paw. The rightmost graph above does not have such a partition.

7.1.16. Theorem. (Krausz [1943]) For a simple graph G, there is a solution to $L(H) = G$ if and only if G decomposes into complete subgraphs, with each vertex of G appearing in at most two in the list.

Proof: We argued above that the condition is necessary. Note that when $G = L(H)$, the vertices of G that belong to only one of the cliques we have defined are those corresponding to edges of H that are incident to leaves.

For sufficiency, let $S_1, \ldots, S_k$ be the vertex sets of the specified complete subgraphs. We construct H such that $G = L(H)$. Isolated vertices of G become isolated edges of H, so we may assume that $\delta(G) \geq 1$. Let $v_1, \ldots, v_l$ be the vertices of G (if any) that appear in exactly one of $S_i, \ldots, S_1n$. Give H one vertex for each set in the list $\mathbf{A} = S_1, \ldots, S_k, \{v_1\}, \ldots, \{v_l\}\}$, and let vertices of H be adjacent if the corresponding sets intersect.

Each vertex of G appears in exactly two sets in $\mathbf{A}$, and no two vertices appear in the same two sets. Hence H is a simple graph with one edge for each vertex of G. If vertices are adjacent in G, then they appear together in some S_i, and the corresponding edges of H share the vertex for S_i. Hence $G = L(H)$. ■

Krausz's characterization does not directly yield an efficient test for line graphs, because there are too many possible decompositions to test. The next characterization tests substructures of fixed size and therefore yields a good algorithm. We say that each triangle T in G is odd or even as defined below.

T is **odd** if $|N(v) \cap V(T)|$ is odd for *some* $v \in V(G)$.
T is **even** if $|N(v) \cap V(T)|$ is even for *every* $v \in V(G)$.
An induced kite is a **double triangle**; it consists of two triangles sharing an edge, and the two vertices not in that edge are nonadjacent.

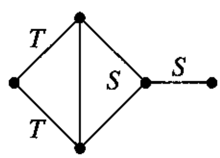
7.1.17. Theorem. (van Rooij and Wilf [1965]) For a simple graph G, there is a solution to $L(H) = G$ if and only if G is claw-free and no double triangle of G has two odd triangles.

Proof: *Necessity.* Suppose that $G = L(H)$. A vertex e in G with neighbors x, y, z corresponds to an edge e in H incident to edges x, y, z. Since e has only two endpoints in H, two of x, y, z are incident at one of them and hence are adjacent in G. This forbids the claw as an induced subgraph of G.

For the other condition, we saw in Example 7.1.15 that the vertices of a double triangle in G must correspond to the edges of a paw in H. In particular, the vertices of one of these triangles in G correspond to the edges of a triangle in H. This triangle must be even, because every edge in H incident to exactly one vertex of a triangle shares an endpoint with exactly two of its edges. Hence for each double triangle in G, at least one of its triangles is even.

Sufficiency. Suppose that G satisfies the specified conditions. We may assume that G is connected; otherwise, we apply the construction to each component. The case where G is claw-free and has a double triangle with both triangles even is very special; there are only three such graphs (Exercise 38). Here we consider only the general case, in which every double triangle of G has exactly one odd triangle.

By Theorem 7.1.16, it suffices to decompose G into complete subgraphs, using each vertex in at most two of them. Let $S_1, \ldots, S_k$ be the maximal complete subgraphs of G that are not even triangles, and let $T_1, \ldots, T_l$ be the edges that belong to one even triangle and no odd triangle. We claim that together these form the desired decomposition **B**.



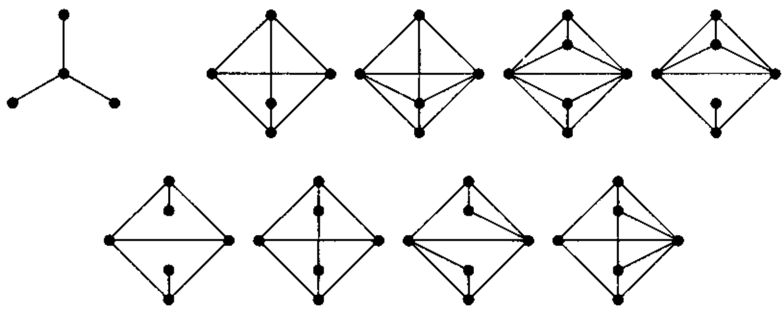
Every edge appears in a maximal complete subgraph, but every triangle in a complete subgraph with more than three vertices is odd. Hence each edge T_j in the list is not in any S_i. Also S_i and $S_{i'}$ share no edge, because G has no double triangles with both triangles odd. Hence the subgraphs in **B** are pairwise edge-disjoint.

If $e \in E(G)$, then e is in some S_i unless the only maximal clique containing e is an even triangle. In this case e is a T_j, since we have forbidden double triangles with both triangles even. Hence **B** is a decomposition.

It remains to show that each $v \in G$ appears in at most two of these subgraphs. Suppose that v belongs to $A, B, C \in$ **B**. Edge-disjointness implies that v has neighbors x, y, z with each belonging to only one of $\{A, B, C\}$. Since G has no induced claw, we may assume that $x \leftrightarrow y$. By edge-disjointness, the triangle vxy cannot belong to a member of **B**. Hence it must be an even triangle. Therefore, z must have exactly one other edge to vxy, say $z \leftrightarrow x$ and $z \not\leftrightarrow y$. But now the same argument shows zvx is an even triangle, and we have a double triangle with both triangles even. ■

Theorem 7.1.17 is close to a forbidden subgraph characterization.

7.1.18. Theorem. (Beineke [1968]) A simple graph G is the line graph of some simple graph if and only if G does not have any of the nine graphs below as an induced subgraph.



Proof: By Theorem 7.1.17, it suffices to show that the eight graphs listed other than $K_{1,3}$ are the vertex-minimal claw-free graphs containing a double triangle with both triangles odd. Each such graph has a double triangle and one or two additional vertices that make the triangles odd by having one or three neighbors in the triangles. The details of showing that this is the full list are requested in Exercise 40. ■

The characterizations in Theorems 7.1.17–7.1.18 yield algorithms to test whether G is a line graph that run in time polynomial in $n(G)$. In fact, there is such an algorithm that runs in linear time (Lehot [1974]) and produces a graph H such that $G = L(H)$ when G is a line graph. This graph H is unique if G has no component that is a triangle (Exercise 39).

EXERCISES

7.1.1. (−) For each graph G below, compute $\chi'(G)$ and draw $L(G)$.

7.1.2. (−) Give an explicit edge-coloring to prove that $\chi'(Q_k) = \Delta(Q_k)$

7.1.3. (−) Determine the edge-chromatic number of $C_n \square K_2$.

7.1.4. (−) Obtain an inequality for $\chi'(G)$ in terms of $e(G)$ and $\alpha'(G)$.

7.1.5. (−) Prove that the Petersen graph is the complement of $L(K_5)$.

7.1.6. (−) Determine the number of triangles in the line graph of the Petersen graph.

7.1.7. (−) Determine whether $\overline{P}_5$ is a line graph. If so, find H such that $L(H) = \overline{P}_5$.

7.1.8. (−) Prove that $L(K_{m,n}) \cong K_m \square K_n$.

• • • • •

7.1.9. Let G be a simple graph. Prove that vertices form a clique in $L(G)$ if and only if the corresponding edges in G have one common endpoint or form a triangle. (Comment: Thus $\omega(L(G)) = \Delta(G)$ unless $\Delta(G) = 2$ and some component of G is a triangle.)

7.1.10. Let G be a simple graph without isolated vertices. Prove that if $L(G)$ is connected and regular, then either G is regular or G is a bipartite graph in which vertices of the same partite set have the same degree. (Ray-Chaudhuri [1967])

7.1.11. (!) Let G be a simple graph.
a) Prove that the number of edges in $L(G)$ is $\sum_{v \in V(G)} \binom{d(v)}{2}$.
b) Prove that G is isomorphic to $L(G)$ if and only if G is 2-regular.

7.1.12. Let G be a connected simple graph. Use part (a) of Exercise 7.1.11 to determine when $e(L(G)) < e(G)$.

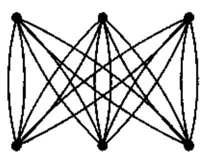
7.1.13. (+) Prove that the graph below is the only simple graph whose line graph is isomorphic to its complement. (Albertson)

7.1.14. (!) Let G be a k-edge-connected simple graph. Prove that $L(G)$ is k-connected and is $2k-2$-edge-connected. (Hint: For a minimum edge cut $[S, \overline{S}]$ in $L(G)$, describe what the cut corresponds to in G and count its edges in terms of the vertices of G.)

7.1.15. (!) Use Tutte's 1-factor Theorem to prove that every connected line graph of even order has a perfect matching. Conclude from this that the edges of a simple connected graph of even size can be partitioned into paths of length 2. (Comment: Exercise 3.3.22 shows that every connected claw-free graph has a perfect matching , but that stronger result is more difficult than this.) (Chartrand–Polimeni–Stewart [1973])

7.1.16. (∗) Let G be a simple graph. Prove that $\gamma(L(G)) \ge \gamma(G)$, where $\gamma(G)$ denotes the genus of G (Definition 6.3.20). (D. Greenwell)

7.1.17. Compute the number of proper 6-edge-colorings of the graph below.



7.1.18. (!) Give an explicit edge-coloring to prove that $\chi'(K_{r,s}) = \Delta(K_{r,s})$.

7.1.19. (!) Prove that for every simple bipartite graph G, there is a $\Delta(G)$-regular simple bipartite graph H that contains G.

7.1.20. (!) Let D be a digraph (loops allowed) such that $d^+(v) \le d$ and $d^-(v) \le d$ for all $v \in V(D)$. Prove that $E(D)$ can be colored using at most d colors so that the edges entering each vertex have distinct colors and the edges exiting each vertex have distinct colors. (Hint: Transform the digraph into another object where a known result applies.)

7.1.21. *Algorithmic proof of Theorem 7.1.7.* Let G be a bipartite graph with maximum degree k. Let f be a proper k-edge-coloring of a subgraph H of G. Let uv be an edge not in H. By using a path alternating in two colors, show that f can be altered and then extended to a proper k-edge-coloring of $H + uv$. Conclude that $\chi'(G) = \Delta(G)$.

7.1.22. Use Brooks' Theorem to an appropriate graph to prove that if G is a simple graph with $\Delta(G) = 3$, then G is 4-edge-colorable. (Comment: The result is a special case of Vizing's Theorem; do not use Vizing's Theorem to prove this.)

7.1.23. (+) Let $K(p, q)$ be the complete p-partite graph with q vertices in each partite set. Let $G[H]$ denote the composition operation, in which each vertex of G expands into a copy of H. Note that $K(p, q) = K(p, d)[\overline{K}_{q/d}]$ when d divides q.

a) Show that if G has a decomposition into copies of F, then $G[\overline{K}_m]$ has a decomposition into copies of $F[\overline{K}_m]$. Show also that the relation "G decomposes into spanning copies of F" is transitive.

b) Cliques of even order decompose into 1-factors. Cliques of odd order decompose into spanning cycles. Use these statements and part (a) to prove that $K(p, q)$ decomposes into 1-factors when pq is even. (Hartman [1997])

7.1.24. (!) Let G and H be nontrivial simple graphs. Use Vizing's Theorem to prove that $\chi'(H) = \Delta(H)$ implies $\chi'(G \,\square\, H) = \Delta(G \,\square\, H)$.

7.1.25. *Kotzig's Theorem for cartesian products of simple graphs.*

a) Use Vizing's Theorem to prove that $\chi'(G \,\square\, K_2) = \Delta(G \,\square\, K_2)$.

b) Let G_1, G_2 be edge-disjoint graphs with vertex set V, and let H_1, H_2 be edge-disjoint graphs with vertex set W. Prove that $(G_1 \cup G_2) \square (H_1 \cup H_2) = (G_1 \square H_2) \cup (G_2 \square H_1)$.

c) Use parts (a) and (b) to prove that $\chi'(G \,\square\, H) = \Delta(G \,\square\, H)$ if both G and H have 1-factors. (Comment: As a result, the product of the Petersen graph with itself is Class 1, which does not follow from Exercise 7.1.24. Here neither factor need be Class 1; there G need not have a 1-factor.) (Kotzig [1979], J. George [1991])

7.1.26. (!) Let G be a regular graph with a cut-vertex. Prove that $\chi'(G) > \Delta(G)$.

7.1.27. *Density conditions for $\chi'(G) > \Delta(G)$.*

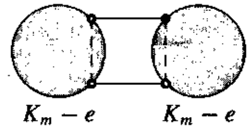
a) Prove that if $n(G) = 2m + 1$ and $e(G) > m \cdot \Delta(G)$, then $\chi'(G) > \Delta(G)$.

b) Prove that if G is obtained from a k-regular graph with $2m+1$ vertices by deleting fewer than $k/2$ edges, then $\chi'(G) > \Delta(G)$.

c) Prove that if G is obtained by subdividing an edge of a regular graph with $2m$ vertices and degree at least 2, then $\chi'(G) > \Delta(G)$.

7.1.28. ($*-$) Prove that the Petersen graph has no overfull subgraph.

7.1.29. Let G be the $m-1$-regular connected graph formed from $2K_m$ by deleting an edge from each component and adding two edges between the components to restore regularity. Prove that G is not 1-factorable if m is odd and greater than 3. (Comment: This shows that the 1-factorization Conjecture (Remark 7.1.12) is sharp.)



7.1.30. (∗!) *Overfull Conjecture ⇒ 1-factorization Conjecture* (Remark 7.1.12).

a) Prove that in a regular graph of even order, an induced subgraph is overfull if and only if the subgraph induced by the other vertices is overfull.

b) Let G be an k-regular graph of order $2m$ having an overfull subgraph. Prove that $k < m$ if m is odd and that $k < m - 1$ if m is even.

7.1.31. Given an edge-coloring of a graph G, let $c(v)$ denote the number of distinct colors appearing on edges incident to v. Among all k-edge-colorings of G, a coloring is **optimal** if it maximizes $\sum_{v \in V(G)} c(v)$.

a) Prove that if no component is an odd cycle, then G has a 2-edge-coloring where both colors appear at each vertex of degree at least 2. (Hint: Use Eulerian circuits.)

b) Let f be an optimal k-edge-coloring of G in which color a appears at least twice at $u \in V(G)$ and color b does not appear at u. Let H be the subgraph of G consisting of edges colored a or b. Prove that the component of H containing u is an odd cycle.

c) Let G be a bipartite graph. Conclude from part (b) that G is $\Delta(G)$-edge-colorable. (Comment: These ideas also lead to a proof of Vizing's Theorem.) (Fournier [1973])

7.1.32. Let G be a bipartite graph with minimum degree k. Prove that G has a k-edge-coloring in which at each vertex v, each color appears $\lceil d(v)/k \rceil$ or $\lfloor d(v)/k \rfloor$ times. (Hint: Use a graph transformation.) (Gupta [1966])

7.1.33. Use Vizing's Theorem to prove that every simple graph with maximum degree Δ has an "equitable" $\Delta + 1$-edge-coloring: a proper edge-coloring with each color used $\lceil e(G)/(\Delta+1) \rceil$ or $\lfloor e(G)/(\Delta+1) \rfloor$ times. (de Werra [1971], McDiarmid [1972])

7.1.34. Use Petersen's Theorem (every $2k$-regular graph has a 2-factor—Theorem 3.3.9) to prove that $\chi'(G) \le 3 \lceil \Delta(G)/2 \rceil$ when G is a loopless graph.

7.1.35. *Bounds on* $\chi'(G)$. Let $\mathbf{P} = \{x, y, z \in V(G)\colon y \in N(x) \cap N(z)$. Prove that the last bound below (Andersen [1977], Goldberg [1977, 1984]) implies the earlier bounds.

$\chi'(G) \le \lfloor 3\Delta(G)/2 \rfloor$. (Shannon [1949])

$\chi'(G) \le \Delta(G) + \mu(G)$. (Vizing [1964, 1965], Gupta [1966])

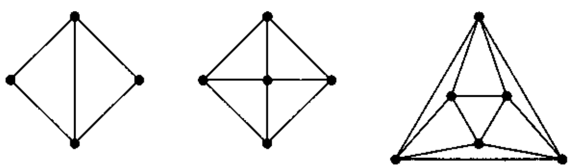
$\chi'(G) \le \max\{\Delta(G), \max_{\mathbf{P}} \left\lfloor \frac{1}{2}(d(x) + d(y) + d(z)) \right\rfloor\}$. (Ore [1967a])

$\chi'(G) \le \max\{\Delta(G), \max_{\mathbf{P}} \left\lfloor \frac{1}{2}(d(x) + \mu(xy) + \mu(yz) + d(z)) \right\rfloor\}$.

7.1.36. (+) For $n \ne 8$, prove that $L(K_n)$ is the only $2n - 4$-regular simple graph of order $\binom{n}{2}$ in which nonadjacent vertices have four common neighbors and adjacent vertices have $n - 2$ common neighbors. (Comment: When $n = 8$, three exceptional graphs satisfy the conditions.) (Chang [1959], Hoffman [1960])

7.1.37. (+) For n, m not both equalling 4, prove that $L(K_{m,n})$ is the only $(n+m-2)$-regular simple graph of order mn in which nonadjacent vertices have two common neighbors, $n\binom{m}{2}$ pairs of adjacent vertices have $m - 2$ common neighbors, and $m\binom{n}{2}$ pairs of adjacent vertices have $n - 2$ common neighbors. (Comment: When $n = m = 4$, there one exceptional graph—Shrikande [1959].) (Moon [1963], Hoffman [1964])

7.1.38. (∗) Let G be a connected, simple, claw-free graph having a double triangle H with each triangle even. Prove that G is one of the three graphs below, and conclude that G is a line graph. (Comment: This completes the proof of Theorem 7.1.17.)



7.1.39. (*) A **Krausz decomposition** of a simple graph H is a partition of $E(H)$ into cliques such that each vertex of H appears in at most two of the cliques.

a) Prove that for a connected simple graph H, two Krausz decompositions of H that have a common clique are identical.

b) Find distinct Krausz decompositions for the graphs in Exercise 7.1.38.

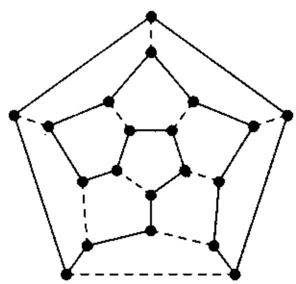
c) Prove that no other connected simple graph except K_3 has two distinct Krausz decompositions (use Exercise 7.1.38 and the proof of Theorem 7.1.17).

d) Conclude that $K_{1,3}$, K_3 is the only pair of nonisomorphic connected simple graphs with isomorphic line graphs. (Whitney [1932a])

7.1.40. (*) Complete the proof of Theorem 7.1.18 by proving that a simple graph with no induced claw has a double triangle with both triangles odd if and only if it contains an induced subgraph among the other eight graphs listed in the theorem statement.

7.2. Hamiltonian Cycles

Studied first by Kirkman [1856], Hamiltonian cycles are named for Sir William Hamilton, who described a game on the graph of the dodecahedron in which one player specifies a 5-vertex path and the other must extend it to a spanning cycle. The game was marketed as the "Traveller's Dodecahedron", a wooden version in which the vertices were named for 20 important cities.



7.2.1. Definition. A **Hamiltonian graph** is a graph with a spanning cycle, also called a **Hamiltonian cycle**.

Until the 1970s, interest in Hamiltonian cycles centered on their relationship to the Four Color Problem (Section 7.3). Later study was stimulated by practical applications and by the issue of complexity (Appendix B).

No easily testable characterization is known for Hamiltonian graphs; we will study necessary conditions and sufficient conditions. Loops and multiple edges are irrelevant; a graph is Hamiltonian if and only if the simple graph obtained by keeping one copy of each non-loop edge is Hamiltonian. Therefore, **in this section we restrict our attention to simple graphs**; this is relevant when discussing conditions involving vertex degrees.

For further material on Hamiltonian cycles, see Chvátal [1985a].

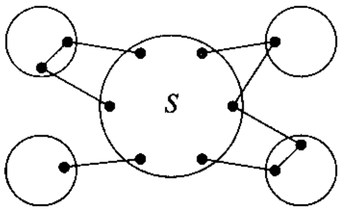
NECESSARY CONDITIONS

Every Hamiltonian graph is 2-connected, because deleting a vertex leaves a subgraph with a spanning path. Bipartite graphs suggest a way to strengthen this necessary condition.

7.2.2. Example. *Bipartite graphs.* A spanning cycle in a bipartite graph visits the two partite sets alternately, so there can be no such cycle unless the partite sets have the same size. Hence $K_{m,n}$ is Hamiltonian only if $m = n$. Alternatively, we can argue that the cycle returns to different vertices of one partite set after each visit to the other partite set. ■

7.2.3. Proposition. If G has a Hamiltonian cycle, then for each nonempty set $S \subseteq V$, the graph $G - S$ has at most $|S|$ components.

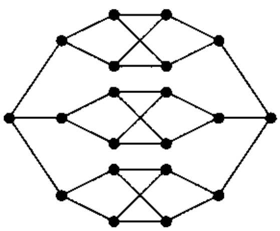
Proof: When leaving a component of $G - S$, a Hamiltonian cycle can go only to S, and the arrivals in S must use distinct vertices of S. Hence S must have at least as many vertices as $G - S$ has components. ■

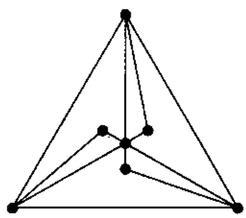


7.2.4. Definition. Let $c(H)$ denote the number of components of a graph H.

Thus the necessary condition is that $c(G - S) \leq |S|$ for all $\emptyset \neq S \subseteq V$. This condition guarantees that G is 2-connected (deleting one vertex leaves at most one component), but it does not guarantee a Hamiltonian cycle.

7.2.5. Example. The graph on the left below is bipartite with partite sets of equal size. However, it fails the necessary condition of Proposition 7.2.3. Hence it is not Hamiltonian.





The graph on the right shows that the necessary condition is not sufficient. This graph satisfies the condition but has no spanning cycle. All edges incident to vertices of degree 2 must be used, but in this graph that requires three edges incident to the central vertex.

The Petersen graph is another non-Hamiltonian graph satisfying the condition. We proved in Example 7.1.9 that $2C_5$ is the only 2-factor of the Petersen graph, so it has no spanning cycle. ■

7.2.6.* Remark. Strengthening a necessary condition may yield a sufficient condition. Perhaps requiring $|S| \geq 2c(G-S)$ for every cutset S would guarantee a spanning cycle. A graph G is **t-tough** if $|S| \geq tc(G-S)$ for every cutset $S \subset V$. The **toughness** of G is the maximum t such that G is t-tough. For example, the toughness of the Petersen graph is 4/3 (Exercise 23).

By Proposition 7.2.3, spanning cycles require toughness at least 1. Chvátal [1974] conjectured that a sufficiently large toughness is sufficient. No value of toughness larger than 1 is necessary, since C_n itself is only 1-tough. For some years it was thought that toughness 2 would be sufficient. Enomoto–Jackson–Katerinis–Saito [1985] constructed non-Hamiltonian graphs with toughness $2-\epsilon$ for each $\epsilon > 0$. Finally, Bauer–Broersma–Veldman [2000] constructed non-Hamiltonian graphs with toughness approaching 9/4. Chvátal's conjecture that some value of toughness suffices remains open. ■

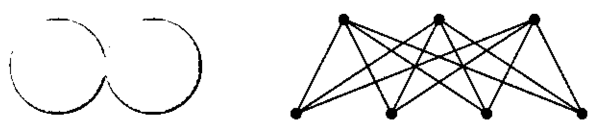
SUFFICIENT CONDITIONS

The number of edges needed to force an n-vertex graph to be Hamiltonian is quite large (Exercises 26–27). Under conditions that "spread out" the edges, we can reduce the number of edges while still guaranteeing Hamiltonian cycles. The simplest such condition is a lower bound on the minimum degree; $\delta(G) \geq n(G)/2$ suffices. We first note that no smaller minimum degree is sufficient.

7.2.7. Example. The graph consisting of cliques of orders $\lfloor (n+1)/2 \rfloor$ and $\lceil (n+1)/2 \rceil$ sharing a vertex has minimum degree $\lfloor (n-1)/2 \rfloor$ but is not Hamiltonian (not even 2-connected).

For odd order, another non-Hamiltonian graph with this minimum degree is the biclique with partite sets of sizes $(n-1)/2$ and $(n+1)/2$.

Proving that $\delta(G) \geq n(G)/2$ forces a spanning cycle thus shows that $\lfloor (n-1)/2 \rfloor$ is the largest value of the minimum degree among non-Hamiltonian graphs with n vertices. ■



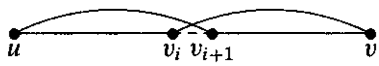
7.2.8. Theorem. (Dirac [1952b]). If G is a simple graph with at least three vertices and $\delta(G) \geq n(G)/2$, then G is Hamiltonian.

Proof: The condition $n(G) \geq 3$ is annoying but must be included, since K_2 is not Hamiltonian but satisfies $\delta(K_2) = n(K_2)/2$.

The proof uses contradiction and extremality. If there is a non-Hamiltonian graph satisfying the hypotheses, then adding edges cannot reduce the minimum degree. Thus we may restrict our attention to maximal non-Hamiltonian graphs with minimum degree at least $n/2$, where "maximal" means that adding any edge joining nonadjacent vertices creates a spanning cycle.

When $u \not\leftrightarrow v$ in G, the maximality of G implies that G has a spanning path $v_1, \ldots, v_n$ from $u = v_1$ to $v = v_n$, because every spanning cycle in $G + uv$ contains the new edge uv. To prove the theorem, it suffices to make a small change in this cycle to avoid using the edge uv; this will build a spanning cycle in G.

If a neighbor of u directly follows a neighbor of v on the path, such as $u \leftrightarrow v_{i+1}$ and $v \leftrightarrow v_i$, then $(u, v_{i+1}, v_{i+2}, \ldots, v, v_i, v_{i-1}, \ldots, v_2)$ is a spanning cycle.



To prove that such a cycle exists, we show that there is a common index in the sets S and T defined by $S = \{i\colon u \leftrightarrow v_{i+1}\}$ and $T = \{i\colon v \leftrightarrow v_i\}$. Summing the sizes of these sets yields

$$|S \cup T| + |S \cap T| = |S| + |T| = d(u) + d(v) \geq n.$$

Neither S nor T contains the index n. Thus $|S \cup T| < n$, and hence $|S \cap T| \geq 1$. We have established a contradiction by finding a spanning cycle in G; hence there is no (maximal) non-Hamiltonian graph satisfying the hypotheses. ■

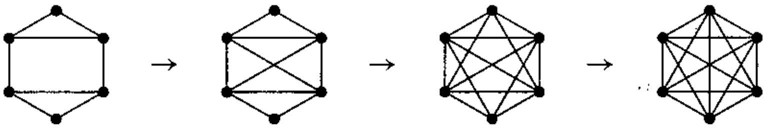
Ore observed that this argument uses $\delta(G) \geq n(G)/2$ only to show that $d(u) + d(v) \geq n$. Therefore, we can weaken the requirement of minimum degree $n/2$ to require only that $d(u) + d(v) \geq n$ whenever $u \not\leftrightarrow v$. We also did not need that G was a maximal non-Hamiltonian graph, only that $G + uv$ was Hamiltonian and thereby provided a spanning u, v-path.

7.2.9. Lemma. (Ore [1960]) Let G be a simple graph. If u, v are distinct nonadjacent vertices of G with $d(u) + d(v) \geq n(G)$, then G is Hamiltonian if and only if $G + uv$ is Hamiltonian.

Proof: One direction is trivial, and the proof of the other direction is the same as for Theorem 7.2.8. ■

Bondy and Chvátal [1976] phrased the essence of Ore's argument in a much more general form that yields sufficient conditions for cycles of length l and other subgraphs. Here we discuss only the application to spanning cycles. Using Lemma 7.2.9 to add edges, we can test whether G is Hamiltonian by testing whether the larger graph is Hamiltonian.

7.2.10. Definition. The **(Hamiltonian) closure** of a graph G, denoted $C(G)$, is the graph with vertex set $V(G)$ obtained from G by iteratively adding edges joining pairs of nonadjacent vertices whose degree sum is at least n, until no such pair remains.



The graph above begins with vertices of degree 2, but its closure is K_6. Ore's Lemma yields the following theorem.

7.2.11. Theorem. (Bondy–Chvátal [1976]) A simple n-vertex graph is Hamiltonian if and only if its closure is Hamiltonian. ■

Fortunately, the closure does not depend on the order in which we choose to add edges when more than one is available.

7.2.12. Lemma. The closure of G is well-defined.

Proof: Let $e_1, \dots, e_r$ and $f_1, \dots, f_s$ be sequences of edges added in forming $C(G)$, the first yielding G_1 and the second G_2. If in either sequence nonadjacent vertices u and v acquire degree summing to at least $n(G)$, then the edge uv must be added before the sequence ends.

Thus f_1, being initially addable to G, must belong to G_1. Similarly, if $f_1, \dots, f_{i-1} \in E(G_1)$, then f_i becomes addable to G_1 and therefore belongs to G_1. Hence neither sequence contains a first edge omitted by the other sequence, and we have $G_1 \subseteq G_2$ and $G_2 \subseteq G_1$. ■

We now have a necessary and sufficient condition to test for Hamiltonian cycles in simple graphs. It doesn't help much, because it requires us to test whether another graph is Hamiltonian! Nevertheless, it does furnish a method for proving sufficient conditions. A condition that forces $C(G)$ to be Hamiltonian also forces a Hamiltonian cycle in G.

For example, the condition may imply $C(G) = K_n$. Chvátal used this method to prove the best possible degree sequence condition for Hamiltonian cycles. Some vertex degrees can be small if others are large enough.

7.2.13. Theorem. (Chvátal [1972]) Let G be a simple graph with vertex degrees $d_1 \le \cdots \le d_n$, where $n \ge 3$. If $i < n/2$ implies that $d_i > i$ or $d_{n-i} \ge n-i$ (**Chvátal's condition**), then G is Hamiltonian.

Proof: Adding edges to form the closure reduces no entry in the degree sequence. Also, G is Hamiltonian if and only if $C(G)$ is Hamiltonian. Thus it suffices to consider the case where $C(G) = G$, which we describe by saying that G is *closed*. In this case, we prove that Chvátal's condition implies that $G = K_n$.

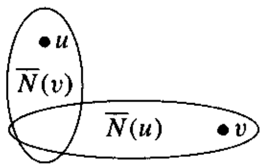
We prove the contrapositive; if G is a closed n-vertex graph that is not a complete graph, then we construct a value of i less than $n/2$ for which Chvátal's condition is violated. Violation means that at least i vertices have degree at most i and at least $n-i$ vertices have degree less than $n-i$.

With $G \neq K_n$, we choose among the pairs of nonadjacent vertices a pair u, v with maximum degree sum. Because G is closed, $u \not\leftrightarrow v$ implies that $d(u) + d(v) < n$. We choose the labels on u, v so that $d(u) \le d(v)$. Since $d(u) + d(v) < n$, we thus have $d(u) < n/2$. Let $i = d(u)$.

We need to find i vertices with degree at most i. Because we chose a nonadjacent pair with maximum degree sum, every vertex of $V - \{v\}$ that is not adjacent to v has degree at most $d(u)$, which equals i. There are $n - 1 - d(v)$ such vertices, and $d(u) + d(v) \le n - 1$ yields $n - 1 - d(v) \ge i$.

We also need $n - i$ vertices with degree less than $n - i$. Every vertex of $V - \{u\}$ that is not adjacent to u has degree at most $d(v)$, and we have $d(v) < n - d(u) = n - i$. There are $n - 1 - d(u)$ such vertices. Since $d(u) \le d(v)$, we can also add u itself to the set of vertices with degree at most $d(v)$. We thus obtain $n - i$ vertices with degree less than $n - i$.

We have proved that $d_i \le i$ and $d_{n-i} < n - i$ for this specially chosen i, which contradicts the hypothesis. ■

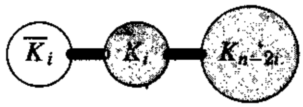


7.2.14. Example. *Non-Hamiltonian graphs with "large" vertex degrees.* Theorem 7.2.13 characterizes the degree sequences of simple graphs that force Hamiltonian cycles. If the degree sequence fails Chvátal's condition at i, then the largest we can make the terms in $d_1, \ldots, d_n$ is

$$\begin{array}{ll} d_j = i & \text{for } j \le i, \\ d_j = n - i - 1 & \text{for } i + 1 \le j \le n - i, \\ d_j = n - 1 & \text{for } j > n - i. \end{array}$$

Let G be a simple graph realizing this degree sequence (if it exists). The i vertices of degree $n - 1$ are adjacent to all others (the central clique in the figure). This already gives i neighbors to the i vertices of degree i, so they form an independent set and have no additional neighbors. With degree $n - i - 1$, each of the remaining $n - 2i$ vertices must be adjacent to all vertices except itself and the independent set. Thus these vertices form a clique. The only possible realization is $(\overline{K}_i + K_{n-2i}) \vee K_i$, shown below.

This graph is not Hamiltonian, because deleting the i vertices of degree $n - 1$ leaves a subgraph with $i + 1$ components. If a simple graph H is non-Hamiltonian and has vertex degrees $d_1' \le \cdots \le d_n'$, then Chvátal's result implies that for some i the graph $(\overline{K}_i + K_{n-2i}) \vee K_i$ with vertex degrees $d_1 \le \cdots \le d_n$ satisfies $d_j \ge d_j'$ for all i. ■



7.2.15. Definition. A **Hamiltonian path** is a spanning path.

Every graph with a spanning cycle has a spanning path, but P_n shows that the converse is not true. We could make arguments like those above to prove sufficient conditions for Hamiltonian paths, but it is easier to use our previous work and prove the new theorem by invoking a theorem about cycles. To do this, we use a standard transformation.

7.2.16. Remark. A graph G has a spanning path if and only if the graph $G \vee K_1$ has a spanning cycle. ■

Remark 7.2.16 applies in several of the exercises. Here we use it to derive the analogue for paths of Chvátal's condition for spanning cycles.

7.2.17. Theorem. Let G be a simple graph with vertex degrees $d_1 \le \cdots \le d_n$. If $i < (n+1)/2$ implies ($d_i \ge i$ or $d_{n+1-i} \ge n-i$), then G has a spanning path.

Proof: Let $G' = G \vee K_1$, let $n' = n+1$, and let $d'_1, \ldots, d'_{n'}$ be the degree sequence of G'. Since a spanning cycle in $G \vee K_1$ becomes a spanning path in G when the extra vertex is deleted, it suffices to show that G' satisfies Chvátal's sufficient condition for Hamiltonian cycles.

Since the new vertex is adjacent to all of $V(G)$, we have $d'_{n'} = n$ and $d'_j = d_j + 1$ for $j < n'$. For $i < n'/2 = (n+1)/2$, the hypothesis on G yields

$$d'_i = d_i + 1 \ge i + 1 > i \quad \text{or} \quad d'_{n'-i} = d_{n+1-i} + 1 \ge n - i + 1 = n' - i.$$

This is precisely Chvátal's sufficient condition, so G' has a spanning cycle, and deleting the extra vertex leaves a spanning path in G. ■

7.2.18.* Remark. The degree requirements can be weakened under conditions such as regularity or high toughness. Every regular simple graph G with vertex degrees at least $n(G)/3$ is Hamiltonian (Jackson [1980]). Only the Petersen graph prevents lowering the threshold to $(n(G)-1)/3$ (Zhu–Liu–Yu [1985], partly simplified in Bondy–Kouider [1988]; see also Exercise 13).

It may be possible to lower the degree condition further when connectivity is high. For example, Tutte [1971] conjectured that every 3-connected 3-regular bipartite graph is Hamiltonian. Horton [1982] found a counterexample with 96 vertices, and the smallest known counterexample has 50 vertices (Georges [1989]), but stronger conditions of this sort may suffice. ■

Our last sufficient condition for Hamiltonian cycles involves connectivity and independence, not degrees. The proof yields a good algorithm that constructs a Hamiltonian cycle or shows that the hypothesis is false.

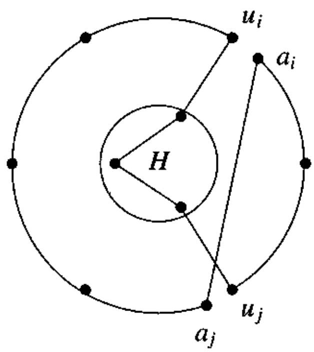
7.2.19. Theorem. (Chvátal–Erdős [1972]) If $\kappa(G) \ge \alpha(G)$, then G has a Hamiltonian cycle (unless $G = K_2$).

Proof: With $G \ne K_2$, the conditions require $\kappa(G) > 1$. Suppose that $\kappa(G) \ge \alpha(G)$. Let $k = \kappa(G)$, and let C be a longest cycle in G. Since $\delta(G) \ge \kappa(G)$, and every graph with $\delta(G) \geq 2$ has a cycle of length at least $\delta(G)+1$ (Proposition 1.2.28), C has at least $k+1$ vertices.

Let H be a component of $G - V(C)$. The cycle C has at least k vertices with edges to H; otherwise, deleting the vertices of C with edges to H contradicts $\kappa(G) = k$. Let $u_1, \dots, u_k$ be k vertices of C with edges to H, in clockwise order.

For $i = 1, \dots, k$, let a_i be the vertex immediately following u_i on C. If any two of these vertices are adjacent, say $a_i \leftrightarrow a_j$, then we construct a longer cycle by using $a_i a_j$, the portions of C from a_i to u_j and a_j to u_i, and a u_i, u_j-path through H (see illustration).

If a_i has a neighbor in H, then we can detour to H between u_i and a_i on C. Thus we also conclude that no a_i has a neighbor in H. Hence $\{a_1, \dots, a_k\}$ plus a vertex of H forms an independent set of size $k+1$. This contradiction implies that C is a Hamiltonian cycle. ∎



7.2.20.* Remark. Most sufficient conditions for Hamiltonian cycles generalize to conditions for long cycles. The **circumference** of a graph is the length of its longest cycle. A weaker form of a sufficient condition for spanning cycles may force a long cycle. Dirac [1952b] proved the first such result: a 2-connected graph with minimum degree k has circumference at least $\min\{n, 2k\}$. Proposition 1.2.28 only guarantees a cycle of length at least $k+1$. Most long-cycle results are more more difficult than the corresponding sufficient conditions for Hamiltonian cycles (see Lemma 8.4.36–Theorem 8.4.37). ∎

CYCLES IN DIRECTED GRAPHS (optional)

The theory of cycles in digraphs is similar to that of cycles in graphs. For a digraph G, let $\delta^-(G) = \min d^-(v)$ and $\delta^+(G) = \min d^+(v)$. The arguments of Chapter 1 using maximal paths guarantee paths of length k and cycles of length $k+1$, where $k = \max\{\delta^-(G), \delta^+(G)\}$.

Every complete graph is Hamiltonian, but orientations of complete graphs are more complicated. The necessary condition of 2-connectedness becomes a necessary condition of strong connectedness for spanning cycles in digraphs. For tournaments, this necessary condition is also sufficient (Exercise 45).