Proposition 15. The Zariski closure of a subset A in $\mathbb{A}^n$ is $\mathcal{Z}(\mathcal{I}(A))$.

Proof: Certainly $A \subseteq \mathcal{Z}(\mathcal{I}(A))$. Suppose V is any algebraic set containing A: $A \subseteq V$. Then $\mathcal{I}(V) \subseteq \mathcal{I}(A)$ and $\mathcal{Z}(\mathcal{I}(A)) \subseteq \mathcal{Z}(\mathcal{I}(V)) = V$, so $\mathcal{Z}(\mathcal{I}(A))$ is the smallest algebraic set containing A.

If $\varphi : V \to W$ is a morphism of algebraic sets, the image $\varphi(V)$ of V need not be an algebraic subset of W, i.e., need not be Zariski closed in W. For example the projection of the hyperbola $V = \mathcal{Z}(xy - 1)$ in $\mathbb{R}^2$ onto the x-axis has image $\mathbb{R}^1 - \{0\}$, which as we have just seen is not an affine algebraic set.

The next result shows that the Zariski closure of the image of a morphism is determined by the kernel of the associated k-algebra homomorphism.

Proposition 16. Suppose $\varphi : V \to W$ is a morphism of algebraic sets and $\widetilde{\varphi} : k[W] \to k[V]$ is the associated k-algebra homomorphism of coordinate rings. Then

(1) The kernel of $\widetilde{\varphi}$ is $\mathcal{I}(\varphi(V))$.

(2) The Zariski closure of $\varphi(V)$ is the zero set in W of $\ker \widetilde{\varphi}$. In particular, the homomorphism $\widetilde{\varphi}$ is injective if and only if $\varphi(V)$ is Zariski dense in W.

Proof: Since $\widetilde{\varphi} = f \circ \varphi$, we have $\widetilde{\varphi}(f) = 0$ if and only if $(f \circ \varphi)(P) = 0$ for all $P \in V$, i.e., $f(Q) = 0$ for all $Q = \varphi(P) \in \varphi(V)$, which is the statement that $f \in \mathcal{I}(\varphi(V))$, proving the first statement. Since the Zariski closure of $\varphi(V)$ is the zero set of $\mathcal{I}(\varphi(V))$ by the previous proposition, the first statement in (2) follows.

If $\widetilde{\varphi}$ is injective then the Zariski closure of $\varphi(V)$ is $\mathcal{Z}(0) = W$ and so $\varphi(V)$ is Zariski dense. Conversely, suppose $\varphi(V)$ is Zariski dense in W, i.e., $\mathcal{Z}(\mathcal{I}(\varphi(V))) = W$. Then $\mathcal{I}(\varphi(V)) = \mathcal{I}(\mathcal{Z}(\mathcal{I}(\varphi(V)))) = \mathcal{I}(W) = 0$ and so $\ker \widetilde{\varphi} = 0$.

By Proposition 16 the ideal of polynomials defining the Zariski closure of the image of a morphism φ is the kernel of the corresponding k-algebra homomorphism $\widetilde{\varphi}$ in Theorem 6. Proposition 8(1) allows us to compute this kernel using Gröbner bases.

Example: (Implicitization)

A morphism $\varphi : \mathbb{A}^n \to \mathbb{A}^m$ is just a map

$$\varphi((a_1, a_2, \dots, a_n)) = (\varphi_1(a_1, a_2, \dots, a_n), \dots, \varphi_m(a_1, a_2, \dots, a_n))$$

where φ_i is a polynomial. If k is an infinite field, then $\mathcal{I}(\mathbb{A}^m)$ and $\mathcal{I}(\mathbb{A}^n)$ are both 0, so we may write $k[\mathbb{A}^m] = k[y_1, \dots, y_m]$ and $k[\mathbb{A}^n] = k[x_1, \dots, x_n]$. The k-algebra homomorphism $\widetilde{\varphi} : k[\mathbb{A}^m] \to k[\mathbb{A}^n]$ corresponding to φ is then defined by mapping y_i to $\varphi_i = \varphi_i(x_1, \dots, x_n)$. The image $\varphi(\mathbb{A}^n)$ consists of the set of points $(b_1, \dots, b_m)$ with

$$\begin{aligned} b_1 &= \varphi_1(a_1, a_2, \dots, a_n) \\ b_2 &= \varphi_2(a_1, a_2, \dots, a_n) \\ &\vdots \\ b_m &= \varphi_m(a_1, a_2, \dots, a_n) \end{aligned}$$

where $a_i \in k$. This is the collection of points in $\mathbb{A}^m$ *parametrized* by the functions $\varphi_1, \dots, \varphi_m$ (with the a_i as parameters). In general such a parametrized collection of points is not an algebraic set. Finding the equations for the smallest algebraic set containing these points is referred to as *implicitization*, since it amounts to finding a ('smallest') collection of equations satisfied by the b_i (the 'implicit' algebraic relations).

By Proposition 16, this algebraic set is the Zariski closure of $\varphi(\mathbb{A}^n)$ and is the zero set of $\ker\widetilde{\varphi}$. By Proposition 8 this kernel is given by $\mathcal{A} \cap k[y_1, \ldots, y_m]$, where $\mathcal{A}$ is the ideal in $k[x_1, \ldots, x_n, y_1, \ldots, y_m]$ generated by the polynomials $y_1 - \varphi_1, \ldots, y_m - \varphi_m$. If we compute the reduced Gröbner basis G for $\mathcal{A}$ with respect to the lexicographic monomial ordering $x_1 > \cdots > x_n > y_1 > \cdots > y_m$, then the polynomials of G lying in $k[y_1, \ldots, y_m]$ generate $\ker\tilde{\pi}$. The zero set of these polynomials defines the Zariski closure of $\varphi(\mathbb{A}^n)$ and therefore give the implicitization.

For an explicit example, consider the points $A = \{(a^2, a^3) \mid a \in \mathbb{R}\}$ in $\mathbb{R}^2$. Using coordinates x, y for $\mathbb{R}^2$ and t for $\mathbb{R}^1$, the ideal $\mathcal{A}$ in $\mathbb{R}[x, y, z, t]$ is $(x - t^2, y - t^3)$. The only element of the reduced Gröbner basis for $\mathcal{A}$ for the ordering $t > x > y$ lying in $\mathbb{R}[x, y]$ is $x^3 - y^2$, so $\mathcal{Z}(x^3 - y^2)$ is the smallest algebraic set in $\mathbb{R}^2$ containing A.

Example: (Projections of Algebraic Sets)

Suppose $V \subseteq \mathbb{A}^n$ is an algebraic set and $m < n$. Let $\pi : V \to \mathbb{A}^m$ be the morphism projecting onto the first m coordinates:

$$\pi((a_1, a_2, \ldots, a_n)) = (a_1, a_2, \ldots, a_m).$$

If we use coordinates $x_1, \ldots, x_n$ in $k[V]$ and coordinates $y_1, \ldots, y_m$ in $k[\mathbb{A}^m]$, the k-algebra homomorphism corresponding to π is given by the map

$$\begin{aligned} \tilde{\pi} : k[y_1, \ldots, y_m] &\longrightarrow k[x_1, \ldots, x_n]/\mathcal{I}(V) \\ y_i &\longmapsto x_i. \end{aligned}$$

Suppose $V = \mathcal{Z}(I)$ and $I = (f_1, \ldots, f_s)$. The Zariski closure of $\pi(V)$ is the zero set of $\ker\tilde{\pi} = \mathcal{A} \cap k[y_1, \ldots, y_m]$ where $\mathcal{A}$ is the ideal in $k[x_1, \ldots, x_n, y_1, \ldots, y_m]$ generated by the polynomials $y_1 - x_1, \ldots, y_m - x_m$ together with a set of generators for $\mathcal{I}(V)$. The polynomials involving only $y_1, \ldots, y_m$ in the reduced Gröbner basis G for $\mathcal{A}$ with respect to the lexicographic monomial ordering $x_1 > \cdots > x_n > y_1 > \cdots > y_m$ are generators for the Zariski closure of $\pi(V)$.

If k is algebraically closed we can actually do better with the help of the Nullstellensatz, which gives $\mathcal{I}(V) = \operatorname{rad} I$. Then it is straightforward to see that we obtain the same zero set if in the ideal $\mathcal{A}$ we replace the generators for $\mathcal{I}(V)$ by the generators $f_1, \ldots, f_s$ of I (cf. Exercise 46).

For an explicit example, consider projection onto the first two coordinates of $V = \mathcal{Z}(xy - z^2, xz - y, x^2 - z)$ in $\mathbb{C}^3$. Using u, v as coordinates in $\mathbb{C}^2$, we find the reduced Gröbner basis G for the ideal $(u - x, v - y, xy - z^2, xz - y, x^2 - z)$ for the ordering $x > y > z > u > v$ contains only the polynomial $u^3 - v$ in $\mathbb{C}[u, v]$. The smallest algebraic set containing $\pi(V)$ is then the cubic $v = u^3$.

Affine Varieties

We next consider the question of whether an algebraic set can be decomposed into smaller algebraic sets and the corresponding algebraic formulation in terms of its coordinate ring.

Definition. A nonempty affine algebraic set V is called *irreducible* if it cannot be written as $V = V_1 \cup V_2$, where V_1 and V_2 are proper algebraic sets in V. An irreducible affine algebraic set is called an affine *variety*.

Equivalently, an algebraic set (which is a closed set in the Zariski topology) is irreducible if it cannot be written as the union of two proper, closed subsets.

Proposition 17.
(1) The affine algebraic set V is irreducible if and only if $\mathcal{I}(V)$ is a prime ideal.
(2) Every nonempty affine algebraic set V may be written uniquely in the form

$$V = V_1 \cup V_2 \cup \cdots \cup V_q$$

where each V_i is irreducible, and $V_i \not\subseteq V_j$ for all $j \neq i$ (i.e., the decomposition is "minimal" or "irredundant").

Proof: Let $I = \mathcal{I}(V)$ and suppose first that $V = V_1 \cup V_2$ is reducible, where V_1 and V_2 are proper closed subsets. Since $V_1 \neq V$, there is some function f_1 that vanishes on V_1 but not on V, i.e., $f_1 \in \mathcal{I}(V_1) - I$. Similarly, there is a function $f_2 \in \mathcal{I}(V_2) - I$. Then $f_1 f_2$ vanishes on $V_1 \cup V_2 = V$, so $f_1 f_2 \in I$ which shows that I is not a prime ideal. Conversely, if I is not a prime ideal, there exists $f_1, f_2 \in k[\mathbb{A}^n]$ such that $f_1 f_2 \in I$ but neither f_1 nor f_2 belongs to I. Let $V_1 = \mathcal{Z}(f_1) \cap V$ and $V_2 = \mathcal{Z}(f_2) \cap V$. Since the intersection of closed sets is closed, V_1 and V_2 are algebraic sets. Since neither f_1 nor f_2 vanishes on V, both V_1 and V_2 are proper subsets of V. Because $f_1 f_2 \in I$, $V \subseteq \mathcal{Z}(f_1 f_2) = \mathcal{Z}(f_1) \cup \mathcal{Z}(f_2)$, and so V is reducible. This proves (1).

To prove (2), let $\mathcal{S}$ be the collection of nonempty algebraic sets that cannot be written as a finite union of irreducible algebraic sets, and suppose by way of contradiction that $\mathcal{S} \neq \emptyset$. Let I_0 be a maximal element of the corresponding set of ideals, $\{\mathcal{I}(V) \mid V \in \mathcal{S}\}$, which exists (by Theorem 2) since $k[\mathbb{A}^n]$ is Noetherian. Then $V_0 = \mathcal{Z}(I_0)$ is a *minimal* element of $\mathcal{S}$. Since $V_0 \in \mathcal{S}$, it cannot be irreducible by the definition of $\mathcal{S}$. On the other hand, if $V_0 = V_1 \cup V_2$ for some proper, closed subsets V_1, V_2 of V_0, then by the minimality of V_0 both V_1 and V_2 may be written as finite unions of irreducible algebraic sets. Then V_0 may be written as a finite union of irreducible algebraic sets, a contradiction. This proves $\mathcal{S} = \emptyset$, i.e., every affine algebraic set has a decomposition into affine varieties.

To prove uniqueness, suppose V has two decompositions into affine varieties (where redundant terms have been removed from each decomposition):

$$V = V_1 \cup V_2 \cup \cdots \cup V_r = U_1 \cup U_2 \cup \cdots \cup U_s.$$

Then V_1 is contained in the union of the U_i. Since $V_1 \cap U_i$ is an algebraic set for each i, we obtain a decomposition of V_1 into algebraic subsets:

$$V_1 = (V_1 \cap U_1) \cup (V_1 \cap U_2) \cup \cdots \cup (V_1 \cap U_s).$$

Since V_1 is irreducible, we must have $V_1 = V_1 \cap U_j$ for some j, i.e., $V_1 \subseteq U_j$. By the symmetric argument we have $U_j \subseteq V_{j'}$ for some j'. Thus $V_1 \subseteq V_{j'}$, so $j' = 1$ and $V_1 = U_j$. Applying a similar argument for each V_i it follows that $r = s$ and that $\{V_1, \ldots, V_r\} = \{U_1, \ldots, U_s\}$. This completes the proof.

Corollary 18. An affine algebraic set V is a variety if and only if its coordinate ring $k[V]$ is an integral domain.

Proof: This follows immediately since $\mathcal{I}(V)$ is a prime ideal if and only if the quotient $k[V] = k[\mathbb{A}^n]/\mathcal{I}(V)$ is an integral domain (Proposition 13 of Chapter 7).