1

The Theorem of Pythagoras

1.1 Arithmetic and Geometry

If there is one theorem that is known to all mathematically educated people, it is surely the theorem of Pythagoras. It will be recalled as a property of right-angled triangles: the square of the hypotenuse equals the sum of the squares of the other two sides (Figure 1.1). The "sum" is of course the sum of areas and the area of a square of side l is l^2, which is why we call it "l squared." Thus Pythagoras' theorem can also be expressed by the equation

$$a^2 + b^2 = c^2, \tag{1}$$

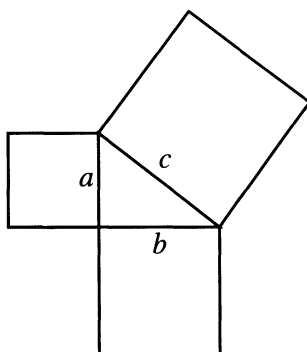where a, b, c are the lengths shown in Figure 1.1.

Figure 1.1: Pythagoras' theorem

1

Conversely, a solution of (1) by positive numbers a, b, c can be realized by a right-angled triangle with sides a, b and hypotenuse c. It is clear that we can draw perpendicular sides a, b for any given positive numbers a, b, and then the hypotenuse c must be a solution of (1) to satisfy Pythagoras' theorem. This converse view of Pythagoras' theorem becomes interesting when we notice that (1) has some very simple solutions. For example,

$$(a,b,c) = (3,4,5), \quad (3^2 + 4^2 = 9 + 16 = 25 = 5^2),$$
$$(a,b,c) = (5,12,13), \quad (5^2 + 12^2 = 25 + 144 = 169 = 13^2).$$

It is thought that in ancient times such solutions may have been used for the construction of right angles. For example, by stretching a closed rope with 12 equally spaced knots one can obtain a $(3,4,5)$ triangle with right angle between the sides 3, 4, as seen in Figure 1.2.
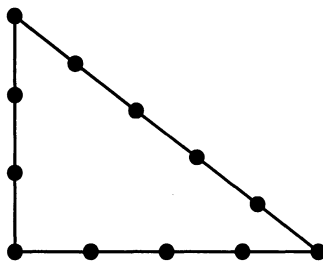

Figure 1.2: Right angle by rope stretching

Whether or not this is a practical method for constructing right angles, the very existence of a geometrical interpretation of a purely arithmetical fact like

$$3^2 + 4^2 = 5^2$$

is quite wonderful. At first sight, arithmetic and geometry seem to be completely unrelated realms. Arithmetic is based on counting, the epitome of a *discrete* (or *digital*) process. The facts of arithmetic can be clearly understood as outcomes of certain counting processes, and one does not expect them to have any meaning beyond this. Geometry, on the other hand, involves *continuous* rather than discrete objects, such as lines, curves, and surfaces. Continuous objects cannot be built from simple elements by discrete processes, and one expects to *see* geometrical facts rather than arrive at them by calculation.

Pythagoras' theorem was the first hint of a hidden, deeper relationship between arithmetic and geometry, and it has continued to hold a key po-

sition between these two realms throughout the history of mathematics. This has sometimes been a position of cooperation and sometimes one of conflict, as followed the discovery that $\sqrt{2}$ is irrational (see Section 1.5). It is often the case that new ideas emerge from such areas of tension, resolving the conflict and allowing previously irreconcilable ideas to interact fruitfully. The tension between arithmetic and geometry is, without doubt, the most profound in mathematics, and it has led to the most profound theorems. Since Pythagoras' theorem is the first of these, and the most influential, it is a fitting subject for our first chapter.

1.2   Pythagorean Triples

Pythagoras lived around 500 BCE (see Section 1.7), but the story of Pythagoras' theorem begins long before that, at least as far back as 1800 BCE in Babylonia. The evidence is a clay tablet, known as Plimpton 322, which systematically lists a large number of integer pairs (a,c) for which there is an integer b satisfying

$$a^2 + b^2 = c^2. \tag{1}$$

A translation of this tablet, together with its interpretation and historical background, was first published by Neugebauer and Sachs (1945) [for a more recent discussion, see van der Waerden (1983), p. 2]. Integer triples (a,b,c) satisfying (1)—for example, $(3,4,5)$, $(5,12,13)$, $(8,15,17)$—are now known as *Pythagorean triples*. Presumably the Babylonians were interested in them because of their interpretation as sides of right-angled triangles, though this is not known for certain. At any rate, the problem of finding Pythagorean triples was considered interesting in other ancient civilizations that are known to have possessed Pythagoras' theorem; van der Waerden (1983) gives examples from China (between 200 BCE and 220 CE) and India (between 500 and 200 BCE). The most complete understanding of the problem in ancient times was achieved in Greek mathematics, between Euclid (around 300 BCE) and Diophantus (around 250 CE).

We now know that the general formula for generating Pythagorean triples is

$$a = (p^2 - q^2)r, \qquad b = 2qpr, \qquad c = (p^2 + q^2)r.$$

It is easy to see that $a^2 + b^2 = c^2$ when a, b, c are given by these formulas, and of course a, b, c will be integers if p, q, r are. Even though the Babylonians did not have the advantage of our algebraic notation, it is plausible

that this formula, or the special case

$$a = p^2 - q^2, \qquad b = 2pq, \qquad c = p^2 + q^2$$

(which gives all solutions a, b, c, without common divisor) was the basis for the triples they listed. Less general formulas have been attributed to Pythagoras himself (around 500 BCE) and Plato [see Heath (1921), Vol. 1, pp. 80–81]; a solution equivalent to the general formula is given in Euclid's *Elements*, Book X (lemma following Prop. 28). As far as we know, this is the first statement of the general solution and the first proof that it is general. Euclid's proof is essentially arithmetical, as one would expect since the problem seems to belong to arithmetic.

However, there is a far more striking solution, which uses the geometric interpretation of Pythagorean triples. This emerges from the work of Diophantus, and it is described in the next section.

EXERCISES

The integer pairs (a,c) in Plimpton 322 are

a	c
119	169
3367	4825
4601	6649
12709	18541
65	97
319	481
2291	3541
799	1249
481	769
4961	8161
45	75
1679	2929
161	289
1771	3229
56	106

Figure 1.3: Pairs in Plimpton 322

1.2.1 For each pair (a,c) in the table, compute $c^2 - a^2$, and confirm that it is a perfect square, b^2. (Computer assistance is recommended.)

You should notice that in most cases b is a "rounder" number than a or c.