6
Complex Numbers

Negative numbers are required to solve the equation $x + 2 = 1$. To solve $2x = 3$, we need the rationals. The equation $x^2 = 2$ has an irrational solution. Finally, the equation $x^2 = -1$ has an *imaginary root* called i.

Numbers of the form $a + bi$ where a and b are real, are called *complex numbers.* Complex numbers with $a = 0$ are also called *imaginary.* The complex number $a + bi$ is often associated with the point (a, b) in the Cartesian plane. The absolute value of a complex number is just its distance $\sqrt{a^2 + b^2}$ from the origin. The angle θ measured counterclockwise from the positive x axis to the line joining (a, b) to the origin is called the *angle* of the complex number $a + bi$; thus $\tan\theta = b/a$.

Complex numbers were introduced by Girolamo Cardano (1501–1576), who used them in his *Ars Magna* (1545) to solve cubic equations. Cardano tells us to multiply $5 + \sqrt{-15}$ by $5 - \sqrt{-15}$, 'putting aside the mental tortures involved' (see T. Richard Witmer's translation of the *Ars Magna*, p. 219).

There are many ways to define the complex numbers, all of them being essentially equivalent. For example, we can define them as ordered pairs of real numbers subject to the multiplication rule

$$(a, b) \cdot (c, d) = (ac - bd, ad + bc).$$

Here $(1, 0)$ plays the role of 1 and $(0, 1)$ the role of i.

Another way to introduce the 'field' of complex numbers is to say that it is the quotient ring $\mathbf{R}[x]/(x^2 + 1)$. The elements of this ring are equivalence classes of polynomials with real coefficients, where two polynomials are said to be equivalent if they differ by a multiple of $x^2 + 1$. Here the role of i is played by the equivalence class whose representative is the polynomial x.

Yet another way to define complex numbers is to say that they are 2×2 matrices with real entries of the form

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

It is easy to see that the set of matrices of this form is closed under addition and multiplication. Here the role of 1 is played by the identity matrix and the role of i is played by

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The advantage of defining complex numbers in this way is that one can use the arithmetic of matrices to give a quick proof of the fact that the complex numbers form a ring. The commutative law of multiplication still needs checking, since it does not hold for matrices in general. Moreover, the inverse of a nonzero matrix of the form

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

is just the matrix

$$\begin{pmatrix} a/k & -b/k \\ b/k & a/k \end{pmatrix},$$

where $k = a^2 + b^2$ is the determinant of the former.

Since a real number is not usually thought of as a matrix, one might well ask how the reals relate to the complex numbers if the latter are conceived as matrices. Well, the mapping $h : \mathbf{R} \to \mathbf{C}$, where

$$h(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

is a 1-to-1 homomorphism. Hence $h(\mathbf{R})$ is an isomorphic image of $\mathbf{R}$ and may be identified with $\mathbf{R}$, and we may say that $\mathbf{R}$ is a subset of $\mathbf{C}$.

Let u and v be any complex numbers. Then

$$|u + v| \leq |u| + |v|,$$

$$|uv| = |u|\,|v|.$$

If we think of the complex numbers as points in the Cartesian plane, the above inequality is just the triangle inequality of Euclid (Book I, 20). The above equality is equivalent to the algebraic identity

$$(ac - bd)^2 + (ad + bc)^2 = (a^2 + b^2)(c^2 + d^2).$$

This identity was known to al-Khazin in 950 AD. (See Part I, Chapter 23.)

Associating $a + bi$ with the point (a, b) in the Cartesian plane, let r be its distance from the origin, and let θ be its angle. Then

$$a + bi = r(\cos\theta + i\sin\theta).$$

If $a' + b'i$ has absolute value r' and angle θ', we may exploit the well-known addition formulas for trigonometric functions to obtain

$$(a + bi)(a' + b'i) = rr'(\cos(\theta + \theta') + i\sin(\theta + \theta')).$$

By induction on the natural number m, this formula leads to the equation

$$(r(\cos\theta + i\sin\theta))^m = r^m(\cos(m\theta) + i\sin(m\theta)),$$

which is known as de Moivre's theorem after Abraham de Moivre (1667–1754), who was the first to make use of it (Part I, Chapter 30).

Exercises

1. Using the matrix definition of complex numbers, verify that $\mathbf{C}$ is a field.

2. What is the multiplicative inverse of $5 + \sqrt{-15}$?

3. Give an algebraic proof of the triangle inequality.

4. Prove the theorem of de Moivre.