Quite frequently, (39) is taken as the starting point of the theory of the logarithm and the exponential function. Writing $u = E(x)$, $v = E(y)$, (26) gives

$$L(uv) = L(E(x) \cdot E(y)) = L(E(x+y)) = x + y,$$

so that

$$L(uv) = L(u) + L(v) \qquad (u > 0, v > 0). \tag{40}$$

This shows that L has the familiar property which makes logarithms useful tools for computation. The customary notation for $L(x)$ is of course $\log x$.

As to the behavior of $\log x$ as $x \to +\infty$ and as $x \to 0$, Theorem 8.6(e) shows that

$$\begin{aligned} \log x &\to +\infty \qquad \text{as } x \to +\infty, \\ \log x &\to -\infty \qquad \text{as } x \to 0. \end{aligned}$$

It is easily seen that

$$x^n = E(nL(x)) \tag{41}$$

if $x > 0$ and n is an integer. Similarly, if m is a positive integer, we have

$$x^{1/m} = E\left(\frac{1}{m} L(x)\right), \tag{42}$$

since each term of (42), when raised to the mth power, yields the corresponding term of (36). Combining (41) and (42), we obtain

$$x^\alpha = E(\alpha L(x)) = e^{\alpha \log x} \tag{43}$$

for any rational α.

We now define x^α, for any real α and any $x > 0$, by (43). The continuity and monotonicity of E and L show that this definition leads to the same result as the previously suggested one. The facts stated in Exercise 6 of Chap. 1, are trivial consequences of (43).

If we differentiate (43), we obtain, by Theorem 5.5,

$$(x^\alpha)' = E(\alpha L(x)) \cdot \frac{\alpha}{x} = \alpha x^{\alpha - 1}. \tag{44}$$

Note that we have previously used (44) only for integral values of α, in which case (44) follows easily from Theorem 5.3(b). To prove (44) directly from the definition of the derivative, if x^α is defined by (33) and α is irrational, is quite troublesome.

The well-known integration formula for x^α follows from (44) if $\alpha \neq -1$, and from (38) if $\alpha = -1$. We wish to demonstrate one more property of $\log x$, namely,

$$\lim_{x \to +\infty} x^{-\alpha} \log x = 0 \tag{45}$$

for every $\alpha > 0$. That is, $\log x \to +\infty$ "slower" than any positive power of x, as $x \to +\infty$.

For if $0 < \varepsilon < \alpha$, and $x > 1$, then

$$\begin{aligned} x^{-\alpha} \log x &= x^{-\alpha} \int_1^x t^{-1}\, dt < x^{-\alpha} \int_1^x t^{\varepsilon-1}\, dt \\ &= x^{-\alpha} \cdot \frac{x^{\varepsilon} - 1}{\varepsilon} < \frac{x^{\varepsilon-\alpha}}{\varepsilon}, \end{aligned}$$

and (45) follows. We could also have used Theorem 8.6(f) to derive (45).

THE TRIGONOMETRIC FUNCTIONS

Let us define

$$(46) \qquad C(x) = \frac{1}{2}[E(ix) + E(-ix)], \qquad S(x) = \frac{1}{2i}[E(ix) - E(-ix)].$$

We shall show that $C(x)$ and $S(x)$ coincide with the functions $\cos x$ and $\sin x$, whose definition is usually based on geometric considerations. By (25), $E(\bar{z}) = \overline{E(z)}$. Hence (46) shows that $C(x)$ and $S(x)$ are real for real x. Also,

$$(47) \qquad E(ix) = C(x) + iS(x).$$

Thus $C(x)$ and $S(x)$ are the real and imaginary parts, respectively, of $E(ix)$, if x is real. By (27),

$$|E(ix)|^2 = E(ix)\overline{E(ix)} = E(ix)E(-ix) = 1,$$

so that

$$(48) \qquad |E(ix)| = 1 \qquad (x \text{ real}).$$

From (46) we can read off that $C(0) = 1$, $S(0) = 0$, and (28) shows that

$$(49) \qquad C'(x) = -S(x), \qquad S'(x) = C(x).$$

We assert that there exist positive numbers x such that $C(x) = 0$. For suppose this is not so. Since $C(0) = 1$, it then follows that $C(x) > 0$ for all $x > 0$, hence $S'(x) > 0$, by (49), hence S is strictly increasing; and since $S(0) = 0$, we have $S(x) > 0$ if $x > 0$. Hence if $0 < x < y$, we have

$$(50) \qquad S(x)(y - x) < \int_x^y S(t)\, dt = C(x) - C(y) \leq 2.$$

The last inequality follows from (48) and (47). Since $S(x) > 0$, (50) cannot be true for large y, and we have a contradiction.

Let x_0 be the smallest positive number such that $C(x_0) = 0$. This exists, since the set of zeros of a continuous function is closed, and $C(0) \neq 0$. We define the number π by

$$\pi = 2x_0. \tag{51}$$

Then $C(\pi/2) = 0$, and (48) shows that $S(\pi/2) = \pm 1$. Since $C(x) > 0$ in $(0, \pi/2)$, S is increasing in $(0, \pi/2)$; hence $S(\pi/2) = 1$. Thus

$$E\left(\frac{\pi i}{2}\right) = i,$$

and the addition formula gives

$$E(\pi i) = -1, \qquad E(2\pi i) = 1; \tag{52}$$

hence

$$E(z + 2\pi i) = E(z) \qquad (z \text{ complex}). \tag{53}$$

8.7 Theorem

(*a*) *The function E is periodic, with period $2\pi i$.*
(*b*) *The functions C and S are periodic, with period 2π.*
(*c*) *If $0 < t < 2\pi$, then $E(it) \neq 1$.*
(*d*) *If z is a complex number with $|z| = 1$, there is a unique t in $[0, 2\pi)$ such that $E(it) = z$.*

Proof By (53), (*a*) holds; and (*b*) follows from (*a*) and (46).

Suppose $0 < t < \pi/2$ and $E(it) = x + iy$, with x, y real. Our preceding work shows that $0 < x < 1$, $0 < y < 1$. Note that

$$E(4it) = (x + iy)^4 = x^4 - 6x^2y^2 + y^4 + 4ixy(x^2 - y^2).$$

If $E(4it)$ is real, it follows that $x^2 - y^2 = 0$; since $x^2 + y^2 = 1$, by (48), we have $x^2 = y^2 = \frac{1}{2}$, hence $E(4it) = -1$. This proves (*c*).

If $0 \leq t_1 < t_2 < 2\pi$, then

$$E(it_2)[E(it_1)]^{-1} = E(it_2 - it_1) \neq 1,$$

by (*c*). This establishes the uniqueness assertion in (*d*).

To prove the existence assertion in (*d*), fix z so that $|z| = 1$. Write $z = x + iy$, with x and y real. Suppose first that $x \geq 0$ and $y \geq 0$. On $[0, \pi/2]$, C decreases from 1 to 0. Hence $C(t) = x$ for some $t \in [0, \pi/2]$. Since $C^2 + S^2 = 1$ and $S \geq 0$ on $[0, \pi/2]$, it follows that $z = E(it)$.

If $x < 0$ and $y \geq 0$, the preceding conditions are satisfied by $-iz$. Hence $-iz = E(it)$ for some $t \in [0, \pi/2]$, and since $i = E(\pi i/2)$, we obtain $z = E(i(t + \pi/2))$. Finally, if $y < 0$, the preceding two cases show that

$-z = E(it)$ for some $t \in (0, \pi)$. Hence $z = -E(it) = E(i(t + \pi))$.
This proves (d), and hence the theorem.

It follows from (d) and (48) that the curve γ defined by

$$\gamma(t) = E(it) \qquad (0 \le t \le 2\pi) \tag{54}$$

is a simple closed curve whose range is the unit circle in the plane. Since $\gamma'(t) = iE(it)$, the length of γ is

$$\int_0^{2\pi} |\gamma'(t)|\, dt = 2\pi,$$

by Theorem 6.27. This is of course the expected result for the circumference of a circle of radius 1. It shows that π, defined by (51), has the usual geometric significance.

In the same way we see that the point $\gamma(t)$ describes a circular arc of length t_0 as t increases from 0 to t_0. Consideration of the triangle whose vertices are

$$z_1 = 0, \qquad z_2 = \gamma(t_0), \qquad z_3 = C(t_0)$$

shows that $C(t)$ and $S(t)$ are indeed identical with cos t and sin t, if the latter are defined in the usual way as ratios of the sides of a right triangle.

It should be stressed that we derived the basic properties of the trigonometric functions from (46) and (25), without any appeal to the geometric notion of angle. There are other nongeometric approaches to these functions. The papers by W. F. Eberlein (*Amer. Math. Monthly*, vol. 74, 1967, pp. 1223–1225) and by G. B. Robison (*Math. Mag.*, vol. 41, 1968, pp. 66–70) deal with these topics.

THE ALGEBRAIC COMPLETENESS OF THE COMPLEX FIELD

We are now in a position to give a simple proof of the fact that the complex field is algebraically complete, that is to say, that every nonconstant polynomial with complex coefficients has a complex root.

8.8 Theorem *Suppose $a_0, \ldots, a_n$ are complex numbers, $n \ge 1$, $a_n \ne 0$,*

$$P(z) = \sum_0^n a_k z^k.$$

Then $P(z) = 0$ for some complex number z.

Proof Without loss of generality, assume $a_n = 1$. Put

$$\mu = \inf |P(z)| \qquad (z \text{ complex}) \tag{55}$$

If $|z| = R$, then

$$|P(z)| \ge R^n[1 - |a_{n-1}| R^{-1} - \cdots - |a_0| R^{-n}]. \tag{56}$$

The right side of (56) tends to ∞ as $R \to \infty$. Hence there exists R_0 such that $|P(z)| > \mu$ if $|z| > R_0$. Since $|P|$ is continuous on the closed disc with center at 0 and radius R_0, Theorem 4.16 shows that $|P(z_0)| = \mu$ for some z_0.

We claim that $\mu = 0$.

If not, put $Q(z) = P(z + z_0)/P(z_0)$. Then Q is a nonconstant polynomial, $Q(0) = 1$, and $|Q(z)| \geq 1$ for all z. There is a smallest integer k, $1 \leq k \leq n$, such that

$$Q(z) = 1 + b_k z^k + \cdots + b_n z^n, \qquad b_k \neq 0. \tag{57}$$

By Theorem 8.7(d) there is a real θ such that

$$e^{ik\theta} b_k = -|b_k|. \tag{58}$$

If $r > 0$ and $r^k |b_k| < 1$, (58) implies

$$|1 + b_k r^k e^{ik\theta}| = 1 - r^k |b_k|,$$

so that

$$|Q(re^{i\theta})| \leq 1 - r^k\{|b_k| - r|b_{k+1}| - \cdots - r^{n-k}|b_n|\}.$$

For sufficiently small r, the expression in braces is positive; hence $|Q(re^{i\theta})| < 1$, a contradiction.

Thus $\mu = 0$, that is, $P(z_0) = 0$.

Exercise 27 contains a more general result.

FOURIER SERIES

8.9 Definition A *trigonometric polynomial* is a finite sum of the form

$$f(x) = a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx) \qquad (x \text{ real}), \tag{59}$$

where $a_0, \ldots, a_N, b_1, \ldots, b_N$ are complex numbers. On account of the identities (46), (59) can also be written in the form

$$f(x) = \sum_{-N}^{N} c_n e^{inx} \qquad (x \text{ real}), \tag{60}$$

which is more convenient for most purposes. It is clear that every trigonometric polynomial is periodic, with period 2π.

If n is a nonzero integer, e^{inx} is the derivative of e^{inx}/in, which also has period 2π. Hence

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}\, dx = \begin{cases} 1 & (\text{if } n = 0), \\ 0 & (\text{if } n = \pm 1, \pm 2, \ldots). \end{cases} \tag{61}$$

Let us multiply (60) by e^{-imx}, where m is an integer; if we integrate the product, (61) shows that

$$c_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-imx} \, dx \tag{62}$$

for $|m| \le N$. If $|m| > N$, the integral in (62) is 0.

The following observation can be read off from (60) and (62): The trigonometric polynomial f, given by (60), is *real* if and only if $c_{-n} = \overline{c_n}$ for $n = 0, \ldots, N$.

In agreement with (60), we define a *trigonometric series* to be a series of the form

$$\sum_{-\infty}^{\infty} c_n e^{inx} \qquad (x \text{ real}); \tag{63}$$

the Nth partial sum of (63) is defined to be the right side of (60).

If f is an integrable function on $[-\pi, \pi]$, the numbers c_m defined by (62) for all integers m are called the *Fourier coefficients* of f, and the series (63) formed with these coefficients is called the *Fourier series* of f.

The natural question which now arises is whether the Fourier series of f converges to f, or, more generally, whether f is determined by its Fourier series. That is to say, if we know the Fourier coefficients of a function, can we find the function, and if so, how?

The study of such series, and, in particular, the problem of representing a given function by a trigonometric series, originated in physical problems such as the theory of oscillations and the theory of heat conduction (Fourier's "Théorie analytique de la chaleur" was published in 1822). The many difficult and delicate problems which arose during this study caused a thorough revision and reformulation of the whole theory of functions of a real variable. Among many prominent names, those of Riemann, Cantor, and Lebesgue are intimately connected with this field, which nowadays, with all its generalizations and ramifications, may well be said to occupy a central position in the whole of analysis.

We shall be content to derive some basic theorems which are easily accessible by the methods developed in the preceding chapters. For more thorough investigations, the Lebesgue integral is a natural and indispensable tool.

We shall first study more general systems of functions which share a property analogous to (61).

8.10 Definition Let $\{\phi_n\}$ $(n = 1, 2, 3, \ldots)$ be a sequence of complex functions on $[a, b]$, such that

$$\int_a^b \phi_n(x) \overline{\phi_m(x)} \, dx = 0 \qquad (n \ne m). \tag{64}$$

Then $\{\phi_n\}$ is said to be an *orthogonal system of functions* on $[a, b]$. If, in addition,

$$\int_a^b |\phi_n(x)|^2 \, dx = 1 \tag{65}$$

for all n, $\{\phi_n\}$ is said to be *orthonormal*.

For example, the functions $(2\pi)^{-\frac{1}{2}}e^{inx}$ form an orthonormal system on $[-\pi, \pi]$. So do the real functions

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \cdots .$$

If $\{\phi_n\}$ is orthonormal on $[a, b]$ and if

$$c_n = \int_a^b f(t)\overline{\phi_n(t)} \, dt \qquad (n = 1, 2, 3, \ldots), \tag{66}$$

we call c_n the nth Fourier coefficient of f relative to $\{\phi_n\}$. We write

$$f(x) \sim \sum_1^\infty c_n \phi_n(x) \tag{67}$$

and call this series the Fourier series of f (relative to $\{\phi_n\}$).

Note that the symbol $\sim$ used in (67) implies nothing about the convergence of the series; it merely says that the coefficients are given by (66).

The following theorems show that the partial sums of the Fourier series of f have a certain minimum property. We shall assume here and in the rest of this chapter that $f \in \mathscr{R}$, although this hypothesis can be weakened.

8.11 Theorem *Let $\{\phi_n\}$ be orthonormal on $[a, b]$. Let*

$$s_n(x) = \sum_{m=1}^n c_m \phi_m(x) \tag{68}$$

be the nth partial sum of the Fourier series of f, and suppose

$$t_n(x) = \sum_{m=1}^n \gamma_m \phi_m(x). \tag{69}$$

Then

$$\int_a^b |f - s_n|^2 \, dx \le \int_a^b |f - t_n|^2 \, dx, \tag{70}$$

and equality holds if and only if

$$\gamma_m = c_m \qquad (m = 1, \ldots, n). \tag{71}$$

That is to say, among all functions t_n, s_n gives the best possible mean square approximation to f.