function on 2×2 matrices. For we showed that for any 2-linear function D

$$D(A) = A_{11}A_{21}D(\epsilon_1, \epsilon_1) + A_{11}A_{22}D(\epsilon_1, \epsilon_2) + A_{12}A_{21}D(\epsilon_2, \epsilon_1) + A_{12}A_{22}D(\epsilon_2, \epsilon_2).$$

If D is alternating, then

$$D(\epsilon_1, \epsilon_1) = D(\epsilon_2, \epsilon_2) = 0$$

and

$$D(\epsilon_2, \epsilon_1) = -D(\epsilon_1, \epsilon_2) = -D(I).$$

If D also satisfies $D(I) = 1$, then

$$D(A) = A_{11}A_{22} - A_{12}A_{21}.$$

Example 4. Let F be a field and let D be any alternating 3-linear function on 3×3 matrices over the polynomial ring $F[x]$.

Let

$$A = \begin{bmatrix} x & 0 & -x^2 \\ 0 & 1 & 0 \\ 1 & 0 & x^3 \end{bmatrix}.$$

If we denote the rows of the 3×3 identity matrix by $\epsilon_1, \epsilon_2, \epsilon_3$, then

$$D(A) = D(x\epsilon_1 - x^2\epsilon_3, \epsilon_2, \epsilon_1 + x^3\epsilon_3).$$

Since D is linear as a function of each row,

$$\begin{aligned} D(A) &= xD(\epsilon_1, \epsilon_2, \epsilon_1 + x^3\epsilon_3) - x^2D(\epsilon_3, \epsilon_2, \epsilon_1 + x^3\epsilon_3) \\ &= xD(\epsilon_1, \epsilon_2, \epsilon_1) + x^4D(\epsilon_1, \epsilon_2, \epsilon_3) - x^2D(\epsilon_3, \epsilon_2, \epsilon_1) - x^5D(\epsilon_3, \epsilon_2, \epsilon_3). \end{aligned}$$

Because D is alternating it follows that

$$D(A) = (x^4 + x^2)D(\epsilon_1, \epsilon_2, \epsilon_3).$$

Lemma. *Let D be a 2-linear function with the property that D(A) = 0 for all 2×2 matrices A over K having equal rows. Then D is alternating.*

Proof. What we must show is that if A is a 2×2 matrix and A' is obtained by interchanging the rows of A, then $D(A') = -D(A)$. If the rows of A are α and β, this means we must show that $D(\beta, \alpha) = -D(\alpha, \beta)$. Since D is 2-linear,

$$D(\alpha + \beta, \alpha + \beta) = D(\alpha, \alpha) + D(\alpha, \beta) + D(\beta, \alpha) + D(\beta, \beta).$$

By our hypothesis $D(\alpha + \beta, \alpha + \beta) = D(\alpha, \alpha) = D(\beta, \beta) = 0$. So

$$0 = D(\alpha, \beta) + D(\beta, \alpha). \quad \blacksquare$$

Lemma. *Let D be an n-linear function on $n \times n$ matrices over K. Suppose D has the property that D(A) = 0 whenever two adjacent rows of A are equal. Then D is alternating.*

Proof. We must show that $D(A) = 0$ when any two rows of A are equal, and that $D(A') = -D(A)$ if A' is obtained by interchanging some two rows of A. First, let us suppose that A' is obtained by interchanging two adjacent rows of A. The reader should see that the argument used in the proof of the preceding lemma extends to the present case and gives us $D(A') = -D(A)$.

Now let B be obtained by interchanging rows i and j of A, where $i < j$. We can obtain B from A by a succession of interchanges of pairs of adjacent rows. We begin by interchanging row i with row $(i + 1)$ and continue until the rows are in the order

$$\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_j, \alpha_i, \alpha_{j+1}, \ldots, \alpha_n.$$

This requires $k = j - i$ interchanges of adjacent rows. We now move α_j to the ith position using $(k - 1)$ interchanges of adjacent rows. We have thus obtained B from A by $k + (k - 1) = 2k - 1$ interchanges of adjacent rows. Thus

$$D(B) = (-1)^{2k-1}D(A) = -D(A).$$

Suppose A is any $n \times n$ matrix with two equal rows, say $\alpha_i = \alpha_j$ with $i < j$. If $j = i + 1$, then A has two equal and adjacent rows and $D(A) = 0$. If $j > i + 1$, we interchange α_{i+1} and α_j and the resulting matrix B has two equal and adjacent rows, so $D(B) = 0$. On the other hand, $D(B) = -D(A)$, hence $D(A) = 0$. ∎

Definition. *If* $n > 1$ *and* A *is an* $n \times n$ *matrix over* K, *we let* $A(i|j)$ *denote the* $(n - 1) \times (n - 1)$ *matrix obtained by deleting the* i*th row and* j*th column of* A. *If* D *is an* $(n - 1)$*-linear function and* A *is an* $n \times n$ *matrix, we put* $D_{ij}(A) = D[A(i|j)]$.

Theorem 1. *Let* $n > 1$ *and let* D *be an alternating* $(n - 1)$*-linear function on* $(n - 1) \times (n - 1)$ *matrices over* K. *For each* j, $1 \leq j \leq n$, *the function* E_j *defined by*

$$E_j(A) = \sum_{i=1}^{n} (-1)^{i+j}A_{ij}D_{ij}(A) \tag{5-4}$$

is an alternating n*-linear function on* $n \times n$ *matrices* A. *If* D *is a determinant function, so is each* E_j.

Proof. If A is an $n \times n$ matrix, $D_{ij}(A)$ is independent of the ith row of A. Since D is $(n - 1)$-linear, it is clear that D_{ij} is linear as a function of any row except row i. Therefore $A_{ij}D_{ij}(A)$ is an n-linear function of A. A linear combination of n-linear functions is n-linear; hence, E_j is n-linear. To prove that E_j is alternating, it will suffice to show that $E_j(A) = 0$ whenever A has two equal and adjacent rows. Suppose $\alpha_k = \alpha_{k+1}$. If $i \neq k$ and $i \neq k + 1$, the matrix $A(i|j)$ has two equal rows, and thus $D_{ij}(A) = 0$. Therefore

$$E_j(A) = (-1)^{k+j}A_{kj}D_{kj}(A) + (-1)^{k+1+j}A_{(k+1)j}D_{(k+1)j}(A).$$

Since $\alpha_k = \alpha_{k+1}$,

$$A_{kj} = A_{(k+1)j} \quad \text{and} \quad A(k|j) = A(k+1|j).$$

Clearly then $E_j(A) = 0$.

Now suppose D is a determinant function. If $I^{(n)}$ is the $n \times n$ identity matrix, then $I^{(n)}(j|j)$ is the $(n-1) \times (n-1)$ identity matrix $I^{(n-1)}$. Since $I^{(n)}_{ij} = \delta_{ij}$, it follows from (5-4) that

$$E_j(I^{(n)}) = D(I^{(n-1)}). \tag{5-5}$$

Now $D(I^{(n-1)}) = 1$, so that $E_j(I^{(n)}) = 1$ and E_j is a determinant function. ▌

Corollary. *Let* K *be a commutative ring with identity and let* n *be a positive integer. There exists at least one determinant function on* $K^{n\times n}$.

Proof. We have shown the existence of a determinant function on 1×1 matrices over K, and even on 2×2 matrices over K. Theorem 1 tells us explicitly how to construct a determinant function on $n \times n$ matrices, given such a function on $(n-1) \times (n-1)$ matrices. The corollary follows by induction. ▌

EXAMPLE 5. If B is a 2×2 matrix over K, we let

$$|B| = B_{11}B_{22} - B_{12}B_{21}.$$

Then $|B| = D(B)$, where D is the determinant function on 2×2 matrices. We showed that this function on $K^{2\times 2}$ is unique. Let

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

be a 3×3 matrix over K. If we define E_1, E_2, E_3 as in (5-4), then

$$E_1(A) = A_{11}\begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} - A_{21}\begin{vmatrix} A_{12} & A_{13} \\ A_{32} & A_{33} \end{vmatrix} + A_{31}\begin{vmatrix} A_{12} & A_{13} \\ A_{22} & A_{23} \end{vmatrix} \tag{5-6}$$

$$E_2(A) = -A_{12}\begin{vmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{vmatrix} + A_{22}\begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix} - A_{32}\begin{vmatrix} A_{11} & A_{13} \\ A_{21} & A_{23} \end{vmatrix} \tag{5-7}$$

$$E_3(A) = A_{13}\begin{vmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{vmatrix} - A_{23}\begin{vmatrix} A_{11} & A_{12} \\ A_{31} & A_{32} \end{vmatrix} + A_{33}\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix}. \tag{5-8}$$

It follows from Theorem 1 that E_1, E_2, and E_3 are determinant functions. Actually, as we shall show later, $E_1 = E_2 = E_3$, but this is not yet apparent even in this simple case. It could, however, be verified directly, by expanding each of the above expressions. Instead of doing this we give some specific examples.

(a) Let $K = R[x]$ and

$$A = \begin{bmatrix} x-1 & x^2 & x^3 \\ 0 & x-2 & 1 \\ 0 & 0 & x-3 \end{bmatrix}.$$

Then

$$E_1(A) = (x-1)\begin{vmatrix} x-2 & 1 \\ 0 & x-3 \end{vmatrix} = (x-1)(x-2)(x-3)$$

$$\begin{aligned} E_2(A) &= -x^2\begin{vmatrix} 0 & 1 \\ 0 & x-3 \end{vmatrix} + (x-2)\begin{vmatrix} x-1 & x^3 \\ 0 & x-3 \end{vmatrix} \\ &= (x-1)(x-2)(x-3) \end{aligned}$$

and

$$\begin{aligned} E_3(A) &= x^3\begin{vmatrix} 0 & x-2 \\ 0 & 0 \end{vmatrix} - \begin{vmatrix} x-1 & x^2 \\ 0 & 0 \end{vmatrix} + (x-3)\begin{vmatrix} x-1 & x^2 \\ 0 & x-2 \end{vmatrix} \\ &= (x-1)(x-2)(x-3). \end{aligned}$$

(b) Let $K = R$ and

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Then

$$\begin{aligned} E_1(A) &= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \\ E_2(A) &= -\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 1 \\ E_3(A) &= -\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 1. \end{aligned}$$

Exercises

1. Each of the following expressions defines a function D on the set of 3×3 matrices over the field of real numbers. In which of these cases is D a 3-linear function?

(a) $D(A) = A_{11} + A_{22} + A_{33}$;
(b) $D(A) = (A_{11})^2 + 3A_{11}A_{22}$;
(c) $D(A) = A_{11}A_{12}A_{33}$;
(d) $D(A) = A_{13}A_{22}A_{32} + 5A_{12}A_{22}A_{32}$;
(e) $D(A) = 0$;
(f) $D(A) = 1$.

2. Verify directly that the three functions E_1, E_2, E_3 defined by (5-6), (5-7), and (5-8) are identical.

3. Let K be a commutative ring with identity. If A is a 2×2 matrix over K, the **classical adjoint** of A is the 2×2 matrix adj A defined by

$$\operatorname{adj} A = \begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix}.$$

If det denotes the unique determinant function on 2×2 matrices over K, show that

(a) $(\text{adj } A)A = A(\text{adj } A) = (\det A)I$;
(b) $\det(\text{adj } A) = \det(A)$;
(c) $\text{adj}(A^t) = (\text{adj } A)^t$.
(A^t denotes the transpose of A.)

4. Let A be a 2×2 matrix over a field F. Show that A is invertible if and only if $\det A \neq 0$. When A is invertible, give a formula for A^{-1}.

5. Let A be a 2×2 matrix over a field F, and suppose that $A^2 = 0$. Show for each scalar c that $\det(cI - A) = c^2$.

6. Let K be a subfield of the complex numbers and n a positive integer. Let $j_1, \ldots, j_n$ and $k_1, \ldots, k_n$ be positive integers not exceeding n. For an $n \times n$ matrix A over K define

$$D(A) = A(j_1, k_1)A(j_2, k_2) \cdots A(j_n, k_n).$$

Prove that D is n-linear if and only if the integers $j_1, \ldots, j_n$ are distinct.

7. Let K be a commutative ring with identity. Show that the determinant function on 2×2 matrices A over K is alternating and 2-linear as a function of the columns of A.

8. Let K be a commutative ring with identity. Define a function D on 3×3 matrices over K by the rule

$$D(A) = A_{11} \det \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} - A_{12} \det \begin{bmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{bmatrix} + A_{13} \det \begin{bmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix}.$$

Show that D is alternating and 3-linear as a function of the columns of A.

9. Let K be a commutative ring with identity and D an alternating n-linear function on $n \times n$ matrices over K. Show that
(a) $D(A) = 0$, if one of the rows of A is 0.
(b) $D(B) = D(A)$, if B is obtained from A by adding a scalar multiple of one row of A to another.

10. Let F be a field, A a 2×3 matrix over F, and (c_1, c_2, c_3) the vector in F^3 defined by

$$c_1 = \begin{vmatrix} A_{12} & A_{13} \\ A_{22} & A_{23} \end{vmatrix}, \qquad c_2 = \begin{vmatrix} A_{13} & A_{11} \\ A_{23} & A_{21} \end{vmatrix}, \qquad c_3 = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix}.$$

Show that
(a) $\text{rank}(A) = 2$ if and only if $(c_1, c_2, c_3) \neq 0$;
(b) if A has rank 2, then (c_1, c_2, c_3) is a basis for the solution space of the system of equations $AX = 0$.

11. Let K be a commutative ring with identity, and let D be an alternating 2-linear function on 2×2 matrices over K. Show that $D(A) = (\det A)D(I)$ for all A. Now use this result (no computations with the entries allowed) to show that $\det(AB) = (\det A)(\det B)$ for any 2×2 matrices A and B over K.

12. Let F be a field and D a function on $n \times n$ matrices over F (with values in F). Suppose $D(AB) = D(A)D(B)$ for all A, B. Show that either $D(A) = 0$ for all A, or $D(I) = 1$. In the latter case show that $D(A) \neq 0$ whenever A is invertible.

13. Let R be the field of real numbers, and let D be a function on 2×2 matrices over R, with values in R, such that $D(AB) = D(A)D(B)$ for all A, B. Suppose also that

$$D\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) \neq D\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right).$$

Prove the following.

(a) $D(0) = 0$;

(b) $D(A) = 0$ if $A^2 = 0$;

(c) $D(B) = -D(A)$ if B is obtained by interchanging the rows (or columns) of A;

(d) $D(A) = 0$ if one row (or one column) of A is 0;

(e) $D(A) = 0$ whenever A is singular.

14. Let A be a 2×2 matrix over a field F. Then the set of all matrices of the form $f(A)$, where f is a polynomial over F, is a commutative ring K with identity. If B is a 2×2 matrix over K, the determinant of B is then a 2×2 matrix over F, of the form $f(A)$. Suppose I is the 2×2 identity matrix over F and that B is the 2×2 matrix over K

$$B = \begin{bmatrix} A - A_{11}I & -A_{12}I \\ -A_{21}I & A - A_{22}I \end{bmatrix}.$$

Show that $\det B = f(A)$, where $f = x^2 - (A_{11} + A_{22})x + \det A$, and also that $f(A) = 0$.

5.3. Permutations and the Uniqueness of Determinants

In this section we prove the uniqueness of the determinant function on $n \times n$ matrices over K. The proof will lead us quite naturally to consider permutations and some of their basic properties.

Suppose D is an alternating n-linear function on $n \times n$ matrices over K. Let A be an $n \times n$ matrix over K with rows $\alpha_1, \alpha_2, \cdots, \alpha_n$. If we denote the rows of the $n \times n$ identity matrix over K by $\epsilon_1, \epsilon_2, \cdots, \epsilon_n$, then

$$\alpha_i = \sum_{j=1}^{n} A(i, j)\epsilon_j, \qquad 1 \leq i \leq n. \tag{5-9}$$

Hence

$$\begin{aligned} D(A) &= D\left(\sum_j A(1, j)\epsilon_j, \alpha_2, \ldots, \alpha_n\right) \\ &= \sum_j A(1, j)D(\epsilon_j, \alpha_2, \ldots, \alpha_n). \end{aligned}$$

If we now replace α_2 by $\sum_k A(2, k)\epsilon_k$, we see that

$$D(\epsilon_j, \alpha_2, \ldots, \alpha_n) = \sum_k A(2, k)D(\epsilon_j, \epsilon_k, \ldots, \alpha_n).$$

Thus

$$D(A) = \sum_{j,k} A(1, j)A(2, k)D(\epsilon_j, \epsilon_k, \ldots, \alpha_n).$$