7 CHAPTER Complex Numbers

7.1 Addition, Multiplication, and Absolute Value

Complex numbers are objects of the form $a + b\sqrt{-1}$, where a and b are real numbers and $\sqrt{-1}$ is... what? Mathematicians worried about this question for several centuries and did not come up with a good answer until the 19th century, by which time complex numbers had become indispensable in virtually all fields of mathematics. Their story is perhaps the supreme illustration of a saying of Hilbert's: "In mathematics, existence means freedom from contradiction."[1] Mathematicians came to believe in complex numbers because they worked, not because they could define them, and finding a definition was not a high priority until *all* concepts of number came under scrutiny.

So let us begin by assuming there is such a thing as $i = \sqrt{-1}$, and see where this leads. As we did when we introduced other new numbers, such as the integers and the reals, we want to retain the

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[1]See Constance Reid's *Hilbert*, p. 98.

J. Stillwell, *Numbers and Geometry*
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properties of the old numbers as far as possible. We therefore assume i is like any other number, except that $i^2 = -1$. Addition of complex numbers does not even involve i^2, so it is completely straightforward:

$$(a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2).$$

If we strip each complex number $a + ib$ down to its essence, the *ordered pair* of real numbers (a, b), then addition of complex numbers is simply separate addition of a and b components:

$$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2), \qquad (+\text{ rule})$$

as one does with direct products (Section 6.6). The a and b components are traditionally called the *real* and *imaginary* parts of $a + ib$.

The interesting properties of complex numbers begin with multiplication, where $i^2 = -1$ becomes involved:

$$(a_1 + ib_1)(a_2 + ib_2) = (a_1a_2 - b_1b_2) + i(b_1a_2 + a_1b_2).$$

In terms of ordered pairs, multiplication is the rule

$$(a_1, b_1)(a_2, b_2) = (a_1a_2 - b_1b_2, b_1a_2 + a_1b_2) \qquad (\times\text{ rule}).$$

This rule is more mysterious, but it gives us a cheap way to define the complex numbers without worrying about $\sqrt{-1}$: simply define them to be ordered pairs of reals (a, b) with addition and multiplication defined by the $+$ and $\times$ rules just given. This was first done by Hamilton in 1833.

Perhaps it seems underhand to define multiplication by the $\times$ rule when one knows it is just the disguised result of assuming $i^2 = -1$, but it isn't! The $\times$ rule was in use long before anyone dreamt of $\sqrt{-1}$. The first hint of it appears in Diophantus' *Arithmetica*, Book III, Problem 19, where he says:

> 65 is naturally divided into two squares in two ways, namely into $7^2 + 4^2$ and $8^2 + 1^2$, which is due to the fact that 65 is the product of 13 and 5, each of which is the sum of two squares.

Apparently he knew that the product of sums of squares is itself a sum of squares, in two ways, which points to the identity:

$$(a_1^2 + b_1^2)(a_2^2 + b_2^2) = (a_1a_2 \pm b_1b_2)^2 + (b_1a_2 \mp a_1b_2)^2.$$

(He had the special case $a_1 = 3$, $b_1 = 2$, $a_2 = 2$, $b_2 = 1$.) This remarkable identity was first observed explicitly by Abū Ja'far al-Khazin around 950 A.D., commenting on this problem of Diophantus, and it was proved in Fibonacci's *The Book of Squares* in 1225.

Diophantus talks about products of sums of squares, but he views $a^2 + b^2$ as the square on the hypotenuse of the right-angled triangle with sides a and b. His view reveals an important aspect of the *geometric interpretation of complex numbers*, which gradually emerged during the 16th, 17th, and 18th centuries and became standard in the 19th. A second aspect, which virtually completes the picture, will be discussed in the next section. For the moment, let us see how much is visible from Diophantus' viewpoint.

The triangle with sides a and b represents the *pair* (a, b), and hence corresponds to what we would call the complex number $a+ib$. We interpret $a + ib$ as the *vertex* of a triangle in the plane with one vertex at the origin and sides a and b parallel to the axes (Figure 7.1), and interpret the set $\mathbb{C}$ of all complex numbers as the plane. The "hypotenuse" $\sqrt{a^2 + b^2}$ of the triangle with sides a and b is what we call the *absolute value* $|a + ib|$ of the corresponding complex number $a + ib$. Diophantus' identity says that this geometrically defined quantity has a simple algebraic property; it is *multiplicative*. The absolute value of a product is the product of the absolute values:

$$|a_1 + ib_1||a_2 + ib_2| = |(a_1 + ib_1)(a_2 + ib_2)|.$$

In fact, the two sides of this equation are just the square roots of the two sides of Diophantus' identity, with the lower signs chosen on the right-hand side.

Of course, there was no reason for Diophantus to speak of a "product of triangles," or even to think of it. All he wanted was a *rule* for taking two triangles and producing a third for which the hypotenuse was the product of the two hypotenuses he started with. Applied

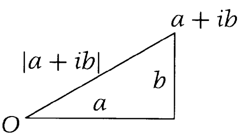

FIGURE 7.1 Geometric meaning of absolute value.