Proposition 2.2.13 (Trichotomy of order for natural numbers). *Let a and b be natural numbers. Then exactly one of the following statements is true: $a < b$, $a = b$, or $a > b$.*

Proof. This is only a sketch of the proof; the gaps will be filled in Exercise 2.2.4.

First we show that we cannot have more than one of the statements $a < b$, $a = b$, $a > b$ holding at the same time. If $a < b$ then $a \neq b$ by definition, and if $a > b$ then $a \neq b$ by definition. If $a > b$ and $a < b$ then by Proposition 2.2.12 we have $a = b$, a contradiction. Thus no more than one of the statements is true.

Now we show that at least one of the statements is true. We keep b fixed and induct on a. When $a = 0$ we have $0 \leq b$ for all b (why?), so we have either $0 = b$ or $0 < b$, which proves the base case. Now suppose we have proven the proposition for a, and now we prove the proposition for $a{+}{+}$. From the trichotomy for a, there are three cases: $a < b$, $a = b$, and $a > b$. If $a > b$, then $a{+}{+} > b$ (why?). If $a = b$, then $a{+}{+} > b$ (why?). Now suppose that $a < b$. Then by Proposition 2.2.12, we have $a{+}{+} \leq b$. Thus either $a{+}{+} = b$ or $a{+}{+} < b$, and in either case we are done. This closes the induction. □

The properties of order allow one to obtain a stronger version of the principle of induction:

Proposition 2.2.14 (Strong principle of induction). *Let m_0 be a natural number, and Let $P(m)$ be a property pertaining to an arbitrary natural number m. Suppose that for each $m \geq m_0$, we have the following implication: if $P(m')$ is true for all natural numbers $m_0 \leq m' < m$, then $P(m)$ is also true. (In particular, this means that $P(m_0)$ is true, since in this case the hypothesis is vacuous.) Then we can conclude that $P(m)$ is true for all natural numbers $m \geq m_0$.*

Remark 2.2.15. In applications we usually use this principle with $m_0 = 0$ or $m_0 = 1$.

Proof. See Exercise 2.2.5. □

Exercise 2.2.1. Prove Proposition 2.2.5. (Hint: fix two of the variables and induct on the third.)

Exercise 2.2.2. Prove Lemma 2.2.10. (Hint: use induction.)

Exercise 2.2.3. Prove Proposition 2.2.12. (Hint: you will need many of the preceding propositions, corollaries, and lemmas.)

Exercise 2.2.4. Justify the three statements marked (why?) in the proof of Proposition 2.2.13.

Exercise 2.2.5. Prove Proposition 2.2.14. (Hint: define $Q(n)$ to be the property that $P(m)$ is true for all $m_0 \leq m < n$; note that $Q(n)$ is vacuously true when $n < m_0$.)

Exercise 2.2.6. Let n be a natural number, and let $P(m)$ be a property pertaining to the natural numbers such that whenever $P(m{+}{+})$ is true, then $P(m)$ is true. Suppose that $P(n)$ is also true. Prove that $P(m)$ is true for all natural numbers $m \leq n$; this is known as the *principle of backwards induction.* (Hint: apply induction to the variable n.)

2.3 Multiplication

In the previous section we have proven all the basic facts that we know to be true about addition and order. To save space and to avoid belaboring the obvious, we will now allow ourselves to use all the rules of algebra concerning addition and order that we are familiar with, without further comment. Thus for instance we may write things like $a + b + c = c + b + a$ without supplying any further justification. Now we introduce multiplication. Just as addition is the iterated increment operation, multiplication is iterated addition:

Definition 2.3.1 (Multiplication of natural numbers). Let m be a natural number. To multiply zero to m, we define $0 \times m := 0$. Now suppose inductively that we have defined how to multiply n to m. Then we can multiply $n{+}{+}$ to m by defining $(n{+}{+}) \times m := (n \times m) + m$.

Thus for instance $0 \times m = 0$, $1 \times m = 0+m$, $2 \times m = 0+m+m$, etc. By induction one can easily verify that the product of two natural numbers is a natural number.

Lemma 2.3.2 (Multiplication is commutative). *Let n, m be natural numbers. Then $n \times m = m \times n$.*

Proof. See Exercise 2.3.1. □

We will now abbreviate $n \times m$ as nm, and use the usual convention that multiplication takes precedence over addition, thus for instance $ab + c$ means $(a \times b) + c$, not $a \times (b + c)$. (We will also use the usual notational conventions of precedence for the other arithmetic operations when they are defined later, to save on using parentheses all the time.)

Lemma 2.3.3 (Natural numbers have no zero divisors). *Let n, m be natural numbers. Then $n \times m = 0$ if and only if at least one of n, m is equal to zero. In particular, if n and m are both positive, then nm is also positive.*

Proof. See Exercise 2.3.2. □

Proposition 2.3.4 (Distributive law). *For any natural numbers a, b, c, we have $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$.*

Proof. Since multiplication is commutative we only need to show the first identity $a(b + c) = ab + ac$. We keep a and b fixed, and use induction on c. Let's prove the base case $c = 0$, i.e., $a(b + 0) = ab + a0$. The left-hand side is ab, while the right-hand side is $ab + 0 = ab$, so we are done with the base case. Now let us suppose inductively that $a(b + c) = ab + ac$, and let us prove that $a(b + (c{+}{+})) = ab + a(c{+}{+})$. The left-hand side is $a((b + c){+}{+}) = a(b+c)+a$, while the right-hand side is $ab+ac+a = a(b+c)+a$ by the induction hypothesis, and so we can close the induction. □

Proposition 2.3.5 (Multiplication is associative). *For any natural numbers a, b, c, we have $(a \times b) \times c = a \times (b \times c)$.*

Proof. See Exercise 2.3.3. □

Proposition 2.3.6 (Multiplication preserves order). *If a, b are natural numbers such that $a < b$, and c is positive, then $ac < bc$.*

Proof. Since $a < b$, we have $b = a + d$ for some positive d. Multiplying by c and using the distributive law we obtain $bc = ac + dc$. Since d is positive, and c is positive, dc is positive, and hence $ac < bc$ as desired. □

Corollary 2.3.7 (Cancellation law). *Let a, b, c be natural numbers such that $ac = bc$ and c is non-zero. Then $a = b$.*

Remark 2.3.8. Just as Proposition 2.2.6 will allow for a "virtual subtraction" which will eventually let us define genuine subtraction, this corollary provides a "virtual division" which will be needed to define genuine division later on.

Proof. By the trichotomy of order (Proposition 2.2.13), we have three cases: $a < b$, $a = b$, $a > b$. Suppose first that $a < b$, then by Proposition 2.3.6 we have $ac < bc$, a contradiction. We can obtain a similar contradiction when $a > b$. Thus the only possibility is that $a = b$, as desired. □

With these propositions it is easy to deduce all the familiar rules of algebra involving addition and multiplication, see for instance Exercise 2.3.4.

Now that we have the familiar operations of addition and multiplication, the more primitive notion of increment will begin to fall by the wayside, and we will see it rarely from now on. In any event we can always use addition to describe incrementation, since $n{+}{+} = n + 1$.

Proposition 2.3.9 (Euclidean algorithm). *Let n be a natural number, and let q be a positive number. Then there exist natural numbers m, r such that $0 \leq r < q$ and $n = mq + r$.*

Remark 2.3.10. In other words, we can divide a natural number n by a positive number q to obtain a quotient m (which is another natural number) and a remainder r (which is less than q). This algorithm marks the beginning of *number theory*, which is a beautiful and important subject but one which is beyond the scope of this text.

Proof. See Exercise 2.3.5. □

Just like one uses the increment operation to recursively define addition, and addition to recursively define multiplication, one can use multiplication to recursively define *exponentiation*:

Definition 2.3.11 (Exponentiation for natural numbers). Let m be a natural number. To raise m to the power 0, we define $m^0 := 1$. Now suppose recursively that m^n has been defined for some natural number n, then we define $m^{n++} := m^n \times m$.

Examples 2.3.12. Thus for instance $x^1 = x^0 \times x = 1 \times x = x$; $x^2 = x^1 \times x = x \times x$; $x^3 = x^2 \times x = x \times x \times x$; and so forth. By induction we see that this recursive definition defines x^n for all natural numbers n.

We will not develop the theory of exponentiation too deeply here, but instead wait until after we have defined the integers and rational numbers; see in particular Proposition 4.3.10.

Exercise 2.3.1. Prove Lemma 2.3.2. (Hint: modify the proofs of Lemmas 2.2.2, 2.2.3 and Proposition 2.2.4.)

Exercise 2.3.2. Prove Lemma 2.3.3. (Hint: prove the second statement first.)

Exercise 2.3.3. Prove Proposition 2.3.5. (Hint: modify the proof of Proposition 2.2.5 and use the distributive law.)

Exercise 2.3.4. Prove the identity $(a+b)^2 = a^2 + 2ab + b^2$ for all natural numbers a, b.

Exercise 2.3.5. Prove Proposition 2.3.9. (Hint: fix q and induct on n.)

Chapter 3

Set theory

Modern analysis, like most of modern mathematics, is concerned with numbers, sets, and geometry. We have already introduced one type of number system, the natural numbers. We will introduce the other number systems shortly, but for now we pause to introduce the concepts and notation of set theory, as they will be used increasingly heavily in later chapters. (We will not pursue a rigourous description of Euclidean geometry in this text, preferring instead to describe that geometry in terms of the real number system by means of the Cartesian co-ordinate system.)

While set theory is not the main focus of this text, almost every other branch of mathematics relies on set theory as part of its foundation, so it is important to get at least some grounding in set theory before doing other advanced areas of mathematics. In this chapter we present the more elementary aspects of axiomatic set theory, leaving more advanced topics such as a discussion of infinite sets and the axiom of choice to Chapter 8. A full treatment of the finer subtleties of set theory (of which there are many!) is unfortunately well beyond the scope of this text.

3.1 Fundamentals

In this section we shall set out some axioms for sets, just as we did for the natural numbers. For pedagogical reasons, we will use a somewhat overcomplete list of axioms for set theory, in the sense that some of the axioms can be used to deduce others, but there is no real harm in doing this. We begin with an informal description of what sets should be.

Definition 3.1.1. (Informal) We define a *set* A to be any unordered collection of objects, e.g., $\{3, 8, 5, 2\}$ is a set. If x is an object, we say that *x is an element of A* or $x \in A$ if x lies in the collection; otherwise we say that $x \notin A$. For instance, $3 \in \{1, 2, 3, 4, 5\}$ but $7 \notin \{1, 2, 3, 4, 5\}$.

This definition is intuitive enough, but it doesn't answer a number of questions, such as which collections of objects are considered to be sets, which sets are equal to other sets, and how one defines operations on sets (e.g., unions, intersections, etc.). Also, we have no axioms yet on what sets do, or what their elements do. Obtaining these axioms and defining these operations will be the purpose of the remainder of this section.

We first clarify one point: we consider sets themselves to be a type of object.

Axiom 3.1 (Sets are objects). *If A is a set, then A is also an object. In particular, given two sets A and B, it is meaningful to ask whether A is also an element of B.*

Example 3.1.2. (Informal) The set $\{3, \{3, 4\}, 4\}$ is a set of three distinct elements, one of which happens to itself be a set of two elements. See Example 3.1.10 for a more formal version of this example. However, not all objects are sets; for instance, we typically do not consider a natural number such as 3 to be a set. (The more accurate statement is that natural numbers can be the *cardinalities* of sets, rather than necessarily being sets themselves. See Section 3.6.)

Remark 3.1.3. There is a special case of set theory, called "pure set theory", in which *all* objects are sets; for instance the number 0 might be identified with the empty set $\emptyset = \{\}$, the number 1 might be identified with $\{0\} = \{\{\}\}$, the number 2 might be identified with $\{0, 1\} = \{\{\}, \{\{\}\}\}$, and so forth. From a logical point of view, pure set theory is a simpler theory, since one only has to deal with sets and not with objects; however, from a conceptual point of view it is often easier to deal with impure set theories in which some objects are not considered to be sets. The two types of theories are more or less equivalent for the purposes of doing mathematics, and so we shall take an agnostic position as to whether all objects are sets or not.

To summarize so far, among all the objects studied in mathematics, some of the objects happen to be sets; and if x is an object and A is a set, then either $x \in A$ is true or $x \in A$ is false. (If A is not a set, we leave the statement $x \in A$ undefined; for instance, we consider the statement $3 \in 4$ to neither be true or false, but simply meaningless, since 4 is not a set.)

Next, we define the notion of equality: when are two sets considered to be equal? We do not consider the order of the elements inside a set to be important; thus we think of $\{3, 8, 5, 2\}$ and $\{2, 3, 5, 8\}$ as the same set. On the other hand, $\{3, 8, 5, 2\}$ and $\{3, 8, 5, 2, 1\}$ are different sets, because the latter set contains an element that the former one does not, namely the element 1. For similar reasons $\{3, 8, 5, 2\}$ and $\{3, 8, 5\}$ are different sets. We formalize this as a definition:

Definition 3.1.4 (Equality of sets). Two sets A and B are *equal*, $A = B$, iff every element of A is an element of B and vice versa. To put it another way, $A = B$ if and only if every element x of A belongs also to B, and every element y of B belongs also to A.

Example 3.1.5. Thus, for instance, $\{1, 2, 3, 4, 5\}$ and $\{3, 4, 2, 1, 5\}$ are the same set, since they contain exactly the same elements. (The set $\{3, 3, 1, 5, 2, 4, 2\}$ is also equal to $\{1, 2, 3, 4, 5\}$; the repetition of 3 and 2 is irrelevant as it does not further change the status of 2 and 3 being elements of the set.)

One can easily verify that this notion of equality is reflexive, symmetric, and transitive (Exercise 3.1.1). Observe that if $x \in A$ and $A = B$, then $x \in B$, by Definition 3.1.4. Thus the "is an element of" relation $\in$ obeys the axiom of substitution (see Section

A.7). Because of this, any new operation we define on sets will also obey the axiom of substitution, as long as we can define that operation purely in terms of the relation $\in$. This is for instance the case for the remaining definitions in this section. (On the other hand, we cannot use the notion of the "first" or "last" element in a set in a well-defined manner, because this would not respect the axiom of substitution; for instance the sets $\{1, 2, 3, 4, 5\}$ and $\{3, 4, 2, 1, 5\}$ are the same set, but have different first elements.)

Next, we turn to the issue of exactly which objects are sets and which objects are not. The situation is analogous to how we defined the natural numbers in the previous chapter; we started with a single natural number, 0, and started building more numbers out of 0 using the increment operation. We will try something similar here, starting with a single set, the *empty set*, and building more sets out of the empty set by various operations. We begin by postulating the existence of the empty set.

Axiom 3.2 (Empty set). *There exists a set $\emptyset$, known as the empty set, which contains no elements, i.e., for every object x we have $x \notin \emptyset$.*

The empty set is also denoted $\{\}$. Note that there can only be one empty set; if there were two sets $\emptyset$ and $\emptyset'$ which were both empty, then by Definition 3.1.4 they would be equal to each other (why?).

If a set is not equal to the empty set, we call it *non-empty*. The following statement is very simple, but worth stating nevertheless:

Lemma 3.1.6 (Single choice). *Let A be a non-empty set. Then there exists an object x such that $x \in A$.*

Proof. We prove by contradiction. Suppose there does not exist any object x such that $x \in A$. Then for all objects x, we have $x \notin A$. Also, by Axiom 3.2 we have $x \notin \emptyset$. Thus $x \in A \iff x \in \emptyset$ (both statements are equally false), and so $A = \emptyset$ by Definition 3.1.4, a contradiction. $\square$

Remark 3.1.7. The above Lemma asserts that given any non-empty set A, we are allowed to "choose" an element x of A which demonstrates this non-emptyness. Later on (in Lemma 3.5.12) we will show that given any finite number of non-empty sets, say $A_1, \ldots, A_n$, it is possible to choose one element $x_1, \ldots, x_n$ from each set $A_1, \ldots, A_n$; this is known as "finite choice". However, in order to choose elements from an infinite number of sets, we need an additional axiom, the *axiom of choice*, which we will discuss in Section 8.4.

Remark 3.1.8. Note that the empty set is *not* the same thing as the natural number 0. One is a set; the other is a number. However, it is true that the *cardinality* of the empty set is 0; see Section 3.6.

If Axiom 3.2 was the only axiom that set theory had, then set theory could be quite boring, as there might be just a single set in existence, the empty set. We now present further axioms to enrich the class of sets available.

Axiom 3.3 (Singleton sets and pair sets). *If a is an object, then there exists a set $\{a\}$ whose only element is a, i.e., for every object y, we have $y \in \{a\}$ if and only if $y = a$; we refer to $\{a\}$ as the singleton set whose element is a. Furthermore, if a and b are objects, then there exists a set $\{a, b\}$ whose only elements are a and b; i.e., for every object y, we have $y \in \{a, b\}$ if and only if $y = a$ or $y = b$; we refer to this set as the pair set formed by a and b.*

Remarks 3.1.9. Just as there is only one empty set, there is only one singleton set for each object a, thanks to Definition 3.1.4 (why?). Similarly, given any two objects a and b, there is only one pair set formed by a and b. Also, Definition 3.1.4 also ensures that $\{a, b\} = \{b, a\}$ (why?) and $\{a, a\} = \{a\}$ (why?). Thus the singleton set axiom is in fact redundant, being a consequence of the pair set axiom. Conversely, the pair set axiom will follow from the singleton set axiom and the pairwise union axiom below (see Lemma 3.1.13). One may wonder why we don't go further and create triplet axioms, quadruplet axioms, etc.; however there will be no need for this once we introduce the pairwise union axiom below.

Examples 3.1.10. Since $\emptyset$ is a set (and hence an object), so is singleton set $\{\emptyset\}$, i.e., the set whose only element is $\emptyset$, is a set (and it is *not* the same set as $\emptyset$, $\{\emptyset\} \neq \emptyset$ (why?). Similarly, the singleton set $\{\{\emptyset\}\}$ and the pair set $\{\emptyset, \{\emptyset\}\}$ are also sets. These three sets are not equal to each other (Exercise 3.1.2).

As the above examples show, we can now create quite a few sets; however, the sets we make are still fairly small (each set that we can build consists of no more than two elements, so far). The next axiom allows us to build somewhat larger sets than before.

Axiom 3.4 (Pairwise union). *Given any two sets A, B, there exists a set $A \cup B$, called the union $A \cup B$ of A and B, whose elements consists of all the elements which belong to A or B or both. In other words, for any object x,*

$$x \in A \cup B \iff (x \in A \text{ or } x \in B).$$

Recall that "or" refers by default in mathematics to *inclusive* or: "X or Y is true" means that "either X is true, or Y is true, or both are true". See Section A.1.

Example 3.1.11. The set $\{1, 2\} \cup \{2, 3\}$ consists of those elements which either lie on $\{1, 2\}$ or in $\{2, 3\}$ or in both, or in other words the elements of this set are simply 1, 2, and 3. Because of this, we denote this set as $\{1, 2\} \cup \{2, 3\} = \{1, 2, 3\}$.

Remark 3.1.12. If A, B, A' are sets, and A is equal to A', then $A \cup B$ is equal to $A' \cup B$ (why? One needs to use Axiom 3.4 and Definition 3.1.4). Similarly if B' is a set which is equal to B, then $A \cup B$ is equal to $A \cup B'$. Thus the operation of union obeys the axiom of substitution, and is thus well-defined on sets.

We now give some basic properties of unions.

Lemma 3.1.13. *If a and b are objects, then $\{a, b\} = \{a\} \cup \{b\}$. If A, B, C are sets, then the union operation is commutative (i.e., $A \cup B = B \cup A$) and associative (i.e., $(A \cup B) \cup C = A \cup (B \cup C)$). Also, we have $A \cup A = A \cup \emptyset = \emptyset \cup A = A$.*

Proof. We prove just the associativity identity $(A \cup B) \cup C = A \cup (B \cup C)$, and leave the remaining claims to Exercise 3.1.3. By Definition 3.1.4, we need to show that every element x of $(A \cup B) \cup C$ is an element of $A \cup (B \cup C)$, and vice versa. So suppose first that x is an element of $(A \cup B) \cup C$. By Axiom 3.4, this means that at least one of $x \in A \cup B$ or $x \in C$ is true. We now divide into two cases. If $x \in C$, then by Axiom 3.4 again $x \in B \cup C$, and so by Axiom 3.4 again we have $x \in A \cup (B \cup C)$. Now suppose instead $x \in A \cup B$, then by Axiom 3.4 again $x \in A$ or $x \in B$. If $x \in A$ then $x \in A \cup (B \cup C)$ by Axiom 3.4, while if $x \in B$ then by consecutive applications of Axiom 3.4 we have $x \in B \cup C$ and hence $x \in A \cup (B \cup C)$. Thus in all cases we see that every element of $(A \cup B) \cup C$ lies in $A \cup (B \cup C)$. A similar argument shows that every element of $A \cup (B \cup C)$ lies in $(A \cup B) \cup C$, and so $(A \cup B) \cup C = A \cup (B \cup C)$ as desired. $\square$

Because of the above lemma, we do not need to use parentheses to denote multiple unions, thus for instance we can write $A \cup B \cup C$ instead of $(A \cup B) \cup C$ or $A \cup (B \cup C)$. Similarly for unions of four sets, $A \cup B \cup C \cup D$, etc.

Remark 3.1.14. While the operation of union has some similarities with addition, the two operations are *not* identical. For instance, $\{2\} \cup \{3\} = \{2, 3\}$ and $2 + 3 = 5$, whereas $\{2\} + \{3\}$ is meaningless (addition pertains to numbers, not sets) and $2 \cup 3$ is also meaningless (union pertains to sets, not numbers).

This axiom allows us to define triplet sets, quadruplet sets, and so forth: if a, b, c are three objects, we define $\{a, b, c\} := \{a\} \cup \{b\} \cup \{c\}$; if a, b, c, d are four objects, then we define $\{a, b, c, d\} := \{a\} \cup \{b\} \cup \{c\} \cup \{d\}$, and so forth. On the other hand, we are not yet in a position to define sets consisting of n objects for any given natural number n; this would require iterating the above construction "n times", but the concept of n-fold iteration has not yet been rigourously defined. For similar reasons, we cannot yet define sets consisting of infinitely many objects, because that would require iterating the axiom of pairwise union infinitely often, and it is not clear at this stage that one can do this rigourously. Later on, we will introduce other axioms of set theory which allow one to construct arbitrarily large, and even infinite, sets.

Clearly, some sets seem to be larger than others. One way to formalize this concept is through the notion of a *subset.*

Definition 3.1.15 (Subsets). Let A, B be sets. We say that A is a *subset* of B, denoted $A \subseteq B$, iff every element of A is also an element of B, i.e.

$$\text{For any object } x, \quad x \in A \quad \Longrightarrow \quad x \in B.$$

We say that A is a *proper subset* of B, denoted $A \subsetneq B$, if $A \subseteq B$ and $A \neq B$.

Remark 3.1.16. Because these definitions involve only the notions of equality and the "is an element of" relation, both of which already obey the axiom of substitution, the notion of subset also automatically obeys the axiom of substitution. Thus for instance if $A \subseteq B$ and $A = A'$, then $A' \subseteq B$.

Examples 3.1.17. We have $\{1,2,4\} \subseteq \{1,2,3,4,5\}$, because every element of $\{1,2,4\}$ is also an element of $\{1,2,3,4,5\}$. In fact we also have $\{1,2,4\} \subsetneq \{1,2,3,4,5\}$, since the two sets $\{1,2,4\}$ and $\{1,2,3,4,5\}$ are not equal. Given any set A, we always have $A \subseteq A$ (why?) and $\emptyset \subseteq A$ (why?).

The notion of subset in set theory is similar to the notion of "less than or equal to" for numbers, as the following Proposition demonstrates (for a more precise statement, see Definition 8.5.1):

Proposition 3.1.18 (Sets are partially ordered by set inclusion). *Let A, B, C be sets. If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$. If $A \subseteq B$ and $B \subseteq A$, then $A = B$. Finally, if $A \subsetneq B$ and $B \subsetneq C$ then $A \subsetneq C$.*

Proof. We shall just prove the first claim. Suppose that $A \subseteq B$ and $B \subseteq C$. To prove that $A \subseteq C$, we have to prove that every element of A is an element of C. So, let us pick an arbitrary element x of A. Then, since $A \subseteq B$, x must then be an element of B. But then since $B \subseteq C$, x is an element of C. Thus every element of A is indeed an element of C, as claimed. $\square$

Remark 3.1.19. There is a relationship between subsets and unions: see for instance Exercise 3.1.7.

Remark 3.1.20. There is one important difference between the subset relation $\subsetneq$ and the less than relation $<$. Given any two distinct natural numbers n, m, we know that one of them is smaller than the other (Proposition 2.2.13); however, given two distinct sets, it is not in general true that one of them is a subset of the other. For instance, take $A := \{2n : n \in \mathbf{N}\}$ to be the set of even natural numbers, and $B := \{2n + 1 : n \in \mathbf{N}\}$ to be the set of odd natural numbers. Then neither set is a subset of the other. This is why we say that sets are only *partially ordered*, whereas the natural numbers are *totally ordered* (see Definitions 8.5.1, 8.5.3).

Remark 3.1.21. We should also caution that the subset relation $\subseteq$ is not the same as the element relation $\in$. The number 2 is an element of $\{1, 2, 3\}$ but not a subset; thus $2 \in \{1, 2, 3\}$, but $2 \not\subseteq \{1, 2, 3\}$. Indeed, 2 is not even a set. Conversely, while $\{2\}$ is a subset of $\{1, 2, 3\}$, it is not an element; thus $\{2\} \subseteq \{1, 2, 3\}$ but $\{2\} \not\in \{1, 2, 3\}$. The point is that the number 2 and the set $\{2\}$ are distinct objects. It is important to distinguish sets from their elements, as they can have different properties. For instance, it is possible to have an infinite set consisting of finite numbers (the set $\mathbf{N}$ of natural numbers is one such example), and it is also possible to have a finite set consisting of infinite objects (consider for instance the finite set $\{\mathbf{N}, \mathbf{Z}, \mathbf{Q}, \mathbf{R}\}$, which has four elements, all of which are infinite).

We now give an axiom which easily allows us to create subsets out of larger sets.

Axiom 3.5 (Axiom of specification). *Let A be a set, and for each $x \in A$, let $P(x)$ be a property pertaining to x (i.e., $P(x)$ is either a true statement or a false statement). Then there exists a set,*