Now we compute the character table of the quaternion group of order 8. We use the usual presentation

$$Q_8 = \langle i, j \mid i^4 = 1,\ i^2 = j^2,\ i^{-1}ji = j^{-1} \rangle$$

and let $k = ij$ and $i^2 = -1$. The conjugacy classes of Q_8 are represented by $1, -1, i$, j and k of sizes 1, 1, 2, 2 and 2, respectively. Since the commutator quotient of Q_8 is the Klein 4-group, there are four characters of degree 1. The one remaining irreducible character must have degree 2 in order that the sum of the squares of the degrees be 8. Let χ_5 be the degree 2 irreducible character of Q_8. One may check that the representation $\varphi : Q_8 \to GL_2(\mathbb{C})$ described explicitly in Example 7 in the second set of examples of Section 18.1 affords χ_5, but we show how the orthogonality relations give the values of χ_5 without knowing these explicit matrices. If φ is an irreducible representation of degree 2, by Schur's Lemma (cf. Exercise 18 in Section 18.1) $\varphi(-1)$ is a 2×2 scalar matrix and so is $\pm$ the identity matrix since -1 has order 2 in Q_8. Hence $\chi_5(-1) = \pm 2$. Let $\chi_5(i) = a$, $\chi_5(j) = b$ and $\chi_5(k) = c$. The orthogonality relations give

$$1 = (\chi_5, \chi_5) = \frac{1}{8}(2^2 + (\pm 2)^2 + 2a\overline{a} + 2b\overline{b} + 2c\overline{c}).$$

Since $a\overline{a}$, $b\overline{b}$ and $c\overline{c}$ are nonnegative real numbers, they must all be zero. Also, since χ_5 is orthogonal to the principal character we get

$$0 = (\chi_1, \chi_5) = \frac{1}{8}(2 + (\pm 2) + 0 + 0 + 0),$$

hence $\chi_5(-1) = -2$. The complete character table of Q_8 is the following:

classes:	1	-1	i	j	k
sizes:	1	1	2	2	2
χ_1	1	1	1	1	1
χ_2	1	1	-1	1	-1
χ_3	1	1	1	-1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

Character Table of $\mathbf{Q_8}$

Observe that D_8 and Q_8 have the same character table, hence

nonisomorphic groups may have the same character table.

Note that the values of the degree 2 representation of Q_8 could also have been easily calculated by applying the second orthogonality relation to each column of the character table. We leave this check as an exercise. Also note that although the degree 2 irreducible characters of D_8 and Q_8 have the same (real number) values the degree 2 representation of D_8 may be realized by real matrices whereas it may be shown that Q_8 has no faithful 2-dimensional representation over $\mathbb{R}$ (cf. Exercise 10 in Section 18.1).

For the next example we construct the character table of S_4. The conjugacy classes of S_4 are represented by 1, (1 2), (1 2 3), (1 2 3 4) and (1 2)(3 4) with sizes 1, 6, 8, 6, and 3 respectively. Since $S_4' = A_4$, there are two characters of degree 1: the principal character and the character whose values are the sign of the permutation.

To obtain a degree 2 irreducible character let V be the normal subgroup of order 4 generated by $(1\,2)(3\,4)$ and $(1\,3)(2\,4)$. Any representation φ of $S_4/V \cong S_3$ gives, by composition with the natural projection $S_4 \to S_4/V$, a representation of S_4; if the former is irreducible, so is the latter. Let φ be the composition of the projection with the irreducible 2-dimensional representation of S_3, and let χ_3 be its character. The classes of 1 and $(1\,2)(3\,4)$ map to the identity in the S_3 quotient, $(1\,2)$ and $(1\,2\,3\,4)$ map to transpositions and $(1\,2\,3)$ maps to a 3-cycle. The values of χ_3 can thus be read directly from the values of the character of degree 2 in the table for S_3.

Since S_4 has 5 irreducible characters and the sum of the squares of the degrees is 24, there must be two remaining irreducible characters, each of degree 3. In Example 2 of Section 18.3 one of these was calculated, call it χ_4. Recall that

$$\chi_4(\sigma) = (\text{the number of fixed points of } \sigma) - 1.$$

The remaining irreducible character, χ_5, is $\chi_4\chi_2$. One can either use Proposition 17 in Section 18.3 or Exercise 13 in Section 18.3 to see that this product is indeed a character. The first orthogonality relation verifies that it is irreducible.

classes:	1	(1 2)	(1 2 3)	(1 2 3 4)	(1 2)(3 4)
sizes:	1	6	8	6	3
χ_1	1	1	1	1	1
χ_2	1	-1	1	-1	1
χ_3	2	0	-1	0	2
χ_4	3	1	0	-1	-1
χ_5	3	-1	0	1	-1

Character Table of S_4

From the character table of S_4 one can easily compute the character table of A_4. Note that A_4 has 4 conjugacy classes. Also $|A_4 : A_4'| = 3$, so A_4 has three characters of degree 1 with $V = A_4'$ in the kernel of each degree 1 representation. The remaining irreducible character must have degree 3. One checks directly from the orthogonality relation applied in A_4 that the character χ_4 of S_4 restricted to A_4 ($= \chi_5|_{A_4}$) is irreducible. This irreducibility check is really necessary since an irreducible representation of a group need not restrict to an irreducible representation of a subgroup (for instance, the irreducible degree 2 representation of S_3 must become reducible when restricted to any proper subgroup, since these are all abelian). The character table of A_4 is the following

classes:	1	(1 2)(3 4)	(1 2 3)	(1 3 2)
sizes:	1	3	4	4
χ_1	1	1	1	1
χ_2	1	1	ζ	ζ^2
χ_3	1	1	ζ^2	ζ
χ_4	3	-1	0	0

Character Table of A_4

where ζ is a primitive cube root of 1 in $\mathbb{C}$.

As a final example we construct the following character table of S_5:

classes:	1	(1 2)	(1 2 3)	(1 2 3 4)	(1 2 3 4 5)	(1 2)(3 4)	(1 2)(3 4 5)
sizes:	1	10	20	30	24	15	20
χ_1	1	1	1	1	1	1	1
χ_2	1	−1	1	−1	1	1	−1
χ_3	4	2	1	0	−1	0	−1
χ_4	4	−2	1	0	−1	0	1
χ_5	5	−1	−1	1	0	1	−1
χ_6	5	1	−1	−1	0	1	1
χ_7	6	0	0	0	1	−2	0

Character Table of S_5

The conjugacy classes and their sizes were computed in Section 4.3. Since $|S_5 : S_5'| = 2$, there are two degree 1 characters: the principal character and the "sign" character.

The natural permutation of S_5 on 5 points gives rise to a permutation character of degree 5. As with S_4 and S_3 the orthogonality relations show that the square of its norm is 2 and it contains the principal character. Thus χ_3 is the permutation character minus the principal character (and, as with the smaller symmetric groups, $\chi_3(\sigma)$ is the number of fixed points of σ minus 1). As argued with S_4, it follows that $\chi_4 = \chi_3\chi_2$ is also an irreducible character.

To obtain χ_5 recall that S_5 has six Sylow 5-subgroups. Its action by conjugation on these gives a faithful permutation representation of degree 6. If ψ is the character of the associated linear representation, then since $\sigma \in S_5$ fixes a Sylow 5-subgroup if and only if it normalizes that subgroup, we have

$$\psi(\sigma) = \text{the number of Sylow 5-subgroups normalized by } \sigma.$$

The normalizer in S_5 of the Sylow 5-subgroup $\langle (1\,2\,3\,4\,5) \rangle$ is $\langle\, (1\,2\,3\,4\,5)\,,\ (2\,3\,5\,4)\,\rangle$ and all normalizers of Sylow 5-subgroups are conjugate in S_5 to this group. This normalizer contains only the identity, 5-cycles, 4-cycles and products of two disjoint transpositions. No other cycle type normalizes any Sylow 5-subgroup so on any other class, ψ is zero. To compute ψ on the remaining three nonidentity classes note (by inspection in S_6) that in any faithful action on 6 points the following hold: an element of order 5 must be a 5-cycle (hence fixes 1 point); any element of order 4 which fixes one point must be a 4-cycle (hence fixes 2 points); an element of order 2 which is the square of an element of order 4 fixes exactly 2 points also. This gives all the values of ψ. Now direct computation shows that

$$\|\psi\|^2 = 2 \qquad \text{and} \qquad (\chi_1, \psi) = 1.$$

Thus $\chi_5 = \psi - \chi_1$ is irreducible of degree 5. By the same theory as for χ_4 one gets that $\chi_6 = \chi_5\chi_2$ is another irreducible character.

Since there are 7 conjugacy classes, there is one remaining irreducible character and its degree is 6. Its values can be obtained immediately from the decomposition of the regular character, ρ (cf. Example 3 in Section 18.2 and Example 4 in Section 18.3):

$$\chi_7 = \frac{\rho - \chi_1 - \chi_2 - 4\chi_3 - 4\chi_4 - 5\chi_5 - 5\chi_6}{6}.$$

A direct calculation by the orthogonality relations checks that χ_7 is irreducible. Note that the values of the character χ_7 were computed without explicitly exhibiting a representation with this character.

EXERCISES

1. Calculate the character tables of $Z_2 \times Z_2$, $Z_2 \times Z_3$ and $Z_2 \times Z_2 \times Z_2$. Explain why the table of $Z_2 \times Z_3$ contains primitive 6^{th} roots of 1.
2. Compute the degrees of the irreducible characters of D_{16}.
3. Compute the degrees of the irreducible characters of A_5. Deduce that the degree 6 irreducible character of S_5 is not irreducible when restricted to A_5. [The conjugacy classes of A_5 are worked out in Section 4.3.]
4. Using the character tables in this section, for each of parts (a) to (d) use the first orthogonality relation to write the specified permutation character (cf. Example 3, Section 18.3) as a sum of irreducible characters:
   (a) the permutation character of the subgroup A_3 of S_3
   (b) the permutation character of the subgroup $\langle (1\ 2\ 3\ 4) \rangle$ of S_4
   (c) the permutation character of the subgroup V_4 of S_4
   (d) the permutation character of the subgroup $\langle (1\ 2\ 3),\ (1\ 2),\ (4\ 5) \rangle$ of S_5 (this subgroup is the normalizer of a Sylow 3-subgroup of S_5).
5. Assume that for any character ψ of a group, ψ^2 is also a character (where $\psi^2(g) = (\psi(g))^2$) — this is a special case of Proposition 17 in Section 18.3. Using the character tables in this section, for each of parts (a) to (e) write out the values of the square, χ^2, of the specified character χ and use the first orthogonality relation to write χ^2 as a sum of irreducible characters:
   (a) $\chi = \chi_3$, the degree 2 character in the table of S_3
   (b) $\chi = \chi_5$, the degree 2 character in the table of Q_8
   (c) $\chi = \chi_5$, the last character in the table of S_4
   (d) $\chi = \chi_4$, the second degree 4 character in the table of S_5
   (e) $\chi = \chi_7$, the last character in the table of S_5.
6. Calculate the character table of A_5.
7. Show that S_6 has an irreducible character of degree 5.
8. Calculate the character table of D_{10}. (This table contains nonreal entries.)
9. Calculate the character table of D_{12}.
10. Calculate the character table of $S_3 \times S_3$.
11. Calculate the character table of $Z_3 \times S_3$.
12. Calculate the character table of $Z_2 \times S_4$.
13. Calculate the character table of $S_3 \times S_4$.
14. Let n be an integer with $n \geq 3$. Show that every irreducible character of D_{2n} has degree 1 or 2 and find the number of irreducible characters of each degree. [The conjugacy classes of D_{2n} were found in Exercises 31 and 32 of Section 4.3 and its commutator subgroup was computed in Section 5.4.]
15. Prove that the character table is an invertible matrix. [Use the orthogonality relations.]
16. For each of A_5 and D_{10} describe which irreducible characters are algebraically conjugate (cf. the exercises in Section 18.3).