10

Infinite Series

10.1 Early Results

Infinite series were present in Greek mathematics, though the Greeks tried to deal with them as finitely as possibly by working with arbitrary finite sums $a_1 + a_2 + \cdots + a_n$ instead of infinite sums $a_1 + a_2 + \cdots$. However, this is just the difference between potential and actual infinity. There is no question that Zeno's paradox of the dichotomy (Section 4.1), for example, concerns the decomposition of the number 1 into the infinite series

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots$$

and that Archimedes found the area of the parabolic segment (Section 4.4) essentially by summing the infinite series

$$1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots = \frac{4}{3}.$$

Both these examples are special cases of the result we express as summation of a geometric series

$$a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1-r} \quad \text{when} \quad |r| < 1.$$

The first examples of infinite series other than geometric series appeared in the Middle Ages. In a book from around 1350, called the *Liber*

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calculationum, Richard Suiseth (or Swineshead, known as the Calculator) used a very lengthy verbal argument to show that

$$\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\frac{4}{2^4}+\cdots=2.$$

[The argument is reproduced in Boyer (1959), p. 78.] At about the same

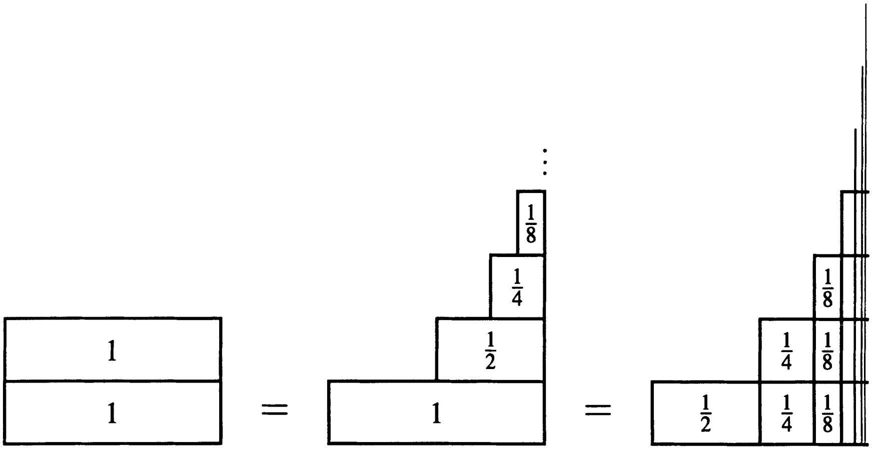

Figure 10.1: Oresme's summation

time, Oresme (1350b), pp. 413–421, summed this and similar series by geometric decomposition as in Figure 10.1, showing

$$2=\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\frac{4}{2^4}+\cdots.$$

Actually Oresme gives only the last picture in the figure, but it seems likely he arrived at it by cutting up an area of two square units as shown, judging from his opening remark: "A finite surface can be made as long as we wish, or as high, by varying the extension without increasing the size." The region constructed by Oresme, incidentally, is perhaps the first example of the phenomenon encountered by Torricelli (Section 9.2) in his hyperbolic solid of revolution—infinite extent but finite content.

Another important discovery of Oresme (1350a) was the divergence of the *harmonic series*

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots.$$

His proof was by an elementary argument that is now standard:

$$\begin{aligned}
&1+\left(\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\cdots\\
>&1+\left(\frac{1}{2}\right)+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\cdots\\
=&1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots.
\end{aligned}$$

Thus by repeatedly doubling the number of terms collected in successive groups, we can indefinitely obtain groups of sum $> \frac{1}{2}$, enabling the sum to grow beyond all bounds.

As mentioned in Section 9.4, Indian mathematicians found the series

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

with its important special case

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

in the fifteenth century. The series for π was the first satisfactory answer to the classical problem of squaring the circle, for although the expression is infinite (as it would have to be, in view of Lindemann's theorem on the transcendence of π), the rule for generating successive terms is as finite and transparent as it could possibly be. It is sad that the Indian series became known in the West too late to have any influence or even, as yet, to obtain proper credit for its discoverer. Rajagopal and Rangachari (1977, 1986) showed that the series for $\tan^{-1} x$, $\sin x$, and $\cos x$ were known in India before 1540, and probably before 1500, but their exact dates and discoverers are uncertain.

Exercises

Oresme's proof by partitioning the harmonic series into

$$1+\left(\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\cdots$$

has the following geometric counterpart.

10.1.1 By referring to Figure 10.2, show that

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} > \text{area under } y=\tfrac{1}{x} \text{ between } x=1 \text{ and } x=n+1.$$

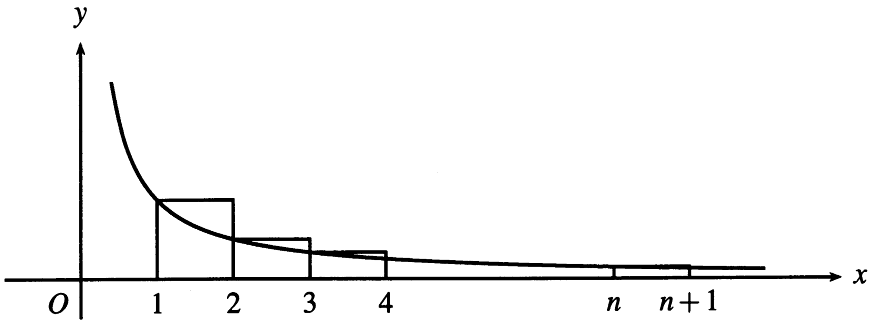

Figure 10.2: Comparing $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$ with an area

10.1.2 Now partition this area under $y = 1/x$ into the pieces between $x = 1$ and $x = 2$, $x = 2$ and $x = 4$, $x = 4$ and $x = 8, \ldots$, and show that *all these pieces have the same area.* (This can even be done without using calculus, if you use the argument of Exercises 4.4.1 and 4.4.2.)

10.1.3 Deduce from Exercise 10.1.2 that the area from $x = 1$ to $x = n$, and hence the sum $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$, tends to infinity.

The area under $y = 1/x$ from $x = 1$ to $x = n+1$ is of course $\log(n+1)$, so Figure 10.2 shows that $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} > \log(n+1)$. As $n \to \infty$, these two functions of n remain about the same size.

10.1.4 By comparing the curved area with suitable rectangles beneath the curve, show that

$$\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} < \log(n+1),$$

and hence that $0 < 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} - \log(n+1) < 1$.

10.1.5 Also show, by a geometric argument, that $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} - \log(n+1)$ increases as n increases, so that it has a finite limit < 1.

The value of the limit is known as *Euler's constant* γ, and γ is approximately 0.577. However, little is known about the nature of γ—not even whether it is irrational.

10.2 Power Series

The Indian series for $\tan^{-1} x$ was the first example, apart from geometric series such as $1+x+x^2+x^3+\cdots = 1/(1-x)$, of a *power series*, that is, the expansion of a function $f(x)$ in powers of x. The idea of power series turned out to be fruitful not only in the representation of functions but even