For each 1/3 turn, we have $\theta = \pm 2\pi/3$. Hence,

$$\cos\frac{\theta}{2} = \cos\frac{\pi}{3} = \frac{1}{2}, \qquad \sin\frac{\theta}{2} = \pm\sin\frac{\pi}{3} = \pm\frac{\sqrt{3}}{2}.$$

The $\sqrt{3}$ in $\sin\frac{\pi}{3}$ neatly cancels the factor $1/\sqrt{3}$ in the axis of rotation, and we find that the eight 1/3 turns are represented by the eight pairs of opposites among the 16 quaternions

$$\pm\frac{1}{2} \pm \frac{\mathbf{i}}{2} \pm \frac{\mathbf{j}}{2} \pm \frac{\mathbf{k}}{2}.$$

Finally, the identity rotation is represented by the pair ± 1, and thus the 12 symmetries of the tetrahedron are represented by the 24 quaternions

$$\pm 1, \quad \pm\mathbf{i}, \quad \pm\mathbf{j}, \quad \pm\mathbf{k}, \quad \pm\frac{1}{2} \pm \frac{\mathbf{i}}{2} \pm \frac{\mathbf{j}}{2} \pm \frac{\mathbf{k}}{2}.$$

The 24-cell

These 24 quaternions all lie at distance 1 from O in $\mathbb{R}^4$, and they are distributed in a highly symmetrical manner. In fact, they are the vertices of a four-dimensional figure analogous to a regular polyhedron—called a *regular polytope*. This particular polytope is called the *24-cell*. Because we cannot directly perceive four-dimensional figures, the best we can do is study the 24-cell via projections of it into $\mathbb{R}^3$ (just as we often study polyhedra, such as the tetrahedron and cube, via projections onto the plane such as Figure 7.7). One such projection is shown in Figure 7.8 (which of course is a projection of a three-dimensional figure onto the plane—but it is easy to visualize what the three-dimensional figure is). This superb drawing is taken from Hilbert and Cohn-Vossen's *Geometry and the Imagination*.

Exercises

The vertices of the 24-cell include the eight unit points (positive and negative) on the four axes in $\mathbb{R}^4$, but the other 16 points have some less obvious properties.

7.7.1 Verify directly that the 16 points $\pm\frac{1}{2} \pm \frac{\mathbf{i}}{2} \pm \frac{\mathbf{j}}{2} \pm \frac{\mathbf{k}}{2}$ are all at distance 1 from the origin in $\mathbb{R}^4$.

7.7.2 Deduce that the distance from the center to any vertex of a four-dimensional cube is equal to the length of its side.

7.7.3 Show also that each of the points $\pm\frac{1}{2} \pm \frac{\mathbf{i}}{2} \pm \frac{\mathbf{j}}{2} \pm \frac{\mathbf{k}}{2}$ is at distance 1 from the four nearest unit points on the axes.