Lemma 17. The exceptional characters associated to Q_i are all distinct from the exceptional characters associated to Q_j for i and j distinct elements of $\{1, 2, 3, 4\}$.

Proof: Let χ be an exceptional character associated to Q_i and let θ be an exceptional character associated to Q_j. By construction, there are distinct irreducible characters ψ and ψ' of Q_i such that $\psi^* - \psi'^* = \chi - \chi'$ and there are distinct irreducible characters λ and λ' of Q_j such that $\lambda^* - \lambda'^* = \theta - \theta'$. Let $\alpha = \psi - \psi'$ and let $\beta = \lambda - \lambda'$. By Lemma 15, α^* is zero on all elements of G whose order is not equal to q_i (including the identity) and β^* is zero on all elements of G whose order is not equal to q_j. Thus clearly $(\alpha^*, \beta^*) = 0$. It follows easily that the two irreducible constituents of α^* are pairwise orthogonal to those of β^* as well. This establishes the lemma.

It is now easy to show that such a simple group G does not exist. By Lemma 16 and properties (i) to (iv) of G we can count the number of exceptional characters:

(i) there are 2 exceptional characters associated to Q_1
(ii) there are 2 exceptional characters associated to Q_2
(iii) there are 4 exceptional characters associated to Q_3
(iv) there are 136 exceptional characters associated to Q_4.

Denote the common degree of the exceptional characters associated to Q_i by d_i for $i = 1, \dots, 4$. By Lemma 17, the exceptional characters account for 144 nonprincipal irreducible characters of G hence these, together with the principal character, are all the irreducible characters of G (the number of conjugacy classes of G is 145). The sum of the squares of the degrees of the irreducible characters is the order of G:

$$1 + 2d_1^2 + 2d_2^2 + 4d_3^2 + 136d_4^2 = 1004913.$$

Simplifying this, we obtain

$$d_1^2 + d_2^2 + 2d_3^2 + 68d_4^2 = 502456. \tag{19.4}$$

Finally, since each nonprincipal irreducible representation of the simple group G is faithful and since the smallest degree of a faithful representation of N_1 is 13, each $d_i \geq 13$. Since $d_4 < \sqrt{502456/68} < 86$ and d_4 divides $|G|$, it follows that

$$d_4 \in \{13,\ 21,\ 27,\ 39,\ 63\}.$$

Furthermore, each $d_i \bigm| |G|$ by Corollary 5 and so there are a small number of possibilities for each d_i. One now checks that equation (4) has no solution (this is particularly easy to do by computer). This contradiction completes the proof.

EXERCISES

Throughout the exercises all representations are over the complex numbers.

1. Let $G = S_3$, let $H = A_3$ and let V be the 3-dimensional $\mathbb{C}H$-module which affords the natural permutation representation of A_3. More explicitly, let V have basis e_1, e_2, e_3 and let $\sigma \in A_3$ act on V by $\sigma e_i = e_{\sigma(i)}$. Let 1 and (1 2) be coset representatives for the left cosets of A_3 in S_3 and write out the explicit matrices described in Theorem 11 for the action of S_3 on the induced module W, for each of the elements of S_3.

2. In each of parts (a) to (f) a character ψ of a subgroup H of a particular group G is specified. Compute the values of the induced character $\mathrm{Ind}_H^G(\psi)$ on all the conjugacy classes of G and use the character tables in Section 1 to write $\mathrm{Ind}_H^G(\psi)$ as a sum of irreducible characters:

(a) ψ is the unique nonprincipal degree 1 character of the subgroup $\langle (1\ 2) \rangle$ of S_3
(b) ψ is the degree 1 character of the subgroup $\langle r \rangle$ of D_8 defined by $\psi(r) = i$, where $i \in \mathbb{C}$ is a square root of -1
(c) ψ is the degree 1 character of the subgroup $\langle r \rangle$ of D_8 defined by $\psi(r) = -1$
(d) ψ is any of the nonprincipal degree 1 characters of the subgroup $V_4 = \langle (1\ 2),\ (3\ 4) \rangle$ of S_4
(e) $\psi = \chi_4$ is the first of the two characters of degree 3 in the character table of $H = S_4$ in Section 1 and H is a subgroup of $G = S_5$
(f) ψ is any of the nonprincipal degree 1 characters of the subgroup $V_4 = \langle (1\ 2),\ (3\ 4) \rangle$ of S_5.

3. Use Proposition 13 to explicitly write out the character table of each of the following groups:
(a) the dihedral group of order 10
(b) the non-abelian group of order 57
(c) the non-abelian group of order 56 which has a normal, elementary abelian Sylow 2-subgroup.

4. Let H be a subgroup of G, let φ be a representation of H and suppose that N is a normal subgroup of G with $N \leq H$ and N contained in the kernel of φ. Prove that N is also contained in the kernel of the induced representation of φ.

5. Let N be a normal subgroup of G and let ψ_1 be the principal character of N. Let Ψ be the induced character $\mathrm{Ind}_N^G(\psi_1)$ so that by the preceding exercise we may consider Ψ as the character of a representation of G/N. Prove that Ψ is the character of the regular representation of G/N.

6. Let Z be any subgroup of the center of G, let $|G : Z| = m$ and let ψ be a character of Z. Prove that
$$\mathrm{Ind}_Z^G(\psi)(g) = \begin{cases} m\psi(g) & \text{if } g \in Z \\ 0 & \text{if } g \notin Z. \end{cases}$$

7. Let φ be a matrix representation of the subgroup H of G and define matrices $\Phi(g)$ for every $g \in G$ by the displayed formula in the statement of Theorem 11. Prove directly that Φ is a representation by showing that $\Phi(xy) = \Phi(x)\Phi(y)$ for all $x, y \in G$.

8. Let G be a Frobenius group with Frobenius kernel Q. Assume that both Q and G/Q are abelian but G is not abelian (i.e., $G \neq Q$). Let $|Q| = n$ and $|G : Q| = m$.
(a) Prove that G/Q is cyclic and show that $G = QC$ for some cyclic subgroup C of G with $C \cap Q = 1$ (i.e., G is a semidirect product of Q and C and $|C| = m$). [Let q be a prime divisor of n and let G/Q act by conjugation on the elementary abelian q-group $\{h \in Q \mid h^q = 1\}$. Apply Exercise 14(e) of Section 18.1 and the definition of a Frobenius group to an irreducible constituent of this $\mathbb{F}_q G/Q$-module.]
(b) Prove that n and m are relatively prime. [If a prime p divides both the order and index of Q, let P be a Sylow p-subgroup of G. Then $P \cap Q \trianglelefteq P$ and $P \cap Q$ is a Sylow p-subgroup of Q. Consider the centralizer in G of the subgroup $Z(P) \cap Q$ (this intersection is nontrivial by Theorem 1 of Section 6.1).]
(c) Show that G has no elements of order qp, where q is any nontrivial divisor of n and p is any nontrivial divisor of m. [Argue as in Proposition 13.]
(d) Prove that the number of nonidentity conjugacy classes of G contained in Q is $(n-1)/m$ and that each of these classes has size m. [Argue as in Proposition 13.]
(e) Prove that no two distinct elements of C are conjugate in G. Deduce that the nonidentity elements of C are representatives for $m-1$ distinct conjugacy classes of G and that each of these classes has size n. Deduce then that every element of $G - Q$

is conjugate to some element of C and that G has $m + (n-1)/m$ conjugacy classes.

(f) Prove that $G' = Q$ and deduce that G has m distinct characters of degree 1. [To show $Q \leq G'$ let $C = \langle x \rangle$ and argue that the map $h \mapsto [h, x] = x^{-1}h^{-1}xh$ is a homomorphism from Q to Q whose kernel is trivial, hence this map is surjective.]

(g) Show that if ψ is any nonprincipal irreducible character of Q, then $\mathrm{Ind}_Q^G(\psi)$ is an irreducible character of G. Show that every irreducible character of G of degree > 1 is equal to $\mathrm{Ind}_Q^G(\psi)$ for some nonprincipal irreducible character ψ of Q. Deduce that every irreducible character of G has degree either 1 or m and the number of irreducible characters of degree m is $(n-1)/m$. [Check that the proof of Proposition 13(3) establishes this more general result with the appropriate changes to the numbers involved.]

9. Use the preceding exercise to explicitly write out the character table of $\langle (1\,2\,3\,4\,5), (2\,3\,5\,4) \rangle$, which is the normalizer in S_5 of a Sylow 5-subgroup (this group is a Frobenius group of order 20).

10. Let N be a normal subgroup of G, let ψ be a character of N and let $g \in G$. Define ψ^g by $\psi^g(h) = \psi(ghg^{-1})$ for all $h \in N$.

(a) Prove that ψ^g is a character of N (ψ and ψ^g are called *G-conjugate* characters of N). Prove that ψ^g is irreducible if and only if ψ is irreducible.

(b) Prove that the map $\psi \mapsto \psi^g$ is a right group action of G on the set of characters of N and N is in the kernel of this action.

(c) Prove that if ψ_1 and ψ_2 are G-conjugate characters of N, then $\mathrm{Ind}_N^G(\psi_1) = \mathrm{Ind}_N^G(\psi_2)$. Prove also that if ψ_1 and ψ_2 are characters of N that are not G-conjugate then $\mathrm{Ind}_N^G(\psi_1) \neq \mathrm{Ind}_N^G(\psi_2)$. [Use the argument in the proof of Proposition 13(3).]

11. Show that if $G = A_4$ and $N = V_4$ is its Sylow 2-subgroup then any two nonprincipal irreducible characters of N are G-conjugate (cf. the preceding exercise).

12. Let $G = D_{2n}$ be presented by its usual generators and relations. Prove that if ψ is any degree 1 character of $H = \langle r \rangle$ such that $\psi \neq \psi^s$, then $\mathrm{Ind}_H^G(\psi)$ is an irreducible character of D_{2n}. Show that every irreducible character of D_{2n} is the induced character of some degree 1 character of $\langle r \rangle$.

13. Prove both parts of Proposition 14.

14. Prove the following result known as *Frobenius Reciprocity*: let $H \leq G$, let ψ be any character of H and let χ be any character of G. Then

$$(\psi\,,\ \chi|_H)_H = (\mathrm{Ind}_H^G(\psi)\,,\ \chi)_G.$$

[Expand the right hand side using the formula for the induced character $\mathrm{Ind}_H^G(\psi)$ or follow the proof of Shapiro's Lemma in Section 17.2.]

15. Assume G were a simple group of order $3^3 \cdot 7 \cdot 13 \cdot 409$ whose Sylow subgroups and their normalizers are described by properties (1) to (5) in this section. Prove that the permutation character of degree 819 obtained from the action of G on the left cosets of the subgroup N_4 decomposes as $\chi_0 + \gamma + \gamma'$, where χ_0 is the principal character of G and γ and γ' are distinct irreducible characters of G of degree 409. [Use Exercise 9 in Section 18.3 to show that this permutation character π has $\|\pi\|^2 = 3$.]