Let $GL(n)$ denote the set of all invertible complex $n \times n$ matrices. Then $GL(n)$ is also a group under matrix multiplication. This group is called the **general linear group.** Theorem 14 is equivalent to the following result.

Corollary. *For each* B *in* GL(n) *there exist unique matrices* N *and* U *such that* N *is in* $T^+(n)$, U *is in* U(n), *and*

$$B = N \cdot U.$$

Proof. By the theorem there is a unique matrix M in $T^+(n)$ such that MB is in $U(n)$. Let $MB = U$ and $N = M^{-1}$. Then N is in $T^+(n)$ and $B = N \cdot U$. On the other hand, if we are given any elements N and U such that N is in $T^+(n)$, U is in $U(n)$, and $B = N \cdot U$, then $N^{-1}B$ is in $U(n)$ and N^{-1} is the unique matrix M which is characterized by the theorem; furthermore U is necessarily $N^{-1}B$. ∎

Example 28. Let x_1 and x_2 be real numbers such that $x_1^2 + x_2^2 = 1$ and $x_1 \neq 0$. Let

$$B = \begin{bmatrix} x_1 & x_2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Applying the Gram-Schmidt process to the rows of B, we obtain the vectors

$$\begin{aligned} \alpha_1 &= (x_1, x_2, 0) \\ \alpha_2 &= (0, 1, 0) - x_2(x_1, x_2, 0) \\ &= x_1(-x_2, x_1, 0) \\ \alpha_3 &= (0, 0, 1). \end{aligned}$$

Let U be the matrix with rows α_1, (α_2/x_1), α_3. Then U is unitary, and

$$U = \begin{bmatrix} x_1 & x_2 & 0 \\ -x_2 & x_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\dfrac{x_2}{x_1} & \dfrac{1}{x_1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now multiplying by the inverse of

$$M = \begin{bmatrix} 1 & 0 & 0 \\ -\dfrac{x_2}{x_1} & \dfrac{1}{x_1} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

we find that

$$\begin{bmatrix} x_1 & x_2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ x_2 & x_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & 0 \\ -x_2 & x_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let us now consider briefly change of coordinates in an inner product space. Suppose V is a finite-dimensional inner product space and that $\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}$ and $\mathcal{B}' = \{\alpha'_1, \ldots, \alpha'_n\}$ are two ordered *orthonormal* bases for V. There is a unique (necessarily invertible) $n \times n$ matrix P such that

$$[\alpha]_{\mathcal{B}'} = P^{-1}[\alpha]_{\mathcal{B}}$$

for every α in V. If U is the unique linear operator on V defined by $U\alpha_j = \alpha'_j$, then P is the matrix of U in the ordered basis $\mathcal{B}$:

$$\alpha'_k = \sum_{j=1}^{n} P_{jk}\alpha_j.$$

Since $\mathcal{B}$ and $\mathcal{B}'$ are orthonormal bases, U is a unitary operator and P is a unitary matrix. If T is any linear operator on V, then

$$[T]_{\mathcal{B}'} = P^{-1}[T]_{\mathcal{B}}P = P^*[T]_{\mathcal{B}}P.$$

Definition. *Let* A *and* B *be complex* n $\times$ n *matrices. We say that* B *is* **unitarily equivalent to** A *if there is an* n $\times$ n *unitary matrix* P *such that* B = $\mathrm{P^{-1}AP}$. *We say that* B *is* **orthogonally equivalent to** A *if there is an* n $\times$ n *orthogonal matrix* P *such that* B = $\mathrm{P^{-1}AP}$.

With this definition, what we observed above may be stated as follows: If $\mathcal{B}$ and $\mathcal{B}'$ are two ordered orthonormal bases for V, then, for each linear operator T on V, the matrix $[T]_{\mathcal{B}'}$ is unitarily equivalent to the matrix $[T]_{\mathcal{B}}$. In case V is a real inner product space, these matrices are orthogonally equivalent, via a real orthogonal matrix.

Exercises

1. Find a unitary matrix which is not orthogonal, and find an orthogonal matrix which is not unitary.

2. Let V be the space of complex $n \times n$ matrices with inner product $(A|B) = \operatorname{tr}(AB^*)$. For each M in V, let T_M be the linear operator defined by $T_M(A) = MA$. Show that T_M is unitary if and only if M is a unitary matrix.

3. Let V be the set of complex numbers, regarded as a *real* vector space.

(a) Show that $(\alpha|\beta) = \operatorname{Re}(\alpha\bar{\beta})$ defines an inner product on V.

(b) Exhibit an (inner product space) isomorphism of V onto R^2 with the standard inner product.

(c) For each γ in V, let M_γ be the linear operator on V defined by $M_\gamma(\alpha) = \gamma\alpha$. Show that $(M_\gamma)^* = M_{\bar{\gamma}}$.

(d) For which complex numbers γ is M_γ self-adjoint?

(e) For which γ is M_γ unitary?

(f) For which γ is M_γ positive?

(g) What is det (M_γ)?

(h) Find the matrix of M_γ in the basis $\{1, i\}$.

(i) If T is a linear operator on V, find necessary and sufficient conditions for T to be an M_γ.

(j) Find a unitary operator on V which is not an M_γ.

4. Let V be R^2, with the standard inner product. If U is a unitary operator on V, show that the matrix of U in the standard ordered basis is either

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

for some real θ, $0 \leq \theta < 2\pi$. Let U_θ be the linear operator corresponding to the first matrix, i.e., U_θ is rotation through the angle θ. Now convince yourself that every unitary operator on V is either a rotation, or reflection about the ϵ_1-axis followed by a rotation.

(a) What is $U_\theta U_\phi$?

(b) Show that $U_\theta^* = U_{-\theta}$.

(c) Let ϕ be a fixed real number, and let $\mathcal{B} = \{\alpha_1, \alpha_2\}$ be the orthonormal basis obtained by rotating $\{\epsilon_1, \epsilon_2\}$ through the angle ϕ, i.e., $\alpha_j = U_\phi \epsilon_j$. If θ is another real number, what is the matrix of U_θ in the ordered basis $\mathcal{B}$?

5. Let V be R^3, with the standard inner product. Let W be the plane spanned by $\alpha = (1, 1, 1)$ and $\beta = (1, 1, -2)$. Let U be the linear operator defined, geometrically, as follows: U is rotation through the angle θ, about the straight line through the origin which is orthogonal to W. There are actually two such rotations —choose one. Find the matrix of U in the standard ordered basis. (Here is one way you might proceed. Find α_1 and α_2 which form an orthonormal basis for W. Let α_3 be a vector of norm 1 which is orthogonal to W. Find the matrix of U in the basis $\{\alpha_1, \alpha_2, \alpha_3\}$. Perform a change of basis.)

6. Let V be a finite-dimensional inner product space, and let W be a subspace of V. Then $V = W \oplus W^\perp$, that is, each α in V is uniquely expressible in the form $\alpha = \beta + \gamma$, with β in W and γ in $W^\perp$. Define a linear operator U by $U\alpha = \beta - \gamma$.

(a) Prove that U is both self-adjoint and unitary.

(b) If V is R^3 with the standard inner product and W is the subspace spanned by $(1, 0, 1)$, find the matrix of U in the standard ordered basis.

7. Let V be a *complex* inner product space and T a *self-adjoint* linear operator on V. Show that

(a) $||\alpha + iT\alpha|| = ||\alpha - iT\alpha||$ for every α in V.

(b) $\alpha + iT\alpha = \beta + iT\beta$ if and only if $\alpha = \beta$.

(c) $I + iT$ is non-singular.

(d) $I - iT$ is non-singular.

(e) Now suppose V is finite-dimensional, and prove that

$$U = (I - iT)(I + iT)^{-1}$$

is a unitary operator; U is called the **Cayley transform** of T. In a certain sense, $U = f(T)$, where $f(x) = (1 - ix)/(1 + ix)$.

8. If θ is a real number, prove that the following matrices are unitarily equivalent

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}.$$

9. Let V be a finite-dimensional inner product space and T a positive linear operator on V. Let p_T be the inner product on V defined by $p_T(\alpha, \beta) = (T\alpha|\beta)$. Let U be a linear operator on V and U^* its adjoint with respect to $(\ |\ )$. Prove that U is unitary with respect to the inner product p_T if and only if $T = U^*TU$.

10. Let V be a finite-dimensional inner product space. For each α, β in V, let $T_{\alpha,\beta}$ be the linear operator on V defined by $T_{\alpha,\beta}(\gamma) = (\gamma|\beta)\alpha$. Show that

(a) $T^*_{\alpha,\beta} = T_{\beta,\alpha}$.
(b) trace $(T_{\alpha,\beta}) = (\alpha|\beta)$.
(c) $T_{\alpha,\beta}T_{\gamma,\delta} = T_{\alpha,(\beta|\gamma)\delta}$.
(d) Under what conditions is $T_{\alpha,\beta}$ self-adjoint?

11. Let V be an n-dimensional inner product space over the field F, and let $L(V, V)$ be the space of linear operators on V. Show that there is a unique inner product on $L(V, V)$ with the property that $||T_{\alpha,\beta}||^2 = ||\alpha||^2||\beta||^2$ for all α, β in V. ($T_{\alpha,\beta}$ is the operator defined in Exercise 10.) Find an isomorphism between $L(V, V)$ with this inner product and the space of $n \times n$ matrices over F, with the inner product $(A|B) = \text{tr}\,(AB^*)$.

12. Let V be a finite-dimensional inner product space. In Exercise 6, we showed how to construct some linear operators on V which are both self-adjoint and unitary. Now prove that there are no others, i.e., that every self-adjoint unitary operator arises from some subspace W as we described in Exercise 6.

13. Let V and W be finite-dimensional inner product spaces having the same dimension. Let U be an isomorphism of V onto W. Show that:

(a) The mapping $T \to UTU^{-1}$ is an isomorphism of the vector space $L(V, V)$ onto the vector space $L(W, W)$.
(b) trace (UTU^{-1}) = trace (T) for each T in $L(V, V)$.
(c) $UT_{\alpha,\beta}U^{-1} = T_{U\alpha,U\beta}$ ($T_{\alpha,\beta}$ defined in Exercise 10).
(d) $(UTU^{-1})^* = UT^*U^{-1}$.
(e) If we equip $L(V, V)$ with inner product $(T_1|T_2)$ = trace $(T_1T_2^*)$, and similarly for $L(W, W)$, then $T \to UTU^{-1}$ is an inner product space isomorphism.

14. If V is an inner product space, a **rigid motion** is any function T from V into V (not necessarily linear) such that $||T\alpha - T\beta|| = ||\alpha - \beta||$ for all α, β in V. One example of a rigid motion is a linear unitary operator. Another example is translation by a fixed vector γ:

$$T_\gamma(\alpha) = \alpha + \gamma$$

(a) Let V be R^2 with the standard inner product. Suppose T is a rigid motion of V and that $T(0) = 0$. Prove that T is linear and a unitary operator.
(b) Use the result of part (a) to prove that every rigid motion of R^2 is composed of a translation, followed by a unitary operator.
(c) Now show that a rigid motion of R^2 is either a translation followed by a rotation, or a translation followed by a reflection followed by a rotation.

15. A unitary operator on R^4 (with the standard inner product) is simply a linear operator which preserves the quadratic form

$$||(x, y, z, t)||^2 = x^2 + y^2 + z^2 + t^2$$

that is, a linear operator U such that $||U\alpha||^2 = ||\alpha||^2$ for all α in R^4. In a certain part of the theory of relativity, it is of interest to find the linear operators T which preserve the form

$$||(x, y, z, t)||_L^2 = t^2 - x^2 - y^2 - z^2.$$

Now $||\quad||_L^2$ does not come from an inner product, but from something called the 'Lorentz metric' (which we shall not go into). For that reason, a linear operator T on R^4 such that $||T\alpha||_L^2 = ||\alpha||_L^2$, for every α in R^4, is called a **Lorentz transformation.**

(a) Show that the function U defined by

$$U(x, y, z, t) = \begin{bmatrix} t + x & y + iz \\ y - iz & t - x \end{bmatrix}$$

is an isomorphism of R^4 onto the real vector space H of all self-adjoint 2×2 complex matrices.

(b) Show that $||\alpha||_L^2 = \det (U\alpha)$.

(c) Suppose T is a (real) linear operator on the space H of 2×2 self-adjoint matrices. Show that $L = U^{-1}TU$ is a linear operator on R^4.

(d) Let M be any 2×2 complex matrix. Show that $T_M(A) = M^*AM$ defines a linear operator T_M on H. (Be sure you check that T_M maps H into H.)

(e) If M is a 2×2 matrix such that $|\det M| = 1$, show that $L_M = U^{-1}T_MU$ is a Lorentz transformation on R^4.

(f) Find a Lorentz transformation which is not an L_M.

8.5. Normal Operators

The principal objective in this section is the solution of the following problem. If T is a linear operator on a finite-dimensional inner product space V, under what conditions does V have an orthonormal basis consisting of characteristic vectors for T? In other words, when is there an *orthonormal* basis $\mathcal{B}$ for V, such that the matrix of T in the basis $\mathcal{B}$ is diagonal?

We shall begin by deriving some necessary conditions on T, which we shall subsequently show are sufficient. Suppose $\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}$ is an orthonormal basis for V with the property

$$T\alpha_j = c_j\alpha_j, \qquad j = 1, \ldots, n. \tag{8-16}$$

This simply says that the matrix of T in the ordered basis $\mathcal{B}$ is the diagonal matrix with diagonal entries $c_1, \ldots, c_n$. The adjoint operator T^* is represented in this same ordered basis by the conjugate transpose matrix, i.e., the diagonal matrix with diagonal entries $\bar{c}_1, \ldots, \bar{c}_n$. If V is a real inner