

than one way to represent a rational as such a ratio, e.g., $6/4 = 3/2$.

Since ten = 10, we will now use 10 instead of ten throughout, as is customary.

Exercise B.1.1. The purpose of this exercise is to demonstrate that the procedure of long addition taught to you in elementary school is actually valid. Let $A = a_n \dots a_0$ and $B = b_m \dots b_0$ be positive integer decimals. Let us adopt the convention that $a_i = 0$ when $i > n$, and $b_i = 0$ when $i > m$; for instance, if $A = 372$, then $a_0 = 2$, $a_1 = 7$, $a_2 = 3$, $a_3 = 0$, $a_4 = 0$, and so forth. Define the numbers c_0, c_1, \dots and $\varepsilon_0, \varepsilon_1, \dots$ recursively by the following *long addition algorithm*.

- We set $\varepsilon_0 := 0$.
- Now suppose that ε_i has already been defined for some $i \geq 0$. If $a_i + b_i + \varepsilon_i < 10$, we set $c_i := a_i + b_i + \varepsilon_i$ and $\varepsilon_{i+1} := 0$; otherwise, if $a_i + b_i + \varepsilon_i \geq 10$, we set $c_i := a_i + b_i + \varepsilon_i - 10$ and $\varepsilon_{i+1} = 1$. (The number ε_{i+1} is the “carry digit” from the i^{th} decimal place to the $(i+1)^{\text{th}}$ decimal place.)

Prove that the numbers c_0, c_1, \dots are all digits, and that there exists an l such that $c_l \neq 0$ and $c_i = 0$ for all $i > l$. Then show that $c_l c_{l-1} \dots c_1 c_0$ is the decimal representation of $A + B$.

Note that one could in fact use this algorithm to *define* addition, but it would look extremely complicated, and to prove even such simple facts as $(a+b)+c = a+(b+c)$ would be rather difficult. This is one of the reasons why we have avoided the decimal system in our construction of the natural numbers. The procedure for long multiplication (or long subtraction, or long division) is even worse to lay out rigourously; we will not do so here.

B.2 The decimal representation of real numbers

We need a new symbol: the *decimal point* “.”.

Definition B.2.1 (Real decimals). A *real decimal* is any sequence of digits, and a decimal point, arranged as

$$\pm a_n \dots a_0.a_{-1}a_{-2} \dots$$

which is finite to the left of the decimal point (so n is a natural number), but infinite to the right of the decimal point, where \pm is either $+$ or $-$, and $a_n \dots a_0$ is a natural number decimal (i.e., either a positive integer decimal, or 0). This decimal is equated to the real number

$$\pm a_n \dots a_0.a_{-1}a_{-2} \dots \equiv \pm 1 \times \sum_{i=-\infty}^n a_i \times 10^i.$$

The series is always convergent (Exercise B.2.1). Next, we show that every real number has at least one decimal representation:

Theorem B.2.2 (Existence of decimal representations). *Every real number x has at least one decimal representation*

$$x = \pm a_n \dots a_0.a_{-1}a_{-2} \dots$$

Proof. We first note that $x = 0$ has the decimal representation 0.000.... Also, once we find a decimal representation for x , we automatically get a decimal representation for $-x$ by changing the sign \pm . Thus it suffices to prove the theorem for positive real numbers x (by Proposition 5.4.4).

Let $n \geq 0$ be any natural number. From the Archimedean property (Corollary 5.4.13) we know that there is a natural number M such that $M \times 10^{-n} > x$. Since $0 \times 10^{-n} \leq x$, we thus see that there must exist a natural number s_n such that $s_n \times 10^{-n} \leq x$ and $s_n + 1 \times 10^{-n} > x$. (If no such natural number existed, one could use induction to conclude that $s \times 10^{-n} \leq x$ for all natural numbers s , contradicting the Archimedean property.)

Now consider the sequence s_0, s_1, s_2, \dots . Since we have

$$s_n \times 10^{-n} \leq x < (s_n + 1) \times 10^{-n}$$

we thus have

$$(10 \times s_n) \times 10^{-(n+1)} \leq x < (10 \times s_n + 10) \times 10^{-(n+1)}.$$

On the other hand, we have

$$s_{n+1} \times 10^{-(n+1)} \leq x < (s_{n+1} + 1) \times 10^{-(n+1)}$$

and hence we have

$$10 \times s_n < s_{n+1} + 1 \text{ and } s_{n+1} < 10 \times s_n + 10.$$

From these two inequalities we see that we have

$$10 \times s_n \leq s_{n+1} \leq 10 \times s_n + 9$$

and hence we can find a digit a_{n+1} such that

$$s_{n+1} = 10 \times s_n + a_n$$

and hence

$$s_{n+1} \times 10^{-(n+1)} = s_n \times 10^{-n} + a_{n+1} \times 10^{-(n+1)}.$$

From this identity and induction, we can obtain the formula

$$s_n \times 10^{-n} = s_0 + \sum_{i=0}^n a_i \times 10^{-i}.$$

Now we take limits of both sides (using Exercise B.2.1) to obtain

$$\lim_{n \rightarrow \infty} s_n \times 10^{-n} = s_0 + \sum_{i=0}^{\infty} a_i \times 10^{-i}.$$

On the other hand, we have

$$x - 10^{-n} \leq s_n \times 10^{-n} \leq x$$

for all n , so by the squeeze test (Corollary 6.4.14) we have

$$\lim_{n \rightarrow \infty} s_n \times 10^{-n} = x.$$

Thus we have

$$x = s_0 + \sum_{i=0}^{\infty} a_i \times 10^{-i}.$$

Since s_0 already has a positive integer decimal representation by Theorem B.1.4, we thus see that x has a decimal representation as desired. \square

There is however one slight flaw with the decimal system: it is possible for one real number to have two decimal representations.

Proposition B.2.3 (Failure of uniqueness of decimal representations). *The number 1 has two different decimal representations: 1.000... and 0.999....*

Proof. The representation $1 = 1.000\dots$ is clear. Now let's compute $0.999\dots$. By definition, this is the limit of the Cauchy sequence

$$0.9, 0.99, 0.999, 0.9999, \dots$$

But this sequence has 1 as a formal limit by Proposition 5.2.8. \square

It turns out that these are the only two decimal representations of 1 (Exercise B.2.2). In fact, as it turns out, all real numbers have either one or two decimal representations - two if the real is a terminating decimal, and one otherwise (Exercise B.2.3).

Exercise B.2.1. If $a_n\dots a_0.a_{-1}a_{-2}\dots$ is a real decimal, show that the series $\sum_{i=-\infty}^n a_i \times 10^i$ is absolutely convergent.

Exercise B.2.2. Show that the only decimal representations

$$1 = \pm a_n \dots a_0.a_{-1}a_{-2}\dots$$

of 1 are $1 = 1.000\dots$ and $1 = 0.999\dots$

Exercise B.2.3. A real number x is said to be a *terminating decimal* if we have $x = n/10^{-m}$ for some integers n, m . Show that if x is a terminating decimal, then x has exactly two decimal representations, while if x is not a terminating decimal, then x has exactly one decimal representation.

Exercise B.2.4. Rewrite the proof of Corollary 8.3.4 using the decimal system.

Index

- ++ (increment), 18, 56
 - on integers, 87
- +C, 342
- α -length, 334
- ε -adherent, 161, 244
 - contually ε -adherent, 161
- ε -close
 - eventual, 115
 - functions, 253
 - local, 253
 - rationals, 99
 - reals, 146
 - sequences, 115, 147
- ε -steady, 110, 146
 - eventually ε -steady, 111, 146
- π , 506, 508
- σ -algebra, 576, 595
- a posteriori*, 20
- a priori*, 20
- Abel's theorem, 484
- absolute convergence
 - for series, 192, 220
 - test, 192
- absolute value
 - for complex numbers, 498
 - for rationals, 98
 - for reals, 129
- absolutely integrable, 617
- absorption laws, 52
- abstraction, 24–25, 390
- addition
 - long, 385
 - of cardinals, 82
 - of functions, 252
 - of complex numbers, 496
 - of integers, 86
 - of natural numbers, 27
 - of rationals, 93
 - of reals, 119
- (countably) additive measure, 576, 592
- adherent point
 - infinite, 286
 - of sequences: *see* limit point of sequences
 - of sets: 245, 402, 435
- alternating series test, 193
- ambient space, 406
- analysis, 1
- and: *see* conjunction
- antiderivative, 340
- approximation to the identity, 466, 469, 523
- Archimedean property, 132
- arctangent: *see* trigonometric functions

- Aristotelian logic, 373–374
- associativity
- of addition in \mathbf{C} , 496
 - of addition in \mathbf{N} , 29
 - of composition, 59–60
 - of multiplication in \mathbf{N} , 34
 - of scalar multiplication, 538
 - of vector addition, 534
- see also:* ring, field, laws of algebra
- asymptotic discontinuity, 268
- Axiom(s)
- in mathematics, 24–25
 - of choice, 40, 73, 229
 - of comprehension: *see Axiom of universal specification*
 - of countable choice, 230
 - of equality, 377
 - of foundation: *see Axiom of regularity*
 - of induction: *see principle of mathematical induction*
 - of infinity, 50
 - of natural numbers: *see Peano axioms*
 - of pairwise union, 42
 - of power set, 66
 - of reflexivity, 377
 - of regularity, 54
 - of replacement, 49
 - of separation, 45
 - of set theory, 38, 40–42, 45, 49–50, 54, 66
 - of singleton sets and pair
- sets, 41
- of specification, 45
- of symmetry, 377
- of substitution, 57, 377
- of the empty set, 40
- of transitivity, 377
- of universal specification, 52
- of union, 67
- ball, 400
- Banach-Tarski paradox, 575, 590
- base of the natural logarithm: *see e*
- basis
- standard basis of row vectors, 535
- bijection, 62
- binomial formula, 189
- Bolzano-Weierstrass theorem, 174
- Boolean algebra, 47, 576, 591
- Boolean logic, 367
- Borel property, 575, 596
- Borel-Cantelli lemma, 615
- bound variable, 180, 369, 376
- boundary (point), 401, 435
- bounded
- from above and below, 269
 - function, 269, 451
 - interval, 244
 - sequence, 113, 150
 - sequence away from zero, 123, 127
- set, 248, 413

- $C, C^0, C^1, C^2, C^k, 556$
- cancellation law
- of addition in $\mathbf{N}, 29$
 - of multiplication in $\mathbf{N}, 35$
 - of multiplication in $\mathbf{Z}, 91$
 - of multiplication in $\mathbf{R}, 126$
- Cantor's theorem, 224
- cardinality
- arithmetic of, 81
 - of finite sets, 80
 - uniqueness of, 80
- Cartesian product, 70–71
- infinite, 228–229
- Cauchy criterion, 197
- Cauchy sequence, 111, 146, 409
- Cauchy-Schwarz inequality, 399, 516
- chain: *see* totally ordered set
- chain rule, 293
- in higher dimensions, 552, 555
- change of variables formula, 346–358
- character, 518
- characteristic function, 602
- choice
- single, 40
 - finite, 73
 - countable, 230
 - arbitrary, 229
- closed
- box, 580
 - interval, 243
 - set, 403, 435
- Clairaut's theorem: *see* interchanging derivatives
- with derivatives
- closure, 245, 403, 435
- cluster point: *see* limit point
- cocountable topology, 438
- coefficient, 476
- cofinite topology, 437
- column vector, 534
- common refinement, 311
- commutativity
- of addition in $\mathbf{C}, 496$
 - of addition in $\mathbf{N}, 29$
 - of addition in vector spaces, 534
 - of convolution, 467, 522
 - of multiplication in $\mathbf{N}, 34$
- see also:* ring, field, laws of algebra
- compactness, 413, 436
- compact support, 465
- comparison principle (or test)
- for finite series, 181
 - for infinite series, 196
 - for sequences, 166
- completeness
- of the space of continuous functions, 454
 - of metric spaces, 410
 - of the reals, 168
- completion of a metric space, 412
- complex numbers $\mathbf{C}, 495$
- complex conjugation, 498
- composition of functions, 59
- conjunction (and), 354
- connectedness, 307, 430
- connected component, 433

- constant
 function, 58, 312
 sequence, 170
- continuity, 261, 420, 436
 and compactness, 427
 and connectedness, 431
 and convergence, 255, 421
- continuum, 242
 hypothesis, 227
- contraction, 558
 mapping theorem, 559
- contrapositive, 362
- convergence
 in L^2 , 517
 of a function at a point, 254, 441
 of sequences, 148, 394, 434
 of series, 190
 pointwise: *see* pointwise convergence
 uniform: *see* uniform convergence
- converse, 362
- convolution, 466, 487, 522
- corollary, 28
- coset, 587
- cosine: *see* trigonometric functions
- cotangent: *see* trigonometric functions
- countability, 208
 of the integers, 212
 of the rationals, 214
- cover, 578
 see also: open cover
- critical point, 572
- de Moivre identities, 507
- de Morgan laws, 47
- decimal
 negative integer, 384
 non-uniqueness of representation, 388
 point, 385
 positive integer, 384
 real, 385–386
- degree, 464
- denumerable: *see* countable
- dense, 465
- derivative, 288
 directional, 544
 in higher dimensions, 542, 544, 546, 551
 partial, 546
 matrix, 551
 total, 542, 544
 uniqueness of, 542
- difference rule, 293
- difference set, 47
- differential matrix: *see* derivative matrix
- differentiability
 at a point, 288
 continuous, 556
 directional, 544
 in higher dimensions, 542
 infinite, 478
 k-fold, 478, 556
- digit, 381
- dilation, 536
- diophantine, 616
- Dirac delta function, 466
- direct sum

- of functions, 75, 425
- discontinuity: *see* singularity
- discrete metric, 393
- disjoint sets, 47
- disjunction (or), 354
 - inclusive vs. exclusive, 354
- distance
 - in \mathbf{C} , 499
 - in \mathbf{Q} , 98
 - in \mathbf{R} , 145, 391
- distributive law
 - for natural numbers, 34
 - for complex numbers, 497
 - see also:* laws of algebra
- divergence
 - of series, 3, 190
 - of sequences, 4
 - see also:* convergence
- divisibility, 237
- division
 - by zero, 3
 - formal (//), 93
 - of functions, 252
 - of rationals, 96
- domain, 55
- dominate: *see* majorize
 - dominated convergence: *see* Lebesgue dominated convergence theorem
- doubly infinite, 244
- dummy variable: *see* bound variable
- e , 491
- Egoroff's theorem, 617
- empty
 - Cartesian product, 73
- function, 58
- sequence, 73
- series, 185
- set, 40, 576, 579
- equality, 377
 - for functions, 58
 - for sets, 39
 - of cardinality, 77
- equivalence
 - of sequences, 116, 281
 - relation, 378
- error-correcting codes, 392
- Euclidean algorithm, 35
- Euclidean metric, 391
- Euclidean space, 391
- Euler's formula, 503, 506
- Euler's number: *see e*
- exponential function, 490, 501
- exponentiation
 - of cardinals, 81
 - with base and exponent in \mathbf{N} , 36
 - with base in \mathbf{Q} and exponent in \mathbf{Z} , 101, 102
 - with base in \mathbf{R} and exponent in \mathbf{Z} , 140
 - with base in \mathbf{R}^+ and exponent in \mathbf{Q} , 142
 - with base in \mathbf{R}^+ and exponent in \mathbf{R} , 177
- expression, 353
- extended real number system
 - \mathbf{R}^* , 137, 153
- extremum: *see* maximum, minimum
- exterior (point), 401, 435

- factorial, 189
 family, 67
 Fatou's lemma, 614
 Fejér kernel, 524
 field, 95
 ordered, 97
 finite intersection property, 419
 finite set, 80
 fixed point theorem, 276, 559
 forward image: *see* image
 Fourier
 coefficients, 520
 inversion formula, 520
 series, 520
 series for arbitrary periods, 531
 theorem, 536
 transform, 520
 fractional part, 512
 free variable, 368
 frequency, 518
 Fubini's theorem, 624
 for finite series, 188
 for infinite series, 217
 see also: interchanging integrals/sums with integrals/sums
 function, 55
 implicit definition, 57
 fundamental theorems of calculus, 338, 341
 geometric series, 190, 196
 formula, 197, 200, 460
 geodesic, 394
 gradient, 550
 graph, 58, 75, 251, 568
 greatest lower bound: *see* least upper bound
 hairy ball theorem, 559
 half-infinite, 244
 half-open, 243
 half-space, 591
 harmonic series, 199
 Hausdorff space, 437, 438
 Hausdorff maximality principle, 240
 Heine-Borel theorem, 414
 for the real line, 248
 Hermitian form, 515
 homogeneity, 516, 535
 hypersurface, 568
 identity map (or operator), 63, 536
 if: *see* implication
 iff (if and only if), 30
 ill-defined, 351, 353
 image
 of sets, 64
 inverse image, 65
 imaginary, 498
 implication (if), 357
 implicit differentiation, 568
 implicit function theorem, 568
 improper integral, 318
 inclusion map, 63
 inconsistent, 227, 228, 502
 index of summation: *see* dummy variable
 index set, 67
 indicator function: *see* characteristic function

- induced
 - metric, 391, 407
 - topology, 407, 435
- induction: *see* Principle of mathematical induction
- infinite
 - interval, 244
 - set, 80
- infimum: *see* supremum
- injection: *see* one-to-one function
- inner product, 514
- integer part, 103, 133, 512
- integers \mathbf{Z}
 - definition, 85
 - identification with rationals, 94
 - interspersing with rationals, 103
- integral test, 332
- integration
 - by parts, 343–345, 484
 - laws, 315, 321
 - piecewise constant, 313, 315
 - Riemann: *see* Riemann integral
- interchanging
 - derivatives with derivatives, 10, 556
 - finite sums with finite sums, 187, 188
 - integrals with integrals, 7, 614, 624
 - limits with derivatives, 9, 463
- limits with integrals, 9, 462, 610, 619
- limits with length, 12
- limits with limits, 8, 9, 450
- limits with sums, 617
- sums with derivatives, 463, 476
- sums with integrals, 459, 476, 613, 614, 616
- sums with sums, 6, 217
- interior (point), 401, 435
- intermediate value theorem, 274, 432
- intersection
 - pairwise, 46
- interval, 243
- intrinsic, 413
- inverse
 - function theorem, 301, 562
 - image, 65
 - in logic, 362
 - of functions, 63
- invertible function: *see* bijection
 - local, 562
- involution, 498
- irrationality, 108
 - of $\sqrt{2}$, 104, 137
- isolated point, 247
- isometry, 406
- jump discontinuity, 268
- $l^1, l^2, l^\infty, L^1, L^2, L^\infty$, 391–393, 516, 617

- equivalence of in finite dimensions, 396
 - see also:* absolutely integrable
 - see also:* supremum as norm
- L'Hôpital's rule, 11, 303
- label, 67
- laws of algebra
 - for complex numbers, 496, 497
 - for integers, 89
 - for rationals, 95
 - for reals, 122
- laws of arithmetic: *see* laws of algebra
- laws of exponentiation, 101, 102, 141, 143, 177, 490
- least upper bound, 134
 - least upper bound property, 135, 158
 - see also* supremum
- Lebesgue dominated convergence theorem, 619
- Lebesgue integral
 - of absolutely integrable functions, 618
 - of nonnegative functions, 608
 - of simple functions, 604
 - upper and lower, 620
 - vs. the Riemann integral, 622
- Lebesgue measurable, 590
- Lebesgue measure, 577
 - motivation of, 575-577
- Lebesgue monotone convergence theorem, 610
- Leibnitz rule, 293, 554
- lemma, 28
- length of interval, 308
- limit
 - at infinity, 286
 - formal (LIM), 118, 150, 412
 - laws, 150, 256, 500
 - left and right, 265
 - limiting values of functions, 5, 254, 441
 - of sequences, 148
 - pointwise, 444
 - uniform, *see* uniform limit
 - uniqueness of, 148, 256, 397, 442
- limit inferior, *see* limit superior
- limit point
 - of sequences, 160, 409
 - of sets, 247
- limit superior, 162
- linear combination, 535
- linearity
 - approximate, 541
 - of convolution, 471, 522
 - of finite series, 186
 - of limits, 151
 - of infinite series, 194
 - of inner product, 515
 - of integration, 315, 321, 606, 612
 - of transformations, 535
- Lipschitz constant, 298
- Lipschitz continuous, 298

- logarithm (natural), 492
 - power series of, 460, 492
- logical connective, 354
- lower bound: *see* upper bound
- majorize, 317, 608
- manifold, 572
- map: *see* function
- matrix, 536
 - identification with linear transformations, 537-540
- maximum, 233, 296
 - local, 296
 - of functions, 252, 271
 - principle, 271, 427
- mean value theorem, 297
- measurability
 - for functions, 597, 598
 - for sets, 590
 - motivation of, 574
 - see also:* Lebesgue measure, outer measure
- meta-proof, 140
- metric, 390
 - ball: *see* ball
 - on **C**, 499
 - on **R**, 391
 - space, 390
 - see also:* distance
- minimum, 233, 296
 - local, 296
 - of a set of natural numbers, 210
 - of functions, 252, 271
- minorize: *see* majorize
- monomial, 518
- monotone (increasing or decreasing)
- convergence: *see* Lebesgue monotone convergence theorem
- function, 276, 336
- measure, 576, 580
- sequence, 159
- morphism: *see* function
- moving bump example, 446, 614
- multiplication
 - of cardinals, 81
 - of complex numbers, 497
 - of functions, 252
 - of integers, 86
 - of matrices, 536, 540
 - of natural numbers, 33
 - of rationals, 93, 94
 - of reals, 120
- Natural numbers **N**
 - are infinite, 80
 - axioms: *see* Peano axioms
 - identification with integers, 87
 - informal definition, 17
 - in set theory: *see* Axiom of infinity
 - uniqueness of, 76
- negation
 - in logic, 355
 - of extended reals, 154
 - of complex numbers, 497
 - of integers, 88
 - of rationals, 93
 - of reals, 121

negative: *see* negation, positive
 neighbourhood, 434
 Newton's approximation, 291, 544
 non-constructive, 229
 non-degenerate, 516
 nowhere differentiable function, 464, 508
 objects, 38
 primitive, 53
 one-to-one function, 61
 one-to-one correspondence: *see* bijection
 onto, 61
 open
 box, 578
 cover, 414
 interval, 243
 set, 403
 or: *see* disjunction
 order ideal, 238
 order topology, 437
 ordered pair, 70
 construction of, 74
 ordered n -tuple, 71
 ordering
 lexicographical, 239
 of cardinals, 227
 of orderings, 240
 of partitions, 310
 of sets, 233
 of the extended reals, 154
 of the integers, 91
 of the natural numbers, 31

of the rationals, 97
 of the reals, 129
 orthogonality, 516
 orthonormal, 519
 oscillatory discontinuity, 268
 outer measure, 579
 non-additivity of, 587, 589
 pair set, 41
 partial function, 69
 partially ordered set, 45, 232
 partial sum, 190
 Parseval identity, 531
 see also: Plancherel formula
 partition, 308
 path-connected, 432
 Peano axioms, 18-21, 23
 perfect matching: *see* bijection
 periodic, 511
 extension, 512
 piecewise
 constant, 312
 constant Riemann-Stieltjes integral, 335
 continuous, 330
 pigeonhole principle, 83
 Plancherel formula (or theorem), 520, 528
 pointwise convergence, 444
 of series, 456
 topology of, 455
 polar representation, 507
 polynomial, 265, 464
 and convolution, 467
 approximation by, 465, 470

- positive
 complex number, 498, 502
 integer, 88
 inner product, 515
 measure, 576, 580
 natural number, 30
 rational, 96
 real, 128
- power series, 476
 formal, 474
 multiplication of, 487
 uniqueness of, 481
- power set, 66
- pre-image: *see* inverse image
- principle of infinite descent, 106
- principle of mathematical induction, 21
 backwards induction, 33
 strong induction, 32, 234
 transfinite, 237
- product rule, *see* Leibnitz rule
- product topology, 455
- projection, 536
- proof
 by contradiction, 352, 363
 abstract examples, 364-367, 375-377
- proper subset, 44
- property, 354
- proposition, 28
- propositional logic, 367
- Pythagoras' theorem, 516
- quantifier, 369
 existential (for some), 371
 negation of, 372
- nested, 372
 universal (for all), 370
- Quotient: *see* division
- Quotient rule, 293, 555
- radius of convergence, 475
- range, 55
- ratio test, 206
- rational numbers **Q**
 definition, 93
 identification with reals, 121
 interspersing with rationals, 103
 interspersing with reals, 132
- real analytic, 478
- real numbers **R**
 are uncountable: *see* uncountability of the reals
 definition, 117
- real part, 498
- real-valued, 455
- rearrangement
 of absolutely convergent series, 202
 of divergent series, 203, 222
 of finite series, 185
 of non-negative series, 200
- reciprocal
 of complex numbers, 499
 of rationals, 95
 of reals, 125
- recursive definitions, 26, 76

- reductio ad absurdum*: *see* proof by contradiction
- relative topology: *see* induced topology
- removable discontinuity: *see* removable singularity
- removable singularity, 259, 268
- restriction of functions, 250
- Riemann hypothesis, 200
- Riemann integrability, 318
- closure properties, 321–326
 - failure of, 332
 - of bounded continuous functions, 328
 - of continuous functions on compacta, 328
 - of monotone functions, 330
 - of piecewise continuous bounded functions, 329
 - of uniformly continuous functions, 326
- Riemann integral, 318
- upper and lower, 317
- Riemann sums (upper and lower), 321
- Riemann zeta function, 199
- Riemann-Stieltjes integral, 336
- ring, 89
- commutative, 89, 497
- Rolle's theorem, 297
- root, 140
- mean square: *see* L^2
 - test, 204
- row vector, 533
- Russell's paradox, 52
- scalar multiplication, 533
- of functions, 252
- Schröder-Bernstein theorem, 227
- sequence, 109
- finite, 74
- series
- finite, 179, 182
 - formal infinite, 189
 - laws, 194, 220
 - of functions, 459
 - on arbitrary sets, 220
 - on countable sets, 216
 - vs. sum, 180
- set
- axioms: *see* axioms of set theory
 - informal definition, 38
- signum function, 258
- simple function, 602
- sine: *see* trigonometric functions
- singleton set, 41
- singularity, 268
- space, 390
- statement, 350
- sub-additive measure, 576, 580
- subset, 44
- subsequence, 172, 408
- substitution: *see* rearrangement
- subtraction
- formal (—), 86
 - of functions, 252
 - of integers, 91
- sum rule, 292
- summation by parts, 484

- sup norm: *see* supremum as norm
 support, 465
 supremum (and infimum)
 as metric, 393
 as norm, 393, 457
 of a set of extended reals, 156, 157
 of a set of reals, 137, 139
 of sequences of reals, 158
 square root, 56
 square wave, 512, 518
 Squeeze test
 for sequences, 167
 Stone-Weierstrass theorem, 472, 522
 strict upper bound, 235
 surjection: *see* onto
 taxi-cab metric, 392
 tangent: *see* trigonometric function
 Taylor series, 480
 Taylor's formula: *see* Taylor series
 telescoping series, 195
 ten, 381
 theorem, 28
 topological space, 433
 totally bounded, 418
 totally ordered set, 45, 233
 transformation: *see* function
 translation invariance, 577, 580, 591
 transpose, 534
 triangle inequality
 in Euclidean space, 399
 in inner product spaces, 516
 in metric spaces, 390
 in **C**, 499
 in **R**, 99
 for finite series, 181, 186
 for integrals, 618
 trichotomy of order
 of extended reals, 155
 for natural numbers, 31
 for integers, 91
 for rationals, 97
 for reals, 129
 trigonometric functions, 503, 509
 and Fourier series, 530
 trigonometric polynomials, 518
 power series, 504, 508
 trivial topology, 437
 two-to-one function, 61
 uncountability, 208
 of the reals, 225
 undecidable, 228
 uniform continuity, 280, 428
 uniform convergence, 447
 and anti-derivatives, 462
 and derivatives, 451
 and integrals, 459
 and limits, 450
 and radius of convergence, 476
 as a metric, 453, 514
 of series, 457
 uniform limit, 447
 of bounded functions, 451

- of continuous functions, 450
- and Riemann integration,
 - 458
- union, 67
 - pairwise, 42
- universal set, 53
- upper bound,
 - of a set of reals, 133
 - of a partially ordered set,
 - 234
- see also:* least upper bound
- variable, 368
- vector space, 534
- vertical line test, 55, 76, 567
- volume, 578
- Weierstrass approximation theorem, 465, 470-471, 521
- Weierstrass example: *see* nowhere differentiable function
- Weierstrass *M*-test, 457
- well-defined, 351
- well-ordered sets, 234
- well ordering principle
 - for natural numbers, 210
 - for arbitrary sets, 241
- Zermelo-Fraenkel(-Choice) axioms, 69
 - see also* axioms of set theory
- zero test
 - for sequences, 167
 - for series, 191