Definition. Define the *tangent space to V at v* to be the linear variety

$$\mathbb{T}_{v,V} = \mathcal{Z}(\{D_v(f)(x_1, \dots, x_n) \mid f \in \mathcal{I}(V)\}).$$

The formal partial derivatives are k-linear and obey the usual product rule for derivatives, so the tangent space may be computed from the generators for $\mathcal{I}(V)$:

$$\text{if} \quad \mathcal{I}(V) = (f_1, f_2, \dots, f_m) \qquad \text{then} \quad \mathbb{T}_{v,V} = \bigcap_{i=1}^{m} \mathcal{Z}(D_v(f_i)).$$

Note that $\mathbb{T}_{v,V}$ is an intersection of vector spaces, so is a vector subspace of k^n.

This definition of the tangent space $\mathbb{T}_{v,V}$, while making apparent the connection with tangents to the variety V, seems to depend on the embedding of V in $\mathbb{A}^n$. In fact the tangent space can be defined entirely in terms of the local ring $\mathcal{O}_{v,V}$, as the next proposition proves.

Proposition 52. Let V be an affine variety over the algebraically closed field k and let v be a point on V with local ring $\mathcal{O}_{v,V}$ and corresponding maximal ideal $\mathfrak{m}_{v,V}$. Then there is a k-vector space isomorphism

$$(\mathbb{T}_{v,V})^* \cong \mathfrak{m}_{v,V}/\mathfrak{m}_{v,V}^2$$

where $(\mathbb{T}_{v,V})^*$ denotes the vector space dual (cf. Section 11.3) of the tangent space $\mathbb{T}_{v,V}$ to V at v.

Proof: Let $(k^n)^*$ denote the n-dimensional vector space dual to k^n. Since each $D_v(f)$ is a linear function, D_v is a linear transformation from $k[x_1, \dots, x_n]$ to $(k^n)^*$.

Let M_v be the maximal ideal in $k[x_1, \dots, x_n]$ generated by the set $x_i - v_i$ for $1 \le i \le n$. The image $M_v/\mathcal{I}(V)$ of M_v in $k[V]$ is the ideal $\mathcal{I}(v)$ of functions on V that are zero at v and $\mathcal{I}(v)^2 = M_v^2 + \mathcal{I}(V)$. Then $\mathcal{O}_{v,V}$ is the localization of $k[V]$ at $\mathcal{I}(v)$; and identifying $\mathcal{I}(v)$ with its image in $\mathcal{O}_{v,V}$ we have $\mathfrak{m}_{v,V} = \mathcal{I}(v)\mathcal{O}_{v,V}$ (Proposition 46(2)). By definition of D_v we have $D_v(x_i - v_i) = x_i$, and since these linear functions form a basis of $(k^n)^*$, it follows that D_v maps M_v surjectively onto $(k^n)^*$. The kernel of D_v consists of the elements of $k[x_1, \dots, x_n]$ whose Taylor expansion at v starts in degree at least 2 and these are just the elements in M_v^2. Hence D_v defines an isomorphism

$$D_v : M_v/M_v^2 \xrightarrow{\sim} (k^n)^*.$$

The tangent space $\mathbb{T}_{v,V}$ is a vector subspace of k^n, so every linear function on k^n restricts to a linear function on $\mathbb{T}_{v,V}$. Composing D_v with this restriction map gives a linear transformation

$$D : M_v \xrightarrow{D_v} (k^n)^* \xrightarrow{\text{res}} (\mathbb{T}_{v,V})^*$$

which is surjective since the individual maps are each surjective. We have already seen that $\mathcal{I}(v)^2 = M_v^2 + \mathcal{I}(V)$, so $\mathcal{I}(v)/\mathcal{I}(v)^2 \cong M_v/(M_v^2 + \mathcal{I}(V))$. It follows by Proposition 46(5) that $\mathfrak{m}_{v,V}/\mathfrak{m}_{v,V}^2 \cong \mathcal{I}(v)/\mathcal{I}(v)^2$. To prove the proposition it is therefore sufficient to show that $\ker D = M_v^2 + \mathcal{I}(V)$, since then

$$\mathfrak{m}_{v,V}/\mathfrak{m}_{v,V}^2 \cong M_v/(M_v^2 + \mathcal{I}(V)) = M_v/\ker D \cong (\mathbb{T}_{v,V})^*.$$

The polynomial f is in $\ker D$ if and only if $D_v(f)$ is zero on $\mathbb{T}_{v,V}$, i.e., if and only if the linear term of the Taylor polynomial of f expanded about v lies in $\mathcal{I}(\mathbb{T}_{v,V})$. Since the linear terms of the functions in $\mathcal{I}(V)$ generate the ideal $\mathcal{I}(\mathbb{T}_{v,V})$, it follows that f is in $\ker D$ if and only if $f - g$ has zero linear term for some g in $\mathcal{I}(V)$. But this is equivalent to $f \in \mathcal{I}(V) + M_v^2$, so $\ker D = \mathcal{I}(V) + M_v^2$, completing the proof of the proposition.

Recall that the *dimension* of a variety V is by definition the transcendence degree of the field $k(V)$ over k. Since each local ring $\mathcal{O}_{v,V}$ has $k(V)$ as its field of fractions, the dimension of V is determined by the transcendence degree over k of the field of fractions of any of its local rings.

Definition. We say V is *nonsingular* at the point $v \in V$ (or v is a *nonsingular point* of V) if the dimension of the k-vector space $\mathbb{T}_{v,V}$ is $\dim V$. Equivalently (by Proposition 52), v is a nonsingular point of V if $\dim_k(\mathfrak{m}_{v,V}/\mathfrak{m}_{v,V}^2) = \dim V$. Otherwise the point v is called a *singular point*. The variety V is *nonsingular* or *smooth* if it is nonsingular at every point.

The geometric picture is that at a nonsingular point v there are as many independent tangents as one would expect: a tangent line on a curve, a tangent plane on a surface, etc.

Whether a variety V is nonsingular at a point v can be determined from properties of the local ring $\mathcal{O}_{v,V}$, namely whether $\dim_k(\mathfrak{m}_{v,V}/\mathfrak{m}_{v,V}^2) = \dim \mathcal{O}_{v,V}$. A local ring having this property is said to be a *regular local ring*. In particular, the notion of singularity does not depend on the embedding of V in a specific affine space. This algebraic interpretation can be used to *define* smoothness for abstract algebraic varieties, where the geometric intuition of tangent planes to surfaces (for example) is not as obvious.

If $f_1, \dots, f_m$ are generators for $\mathcal{I}(V)$ defining V in $\mathbb{A}^n$, then the dimension of V can be determined from a Gröbner basis for $\mathcal{I}(V)$ (cf. Exercise 29). Determining the dimension of the tangent space $\mathbb{T}_{v,V}$ as a vector space over k is a linear algebra problem: this vector space is the set of solutions of the m linear equations $D_v(f_i)(x_1, \dots, x_n) = 0$. If r is the rank of the $m \times n$ matrix of coefficients $\partial f_i/\partial x_j(v)$ of this system of equations, then $\mathbb{T}_{v,V}$ is a vector space of dimension $n-r$. Using this it is not too difficult to establish the following:

1. We have $\dim V \le \dim_k(\mathbb{T}_{v,V}) \le n$ for every point v in $V \subseteq \mathbb{A}^n$.

2. The set of singular points of V is a proper Zariski closed subset of V. The set of nonsingular points of V is a nonempty open subset of V; in particular the nonsingular points of V are dense in V (so "most" points of V are nonsingular).

We also state without proof the following result which further relates the local geometry of V to the algebraic properties of the local rings of V:

3. If v is a nonsingular point, then the local ring $\mathcal{O}_{v,V}$ is a Unique Factorization Domain; in particular, $\mathcal{O}_{v,V}$ is integrally closed (cf. Example 3 following Corollary 25).

The variety V is said to be *factorial* if $\mathcal{O}_{v,V}$ is a U.F.D. for every point $v \in V$, and is said to be a *normal* variety if $\mathcal{O}_{v,V}$ is integrally closed for every $v \in V$ (which by Proposition 49 is equivalent to $k[V]$ being integrally closed). By (3) above we have

$$\textit{smooth varieties} \quad \subseteq \quad \textit{factorial varieties} \quad \subseteq \quad \textit{normal varieties.}$$

In general each of the above containments is proper. In the case when V has dimension 1, i.e., V is an *affine curve*, however, these three properties are in fact equivalent: we shall prove later that an irreducible affine curve is smooth if and only if it is normal or factorial (cf. Corollary 13 in Section 16.2). It follows that over an algebraically closed field k,

an irreducible affine curve C is smooth if and only if $k[C]$ is integrally closed.

For any irreducible affine curve C the integral closure, S, of $k[V]$ in $k(V)$ is also the coordinate ring of an irreducible affine curve $\widetilde{C}$. Then S is integral over $k[V]$ and, by Theorem 30 and Corollary 27 it follows that there is a morphism from the smooth curve $\widetilde{C}$ onto C that has finite fibers. The curve $\widetilde{C}$ is called the *normalization* or the *nonsingular model* of C, and one can show that it is unique up to isomorphism. Note how the existence of a smooth curve mapping finitely to C (a problem in "geometry") is solved by the existence of integral closures in ring extensions (a problem in "algebra").

We shall give another characterization of smoothness for irreducible affine curves at the end of Section 16.2.

EXERCISES

As usual R is a commutative ring with 1 and D is a multiplicatively closed set in R.

1. Suppose M is a finitely generated R-module. Prove that $D^{-1}M = 0$ if and only if $dM = 0$ for some $d \in D$.
2. Let I be an ideal in R, let D be a multiplicatively closed subset of R with ring of fractions $D^{-1}R$, and let ${}^{c}({}^{e}I) = R$ be the saturation of I with respect to D.
   (a) Prove that ${}^{c}({}^{e}I) = R$ if and only if ${}^{e}I = D^{-1}R$ if and only if $I \cap D \neq \emptyset$.
   (b) Prove that $I = {}^{c}({}^{e}I)$ is saturated if and only if for every $d \in D$, if $da \in I$ then $a \in I$.
   (c) Prove that extension and contraction define inverse bijections between the ideals of R saturated with respect to D and the ideals of $D^{-1}R$.
   (d) Let $I = (2x, 3y) \subset \mathbb{Z}[x, y]$. Show the saturation of I with respect to $\mathbb{Z} - \{0\}$ is (x, y).
3. If I is an ideal in the commutative ring R let $\varphi : R[x_1, \dots, x_n] \cong (R/I)[x_1, \dots, x_n]$ be the ring homomorphism with kernel $I[x_1, \dots, x_n]$ given by reducing coefficients modulo I. If $\overline{A}$ is an ideal in $(R/I)[x_1, \dots, x_n]$, let A denote the inverse image of $\overline{A}$ under φ.
   (a) For any $i \geq 1$ show that the inverse image under φ of the subring $(R/I)[x_1, \dots, x_i]$ is $R[x_1, \dots, x_i] + I[x_1, \dots. x_n]$.
   (b) Prove that $\varphi(A \cap R[x_1, \dots, x_i]) = \overline{A} \cap (R/I)[x_1, \dots, x_i]$
4. Let $f = y^5 - z^4$, viewed as a polynomial in y with coefficients in $\mathbb{Q}[z]$.
   (a) Prove that f has no roots in $\mathbb{Q}[z]$.
   (b) Suppose $f = (y^2 + ay + b)(y^3 + cy^2 + dy + e)$. Show that a, b, c, d, e satisfy the system of equations

   $$a + c = 0, \quad ac + b + d = 0, \quad ad + bc + e = 0, \quad ae + bd = 0, \quad be - z^4 = 0.$$

   Deduce that $e^5 = z^{12}$ and conclude that f is irreducible in $\mathbb{Q}[y, z]$. [Use elimination.]