3.18 Examples

(*a*) Let $\{s_n\}$ be a sequence containing all rationals. Then every real number is a subsequential limit, and

$$\limsup_{n\to\infty} s_n = +\infty, \qquad \liminf_{n\to\infty} s_n = -\infty.$$

(*b*) Let $s_n = (-1^n)/[1 + (1/n)]$. Then

$$\limsup_{n\to\infty} s_n = 1, \qquad \liminf_{n\to\infty} s_n = -1.$$

(*c*) For a real-valued sequence $\{s_n\}$, $\lim_{n\to\infty} s_n = s$ if and only if

$$\limsup_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = s.$$

We close this section with a theorem which is useful, and whose proof is quite trivial:

3.19 Theorem *If $s_n \le t_n$ for $n \ge N$, where N is fixed, then*

$$\liminf_{n\to\infty} s_n \le \liminf_{n\to\infty} t_n,$$

$$\limsup_{n\to\infty} s_n \le \limsup_{n\to\infty} t_n.$$

SOME SPECIAL SEQUENCES

We shall now compute the limits of some sequences which occur frequently. The proofs will all be based on the following remark: If $0 \le x_n \le s_n$ for $n \ge N$, where N is some fixed number, and if $s_n \to 0$, then $x_n \to 0$.

3.20 Theorem

(*a*) *If $p > 0$, then* $\lim_{n\to\infty} \dfrac{1}{n^p} = 0$.

(*b*) *If $p > 0$, then* $\lim_{n\to\infty} \sqrt[n]{p} = 1$.

(*c*) $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

(*d*) *If $p > 0$ and α is real, then* $\lim_{n\to\infty} \dfrac{n^\alpha}{(1+p)^n} = 0$.

(*e*) *If $|x| < 1$, then* $\lim_{n\to\infty} x^n = 0$.

Proof

(*a*) Take $n > (1/\varepsilon)^{1/p}$. (Note that the archimedean property of the real number system is used here.)

(*b*) If $p > 1$, put $x_n = \sqrt[n]{p} - 1$. Then $x_n > 0$, and, by the binomial theorem,

$$1 + nx_n \le (1 + x_n)^n = p,$$

so that

$$0 < x_n \le \frac{p-1}{n}.$$

Hence $x_n \to 0$. If $p = 1$, (*b*) is trivial, and if $0 < p < 1$, the result is obtained by taking reciprocals.

(*c*) Put $x_n = \sqrt[n]{n} - 1$. Then $x_n \ge 0$, and, by the binomial theorem,

$$n = (1 + x_n)^n \ge \frac{n(n-1)}{2} x_n^2.$$

Hence

$$0 \le x_n \le \sqrt{\frac{2}{n-1}} \qquad (n \ge 2).$$

(*d*) Let k be an integer such that $k > \alpha$, $k > 0$. For $n > 2k$,

$$(1 + p)^n > \binom{n}{k} p^k = \frac{n(n-1)\cdots(n-k+1)}{k!} p^k > \frac{n^k p^k}{2^k k!}.$$

Hence

$$0 < \frac{n^\alpha}{(1+p)^n} < \frac{2^k k!}{p^k} n^{\alpha - k} \qquad (n > 2k).$$

Since $\alpha - k < 0$, $n^{\alpha - k} \to 0$, by (*a*).

(*e*) Take $\alpha = 0$ in (*d*).

SERIES

In the remainder of this chapter, all sequences and series under consideration will be complex-valued, unless the contrary is explicitly stated. Extensions of some of the theorems which follow, to series with terms in R^k, are mentioned in Exercise 15.

3.21 Definition Given a sequence $\{a_n\}$, we use the notation

$$\sum_{n=p}^{q} a_n \qquad (p \le q)$$

to denote the sum $a_p + a_{p+1} + \cdots + a_q$. With $\{a_n\}$ we associate a sequence $\{s_n\}$, where

$$s_n = \sum_{k=1}^{n} a_k .$$

For $\{s_n\}$ we also use the symbolic expression

$$a_1 + a_2 + a_3 + \cdots$$

or, more concisely,

$$\sum_{n=1}^{\infty} a_n . \tag{4}$$

The symbol (4) we call an *infinite series*, or just a *series*. The numbers s_n are called the *partial sums* of the series. If $\{s_n\}$ converges to s, we say that the series *converges*, and write

$$\sum_{n=1}^{\infty} a_n = s.$$

The number s is called the sum of the series; but it should be clearly understood that *s is the limit of a sequence of sums*, and is not obtained simply by addition.

If $\{s_n\}$ diverges, the series is said to diverge.

Sometimes, for convenience of notation, we shall consider series of the form

$$\sum_{n=0}^{\infty} a_n . \tag{5}$$

And frequently, when there is no possible ambiguity, or when the distinction is immaterial, we shall simply write Σa_n in place of (4) or (5).

It is clear that every theorem about sequences can be stated in terms of series (putting $a_1 = s_1$, and $a_n = s_n - s_{n-1}$ for $n > 1$), and vice versa. But it is nevertheless useful to consider both concepts.

The Cauchy criterion (Theorem 3.11) can be restated in the following form:

3.22 Theorem *Σa_n converges if and only if for every $\varepsilon > 0$ there is an integer N such that*

$$\left| \sum_{k=n}^{m} a_k \right| \le \varepsilon \tag{6}$$

if $m \ge n \ge N$.

In particular, by taking $m = n$, (6) becomes

$$|a_n| \le \varepsilon \qquad (n \ge N).$$

In other words:

3.23 Theorem *If Σa_n converges, then* $\lim_{n\to\infty} a_n = 0$.

The condition $a_n \to 0$ is not, however, sufficient to ensure convergence of Σa_n. For instance, the series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges; for the proof we refer to Theorem 3.28.

Theorem 3.14, concerning monotonic sequences, also has an immediate counterpart for series.

3.24 Theorem *A series of nonnegative*[1] *terms converges if and only if its partial sums form a bounded sequence.*

We now turn to a convergence test of a different nature, the so-called "comparison test."

3.25 Theorem

(*a*) *If $|a_n| \le c_n$ for $n \ge N_0$, where N_0 is some fixed integer, and if Σc_n converges, then Σa_n converges.*
(*b*) *If $a_n \ge d_n \ge 0$ for $n \ge N_0$, and if Σd_n diverges, then Σa_n diverges.*

Note that (*b*) applies only to series of nonnegative terms a_n.

Proof Given $\varepsilon > 0$, there exists $N \ge N_0$ such that $m \ge n \ge N$ implies

$$\sum_{k=n}^{m} c_k \le \varepsilon,$$

by the Cauchy criterion. Hence

$$\left| \sum_{k=n}^{m} a_k \right| \le \sum_{k=n}^{m} |a_k| \le \sum_{k=n}^{m} c_k \le \varepsilon,$$

and (*a*) follows.

Next, (*b*) follows from (*a*), for if Σa_n converges, so must Σd_n [note that (*b*) also follows from Theorem 3.24].

[1] The expression "nonnegative" always refers to *real* numbers.

The comparison test is a very useful one; to use it efficiently, we have to become familiar with a number of series of nonnegative terms whose convergence or divergence is known.

SERIES OF NONNEGATIVE TERMS

The simplest of all is perhaps the geometric series.

3.26 Theorem *If $0 \le x < 1$, then*

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

If $x \ge 1$, the series diverges.

Proof If $x \ne 1$,

$$s_n = \sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}.$$

The result follows if we let $n \to \infty$. For $x = 1$, we get

$$1 + 1 + 1 + \cdots,$$

which evidently diverges.

In many cases which occur in applications, the terms of the series decrease monotonically. The following theorem of Cauchy is therefore of particular interest. The striking feature of the theorem is that a rather "thin" subsequence of $\{a_n\}$ determines the convergence or divergence of Σa_n.

3.27 Theorem *Suppose $a_1 \ge a_2 \ge a_3 \ge \cdots \ge 0$. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series*

$$\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots \tag{7}$$

converges.

Proof By Theorem 3.24, it suffices to consider boundedness of the partial sums. Let

$$s_n = a_1 + a_2 + \cdots + a_n,$$
$$t_k = a_1 + 2a_2 + \cdots + 2^k a_{2^k}.$$

For $n < 2^k$,

$$\begin{aligned} s_n &\le a_1 + (a_2 + a_3) + \cdots + (a_{2^k} + \cdots + a_{2^{k+1}-1}) \\ &\le a_1 + 2a_2 + \cdots + 2^k a_{2^k} \\ &= t_k, \end{aligned}$$

so that

$$s_n \le t_k. \tag{8}$$

On the other hand, if $n > 2^k$,

$$\begin{aligned} s_n &\ge a_1 + a_2 + (a_3 + a_4) + \cdots + (a_{2^{k-1}+1} + \cdots + a_{2^k}) \\ &\ge \tfrac{1}{2}a_1 + a_2 + 2a_4 + \cdots + 2^{k-1}a_{2^k} \\ &= \tfrac{1}{2}t_k, \end{aligned}$$

so that

$$2s_n \ge t_k. \tag{9}$$

By (8) and (9), the sequences $\{s_n\}$ and $\{t_k\}$ are either both bounded or both unbounded. This completes the proof.

3.28 Theorem $\sum \frac{1}{n^p}$ *converges if* $p > 1$ *and diverges if* $p \le 1$.

Proof If $p \le 0$, divergence follows from Theorem 3.23. If $p > 0$, Theorem 3.27 is applicable, and we are led to the series

$$\sum_{k=0}^{\infty} 2^k \cdot \frac{1}{2^{kp}} = \sum_{k=0}^{\infty} 2^{(1-p)k}.$$

Now, $2^{1-p} < 1$ if and only if $1 - p < 0$, and the result follows by comparison with the geometric series (take $x = 2^{1-p}$ in Theorem 3.26).

As a further application of Theorem 3.27, we prove:

3.29 Theorem *If* $p > 1$,

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p} \tag{10}$$

converges; if $p \le 1$, *the series diverges.*

Remark "$\log n$" denotes the logarithm of n to the base e (compare Exercise 7, Chap. 1); the number e will be defined in a moment (see Definition 3.30). We let the series start with $n = 2$, since $\log 1 = 0$.

Proof The monotonicity of the logarithmic function (which will be discussed in more detail in Chap. 8) implies that $\{\log n\}$ increases. Hence $\{1/n \log n\}$ decreases, and we can apply Theorem 3.27 to (10); this leads us to the series

$$\sum_{k=1}^{\infty} 2^k \cdot \frac{1}{2^k(\log 2^k)^p} = \sum_{k=1}^{\infty} \frac{1}{(k \log 2)^p} = \frac{1}{(\log 2)^p} \sum_{k=1}^{\infty} \frac{1}{k^p}, \tag{11}$$

and Theorem 3.29 follows from Theorem 3.28.

This procedure may evidently be continued. For instance,

$$\sum_{n=3}^{\infty} \frac{1}{n \log n \log \log n} \tag{12}$$

diverges, whereas

$$\sum_{n=3}^{\infty} \frac{1}{n \log n (\log \log n)^2} \tag{13}$$

converges.

We may now observe that the terms of the series (12) differ very little from those of (13). Still, one diverges, the other converges. If we continue the process which led us from Theorem 3.28 to Theorem 3.29, and then to (12) and (13), we get pairs of convergent and divergent series whose terms differ even less than those of (12) and (13). One might thus be led to the conjecture that there is a limiting situation of some sort, a "boundary" with all convergent series on one side, all divergent series on the other side—at least as far as series with monotonic coefficients are concerned. This notion of "boundary" is of course quite vague. The point we wish to make is this: No matter how we make this notion precise, the conjecture is false. Exercises 11(*b*) and 12(*b*) may serve as illustrations.

We do not wish to go any deeper into this aspect of convergence theory, and refer the reader to Knopp's "Theory and Application of Infinite Series," Chap. IX, particularly Sec. 41.

THE NUMBER e

3.30 Definition $e = \sum_{n=0}^{\infty} \frac{1}{n!}$.

Here $n! = 1 \cdot 2 \cdot 3 \cdots n$ if $n \geq 1$, and $0! = 1$.

Since

$$\begin{aligned} s_n &= 1 + 1 + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \cdots + \frac{1}{1\cdot 2\cdots n} \\ &< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} < 3, \end{aligned}$$

the series converges, and the definition makes sense. In fact, the series converges very rapidly and allows us to compute e with great accuracy.

It is of interest to note that e can also be defined by means of another limit process; the proof provides a good illustration of operations with limits:

3.31 Theorem $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$

Proof Let

$$s_n = \sum_{k=0}^{n}\frac{1}{k!}, \qquad t_n = \left(1+\frac{1}{n}\right)^n.$$

By the binomial theorem,

$$\begin{aligned} t_n = 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots \\ + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right). \end{aligned}$$

Hence $t_n \le s_n$, so that

$$\limsup_{n\to\infty} t_n \le e, \tag{14}$$

by Theorem 3.19. Next, if $n \ge m$,

$$t_n \ge 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \cdots + \frac{1}{m!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{m-1}{n}\right).$$

Let $n\to\infty$, keeping m fixed. We get

$$\liminf_{n\to\infty} t_n \ge 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{m!},$$

so that

$$s_m \le \liminf_{n\to\infty} t_n .$$

Letting $m\to\infty$, we finally get

$$e \le \liminf_{n\to\infty} t_n . \tag{15}$$

The theorem follows from (14) and (15).

The rapidity with which the series $\sum \frac{1}{n!}$ converges can be estimated as follows: If s_n has the same meaning as above, we have

$$e - s_n = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \cdots$$
$$< \frac{1}{(n+1)!}\left\{1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots\right\} = \frac{1}{n!n}$$

so that

(16) $$0 < e - s_n < \frac{1}{n!n}.$$

Thus s_{10}, for instance, approximates e with an error less than 10^{-7}. The inequality (16) is of theoretical interest as well, since it enables us to prove the irrationality of e very easily.

3.32 Theorem *e is irrational.*

Proof Suppose e is rational. Then $e = p/q$, where p and q are positive integers. By (16),

(17) $$0 < q!(e - s_q) < \frac{1}{q}.$$

By our assumption, $q!e$ is an integer. Since

$$q!s_q = q!\left(1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{q!}\right)$$

is an integer, we see that $q!(e - s_q)$ is an integer.

Since $q \geq 1$, (17) implies the existence of an integer between 0 and 1. We have thus reached a contradiction.

Actually, e is not even an algebraic number. For a simple proof of this, see page 25 of Niven's book, or page 176 of Herstein's, cited in the Bibliography.

THE ROOT AND RATIO TESTS

3.33 Theorem (Root Test) *Given* Σa_n, *put* $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$.

Then

(*a*) *if* $\alpha < 1$, Σa_n *converges;*
(*b*) *if* $\alpha > 1$, Σa_n *diverges;*
(*c*) *if* $\alpha = 1$, *the test gives no information.*

Proof If $\alpha < 1$, we can choose β so that $\alpha < \beta < 1$, and an integer N such that

$$\sqrt[n]{|a_n|} < \beta$$

for $n \geq N$ [by Theorem 3.17(*b*)]. That is, $n \geq N$ implies

$$|a_n| < \beta^n.$$

Since $0 < \beta < 1$, $\Sigma\beta^n$ converges. Convergence of Σa_n follows now from the comparison test.

If $\alpha > 1$, then, again by Theorem 3.17, there is a sequence $\{n_k\}$ such that

$$\sqrt[n_k]{|a_{n_k}|} \to \alpha.$$

Hence $|a_n| > 1$ for infinitely many values of n, so that the condition $a_n \to 0$, necessary for convergence of Σa_n, does not hold (Theorem 3.23).

To prove (*c*), we consider the series

$$\sum \frac{1}{n}, \sum \frac{1}{n^2}.$$

For each of these series $\alpha = 1$, but the first diverges, the second converges.

3.34 Theorem (Ratio Test) *The series* Σa_n

(*a*) *converges if* $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$,

(*b*) *diverges if* $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ *for all* $n \geq n_0$, *where* n_0 *is some fixed integer.*

Proof If condition (*a*) holds, we can find $\beta < 1$, and an integer N, such that

$$\left|\frac{a_{n+1}}{a_n}\right| < \beta$$

for $n \geq N$. In particular,

$$\begin{aligned} &|a_{N+1}| < \beta|a_N|, \\ &|a_{N+2}| < \beta|a_{N+1}| < \beta^2|a_N|, \\ &\dots\dots\dots\dots\dots\dots\dots \\ &|a_{N+p}| < \beta^p|a_N|. \end{aligned}$$