20.5.1 Show that $i^2 = j^2 = k^2 = ijk = -1$, and also that $ij = k$, etc.

It follows that the arbitrary quaternion $\alpha + \beta i + \gamma j + \delta k$ is represented by the complex matrix

$$\alpha 1 + \beta i + \gamma j + \delta k = \begin{pmatrix} \alpha + i\beta & \gamma + i\delta \\ -\gamma + i\delta & \alpha - i\beta \end{pmatrix}.$$

A nice feature of this representation is that the square of the absolute value of a quaternion is simply the determinant of the corresponding matrix. Since the square of the absolute value turns up so often, it is also given a name: the *norm*.

20.5.2 Show that

$$\det \begin{pmatrix} \alpha + i\beta & \gamma + i\delta \\ -\gamma + i\delta & \alpha - i\beta \end{pmatrix} = \alpha^2 + \beta^2 + \gamma^2 + \delta^2.$$

The multiplicative property of the norm then follows from the multiplicative property of determinants: $\det AB = \det A \det B$ for any 2×2 matrices A and B. The other algebraic properties of quaternions also follow from properties of matrices which are familiar today: addition is associative and commutative; multiplication is associative but not commutative; the distributive law holds, and every matrix with nonzero determinant has a multiplicative inverse.

The quaternions have a *conjugation* operation analogous to conjugation in $\mathbb{C}$. The conjugate of $q = \alpha + \beta i + \gamma j + \delta k$ is defined to be $\bar{q} = \alpha - \beta i - \gamma j - \delta k$.

20.5.3 Show that $q\bar{q} = \alpha^2 + \beta^2 + \gamma^2 + \delta^2$ (the norm $|q|^2$ of q), and hence express the multiplicative inverse of q in terms of $\bar{q}$ and $|q|$.

20.6   Octonions

Hamilton and his friend John Graves had long discussed the problem of defining multiplication for triples and other n-tuples of real numbers. The discovery of quaternions evidently catalyzed Graves' own thinking about n-tuples, because by December 1843 he was able to tell Hamilton of an interesting discovery of his own: a system of octuples with a multiplicative absolute value, which he called the *octaves*. Hamilton congratulated Graves on his discovery, but pointed out that octaves were not quite as nice as quaternions, because their multiplication was not only noncommutative, but also nonassociative. He agreed to arrange for publication of Graves' discovery, but failed to follow up, with the result that the octaves were re-discovered by Cayley (1845) before Graves' priority was generally known.

As a consequence, they have often been called *Cayley numbers* or *Cayley–Graves* numbers. Today they are generally called the *octonions*, and the set of them is called $\mathbb{O}$.

The octonions are octuples of real numbers with the usual vector addition and scalar multiplication. The standard basis vectors $(1,0,0,0,0,0,0,0)$, $(0,1,0,0,0,0,0,0)$, ..., $(0,0,0,0,0,0,0,1)$ are called *1, i, j, k, l, m, n, o*, respectively, so any octonion can be written in the form

$$\alpha + \beta i + \gamma j + \delta k + \varepsilon l + \zeta m + \eta n + \theta o.$$

They satisfy the distributive axiom, so the value of any octonion product is determined by the products of the "imaginary units" *1, i, j, k, l, m, n, o*. The square of each imaginary unit is -1, and Figure 20.1 gives a description of all products of distinct basis vectors. The product of any two basis vectors
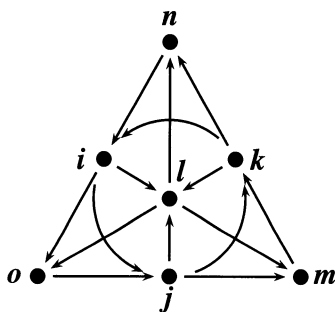

Figure 20.1: Products of octonion basis vectors

is the third vector in the "line" containing them, with a plus or minus sign determined by the arrow and the position of the two vectors in the product. The "lines" include the circle through *i, j*, and *k*, and in fact all the "lines" are supposed to be like this—you should imagine adding a third segment to each of them, joining the end points.

A much simpler description of octonion multiplication was given by Dickson (1914), p. 15. Dickson's description is a generalization of Hamilton's definition of multiplication of pairs, and in fact it shows that the *same construction* produces $\mathbb{C}$ from $\mathbb{R}$, $\mathbb{H}$ from $\mathbb{C}$, and $\mathbb{O}$ from $\mathbb{H}$. Each system consists of ordered pairs (a, b) from the previous system, and pairs are multiplied by the rule

$$(a_1, b_1) \times (a_2, b_2) = (a_1 a_2 - \overline{b_2} b_1, b_2 a_1 + b_1 \overline{a_2}),$$

where $^-$ denotes the conjugation operation, which changes the sign of all the imaginary units. (Thus conjugation has no effect on a real number.) In particular, octonions can be viewed as pairs (a, b) of quaternions a and b. In this case it is important to observe the precise order of products in the definition, because the quaternion product is generally not commutative.

The octonion $p = \alpha + \beta i + \gamma j + \delta k + \varepsilon l + \zeta m + \eta n + \theta o$ has the square of its absolute value equal to $p\overline{p} = \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \varepsilon^2 + \zeta^2 + \eta^2 + \theta^2$, so the multiplicative property of absolute value gives an identity expressing the product of two sums of eight squares as a sum of eight squares. After discovering this, Graves searched the literature for such identities, and uncovered Euler's four-square identity from 1748 (though actually a later appearance of it), and also his own identity in a paper of Degen (1822). Thus the octonions, like the complex numbers and quaternions, gave the first intimation of their existence in the theory of sums of squares.

EXERCISES

The Dickson formula

$$(a_1, b_1) \times (a_2, b_2) = (a_1 a_2 - \overline{b_2} b_1, b_2 a_1 + b_1 \overline{a_2})$$

can be taken as the definition of multiplication for the octonions, but first we should check that the formula gives the correct definition of multiplication for quaternions. This can be done using Cayley's representation of the quaternions by 2×2 matrices of complex numbers (Exercises 20.5.1 and 20.5.2). Each quaternion $\alpha + \beta i + \gamma j + \delta k$ is represented by the complex matrix

$$\begin{pmatrix} \alpha + i\beta & \gamma + i\delta \\ -\gamma + i\delta & \alpha - i\beta \end{pmatrix},$$

which we may call $M(\alpha + i\beta, \gamma + i\delta)$, because it is determined by the pair of complex numbers $\alpha + i\beta$, $\gamma + i\delta$. Then if we write this pair more simply as

$$a = \alpha + i\beta,$$
$$b = \gamma + i\delta,$$

it suffices to prove that the product according to Dickson:

$$(a_1, b_1) \times (a_2, b_2) = (a_1 a_2 - \overline{b_2} b_1, b_2 a_1 + b_1 \overline{a_2}),$$

corresponds to the product according to Cayley's matrices. That is, we have to show that

$$M(a_1, b_1) M(a_2, b_2) = M(a_1 a_2 - \overline{b_2} b_1, b_2 a_1 + b_1 \overline{a_2}).$$

20.6.1 Show that
$$M(a,b) = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}.$$

Hence compute $M(a_1,b_1)M(a_2,b_2)$ for any complex numbers a_1, b_1, a_2, b_2, and show that it equals $M(a_1a_2 - \overline{b_2}b_1, b_2a_1 + b_1\overline{a_2})$.

Figure 20.1 for the products of the octonion units is due to Freudenthal (1951), and it shows $ij = k$ as we would expect, because these i, j, k behave the same as the quaternion units. Since the "line" $i \to j \to k$ is closed by an arrow from k to i, it also shows $kj = i$, and likewise (using the invisible arrow from m to o) $jm = o$ and $mo = j$.

20.6.2 Check that the same products result from Dickson's multiplication formula when i,j,k,l,m,n,o are defined in terms of the quaternion units $\mathbf{i}, \mathbf{j}, \mathbf{k}$ by

$$l = (0,1),$$
$$\begin{aligned} i &= (\mathbf{i},0), & m &= (0,\mathbf{i}), \\ j &= (\mathbf{j},0), & n &= (0,\mathbf{j}), \\ k &= (\mathbf{k},0), & o &= (0,\mathbf{k}). \end{aligned}$$

20.7   Why $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ Are Special

The pre-established harmony between the two-square, four-square, and eight-square identities and the norms on $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ suggest that $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ are not just random curiosities, but actually very special structures. In fact, they are unique. If we define a *hypercomplex number system* to consist of n-tuples of real numbers ($n \geq 2$) with vector addition, a distributive multiplication and a multiplicative absolute value, then

- $\mathbb{C}$ is the only hypercomplex number system for which the multiplication is commutative and associative. This was proved by Weierstrass (1884).

- $\mathbb{H}$ is the only other hypercomplex number system for which the multiplication is associative. This was proved by Frobenius (1878).

- $\mathbb{O}$ is the only other hypercomplex number system. This was proved by Hurwitz (1898). (In the process, Hurwitz proved that there are no n-square identities except for $n = 1,2,4,8$.)

Since that time, it has been found that $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ have relationships with many other "exceptional" structures in mathematics. One of the most

remarkable is their relationship with projective geometry, via the theorems of Pappus and Desargues.

The theorem of Pappus is a theorem of classical geometry that belongs to projective geometry, seemingly by accident. As mentioned in the exercises to Section 8.7, it states that *if the vertices of a hexagon ABCDEF lie alternately on two straight lines, then the intersections of opposite sides of the hexagon lie on a line* (Figure 20.2).
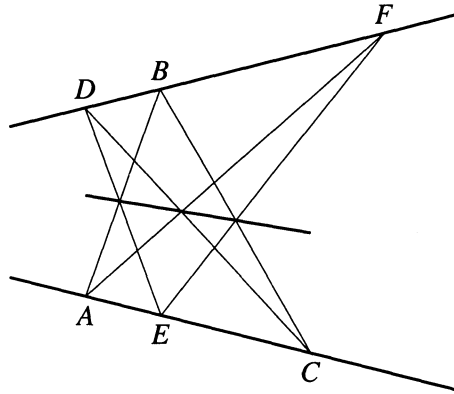

Figure 20.2: Pappus' theorem

This theorem is meaningful in projective geometry, because it involves only points and lines and whether they meet or not, yet its *proof* involves the concept of distance. Desargues' theorem in the plane is like this too, as mentioned in Section 8.3; it is a projective theorem without a projective proof, and this is even more puzzling because Desargues' theorem in space *does* have a projective proof.

An amazing explanation of these phenomena was uncovered by the work of von Staudt (1847) and Hilbert (1899). In 1847 von Staudt gave geometric constructions of $+$ and $\times$, allowing each projective plane to be "coordinatized" by hypercomplex numbers. Then in 1899 Hilbert made the wonderful discovery that the geometry of a projective plane is tied to the algebra of the corresponding hypercomplex number system:

- *Pappus' theorem holds $\Leftrightarrow$ the system is commutative.*

- *Desargues' theorem holds $\Leftrightarrow$ the system is associative.*

Conversely, any hypercomplex number system R yields a projective plane RP^2, by construction of homogeneous coordinates essentially as in Section 8.5. Then by Hilbert's theorem,

- $\mathbb{R}P^2$ and $\mathbb{C}P^2$ satisfy Pappus,

- $\mathbb{H}P^2$ satisfies Desargues but not Pappus, and

- $\mathbb{O}P^2$ satisfies neither.

The results of Hilbert explain why the theorems of Pappus and Desargues' do not have projective proofs. It is because these theorems *do not hold for all projective planes*, only for those with enough algebraic structure. This is a remarkable contribution of algebra to geometry, but it also gives insight in the opposite direction. It can seriously be said that Pappus' theorem "explains" why $\mathbb{R}$ and $\mathbb{C}$ have commutative multiplication, because it is simpler (takes fewer axioms) to describe a projective plane satisfying Pappus' theorem than to describe a field. This is possibly the most remarkable aspect of Hilbert's work on the foundations of geometry. It shows that the long historical trend of turning geometry into algebra—which began with Fermat and Descartes—may conceivably be coming to an end.

EXERCISES

Freudenthal's diagram of octonion units (Figure 20.1) itself has the structure of a projective plane, which is why we used the name "lines" for the collinear (or co-circular) triples of points in it.

20.7.1 Check that the seven "points" (octonion units) and seven "lines" of Freudenthal's diagram have the following properties.

- Through any two "points" there is exactly one "line."
- Any two "lines" have exactly one "point" in common.

Such a structure is called a *finite projective plane*, and this one is often called the *Fano* plane after its discoverer. The diagram makes it easy to show that $\mathbb{O}$ is not associative.

20.7.2 Find a triple of octonion units a, b, c such that $a(bc) \neq (ab)c$.

The weakening of multiplicative structure as we construct hypercomplex number systems of higher dimension (losing commutativity with $\mathbb{H}$ and associativity with $\mathbb{O}$) is a hint that we can't go on constructing hypercomplex systems indefinitely. In fact, the 16-dimensional system of pairs of octonions, with the Dickson

multiplication rule, does not have a multiplicative absolute value. This is because it includes "zero divisors"—nonzero elements whose product is the zero element $(0,0)$.

20.7.3 Show that the nonzero pairs (i,n), (k,l) of octonion units have Dickson product $(0,0)$. Also, find another pair (a,b) of octonion units such that $(i,n)(a,b) = (0,0)$.

20.7.4 Show that in any system with a multiplicative absolute value $|\ |$, $x \neq 0$ and $y \neq 0$ imply $xy \neq 0$. (Hence the system of pairs of octonions does not have a multiplicative absolute value.)

20.8   Biographical Notes: Hamilton

The world of mathematics is one of logic and order, so mathematicians tend to look for order in their personal lives. Usually they find it (it is hard to do mathematics otherwise!), even though the human world is not very orderly. But sometimes they don't, and the result can be both a mathematical and human tragedy. One such case was Galois; another was Hamilton.

William Rowan Hamilton (Figure 20.3) was born in Dublin at midnight, between August 3 and 4, in 1805. His father, Archibald, a lawyer, and his mother, Sarah, cared for him until he was three years old, but then they got into financial difficulties, and young William was sent to live with Archibald's brother James and his wife, Sydney. Uncle James Hamilton was Anglican curate and schoolmaster in Trim, about 40 miles from Dublin, a devoted father figure and educator, but with highly eccentric methods of instruction. Here is how he taught William to spell at age three:

> James printed on cards every word he has yet spelled; he began with every monosyllable in which *A* is the principal letter, and so on alphabetically, never beginning a new set till he could spell them off book and on book; every spelling book and dictionary was searched . . . so that he is now completely grounded in words that most children are very deficient in, and indeed many grown people . . . he is going through them now for the last time, and James is now preparing words of two syllables. [Letter from Sydney to Sarah Hamilton, 17 October 1808, in Graves (1975), vol. 1, p. 31.]

At this time, William was also taught addition, subtraction, and multiplication of numbers up to 10, but mathematics did not play a big part in his childhood. Uncle James was primarily a classicist with an interest in Asian languages, and William was the ideal pupil. He began learning Hebrew at three, followed by Latin and Greek by age five, Italian and French at eight, and Arabic, Sanscrit, and Persian by age ten. Only then do we hear of mathematics again, when William reports in a letter to his sister Grace that "I have done near half the first book of Euclid with uncle"—a pretty ordinary accomplishment by the standards of the time.

Figure 20.3: Sir William Rowan Hamilton

Hamilton reached a turning point in his intellectual life at age 13. He seems to have decided that he knew enough languages, because he stopped picking up new ones and wrote a small book on Syriac grammar for the benefit of other learners. At the same time, he met another boy who could beat him in an intellectual contest, the American calculating prodigy Zerah Colburn. Hamilton was consistently outclassed by Colburn's feats, such as calculating the number of minutes in 1811 years, and factorizing numbers in the billions. But far from being discouraged by the experience, he wanted to know more. When Colburn retired from the mental arithmetic game and returned two years later as an actor, Hamilton asked him about