Definition. If G is a profinite group and A is a discrete G-module, the cohomology groups $H^n(G, A)$ computed using continuous cochains are called the *profinite* or *continuous* cohomology groups. When $G \doteq \mathrm{Gal}(K/F)$ is the Galois group of a field extension K/F then the *Galois cohomology groups* $H^n(G, A)$ will always mean the cohomology groups computed using continuous cochains.

When G is a finite group, every G-module is a discrete G-module so the discrete and continuous cohomology groups of G are the same. When G is infinite, this need not be the case as shown by the example mentioned previously of the free G-module $\mathbb{Z}G$ when G is an infinite profinite group. All the major results in this section remain valid for the continuous cohomology' groups when "G-module" is replaced by "discrete G-module" and "subgroup" is replaced by "closed subgroup." For example, the Long Exact Sequence in Group Cohomology remains true as stated, the restriction homomorphism requires the subgroup H of G to be a closed subgroup (so that the restriction of a continuous map on G^n to H^n remains continuous), Proposition 26 requires H to be closed, etc.

We can write $G = \varprojlim(G/N)$ and $A = \cup A^N$ where N runs over the open normal subgroups of G (necessarily of finite index in G since G is compact). Then A^N is a discrete G/N-module and it is not difficult to show that

$$H^n(G, A) = \varinjlim_N H^n(G/N, A^N) \tag{17.19}$$

where the cohomology groups are continuous cohomology and the direct limit is taken over the collection of all open normal subgroups N of G (cf. Exercise 24). Since G/N is a finite group, the continuous cohomology groups $H^n(G/N, A^N)$ in this direct limit are just the (discrete) cohomology groups considered earlier in this section. The computation of the continuous cohomology for a profinite group G can therefore always be reduced to the consideration of finite group cohomology where there is no distinction between the continuous and discrete theories.

EXERCISES

1. Let $F_n = \mathbb{Z}G \otimes_{\mathbb{Z}} \mathbb{Z}G \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mathbb{Z}G$ ($n+1$ factors) for $n \geq 0$ with G-action defined on simple tensors by $g \cdot (g_0 \otimes g_1 \otimes \cdots \otimes g_n) = (gg_0) \otimes g_1 \otimes \cdots \otimes g_n$.
   (a) Prove that F_n is a free $\mathbb{Z}G$-module of rank $|G|^n$ with $\mathbb{Z}G$ basis $1 \otimes g_1 \otimes g_2 \otimes \cdots \otimes g_n$ with $g_i \in G$.

   Denote the basis element $1 \otimes g_1 \otimes g_2 \otimes \cdots \otimes g_n$ in (a) by $(g_1, g_2, \ldots, g_n)$ and define the G-module homomorphisms d_n for $n \geq 1$ on these basis elements by $d_1(g_1) = g_1 - 1$ and

   $$\begin{aligned} d_n(g_1, \ldots, g_n) = g_1 \cdot (g_2, \ldots, g_n) &+ \sum_{i=1}^{n-1} (-1)^i (g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_n) \\ &+ (-1)^n (g_1, \ldots, g_{n-1}), \end{aligned}$$

   for $n \geq 2$. Define the $\mathbb{Z}$-module *contracting homomorphisms*

   $$\mathbb{Z} \xrightarrow{s_{-1}} F_0 \xrightarrow{s_0} F_1 \xrightarrow{s_1} F_2 \xrightarrow{s_2} \cdots$$

   on a $\mathbb{Z}$ basis by $s_{-1}(1) = 1$ and $s_n(g_0 \otimes \cdots \otimes g_n) = 1 \otimes g_0 \otimes \ldots \otimes g_n$.

(b) Prove that

$$\epsilon s_{-1} = 1, \qquad d_1 s_0 + s_{-1}\epsilon = 1, \qquad d_{n+1}s_n + s_{n-1}d_n = 1, \text{ for all } n \geq 1$$

where the map aug : $F_0 \to \mathbb{Z}$ is the augmentation map $\text{aug}(\sum_{g\in G}\alpha_g g) = \sum_{g\in G}\alpha_g$.

(c) Prove that the maps s_n are a chain homotopy (cf. Exercise 4 in Section 1) between the identity (chain) map and the zero (chain) map from the chain

$$\cdots \longrightarrow F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} F_0 \xrightarrow{\text{aug}} \mathbb{Z} \longrightarrow 0 \qquad (*)$$

of $\mathbb{Z}$-modules to itself.

(d) Deduce from (c) that all $\mathbb{Z}$-module homology groups of $(*)$ are zero, i.e., $(*)$ is an exact sequence of $\mathbb{Z}$-modules. Conclude that $(*)$ is a projective G-module resolution of $\mathbb{Z}$.

2. Let P_n denote the free $\mathbb{Z}$-module with basis $(g_0, g_1, g_2, \ldots, g_n)$ with $g_i \in G$ and define an action of G on P_n by $g\cdot(g_0, g_1, \ldots, g_n) = (gg_0, gg_1, \ldots, gg_n)$. For $n \geq 1$ define

$$d_n(g_0, g_1, g_2, \ldots, g_n) = \sum_{i=0}^{n}(-1)^i(g_0, \ldots, \hat{g_i}, \ldots, g_n),$$

where $(g_0, \ldots, \hat{g_i}, \ldots, g_n)$ denotes the term $(g_0, g_1, g_2, \ldots, g_n)$ with g_i deleted.

(a) Prove that P_n is a free $\mathbb{Z}G$-module with basis $(1, g_1, g_2, \ldots, g_n)$ where $g_i \in G$.

(b) Prove that $d_{n-1} \circ d_n = 0$ for $n \geq 1$. [Show that the term $(g_0, \ldots, \hat{g_j}, \ldots, \hat{g_k}, \ldots, g_n)$ missing the entries g_j and g_k occurs twice in $d_{n-1} \circ d_n(g_0, g_1, g_2, \ldots, g_n)$, with opposite signs.]

(c) Prove that $\varphi : P_n \to F_n$ defined by

$$\varphi((g_0, g_1, g_2, \ldots, g_n)) = g_0 \otimes (g_0^{-1}g_1) \otimes (g_1^{-1}g_2)\ldots \otimes (g_{n-1}^{-1}g_n)$$

is a G-module isomorphism with inverse $\psi : P_n \to F_n$ given by

$$\psi(g_0 \otimes g_1 \otimes \ldots \otimes g_n) = (g_0, g_0g_1, g_0g_1g_2, \ldots, g_0g_1g_2\cdots g_n).$$

(d) Prove that if $\epsilon(g_0) = 1$ for all $g_0 \in G$ then

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0 \qquad (**)$$

is a free G-module resolution of $\mathbb{Z}$. [Show that the isomorphisms in (c) take the G-module resolutions $(**)$ and $(*)$ of the previous exercise into each other.]

3. Let F_n and P_n be as in the previous two exercises and let A be a G-module.

(a) Prove that $\text{Hom}_{\mathbb{Z}G}(F_n, A)$ can be identified with the collection $C^n(G, A)$ of maps from $G \times G \times \cdots \times G$ (n copies) to A and that under this identification the associated coboundary maps from $C^n(G, A)$ to $C^{n+1}(G, A)$ are given by equation (18).

(b) Prove that $\text{Hom}_{\mathbb{Z}G}(P_n, A)$ can be identified with the collection of maps f from $n+1$ copies $G \times G \times \cdots \times G$ to A that satisfy $f(gg_0, gg_1, \ldots, gg_n) = gf(g_0, g_1, \ldots, g_n)$.

The group $C^n(G, A)$ is sometimes called the group of *inhomogeneous n-cochains of G in A*, and the group in (b) of the previous exercise is called the group of *homogeneous n-cochains of G in A*. The inhomogeneous cochains are easier to describe since there is no restriction on the maps from G^n to A, but the coboundary map d_n on homogeneous cochains is less complicated (and more naturally suggested in topological contexts) than the coboundary map on inhomogeneous cochains. The results of the previous exercises show that the cohomology groups $H^n(G, A)$ defined using either homogeneous or inhomogeneous cochains are the same and indicate the origin of the coboundary maps d_n used in the text. Historically, $H^n(G, A)$ was originally defined using homogeneous cochains.

4. Suppose H is a normal subgroup of the group G and A is a G-module. For every $g \in G$ prove that the map $f(a) = ga$ for $a \in A^H$ defines an automorphism of the subgroup A^H.

5. Suppose the G-module A decomposes as a direct sum $A = A_1 \oplus A_2$ of G-submodules. Prove that for all $n \geq 0$, $H^n(G, A) \cong H^n(G, A_1) \oplus H^n(G, A_2)$.

6. Suppose $0 \to A \to M_1 \to M_2 \to \cdots \to M_k \to C \to 0$ is an exact sequence of G-modules where $M_1, M_2, \ldots, M_k$ are cohomologically trivial. Prove that $H^{n+k}(G, A) \cong H^n(G, C)$ for all $n \geq 1$. [Decompose the exact sequence into a succession of short exact sequences and use Corollary 22. For example, if $0 \to A \xrightarrow{\alpha} M_1 \xrightarrow{\beta} M_2 \xrightarrow{\gamma} C \to 0$ is exact, show that $0 \to A \to M_1 \to B \to 0$ and $0 \to B \to M_2 \to C \to 0$ are both exact, where $B = M_1/\operatorname{image} \alpha = M_1/\ker \beta \cong \operatorname{image} \beta = \ker \gamma$.]

7. (*Adjoint Associativity*) Let R, S and T be rings with 1, let P be a left S-module, let N be a (T, S)-bimodule, and let A be a left T-module. Prove that

$$\Phi : \operatorname{Hom}_S(P, \operatorname{Hom}_T(N, A)) \longrightarrow \operatorname{Hom}_T(N \otimes_S P, A)$$

defined by $\Phi(f)(n \otimes p) = f(p)(n)$ is an isomorphism of abelian groups. (See also Theorem 43 in Section 10.5).

8. Suppose G is cyclic of order m with generator σ and let $N = 1 + \sigma + \sigma^2 + \cdots + \sigma^{m-1} \in \mathbb{Z}G$.
   (a) Prove that the *augmentation* map $\operatorname{aug}(\sum_{i=0}^{m-1} a_i \sigma^i) = \sum_{i=0}^{m-1} a_i$ is a G-module homomorphism from $\mathbb{Z}G$ to $\mathbb{Z}$.
   (b) Prove that multiplication by N and by $\sigma - 1$ in $\mathbb{Z}G$ define a free G-module resolution of $\mathbb{Z}$: $\ldots \xrightarrow{\sigma-1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{\sigma-1} \ldots \xrightarrow{N} \mathbb{Z}G \xrightarrow{\sigma-1} \mathbb{Z}G \xrightarrow{\text{aug}} \mathbb{Z} \longrightarrow 0$.

9. Suppose G is an infinite cyclic group with generator σ.
   (a) Prove that multiplication by $\sigma - 1 \in \mathbb{Z}G$ defines a free G-module resolution of $\mathbb{Z}$: $0 \longrightarrow \mathbb{Z}G \xrightarrow{\sigma-1} \mathbb{Z}G \longrightarrow \mathbb{Z} \longrightarrow 0$.
   (b) Show that $H^0(G, A) \cong A^G$, that $H^1(G, A) \cong A/(\sigma-1)A$, and that $H^n(G, A) = 0$ for all $n \geq 2$. Deduce that $H^1(G, \mathbb{Z}G) \cong \mathbb{Z}$ (so free modules need not be cohomologically trivial).

10. Suppose H is a subgroup of finite index m in the group G and A is an H-module. Let $x_1, \ldots, x_m$ be a set of left coset representatives for H in G: $G = x_1 H \cup \cdots \cup x_m H$.
   (a) Prove that $\mathbb{Z}G = \bigoplus_{i=1}^m x_i \mathbb{Z}H = \bigoplus_{i=1}^m \mathbb{Z}H x_i^{-1}$ and $\mathbb{Z}G \otimes_{\mathbb{Z}H} A = \bigoplus_{i=1}^m (x_i \otimes A)$ as abelian groups.
   (b) Let $f_{i,a}$ be the function from $\mathbb{Z}G$ to A defined by

$$f_{i,a}(x) = \begin{cases} ha & \text{if } x = hx_i^{-1} \text{ with } h \in H \\ 0 & \text{otherwise.} \end{cases}$$

   Prove that $f_{i,a} \in M_H^G(A) = \operatorname{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, A)$, i.e., $f_{i,a}(h'x) = h' f_{i.a}(x)$ for $h' \in H$.
   (c) Prove that the map $\varphi(f) = \sum_{i=1}^m x_i \otimes f(x_i^{-1})$ from $M_H^G(A)$ to $\mathbb{Z}G \otimes_{\mathbb{Z}H} A$ is a G-module homomorphism. [Write $x_i^{-1} g = h_i x_{i'}^{-1}$ for $i = 1, \ldots, m$ and observe that $x_i \otimes f(x_i^{-1} g) = x_i \otimes h_i f(x_{i'}^{-1}) = x_i h_i \otimes f(x_{i'}^{-1}) = g x_{i'} \otimes f(x_{i'}^{-1})$.]
   (d) Prove that φ gives a G-module isomorphism $\varphi : M_H^G(A) \cong \mathbb{Z}G \otimes_{\mathbb{Z}H} A$. [For the injectivity observe that an H-module homomorphism is 0 if and only if $f(x_i^{-1}) = 0$ for $i = 1, \ldots, m$. For the surjectivity prove that $\varphi(f_{i,a}) = x_i \otimes a$.]

11. Prove that the isomorphism $M_H^G(A) \cong \mathbb{Z}G \otimes_{\mathbb{Z}H} A$ in (d) of the previous exercise need not hold if H is not of finite index in G. [If G is an infinite cyclic group show that Shapiro's Lemma implies $H^1(G, M_1^G(\mathbb{Z})) = 0$ while $H^1(G, \mathbb{Z}G) \cong \mathbb{Z}$ by Exercise 9.]