Figure 14.1: Bombelli's manuscript

It is possible to work backwards and concoct a cubic equation with an "obvious" solution that can be reconciled with the hideous solution in the Cardano formula. Here is an example.

14.3.2 Check that $(3+\sqrt{-1})^3 = 18+26\sqrt{-1}$.

14.3.3 Hence explain why

$$6 = (3+\sqrt{-1})+(3-\sqrt{-1}) = \sqrt[3]{18+26\sqrt{-1}}+\sqrt[3]{18-26\sqrt{-1}}.$$

14.3.4 Find p and q such that

$$18 = \frac{q}{2} \quad \text{and} \quad 26\sqrt{-1} = \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}.$$

14.3.5 Check that 6 is a solution of the equation $x^3 = px + q$ for the values of p and q found in Exercise 14.3.4.

14.4 Wallis' Attempt at Geometric Representation

Despite Bombelli's successful use of complex numbers, most mathematicians regarded them as impossible, and of course even today we call them "imaginary" and use the symbol i for the imaginary unit $\sqrt{-1}$. The first attempt to give complex numbers a concrete interpretation was made by Wallis (1673). This attempt was unsatisfactory, as we shall see, but nevertheless an interesting "near miss." Wallis wanted to give a geometric interpretation to the roots of the quadratic equation, which we shall write as

$$x^2 + 2bx + c^2 = 0, \qquad b, c \geq 0.$$

The roots are

$$x = -b \pm \sqrt{b^2 - c^2}$$

and hence real when $b \geq c$. In this case the roots can be represented by points P_1, P_2 on the real number line which are determined by the geometric construction in Figure 14.2. When $b < c$, lines of length b attached to Q are too short to reach the number line, so the points P_1, P_2 "cannot be had in the line," and Wallis seeks them "out of that line ... (in the same Plain)." He is on the right track, but he arrives at unsuitable positions for P_1, P_2 by sticking too closely to his first construction.

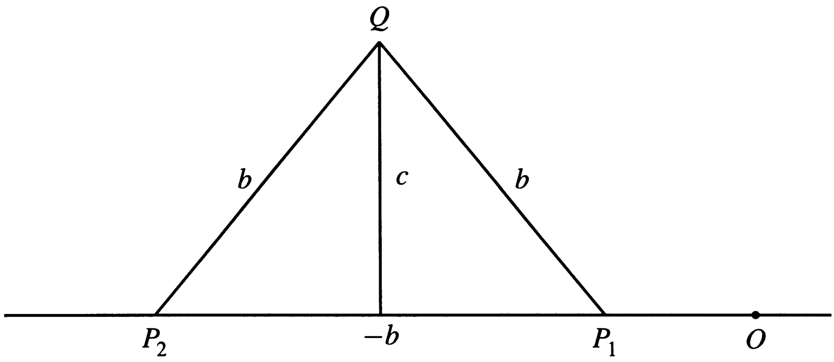

Figure 14.2: Wallis' construction of real roots

Figure 14.3 compares his representation of $P_1, P_2 = -b \pm i\sqrt{c^2 - b^2}$ when $b < c$ with the modern representation. Apparently Wallis thought $+$ and $-$ should continue to correspond to "right" and "left," though this has the unacceptable consequence that $i = -i$ (let $b \to 0$ in his representation). This was an understandable oversight, since in Wallis' time even negative numbers were still under suspicion, and there was confusion about the meaning of $(-1) \times (-1)$, for example. Confusion was compounded by the introduction of square roots, and as late as 1770 Euler gave a "proof" in his *Algebra* that $\sqrt{-2} \times \sqrt{-3} = \sqrt{6}$ [Euler (1770), p. 43].

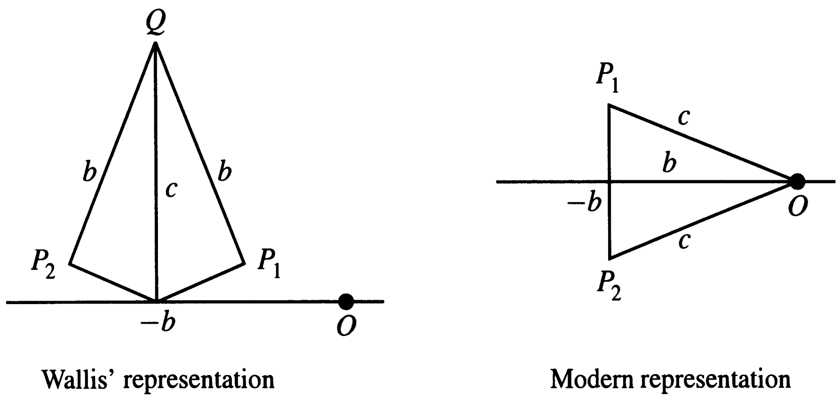

Figure 14.3: Construction of complex roots

EXERCISES

The claim that $\sqrt{-2} \times \sqrt{-3} = \sqrt{6}$ is wrong only if one uses the convention that $\sqrt{6}$ means the *positive* square root of 6, as we normally do today. It is not unreasonable to let $\sqrt{6}$ denote the *pair* $\pm\sqrt{6}$ of square roots of 6, in which case Euler's claim is correct.

14.4.1 Supposing that $\sqrt{-2}$ denotes the pair of square roots of -2, that $\sqrt{-3}$ denotes the pair of square roots of -3, and that $\sqrt{-2} \times \sqrt{-3}$ denotes all possible products, show that

$$\sqrt{-2} \times \sqrt{-3} = \sqrt{6}.$$

14.4.2 Is it also true (as in the usual interpretation) that

$$\sqrt{-2} \times \sqrt{-3} = -\sqrt{6}?$$

14.5 Angle Division

In Section 6.6 we saw how Viète related angle trisection to the solution of cubic equations, and how Leibniz (1675) and de Moivre (1707) solved the angle n-section equation by the Cardano-type formula

$$x = \frac{1}{2}\sqrt[n]{y+\sqrt{y^2-1}} + \frac{1}{2}\sqrt[n]{y-\sqrt{y^2-1}}. \tag{1}$$

We also saw how this and Viète's formulas for $\cos n\theta$ and $\sin n\theta$ could easily be explained by the formula

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \tag{2}$$

usually associated with de Moivre. Actually, de Moivre never stated (2) explicitly. The closest he came was to give a formula for $(\cos\theta + i\sin\theta)^{1/n}$ [de Moivre (1730); see Smith (1959) for a series of extracts from the work of de Moivre on angle division]. It seems that the clues in the algebra of circular functions were not strong enough to reveal (2) until a deeper reason for it had been brought to light by calculus.

Complex numbers made their entry into the theory of circular functions in a paper on integration by Johann Bernoulli (1702). Observing that $\sqrt{-1} = i$ makes possible the partial fraction decomposition

$$\frac{1}{1+z^2} = \frac{1/2}{1+zi} + \frac{1/2}{1-zi},$$

Bernoulli saw that integration would give an expression for $\tan^{-1} z$ as an imaginary logarithm, though he did not write down the expression in question and was evidently puzzled as to what it could mean. In Section 16.1 we shall see how Euler clarified Johann Bernoulli's discovery and developed it into the beautiful theory of complex logarithms and exponentials. What is relevant here is that Johann Bernoulli (1712) took up the idea again, and this time he carried out the integration to obtain an algebraic relation between $\tan n\theta$ and $\tan\theta$. His argument is as follows. Given

$$y = \tan n\theta, \qquad x = \tan\theta,$$

we have

$$n\theta = \tan^{-1} y = n\tan^{-1} x;$$

hence, taking differentials

$$\frac{dy}{1+y^2} = \frac{n\,dx}{1+x^2}$$

or

$$dy\left(\frac{1}{y+i} - \frac{1}{y-i}\right) = n\,dx\left(\frac{1}{x+i} - \frac{1}{x-i}\right).$$

Integration gives

$$\log(y+i) - \log(y-i) = n\log(x+i) - n\log(x-i),$$

that is,

$$\log\frac{y+i}{y-i} = \log\left(\frac{x+i}{x-i}\right)^n,$$

whence

$$(x-i)^n(y+i) = (x+i)^n(y-i). \tag{3}$$

This formula was the first of the de Moivre type actually to use i explicitly and the first example of a phenomenon later articulated by Hadamard: the shortest route between two truths in the real domain sometimes passes through the complex domain. Solving (3) for y as a function of x expresses $\tan n\theta$ as a rational function of $\tan\theta$, which is difficult to obtain using real formulas alone. In fact, it is easy to show from (3) that y is the quotient of the polynomials consisting of alternate terms in $(x+1)^n$, provided with alternate $+$ and $-$ signs (see exercises).

During the eighteenth century, mathematicians were ambivalent about $\sqrt{-1}$. They were willing to use it en route to results about real numbers but doubted whether it had a concrete meaning of its own. Cotes (1714) even used $a+\sqrt{-1}b$ to represent the point (a,b) in the plane (as Euler did later), apparently without noticing that (a,b) was a valid *interpretation* of $a+\sqrt{-1}b$. Since results about $\sqrt{-1}$ were suspect, they were often left unstated when it was possible to state an equivalent result about reals. This may explain why de Moivre stated (1) but not (2). Another example of the avoidance of results about $\sqrt{-1}$ is the remarkable theorem on the regular n-gon discovered by Cotes in 1716 and published posthumously in Cotes (1722):

If $A_0, \ldots, A_{n-1}$ are equally spaced points on the unit circle with center O, and if P is a point on OA_0 such that $OP = x$, then (Figure 14.4)

$$PA_0 \cdot PA_1 \cdot \cdots \cdot PA_{n-1} = 1 - x^n.$$

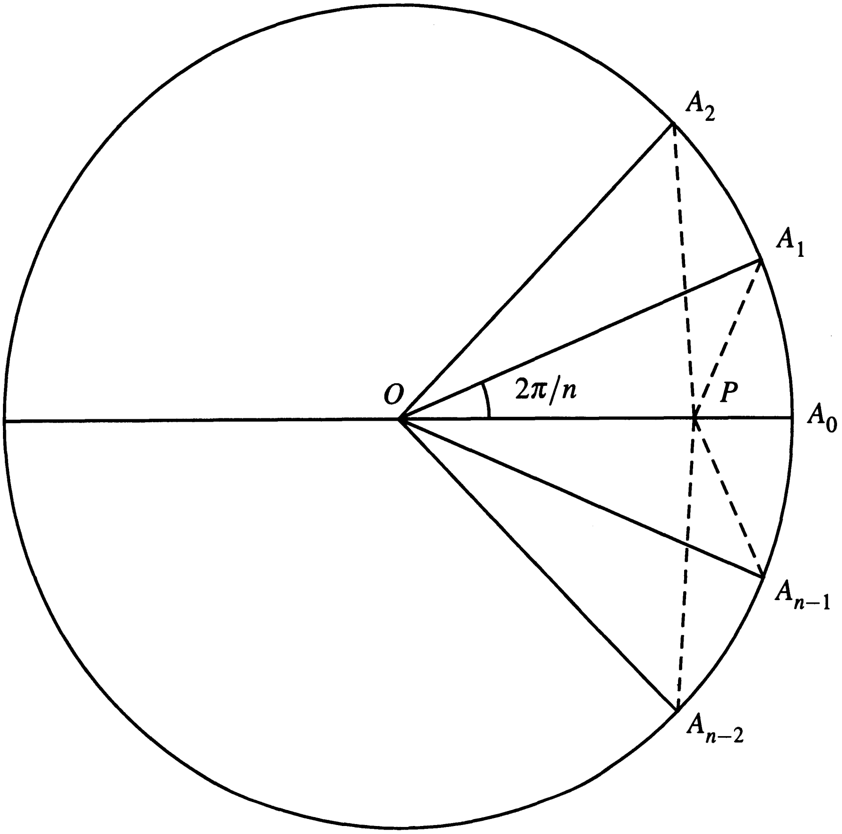

Figure 14.4: Cotes' theorem

This theorem not only relates the regular n-gon to the polynomial $x^n - 1$ but in fact geometrically realizes the *factorization of $x^n - 1$ into real linear and quadratic factors*. By symmetry one has $PA_1 = PA_{n-1}, \ldots$, hence

$$PA_0 \cdots PA_1 \cdots PA_{n-1} = \begin{cases} PA_0 \cdot PA_1^2 \cdot PA_2^2 \cdots PA_{(n-1)/2}^2 & n \text{ odd}, \\ PA_0 \cdot PA_1^2 \cdot PA_2^2 \cdots PA_{n/2-1}^2 PA_{n/2} & n \text{ even}. \end{cases}$$

$PA_0 = 1 - x$ is a real linear factor, as is $PA_{n/2}$ when n is even, and it follows from the cosine rule in triangle OPA_k that

$$PA_k^2 = 1 - 2x\cos\frac{2k\pi}{n} + x^2.$$

The easiest route from here to the theorem is by splitting PA_k^2 into complex linear factors and using de Moivre's theorem, though we can only speculate that this was Cotes' method, since he stated the theorem without proof. There is a second half to Cotes' theorem, which similarly decomposes $1+x^n$ into real linear and quadratic factors. These factorizations were needed to integrate $1/(1\pm x^n)$ by resolution into partial fractions, which was in fact Cotes' main objective. Such problems were then high on the mathematical agenda, and they motivated subsequent research into the factorization of polynomials, in particular the first attempts to prove the fundamental theorem of algebra.

EXERCISES

Johann Bernoulli's formula relating $y = \tan n\theta$ to $x = \tan\theta$ is false for some values of n, because it neglects a possible constant of integration. The result of integration should be

$$\log(y+i) - \log(y-i) = n\log(x+i) - n\log(x-i) + C,$$

for some C, leading to

$$\frac{y+i}{y-i} = D\frac{(x+i)^n}{(x-i)^n}, \qquad (*)$$

for some constant D (equal to e^C). Sometimes $D = 1$ gives the correct formula, but sometimes we need $D = -1$.

14.5.1 Show that $D = 1$ gives the correct formula when $n = 1$.

14.5.2 Using formulas for $\sin 2\theta$ and $\cos 2\theta$, or otherwise, show that

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta},$$

and check that this follows from (*) for $D = -1$, but not for $D = 1$.

14.5.3 Use the formula in Exercise 14.5.2 to express $\tan 4\theta$ in terms of $\tan 2\theta$, and hence in terms of $\tan\theta$.

14.5.4 Letting $y = \tan 4\theta$ and $x = \tan\theta$, express the result of Exercise 14.5.3 as

$$y = \frac{4x-4x^3}{x^4-6x^2+1},$$

and check that this follows from (*) when $D = -1$.