This proof falls short of modern standards of rigour. In general, two circles may meet in two points, touch at one point, or not meet at all. In the present situation, they do, in fact, meet in two points; but this does not follow from Euclid's explicit assumptions.

Book I concludes with proofs of the Theorem of Pythagoras and its converse. Euclid is careful to show that there is a square on the hypotenuse, before discussing its properties. This is interesting, in view of Legendre's later proof that the existence of such a square implies Euclid's Postulate V.

To prove the Theorem of Pythagoras, Euclid uses a theory of area. Nowadays we are tempted to define the area of a rectangle as 'length times width'. This presupposes a theory which explains what it means to multiply two irrationals. Euclid approached the question of area from a more elementary point of view. He began with the idea that two polygons have the same area if they first can be dissected into triangles which can be reassembled, as in a jigsaw puzzle, to form a polygon exactly like the second polygon. It is only in Book VI, after Euclid has presented Eudoxus's theory of irrationals, that the length times width formula is justified.

However, in Book II, Euclid gives geometric treatments of certain basic algebraic identities, such as $a(b+c) = ab+ac$, using the areas of rectangles to handle products. He also gives a proof of a statement equivalent to what we now call the Law of Cosines.

Book III discusses the basic properties of the circle. Euclid goes to great length to give rigorous proofs. For example, in spite of the fact that it is 'obvious from the diagram', Euclid offers a demonstration of the fact that the points on a chord of a circle lie in the interior of the circle. Euclid is not always successful in his attempt at rigour, but it is clear that he does understand the need for it.

Book IV gives constructions for various regular polygons. It culminates with a treatment of the regular 15-gon. This achievement remained unsurpassed until 1796, when Carl Friedrich Gauss (1777–1855) found a construction for the regular 17-gon.

In Book V, Euclid uses Eudoxus's definitions of proportion to deduce an arithmetic for line segments. The 'commutativity of multiplication' is the subject of Proposition 16.

In Book VI, Euclid uses the material of Book V to derive the basic properties of similar triangles. The book concludes with the theorem that the length of a circular arc is proportional to the angle it subtends at the center of the circle. In talking about 'arclength', Euclid is implicitly presupposing the 'completeness' of the plane.

Books VII to IX present some elementary theorems of number theory. Included are proofs for Euclid's Algorithm (VII 2), the unique factorization of square-free integers (IX 14), the infinitude of primes (IX 20), the formula for the sum of a geometric progression (IX 35), and the formula for even perfect numbers (IX 36).

Book X is occupied with what we might call 'field extensions of degree 4 over rationals'. Euclid is interested in knowing when an expression like $\sqrt{7+2\sqrt{6}}$, which looks like it has 'degree 4' is actually equal to an expression like $1+\sqrt{6}$, which involves only one 'layer' of square roots.

Book XI derives the basic theorems of solid geometry. A 'cone' is defined in terms of the revolution of a right triangle. A 'cube' is 'a solid figure contained by six equal squares'. Proposition XI 21 says that 'any solid angle is contained by plane angles [whose sum is] less than four right angles'. This proposition is used at the end of Book XIII to show that there are at most 5 regular polyhedra.

Book XII is the masterpiece of Eudoxus. Without the help of calculus, he manages to give a rigorous treatment of the volumes of the pyramid, cone and sphere.

Book XIII is the apex of the *Elements*. For each of the five regular polyhedra, Euclid derives the ratio of its side to the radius of the sphere in which it is inscribed. Although Euclid failed to give a complete theory of regular polygons – for example, the construction of the regular 17-gon is missing – he succeeded in giving a complete theory of regular polyhedra.

Euclid's *Elements*, or watered down versions of it, was used for over 2,000 years in universities and schools to teach not only geometry but also rigorous thinking. Not long after World War II, a reaction against this program set in and educators decided that geometry was not the appropriate place for training in logic. Anyway, they argued, Euclid was not rigorous enough and Hilbert's rigorous treatment (Chapter 17) was too cumbersome. So geometry was swept away in favour of 'New Mathematics'. The French mathematician Dieudonné, one of the founding members of the Bourbaki group, suggested that linear algebra should replace what he contemptuously called 'the theory of the triangle'.

Exercises

1. True or false? If two triangles have the same area, you can cut one of them up into little triangles, which can then be placed side by side to form a triangle congruent to the second triangle. Give a reference or a reason for your answer.

2. How did Euclid construct the regular 15-gon?