Let

$$\beta_j = |f(\alpha_j, \alpha_j)|^{-1/2}\alpha_j, \qquad 1 \le j \le r$$
$$\beta_j = \alpha_j, \qquad j > r.$$

Then $\{\beta_1, \ldots, \beta_n\}$ is a basis with the stated properties.

Let p be the number of basis vectors β_j for which $f(\beta_j, \beta_j) = 1$; we must show that the number p is independent of the particular basis we have, satisfying the stated conditions. Let V^+ be the subspace of V spanned by the basis vectors β_j for which $f(\beta_j, \beta_j) = 1$, and let V^- be the subspace spanned by the basis vectors β_j for which $f(\beta_j, \beta_j) = -1$. Now $p = \dim V^+$, so it is the uniqueness of the dimension of V^+ which we must demonstrate. It is easy to see that if α is a non-zero vector in V^+, then $f(\alpha, \alpha) > 0$; in other words, f is positive definite on the subspace V^+. Similarly, if α is a non-zero vector in V^-, then $f(\alpha, \alpha) < 0$, i.e., f is negative definite on the subspace V^-. Now let $V^\perp$ be the subspace spanned by the basis vectors β_j for which $f(\beta_j, \beta_j) = 0$. If α is in $V^\perp$, then $f(\alpha, \beta) = 0$ for all β in V.

Since $\{\beta_1, \ldots, \beta_n\}$ is a basis for V, we have

$$V = V^+ \oplus V^- \oplus V^\perp.$$

Furthermore, we claim that if W is any subspace of V on which f is positive definite, then the subspaces W, V^-, and $V^\perp$ are independent. For, suppose α is in W, β is in V^-, γ is in $V^\perp$, and $\alpha + \beta + \gamma = 0$. Then

$$0 = f(\alpha, \alpha + \beta + \gamma) = f(\alpha, \alpha) + f(\alpha, \beta) + f(\alpha, \gamma)$$
$$0 = f(\beta, \alpha + \beta + \gamma) = f(\beta, \alpha) + f(\beta, \beta) + f(\beta, \gamma).$$

Since γ is in $V^\perp$, $f(\alpha, \gamma) = f(\beta, \gamma) = 0$; and since f is symmetric, we obtain

$$0 = f(\alpha, \alpha) + f(\alpha, \beta)$$
$$0 = f(\beta, \beta) + f(\alpha, \beta)$$

hence $f(\alpha, \alpha) = f(\beta, \beta)$. Since $f(\alpha, \alpha) \ge 0$ and $f(\beta, \beta) \le 0$, it follows that

$$f(\alpha, \alpha) = f(\beta, \beta) = 0.$$

But f is positive definite on W and negative definite on V^-. We conclude that $\alpha = \beta = 0$, and hence that $\gamma = 0$ as well.

Since

$$V = V^+ \oplus V^- \oplus V^\perp$$

and W, V^-, $V^\perp$ are independent, we see that $\dim W \le \dim V^+$. That is, if W is any subspace of V on which f is positive definite, the dimension of W cannot exceed the dimension of V^+. If $\mathcal{B}_1$ is another ordered basis for V which satisfies the conditions of the theorem, we shall have corresponding subspaces V_1^+, V_1^-, and $V_1^\perp$; and, the argument above shows that $\dim V_1^+ \le \dim V^+$. Reversing the argument, we obtain $\dim V^+ \le \dim V_1^+$, and consequently

$$\dim V^+ = \dim V_1^+. \quad \blacksquare$$

There are several comments we should make about the basis $\{\beta_1, \ldots, \beta_n\}$ of Theorem 5 and the associated subspaces V^+, V^-, and $V^\perp$. First, note that $V^\perp$ is exactly the subspace of vectors which are 'orthogonal' to all of V. We noted above that $V^\perp$ is contained in this subspace; but,

$$\dim V^\perp = \dim V - (\dim V^+ + \dim V^-) = \dim V - \operatorname{rank} f$$

so every vector α such that $f(\alpha, \beta) = 0$ for all β must be in $V^\perp$. Thus, the subspace $V^\perp$ is unique. The subspaces V^+ and V^- are not unique; however, their dimensions are unique. The proof of Theorem 5 shows us that dim V^+ is the largest possible dimension of any subspace on which f is positive definite. Similarly, dim V^- is the largest dimension of any subspace on which f is negative definite. Of course

$$\dim V^+ + \dim V^- = \operatorname{rank} f.$$

The number

$$\dim V^+ - \dim V^-$$

is often called the **signature** of f. It is introduced because the dimensions of V^+ and V^- are easily determined from the rank of f and the signature of f.

Perhaps we should make one final comment about the relation of symmetric bilinear forms on real vector spaces to inner products. Suppose V is a finite-dimensional real vector space and that V_1, V_2, V_3 are subspaces of V such that

$$V = V_1 \oplus V_2 \oplus V_3.$$

Suppose that f_1 is an inner product on V_1, and f_2 is an inner product on V_2. We can then define a symmetric bilinear form f on V as follows: If α, β are vectors in V, then we can write

$$\alpha = \alpha_1 + \alpha_2 + \alpha_3 \quad \text{and} \quad \beta = \beta_1 + \beta_2 + \beta_3$$

with α_j and β_j in V_j. Let

$$f(\alpha, \beta) = f_1(\alpha_1, \beta_1) - f_2(\alpha_2, \beta_2).$$

The subspace $V^\perp$ for f will be V_3, V_1 is a suitable V^+ for f, and V_2 is a suitable V^-. One part of the statement of Theorem 5 is that every symmetric bilinear form on V arises in this way. The additional content of the theorem is that an inner product is represented in some ordered basis by the identity matrix.

Exercises

1. The following expressions define quadratic forms q on R^2. Find the symmetric bilinear form f corresponding to each q.

(a) ax_1^2.

(b) bx_1x_2.

(c) cx_2^2.

(d) $2x_1^2 - \frac{1}{3}x_1x_2$.

(e) $x_1^2 + 9x_2^2$.

(f) $3x_1x_2 - x_2^2$.

(g) $4x_1^2 + 6x_1x_2 - 3x_2^2$.

2. Find the matrix, in the standard ordered basis, and the rank of each of the bilinear forms determined in Exercise 1. Indicate which forms are non-degenerate.

3. Let $q(x_1, x_2) = ax_1^2 + bx_1x_2 + cx_2^2$ be the quadratic form associated with a symmetric bilinear form f on R^2. Show that f is non-degenerate if and only if $b^2 - 4ac \neq 0$.

4. Let V be a finite-dimensional vector space over a subfield F of the complex numbers, and let S be the set of all symmetric bilinear forms on V.

(a) Show that S is a subspace of $L(V, V, F)$.

(b) Find dim S.

Let Q be the set of all quadratic forms on V.

(c) Show that Q is a subspace of the space of all functions from V into F.

(d) Describe explicitly an isomorphism T of Q onto S, without reference to a basis.

(e) Let U be a linear operator on V and q an element of Q. Show that the equation $(U^\dagger q)(\alpha) = q(U\alpha)$ defines a quadratic form $U^\dagger q$ on V.

(f) If U is a linear operator on V, show that the function $U^\dagger$ defined in part (e) is a linear operator on Q. Show that $U^\dagger$ is invertible if and only if U is invertible.

5. Let q be the quadratic form on R^2 given by

$$q(x_1, x_2) = ax_1^2 + 2bx_1x_2 + cx_2^2, \qquad a \neq 0.$$

Find an invertible linear operator U on R^2 such that

$$(U^\dagger q)(x_1, x_2) = ax_1^2 + \left(c - \frac{b^2}{a}\right)x_2^2.$$

(*Hint:* To find U^{-1} (and hence U), complete the square. For the definition of $U^\dagger$, see part (e) of Exercise 4.)

6. Let q be the quadratic form on R^2 given by

$$q(x_1, x_2) = 2bx_1x_2.$$

Find an invertible linear operator U on R^2 such that

$$(U^\dagger q)(x_1, x_2) = 2bx_1^2 - 2bx_2^2.$$

7. Let q be the quadratic form on R^3 given by

$$q(x_1, x_2, x_3) = x_1x_2 + 2x_1x_3 + x_3^2.$$

Find an invertible linear operator U on R^3 such that

$$(U^\dagger q)(x_1, x_2, x_3) = x_1^2 - x_2^2 + x_3^2.$$

(*Hint:* Express U as a product of operators similar to those used in Exercises 5 and 6.)

8. Let A be a symmetric $n \times n$ matrix over R, and let q be the quadratic form on R^n given by

$$q(x_1, \ldots, x_n) = \sum_{i,j} A_{ij} x_i x_j.$$

Generalize the method used in Exercise 7 to show that there is an invertible linear operator U on R^n such that

$$(U^\dagger q)(x_1, \ldots, x_n) = \sum_{i=1}^{n} c_i x_i^2$$

where c_i is 1, -1, or 0, $i = 1, \ldots, n$.

9. Let f be a symmetric bilinear form on R^n. Use the result of Exercise 8 to prove the existence of an ordered basis $\mathfrak{B}$ such that $[f]_{\mathfrak{B}}$ is diagonal.

10. Let V be the real vector space of all 2×2 (complex) Hermitian matrices, that is, 2×2 complex matrices A which satisfy $A_{ij} = \overline{A_{ji}}$.

    (a) Show that the equation $q(A) = \det A$ defines a quadratic form q on V.

    (b) Let W be the subspace of V of matrices of trace 0. Show that the bilinear form f determined by q is negative definite on the subspace W.

11. Let V be a finite-dimensional vector space and f a non-degenerate symmetric bilinear form on V. Show that for each linear operator T on V there is a unique linear operator T' on V such that $f(T\alpha, \beta) = f(\alpha, T'\beta)$ for all α, β in V. Also show that

$$(T_1 T_2)' = T_2' T_1'$$
$$(c_1 T_1 + c_2 T_2)' = c_1 T_1' + c_2 T_2'$$
$$(T')' = T.$$

How much of the above is valid without the assumption that T is non-degenerate?

12. Let F be a field and V the space of $n \times 1$ matrices over F. Suppose A is a fixed $n \times n$ matrix over F and f is the bilinear form on V defined by $f(X, Y) = X^t A Y$. Suppose f is symmetric and non-degenerate. Let B be an $n \times n$ matrix over F and T the linear operator on V sending X into BX. Find the operator T' of Exercise 11.

13. Let V be a finite-dimensional vector space and f a non-degenerate symmetric bilinear form on V. Associated with f is a 'natural' isomorphism of V onto the dual space V^*, this isomorphism being the transformation L_f of Section 10.1. Using L_f, show that for each basis $\mathfrak{B} = \{\alpha_1, \ldots, \alpha_n\}$ of V there exists a unique basis $\mathfrak{B}' = \{\alpha_1', \ldots, \alpha_n'\}$ of V such that $f(\alpha_i, \alpha_j') = \delta_{ij}$. Then show that for every vector α in V we have

$$\alpha = \sum_i f(\alpha, \alpha_i') \alpha_i = \sum_i f(\alpha_i, \alpha) \alpha_i'.$$

14. Let $V, f, \mathfrak{B}$, and $\mathfrak{B}'$ be as in Exercise 13. Suppose T is a linear operator on V and that T' is the operator which f associates with T as in Exercise 11. Show that

    (a) $[T']_{\mathfrak{B}'} = [T]_{\mathfrak{B}}^t$.

    (b) $\text{tr}\,(T) = \text{tr}\,(T') = \sum_i f(T\alpha_i, \alpha_i')$.

15. Let $V, f, \mathfrak{B}$, and $\mathfrak{B}'$ be as in Exercise 13. Suppose $[f]_{\mathfrak{B}} = A$. Show that

$$\alpha_i' = \sum_j (A^{-1})_{ij} \alpha_j = \sum_j (A^{-1})_{ji} \alpha_j.$$

16. Let F be a field and V the space of $n \times 1$ matrices over F. Suppose A is an invertible, symmetric $n \times n$ matrix over F and that f is the bilinear form on V defined by $f(X, Y) = X^t A Y$. Let P be an invertible $n \times n$ matrix over F and $\mathfrak{B}$ the basis for V consisting of the columns of P. Show that the basis $\mathfrak{B}'$ of Exercise 13 consists of the columns of the matrix $A^{-1}(P^t)^{-1}$.

17. Let V be a finite-dimensional vector space over a field F and f a symmetric bilinear form on V. For each subspace W of V, let $W^{\perp}$ be the set of all vectors α in V such that $f(\alpha, \beta) = 0$ for every β in W. Show that

  (a) $W^{\perp}$ is a subspace.
  (b) $V = \{0\}^{\perp}$.
  (c) $V^{\perp} = \{0\}$ if and only if f is non-degenerate.
  (d) rank $f = \dim V - \dim V^{\perp}$.
  (e) If $\dim V = n$ and $\dim W = m$, then $\dim W^{\perp} \geq n - m$. (*Hint:* Let $\{\beta_1, \ldots, \beta_m\}$ be a basis of W and consider the mapping

$$\alpha \rightarrow (f(\alpha, \beta_1), \ldots, f(\alpha, \beta_m))$$

of V into F^m.)
  (f) The restriction of f to W is non-degenerate if and only if

$$W \cap W^{\perp} = \{0\}.$$

  (g) $V = W \oplus W^{\perp}$ if and only if the restriction of f to W is non-degenerate.

18. Let V be a finite-dimensional vector space over C and f a non-degenerate symmetric bilinear form on V. Prove that there is a basis $\mathfrak{B}$ of V such that $\mathfrak{B}' = \mathfrak{B}$. (See Exercise 13 for a definition of $\mathfrak{B}'$.)

10.3. Skew-Symmetric Bilinear Forms

Throughout this section V will be a vector space over a subfield F of the field of complex numbers. A bilinear form f on V is called **skew-symmetric** if $f(\alpha, \beta) = -f(\beta, \alpha)$ for all vectors α, β in V. We shall prove one theorem concerning the simplification of the matrix of a skew-symmetric bilinear form on a finite-dimensional space V. First, let us make some general observations.

Suppose f is *any* bilinear form on V. If we let

$$g(\alpha, \beta) = \tfrac{1}{2}[f(\alpha, \beta) + f(\beta, \alpha)]$$
$$h(\alpha, \beta) = \tfrac{1}{2}[f(\alpha, \beta) - f(\beta, \alpha)]$$

then it is easy to verify that g is a symmetric bilinear form on V and h is a skew-symmetric bilinear form on V. Also $f = g + h$. Furthermore, this expression for V as the sum of a symmetric and a skew-symmetric form is unique. Thus, the space $L(V, V, F)$ is the direct sum of the subspace of symmetric forms and the subspace of skew-symmetric forms.

If V is finite-dimensional, the bilinear form f is skew-symmetric if and only if its matrix A in some (or every) ordered basis is skew-symmetric, $A^t = -A$. This is proved just as one proves the corresponding fact about