group, say $G = \langle g_1, g_2, \ldots, g_n \rangle$, and let $\mathcal{S}$ be the set of all proper subgroups of G. Then $\mathcal{S}$ is partially ordered by inclusion. Let $\mathcal{C}$ be a chain in $\mathcal{S}$.

(a) Prove that the union, H, of all the subgroups in $\mathcal{C}$ is a subgroup of G.

(b) Prove that H is a *proper* subgroup. [If not, each g_i must lie in H and so must lie in some element of the chain $\mathcal{C}$. Use the definition of a chain to arrive at a contradiction.]

(c) Use Zorn's Lemma to show that $\mathcal{S}$ has a maximal element (which is, by definition, a maximal subgroup).

18. Let p be a prime and let $Z = \{z \in \mathbb{C} \mid z^{p^n} = 1 \text{ for some } n \in \mathbb{Z}^+\}$ (so Z is the multiplicative group of all p-power roots of unity in $\mathbb{C}$). For each $k \in \mathbb{Z}^+$ let $H_k = \{z \in Z \mid z^{p^k} = 1\}$ (the group of p^kth roots of unity). Prove the following:

(a) $H_k \le H_m$ if and only if $k \le m$

(b) H_k is cyclic for all k (assume that for any $n \in \mathbb{Z}^+$, $\{e^{2\pi i t/n} \mid t = 0, 1, \ldots, n-1\}$ is the set of all n^{th} roots of 1 in $\mathbb{C}$)

(c) every proper subgroup of Z equals H_k for some $k \in \mathbb{Z}^+$ (in particular, every proper subgroup of Z is finite and cyclic)

(d) Z is not finitely generated.

19. A nontrivial abelian group A (written multiplicatively) is called *divisible* if for each element $a \in A$ and each nonzero integer k there is an element $x \in A$ such that $x^k = a$, i.e., each element has a k^{th} root in A (in additive notation, each element is the k^{th} multiple of some element of A).

(a) Prove that the additive group of rational numbers, $\mathbb{Q}$, is divisible.

(b) Prove that no finite abelian group is divisible.

20. Prove that if A and B are nontrivial abelian groups, then $A \times B$ is divisible if and only if both A and B are divisible groups.

2.5 THE LATTICE OF SUBGROUPS OF A GROUP

In this section we describe a graph associated with a group which depicts the relationships among its subgroups. This graph, called the lattice[2] of subgroups of the group, is a good way of "visualizing" a group — it certainly illuminates the structure of a group better than the group table. We shall be using lattice diagrams, or parts of them, to describe both specific groups and certain properties of general groups throughout the chapters on group theory. Moreover, the lattice of subgroups of a group will play an important role in Galois Theory.

The lattice of subgroups of a given finite group G is constructed as follows: plot all subgroups of G starting at the bottom with 1, ending at the top with G and, roughly speaking, with subgroups of larger order positioned higher on the page than those of smaller order. Draw paths upwards between subgroups using the rule that there will be a line upward from A to B if $A \le B$ and there are no subgroups properly between A and B. Thus if $A \le B$ there is a path (possibly many paths) upward from A to B passing through a chain of intermediate subgroups (and a path downward from B to A if $B \ge A$). The initial positioning of the subgroups on the page, which is, a priori, somewhat arbitrary, can often (with practice) be chosen to produce a simple picture. Notice that for any pair of subgroups H and K of G the unique smallest subgroup

[2] The term "lattice" has a precise mathematical meaning in terms of partially ordered sets.

which contains both of them, namely $\langle H, K \rangle$ (called the *join* of H and K), may be read off from the lattice as follows: trace paths upwards from H and K until a common subgroup A which contains H and K is reached (note that G itself always contains all subgroups so at least one such A exists). To ensure that $A = \langle H, K \rangle$ make sure there is no $A_1 \leq A$ (indicated by a downward path from A to A_1) with both H and K contained in A_1 (otherwise replace A with A_1 and repeat the process to see if $A_1 = \langle H, K \rangle$). By a symmetric process one can read off the largest subgroup of G which is contained in both H and K, namely their intersection (which is a subgroup by Proposition 8).

There are some limitations to this process, in particular it cannot be carried out per se for infinite groups. Even for finite groups of relatively small order, lattices can be quite complicated (see the book *Groups of Order* 2^n, $n \leq 6$ by M. Hall and J. Senior, Macmillan, 1964, for some hair-raising examples). At the end of this section we shall describe how parts of a lattice may be drawn and used even for infinite groups.

Note that isomorphic groups have the same lattices (i.e., the same directed graphs). Nonisomorphic groups may also have identical lattices (this happens for two groups of order 16 — see the following exercises). Since the lattice of subgroups is only part of the data we shall carry in our descriptors of a group, this will not be a serious drawback (indeed, it might even be useful in seeing when two nonisomorphic groups have some common properties).

Examples

Except for the cyclic groups (Example 1) we have not proved that the following lattices are correct (e.g., contain all subgroups of the given group or have the right joins and intersections). For the moment we shall take these facts as given and, as we build up more theory in the course of the text, we shall assign as exercises the proofs that these are indeed correct.

(1) For $G = Z_n \cong \mathbb{Z}/n\mathbb{Z}$, by Theorem 7 the lattice of subgroups of G is the lattice of divisors of n (that is, the divisors of n are written on a page with n at the bottom, 1 at the top and paths upwards from a to b if $b \mid a$). Some specific examples for various values of n follow.

$$\begin{array}{l}
\mathbb{Z}/2\mathbb{Z} = \langle 1 \rangle \\
\quad | \\
\langle 2 \rangle = \{0\}
\end{array}
\qquad\qquad
\begin{array}{l}
\mathbb{Z}/4\mathbb{Z} = \langle 1 \rangle \quad \text{(note: } \langle 1 \rangle = \langle 3 \rangle\text{)} \\
\quad | \\
\langle 2 \rangle \\
\quad | \\
\langle 4 \rangle = \{0\}
\end{array}$$

$$\begin{array}{l}
\mathbb{Z}/8\mathbb{Z} = \langle 1 \rangle \quad \text{(note: } \langle 1 \rangle = \langle 3 \rangle = \langle 5 \rangle = \langle 7 \rangle\text{)} \\
\quad | \\
\langle 2 \rangle \\
\quad | \\
\langle 4 \rangle \\
\quad | \\
\langle 8 \rangle = \{0\}
\end{array}$$

In general, if p is a prime, the lattice of $\mathbb{Z}/p^n\mathbb{Z}$ is

$$\begin{tikzcd}[row sep=small]
\mathbb{Z}/p^n\mathbb{Z} = \langle 1 \rangle \arrow[d, dash] \\
\langle p \rangle \arrow[d, dash] \\
\langle p^2 \rangle \arrow[d, dash] \\
\langle p^3 \rangle \arrow[d, dash] \\
\vdots \arrow[d, dash] \\
\langle p^{n-1} \rangle \arrow[d, dash] \\
\langle p^n \rangle = \{0\}
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
& \mathbb{Z}/6\mathbb{Z} \arrow[dl, dash] \arrow[ddr, dash] & \\
\langle 2 \rangle \arrow[ddr, dash] & & \\
& & \langle 3 \rangle \arrow[dl, dash] \\
& \langle 6 \rangle = \{0\} &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
& & \mathbb{Z}/12\mathbb{Z} \arrow[dl, dash] \arrow[ddr, dash] & \\
& \langle 2 \rangle \arrow[dl, dash] \arrow[ddr, dash] & & \\
\langle 4 \rangle \arrow[ddr, dash] & & & \langle 3 \rangle \arrow[dl, dash] \\
& & \langle 6 \rangle \arrow[dl, dash] & \\
& \langle 12 \rangle = \{0\} & &
\end{tikzcd}$$

(2) The *Klein 4-group (Viergruppe)*, V_4, is the group of order 4 with multiplication table

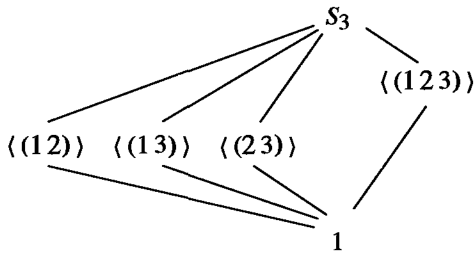
$\cdot$	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1

and lattice

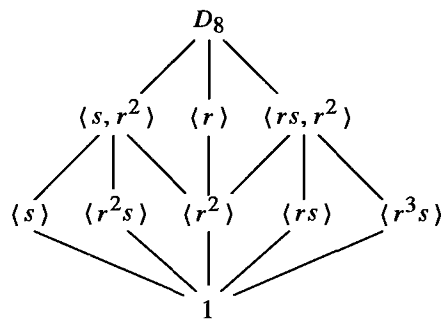
$$\begin{tikzcd}
& V_4 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\langle a \rangle \arrow[dr, dash] & \langle b \rangle \arrow[d, dash] & \langle c \rangle \arrow[dl, dash] \\
& 1 &
\end{tikzcd}$$

Note that V_4 is abelian and is not isomorphic to Z_4 (why?). We shall see that D_8 has an isomorphic copy of V_4 as a subgroup, so it will not be necessary to check that the associative law holds for the binary operation defined above.

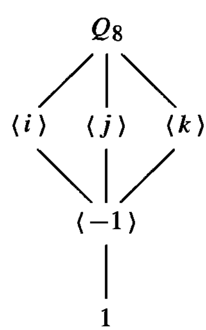
(3) The lattice of S_3 is



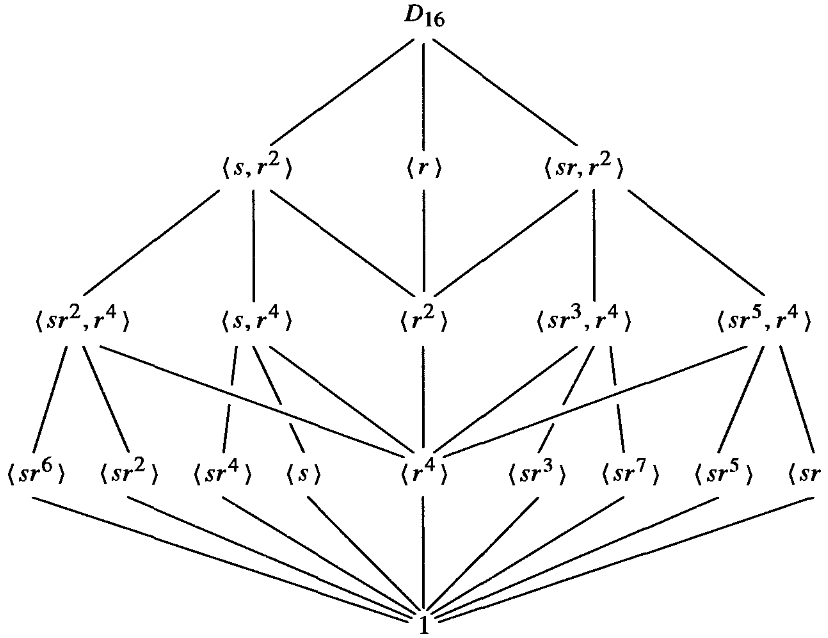
(4) Using our usual notation for $D_8 = \langle r, s \rangle$, the lattice of D_8 is



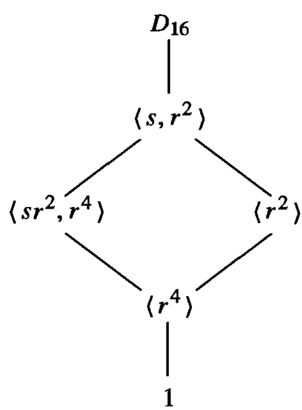
(5) The lattice of subgroups of Q_8 is



(6) The lattice of D_{16} is not a planar graph (cannot be drawn on a plane without lines crossing). One way of drawing it is



In many instances in both theoretical proofs and specific examples we shall be interested only in information concerning two (or some small number of) subgroups of a given group and their interrelationships. To depict these graphically we shall draw a *sublattice* of the entire group lattice which contains the relevant joins and intersections. An unbroken line in such a sublattice will not, in general, mean that there is no subgroup in between the endpoints of the line. These partial lattices for groups will also be used when we are dealing with infinite groups. For example, if we wished to discuss only the relationship between the subgroups $\langle sr^2, r^4 \rangle$ and $\langle r^2 \rangle$ of D_{16} we would draw the sublattice



Note that $\langle s, r^2 \rangle$ and $\langle r^4 \rangle$ are precisely the join and intersection, respectively, of these two subgroups in D_{16}.

Finally, given the lattice of subgroups of a group, it is relatively easy to compute normalizers and centralizers. For example, in D_8 we can see that $C_{D_8}(s) = \langle s, r^2 \rangle$ because we first calculate that $r^2 \in C_{D_8}(s)$ (see Section 2). This proves $\langle s, r^2 \rangle \le C_{D_8}(s)$ (note that an element always belongs to its own centralizer). The only subgroups which contain $\langle s, r^2 \rangle$ are that subgroup itself and all of D_8. We cannot have $C_{D_8}(s) = D_8$ because r does not commute with s (i.e., $r \notin C_{D_8}(s)$). This leaves only the claimed possibility for $C_{D_8}(s)$.

EXERCISES

1. Let H and K be subgroups of G. Exhibit all possible sublattices which show only G, 1, H, K and their joins and intersections. What distinguishes the different drawings?
2. In each of (a) to (d) list all subgroups of D_{16} that satisfy the given condition.
   (a) Subgroups that are contained in $\langle sr^2, r^4 \rangle$
   (b) Subgroups that are contained in $\langle sr^7, r^4 \rangle$
   (c) Subgroups that contain $\langle r^4 \rangle$
   (d) Subgroups that contain $\langle s \rangle$.
3. Show that the subgroup $\langle s, r^2 \rangle$ of D_8 is isomorphic to V_4.
4. Use the given lattice to find all pairs of elements that generate D_8 (there are 12 pairs).
5. Use the given lattice to find all elements $x \in D_{16}$ such that $D_{16} = \langle x, s \rangle$ (there are 16 such elements x).
6. Use the given lattices to help find the centralizers of every element in the following groups:
   (a) D_8 (b) Q_8 (c) S_3 (d) D_{16}.
7. Find the center of D_{16}.
8. In each of the following groups find the normalizer of each subgroup:
   (a) S_3 (b) Q_8.
9. Draw the lattices of subgroups of the following groups:
   (a) $\mathbb{Z}/16\mathbb{Z}$ (b) $\mathbb{Z}/24\mathbb{Z}$ (c) $\mathbb{Z}/48\mathbb{Z}$. [See Exercise 6 in Section 3.]
10. Classify groups of order 4 by proving that if $|G| = 4$ then $G \cong Z_4$ or $G \cong V_4$. [See Exercise 36, Section 1.1.]
11. Consider the group of order 16 with the following presentation:

$$QD_{16} = \langle \sigma, \tau \mid \sigma^8 = \tau^2 = 1,\ \sigma\tau = \tau\sigma^3 \rangle$$

(called the *quasidihedral* or *semidihedral* group of order 16). This group has three subgroups of order 8: $\langle \tau, \sigma^2 \rangle \cong D_8$, $\langle \sigma \rangle \cong Z_8$ and $\langle \sigma^2, \sigma\tau \rangle \cong Q_8$ and every proper subgroup is contained in one of these three subgroups. Fill in the missing subgroups in the lattice of all subgroups of the quasidihedral group on the following page, exhibiting each subgroup with at most two generators. (This is another example of a nonplanar lattice.)

The next three examples lead to two nonisomorphic groups that have the same lattice of subgroups.

12. The group $A = Z_2 \times Z_4 = \langle a, b \mid a^2 = b^4 = 1,\ ab = ba \rangle$ has order 8 and has three subgroups of order 4: $\langle a, b^2 \rangle \cong V_4$, $\langle b \rangle \cong Z_4$ and $\langle ab \rangle \cong Z_4$ and every proper