10.32 Example For $0 \le u \le \pi$, $0 \le v \le 2\pi$, define

$$\Sigma(u, v) = (\sin u \cos v, \sin u \sin v, \cos u).$$

Then Σ is a 2-surface in R^3, whose parameter domain is a rectangle $D \subset R^2$, and whose range is the unit sphere in R^3. Its boundary is

$$\partial\Sigma = \Sigma(\partial D) = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$$

where

$$\gamma_1(u) = \Sigma(u, 0) = (\sin u, 0, \cos u),$$
$$\gamma_2(v) = \Sigma(\pi, v) = (0, 0, -1),$$
$$\gamma_3(u) = \Sigma(\pi - u, 2\pi) = (\sin u, 0, -\cos u),$$
$$\gamma_4(v) = \Sigma(0, 2\pi - v) = (0, 0, 1),$$

with $[0, \pi]$ and $[0, 2\pi]$ as parameter intervals for u and v, respectively.

Since γ_2 and γ_4 are constant, their derivatives are 0, hence the integral of any 1-form over γ_2 or γ_4 is 0. [See Example 1.12(*a*).]

Since $\gamma_3(u) = \gamma_1(\pi - u)$, direct application of (35) shows that

$$\int_{\gamma_3} \omega = -\int_{\gamma_1} \omega$$

for every 1-form ω. Thus $\int_{\partial\Sigma} \omega = 0$, and we conclude that $\partial\Sigma = 0$.

(In geographic terminology, $\partial\Sigma$ starts at the north pole N, runs to the south pole S along a meridian, pauses at S, returns to N along the same meridian, and finally pauses at N. The two passages along the meridian are in opposite directions. The corresponding two line integrals therefore cancel each other. In Exercise 32 there is also one curve which occurs twice in the boundary, but without cancellation.)

STOKES' THEOREM

10.33 Theorem *If Ψ is a k-chain of class $\mathscr{C}''$ in an open set $V \subset R^m$ and if ω is a $(k-1)$-form of class $\mathscr{C}'$ in V, then*

$$\int_{\Psi} d\omega = \int_{\partial\Psi} \omega. \tag{91}$$

The case $k = m = 1$ is nothing but the fundamental theorem of calculus (with an additional differentiability assumption). The case $k = m = 2$ is Green's theorem, and $k = m = 3$ gives the so-called "divergence theorem" of Gauss. The case $k = 2$, $m = 3$ is the one originally discovered by Stokes. (Spivak's book describes some of the historical background.) These special cases will be discussed further at the end of the present chapter.

Proof It is enough to prove that

(92) $$\int_{\Phi} d\omega = \int_{\partial\Phi} \omega$$

for every oriented k-simplex Φ of class $\mathscr{C}''$ in V. For if (92) is proved and if $\Psi = \Sigma\Phi_i$, then (87) and (89) imply (91).

Fix such a Φ and put

(93) $$\sigma = [\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_k].$$

Thus σ is the oriented affine k-simplex with parameter domain Q^k which is defined by the identity mapping. Since Φ is also defined on Q^k (see Definition 10.30) and $\Phi \in \mathscr{C}''$, there is an open set $E \subset R^k$ which contains Q^k, and there is a $\mathscr{C}''$-mapping T of E into V such that $\Phi = T \circ \sigma$. By Theorems 10.25 and 10.22(c), the left side of (92) is equal to

$$\int_{T\sigma} d\omega = \int_{\sigma} (d\omega)_T = \int_{\sigma} d(\omega_T).$$

Another application of Theorem 10.25 shows, by (89), that the right side of (92) is

$$\int_{\partial(T\sigma)} \omega = \int_{T(\partial\sigma)} \omega = \int_{\partial\sigma} \omega_T.$$

Since ω_T is a $(k-1)$-form in E, we see that *in order to prove* (92) *we merely have to show that*

(94) $$\int_{\sigma} d\lambda = \int_{\partial\sigma} \lambda$$

for the special simplex (93) *and for every* $(k-1)$*-form* λ *of class* $\mathscr{C}'$ *in* E.

If $k = 1$, the definition of an oriented 0-simplex shows that (94) merely asserts that

(95) $$\int_0^1 f'(u)\, du = f(1) - f(0)$$

for every continuously differentiable function f on $[0, 1]$, which is true by the fundamental theorem of calculus.

From now on we assume that $k > 1$, fix an integer r $(1 \le r \le k)$, and choose $f \in \mathscr{C}'(E)$. It is then enough to prove (94) for the case

(96) $$\lambda = f(\mathbf{x})\, dx_1 \wedge \cdots \wedge dx_{r-1} \wedge dx_{r+1} \wedge \cdots \wedge dx_k$$

since every $(k-1)$-form is a sum of these special ones, for $r = 1, \ldots, k$.

By (85), the boundary of the simplex (93) is

$$\partial\sigma = [\mathbf{e}_1, \ldots, \mathbf{e}_k] + \sum_{i=1}^{k} (-1)^i \tau_i$$

where

$$\tau_i = [\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_{i-1}, \mathbf{e}_{i+1}, \ldots, \mathbf{e}_k]$$

for $i = 1, \ldots, k$. Put

$$\tau_0 = [\mathbf{e}_r, \mathbf{e}_1, \ldots, \mathbf{e}_{r-1}, \mathbf{e}_{r+1}, \ldots, \mathbf{e}_k].$$

Note that τ_0 is obtained from $[\mathbf{e}_1, \ldots, \mathbf{e}_k]$ by $r - 1$ successive interchanges of $\mathbf{e}_r$ and its left neighbors. Thus

$$\partial\sigma = (-1)^{r-1}\tau_0 + \sum_{i=1}^{k} (-1)^i \tau_i. \tag{97}$$

Each τ_i has Q^{k-1} as parameter domain.

If $\mathbf{x} = \tau_0(\mathbf{u})$ and $\mathbf{u} \in Q^{k-1}$, then

$$x_j = \begin{cases} u_j & (1 \le j < r), \\ 1 - (u_1 + \cdots + u_{k-1}) & (j = r), \\ u_{j-1} & (r < j \le k). \end{cases} \tag{98}$$

If $1 \le i \le k$, $\mathbf{u} \in Q^{k-1}$, and $\mathbf{x} = \tau_i(\mathbf{u})$, then

$$x_j = \begin{cases} u_j & (1 \le j < i), \\ 0 & (j = i), \\ u_{j-1} & (i < j \le k). \end{cases} \tag{99}$$

For $0 \le i \le k$, let J_i be the Jacobian of the mapping

$$(u_1, \ldots, u_{k-1}) \to (x_1, \ldots, x_{r-1}, x_{r+1}, \ldots, x_k) \tag{100}$$

induced by τ_i. When $i = 0$ and when $i = r$, (98) and (99) show that (100) is the identity mapping. Thus $J_0 = 1$, $J_r = 1$. For other i, the fact that $x_i = 0$ in (99) shows that J_i has a row of zeros, hence $J_i = 0$. Thus

$$\int_{\tau_i} \lambda = 0 \qquad (i \ne 0, i \ne r), \tag{101}$$

by (35) and (96). Consequently, (97) gives

$$\begin{aligned} \int_{\partial\sigma} \lambda &= (-1)^{r-1} \int_{\tau_0} \lambda + (-1)^r \int_{\tau_r} \lambda \\ &= (-1)^{r-1} \int \qquad [f(\tau_0(\mathbf{u})) - f(\tau_r(\mathbf{u}))]\, d\mathbf{u}. \end{aligned} \tag{102}$$

On the other hand,

$$\begin{aligned} d\lambda &= (D_r f)(\mathbf{x}) dx_r \wedge dx_1 \wedge \cdots \wedge dx_{r-1} \wedge dx_{r+1} \wedge \cdots \wedge dx_k \\ &= (-1)^{r-1}(D_r f)(\mathbf{x})\, dx_1 \wedge \cdots \wedge dx_k \end{aligned}$$

so that

$$\int_\sigma d\lambda = (-1)^{r-1} \int_{Q^k} (D_r f)(\mathbf{x})\, d\mathbf{x}. \tag{103}$$

We evaluate (103) by first integrating with respect to x_r, over the interval

$$[0, 1 - (x_1 + \cdots + x_{r-1} + x_{r+1} + \cdots + x_k)],$$

put $(x_1, \ldots, x_{r-1}, x_{r+1}, \ldots, x_k) = (u_1, \ldots, u_{k-1})$, and see with the aid of (98) that the integral over Q^k in (103) is equal to the integral over Q^{k-1} in (102). Thus (94) holds, and the proof is complete.

CLOSED FORMS AND EXACT FORMS

10.34 Definition Let ω be a k-form in an open set $E \subset R^n$. If there is a $(k-1)$-form λ in E such that $\omega = d\lambda$, then ω is said to be *exact in E.*

If ω is of class $\mathscr{C}'$ and $d\omega = 0$, then ω is said to be *closed.*

Theorem 10.20(*b*) shows that every exact form of class $\mathscr{C}'$ is closed.

In certain sets E, for example in convex ones, the converse is true; this is the content of Theorem 10.39 (usually known as *Poincaré's lemma*) and Theorem 10.40. However, Examples 10.36 and 10.37 will exhibit closed forms that are not exact.

10.35 Remarks

(*a*) Whether a given k-form ω is or is not closed can be verified by simply differentiating the coefficients in the standard presentation of ω. For example, a 1-form

$$\omega = \sum_{i=1}^{n} f_i(\mathbf{x})\, dx_i, \tag{104}$$

with $f_i \in \mathscr{C}'(E)$ for some open set $E \subset R^n$, is closed if and only if the equations

$$(D_j f_i)(\mathbf{x}) = (D_i f_j)(\mathbf{x}) \tag{105}$$

hold for all i, j in $\{1, \ldots, n\}$ and for all $\mathbf{x} \in E$.

Note that (105) is a "pointwise" condition; it does not involve any global properties that depend on the shape of E.

On the other hand, to show that ω is exact in E, one has to prove the existence of a form λ, defined in E, such that $d\lambda = \omega$. This amounts to solving a system of partial differential equations, not just locally, but in all of E. For example, to show that (104) is exact in a set E, one has to find a function (or 0-form) $g \in \mathscr{C}'(E)$ such that

$$(D_i g)(\mathbf{x}) = f_i(\mathbf{x}) \qquad (\mathbf{x} \in E, 1 \le i \le n). \tag{106}$$

Of course, (105) is a necessary condition for the solvability of (106).

(*b*) Let ω be an *exact* k-form in E. Then there is a $(k-1)$-form λ in E with $d\lambda = \omega$, and Stokes' theorem asserts that

$$\int_\Psi \omega = \int_\Psi d\lambda = \int_{\partial\Psi} \lambda \tag{107}$$

for every k-chain Ψ of class $\mathscr{C}''$ in E.

If Ψ_1 and Ψ_2 are such chains, and if they have the same boundaries, it follows that

$$\int_{\Psi_1} \omega = \int_{\Psi_2} \omega.$$

In particular, *the integral of an exact k-form in E is* 0 *over every k-chain in E whose boundary is* 0.

As an important special case of this, note that integrals of exact 1-forms in E are 0 over closed (differentiable) curves in E.

(*c*) Let ω be a *closed* k-form in E. Then $d\omega = 0$, and Stokes' theorem asserts that

$$\int_{\partial\Psi} \omega = \int_\Psi d\omega = 0 \tag{108}$$

for every $(k+1)$-chain Ψ of class $\mathscr{C}''$ in E.

In other words, *integrals of closed k-forms in E are* 0 *over k-chains that are boundaries of $(k+1)$-chains in E.*

(*d*) Let Ψ be a $(k+1)$-chain in E and let λ be a $(k-1)$-form in E, both of class $\mathscr{C}''$. Since $d^2\lambda = 0$, two applications of Stokes' theorem show that

$$\int_{\partial\partial\Psi} \lambda = \int_{\partial\Psi} d\lambda = \int_\Psi d^2\lambda = 0. \tag{109}$$

We conclude that $\partial^2\Psi = 0$. In other words, *the boundary of a boundary is* 0.

See Exercise 16 for a more direct proof of this.

10.36 Example Let $E = R^2 - \{\mathbf{0}\}$, the plane with the origin removed. The 1-form

$$\eta = \frac{x\,dy - y\,dx}{x^2 + y^2} \tag{110}$$

is *closed* in $R^2 - \{\mathbf{0}\}$. This is easily verified by differentiation. Fix $r > 0$, and define

$$\gamma(t) = (r \cos t, r \sin t) \qquad (0 \le t \le 2\pi). \tag{111}$$

Then γ is a curve (an "oriented 1-simplex") in $R^2 - \{\mathbf{0}\}$. Since $\gamma(0) = \gamma(2\pi)$, we have

$$\partial\gamma = 0. \tag{112}$$

Direct computation shows that

$$\int_\gamma \eta = 2\pi \neq 0. \tag{113}$$

The discussion in Remarks 10.35(*b*) and (*c*) shows that we can draw two conclusions from (113):

First, η *is not exact in* $R^2 - \{\mathbf{0}\}$, for otherwise (112) would force the integral (113) to be 0.

Secondly, γ *is not the boundary of any* 2*-chain in* $R^2 - \{\mathbf{0}\}$ (of class $\mathscr{C}''$), for otherwise the fact that η is closed would force the integral (113) to be 0.

10.37 Example Let $E = R^3 - \{\mathbf{0}\}$, 3-space with the origin removed. Define

$$\zeta = \frac{x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}} \tag{114}$$

where we have written (x, y, z) in place of (x_1, x_2, x_3). Differentiation shows that $d\zeta = 0$, so that ζ is a closed 2-form in $R^3 - \{\mathbf{0}\}$.

Let Σ be the 2-chain in $R^3 - \{\mathbf{0}\}$ that was constructed in Example 10.32; recall that Σ is a parametrization of the unit sphere in R^3. Using the rectangle D of Example 10.32 as parameter domain, it is easy to compute that

$$\int_\Sigma \zeta = \int_D \sin u\,du\,dv = 4\pi \neq 0. \tag{115}$$

As in the preceding example, we can now conclude that ζ is *not* exact in $R^3 - \{\mathbf{0}\}$ (since $\partial\Sigma = 0$, as was shown in Example 10.32) and that the sphere Σ is not the boundary of any 3-chain in $R^3 - \{\mathbf{0}\}$ (of class $\mathscr{C}''$), although $\partial\Sigma = 0$.

The following result will be used in the proof of Theorem 10.39.

10.38 Theorem *Suppose E is a convex open set in R^n, $f \in \mathscr{C}'(E)$, p is an integer, $1 \le p \le n$, and*

$$(D_j f)(\mathbf{x}) = 0 \qquad (p < j \le n, \mathbf{x} \in E). \tag{116}$$

Then there exists an $F \in \mathscr{C}'(E)$ such that

$$(D_p F)(\mathbf{x}) = f(\mathbf{x}), \qquad (D_j F)(\mathbf{x}) = 0 \qquad (p < j \le n, \mathbf{x} \in E). \tag{117}$$

Proof Write $\mathbf{x} = (\mathbf{x}', x_p, \mathbf{x}'')$, where

$$\mathbf{x}' = (x_1, \ldots, x_{p-1}), \ \mathbf{x}'' = (x_{p+1}, \ldots, x_n).$$

(When $p = 1$, $\mathbf{x}'$ is absent; when $p = n$, $\mathbf{x}''$ is absent.) Let V be the set of all $(\mathbf{x}', x_p) \in R^p$ such that $(\mathbf{x}', x_p, \mathbf{x}'') \in E$ for some $\mathbf{x}''$. Being a projection of E, V is a convex open set in R^p. Since E is convex and (116) holds, $f(\mathbf{x})$ does not depend on $\mathbf{x}''$. Hence there is a function φ, with domain V, such that

$$f(\mathbf{x}) = \varphi(\mathbf{x}', x_p)$$

for all $\mathbf{x} \in E$.

If $p = 1$, V is a segment in R^1 (possibly unbounded). Pick $c \in V$ and define

$$F(\mathbf{x}) = \int_c^{x_1} \varphi(t)\, dt \qquad (\mathbf{x} \in E).$$

If $p > 1$, let U be the set of all $\mathbf{x}' \in R^{p-1}$ such that $(\mathbf{x}', x_p) \in V$ for some x_p. Then U is a convex open set in R^{p-1}, and there is a function $\alpha \in \mathscr{C}'(U)$ such that $(\mathbf{x}', \alpha(\mathbf{x}')) \in V$ for every $\mathbf{x}' \in U$; in other words, the graph of α lies in V (Exercise 29). Define

$$F(\mathbf{x}) = \int_{\alpha(\mathbf{x}')}^{x_p} \varphi(\mathbf{x}', t)\, dt \qquad (\mathbf{x} \in E).$$

In either case, F satisfies (117).

(*Note:* Recall the usual convention that $\int_a^b$ means $-\int_b^a$ if $b < a$.)

10.39 Theorem *If $E \subset R^n$ is convex and open, if $k \ge 1$, if ω is a k-form of class $\mathscr{C}'$ in E, and if $d\omega = 0$, then there is a $(k-1)$-form λ in E such that $\omega = d\lambda$.*

Briefly, closed forms are exact in convex sets.

Proof For $p = 1, \ldots, n$, let Y_p denote the set of all k-forms ω, of class $\mathscr{C}'$ in E, whose standard presentation

$$\omega = \sum_I f_I(\mathbf{x})\, dx_I \tag{118}$$

does not involve $dx_{p+1}, \ldots, dx_n$. In other words, $I \subset \{1, \ldots, p\}$ if $f_I(\mathbf{x}) \ne 0$ for some $\mathbf{x} \in E$.

We shall proceed by induction on p.

Assume first that $\omega \in Y_1$. Then $\omega = f(\mathbf{x})\, dx_1$. Since $d\omega = 0$, $(D_j f)(\mathbf{x}) = 0$ for $1 < j \le n$, $\mathbf{x} \in E$. By Theorem 10.38 there is an $F \in \mathscr{C}'(E)$ such that $D_1 F = f$ and $D_j F = 0$ for $1 < j \le n$. Thus

$$dF = (D_1 F)(\mathbf{x})\, dx_1 = f(\mathbf{x})\, dx_1 = \omega.$$

Now we take $p > 1$ and make the following induction hypothesis: *Every closed k-form that belongs to Y_{p-1} is exact in E.*

Choose $\omega \in Y_p$ so that $d\omega = 0$. By (118),

$$\sum_I \sum_{j=1}^{n} (D_j f_I)(\mathbf{x})\, dx_j \wedge dx_I = d\omega = 0. \tag{119}$$

Consider a fixed j, with $p < j \le n$. Each I that occurs in (118) lies in $\{1, \ldots, p\}$. If I_1, I_2 are two of these k-indices, and if $I_1 \ne I_2$, then the $(k+1)$-indices (I_1, j), (I_2, j) are distinct. Thus there is no cancellation, and we conclude from (119) that every coefficient in (118) satisfies

$$(D_j f_I)(\mathbf{x}) = 0 \qquad (\mathbf{x} \in E, p < j \le n). \tag{120}$$

We now gather those terms in (118) that contain dx_p and rewrite ω in the form

$$\omega = \alpha + \sum_{I_0} f_I(\mathbf{x})\, dx_{I_0} \wedge dx_p, \tag{121}$$

where $\alpha \in Y_{p-1}$, each I_0 is an increasing $(k-1)$-index in $\{1, \ldots, p-1\}$, and $I = (I_0, p)$. By (120), Theorem 10.38 furnishes functions $F_I \in \mathscr{C}'(E)$ such that

$$D_p F_I = f_I, \qquad D_j F_I = 0 \qquad (p < j \le n). \tag{122}$$

Put

$$\beta = \sum_{I_0} F_I(\mathbf{x})\, dx_{I_0} \tag{123}$$

and define $\gamma = \omega - (-1)^{k-1}\, d\beta$. Since β is a $(k-1)$-form, it follows that

$$\begin{aligned}\gamma &= \omega - \sum_{I_0} \sum_{j=1}^{p} (D_j F_I)(\mathbf{x})\, dx_{I_0} \wedge dx_j \\ &= \alpha - \sum_{I_0} \sum_{j=1}^{p-1} (D_j F_I)(\mathbf{x})\, dx_{I_0} \wedge dx_j,\end{aligned}$$

which is clearly in Y_{p-1}. Since $d\omega = 0$ and $d^2\beta = 0$, we have $d\gamma = 0$. Our induction hypothesis shows therefore that $\gamma = d\mu$ for some $(k-1)$-form μ in E. If $\lambda = \mu + (-1)^{k-1}\beta$, we conclude that $\omega = d\lambda$.

By induction, this completes the proof.