$A \vdash A$ (inferring a formula from itself);

$$\frac{\Lambda \vdash A \qquad \Gamma, A \vdash B}{\Gamma, \Lambda \vdash B}$$ (replacing an assumption by others which entail it);

$$\frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}$$ (interchanging two assumptions);

$$\frac{\Gamma \vdash B}{\Gamma, A \vdash B}$$ (introducing a superfluous assumption);

$$\frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}$$ (contracting two identical assumptions into one).

These so-called *structural rules* were formally introduced by Gerhard Gentzen (1919–1945). Except for the axiom $A \vdash A$, they are often tacitly understood and not mentioned in actual arguments.

We will show how to establish deductions of the form $\Gamma \vdash B$ by looking at a few examples.

EXAMPLE 1. To prove that $A \vee B, \neg B \vdash A$.

Informally, we would argue as follows. We are given the assumptions $A \vee B$ and $B \Rightarrow \perp$. Suppose A. Then surely A, by the axiom $A \vdash A$. Suppose B. Then $\perp$ by modus ponens from the second given assumption. Therefore A, by the axiom $\perp \vdash A$. Since A in either case, we invoke the argument by cases and infer that A holds in view of the given assumptions.

It is customary to rewrite such an argument more formally in a vertical fashion:

1	(1)	$A \vee B$	given
2	(2)	$B \Rightarrow \perp$	given
3	(3)	A	assumed
2,3	(4)	A	introducing a superfluous hypothesis
5	(5)	B	assumed
2,5	(6)	$\perp$	MP 2,5
2,5	(7)	A	by axiom $\perp \vdash A$
2,1	(8)	A	AC 4,7 replacing 3 and 5 by 1
1,2	(9)	A	interchanging two arguments

Note that the middle column contains the formulas given, assumed or inferred at different stages of the argument, numbered consecutively; the left column lists the numbers of all the hypotheses, given or assumed, upon which the formula in the middle column depends, and the right column indicates the justification for writing it down. The first two entries in the last line say precisely that $A \vee B, B \Rightarrow \perp \vdash A$, as was to be proved.

EXAMPLE 2. To prove that $A \vdash \neg\neg A$.

Here is the informal argument: we are given A. Suppose $A \Rightarrow \perp$. Then $\perp$ by modus ponens. Therefore, $(A \Rightarrow \perp) \Rightarrow \perp$ by the deduction rule.

Formally:

1	(1)	A	given
2	(2)	$A \Rightarrow \perp$	assumed
1,2	(3)	$\perp$	MP 1,2
1	(4)	$(A \Rightarrow \perp) \Rightarrow \perp$	DR 2,3

EXAMPLE 3. To prove that $\vdash A \Rightarrow (B \Rightarrow A)$.

We shall give the formal argument only.

1	(1)	A	assumed
2	(2)	B	assumed
1,2	(3)	A	introducing a superfluous hypothesis
1	(4)	$B \Rightarrow A$	DR 2,3
	(5)	$A \Rightarrow (B \Rightarrow A)$	DR 1,4

After some practice, the student may stop mentioning the structural rules, such as line (3) above, or lines (4) and (9) in Example 1.

EXAMPLE 4. To prove that $A \Rightarrow B \vdash \neg B \Rightarrow \neg A$.

1	(1)	$A \Rightarrow B$	given
2	(2)	$B \Rightarrow \bot$	assumed
3	(3)	A	assumed
1,3	(4)	B	MP 1,3
1,2,3	(5)	$\bot$	MP 2,4
1,2	(6)	$A \Rightarrow \bot$	DR 3,5
1	(7)	$\neg B \Rightarrow \neg A$	DR 2,6

EXAMPLE 5. To prove that $\vdash \neg\neg(A \vee \neg A)$.

1	(1)	$(A \vee \neg A) \Rightarrow \bot$	assumed
2	(2)	A	assumed
2	(3)	$A \vee \neg A$	axiom $A \vdash A \vee B$
1,2	(4)	$\bot$	MP 1,3
1	(5)	$A \Rightarrow \bot$	DR 2,4
1	(6)	$A \vee \neg A$	axiom $B \vdash A \vee B$
1	(7)	$\bot$	MP 1,6
	(8)	$((A \vee \neg A) \Rightarrow \bot) \Rightarrow \bot$	DR 1,7

Exercises

Prove the following.

1. $A \vee \neg A \vdash \neg\neg A \Rightarrow A$.

2. $((A \Rightarrow B) \Rightarrow A) \vdash \neg\neg A$.

3. $C \Rightarrow (A \wedge B) \vdash (C \Rightarrow A) \wedge (C \Rightarrow B)$.

4. $(C \Rightarrow A) \wedge (C \Rightarrow B) \vdash C \Rightarrow (A \wedge B)$.

5. $A \Rightarrow (B \Rightarrow C) \vdash (A \wedge B) \Rightarrow C$.

6. $(A \wedge B) \Rightarrow C \vdash A \Rightarrow (B \Rightarrow C)$.

Note that the classical result $((A \Rightarrow B) \Rightarrow A) \vdash A$ follows from (2) if $\neg\neg A \vdash A$, but it does not hold intuitionistically.

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How to Interpret Intuitionistic Logic

To explain the subtle difference between intuitionistic and classical logic, we shall present an intuitionistic interpretation of the logical connectives that goes back to Brouwer, Heyting and Kolmogorov. It involves talking about *reasons* for a formula. A *reason* for A may be thought of as a proof of A from some suitable assumption. We would like to say

- there is exactly one reason for $\top$ (namely, quoting the axiom);
- there is no reason for $\bot$;
- a reason for $A \wedge B$ consists of a reason for A and a reason for B;
- a reason for $A \vee B$ is a reason for A or a reason for B;
- a reason for $B \Rightarrow C$ is a rule for converting a reason for B into a reason for C.

Now compare these statements with the following statements about sets (see Chapter 13, where 0 was defined as the empty set and 1 as $\{0\}$):

- there is exactly one element of 1;
- there is no element of 0;
- an element of $A \times B$ is a pair of elements of A and B, respectively;
- an element of $A + B$ is an element of A or an element of B;

- an element of C^B is a function that converts an element of B into an element of C.

Comparing intuitionistic logic with the arithmetic of sets, we are led to the following analogies:

$$\begin{array}{cc} \top & 1 \\ \bot & 0 \\ A \wedge B & A \times B \\ A \vee B & A + B \\ B \Rightarrow C & C^B \end{array}$$

Moreover, a deduction from A to B, namely, an argument showing that $A \vdash B$, corresponds to a mapping $A \to B$. If there is a deduction from A to B and a deduction from B to A, we shall write $A \vdash\dashv B$. This corresponds to mappings $A \to B$ and $B \to A$ and we may write $A \leftrightarrow B$. Frequently these two mappings are inverse to one another, so we have a one-to-one correspondence between A and B, that is, $A \cong B$.

For example, we can prove intuitionistically that

$$C \Rightarrow (A \wedge B) \vdash\dashv (C \Rightarrow A) \wedge (C \Rightarrow B),$$

$$A \Rightarrow (B \Rightarrow C) \vdash\dashv (A \wedge B) \Rightarrow C,$$

$$(A \vee B) \Rightarrow C \vdash\dashv (A \Rightarrow C) \wedge (B \Rightarrow C).$$

These equivalences correspond to the following one-to-one mappings between sets:

$$(A \times B)^C \cong A^C \times B^C,$$
$$(C^B)^A \cong C^{A \times B},$$
$$C^{A+B} \cong C^A \times C^B.$$

As we saw in Chapter 13, these are just generalizations of the familiar laws of arithmetic:

$$(a \times b)^c = a^c \times b^c,$$
$$(c^b)^a = c^{a \times b},$$
$$c^{a+b} = c^a \times c^b.$$

One cannot but be impressed by the remarkable unity pervading logic, set theory and arithmetic.

A word of warning: $A \leftrightarrow B$ does not always mean $A \cong B$. For example, the intuitionistic equivalence

$$A \Rightarrow (B \Rightarrow A) \vdash\dashv \top$$