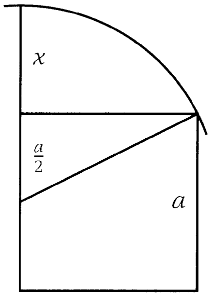

FIGURE 2.20 Geometric solution of a quadratic equation.

In the *Elements*, Book II, Proposition 11, Euclid solves the equation $x^2 = (a - x)a$ by the construction shown in Figure 2.20.

2.6.4. Use Pythagoras' theorem and algebra to check that this is a correct solution.

2.7 Volume

The theory of volume looks much the same as the theory of area at first. The unit of volume is the cube with sides of unit length, and this leads easily to the volume formula for a *cuboid*, the figure whose faces are rectangles. For a cuboid with integer sides we see immediately that volume = width × height × depth by cutting the cuboid into unit cubes. The same formula follows for a cuboid with rational sides by cutting into equal fractional cubes, as we did for rectangles in Section 2.5. Finally, the formula is true for irrational sides either by definition of the product of three lengths (as the Greeks would have it), or by definition of the product of irrational numbers (as we prefer today).

From the cuboid, we can obtain certain other volumes by cutting and pasting, for example, the volume of the *parallelepiped* and the *triangular prism*. The parallelepiped (pronounced "parallel epi ped" where "epi" rhymes with "peppy"), is the three-dimensional figure analogous to the parallelogram, and the triangular prism is obtained by cutting a parallelepiped in half (Figure 2.21).

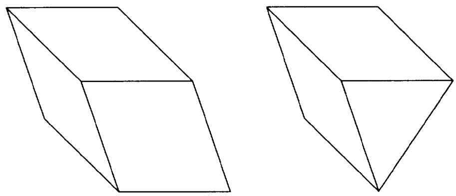

FIGURE 2.21 Parallelepiped and triangular prism.

If we cut a prism off the left end of a cuboid and paste it to the right end (Figure 2.22), we obtain a parallelepiped with a rectangular base and rectangular ends, and volume equal to that of the cuboid, namely, base area $\times$ height.

By similarly cutting a prism off the front and pasting it to the back, we obtain a parallelepiped with only the top and bottom rectangular, but still with volume equal to base area $\times$ height. Finally, one more cut and paste gives the general parallelepiped, whose faces are arbitrary parallelograms and whose volume is still base area $\times$ height. The same is true of a general triangular prism, because it is obtained by cutting a parallelepiped in half.

So far, so good, but these are not typical three-dimensional figures. They are figures of constant cross section, and all we have done so far is operate within their cross sections the way we did in the plane with parallellograms and triangles.

What we really need to know is the volume of a *tetrahedron*, the three-dimensional counterpart of the triangle, because all polyhedra can be built from tetrahedra. The Greeks were unable to cut and paste the tetrahedron into a cuboid, but they found its volume by various ingenious infinite constructions. Perhaps the most elegant is

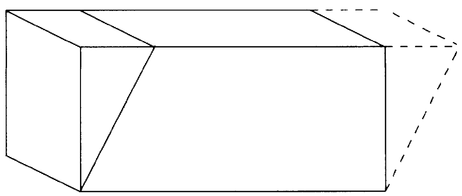

FIGURE 2.22 Volume of a parallelepiped.

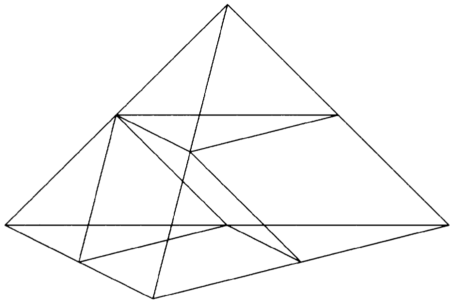

FIGURE 2.23 Pieces of a tetrahedron.

the following, which comes from Euclid. He fills up the tetrahedron with infinitely many prisms.

Figure 2.23 shows a tetrahedron cut into two smaller tetrahedra, the same shape as the original but half its height, and two triangular prisms of equal volume. Each prism has volume $\frac{1}{8}$base $\times$ height of the tetrahedron, so their combined volume is $\frac{1}{4}$base$\times$height. Now if each half-size tetrahedron is cut in a similar way, we get half-size prisms; hence each of them is $\frac{1}{8}$ the volume of each original prism. Because there are four half-size prisms, their combined volume is $\frac{1}{4}$ the combined volume of the original two. Continuing this process inside the quarter-size tetrahedra, we get eight quarter-size prisms, with combined volume $\frac{1}{16}$ the combined volume of the original two, and so on.

The total volume of the prisms is therefore

$$\left(\frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+\cdots\right)\text{base}\times\text{height}=\frac{1}{3}\text{base}\times\text{height},$$

by the formula for the sum of the geometric series. But the prisms exhaust the volume of the tetrahedron – they include all points inside the faces of the tetrahedron because the size of the little tetrahedra shrinks toward zero, and hence the volume of the tetrahedron itself is $\frac{1}{3}$base $\times$ height.

With such a simple result, it is all the more mysterious that we cannot derive it by cutting and pasting finitely often, as with the area of a triangle. But this is really the case; it can be proved that it is impossible to convert a regular tetrahedron into a cube by cutting it into a finite number of polyhedral pieces. This remarkable result, which was not discovered until 1900, will be proved in Chapter 5.

Exercises

Some of the claims about volumes in the dissection of the tetrahedron should perhaps be checked more carefully.

2.7.1. Explain why the two prisms in Figure 2.23 have equal volume, and why the volume of each is $\frac{1}{8}$base $\times$ height of the tetrahedron.

2.7.2. Show that the half-size, quarter-size tetrahedra, ... all lie against the leftmost edge of the tetrahedron, and hence justify the claim that the prisms fill the inside of the tetrahedron.

After the cube and the regular tetrahedron, the next simplest polyhedron is the regular *octahedron*, which is bounded by eight equilateral triangles.

2.7.3. Sketch a regular octahedron, and show that pasting regular tetrahedra on two of its opposite faces gives a parallelepiped.

2.7.4. If one of the triangular faces of the octahedron is taken as the "base" and the distance to the opposite face as the "height," deduce from Exercise 2.7.3 that

$$\text{volume of octahedron} = \frac{4}{3}\,\text{base} \times \text{height}.$$

2.7.5. Also deduce from Exercise 2.7.3 that space may be filled with a mixture of regular tetrahedra and octahedra.

Another way to see the space-filling property, though probably harder, is to prove the following result.

2.7.6.* Show that both the regular tetrahedron and the regular octahedron may be cut into half-sized regular tetrahedra and octahedra.

2.8* The Whole and the Part

There is an unconscious assumption in cutting and pasting, which we made by speaking of "the" area of a polygon. We are assuming that area is *conserved* in some sense; that if we repeatedly cut and paste, we never get a polygon larger or smaller than the one we started with. This is a blatantly physical assumption, like conservation of mass, and a conscientious geometer would avoid it if possible. To help decide whether we can, let us analyze the process of cutting and pasting more closely.

It is easiest to see the difficulty if we continue to assume that lengths and areas are numbers. There is more than one way to cut a polygon into triangles; what if different ways lead to different numbers? In fact, there is an even more alarming possibility: what if the area $\frac{1}{2}$base $\times$ height of a triangle depends on which side we choose to be the base?

The latter possibility can be ruled out, with some difficulty, by Euclid's theory of similar triangles. Triangles are called *similar* if they have the same angles, and Euclid proved that the corresponding sides of similar triangles are proportional. He might also have taken this as an axiom, because it is similar to his axioms about congruence for triangles.

Anyway, assuming proportionality of similar triangles, we can prove the following.

Constancy of base $\times$ height *In any triangle, the product of any side by the corresponding height is constant.*

Proof Take a triangle ABC and the perpendiculars AD and BE shown in Figure 2.24. Thus if we take BC as the base, AD is the height, and if we take AC as the base, BE is the height. We wish to prove

$$BC \times AD = AC \times BE.$$

To do this we show that triangles ADC and BEC have equal angles. They have angle C in common, and they have right angles at D and

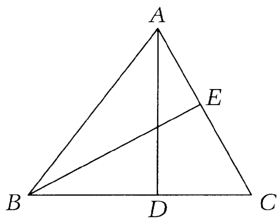

FIGURE 2.24 The constancy of base $\times$ height.

E, respectively, hence their remaining angles are also equal because any triangle has angle sum equal to two right angles (Section 2.3).

It follows, by the proportionality of corresponding sides, that

$$\frac{BC}{BE} = \frac{AC}{AD},$$

and therefore

$$BC \times AD = AC \times BE.$$

Thus any two values of base $\times$ height for the same triangle are the same. □

The possibility of different areas arising from different subdivisions of the same polygon is more difficult to rule out, and as far as I know this was not done until modern times. For the Greeks, of course, polygons Π and Π' that could be cut and pasted onto each other had equal area *by definition*. Their problem was to show that a polygon could not be cut and pasted onto one that was intuitively "smaller," namely, a *part of itself*. The Greeks could not prove this, and as a last resort Euclid made it one of his assumptions: "the whole is greater than the part."

When areas are numbers, or *numerical areas* as we shall call them, the problem becomes solvable. Hilbert (1899) proved that different subdivisions of the same polygon give the same numerical area. In my opinion, this clinches the case for using numbers in geometry. The main steps in Hilbert's proof are covered in Exercises 2.8.1 to 2.8.3.

Exercises

There are a few preliminaries to Hilbert's proof that can be skipped, because they involve results in the exercises to Section 2.5, and the following result in the same vein: any two subdivisions Σ and Σ' of the same polygon have a *common refinement*, a subdivision Σ'' such that each piece in Σ or Σ' is a union of pieces in Σ''.

The crux of the problem is to prove that any subdivision of triangle Δ into triangles $\Delta_1, \Delta_2, \ldots, \Delta_n$ gives the same numerical area. This comes

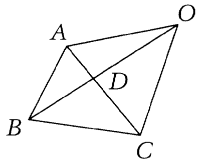

FIGURE 2.25 Areas that the edges span with O.

from proving that *the sum of the* $\frac{1}{2}$*base* $\times$ *height values for the* Δ_k *equals the* $\frac{1}{2}$*base* $\times$ *height value for* Δ.

Hilbert proved this very elegantly using a concept of *signed area* $[ABC]$. $[ABC] = \frac{1}{2}$base $\times$ height of triangle ABC when the vertices A, B, and C occur in clockwise order. $[ABC] = -\frac{1}{2}$base $\times$ height when A, B, and C occur in counterclockwise order.

2.8.1. Show that $[ABC] = [BCA] = [CAB] = -[ACB] = -[CBA] = -[BAC]$.

The advantage of $[ABC]$ over unsigned area is that it allows the area of any triangle to be expressed as the sum of areas that its edges "span" with a common origin O, as shown in Figure 2.25.

2.8.2. Show that $[OAB] = [OAD] + [ABD]$, $[OBC] = [ODC] + [DBC]$, and hence

$$[ABC] = [OAB] + [OBC] + [OCA].$$

Check that the same result holds for other positions of O outside triangle ABC.

If we have a triangle $\Delta = ABC$ cut into triangles $\Delta_k = A_kB_kC_k$ we use the previous exercise to write $[A_kB_kC_k] = [OA_kB_k] + [OB_kC_k] + [OC_kA_k]$ for each k, expressing the area of Δ_k as the sum of the areas its edges span with O.

2.8.3. Show that if the equations $[A_kB_kC_k] = [OA_kB_k] + [OB_kC_k] + [OC_kA_k]$ are added, the areas spanned by edges inside Δ cancel out. Thus the only terms remaining on the right-hand side are of the form $[OEF]$, where EF is a segment of one of the sides of Δ. Deduce that

$$[A_1B_1C_1] + [A_2B_2C_2] + \cdots + [A_kB_kC_k] = [ABC],$$

and conclude that any subdivision of Δ gives the same numerical area.

We say that polygons Π and Π' are *equidecomposable* if Π may be cut into polygonal pieces that can be pasted together to form Π'. Two equidecomposable polygons have the same numerical area, because their areas are the sums of the numerical areas of the same pieces.

More surprisingly, any two polygons of equal numerical area are equidecomposable. The result is not hard, but it was not noticed until the 19th century, possibly because the Greek idea that equal area meant equidecomposability *by definition* lingered until then. The proof can be broken down into the following steps:

1. Show that each polygon can be cut into a finite number of triangles.
2. Show that any triangle can be cut and pasted into a rectangle with given base, say 1.
3. Given any polygon Π, cut it into triangles, and cut and paste each triangle into a rectangle of base 1.
4. Stack up the rectangles, obtaining a single rectangle R of base 1, equidecomposable with Π.
5. Do the same with the other polygon Π', obtaining the same rectangle R (because Π' has the same area as Π).
6. Conclude, by running the construction from Π to R, and then the construction from Π' to R in reverse, that Π is equidecomposable with Π'.

After making this breakdown, the only steps that need any work are the first two.

2.8.4. Explain how to do Step 1.

Step 2 is the hardest, and is best broken down into two substeps.

2.8.5. Show that any triangle may be cut and pasted into a rectangle.

2.8.6. Use Figure 2.26 to explain how any rectangle may be cut and pasted into a rectangle of width 1.

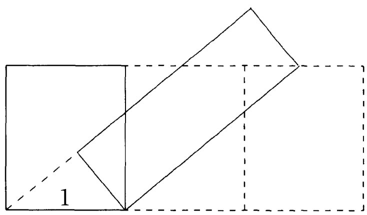

FIGURE 2.26 Equidecomposable rectangles.

2.9 Discussion

The Pythagorean Influence

The Pythagorean philosophy that "all is number" has come down to us through legends rather than hard evidence from Pythagoras' time. Nevertheless, these legends were persistent enough to influence the development of mathematics and physics until the present day. The story goes that the Pythagoreans came to believe in the power of numbers through discovering their role in music. They found, by studying the sounds of plucked strings, that the most harmonious notes were produced by strings whose lengths were in simple integer ratios. Given that the strings are of the same material, thickness, and tension, the most harmonious pairs of notes occur when the ratio of lengths is 2:1 (the octave), 3:2 (the "perfect fifth"), and 4:3 (the "perfect fourth").

Even today one must admit that this is a discovery good enough to build a dream on—if the subjective experience of harmony can be explained by numbers, perhaps *anything* can. Whether or not the Pythagoreans actually thought this, the idea sooner or later caught on and inspired other persistent dreams. The best known was the "harmony of the spheres," which tried to explain the position of the planets by numbers. It haunted astronomy from Aristotle (around 350 B.C.) to Johannes Kepler (1571–1630), until Isaac Newton came up with a better idea, the theory of gravitation.

If the Pythagoreans really believed that "all is number" (meaning natural numbers and their ratios), then it is, of course, ironic that their own philosophy should be brought down by the Pythagorean theorem and the irrationality of $\sqrt{2}$. An even greater irony, however, is that *Pythagorean music theory itself is fundamentally irrational*. This fact comes to light as soon as one tries to compare the "size" of the octave and the perfect fifth. According to another legend, Pythagoras himself tried to do this, finding that the interval of 7 octaves is very close, but not equal, to the interval of 12 perfect fifths. The pitch of a string is lowered by 7 octaves when its length is multiplied by $2^7 = 128$, and by 12 perfect fifths when its length is multiplied by $(3/2)^{12} = 129.746\ldots$.