We shall not give a proof here, but we remark that the proof of this theorem depends on the axiom of choice and is thus unacceptable to intuitionists. The converse of this theorem is called the soundness theorem; its proof is straightforward and is acceptable to intuitionists. As a corollary of the completeness-soundness result, L is consistent if and only if it has a model.

If n is a natural number, let S^n0 be the expression in L formed by placing the letter S n times before the symbol 0. This expression will normally be interpreted as the natural number n. We call a model *ω-complete* if, for any formula $A(x)$ of L, x being of type N, whenever $A(S^n0)$ is true in that model for each natural number n, then $\forall_{x\in N}A(x)$ is also true in that model.

In 1931, Gödel proved his *incompleteness theorem* for arithmetic, which may be expressed in our terminology as follows:

Theorem 28.2. (Incompleteness)
There is a formula in L which is true in any ω-complete model, but not provable in L, assuming L to be consistent.

Combining this with the completeness theorem, we may conclude that some models are not ω-complete.

To an intuitionist, the notion of truth is equivalent to that of knowability, which we shall here interpret as provability. Thus we may conclude that the world in which we live is ω-complete provided, whenever $A(S^n0)$ has a proof for each n, then $\forall_{x\in N}A(x)$ does also. However, there is no particular reason to believe that this is the case. Even if, for each n, the formula $A(S^n0)$ showed up as the last line of a proof, it would not guarantee that $\forall_{x\in N}A(x)$ showed up as the last line of a proof. Hence the intuitionist has no particular reason to think of the world we live in as ω-complete.

Platonists, of whom Gödel was one, see truth as the property of an eternal and immutable reality which is independent of finite human minds. A classical Platonist believes that the real world contains an infinite collection of natural numbers. Now if it is true of each of these numbers that it has a property A, then it is true that they all have property A, whence the real world is an ω-complete model of L. Since Gödel was a classical Platonist, he concluded from his incompleteness theorem that there is a formula of arithmetic which is eternally true, but which is not the last line of a proof in L. (Note that for Platonists, the language of arithmetic is about the real world, hence it is consistent.)

29

Proof of Gödel's Incompleteness Theorem

We shall now sketch a proof of the incompleteness result in a manner acceptable to intuitionists. We begin with a lemma.

Lemma 29.1. (Gödel's Lemma)
Suppose $R(m, n)$ is a recursive relation between the natural numbers m and n. That is, assume that, for any two numbers m and n, there is a finite effective procedure for deciding whether they are in the relation R. Then there is a formula $F(x, y)$ in L, with x and y of type N, such that

- *if $R(m, n)$ then $\vdash F(S^m 0, S^n 0)$,*
- *if not $R(m, n)$ then $\vdash \neg F(S^m 0, S^n 0)$.*

(Here $\vdash$ means 'there is a proof in L that'.)

As an example of this lemma, let R be the relation 'is 1 greater than'. Let $F(x, y)$ be the formula $x = Sy$. Then, if m is 1 greater than n, it is provable in L that $S^m 0 = SS^n 0$, and if m is not greater than n, it is provable in L that $S^m 0 \neq SS^n 0$.

We shall not prove this lemma here, but it should seem reasonable, inasmuch as L is meant to capture ordinary number theory.

Theorem 29.2.
(Gödel's Incompleteness Theorem (Semantic Version))
If L is consistent, there is a formula in L which is true in any ω-complete model but not provable in L.

Proof: We begin by enumerating all expressions in L of type PN, that is, of the form $\{x \in N | A(x)\}$, where $A(x)$ is a formula in L, with x of type