10. Prove that for k a finite field the Zariski topology is the same as the discrete topology: every subset is closed (and open).

11. Let V be a variety in $\mathbb{A}^n$ and let U_1 and U_2 be two subsets of $\mathbb{A}^n$ that are open in the Zariski topology. Prove that if $V \cap U_1 \neq \emptyset$ and $V \cap U_2 \neq \emptyset$ then $V \cap U_1 \cap U_2 \neq \emptyset$. Conclude that *any* nonempty open subset of a variety is *everywhere dense* in the Zariski topology (i.e., its closure is all of V).

12. Use the fact that nonempty open sets of an affine variety are everywhere dense to prove that an affine variety is connected in the Zariski topology. (A topological space is *connected* if it is not the union of two disjoint, proper, open subsets.)

13. Prove that the affine algebraic set V is connected in the Zariski topology if and only if $k[V]$ is not a direct sum of two nonzero ideals. Deduce from this that a variety is connected in the Zariski topology.

14. Prove that if k is an infinite field, then the varieties in $\mathbb{A}^1$ are the empty set, the whole space, and the one point subsets. What are the varieties in $\mathbb{A}^1$ in the case of a finite field k?

15. Suppose V is a hypersurface in $\mathbb{A}^n$ and $\mathcal{I}(V) = (f)$ for some nonconstant polynomial $f \in k[x_1, x_2, \ldots, x_n]$. Prove that V is a variety if and only if f is irreducible.

16. Suppose $V \subseteq \mathbb{A}^n$ is an affine variety and $f \in k[V]$. Prove that the *graph* of f (cf. Exercise 25 in Section 1) is an affine variety.

17. Prove that any permutation of the elements of a field k is a continuous map from $\mathbb{A}^1$ to itself in the Zariski topology on $\mathbb{A}^1$. Deduce that if k is an infinite field, there are Zariski continuous maps from $\mathbb{A}^1$ to itself that are not polynomials.

18. Let V be an affine algebraic set in $\mathbb{A}^n$ over $k = \mathbb{C}$.
    (a) Prove that morphisms of algebraic sets over $\mathbb{C}$ are continuous in the Euclidean topology (the topology on $\mathbb{C}^n$ obtained by identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ with its usual Euclidean topology).
    (b) Prove that V is a closed set in the Euclidean topology on $\mathbb{C}^n$ (so the Zariski closed sets of $\mathbb{A}^n$ over $\mathbb{C}$ are also Euclidean closed).
    (c) Give an example of a set that is closed in the Euclidean topology but is not closed in the Zariski topology, i.e., is not an affine algebraic set (so the Euclidean topology is "finer" than the Zariski topology).

19. Give an example of an injective k-algebra homomorphism $\widetilde{\varphi} : k[W] \to k[V]$ whose associated morphism $\varphi : V \to W$ is not surjective.

20. Suppose $\varphi : V \to W$ is a surjective morphism of affine algebraic sets. Prove that if V is a variety then W is a variety.

21. Let V be an algebraic set in $\mathbb{A}^n$ and let $f \in k[V]$. Define $V_f = \{v \in V \mid f(v) \neq 0\}$.
    (a) Show that V_f is a Zariski open set in V (called a *principal open set* in V).
    (b) Let J be the ideal in $k[x_1, \ldots, x_n, x_{n+1}]$ generated by $\mathcal{I}(V)$ and $x_{n+1}f - 1$, and let $W = \mathcal{Z}(J) \subseteq \mathbb{A}^{n+1}$. Show that $J = \mathcal{I}(W)$ and that the map $\pi : \mathbb{A}^{n+1} \to \mathbb{A}^n$ by projection onto the first n coordinates is a Zariski continuous bijection from W onto V_f (so the principal *open* set V_f in V may be embedded as a *closed* set in some (larger) affine space).
    (c) If U is any open set in V show that $U = V_{f_1} \cup \cdots \cup V_{f_m}$ for some $f_1, \ldots, f_m \in k[V]$. (This shows that the principal open sets form a *base* for the Zariski topology.)

22. Prove that $GL_n(k)$ is an open affine algebraic set in $\mathbb{A}^{n^2}$ and can be embedded as a closed affine algebraic set in $\mathbb{A}^{n^2+1}$. In particular, deduce that the set $k^\times$ of nonzero elements in

$\mathbb{A}^1$ embeds into $\mathbb{A}^2$ as the hyperbola $xy = 1$. [Use the preceding exercise.]

23. Show that if k is infinite then $\{(a, a^2, a^3) \mid a \in k\} \subset \mathbb{A}^3$ is an affine algebraic variety. If k is finite show that this set is always reducible.

24. Let $V = \mathcal{Z}(xz - y^2, yz - x^3, z^2 - x^2y) \subset \mathbb{A}^3$. Show that if k is infinite then V is an affine variety. [Use Exercise 26 of Section 1 and Exercise 20.]

25. Suppose $f(x) = x^3 + ax^2 + bx + c$ is an irreducible cubic in $\mathbb{Q}[x]$ of discriminant D. Let $I = (x + y + z + a, xy + xz + yz - b, xyz + c)$ in $\mathbb{Q}[x, y, z]$.

(a) Prove that I is a prime ideal if and only if D is not a square in $\mathbb{Q}$, in which case I is a maximal ideal and $\mathbb{Q}[x, y, z]/I$ is a splitting field for $f(x)$ over $\mathbb{Q}$.

(b) If $D = r^2$, prove that the primary decomposition of I is $I = Q_+ \cap Q_-$ where $Q_\pm = (I, (x - y)(x - z)(y - z) \pm r)$. Prove Q_+ and Q_- are maximal ideals, and $\mathbb{Q}[x, y, z]$ modulo Q_+ or Q_- is a splitting field for $f(x)$ over $\mathbb{Q}$.

26. A topological space X is called *quasicompact* if whenever any collection of closed subsets V_i of X has empty intersection, then some finite number of these also has empty intersection, i.e.,

$$\text{whenever } \bigcap_i V_i = \emptyset \text{ there exists } V_{i_1}, V_{i_2}, \ldots, V_{i_N} \text{ such that } \bigcap_{t=1}^{N} V_{i_t} = \emptyset.$$

Prove that every affine algebraic set is quasicompact. [Translate the definition into a property of ideals in $k[x_1, \ldots, x_n]$.] (A quasicompact and Hausdorff space is called *compact*.)

27. When k is an infinite field prove that the Zariski topology on k^2 is not the same as taking the Zariski topology on k and then forming the product topology on $k \times k$. [By Exercise 14 of Section 1, in the product topology on $k \times k$ the Zariski closed sets in $k \times k$ are finite unions of sets of the form $\{a\} \times \{b\}$, $\{a\} \times k$ and $k \times \{b\}$, for any $a, b \in k$.]

28. Prove that each of the following rings have infinitely many minimal prime ideals, and that (0) is not the intersection of any finite number of these (so (0) does not have a primary decomposition in these rings):

(a) the infinite direct product ring $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \cdots$ (which is a Boolean ring, cf. Exercise 23 in Section 7.4).

(b) $k[x_1, x_2, \ldots]/(x_1x_2, x_3x_4, \ldots, x_{2i-1}x_{2i}, \ldots)$, where $x_1, x_2, \ldots$ are independent variables over the field k.

29. Suppose that A and B are ideals with $AB \subseteq Q$ for a primary ideal Q. Prove that if $A \nsubseteq Q$ then $B \subset \operatorname{rad} Q$.

30. Let Q be a P-primary ideal and suppose A is an ideal not contained in Q. Define $A' = \{r \in R \mid rA \subseteq Q\}$ to be the elements of R that when multiplied by elements of A give elements of Q. Prove that A' is a P-primary ideal.

31. Prove that if Q_1 and Q_2 are primary ideals belonging to the same prime ideal P, then $Q_1 \cap Q_2$ is a primary ideal belonging to P. Conclude that a finite intersection of P-primary ideals is again P-primary.

32. Prove that if Q_1 and Q_2 are primary ideals belonging to the same *maximal* ideal M, then $Q_1 + Q_2$ and Q_1Q_2 are primary ideals belonging to M. Conclude that finite sums and finite products of M-primary ideals are again M-primary.

33. Let $I = (x^2, xy, xz, yz)$ in $k[x, y, z]$. Prove that a primary decomposition of I is $I = (x, y) \cap (x, z) \cap (x, y, z)^2$, determine the isolated and embedded primes of I, and find rad I.

34. Suppose $\varphi : R \to S$ is a surjective ring homomorphism. Prove that an ideal Q in R containing the kernel of φ is primary if and only if $\varphi(Q)$ is primary in S, and when this is the case the prime associated to $\varphi(Q)$ is the image $\varphi(P)$ of the prime P associated to Q.

35. Suppose $\varphi : R \to S$ is a ring homomorphism.

(a) Suppose I is an ideal of R containing $\ker\varphi$ with minimal primary decomposition $I = Q_1 \cap \cdots \cap Q_m$ with rad $Q_i = P_i$. If φ is a surjective homomorphism prove that $\varphi(I) = \varphi(Q_1) \cap \cdots \cap \varphi(Q_m)$, where rad $\varphi(Q_i)$ is given by $\varphi(P_i)$, is a minimal primary decomposition of $\varphi(I)$. [Use the previous exercise.]

(b) Suppose I is an ideal of S with minimal primary decomposition $I = Q_1 \cap \cdots \cap Q_m$ with rad $Q_i = P_i$. Prove that $\varphi^{-1}(I) = \varphi^{-1}(Q_1) \cap \cdots \cap \varphi^{-1}(Q_m)$, where rad $\varphi^{-1}(Q_i)$ is given by $\varphi^{-1}(P_i)$, is a primary decomposition of $\varphi^{-1}(I)$, and is minimal if φ is surjective.

36. Let $I = (xy, x - yz)$ in $k[x, y, z]$. Prove that $(x, z) \cap (y^2, x - yz)$ is a minimal primary decomposition of I. [Consider the ring homomorphism $\varphi : k[x, y, z] \to k[y, z]$ given by mapping x to yz, y to y, and z to z and use the previous exercise.]

37. Prove that a prime ideal P contains the ideal I if and only if P contains one of the associated primes of a minimal primary decomposition of I. [Use Exercise 3 and Exercise 11 in Section 7.4.]

38. Show that every associated prime ideal for a radical ideal is isolated. [Suppose that $P_2 = \text{rad } Q_2 \subseteq P_1 = \text{rad } Q_1$ in the decomposition of Theorem 21 for the radical ideal I. Show that if $a \in Q_2 \cap \cdots \cap Q_m \subseteq P_2$ then $a^n \in I$ for some $n \geq 1$, conclude that $a \in Q_1$ and derive a contradiction to the minimality of the primary decomposition.]

39. Fix an element a in the ring R. For any ideal I in the ring R let $I_a = \{r \in R \mid ar \in I\}$.

(a) Prove that I_a is an ideal and $I_a = R$ if and only if $a \in I$.

(b) Prove that $(I \cap J)_a = I_a \cap J_a$ for ideals I and J.

(c) Suppose that Q is a P-primary ideal and that $a \notin Q$. Prove that Q_a is a P-primary ideal and that $Q_a = Q$ if $a \notin P$.

40. With notation as in the previous exercise, suppose $I = Q_1 \cap \cdots \cap Q_m$ is a minimal primary decomposition of the ideal I and let P_i be the prime ideal associated to Q_i.

(a) Prove that $I_a = (Q_1)_a \cap \cdots \cap (Q_m)_a$ and that $\text{rad}(I_a) = \text{rad}((Q_1)_a) \cap \cdots \cap \text{rad}((Q_m)_a)$.

(b) Prove that $\text{rad}(I_a)$ is the intersection of the prime ideals P_i for which $a \notin Q_i$. [Use the previous exercise.]

(c) Prove that if $\text{rad}(I_a)$ is a prime ideal then $\text{rad}(I_a) = P_j$ for some j. [Use the fact that prime ideals are irreducible.]

(d) For each $i = 1, \ldots, m$, prove that $\text{rad}(I_a) = P_i$ for some $a \in R$. [Show there exists an $a \in R$ with $a \notin Q_i$ but $a \in Q_j$ for all $j \neq i$.]

(e) Show from (c) and (d) that the associated primes for a minimal primary decomposition are precisely the collection of prime ideals among the ideals $\text{rad}(I_a)$ for $a \in R$, and conclude that they are uniquely determined by I independent of the minimal primary decomposition.

41. Let $P_1, \ldots, P_m$ be the associated prime ideals of the ideal (0) in the Noetherian ring R.

(a) Show that $P_1 \cap \cdots \cap P_m$ is the collection of nilpotent elements in R. [Apply Corollary 22 to $I = (0)$.]

(b) Show that $P_1 \cup \cdots \cup P_m$ is the collection of zero divisors in R. [Let $I = (0)$ in the previous exercise and show that the set of zero divisors is given by the set $\cup_{a \in R - \{0\}} (0)_a = \cup_{a \in R - \{0\}} \text{rad}((0)_a)$.]

42. Suppose R is a Noetherian ring. Prove that R is either an integral domain, has nonzero nilpotent elements, or has at least two minimal prime ideals. [Use the previous exercise.]

43. Prove that the ideal I in the Noetherian ring R is radical if and only if the primary compo-