Definition. The ring $\mathcal{T}(M)$ is called the *tensor algebra* of M.

Proposition 32. Let V be a finite dimensional vector space over the field F with basis $\mathcal{B} = \{v_1, \ldots, v_n\}$. Then the k-tensors

$$v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k} \qquad \text{with } v_{i_j} \in \mathcal{B}$$

are a vector space basis of $\mathcal{T}^k(V)$ over F (with the understanding that the basis vector is the element $1 \in F$ when $k = 0$). In particular, $\dim_F(\mathcal{T}^k(V)) = n^k$.

Proof: This follows immediately from Proposition 16 of Section 2.

Theorem 31 and Proposition 32 show that the space $\mathcal{T}(V)$ may be regarded as the *noncommutative polynomial algebra* over F in the (noncommuting) variables $v_1, \ldots, v_n$. The analogous result also holds for finitely generated free modules over any commutative ring (using Corollary 19 in Section 10.4).

Examples

(1) Let $R = \mathbb{Z}$ and let $M = \mathbb{Q}/\mathbb{Z}$. Then $(\mathbb{Q}/\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z}) = 0$ (Example 4 following Corollary 12 in Section 10.4). Thus $\mathcal{T}(\mathbb{Q}/\mathbb{Z}) = \mathbb{Z} \oplus (\mathbb{Q}/\mathbb{Z})$, where addition is componentwise and the multiplication is given by $(r, \overline{p})(s, \overline{q}) = (rs, \overline{rq + sp})$. The ring $R/(x)$ of Exercise 4(d) in Section 9.3 is isomorphic to $\mathcal{T}(\mathbb{Q}/\mathbb{Z})$.

(2) Let $R = \mathbb{Z}$ and let $M = \mathbb{Z}/n\mathbb{Z}$. Then $(\mathbb{Z}/n\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$ (Example 3 following Corollary 12 in Section 10.4). Thus $\mathcal{T}^i(M) \cong M$ for all $i > 0$ and so $\mathcal{T}(\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z} \oplus (\mathbb{Z}/n\mathbb{Z}) \oplus (\mathbb{Z}/n\mathbb{Z}) \cdots$. It follows easily that $\mathcal{T}(\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}[x]/(nx)$.

Since $\mathcal{T}^i(M)\mathcal{T}^j(M) \subseteq \mathcal{T}^{i+j}(M)$, the tensor algebra $\mathcal{T}(M)$ has a natural "grading" or "degree" structure reminiscent of a polynomial ring.

Definition.

(1) A ring S is called a *graded ring* if it is the direct sum of additive subgroups: $S = S_0 \oplus S_1 \oplus S_2 \oplus \cdots$ such that $S_iS_j \subseteq S_{i+j}$ for all $i, j \geq 0$. The elements of S_k are said to be *homogeneous of degree k*, and S_k is called the *homogeneous component of S of degree k*.

(2) An ideal I of the graded ring S is called a *graded ideal* if $I = \oplus_{k=0}^{\infty}(I \cap S_k)$.

(3) A ring homomorphism $\varphi : S \to T$ between two graded rings is called a *homomorphism of graded rings* if it respects the grading structures on S and T, i.e., if $\varphi(S_k) \subseteq T_k$ for $k = 0, 1, 2, \ldots$.

Note that $S_0S_0 \subseteq S_0$, which implies that S_0 is a subring of the graded ring S and then S is an S_0-module. If S_0 is in the center of S and it contains an identity of S, then S is an S_0-algebra. Note also that the ideal I is graded if whenever a sum $i_{k_1} + \cdots + i_{k_n}$ of homogeneous elements with distinct degrees $k_1, \ldots, k_n$ is in I then each of the individual summands $i_{k_1}, \ldots, i_{k_n}$ is itself in I.

Example

The polynomial ring $S = R[x_1, x_2, \dots, x_n]$ in n variables over the commutative ring R is an example of a graded ring. Here $S_0 = R$ and the homogeneous component of degree k is the subgroup of all R-linear combinations of monomials of degree k.

The ideal I generated by $x_1, \dots, x_n$ is a graded ideal: every polynomial with zero constant term may be written uniquely as a sum of homogeneous polynomials of degree $k > 1$, and each of these has zero constant term hence lies in I. More generally, an ideal is a graded ideal if and only if it can be generated by homogeneous polynomials (cf. Exercise 17 in Section 9.1).

Not every ideal of a graded ring need be a graded ideal. For example in the graded ring $\mathbb{Z}[x]$ the principal ideal J generated by $1 + x$ is not graded: $1 + x \in J$ and $1 \notin J$ so $1 + x$ cannot be written as a sum of homogeneous polynomials each of which belongs to J.

The next result shows that quotients of graded rings by graded ideals are again graded rings.

Proposition 33. Let S be a graded ring, let I be a graded ideal in S and let $I_k = I \cap S_k$ for all $k \geq 0$. Then S/I is naturally a graded ring whose homogeneous component of degree k is isomorphic to S_k/I_k.

Proof: The map

$$\begin{aligned} S = \oplus_{k=0}^{\infty} S_k &\longrightarrow \oplus_{k=0}^{\infty}(S_k/I_k) \\ (\dots, s_k, \dots) &\longmapsto (\dots, s_k \bmod I_k, \dots) \end{aligned}$$

is surjective with kernel $I = \oplus_{k=0}^{\infty} I_k$ and defines an isomorphism of graded rings. The details are left for the exercises.

Symmetric Algebras

The first application of Proposition 33 is in the construction of a commutative quotient ring of $\mathcal{T}(M)$ through which R-module homomorphisms from M to any *commutative* R-algebra must factor. This gives an "abelianized" version of Theorem 31. The construction is analogous to forming the commutator quotient G/G' of a group (cf. Section 5.4).

Definition. The *symmetric algebra* of an R-module M is the R-algebra obtained by taking the quotient of the tensor algebra $\mathcal{T}(M)$ by the ideal $\mathcal{C}(M)$ generated by all elements of the form $m_1 \otimes m_2 - m_2 \otimes m_1$, for all $m_1, m_2 \in M$. The symmetric algebra $\mathcal{T}(M)/\mathcal{C}(M)$ is denoted by $\mathcal{S}(M)$.

The tensor algebra $\mathcal{T}(M)$ is generated as a ring by $R = \mathcal{T}^0(M)$ and $M = \mathcal{T}^1(M)$, and these elements commute in the quotient ring $\mathcal{S}(M)$ by definition. It follows that the symmetric algebra $\mathcal{S}(M)$ is a commutative ring. The ideal $\mathcal{C}(M)$ is generated by homogeneous tensors of degree 2 and it follows easily that $\mathcal{C}(M)$ is a graded ideal. Then by Proposition 33 the symmetric algebra is a graded ring whose homogeneous component of degree k is $\mathcal{S}^k(M) = \mathcal{T}^k(M)/\mathcal{C}^k(M)$. Since $\mathcal{C}(M)$ consists of k-tensors with $k \geq 2$, we have $\mathcal{C}(M) \cap M = 0$ and so the image of $M = \mathcal{T}^1(M)$ in $\mathcal{S}(M)$ is isomorphic to M. Identifying M with its image we see that $\mathcal{S}^1(M) = M$ and the symmetric algebra contains M. In a similar way $\mathcal{S}^0(M) = R$, so the symmetric algebra is also an R-algebra. The R-module $\mathcal{S}^k(M)$ is called the k^{th} *symmetric power* of M.

The first part of the next theorem shows that the elements of the k^{th} symmetric power of M can be considered as finite sums of simple tensors $m_1 \otimes \cdots \otimes m_k$ where tensors with the order of the factors permuted are identified. Recall also from Section 4 that a k-multilinear map $\varphi : M \times \cdots \times M \to N$ is said to be *symmetric* if $\varphi(m_1, \ldots, m_k) = \varphi(m_{\sigma(1)}, \ldots, m_{\sigma(k)})$ for all permutations σ of $1, 2, \ldots, k$. (The definition is the same for modules over any commutative ring R as for vector spaces.)

Theorem 34. Let M be an R-module over the commutative ring R and let $\mathcal{S}(M)$ be its symmetric algebra.

(1) The k^{th} symmetric power, $\mathcal{S}^k(M)$, of M is equal to $M \otimes \cdots \otimes M$ (k factors) modulo the submodule generated by all elements of the form

$$(m_1 \otimes m_2 \otimes \cdots \otimes m_k) - (m_{\sigma(1)} \otimes m_{\sigma(2)} \otimes \cdots \otimes m_{\sigma(k)})$$

for all $m_i \in M$ and all permutations σ in the symmetric group S_k.

(2) *(Universal Property for Symmetric Multilinear Maps)* If $\varphi : M \times \cdots \times M \to N$ is a symmetric k-multilinear map over R then there is a unique R-module homomorphism $\Phi : \mathcal{S}^k(M) \to N$ such that $\varphi = \Phi \circ \iota$, where

$$\iota : M \times \cdots \times M \to \mathcal{S}^k(M)$$

is the map defined by

$$\iota(m_1, \ldots, m_k) = m_1 \otimes \cdots \otimes m_n \bmod \mathcal{C}(M).$$

(3) *(Universal Property for maps to commutative R-algebras)* If A is any commutative R-algebra and $\varphi : M \to A$ is an R-module homomorphism, then there is a unique R-algebra homomorphism $\Phi : \mathcal{S}(M) \to A$ such that $\Phi|_M = \varphi$.

Proof: The k-tensors $\mathcal{C}^k(M)$ in the ideal $\mathcal{C}(M)$ are finite sums of elements of the form

$$m_1 \otimes \ldots \otimes m_{i-1} \otimes (m_i \otimes m_{i+1} - m_{i+1} \otimes m_i) \otimes m_{i+2} \otimes \ldots \otimes m_k$$

with $m_1, \ldots, m_k \in M$ (where $k \geq 2$ and $1 \leq i < k$). This product gives a difference of two k-tensors which are equal except that two entries (in positions i and $i+1$) have been transposed, i.e., gives the element in (1) of the theorem corresponding to the transposition $(i\ i{+}1)$ in the symmetric group S_k. Conversely, since any permutation σ in S_k can be written as a product of such transpositions it is easy to see that every element in (1) can be written as a sum of elements of the form above. This gives (1).

The proofs of (2) and (3) are very similar to the proofs of the corresponding "asymmetric" results (Corollary 16 of Section 10.4 and Theorem 31) noting that $\mathcal{C}^k(M)$ is contained in the kernel of any symmetric map from $\mathcal{T}^k(M)$ to N by part (1).