use Exercise 37 in Section 9.6. Show (b) implies (d). For (d) implies (b) show the product $m_{a_1,k}(x_i)\dots m_{a_N,k}(x_i)$ of the minimal polynomials of the i^{th} coordinates $a_1,\dots,a_N$ of the points in $\mathcal{Z}_{\bar{k}}(I)$ is a nonzero polynomial in $\mathcal{I}(\mathcal{Z}_{\bar{k}}(I))$ and apply Corollary 33.]

27. Let I be a zero-dimensional ideal in $k[x_1,\dots,x_n]$ and let I' be the ideal generated by I in $\bar{k}[x_1,\dots,x_n]$ where $\bar{k}$ is the algebraic closure of k. Let $\mathcal{Z}(I)$ be the zero set of I in k^n and let $\mathcal{Z}_{\bar{k}}(I)$ be the zero set of I (equivalently, of I') in $\bar{k}^n$.

(a) Prove that $|\mathcal{Z}_{\bar{k}}(I)| = \dim_{\bar{k}}\bar{k}[x_1,\dots,x_n]/\operatorname{rad} I'$. [Show that $\operatorname{rad} I'$ is the product of the maximal ideals corresponding to the points in $V_{\bar{k}}$ and use the Chinese Remainder Theorem.]

(b) Show $|\mathcal{Z}(I)| \le \dim_k k[x_1,\dots,x_n]/I$. [One approach: use Exercise 43 in Section 1 and observe that $\dim_{\bar{k}}\bar{k}[x_1,\dots,x_n]/\operatorname{rad} I' \le \dim_{\bar{k}}\bar{k}[x_1,\dots,x_n]/I'$.]

28. Suppose I is a zero-dimensional ideal in $k[x_1,\dots,x_n]$, and suppose $I\cap k[x_i]$ is generated by the nonzero polynomial h_i (cf. Exercise 26). Let r_i be the product of the irreducible factors of h_i (the 'squarefree part' of h_i).

(a) Prove that $I + (r_1,\dots,r_n) \subseteq \operatorname{rad} I$.

(b) *(Radicals of zero-dimensional ideals for perfect fields)* If k is a perfect field, prove that $\operatorname{rad} I = I + (r_1,\dots,r_n)$. [Use induction on n. Write $r_1 = p_1\dots p_t$ with distinct irreducibles p_i in $k[x_1]$. If $J = I + (r_1,\dots,r_n)$ show that $J = J_1\cap\dots\cap J_t$ where $J_t = J + (p_t)$. Show for each i that reduction modulo p_i induces an isomorphism $k[x_1,\dots,x_n]/J_i \cong K[x_2,\dots,x_n]/J_i'$ where K is the extension field $k[x]/(p_i)$ and $J_i' \subseteq K[x_2,\dots,x_n]$ is the reduction of the ideal J_i modulo (p_i). Use Exercise 11 of Section 13.5 to show that the image of r_j in $J_i'\cap K[x_j]$ remains a nonzero squarefree polynomial for each $j = 2,\dots,n$ since k is perfect. Conclude by induction that J_i' is a radical ideal. Deduce that J_i is a radical ideal, and finally that J is a radical ideal.]

(c) Find the radicals of (x^7+x+y^3, x^4+y^3+y), (x^3-xy^2+x, x^2y+y^3), and (x^4+y^3, x^3-xy+y^2) in $\mathbb{Q}[x,y]$ and of $(x^2+y^2z, x^2y^2+z^3, y^2+z^2)$ in $\mathbb{Q}[x,y,z]$.

(d) Let $k = \mathbb{F}_p(t)$. Show that $I = (x^p+t, y^p-t)$ is a zero-dimensional ideal in $k[x,y]$ such that both $I\cap k[x]$ and $I\cap k[y]$ contain nonzero squarefree polynomials, but that I is not a radical ideal (so the result in (b) need not hold if k is not perfect). [Show that $x+y\in\operatorname{rad} I$ but $x+y\notin I$.]

15.4 LOCALIZATION

The idea of "localization at a prime" in a ring is an extremely powerful and pervasive tool in algebra for isolating the behavior of the ideals in a ring. It is an algebraic analogue of the familiar idea of localizing at a point when considering questions of, for example, the differentiability of a function $f(x)$ on the real line. In fact one of the important applications (and also one of the original motivations for the development) of this technique is to translate such "local" properties in the geometry of affine algebraic spaces to corresponding properties of their coordinate rings.

We first consider a very general construction of "rings of fractions." Let D be a multiplicatively closed subset of R containing 1 (i.e., $1\in D$ and $ab\in D$ if $a, b\in D$). The next result constructs a new ring $D^{-1}R$ which is the "smallest" ring in which the elements of D become units. This generalizes the construction of rings of fractions in Section 7.5 by allowing D to contain zero or zero divisors, and so in this case R need not embed as a subring of $D^{-1}R$.

Theorem 36. Let R be a commutative ring with 1 and let D be a multiplicatively closed subset of R containing 1. Then there is a commutative ring $D^{-1}R$ and a ring homomorphism $\pi : R \to D^{-1}R$ satisfying the following universal property: for any homomorphism $\psi : R \to S$ of commutative rings that sends 1 to 1 such that $\psi(d)$ is a unit in S for every $d \in D$, there is a unique homomorphism $\Psi : D^{-1}R \to S$ such that $\Psi \circ \pi = \psi$.

Proof: The proof is very similar to the proof of Theorem 15 in Section 7.5. In this case we define a relation on $R \times D$ by

$$(r, d) \sim (s, e) \quad \text{if and only if} \quad x(er - ds) = 0 \quad \text{for some } x \in D.$$

This relation is clearly reflexive and symmetric. If $(r, d) \sim (s, e)$ and $(s, e) \sim (t, f)$ then $x(er - ds) = 0$ and $y(fs - et) = 0$ for some $x, y \in D$. Multiplying the first equation by fy and the second by dx and adding gives $exy(fr - dt) = 0$. Since D is closed under multiplication, $(r, d) \sim (t, f)$ and so $\sim$ is transitive.

Let r/d denote the equivalence class of (r, d) under $\sim$ and let $D^{-1}R$ be the set of these equivalence classes. Define addition and multiplication in $D^{-1}R$ by

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \qquad \text{and} \qquad \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$$

It is an exercise to check that these operations are well defined and make $D^{-1}R$ into a commutative ring with $1 = 1/1$. For each $d \in D$, $d/1$ is a unit in $D^{-1}R$ (even in the degenerate case when $D^{-1}R$ is the zero ring).

Finally, define $\pi : R \to D^{-1}R$ by $\pi(r) = r/1$. It follows easily that π is a ring homomorphism. Suppose that $\psi : R \to S$ is a homomorphism of commutative rings that sends 1 to 1 such that $\psi(d)$ is a unit in S for every $d \in D$. Define

$$\Psi : D^{-1}R \to S \qquad \text{by} \qquad \Psi\left(\frac{r}{d}\right) = \psi(r)\psi(d)^{-1}.$$

This map is well defined because if $r/d = s/e$ then $x(er - ds) = 0$ for some $x \in D$. Then $\psi(x)(\psi(er) - \psi(ds)) = 0$ in S, so $\psi(er) - \psi(ds) = 0$ since $\psi(x)$ is a unit in S, and therefore $\psi(r)\psi(d)^{-1} = \psi(s)\psi(e)^{-1}$. It is immediate that Ψ is a ring homomorphism and $\Psi \circ \pi = \psi$.

Finally, Ψ is unique because every element of $D^{-1}R$ can be written as a product $(r/1)(d/1)^{-1}$. The value of Ψ on each element of the form $x/1$ is uniquely determined by ψ, namely $\Psi(x/1) = \Psi(\pi(x)) = \psi(x)$. Since Ψ is a ring homomorphism, its value on u^{-1} for any unit u is uniquely determined by $\Psi(u)$. Thus Ψ is uniquely determined on every element of $D^{-1}R$, completing the proof.

Corollary 37. In the notation of Theorem 36,

(1) $\ker \pi = \{r \in R \mid xr = 0 \text{ for some } x \in D\}$; in particular, $\pi : R \to D^{-1}R$ is an injection if and only if D contains no zero divisors of R, and

(2) $D^{-1}R = 0$ if and only if $0 \in D$, hence if and only if D contains nilpotent elements.

Proof: By definition, we have $\pi(r) = 0$ if and only if $(r, 1) \sim (0, 1)$, i.e., if and only if $xr = 0$ for some $x \in D$, which is (1). For (2), note that $D^{-1}R = 0$ if and only if the 1 of this ring is zero, i.e., $(1, 1) \sim (0, 1)$. This occurs if and only if $x1 = 0$ for some $x \in D$, i.e., if and only if $0 \in D$.

Definition. The ring $D^{-1}R$ is called the *ring of fractions of R with respect to D* or the *localization of R at D.*

Examples

(1) Let R be an integral domain and let $D = R - \{0\}$. Then $D^{-1}R$ is the field of fractions, Q, of R described in Section 7.5. More generally, if D is any multiplicatively closed subset of $R - \{0\}$, then $D^{-1}R$ is the subring of Q consisting of elements r/d with $r \in R$ and $d \in D$.

(2) Let R be any commutative ring with 1 and let f be any element of R. Let D be the multiplicative set $\{f^n \mid n \geq 0\}$ of nonnegative powers of f in R. Define $R_f = D^{-1}R$. Note that $R_f = 0$ if and only if f is nilpotent. If f is not nilpotent, then f becomes a unit in R_f. It is not difficult to see that

$$R_f \cong R[x]/(xf - 1),$$

where $R[x]$ is the polynomial ring in the variable x (cf. the exercises). Note also that R_f and R_{f^n} are naturally isomorphic for any $n \geq 1$ since both f and f^n are units in both rings. If f is a zero divisor then $\pi : R \to R_f$ does not embed R into R_f. For example, let $R = k[x, y]/(xy)$, and take $f = x$. Then x is a unit in R_x and y is mapped to 0 by the first part of the corollary (explicitly: $y = xy/x = 0$ in R_x). In this case $\pi(R) = k[x] \subset R_f = k[x, x^{-1}]$.

(3) (*Localizing at a Prime*) Let P be a prime ideal in any ring R and let $D = R - P$. By definition of a prime ideal D is multiplicatively closed. Passing to the ring $D^{-1}R$ in this case is called *localizing R at P* and the ring $D^{-1}R$ is denoted by R_P. Every element of R not in P becomes a unit in R_P. For example, if $R = \mathbb{Z}$ and $P = (p)$ is a prime ideal, then

$$\mathbb{Z}_{(p)} = \{\frac{a}{b} \in \mathbb{Q} \mid p \nmid b\} \subseteq \mathbb{Q}$$

and every integer b not divisible by p is a unit.

(4) If V is any nonempty set and k is a field, let R be any ring of k-valued functions on V containing the constant functions (for instance, the ring of all continuous real valued functions on the closed interval $[0, 1]$). For any $a \in V$ let M_a be the ideal of functions in R that vanish at a. Then M_a is the kernel of the ring homomorphism from R to the field k given by evaluating each function in R at a. Since R contains the constant functions, evaluation is surjective and so M_a is a maximal (hence also prime) ideal. The localization of R at this prime ideal is then

$$R_{M_a} = \left\{\frac{f}{g} \mid f, g \in R,\ g(a) \neq 0\right\}.$$

Each function in R_{M_a} can then be evaluated at a by $(f/g)(a) = f(a)/g(a)$, and this value does not depend on the choice of representative for the class f/g, so R_{M_a} becomes a ring of k-valued "rational functions" defined at a.

We next consider extensions and contractions of ideals with respect to the map $\pi : R \to D^{-1}R$ in Theorem 36. To ease some of the notation, if I is an ideal of R, let ${}^eI$ denote the extension of I to $D^{-1}R$ (instead of the more cumbersome $D^{-1}R\,\pi(I)$), and if J is an ideal of $D^{-1}R$, let ${}^cJ$ denote the contraction of J to R.