Since (A) and (B) must be rejected, it follows from Aristotle's 'Law of the Excluded Middle' that (C) the area of the circle equals that of a right triangle whose legs equal the radius and circumference of that circle. In other words,

$$\text{circle area} = \tfrac{1}{2}\ \text{circumference} \times \text{radius}.$$

The expression on the right-hand side is of course πr^2 (see (6) above).

Archimedes's proof of this formula was the culmination of about two hundred years of previous work on the circle, beginning with Antiphon (425 BC).

Archimedes, working in Syracuse, would communicate his results to the mathematicians back in Alexandria. He became annoyed when, suspiciously often, they claimed that they had made the same discoveries. To fool them, Archimedes included some false results in a book on the sphere and the cylinder, but history does not reveal the outcome. To challenge the mathematicians at Alexandria, Archimedes posed the following problem (which we have 'translated' into modern algebraic notation).

The sungod had a herd of cattle consisting of w white bulls, g grey bulls, b brown bulls and s spotted bulls, as well as w', g', b' and s' cows of matching colours. What was the total number of bulls and cows if

$$s = w - 5g/6 = g - 9b/20 = b - 13w/42,$$

$$w' = 7(g + g')/12, \quad g' = 9(b + b')/20,$$
$$b' = 11(s + s')/30, \quad s' = 13(w + w')/42,$$

and $w + g$ is a square and $b + s$ is a triangular number?

It is a curious triumph of tradition that Archimedes used the Egyptian method for representing the fractions appearing in this problem as sums of reciprocals of positive integers. Aside from the last two restrictions, we have here seven equations in eight unknowns, which cannot be solved by algebraic methods alone. However, if one is looking for positive integer solutions, such problems are called 'Diophantine', after the mathematician Diophantus, who will appear about 500 years after Archimedes. We sketch how a solution may proceed, though the reader will have to fill in many details.

From the equations

$$s = w - 5g/6 = g - 9b/20 = b - 13w/42$$

we find that, for some positive integer m,

$$w = 2226m, \quad g = 1602m, \quad b = 1580m, \quad s = 891m.$$

From the next four equations, we find that there is some natural number k such that

$$m = 4657k, \quad w' = 7,206,360k, \quad g' = 4,893,246k,$$

$$b' = 3,515,820k, \quad s' = 5,439,213k.$$

If $w+g$ is a square then $4(957)(4657)k$ is a square. Since 4657 is prime and 957 is the product of two distinct primes, it is easy to show that this occurs only if k has the form $(957)(4657)t^2$, where t is an integer. If $b+s$ is triangular, then $2471m$ has the form $\frac{1}{2}n(n+1)$. Thus $8(2471)(4657)(957)(4657)t^2$ has the form $4n^2+4n$. In other words, to find an integer solution to Archimedes's Cattle Problem, we have to find an integer solution to

$$(2n+1)^2 - 8(2471)(957)(4657^2)t^2 = 1.$$

The mathematicians at Alexandria were not able to solve this problem. Indeed, it was not solved until 1965, when H. C. Williams, R. A. German and C. R. Zarnke used a computer to generate the 206,545 digit answer. The answer was published for the first time in 1980–1981. The reader can find the full 206,545 digit answer printed in Harry L. Nelson's article 'A Solution to Archimedes' Cattle Problem', *Journal of Recreational Mathematics* 13, pp. 164–176.

It is impossible to do full justice here to Archimedes's important contributions to physics. Let us only mention that he developed the theory of the lever and investigated the properties of floating bodies.

Exercises

1. Let a and b be positive real numbers. Archimedes proved that $x^3 - ax^2 + (4/9)a^2b$ has a positive root if and only if $a > 3b$. Do the same.

2. If, in a cube of side 1, two cylinders, each of diameter 1, are constructed so that their axes are perpendicular, show that the volume common to these cylinders is $2/3$.

3. Prove that a regular 2^n-gon circumscribed about a circle has an area less than $1 + 1/2^{n-2}$ of that of the circle.

4. Find the least positive integer solution of $x^2 - 5y^2 = 1$.

5. Let B be a point in the straight line segment AC. Construct three semi-circles with diameters AB, BC and AC, all on the same side of AC. The area which is in the semi-circle on AC but not in either of the two smaller semi-circles is called the 'arbelos' (or 'shoe-maker's knife'). Archimedes found the area of the arbelos, in terms of AB and BC. Do the same.

6. Justify the existence of the numbers m and k in the above solution of the Cattle Problem.

7. If Archimedes were alive today, would he have a moral obligation to help his country design nuclear weapons or would he have a moral obligation not to help them design nuclear weapons? Support your answer with reasons related to the role of science in human history.

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Alexandria from 200 BC to 500 AD

In this chapter we discuss the more important mathematicians who worked in Alexandria after 200 BC:

- Hipparchus of Nicea, born about 180 BC,
- Heron of Alexandria, about 60 AD,
- Menelaus of Alexandria, about 100 AD,
- Ptolemy of Alexandria, died in 168 AD,
- Diophantus, about 250 AD,
- Pappus, about 320 AD.

Less significant as mathematicians, but nonetheless important in the history of the subject are

- Nicomachus of Gerasa, about 100 AD,
- Hypatia, died in 415 AD,
- Proclus, 410 – 485 AD,
- Boethius, 475 – 524 AD.

Hipparchus came from Nicea, a town near present day Istanbul, which was to be the site of the great Church Council of 325 AD. Hipparchus made many contributions to astronomy. He calculated the duration of the year