an inner product on V, and W is finite-dimensional, there is a particular subspace which one would probably call the 'natural' complementary subspace for W. This is the orthogonal complement of W. But, if V has no structure in addition to its vector space structure, there is no way of selecting a subspace W' which one could call the natural complementary subspace for W. However, one can construct from V and W a vector space V/W, known as the 'quotient' of V and W, which will play the role of the natural complement to W. This quotient space is not a subspace of V, and so it cannot actually be a subspace complementary to W; but, it is a vector space defined only in terms of V and W, and has the property that it is isomorphic to any subspace W' which is complementary to W.

Let W be a subspace of the vector space V. If α and β are vectors in V, we say that α is **congruent to β modulo W**, if the vector $(\alpha - \beta)$ is in the subspace W. If α is congruent to β modulo W, we write

$$\alpha \equiv \beta, \qquad \text{mod } W.$$

Now congruence modulo W is an equivalence relation on V.

(1) $\alpha \equiv \alpha$, mod W, because $\alpha - \alpha = 0$ is in W.

(2) If $\alpha \equiv \beta$, mod W, then $\beta \equiv \alpha$, mod W. For, since W is a subspace of V, the vector $(\alpha - \beta)$ is in W if and only if $(\beta - \alpha)$ is in W.

(3) If $\alpha \equiv \beta$, mod W, and $\beta \equiv \gamma$, mod W, then $\alpha \equiv \gamma$, mod W. For, if $(\alpha - \beta)$ and $(\beta - \gamma)$ are in W, then $\alpha - \gamma = (\alpha - \beta) + \beta - \gamma)$ is in W.

The equivalence classes for this equivalence relation are known as the **cosets** of W. What is the equivalence class (coset) of a vector α? It consists of all vectors β in V such that $(\beta - \alpha)$ is in W, that is, all vectors β of the form $\beta = \alpha + \gamma$, with γ in W. For this reason, the coset of the vector α is denoted by

$$\alpha + W.$$

It is appropriate to think of the coset of α relative to W as the set of vectors obtained by translating the subspace W by the vector α. To picture these cosets, the reader might think of the following special case. Let V be the space R^2, and let W be a one-dimensional subspace of V. If we picture V as the Euclidean plane, W is a straight line through the origin. If $\alpha = (x_1, x_2)$ is a vector in V, the coset $\alpha + W$ is the straight line which passes through the point (x_1, x_2) and is parallel to W.

The collection of all cosets of W will be denoted by V/W. We now define a vector addition and scalar multiplication on V/W as follows:

$$\begin{aligned}(\alpha + W) + (\beta + W) &= (\alpha + \beta) + W \\ c(\alpha + W) &= (c\alpha) + W.\end{aligned}$$

In other words, the sum of the coset of α and the coset of β is the coset of $(\alpha + \beta)$, and the product of the scalar c and the coset of α is the coset of the vector $c\alpha$. Now many different vectors in V will have the same coset relative to W, and so we must verify that the sum and product above depend only upon the cosets involved. What this means is that we must show the following:

(a) If $\alpha \equiv \alpha'$, mod W, and $\beta \equiv \beta'$, mod W, then

$$\alpha + \beta \rightarrow \alpha' + \beta', \quad \text{mod } W.$$

(2) If $\alpha \equiv \alpha'$, mod W, then $c\alpha \equiv c\alpha'$, mod W.

These facts are easy to verify. (1) If $\alpha - \alpha'$ is in W and $\beta - \beta'$ is in W, then since $(\alpha + \beta) - (\alpha' - \beta') = (\alpha - \alpha') + (\beta - \beta')$, we see that $\alpha + \beta$ is congruent to $\alpha' - \beta'$ modulo W. (2) If $\alpha - \alpha'$ is in W and c is any scalar, then $c\alpha - c\alpha' = c(\alpha - \alpha')$ is in W.

It is now easy to verify that V/W, with the vector addition and scalar multiplication defined above, is a vector space over the field F. One must directly check each of the axioms for a vector space. Each of the properties of vector addition and scalar multiplication follows from the corresponding property of the operations in V. One comment should be made. The zero vector in V/W will be the coset of the zero vector in V. In other words, W is the zero vector in V/W.

The vector space V/W is called the **quotient** (or difference) of V and W. There is a natural linear transformation Q from V onto V/W. It is defined by $Q(\alpha) = \alpha + W$. One should see that we have defined the operations in V/W just so that this transformation Q would be linear. Note that the null space of Q is exactly the subspace W. We call Q the **quotient transformation** (or **quotient mapping**) of V onto V/W.

The relation between the quotient space V/W and subspaces of V which are complementary to W can now be stated as follows.

Theorem. *Let* W *be a subspace of the vector space* V, *and let* Q *be the quotient mapping of* V *onto* V/W. *Suppose* W′ *is a subspace of* V. *Then* V = W ⊕ W′ *if and only if the restriction of* Q *to* W′ *is an isomorphism of* W′ *onto* V/W.

Proof. Suppose $V = W \oplus W'$. This means that each vector α in V is uniquely expressible in the form $\alpha = \gamma + \gamma'$, with γ in W and γ' in W'. Then $Q\alpha = Q\gamma + Q\gamma' = Q\gamma'$, that is $\alpha + W = \gamma' + W$. This shows that Q maps W' onto V/W, i.e., that $Q(W') = V/W$. Also Q is 1:1 on W'; for suppose γ_1' and γ_2' are vectors in W' and that $Q\gamma_1' = Q\gamma_2'$. Then $Q(\gamma_1' - \gamma_2') = 0$ so that $\gamma_1' - \gamma_2'$ is in W. This vector is also in W', which is disjoint from W; hence $\gamma_1' - \gamma_2' = 0$. The restriction of Q to W' is therefore a one-one linear transformation of W' onto V/W.

Suppose W' is a subspace of V such that Q is one-one on W' and $Q(W') = V/W$. Let α be a vector in V. Then there is a vector γ' in W' such that $Q\gamma' = Q\alpha$, i.e., $\gamma' + W = \alpha + W$. This means that $\alpha = \gamma + \gamma'$ for some vector γ in W. Therefore $V = W + W'$. To see that W and W'

are disjoint, suppose γ is in both W and W'. Since γ is in W, we have $Q\gamma = 0$. But Q is 1:1 on W', and so it must be that $\gamma = 0$. Thus we have $V = W \oplus W'$. ∎

What this theorem really says is that W' is complementary to W if and only if W' is a subspace which contains exactly one element from each coset of W. It shows that when $V = W \oplus W'$, the quotient mapping Q 'identifies' W' with V/W. Briefly $(W \oplus W')/W$ is isomorphic to W' in a 'natural' way.

One rather obvious fact should be noted. If W is a subspace of the finite-dimensional vector space V, then

$$\dim W + \dim (V/W) = \dim V.$$

One can see this from the above theorem. Perhaps it is easier to observe that what this dimension formula says is

$$\text{nullity } (Q) + \text{rank } (Q) = \dim V.$$

It is not our object here to give a detailed treatment of quotient spaces. But there is one fundamental result which we should prove.

Theorem. *Let* V *and* Z *be vector spaces over the field* F. *Suppose* T *is a linear transformation of* V *onto* Z. *If* W *is the null space of* T, *then* Z *is isomorphic to* V/W.

Proof. We define a transformation U from V/W into Z by $U(\alpha + W) = T\alpha$. We must verify that U is well defined, i.e., that if $\alpha + W = \beta + W$ then $T\alpha = T\beta$. This follows from the fact that W is the null space of T; for, $\alpha + W = \beta + W$ means $\alpha - \beta$ is in W, and this happens if and only if $T(\alpha - \beta) = 0$. This shows not only that U is well defined, but also that U is one-one.

It is now easy to verify that U is linear and sends V/W onto Z, because T is a linear transformation of V onto Z. ∎

A.5. Equivalence Relations in Linear Algebra

We shall consider some of the equivalence relations which arise in the text of this book. This is just a sampling of such relations.

(1) Let m and n be positive integers and F a field. Let X be the set of all $m \times n$ matrices over F. Then row-equivalence is an equivalence relation on the set X. The statement 'A is row-equivalent to B' means that A can be obtained from B by a finite succession of elementary row operations. If we write $A \sim B$ for A is row-equivalent to B, then it is not difficult to check the properties (i) $A \sim A$; (ii) if $A \sim B$, then $B \sim A$;

(iii) if $A \sim B$ and $B \sim C$, then $A \sim C$. What do we know about this equivalence relation? Actually, we know a great deal. For example, we know that $A \sim B$ if and only if $A = PB$ for some invertible $m \times m$ matrix P; or, $A \sim B$ if and only if the homogeneous systems of linear equations $AX = 0$ and $BX = 0$ have the same solutions. We also have very explicit information about the equivalence classes for this relation. Each $m \times n$ matrix A is row-equivalent to one and only one row-reduced echelon matrix. What this says is that each equivalence class for this relation contains precisely one row-reduced echelon matrix R; the equivalence class determined by R consists of all matrices $A = PR$, where P is an invertible $m \times m$ matrix. One can also think of this description of the equivalence classes in the following way. Given an $m \times n$ matrix A, we have a rule (function) f which associates with A the row-reduced echelon matrix $f(A)$ which is row-equivalent to A. Row-equivalence is completely determined by f. For, $A \sim B$ if and only if $f(A) = f(B)$, i.e., if and only if A and B have the same row-reduced echelon form.

(2) Let n be a positive integer and F a field. Let X be the set of all $n \times n$ matrices over F. Then similarity is an equivalence relation on X; each $n \times n$ matrix A is similar to itself; if A is similar to B, then B is similar to A; if A is similar to B and B is similar to C, then A is similar to C. We know quite a bit about this equivalence relation too. For example, A is similar to B if and only if A and B represent the same linear operator on F^n in (possibly) different ordered bases. But, we know something much deeper than this. Each $n \times n$ matrix A over F is similar (over F) to one and only one matrix which is in rational form (Chapter 7). In other words, each equivalence class for the relation of similarity contains precisely one matrix which is in rational form. A matrix in rational form is determined by a k-tuple $(p_1, \ldots, p_k)$ of monic polynomials having the property that p_{j+1} divides p_j, $j = 1, \ldots, k - 1$. Thus, we have a function f which associates with each $n \times n$ matrix A a k-tuple $f(A) = (p_1, \ldots, p_k)$ satisfying the divisibility condition p_{j+1} divides p_j. And, A and B are similar if and only if $f(A) = f(B)$.

(3) Here is a special case of Example 2 above. Let X be the set of 3×3 matrices over a field F. We consider the relation of similarity on X. If A and B are 3×3 matrices over F, then A and B are similar if and only if they have the same characteristic polynomial and the same minimal polynomial. Attached to each 3×3 matrix A, we have a pair (f, p) of monic polynomials satisfying

(a) $\deg f = 3$,
(b) p divides f,

f being the characteristic polynomial for A, and p the minimal polynomial for A. Given monic polynomials f and p over F which satisfy (a) and (b), it is easy to exhibit a 3×3 matrix over F, having f and p as its charac-