(4) The special case of Example 3 when Π is the regular permutation representation of G is worth recording: if φ is the regular representation of G (afforded by the module FG) and ρ is its character:

$$\rho(g) = \begin{cases} 0 & \text{if } g \neq 1 \\ |G| & \text{if } g = 1. \end{cases}$$

The character of the regular representation of G is called the *regular character* of G. Note that this provides specific examples where a character takes on the value 0 and is not a group homomorphism from G into either F or $F^{\times}$.

(5) Let $\varphi : D_{2n} \to GL_2(\mathbb{R})$ be the explicit matrix representation described in Example 6 in the second set of examples of Section 1. If χ is the character of φ then, by taking traces of the given 2×2 matrices one sees that $\chi(r) = 2\cos(2\pi/n)$ and $\chi(s) = 0$. Since φ takes the identity of D_{2n} to the 2×2 identity matrix, $\chi(1) = 2$.

(6) Let $\varphi : Q_8 \to GL_2(\mathbb{C})$ be the explicit matrix representation described in Example 7 in the second set of examples of Section 1. If χ is the character of φ then, by taking traces of the given 2×2 matrices, $\chi(i) = 0$ and $\chi(j) = 0$. Since the element $-1 \in Q_8$ maps to minus the 2×2 identity matrix, $\chi(-1) = -2$. Since φ takes the identity of Q_8 to the 2×2 identity matrix, $\chi(1) = 2$.

(7) Let $\varphi : Q_8 \to GL_4(\mathbb{R})$ be the matrix representation described in Example 8 in the second set of examples of Section 1. If χ is the character of φ then, by inspection of the matrices exhibited, $\chi(i) = \chi(j) = 0$. Since φ takes the identity of Q_8 to the 4×4 identity matrix, $\chi(1) = 4$.

For $n \times n$ matrices A and B, direct computation shows that $\operatorname{tr} AB = \operatorname{tr} BA$. If A is invertible, this implies that

$$\operatorname{tr} A^{-1}BA = \operatorname{tr} B.$$

Thus the character of a representation is independent of the choice of basis of the vector space affording it, i.e.,

$$\textit{equivalent representations have the same character.} \tag{18.1}$$

Let φ be a representation of G of degree n with character χ. Since $\varphi(g^{-1}xg)$ is $\varphi(g)^{-1}\varphi(x)\varphi(g)$ for all $g, x \in G$, taking traces shows that

$$\textit{the character of a representation is a class function.} \tag{18.2}$$

Since the trace of the $n \times n$ identity matrix is n and φ takes the identity of G to the identity linear transformation (or matrix),

$$\chi(1) \textit{ is the degree of } \varphi. \tag{18.3}$$

If V is an FG-module whose corresponding representation has character χ, then each element of the group ring FG acts as a linear transformation from V to V. Thus each $\sum_{g\in G} \alpha_g g \in FG$ has a trace when it is considered as a linear transformation from V to V. The trace of $g \in G$ acting on V is, by definition, $\chi(g)$. Since the trace of any linear combination of matrices is the linear combination of the traces, the trace of $\sum_{g\in G} \alpha_g g$ acting on V is $\sum_{g\in G} \alpha_g \chi(g)$. Note that this trace function on FG is the unique extension of the character χ of G to an F-linear transformation from FG to F. In this way we shall consider characters of G as also being defined on the group ring FG.

Notice in Example 3 above that if the field F has characteristic $p > 0$, the values of the character mod p might be zero even though the number of fixed points is nonzero. In order to circumvent such anomalies and to use the consequences of Wedderburn's Theorem obtained when F is algebraically closed we again specialize the field to be the complex numbers (or any algebraically closed field of characteristic 0). By the results of the previous section

$$\mathbb{C}G \cong M_{n_1}(\mathbb{C}) \times M_{n_2}(\mathbb{C}) \times \cdots \times M_{n_r}(\mathbb{C}). \tag{18.4}$$

For the remainder of this section fix the following notation:

$$\begin{gathered} M_1, M_2, \dots, M_r \text{ are the inequivalent irreducible } \mathbb{C}G\text{-modules,} \\ \chi_i \text{ is the character afforded by } M_i, \quad 1 \le i \le r. \end{gathered} \tag{18.5}$$

Thus r is the number of conjugacy classes of G and we may relabel $M_1, \dots, M_r$ if necessary so that the degree of χ_i is n_i for all i (which is also the dimension of M_i over $\mathbb{C}$).

Now every (finite dimensional) $\mathbb{C}G$-module M is isomorphic (equivalent) to a direct sum of irreducible modules:

$$M \cong a_1 M_1 \oplus a_2 M_2 \oplus \cdots \oplus a_r M_r, \tag{18.6}$$

where a_i is a nonnegative integer indicating the multiplicity of the irreducible module M_i in this direct sum decomposition, i.e.,

$$a_i M_i = \overbrace{M_i \oplus \cdots \oplus M_i}^{a_i \text{ times}}.$$

Note that if the representation φ is afforded by the module M and $M = M_1 \oplus M_2$, then we may choose a basis of M consisting of a basis of M_1 together with a basis of M_2. The matrix representation with respect to this basis is of the form

$$\varphi(g) = \begin{pmatrix} \varphi_1(g) & 0 \\ 0 & \varphi_2(g) \end{pmatrix}$$

where φ_i is the representation afforded by M_i, $i = 1, 2$. One sees immediately that if ψ is the character of φ and ψ_i is the character of φ_i, then $\psi(g) = \psi_1(g) + \psi_2(g)$, i.e., $\psi = \psi_1 + \psi_2$. By induction we obtain:

the character of a representation is the sum of the characters of the constituents appearing in a direct sum decomposition. (18.7)

If ψ is the character afforded by the module M in (6) above, this gives

$$\psi = a_1\chi_1 + a_2\chi_2 + \cdots + a_r\chi_r. \tag{18.8}$$

Thus every (complex) character is a nonnegative integral sum of irreducible (complex) characters. Conversely, by taking direct sums of modules one sees that every such sum of characters is the character of some complex representation of G.

We next prove that the correspondence between characters and equivalence classes of complex representations is *bijective*. Let $z_1, z_2, \dots, z_r$ be the primitive central idempotents of $\mathbb{C}G$ described in the preceding section. Since these are orthogonal (or equivalently, since they are the r-tuples in the decomposition of $\mathbb{C}G$ into a direct product of r

subrings which have a 1 in one position and zeros elsewhere), $z_1, \dots, z_r$ are $\mathbb{C}$-linearly independent elements of $\mathbb{C}G$. As above, each irreducible character χ_i is a function on $\mathbb{C}G$. By Proposition 8(3) we have

(a) if $j \neq i$ then $z_j M_i = 0$, i.e., z_j acts as the zero matrix on M_j, hence $\chi_j(z_i) = 0$, and

(b) z_i acts as the identity on M_i, hence $\chi_i(z_i) = n_i$.

Thus $\chi_1, \dots, \chi_r$ are multiples of the dual basis to the independent set $z_1, \dots, z_r$, hence are linearly independent functions. Now if the $\mathbb{C}G$-module M described in (6) above can be decomposed in a different fashion into irreducibles, say,

$$M \cong b_1 M_1 \oplus b_2 M_2 \oplus \cdots \oplus b_r M_r,$$

then we would obtain a relation

$$a_1\chi_1 + a_2\chi_2 + \cdots + a_r\chi_r = b_1\chi_1 + b_2\chi_2 + \cdots + b_r\chi_r.$$

By linear independence of the irreducible characters, $b_i = a_i$ for all $i \in \{1, \dots, r\}$. Thus, in any decomposition of M into a direct sum of irreducibles, the multiplicity of the irreducible M_i is the same, $1 \leq i \leq r$. In particular,

two representations are equivalent if and only if they have the same character. (18.9)

This uniqueness can be seen in an alternate way. First, use Proposition 8(2) to decompose an arbitrary finite dimensional $\mathbb{C}G$-module M uniquely as

$$M = z_1 M \oplus z_2 M \oplus \cdots \oplus z_r M.$$

By part (4) of the same proposition, $z_i M$ is a direct sum of simple modules, each of which is isomorphic to M_i. The multiplicity of M_i in a direct sum decomposition of $z_i M$ is, by counting dimensions, equal to $\dfrac{\dim z_i M}{\dim M_i}$. This proves that the multiplicity of M_i in any direct sum decomposition of M into simple submodules is uniquely determined.

Note that, as with decompositions of $F[x]$-modules into cyclic submodules, a $\mathbb{C}G$-module may have many direct sum decompositions into irreducibles — only the multiplicities are unique (see also the exercises). More precisely, comparing with the Jordan canonical form of a single linear transformation, the direct summand $a_i M_i = M_i \oplus \cdots \oplus M_i$ (a_i times) which equals the submodule $z_i M$ is the analogue of the generalized eigenspace corresponding to a single eigenvalue. This submodule of M is unique (as is a generalized eigenspace) and is called the χ_i^{th} *isotypic component* of M. Within the χ_i^{th} isotypic component, the summands M_i are analogous to the 1-dimensional eigenspaces and, just as with the eigenspace of an endomorphism there is no unique basis for the eigenspace. If $G = \langle g \rangle$ is a finite cyclic group, the isotypic components of G are the same as the generalized eigenspaces of g.

Observe that the vector space of all (complex valued) class functions on G has a basis consisting of the functions which are 1 on a given class and zero on all other classes. There are r of these, where r is the number of conjugacy classes of G, so the dimension of the complex vector space of class functions is r. Since the number of (complex) irreducible characters of G equals the number of conjugacy classes and these are linearly independent class functions, we see that

the irreducible characters are a basis for the space of all complex class functions. (18.10)

The next step in the theory of characters is to put an Hermitian inner product structure on the space of class functions and prove that the irreducible characters form an orthonormal basis with respect to this inner product. For class functions θ and ψ define

$$(\theta, \psi) = \frac{1}{|G|} \sum_{g \in G} \theta(g)\overline{\psi(g)}$$

(where the bar denotes complex conjugation). One easily checks that $(\ ,\ )$ is Hermitian: for $\alpha, \beta \in \mathbb{C}$

(a) $(\alpha\theta_1 + \beta\theta_2, \psi) = \alpha(\theta_1, \psi) + \beta(\theta_2, \psi)$,
(b) $(\theta, \alpha\psi_1 + \beta\psi_2) = \overline{\alpha}(\theta, \psi_1) + \overline{\beta}(\theta, \psi_2)$, and
(c) $(\theta, \psi) = \overline{(\psi, \theta)}$.

Our principal aim is to show that the irreducible characters form an orthonormal basis for the space of complex class functions with respect to this Hermitian form (we already know that they are a basis). This fact will follow from the orthogonality of the primitive central idempotents, once we have explicitly determined these in the next proposition.

Proposition 13. Let $z_1, \ldots, z_r$ be the orthogonal primitive central idempotents in $\mathbb{C}G$ labelled in such a way that z_i acts as the identity on the irreducible $\mathbb{C}G$-module M_i, and let χ_i be the character afforded by M_i. Then

$$z_i = \frac{\chi_i(1)}{|G|} \sum_{g \in G} \chi_i(g^{-1})g.$$

Proof: Let $z = z_i$ and write

$$z = \sum_{g \in G} \alpha_g g.$$

Recall from Example 4 in this section that if ρ is the regular character of G then

$$\rho(g) = \begin{cases} 0 & \text{if } g \neq 1 \\ |G| & \text{if } g = 1 \end{cases} \tag{18.11}$$

and recall from the last example in Section 2 that

$$\rho = \sum_{j=1}^{r} \chi_j(1)\chi_j. \tag{18.12}$$

To find the coefficient α_g, apply ρ to zg^{-1} and use linearity of ρ together with equation (11) to obtain

$$\rho(zg^{-1}) = \alpha_g |G|.$$