If M is a finitely generated module over the Dedekind Domain R as in Theorem 22, then the isomorphism type of M as an R-module is determined by the *rank* n, the prime powers $P_i^{e_i}$ for $i = 1, \ldots, s$ (called the *elementary divisors* of M, and the class of the ideal I in the class group of R (called the *Steinitz class* of M). Note that a P.I.D. is the same as a Dedekind Domain whose class number is 1, in which case every nonzero ideal I of R is isomorphic as an R-module simply to R. In this case, Theorem 22 reduces to the elementary divisor form of the structure theorem for finitely generated modules over P.I.D.s in Chapter 12. There is also an invariant factor version of the description of the torsion R-modules in Theorem 22 (cf. Exercise 14).

The next result extends the characterization of finitely generated projective modules over P.I.D.s (Exercise 21 in Section 12.1) to Dedekind Domains.

Corollary 23. A finitely generated module over a Dedekind Domain is projective if and only if it is torsion free.

Proof: We showed that a finitely generated torsion free R-module is projective in the proof of Theorem 22, so by the decomposition of M in Theorem 22, M is projective if and only if $\mathrm{Tor}(M)$ is projective (cf. Exercise 3 in Section 10.5). To complete the proof it suffices to show that no nonzero torsion R-module is projective, which is left as an exercise (cf. Exercise 15).

EXERCISES

1. If R is an integral domain, show that every fractional ideal of R is invertible if and only if every integral ideal of R is invertible.
2. Suppose R is an integral domain with fraction field K and $A_1, A_2, \ldots, A_n$ are fractional ideals of R whose product is a nonzero principal fractional ideal: $A_1 A_2 \cdots A_n = Rx$ for some $0 \neq x \in K$. For each $i = 1, \ldots, n$ prove that A_i is an invertible fractional ideal with inverse $(x^{-1})A_1 \cdots A_{i-1}A_{i+1} \cdots A_n$.
3. Suppose R is an integral domain with fraction field K and P is a nonzero prime ideal in R. Show that the fractional ideals of R_P in K are the R_P-modules of the form AR_P where A is a fractional ideal of R.
4. Suppose R is an integral domain with fraction field K and A is a fractional ideal of R in K. Let $A' = \{x \in K \mid xA \subseteq R\}$ as in Proposition 9.
   (a) For any prime ideal P in R prove that the localization $(A')_P$ of A' at P is a fractional ideal of R_P in K.
   (b) If A is a finitely generated R-module, prove that $(A')_P = (A_P)'$ where $(A_P)'$ is the fractional R_P ideal $\{x \in K \mid xA_P \subseteq R_P\}$ corresponding to the localization A_P.
5. If Q_1 is a P_1-primary ideal and Q_2 is a P_2-primary ideal where P_1 and P_2 are comaximal ideals in a Noetherian ring R, prove that Q_1 and Q_2 are also comaximal. [Use Proposition 14 in Section 15.2.]
6. Suppose R is a Dedekind Domain with fraction field K.
   (a) Prove that every nonzero fractional ideal of R in K can be written uniquely as the product of distinct prime powers $P_1^{a_1} \cdots P_n^{a_n}$ where the a_i are nonzero integers, possibly negative.

(b) If $0 \neq x \in K$, let $P^{\nu_P(x)}$ be the power of the prime P in the factorization of the principal ideal (x) as in (a) (where $\nu_P(x) = 0$ if P is not one of the primes occurring). Prove ν_P is a valuation on K with valuation ring R_P, the localization of R at P.

7. Suppose R is a Noetherian integral domain that is not a field. Prove that R is a Dedekind Domain if and only if for every maximal ideal M of R there are no ideals I of R with $M^2 \subset I \subset M$. [Use Exercise 12 in Section 1 and Theorems 7 and 15.]

8. Suppose R is a Noetherian integral domain with Krull dimension 1. Prove that every nonzero ideal I in R can be written uniquely as a product of primary ideals whose radicals are all distinct. [Cf. the proof of Theorem 15. Use the uniqueness of the primary components belonging to the isolated primes in a minimal primary decomposition (Theorem 21 in Section 15.2).]

9. Suppose R is an integral domain. Prove that R_P is a D.V.R. for every nonzero prime ideal P if and only if R_M is a D.V.R. for every nonzero maximal ideal.

10. Suppose R is a Noetherian integral domain that is not a field. Prove that R is a Dedekind Domain if and only if nonzero primes M are maximal and every M-primary ideal is a power of M.

11. If I and J are nonzero ideals in the Dedekind Domain R show there exists an integral ideal I_1 in R that is relatively prime to both I and J such that $I_1 I$ is a principal ideal in R.

12. If I and J are nonzero fractional ideals for the Dedekind Domain R prove there are elements $\alpha, \beta \in K$ such that αI and βJ are nonzero integral ideals in R are relatively prime.

13. Suppose I and J are nonzero ideals in the Dedekind Domain R. Prove that there is an ideal $I_1 \cong I$ that is relatively prime to J. [Use Corollary 19 to find an ideal I_2 with $I_2 I = (a)$ and $(I_2, J) = R$. If $I_2 = P_1^{e_1} \cdots P_n^{e_n}$, choose $b \in R$ with $b \in P_i^{e_i} - P_i^{e_i+1}$ and $b \equiv 1 \bmod P$ for every prime P dividing J. Show that $(b) = I_2 I_1$ for some ideal I_1 and consider $(a) I_1$ to prove that $I_1 \cong I$.]

14. Prove that every finitely generated torsion module over a Dedekind Domain R is isomorphic to a direct sum $R/I_1 \oplus R/I_2 \oplus \cdots \oplus R/I_n$ with unique nonzero ideals $I_1, \ldots, I_n$ of R satisfying $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n$ (called the *invariant factors* of M). [cf. Section 12.1.]

15. If P is a nonzero prime ideal in the Dedekind Domain R prove that R/P^n is not a projective R-module for any $n \geq 1$. [Consider the exact sequence $0 \to P^n/P^{n+1} \to R/P^{n+1} \to R/P^n \to 0$.] Conclude that if $M \neq 0$ is a finitely generated torsion R-module then M is not projective. [cf. Exercise 3, Section 10.5.]

16. Prove that the class number of the Dedekind Domain R is 1 if and only if every finitely generated projective R-module is free.

17. Suppose R is a Dedekind Domain.

(a) Show that $I \sim J$ if and only if $I \cong J$ as R-modules defines an equivalence relation on the set of nonzero fractional ideals of R. Let $C(R)$ be the corresponding set of R-module isomorphism classes and let $[I] \in C(R)$ denote the equivalence class containing the fractional ideal I of R.

(b) Show that the multiplication $[I][J] = [I \oplus J]$ gives a well defined binary operation with respect to which $C(R)$ is an abelian group with identity $1 = [R]$.

(c) Prove that the abelian group $C(R)$ in (b) is isomorphic to the class group of R.

18. If R is a Dedekind Domain and I is any nonzero ideal, prove that R/I contains only finitely many ideals. In particular, show that R/I is an Artinian ring.

19. Suppose I is a nonzero fractional ideal in the Dedekind Domain R. Explicitly exhibit I as a direct summand of a free R-module to show that I is projective. [Consider $I \oplus I^{-1}$

and use Proposition 21.]

20. Suppose I and J are two nonzero fractional ideals in the Dedekind Domain R and that $I^n = J^n$ for some $n \neq 0$. Prove that $I = J$.

21. Suppose K is an algebraic number field and $\mathcal{O}_K$ is the ring of integers in K. If P is a nonzero prime ideal in $\mathcal{O}_K$ prove that $P = (p, \pi)$ for some prime $p \in \mathbb{Z}$ and algebraic integer $\pi \in \mathcal{O}_K$.

22. Suppose $K = \mathbb{Q}(\sqrt{D})$ is a quadratic extension of $\mathbb{Q}$ where D is a squarefree integer and $\mathcal{O}_K$ is the ring of integers in K.

(a) Prove that $|\mathcal{O}_K/(p)| = p^2$. [Observe that $\mathcal{O}_K \cong \mathbb{Z}^2$ as an abelian group.]

(b) Use Corollary 16 to show that there are 3 possibilities for the prime ideal factorization of (p) in $\mathcal{O}_K$:
(i) $(p) = P$ is a prime ideal with $|\mathcal{O}_K/P| = p^2$,
(ii) $(p) = P_1P_2$ with distinct prime ideals P_1, P_2 and $|\mathcal{O}_K/P_1| = |\mathcal{O}_K/P_2| = p$,
(iii) $(p) = P^2$ for some prime ideal P with $|\mathcal{O}_K/P| = p$.

(In cases (i), (ii), and (iii) the prime p is said to be *inert*, *split*, or *ramified* in $\mathcal{O}_K$, respectively. The set of ramified primes is finite: the primes p dividing D if $D \equiv 1, 2 \bmod 4$; $p = 2$ and the primes p dividing D if $D \equiv 3 \bmod 4$. Cf. Exercise 31 in Section 15.5.)

(c) Determine the prime ideal factorizations of the primes $p = 2, 3, 5, 7, 11$ in the ring of integers $\mathcal{O}_K = \mathbb{Z}[\sqrt{-5}]$ of $K = \mathbb{Q}(\sqrt{-5})$.

23. Let $\mathcal{O}$ be the ring of integers in the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$.

(a) Show that the infinite sequence of ideals in $\mathcal{O}$ $(2) \subseteq (\sqrt{2}) \subseteq (\sqrt[4]{2}) \subseteq (\sqrt[8]{2}) \subseteq \cdots$ is strictly increasing, and so $\mathcal{O}$ is not Noetherian.

(b) Show that $\mathcal{O}$ has Krull dimension 1. [Use Theorem 26 in Section 15.3.]

(c) Let K be a number field and let I be any ideal in $\mathcal{O}_K$. Show that there is some finite extension L of K such that I becomes principal when extended to $\mathcal{O}_L$, i.e., the ideal $I\mathcal{O}_L$ is principal (where L depends on I)—you may use the theorem that the class group of K is a finite group. [cf. Exercise 20.]

(d) Prove that $\mathcal{O}$ is a Bezout Domain (cf. Section 8.1).

24. Suppose F and K are algebraic number fields with $\mathbb{Q} \subseteq F \subseteq K$, with rings of integers $\mathcal{O}_F$ and $\mathcal{O}_K$, respectively. Since $\mathcal{O}_F \subseteq \mathcal{O}_K$, the ring $\mathcal{O}_K$ is naturally a module over $\mathcal{O}_F$.

(a) Prove $\mathcal{O}_K$ is a torsion free $\mathcal{O}_F$-module of rank $n = [K : F]$. [Compute ranks over $\mathbb{Z}$.] If $\mathcal{O}_K$ is *free* over $\mathcal{O}_F$ then $\mathcal{O}_K$ is said to have a *relative integral basis* over $\mathcal{O}_F$.

(b) Prove that if F has class number 1 then $\mathcal{O}_K$ has a relative integral basis over $\mathcal{O}_F$.

If $K = \mathbb{Q}(\sqrt{-5}, \sqrt{2})$ then the ring of integers $\mathcal{O}_K$ is given by

$$\mathcal{O}_K = \mathbb{Z} + \mathbb{Z}\sqrt{-5} + \mathbb{Z}\sqrt{-10} + \mathbb{Z}\omega \qquad \text{where } \omega = (\sqrt{-10} + \sqrt{2})/2.$$

(c) If $F_1 = \mathbb{Q}(\sqrt{2})$ prove that $\mathcal{O}_K$ has a relative integral basis over $\mathcal{O}_{F_1}$ and find an explicit basis $\{\alpha, \beta\}$: $\mathcal{O}_K = \mathcal{O}_{F_1} \cdot \alpha + \mathcal{O}_{F_1} \cdot \beta$.

(d) If $F_2 = \mathbb{Q}(\sqrt{-5})$, show that $P_3 = (3, 1 + \sqrt{-5}) = (3, 5 - \sqrt{-5})$ is a prime ideal of $\mathcal{O}_{F_2}$ that is not principal and that $\mathcal{O}_K = \mathcal{O}_{F_2} \cdot 1 + (1/3)P_3 \cdot \omega$. [Check that $\sqrt{-10} = -(5 - \sqrt{-5})\omega/3$.] Conclude that the Steinitz class of $\mathcal{O}_K$ as a module over $\mathcal{O}_{F_2}$ is the nontrivial class of P_3 in the class group of $\mathcal{O}_{F_2}$ and so there is no relative integral basis of $\mathcal{O}_K$ over $\mathcal{O}_{F_2}$.

(e) Determine whether $\mathcal{O}_K$ has a relative integral basis over the ring of integers of the remaining quadratic subfield $F_3 = \mathbb{Q}(\sqrt{-10})$ of K.

25. Suppose C is a nonsingular irreducible affine curve over an algebraically closed field k. Prove that the coordinate ring $k[C]$ is a Dedekind Domain.