In summary, we may not agree with Zeno that there is no motion, but we must credit him with probing into the very foundation of mathematics and physics.

Empedocles is important in experimental science and cosmology. He demonstrated that air is a substance by pushing an inverted tumbler into a tub of water; the water did not rush in to fill the apparent vacuum. He recognized not only the four traditional substances, earth, water, air and fire, but postulated two other: love and strife. He believed that, as long as love prevails, the cosmos contracts, but when strife prevails, it expands. Thus he seems to have anticipated modern astronomers in realizing that our universe is in an expanding phase. As his last experiment, he leaped into a volcano to demonstrate his immortality.

Democritus finally did away with substances and replaced them with atoms. These were assumed to be physically but not geometrically, indivisible. They were indestructible and constantly moving, the space between them being empty. The number of atoms was infinite, but they differed in shape and size. In the 19th century, people thought they had identified the atoms of Democritus; but then, in the early 20th century, Rutherford showed that the so-called atoms were divisible after all. It might have been wiser to reserve the word 'atom' for our electrons and quarks, if these indeed turn out to be the ultimate constituents of matter.

Democritus was a determinist, he believed that everything that happens must happen. He wrote books on geometry, which have been lost. He is said to have emphasized the importance of proofs and to have discovered (or rediscovered) how to calculate the volume of a pyramid or cone by taking one third of the base area times the height. It is curious that Marx wrote his doctoral dissertation on Democritus and not on Heraclitus.

12

Mathematics in Athens

After an alliance of Greek cities defeated the Persians in 490 BC, Athens became, for over a hundred years, a great center of civilization. There are things about it we do not like: Athens exacted tribute from its allies, and the leisurely life of its leading citizens was based upon slavery. Nonetheless, one can safely say that the degree of civilization achieved by the Athenians around 400 BC has rarely, if ever, been surpassed in the history of the world. Because we must confine our attention to mathematics, we shall touch on only one of the many areas in which cultural development took place.

One of the first mathematicians who worked in Athens was the sophist Hippias (420 BC), from Elis on the west coast of Greece. In a dialogue sometimes ascribed to Plato, we hear Socrates (469–399 BC) teasing Hippias about his mathematics:

> Socrates: And tell me, Hippias, are you not a skilful calculator and arithmetician?
>
> Hippias: Yes, Socrates, assuredly I am.
>
> Socrates: And if someone were to ask you what is the sum of 3 multiplied by 700, you would tell him the true answer in a moment, if you pleased?
>
> Hippias: Certainly I should.
>
> Socrates: Is not that because you are the wisest and ablest of men in these matters?
>
> Hippias: Yes.

Hippias discovered a curve called the *quadratrix*, which can be used for trisecting an arbitrary angle, and also for constructing a square equal in area to a given circle. He described the quadratix as follows: imagine that side AD of the unit square $ABCD$ moves down at a rate of 1 unit per second towards the side BC (on the 'bottom' of the square). Imagine that side AB rotates about B at a rate of $1/4$ revolution per second towards BC, so that, after 1 second, both AD and AB coincide with BC. At any time t $(0 \le t \le 1)$, the two moving sides meet at a point P. The set or *locus* of these points P is the quadratrix.

In terms of our modern analytic geometry and trigonometry, we would put it this way: the point P has coordinates

$$((1 - t)/\tan(90°(1 - t)),\ \ 1 - t)$$

so the equation of the quadratrix is $y = x\tan(90°y)$, with $0 \le y$.

To divide an angle of, say, 60° into 3 equal parts, it is enough to place it in standard position, with its vertex on the origin, and one arm along the positive x axis. If the other arm meets the quadratrix at (a, b), we find the point P where $y = b/3$ meets the quadratrix. The angle between the line joining P to the origin and the positive x axis is 20°.

Furthermore, as y tends to 0, $x = y/\tan(90°y)$ tends to $2/\pi$, and so the quadratrix can be used to 'square the circle'.

This highly ingenious method was criticized — by Plato, it seems — on the grounds that it is more elegant to use only straight lines and circles in the solution of mathematical problems. One ought to carry out geometric constructions using only a ruler (for drawing straight lines) and a compass (for drawing circles); using a quadratrix was considered to be cheating.

However, as Pierre Wantzel (1814–1848) was the first to prove, it is not possible to trisect an arbitrary angle using only straight lines and circles. One has to use some other tool — such as the quadratrix. Hippias was right and Plato was wrong; although by insisting on ruler and compass constructions he raised an interesting and challenging problem, which we shall discuss in Chapter 14.

In order to understand the Greek contribution to the beginnings of analysis, it is important to know how they attacked, and finally solved, the problem of the area of the circle. Antiphon the sophist (425 BC) was one of the early Athenian mathematicians who worked on this problem. He suggested that the area of the circle be calculated in terms of the regular polygons inscribed in it. (The *regular m-gon* is the polygon with m equal sides and angles.)

Using the assumption that the area of the union of pairwise disjoint sets equals the sum of their areas, it is not hard to show that an inscribed square takes up more than $1/2$ the area of a circle, and an inscribed regular octagon takes up more than $3/4$ of the area of the circle. Indeed, as the