to the duality between cycles and bonds in Theorem 6.1.14 and Corollary 8.2.42 and to the use of determinants in the Matrix Tree Theorem (Theorem 2.2.12). For further discussion of these vector spaces, see Biggs [1993, Part 1].

Groups arise in studying graph isomorphism, embeddings, and enumeration. The automorphisms of a graph form a group of permutations of its vertices. Group-theoretic ideas lead to algorithms for testing isomorphism and to constructions for embedding on surfaces. Conversely, every group can be modeled using graphs. An introduction to this interplay appears in White [1973]; see also Gross–Yellen [1999, Chapters 13–15].

We restrict our attention to eigenvalues of adjacency matrices. We interpret the characteristic polynomial in terms of subgraphs, relate the eigenvalues to other graph parameters, and characterize the sets of eigenvalues for bipartite graphs and regular graphs. We close with applications to expander graphs and the "Friendship Theorem". An encyclopedic discussion of graph eigenvalues appears in Cvetković–Doob–Sachs [1979]. Chung [1997] presents the modern approach, modifying the adjacency matrix in a way that normalizes the eigenvalues and yields analogous results that hold more generally. For our brief presentation, we use the classical version.

THE CHARACTERISTIC POLYNOMIAL

8.6.1. Definition. The **eigenvalues** of a matrix A are the numbers λ such that $Ax = \lambda x$ has a nonzero solution vector; each such solution is an **eigenvector** associated with λ. The **eigenvalues** of a graph are the eigenvalues of its adjacency matrix A. These are the roots $\lambda_1, \dots, \lambda_n$ of the **characteristic polynomial** $\phi(G;\lambda) = \det(\lambda I - A) = \prod_{i=1}^n (\lambda - \lambda_i)$. The **spectrum** is the list of distinct eigenvalues with their multiplicities $m_1, \dots, m_t$; we write Spec $(G) = \binom{\lambda_1 \cdots \lambda_t}{m_1 \cdots m_t}$.

8.6.2. Remark. *Elementary properties of eigenvalues.*

0) The eigenvalues are the values λ such that the square matrix $\lambda I - A$ is singular, which is equivalent to $\det(\lambda I - A) = 0$.

1) $\sum \lambda_i = \text{Trace}\, A$. The **trace** is the sum of the diagonal elements and is the coefficient of λ^{n-1} in $\det(\lambda I - A)$. Since $\det(\lambda I - A) = \prod_{i=1}^n (\lambda - \lambda_i)$, that coefficient is also $\sum \lambda_i$. For simple graphs, it is 0.

2) $\prod \lambda_i = (-1)^n \phi(G; 0) = \det A = \sum_\sigma \text{sign}(\sigma) \prod_{i=1}^n a_{i,\sigma(i)}$, where the sum runs over permutations σ of $[n]$.

3) For a symmetric real n-by-n matrix A and $\lambda \in \mathbb{R}$, the multiplicity of λ as an eigenvalue of A is $n - \text{rank}(\lambda I - A)$.

4) Adding c to the diagonal shifts the eigenvalues by c, since $\alpha + c$ is a root of $\det(\lambda I - (cI + A))$ if and only if α is a root of $\det(\lambda I - A)$.

8.6.3. Example. *Spectra of cliques and bicliques.* The adjacency matrix of K_n is $J - I$, where J is the matrix of all 1s. Hence the eigenvalues of K_n are 1 less than those of J. Since Spec $J = \binom{n \quad 0}{1 \quad n-1}$, we have Spec $K_n = \binom{n-1 \quad -1}{1 \quad n-1}$.

The adjacency matrix of $K_{m,n}$ has rank 2, so it has two nonzero eigenvalues λ_1, λ_2. The trace is 0, so $\lambda_1 = -\lambda_2$; call this constant b. Hence $\phi(K_{m,n}, \lambda) = \lambda^n - b^2\lambda^{n-2}$. We compute b using $\phi(G;\lambda) = \det(\lambda I - A)$. Since λ appears only on the diagonal, contributions in the permutation expansion to the coefficient of λ^{n-2} arise only from permutations that use $n-2$ positions on the diagonal. The remaining two positions must be $-a_{i,j}$ and $-a_{j,i}$ for some i, j. There are mn nonzero contributions of this form, all negative. Hence $b^2 = mn$, and $\mathrm{Spec}\,(K_{m,n}) = \binom{\sqrt{mn} \quad 0 \quad -\sqrt{mn}}{1 \quad m+n-2 \quad 1}$. ■

We index the coefficients of the characteristic polynomial so that $\phi(G;\lambda) = \sum_{i=0}^{n} c_i\lambda^{n-i}$. Since $\phi(G;\lambda) = \det(\lambda I - A)$, always $c_0 = 1$ and $c_1 = -\mathrm{Trace}\,A = 0$. Our computation of c_2 for $K_{m,n}$ extends to all graphs.

8.6.4. Definition. A **principal submatrix** of a square matrix A is a submatrix selecting rows and columns with the same indices.

Since contributions to $c_2\lambda^{n-2}$ involve $n-2$ factors of λ from the diagonal, the coefficient c_2 is the sum of the principal 2×2 subdeterminants of $-A$. For a simple graph, $-a_{i,j}$ is -1 when $v_i \leftrightarrow v_j$ and 0 otherwise, so $c_2 = -e(G)$.

Similarly, c_3 is the sum of the principal 3×3 subdeterminants of $-A$. For triple i, j, k, the determinant depends only on the number of edges among v_i, v_j, v_k. The determinant is 0 unless they form a triangle, and then it is -2. Hence c_3 is -2 times the number of 3-cycles in G.

Since principal submatrices are the adjacency matrices of induced subgraphs, in general we have $c_i = (-1)^i \sum_{|S|=i} \det A(G[S])$.

8.6.5. Theorem. (Harary [1962b]) Given a simple graph G, let $\mathbf{H}$ be the set of spanning subgraphs in which every component is an edge or a cycle. If $k(H)$ and $s(H)$ denote the number of components of H and the number of components that are cycles, respectively, then $\det A(G) = \sum_{H\in\mathbf{H}} (-1)^{n(H)-k(H)} 2^{s(H)}$

Proof: The determinant formula is $\det A = \sum_\sigma (-1)^{t(\sigma)} \prod a_{i,\sigma(i)}$, where the sum is over permutations of $[n]$ and $t(\sigma)$ is the number of row exchanges (transpositions) needed to put the positions $i, \sigma(i)$ on the diagonal. When A is a 0,1-matrix, the contribution from σ is nonzero if and only if these entries all equal 1.

We view such a σ as a vertex permutation mapping each v_i to $v_{\sigma(i)}$. This partitions $V(G)$ into orbits. Since $a_{i,\sigma(i)} = 1$ means $v_i \leftrightarrow v_{\sigma(i)}$, there are no orbits of size 1, orbits of size 2 correspond to edges, and longer orbits correspond to cycles. Thus the permutation makes a nonzero contribution when it describes a spanning subgraph H of G in which the components are edges and cycles.

The sign of the contribution is determined by the number of transpositions needed to move the entries to the diagonal. Row exchanges move one element of an orbit at a time to the diagonal, but the last switch moves the last two elements to the diagonal. Hence $t(\sigma) = n(H) - k(H)$. Finally, each cycle of length at least 3 in H can appear in one of two ways in the permutation matrix, since we can follow the cycle in one of two directions. Hence the number of permutations that give rise to H is $2^{s(H)}$. ■

8.6.6. Corollary. (Sachs [1967]) Let $\mathbf{H}_i$ denote the collection of i-vertex subgraphs of a simple graph G whose components are edges or cycles. The characteristic polynomial of G is $\sum c_i\lambda^{n-i}$, where $c_i = \sum_{H\in\mathbf{H}_i}(-1)^{k(H)}2^{s(H)}$.

Proof: This follows from Theorem 8.6.5 and the earlier observation that $c_i = (-1)^i\sum_{|S|=i}\det A(G[S])$. ■

This formula leads to a recursive expression for the characteristic polynomial (Exercise 5). The formula can be used to construct nonisomorphic trees with the same characteristic polynomial (and only eight vertices) (Exercise 7).

We next discuss the properties of eigenvalues for bipartite graphs.

8.6.7. Proposition. The (i, j)th entry of A^k counts the v_i, v_j-walks of length k. The eigenvalues of A^k are the kth powers of the eigenvalues of A.

Proof: The statement about walks holds easily by induction on k (Exercise 1.2.30). For the second statement, $Ax = \lambda x$ implies $A^kx = \lambda^kx$, by repeated multiplication. Using an arbitrary eigenvector x ensures that the multiplicities of the eigenvalues don't change. ■

8.6.8. Lemma. If G is bipartite and λ is an eigenvalue of G with multiplicity m, then $-\lambda$ is also an eigenvalue with multiplicity m.

Proof: Adding isolated vertices to give the partite sets equal size merely adds rows and columns of 0's to the adjacency matrix, which does not change the rank and hence changes the spectrum only by including one extra 0 for each vertex added. Hence we may assume that the partite sets have equal sizes.

Since G is bipartite, we can permute the rows and columns of A to obtain the form $A = \begin{pmatrix} 0 & B \\ B^T & 0\end{pmatrix}$, where B is square. If λ is an eigenvalue associated with eigenvector $v = \binom{x}{y}$ (partitioned according to the bipartition of G), then $\lambda v = Av = \begin{pmatrix} 0 & B \\ B^T & 0\end{pmatrix}\binom{x}{y} = \binom{By}{B^Tx}$. Hence $By = \lambda x$ and $B^Tx = \lambda y$.

Let $v' = \binom{x}{-y}$. We compute $Av' = \binom{B(-y)}{B^Tx} = \binom{-\lambda x}{\lambda y} = -\lambda v'$. Hence v' is an eigenvector of A for the eigenvalue $-\lambda$. Furthermore, m independent eigenvectors for λ yield m independent eigenvectors for $-\lambda$ in this way. Hence $-\lambda$ is an eigenvector of A with the same multiplicity as λ. ■

8.6.9. Theorem. The following are equivalent statements about a graph G.

A) G is bipartite.

B) The eigenvalues of G occur in pairs λ_i, λ_j such that $\lambda_i = -\lambda_j$.

C) $\phi(G;\lambda)$ is a polynomial in λ^2.

D) $\sum_{i=1}^n \lambda_i^{2t-1} = 0$ for any positive integer t.

Proof: We proved A $\Rightarrow$ B in the lemma.

B $\Leftrightarrow$ C: $(\lambda-\lambda_i)(\lambda-\lambda_j) = (\lambda^2 - a)$ if and only if $\lambda_j = -\lambda_i$. Hence the roots occur in such pairs if and only if $\phi(G;\lambda)$ is a product of linear factors in λ^2.

B $\Rightarrow$ D: If $\lambda_j = -\lambda_i$, then $\lambda_j^{2t-1} = -\lambda_i^{2t-1}$.

D $\Rightarrow$ A: Because $\sum\lambda_i^k$ counts the closed k-walks in the graph (from each starting vertex), condition D forbids closed walks of odd length. This forbids odd cycles, since an odd cycle is an odd closed walk, and hence G is bipartite. ■

LINEAR ALGEBRA OF REAL SYMMETRIC MATRICES

Relating eigenvalues to other parameters requires several results from linear algebra, including the Spectral Theorem and Cayley–Hamilton Theorem for real symmetric matrices. These are usually stated in more generality, but adjacency matrices are real and symmetric, and here the theorems have shorter proofs. We begin with a lemma that follows *from* the Spectral Theorem when the latter is proved using complex matrices. The proofs of these results may be skipped, especially by readers well-versed in linear algebra.

8.6.10. Lemma. If $f(x) = x^T Ax$, where A is a real symmetric matrix, then f attains its maximum and minimum over unit vectors x at eigenvectors of A, where it equals the corresponding eigenvalues.

Proof: The function f is continuous in $x_1, \dots, x_n$. For constrained optimization, we use Lagrangian multipliers. Given the constraint $x^T x = 1$, we let $g(x) = x^T x - 1$. Forming $L(x, \lambda) = f(x) - \lambda g(x)$, the extreme values occur where all partial derivatives of L are 0. With respect to λ, this yields $x^T x = 1$.

Let ∇ denote the vector of partial derivatives with respect to $x_1, \dots, x_n$. We compute $\nabla L(x, \lambda) = \nabla f(x) - \lambda \nabla g(x) = 2Ax - 2\lambda x$. The statement $\nabla f(x) = 2Ax$ uses the symmetry of A. We have $\nabla L = 0$ precisely when $Ax = \lambda x$, which requires x to be an eigenvector of A for eigenvalue λ. This yields $f(x) = x^T Ax = \lambda x^T x = \lambda$. ■

Since our variables in the optimization are real, we have found at least one real eigenvector and eigenvalue. We can use this inductively to show that all eigenvectors have this property.

8.6.11. Theorem. (Spectral Theorem) A real symmetric $n \times n$ matrix has real eigenvalues and n orthonormal eigenvectors.

Proof: We use induction on n. The claim is trivial for $n = 1$; consider $n > 1$. Let v_n be the eigenvector maximizing $x^T Ax$. Let W be the orthogonal complement of the space spanned by v_n; it has dimension $n - 1$. If $w \in W$, then $v_n^T Aw = w^T Av_n = \lambda_n w^T v_n = 0$. Hence $Aw \in W$. Viewing multiplication by A as a mapping f_A, we have $f_A \colon W \to W$.

Let S be a matrix whose columns are the vectors of an orthonormal basis of $\mathbb{R}^n$ with v_n as the last column. Since the basis is orthonormal, $S^{-1} = S^T$. The matrix for f_A with respect to this basis is $S^T AS$. Since the basis is orthonormal and v_n is an eigenvector, the last column of $S^T AS$ is 0, except for λ_n in the last position. Furthermore, the matrix is symmetric. Hence its first $n - 1$ rows and columns form the matrix A' for f_A on W with respect to this basis.

By the induction hypothesis, A' has orthonormal eigenvectors $v_1, \dots, v_{n-1}$, with real eigenvalues. Using S, we convert these back into real eigenvectors for A. They have the same real eigenvalues, and they form an orthonormal set. ■

Next we consider polynomial functions of a matrix. Viewed as members of $\mathbb{R}^{n^2}$, the matrices $I, A, A^2, \dots, A^{n^2}$ cannot be independent, since there are $n^2 + 1$

of them. Using an equation of linear dependence, we obtain a polynomial p such that $p(A)$ is the zero matrix. The characteristic polynomial itself suffices. This holds for all A, but again we consider only real symmetric matrices.

8.6.12. Theorem. (Cayley–Hamilton Theorem) If $\phi(\lambda)$ is the characteristic polynomial of a real symmetric matrix A, then $\phi(A)$ is the zero matrix (A "satisfies" its own characteristic polynomial).

Proof: Let the eigenvalues of A be $\lambda_1, \ldots, \lambda_n$, so $\phi(\lambda) = \prod_{i=1}^{n}(\lambda - \lambda_i)$. Since powers of A commute, the matrix polynomial obtained by using A for λ factors as $\phi(A) = \prod_{i=1}^{n}(A - \lambda_i I)$. To prove that $\phi(A) = 0$, we need only show that the matrix $\phi(A)$ maps every vector to 0. Write an arbitrary vector x as a linear combination of the basis of eigenvectors guaranteed by the Spectral Theorem. Applying $A - \lambda_i I$ kills the coefficient of v_i. Successively multiplying by all the factors $A - \lambda_i I$ produces the zero vector. ∎

8.6.13. Definition. The **minimum polynomial** ψ of a matrix A is the polynomial of minimum degree satisfied by A and having leading coefficient 1. When A is the adjacency matrix of G, we call this the **minimum polynomial** $\psi(G; \lambda)$ of G.

The minimum polynomial is unique: if A satisfies two such polynomials of the same degree, then A satisfies their difference, which has lower degree.

8.6.14. Theorem. The minimum polynomial of A is $\psi(A) = \prod_{i=1}^{t}(\lambda - \lambda_i)$, where $\{\lambda_1, \ldots, \lambda_t\}$ are the distinct eigenvalues of A.

Proof: The minimum polynomial divides every polynomial satisfied by A, since otherwise the remainder would be a polynomial of lower degree satisfied by A. The Cayley–Hamilton Theorem now implies that ψ divides ϕ and must be the product of some of its factors. Killing the vectors in the subspace of eigenvectors for eigenvalue λ_i requires a factor of the form $A - \lambda_i I$. This factor kills all vectors in that subspace, so we only need one copy of each such factor. ∎

8.6.15. Lemma. (Sylvester's Law of Inertia) Let A be a real symmetric matrix. If $x^T Ax$ can be written as a sum of N products of linear expressions, that is $x^T Ax = \sum_{m=1}^{N}(\sum_{i\in S_m} a_{i,m}x_i)(\sum_{j\in T_m} b_{j,m}x_j)$, then N is at least the maximum of the number of positive and the number of negative eigenvalues of A.

Proof: (Tverberg [1982]) Write the linear expressions as $u_m(x)$ and $v_m(x)$. For each m, we have $u_m(x)v_m(x) = L_m^2(x) - M_m^2(x)$, where $L = \frac{1}{2}(u + v)$ and $M = \frac{1}{2}(u - v)$ are also linear combinations of $x_1, \ldots, x_n$. This expresses the quadratic form as $x^T Ax = \sum_{m=1}^{N}[L_m^2(x) - M_m^2(x)]$.

On the other hand, A is a real symmetric matrix and thus has orthonormal eigenvectors $w^1, \ldots, w^n$. Using this, we write $x^T Ax = x^T S\Lambda S^T x$, where Λ is the diagonal matrix of eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$ and S has columns $w^1, \ldots, w^n$. If S has p positive and q negative eigenvalues, then this becomes $x^T Ax = \sum_{i=1}^{p}(y^i \cdot x)^2 - \sum_{i=n-q+1}^{n}(z^i \cdot x)^2$, where each y^i or z^i is $|\lambda_i|^{1/2}\, w^i$.

Now we consider a homogeneous system of linear equations. We require $L_m(x) = 0$ for $1 \le m \le N$, also $z^i \cdot x = 0$ for $n-q < i \le n$, and $w^i \cdot x = 0$ for $p < i \le n-q$. This places $N + n - p$ homogeneous linear constraints on n variables. If $N < p$, then these equations have a nonzero simultaneous solution x'. Setting x to x' in the two expressions for $x^T Ax$ yields $\sum_{i=1}^{p}(y^i \cdot x')^2 = -\sum_{m=1}^{N} M_m^2(x')$. Since x' is orthogonal to all eigenvectors with nonpositive eigenvalues, the left side is positive, while the right is nonpositive. The contradiction yields $N \ge p$; an analogous argument yields $N \ge q$. ■

EIGENVALUES AND GRAPH PARAMETERS

Eigenvalues provide bounds on various parameters, or alternatively graph parameters yield bounds on the eigenvalues. Our first result uses only the minimum polynomial.

8.6.16. Theorem. The diameter of a graph G is less than the number of distinct eigenvalues of G.

Proof: Let A be the adjacency matrix; A satisfies a polynomial of degree r if and only if some linear combination of $A^0, \dots, A^r$ is 0. Since the number of distinct eigenvalues is the degree of the minimum polynomial, we need only show that $A^0, \dots, A^k$ are linearly independent when $k \le \text{diam}(G)$.

It suffices to show for $k \le \text{diam}(G)$ that A^k is not a linear combination of $A^0, \dots, A^{k-1}$. Choose $v_i, v_j \in V(G)$ such that $d(v_i, v_j) = k$. By counting walks, we have $A_{i,j}^k \ne 0$ but $A_{i,j}^t = 0$ for $t < k$. Therefore, A^k is not a linear combination of the smaller powers. ■

Since the Spectral Theorem guarantees real eigenvalues, we can index our eigenvalues as $\lambda_1 \ge \cdots \ge \lambda_n$. We also refer to λ_1 and λ_n as $\lambda_{\max}(G)$ and $\lambda_{\min}(G)$.

8.6.17. Lemma. If G' is an induced subgraph of G, then

$$\lambda_{\min}(G) \le \lambda_{\min}(G') \le \lambda_{\max}(G') \le \lambda_{\max}(G) .$$

Proof: Since A is a real symmetric matrix, Lemma 8.6.10 yields $\lambda_{\min}(A) \le x^T Ax \le \lambda_{\max}(A)$ for every unit vector x. Consider the adjacency matrix A' of G'. By permuting the vertices of G, we can view A' as an upper left principal submatrix of $A = A(G)$. Let z' be the unit eigenvector of A' such that $A'z' = \lambda_{\max}(G')z'$. Let z be the unit vector in R_n obtained by appending zeros to z'. Then $\lambda_{\max}(G') = z'^T A'z' = z^T Az \le \lambda_{\max}(G)$. Similarly, $\lambda_{\min}(G') \ge \lambda_{\min}(G)$. ■

The behavior of the extreme eigenvalues under vertex deletion is a special case of the "Interlacing Theorem": If G has eigenvalues $\lambda_1 \ge \cdots \ge \lambda_n$ and $G - x$ has eigenvalues $\mu_1 \ge \cdots \ge \mu_{n-1}$, then $\lambda_1 \ge \mu_1 \ge \lambda_2 \ge \cdots \ge \mu_{n-1} \ge \lambda_n$. We will not need this and hence omit the proof, which involves only linear algebra.

8.6.18. Lemma. For every graph G, $\delta(G) \le \frac{2e(G)}{n(G)} \le \lambda_{\max}(G) \le \Delta(G)$.

Proof: Let x be an eigenvector for eigenvalue λ, and let $x_j = \max_i x_i$ be the largest coordinate value in x. Then $\lambda \le \Delta(G)$ follows from

$$\lambda x_j = (Ax)_j = \sum_{v_i \in N(v_j)} x_i \le d(v_j)x_j \le \Delta(G)x_j .$$

For the lower bound, we apply Lemma 8.6.10 to the unit vector with equal coordinates. Since the sum of the entries in the adjacency matrix is twice the number of edges of G, we have

$$\lambda_{\max} \ge \frac{\mathbf{1}_n^T}{\sqrt{n}} A \frac{\mathbf{1}_n}{\sqrt{n}} = \frac{1}{n}\sum\sum a_{ij} = \frac{2e(G)}{n}. \qquad \blacksquare$$

Lemma 8.6.18 enables us to improve the trivial bound $\chi(G) \le 1 + \Delta(G)$ given by the greedy coloring algorithm. Replacing $\Delta(G)$ with the average degree is too small; $K_n + K_1$ has chromatic number n and average degree less than $n-1$. Since $\lambda_{\max}$ is always at least the average degree, $1 + \lambda_{\max}(G)$ has a chance to work and can't be much improved.

8.6.19. Theorem. (Wilf [1967]) For every graph G, $\chi(G) \le 1 + \lambda_{\max}(G)$.

Proof: If $\chi(G) = k$, then we can successively delete vertices without reducing the chromatic number until we obtain a subgraph H such that $\chi(H - v) = k - 1$ for all $v \in V(H)$. As observed in Lemma 5.1.18, $\delta(H) \ge k - 1$. Since H is an induced subgraph of G, Lemma 8.6.18 and then Lemma 8.6.17 yield

$$k \le 1 + \delta(H) \le 1 + \lambda_{\max}(H) \le 1 + \lambda_{\max}(G) . \qquad \blacksquare$$

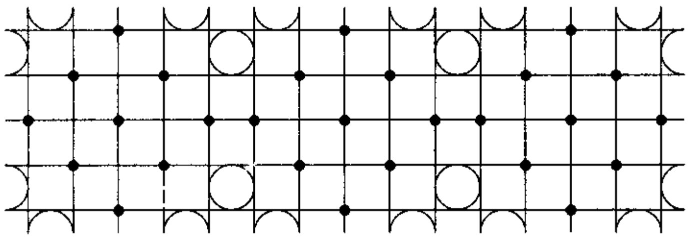
Sylvester's Law of Inertia yields a lower bound on the number of bicliques needed to decompose a graph. Because stars are bicliques and every subgraph of a star is a star, the number of bicliques needed is at most the vertex cover number $\beta(G) = n(G) - \alpha(G)$. Erdős conjectured that equality almost always holds, but this remains open. Graphs with special structure may have efficient partitions using other bicliques. The general lower bound using eigenvalues appears explicitly in Reznick–Tiwari–West [1985], but it is implicit in earlier work decomposing the complete graph (Tverberg [1982], Peck [1984]).

8.6.20. Theorem. For a simple graph G, the number of bicliques needed to decompose G is at least the maximum of the number of positive and number of negative eigenvalues of the adjacency matrix $A(G)$.

Proof: When G decomposes into subgraphs $G_1, \ldots, G_t$, we may write $A(G) = \sum_{i=1}^t B_i$, where B_i is the adjacency matrix of the spanning subgraph of G with edge set $E(G_i)$. When G_i is the biclique with bipartition S_i, T_i, we have $x^T B_i x = 2\sum_{j\in S_i} x_j \sum_{k\in T_i} x_k$. Writing these linear expressions as $u_i(x) = \sqrt{2}\sum_{j\in S_i} x_j$ and $v_i(x) = \sum_{k\in T_i} x_k$, we have $x^T Ax = \sum_{i=1}^t x^T B_i x = \sum_{i=1}^t u_i(x)v_i(x)$. Sylvester's Law of Inertia (Lemma 8.6.15) now yields the claimed lower bound. $\blacksquare$

8.6.21. Example. *Biclique decomposition of* $C_{(2t+1)n} \square C_n$. There are simple formulas for the eigenvalues of a cycle (Exercise 6) and for computing the eigenvalues of a cartesian product from the eigenvalues of the factors (Exercise 10). These yield simple formulas for the numbers of positive and negative eigenvalues of $C_m \square C_n$ when m is an odd multiple of n. In particular, $C_{(2t+1)n} \square C_n$ has $(2t+1)(n^2+1)/2$ positive eigenvalues and $(2t+1)(n^2-1)/2$ negative eigenvalues when n is odd (0 is not an eigenvalue).

Furthermore, such a product decomposes into $(2t+1)(n^2+1)/2$ bicliques, consisting of $(2t+1)(n-1)/2$ 4-cycles and $(2t+1)(n+1)/2$ stars (Kratzke–West). Note that 4-cycles and stars are the only subgraphs of $C_m \square C_n$ that are bicliques. The optimal decomposition of $C_{15} \square C_5$ appears below. Edges wrap around from top to bottom and right to left, and all grid points indicate vertices. The heavy dots indicate vertices that are centers of stars in the decomposition, and the circles indicate 4-cycles in the decomposition. ■



EIGENVALUES OF REGULAR GRAPHS

Like bipartite graphs, regular graphs can be characterized using spectra. The n-vector $\mathbf{1}_n$ with all coordinates 1 plays a special role in this and many other arguments involving eigenvalues, as does the matrix J of all 1s.

8.6.22. Theorem. The eigenvalue of G with largest absolute value is $\Delta(G)$ if and only if some component of G is $\Delta(G)$-regular. The multiplicity of $\Delta(G)$ as an eigenvalue is the number of $\Delta(G)$-regular components.

Proof: Let A be the adjacency matrix. The ith entry of $A\mathbf{1}_n$ is $d(v_i)$. When G is k-regular, we obtain $A\mathbf{1}_n = k\mathbf{1}_n$, and thus k is an eigenvalue with eigenvector $\mathbf{1}_n$. In general, let x be an eigenvector for eigenvalue λ, and let x_j be a coordinate of largest absolute value among coordinates of x corresponding to the vertices of some component H of G. For the jth coordinate of Ax, we have

$$|\lambda|\,|x_j| = |(Ax)_j| = \left|\sum_{v_i \in N(v_j)} x_i\right| \le d(v_j)\,|x_j| \le \Delta(G)\,|x_j| \,.$$

Hence $|\lambda| \le \Delta(G)$. Equality requires $d(v_j) = \Delta(G)$ and $x_i = x_j$ for all $v_i \in N(v_j)$. We can iterate this argument to reach all coordinates for vertices in H. Hence the eigenvalue associated with x has absolute value as large as $\Delta(G)$ only if H is $\Delta(G)$-regular.

Thus the eigenvalue associated with an eigenvector x has absolute value as large as $\Delta(G)$ if and only if (1) each component of G containing a vertex where x is nonzero is $\Delta(G)$-regular, and (2) x is constant on the coordinates corresponding to each such component. We can choose the constant independently for each $\Delta(G)$-regular component, so the dimension of the space of eigenvectors associated with $\Delta(G)$ is the number of $\Delta(G)$-regular components. ■

When G is connected and not regular, it remains true that eigenvalues of largest absolute value have multiplicity 1 and that coordinates of the associated eigenvector have the same sign. This is related to the Perron–Frobenius Theorem of linear algebra and uses arguments like those above; we omit the proof.

Powers of the adjacency matrix yield another characterization.

8.6.23. Theorem. (Hoffman [1963]) A graph G is regular and connected if and only if J is a linear combination of powers of $A(G)$.

Proof: *Sufficiency.* If J can be so expressed, then for each i, j we have $(A^k)_{ij} \neq 0$ for some $k \geq 0$, which requires a v_i, v_j-walk of length k. Hence G is connected. For regularity, consider the matrices JA and AJ. The i,jth position of AJ is $d(v_i)$ (constant on rows), and the i,jth position of JA is $d(v_j)$ (constant on columns). Since J is a linear combination of powers of A, each of which commutes with A, we have $JA = AJ$. Thus the i,jth position is both $d(v_i)$ and $d(v_j)$ and the graph is regular.

Necessity. Since G is k-regular, k is an eigenvalue, and the minimum polynomial is $\psi(G;\lambda) = (\lambda - k)g(\lambda)$ for some polynomial g. Since $\psi(G;A) = 0$, we have $Ag(A) = kg(A)$. Hence each column of $g(A)$ is an eigenvector of A with eigenvalue k. Since G is regular and connected, each such eigenvector is a multiple of $\mathbf{1}_n$. Hence the columns of $g(A)$ are constant. However, $g(A)$ is a linear combination of powers of a symmetric matrix and therefore must itself be symmetric. Hence the columns are equal and $g(A)$ is a multiple of J. ■

When G is simple and regular, $\overline{G}$ is also regular, and the eigenvalues of $\overline{G}$ can be obtained from the eigenvalues of G. This rests on the matrix expression for complementation: $A(\overline{G}) = J - I - A(G)$.

8.6.24. Lemma. $\phi(\overline{G};\lambda) = (-1)^n \det[(-\lambda - 1)I - A(G) + J]$.

Proof: Direct computation yields $\det(\lambda I - A(\overline{G})) = \det(\lambda I - (J - I - A)) = \det[(\lambda+1)I - J + A] = (-1)^n \det[(-\lambda - 1)I - A + J]$. ■

8.6.25. Theorem. If a simple graph G is k-regular, then G and $\overline{G}$ have the same eigenvectors. The eigenvalue associated with $\mathbf{1}_n$ is k in G and $n-k-1$ in $\overline{G}$. If $x \neq \mathbf{1}_n$ is an eigenvector of G for eigenvalue λ of G, then its associated eigenvalue in $\overline{G}$ is $-1-\lambda$.

Proof: Since $\overline{G}$ is $n-k-1$-regular, $\mathbf{1}_n$ is an eigenvector for both G and $\overline{G}$, with eigenvalue k for G and $n-k-1$ for $\overline{G}$. Let x be another eigenvector of G in an orthonormal basis of eigenvectors, and let $\overline{A} = A(\overline{G})$. Since $\mathbf{1}_n \cdot x = 0$, $\sum x_i = 0$. We compute $\overline{A}x = Jx - x - Ax = 0 - x - Ax = (-1-\lambda)x$. ■

This yields a lower bound on the smallest eigenvalue of a regular graph and another derivation of the spectrum of K_n.

8.6.26. Corollary. For a k-regular simple graph, $\lambda_n \geq k - n$.

Proof: If G is k-regular and $\lambda_1 \geq \cdots \geq \lambda_n$, then the eigenvalues of $\overline{G}$ are $(n-k-1, -1-\lambda_n, \ldots, -1-\lambda_2)$, by Theorems 8.6.22–8.6.25. In particular, $n-k-1 \geq -\lambda_n - 1$. ■

The eigenvalues of a connected regular simple graph G can be used to count its spanning trees. The eigenvalues need not be rational, yet the result $\tau(G)$ is an integer. The Matrix Tree Theorem (Theorem 2.2.12) says that $\tau(G)$ equals each minor of $Q = D - A$, where A is the adjacency matrix and D is the diagonal matrix of degrees. When G is k-regular, $D = kI$. Letting $\mathrm{Adj}\,Q$ denote the adjugate matrix of Q (the matrix of signed cofactors), the Matrix Tree Theorem is the statement that $\mathrm{Adj}\,Q = \tau(G)J$. Using Cayley's Formula (Theorem 2.2.3) for spanning trees in K_n, we have $\mathrm{Adj}\,(nI - J) = n^{n-2}J$.

8.6.27. Lemma. Let D be the diagonal matrix of vertex degrees in a simple graph G, let $A = A(G)$, and let $Q = D - A$. The number of spanning trees of G is $\tau(G) = \det(J + Q)/n^2$.

Proof: Observe that $J^2 = nJ$, $JQ = 0$, and $\mathrm{Adj}\,(AB) = \mathrm{Adj}\,(A)\,\mathrm{Adj}\,(B)$. We apply this using $J + Q$ and the matrix $nI - J$ that arises from K_n. We have

$$\mathrm{Adj}\,(nI - J)\,\mathrm{Adj}\,(J + Q) = \mathrm{Adj}\,[(nI - J)(J + Q)] = \mathrm{Adj}\,(nQ),$$

since $J^2 = nJ$ and $JQ = 0$. We have computed that $\mathrm{Adj}\,(nI - J) = n^{n-2}J$. Also, $\mathrm{Adj}\,(nQ) = n^{n-1}\,\mathrm{Adj}\,Q$ for any matrix Q. Canceling common factors of n yields $J\mathrm{Adj}\,(J + Q) = n\tau(G)J$. Multiplying both sides of this on the right by $(J + Q)^T$ yields $J(\det(J + Q)I) = n\tau(G)nJ$. Both sides are multiples of J, so the desired equality holds. ■

We can now compute $\tau(G)$ from the eigenvalues if G is regular. (This analysis extends to all graphs when the modified system of eigenvalues is used.)

8.6.28. Theorem. If G is a k-regular connected simple n-vertex graph with spectrum $\binom{k\ \ \lambda_2 \cdots \lambda_t}{1\ \ m_2 \cdots m_t}$, then $\tau(G) = n^{-1}\phi'(G;k) = n^{-1}\prod_{j=2}^{t}(k - \lambda_j)^{m_j}$.

Proof: Since $J + Q = J + kI - A$, the determinant of $J + Q$ is the value at k of the characteristic polynomial of $A - J$. Since G is k-regular and connected, it has $\mathbf{1}_n$ as an eigenvector with eigenvalue k, and the other eigenvectors are orthogonal to $\mathbf{1}_n$. Every such eigenvector of A is also an eigenvector of $A - J$, with the same eigenvalue, since $(A - J)x = Ax - Jx = Ax = \lambda x$.

Also, $\mathbf{1}_n$ is an eigenvector of $A - J$ with eigenvalue $k - n$. This produces a full set of eigenvalues for $A - J$. Evaluating the characteristic polynomial at k yields $\det(J + Q) = n\prod_{j=2}^{t}(k - \lambda_j)$. The product is $\phi'(G;k)$, since $\phi(G;\lambda)$ has $\lambda - k$ as a non-repeated factor when G is k-regular and connected. By Lemma 8.6.27, we obtain $\tau(G)$ upon dividing by n^2. ■

The results in Lemma 8.6.24–Theorem 8.6.28 were extended to arbitrary (non-regular) graphs in Kelmans [1967b] (also Kelmans–Chelnokov [1974]) using the eigenvalues of the Laplacian matrix of the graph. This is the matrix Q used above. Another method for counting spanning trees appears in Kelmans [1965, 1966], and another variation on the Matrix Tree Theorem appears in Hartsfield–Kelmans–Shen [1996].

EIGENVALUES AND EXPANDERS

Many applications in computer science require "expander graphs". Walters [1996] collects many definitions that have been used for such graphs. The basic notion of expansion is that all small sets should have large neighborhoods. The aim is to establish good connectivity properties without having many edges.

8.6.29. Definition. An (n, k, c)-**expander** is an X, Y-bigraph G with $|X| = |Y| = n$ such that $\Delta(G) \le k$ and that $|N(S)| \ge (1 + c(1 - |S|/n)) \cdot |S|$ for every $S \subseteq X$ with $|S| \le n/2$. An (n, k, c)-**magnifier** is an n-vertex graph G such that $\Delta(G) \le k$ and that $\left|N(S) \cap \overline{S}\right| \ge c \cdot |S|$ for every $S \subseteq V(G)$ with $|S| \le n/2$. An n-**superconcentrator** is an acyclic digraph with n sources and n sinks such that for every set A of sources and every set B of $|A|$ sinks, there are $|A|$ disjoint A, B-paths.

Expanders appear in the parallel sorting network of Ajtai, Komlós, and Szemerédi [1983]. The condition for expansion strengthens Hall's Condition; we have not one matching but many. This facilitates using expanders to construct superconcentrators. Applications of superconcentrators are discussed in Alon [1986a]. The bound on maximum degree makes the number of edges linear in n, thereby limiting the cost of constructing the network.

Probabilistic methods (Exercise 22) yield the *existence* of expanders (and superconcentrators) with large n and bounded average degree (Pinsker [1973]), Pippenger [1977], Chung [1978b]). Margulis [1973] used algebraic ideas to construct an explicit example (see also Gabber–Galil [1981]).

Although an appropriately generated random graph will almost always have good expansion properties, it is hard to measure expansion. Tanner [1984] and Alon–Milman [1984, 1985] independently used eigenvalues to remedy this. They proved that graphs have good expansion properties when the two largest eigenvalues are far apart. Since eigenvalues are easy to compute (or approximate), we can generate a graph randomly and then compute its eigenvalues to check the amount of expasion.

We consider only the special case of regular graphs. Expanders are more useful than magnifiers in applications, but it is easy to obtain an $(n, (k+1), c)$-expander from an (n, k, c)-magnifier (Exercise 21). Hence we consider the relationship between eigenvalues and magnification. Our presentation follows that of Alon–Spencer [1992, p119ff], which discusses additional properties of the eigenvalues of regular (and random) graphs.

8.6.30. Theorem. If G is a k-regular n-vertex graph with second-largest eigenvalue λ, and S is a nonempty proper subset of $V(G)$, then

$$\left|[S,\overline{S}]\right| \ge (k-\lambda)\,|S|\,\left|\overline{S}\right|/n.$$

Proof: Since G is k-regular, $\lambda_{\max}(G) = k$. The claim is trivial if $k-\lambda = 0$, so we may assume that G is connected. We compute

$$x^T(kI-A)x \;=\; k\sum x_i^2 - 2\sum_{ij\in E(G)} x_ix_j \;=\; \sum_{ij\in E(G)} (x_i - x_j)^2.$$

Now let $s = |S|$ and set $x_i = -(n-s)$ for $i \in S$ and $x_i = s$ for $i \notin S$. The sum on the right above becomes $n^2\left|[S,\overline{S}]\right|$.

Because $|S| = s$ implies $\sum x_i = 0$, the vector x is orthogonal to the eigenvector $\mathbf{1}_n$ of A with eigenvalue k. The eigenvector $\mathbf{1}_n$ is also the eigenvector of $kI - A$ for its smallest eigenvalue 0. Using Lemma 8.6.10 and Theorem 8.6.11, the minimum of $\frac{x^T(kI-A)x}{x^Tx}$ over vectors orthogonal to 1_n is the next smallest eigenvalue of $kI - A$, which is $k - \lambda$. Hence

$$x^T(kI-A)x \;\ge\; (k-\lambda)x^Tx \;=\; (k-\lambda)\left(s(n-s)^2 + (n-s)s^2\right) \;=\; (k-\lambda)s(n-s)n.$$

Since $x^T(kI-A)x = n^2\left|[S,\overline{S}]\right|$, we have $\left|[S,\overline{S}]\right| \ge (k-\lambda)s(n-s)/n$. ■

8.6.31. Corollary. If G is a k-regular n-vertex graph with second-largest eigenvalue λ, then G is an (n,k,c)-magnifier, where $c = (k-\lambda)/2k$.

Proof: If S is a set of $s \le n/2$ vertices in G, then Theorem 8.6.30 yields $\left|[S,\overline{S}]\right| \ge$ $k-\lambda)s(n-s)/n$. Each vertex of $\overline{S}$ receives at most k of these edges, so S must have at least $(k-\lambda)s(n-s)/(nk)$ neighbors in $\overline{S}$. Since $(n-s)/n \ge 1/2$, the result follows. ■

Greater separation between the two largest eigenvalues yields greater magnification. Alon and Milman [1984] improved the lower bound to $c \ge (2k-2\lambda)/(3k-2\lambda)$. Alon [1986b] proved a partial converse: If a k-regular graph G is an (n,k,c)-magnifier, then the separation $k-\lambda$ is at least $c^2/(4+2c^2)$.

Explicit constructions of regular graphs are known with separation between λ_1 and λ_2 nearly as large as possible. The second largest eigenvalue of a k-regular graph with diameter d is at least $2\sqrt{k}-1(1-O(1/d))$ (see Nilli [1991]). Lubotzky–Phillips–Sarnak [1986] and Margulis [1988] constructed infinite families of regular graphs where the degree k is 1 more than a prime congruent to 1 mod 4 and the second largest eigenvalue is at most $2\sqrt{k}-1$.

STRONGLY REGULAR GRAPHS

We close with an application to a special class of regular graphs.

8.6.32. Definition. A simple n-vertex graph G is **strongly regular** if there are parameters k, λ, μ such that G is k-regular, every adjacent pair of vertices have λ common neighbors, and every nonadjacent pair of vertices have μ common neighbors.

Properties of eigenvalues of strongly regular graphs provide a short proof of a curious result called the "Friendship Theorem". This theorem says that at any party at which every pair of people have exactly one common acquaintance, there is one person who knows everyone (presumably the host). The resulting graph of the acquaintance relation consists of some number of triangles sharing a vertex. Another motivation for studying strongly regular graphs is their connection with the theory of designs. Strongly regular graphs with $\lambda = \mu$ correspond to symmetric balanced incomplete block designs. Other regular graphs with rich algebraic structure appear in Biggs [1993, part 3].

8.6.33. Theorem. If G is a strongly regular graph with n vertices and parameters k, λ, μ, then $\overline{G}$ is strongly regular with parameters $k' = n - k - 1$, $\lambda' = n - 2 - 2k + \mu$, and $\mu' = n - 2k + \lambda$.

Proof: For each adjacent pair $v \leftrightarrow w$ in G, there are $2(k-1) - \lambda$ other vertices in $N(v) \cup N(w)$, so v and w have $n - 2 - 2(k-1) + \lambda$ common nonneighbors. When $v \not\leftrightarrow w$ there are $2k - \mu$ vertices in $N(v) \cup N(w)$ and thus $n - 2k + \mu$ common nonneighbors. ■

8.6.34. Theorem. If G is a strongly regular graph with n vertices and parameters k, λ, μ, then $k(k - \lambda - 1) = \mu(n - k - 1)$.

Proof: We count induced copies of P_3 with a fixed vertex v as an endpoint. The middle vertex w can be picked in k ways. For each such w, the third vertex can be any neighbor of w not adjacent to v. With v unavailable, there are always $k - \lambda - 1$ ways to pick the third vertex. On the other hand, the third vertex can be picked in $n - k - 1$ ways as a nonneighbor of v, and for each such choice there are μ common neighbors with v that can serve as w. ■

8.6.35. Example. *Degenerate cases:* $\mu = 0$ *or* $\lambda = k - 1$ *or* $k = n - 1$. We show that such a strongly regular graph is a disjoint union of $k+1$-cliques. By Theorem 8.6.34, $\lambda = k - 1$ if and only if $\mu = 0$ or $k = n - 1$. Hence we may assume that $\lambda = k - 1$. Now every neighbor of v is adjacent to every other, which forbids an induced P_3 and forces G to be a disjoint union of cliques. ■

Henceforth, we assume that $\mu > 0$ and $\lambda < k - 1$. Theorem 8.6.34 states a necessary condition on the set of parameters for a strongly regular graph. Another necessary condition arises from the eigenvalues.

8.6.36. Theorem. (Integrality Condition) If G is strongly regular with n vertices and parameters k, λ, μ, then the two numbers below are nonnegative integers.

$$\frac{1}{2}\left(n - 1 \pm \frac{(n-1)(\mu - \lambda) - 2k}{\sqrt{(\mu - \lambda)^2 + 4(k - \mu)}}\right)$$

Proof: These are nonnegative integers because they are multiplicities of eigenvalues. Consider A^2. The ijth entry of A^2 is k if $i = j$, is λ if $v_i \leftrightarrow v_j$, and is μ if $v_i \not\leftrightarrow v_j$. Since $v_i \leftrightarrow v_j$ marks the 1s in the adjacency matrix and $v_i \not\leftrightarrow v_j$ marks the 1s in the adjacency matrix of the complement, we have $A^2 = kI + \lambda A + \mu(J - I - A)$. Rearranging terms yields $A^2 = (k-\mu)I + (\lambda-\mu)A + \mu J$.

Multiplying $\mathbf{1}_n$ by both expressions for A^2 yields

$$k^2\mathbf{1}_n = (k-\mu)\mathbf{1}_n + (\lambda-\mu)k\mathbf{1}_n + \mu n\mathbf{1}_n,$$

which yields another proof of $k(k-\lambda-1) = \mu(n-k-1)$. Let x be an eigenvector for another eigenvalue $\theta \neq k$. Since x is orthogonal to $\mathbf{1}_n$, we have $Jx = \mathbf{0}_n$. Multiplying x by both expressions for A^2 produces $\theta^2 - (\lambda-\mu)\theta - (k-\mu) = 0$. This quadratic equation for θ has two roots r, s, which must be the values of all the other eigenvalues. The values are $\frac{1}{2}\left(\lambda - \mu \pm \sqrt{(\lambda-\mu)^2 + 4(k-\mu)}\right)$.

Now let a and b be the multiplicities of the eigenvalues r and s. Example 8.6.35 describes all cases when $\mu = 0$. Hence we may assume that $\mu > 0$, and thus G is connected. Because G is connected, eigenvalue k has multiplicity 1, and we have $1 + a + b = n$. Since the eigenvalues sum to 0, we have $k + ra + sb = 0$. The solution to these two linear equations for a and b is $a = -\frac{k+s(n-1)}{r-s}$ and $b = \frac{k+r(n-1)}{r-s}$. These are the values claimed above to be nonnegative integers. ■

The argument above can also be traced in the opposite direction.

8.6.37. Theorem. A k-regular connected graph G is strongly regular with parameters k, λ, μ if and only if it has exactly three eigenvalues $k > r > s$ and these satisfy $r + s = \lambda - \mu$ and $rs = -(k - \mu)$. ■

8.6.38. Example. *Classes of strongly regular graphs.* We consider two cases: $(n-1)(\mu-\lambda) = 2k$ and $(n-1)(\mu-\lambda) \neq 2k$. Excluding the trivial values, the first case requires $\mu = \lambda + 1$, because $0 < 2k < 2n - 2$. By Theorem 8.6.33, G and $\overline{G}$ are thus strongly regular graphs with the same parameters. In this case, we also know that $n = 4\mu + 1$ and that n is the sum of two perfect squares. Furthermore, the eigenvalues r and s have the same multiplicity.

In the second case, rationality requires $(\mu-\lambda)^2 + 4(k-\mu) = d^2$ for some positive integer d, and d must divide $(n-1)(\mu-\lambda) - 2k$. Here the eigenvalues must be integers. Various such examples are known. In the special case $\lambda = 0$ and $\mu = 2$, three such graphs are known, but it is not known whether the list is finite! The known examples, listing the parameters (n, k, λ, μ), are the square $(4, 2, 0, 2)$, the Clebsch graph $(16, 5, 0, 2)$, and the Gewirtz graph $(56, 10, 0, 2)$ (see Cameron–van Lint [1991], p43). The Clebsch graph arises in Exercise 23. Other strongly regular graphs appear in Exercises 24–26. ■

Finally, we prove the Friendship Theorem. It is startling that such a combinatorial-sounding result seems to have no short combinatorial proof. There do exist proofs avoiding eigenvalues (see Hammersley [1983]), but they require complicated numerical arguments to eliminate regular graphs.