Lemma 7. If $|\mathcal{K}|$ is a power of a prime for some nonidentity conjugacy class $\mathcal{K}$ of G, then G is not a non-abelian simple group.

Proof: Suppose to the contrary that G is a non-abelian simple group and let $|\mathcal{K}| = p^c$. Let $g \in \mathcal{K}$. If $c = 0$ then $g \in Z(G)$, contrary to a non-abelian simple group having a trivial center. As above, let $\chi_1, \dots, \chi_r$ be all the irreducible characters of G with χ_1 the principal character and let ρ be the regular character of G. By decomposing ρ into irreducibles we obtain

$$0 = \rho(g) = 1 + \sum_{i=2}^{r} \chi_i(1)\chi_i(g). \tag{19.3}$$

If $p \mid \chi_j(1)$ for every $j > 1$ with $\chi_j(g) \neq 0$, then write $\chi_j(1) = pd_j$. In this case (3) becomes

$$0 = 1 + p\sum_j d_j\chi_j(g).$$

Thus $\sum_j d_j\chi_j(g) = -1/p$ is an algebraic integer, a contradiction. This proves there is some j such that p does not divide $\chi_j(1)$ and $\chi_j(g) \neq 0$. If φ is a representation whose character is χ_j, then φ is faithful (because G is assumed to be simple) and, by Lemma 6, $\varphi(g)$ is a scalar matrix. Since $\varphi(g)$ commutes with all matrices, $\varphi(g) \in Z(\varphi(G))$. This forces $g \in Z(G)$, contrary to G being a non-abelian simple group. The proof of the lemma is complete.

We now prove Burnside's Theorem. Let G be a group of order $p^a q^b$ for some primes p and q. If $p = q$ or if either exponent is 0 then G is nilpotent hence solvable. Thus we may assume this is not the case. Proceeding by induction let G be a counterexample of minimal order. If G has a proper, nontrivial normal subgroup N, then by induction both N and G/N are solvable, hence so is G (cf. Section 3.4 or Proposition 6.10). Thus we may assume G is a non-abelian simple group. Let $P \in Syl_p(G)$. By Theorem 8 of Chapter 4 there exists $g \in Z(P)$ with $g \neq 1$. Since $P \leq C_G(g)$, the order of the conjugacy class of g (which equals $|G : C_G(g)|$) is prime to p, i.e., is a power of q. This violates Lemma 7 and so completes the proof of Burnside's Theorem.

Philip Hall's Theorem

Recall that a subgroup of a finite group is called a *Hall subgroup* if its order and index are relatively prime. For any subgroup H of a group G a subgroup K such that $G = HK$ and $H \cap K = 1$ is called a *complement* to H in G.

Theorem 8. (P. Hall) Let G be a group of order $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}$ where $p_1, \dots, p_t$ are distinct primes. If for each $i \in \{1, \dots, t\}$ there exists a subgroup H_i of G with $|G : H_i| = p_i^{\alpha_i}$, then G is solvable.

Hall's Theorem can also be phrased: if for each $i \in \{1, \dots, t\}$ a Sylow p_i-subgroup of G has a complement, then G is solvable. The converse to Hall's Theorem is also true — this was Exercise 33 in Section 6.1.

We shall first need some elementary lemmas.

Lemma 9. If G is solvable of order > 1, then there exists $P \trianglelefteq G$ with P a nontrivial p-group for some prime p.

Proof: This is a special case of the exercise on minimal normal subgroups of solvable groups at the end of Section 6.1. One can see this easily by letting P be a nontrivial Sylow subgroup of the last nontrivial term, $G^{(n-1)}$, in the derived series of G (where G has solvable length n). In this case $G^{(n-1)}$ is abelian so P is a characteristic subgroup of $G^{(n-1)}$, hence is normal in G.

Lemma 10. Let G be a group of order $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}$ where $p_1, \ldots, p_t$ are distinct primes. Suppose there are subgroups H and K of G such that for each $i \in \{1, \ldots, t\}$, either $p_i^{\alpha_i}$ divides $|H|$ or $p_i^{\alpha_i}$ divides $|K|$. Then $G = HK$ and $|H \cap K| = (|H|, |K|)$.

Proof: Fix some $i \in \{1, \ldots, t\}$ and suppose first that $p_i^{\alpha_i}$ divides the order of H. Since HK is a disjoint union of right cosets of H and each of these right cosets has order equal to $|H|$, it follows that $p_i^{\alpha_i}$ divides $|HK|$. Similarly, if $p_i^{\alpha_i}$ divides $|K|$, since HK is a disjoint union of left cosets of K, again $p_i^{\alpha_i}$ divides $|HK|$. Thus $|G| \mid |HK|$ and so $G = HK$. Since

$$|HK| = \frac{|H||K|}{|H \cap K|},$$

it follows that $|H \cap K| = (|H|, |K|)$.

We now begin the proof of Hall's Theorem, proceeding by induction on $|G|$. Note that if $t = 1$ the hypotheses are trivially satisfied for any group ($H_1 = 1$) and if $t = 2$ the hypotheses are again satisfied for any group by Sylow's Theorem (H_1 is a Sylow p_2-subgroup of G and H_2 is a Sylow p_1-subgroup of G). If $t = 1$, G is nilpotent, hence solvable and if $t = 2$, G is solvable by Burnside's Theorem. Assume therefore that $t \geq 3$.

Fix i and note that by the preceding lemma, for all $j \in \{1, \ldots, t\} - \{i\}$,

$$|H_i : H_i \cap H_j| = p_j^{\alpha_j}.$$

Thus every Sylow p_j-subgroup of H_i has a complement in H_i: $H_j \cap H_i$. By induction H_i is solvable.

By Lemma 9 we may choose $P \trianglelefteq H_1$ with $|P| = p_i^a > 1$ for some $i > 1$. Since $t \geq 3$ there exists an index $j \in \{1, \ldots, t\} - \{1, i\}$. By Lemma 10

$$|H_1 \cap H_j| = p_2^{\alpha_2} \cdots p_{j-1}^{\alpha_{j-1}} p_{j+1}^{\alpha_{j+1}} \cdots p_t^{\alpha_t}.$$

Thus $H_1 \cap H_j$ contains a Sylow p_i-subgroup of H_1. Since P is a normal p_i-subgroup of H_1, P is contained in every Sylow p_i-subgroup of H_1 and so $P \leq H_1 \cap H_j$. By Lemma 10, $G = H_1 H_j$ so each $g \in G$ may be written $g = h_1 h_j$ for some $h_1 \in H_1$ and $h_j \in H_j$. Then

$$g H_j g^{-1} = (h_1 h_j) H_j (h_1 h_j)^{-1} = h_1 H_j h_1^{-1}$$

and so

$$\bigcap_{g \in G} g H_j g^{-1} = \bigcap_{h_1 \in H_1} h_1 H_j h_1^{-1}.$$

Now $P \le H_j$ and $h_1 P h_1^{-1} = P$ for all $h_1 \in H_1$. Thus

$$1 \neq P \le \bigcap_{h_1 \in H_1} h_1 H_j h_1^{-1}.$$

Thus $N = \bigcap_{g \in G} g H_j g^{-1}$ is a nontrivial, proper normal subgroup of G. It follows that both N and G/N satisfy the hypotheses of the theorem (cf. the exercises in Section 3.3). Both N and G/N are solvable by induction, so G is solvable. This completes the proof of Hall's Theorem.

EXERCISES

1. Show that every character of the symmetric group S_n is integer valued, for all n (i.e., $\psi(g) \in \mathbb{Z}$ for all $g \in S_n$ and all characters ψ of S_n). [See Exercise 22 in Section 18.3.]
2. Let G be a finite group with the property that every maximal subgroup has either prime or prime squared index. Prove that G is solvable. (The simple group $GL_3(\mathbb{F}_2)$ has the property that every maximal subgroup has index either 7 or 8, i.e., either prime or prime cubed index — cf. Section 6.2.). [Let p be the largest prime dividing $|G|$ and let P be a Sylow p-subgroup of G. If $P \trianglelefteq G$, apply induction to G/P. Otherwise let M be a maximal subgroup containing $N_G(P)$. Use Exercise 51 in Section 4.5 to show that $p = 3$ and deduce that $|G| = 2^a 3^b$.]
3. Assume G is a finite group that possesses an abelian subgroup H whose index is a power of a prime. Prove that G is solvable.
4. Repeat the preceding exercise with the word "abelian" replaced by "nilpotent."
5. Use the ideas in the proof of Philip Hall's Theorem to prove Burnside's $p^a q^b$ Theorem in the special case when all Sylow subgroups are abelian (without use of character theory.)

19.3 INTRODUCTION TO THE THEORY OF INDUCED CHARACTERS

Let G be a finite group, let H be a subgroup of G and let φ be a representation of the subgroup H over an arbitrary field F. In this section we show how to obtain a representation of G, called the induced representation, from the representation φ of its subgroup. We also determine a formula for the character of this induced representation, the induced character, in terms of the character of φ and we illustrate this formula by computing some induced characters in specific groups. Finally, we apply the theory of induced characters to prove that there are no simple groups of order $3^3 \cdot 7 \cdot 13 \cdot 409$, a group order which was discussed at the end of Section 6.2 in the context of the existence problem for simple groups. The theory of induced representations and induced characters marks the beginning of more advanced representation theory. This section is intended as an introduction rather than as a comprehensive treatment, and the results we have included were chosen to serve this purpose.

First observe that it may not be possible to extend a representation φ of the subgroup H to a representation Φ of G in such a way that $\Phi|_H = \varphi$. For example, $A_3 \le S_3$ and A_3 has a faithful representation of degree 1 (cf. Section 1). Since every degree 1 representation of S_3 contains $A_3 = S_3'$ in its kernel, this representation of A_3 cannot be extended to a representation of S_3. For another example of a representation of a