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The Real Numbers

There are two well-known ways of constructing the reals from the rationals: the Dedekind cut approach, which goes back to Eudoxus, and the Cauchy sequence approach.

Eudoxus, a member of Plato's Academy, was interested in defining the notion of proportion for geometric quantities. What he did can be interpreted in modern terms as defining *equality* between two real numbers (ratios of geometric quantities) α and β as follows: $\alpha = \beta$ iff the set of all rationals below α is the same as the set of all rationals below β, and similarly for the sets of rationals above α and β. Dedekind exploited this idea further by defining the real number α as the pair (L, U) of sets of rationals below and above α, respectively. Such pairs can be described without mentioning α; e.g., L might be the set of all rationals x for which $x^2 < 2$ and U the set of rationals y for which $y^2 > 2$. We shall not develop this idea further, as it is discussed in many algebra books.

Analysts prefer a construction of the reals proposed by Cauchy, according to which the real number α is defined as the set of all sequences of rational numbers which converge to α. Again, this can be done without mentioning α. A *Cauchy sequence* is a sequence $\{a_n | n \in \mathbf{N}\}$ of rational numbers such that $|a_m - a_n|$ can be made as small as one likes by taking m and n sufficiently large. Two Cauchy sequences $\{a_n | n \in \mathbf{N}\}$ and $\{b_n | n \in \mathbf{N}\}$ are said to be *equivalent* if $|a_n - b_n|$ can be made as small as one likes by taking n sufficiently large. A *real number* is then defined as an equivalence class of Cauchy sequences.

An amusing construction of the real numbers, which is not so well-known, bypasses the rationals altogether and defines a real number as an equivalence class of certain mappings $f : \mathbf{Z} \to \mathbf{Z}$. Call f *almost linear* if the set of all $|f(m+n) - f(m) - f(n)|$, where $m, n \in \mathbf{Z}$, is bounded. Call two almost linear mappings $f, g : \mathbf{Z} \to \mathbf{Z}$ *equivalent* if the set of all $|f(n) - g(n)|$, where $n \in \mathbf{Z}$, is bounded. Define a *real number* to be an equivalence class of almost linear mappings $\mathbf{Z} \to \mathbf{Z}$. The real number α will be the equivalence class of the mapping f for which $f(n) = [\alpha n]$ is the greatest integer $\leq \alpha n$. Addition and multiplication of the real numbers corresponding to the almost linear mappings f and g are easily defined as the equivalence classes of $f+g$ and $f \circ g$, respectively, where $(f+g)(n) = f(n) + g(n)$ and $(f \circ g)(n) = f(g(n))$. This definition is due to Steve Schanuel.

Instead of constructing the real numbers, one may describe the field of real numbers axiomatically as a *complete ordered field.* We already know what is meant by a field, so we only have to define the words 'ordered' and 'complete'.

A field F is *ordered* if it has a subset P such that

1. $x, y \in P \Rightarrow x + y \in P$,
2. $x, y \in P \Rightarrow xy \in P$,
3. exactly one of the following holds: $x = 0$, or $x \in P$, or $-x \in P$.

Note that by (3) either 1 or -1 is an element of P. Since $(-1)(-1) = 1$, it follows from (2) that $1 \in P$. The elements of P are the *positive* elements of the field. The existence of P allows us to define an *order relation* on the field F:

$$x \leq y \Longleftrightarrow x = y \text{ or } y - x \in P.$$

From this definition we obtain the following propositions:

$x \leq x$	reflexivity,
$x \leq y$ and $y \leq z \Rightarrow x \leq z$	transitivity,
$x \leq y$ and $y \leq x \Rightarrow x = y$	antisymmetry,
$x \leq y$ or $y \leq x$	dichotomy.

It is because $\leq$ has these four properties that it is called an *order relation.* $\mathbf{Q}$ and $\mathbf{R}$ are ordered fields, but $\mathbf{C}$ is not. To see that $\mathbf{C}$ is not ordered, consider the element i. If $i \in P$ then $-1 = i \cdot i \in P$. This is impossible since $1 \in P$. If $-i \in P$ we again have that $-1 = (-i)(-i) \in P$. So neither i nor $-i$ is in P, and this contradicts (3) above.

An ordered field is *complete* if every nonempty set of positive elements has a greatest lower bound. For example, $\sqrt{2}$ is the greatest lower bound of all positive reals r such that $2 \leq r^2$. Since $\sqrt{2}$ is not rational, we may conclude from this example that $\mathbf{Q}$ is not a complete ordered field. There is really only one complete ordered field in the sense of the following:

Theorem 5.1. *Any two complete ordered fields are isomorphic.*

For the proof, see Chapter 4.3 of Birkhoff and Mac Lane [1977].

Exercises

1. Pick any of the three definitions of the reals mentioned here and prove that they form a field.

2. Prove the four properties of the order relation defined above.