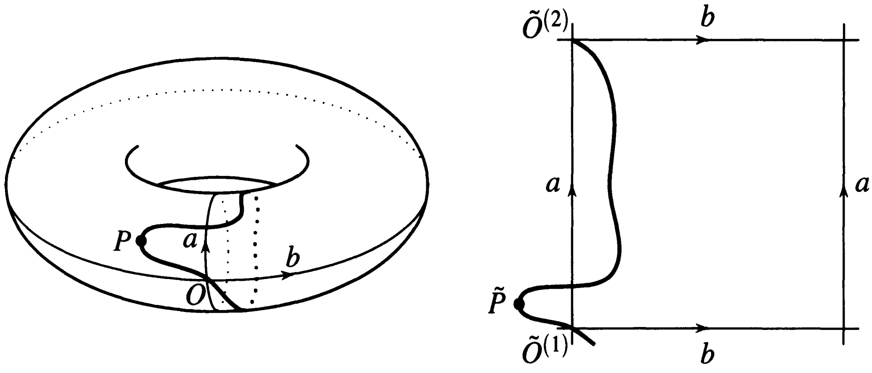

Figure 22.15: Plotting on the covering surface

Thus from the topological notion of homotopic closed paths we have arrived once again at ordinary geometry. We have also arrived at a group called the *fundamental group* of S. Geometrically, it is the group of motions of $\tilde{S}$ that map the tessellation onto itself (which includes mapping each edge to a like-labeled edge). Topologically, it is the group of homotopy classes of closed paths, with a common initial point O, on S. The product of homotopy classes is defined by successive traversal of representative paths.

The fundamental group was first defined topologically by Poincaré (1895). Poincaré defined it for much more general figures, whose universal covers are not so apparent, so the interpretation as a covering motion group did not emerge until later. As we know, Poincaré had already studied groups of motions of tessellations (1882). He reconsidered these earlier results from a topological viewpoint (1904), arriving at the interpretation just given. This paper was very influential on the later work of Dehn (1912) and Nielsen (1927) and has been indirectly responsible for a recent surge of interest in hyperbolic geometry.

The more general notion of fundamental group in Poincaré (1895) has also been influential outside topology. It turns out, for example, that for any "reasonably described" figure $\mathscr{F}$ it is possible to compute generators and defining relations for the fundamental group of $\mathscr{F}$. The defining relations of a fundamental group can be quite arbitrary [in fact, *completely* arbitrary, as was shown by Dehn (1910) and Seifert and Threlfall (1934), p. 180]. So the question arises: can the properties of a group be determined from its defining relations? One would like to know, for example, when two different sets of relations define the same group. The latter question was raised by Tietze (1908) in the first paper to follow up Poincaré's work. Tietze made the remarkable conjecture—which could not even be precisely formulated at the time—that the problem is unsolvable. The *isomorphism problem for groups*, as it came to be known, was indeed shown to be unsolvable by Adyan (1957), in the sense that no algorithm can settle the question for *all* finite sets of defining relations. Adyan's result was based on the development of a theory of algorithms that is outlined in the next chapter.

By combining Adyan's result with some of Tietze (1908) and the result of Seifert and Threlfall mentioned above, Markov (1958) was able to show the unsolvability of the *homeomorphism problem*. This is the problem of deciding, given "reasonably described" figures $\mathscr{F}_1$ and $\mathscr{F}_2$, whether $\mathscr{F}_1$ is homeomorphic to $\mathscr{F}_2$. [A complete proof of the unsolvability of the isomorphism problem and homeomorphism problem may be found in Stillwell (1993), and its history may be found in Stillwell (1982).] Thus Poincaré's construction of the fundamental group led in the end to quite an unexpected conclusion: the basic problem of topology is unsolvable.

22.8 Biographical Notes: Poincaré

Henri Poincaré (Figure 22.16) was born in Nancy in 1854 and died in Paris in 1912. His father, Leon, was a physician and professor of medicine at the University of Nancy, and Henri grew up in a comfortable academic environment. He and his younger sister, Aline, were at first educated by their mother, and Poincaré later traced his mathematical ability to his maternal grandmother. At the age of five he suffered an attack of diphtheria, which weakened his health and excluded him from the more boisterous childhood games. He made up for this by organizing charades and playlets, and he later became a keen dancer. Many photographs of Poincaré and his family may be seen in the centenary volume (1955), which forms the second half of vol. 11 of Poincaré's *Oeuvres*.

Being excluded from most games, Poincaré had ample time to read and study, and when he began attending school, at the age of eight, he made rapid progress. His ability first showed in French composition, but by the end of his school career his awesome mathematical talent was also apparent. He won first prize in a nationwide mathematics competition and topped the entrance exam to the École Polytechnique in 1873. This, inci-

Figure 22.16: Henri Poincaré

dentally, was despite the Franco-Prussian War (1870–1871), during which Poincaré's home province of Lorraine bore the brunt of the German invasion. Poincaré accompanied his father on ambulance rounds at this time, becoming a fervent French patriot as a result. However, he never held German mathematicians responsible for the brutalities of their compatriots. He learned German during the war in order to read the news, and he later put the knowledge to good use in communicating with his German colleagues Fuchs and Klein.

At the École Polytechnique, Poincaré continued to do well, though clumsiness in drawing and experimental work cost him first place. [His marks in drawing, though mediocre, were never zero, despite oft-told tales to that effect. Poincaré's results may be seen in the centenary volume (1955).] Curiously, he planned to become an engineer at this stage and studied at the École des Mines from 1875 to 1879, at the same time writing a doctoral thesis in mathematics. He worked briefly as a mining engineer before becoming an instructor in mathematics at the University of Caen in 1879. It was at Caen that Poincaré made his first important discovery: the occurrence of noneuclidean geometry in the theory of complex functions. He had been thinking about periodicity with respect to linear fractional transformations, after encountering functions with this property in the work of Lazarus Fuchs. The functions in question arose from differential equations, and Poincaré had been struggling to understand them analytically when he was struck by an unexpected geometric inspiration:

> Just at this time I left Caen, where I was then living, to go on a geological excursion under the auspices of the school of mines. The changes of travel made me forget my mathematical work. Having reached Coutances, we entered an omnibus to go some place or other. At the moment when I put my foot on the step the idea came to me, without anything in my former thoughts seeming to have paved the way for it, that the transformations I had used to define the Fuchsian functions were identical with those of Non-Euclidean geometry.
>
> [Poincaré (1918); translation from Halsted, 1929, p. 387)]

The discovery of the underlying geometry (and topology, which soon followed) put Fuchsian functions in a completely new light, rather like the illumination of elliptic functions by Riemann's discovery that they belonged to the torus. For the next few years Poincaré worked feverishly to develop these ideas, in friendly competition with Klein. There were some reservations about his style—undisciplined and lacking in rigor, though very readable—but his brilliance was not contested. He was appointed to a chair at the University of Paris in 1881 and remained there, winning ever higher honors, until the end of his life. In 1881 he was married to Louise Poulain; they had a son and three daughters.

Poincaré's work on Fuchsian functions led him to topology, as we have seen in Sections 22.6 and 22.7. So did another of his great inventions, the qualitative theory of differential equations. He used this theory, which deals with such questions as the long-term stability of a mechanical system, in his *Les méthodes nouvelles de la mécanique céleste* (1892, 1893, 1899), probably the greatest advance in celestial mechanics since Newton. Poincaré's topological ideas not only breathed new life into complex analysis and mechanics; they amounted to the creation of a major new field, algebraic topology. In papers between 1892 and 1904, Poincaré built up an arsenal of techniques and concepts that were to keep topologists going for the next 30 years. It was not until Hurewicz discovered higher-dimensional analogues of the fundamental group in 1933 that a significant new weapon was added to Poincaré's arsenal. Recently, as mentioned in Section 22.1, there has been a return to geometric methods in topology. It would be fitting if these methods finally succeed in resolving the main unsolved problem left by Poincaré, the so-called *Poincaré conjecture*. The conjecture states that the 3-sphere (the space obtained by completing $\mathbb{R}^3$ by a point at infinity, as the plane was completed to an ordinary sphere in Section 15.2) is the only closed three-dimensional space whose fundamental group is trivial.

Poincaré was perhaps the last mathematician to have a general grasp of all branches of mathematics. Like Euler, he wrote fluently and copiously on all parts of mathematics, and in fact he surpassed Euler in his popular writing. He wrote many volumes on science and its philosophy, which were best-sellers in the early part of this century. Poincaré would perhaps have been as prolific as Euler if ill health had not overtaken him in his fifties. In 1911 he took the unusual step of publishing an unfinished paper, on periodic solutions of the three-body problem, believing he might not live to complete the proof. "Poincaré's last theorem" was indeed still open when he died in 1912, but the proof was completed in 1913 by the American mathematician G.D. Birkhoff.

23

Sets, Logic, and Computation

23.1 An Explanation

In any survey of the history of mathematics it is hard to ignore the twentieth century. At the very least, we have to admit that our conception of classical mathematics is influenced by the mathematical ideas that are fashionable today. Indeed, it can be argued that most of classical mathematics becomes clearer when presented in modern terms, and that this is the best way to make it accessible to mathematicians who are not professional historians. As will be evident by now, this is the point of view I have adopted in this book.

At the same time, twentieth-century mathematics is far more than a mere instrument for viewing the mathematics of the past. It includes many results that are themselves historic and hence eligible for inclusion in our survey. Some of them have been mentioned in previous chapters, particularly where they answer classical questions. The problem is that twentieth-century mathematics is so vast that no one can grasp it all, and even some single results are based on theories too large to be explained in a book of this size. Under these circumstances, no chapter on twentieth-century mathematics is likely to be representative, and the reader is entitled to an explanation of the author's choice of topics.

I believe that the choice of sets, logic, and computation is appropriate for the following reasons:

(i) The three topics are linked, historically and logically.