Using Equation (9.26) of Example 1 we obtain the formula

$$h'(x) = \frac{\dfrac{\partial g}{\partial y}\dfrac{\partial f}{\partial x} - \dfrac{\partial f}{\partial y}\dfrac{\partial g}{\partial x}}{\dfrac{\partial g}{\partial y}}.$$

The partial derivatives on the right are to be evaluated at the point $(x, Y(x))$. Note that the numerator can also be expressed as a Jacobian determinant, giving us

$$h'(x) = \frac{\partial(f, g)/\partial(x, y)}{\partial g/\partial y}.$$

EXAMPLE 3. The two equations $2x = v^2 - u^2$ and $y = uv$ define u and v as functions of x and y. Find formulas for $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$.

Solution. If we hold y fixed and differentiate the two equations in question with respect to x, remembering that u and v are functions of x and y, we obtain

$$2 = 2v\frac{\partial v}{\partial x} - 2u\frac{\partial u}{\partial x} \qquad \text{and} \qquad 0 = u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x}.$$

Solving these simultaneously for $\partial u/\partial x$ and $\partial v/\partial x$ we find

$$\frac{\partial u}{\partial x} = -\frac{u}{u^2 + v^2} \qquad \text{and} \qquad \frac{\partial v}{\partial x} = \frac{v}{u^2 + v^2}.$$

On the other hand, if we hold x fixed and differentiate the two given equations with respect to y we obtain the equations

$$0 = 2v\frac{\partial v}{\partial y} - 2u\frac{\partial u}{\partial y} \qquad \text{and} \qquad 1 = u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y}.$$

Solving these simultaneously we find

$$\frac{\partial u}{\partial y} = \frac{v}{u^2 + v^2} \qquad \text{and} \qquad \frac{\partial v}{\partial y} = \frac{u}{u^2 + v^2}.$$

EXAMPLE 4. Let u be defined as a function of x and y by means of the equation

$$u = F(x + u, yu).$$

Find $\partial u/\partial x$ and $\partial u/\partial y$ in terms of the partial derivatives of F.

Solution. Suppose that $u = g(x, y)$ for all (x, y) in some open set S. Substituting $g(x, y)$ for u in the original equation we must have

$$g(x, y) = F[u_1(x, y), u_2(x, y)], \tag{9.27}$$

where $u_1(x, y) = x + g(x, y)$ and $u_2(x, y) = y\,g(x, y)$. Now we hold y fixed and differentiate both sides of (9.27) with respect to x, using the chain rule on the right, to obtain

$$\frac{\partial g}{\partial x} = D_1F \frac{\partial u_1}{\partial x} + D_2F \frac{\partial u_2}{\partial x}. \tag{9.28}$$

But $\partial u_1/\partial x = 1 + \partial g/\partial x$, and $\partial u_2/\partial x = y\,\partial g/\partial x$. Hence (9.28) becomes

$$\frac{\partial g}{\partial x} = D_1F \cdot \left(1 + \frac{\partial g}{\partial x}\right) + D_2F \cdot \left(y \frac{\partial g}{\partial x}\right).$$

Solving this equation for $\partial g/\partial x$ (and writing $\partial u/\partial x$ for $\partial g/\partial x$) we obtain

$$\frac{\partial u}{\partial x} = \frac{-D_1F}{D_1F + y\,D_2F - 1}.$$

In a similar way we find

$$\frac{\partial g}{\partial y} = D_1F \frac{\partial u_1}{\partial y} + D_2F \frac{\partial u_2}{\partial y} = D_1F \frac{\partial g}{\partial y} + D_2F\left(y \frac{\partial g}{\partial y} + g(x, y)\right).$$

This leads to the equation

$$\frac{\partial u}{\partial y} = \frac{-g(x, y)\,D_2F}{D_1F + y\,D_2F - 1}.$$

The partial derivatives D_1F and D_2F are to be evaluated at the point $(x + g(x, y), y\,g(x, y))$.

EXAMPLE 5. When u is eliminated from the two equations $x = u + v$ and $y = uv^2$, we get an equation of the form $F(x, y, v) = 0$ which defines v implicitly as a function of x and y, say $v = h(x, y)$. Prove that

$$\frac{\partial h}{\partial x} = \frac{h(x, y)}{3h(x, y) - 2x}$$

and find a similar formula for $\partial h/\partial y$.

Solution. Eliminating u from the two given equations, we obtain the relation

$$xv^2 - v^3 - y = 0.$$

Let F be the function defined by the equation

$$F(x, y, v) = xv^2 - v^3 - y.$$

The discussion in Section 9.6 is now applicable and we can write

$$\frac{\partial h}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial v} \quad \text{and} \quad \frac{\partial h}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial v}. \tag{9.29}$$

But $\partial F/\partial x = v^2$, $\partial F/\partial v = 2xv - 3v^2$, and $\partial F/\partial y = -1$. Hence the equations in (9.29) become

$$\frac{\partial h}{\partial x} = -\frac{v^2}{2xv - 3v^2} = -\frac{v}{2x - 3v} = \frac{h(x, y)}{3h(x, y) - 2x}$$

and

$$\frac{\partial h}{\partial y} = -\frac{-1}{2xv - 3v^2} = \frac{1}{2xh(x, y) - 3h^2(x, y)}.$$

EXAMPLE 6. The equation $F(x, y, z) = 0$ defines z implicitly as a function of x and y, say $z = f(x, y)$. Assuming that $\partial^2 F/(\partial x\, \partial z) = \partial^2 F/(\partial z\, \partial x)$, show that

$$\frac{\partial^2 f}{\partial x^2} = -\frac{\left(\frac{\partial^2 F}{\partial z^2}\right)\left(\frac{\partial F}{\partial x}\right)^2 - 2\left(\frac{\partial^2 F}{\partial x\, \partial z}\right)\left(\frac{\partial F}{\partial z}\right)\left(\frac{\partial F}{\partial x}\right) + \left(\frac{\partial F}{\partial z}\right)^2\left(\frac{\partial^2 F}{\partial x^2}\right)}{\left(\frac{\partial F}{\partial z}\right)^3}, \tag{9.30}$$

where the partial derivatives on the right are to be evaluated at $(x, y, f(x, y))$.

Solution. By Equation (9.20) of Section 9.6 we have

(9.31) $\qquad \partial f \qquad \partial F/\partial x$

*a x-- i @ % **

We must remember that this quotient really means

$$-\frac{D_1F[x, y, f(x, y)]}{D_3F[x, y, f(x, y)]}.$$

Let us introduce $G(x, y) = D_1F[x, y, f(x, y)]$ and $H(x, y) = D_3F[x, y, f(x, y)]$. Our object is to evaluate the partial derivative with respect to x of the quotient

$$\frac{\partial f}{\partial x} = -\frac{G(x, y)}{H(x, y)},$$

holding y fixed. The rule for differentiating quotients gives us

$$\frac{\partial^2 f}{\partial x^2} = -\frac{H\frac{\partial G}{\partial x} - G\frac{\partial H}{\partial x}}{H^2}. \tag{9.32}$$

Since G and H are composite functions, we use the chain rule to compute the partial derivatives $\partial G/\partial x$ and $\partial H/\partial x$. For $\partial G/\partial x$ we have

$$\begin{aligned}\frac{\partial G}{\partial x} &= D_1(D_1F)\cdot 1 + D_2(D_1F)\cdot 0 + D_3(D_1F)\cdot\frac{\partial f}{\partial x}\\ &= \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial z\, \partial x}\frac{\partial f}{\partial x}.\end{aligned}$$

Similarly, we find

$$\frac{\partial H}{\partial x} = D_1(D_3F) \cdot 1 + D_2(D_3F) \cdot 0 + D_3(D_3F) \cdot \frac{\partial f}{\partial x}$$

$$= \frac{\partial^2 F}{\partial x \, \partial z} + \frac{\partial^2 F}{\partial z^2} \frac{\partial f}{\partial x} .$$

Substituting these in (9.32) and replacing $\partial f / \partial x$ by the quotient in (9.31) we obtain the formula in (9.30).

9.8 Exercises

In the exercises in this section you may assume the existence and continuity of all derivatives under consideration.

1. The two equations $x + y = uv$ and $xy = u - v$ determine x and y implicitly as functions of u and v, say $x = X(u, v)$ and $y = Y(u, v)$. Show that $\partial X / \partial u = (xv - 1)/(x - y)$ if $x \neq y$, and find similar formulas for $\partial X / \partial v$, $\partial Y / \partial u$, $\partial Y / \partial v$.
2. The two equations $x + y = uv$ and $xy = u - v$ determine x and v as functions of u and y, say $x = X(u, y)$ and $v = V(u, y)$. Show that $\partial X / \partial u = (u + v)/(1 + yu)$ if $1 + yu \neq 0$, and find similar formulas for $\partial X / \partial y$, $\partial V / \partial u$, $\partial V / \partial y$.
3. The two equations $F(x, y, u, v) = 0$ and $G(x, y, u, v) = 0$ determine x and y implicitly as functions of u and v, say $x = X(u, v)$ and $y = Y(u, v)$. Show that

$$\frac{\partial X}{\partial u} = -\frac{\partial(F, G)/\partial(y, u)}{\partial(F, G)/\partial(x, y)}$$

   at points at which the Jacobian $\partial(F, G)/\partial(x, y) \neq 0$, and find similar formulas for the partial derivatives $\partial X / \partial v$, $\partial Y / \partial u$, and $\partial Y / \partial v$.
4. The intersection of the two surfaces given by the Cartesian equations $2x^2 + 3y^2 - z^2 = 25$ and $x^2 + y^2 = z^2$ contains a curve C passing through the point $P = (\sqrt{7}, 3, 4)$. These equations may be solved for x and y in terms of z to give a parametric representation of C with z as parameter.
   (a) Find a unit tangent vector T to C at the point P without using an explicit knowledge of the parametric representation.
   (b) Check the result in part (a) by determining a parametric representation of C with z as parameter.
5. The three equations $F(u, v) = 0$, $u = xy$, and $v = \sqrt{x^2 + z^2}$ define a surface in xyz-space. Find a normal vector to this surface at the point $x = 1$, $y = 1$, $z = \sqrt{3}$ if it is known that $D_1F(1, 2) = 1$ and $D_2F(1, 2) = 2$.
6. The three equations

$$x^2 - y \cos(uv) + z^2 = 0,$$

$$x^2 + y^2 - \sin(uv) + 2z^2 = 2,$$

$$xy - \sin u \cos v + z = 0,$$

   define x, y, and z as functions of u and v. Compute the partial derivatives $\partial x / \partial u$ and $\partial x / \partial v$ at the point $x = y = 1$, $u = \pi/2$, $v = 0$, $z = 0$.

7. The equation $f(y/x, z/x) = 0$ defines z implicitly as a function of x and y, say $z = g(x, y)$. Show that

$$x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y} = g(x, y)$$

at those points at which $D_2 f[y/x, g(x, y)/x]$ is not zero.

8. Let F be a real-valued function of two real variables and assume that the partial derivatives D_1F and D_2F are never zero. Let u be another real-valued function of two real variables such that the partial derivatives $\partial u/\partial x$ and $\partial u/\partial y$ are related by the equation $F(\partial u/\partial x, \partial u/\partial y) = 0$. Prove that a constant n exists such that

$$\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} = \left(\frac{\partial^2 u}{\partial x\, \partial y}\right)^n,$$

and find n. Assume that $\partial^2 u/(\partial x\, \partial y) = \partial^2 u/(\partial y\, \partial x)$.

9. The equation $x + z + (y + z)^2 = 6$ defines z implicitly as a function of x and y, say $z = f(x, y)$. Compute the partial derivatives $\partial f/\partial x$, $\partial f/\partial y$, and $\partial^2 f/(\partial x\, \partial y)$ in terms of x, y, and z.

10. The equation $\sin(x + y) + \sin(y + z) = 1$ defines z implicitly as a function of x and y, say $z = f(x, y)$. Compute the second derivative $D_{1,2}f$ in terms of x, y, and z.

11. The equation $F(x + y + z, x^2 + y^2 + z^2) = 0$ defines z implicitly as a function of x and y, say $z = f(x, y)$. Determine the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ in terms of x, y, z and the partial derivatives D_1F and D_2F.

12. Let f and g be functions of one real variable and define $F(x, y) = f[x + g(y)]$. Find formulas for all the partial derivatives of F of first and second order, expressed in terms of the derivatives of f and g. Verify the relation

$$\frac{\partial F}{\partial x}\frac{\partial^2 F}{\partial x\, \partial y} = \frac{\partial F}{\partial y}\frac{\partial^2 F}{\partial x^2}.$$

9.9 Maxima, minima, and saddle points

A surface that is described explicitly by an equation of the form $z = f(x, y)$ can be thought of as a level surface of the scalar field F defined by the equation

$$F(x, y, z) = f(x, y) - z.$$

If f is differentiable, the gradient of this field is given by the vector

$$\nabla F = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} - \boldsymbol{k}.$$

A linear equation for the tangent plane at a point $P_1 = (x_1, y_1, z_1)$ can be written in the form

$$z - z_1 = A(x - x_1) + B(y - y_1),$$

where

$$A = D_1 f(x_1, y_1) \quad \text{and} \quad B = D_2 f(x_1, y_1).$$

When both coefficients A and B are zero, the point P_1 is called *a stationary point* of the surface and the point (x_1, y_1) is called a *stationary point* or a *critical point* of the function f.

The tangent plane is horizontal at a stationary point. The stationary points of a surface are usually classified into three categories : maxima, minima, and saddle points. If the surface is thought of as a mountain landscape, these categories correspond, respectively, to mountain tops, bottoms of valleys, and mountain passes.

The concepts of maxima, minima, and saddle points can be introduced for arbitrary scalar fields defined on subsets of R”.

DEFINITION. *A scalar field f is said to have an absolute maximum at a point a of a set S in $\mathbf{R}^n$ if*

$$f(\boldsymbol{x}) \le f(\boldsymbol{a}) \tag{9.33}$$

for all $\boldsymbol{x}$ in S. The number f (a) is called the absolute maximum value off on S. The function f is said to have a relative maximum at a if the inequality in (9.33) is satisjiedfor every $\boldsymbol{x}$ in some n-ball B(a) lying in S.

In other words, a relative maximum at ***a*** is the absolute maximum in some neighborhood of ***a.*** The terms ***absolute minimum*** and ***relative minimum*** are defined in an analogous fashion, using the inequality opposite to that in (9.33). The adjectives ***global*** and ***local*** are sometimes used in place of ***absolute*** and ***relative,*** respectively.

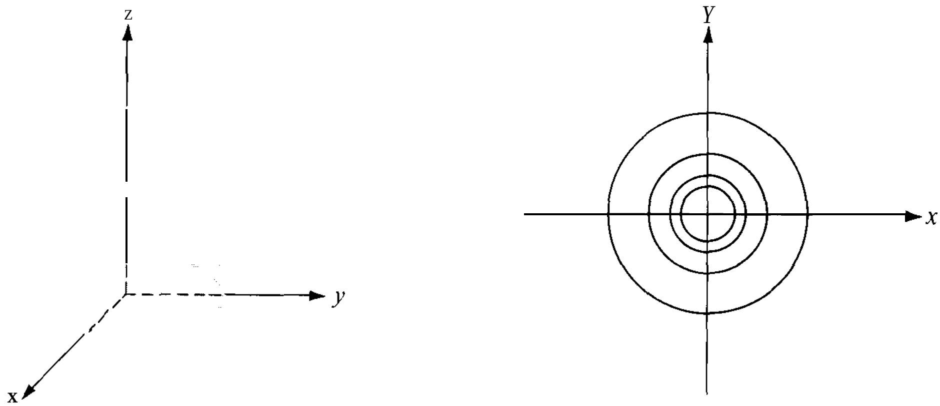
DEFINITION. ***A number which is either a relative maximum or a relative minimum off is called an extremum off.***

Iff has an extremum at an interior point ***a*** and is differentiable there, then all first-order partial derivatives $D_1f(\boldsymbol{a}), \ldots, D_nf(\boldsymbol{a})$ must be zero. In other words, ***Vf(a)*** = 0. (This is easily proved by holding each component fixed and reducing the problem to the one-dimensional case.) In the case $n = 2$, this means that there is a horizontal tangent plane to the surface z = $f(x$, y) at the point ***(a, f (a)).*** On the other hand, it is easy to find examples in which the vanishing of all partial derivatives at ***a*** does not necessarily imply an extremum at ***a.*** This occurs at the so-called ***saddlepoints*** which are defined as follows.

DEFINITION. *Assume f is differentiable at a. If ∇f (a) = 0 the point a is called a stationary point of f. A stationary point is called a saddle point if every n-ball B(a) contains points $\boldsymbol{x}$ such that f (x) $<$ f(a) and other points such that f (x) > f (a).*

The situation is somewhat analogous to the one-dimensional case in which stationary points of a function are classified as maxima, minima, and points of inflection. The following examples illustrate several types of stationary points. In each case the stationary point in question is at the origin.

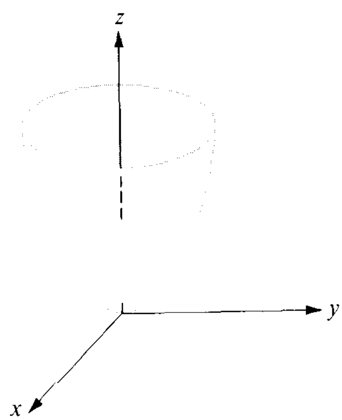
EXAMPLE 1. ***Relative maximum.*** $z = f(x, y) = 2 - x^2 - y^2$. This surface is a paraboloid of revolution. In the vicinity of the origin it has the shape shown in Figure 9.3(a). Its



(a) $z = 2 - x^2 - y^2$

(b) Level curves: $x^2 + y^2 = c$

Example 1. Relative maximum at the origin.



(c) $z = x^2 + y^2$

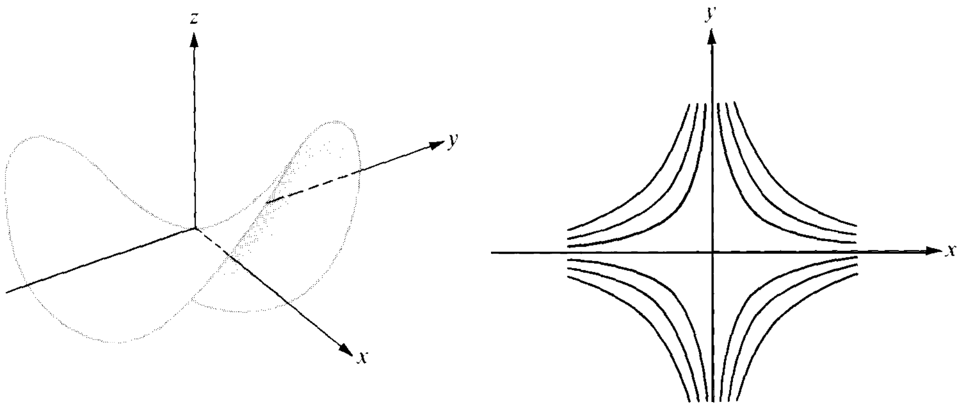
Example 2. Relative minimum at the origin.

FIGURE 9.3 Examples 1 and 2.

level curves are circles, some of which are shown in Figure 9.3(b). Since $f(x, y) = 2 - (x^2 + y^2) \leq 2 = f(0, 0)$ for all (x, y), it follows that f not only has a relative maximum at $(0, 0)$ but also an *absolute* maximum there. Both partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ vanish at the origin.

EXAMPLE 2. *Relative minimum.* $z = f(x, y) = x^2 + y^2$. This example, another paraboloid of revolution, is essentially the same as Example 1, except that there is a minimum at the origin rather than a maximum. The appearance of the surface near the origin is illustrated in Figure 9.3(c) and some of the level curves are shown in Figure 9.3(b).

EXAMPLE 3. ***Saddle point.*** $z = f(x, y) = xy$. This surface is a hyperbolic paraboloid. Near the origin the surface is saddle shaped, as shown in Figure 9.4(a). Both partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ are zero at the origin but there is neither a relative maximum nor a relative minimum there. In fact, for points (x, y) in the first or third quadrants, x and y have the same sign, giving us $f(x, y) > 0 = f(0, 0)$, whereas for points in the second and fourth quadrants x and y have opposite signs, giving us $f(x, y) < 0 = f(0, 0)$. Therefore, in every neighborhood of the origin there are points at which the function is less than $f(0, 0)$ and points at which the function exceeds $f(0, 0)$, so the origin is a saddle



(a) $z = xy$ (b) Level curves: $xy = c$

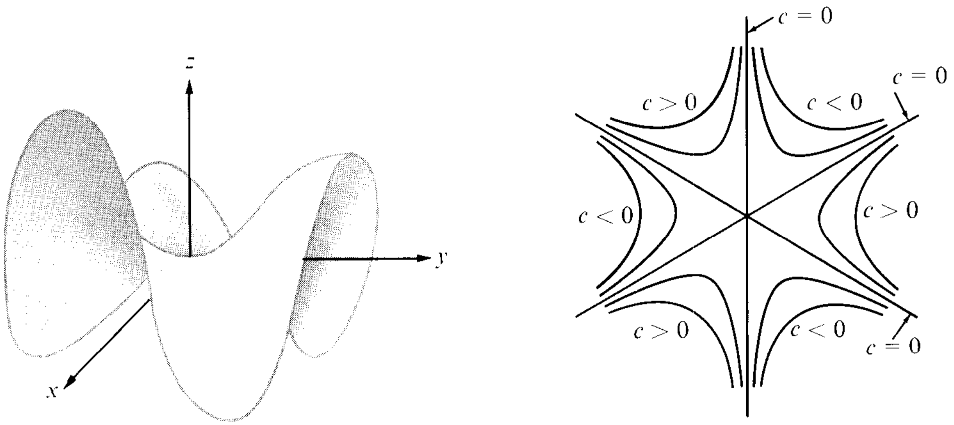
FIGURE 9.4 Example 3. Saddle point at the origin.

point. The presence of the saddle point is also revealed by Figure 9.4(b), which shows some of the level curves near $(0, 0)$. These are hyperbolas having the x- and y-axes as asymptotes.

EXAMPLE 4. ***Saddle point.*** $z = f(x, y) = x^3 - 3xy^2$. Near the origin, this surface has the appearance of a mountain pass in the vicinity of three peaks. This surface, sometimes referred to as a "monkey saddle," is shown in Figure 9.5(a). Some of the level curves are illustrated in Figure 9.5(b). It is clear that there is a saddle point at the origin.

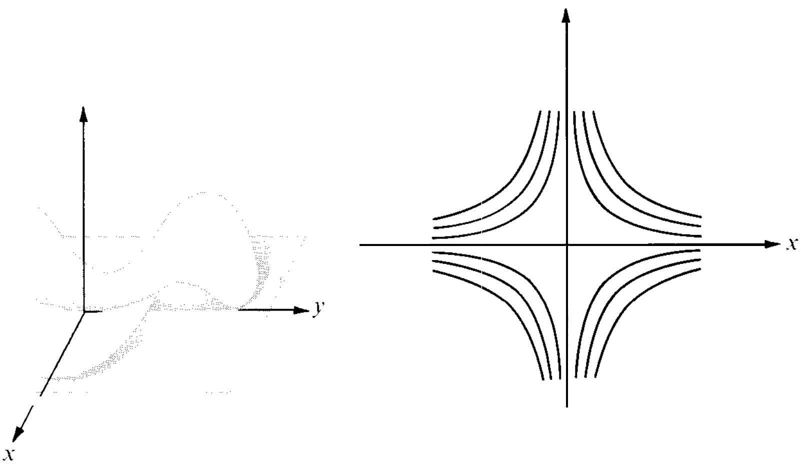
EXAMPLE 5. ***Relative minimum.*** $z = f(x, y) = x^2y^2$. This surface has the appearance of a valley surrounded by four mountains, as suggested by Figure 9.6(a). There is an absolute minimum at the origin, since $f(x, y) \geq f(0, 0)$ for all (x, y). The level curves [shown in Figure 9.6(b)] are hyperbolas having the x- and y-axes as asymptotes. Note that these level curves are similar to those in Example 3. In this case, however, the function assumes only nonnegative values on all its level curves.

EXAMPLE 6. ***Relative maximum.*** $z = f(x, y) = 1 - x^2$. In this case the surface is a cylinder with generators parallel to the y-axis, as shown in Figure 9.7(a). Cross sections cut by planes parallel to the x-axis are parabolas. There is obviously an absolute maximum at the origin because $f(x, y) = 1 - x^2 \leq 1 = f(0, 0)$ for all (x, y). The level curves form a family of parallel straight lines as shown in Figure 9.7(b).



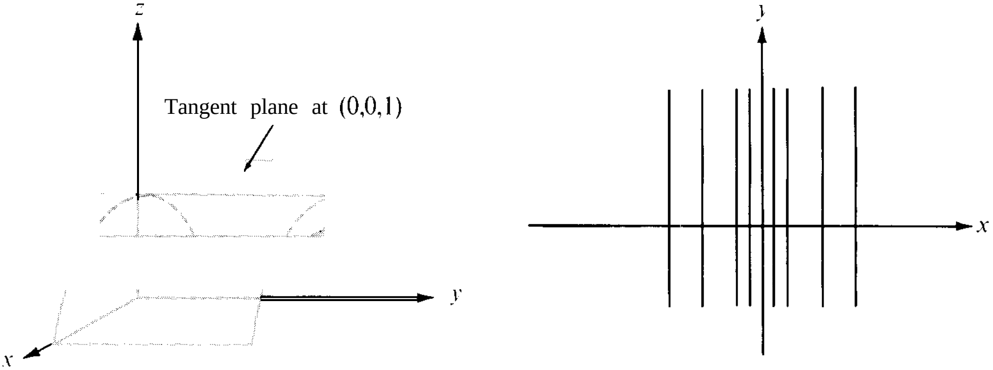
(a) $z = x^3 - 3xy^2$. (b) Level curves: $x^3 - 3xy^2 = c$.

FIGURE 9.5 Example 4. Saddle point at the origin.



(a) $z = x^2y^2$ (b) Level curves: $x^2y^2 = c$

FIGURE 9.6 Example 5. Relative minimum at the origin.



(a) $z = 1 - x^2$ (b) Level curves: $1 - x^2 = c$

FIGURE 9.7 Example 6. Relative maximum at the origin.

9.10 Second-order Taylor formula for scalar fields

If a differentiable scalar field f has a stationary point at a, the nature of the stationary point is determined by the algebraic sign of the difference f'(x) -f(a) for x near a. If $x = a + y$, we have the first-order Taylor formula

$$f(a + y) - f(a) = \nabla f(a) \cdot y + \|y\| E(a, y), \qquad \text{where } E(a, y) \to 0 \text{ as } y \to 0.$$

At a stationary point, $\nabla f(a) = 0$ and the Taylor formula becomes

$$f(a + y) - f(a) = \|y\| E(a, y).$$

To determine the algebraic sign of $f(a + y) - f(a)$ we need more information about the error term $\|y\| E(a, y)$. The next theorem shows that if f has continuous second-order partial derivatives at a, the error term is equal to a quadratic form,

$$\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} D_{ij}f(a)y_iy_j$$

plus a term of smaller order than $\|y\|^2$. The coefficients of the quadratic form are the second-order partial derivatives $D_{ij}f = D_i(D_jf)$, evaluated at a. The n x n matrix of second-order derivatives $D_{ij}f(x)$ is called the ***Hessian matrix?*** and is denoted by ***H(x).*** Thus, we have

$$H(x) = [D_{ij}f(x)]_{i,j=1}^{n}$$

whenever the derivatives exist. The quadratic form can be written more simply in matrix notation as follows:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} D_{ij}f(a)y_iy_j = yH(a)y^t,$$

where y = $(y_1, \ldots, y_n)$ is considered as a 1 x n row matrix, and y^t is its transpose, an n x 1 column matrix. When the partial derivatives $D_{ij}f$ are continuous we have $D_{ij}f = D_{ji}f$ and the matrix $H(u)$ is symmetric.

Taylor's formula, giving a quadratic approximation to $f(a + y) - f(u)$, now takes the following form.

THEOREM **9.4.** SECOND-ORDER TAYLOR FORMULA FOR SCALAR FIELDS. ***Let f be a scalar field with continuous second-order partial derivatives $D_{ij}f$ in an n-ball B(u). Then for all y in $\mathbf{R}^n$ such that $a + y \in B(u)$ we have***

$$(9.34) \qquad f(a + y) - f(a) = \nabla f(a) \cdot y + \frac{1}{2!} yH(a + cy)y^t, \qquad \textit{where} \quad 0 < c < 1.$$

† Named for Ludwig Otto Hesse (1811–1874), a German mathematician who made many contributions to the theory of surfaces.

This can also be written in the form

(9.35) $$f(\boldsymbol{a}+\boldsymbol{y})-f(\boldsymbol{a})=\nabla f(\boldsymbol{a})\cdot \boldsymbol{y}+\frac{1}{2!}\boldsymbol{y}H(\boldsymbol{a})\boldsymbol{y}^t+\|\boldsymbol{y}\|^2 E_2(\boldsymbol{a},\boldsymbol{y}),$$

where $E_2(\boldsymbol{a}, \boldsymbol{y}) \to \boldsymbol{0}$ as $\boldsymbol{y} \to \boldsymbol{0}$.

Proof. Keep $\boldsymbol{y}$ fixed and define $g(u)$ for real u by the equation

$$g(u)=f(\boldsymbol{a}+u\boldsymbol{y}) \qquad \text{for} \quad -1 \le u \le 1.$$

Then $f(\boldsymbol{a}+\boldsymbol{y})-f(\boldsymbol{a})=g(1)-g(0)$. We will prove the theorem by applying the second-order Taylor formula to g on the interval $[0, 1]$. We obtain

(9.36) $$g(1)-g(0)=g'(0)+\frac{1}{2!}g''(c), \qquad \text{where } 0<c<1.$$

Here we have used Lagrange's form of the remainder (see Section 7.7 of Volume I).

Since g is a composite function given by $g(u)=f[\boldsymbol{r}(u)]$, where $\boldsymbol{r}(u)=\boldsymbol{a}+u\boldsymbol{y}$, we can compute its derivative by the chain rule. We have $\boldsymbol{r}'(u)=\boldsymbol{y}$ so the chain rule gives us

$$g'(u)=\nabla f[\boldsymbol{r}(u)]\cdot \boldsymbol{r}'(u)=\nabla f[\boldsymbol{r}(u)]\cdot \boldsymbol{y}=\sum_{j=1}^{n} D_j f[\boldsymbol{r}(u)]y_j,$$

provided $\boldsymbol{r}(u) \in B(\boldsymbol{a})$. In particular, $g'(0)=\nabla f(\boldsymbol{a})\cdot \boldsymbol{y}$. Using the chain rule once more we find

$$g''(u)=\sum_{i=1}^{n} D_i\Big(\sum_{j=1}^{n} D_j f[\boldsymbol{r}(u)]y_j\Big)y_i=\sum_{i=1}^{n}\sum_{j=1}^{n} D_{ij}f[\boldsymbol{r}(u)]y_iy_j=\boldsymbol{y}H[\boldsymbol{r}(u)]\boldsymbol{y}^t.$$

Hence $g''(c)=\boldsymbol{y}H(\boldsymbol{a}+c\boldsymbol{y})\boldsymbol{y}^t$, so Equation (9.36) becomes (9.34).

To prove (9.35) we define $E_2(\boldsymbol{a}, \boldsymbol{y})$ by the equation

(9.37) $$\|\boldsymbol{y}\|^2 E_2(\boldsymbol{a},\boldsymbol{y})=\frac{1}{2!}\boldsymbol{y}\{H(\boldsymbol{a}+c\boldsymbol{y})-H(\boldsymbol{a})\}\boldsymbol{y}^t \qquad \text{if } \boldsymbol{y}\ne \boldsymbol{O},$$

and let $E_2(\boldsymbol{a}, \boldsymbol{0})=0$. Then Equation (9.34) takes the form

$$f(\boldsymbol{a}+\boldsymbol{y})-f(\boldsymbol{a})=\nabla f(\boldsymbol{a})\cdot \boldsymbol{y}+\frac{1}{2!}\boldsymbol{y}H(\boldsymbol{a})\boldsymbol{y}^t+\|\boldsymbol{y}\|^2 E_2(\boldsymbol{a},\boldsymbol{y}).$$

To complete the proof we need to show that $E_2(\boldsymbol{a}, \boldsymbol{y}) \to \boldsymbol{0}$ as $\boldsymbol{y} \to \boldsymbol{0}$.

From (9.37) we find that

$$\begin{aligned}\|\boldsymbol{y}\|^2 |E_2(\boldsymbol{a},\boldsymbol{y})| &= \frac{1}{2}\Big|\sum_{i=1}^{n}\sum_{j=1}^{n}\{D_{ij}f(\boldsymbol{a}+c\boldsymbol{y})-D_{ij}f(\boldsymbol{a})\}y_iy_j\Big| \\ &\le \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}|D_{ij}f(\boldsymbol{a}+c\boldsymbol{y})-D_{ij}f(\boldsymbol{a})|\,\|\boldsymbol{y}\|^2.\end{aligned}$$