problem.)

The proofs of these Lemmas are left to the reader.

The general cubic equation (after division by the leading coefficient) has the form

$$x^3 + ax^2 + bx + c = 0.$$

(We assume here that a, b, and c are real.) Putting $x = y + k$, this equation becomes

$$y^3 + (3k + a)y^2 + (\ldots)y + (\ldots) = 0.$$

We choose $k = -a/3$, so that the square term disappears. (Tartaglia was the discoverer of this trick.) The resulting equation has the form

$$y^3 - 3py - 2q = 0$$

with p and q real and where the numbers -3 and -2 are introduced for convenience only. It is this *reduced* equation which we now want to solve.

If we put $y = u + v$, then the reduced equation becomes

$$u^3 + v^3 + 3(uv - p)(u + v) - 2q = 0.$$

To make this as simple as possible, we choose $v = p/u$ (see Lemma 25.5). The equation then becomes

$$u^3 + v^3 = 2q.$$

Since, however, $u^3v^3 = p^3$, we are back with the Babylonian problem of seeking two numbers with given sum and product. Clearly u^3 and v^3 are solutions of

$$t^2 - 2qt + p^3 = 0.$$

Therefore we have, say,

$$u^3 = q + \sqrt{q^2 - p^3},$$

$$v^3 = q - \sqrt{q^2 - p^3}.$$

Thus $y = u + v = u + p/u$, where u is a cube root of

$$q + \sqrt{q^2 - p^3}.$$

If one of these cube roots is u_1, then the others are $u_1\omega$ and $u_1\omega^2$. Let $v_1 = p/u_1$. Then $u_1\omega + p/(u_1\omega) = u_1\omega + v_1\omega^2$ and $u_1\omega^2 + p/(u_1\omega^2) = u_1\omega^2 + v_1\omega$. Hence the reduced cubic has the following three solutions only:

$$u_1 + v_1, \quad u_1\omega + v_1\omega^2, \quad u_1\omega^2 + v_1\omega.$$

To discuss these solutions in detail, we consider three cases.

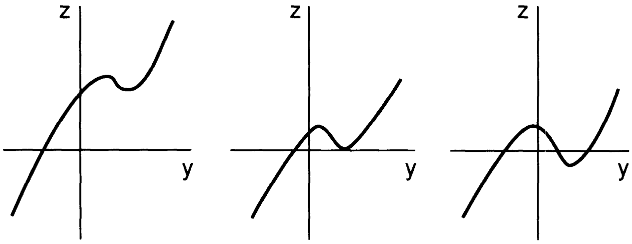

FIGURE 25.1. Cubic polynomials

Case 1. $q^2 - p^3 > 0$. Say $q^2 - p^3 = r^2$, where r is real. Let u_1 be the real cube root of $q + r$, so that v_1 is the real cube root of $q - r$ (the product u_1v_1 must be real). Then the solutions are the real number $u_1 + v_1$ and the imaginary 'conjugates' $u_1\omega + v_1\omega^2$ and $u_1\omega^2 + v_1\omega$. (Note that ω^2 is the *conjugate* of ω, that is, it may be obtained from ω by replacing i with $-i$.)

Case 2. $q^2 - p^3 = 0$. Then we can take u_1 as the real cube root of q, and v_1 the same. The equation then has only two distinct roots, namely, $2u_1$ and $-u_1$ (since $\omega + \omega^2 = -1$), but we say that the latter root occurs twice.

Case 3. $q^2 - p^3 < 0$. Say $q^2 - p^3 = -r^2$, where r is a positive real number. Then $u^3 = q + ir$ and $v^3 = q - ir$. Let $u_1 = a + bi$ with a and b real. Then $(a^2 + b^2)^3 = q^2 + r^2 = p^3$, so that $a^2 + b^2 = p$ and hence $v_1 = p/u_1 = a - bi$. We calculate

$$u_1\omega + v_1\omega^2 = a(\omega + \omega^2) + bi(\omega - \omega^2) = -a - b\sqrt{3}.$$

Similarly, $u_1\omega^2 + v_1\omega = -a + b\sqrt{3}$. It is not hard to show that $a = \sqrt{p}\cos((\arctan r/q)/3)$ and $b = \sqrt{p}\sin((\arctan r/q)/3)$. (Note that, in Case 3, $p > 0$.)

If $z = y^3 - 3py - 2q$, the three cases are illustrated by the three graphs in Figure 25.1. However, it should be borne in mind that these graphs were not available in the Renaissance, as analytic geometry had not yet been invented.

The general quartic equation has the form

$$x^4 + ax^3 + bx^2 + cx + d = 0,$$

which we may write

$$x^4 + ax^3 = -bx^2 - cx - d.$$

Adding $a^2x^2/4$ to both sides of this equation, we obtain

$$\left(x^2 + \frac{1}{2}ax\right)^2 = \left(\frac{1}{4}a^2 - b\right)x^2 - cx - d.$$

In an attempt to get a perfect square on both sides of the equation, we add $t(x^2 + \frac{1}{2}ax) + \frac{1}{4}t^2$ to both sides:

$$\left(x^2 + \frac{1}{2}ax + \frac{1}{2}t\right)^2 = \left(\frac{1}{4}a^2 - b + t\right)x^2 + \left(-c + \frac{1}{2}at\right)x - d + \frac{1}{4}t^2.$$

Now $Ax^2 + Bx + C$ is a perfect square when $B^2 - 4AC = 0$. In fact, if $A \neq 0$, we can then write

$$Ax^2 + Bx + C = (\sqrt{A}x + B/(2\sqrt{A}))^2.$$

So, to get a square, it will suffice to pick t so that

$$-d + \frac{1}{4}t^2 = \frac{\left(-c + \frac{1}{2}at\right)^2}{4\left(\frac{1}{4}a^2 - b + t\right)},$$

that is,

$$t^3 - bt^2 + (ac - 4d)t + 4bd - a^2d - c^2 = 0.$$

But this is a cubic equation! In practice, this associated cubic equation can often be solved by trial and error. We only need one value of t.

We see that if x is such that $x^4 + ax^3 + bx^2 + cx + d = 0$, then there is a t, which we can determine by finding a real root of the above cubic equation, such that

$$\left(x^2 + \frac{1}{2}ax + \frac{1}{2}t\right)^2 = (\sqrt{A}x + B/2\sqrt{A})^2,$$

with $A = \frac{1}{4}a^2 - b + t$, $B = -c + \frac{1}{2}at$ and $C = -d + \frac{1}{4}t^2$.
This gives us

$$x^2 + \frac{1}{2}ax + \frac{1}{2}t = \pm(\sqrt{A}x + B/2\sqrt{A}),$$

which is a quadratic equation in x.

For several centuries people tried to find similar methods for solving equations of degree greater than 4. They failed. It was only in the 19th century that it was shown by Ruffini, Abel and Galois that the general equation of degree 5 or more cannot be solved by 'radicals' (e.g. fifth roots). Of course, special cases can be solved by radicals, for example $x^5 + x = 34$.

Exercises

1. Prove all the Lemmas in this chapter.

2. Solve the following equations, obtaining exact answers. Do not use decimal approximations. Simplify answers.

   (a) $x^3 + x^2 - 2 = 0$,

   (b) $x^3 + 9x - 2 = 0$,

   (c) $y^3 - 3y + 1 = 0$,

   (d) $y^3 - 7y - 7 = 0$,

   (e) $x^3 + 2x^2 + 10x = 20$,

   (f) $x^3 + 3x^2 = 5$,

   (g) $x^3 + 6x^2 + 8x = 1000$,

   (h) $x^4 + x^3 - 6x^2 - x + 1 = 0$,

   (i) $x^3 = 15x + 4$.

3. Why cannot a fourth degree equation have five or more distinct complex roots?