but then three more were discovered in an Arabic translation going back to the ninth century. (See Jacques Sesiano, *Books IV to VII of Diophantus's Arithmetica.*)

These 13 books consisted of solutions to algebraic problems. The solutions are all rational numbers. Some of the equations or systems of equations are indeterminate and often there is more than one rational solution. Diophantus, however, is usually content to give just one solution. Note that what we now call a 'Diophantine equation' is one whose unknowns are not just rational but integers. Diophantus, however, accepted any rational solution.

As an example, let us consider problem 9 of Book II. The problem is 'to divide a given number which is the sum of two squares into two other squares.' That is, given rationals a and b, find a nontrivial rational solution of

$$x^2 + y^2 = a^2 + b^2.$$

Diophantus takes the special case where $a = 2$ and $b = 3$, but his solution is easily generalized. He writes:

> Take $(x + 2)^2$ as the first square and $(mx - 3)^2$ as the second (where m is an integer), say $(2x - 3)^2$. Therefore $(x^2 + 4x + 4) + (4x^2 + 9 - 12x) = 13$, or $5x^2 + 13 - 8x = 13$. Therefore $x = 8/5$, and the required squares are $324/25$ and $1/25$.

Note that, to get the general solution, m should be any rational number.

An interesting question is whether Diophantus was aware of the algebraic rules that lay behind many of his solutions. In Book III, problem 19, Diophantus notes that 65 is a sum of two squares in two ways since 65 'is the product of 13 and 5, each of which numbers is the sum of two squares'. From this we can deduce that he knew that the product of two integers, each of which is a sum of two squares, is itself a sum of two squares, and in two ways. Did Diophantus also know the stronger proposition that

$$(a^2 + b^2)(c^2 + d^2) = (ac \mp bd)^2 + (ad \pm bc)^2 \ ?$$

Basing himself just on the remark which we have quoted from Book III, problem 19, T. L. Heath conjectured that Diophantus did know this identity (see page 105 in Heath's translation of the *Arithmetica*). The person who first published the algebraic identity, however, was Abu Jafar al-Khazin (950 AD), and, later, Fibonacci gave it in his *Liber Quadratorum* (1225 AD).

Diophantus was the first to make systematic use of a symbolic notation for algebraic expressions. He denoted $+$ by juxtaposition, $-$ by the symbol $\pitchfork$ and $=$ by $\wr$. He wrote

$$K^{v} \text{ for } x^3,$$
$$\Delta^{v} \text{ for } x^2,$$

$$\varsigma \text{ for } x^1 \ (\varsigma\varsigma \text{ for the plural}),$$
$$\overset{\circ}{M} \text{ for } x^0.$$

For example,

$$\Delta^{\upsilon}\overline{\gamma}\ \overset{\circ}{M}\ \overline{\iota\beta}$$

stands for $3x^2 + 12$, while

$$K^{\upsilon}\overline{\alpha}\varsigma\varsigma\overline{\eta} \pitchfork \Delta^{\upsilon}\overline{\epsilon}\ \overset{\circ}{M}\ \overline{\alpha} \wr \varsigma\overline{\alpha}$$

represents the equation

$$(x^3 + 8x) - (5x^2 + 1) = x.$$

In 320 AD, the Roman Empire had its first Christian emperor, Constantine, after whom the Eastern Roman capital was named Constantinople. In Alexandria, Athanasius was defending the divinity of Jesus against Arius, who asserted that Jesus was *like* God, but not *equal* to him. (The difference between these two words in Greek consisted of one letter, the Greek letter iota. We have preserved this difference in Mathematics, when we distinguish between 'homeomorphism' and 'homomorphism'.)

Meanwhile in Alexandria, Pappus was writing his encyclopaedic *Collection* of earlier mathematical works. The school of mathematics had declined, and Pappus was its last, lone member.

The 'Theorem of Pappus' appears in Book VII of the *Collection*. It is far more important than Pappus realized. It expresses the commutativity of multiplication and is fundamental to projective geometry. Hilbert made use of it as a key theorem in his presentation of Euclidean geometry. The theorem of Pappus can be proved with the help of the theorem of Menelaus as follows.

Theorem of Pappus: Given points ABC on one line, $A'B'C'$ on another, the three points of intersection $P = BC' \cap CB'$, $Q = AB' \cap BA'$ and $R = CA' \cap AC'$ are collinear (Figure 20.1).

In stating this result, we have assumed that BC' and CB' etc., are not parallel. We shall also assume that ABC and $A'B'C'$ meet at a point X and that none of the other lines in the diagram are parallel.

Proof: Let $A'B \cap B'C = U$, $AC' \cap A'B = V$ and $B'C \cap AC' = W$. We apply Menelaus's Theorem five times to the triangle UVW. Since $A'CR$, $BC'P$, $AB'Q$, $A'B'C'$, and ABC are all collinear, we have

$$VR \cdot WC \cdot UA' = RW \cdot CU \cdot A'V,$$

$$VC' \cdot WP \cdot UB = C'W \cdot PU \cdot BV,$$

$$VA \cdot WB' \cdot UQ = AW \cdot B'U \cdot QV,$$

$$VC' \cdot WB' \cdot UA' = C'W \cdot B'U \cdot A'V,$$