15.4 Topology of Complex Projective Curves

To understand the complete structure of the complex projective curve defined by $y^2 = x$ we need to know its behavior at infinity. At ∞ there is another branch point like the one at 0 (just replace x by $1/u$ and y by $1/v$ and notice that we are looking at $v^2 = u$ near $y = 0$, $v = 0$—the same situation as before). The topological nature of the relation between x and y can then be captured by the model seen in Figure 15.9. A sphere (the x sphere) is covered by two spheres (like skins of an onion), slit along a line from 0 to ∞ and cross-joined. The slit from 0 to ∞ is arbitrary, but the cross-joining is necessary to produce the branch point structure at 0 and ∞.

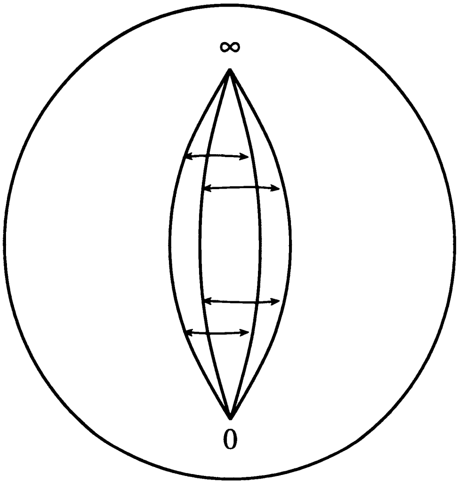

Figure 15.9: Covering the sphere

The covering of the x sphere by this two sheeted surface expresses the "covering projection map" $(x,y) \mapsto x$ from a general point on the curve $y^2 = x$ to its x coordinate and shows that it is two-to-one except at the branch points 0, ∞. The two-sheeted surface itself captures the intrinsic topological structure of the curve, and this structure can be more readily seen by separating the two skins from the x sphere and each other, then joining the required edges (Figure 15.10). Edges to be joined are labeled by the same letters, and we see that the resulting surface is topologically a sphere.

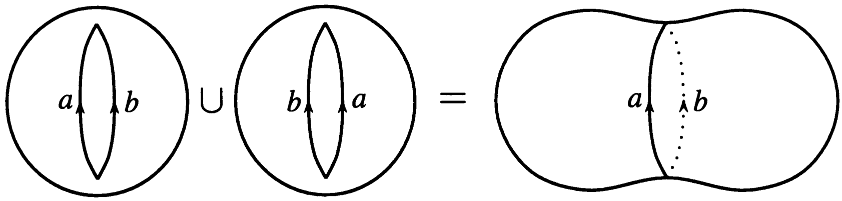

Figure 15.10: Joining the separated sheets

This result could have been obtained more directly by projecting each point (x,y) on the curve to y, since this is a one-to-one continuous map between the curve and the y-axis, which we know to be topologically a sphere (when ∞ is included). The curve here was modelled by cutting and joining sheets on the sphere because this method extends to all algebraic curves. The Newton–Puiseux theory implies that any algebraic relation $p(x,y) = 0$ can be modeled by a finite-sheeted covering of the sphere, with finitely many branch points. The most general branch point structure is given by a prescription for cross-joining (permuting) the sheets, and by slitting the sheets between branch points (or, if necessary, to an auxiliary point) they can be rejoined to produce the prescribed branching behavior.

The most interesting case of this method is the cubic curve

$$y^2 = x(x-\alpha)(x-\beta).$$

This relation defines a covering in the x sphere which is two-sheeted, since for each x there are $+$ and $-$ values for y, with branch points at 0, α, β, and ∞. (The branch point at ∞ is explained in the exercises below.) Thus if we slit the sheets from 0 to α and from β to ∞, the required joining is like that shown in Figure 15.11. We find, as Riemann did, that the surface is a torus, and hence *not* topologically the same as a sphere. This discovery proved to be a revelation for the understanding of cubic curves and elliptic functions, as we see in the next chapter.

One quickly sees that by considering relations of the form

$$y^2 = (x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_{2n})$$

it is possible to obtain Riemann surfaces of the form shown in Figure 15.12. These surfaces are distinguished topologically from each other by the number of "holes": 0 for the sphere, 1 for the torus, and so on. This simple topological invariant turns out to be the *genus* that also determines the type

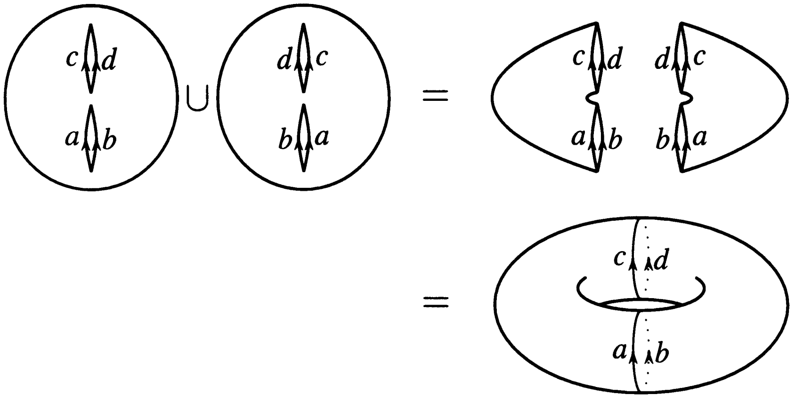

Figure 15.11: Joining the sheets of a cubic curve

of functions that can parameterize the corresponding complex curve. Other geometric and analytic properties of genus will unfold over the next few chapters. The topological importance of genus was established by Möbius (1863), when he showed that any closed surface in ordinary space is topologically equivalent to one of the form seen in Figure 15.12.

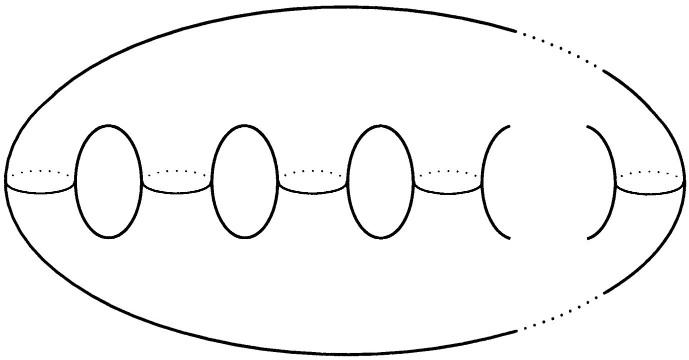

Figure 15.12: A general Riemann surface

EXERCISES

We can transfer the "one-dimensional branch point" (Figure 15.8) to infinity to see the topology of the real projective curve $y^2 = x$.

15.4.1 Explain why the real projective curve $y^2 = x$ has a branch point at infinity like the one at 0, and hence conclude that this curve is topologically a circle.

The explanation of the branch point at infinity of a cubic curve goes as follows.

15.4.2 Use the substitution $x = 1/u, y = 1/v$ to show that the curve

$$y^2 = x(x-\alpha)(x-\beta)$$

behaves at infinity like the curve

$$v^2 = u^3(1-u\alpha)^{-1}(1-u\beta)^{-1}$$

does at 0, which in turn is qualitatively like the behavior of

$$v = u^{3/2}.$$

15.4.3 Show, by considering the points lying above $u = e^{i\theta}$, that $v = u^{3/2}$ has a branch point at 0 like that of $v = u^{1/2}$.

15.5 Biographical Notes: Riemann

Bernhard Riemann (Figure 15.13) was born in the village of Breselenz, near Hannover, in 1826, and died at Selasca in Italy in 1866. He was the second of six children of Friedrich Riemann, a Protestant minister, and Charlotte Ebell. Up to the age of 13 he was taught by his father, with the help of the village schoolmaster, but he showed such a grasp of mathematics that sometimes they were unable to follow him. In 1840 Riemann went to live with his grandmother in Hannover in order to attend secondary school. After her death in 1842 he continued his studies at a school in Lüneburg, which was nearer to home, his father having moved to a new parish in the village of Quickborn. In Lüneburg it was his good fortune to have a headmaster who recognized his talent and gave him books by Euler and Legendre to read. The story goes that he mastered Legendre's 800-page *Théorie des Nombres* in six days.

The bright side of Riemann's life, which we have seen so far, was not unlike Abel's. But, as with Abel, there was a darker side as well. Riemann's family was also poor and suffered from tuberculosis. His mother, three sisters, and Riemann himself eventually died from the disease. At least Riemann was spared the family discord and very early death that made Abel's life so tragic. At all times he maintained a close and loving relationship with his family, he lived long enough to marry and become a father,

Figure 15.13: Bernhard Riemann

and he also had time to develop his major ideas to maturity and to gain a significant following. Riemann's published work—just a single volume—is in fact less copious than that of any important mathematician who lived to his fortieth year. But no other single volume has had such an impact on modern mathematics.

Riemann's career as a mathematician began soon after he entered the University of Göttingen in 1846. He intended to follow in his father's footsteps by studying theology but, like Euler and the Bernoullis before him, he found the call of mathematics too strong and obtained his father's permission to switch fields. The switch to mathematics was in recognition of where his greatest talent lay, not because of disdain for theology or philosophy. In fact, Riemann was deeply pious and well read in philosophy—so much so that readers ever since have lamented the influence of German philosophical writing on his style.

Göttingen in 1846 was not the mecca for mathematicians one would have expected it to be with the great Gauss in the chair of mathematics. Professors kept aloof from students and did not encourage original thinking or lecture on current research. Even Gauss himself taught only elementary courses. After a year, Riemann transferred to the University of Berlin, where the atmosphere was more democratic and where Jacobi, Dirichlet, Steiner, and Eisenstein shared their latest ideas. Riemann was too shy to immerse himself fully in this radically different environment, but he became friendly with Eisenstein, who was just three years his senior, and learned a great deal from Dirichlet. Riemann's later work made highly original use of some of Dirichlet's ideas, in particular a quasi-physical principle (actually first stated by Kelvin) Riemann called *Dirichlet's principle.* Among the remarkable conclusions he drew from this principle was the theorem that curves of topological genus 0 are precisely those that can be parameterized by rational functions.

Dirichlet's forté was the use of analysis in pure mathematics, particularly in number theory, and Riemann too has been broadly classified as an analyst. However, he was not a specialist as analysts usually are today. His field was all of mathematics, seen from the analytic viewpoint. He saw where analysis could be used to illuminate mathematics from number theory to geometry, but he also saw where analysis itself was in need of illumination from outside. The concept of a Riemann surface, and the topological concept of genus in particular, made many previously hard-won results of analysis almost obvious. A vivid example of the illumination of analysis by topology is Riemann's explanation of the double periodicity of elliptic functions, which we shall see in Section 16.4.

Riemann surfaces were introduced in Riemann's doctoral thesis [Riemann (1851)]. He had returned to Göttingen in 1849 and, after gaining his doctorate, began working to qualify for a Privatdozent (lecturing) position. One of the requirements was an essay, which he met with a memoir on Fourier series in which he introduced the "Riemann integral" concept. The Riemann integral is not really one of Riemann's best ideas—although it is the one best known to students today—since the integral later introduced by Lebesgue is far better suited to the subject (see Chapter 23). The other requirement was a lecture, for which he had to submit three titles to the university faculty. Gauss chose the third, which was the most difficult, on the foundations of geometry. However, Riemann rose brilliantly to the occasion, and his lecture *Über die Hypothesen, welche der Geome-*

trie zu Grunde liegen became one of the classics of mathematics [Riemann (1854b)]. In it he introduced the main ideas of modern differential geometry: n-dimensional spaces, metrics and curvature, and the way in which curvature controls global geometric properties of a space. In the special case of two dimensions, these ideas had already been grasped by Gauss (see Chapter 17), so it was a joy and a revelation to Gauss, then in the last year of his life, to see how much further Riemann had carried them.

Riemann succeeded in becoming a lecturer and had the satisfaction of attracting an unexpectedly large class (eight students!). During the next few years he developed the material for perhaps his greatest work [Riemann (1857)], which did for algebraic geometry what he had earlier [Riemann (1854b)] done for differential geometry. One of his students at this time was Dedekind, who later recast Riemann's theory into the more algebraic form that is used today. Dedekind also coedited Riemann's collected works and wrote an essay on Riemann's life [Dedekind (1876)], which is the main biographical source for this section. The lecturer's position was very productive mathematically, but it brought in only voluntary fees from students, and Riemann was close to starvation. Other setbacks he suffered were the death of his father and sister Clara and a nervous breakdown brought on by overwork.

When Gauss died in 1855 and was succeeded by Dirichlet there was an unsuccessful move to appoint Riemann as associate professor. This move failed, but Riemann was granted a regular salary, and when Dirichlet died in 1859 Riemann succeeded him. In 1862 he married Elise Koch, a friend of his sisters, and their daughter, Ida, was born in Pisa in 1863. Riemann had begun traveling to Italy for the sake of his health in 1862, and he spent much time there during his remaining years. He loved Italy and its art treasures and also received a warm reception from Italian mathematicians. Two of his friends in Pisa, Enrico Betti and Eugenio Beltrami, were inspired by Riemann's ideas to make important contributions to topology and differential geometry. Beltrami saw how Riemann's concept of curved space could be used as a basis for noneuclidean geometry, a revolutionary discovery that even Riemann may not have anticipated (see Chapter 18).

Riemann's sojourn in Italy was all too short. He died at Selasca on Lake Maggiore in the summer of 1866, with his wife beside him. Dedekind described his last days as follows (not in his usual style, but no doubt sensitive to the feelings of Riemann's widow):

> One the day before his death he lay beneath a fig tree, filled

> with joy at the glorious landscape, writing his last work, unfortunately left incomplete. His end came gently, without struggle or death agony; it seemed as though he followed with interest the parting of the soul from his body; his wife had to give him bread and wine, he asked her to convey his love to those at home, saying "Kiss our child." She said the Lord's prayer with him, he could no longer speak; at the work "Forgive our trespasses" he raised his eyes devoutly, she felt his hand in hers becoming colder, and after a few more breaths his pure, noble heart ceased to beat. The gentle mind implanted in him in his father's house stayed with him all his life, and he remained true to his God as his father had, though not in the same form.
>
> [Dedekind (1876)]

It was said of Abel that he left enough to keep mathematicians busy for 500 years, and the same might be said of Riemann. Today, more than 130 years after Riemann's death, the major unsolved problem in pure mathematics is the so-called *Riemann hypothesis*, a conjecture made casually by Riemann (1859) in his paper on the distribution of prime numbers. Riemann considered Euler's function (discussed in Section 10.7),

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots,$$

introducing the zeta notation for it, and extended it to complex values of s. He observed that if $\zeta(s) = 0$, then $0 \leq \mathrm{Re}(s) \leq 1$, and added that it was quite likely that all zeros of $\zeta(s)$ had real part $1/2$. He did not pursue the matter further, since his initial observation was enough for his purpose, which was to derive an infinite series for $F(x)$, the number of primes less than a positive integer x. Mathematicians later realized that Riemann's hypothesis governs the distribution of prime numbers to an extraordinary extent, which is why its proof is so eagerly sought. Since all the efforts of the best mathematicians have failed so far, perhaps only another Riemann will succeed.