translates into

$$(A^B)^A \leftrightarrow 1,$$

with mappings in both directions; but the 1-to-1 correspondence $(A^B)^A \cong 1$ holds only if $A \cong 1$ or $A \cong 0$ or $B \cong 0$.

We should also point out that the interpretation of $A \vee B$ advocated here does not work for classical logic. As we see in Chapter 28, there is a formula G in number theory for which one can prove neither G nor $\neg G$. According to the intended interpretation of $A \vee B$, $G \vee \neg G$ has no proof. Yet, classically, $G \vee \neg G$ is just a case of Aristotle's axiom of the excluded third and thus quoting this axiom would constitute a proof.

Exercises

1. Take any of the one-to-one correspondences of Chapter 13, translate it into a statement of intuitionistic logic and prove the latter.

2. Take any intuitionistic theorem of the form $A \vdash B$ and find the corresponding mapping $A \to B$ between sets.

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Intuitionistic Predicate Calculus

We shall be dealing with formulas A which contain so-called *free variables* such as x in $x^2+2=0$ or x and y in $xy+x=y$. To indicate which variables may be present we often write $A(x)$ or $A(x,y)$ instead of just A.

From $A(x)$ one may obtain the formulas $\forall_x A(x)$ and $\exists_x A(x)$ in which x is no longer free; it is *bound* to the *universal quantifier* $\forall_x$, meaning *for all* x, or to the *existential quantifier* $\exists_x$, meaning *for some* x. Similarly, one may form $\forall_x\forall_y A(x,y)$, $\forall_x\exists_y A(x,y)$, etc.

From $A(x)$ one may also obtain the formula $A(t)$, the result of substituting the *term* t for x. Here t may be 5 or y or $x+3y$ or whatever — it may even be x. However, if $A(x)$ is $\exists_y B(x,y)$, we are not supposed to substitute y or $x+3y$ for x, because y is no longer free in $\exists_y B(y,y)$ or $\exists_y B(3x+y,y)$. This prohibition is spelled out in the following definition: t is *substitutable* for x in $A(x)$ provided any free variable in t (maybe t itself) remains free in $A(t)$. This definition is needed for stating the following two axioms:

$$\forall_x A(x) \vdash A(t) \quad \text{(universal specification)},$$

$$A(t) \vdash \exists_x A(x) \quad \text{(existential generalization)},$$

— subject to the restriction that t is substitutable for x in $A(x)$.

Were it not for this restriction, we would have as a special case of the first axiom

$$\forall_x\exists_y B(x,y) \vdash \exists_y B(y,y)$$

and from 'everybody blames somebody' we could infer that 'somebody blames himself'.

In addition to the above two axioms, we shall also adopt the following two rules of inference:

$$\frac{\Gamma \vdash A(x)}{\Gamma \vdash \forall_x A(x)} \quad \text{(universal generalization)},$$

provided x is not free in Γ;

$$\frac{\Gamma,\ A(x) \vdash B}{\Gamma,\ \exists_x A(x) \vdash B} \quad \text{(existential specification)},$$

provided x is not free in Γ or B.

We abbreviate the names of these axioms and rules of inference by US, EG, UG, and ES, respectively. The reason for the restriction on UG, for example, is to avoid inferring from 'x is afraid' that 'everyone is afraid'.

We shall present some examples to illustrate arguments involving quantifiers.

EXAMPLE 1. To prove $\forall_x(F(x) \wedge G(x)) \vdash \forall_x F(x) \wedge \forall_x G(x)$.

1	(1)	$\forall_x(F(x) \wedge G(x))$	given
1	(2)	$F(x) \wedge G(x)$	US 1
1	(3)	$F(x)$	axiom for $\wedge$, 2
1	(4)	$\forall_x F(x)$	UG 3 (x not free in 1)
1	(5)	$G(x)$	axiom for $\wedge$, 2
1	(6)	$\forall_x G(x)$	UG 5 (x not free in 1)
1	(7)	$\forall_x F(x) \wedge \forall_x G(x)$	axiom for $\wedge$, 4,6

EXAMPLE 2. To prove $\exists_y \forall_x F(x, y) \vdash \forall_x \exists_y F(x, y)$.

1	(1)	$\exists_y \forall_x F(x, y)$	given
2	(2)	$\forall_x F(x, y)$	assumed
2	(3)	$F(x, y)$	US 2
2	(4)	$\exists_y F(x, y)$	EG 3
1	(5)	$\exists_y F(x, y)$	ES 4 (y not free in 4)
1	(6)	$\forall_x \exists_y F(x, y)$	UG 5 (x not free in 1)

EXAMPLE 3. To prove $\neg\exists_x F(x) \vdash \forall_x \neg F(x)$.

1	(1)	$\exists_x F(x) \Rightarrow \perp$	given
2	(2)	$F(x)$	assumed
2	(3)	$\exists_x F(x)$	EG 2
1,2	(4)	$\perp$	MP 1,2
1	(5)	$F(x) \Rightarrow \perp$	DR 4,2
1	(6)	$\forall_x (F(x) \Rightarrow \perp)$	UG 5 (x not free in 1)

Exercises

1. Prove the converse of Example 1.
2. What goes wrong if you try to prove the converse of Example 2?
3. Can you prove the converse of Example 3?
4. Prove that $\neg\forall_x \neg F(x) \vdash \neg\neg\exists_x F(x)$, using Example 3 above and Example 4 of Chapter 24. Classically, but not intuitionistically, one can infer from this that $\neg\forall_x \neg F(x) \vdash \exists_x F(x)$. If we could prove $\neg\forall_x \neg F(x)$, we would have a nonconstructive proof of $\exists_x F(x)$.
5. Prove that $\exists_x \forall_y F(x, y) \vdash \exists_x F(x, x)$.

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Intuitionistic Type Theory

This chapter is an adaptation of the appendix in Couture and Lambek [1991], giving a brief overview of a recent formulation of type theory in Lambek and Scott [1986], which is adequate for elementary mathematics, including arithmetic and analysis, when treated constructively. As far as we know, the only proofs in these disciplines which are essentially nonconstructive depend on the *axiom of choice*. One formulation of this axiom asserts that, for any nonempty collection of nonempty sets, there exists a set containing exactly one element from each of the given sets.

From basic types 1, Ω and N one builds others by two processes: if A is a type so is PA; if A and B are types, so is $A \times B$. The intended meaning of these types is as follows:

- 1 is the type of a specified single entity (introduced for convenience);
- Ω is the type of truth values or propositions (here there are more than two truth values);
- N is the type of natural numbers;
- PA is the type of sets of entities of type A;
- $A \times B$ is the type of pairs of entities of types A and B, respectively.

We allow arbitrarily many variables of each type and write $x \in A$ to mean that x is a variable of type A. In addition, we construct terms of different types inductively as follows: