It remains to prove that the idempotents in part (3) are primitive. If $e_i = a + b$, for some orthogonal idempotents a and b, then we shall see that

$$L_i = Re_i = Ra \oplus Rb.$$

This will contradict the fact that L_i is a simple R-module. To establish the above direct sum note first that since $ab = ba = 0$, we have $ae_i = a \in Re_i$ and $be_i = b \in Re_i$. For all $r \in R$ we have $re_i = ra + rb$, hence $Re_i = Ra + Rb$. Moreover, $Ra \cap Rb = 0$ because if $ra = sb$ for some $r, s \in R$, then $ra = raa = sba = 0$ (recall $a = a^2$ and $ba = 0$). This completes all parts of the proof.

Proposition 8. Let $R = R_1 \times R_2 \times \cdots \times R_r$, where R_i is the ring of $n_i \times n_i$ matrices over the division ring Δ_i, for $i = 1, 2, \ldots, r$.

(1) Identify R_i with the i^{th} component of the direct product. Let z_i be the r-tuple with the identity of R_i in position i and zero in all other positions. Then $R_i = z_i R$ and for any $a \in R_i$, $z_i a = a$ and $z_j a = 0$ for all $j \neq i$. The elements $z_1, \ldots, z_r$ are all of the primitive central idempotents of R. They are pairwise orthogonal and $\sum_{i=1}^{r} z_i = 1$.

(2) Let N be any left R-module and let $z_i N = \{z_i x \mid x \in N\}$, $1 \leq i \leq r$. Then $z_i N$ is a left R-submodule of N, each $z_i N$ is an R_i-module on which R_j acts trivially for all $j \neq i$, and

$$N = z_1 N \oplus z_2 N \oplus \cdots \oplus z_r N.$$

(3) The simple R-modules are the simple R_i-modules on which R_j acts trivially for $j \neq i$ in the following sense. Let M_i be the unique simple R_i-module (cf. Proposition 6). We may consider M_i as an R-module by letting R_j act trivially for all $j \neq i$. Then $M_1, \ldots, M_r$ are pairwise nonisomorphic simple R-modules and any simple R-module is isomorphic to one of $M_1, \ldots, M_r$. Explicitly, the R-module M_i is isomorphic to the simple left ideal $(0, \ldots, 0, L^{(i)}, 0, \ldots, 0)$ of all elements of R whose i^{th} component, $L^{(i)}$, consists of matrices with arbitrary entries in the first column and zeros elsewhere.

(4) For any R-module N the R-submodule $z_i N$ is a direct sum of simple R-modules, each of which is isomorphic to the module M_i in (3). In particular, if M is a simple R-module, then there is a unique i such that $z_i M = M$ and for this index i we have $M \cong M_i$; for all $j \neq i$, $z_j M = 0$.

(5) If each Δ_i equals the field F, then R is a vector space over F of dimension $\sum_{i=1}^{r} n_i^2$ and $\dim_F Z(R) = r$.

Proof: In part (1) since multiplication in the direct product of rings is componentwise it is clear that z_i times the element $(a_1, \ldots, a_r)$ of R is the r-tuple with a_i in position i and zeros elsewhere. Thus $R_i = z_i R$, z_i is the identity in R_i and $z_i a = 0$ if $a \in R_j$ for any $j \neq i$. It is also clear that $z_1, \ldots, z_r$ are pairwise orthogonal central idempotents whose sum is the identity of R. The central idempotents of R are, by definition, the idempotents in $Z(R) = F_1 \times F_2 \times \cdots \times F_r$, where F_i is the center of R_i. By Proposition 6, F_i is the field $Z(\Delta_i)$. If $w = (w_1, \ldots, w_r)$ is any central idempotent then $w_i \in F_i$ for all i, and since $w^2 = w$ we have $w_i^2 = w_i$ in the field F_i. Since 0 and 1 are the only solutions to $x^2 = x$ in a field, the only central idempotents in R are r-tuples whose entries are 0's and 1's. Thus $z_1, \dots, z_r$ are primitive central idempotents and since every central idempotent is a sum of these, they are the complete set of primitive central idempotents of R. This proves (1).

To prove (2) let N be any left R-module. First note that for any $z \in Z(R)$ the set $\{zx \mid x \in N\}$ is an R-submodule of N. In particular, $z_i N$ is an R-submodule. Let $z_i x \in z_i N$ and let $a \in R_j$ for some $j \neq i$. By (1) we have that $a = az_j$ and so $az_i x = (az_j)(z_i x) = az_i z_j x = 0$ because $z_i z_j = 0$. Thus the R-submodule $z_i N$ is acted on trivially by R_j for all $j \neq i$. For each $x \in N$ we have by (1) that $x = 1x = z_1 x + \cdots + z_r x$, hence $N = z_1 N + \cdots + z_r N$. Finally, this sum is direct because if, for instance, $x \in z_1 N \cap (z_2 N + \cdots + z_r N)$, then $x = z_1 x$ whereas z_1 times any element of $z_2 N + \cdots + z_r N$ is zero. This proves (2).

In part (3) first note that an R_i-module M becomes an R-module when R_j is defined to act trivially on M for all $j \neq i$. For such a module M the R-submodules are the same as the R_i-submodules. Thus M_i is a simple R-module for each i since it is a simple R_i-module.

Next, let M be a simple R-module. By (2), $M = z_1 M \oplus \cdots \oplus z_r M$. Since M has no nontrivial proper R-submodules, there must be a unique i such that $M = z_i M$ and $z_j M = 0$ for all $j \neq i$. Thus the simple R-module M is annihilated by R_j for all $j \neq i$. This implies that the R-submodules of M are the same as the R_i-submodules of M, so M is therefore a simple R_i-module. By Proposition 6, M is isomorphic as an R_i-module to M_i. Since R_j acts trivially on both M and M_i for all $j \neq i$, it follows that the R_i-module isomorphism may be viewed as an R-module isomorphism as well.

Suppose $i \neq j$ and suppose $\varphi : M_i \to M_j$ is an R-module isomorphism. If $s_i \in M_i$ then $s_i = z_i s_i$ so

$$\varphi(s_i) = \varphi(z_i s_i) = z_i \varphi(s_i) = 0,$$

since $\varphi(s_i) \in M_j$ and z_i acts trivially on M_j. This contradicts the fact that φ is an isomorphism and proves that $M_1, \dots, M_r$ are pairwise nonisomorphic simple R-modules.

Finally, the left ideal of R described in (3) is acted on trivially by R_j for all $j \neq i$ and, by Proposition 6, it is up to isomorphism the unique simple R_i-module. This left ideal is therefore a simple R-module which is isomorphic to M_i. This proves (3).

For part (4) we have already proved that if M is any simple R-module then there is a unique i such that $z_i M = M$ and $z_j M = 0$ for all $j \neq i$. Furthermore, we have shown that for this index i the simple R-module M is isomorphic to M_i. Now let N be any R-module. Then $z_i N$ is a module over R_i which is acted on trivially by R_j for all $j \neq i$. By Wedderburn's Theorem $z_i N$ is a direct sum of simple R-modules. Since each of these simple summands is acted on trivially by R_j for all $j \neq i$, each is isomorphic to M_i. This proves (4).

In part (5) if each Δ_i equals the field F, then as an F-vector space

$$R \cong M_{n_1}(F) \oplus M_{n_2}(F) \oplus \cdots \oplus M_{n_r}(F).$$

Each matrix ring $M_{n_i}(F)$ has dimension n_i^2 over F, hence R has dimension $\sum_{i=1}^{r} n_i^2$ over F. Furthermore, the center of each $M_{n_i}(F)$ is 1-dimensional (since by Proposition 6(2) it is isomorphic to F), hence $Z(R)$ has dimension r over F. This completes the proof of the proposition.

We now apply Wedderburn's Theorem (and the above ring-theoretic calculations) to the group algebra FG. First of all, in order to apply Wedderburn's Theorem we need the characteristic of F not to divide $|G|$. In fact, since we shall be dealing with numerical data in the sections to come it will be convenient to have the characteristic of F equal to 0. Secondly, it will simplify matters if we force all the division rings which will appear in the Wedderburn decomposition of FG to equal the field F — we shall prove that imposing the condition that F be algebraically closed is sufficient to ensure this. To simplify notation we shall therefore take $F = \mathbb{C}$ for most of the remainder of the text. The reader can easily check that any algebraically closed field of characteristic 0 (e.g., the field of all algebraic numbers) can be used throughout in place of $\mathbb{C}$.

By Corollary 5 the ring $\mathbb{C}G$ is semisimple so by Wedderburn's Theorem

$$\mathbb{C}G \cong R_1 \times R_2 \times \cdots \times R_r$$

where R_i is the ring of $n_i \times n_i$ matrices over some division ring Δ_i. Thinking of the elements of this direct product as $n \times n$ block matrices ($n = \sum_{i=1}^{r} n_i$) where the i^{th} block has entries from Δ_i, the field $\mathbb{C}$ appears in this direct product as scalar matrices and is contained in the center of $\mathbb{C}G$. Note that each Δ_i is a vector space over $\mathbb{C}$ of dimension $\leq n$. The next result shows that this implies each $\Delta_i = \mathbb{C}$.

Proposition 9. If Δ is a division ring that is a finite dimensional vector space over an algebraically closed field F and $F \subseteq Z(\Delta)$, then $\Delta = F$.

Proof: Since $F \subseteq Z(\Delta)$, for each $\alpha \in \Delta$ the division ring generated by α and F is a field. Also, since Δ is finite dimensional over F the field $F(\alpha)$ is a finite extension of F. Because F is algebraically closed it has no nontrivial finite extensions, hence $F(\alpha) = F$ for all $\alpha \in \Delta$, i.e., $\Delta = F$.

This proposition proves that each R_i in the Wedderburn decomposition of $\mathbb{C}G$ is a matrix ring over $\mathbb{C}$:

$$R_i = M_{n_i}(\mathbb{C}).$$

Now Proposition 8(5) implies that

$$\sum_{i=1}^{r} n_i^2 = |G|.$$

The final application in this section is to prove that r (= the number of Wedderburn components in $\mathbb{C}G$) equals the number of conjugacy classes of G. To see this, first note that Proposition 8(5) asserts that $r = \dim_{\mathbb{C}} Z(\mathbb{C}G)$. We compute this dimension in another way.

Let $\mathcal{K}_1, \ldots, \mathcal{K}_s$ be the distinct conjugacy classes of G (recall that these partition G). For each conjugacy class $\mathcal{K}_i$ of G let

$$X_i = \sum_{g \in \mathcal{K}_i} g \quad \in \mathbb{C}G.$$

Note that X_i and X_j have no common terms for $i \neq j$, hence they are linearly independent elements of $\mathbb{C}G$. Furthermore, since conjugation by a group element permutes the