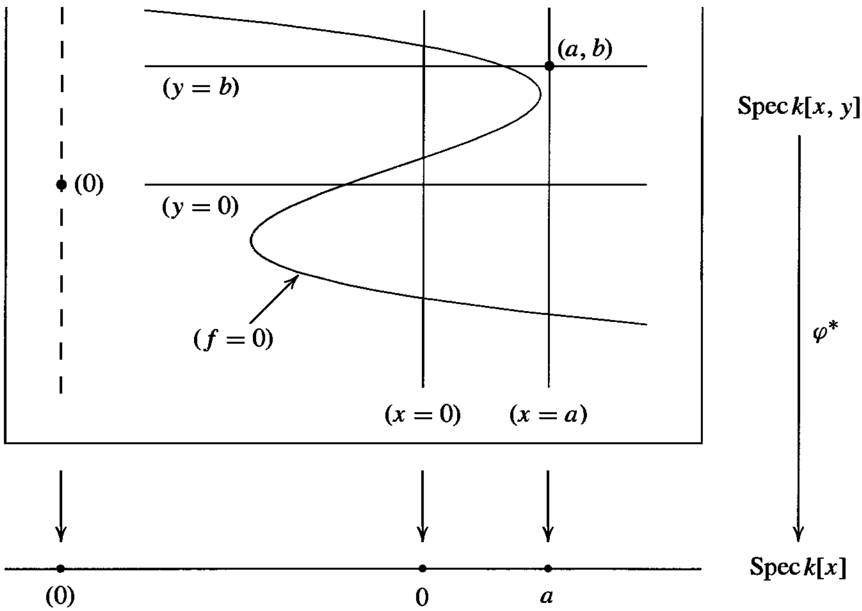


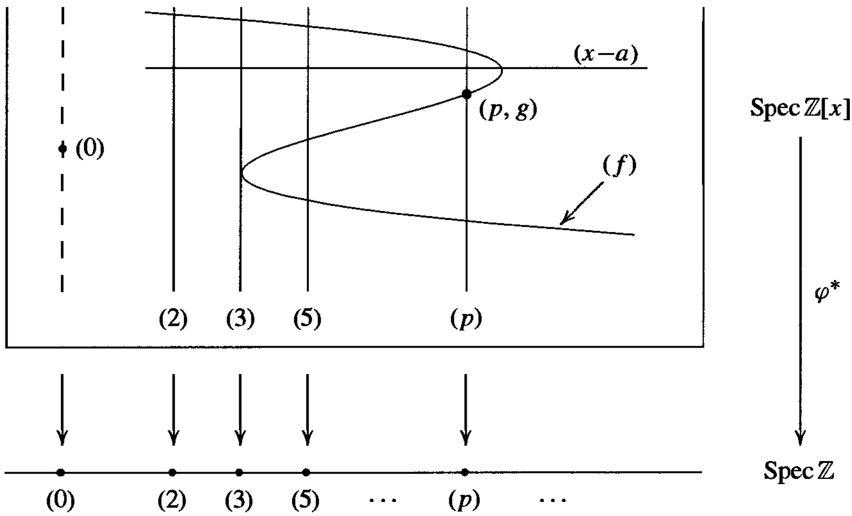
with the y-axis in $\mathbb{A}^2$; the prime $(y) \in \operatorname{Spec} k[x, y]$ similarly corresponds to the x-axis. The prime $(f) \in \operatorname{Spec} k[x, y]$ corresponds to the irreducible curve $f(x, y) = 0$ in $\mathbb{A}^2$; the points $(a, b) \in \mathbb{A}^2$ lying on this curve correspond to the maximal ideals $(x - a, y - b) \in \operatorname{Spec} k[x, y]$ containing (f). The closed point $(x - a, y - b) \in \operatorname{Spec} k[x, y]$ corresponds to the "geometric point" $(a, b) \in \mathbb{A}^2$.

Note that $\operatorname{Spec} k[x, y]$ captures all of the geometry of algebraic sets in $\mathbb{A}^2$: every algebraic set in $\mathbb{A}^2$ is the finite union of some subset of the irreducible algebraic sets corresponding to the elements of $\operatorname{Spec} k[x, y]$ pictured above. With the exception of the everywhere dense point (0), the "geometric" picture of $\operatorname{Spec} k[x, y]$ is precisely the usual geometry of the affine plane $\mathbb{A}^2$. When k is not algebraically closed the situation is slightly more complicated, but the picture is similar, cf. Exercise 4.

(3) The situation for $\operatorname{Spec} \mathbb{Z}[x]$, viewed as fibered over $\operatorname{Spec} \mathbb{Z}$ by the natural inclusion $\mathbb{Z} \to \mathbb{Z}[x]$ is very similar to the situation of $\operatorname{Spec} k[x, y]$ in the previous example. The elements of $\operatorname{Spec} \mathbb{Z}[x]$ were discussed in Example 2 following Proposition 54 and can be pictured as in the diagram on the following page.

The element (0) is Zariski dense in $\operatorname{Spec} \mathbb{Z}[x]$. The closure of (p) consists of (p) and all the closed points (p, g) where g is a monic polynomial in $\mathbb{Z}[x]$ that is irreducible mod p. The closure of (f) consists of (f) together with the maximal ideals (p, g) that contain (f), which is the same as saying that the image of f in the quotient $\mathbb{Z}[x]/(p, g)$ is 0, i.e., the irreducible polynomial g is a factor of f mod p. The closed points, $\operatorname{mSpec} \mathbb{Z}[x]$, are the maximal ideals (p, g).

Note that the maximal ideals (p, g) containing (f) are precisely the closed points in $\operatorname{mSpec} \mathbb{Z}[x]$ in the diagram above where the "function" f on $\operatorname{Spec} \mathbb{Z}[x]$ (taking the prime P to $f(P) = f \bmod P \in \mathbb{Z}[x]/P$) is zero. For example, the polynomial $f = x^3 - 4x^2 + x - 9 \in \mathbb{Z}[x]$ fits the diagram above: f is irreducible in $\mathbb{Z}[x]$, and



over $\mathbb{F}_p$ factors into irreducibles as follows:

$$f \equiv x^3 + x + 1 \bmod 2$$
$$f \equiv x(x+1)^2 \bmod 3$$
$$f \equiv (x+1)(x+2)(x+3) \bmod 5.$$

There is one point in the fiber over (2) intersecting (f), namely the closed point $(2, x^3 + x + 1)$. There are two closed points in the fiber over (3) given by $(3, x)$ and $(3, x + 1)$ (with some "multiplicity" at the latter point). Over (5) there are three closed points: $(5, x + 1)$, $(5, x + 2)$, and $(5, x + 3)$. For the diagram above, the prime p might be $p = 53$, since this is the first prime p greater than 5 for which this polynomial has three irreducible factors mod p. Note that while the prime (f) is drawn as a smooth curve in this diagram to emphasize the geometric similarity with the structure of Spec $k[x, y]$ in the previous example, the fibers above the primes in Spec $\mathbb{Z}$ are discrete, so some care should be exercised. For example, since f factors as $(x+2)(x^2+x+6) \bmod 7$, the intersection of (f) with the fiber above (7) contains only the two points $(7, x + 2)$ and $(7, x^2 + x + 6)$, each with multiplicity one.

The possible number of closed points in (f) lying in a fiber over $(p) \in \operatorname{Spec}\mathbb{Z}$ is controlled by the Galois group of the polynomial f over $\mathbb{Q}$ (cf. Section 14.8). For example, $f = x^4 + 1$ has one closed point in the fiber above (2) and either two or four closed points in a fiber above (p) for p odd (cf. Exercise 8).

The space Spec R together with its Zariski topology gives a geometric generalization for arbitrary commutative rings of the points in a variety V. We now consider the question of generalizing the ring of rational functions on V.

When V is a variety over the algebraically closed field k the elements in the quotient field $k(V)$ of the coordinate ring $k[V]$ define the rational functions on V. Each element α in $k(V)$ can in general be written as a quotient a/f of elements $a, f \in k[V]$ in many different ways. The set of points U at which α is regular is an open subset of V; by definition, it consists of all the points $v \in V$ where α can be represented by some quotient a/f with $f(v) \neq 0$, and then the representative a/f defines an element in the local ring $\mathcal{O}_{v,V}$. Note also that the same representative a/f defines α not only at v, but also at all the other points where f is nonzero, namely on the open subset $V_f = \{w \in V \mid f(w) \neq 0\}$ of V. These open sets V_f (called principal open sets, cf. Exercise 21 in Section 2) for the various possible representatives a/f for α give an open cover of U. The example of the function $\alpha = \bar{x}/\bar{y}$ for $V = \mathcal{Z}(xz - yw) \subset \mathbb{A}^4$ preceding Proposition 51 shows that in general a single representative for α does not suffice to determine all of U — for this example, $U = V_{\bar{y}} \cup V_{\bar{z}}$, and U is not covered by any single V_f (cf. Exercise 25 of Section 4).

This interpretation of rational functions as functions that are regular on open subsets of V can be generalized to Spec R. We first define the analogues X_f in $X = \text{Spec } R$ of the sets V_f and establish their basic properties.

Definition. For any $f \in R$ let X_f denote the collection of prime ideals in $X = \text{Spec } R$ that do not contain f. Equivalently, X_f is the set of points of Spec R at which the value of $f \in R$ is nonzero. The set X_f is called a *principal* (or *basic*) *open set* in Spec R.

Since X_f is the complement of the Zariski closed set $\mathcal{Z}(f)$ it is indeed an open set in Spec R as the name implies. Some basic properties of the principal open sets are indicated in the next proposition. Recall that a map between topological spaces is a *homeomorphism* if it is continuous and bijective with continuous inverse.

Proposition 56. Let $f \in R$ and let X_f be the corresponding principal open set in $X = \text{Spec } R$. Then

(1) $X_f = X$ if and only if f is a unit, and $X_f = \emptyset$ if and only if f is nilpotent,
(2) $X_f \cap X_g = X_{fg}$,
(3) $X_f \subseteq X_{g_1} \cup \cdots \cup X_{g_n}$ if and only if $f \in \text{rad}(g_1, \dots, g_n)$; in particular $X_f = X_g$ if and only if $\text{rad}(f) = \text{rad}(g)$,
(4) the principal open sets form a basis for the Zariski topology on Spec R, i.e., every Zariski open set in X is the union of some collection of principal open sets X_f,
(5) the natural map from R to R_f induces a homeomorphism from Spec R_f to X_f, where R_f is the localization of R at f,
(6) the spectrum of any ring is quasicompact (i.e., every open cover has a finite subcover); in particular, X_f is quasicompact, and
(7) if $\varphi : R \to S$ is any homomorphism of rings (with $\varphi(1_R) = 1_S$) then under the induced map $\varphi^* : Y = \text{Spec } S \to \text{Spec } R$ the full preimage of the principal open set X_f in X is the principal open set $Y_{\varphi(f)}$ in Y.

Proof: Parts (1), (2) and (7) are left as easy exercises. For (3), observe that, by definition, $X_{g_1} \cup \cdots \cup X_{g_n}$ consists of the primes P not containing at least one of $g_1, \dots, g_n$. Hence $X_{g_1} \cup \cdots \cup X_{g_n}$ is the complement of the closed set $\mathcal{Z}((g_1, \dots, g_n))$ consisting of the primes P that contain the ideal generated by $g_1, \dots, g_n$. If $(g_1, \dots, g_n) = R$ then $X_{g_1} \cup \cdots \cup X_{g_n} = X$ and there is nothing to prove. Otherwise, $X_f \subseteq X_{g_1} \cup \cdots \cup X_{g_n}$ if and only if every prime P with $f \notin P$ also satisfies $P \notin \mathcal{Z}((g_1, \dots, g_n))$. This latter condition is equivalent to the statement that if the prime P contains the ideal