Definition. If A is a central simple F-algebra then a field L containing F is said to *split* A if $A \otimes_F L \cong M_m(L)$ for some $m \geq 1$.

It follows from (II) that every maximal commutative subalgebra of Δ is a field E with $E = E^c = E^{opp}$; if $[E : F] = m$ we obtain $\dim_F \Delta = m^2$. Applying (II) to $A = \Delta$ and $B = E$ we also see that $\Delta \otimes_F E \cong M_m(E)$. It can also be shown that a maximal subfield E of the central simple F-algebra A also satisfies $E = E^c = E^{opp}$ and so again by (II) it follows that $A \otimes_F E \cong M_r(E)$ ($r^2 = \dim_F A$).

If $A = M_r(\Delta)$ then the field L splits A if and only if L splits Δ, as follows. If $\Delta \otimes_F L \cong M_n(L)$ then

$$A \otimes_F L \cong M_r(\Delta) \otimes_F L \cong M_r(\Delta \otimes_F L) \cong M_r(M_n(L)) \cong M_{rn}(L).$$

Conversely if $A \otimes_F L \cong M_n(L)$ then

$$M_n(L) \cong M_r(\Delta) \otimes_F L \cong M_r(\Delta \otimes_F L).$$

By (II) and (III), $\Delta \otimes_F L \cong M_s(\Delta')$ for some division ring Δ'. Together with the previous isomorphism, the uniqueness statement in (III) shows that $\Delta' \cong L$ and then the isomorphism $\Delta \otimes_F L \cong M_s(L)$ shows that L splits Δ.

We see from the discussion above that a maximal commutative subfield of Δ splits both Δ and $A \cong M_r(\Delta)$ for any $r \geq 1$. It is not too difficult to show from this that every central simple F-algebra of finite dimension over F can be split by a finite Galois extension of F.

Applying (I) by taking A to be the crossed product algebra B_f and taking $B = K$ shows that $K = K^c = K^{opp}$ and $B_f \otimes_F K \cong M_n(K)$. In particular, the crossed product algebras B_f are always split by K.

Example

In the example of the Hamilton Quaternions above we have $B_f \otimes_{\mathbb{R}} \mathbb{C} \cong M_2(\mathbb{C})$. We have $B_f \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C} + \mathbb{C}i + \mathbb{C}j + \mathbb{C}k$ and an explicit isomorphism φ to $M_2(\mathbb{C})$ is given by

$$\varphi(i) = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix} \qquad \varphi(j) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and extending $\mathbb{C}$-linearly.

By (III) every central simple F-algebra A is isomorphic as an F-algebra to $M_r(\Delta)$ for some division ring Δ uniquely determined up to F-isomorphism, called the *division ring part* of A.

Definition. Two central simple F-algebras A and B are *similar* if $A \cong M_r(\Delta)$ and $B \cong M_s(\Delta)$ for the same division ring Δ, i.e., if A and B have the same division ring parts.

Let $[A]$ denote the similarity class of A. By (II), if A and B are central simple F-algebras then $A \otimes_F B$ is again a central simple F-algebra, so we may define a multiplication on similarity classes by $[A][B] = [A \otimes_F B]$. The class $[F]$ is an identity for this multiplication and associativity of the tensor product shows that the multiplication is associative. By (Ib) applied with $B = A$ (so then $B^c = F$ since A is central) we have $[A][A^{opp}] = [F]$, so inverses exist with this multiplication.

Definition. The group of similarity classes of central simple F-algebras with multiplication $[A][B] = [A \otimes_F B]$ is called the *Brauer group* of F and is denoted $Br(F)$.

If L is any extension field of F then by (II) the algebra $A \otimes_F L$ is a central simple L-algebra. It is easy to check that the map $[A] \to [A \otimes_F L]$ is a well defined homomorphism from $Br(F)$ to $Br(L)$. The kernel of this homomorphism consists of the classes of the algebras A with $A \otimes_F L \cong M_m(L)$ for some $m \geq 1$, i.e., the algebras A that are split by L.

Definition. If L/F is a field extension then the *relative Brauer group* $Br(L/F)$ is the group of similarity classes of central simple F-algebras that are split by L. Equivalently, $Br(L/F)$ is the kernel of the homomorphism $[A] \to [A \otimes_F L]$ from $Br(F)$ to $Br(L)$.

The following theorem summarizes some major results in this area and shows the fundamental connection between Brauer groups and the crossed product algebras constructed above.

Theorem 42. Suppose K/F is a Galois extension of degree n with $G = \mathrm{Gal}(K/F)$.

(1) The central simple F-algebra A with $\dim_F A = n^2$ is split by K if and only if $A \otimes_F K \cong M_n(K)$ if and only if A is isomorphic to a crossed product algebra B_f as in (39) and (40).

(2) There is a bijection between the F-isomorphism classes of central simple F-algebras A with $A \otimes_F K \cong M_n(K)$ and the elements of $H^2(G, K^\times)$. Under this bijection the class $c \in H^2(G, K^\times)$ containing the normalized cocycle f corresponds to the isomorphism class of the crossed product algebra B_f defined in (39) and (40), and the trivial cohomology class corresponds to $M_n(F)$.

(3) Every central simple F-algebra of finite dimension over F and split by K is similar to one of dimension n^2 split by K. The bijection in (2) also establishes a bijection between $Br(K/F)$ and $H^2(G, K^\times)$ which is also an isomorphism of groups.

(4) There is a bijection between the collection of F-isomorphism classes of central simple division algebras over F that are split by K and $H^2(G, K^\times)$.

As previously mentioned, every central simple F-algebra of finite dimension over F can be split by some finite Galois extension of F, and it follows that

$$Br(F) = \bigcup_K Br(K/F)$$

where the union is over all finite Galois extensions of F. It follows that there is a bijection between $Br(F)$ and $H^2(\mathrm{Gal}(F^s/F), (F^s)^\times)$ where F^s denotes a separable algebraic closure of F. Here $\mathrm{Gal}(F^s/F)$ is considered as a profinite group and the cohomology group refers to continuous Galois cohomology.

One consequence of this result and Theorem 42 is that a full set of representatives for the F-isomorphism classes of central simple division algebras Δ over F can be obtained from the division algebra parts of the crossed product algebras for finite Galois extensions of F. Those division algebras that are split over K occur for the crossed product algebras for K/F.

Example

Since $H^2(\mathrm{Gal}(\mathbb{F}_{q^d}/\mathbb{F}_q), \mathbb{F}_{q^d}^{\times}) = 0$ (cf. Exercise 10), we have $Br(\mathbb{F}_{q^d}/\mathbb{F}_q) = 0$ and hence also $Br(\mathbb{F}_q) = 0$. As a consequence, every finite division algebra is a field (cf. Exercise 13 in Section 13.6 for a direct proof), and every finite central simple algebra $\mathbb{F}_q$-algebra is isomorphic to a full matrix ring $M_r(\mathbb{F}_q)$.

EXERCISES

1. Let $A = \{1, a, b, c\}$ be the Klein 4-group and let $G = \langle g \rangle$ be the cyclic group of order 2 acting trivially on A.
   (a) Prove that $|C^2(G, A)| = 2^8$.
   (b) Show that coboundaries are constant functions, and deduce that $|B^2(G, A)| = 4$.
   (c) Use the cocycle condition to show that $|Z^2(G, A)| \leq 2^4$.
   (d) If $E = Z_4 \times Z_2 = \langle x \rangle \times \langle y \rangle$, prove that the extensions $1 \to A \xrightarrow{\iota_i} E \xrightarrow{\pi} G \to 1$ defined by $\pi(x) = g$, $\pi(y) = 1$ and $\iota_1(a) = x^2$, $\iota_1(b) = y$ (respectively, $\iota_2(b) = x^2$, $\iota_2(a) = y$, and $\iota_3(c) = x^2$, $\iota_3(a) = y$), together with the split extension $Z_2 \times Z_2 \times Z_2$ give 4 inequivalent extensions of Z_2 by the Klein 4-group. Deduce that $H^2(G, A)$ has order 4 by explicitly exhibiting the corresponding cocycles.
2. Let $A = \mathbb{Z}/4\mathbb{Z}$ and let G be the cyclic group of order 2 acting trivially on A.
   (a) Prove that $|C^2(G, A)| = 2^8$.
   (b) Use the coboundary condition to show that $|B^2(G, A)| = 2^3$.
   (c) Use the cocycle condition to show that $|Z^2(G, A)| \leq 2^4$.
   (d) Show that $|H^2(G, A)| = 2$ by exhibiting two inequivalent extensions of G by A and their corresponding cocycles.
3. Let $A = \mathbb{Z}/4\mathbb{Z}$ and let G be the cyclic group of order 2 acting by inversion on A.
   (a) Show that there are four coboundaries and that only the zero coboundary is normalized.
   (b) Prove by a direct computation of cocycle and coboundary groups that $|H^2(G, A)| = 2$.
   (c) Exhibit two distinct cohomology classes and their corresponding extension groups.
   (d) Show that for a given extension of G by A with extension group isomorphic to D_8 there are four normalized sections, all of which have the zero 2-cocycle as their factor set.
   (e) Show that for a given extension of G by A with extension group isomorphic to Q_8 there are sixteen sections, four of which are normalized, and all of the latter have the same factor set.
4. For a *non*-normalized 2-cocycle f one defines the extension group E_f on the set $A \times G$ by the same binary operation in equation (34). Verify two of the group axioms in this case by showing that identity is now $(-f(1, 1), 1)$ and inverses are given by
$$(a, x)^{-1} = (-x^{-1} \cdot a - f(x^{-1}, x) - f(1, 1), x^{-1}).$$
   (Verification of the associative law is essentially the same as for normalized 2-cocycles.) Prove also that the set $A^{**} = \{(a - f(1, 1), 1) \mid a \in A\}$ is a subgroup of E_f and the map $\iota^{**} : a \mapsto (a - f(1, 1), 1)$ is an isomorphism from A to A^{**}. Show that this extension E_f, with the injection ι^{**} and the usual projection map π^* onto G, is equivalent to an extension derived from a normalized cocycle in the same class as f.
5. Show that the set of equivalences of a given extension $1 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1$ with itself form a group under composition, and that this group is isomorphic to the stability group