Hint: Given N, let $p_1, \ldots, p_k$ be those primes that divide at least one integer $\leq N$. Then

$$\begin{aligned}\sum_{n=1}^{N} \frac{1}{n} &\leq \prod_{j=1}^{k} \left(1 + \frac{1}{p_j} + \frac{1}{p_j^2} + \cdots\right) \\ &= \prod_{j=1}^{k} \left(1 - \frac{1}{p_j}\right)^{-1} \\ &\leq \exp \sum_{j=1}^{k} \frac{2}{p_j}.\end{aligned}$$

The last inequality holds because

$$(1 - x)^{-1} \leq e^{2x}$$

if $0 \leq x \leq \frac{1}{2}$.

(There are many proofs of this result. See, for instance, the article by I. Niven in *Amer. Math. Monthly*, vol. 78, 1971, pp. 272–273, and the one by R. Bellman in *Amer. Math. Monthly*, vol. 50, 1943, pp. 318–319.)

11. Suppose $f \in \mathscr{R}$ on $[0, A]$ for all $A < \infty$, and $f(x) \to 1$ as $x \to +\infty$. Prove that

$$\lim_{t \to 0} t \int_0^\infty e^{-tx} f(x)\, dx = 1 \qquad (t > 0).$$

12. Suppose $0 < \delta < \pi$, $f(x) = 1$ if $|x| \leq \delta$, $f(x) = 0$ if $\delta < |x| \leq \pi$, and $f(x + 2\pi) = f(x)$ for all x.

(*a*) Compute the Fourier coefficients of f.

(*b*) Conclude that

$$\sum_{n=1}^{\infty} \frac{\sin (n\delta)}{n} = \frac{\pi - \delta}{2} \qquad (0 < \delta < \pi).$$

(*c*) Deduce from Parseval's theorem that

$$\sum_{n=1}^{\infty} \frac{\sin^2 (n\delta)}{n^2 \delta} = \frac{\pi - \delta}{2}.$$

(*d*) Let $\delta \to 0$ and prove that

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}.$$

(*e*) Put $\delta = \pi/2$ in (*c*). What do you get?

13. Put $f(x) = x$ if $0 \leq x < 2\pi$, and apply Parseval's theorem to conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

14. If $f(x) = (\pi - |x|)^2$ on $[-\pi, \pi]$, prove that

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nx$$

and deduce that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \qquad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

(A recent article by E. L. Stark contains many references to series of the form $\sum n^{-s}$, where s is a positive integer. See *Math. Mag.*, vol. 47, 1974, pp. 197–202.)

15. With D_n as defined in (77), put

$$K_N(x) = \frac{1}{N+1} \sum_{n=0}^{N} D_n(x).$$

Prove that

$$K_N(x) = \frac{1}{N+1} \cdot \frac{1 - \cos (N+1)x}{1 - \cos x}$$

and that

(*a*) $K_N \geq 0$,

(*b*) $\displaystyle \frac{1}{2\pi} \int_{-\pi}^{\pi} K_N(x)\, dx = 1$,

(*c*) $\displaystyle K_N(x) \leq \frac{1}{N+1} \cdot \frac{2}{1 - \cos \delta} \qquad$ if $0 < \delta \leq |x| \leq \pi$.

If $s_N = s_N(f; x)$ is the Nth partial sum of the Fourier series of f, consider the arithmetic means

$$\sigma_N = \frac{s_0 + s_1 + \cdots + s_N}{N+1}.$$

Prove that

$$\sigma_N(f; x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x - t) K_N(t)\, dt,$$

and hence prove Fejér's theorem:

If f is continuous, with period 2π, then $\sigma_N(f; x) \to f(x)$ uniformly on $[-\pi, \pi]$.

Hint: Use properties (*a*), (*b*), (*c*) to proceed as in Theorem 7.26.

16. Prove a pointwise version of Fejér's theorem:

If $f \in \mathscr{R}$ and $f(x+)$, $f(x-)$ exist for some x, then

$$\lim_{N \to \infty} \sigma_N(f; x) = \tfrac{1}{2}[f(x+) + f(x-)].$$

17. Assume f is bounded and monotonic on $[-\pi, \pi)$, with Fourier coefficients c_n, as given by (62).

(*a*) Use Exercise 17 of Chap. 6 to prove that $\{nc_n\}$ is a bounded sequence.

(*b*) Combine (*a*) with Exercise 16 and with Exercise 14(*e*) of Chap. 3, to conclude that

$$\lim_{N\to\infty} s_N(f; x) = \tfrac{1}{2}[f(x+) + f(x-)]$$

for every x.

(*c*) Assume only that $f \in \mathscr{R}$ on $[-\pi, \pi]$ and that f is monotonic in some segment $(\alpha, \beta) \subset [-\pi, \pi]$. Prove that the conclusion of (*b*) holds for every $x \in (\alpha, \beta)$.

(This is an application of the localization theorem.)

18. Define

$$f(x) = x^3 - \sin^2 x \tan x$$
$$g(x) = 2x^2 - \sin^2 x - x \tan x.$$

Find out, for each of these two functions, whether it is positive or negative for all $x \in (0, \pi/2)$, or whether it changes sign. Prove your answer.

19. Suppose f is a continuous function on R^1, $f(x + 2\pi) = f(x)$, and α/π is irrational. Prove that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x + n\alpha) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, dt$$

for every x. *Hint:* Do it first for $f(x) = e^{ikx}$.

20. The following simple computation yields a good approximation to Stirling's formula.

For $m = 1, 2, 3, \ldots$, define

$$f(x) = (m + 1 - x) \log m + (x - m) \log (m + 1)$$

if $m \le x \le m + 1$, and define

$$g(x) = \frac{x}{m} - 1 + \log m$$

if $m - \frac{1}{2} \le x < m + \frac{1}{2}$. Draw the graphs of f and g. Note that $f(x) \le \log x \le g(x)$ if $x \ge 1$ and that

$$\int_1^n f(x)\, dx = \log (n!) - \tfrac{1}{2} \log n > -\tfrac{1}{8} + \int_1^n g(x)\, dx.$$

Integrate $\log x$ over $[1, n]$. Conclude that

$$\tfrac{7}{8} < \log (n!) - (n + \tfrac{1}{2}) \log n + n < 1$$

for $n = 2, 3, 4, \ldots$. (*Note:* $\log \sqrt{2\pi} \sim 0.918\ldots$.) Thus

$$e^{7/8} < \frac{n!}{(n/e)^n \sqrt{n}} < e.$$

21. Let

$$L_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_n(t)|\,dt \qquad (n = 1, 2, 3, \ldots).$$

Prove that there exists a constant $C > 0$ such that

$$L_n > C \log n \qquad (n = 1, 2, 3, \ldots),$$

or, more precisely, that the sequence

$$\left\{L_n - \frac{4}{\pi^2}\log n\right\}$$

is bounded.

22. If α is real and $-1 < x < 1$, prove Newton's binomial theorem

$$(1 + x)^\alpha = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha - 1)\cdots(\alpha - n + 1)}{n!} x^n.$$

Hint: Denote the right side by $f(x)$. Prove that the series converges. Prove that

$$(1 + x)f'(x) = \alpha f(x)$$

and solve this differential equation.

Show also that

$$(1 - x)^{-\alpha} = \sum_{n=0}^{\infty} \frac{\Gamma(n + \alpha)}{n!\,\Gamma(\alpha)} x^n$$

if $-1 < x < 1$ and $\alpha > 0$.

23. Let γ be a continuously differentiable *closed* curve in the complex plane, with parameter interval $[a, b]$, and assume that $\gamma(t) \neq 0$ for every $t \in [a, b]$. Define the *index* of γ to be

$$\operatorname{Ind}(\gamma) = \frac{1}{2\pi i}\int_a^b \frac{\gamma'(t)}{\gamma(t)}\,dt.$$

Prove that $\operatorname{Ind}(\gamma)$ is always an integer.

Hint: There exists φ on $[a, b]$ with $\varphi' = \gamma'/\gamma$, $\varphi(a) = 0$. Hence $\gamma\exp(-\varphi)$ is constant. Since $\gamma(a) = \gamma(b)$ it follows that $\exp\varphi(b) = \exp\varphi(a) = 1$. Note that $\varphi(b) = 2\pi i \operatorname{Ind}(\gamma)$.

Compute $\operatorname{Ind}(\gamma)$ when $\gamma(t) = e^{int}$, $a = 0$, $b = 2\pi$.

Explain why $\operatorname{Ind}(\gamma)$ is often called the *winding number* of γ around 0.

24. Let γ be as in Exercise 23, and assume in addition that the range of γ does not intersect the negative real axis. Prove that $\operatorname{Ind}(\gamma) = 0$. *Hint:* For $0 \le c < \infty$, $\operatorname{Ind}(\gamma + c)$ is a continuous integer-valued function of c. Also, $\operatorname{Ind}(\gamma + c) \to 0$ as $c \to \infty$.

25. Suppose γ_1 and γ_2 are curves as in Exercise 23, and

$$|\gamma_1(t) - \gamma_2(t)| < |\gamma_1(t)| \qquad (a \le t \le b).$$

Prove that Ind $(\gamma_1) =$ Ind (γ_2).

Hint: Put $\gamma = \gamma_2/\gamma_1$. Then $|1 - \gamma| < 1$, hence Ind $(\gamma) = 0$, by Exercise 24. Also,

$$\frac{\gamma'}{\gamma} = \frac{\gamma_2'}{\gamma_2} - \frac{\gamma_1'}{\gamma_1}.$$

26. Let γ be a *closed* curve in the complex plane (not necessarily differentiable) with parameter interval $[0, 2\pi]$, such that $\gamma(t) \ne 0$ for every $t \in [0, 2\pi]$.

Choose $\delta > 0$ so that $|\gamma(t)| > \delta$ for all $t \in [0, 2\pi]$. If P_1 and P_2 are trigonometric polynomials such that $|P_j(t) - \gamma(t)| < \delta/4$ for all $t \in [0, 2\pi]$ (their existence is assured by Theorem 8.15), prove that

$$\text{Ind}\,(P_1) = \text{Ind}\,(P_2)$$

by applying Exercise 25.

Define this common value to be Ind (γ).

Prove that the statements of Exercises 24 and 25 hold without any differentiability assumption.

27. Let f be a continuous complex function defined in the complex plane. Suppose there is a positive integer n and a complex number $c \ne 0$ such that

$$\lim_{|z| \to \infty} z^{-n} f(z) = c.$$

Prove that $f(z) = 0$ for at least one complex number z.

Note that this is a generalization of Theorem 8.8.

Hint: Assume $f(z) \ne 0$ for all z, define

$$\gamma_r(t) = f(re^{it})$$

for $0 \le r < \infty$, $0 \le t \le 2\pi$, and prove the following statements about the curves γ_r:

(*a*) Ind $(\gamma_0) = 0$.

(*b*) Ind $(\gamma_r) = n$ for all sufficiently large r.

(*c*) Ind (γ_r) is a continuous function of r, on $[0, \infty)$.

[In (*b*) and (*c*), use the last part of Exercise 26.]

Show that (*a*), (*b*), and (*c*) are contradictory, since $n > 0$.

28. Let $\bar{D}$ be the closed unit disc in the complex plane. (Thus $z \in \bar{D}$ if and only if $|z| \le 1$.) Let g be a continuous mapping of $\bar{D}$ into the unit circle T. (Thus, $|g(z)| = 1$ for every $z \in \bar{D}$.)

Prove that $g(z) = -z$ for at least one $z \in T$.

Hint: For $0 \le r \le 1$, $0 \le t \le 2\pi$, put

$$\gamma_r(t) = g(re^{it}),$$

and put $\psi(t) = e^{-it}\gamma_1(t)$. If $g(z) \ne -z$ for every $z \in T$, then $\psi(t) \ne -1$ for every $t \in [0, 2\pi]$. Hence Ind $(\psi) = 0$, by Exercises 24 and 26. It follows that Ind $(\gamma_1) = 1$. But Ind $(\gamma_0) = 0$. Derive a contradiction, as in Exercise 27.

29. Prove that every continuous mapping f of $\bar{D}$ into $\bar{D}$ has a fixed point in $\bar{D}$.

(This is the 2-dimensional case of Brouwer's fixed-point theorem.)

Hint: Assume $f(z) \neq z$ for every $z \in \bar{D}$. Associate to each $z \in \bar{D}$ the point $g(z) \in T$ which lies on the ray that starts at $f(z)$ and passes through z. Then g maps $\bar{D}$ into T, $g(z) = z$ if $z \in T$, and g is continuous, because

$$g(z) = z - s(z)[f(z) - z],$$

where $s(z)$ is the unique nonnegative root of a certain quadratic equation whose coefficients are continuous functions of f and z. Apply Exercise 28.

30. Use Stirling's formula to prove that

$$\lim_{x \to \infty} \frac{\Gamma(x + c)}{x^c \Gamma(x)} = 1$$

for every real constant c.

31. In the proof of Theorem 7.26 it was shown that

$$\int_{-1}^{1} (1 - x^2)^n \, dx \geq \frac{4}{3\sqrt{n}}$$

for $n = 1, 2, 3, \ldots$. Use Theorem 8.20 and Exercise 30 to show the more precise result

$$\lim_{n \to \infty} \sqrt{n} \int_{-1}^{1} (1 - x^2)^n \, dx = \sqrt{\pi}.$$

9

FUNCTIONS OF SEVERAL VARIABLES

LINEAR TRANSFORMATIONS

We begin this chapter with a discussion of sets of vectors in euclidean n-space R^n. The algebraic facts presented here extend without change to finite-dimensional vector spaces over any field of scalars. However, for our purposes it is quite sufficient to stay within the familiar framework provided by the euclidean spaces.

9.1 Definitions

(*a*) A nonempty set $X \subset R^n$ is a *vector space if* $\mathbf{x} + \mathbf{y} \in X$ and $c\mathbf{x} \in X$ for all $\mathbf{x} \in X$, $\mathbf{y} \in X$, and for all scalars c.
(*b*) If $\mathbf{x}_1, \ldots, \mathbf{x}_k \in R^n$ and $c_1, \ldots, c_k$ are scalars, the vector

$$c_1\mathbf{x}_1 + \cdots + c_k\mathbf{x}_k$$

is called a *linear combination* of $\mathbf{x}_1, \ldots, \mathbf{x}_k$. If $S \subset R^n$ and if E is the set of all linear combinations of elements of S, we say that S *spans* E, or that E *is the span of* S.

Observe that every span is a vector space.

(*c*) A set consisting of vectors $\mathbf{x}_1, \dots, \mathbf{x}_k$ (we shall use the notation $\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ for such a set) is said to be *independent* if the relation $c_1\mathbf{x}_1 + \cdots + c_k\mathbf{x}_k = \mathbf{0}$ implies that $c_1 = \cdots = c_k = 0$. Otherwise $\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ is said to be *dependent*.

Observe that no independent set contains the null vector.

(*d*) If a vector space X contains an independent set of r vectors but contains no independent set of $r + 1$ vectors, we say that X has *dimension* r, and write: $\dim X = r$.

The set consisting of $\mathbf{0}$ alone is a vector space; its dimension is 0.

(*e*) An independent subset of a vector space X which spans X is called a *basis* of X.

Observe that if $B = \{\mathbf{x}_1, \dots, \mathbf{x}_r\}$ is a basis of X, then every $\mathbf{x} \in X$ has a unique representation of the form $\mathbf{x} = \Sigma c_j\mathbf{x}_j$. Such a representation exists since B spans X, and it is unique since B is independent. The numbers $c_1, \dots, c_r$ are called the *coordinates of* $\mathbf{x}$ with respect to the basis B.

The most familiar example of a basis is the set $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$, where $\mathbf{e}_j$ is the vector in R^n whose jth coordinate is 1 and whose other coordinates are all 0. If $\mathbf{x} \in R^n$, $\mathbf{x} = (x_1, \dots, x_n)$, then $\mathbf{x} = \Sigma x_j\mathbf{e}_j$. We shall call

$$\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$$

the *standard basis* of R^n.

9.2 Theorem *Let r be a positive integer. If a vector space X is spanned by a set of r vectors, then* $\dim X \le r$.

Proof If this is false, there is a vector space X which contains an independent set $Q = \{\mathbf{y}_1, \dots, \mathbf{y}_{r+1}\}$ and which is spanned by a set S_0 consisting of r vectors.

Suppose $0 \le i < r$, and suppose a set S_i has been constructed which spans X and which consists of all $\mathbf{y}_j$ with $1 \le j \le i$ plus a certain collection of $r - i$ members of S_0, say $\mathbf{x}_1, \dots, \mathbf{x}_{r-i}$. (In other words, S_i is obtained from S_0 by replacing i of its elements by members of Q, without altering the span.) Since S_i spans X, $\mathbf{y}_{i+1}$ is in the span of S_i; hence there are scalars $a_1, \dots, a_{i+1}, b_1, \dots, b_{r-i}$, with $a_{i+1} = 1$, such that

$$\sum_{j=1}^{i+1} a_j\mathbf{y}_j + \sum_{k=1}^{r-i} b_k\mathbf{x}_k = \mathbf{0}.$$

If all b_k's were 0, the independence of Q would force all a_j's to be 0, a contradiction. It follows that some $\mathbf{x}_k \in S_i$ is a linear combination of the other members of $T_i = S_i \cup \{\mathbf{y}_{i+1}\}$. Remove this $\mathbf{x}_k$ from T_i and call the remaining set S_{i+1}. Then S_{i+1} spans the same set as T_i, namely X, so that S_{i+1} has the properties postulated for S_i with $i + 1$ in place of i.