THEOREM 12.2. ***Let r and R be smoothly equivalent functions, as described in Theorem 12.1. If the surface integral $\iint_{r(A)} f\, dS$ exists, the surface integral $\iint_{R(B)} f\, dS$ also exists and we have***

$$\iint_{r(A)} f\, dS = \iint_{R(B)} f\, dS .$$

Proof. By the definition of a surface integral we have

$$\iint_{r(A)} f\, dS = \iint_{A} f[\mathbf{r}(u, v)] \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| du\, dv .$$

Now we use the mapping G of Theorem 12.1 to transform this into a double integral over the region B in the St-plane. The transformation formula for double integrals states that

$$\iint_{A} f[\mathbf{r}(u, v)] \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| du\, dv = \iint_{B} f\{\mathbf{r}[\mathbf{G}(s, t)]\} \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| \left| \frac{\partial(U, V)}{\partial(s, t)} \right| ds\, dt ,$$

where the derivatives $\partial \mathbf{r}/\partial u$ and $\partial \mathbf{r}/\partial v$ on the right are to be evaluated at $(U(s, t), V(s, t))$. Because of Equation (12.18), the integral over B is equal to

$$\iint_{B} f[\mathbf{R}(s, t)] \left\| \frac{\partial \mathbf{R}}{\partial s} \times \frac{\partial \mathbf{R}}{\partial t} \right\| ds\, dt .$$

This, in turn, is the definition of the surface integral $\iint_{R(B)} f\, dS$. The proof is now complete.

12.9 Other notations for surface integrals

If S = $\mathbf{r}(T)$ is a parametric surface, the fundamental vector product $\mathbf{N} = \partial \mathbf{r}/\partial u \times \partial \mathbf{r}/\partial v$ is normal to S at each regular point of the surface. At each such point there are ***two*** unit normals, a unit normal $\mathbf{n}_1$ which has the same direction as $\mathbf{N}$, and a unit normal $\mathbf{n}_2$ which has the opposite direction. Thus,

$$\mathbf{n}_1 = \frac{\mathbf{N}}{\|\mathbf{N}\|} \quad \text{and} \quad \mathbf{n}_2 = -\mathbf{n}_1 .$$

Let $\mathbf{n}$ be one of the two normals $\mathbf{n}_1$ or $\mathbf{n}_2$. Let $\mathbf{F}$ be a vector field defined on S and assume the surface integral $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$ exists. Then we can write

$$(12.19) \qquad \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_T \mathbf{F}[\mathbf{r}(u, v)] \cdot \mathbf{n}(u, v) \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| du\, dv$$

$$= \pm \iint_T \mathbf{F}[\mathbf{r}(u, v)] \cdot \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\, du\, dv ,$$

where the + sign is used if $\mathbf{n} = \mathbf{n}_1$ and the $-$ sign is used if $\mathbf{n} = \mathbf{n}_2$.

Suppose now we express $\boldsymbol{F}$ and $\boldsymbol{r}$ in terms of their components, say

$$\boldsymbol{F}(x, y, z) = P(x, y, z)\boldsymbol{i} + Q(x, y, z)\boldsymbol{j} + R(x, y, z)\boldsymbol{k}$$

and

$$\boldsymbol{r}(u, v) = X(u, v)\boldsymbol{i} + Y(u, v)\boldsymbol{j} + Z(u, v)\boldsymbol{k}.$$

Then the fundamental vector product of $\boldsymbol{r}$ is given by

$$\boldsymbol{N} = \frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} = \frac{\partial(Y, Z)}{\partial(u, v)}\boldsymbol{i} + \frac{\partial(Z, X)}{\partial(u, v)}\boldsymbol{j} + \frac{\partial(X, Y)}{\partial(u, v)}\boldsymbol{k}.$$

If $\boldsymbol{n} = \boldsymbol{n}_1$, Equation (12.19) becomes

$$\text{(12.20)} \quad \iint_S \boldsymbol{F}\cdot\boldsymbol{n}\, dS = \iint_T P[\boldsymbol{r}(u, v)]\frac{\partial(Y, Z)}{\partial(u, v)}\, du\, dv + \iint_T Q[\boldsymbol{r}(u, v)]\frac{\partial(Z, X)}{\partial(u, v)}\, du\, dv + \iint_T R[\boldsymbol{r}(u, v)]\frac{\partial(X, Y)}{\partial(u, v)}\, du\, dv;$$

if $\boldsymbol{n} = \boldsymbol{n}_2$, each double integral on the right must be replaced by its negative. The sum of the double integrals on the right is often written more briefly as

$$\text{(12.21)} \quad \iint_S P(x, y, z)\, dy \wedge dz + \iint_S Q(x, y, z)\, dz \wedge dx + \iint_S R(x, y, z)\, dx \wedge dy,$$

or even more briefly as

$$\text{(12.22)} \quad \iint_S P\, dy \wedge dz + Q\, dz \wedge dx + R\, dx \wedge dy.$$

The integrals which appear in (12.21) and (12.22) are also referred to as surface integrals. Thus, for example, the surface integral $\iint_S P\, dy \wedge dz$ is defined by the equation

$$\text{(12.23)} \quad \iint_S P\, dy \wedge dz = \iint_T P[\boldsymbol{r}(u, v)]\frac{\partial(Y, Z)}{\partial(u, v)}\, du\, dv.$$

This notation is suggested by the formula for changing variables in a double integral.

Despite similarity in notation, the integral on the left of (12.23) is ***not*** a double integral. First of all, P is a function of three variables. Also, we must take into consideration the order in which the symbols dy and dz appear in the surface integral, because

$$\frac{\partial(Y, Z)}{\partial(u, v)} = -\frac{\partial(Z, Y)}{\partial(u, v)}$$

and hence

$$\iint_S P\, dy \wedge dz = -\iint_S P\, dz \wedge dy.$$

In this notation, formula (12.20) becomes

$$\iint_S \boldsymbol{F}\cdot\boldsymbol{n}\,dS = \iint_S P\,dy\wedge dz + Q\,dz\wedge dx + R\,dx\wedge dy \tag{12.24}$$

if $\boldsymbol{n} = \boldsymbol{n}_1$. If $\boldsymbol{n} = \boldsymbol{n}_2$ the integral on the right must be replaced by its negative. This formula resembles the following formula for line integrals:

$$\int_C \boldsymbol{F}\cdot d\boldsymbol{\alpha} = \int_C P\,dx + Q\,dy + R\,dz.$$

If the unit normal $\boldsymbol{n}$ is expressed in terms of its direction cosines, say

$$\boldsymbol{n} = \cos\alpha\,\boldsymbol{i} + \cos\beta\,\boldsymbol{j} + \cos\gamma\,\boldsymbol{k},$$

then $\boldsymbol{F}\cdot\boldsymbol{n} = P\cos\alpha + Q\cos\beta + R\cos\gamma$, and we can write

$$\iint_S \boldsymbol{F}\cdot\boldsymbol{n}\,dS = \iint_S (P\cos\alpha + Q\cos\beta + R\cos\gamma)\,dS.$$

This equation holds when $\boldsymbol{n}$ is either $\boldsymbol{n}_1$ or $\boldsymbol{n}_2$. The direction cosines will depend on the choice of the normal. If $\boldsymbol{n} = \boldsymbol{n}_1$ we can use (12.24) to write

(12.25)

$$\iint_S (P\cos\alpha + Q\cos\beta + R\cos\gamma)\,dS = \iint_S P\,dy\wedge dz + Q\,dz\wedge dx + R\,dx\wedge dy.$$

If $\boldsymbol{n} = \boldsymbol{n}_2$ we have, instead,

(12.26)

$$\iint_S (P\cos\alpha + Q\cos\beta + R\cos\gamma)\,dS = -\iint_S P\,dy\wedge dz + Q\,dz\wedge dx + R\,dx\wedge dy.$$

12.10 Exercises

1. Let S denote the hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$, and let $\boldsymbol{F}(x, y, z) = x\boldsymbol{i} + y\boldsymbol{j}$. Let $\boldsymbol{n}$ be the unit outward normal of S. Compute the value of the surface integral $\iint_S \boldsymbol{F}\cdot\boldsymbol{n}\,dS$, using:

   (a) the vector representation $\boldsymbol{r}(u, v) = \sin u\cos v\,\boldsymbol{i} + \sin u\sin v\,\boldsymbol{j} + \cos u\,\boldsymbol{k}$,

   (b) the explicit representation $z = \sqrt{1 - x^2 - y^2}$.
2. Show that the moment of inertia of a homogeneous spherical shell about a diameter is equal to $\frac{2}{3}ma^2$, where m is the mass of the shell and a is its radius.
3. Find the center of mass of that portion of the homogeneous hemispherical surface $x^2 + y^2 + z^2 = a^2$ lying above the first quadrant in the xy-plane.

4. Let S denote the plane surface whose boundary is the triangle with vertices at (1, 0, 0), (0, 1, 0), and (0, 0, 1), and let $\boldsymbol{F}(x, y, z) = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$. Let $\boldsymbol{n}$ denote the unit normal to S having a nonnegative z-component. Evaluate the surface integral $\iint_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS$, using:
(a) the vector representation $\boldsymbol{r}(u, v) = (u + v)\boldsymbol{i} + (u - v)\boldsymbol{j} + (1 - 2u)\boldsymbol{k}$,
(b) an explicit representation of the form $z = f(x, y)$.
5. Let S be a parametric surface described by the explicit formula $z = f(x, y)$, where (x, y) varies over a plane region T, the projection of S in the xy-plane. Let $\boldsymbol{F} = P\boldsymbol{i} + Q\boldsymbol{j} + R\boldsymbol{k}$ and let $\boldsymbol{n}$ denote the unit normal to S having a nonnegative z-component. Use the parametric representation $\boldsymbol{r}(x, y) = x\boldsymbol{i} + y\boldsymbol{j} + f(x, y)\boldsymbol{k}$ and show that

$$\iint_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS = \iint_T \left(-P\frac{\partial f}{\partial x} - Q\frac{\partial f}{\partial y} + R \right) dx\, dy,$$

where each of P, Q, and R is to be evaluated at $(x, y, f(x, y))$.
6. Let S be as in Exercise 5, and let φ be a scalar field. Show that:

(a) $$\iint_S \varphi(x, y, z)\, dS = \iint_T \varphi[x, y, f(x, y)]\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dx\, dy.$$

(b) $$\iint_S \varphi(x, y, z)\, dy \wedge dz = -\iint_T \varphi[x, y, f(x, y)]\frac{\partial f}{\partial x}\, dx\, dy.$$

(c) $$\iint_S \varphi(x, y, z)\, dz \wedge dx = -\iint_T \varphi[x, y, f(x, y)]\frac{\partial f}{\partial y}\, dx\, dy.$$

7. If S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$, compute the value of the surface integral

$$\iint_S xz\, dy \wedge dz + yz\, dz \wedge dx + x^2\, dx \wedge dy.$$

Choose a representation for which the fundamental vector product points in the direction of the outward normal.
8. The cylinder $x^2 + y^2 = 2x$ cuts out a portion of a surface S from the upper nappe of the cone $x^2 + y^2 = z^2$. Compute the value of the surface integral

$$\iint_S (x^4 - y^4 + y^2z^2 - z^2x^2 + 1)\, dS.$$

9. A homogeneous spherical shell of radius a is cut by one nappe of a right circular cone whose vertex is at the center of the sphere. If the vertex angle of the cone is α, where $0 < \alpha < \pi$, determine (in terms of a and α) the center of mass of the portion of the spherical shell that lies inside the cone.
10. A homogeneous paper rectangle of base $2\pi a$ and altitude h is rolled to form a circular cylindrical surface S of radius a. Calculate the moment of inertia of S about an axis through a diameter of the circular base.

11. Refer to Exercise 10. Calculate the moment of inertia of S about an axis which is in the plane of the base and is tangent to the circular edge of the base.
12. A fluid flow has flux density vector $\boldsymbol{F}(x, y, z) = x\boldsymbol{i} - (2x + y)\boldsymbol{j} + z\boldsymbol{k}$. Let S denote the hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$, and let $\boldsymbol{n}$ denote the unit normal that points out of the sphere. Calculate the mass of fluid flowing through S in unit time in the direction of $\boldsymbol{n}$.
13. Solve Exercise 12 if S also includes the planar base of the hemisphere. On the lower base the unit normal is $-\boldsymbol{k}$.
14. Let S denote the portion of the plane $x + y + z = t$ cut off by the unit sphere $x^2 + y^2 + z^2 = 1$. Let $\varphi(x, y, z) = 1 - x^2 - y^2 - z^2$ if (x, y, z) is inside this sphere, and let $\varphi(x, y, z)$ be 0 otherwise. Show that

$$\iint_S \varphi(x, y, z)\, dS = \begin{cases} \dfrac{\pi}{18}(3 - t^2)^2 & \text{if } |t| \leq \sqrt{3}, \\ 0 & \text{if } |t| > \sqrt{3}. \end{cases}$$

[Hint: Introduce new coordinates (x_1, y_1, z_1) with the z_1-axis normal to the plane $x + y + z = t$. The use the polar coordinates in the x_1y_1-plane as parameters for S.]

12.11 The theorem of Stokes

The rest of this chapter is devoted primarily to two generalizations of the second fundamental theorem of calculus involving surface integrals. They are known, respectively, as ***Stokes' theorem*†** and the ***divergence theorem.*** This section treats Stokes' theorem. The divergence theorem is discussed in Section 12.19.

Stoke's theorem is a direct extension of Green's theorem which states that

$$\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy = \int_C P\, dx + Q\, dy,$$

where S is a plane region bounded by a simple closed curve C traversed in the positive (counterclockwise) direction. Stokes' theorem relates a surface integral to a line integral and can be stated as follows.

THEOREM 12.3 STOKES' THEOREM. *Assume that S is a smooth simple parametric surface, say S = r(T), where T is a region in the uv-plane bounded by a piecewise smooth Jordan curve Γ. (See Figure 12.12.) Assume also that r is a one-to-one mapping whose components have continuous second-order partial derivatives on some open set containing T ∪ Γ. Let C denote the image of Γ under r, and let P, Q, and R be continuously differentiable scalar fields on S. Then we have*

$$(12.27)\quad \iint_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy \wedge dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dz \wedge dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy = \int_C P\, dx + Q\, dy + R\, dz.$$

† In honor of G. G. Stokes (1819–1903), an Irish mathematician who made many fundamental contributions to hydrodynamics and optics.

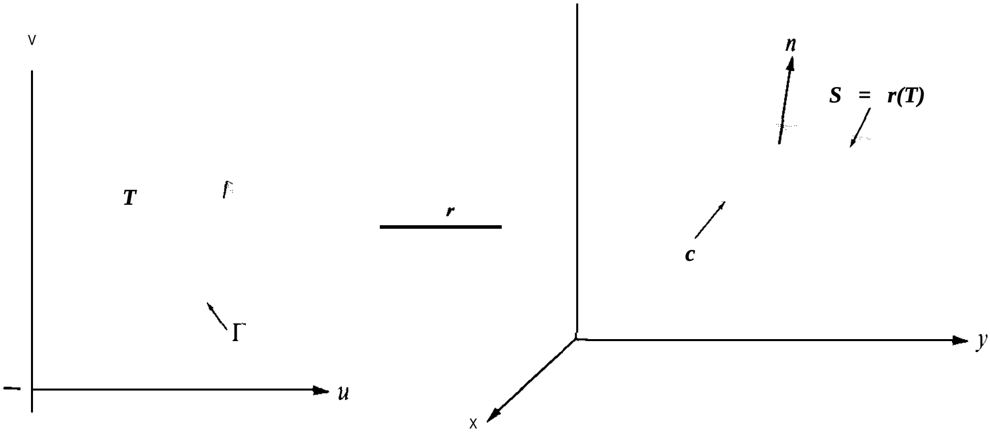

FIGURE 12.12 An example of a surface to which Stokes' theorem is applicable.

The curve Γ is traversed in the positive (counterclockwise) direction and the curve C is traversed in the direction inherited from Γ through the mapping function r.

Proof. To prove the theorem it suffices to establish the following three formulas,

$$\int_C P\,dx = \iint_S \left(-\frac{\partial P}{\partial y}\,dx \wedge dy + \frac{\partial P}{\partial z}\,dz \wedge dx\right), \tag{12.28}$$

$$\int_C Q\,dy = \iint_S \left(-\frac{\partial Q}{\partial z}\,dy \wedge dz + \frac{\partial Q}{\partial x}\,dx \wedge dy\right),$$

$$\int_C R\,dz = \iint_S \left(-\frac{\partial R}{\partial x}\,dz \wedge dx + \frac{\partial R}{\partial y}\,dy \wedge dz\right).$$

Addition of these three equations gives the formula (12.27) in Stokes' theorem. Since the three are similar, we prove only Equation (12.28).

The plan of the proof is to express the surface integral as a double integral over T. Then we use Green's theorem to express the double integral over T as a line integral over Γ. Finally, we show that this line integral is equal to $\int_C P\,dx$.

We write

$$\boldsymbol{r}(u, v) = X(u, v)\boldsymbol{i} + Y(u, v)\boldsymbol{j} + Z(u, v)\boldsymbol{k}$$

and express the surface integral over S in the form

$$\iint_S \left(-\frac{\partial P}{\partial y}\,dx \wedge dy + \frac{\partial P}{\partial z}\,dz \wedge dx\right) = \iint_T \left\{-\frac{\partial P}{\partial y}\frac{\partial(X, Y)}{\partial(u, v)} + \frac{\partial P}{\partial z}\frac{\partial(Z, X)}{\partial(u, v)}\right\} du\,dv.$$

Now let p denote the composite function given by

$$p(u, v) = P[X(u, v), Y(u, v), Z(u, v)] .$$

The last integrand can be written as

$$-\frac{\partial P}{\partial y}\frac{\partial(X, Y)}{\partial(u, v)} + \frac{\partial P}{\partial z}\frac{\partial(Z, X)}{\partial(u, v)} = \frac{\partial}{\partial u}\left(p\frac{\partial X}{\partial v}\right) - \frac{\partial}{\partial v}\left(p\frac{\partial X}{\partial u}\right). \tag{12.29}$$

The verification of (12.29) is outlined in Exercise 13 of Section 12.13. Applying Green's theorem to the double integral over T we obtain

$$\iint_T \left\{\frac{\partial}{\partial u}\left(p\frac{\partial X}{\partial v}\right) - \frac{\partial}{\partial v}\left(p\frac{\partial X}{\partial u}\right)\right\} du\, dv = \int_\Gamma p\frac{\partial X}{\partial u}\, du + p\frac{\partial X}{\partial v}\, dv,$$

where Γ is traversed in the positive direction. We parametrize Γ by a function γ defined on an interval $[a, b]$ and let

$$\boldsymbol{\alpha}(t) = \boldsymbol{r}[\boldsymbol{\gamma}(t)] \tag{12.30}$$

be a corresponding parametrization of C. Then by expressing each line integral in terms of its parametric representation we find that

$$\int_\Gamma p\frac{\partial X}{\partial u}\, du + p\frac{\partial X}{\partial v}\, dv = \int_C P\, dx,$$

which completes the proof of (12.28).

12.12 The curl and divergence of a vector field

The surface integral which appears in Stokes' theorem can be expressed more simply in terms of the ***curl*** of a vector field. Let $\boldsymbol{F}$ be a differentiable vector field given by

$$\boldsymbol{F}(x, y, z) = P(x, y, z)\boldsymbol{i} + Q(x, y, z)\boldsymbol{j} + R(x, y, z)\boldsymbol{k}.$$

The curl of $\boldsymbol{F}$ is another vector field defined by the equation

$$\operatorname{curl} \boldsymbol{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\boldsymbol{k}. \tag{12.31}$$

The components of curl $\boldsymbol{F}$ are the functions appearing in the surface integral in Stokes' formula (12.27). Therefore, this surface integral can be written as

$$\iint_S (\operatorname{curl} \boldsymbol{F}) \cdot \boldsymbol{n}\, dS,$$

where $\boldsymbol{n}$ is the unit normal vector having the same direction as the fundamental vector product of the surface; that is,

$$\boldsymbol{n} = \frac{\dfrac{\partial \boldsymbol{r}}{\partial u} \times \dfrac{\partial \boldsymbol{r}}{\partial v}}{\left\| \dfrac{\partial \boldsymbol{r}}{\partial u} \times \dfrac{\partial \boldsymbol{r}}{\partial v} \right\|}.$$

The line integral in Stokes' formula (12.27) can be written as $\int_C \boldsymbol{F} \cdot d\boldsymbol{\alpha}$, where $\boldsymbol{\alpha}$ is the representation of C given by (12.30). Thus, Stokes' theorem takes the simpler form

$$\iint_S (\operatorname{curl} \boldsymbol{F}) \cdot \boldsymbol{n}\, dS = \int_C \boldsymbol{F} \cdot d\boldsymbol{\alpha}.$$

For the special case in which S is a region in the xy-plane and $\boldsymbol{n} = \boldsymbol{k}$, this formula reduces to Green's theorem,

$$\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy = \int_C P\, dx + Q\, dy.$$

Equation (12.31) defining the curl can be easily remembered by writing it as an expansion of a 3 x 3 determinant,

$$\operatorname{curl} \boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\boldsymbol{k}.$$

This determinant is to be expanded in terms of first-row minors, but each "product" such as $\partial/\partial y$ times $\boldsymbol{R}$ is to be interpreted as a partial derivative $\partial R/\partial y$. We can also write this formula as a cross product,

$$\operatorname{curl} \boldsymbol{F} = \nabla \times \boldsymbol{F},$$

where the symbol ∇ is treated as though it were a vector,

$$\nabla = \frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}.$$

If we form the "dot product" $\nabla \cdot \boldsymbol{F}$ in a purely formal way, again interpreting products such as $\partial/\partial x$ times $\boldsymbol{P}$ as $\partial P/\partial x$, we find that

(12.32) $$\nabla \cdot \boldsymbol{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Equation (12.32) defines a scalar field called the ***divergence*** of $\boldsymbol{F}$, also written as div $\boldsymbol{F}$.

We have already used the symbol $\nabla\varphi$ to denote the gradient of a scalar field φ, given by

$$\nabla\varphi = \frac{\partial\varphi}{\partial x}\boldsymbol{i} + \frac{\partial\varphi}{\partial y}\boldsymbol{j} + \frac{\partial\varphi}{\partial z}\boldsymbol{k}.$$

This formula can be interpreted as formal multiplication of the symbolic vector ∇ by the scalar field φ. Thus, the gradient, the divergence, and the curl can be represented symbolically by the three products $\nabla\varphi$, $\nabla \cdot \boldsymbol{F}$, and $\nabla \times \boldsymbol{F}$, respectively.

Some of the theorems proved earlier can be expressed in terms of the curl. For example, in Theorem 10.9 we proved that a vector field $\boldsymbol{f} = (f_1, \ldots, f_n)$, continuously differentiable on an open convex set S in n-space, is a gradient on S if and only if the partial derivatives of the components of $\boldsymbol{f}$ satisfy the relations

$$D_k f_j(\boldsymbol{x}) = D_j f_k(\boldsymbol{x}) \qquad (j, k = 1, 2, \ldots, n) \tag{12.33}$$

In the 3-dimensional case Theorem 10.9 can be restated as follows.

THEOREM 12.4. *Let $\boldsymbol{F} = P\boldsymbol{i} + Q\boldsymbol{j} + R\boldsymbol{k}$ be a continuously differentiable vector field on an open convex set S in 3-space. Then $\boldsymbol{F}$ is a gradient on S if and only if we have*

$$\operatorname{curl} \boldsymbol{F} = \boldsymbol{O} \quad \text{on } S. \tag{12.34}$$

Proof. In the 3-dimensional case the relations (12.33) are equivalent to the statement that curl $\boldsymbol{F} = \boldsymbol{O}$.

12.13 Exercises

In each of Exercises 1 through 4, transform the surface integral $\iint (\operatorname{curl} \boldsymbol{F}) \cdot \boldsymbol{n}\, dS$ to a line integral by the use of Stokes' theorem, and then evaluate the line integral!

1. $\boldsymbol{F}(x, y, z) = y^2\boldsymbol{i} + xy\boldsymbol{j} + xz\boldsymbol{k}$, where S is the hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$, and $\boldsymbol{n}$ is the unit normal with a nonnegative z-component.
2. $\boldsymbol{F}(x, y, z) = y\boldsymbol{i} + z\boldsymbol{j} + x\boldsymbol{k}$, where S is the portion of the paraboloid $z = 1 - x^2 - y^2$ with $z \geq 0$, and $\boldsymbol{n}$ is the unit normal with a nonnegative z-component.
3. $\boldsymbol{F}(x, y, z) = (y - z)\boldsymbol{i} + yz\boldsymbol{j} - xz\boldsymbol{k}$, where S consists of the five faces of the cube $0 \leq x \leq 2$, $0 \leq y \leq 2$, $0 \leq z \leq 2$ not in the xy-plane. The unit normal $\boldsymbol{n}$ is the outward normal.
4. $\boldsymbol{F}(x, y, z) = xz\boldsymbol{i} - y\boldsymbol{i} + x^2y\boldsymbol{k}$, where S consists of the three faces not in the xz-plane of the tetrahedron bounded by the three coordinate planes and the plane $3x + y + 3z = 6$. The normal $\boldsymbol{n}$ is the unit normal pointing out of the tetrahedron,

In Exercises 5 through 10, use Stokes' theorem to show that the line integrals have the values given. In each case, explain how to traverse C to arrive at the given answer.

5. $\int_C y\,dx + z\,dy + x\,dz = \pi a^2\sqrt{3}$, where C is the curve of intersection of the sphere $x^2 + y^2 + z^2 = a^2$ and the plane $x + y + z = 0$.
6. $\int_C (y + z)\,dx + (z + x)\,dy + (x + y)\,dz = 0$, where C is the curve of intersection of the cylinder $x^2 + y^2 = 2y$ and the plane $y = z$.
7. $\int_C y^2\,dx + xy\,dy + xz\,dz = 0$, where C is the curve of Exercise 6.

8. $\int_C (y - z)\, dx + (z - x)\, dy + (x - y)\, dz = 2\pi a(a + b)$, where C is the intersection of the cylinder $x^2 + y^2 = a^2$ and the plane $x/a + z/b = 1$, $a > 0$, $b > 0$.
9. $\int_C (y^2 + z^2)\, dx + (x^2 + z^2)\, dy + (x^2 + y^2)\, dz = 2\pi ab^2$, where C is the intersection of the hemisphere $x^2 + y^2 + z^2 = 2ax$, $z > 0$, and the cylinder $x^2 + y^2 = 2bx$, where $0 < b < a$.
10. $\int_C (y^2 - z^2)\, dx + (z^2 - x^2)\, dy + (x^2 - y^2)\, dz = 9a^3/2$, where C is the curve cut from the boundary of the cube $0 \le x \le a$, $0 \le y \le a$, $0 \le z \le a$ by the plane $x + y + z = 3a/2$.
11. If $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ and $P\boldsymbol{i} + Q\boldsymbol{j} + R\boldsymbol{k} = \boldsymbol{a} \times \boldsymbol{r}$, where $\boldsymbol{a}$ is a constant vector, show that $\int_C P\, dx + Q\, dy + R\, dz = 2 \iint_S \boldsymbol{a} \cdot \boldsymbol{n}\, dS$, where C is a curve bounding a parametric surface S and $\boldsymbol{n}$ is a suitable normal to S.
12. Let $\boldsymbol{F} = P\boldsymbol{i} + Q\boldsymbol{j} + R\boldsymbol{k}$, where $P = -y/(x^2 + y^2)$, $Q = x/(x^2 + y^2)$, $R = z$, and let D be the torus generated by rotating the circle $(x - 2)^2 + z^2 = 1$, $y = 0$, about the z-axis. Show that curl $\boldsymbol{F} = \boldsymbol{0}$ in D but that $\int_C P\, dx + Q\, dy + R\, dz$ is not zero if the curve C is the circle $x^2 + y^2 = 4$, $z = 0$.
13. This exercise outlines a proof of Equation (12.29), used in the proof of Stokes' theorem.
(a) Use the formula for differentiating a product to show that

$$\frac{\partial}{\partial u}\left(p\frac{\partial X}{\partial v}\right) - \frac{\partial}{\partial v}\left(p\frac{\partial X}{\partial u}\right) = \frac{\partial p}{\partial u}\frac{\partial X}{\partial v} - \frac{\partial p}{\partial v}\frac{\partial X}{\partial u}.$$

(b) Now let $p(u, v) = P[X(u, v), Y(u, v), Z(u, v)]$. Compute $\partial p/\partial u$ and $\partial p/\partial v$ by the chain rule and use part (a) to deduce Equation (12.29),

$$\frac{\partial}{\partial u}\left(p\frac{\partial X}{\partial v}\right) - \frac{\partial}{\partial v}\left(p\frac{\partial X}{\partial u}\right) = -\frac{\partial P}{\partial y}\frac{\partial(X, Y)}{\partial(u, v)} + \frac{\partial P}{\partial z}\frac{\partial(Z, X)}{\partial(u, v)}.$$

12.14 Further properties of the curl and divergence

The curl and divergence of a vector field are related to the Jacobian matrix. If $\boldsymbol{F} = P\boldsymbol{i} + Q\boldsymbol{j} + R\boldsymbol{k}$, the Jacobian matrix of $\boldsymbol{F}$ is

$$\boldsymbol{D}\boldsymbol{F}(x, y, z) = \begin{bmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} & \frac{\partial P}{\partial z} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} & \frac{\partial Q}{\partial z} \\ \frac{\partial R}{\partial x} & \frac{\partial R}{\partial y} & \frac{\partial R}{\partial z} \end{bmatrix} = \begin{bmatrix} \nabla P \\ \nabla Q \\ \nabla R \end{bmatrix}$$

The trace of this matrix (the sum of its diagonal elements) is the divergence of $\boldsymbol{F}$.

Every real matrix A can be written as a sum of a symmetric matrix, $\frac{1}{2}(A + A^t)$, and a skew-symmetric matrix, $\frac{1}{2}(A - A^t)$. When A is the Jacobian matrix $\boldsymbol{D}\boldsymbol{F}$, the skew-symmetric part becomes

$$\frac{1}{2}\begin{bmatrix} 0 & \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} & \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \\ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} & 0 & \frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y} \\ \frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} & \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} & 0 \end{bmatrix}. \tag{12.35}$$

The nonzero elements of this matrix are the component of curl $\boldsymbol{F}$ and their negatives. If the Jacobian matrix $D\boldsymbol{F}$ is symmetric, each entry in (12.35) is zero and curl $\boldsymbol{F} = \boldsymbol{0}$.

EXAMPLE 1. Let $\boldsymbol{F}(x, y, z) = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$. Then we have

$$P(x, y, z) = x, \qquad Q(x, y, z) = y, \qquad R(x, y, z) = z,$$

and the corresponding Jacobian matrix is the 3×3 identity matrix. Therefore

$$\operatorname{div} \boldsymbol{F} = 3 \qquad \text{and} \qquad \operatorname{curl} \boldsymbol{F} = \boldsymbol{0}.$$

More generally, if $\boldsymbol{F}(x, y, z) = f(x)\boldsymbol{i} + g(y)\boldsymbol{j} + h(z)\boldsymbol{k}$, the Jacobian matrix has the elements $f'(x), g'(y), h'(z)$ on the main diagonal and zeros elsewhere, so

$$\operatorname{div} \boldsymbol{F} = f'(x) + g'(y) + h'(z) \qquad \text{and} \qquad \operatorname{curl} \boldsymbol{F} = \boldsymbol{0}.$$

EXAMPLE 2. Let $\boldsymbol{F}(x, y, z) = xy^2z^2\boldsymbol{i} + z^2 \sin y\,\boldsymbol{j} + x^2e^y\boldsymbol{k}$. The Jacobian matrix is

$$\begin{bmatrix} y^2z^2 & 2xyz^2 & 2xy^2z \\ 0 & z^2 \cos y & 2z \sin y \\ 2xe^y & x^2e^y & 0 \end{bmatrix}.$$

$\partial i/\partial x \quad \partial i/\partial y \quad \partial i/\partial z$
$\partial j/\partial x \quad \partial j/\partial y \quad \partial j/\partial z$
$\partial k/\partial x \quad \partial k/\partial y \quad \partial k/\partial y$

Therefore,

$$\operatorname{div} \boldsymbol{F} = y^2z^2 + z^2 \cos y$$

and

$$\operatorname{curl} \boldsymbol{F} = (x^2e^y - 2z \sin y)\boldsymbol{i} + (2xy^2z - 2xe^y)\boldsymbol{j} - 2xyz^2\boldsymbol{k}.$$

EXAMPLE 3. ***The divergence and curl of a gradient.*** Suppose $\boldsymbol{F}$ is a gradient, say $\boldsymbol{F} = \operatorname{grad} \varphi = \partial\varphi/\partial x\, \boldsymbol{i} + \partial\varphi/\partial y\, \boldsymbol{j} + \partial\varphi/\partial z\, \boldsymbol{k}$. The Jacobian matrix is

$$\begin{bmatrix} \dfrac{\partial^2\varphi}{\partial x^2} & \dfrac{\partial^2\varphi}{\partial y\,\partial x} & \dfrac{\partial^2\varphi}{\partial z\,\partial x} \\[2ex] \dfrac{\partial^2\varphi}{\partial x\,\partial y} & \dfrac{\partial^2\varphi}{\partial y^2} & \dfrac{\partial^2\varphi}{\partial z\,\partial y} \\[2ex] \dfrac{\partial^2\varphi}{\partial x\,\partial z} & \dfrac{\partial^2\varphi}{\partial y\,\partial z} & \dfrac{\partial^2\varphi}{\partial z^2} \end{bmatrix}. \tag{12.36}$$

Therefore,

$$\operatorname{div} \boldsymbol{F} = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}.$$

The expression on the right is called the ***Laplacian*** of φ and is often written more briefly as $\nabla^2\varphi$. Thus, the divergence of a gradient $\nabla\varphi$ is the Laplacian of φ. In symbols, this is written

$$\operatorname{div} (\nabla\varphi) = \nabla^2\varphi. \tag{12.37}$$

When $\nabla^2\varphi = 0$, the function φ is called ***harmonic.*** Equation (12.37) shows that the gradient of a harmonic function has zero divergence. When the mixed partial derivatives in matrix (12.36) are continuous, the matrix is symmetric and curl $\boldsymbol{F}$ is zero. In other words,

$$\operatorname{curl}\,(\operatorname{grad}\,\varphi) = \boldsymbol{0}$$

for every scalar field φ with continuous second-order mixed partial derivatives. This example shows that the condition curl $\boldsymbol{F} = \boldsymbol{0}$ is necessary for a continuously differentiable vector field $\boldsymbol{F}$ to be a gradient. In other words, if curl $\boldsymbol{F} \neq \boldsymbol{0}$ on an open set S, then $\boldsymbol{F}$ is not a gradient on S. We know also, from Theorem 12.4 that if curl $\boldsymbol{F} = \boldsymbol{0}$ on an open ***convex*** set S, then $\boldsymbol{F}$ is a gradient on S. A field with zero curl is called ***irrotational.***

EXAMPLE 4. ***A vector field with zero divergence and zero curl.*** Let S be the set of all $(x, y) \neq (0, 0)$, and let

$$\boldsymbol{F}(x, y) = -\frac{y}{x^2 + y^2}\,\boldsymbol{i} + \frac{x}{x^2 + y^2}\,\boldsymbol{j}$$

if $(x, y) \in S$. From Example 2 in Section 10.16 we know that $\boldsymbol{F}$ ***is not a gradient on*** S (although $\boldsymbol{F}$ is a gradient on every rectangle not containing the origin). The Jacobian matrix is

$$D\boldsymbol{F}(x, y) = \begin{bmatrix} \dfrac{2xy}{(x^2 + y^2)^2} & \dfrac{y^2 - x^2}{(x^2 + y^2)^2} & 0 \\[2ex] \dfrac{y^2 - x^2}{(x^2 + y^2)^2} & \dfrac{-2xy}{(x^2 + y^2)^2} & 0 \\[2ex] 0 & 0 & 0 \end{bmatrix},$$

and we see at once that div $\boldsymbol{F} = 0$ and curl $\boldsymbol{F} = \boldsymbol{0}$ on S.

EXAMPLE 5. ***The divergence and curl of a curl.*** If $\boldsymbol{F} = P\boldsymbol{i} + Q\boldsymbol{j} + R\boldsymbol{k}$, the curl of $\boldsymbol{F}$ is a new vector field and we can compute *its* divergence and curl. The Jacobian matrix of curl $\boldsymbol{F}$ is

$$\begin{bmatrix} \dfrac{\partial^2 R}{\partial x\,\partial y} - \dfrac{\partial^2 Q}{\partial x\,\partial z} & \dfrac{\partial^2 R}{\partial y^2} - \dfrac{\partial^2 Q}{\partial y\,\partial z} & \dfrac{\partial^2 R}{\partial z\,\partial y} - \dfrac{\partial^2 Q}{\partial z^2} \\[2ex] \dfrac{\partial^2 P}{\partial x\,\partial z} - \dfrac{\partial^2 R}{\partial x^2} & \dfrac{\partial^2 P}{\partial y\,\partial z} - \dfrac{\partial^2 R}{\partial y\,\partial x} & \dfrac{\partial^2 P}{\partial z^2} - \dfrac{\partial^2 R}{\partial z\,\partial x} \\[2ex] \dfrac{\partial^2 Q}{\partial x^2} - \dfrac{\partial^2 P}{\partial x\,\partial y} & \dfrac{\partial^2 Q}{\partial y\,\partial x} - \dfrac{\partial^2 P}{\partial y^2} & \dfrac{\partial^2 Q}{\partial z\,\partial x} - \dfrac{\partial^2 P}{\partial z\,\partial y} \end{bmatrix}$$

If we assume that all the mixed partial derivatives are continuous, we find that

$$\operatorname{div}\,(\operatorname{curl}\,\boldsymbol{F}) = 0$$

and

$$\operatorname{curl}\,(\operatorname{curl}\,\boldsymbol{F}) = \operatorname{grad}\,(\operatorname{div}\,\boldsymbol{F}) - \nabla^2\boldsymbol{F}, \tag{12.38}$$