9 CHAPTER Elementary Functions

9.1 Algebraic and Transcendental Functions

The main theme of this book has been the tension between arithmetic and geometry and its creative role in the development of mathematics. The story of $\sqrt{2}$ is an excellent example of such tension and its beneficial effects: geometry confronted arithmetic with the diagonal of the unit square, arithmetic expanded its concept of number in response, and the new number $\sqrt{2}$ proved its worth by giving new insight into the old numbers, for example, by generating integer solutions of the equation $x^2 - 2y^2 = 1$ (Section 8.5). In other cases, geometry was not so much a source of conflict with arithmetic as a source of immediate insight; for example, in generating Pythagorean triples by the chord construction (Section 4.3) or in guaranteeing unique prime factorization in the Gaussian integers by the triangle inequality (Section 7.5).

From the other side, arithmetic confronted geometry with the problem of describing curves in terms of numbers, addition, and multiplication. Geometry responded with coordinates and polynomial equations, which were a brilliant success with conic sections and many other curves. This development went hand in hand with another expansion of arithmetic—into the algebra of polynomials and rational functions.

The most complete fusion of arithmetic and geometry was obtained with the concept of *algebraic curve* —a curve defined by a polynomial equation $p(x, y) = 0$ in the two variables x and y—and the corresponding concept of an *algebraic function* y of x. (The symmetry of the definition shows that the inverse function x of y is also algebraic. Strictly speaking, neither may be a true function—for example, more than one value of y may correspond to the same value of x—but this is not important here.) Is algebra then the perfect reconciliation of arithmetic and geometry?

Not quite. Geometry has more challenges to offer, such as the concept of angle and the related sine and cosine functions. The function $\sin x$ is *not* an algebraic function of x and, equivalently, the curve $y = \sin x$ is not an algebraic curve. The reason, already seen in the exercises to Section 5.2, is that $y = \sin x$ does not contain the x-axis, yet it meets the x-axis at infinitely many points. No algebraic curve has this property, because its intersections with the x-axis satisfy the polynomial equation $p(x, 0) = 0$, which has only finitely many solutions.

Thus $y = \sin x$ is not an algebraic curve and $\sin x$ is not an algebraic function. Functions that are not algebraic are called *transcendental*, because they "transcend" algebra. Thinking back to the definition of cos and sin, one recalls that they were defined as functions of arc length of the unit circle, and *arc length* was defined as the least upper bound of the lengths of polygons. The least upper bound exists, by the completeness of the real numbers, but it would be nice to have a more explicit description of its value. We can now see that an algebraic description, at any rate, is out of the question. The arc length of the circle cannot be an algebraic function (of the coordinates of its endpoints, say), otherwise the sine function would also be algebraic.

Geometry creates many transcendental functions. Any nonconstant periodic function, such as the arc length of a closed curve, must be transcendental for the same reason as the sine; its graph meets a straight line in infinitely many points. In fact, the arc length function of almost any algebraic curve is transcendental, and very