his computational methods, and found he was able to simplify them. This was probably his first mathematical research.

In 1823 Hamilton entered Trinity College, Dublin, beginning an academic career of extraordinary distinction in both science and classics. Over the next three years he laid the foundations of his brilliant mathematical life but also, alas, for his miserable personal life. Hamilton was a romantic—he loved *Romeo and Juliet* and the poetry of Wordsworth—and on 17 August 1824 he met the lady of his dreams, Catherine Disney.

Her family were friends of his uncle James, and some of her brothers in fact became friends of Hamilton at Trinity College. Hamilton fell in love with Catherine at first sight, and she apparently reciprocated his feeling; but the boy who knew all the words, in all the languages, did not manage to convey his love to her. Perhaps he thought it improper to express such feelings before he had any prospect of marriage, or before he was sure how she felt; but at any rate his hesitation was fatal. In February 1825 Catherine became engaged to an older and wealthier suitor, encouraged by her family, and on 25 May they were married. Hamilton despaired almost to the point of suicide, and never really recovered. Only his mathematical spirit was not crushed.

On this occasion he rebounded with his first important mathematical paper, his *Theory of Systems of Rays* presented to the Royal Irish Academy in 1827. This paper on optics was followed by his appointment as Professor of Astronomy and Director of Dunsink Observatory, an amazing achievement for a 22-year-old. His fame grew, and over the next few years he became friends with several men who were to influence his intellectual life: the poets Wordsworth and Coleridge, the mathematicians John and Charles Graves, and their brother Robert, who eventually wrote Hamilton's biography.

The scene was also set for his next disaster of the heart. Among Hamilton's students at the observatory in 1830 was a young aristocrat and astronomy enthusiast named Lord Adare. From time to time he invited Hamilton to his family home, Adare Manor in County Limerick. There in 1831 Hamilton met the second love of his life, Ellen de Vere, a beautiful and intelligent 18-year-old whose appreciation of romantic poetry surpassed even his own.

They seemed perfect for each other, and this time he had money, position, and the support of her family. How could he fail? Only by giving up at the first sign of difficulty! Ellen dropped a casual remark that "she could not live happily anywhere but at Curragh" (her home). Hamilton took this as a polite but firm rebuff—and that was the end of the courtship. He retired to nurse his broken heart again, writing an excruciating sonnet entitled *To E. de V. On her saying that she could not live happily anywhere but at Curragh.* In due course Ellen married another, and of course left Curragh.

Hamilton returned to mathematics to ease the pain, and in 1832 lifted his theory of optics to a new level. A supplement to his *Theory of Systems of Rays* in 1832 presented a sensational and unprecedented discovery: a new physical phenomenon predicted by pure mathematics. This was the previously unobserved *conical refraction*, in which a single ray of light entering a slab of suitable crystalline material diverges as a hollow cone. Hamilton's prediction was verified experimentally by Humphrey Lloyd at Trinity College, and was the first of many such predictions. Two of the best-known ones are the prediction of electromagnetic waves from Maxwell's equations of 1864, and the bending of light predicted by Einstein's general theory of relativity in 1915. As in the latter cases, Hamilton's success was no fluke. It was based on a deep and powerful mathematical theory that generalizes to other situations, and is now known as *Hamiltonian dynamics*.

Having regained some self-confidence, Hamilton in 1832 found what he called a "dim perspective of possible marriage" in Helen Bayly, who lived near him and was two years his senior. Dim it was, but this time he steeled himself to resist all opposition. Despite Helen's fragile health (which she warned him about herself) and the total opposition of his family, they were married on 9 April 1833. They spent their honeymoon at the cottage of Helen's widowed mother, where Hamilton continued working on his mathematical papers.

When they returned to his home at Dunsink observatory, Hamilton's sisters, who had previously kept house for him, had moved out. His domestic life descended into chaos, as Helen was frequently ill or absent entirely, and Hamilton came to depend on alcohol for consolation. Despite this, his mathematical work continued unabated. He was knighted in 1835, elected president of the Royal Irish Academy in 1837, and (as we know) discovered quaternions in 1843.

It is probably true that Hamilton spent too much time on quaternions. He did little else until his death in 1865, and few mathematicians shared his enthusiasm. Nevertheless quaternions changed the course of mathematics, though not in the way Hamilton intended. In the 1880s Josiah Willard Gibbs and Oliver Heaviside created what we now know as vector analysis, essentially by separating the real ("scalar") part of a quaternion from its imaginary ("vector") part. Hamilton's followers were outraged to see the simple and elegant quaternions torn limb from limb, but the idea caught on with physicists and engineers, and it still holds sway today.

There are at least three biographies of Hamilton, all worth reading. Graves' three volumes [Graves (1975)] are still valuable, if only for the large amount of correspondence they contain. Hankins (1980) is entertaining and authoritative, with good coverage of the mathematics. O'Donnell (1983) throws more light on Hamilton's psychology and is refreshingly skeptical about his childhood precocity with languages. For more on the remarkable metamorphosis of quaternions into vector analysis, see Crowe (1967).

21

Algebraic Number Theory

21.1 Algebraic Numbers

The integers are the simplest objects in mathematics but, as history shows, their secrets are deeply hidden. A vast range of mathematical disciplines—such as geometry, algebra, and analysis—has been called upon to clarify the apparently simple concept of integer. In particular, a broader *concept of number* itself seems to be useful. We have seen in Section 5.4, for example, how integer solutions of the Pell equation $x^2 - Ny^2 = 1$ can be produced with the help of irrational numbers of the form $a + b\sqrt{N}$, and in Section 10.6 how the number $(1+\sqrt{5})/2$ helps explain the mysterious sequence of Fibonacci numbers. These are examples of the way *algebraic* numbers help elucidate the behavior of integers.

In the nineteenth century, a powerful theory of algebraic numbers was developed, with the aim of throwing more light on ordinary number theory. It was very successful in this respect, but it also developed a life of its own, and in the twentieth century its concepts were appropriated by the abstract theories of rings, fields, and vector spaces. Later in the chapter we sketch how this happened, but our main goal is to explain algebraic number theory itself, the inspiration for this whole development.

First we should state the definition: an *algebraic number* is one that satisfies an equation of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0 \quad \text{where } a_0, a_1, \ldots, a_n \in \mathbb{Z}.$$

The symbol $\mathbb{Z}$ for integers comes from the German word "Zahlen," meaning "numbers." We sometimes call these the "ordinary," or *rational*, in-

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