Definition. A finite group G is called a *Frobenius group* with *Frobenius kernel* Q if Q is a proper, nontrivial normal subgroup of G and $C_G(x) \leq Q$ for all nonidentity elements x of Q.

In view of the application to simple groups mentioned at the beginning of this section we shall restrict attention to Frobenius groups G of order $q^a p$, where p and q are distinct primes, such that the Frobenius kernel Q is an elementary abelian q-group of order q^a and the cyclic group G/Q acts irreducibly by conjugation on Q. In other words, we shall assume Q is a direct product of cyclic groups of order q and the only normal subgroups of G that are contained in Q are 1 and Q, i.e., Q is a minimal normal subgroup of G. For example, A_4 is a Frobenius group of this type with Frobenius kernel V_4, its Sylow 2-subgroup. Also, if p and q are distinct primes with $p < q$ and G is a non-abelian group of order pq (one always exists if $p \mid q - 1$) then G is a Frobenius group whose Frobenius kernel is its Sylow q-subgroup (which is normal by Sylow's Theorem). We essentially determine the character table of these Frobenius groups. Analogous results on more general Frobenius groups appear in the exercises.

Proposition 13. Let G be a Frobenius group of order $q^a p$, where p and q are distinct primes, such that the Frobenius kernel Q is an elementary abelian q-group of order q^a and the cyclic group G/Q acts irreducibly by conjugation on Q. Then the following hold:

(1) $G = QP$ where P is a Sylow p-subgroup of G. Every nonidentity element of G has order p or q. Every element of order p is conjugate to an element of P and every element of order q belongs to Q. The nonidentity elements of P represent the $p - 1$ distinct conjugacy classes of elements of order p and each of these classes has size q^a. There are $(q^a - 1)/p$ distinct conjugacy classes of elements of order q and each of these classes has size p.

(2) $G' = Q$ so the number of degree 1 characters of G is p and every degree 1 character contains Q in its kernel.

(3) If ψ is any nonprincipal irreducible character of Q, then $\mathrm{Ind}_Q^G(\psi)$ is an irreducible character of G. Moreover, every irreducible character of G of degree > 1 is equal to $\mathrm{Ind}_Q^G(\psi)$ for some nonprincipal irreducible character ψ of Q. Every irreducible character of G has degree either 1 or p and the number of irreducible characters of degree p is $(q^a - 1)/p$.

Proof: Note that QP equals G by order consideration. By definition of a Frobenius group and because Q is abelian, $C_G(h) = Q$ for every nonidentity element h of Q. If x were an element of order pq, then x^p would be an element of order q, hence would lie in the unique Sylow q-subgroup Q of G. But then x would commute with x^p and so x would belong to $C_G(x^p) = Q$, a contradiction. Thus G has no elements of order pq. By Sylow's Theorem every element of order p is conjugate to an element of P and every element of order q lies in Q. No two distinct elements of P are conjugate in G because if $g^{-1}xg = y$ for some $x, y \in P$ then $\overline{g^{-1}xg} = \overline{y}$ in the abelian group $\overline{G} = G/Q$ and so $\overline{x} = \overline{y}$. Then $x = y$ because $\overline{P} \cong P$. Thus there are exactly $p - 1$ conjugacy classes of elements of order p and these are represented by the nonidentity elements of P. If x is a nonidentity element of P, then $C_G(x) = P$ and so the conjugacy class of

x consists of $|G : P| = q^a$ elements. Finally, if h is a nonidentity element of Q, then $C_G(h) = Q$ and the conjugacy class of h is $\{h, h^x, \dots, h^{x^{p-1}}\}$, where $P = \langle x \rangle$. This proves all parts of (1).

Since G/Q is abelian, $G' \le Q$. Since G is non-abelian and Q is, by hypothesis, a minimal normal subgroup of G we must have $G' = Q$. Part (2) now follows from Corollary 11 in Section 18.2.

Let ψ be a nonprincipal irreducible character of Q and let $\Psi = \mathrm{Ind}_Q^G(\psi)$. We use the orthogonality relations to show that Ψ is irreducible. Let $1, x, \dots, x^{p-1}$ be coset representatives for Q in G. By Corollary 12, Ψ is zero on $G - Q$ so

$$\begin{aligned}
\|\Psi\|^2 &= \frac{1}{|G|}\sum_{h\in Q}\Psi(h)\overline{\Psi(h)} \\
&= \frac{1}{|G|}\sum_{h\in Q}\sum_{i=0}^{p-1}\psi(x^ihx^{-i})\overline{\psi(x^ihx^{-i})} \\
&= \frac{p}{|G|}\sum_{h\in Q}\psi(h)\overline{\psi(h)} \\
&= \frac{p|Q|}{|G|} = 1,
\end{aligned}$$

where the second line follows from the definition of the induced character Ψ, the third line follows because each element of Q appears exactly p times in the sum in the second line, and the last line follows from the first orthogonality relation in Q because ψ is an irreducible character of Q. This proves Ψ is an irreducible character of G.

We prove that every irreducible character of G of degree > 1 is the induced character of some nonprincipal degree 1 character of Q by counting the number of distinct irreducible characters of G obtained this way. By parts (1) and (2) the number of irreducible characters of G (= the number of conjugacy classes) is $p + (q^a - 1)/p$ and the number of degree 1 characters is p. Thus the number of irreducible characters of G of degree > 1 is $(q^a - 1)/p$. The group P acts on the set $\mathcal{C}$ of nonprincipal irreducible characters of Q as follows: for each $\psi \in \mathcal{C}$ and each $x \in P$ let ψ^x be defined by

$$\psi^x(h) = \psi(xhx^{-1}) \qquad \text{for all } h \in Q.$$

Since ψ is a nontrivial homomorphism from Q into $\mathbb{C}^\times$ (recall that all irreducible characters of the abelian group Q have degree 1) it follows easily that ψ^x is also a homomorphism. Thus $\psi^x \in \mathcal{C}$ and so P permutes the elements of $\mathcal{C}$. Now let x be a generator for the cyclic group P. Then $1, x, \dots, x^{p-1}$ are representatives for the left cosets of Q in G. By Corollary 12 applied with this set of coset representatives we see that if $\psi \in \mathcal{C}$ then the value of $\mathrm{Ind}_Q^G(\psi)$ on any element h of Q is given by the sum $\psi(h) + \psi^x(h) + \cdots + \psi^{x^{p-1}}(h)$. Thus when the induced character $\mathrm{Ind}_Q^G(\psi)$ is restricted to Q it decomposes into irreducible characters of Q as

$$\mathrm{Ind}_Q^G(\psi)|_Q = \psi + \psi^x + \cdots + \psi^{x^{p-1}}.$$

If ψ_1 and ψ_2 are in different orbits of the action of P on $\mathcal{C}$ then the induced characters $\mathrm{Ind}_Q^G(\psi_1)$ and $\mathrm{Ind}_Q^G(\psi_2)$ restrict to distinct characters of Q (they have no irreducible constituents in common). Thus characters induced from elements of distinct orbits of P on $\mathcal{C}$ are distinct irreducible characters of G. The abelian group Q has $q^a - 1$ nonprincipal irreducible characters (i.e., $|\mathcal{C}| = q^a - 1$) and $|P| = p$ so there are at least $(q^a - 1)/p$ orbits of P on $\mathcal{C}$ and hence at least this number of distinct irreducible characters of G of degree p. Since G has exactly $(q^a - 1)/p$ irreducible characters of degree > 1, every irreducible character of G of degree > 1 must have degree p and must be an induced character from some element of $\mathcal{C}$. The proof is complete.

For the final example we shall require two properties of induced characters. These properties are listed in the next proposition and the proofs are straightforward exercises which follow easily from the formula for induced characters or from the definition of induced modules together with properties of tensor products.

Proposition 14. Let G be a group, let H be a subgroup of G and let ψ and ψ' be characters of H.

(1) *(Induction of characters is additive)* $\mathrm{Ind}_H^G(\psi + \psi') = \mathrm{Ind}_H^G(\psi) + \mathrm{Ind}_H^G(\psi')$.

(2) *(Induction of characters is transitive)* If $H \leq K \leq G$ then

$$\mathrm{Ind}_K^G(\mathrm{Ind}_H^K(\psi)) = \mathrm{Ind}_H^G(\psi).$$

It follows from part (1) of Proposition 14 that if $\sum_{i=1}^s n_i\psi_i$ is any integral linear combination of characters of H with $n_i \geq 0$ for all i then

$$\mathrm{Ind}_H^G\left(\sum_{i=1}^s n_i\psi_i\right) = \sum_{i=1}^s n_i \,\mathrm{Ind}_H^G(\psi_i). \tag{$*$}$$

A class function of H of the form $\sum_{i=1}^s n_i\psi_i$, where the coefficients are any integers (not necessarily nonnegative) is called a *generalized character* or *virtual character* of H. For a generalized character of H we define its induced generalized character of G by equation ($*$), allowing now negative coefficients n_i as well. In this way the function Ind_H^G becomes a group homomorphism from the additive group of generalized characters of H to the additive group of generalized characters of G (which maps characters to characters). This implies that the formula for induced characters in Corollary 12 holds also if ψ is a generalized character of H.

Application to Groups of Order $3^3 \cdot 7 \cdot 13 \cdot 409$

We now conclude with a proof of the following result:

there are no simple groups of order $3^3 \cdot 7 \cdot 13 \cdot 409$.

As mentioned at the beginning of this section, simple groups of this order were discussed at the end of Section 6.2 in the context of the existence problem for simple groups. It is possible to prove that there are no simple groups of this order by arguments involving a permutation representation of degree 819 (cf. the exercises in Section 6.2). We include a character-theoretic proof of this since the methods illustrate some important ideas in the theory of finite groups. The approach is based on M. Suzuki's seminal paper *The nonexistence of a certain type of simple group of odd order*, Proc. Amer. Math. Soc.,