Proof. Let α be a vector in W_j, β a vector in W_i, and suppose $i \neq j$. Then $c_j(\alpha|\beta) = (T\alpha|\beta) = (\alpha|T^*\beta) = (\alpha|\bar{c}_i\beta)$. Hence $(c_j - c_i)(\alpha|\beta) = 0$, and since $c_j - c_i \neq 0$, it follows that $(\alpha|\beta) = 0$. Thus W_j is orthogonal to W_i when $i \neq j$. From the fact that V has an orthonormal basis consisting of characteristic vectors (cf. Theorems 18 and 22 of Chapter 8), it follows that $V = W_1 + \cdots + W_k$. If α_j belongs to V_j $(1 \leq j \leq k)$ and $\alpha_1 + \cdots + \alpha_k = 0$, then

$$\begin{aligned} 0 &= (\alpha_i|\sum_j \alpha_j) = \sum_j (\alpha_i|\alpha_j) \\ &= ||\alpha_i||^2 \end{aligned}$$

for every i, so that V is the direct sum of $W_1, \ldots, W_k$. Therefore $E_1 + \cdots + E_k = I$ and

$$\begin{aligned} T &= TE_1 + \cdots + TE_k \\ &= c_1E_1 + \cdots + c_kE_k. \quad \blacksquare \end{aligned}$$

The decomposition (9-11) is called the **spectral resolution** of T. This terminology arose in part from physical applications which caused the **spectrum** of a linear operator on a finite-dimensional vector space to be defined as the set of characteristic values for the operator. It is important to note that the orthogonal projections $E_1, \ldots, E_k$ are canonically associated with T; in fact, they are polynomials in T.

Corollary. *If* $e_j = \prod_{i \neq j} \left(\frac{x - c_i}{c_j - c_i}\right)$, *then* $E_j = e_j(T)$ *for* $1 \leq j \leq k$.

Proof. Since $E_iE_j = 0$ when $i \neq j$, it follows that

$$T^2 = c_1^2E_1 + \cdots + c_k^2E_k$$

and by an easy induction argument that

$$T^n = c_1^nE_1 + \cdots + c_k^nE_k$$

for every integer $n \geq 0$. For an arbitrary polynomial

$$f = \sum_{n=0}^{r} a_nx^n$$

we have

$$\begin{aligned} f(T) &= \sum_{n=0}^{r} a_nT^n \\ &= \sum_{n=0}^{r} a_n \sum_{j=1}^{k} c_j^nE_j \\ &= \sum_{j=1}^{k} \left(\sum_{n=0}^{r} a_nc_j^n\right) E_j \\ &= \sum_{j=1}^{k} f(c_j)E_j. \end{aligned}$$

Since $e_j(c_m) = \delta_{jm}$, it follows that $e_j(T) = E_j$. $\blacksquare$

Because $E_1, \ldots, E_k$ are canonically associated with T and

$$I = E_1 + \cdots + E_k$$

the family of projections $\{E_1, \ldots, E_k\}$ is called the **resolution of the identity defined by** T.

There is a comment that should be made about the proof of the spectral theorem. We derived the theorem using Theorems 18 and 22 of Chapter 8 on the diagonalization of self-adjoint and normal operators. There is another, more algebraic, proof in which it must first be shown that the minimal polynomial of a normal operator is a product of distinct prime factors. Then one proceeds as in the proof of the primary decomposition theorem (Theorem 12, Chapter 6). We shall give such a proof in the next section.

In various applications it is necessary to know whether one may compute certain functions of operators or matrices, e.g., square roots. This may be done rather simply for diagonalizable normal operators.

Definition. *Let* T *be a diagonalizable normal operator on a finite-dimensional inner product space and*

$$\mathrm{T} = \sum_{j=1}^{k} \mathrm{c}_j \mathrm{E}_j$$

its spectral resolution. Suppose f *is a function whose domain includes the spectrum of* T *that has values in the field of scalars. Then the linear operator* f(T) *is defined by the equation*

$$\mathrm{f(T)} = \sum_{j=1}^{k} \mathrm{f(c}_j)\mathrm{E}_j. \tag{9-12}$$

Theorem 10. *Let* T *be a diagonalizable normal operator with spectrum* S *on a finite-dimensional inner product space* V. *Suppose* f *is a function whose domain contains* S *that has values in the field of scalars. Then* f(T) *is a diagonalizable normal operator with spectrum* f(S). *If* U *is a unitary map of* V *onto* V′ *and* $\mathrm{T}' = \mathrm{UTU}^{-1}$, *then* S *is the spectrum of* T′ *and*

$$\mathrm{f(T')} = \mathrm{Uf(T)U}^{-1}.$$

Proof. The normality of $f(T)$ follows by a simple computation from (9-12) and the fact that

$$f(T)^* = \sum_j \overline{f(c_j)} E_j.$$

Moreover, it is clear that for every α in $E_j(V)$

$$f(T)\alpha = f(c_j)\alpha.$$

Thus, the set $f(S)$ of all $f(c)$ with c in S is contained in the spectrum of $f(T)$. Conversely, suppose $\alpha \neq 0$ and that

$$f(T)\alpha = b\alpha.$$

Then $\alpha = \sum_j E_j\alpha$ and

$$\begin{aligned} f(T)\alpha &= \sum_j f(T)E_j\alpha \\ &= \sum_j f(c_j)E_j\alpha \\ &= \sum_j bE_j\alpha. \end{aligned}$$

Hence,

$$\begin{aligned} \|\sum_j (f(c_j) - b)E_j\alpha\|^2 &= \sum_j |f(c_j) - b|^2\|E_j\alpha\|^2 \\ &= 0. \end{aligned}$$

Therefore, $f(c_j) = b$ or $E_j\alpha = 0$. By assumption, $\alpha \neq 0$, so there exists an index i such that $E_i\alpha \neq 0$. It follows that $f(c_i) = b$ and hence that $f(S)$ is the spectrum of $f(T)$. Suppose, in fact, that

$$f(S) = \{b_1, \ldots, b_r\}$$

where $b_m \neq b_n$ when $m \neq n$. Let X_m be the set of indices i such that $1 \leq i \leq k$ and $f(c_i) = b_m$. Let $P_m = \sum_i E_i$, the sum being extended over the indices i in X_m. Then P_m is the orthogonal projection of V on the subspace of characteristic vectors belonging to the characteristic value b_m of $f(T)$, and

$$f(T) = \sum_{m=1}^{r} b_m P_m$$

is the spectral resolution of $f(T)$.

Now suppose U is a unitary transformation of V onto V' and that $T' = UTU^{-1}$. Then the equation

$$T\alpha = c\alpha$$

holds if and only if

$$T'U\alpha = cU\alpha.$$

Thus S is the spectrum of T', and U maps each characteristic subspace for T onto the corresponding subspace for T'. In fact, using (9-12), we see that

$$T' = \sum_j c_j E'_j, \qquad E'_j = UE_jU^{-1}$$

is the spectral resolution of T'. Hence

$$\begin{aligned} f(T') &= \sum_j f(c_j)E'_j \\ &= \sum_j f(c_j)UE_jU^{-1} \\ &= U\left(\sum_j f(c_j)E_j\right)U^{-1} \\ &= Uf(T)U^{-1}. \quad \blacksquare \end{aligned}$$

In thinking about the preceding discussion, it is important for one to keep in mind that the spectrum of the normal operator T is the set

$$S = \{c_1, \ldots, c_k\}$$

of distinct characteristic values. When T is represented by a diagonal matrix in a basis of characteristic vectors, it is necessary to repeat each value c_j as many times as the dimension of the corresponding space of characteristic vectors. This is the reason for the change of notation in the following result.

Corollary. *With the assumptions of Theorem* 10, *suppose that* T *is represented in the ordered basis* $\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}$ *by the diagonal matrix* D *with entries* $d_1, \ldots, d_n$. *Then, in the basis* $\mathcal{B}$, $f(T)$ *is represented by the diagonal matrix* $f(D)$ *with entries* $f(d_1), \ldots, f(d_n)$. *If* $\mathcal{B}' = \{\alpha_1', \ldots, \alpha_n'\}$ *is any other ordered basis and* P *the matrix such that*

$$\alpha_j' = \sum_i P_{ij}\alpha_i$$

then $P^{-1}f(D)P$ *is the matrix of* $f(T)$ *in the basis* $\mathcal{B}'$.

Proof. For each index i, there is a unique j such that $1 \leq j \leq k$, α_i belongs to $E_j(V)$, and $d_i = c_j$. Hence $f(T)\alpha_i = f(d_i)\alpha_i$ for every i, and

$$\begin{aligned} f(T)\alpha_j' &= \sum_i P_{ij}f(T)\alpha_i \\ &= \sum_i d_i P_{ij}\alpha_i \\ &= \sum_i (DP)_{ij}\alpha_i \\ &= \sum_i (DP)_{ij} \sum_k P_{ki}^{-1}\alpha_k' \\ &= \sum_k (P^{-1}DP)_{kj}\alpha_k'. \quad \blacksquare \end{aligned}$$

It follows from this result that one may form certain functions of a normal matrix. For suppose A is a normal matrix. Then there is an invertible matrix P, in fact a unitary P, such that PAP^{-1} is a diagonal matrix, say D with entries $d_1, \ldots, d_n$. Let f be a complex-valued function which can be applied to $d_1, \ldots, d_n$, and let $f(D)$ be the diagonal matrix with entries $f(d_1), \ldots, f(d_n)$. Then $P^{-1}f(D)P$ is independent of D and just a function of A in the following sense. If Q is another invertible matrix such that QAQ^{-1} is a diagonal matrix D', then f may be applied to the diagonal entries of D' and

$$P^{-1}f(D)P = Q^{-1}f(D')Q.$$

Definition. *Under the above conditions,* $f(A)$ *is defined as* $P^{-1}f(D)P$.

The matrix $f(A)$ may also be characterized in a different way. In doing this, we state without proof some of the results on normal matrices that one obtains by formulating the matrix analogues of the preceding theorems.

Theorem 11. *Let* A *be a normal matrix and* $c_1, \ldots, c_k$ *the distinct complex roots of det* $(xI - A)$. *Let*

$$e_i = \prod_{j \neq i} \left(\frac{x - c_j}{c_i - c_j} \right)$$

and $E_i = e_i(A)$ $(1 \leq i \leq k)$. *Then* $E_iE_j = 0$ *when* $i \neq j$, $E_i^2 = E_i$, $E_i^* = E_i$, *and*

$$I = E_1 + \cdots + E_k.$$

If f *is a complex-valued function whose domain includes* $c_1, \ldots, c_k$, *then*

$$f(A) = f(c_1)E_1 + \cdots + f(c_k)E_k;$$

in particular, $A = c_1E_1 + \cdots + c_kE_k$.

We recall that an operator on an inner product space V is non-negative if T is self-adjoint and $(T\alpha|\alpha) \geq 0$ for every α in V.

Theorem 12. *Let* T *be a diagonalizable normal operator on a finite-dimensional inner product space* V. *Then* T *is self-adjoint, non-negative, or unitary according as each characteristic value of* T *is real, non-negative, or of absolute value* 1.

Proof. Suppose T has the spectral resolution $T = c_1E_1 + \cdots + c_kE_k$, then $T^* = \bar{c}_1E_1 + \cdots + \bar{c}_kE_k$. To say T is self-adjoint is to say $T = T^*$, or

$$(c_1 - \bar{c}_1)E_1 + \cdots + (c_k - \bar{c}_k)E_k = 0.$$

Using the fact that $E_iE_j = 0$ for $i \neq j$, and the fact that no E_j is the zero operator, we see that T is self-adjoint if and only if $c_j = \bar{c}_j$, $j = 1, \ldots, k$. To distinguish the normal operators which are non-negative, let us look at

$$\begin{aligned}(T\alpha|\alpha) &= \left(\sum_{j=1}^{k} c_jE_j\alpha \Big| \sum_{i=1}^{k} E_i\alpha \right) \\ &= \sum_i \sum_j c_j(E_j\alpha|E_i\alpha) \\ &= \sum_j c_j||E_j\alpha||^2.\end{aligned}$$

We have used the fact that $(E_j\alpha|E_i\alpha) = 0$ for $i \neq j$. From this it is clear that the condition $(T\alpha|\alpha) \geq 0$ is satisfied if and only if $c_j \geq 0$ for each j. To distinguish the unitary operators, observe that

$$\begin{aligned}TT^* &= c_1c_1E_1 + \cdots + c_kc_kE_k \\ &= |c_1|^2E_1 + \cdots + |c_k|^2E_k.\end{aligned}$$

If $TT^* = I$, then $I = |c_1|^2E_1 + \cdots + |c_k|^2E_k$, and operating with E_j

$$E_j = |c_j|^2E_j.$$

Since $E_j \neq 0$, we have $|c_j|^2 = 1$ or $|c_j| = 1$. Conversely, if $|c_j|^2 = 1$ for each j, it is clear that $TT^* = I$. ▌

It is important to note that this is a theorem about normal operators. If T is a general linear operator on V which has real characteristic values, it does not follow that T is self-adjoint. The theorem states that if T has real characteristic values, and *if* T is diagonalizable and normal, then T is self-adjoint. A theorem of this type serves to strengthen the analogy between the adjoint operation and the process of forming the conjugate of a complex number. A complex number z is real or of absolute value 1 according as $z = \bar{z}$, or $\bar{z}z = 1$. An operator T is self-adjoint or unitary according as $T = T^*$ or $T^*T = I$.

We are going to prove two theorems now, which are the analogues of these two statements:

(1) Every non-negative number has a unique non-negative square root.

(2) Every complex number is expressible in the form ru, where r is non-negative and $|u| = 1$. This is the polar decomposition $z = re^{i\theta}$ for complex numbers.

Theorem 13. *Let* V *be a finite-dimensional inner product space and* T *a non-negative operator on* V. *Then* T *has a unique non-negative square root, that is, there is one and only one non-negative operator* N *on* V *such that* $N^2 = T$.

Proof. Let $T = c_1E_1 + \cdots + c_kE_k$ be the spectral resolution of T. By Theorem 12, each $c_j \geq 0$. If c is any non-negative real number, let $\sqrt{c}$ denote the non-negative square root of c. Then according to Theorem 11 and (9-12) $N = \sqrt{T}$ is a well-defined diagonalizable normal operator on V. It is non-negative by Theorem 12, and, by an obvious computation, $N^2 = T$.

Now let P be a non-negative operator on V such that $P^2 = T$. We shall prove that $P = N$. Let

$$P = d_1F_1 + \cdots + d_rF_r$$

be the spectral resolution of P. Then $d_j \geq 0$ for each j, since P is non-negative. From $P^2 = T$ we have

$$T = d_1^2F_1 + \cdots + d_r^2F_r.$$

Now $F_1, \ldots, F_r$ satisfy the conditions $I = F_1 + \cdots + F_r$, $F_iF_j = 0$ for $i \neq j$, and no F_j is 0. The numbers $d_1^2, \ldots, d_r^2$ are distinct, because distinct non-negative numbers have distinct squares. By the uniqueness of the spectral resolution of T, we must have $r = k$, and (perhaps reordering) $F_j = E_j$, $d_j^2 = c_j$. Thus $P = N$. ▌