Proof. We already know that the integers $\mathbf{Z}$ are countable, which implies that the non-zero integers $\mathbf{Z} - \{0\}$ are countable (why?). By Corollary 8.1.14, the set

$$\mathbf{Z} \times (\mathbf{Z} - \{0\}) = \{(a, b) : a, b \in \mathbf{Z}, b \neq 0\}$$

is thus countable. If one lets $f : \mathbf{Z} \times (\mathbf{Z} - \{0\}) \to \mathbf{Q}$ be the function $f(a, b) := a/b$ (note that f is well-defined since we prohibit b from being equal to 0), we see from Corollary 8.1.9 that $f(\mathbf{Z} \times (\mathbf{Z} - \{0\}))$ is at most countable. But we have $f(\mathbf{Z} \times (\mathbf{Z} - \{0\})) = \mathbf{Q}$ (why? This is basically the definition of the rationals $\mathbf{Q}$). Thus $\mathbf{Q}$ is at most countable. However, $\mathbf{Q}$ cannot be finite, since it contains the infinite set $\mathbf{N}$. Thus $\mathbf{Q}$ is countable. □

Remark 8.1.16. Because the rationals are countable, we know *in principle* that it is possible to arrange the rational numbers as a sequence:

$$\mathbf{Q} = \{a_0, a_1, a_2, a_3, \ldots\}$$

such that every element of the sequence is different from every other element, and that the elements of the sequence exhaust $\mathbf{Q}$ (i.e., every rational number turns up as one of the elements a_n of the sequence). However, it is quite difficult (though not impossible) to actually try and come up with an explicit sequence $a_0, a_1, \ldots$ which does this; see Exercise 8.1.10.

Exercise 8.1.1. Let X be a set. Show that X is infinite if and only if there exists a proper subset $Y \subsetneq X$ of X which has the same cardinality as X.

Exercise 8.1.2. Prove Proposition 8.1.4. (Hint: you can either use induction, or use the principle of infinite descent, Exercise 4.4.2, or use the least upper bound (or greatest lower bound) principle, Theorem 5.5.9.) Does the well-ordering principle work if we replace the natural numbers by the integers? What if we replace the natural numbers by the positive rationals? Explain.

Exercise 8.1.3. Fill in the gaps marked (?) in Proposition 8.1.5.

Exercise 8.1.4. Prove Proposition 8.1.8. (Hint: the basic problem here is that f is not assumed to be one-to-one. Define A to be the set

$$A := \{n \in \mathbf{N} : f(m) \neq f(n) \text{ for all } 0 \leq m < n\};$$

informally speaking, A is the set of natural numbers n for which $f(n)$ does not appear in the sequence $f(0), f(1), \ldots f(n)$. Prove that when f is restricted to A, it becomes a bijection from A to $f(\mathbf{N})$. Then use Proposition 8.1.5.)

Exercise 8.1.5. Use Proposition 8.1.8 to prove Corollary 8.1.9.

Exercise 8.1.6. Let A be a set. Show that A is at most countable if and only if there exists an injective map $f : A \to \mathbf{N}$ from A to $\mathbf{N}$.

Exercise 8.1.7. Prove Proposition 8.1.10. (Hint: by hypothesis, we have a bijection $f : \mathbf{N} \to X$, and a bijection $g : \mathbf{N} \to Y$. Now define $h : \mathbf{N} \to X \cup Y$ by setting $h(2n) := f(n)$ and $h(2n+1) := g(n)$ for every natural number n, and show that $h(\mathbf{N}) = X \cup Y$. Then use Corollary 8.1.9, and show that $X \cup Y$ cannot possibly be finite.)

Exercise 8.1.8. Use Corollary 8.1.13 to prove Corollary 8.1.14.

Exercise 8.1.9. Suppose that I is an at most countable set, and for each $\alpha \in I$, let A_α be an at most countable set. Show that the set $\bigcup_{\alpha \in I} A_\alpha$ is also at most countable. In particular, countable unions of countable sets are countable.

Exercise 8.1.10. Find a bijection $f : \mathbf{N} \to \mathbf{Q}$ from the natural numbers to the rationals. (Warning: this is actually rather tricky to do explicitly; it is difficult to get f to be simultaneously injective and surjective.)

8.2 Summation on infinite sets

We now introduce the concept of summation on *countable sets*, which will be well-defined provided that the sum is absolutely convergent.

Definition 8.2.1 (Series on countable sets). Let X be a countable set, and let $f : X \to \mathbf{R}$ be a function. We say that the series $\sum_{x \in X} f(x)$ is *absolutely convergent* iff for some bijection $g : \mathbf{N} \to X$, the sum $\sum_{n=0}^{\infty} f(g(n))$ is absolutely convergent. We then define the sum of $\sum_{x \in X} f(x)$ by the formula

$$\sum_{x \in X} f(x) = \sum_{n=0}^{\infty} f(g(n)).$$

From Proposition 7.4.3 (and Proposition 3.6.4), one can show that these definitions do not depend on the choice of g, and so are well defined.

We can now give an important theorem about double summations.

Theorem 8.2.2 (Fubini's theorem for infinite sums). *Let $f : \mathbf{N}\times\mathbf{N} \to \mathbf{R}$ be a function such that $\sum_{(n,m)\in\mathbf{N}\times\mathbf{N}} f(n,m)$ is absolutely convergent. Then we have*

$$\begin{aligned}\sum_{n=0}^{\infty}\Big(\sum_{m=0}^{\infty} f(n,m)\Big) &= \sum_{(n,m)\in\mathbf{N}\times\mathbf{N}} f(n,m)\\ &= \sum_{(m,n)\in\mathbf{N}\times\mathbf{N}} f(n,m)\\ &= \sum_{m=0}^{\infty}\Big(\sum_{n=0}^{\infty} f(n,m)\Big).\end{aligned}$$

In other words, we can switch the order of infinite sums *provided that the entire sum is absolutely convergent.* You should go back and compare this with Example 1.2.5.

Proof. (A sketch only; this proof is considerably more complex than the other proofs, and is optional reading.) The second equality follows easily from Proposition 7.4.3 (and Proposition 3.6.4). We shall just prove the first equality, as the third is very similar (basically one switches the rôle of n and m).

Let us first consider the case when $f(n,m)$ is always non-negative (we will deal with the general case later). Write

$$L := \sum_{(n,m)\in\mathbf{N}\times\mathbf{N}} f(n,m);$$

our task is to show that the series $\sum_{n=0}^{\infty}(\sum_{m=0}^{\infty} f(n,m))$ converges to L.

One can easily show that $\sum_{(n,m)\in X} f(n,m) \leq L$ for all finite sets $X \subset \mathbf{N}\times\mathbf{N}$. (Why? Use a bijection g between $\mathbf{N}\times\mathbf{N}$ and $\mathbf{N}$, and then use the fact that $g(X)$ is finite, hence bounded.) In particular, for every $n \in \mathbf{N}$ and $M \in \mathbf{N}$ we have $\sum_{m=0}^{M} f(n,m) \leq L$, which implies by Proposition 6.3.8 that $\sum_{m=0}^{\infty} f(n,m)$ is convergent for each m. Similarly, for any $N \in \mathbf{N}$ and $M \in \mathbf{N}$ we have (by Corollary 7.1.14)

$$\sum_{n=0}^{N} \sum_{m=0}^{M} f(n,m) \leq \sum_{(n,m) \in X} f(n,m) \leq L$$

where X is the set $\{(n,m) \in \mathbf{N} \times \mathbf{N} : n \leq N, m \leq M\}$ which is finite by Proposition 3.6.14. Taking suprema of this as $M \to \infty$ we have (by limit laws, and an induction on N)

$$\sum_{n=0}^{N} \sum_{m=0}^{\infty} f(n,m) \leq L.$$

By Proposition 6.3.8, this implies that $\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} f(n,m)$ converges, and

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} f(n,m) \leq L.$$

To finish the proof, it will suffice to show that

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} f(n,m) \geq L - \varepsilon$$

for every $\varepsilon > 0$. (Why will this be enough? Prove by contradiction.) So, let $\varepsilon > 0$. By definition of L, we can then find a finite set $X \subseteq \mathbf{N} \times \mathbf{N}$ such that $\sum_{(n,m) \in X} f(n,m) \geq L - \varepsilon$. (Why?) This set, being finite, must be contained in some set of the form $Y := \{(n,m) \in \mathbf{N} \times \mathbf{N} : n \leq N; m \leq M\}$. (Why? use induction.) Thus by Corollary 7.1.14

$$\sum_{n=0}^{N} \sum_{m=0}^{M} f(n,m) = \sum_{(n,m) \in Y} f(n,m) \geq \sum_{(n,m) \in X} f(n,m) \geq L - \varepsilon$$

and hence

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} f(n,m) \geq \sum_{n=0}^{N}\sum_{m=0}^{\infty} f(n,m) \geq \sum_{n=0}^{N}\sum_{m=0}^{M} f(n,m) \geq L-\varepsilon$$

as desired.

This proves the claim when the $f(n,m)$ are all non-negative. A similar argument works when the $f(n,m)$ are all non-positive (in fact, one can simply apply the result just obtained to the function $-f(n,m)$, and then use limit laws to remove the $-$. For the general case, note that any function $f(n,m)$ can be written (why?) as $f_+(n,m) + f_-(n,m)$, where $f_+(n,m)$ is the positive part of $f(n,m)$ (i.e., it equals $f(n,m)$ when $f(n,m)$ is positive, and 0 otherwise), and f_- is the negative part of $f(n,m)$ (it equals $f(n,m)$ when $f(n,m)$ is negative, and 0 otherwise). It is easy to show that if $\sum_{(n,m)\in \mathbf{N}\times\mathbf{N}} f(n,m)$ is absolutely convergent, then so are $\sum_{(n,m)\in \mathbf{N}\times\mathbf{N}} f_+(n,m)$ and $\sum_{(n,m)\in \mathbf{N}\times\mathbf{N}} f_-(n,m)$. So now one applies the results just obtained to f_+ and to f_- and adds them together using limit laws to obtain the result for a general f. □

There is another characterization of absolutely convergent series.

Lemma 8.2.3. *Let X be an at most countable set, and let $f : X \to \mathbf{R}$ be a function. Then the series $\sum_{x\in X} f(x)$ is absolutely convergent if and only if*

$$\sup\left\{\sum_{x\in A} |f(x)| : A \subseteq X, A \text{ finite}\right\} < \infty.$$

Proof. See Exercise 8.2.1. □

Inspired by this lemma, we may now define the concept of an absolutely convergent series even when the set X could be uncountable. (We give some examples of uncountable sets in the next section.)

Definition 8.2.4. Let X be a set (which could be uncountable), and let $f : X \to \mathbf{R}$ be a function. We say that the series $\sum_{x\in X} f(x)$ is absolutely convergent iff

$$\sup\left\{\sum_{x\in A} |f(x)| : A \subseteq X, A \text{ finite}\right\} < \infty.$$

Note that we have not yet said what the series $\sum_{x\in X} f(x)$ is equal to. This shall be accomplished by the following lemma.

Lemma 8.2.5. *Let X be a set (which could be uncountable), and let $f : X \to \mathbf{R}$ be a function such that the series $\sum_{x\in X} f(x)$ is absolutely convergent. Then the set $\{x \in X : f(x) \neq 0\}$ is at most countable.*

Proof. See Exercise 8.2.2. □

Because of this, we can define the value of $\sum_{x\in X} f(x)$ for any absolutely convergent series on an uncountable set X by the formula

$$\sum_{x\in X} f(x) := \sum_{x\in X: f(x)\neq 0} f(x),$$

since we have replaced a sum on an uncountable set X by a sum on the countable set $\{x \in X : f(x) \neq 0\}$. (Note that if the former sum is absolutely convergent, then the latter one is also.) Note also that this definition is consistent with the definitions we already have for series on countable sets.

We give some laws for absolutely convergent series on arbitrary sets.

Proposition 8.2.6 (Absolutely convergent series laws). *Let X be an arbitrary set (possibly uncountable), and let $f : X \to \mathbf{R}$ and $g : X \to \mathbf{R}$ be functions such that the series $\sum_{x\in X} f(x)$ and $\sum_{x\in X} g(x)$ are both absolutely convergent.*

(a) *The series $\sum_{x\in X}(f(x) + g(x))$ is absolutely convergent, and*

$$\sum_{x\in X}(f(x) + g(x)) = \sum_{x\in X} f(x) + \sum_{x\in X} g(x).$$

(b) *If c is a real number, then $\sum_{x\in X} cf(x)$ is absolutely convergent, and*

$$\sum_{x\in X} cf(x) = c\sum_{x\in X} f(x).$$

(c) *If $X = X_1 \cup X_2$ for some disjoint sets X_1 and X_2, then $\sum_{x\in X_1} f(x)$ and $\sum_{x\in X_2} f(x)$ are absolutely convergent, and*

$$\sum_{x\in X_1\cup X_2} f(x) = \sum_{x\in X_1} f(x) + \sum_{x\in X_2} f(x).$$

Conversely, if $h : X \to \mathbf{R}$ is such that $\sum_{x\in X_1} h(x)$ and $\sum_{x\in X_2} h(x)$ are absolutely convergent, then $\sum_{x\in X_1\cup X_2} h(x)$ is also absolutely convergent, and

$$\sum_{x\in X_1\cup X_2} h(x) = \sum_{x\in X_1} h(x) + \sum_{x\in X_2} h(x).$$

(d) *If Y is another set, and $\phi : Y \to X$ is a bijection, then $\sum_{y\in Y} f(\phi(y))$ is absolutely convergent, and*

$$\sum_{y\in Y} f(\phi(y)) = \sum_{x\in X} f(x).$$

Proof. See Exercise 8.2.3. □

Recall in Example 7.4.4 that if a series was conditionally convergent, but not absolutely convergent, then its behaviour with respect to rearrangements was bad. We now analyze this phenomenon further.

Lemma 8.2.7. *Let $\sum_{n=0}^{\infty} a_n$ be a series of real numbers which is conditionally convergent, but not absolutely convergent. Define the sets $A_+ := \{n \in \mathbf{N} : a_n \geq 0\}$ and $A_- := \{n \in \mathbf{N} : a_n < 0\}$, thus $A_+ \cup A_- = \mathbf{N}$ and $A_+ \cap A_- = \emptyset$. Then both of the series $\sum_{n\in A_+} a_n$ and $\sum_{n\in A_-} a_n$ are not conditionally convergent (and thus not absolutely convergent).*

Proof. See Exercise 8.2.4.

We are now ready to present a remarkable theorem of Georg Riemann (1826–1866), which asserts that a series which converges conditionally but not absolutely can be rearranged to converge to any value one pleases!

Theorem 8.2.8. *Let $\sum_{n=0}^{\infty} a_n$ be a series which is conditionally convergent, but not absolutely convergent, and let L be any real number. Then there exists a bijection $f : \mathbf{N} \to \mathbf{N}$ such that $\sum_{m=0}^{\infty} a_{f(m)}$ converges conditionally to L.*

Proof. (Optional) We give a sketch of the proof, leaving the details to be filled in in Exercise 8.2.5. Let A_+ and A_- be the sets in Lemma 8.2.7; from that lemma we know that $\sum_{n \in A_+} a_n$ and $\sum_{n \in A_-} a_n$ are both absolutely divergent. In particular A_+ and A_- are infinite (why?). By Proposition 8.1.5 we can then find increasing bijections $f_+ : \mathbf{N} \to A_+$ and $f_- : \mathbf{N} \to A_-$. Thus the sums $\sum_{m=0}^{\infty} a_{f_+(m)}$ and $\sum_{m=0}^{\infty} a_{f_-(m)}$ are both absolutely divergent (why?). The plan shall be to select terms from the divergent series $\sum_{m=0}^{\infty} a_{f_+(m)}$ and $\sum_{m=0}^{\infty} a_{f_-(m)}$ in a well-chosen order in order to keep their difference converging towards L.

We define the sequence $n_0, n_1, n_2, \ldots$ of natural numbers recursively as follows. Suppose that j is a natural number, and that n_i has already been defined for all $i < j$ (this is vacuously true if $j = 0$). We then define n_j by the following rule:

(I) If $\sum_{0 \leq i < j} a_{n_i} < L$, then we set

$$n_j := \min\{n \in A_+ : n \neq n_i \text{ for all } i < j\}.$$

(II) If instead $\sum_{0 \leq i < j} a_{n_i} \geq L$, then we set

$$n_j := \min\{n \in A_- : n \neq n_i \text{ for all } i < j\}.$$

Note that this recursive definition is well-defined because A_+ and A_- are infinite, and so the sets $\{n \in A_+ : n \neq n_i \text{ for all } i < j\}$ and $n_j := \min\{n \in A_- : n \neq n_i \text{ for all } i < j\}$ are never empty. (Intuitively, we add a non-negative number to the series whenever the partial sum is too low, and add a negative number when the sum is too high.) One can then verify the following claims:

- The map $j \mapsto n_j$ is injective. (Why?)
- Case I occurs an infinite number of times, and Case II also occurs an infinite number of times. (Why? prove by contradiction.)
- The map $j \mapsto n_j$ is surjective. (Why?)
- We have $\lim_{j\to\infty} a_{n_j} = 0$. (Why? Note from Corollary 7.2.6 that $\lim_{n\to\infty} a_n = 0$.)
- We have $\lim_{j\to\infty} \sum_{0\leq i<j} a_{n_i} = L$. (Why?)

The claim then follows by setting $f(i) := n_i$ for all i. □

Exercise 8.2.1. Prove Lemma 8.2.3. (Hint: you may find Exercise 3.6.3 to be useful.)

Exercise 8.2.2. Prove Lemma 8.2.5. (Hint: first show if M is the quantity

$$M := \sup\{\sum_{x\in A} |f(x)| : A \subseteq X, A \text{ finite}\}$$

then the sets $\{x \in X : |f(x)| > 1/n\}$ are finite with cardinality at most Mn for every positive integer n. Then use Exercise 8.1.9.)

Exercise 8.2.3. Prove Proposition 8.2.6. (Hint: you may of course use all the results from Chapter 7 to do this.)

Exercise 8.2.4. Prove Lemma 8.2.7. (Hint: prove by contradiction, and use limit laws.)

Exercise 8.2.5. Explain the gaps marked (why?) in the proof of Theorem 8.2.8.

Exercise 8.2.6. Let $\sum_{n=0}^\infty a_n$ be a series which is conditionally convergent, but not absolutely convergent. Show that there exists a bijection $f : \mathbf{N} \to \mathbf{N}$ such that $\sum_{m=0}^\infty a_{f(m)}$ diverges to $+\infty$, or more precisely that

$$\liminf_{N\to\infty} \sum_{m=N}^{\infty} a_{f(m)} = \limsup_{N\to\infty} \sum_{m=N}^{\infty} a_{f(m)} = +\infty.$$

(Of course, a similar statement holds with $+\infty$ replaced by $-\infty$.)

8.3 Uncountable sets

We have just shown that a lot of infinite sets are countable - even such sets as the rationals, for which it is not obvious how to arrange as a sequence. After such examples, one may begin to hope that other infinite sets, such as the real numbers, are also countable - after all, the real numbers are nothing more than (formal) limits of the rationals, and we've already shown the rationals are countable, so it seems plausible that the reals are also countable.

It was thus a great shock when Georg Cantor (1845–1918) showed in 1873 that certain sets - including the real numbers **R** are in fact uncountable - no matter how hard you try, you cannot arrange the real numbers **R** as a sequence $a_0, a_1, a_2, \ldots$. (Of course, the real numbers **R** can *contain* many infinite sequences, e.g., the sequence $0, 1, 2, 3, 4, \ldots$. However, what Cantor proved is that no such sequence can ever *exhaust* the real numbers; no matter what sequence of real numbers you choose, there will always be some real numbers that are not covered by that sequence.)

Recall from Remark 3.4.10 that if X is a set, then the *power set* of X, denoted $2^X := \{A : A \subseteq X\}$, is the set of all subsets of X. Thus for instance $2^{\{1,2\}} = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$. The reason for the notation 2^X is given in Exercise 8.3.1.

Theorem 8.3.1 (Cantor's theorem). *Let X be an arbitrary set (finite or infinite). Then the sets X and 2^X cannot have equal cardinality.*

Proof. Suppose for sake of contradiction that the sets X and 2^X had equal cardinality. Then there exists a bijection $f : X \to 2^X$ between X and the power set of X. Now consider the set

$$A := \{x \in X : x \notin f(x)\}.$$

Note that this set is well-defined since $f(x)$ is an element of 2^X and is hence a subset of X. Clearly A is a subset of X, hence is an element of 2^X. Since f is a bijection, there must therefore exist $x \in X$ such that $f(x) = A$. There are now two cases, depending on whether $x \in A$ or $x \notin A$. If $x \in A$, then by definition of A we have $x \notin f(x)$, hence $x \notin A$, a contradiction. But if $x \notin A$, then $x \notin f(x)$, hence by definition of A we have $x \in A$, a contradiction. Thus in either case we have a contradiction. □

Remark 8.3.2. The reader should compare the proof of Cantor's theorem with the statement of Russell's paradox (Section 3.2). The point is that a bijection between X and 2^X would come dangerously close to the concept of a set X "containing itself"

Corollary 8.3.3. $2^{\mathbf{N}}$ *is uncountable.*

Proof. By Theorem 8.3.1, $2^{\mathbf{N}}$ cannot have equal cardinality with $\mathbf{N}$, hence is either uncountable or finite. However, $2^{\mathbf{N}}$ contains as a subset the set of singletons $\{\{n\} : n \in \mathbf{N}\}$, which is clearly bijective to $\mathbf{N}$ and hence countably infinite. Thus $2^{\mathbf{N}}$ cannot be finite (by Proposition 3.6.14), and is hence uncountable. □

Cantor's theorem has the following important (and unintuitive) consequence.

Corollary 8.3.4. $\mathbf{R}$ *is uncountable.*

Proof. Let us define the map $f : 2^{\mathbf{N}} \to \mathbf{R}$ by the formula

$$f(A) := \sum_{n \in A} 10^{-n}.$$

Observe that since $\sum_{n=0}^{\infty} 10^{-n}$ is an absolutely convergent series (by Lemma 7.3.3), the series $\sum_{n \in A} 10^{-n}$ is also absolutely convergent (by Proposition 8.2.6(c)). Thus the map f is well defined. We now claim that f is injective. Suppose for sake of contradiction that there were two distinct sets $A, B \in 2^{\mathbf{N}}$ such that $f(A) = f(B)$. Since $A \neq B$, the set $(A\backslash B) \cup (B\backslash A)$ is a non-empty subset of $\mathbf{N}$. By the well-ordering principle (Proposition 8.1.4), we can then define the minimum of this set, say $n_0 := \min(A\backslash B) \cup (B\backslash A)$. Thus n_0 either lies in $A\backslash B$ or $B\backslash A$. By symmetry we may assume it lies in $A \backslash B$. Then $n_0 \in A$, $n_0 \notin B$, and for all $n < n_0$ we either have $n \in A, B$ or $n \notin A, B$. Thus

$$\begin{aligned}
0 &= f(A) - f(B) \\
&= \sum_{n \in A} 10^{-n} - \sum_{n \in B} 10^{-n} \\
&= \Big(\sum_{n < n_0 : n \in A} 10^{-n} + 10^{-n_0} + \sum_{n > n_0 : n \in A} 10^{-n} \Big) \\
&\quad - \Big(\sum_{n < n_0 : n \in B} 10^{-n} + \sum_{n > n_0 : n \in B} 10^{-n} \Big) \\
&= 10^{-n_0} + \sum_{n > n_0 : n \in A} 10^{-n} - \sum_{n > n_0 : n \in B} 10^{-n} \\
&\geq 10^{-n_0} + 0 - \sum_{n > n_0} 10^{-n} \\
&\geq 10^{-n_0} - \frac{1}{9} 10^{-n_0} \\
&> 0,
\end{aligned}$$

a contradiction, where we have used the geometric series lemma (Lemma 7.3.3) to sum

$$\sum_{n > n_0} 10^{-n} = \sum_{m=0}^{\infty} 10^{-(n_0+1+m)} = 10^{-n_0-1} \sum_{m=0}^{\infty} 10^{-m} = \frac{1}{9} 10^{-n_0}.$$

Thus f is injective, which means that $f(2^{\mathbf{N}})$ has the same cardinality as $2^{\mathbf{N}}$ and is thus uncountable. Since $f(2^{\mathbf{N}})$ is a subset of $\mathbf{R}$, this forces $\mathbf{R}$ to be uncountable also (otherwise this would contradict Corollary 8.1.7), and we are done. □

Remark 8.3.5. We will give another proof of this result using measure theory in Exercise 18.2.6.

Remark 8.3.6. Corollary 8.3.4 shows that the reals have strictly larger cardinality than the natural numbers (in the sense of Exercise 3.6.7). One could ask whether there exist any sets which have strictly larger cardinality than the natural numbers, but strictly smaller cardinality than the reals. The *Continuum Hypothesis* asserts that no such sets exist. Interestingly, it was shown in separate works of Kurt Gödel (1906–1978) and Paul Cohen (1934–) that this hypothesis is independent of the other axioms of set theory; it can neither be proved nor disproved in that set of axioms (unless those axioms are inconsistent, which is highly unlikely).

Exercise 8.3.1. Let X be a finite set of cardinality n. Show that 2^X is a finite set of cardinality 2^n. (Hint: use induction on n.)

Exercise 8.3.2. Let A, B, C be sets such that $A \subseteq B \subseteq C$, and suppose that there is a bijection $f : C \to A$. Define the sets $D_0, D_1, D_2, \ldots$ recursively by setting $D_0 := B \backslash A$, and then $D_{n+1} := f(D_n)$ for all natural numbers n. Prove that the sets $D_0, D_1, \ldots$ are all disjoint from each other (i.e., $D_n \cap D_m = \emptyset$ whenever $n \neq m$). Also show that if $g : A \to B$ is the function defined by setting $g(x) := f(x)$ when $x \in \bigcup_{n=0}^{\infty} D_n$, and $g(x) := x$ when $x \notin \bigcup_{n=0}^{\infty} D_n$, then g does indeed map A to B and is a bijection between the two. In particular, A and B have the same cardinality.

Exercise 8.3.3. Recall from Exercise 3.6.7 that a set A is said to have lesser or equal cardinality than a set B iff there is an injective map $f : A \to B$ from A to B. Using Exercise 8.3.2, show that if A, B are sets such that A has lesser or equal cardinality to B and B has lesser or equal cardinality to A, then A and B have equal cardinality. (This is known as the *Schröder-Bernstein theorem*, after Ernst Schröder (1841–1902) and Felix Bernstein (1878–1956).)

Exercise 8.3.4. Let us say that a set A has *strictly lesser cardinality* than a set B if A has lesser than or equal cardinality to B (in the sense of Exercise 3.6.7) but A does not have equal cardinality to B. Show that for any set X, that X has strictly lesser cardinality than 2^X. Also, show that if A has strictly lesser cardinality than B, and B has strictly lesser cardinality than C, then A has strictly lesser cardinality than C.

Exercise 8.3.5. Show that no power set (i.e., a set of the form 2^X for some set X) can be countably infinite.

8.4 The axiom of choice

We now discuss the final axiom of the standard Zermelo-Fraenkel-Choice system of set theory, namely the *axiom of choice*. We have delayed introducing this axiom for a while now, to demonstrate that a large portion of the foundations of analysis can be constructed without appealing to this axiom. However, in many further developments of the theory, it is very convenient (and in some cases even essential) to employ this powerful axiom. On the other hand, the axiom of choice can lead to a number of unintuitive consequences (for instance the *Banach-Tarski paradox*, a simplified version of which we will encounter in Section 18.3), and can lead to proofs that are philosophically somewhat unsatisfying. Nevertheless, the axiom is almost universally accepted by mathematicians. One reason for this confidence is a theorem due to the great logician Kurt Gödel, who showed that a result proven using the axiom of choice will never contradict a result proven without the axiom of choice (unless all the other axioms of set theory are themselves inconsistent, which is highly unlikely). More precisely, Gödel demonstrated that the axiom of choice is *undecidable*; it can neither be proved nor disproved from the other axioms of set theory, so long as those axioms are themselves consistent. (From a set of inconsistent axioms one can prove that every statement is both true and false.) In practice, this means that any "real-life" application of analysis (more precisely, any application involving only "decidable" questions) which can be rigourously supported using the axiom of choice, can also be rigourously supported without the axiom of choice, though in many cases it would take a much more complicated and lengthier argument to do so if one were not allowed to use the axiom of choice. Thus one can view the axiom of choice as a convenient and safe labour-saving device in analysis. In other disciplines of mathematics, notably in set theory in which many of the questions are not decidable, the issue of whether to accept the axiom of choice is more open to debate, and involves some philosophical concerns as well as mathematical and logical ones. However, we will not discuss these issues in this text.

We begin by generalizing the notion of finite Cartesian products from Definition 3.5.7 to infinite Cartesian products.

Definition 8.4.1 (Infinite Cartesian products). Let I be a set (possibly infinite), and for each $\alpha \in I$ let X_α be a set. We then