

3. Let φ be the linear transformation $\varphi : \mathbb{R}^4 \rightarrow \mathbb{R}^1$ such that

$$\begin{aligned}\varphi((1, 0, 0, 0)) &= 1 & \varphi((1, -1, 0, 0)) &= 0 \\ \varphi((1, -1, 1, 0)) &= 1 & \varphi((1, -1, 1, -1)) &= 0.\end{aligned}$$

Determine $\varphi((a, b, c, d))$.

4. Prove that the space of real-valued functions on the closed interval $[a, b]$ is an infinite dimensional vector space over \mathbb{R} , where $a < b$.
5. Prove that the space of continuous real-valued functions on the closed interval $[a, b]$ is an infinite dimensional vector space over \mathbb{R} , where $a < b$.
6. Let V be a vector space of finite dimension. If φ is any linear transformation from V to V prove there is an integer m such that the intersection of the image of φ^m and the kernel of φ^m is $\{0\}$.
7. Let φ be a linear transformation from a vector space V of dimension n to itself that satisfies $\varphi^2 = 0$. Prove that the image of φ is contained in the kernel of φ and hence that the rank of φ is at most $n/2$.
8. Let V be a vector space over F and let φ be a linear transformation of the vector space V to itself. A nonzero element $v \in V$ satisfying $\varphi(v) = \lambda v$ for some $\lambda \in F$ is called an *eigenvector* of φ with *eigenvalue* λ . Prove that for any fixed $\lambda \in F$ the collection of eigenvectors of φ with eigenvalue λ together with 0 forms a subspace of V .
9. Let V be a vector space over F and let φ be a linear transformation of the vector space V to itself. Suppose for $i = 1, 2, \dots, k$ that $v_i \in V$ is an eigenvector for φ with eigenvalue $\lambda_i \in F$ (cf. the preceding exercise) and that all the eigenvalues λ_i are distinct. Prove that v_1, v_2, \dots, v_k are linearly independent. [Use induction on k : write a linear dependence relation among the v_i and apply φ to get another linear dependence relation among the v_i involving the eigenvalues — now subtract a suitable multiple of the first linear relation to get a linear dependence relation on fewer elements.] Conclude that any linear transformation on an n -dimensional vector space has at most n distinct eigenvalues.

In the following exercises let V be a vector space of arbitrary dimension over a field F .

10. Prove that any vector space V has a basis (by convention the null set is the basis for the zero space). [Let \mathcal{S} be the set of subsets of V consisting of linearly independent vectors, partially ordered under inclusion; apply Zorn's Lemma to \mathcal{S} and show a maximal element of \mathcal{S} is a basis.]
11. Refine your argument in the preceding exercise to prove that any set of linearly independent vectors of V is contained in a basis of V .
12. If F is a field with a finite or countable number of elements and V is an infinite dimensional vector space over F with basis \mathcal{B} , prove that the cardinality of V equals the cardinality of \mathcal{B} . Deduce in this case that any two bases of V have the same cardinality.
13. Prove that as vector spaces over \mathbb{Q} , $\mathbb{R}^n \cong \mathbb{R}$, for all $n \in \mathbb{Z}^+$ (note that, in particular, this means \mathbb{R}^n and \mathbb{R} are isomorphic as additive abelian groups).
14. Let \mathcal{A} be a basis for the infinite dimensional space V . Prove that V is isomorphic to the *direct sum* of copies of the field F indexed by the set \mathcal{A} . Prove that the *direct product* of copies of F indexed by \mathcal{A} is a vector space over F and it has strictly larger dimension than the dimension of V (see the exercises in Section 10.3 for the definitions of direct sum and direct product of infinitely many modules).

11.2 THE MATRIX OF A LINEAR TRANSFORMATION

Throughout this section let V, W be vector spaces over the same field F , let $\mathcal{B} = \{v_1, v_2, \dots, v_n\}$ be an (ordered) basis of V , let $\mathcal{E} = \{w_1, w_2, \dots, w_m\}$ be an (ordered) basis of W and let $\varphi \in \text{Hom}(V, W)$ be a linear transformation from V to W . For each $j \in \{1, 2, \dots, n\}$ write the image of v_j under φ in terms of the basis \mathcal{E} :

$$\varphi(v_j) = \sum_{i=1}^m \alpha_{ij} w_i. \quad (11.3)$$

Let $M_{\mathcal{B}}^{\mathcal{E}}(\varphi) = (a_{ij})$ be the $m \times n$ matrix whose i, j entry is α_{ij} (that is, use the coefficients of the w_i 's in the above computation of $\varphi(v_j)$ for the j^{th} column of this matrix). The matrix $M_{\mathcal{B}}^{\mathcal{E}}(\varphi)$ is called the *matrix of φ with respect to the bases \mathcal{B}, \mathcal{E}* . The domain **basis** is the lower and the codomain basis the upper letters appearing after the “*M.*” Given this matrix, we can recover the linear transformation φ as follows: to compute $\varphi(v)$ for $v \in V$, write v in terms of the basis \mathcal{B} :

$$v = \sum_{i=1}^n \alpha_i v_i, \quad \alpha_i \in F,$$

and then calculate the product of the $m \times n$ and $n \times 1$ matrices

$$M_{\mathcal{B}}^{\mathcal{E}}(\varphi) \times \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{pmatrix}.$$

The image of v under φ is given by

$$\varphi(v) = \sum_{i=1}^m \beta_i w_i,$$

i.e., the column vector of coordinates of $\varphi(v)$ with respect to the basis \mathcal{E} are obtained by multiplying the matrix $M_{\mathcal{B}}^{\mathcal{E}}(\varphi)$ by the column vector of coordinates of v with respect to the basis \mathcal{B} (sometimes denoted $[\varphi(v)]_{\mathcal{E}} = M_{\mathcal{B}}^{\mathcal{E}}(\varphi)[v]_{\mathcal{B}}$).

Definition. The $m \times n$ matrix $A = (a_{ij})$ associated to the linear transformation φ above is said to *represent* the linear transformation φ with respect to the bases \mathcal{B}, \mathcal{E} . Similarly, φ is the linear transformation represented by A with respect to the bases \mathcal{B}, \mathcal{E} .

Examples

- (1) Let $V = \mathbb{R}^3$ with the standard basis $\mathcal{B} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ and let $W = \mathbb{R}^2$ with the standard basis $\mathcal{E} = \{(1, 0), (0, 1)\}$. Let φ be the linear transformation $\varphi(x, y, z) = (x + 2y, x + y + z)$. Since $\varphi(1, 0, 0) = (1, 1)$, $\varphi(0, 1, 0) = (2, 1)$, $\varphi(0, 0, 1) = (0, 1)$, the matrix $A = M_{\mathcal{B}}^{\mathcal{E}}(\varphi)$ is the matrix $\begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$.

(2) Let $V = W$ be the 2-dimensional space of solutions of the differential equation $y'' - 3y' + 2y = 0$ over \mathbb{C} and let $\mathcal{B} = \mathcal{E}$ be the basis $v_1 = e^t$, $v_2 = e^{2t}$. Since the coefficients of this equation are constants it is easy to check that if y is a solution then its derivative y' is also a solution. It follows that the map $\varphi = d/dt = \text{differentiation (with respect to } t\text{)}$ is a linear transformation from V to itself. Since $\varphi(v_1) = d(e^t)/dt = e^t = v_1$ and $\varphi(v_2) = d(e^{2t})/dt = 2e^{2t} = 2v_2$ we see that the corresponding matrix with respect to these bases is the diagonal matrix $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

(3) Let $V = W = \mathbb{Q}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{Q}\}$ be the usual 3-dimensional vector space of ordered 3-tuples with entries from the field $F = \mathbb{Q}$ of rational numbers and suppose φ is the linear transformation

$$\varphi(x, y, z) = (9x + 4y + 5z, -4x - 3z, -6x - 4y - 2z), \quad x, y, z \in \mathbb{Q}$$

from V to itself. Take the standard basis $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$ for V and for $W = V$. Since $\varphi(1, 0, 0) = (9, -4, -6)$, $\varphi(0, 1, 0) = (4, 0, -4)$, $\varphi(0, 0, 1) = (5, -3, -2)$, the matrix A representing this linear transformation with respect to these bases is

$$A = \begin{pmatrix} 9 & 4 & 5 \\ -4 & 0 & -3 \\ -6 & -4 & -2 \end{pmatrix}.$$

Theorem 10. Let V be a vector space over F of dimension n and let W be a vector space over F of dimension m , with bases \mathcal{B} , \mathcal{E} respectively. Then the map $\text{Hom}_F(V, W) \rightarrow M_{m \times n}(F)$ from the space of linear transformations from V to W to the space of $m \times n$ matrices with coefficients in F defined by $\varphi \mapsto M_{\mathcal{B}}^{\mathcal{E}}(\varphi)$ is a vector space isomorphism. In particular, there is a bijective correspondence between linear transformations and their associated matrices with respect to a fixed choice of bases.

Proof: The columns of the matrix $M_{\mathcal{B}}^{\mathcal{E}}(\varphi)$ are determined by the action of φ on the basis \mathcal{B} as in equation (3). This shows in particular that the map $\varphi \mapsto M_{\mathcal{B}}^{\mathcal{E}}(\varphi)$ is an F -linear map since φ is F -linear. This map is *surjective* since given a matrix M , the map φ defined by equation (3) on a basis and then extended by linearity is a linear transformation with matrix M . The map is *injective* since two linear transformations agreeing on a basis are the same.

Note that different choices of bases give rise to different isomorphisms, so in the same sense that there is no natural choice of basis for a vector space, there is no natural isomorphism between $\text{Hom}_F(V, W)$ and $M_{m \times n}(F)$.

Corollary 11. The dimension of $\text{Hom}_F(V, W)$ is $(\dim V)(\dim W)$.

Proof: The dimension of $M_{m \times n}(F)$ is mn .

Definition. An $m \times n$ matrix A is called *nonsingular* if $Ax = 0$ with $x \in F^n$ implies $x = 0$.

The connection of the term *nonsingular* applied to matrices and to linear transformations is the following: let $A = M_{\mathcal{B}}^{\mathcal{E}}(\varphi)$ be the matrix associated to the linear transformation φ (with some choice of bases \mathcal{B} , \mathcal{E}). Then independently of the choice of bases, the $m \times n$ matrix A is nonsingular if and only if the linear transformation φ is a nonsingular linear transformation from the n -dimensional space V to the m -dimensional space W (cf. the exercises).

Assume now that U , V and W are all finite dimensional vector spaces over F with ordered bases \mathcal{D} , \mathcal{B} and \mathcal{E} respectively, where \mathcal{B} and \mathcal{E} are as before and suppose $\mathcal{D} = \{u_1, u_2, \dots, u_k\}$. Assume $\psi : U \rightarrow V$ and $\varphi : V \rightarrow W$ are linear transformations. Their composite, $\varphi \circ \psi$, is a linear transformation from U to W , so we can compute its matrix with respect to the appropriate bases; namely, $M_{\mathcal{D}}^{\mathcal{E}}(\varphi \circ \psi)$ is found by computing

$$\varphi \circ \psi(u_j) = \sum_{i=1}^m \gamma_{ij} w_i$$

and putting the coefficients γ_{ij} down the j^{th} column of $M_{\mathcal{D}}^{\mathcal{E}}(\varphi \circ \psi)$. Next, compute the matrices of ψ and φ separately:

$$\psi(u_j) = \sum_{p=1}^n \alpha_{pj} v_p \quad \text{and} \quad \varphi(v_p) = \sum_{i=1}^m \beta_{ip} w_i$$

so that $M_{\mathcal{D}}^{\mathcal{B}}(\psi) = (\alpha_{pj})$ and $M_{\mathcal{B}}^{\mathcal{E}}(\varphi) = (\beta_{ip})$.

Using these coefficients we can find an expression for the γ 's in terms of the α 's and β 's as follows:

$$\begin{aligned} \varphi \circ \psi(u_j) &= \varphi \left(\sum_{p=1}^n \alpha_{pj} v_p \right) \\ &= \sum_{p=1}^n \alpha_{pj} \varphi(v_p) \\ &= \sum_{p=1}^n \alpha_{pj} \sum_{i=1}^m \beta_{ip} w_i \\ &= \sum_{p=1}^n \sum_{i=1}^m \alpha_{pj} \beta_{ip} w_i. \end{aligned}$$

By interchanging the order of summation in the above double sum we see that γ_{ij} , which is the coefficient of w_i in the above expression, is

$$\gamma_{ij} = \sum_{p=1}^n \alpha_{pj} \beta_{ip}.$$

Computing the product of the matrices for φ and ψ (in that order) we obtain

$$(\beta_{ij})(\alpha_{ij}) = (\delta_{ij}), \quad \text{where} \quad \delta_{ij} = \sum_{p=1}^m \beta_{ip} \alpha_{pj}.$$