Principle of Least Action. In this they followed Plato, who had Socrates argue in *The Protagoras* that the way we ought to behave is to maximize pleasure minus pain, that virtue consists in knowing when to forego a present gain in favour of a larger future gain (temperance) and when to face the danger of an immediate loss against the expectation of ultimate gain (courage). However, Socrates seems to have had second thoughts about this later. When Crito urged him to flee from his prison so as to escape the death penalty and be able to bring up his sons, Socrates refused, believing it was his moral duty to remain in jail. He replied thus:

> Whatever the popular view is, and whether the alternative is pleasanter than the present one or even harder to bear, the fact remains that to do wrong is in every sense bad and dishonourable (Plato's *Crito* 49b; see also *Laws* 707d).

Leibniz's most original contribution to philosophy is probably the system incorporated in his *Monadology*. In this work he proposes that the universe is made up of certain ultimate elements, called *monads*, which are capable of perception. Human souls are monads with memory and reason.

If Leibniz was influenced by his mathematical background in formulating this model of the universe, then he might have thought of a monad as a point together with the set of all points infinitely close to it. In fact, Robinson attributed this technical meaning to the word 'monad' in his nonstandard analysis.

On the other hand, Bertrand Russell thought of the monads as points, with arrows connecting different points (see his *Mysticism and Logic*). Each monad has a physical aspect, consisting of all arrows emerging from it, and a mental aspect, consisting of all arrows converging on it.

Leibniz asserted that a monad has no windows. (Its contact with the rest of the universe is via a 'pre-established harmony'.) This suggests that anything we know about a monad must be deducible from the 'arrows' relating it to all the other monads in the universe.

One is reminded here of the 20th century notion of a category (see Part II, Chapter 31). For instance, in the category of groups, we have as points all groups, and as arrows all group homomorphisms. If we want to know the elements of a group G, we need not look 'inside' the group at all, but only at the arrows from the free group in one generator to G.

Exercise

1. Suppose the half-plane $y > 0$ is a rough field where you would walk at u km/h, and the half-plane $y < 0$ is a paved surface where you would walk at v km/h. You are at point $(0, -6)$ and want to walk to the point (a, b). How should you go to get there fastest?

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The Eighteenth Century

We shall only mention some of the most important mathematicians of the eighteenth century:

- Brook Taylor (1685–1731),
- Colin Maclaurin (1689–1746),
- Abraham de Moivre (1667–1754),
- Leonhard Euler (1707–1783),
- Joseph Louis Lagrange (1736–1813),
- Pierre Simon Laplace (1749–1827),
- Adrien Marie Legendre (1752–1833).

Brook Taylor, an ardent admirer of Newton, discovered the *Taylor series*

$$f(a+x) = f(a) + xf'(a) + x^2 f''(a)/2! + ...,$$

publishing it in 1715.

Colin Maclaurin, a Scotsman, is best known for the special case $a = 0$ of Taylor's series. This appeared in his *Treatise of Fluxions* (1742). In his book, Maclaurin tried to be sufficiently rigorous to answer Berkeley's objections to the Calculus, but he did not even get to the point of demonstrating conditions under which his *Maclaurin series* converges.

Abraham de Moivre was born in France, but lived in England. He is famous for his formula

$$(\cos x + i \sin x)^n = \cos nx + i \sin nx ,$$

which is easily proved, for natural numbers n, by mathematical induction. De Moivre published an important book on the theory of probability, called the *Doctrine of Chances.*

Society failed de Moivre: in spite of letters of recommendation from both Newton and Leibniz, he was never given a proper job in mathematics. He had to earn a meagre living by private tutoring and answering gamblers' questions on probability. It is said that, as he approached the end of his life, de Moivre slept fifteen minutes longer each day. When he reached a full twenty four hours, he died.

Although Leonhard Euler was Swiss, he spent part of his professional life in Berlin and most of it in St. Petersburg. Towards the end of his life he became blind, but this did not slow down his mathematical output. He found many interesting and exciting results in mathematics. Indeed, it has been said that Euler picked all the raisins out of the mathematical cake. Some of his results are the following:

1. If a convex polyhedron has V vertices, F faces and E edges, then $V + F - E = 2$. For example, a cube has 8 vertices, 6 faces and 12 edges; we have $8 + 6 - 12 = 2$. (Descartes came close to this formula, but he did not actually state it.)

2. $e^{i\pi} = -1$, where e is the 'Euler number':

$$e = \lim_{n \to \infty} (1 + 1/n)^n .$$

3. $1/1^2 + 1/2^2 + 1/3^2 + 1/4^2 + \cdots = \pi/6$.

   Euler's proof of this was not rigorous but, before Euler, no one even guessed that the sum of the series was $\pi/6$.

4. Every even perfect number has the form $2^{n-1}(2^n - 1)$ where $2^n - 1$ is prime.

5. If n is a positive integer, let $\phi(n)$ be the number of natural numbers less than or equal to n and relatively prime to it. Then, if a is a positive integer relatively prime to n, it follows that n is a factor of $a^{\phi(n)} - 1$. Fermat's Little Theorem is a corollary of this.

6. The circumcenter, orthocenter and centroid of a triangle are collinear. The line that passes through them is called the *Euler line.*