ner: the *Hilbert basis theorem* showed the existence of the invariants above the quadratic level, without needing to calculate them!

Gordan was at first incredulous and exclaimed, "This is not mathematics, it is theology!", but eventually Hilbert's idea was developed further, to calculate the invariants, and Gordan had to concede that it was mathematics after all. Hilbert, for his part, moved on to conquer other worlds. In fact, this became his modus operandi for most of his career: investigate a topic thoroughly for a few years, turn it upside down, then do something completely different.

Hilbert's triumph in invariant theory secured his position in Königsberg, and in 1892 he married Käthe Jerosch, a very capable woman who acted as secretary and research assistant for many of his works. In particular, she compiled the bibliography for his massive *Zahlbericht* ("Number Report") of 1897, the work in which algebraic number theory came of age. Hilbert was commissioned by the German Union of Mathematicians in 1893 to write a report on algebraic number theory, and the report became a 300-page book [Hilbert (1897)], looking back to quadratic forms and Fermat's last theorem, and forward to *class field theory*, a major topic of the twentieth century.

The mathematical public, which had not been ready when Dedekind presented algebraic number theory a few years earlier, now saw the point, and Klein invited Hilbert to Göttingen, where he held a chair in mathematics from 1895 until the end of his life.

After the *Zahlbericht*, Hilbert turned to the foundations of geometry, which we have touched on in Sections 1.6, 2.1, 19.5, and 20.7. Again he scored several triumphs—finally filling the gaps in Euclid, discovering the algebraic meaning of the Pappus and Desargues theorems—but also leaving some unfinished business. Hilbert realized that modelling Euclid's geometry by real number coordinates was not exactly a proof that geometry is consistent; one still needs to prove that the theory of real numbers is consistent. Hilbert found this far from obvious and made it second on his list of mathematical problems presented in Paris in 1900. Then he dropped the subject in favor of mathematical physics.

However, no one found a consistency proof for the theory of real numbers, and by the 1920s Hilbert felt compelled to return to the subject. *Hilbert's program*, as it became known, called first for a formal language of mathematics, in which the concept of proof itself was mathematically definable, by precise rules for manipulating formulas. This phase of the program was in fact feasible, and was essentially carried out by Whitehead and Russell in their *Principia Mathematica* of 1910. The hard part, however, was proving that the rules of proof could not lead to a contradiction. This is where Hilbert's program stalled, and in 1931 Gödel showed that it could never be completed. His famous *incompleteness theorems* (Chapter 23) showed that such a consistency proof does not exist, and that enlarging the formal language by new axioms only puts the consistency proof further out of reach.

To his credit, Hilbert was among the first to publicize Gödel's work. The first complete proofs of Gödel's theorems are in the book of Hilbert and Bernays (1938). But it was Hilbert's misfortune to end his career, not only with the failure of one of his mathematical dreams, but also with his mathematical community in ruins. The eclipse of Göttingen began in 1933, when the Nazis came to power in Germany and began dismissing Jewish professors. In a few years, most of Germany's mathematical talent had fled, leaving the elderly and frail Hilbert in Göttingen virtually alone. He died on 14 February 1943.

One of the Jewish mathematicians forced to leave Göttingen in 1933 was Emmy Noether (Figure 21.5), who was in many ways a natural successor of Dedekind and Hilbert. Emmy Noether was born in 1882 in Erlangen and died in 1935 in Bryn Mawr, Pennsylvania. She was the oldest of four children of the mathematician Max Noether and of Ida Kaufmann. As a child she loved music, dance, and languages and planned to become a language teacher, qualifying as a teacher of English and French in 1900.

At this time in Germany, women were permitted to study at universities only unofficially, and very few did so, since the permission of the lecturer was also required. However, a few teachers were permitted to attend for purposes of "further education," and in 1900 Emmy Noether became one of them, studying mathematics at the University of Erlangen. Here she met the "king of invariants," Paul Gordan, and wrote a thesis under his supervision in 1907. It was on invariant theory, naturally, and Emmy later described it as "crap," but it was not a complete waste of time. Physicists today admire one of her early results, on the invariants of mechanical systems.

In 1910 Gordan retired and there was a reshuffle of positions, leading to the appointment of Ernst Fischer in 1911. Fischer is not well known today, but it seems that Noether's algebraic talent suddenly blossomed through working with him. She dropped the computational approach of Gordan

Figure 21.5: Emmy Noether

and rapidly mastered the conceptual approach of Dedekind and Hilbert, so much so that Hilbert invited her to Göttingen in 1915. Getting a position was another matter—Hilbert is said to have ridiculed Göttingen's exclusion of women professors by saying "this is a university, not a bathing establishment"—but she was eventually granted an unofficial chair in 1922.

In the 1920s Noether was at the height of her powers, and she found students worthy of her ability. Among them were Emil Artin, who solved a couple of Hilbert's problems, and B. L. van der Waerden, who brought Noether's ideas to the world in his *Moderne Algebra* of 1930. Noether herself modestly used to claim that "es steht schon bei Dedekind" ("it's already in Dedekind") and encouraged her students to see for themselves by reading all of Dedekind's supplements. Thus, despite the highly abstract nature of Noether's algebra, her students were made aware of its direct descent from the number theory of Gauss and Dirichlet. In van der Waerden's *Algebra* this connection was unfortunately broken, and many in the next generation of students grew up unaware of it. In recent years there has been a welcome reversal of this trend; in particular, the *Algebra* of Emil Artin's son Michael uses number theory to illustrate the theory of ideals [Artin (1991)].

22

Topology

22.1 Geometry and Topology

Topology is concerned with those properties that remain invariant under continuous transformations. In the context of Klein's Erlanger Programm (where it receives a brief mention under its old name of *analysis situs*) it is the "geometry" of groups of continuous invertible transformations, or *homeomorphisms*. The "space" to which transformations are applied and indeed the meaning of "continuous" remain somewhat open. When these terms are interpreted in the most general way, as subject only to certain axioms (which we shall not bother to state here), one has *general topology*. The theorems of general topology, important in fields ranging from set theory to analysis, are not very geometric in flavor. *Geometric topology*, which concerns us in this chapter, is obtained when the transformations are ordinary continuous functions on $\mathbb{R}^n$ or on certain subsets of $\mathbb{R}^n$.

Geometric topology is more recognizably "geometric" than general topology, though the "geometry" is necessarily of a discrete and combinatorial kind. Ordinary geometric quantities—such as length, angle, and curvature—admit continuous variation and hence cannot be invariant under continuous transformations. The kind of quantities that are topologically invariant are such things as the number of "pieces" of a figure or the number of "holes" in it. It turns out, though, that the combinatorial structures of topology can often be reflected by combinatorial structures in ordinary geometry, such as polyhedra and tessellations. In the case of surface topology, this geometric modelling of topological structure is so complete that topology becomes essentially a part of ordinary geometry. "Ordinary" here