EXERCISES

The simplest heat equation is the one-dimensional version,

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2},$$

for the temperature T at time t and position x along an infinite straight wire. This equation may be derived from *Newton's law of cooling*, which asserts that the rate of heat flow between two points is proportional to their temperature difference.

Thus the approximate difference $\frac{\partial T}{\partial x} dx$ between T at x and $x + dx$ will induce heat to flow from $x + dx$ to x at a rate proportional to $\frac{\partial T}{\partial x} dx$. However, at the same time, heat will flow from $x - dx$ to x at approximately the same rate. To find the net flow toward x, and hence the rate $\frac{\partial T}{\partial t}$ of temperature increase, we need to take into account the rate of change of $\frac{\partial T}{\partial x}$, namely $\frac{\partial^2 T}{\partial x^2}$.

13.4.1 By pursuing this line of argument, give a plausible derivation of the heat equation

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}.$$

Sines and cosines arise from the heat equation when one solves it by the method of *separation of variables*.

13.4.2 Suppose the heat equation has a solution of the form $T(x,t) = X(x)Y(t)$, where X and Y are functions of the single variables x and t, respectively. Show that

$$\frac{1}{Y(t)} \frac{dY(t)}{dt} = \frac{\kappa}{X(x)} \frac{d^2 X(x)}{dx^2} = \text{constant}.$$

13.4.3 Now explain how sines and cosines are involved in solving for $X(x)$.

13.5 Hydrodynamics

The properties of fluid flow have been investigated since ancient times, initially in connection with practical questions such as water supply and water-powered machinery. However, nothing like a mathematical theory was obtained before the Renaissance, and until the advent of calculus it was only possible to deal with fairly coarse macroscopic quantities such as the average speed of emission from an opening in a container. Newton (1687), Book II, introduced infinitesimal methods into the study of fluids, but much of his reasoning is incomplete, based on inappropriate mathematical models, or simply wrong. As late as 1738, when the field of hydrodynamics finally got its name in the classic *Hydrodynamica* of Daniel

Bernoulli, the basic infinitesimal laws of fluid motion had still not been discovered.

The first important law was discovered by Clairaut (1740), in a context that in fact was essentially static. Clairaut was interested in one of the burning questions of the time, the shape (or "figure") of the earth. Newton had argued that the earth must bulge somewhat at the equator as a result of its spin. Natural as this seems now (and indeed then, since the phenomenon was clearly observable in Jupiter and Saturn), it was opposed by the anti-Newtonian Cassini, who argued for a spindle-shaped earth, elongated toward the poles. Clairaut actually took part in an expedition to Lapland that confirmed Newton's conjecture by measurement, but he also attacked the problem theoretically by studying the conditions for the equilibrium of a fluid mass.

He considered the vector field of force acting on the fluid and observed that it must be what we now call a *conservative*, or *potential* field. That is, the integral of the force around any closed path must be zero; otherwise the fluid would circulate. The condition he actually formulated was the equivalent one that the integral between any two points be independent of the path. In the special two-dimensional case where there are components P, Q of force in the x and y directions, the quantity to be integrated is

$$P\,dx + Q\,dy.$$

Clairaut argued that for the integral to be path-independent, this quantity must be a complete differential

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

Consequently, $P = \partial f/\partial x$, $Q = \partial f/\partial y$ and P, Q satisfy the condition

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}. \tag{1}$$

This condition is indeed necessary, but the existence of the potential f involved more mathematical subtleties than could have been foreseen at the time. Clairaut derived the corresponding equations for the components P, Q, R in the physically more natural three-dimensional case and went as far as studying the equipotential surfaces $f =$ constant. He also found a satisfying solution to the problem of the figure of the earth. When the force at a point is the resultant of gravity and the rotational force, then an ellipsoid of revolution is an equilibrium figure, with the axis of rotation being the shorter axis of the ellipse [Clairaut (1743), p. 194].

The two-dimensional equation (1), despite being physically special if not unnatural, turned out to have a deep mathematical significance. This was discovered in the dynamic situation, with P, Q taken to be components of velocity rather than force. In this case, (1) still holds when the flow is independent and irrotational as d'Alembert (1752) showed by an argument similar to Clairaut's. The crucial additional fact that now emerges is that P, Q satisfy a second relation

$$\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}=0 \tag{2}$$

derived by d'Alembert as a consequence of the incompressibility of the fluid. He considered an infinitesimal rectangle of fluid with corners at the points (x,y), $(x+dx,y)$, $(x,y+dy)$, $(x+dx,y+dy)$, and the parallelogram into which it is carried in an infinitesimal time interval by the known velocities (P,Q), $(P+(\partial P/\partial x)dx, Q+(\partial Q/\partial x)dx)$, ... Equating the areas of these two parallelograms leads to (2). In the three-dimensional case one similarly gets

$$\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}=0,$$

but the significance of (1) and (2), as d'Alembert discovered, is that they can be combined into a single fact about the complex function $P+iQ$. This flash of inspiration became the basis for the theory of complex functions developed in the nineteenth century by Cauchy and Riemann (see Section 16.1).

Exercises

To understand the concept of irrotational flow more directly, it helps to consider a flow that is clearly *rotational*, for example a rigid rotation of the plane about the origin at constant angular velocity ω.

13.5.1 For this flow, show that the velocity components at the point (x,y) are

$$P=-\omega y, \quad Q=\omega x,$$

and deduce that $\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}=-2\omega$.

Thus the quantity $\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}$ is a measure of the amount of rotation of the flow. It is, in fact, sometimes called the "rotation" but it is more often called the *curl*, a term James Clerk Maxwell introduced in 1870.

The quantity $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$ is called the *divergence* because it measures the amount of "expansion"of the fluid. As one would expect, the divergence is zero for the rigid flow above.

13.5.2 Check that the divergence is zero for the rigid rotation about the origin.

A more direct way to see that divergence is zero for any incompressible flow in the plane is to consider a *fixed* rectangle, with fluid flowing through it.

Consider the rectangle with corners fixed in the plane at (x,y), $(x+dx,y)$, $(x,y+dy)$, $(x+dx,y+dy)$, and consider the instantaneous flux of fluid through it. Fluid flows in the x end at speed P, hence the influx is proportional to $P\,dy$, and it flows out the $x+dx$ end at speed $P+(\partial P/\partial x)\,dx$, etc.

13.5.3 Show that the net influx of fluid is

$$-\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)dx\,dy,$$

and hence that the divergence is zero for incompressible flow.

13.5.4 Show similarly that

$$\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}=0$$

for an incompressible flow in three dimensions.

13.6 Biographical Notes: The Bernoullis

Undoubtedly the most outstanding family in the history of mathematics was the Bernoulli family of Basel, which included at least eight excellent mathematicians between 1650 and 1800. Three of these, the brothers Jakob (1654–1705) and Johann (1667–1754) and Johann's son Daniel (1700–1782), were among the great mathematicians of all time, as one may guess from their contributions already mentioned in this chapter. In fact, all the mathematicians Bernoulli were important in the history of mechanics. One can trace their influence in this field in Szabó (1977), which also contains portraits of most of them, and in Truesdell (1954, 1960). However, Jakob, Johann, and Daniel are of interest from a wider point of view, in mathematics, as well as in their personal lives. The Bernoulli family, with all its mathematical talent, also had more than its share of arrogance and jealousy, which turned brother against brother and father against son. In three successive generations, fathers tried to steer their sons into non-mathematical careers, only to see them gravitate back to mathematics. The fiercest conflict occurred among Jakob, Johann, and Daniel.

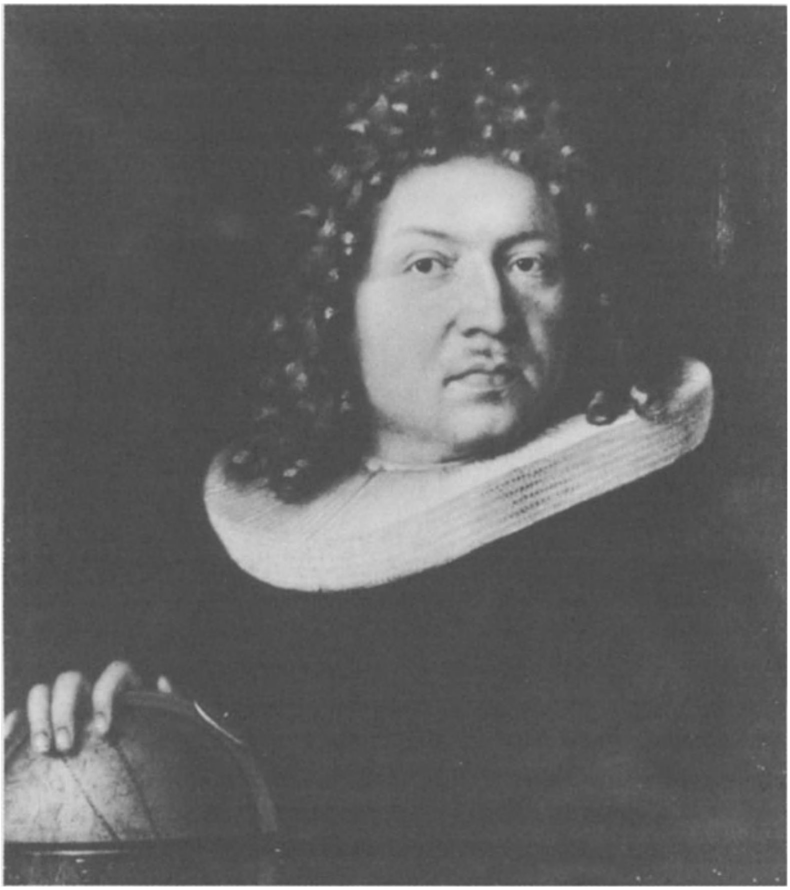

Figure 13.5: Portrait of Jakob Bernoulli by Nicholas Bernoulli

Jakob, the first mathematician in the family, was the oldest son of Nicholas Bernoulli, a successful pharmacist and civic leader in Basel, and Margaretha Schönauer, the daughter of another wealthy pharmacist. There were three other sons: Nicholas, who became an artist and in 1686 painted the portrait of James seen here (Figure 13.5); Johann; and Hieronymus, who took over the family business. Their father's wish was that Jakob should study theology, which he initially did, obtaining his licentiate in 1676. However, Jakob also began to teach himself mathematics and astronomy, and he traveled to France in 1677 to study with the followers of Descartes. In 1681 his astronomy brought him into conflict with the theologians. Inspired by the appearance of a great comet in 1680, he published a pamphlet that proposed laws governing the behavior of comets and claiming that their appearances could be predicted. His theory was not actually correct (this was six years before *Principia*), but it certainly clashed with the theology of the time, which exploited the unexpectedness of comets in claiming they were signs of divine displeasure. Jakob decided that his future was in mathematics rather than theology, and he adopted the motto *Invito Patre, Sidera verso* (Against my father's will, I will turn to the stars). He made a second study tour, to the Netherlands and England, where he met Hooke and Boyle, and began to lecture on mechanics in Basel in 1683.

He married Judith Stepanus in 1684, and they eventually had a son and daughter, neither of whom became a mathematician. In a sense, the mathematical heir of Jakob was his nephew Nicholas (son of the painter), who carried on one of Jakob's most original lines of research, probability theory. He arranged for the posthumous publication of Jakob's book on the subject, the *Ars conjectandi* [Jakob Bernoulli (1713)], which contains the first proof of a law of large numbers. Jakob Bernoulli's law described the behavior of long sequences of trials for which a positive outcome has a fixed probability p (such trials are now called Bernoulli trials). In a precise sense, the proportion of successful trials will be "close" to p for "almost all" sequences.

In 1687 Jakob became professor of mathematics in Basel and, together with Johann (whom he had been secretly teaching mathematics), set about mastering the new methods of calculus that were then appearing in the papers of Leibniz. This proved to be difficult, perhaps more for Jakob than Johann, but by the 1690s the brothers equaled Leibniz himself in the brilliance of their discoveries. Jakob, the self-taught mathematician, was the slower but more penetrating of the two. He sought to get to the bottom of every problem, whereas Johann was content with any solution, the quicker the better.

Johann was the tenth child of the family, and his father intended him to have a business career. When his lack of aptitude for business became clear, he was allowed to enter the University of Basel in 1683 and became a master of arts in 1685. During this time he also attended his brother's lectures and, as mentioned earlier, learned mathematics from him privately. Their rivalry did not come to the surface until the catenary contest of 1690, but Jakob may have felt uneasy about his younger brother's talent as early as 1685. In that year he persuaded Johann to take up the study of medicine, making the highly optimistic forecast that it offered great opportunities for the application of mathematics. Johann went into medicine quite seriously, obtaining a licentiate in 1690 and a doctorate in 1695, but by that time he was more famous as a mathematician. With the help of Huygens he gained the chair of mathematics in Groningen, and thus became free to concentrate on his true calling.

The great applications of mathematics to medicine did not eventuate, though Johann Bernoulli did make an amusing application of geometric series which still circulates today as a piece of physiological trivia. In his *De nutritione* [Johann Bernoulli (1699)] he used the assumption that a fixed proportion of bodily substance, homogeneously distributed, is lost each day and replaced by nutrition, to calculate that almost all the material in the body would be renewed in three years. This result provoked a serious theological dispute at the time, since it implied the impossibility of resurrecting the body from all its past substance.

Johann Bernoulli made several important contributions to calculus in the 1690s, outside mechanics. One was the first textbook in the subject, the *Analyse des infiniment petits*. This was published under the name of his student, the Marquis l'Hôpital (1696), apparently in return for generous financial compensation. Another contribution, made jointly with Leibniz, was the technique of partial differentiation. The two kept this discovery secret for 20 years in order to use it as a "secret weapon" in problems about families of curves [see Engelsman (1984)]. Other discoveries still remain outside the territory usually explored in calculus, for example,

$$\int_0^1 x^x\,dx = 1 - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \cdots.$$

This startling result of Johann Bernoulli (1697) can be proved using a suitable series expansion of x^x and integration by parts (see exercises).

The rivalry between Jakob and Johann turned to open hostility in 1697 over the *isoperimetric problem*, the problem of finding the curve of given length which encloses the greatest area. Jakob correctly recognized that this was a calculus of variations problem but withheld his solution, whereas Johann persisted in publicizing an incorrect solution and claiming that Jakob had no solution at all. Jakob presented his solution to the Paris Academy in 1701, but it somehow remained in a sealed envelope until after his death. Even when the solution was made public in 1706, Johann refused to admit his own error or the superiority of Jakob's analysis.

Johann was married to Dorothea Falkner, the daughter of a parliamentary deputy in Basel, and through his father-in-law's influence was awarded the chair of Greek in Basel in 1705. This enabled him to return to Basel from Groningen, but his real goal was the chair of mathematics, not Greek. Jakob was then in ill health, and his last days were embittered by the belief that Johann was plotting to take his place, using the Greek offer as a stepping stone. This is precisely what happened, for when Jakob died in 1705 Johann became the professor of mathematics.

With the death of Jakob and the virtual retirement of Leibniz and Newton, Johann enjoyed about 20 years as the leading mathematician in the world. He was particularly proud of his successful defense of Leibniz against the supporters of Newton:

> When in England war was declared against M. Leibniz for the honour of the first invention of the new calculus of the infinitely small, I was despite my wishes involved in it; I was pressed to take part. After the death of M. Leibniz the contest fell to me alone. A crowd of English antagonists fell upon my body. It was my lot to meet the attacks of Messrs Keil, Taylor, Pemberton, Robins and others. In short I alone like the famous Horatio Cocles kept at bay at the bridge the entire English army. [Translation by Pearson (1978), p. 235]

His portrait from this era shows the Bernoulli arrogance at its peak (Figure 13.6).

Johann Bernoulli finally met his match at the hands of his own pupil Euler in 1727. There was no open warfare, just a polite exchange of correspondence on the logarithms of negative numbers, but it revealed that Johann Bernoulli understood some of his own results less well than Euler did. Johann Bernoulli persisted in his stubborn misunderstanding for another 20 years, while Euler went on to develop his brilliant theory of complex logarithms and exponentials (see Section 16.1). Johann Bernoulli seems not to have minded his pupil's success at all; instead, he became consumed with jealousy over the success of his son Daniel.

Daniel Bernoulli (Figure 13.7) was the middle of Johann's three sons, all of whom became mathematicians. The oldest, Nicholas (called Nicholas II by historians to distinguish him from the first mathematician Nicholas), died of a fever in St. Petersburg in 1725 at the age of 30. The youngest,