Exercise 9.5.1. Let E be a subset of $\mathbf{R}$, let $f : E \to \mathbf{R}$ be a function, and let x_0 be an adherent point of E. Write down a definition of what it would mean for the limit $\lim_{x \to x_0; x \in E} f(x)$ to exist and equal $+\infty$ or $-\infty$. If $f : \mathbf{R} \backslash \{0\} \to \mathbf{R}$ is the function $f(x) := 1/x$, use your definition to conclude $f(0+) = +\infty$ and $f(0-) = -\infty$. Also, state and prove some analogue of Proposition 9.3.9 when $L = +\infty$ or $L = -\infty$.

9.6 The maximum principle

In the previous two sections we saw that a large number of functions were continuous, though certainly not all functions were continuous. We now show that continuous functions enjoy a number of other useful properties, especially if their domain is a closed interval. It is here that we shall begin exploiting the full power of the Heine-Borel theorem (Theorem 9.1.24).

Definition 9.6.1. Let X be a subset of $\mathbf{R}$, and let $f : X \to \mathbf{R}$ be a function. We say that f is *bounded from above* if there exists a real number M such that $f(x) \leq M$ for all $x \in X$. We say that f is *bounded from below* if there exists a real number M such that $f(x) \geq -M$ for all $x \in X$. We say that f is *bounded* if there exists a real number M such that $|f(x)| \leq M$ for all $x \in X$.

Remark 9.6.2. A function is bounded if and only if it is bounded both from above and below. (Why? Note that one part of the "if and only if" is slightly trickier than the other.) Also, a function $f : X \to \mathbf{R}$ is bounded if and only if its image $f(X)$ is a bounded set in the sense of Definition 9.1.22 (why?).

Not all continuous functions are bounded. For instance, the function $f(x) := x$ on the domain $\mathbf{R}$ is continuous but unbounded (why?), although it is bounded on some smaller domains, such as $[1, 2]$. The function $f(x) := 1/x$ is continuous but unbounded on $(0, 1)$ (why?), though it is continuous and bounded on $[1, 2]$ (why?). However, if the domain of the continuous function is a closed and bounded interval, then we do have boundedness:

Lemma 9.6.3. *Let $a < b$ be real numbers, and let $f : [a, b] \to \mathbf{R}$ be a function continuous on $[a, b]$. Then f is a bounded function.*

Proof. Suppose for sake of contradiction that f is not bounded. Thus for every real number M there exists an element $x \in [a, b]$ such that $|f(x)| \geq M$.

In particular, for every natural number n, the set $\{x \in [a, b] : |f(x)| \geq n\}$ is non-empty. We can thus choose[2] a sequence $(x_n)_{n=0}^{\infty}$ in $[a, b]$ such that $|f(x_n)| \geq n$ for all n. This sequence lies in $[a, b]$, and so by Theorem 9.1.24 there exists a subsequence $(x_{n_j})_{j=0}^{\infty}$ which converges to some limit $L \in [a, b]$, where $n_0 < n_1 < n_2 < \ldots$ is an increasing sequence of natural numbers. In particular, we see that $n_j \geq j$ for all $j \in \mathbf{N}$ (why? use induction).

Since f is continuous on $[a, b]$, it is continuous at L, and in particular we see that

$$\lim_{j \to \infty} f(x_{n_j}) = f(L).$$

Thus the sequence $(f(x_{n_j}))_{j=0}^{\infty}$ is convergent, and hence it is bounded. On the other hand, we know from the construction that $|f(x_{n_j})| \geq n_j \geq j$ for all j, and hence the sequence $(f(x_{n_j}))_{j=0}^{\infty}$ is not bounded, a contradiction. $\square$

Remark 9.6.4. There are two things about this proof that are worth noting. Firstly, it shows how useful the Heine-Borel theorem (Theorem 9.1.24) is. Secondly, it is an indirect proof; it doesn't say *how* to find the bound for f, but it shows that having f unbounded leads to a contradiction.

We now improve Lemma 9.6.3 to say something more.

Definition 9.6.5 (Maxima and minima). Let $f : X \to \mathbf{R}$ be a function, and let $x_0 \in X$. We say that f *attains its maximum at* x_0 if we have $f(x_0) \geq f(x)$ for all $x \in X$ (i.e., the value of f at

[2]Strictly speaking, this requires the axiom of choice, as in Lemma 8.4.5. However, one can also proceed without the axiom of choice, by defining $x_n := \sup\{x \in [a, b] : |f(x)| \geq n\}$, and using the continuity of f to show that $|f(x_n)| \geq n$. We leave the details to the reader.

the point x_0 is larger than or equal to the value of f at any other point in X). We say that f *attains its minimum at* x_0 if we have $f(x_0) \leq f(x)$.

Remark 9.6.6. If a function attains its maximum somewhere, then it must be bounded from above (why?). Similarly if it attains its minimum somewhere, then it must be bounded from below. These notions of maxima and minima are *global*; local versions will be defined in Definition 10.2.1.

Proposition 9.6.7 (Maximum principle). *Let $a < b$ be real numbers, and let $f : [a,b] \to \mathbf{R}$ be a function continuous on $[a,b]$. Then f attains its maximum at some point $x_{max} \in [a,b]$, and also attains its minimum at some point $x_{min} \in [a,b]$.*

Remark 9.6.8. Strictly speaking, "maximum principle" is a misnomer, since the principle also concerns the minimum. Perhaps a more precise name would have been "extremum principle"; the word "extremum" is used to denote either a maximum or a minimum.

Proof. We shall just show that f attains its maximum somewhere; the proof that it attains its minimum also is similar but is left to the reader.

From Lemma 9.6.3 we know that f is bounded, thus there exists an M such that $-M \leq f(x) \leq M$ for each $x \in [a,b]$. Now let E denote the set

$$E := \{f(x) : x \in [a,b]\}.$$

(In other words, $E := f([a,b])$.) By what we just said, this set is a subset of $[-M, M]$. It is also non-empty, since it contains for instance the point $f(a)$. Hence by the least upper bound principle, it has a supremum $\sup(E)$ which is a real number.

Write $m := \sup(E)$. By definition of supremum, we know that $y \leq m$ for all $y \in E$; by definition of E, this means that $f(x) \leq m$ for all $x \in [a,b]$. Thus to show that f attains its maximum somewhere, it will suffice to find an $x_{max} \in [a,b]$ such that $f(x_{max}) = m$. (Why will this suffice?)

Let $n \geq 1$ be any integer. Then $m - \frac{1}{n} < m = \sup(E)$. As $\sup(E)$ is the least upper bound for E, $m - \frac{1}{n}$ cannot be an upper bound for E, thus there exists a $y \in E$ such that $m - \frac{1}{n} < y$. By definition of E, this implies that there exists an $x \in [a, b]$ such that $m - \frac{1}{n} < f(x)$.

We now choose a sequence $(x_n)_{n=1}^{\infty}$ by choosing, for each n, x_n to be an element of $[a, b]$ such that $m - \frac{1}{n} < f(x_n)$. (Again, this requires the axiom of choice; however it is possible to prove this principle without the axiom of choice. For instance, you will see a better proof of this proposition using the notion of *compactness* in Proposition 13.3.2.) This is a sequence in $[a, b]$; by the Heine-Borel theorem (Theorem 9.1.24), we can thus find a subsequence $(x_{n_j})_{j=1}^{\infty}$, where $n_1 < n_2 < \ldots$, which converges to some limit $x_{max} \in [a, b]$. Since $(x_{n_j})_{j=1}^{\infty}$ converges to x_{max}, and f is continuous at x_{max}, we have as before that

$$\lim_{j \to \infty} f(x_{n_j}) = f(x_{max}).$$

On the other hand, by construction we know that

$$f(x_{n_j}) > m - \frac{1}{n_j} \geq m - \frac{1}{j},$$

and so by taking limits of both sides we see that

$$f(x_{max}) = \lim_{j \to \infty} f(x_{n_j}) \geq \lim_{j \to \infty} m - \frac{1}{j} = m.$$

On the other hand, we know that $f(x) \leq m$ for all $x \in [a, b]$, so in particular $f(x_{max}) \leq m$. Combining these two inequalities we see that $f(x_{max}) = m$ as desired. $\square$

Note that the maximum principle does not prevent a function from attaining its maximum or minimum at more than one point. For instance, the function $f(x) := x^2$ on the interval $[-2, 2]$ attains its maximum at two different points, at -2 and at 2.

Let us write $\sup_{x \in [a,b]} f(x)$ as short-hand for $\sup\{f(x) : x \in [a, b]\}$, and similarly define $\inf_{x \in [a,b]} f(x)$. The maximum principle thus asserts that $m := \sup_{x\in[a,b]} f(x)$ is a real number and is the *maximum value* of f on $[a,b]$, i.e., there is at least one point x_{max} in $[a,b]$ for which $f(x_{max}) = m$, and for every other $x \in [a,b]$, $f(x)$ is less than or equal to m. Similarly $\inf_{x\in[a,b]} f(x)$ is the minimum value of f on $[a,b]$.

We now know that on a closed interval, every continuous function is bounded and attains its maximum at least once and minimum at least once. The same is not true for open or infinite intervals; see Exercise 9.6.1.

Remark 9.6.9. You may encounter a rather different "maximum principle" in complex analysis or partial differential equations, involving analytic functions and harmonic functions respectively, instead of continuous functions. Those maximum principles are not directly related to this one (though they are also concerned with whether maxima exist, and where the maxima are located).

Exercise 9.6.1. Give examples of

(a) a function $f : (1,2) \to \mathbf{R}$ which is continuous and bounded, attains its minimum somewhere, but does not attain its maximum anywhere;

(b) a function $f : [0,\infty) \to \mathbf{R}$ which is continuous, bounded, attains its maximum somewhere, but does not attain its minimum anywhere;

(c) a function $f : [-1,1] \to \mathbf{R}$ which is bounded but does not attain its minimum anywhere or its maximum anywhere.

(d) a function $f : [-1,1] \to \mathbf{R}$ which has no upper bound and no lower bound.

Explain why none of the examples you construct violate the maximum principle. (Note: read the assumptions *carefully*!)

9.7 The intermediate value theorem

We have just shown that a continuous function attains both its maximum value and its minimum value. We now show that f also attains every value in between. To do this, we first prove a very intuitive theorem:

Theorem 9.7.1 (Intermediate value theorem). *Let $a < b$, and let $f : [a, b] \to \mathbf{R}$ be a continuous function on $[a, b]$. Let y be a real number between $f(a)$ and $f(b)$, i.e., either $f(a) \leq y \leq f(b)$ or $f(a) \geq y \geq f(b)$. Then there exists $c \in [a, b]$ such that $f(c) = y$.*

Proof. We have two cases: $f(a) \leq y \leq f(b)$ or $f(a) \geq y \geq f(b)$. We will assume the former, that $f(a) \leq y \leq f(b)$; the latter is proven similarly and is left to the reader.

If $y = f(a)$ or $y = f(b)$ then the claim is easy, as one can simply set $c = a$ or $c = b$, so we will assume that $f(a) < y < f(b)$. Let E denote the set

$$E := \{x \in [a, b] : f(x) < y\}.$$

Clearly E is a subset of $[a, b]$, and is hence bounded. Also, since $f(a) < y$, we see that a is an element of E, so E is non-empty. By the least upper bound principle, the supremum

$$c := \sup(E)$$

is thus finite. Since E is bounded by b, we know that $c \leq b$; since E contains a, we know that $c \geq a$. Thus we have $c \in [a, b]$. To complete the proof we now show that $f(c) = y$. The idea is to work from the left of c to show that $f(c) \leq y$, and to work from the right of c to show that $f(c) \geq y$.

Let $n \geq 1$ be an integer. The number $c - \frac{1}{n}$ is less than $c = \sup(E)$ and hence cannot be an upper bound for E. Thus there exists a point, call it x_n, which lies in E and which is greater than $c - \frac{1}{n}$. Also $x_n \leq c$ since c is an upper bound for E. Thus

$$c - \frac{1}{n} \leq x_n \leq c.$$

By the squeeze test (Corollary 6.4.14) we thus have $\lim_{n\to\infty} x_n = c$. Since f is continuous at c, this implies that $\lim_{n\to\infty} f(x_n) = f(c)$. But since x_n lies in E for every n, we have $f(x_n) < y$ for every n. By the comparison principle (Lemma 6.4.13) we thus have $f(c) \leq y$. Since $f(b) > f(c)$, we conclude $c \neq b$.

Since $c \neq b$ and $c \in [a, b]$, we must have $c < b$. In particular there is an $N > 0$ such that $c + \frac{1}{n} < b$ for all $n > N$ (since $c + \frac{1}{n}$ converges to c as $n \to \infty$). Since c is the supremum of E and $c + \frac{1}{n} > c$, we thus have $c + \frac{1}{n} \notin E$ for all $n > N$. Since $c + \frac{1}{n} \in [a, b]$, we thus have $f(c + \frac{1}{n}) \geq y$ for all $n \geq N$. But $c + \frac{1}{n}$ converges to c, and f is continuous at c, thus $f(c) \geq y$. But we already knew that $f(c) \leq y$, thus $f(c) = y$, as desired. ☐

The intermediate value theorem says that if f takes the values $f(a)$ and $f(b)$, then it must also take all the values in between. Note that if f is not assumed to be continuous, then the intermediate value theorem no longer applies. For instance, if $f : [-1, 1] \to \mathbf{R}$ is the function

$$f(x) := \begin{cases} -1 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$$

then $f(-1) = -1$, and $f(1) = 1$, but there is no $c \in [-1, 1]$ for which $f(c) = 0$. Thus if a function is discontinuous, it can "jump" past intermediate values; however continuous functions cannot do so.

Remark 9.7.2. A continuous function may take an intermediate value multiple times. For instance, if $f : [-2, 2] \to \mathbf{R}$ is the function $f(x) := x^3 - x$, then $f(-2) = -6$ and $f(2) = 6$, so we know that there exists a $c \in [-2, 2]$ for which $f(c) = 0$. In fact, in this case there exists three such values of c: we have $f(-1) = f(0) = f(1) = 0$.

Remark 9.7.3. The intermediate value theorem gives another way to show that one can take n^{th} roots of a number. For instance, to construct the square root of 2, consider the function $f : [0, 2] \to \mathbf{R}$ defined by $f(x) = x^2$. This function is continuous, with $f(0) = 0$ and $f(2) = 4$. Thus there exists a $c \in [0, 2]$ such that $f(c) = 2$, i.e., $c^2 = 2$. (This argument does not show that there is just one square root of 2, but it does prove that there is *at least* one square root of 2.)

Corollary 9.7.4 (Images of continuous functions). *Let $a < b$, and let $f : [a, b] \to \mathbf{R}$ be a continuous function on $[a, b]$. Let $M := \sup_{x \in [a,b]} f(x)$ be the maximum value of f, and let $m := \inf_{x \in [a,b]} f(x)$ be the minimum value. Let y be a real number between m and M (i.e., $m \leq y \leq M$). Then there exists a $c \in [a, b]$ such that $f(c) = y$. Furthermore, we have $f([a, b]) = [m, M]$.*

Proof. See Exercise 9.7.1. □

Exercise 9.7.1. Prove Corollary 9.7.4. (Hint: you may need Exercise 9.4.6 in addition to the intermediate value theorem.)

Exercise 9.7.2. Let $f : [0, 1] \to [0, 1]$ be a continuous function. Show that there exists a real number x in $[0, 1]$ such that $f(x) = x$. (Hint: apply the intermediate value theorem to the function $f(x) - x$.) This point x is known as a *fixed point* of f, and this result is a basic example of a *fixed point theorem*, which play an important rôle in certain types of analysis.

9.8 Monotonic functions

We now discuss a class of functions which is distinct from the class of continuous functions, but has somewhat similar properties: the class of monotone (or monotonic) functions.

Definition 9.8.1 (Monotonic functions). Let X be a subset of $\mathbf{R}$, and let $f : X \to \mathbf{R}$ be a function. We say that f is *monotone increasing* iff $f(y) \geq f(x)$ whenever $x, y \in X$ and $y > x$. We say that f is *strictly monotone increasing* iff $f(y) > f(x)$ whenever $x, y \in X$ and $y > x$. Similarly, we say f is *monotone decreasing* iff $f(y) \leq f(x)$ whenever $x, y \in X$ and $y > x$, and *strictly monotone decreasing* iff $f(y) < f(x)$ whenever $x, y \in X$ and $y > x$. We say that f is *monotone* if it is monotone increasing or monotone decreasing, and *strictly monotone* if it is strictly monotone increasing or strictly monotone decreasing.

Examples 9.8.2. The function $f(x) := x^2$, when restricted to the domain $[0, \infty)$, is strictly monotone increasing (why?), but when restricted instead to the domain $(-\infty, 0]$, is strictly monotone decreasing (why?). Thus the function is strictly monotone on both $(-\infty, 0]$ and $[0, \infty)$, but is not strictly monotone (or monotone) on the full real line $(-\infty, \infty)$. Note that if a function is strictly monotone on a domain X, it is automatically monotone as well on the same domain X. The constant function $f(x) := 6$, when restricted to an arbitrary domain $X \subseteq \mathbf{R}$, is both monotone increasing and monotone decreasing, but is not strictly monotone (unless X consists of at most one point - why?).

Continuous functions are not necessarily monotone (consider for instance the function $f(x) = x^2$ on $\mathbf{R}$), and monotone functions are not necessarily continuous; for instance, consider the function $f : [-1, 1] \to \mathbf{R}$ defined earlier by

$$f(x) := \begin{cases} -1 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0. \end{cases}$$

Monotone functions obey the maximum principle (Exercise 9.8.1), but not the intermediate value principle (Exercise 9.8.2). On the other hand, it is possible for a monotone function to have many, many discontinuities (Exercise 9.8.5).

If a function is both strictly monotone and continuous, then it has many nice properties. In particular, it is invertible:

Proposition 9.8.3. *Let $a < b$ be real numbers, and let $f : [a, b] \to \mathbf{R}$ be a function which is both continuous and strictly monotone increasing. Then f is a bijection from $[a, b]$ to $[f(a), f(b)]$, and the inverse $f^{-1} : [f(a), f(b)] \to [a, b]$ is also continuous and strictly monotone increasing.*

Proof. See Exercise 9.8.4. □

There is a similar Proposition for functions which are strictly monotone decreasing; see Exercise 9.8.4.

Example 9.8.4. Let n be a positive integer and $R > 0$. Since the function $f(x) := x^n$ is strictly increasing on the interval $[0, R]$, we see from Proposition 9.8.3 that this function is a bijection from

$[0, R]$ to $[0, R^n]$, and hence there is an inverse from $[0, R^n]$ to $[0, R]$. This can be used to give an alternate means to construct the n^{th} root $x^{1/n}$ of a number $x \in [0, R]$ than what was done in Lemma 5.6.5.

Exercise 9.8.1. Explain why the maximum principle remains true if the hypothesis that f is continuous is replaced with f being monotone, or with f being strictly monotone. (You can use the same explanation for both cases.)

Exercise 9.8.2. Give an example to show that the intermediate value theorem becomes false if the hypothesis that f is continuous is replaced with f being monotone, or with f being strictly monotone. (You can use the same counterexample for both cases.)

Exercise 9.8.3. Let $a < b$ be real numbers, and let $f : [a, b] \to \mathbf{R}$ be a function which is both continuous and one-to-one. Show that f is strictly monotone. (Hint: divide into the three cases $f(a) < f(b)$, $f(a) = f(b)$, $f(a) > f(b)$. The second case leads directly to a contradiction. In the first case, use contradiction and the intermediate value theorem to show that f is strictly monotone increasing; in the third case, argue similarly to show f is strictly monotone decreasing.)

Exercise 9.8.4. Prove Proposition 9.8.3. (Hint: to show that f^{-1} is continuous, it is easiest to use the "epsilon-delta" definition of continuity, Proposition 9.4.7(c).) Is the proposition still true if the continuity assumption is dropped, or if strict monotonicity is replaced just by monotonicity? How should one modify the proposition to deal with strictly monotone decreasing functions instead of strictly monotone increasing functions?

Exercise 9.8.5. In this exercise we give an example of a function which has a discontinuity at every rational point, but is continuous at every irrational. Since the rationals are countable, we can write them as $\mathbf{Q} = \{q(0), q(1), q(2), \ldots\}$, where $q : \mathbf{N} \to \mathbf{Q}$ is a bijection from $\mathbf{N}$ to $\mathbf{Q}$. Now define a function $g : \mathbf{Q} \to \mathbf{R}$ by setting $g(q(n)) := 2^{-n}$ for each natural number n; thus g maps $q(0)$ to 1, $q(1)$ to 2^{-1}, etc. Since $\sum_{n=0}^{\infty} 2^{-n}$ is absolutely convergent, we see that $\sum_{r \in \mathbf{Q}} g(r)$ is also absolutely convergent. Now define the function $f : \mathbf{R} \to \mathbf{R}$ by

$$f(x) := \sum_{r \in \mathbf{Q} : r < x} g(r).$$

Since $\sum_{r\in\mathbf{Q}} g(r)$ is absolutely convergent, we know that $f(x)$ is well-defined for every real number x.

(a) Show that f is strictly monotone increasing. (Hint: you will need Proposition 5.4.14.)

(b) Show that for every rational number r, f is discontinuous at r. (Hint: since r is rational, $r = q(n)$ for some natural number n. Show that $f(x) \geq f(r) + 2^{-n}$ for all $x > r$.)

(c) Show that for every irrational number x, f is continuous at x. (Hint: first demonstrate that the functions

$$f_n(x) := \sum_{r\in\mathbf{Q}:r<x,g(r)\geq 2^{-n}} g(r)$$

are continuous at x, and that $|f(x) - f_n(x)| \leq 2^{-n}$.)

9.9 Uniform continuity

We know that a continuous function on a closed interval $[a, b]$ remains bounded (and in fact attains its maximum and minimum, by the maximum principle). However, if we replace the closed interval by an open interval, then continuous functions need not be bounded any more. An example is the function $f : (0, 2) \to \mathbf{R}$ defined by $f(x) := 1/x$. This function is continuous at every point in $(0, 2)$, and is hence continuous at $(0, 2)$, but is not bounded. Informally speaking, the problem here is that while the function is indeed continuous at every point in the open interval $(0, 2)$, it becomes "less and less" continuous as one approaches the endpoint 0.

Let us analyze this phenomenon further, using the "epsilon-delta" definition of continuity - Proposition 9.4.7(c). We know that if $f : X \to \mathbf{R}$ is continuous at a point x_0, then for every $\varepsilon > 0$ there exists a δ such that $f(x)$ will be ε-close to $f(x_0)$ whenever $x \in X$ is δ-close to x_0. In other words, we can force $f(x)$ to be ε-close to $f(x_0)$ if we ensure that x is sufficiently close to x_0. One way of thinking about this is that around every point x_0 there is an "island of stability" $(x_0 - \delta, x_0 + \delta)$, where the function $f(x)$ doesn't stray by more than ε from $f(x_0)$.

Example 9.9.1. Take the function $f(x) := 1/x$ mentioned above at the point $x_0 = 1$. In order to ensure that $f(x)$ is 0.1-close to $f(x_0)$, it suffices to take x to be $1/11$-close to x_0, since if x is $1/11$-close to x_0 then $10/11 < x < 12/11$, and so $11/12 < f(x) < 11/10$, and so $f(x)$ is 0.1-close to $f(x_0)$. Thus the "δ" one needs to make $f(x)$ 0.1-close to $f(x_0)$ is about $1/11$ or so, at the point $x_0 = 1$.

Now let us look instead at the point $x_0 = 0.1$. The function $f(x) = 1/x$ is still continuous here, but we shall see the continuity is much worse. In order to ensure that $f(x)$ is 0.1-close to $f(x_0)$, we need x to be $1/1010$-close to x_0. Indeed, if x is $1/1010$ close to x_0, then $10/101 < x < 102/1010$, and so $9.901 < f(x) < 10.1$, so $f(x)$ is 0.1-close to $f(x_0)$. Thus one needs a much smaller "δ" for the same value of ε - i.e., $f(x)$ is much more "unstable" near 0.1 than it is near 1, in the sense that there is a much smaller "island of stability" around 0.1 as there is around 1 (if one is interested in keeping $f(x)$ 0.1-stable).

On the other hand, there are other continuous functions which do not exhibit this behavior. Consider the function $g : (0, 2) \to \mathbf{R}$ defined by $g(x) := 2x$. Let us fix $\varepsilon = 0.1$ as before, and investigate the island of stability around $x_0 = 1$. It is clear that if x is 0.05-close to x_0, then $g(x)$ is 0.1-close to $g(x_0)$; in this case we can take δ to be 0.05 at $x_0 = 1$. And if we move x_0 around, say if we set x_0 to 0.1 instead, the δ does not change - even when x_0 is set to 0.1 instead of 1, we see that $g(x)$ will stay 0.1-close to $g(x_0)$ whenever x is 0.05-close to x_0. Indeed, the same δ works for every x_0. When this happens, we say that the function g is *uniformly continuous*. More precisely:

Definition 9.9.2 (Uniform continuity). Let X be a subset of $\mathbf{R}$, and let $f : X \to \mathbf{R}$ be a function. We say that f is *uniformly continuous* if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $f(x)$ and $f(x_0)$ are ε-close whenever $x, x_0 \in X$ are two points in X which are δ-close.

Remark 9.9.3. This definition should be compared with the notion of continuity. From Proposition 9.4.7(c), we know that a function f is *continuous* if for every $\varepsilon > 0$, and every $x_0 \in X$, there is a $\delta > 0$ such that $f(x)$ and $f(x_0)$ are ε-close whenever $x \in X$ is δ-close to x_0. The difference between uniform continuity and continuity is that in uniform continuity one can take a single δ which works for all $x_0 \in X$; for ordinary continuity, each $x_0 \in X$ might use a different δ. Thus every uniformly continuous function is continuous, but not conversely.

Example 9.9.4. (Informal) The function $f : (0, 2) \to \mathbf{R}$ defined by $f(x) := 1/x$ is continuous on $(0, 2)$, but not uniformly continuous, because the continuity (or more precisely, the dependence of δ on ε) becomes worse and worse as $x \to 0$. (We will make this more precise in Example 9.9.10.)

Recall that the notions of adherent point and of continuous function had several equivalent formulations; both had "epsilon-delta" type formulations (involving the notion of ε-closeness), and both had "sequential" formulations (involving the convergence of sequences); see Lemma 9.1.14 and Proposition 9.3.9. The concept of uniform continuity can similarly be phrased in a sequential formulation, this time using the concept of *equivalent sequences* (cf. Definition 5.2.6, but we now generalize to sequences of real numbers instead of rationals, and no longer require the sequences to be Cauchy):

Definition 9.9.5 (Equivalent sequences)**.** Let m be an integer, let $(a_n)_{n=m}^\infty$ and $(b_n)_{n=m}^\infty$ be two sequences of real numbers, and let $\varepsilon > 0$ be given. We say that $(a_n)_{n=m}^\infty$ is *ε-close* to $(b_n)_{n=m}^\infty$ iff a_n is ε-close to b_n for each $n \geq m$. We say that $(a_n)_{n=m}^\infty$ is *eventually ε-close* to $(b_n)_{n=m}^\infty$ iff there exists an $N \geq m$ such that the sequences $(a_n)_{n=N}^\infty$ and $(b_n)_{n=N}^\infty$ are ε-close. Two sequences $(a_n)_{n=m}^\infty$ and $(b_n)_{n=m}^\infty$ are *equivalent* iff for each $\varepsilon > 0$, the sequences $(a_n)_{n=m}^\infty$ and $(b_n)_{n=m}^\infty$ are eventually ε-close.

Remark 9.9.6. One could debate whether ε should be assumed to be rational or real, but a minor modification of Proposition 6.1.4 shows that this does not make any difference to the above definitions.