4

The Rationals

In this chapter we shall construct the field of rationals from the ring of integers. Our argument will use only properties (1) to (9) of the integers (Part II, Chapter 3), and hence the *same* argument will produce a field from *any* integral domain.

One's first idea is to say that a rational number a/b is a pair of integers (a, b), where $b \neq 0$. But this does not work, because then $2/3 \neq 4/6$. So we shall begin by introducing an equivalence relation on pairs of integers (a, b) such that $b \neq 0$. We define

$$(a, b) \equiv (c, d) \Longleftrightarrow ad = bc,$$

it being assumed, of course, that b and d are nonzero. This is an *equivalence relation*, that is

$$(a, b) \equiv (a, b) \qquad \text{(reflexivity)},$$

$$\text{if } (a, b) \equiv (c, d), \text{ then } (c, d) \equiv (a, b) \qquad \text{(symmetry)},$$

$$\text{if } (a, b) \equiv (c, d) \text{ and } (c, d) \equiv (e, f), \text{ then } (a, b) \equiv (e, f) \qquad \text{(transitivity)}.$$

These three properties of an equivalence relation are easily checked; we shall verify only transitivity.

Given $ad = bc$ and $cf = de$, we wish to show that $af = be$. Making free use of associativity and commutativity, we calculate

$$afd = adf = bcf = bde = bed.$$

As we are given that $d \neq 0$, we may use the cancellation law (Exercise 7 of Chapter 3) to infer that $af = be$, as was required to be shown.

Assuming that $b \neq 0$, we define the *ratio* (rational number) a/b as the equivalence class of (a, b), that is, as the set of all pairs (c, d), with $d \neq 0$, such that $ad = bc$. Note that

$$a/b = c/d \Longleftrightarrow (a, b) \equiv (c, d) \Leftrightarrow ad = bc.$$

If $\mathbf{Q}$ is the set of such ratios, we shall obtain a field $(\mathbf{Q}, \underline{0}, -, +, \underline{1}, \cdot)$ where we have underlined $\underline{0}$ and $\underline{1}$ to distinguish them, at least temporarily, from the integers 0 and 1.

Here is how we attempt to define the operations of $\mathbf{Q}$:

$$\begin{aligned} \underline{0} &= 0/1, \\ -(a/b) &= (-a)/b, \\ a/b + c/d &= (ad + bc)/(bd), \\ \underline{1} &= 1/1, \\ a/b \cdot c/d &= (ac)/(bd). \end{aligned}$$

Note that, because of axiom (9) for an integral domain, $bd \neq 0$. There is no problem with $\underline{0}$ and $\underline{1}$, but we must check that the other operations are *well-defined.* We shall do so here for the operation $+$.

Thus, suppose that $a/b = a'/b'$ and $c/d = c'/d'$. We must show that

$$a/b + c/d = a'/b' + c'/d',$$

that is,

$$(ad + bc)/(bd) = (a'd' + b'c')/(b'd'),$$

that is,

$$(ad + bc)(b'd') = (bd)(a'd' + b'c').$$

The reader is invited to verify that this is indeed the case.

The ratios with operations $\underline{0}$, $-$, $+$, $\underline{1}$, and $\cdot$ as defined above form a field. To prove this one must check all the axioms of a field. For example, we shall check (4) and (10). To prove (4) for ratios we argue as follows:

$$\begin{aligned} a/b + c/d &= (ad + bc)/(bd) \\ &= (da + cb)/(db) \quad \text{(commutativity of $\cdot$ for integers)} \\ &= (cb + da)/(db) \quad \text{(commutativity of $+$ for integers)} \\ &= c/d + a/b. \end{aligned}$$

Here we made use of the commutativity of $\cdot$ and $+$ for integers. To prove (10), we may assume that $a/b \neq \underline{0}$, that is, $a1 \neq b0$, that is, $a \neq 0$ (see Exercise 2 of Chapter 3). We claim that $(a/b) \cdot (b/a) = 1$. Indeed,

$$(a/b) \cdot (b/a) = (a \cdot b)/(b \cdot a) = 1/1,$$

since

$$(a \cdot b) \cdot 1 = a \cdot b = b \cdot a = (b \cdot a) \cdot 1.$$

We have thus constructed the field $\mathbf{Q}$ from the ring $\mathbf{Z}$. More generally, the same construction leads from any integral domain to a field, called its *field of quotients.*

What is the relationship between $\mathbf{Q}$ and the ring $\mathbf{Z}$ from which it is constructed? Strictly speaking, the set $\mathbf{Q}$ does not contain the set $\mathbf{Z}$. However, $\mathbf{Q}$ does contain a subset, consisting of the ratios of the form $a/1$, which is *isomorphic* to $\mathbf{Z}$. To make this notion more precise, consider the mapping $h : \mathbf{Z} \to \mathbf{Q}$ such that $h(a) = a/1$. Then h is a *homomorphism*, that is, it preserves the operations of $\mathbf{Z}$:

$$\begin{aligned} h(a+b) &= h(a) + h(b), \\ h(ab) &= h(a)h(b), \\ h(-a) &= -h(a), \\ h(0) &= \underline{0}, \\ h(1) &= \underline{1}. \end{aligned}$$

Furthermore, h is a 1-to-1 mapping (an *injection*). The set $h(\mathbf{Z})$ is thus an *isomorphic image* of $\mathbf{Z}$. We also say that h *embeds* $\mathbf{Z}$ in $\mathbf{Q}$. It is a mathematical convention to identify 2 and $2/1$ and to say that $\mathbf{Z}$ is contained in $\mathbf{Q}$.

Exercises

1. Show that $-(-(a/b)) = a/b$.

2. Show that $((a/b)^{-1})^{-1} = a/b$ (assuming $a, b \neq 0$).

3. Prove that $a/(b/c) = (ac)/b$.

4. Define $\max(a/b, c/d)$.

5. What definition might one use to construct the integers out of the natural numbers?

6. If b is a nonzero integer, let $b\mathbf{Z} = \{bx | x \in \mathbf{Z}\}$. Consider any mapping $f : b\mathbf{Z} \to \mathbf{Z}$ such that $f(bx + by) = f(bx) + f(by)$ for all $x, y \in \mathbf{Z}$. Show that $f(0) = 0$ and $f(-bx) = -f(bx)$. If g is any other function

like f (that is, $g : c\mathbf{Z} \to \mathbf{Z}$ for $c \neq 0$ and $g(cx + cy) = g(cx) + g(cy)$) define $f \equiv g$ to mean that $f(bcx) = g(bcx)$ for all integers x. Show that $\equiv$ is an equivalence relation and that the equivalence class $[f]$ can be taken as a definition of the quotient $f(b)/b$.