5. Suppose R is a U.F.D. with field of fractions F and $p \in R[x]$ is a monic polynomial.
   (a) Show that the ideal $pR[x]$ generated by p in $R[x]$ is prime if and only if the ideal $pF[x]$ generated by p in $F[x]$ is prime. [Use Gauss' Lemma.]
   (b) Show that $pR[x]$ is saturated, i.e., that $pF[x] \cap R[x] = pR[x]$.
6. Show that $I = (y^3 - xz, xy^2 - z^2)$ is not a prime ideal in $\mathbb{Q}[x, y, z]$ and find explicit elements $a, b \in \mathbb{Q}[x, y, z]$ with $ab \in I$ but $a \notin I$ and $b \notin I$.
7. Show that $P = (y^3 - xz, xy^2 - z^2, x^2 - yz)$ is a prime ideal in $\mathbb{Q}[x, y, z]$.
8. Show that $P = (x^2 - yz, w^2 - x^4 z)$ is a prime ideal in $\mathbb{Q}[x, y, z, w]$.
9. Show that $P = (xz^2 - w^3, xw^2 - y^4, y^4z^2 - w^5)$ is a prime ideal in $\mathbb{Q}[x, y, z, w]$.
10. Show that $I = (xy - w^3, y^2 - zw)$ is not a prime ideal in $\mathbb{Q}[x, y, z, w]$ and find a, b with $ab \in I$ but $a, b \notin I$.
11. Let R_P be the localization of R at the prime P. Prove that if Q is a P-primary ideal of R then $Q = {}^c({}^eQ)$ with respect to the extension and contraction of Q to R_P. Show the same result holds if Q is P'-primary for some prime P' contained in P.
12. Let $R = \mathbb{R}[x, y, z]/(xy - z^2)$, let $P = (\bar{x}, \bar{z})$ be the prime ideal generated by the images of x and y in R, and let R_P be the localization of R at P. Prove that $P^2 R_P \cap R = (\bar{x})$ and is strictly larger than P^2.
13. Prove that if N and N' are two R-submodules of an R-module M with $N_P = N'_P$ in the localization M_P for every prime ideal P of R (or just for every maximal ideal) then $N = N'$.
14. Suppose $\varphi : M \to N$ is an R-module homomorphism. Prove that φ is injective (respectively, surjective) if and only if the induced R_P-module homomorphism $\varphi : M_P \to N_P$ is injective (respectively, surjective) for every prime ideal P of R (or just for every maximal ideal of R).
15. Let $R = \mathbb{Z}[\sqrt{-5}\,]$ be the ring of integers in the quadratic field $\mathbb{Q}(\sqrt{-5}\,)$ and let I be the prime ideal $(2, 1 + \sqrt{-5}\,)$ of R generated by 2 and $1 + \sqrt{-5}$ (cf. Exercise 5, Section 8.2). Recall that every nonzero prime ideal P of R contains a prime $p \in \mathbb{Z}$.
   (a) If P is a prime ideal of R not containing 2 prove that $I_P = R_P$.
   (b) If P is a prime ideal of R containing 2 prove that $P = I$ and that $I_P = (1 + \sqrt{-5}\,)R_P$.
   (c) Prove that $I_P \cong R_P$ as R_P-modules for every prime ideal P of R but that I and R are not isomorphic R-modules. (This example shows that it is important in Exercise 14 to be *given* the R-module homomorphism φ.) [Observe that $I \cong R$ as R-modules if and only if I is a *principal* ideal.]
16. Prove that localization commutes with tensor products: there is a unique isomorphism of $D^{-1}R$-modules $\varphi : (D^{-1}M) \otimes_{D^{-1}R} (D^{-1}N) \cong D^{-1}(M \otimes_R N)$ with $\varphi((m/d) \otimes (n/d'))$ given by $(m \otimes n)/dd'$ for any R-modules M, N, and multiplicatively closed set D in R.
17. Prove that the R-module A is a flat R-module if and only if A_P is a flat R_P-module for every prime ideal P of R (or just for every maximal ideal of R). [Use Proposition 41, Exercises 14 and 16, and the exactness properties of localization.]
18. In the notation of Example 2 following Corollary 37, prove that $R_f \cong R[x]/(fx - 1)$ if f is not nilpotent in R. [Show that the map $\varphi : R[x] \to R_f$ defined by $\varphi(r) = r/1$ and $\varphi(x) = 1/f$ gives a surjective ring homomorphism and the universal property in Theorem 36 gives an inverse.]
19. Prove that if R is an integrally closed integral domain and D is any multiplicatively closed subset of R containing 1, then $D^{-1}R$ is integrally closed.

20. Suppose that R is a subring of the ring S with $1 \in R$ and that S is integral over R. If D is any multiplicatively closed subset of R, prove that $D^{-1}S$ is integral over $D^{-1}R$.

21. Suppose $\varphi : R \to S$ is a ring homomorphism and D' is a multiplicatively closed subset of S. Let $D = \varphi^{-1}(D')$. Prove that D is a multiplicatively closed subset of R and that the map $\varphi' : D^{-1}R \to D'^{-1}S$ given by $\varphi'(r/d) = \varphi(r)/\varphi(d)$ is a ring homomorphism.

22. Suppose $P \subseteq Q$ are prime ideals in R and let R_Q be the localization of R at Q. Prove that the localization R_P is isomorphic to the localization of R_Q at the prime ideal PR_Q (cf. the preceding exercise).

23. Let $\varphi : A \to B$ be a homomorphism of commutative rings with $\varphi(1_A) = 1_B$, and let P be a prime ideal of A. Let contraction and extension of ideals with respect to φ be denoted by superscripts c and e respectively. Prove that P is the contraction of a prime ideal in B if and only if $P = (P^e)^c$. [Localize B at $\varphi(A - P)$.]

24. *(The Going-down Theorem)* Let S be an integral domain, let R be an integrally closed subring of S containing 1_S, and let k be the field of fractions of R. Suppose that $P_2 \subseteq P_1$ are prime ideals in R and that Q_1 is a prime ideal in S with $Q_1 \cap R = P_1$. Let S_{Q_1} be the localization of S at Q_1.

(a) Show that $P_2 \subseteq P_2 S_{Q_1} \cap R$.

(b) Suppose that $a \in P_2 S_{Q_1} \cap R$ and write $a = s/d$ with $s \in P_2 S$ and $d \in S$, $d \notin Q_1$. If the minimal polynomial of s over k is $x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with $a_0, \ldots, a_{n-1} \in P_2$ (cf. Exercise 12 in Section 3) show that the minimal polynomial of d over k is $x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0$ where $b_i = a_i / a^{n-i}$ and conclude that $b_i \in R$. [Use Exercise 10 in Section 3.]

(c) Show that $a \in P_2$ and conclude that $P_2 S_{Q_1} \cap R = P_2$. [Show $a \notin P_2$ implies $b_i \in P_2$ for $i = 0, 1, \ldots, n-1$, which would imply $d^n \in P_2 S \subseteq P_1 S \subseteq Q_1$ and so $d \in Q_1$.]

(d) Prove that $P_2 S_{Q_1}$ is contained in a prime ideal P of S_{Q_1} with $P \cap R = P_2$. [Use (c) and the previous exercise for $\varphi : R \to S_{Q_1}$.]

(e) Let $Q_2 = P \cap S$. Prove that $Q_2 \subseteq Q_1$ and that $Q_2 \cap R = P_2$.

(f) Use induction together with the previous result to prove the Going-down Theorem: Theorem 26(4).

25. Let k be an algebraically closed field and let $V = \mathcal{Z}(xz - yw) \subset \mathbb{A}^4$. Prove that the set of points v where $f = \bar{x}/\bar{y} \in k(V)$ is regular is precisely the set of points (x, y, z, w) where $y \neq 0$ or $z \neq 0$. [If $f = \bar{a}/\bar{b}$ show that $ay - bx \in (xz - yw)$ as polynomials in $k[x, y, z, w]$ and conclude that $b \in (y, z)$.] Prove that there is no function $a/b \in k(V)$ with $b(v) \neq 0$ for every v where f is regular.

26. *(Differentials of Morphisms)* Let $\varphi : V \to W$ be a morphism of affine varieties over the algebraically closed field k and suppose $\varphi(v) = w$.

(a) Show that φ induces a linear map from the k-vector space M_w/M_w^2 to the k-vector space M_v/M_v^2, and use this to show that φ induces a linear map $d\varphi$ (called the *differential* of φ) from the k-vector space $\mathbb{T}_{v,V}$ to the k-vector space $\mathbb{T}_{w,W}$.

(b) Prove that if $V \subseteq \mathbb{A}^n$, $W \subseteq \mathbb{A}^m$ and $\varphi = (F_1(x_1, \ldots, x_n), \ldots, F_m(x_1, \ldots, x_n))$ then $d\varphi : \mathbb{T}_{v,V} \to \mathbb{T}_{w,W}$ is given explicitly by

$$(d\varphi)(a_1, \ldots, a_n) = (D_v(F_1)(a_1, \ldots, a_n), \ldots, D_v(F_m)(a_1, \ldots, a_n)).$$

[If $g = g(y_1, \ldots, y_m)$ show that the chain rule implies

$$\frac{\partial (g \circ \varphi)}{\partial x_i}(v) = \sum_{j=1}^{m} \frac{\partial g}{\partial y_j}(w)\, \frac{\partial F_j}{\partial x_i}(v),$$

so that $D_v(g \circ \varphi)(a_1, \ldots, a_n) = D_w(g)(b_1, \ldots, b_m)$ where $b_j = D_v(F_j)(a_1, \ldots, a_n)$. Then use the fact that $g \circ \varphi \in \mathcal{I}(V)$ if $g \in \mathcal{I}(W)$.]

(c) If $\psi : U \to V$ is another morphism with $\psi(u) = v$, prove that the associated $d(\varphi \circ \psi) : \mathbb{T}_{u,U} \to \mathbb{T}_{w,W}$ is the same as $d\varphi \circ d\psi$.

(d) Prove that if φ is an isomorphism then $d\varphi$ is a vector space isomorphism from $\mathbb{T}_{v,V}$ to $\mathbb{T}_{w,W}$ for every $\varphi(v) = w$.

27. Let $V = \mathbb{A}^1$ and $W = \mathcal{Z}(xz - y^2, yz - x^3, z^2 - x^2y) \subset \mathbb{A}^3$. Let $\varphi : V \to W$ be the surjective morphism $\varphi(t) = (t^3, t^4, t^5)$ (cf. Exercise 26 in Section 1). For each $t \in \mathbb{A}^1$ describe the differential $d\varphi : \mathbb{T}_{t,\mathbb{A}^1} \to \mathbb{T}_{(t^3,t^4,t^5),W}$ in the previous exercise explicitly; in particular prove that $d\varphi$ is an isomorphism of vector spaces for all $t \neq 0$ and is the zero map for $t = 0$. Use this to prove that V and W are not isomorphic.

28. If k is a field, the quotient $k[x]/(x^2)$ is called the *ring of dual numbers* over k. If V is an affine algebraic set over k, show that a k-algebra homomorphism from $k[V]$ to $k[x]/(x^2)$ is equivalent to specifying a point $v \in V$ with $\mathcal{O}_{v,V}/\mathfrak{m}_{v,V} = k$ (called a *k-rational point* of V) together with an element in the tangent space $\mathbb{T}_{v,V}$ of V at v.

29. (*Computing the dimension of a variety*) Let P be a prime ideal in $k[x_1, \ldots, x_n]$, set $P_0 = 0$ and let $P_i = P \cap k[x_1, \ldots, x_i]$. Define the varieties $V_i = \mathcal{Z}(P_i) \subseteq \mathbb{A}^i$ with V_0 the zero dimensional variety consisting of a single point and coordinate ring k.

(a) Show that $\dim V_{i-1} \leq \dim V_i \leq \dim V_{i-1}+1$. [First exhibit an injection from $k[V_{i-1}]$ into $k[V_i]$; then show that $k[V_i]$ is a k-algebra generated by $k[V_{i-1}]$ and one additional generator.]

(b) If the ideal generated by P_{i-1} in $k[x_1, \ldots, x_i]$ equals P_i, show that $V_i \cong V_{i-1} \times \mathbb{A}^1$ and deduce that $\dim V_i = \dim V_{i-1} + 1$.

(c) If the ideal generated by P_{i-1} in $k[x_1, \ldots, x_i]$ is properly contained in P_i, show that $\dim V_i = \dim V_{i-1}$.

(d) Show that $\dim V$ equals the number of $i \in \{1, 2, \ldots, n\}$ such that the ideal generated by P_{i-1} in $k[x_1, \ldots, x_i]$ equals the ideal P_i. Deduce that if G is the reduced Gröbner basis for P with respect to the lexicographic monomial ordering $x_n > \cdots > x_1$ and $G_i = G \cap k[x_1, \ldots, x_i]$ where $G_0 = \emptyset$, and N is the number of i with $G_i \neq G_{i-1}$ for $1 \leq i \leq n$, then $\dim V = n - N$.

The following eleven exercises introduce the notion of the *support* of an R-module M and its relation to the associated primes of M. Cf. also Exercises 29 to 35 in Section 1 and Exercises 25 to 30 in Section 5.

Definition. If M is an R-module, then the set of prime ideals P of R for which the localization M_P is nonzero is called the *support* of M, denoted $\operatorname{Supp}(M)$.

30. Prove that $M = 0$ if and only if $\operatorname{Supp}(M) = \emptyset$. [Use Proposition 47.]

31. If $0 \to L \to M \to N \to 0$ is an exact sequence of R-modules, prove that the localization M_P is nonzero if and only if one of the localizations N_P and L_P is nonzero and deduce that $\operatorname{Supp}(M) = \operatorname{Supp}(L) \cup \operatorname{Supp}(N)$. In particular, if $M = M_1 \oplus \cdots \oplus M_n$ prove that $\operatorname{Supp}(M) = \operatorname{Supp}(M_1) \cup \cdots \cup \operatorname{Supp}(M_n)$.

32. Suppose $P \subseteq Q$ are prime ideals in R and that M is an R-module. Prove that the localization of the R-module M_Q at P is the localization M_P, i.e., $(M_Q)_P = M_P$. [Argue directly, or use Proposition 41 and the associativity of the tensor product.]

33. Suppose $P \subseteq Q$ are prime ideals in R and that M is an R-module. Prove that if $P \in \operatorname{Supp}(M)$ then $Q \in \operatorname{Supp}(M)$. [Use the previous exercise.]

34. **(a)** Suppose $M = Rm$ is a cyclic R-module. Prove that $M_P = 0$ if and only if there is