**Examples**

(1) If $G = \langle g \rangle$ is cyclic of order $n \in \mathbb{Z}^+$, then the elements of $FG$ are of the form

$$\sum_{i=0}^{n-1} \alpha_i g^i.$$

The map $F[x] \to F\langle g \rangle$ which sends $x^k$ to $g^k$ for all $k \geq 0$ extends by $F$-linearity to a surjective ring homomorphism with kernel equal to the ideal generated by $x^n - 1$. Thus

$$F\langle g \rangle \cong F[x]/(x^n - 1).$$

This is an isomorphism of $F$-algebras, i.e., is a ring isomorphism which is $F$-linear.

(2) Under the notation of the preceding example let $r = 1 + g + g^2 + \cdots + g^{n-1}$, so $r$ is a nonzero element of $F\langle g \rangle$. Note that $rg = g + g^2 + \cdots + g^{n-1} + 1 = r$, hence $r(1 - g) = 0$. Thus the ring $F\langle g \rangle$ contains zero divisors (provided $n > 1$). More generally, if $G$ is any group of order $> 1$, then for any nonidentity element $g \in G$, $F\langle g \rangle$ is a subring of $FG$, so $FG$ also contains zero divisors.

(3) Let $G = S_3$ and $F = \mathbb{Q}$. The elements $r = 5(1\,2) - 7(1\,2\,3)$ and $s = -4(1\,2\,3) + 12(1\,3\,2)$ are typical members of $\mathbb{Q}S_3$. Their sum and product are seen to be

$$\begin{aligned} r + s &= 5(1\ 2) - 11(1\ 2\ 3) + 12(1\ 3\ 2) \\ rs &= -20(2\ 3) + 28(1\ 3\ 2) + 60(1\ 3) - 84 \end{aligned}$$

(recall that products (compositions) of permutations are computed from right to left). An explicit example of a sum and product of two elements in the group ring $\mathbb{Q}D_8$ appears in Section 7.2.

Before giving specific examples of representations we discuss the correspondence between representations of $G$ and $FG$-modules (after which we can simultaneously give examples of both). This discussion closely parallels the treatment of $F[x]$-modules in Section 10.1.

Suppose first that $\varphi : G \to GL(V)$ is a representation of $G$ on the vector space $V$ over $F$. As above, write $G = \{g_1, \dots, g_n\}$, so for each $i \in \{1, \dots, n\}$, $\varphi(g_i)$ is a linear transformation from $V$ to itself. Make $V$ into an $FG$-module by defining the action of a ring element on an element of $V$ as follows:

$$\left(\sum_{i=1}^{n} \alpha_i g_i\right) \cdot v = \sum_{i=1}^{n} \alpha_i \varphi(g_i)(v), \qquad \text{for all } \sum_{i=1}^{n} \alpha_i g_i \in FG,\ v \in V.$$

We verify a special case of axiom 2(b) of a module (see Section 10.1) which shows precisely where the fact that $\varphi$ is a group homomorphism is needed:

$$\begin{aligned} (g_i g_j) \cdot v &= \varphi(g_i g_j)(v) && \text{(by definition of the action)} \\ &= (\varphi(g_i) \circ \varphi(g_j))(v) && \text{(since } \varphi \text{ is a group homomorphism)} \\ &= \varphi(g_i)(\varphi(g_j)(v)) && \text{(by definition of a composition of linear transformations)} \\ &= g_i \cdot (g_j \cdot v) && \text{(by definition of the action).} \end{aligned}$$

This argument extends by linearity to arbitrary elements of $FG$ to prove that axiom 2(b) of a module holds in general. It is an exercise to check that the remaining module axioms hold.

Note that $F$ is a subring of $FG$ and the action of the field element $\alpha$ on a vector is the same as the action of the ring element $\alpha 1$ on a vector i.e., the $FG$-module action extends the $F$ action on $V$.

Suppose now that conversely we are given an $FG$-module $V$. We obtain an associated vector space over $F$ and representation of $G$ as follows. Since $V$ is an $FG$-module, it is an $F$-module, i.e., it is a vector space over $F$. Also, for each $g \in G$ we obtain a map from $V$ to $V$, denoted by $\varphi(g)$, defined by

$$\varphi(g)(v) = g \cdot v \qquad \text{for all } v \in V,$$

where $g \cdot v$ is the given action of the ring element $g$ on the element $v$ of $V$. Since the elements of $F$ commute with each $g \in G$ it follows by the axioms for a module that for all $v, w \in V$ and all $\alpha, \beta \in F$ we have

$$\begin{aligned}\varphi(g)(\alpha v + \beta w) &= g \cdot (\alpha v + \beta w)\\ &= g \cdot (\alpha v) + g \cdot (\beta w)\\ &= \alpha(g \cdot v) + \beta(g \cdot w)\\ &= \alpha\varphi(g)(v) + \beta\varphi(g)(w),\end{aligned}$$

that is, for each $g \in G$, $\varphi(g)$ is a linear transformation. Furthermore, it follows by axiom 2(b) of a module that

$$\varphi(g_i g_j)(v) = (\varphi(g_i) \circ \varphi(g_j))(v)$$

(this is essentially the calculation above with the steps reversed). This proves that $\varphi$ is a group homomorphism (in particular, $\varphi(g^{-1}) = \varphi(g)^{-1}$, so every element of $G$ maps to a nonsingular linear transformation, i.e., $\varphi : G \to GL(V)$).

This discussion shows there is a bijection between $FG$-modules and pairs $(V, \varphi)$:

$$\left\{ V \text{ an } FG\text{-module} \right\} \longleftrightarrow \left\{ \begin{array}{c} V \text{ a vector space over } F \\ \text{and} \\ \varphi : G \to GL(V) \text{ a representation} \end{array} \right\}.$$

Giving a representation $\varphi : G \to GL(V)$ on a vector space $V$ over $F$ is therefore equivalent to giving an $FG$-module $V$. Under this correspondence we shall say that the module $V$ *affords* the representation $\varphi$ of $G$.

Recall from Section 10.1 that if a vector space $M$ is made into an $F[x]$-module via the linear transformation $T$, then the $F[x]$-submodules of $M$ are precisely the $T$-stable subspaces of $M$. In the current situation if $V$ is an $FG$-module affording the representation $\varphi$, then a subspace $U$ of $V$ is called *$G$-invariant* or *$G$-stable* if $g \cdot u \in U$ for all $g \in G$ and all $u \in U$ (i.e., if $\varphi(g)(u) \in U$ for all $g \in G$ and all $u \in U$). It follows easily that

*the $FG$-submodules of $V$ are precisely the $G$-stable subspaces of $V$.*

## Examples

(1) Let $V$ be a 1-dimensional vector space over $F$ and make $V$ into an $FG$-module by letting $gv = v$ for all $g \in G$ and $v \in V$. This module affords the representation $\varphi : G \to GL(V)$ defined by $\varphi(g) = I$ = the identity linear transformation, for all $g \in G$. The corresponding matrix representation (with respect to any basis of $V$) is the homomorphism of $G$ into $GL_1(F)$ which sends every group element to the $1 \times 1$ identity matrix. We shall henceforth refer to this as the *trivial representation* of $G$. The trivial representation has degree 1 and if $|G| > 1$, it is not faithful.

(2) Let $V = FG$ and consider this ring as a left module over itself. Then $V$ affords a representation of $G$ of degree equal to $|G|$. If we take the elements of $G$ as a basis of $V$, then each $g \in G$ permutes these basis elements under the left regular permutation representation:

$$g \cdot g_i = gg_i.$$

With respect to this basis of $V$ the matrix of the group element $g$ has a 1 in row $i$ and column $j$ if $gg_j = g_i$, and has 0's in all other positions. This (linear or matrix) representation is called the *regular representation* of $G$. Note that each nonidentity element of $G$ induces a nonidentity permutation on the basis of $V$ so the regular representation is always faithful.

(3) Let $n \in \mathbb{Z}^+$, let $G = S_n$ and let $V$ be an $n$-dimensional vector space over $F$ with basis $e_1, e_2, \dots, e_n$. Let $S_n$ act on $V$ by defining for each $\sigma \in S_n$

$$\sigma \cdot e_i = e_{\sigma(i)}, \qquad 1 \le i \le n$$

i.e., $\sigma$ acts by permuting the subscripts of the basis elements. This provides an (injective) homomorphism of $S_n$ into $GL(V)$ (i.e., a faithful representation of $S_n$ of degree $n$), hence makes $V$ into an $FS_n$-module. As in the preceding example, the matrix of $\sigma$ with respect to the basis $e_1, \dots, e_n$ has a 1 in row $i$ and column $j$ if $\sigma \cdot e_j = e_i$ (and has 0 in all other entries). Thus $\sigma$ has a 1 in row $i$ and column $j$ if $\sigma(j) = i$.

For an example of the ring action, consider the action of $FS_3$ on the 3-dimensional vector space over $F$ with basis $e_1, e_2, e_3$. Let $\sigma$ be the transposition $(1\,2)$, let $\tau$ be the 3-cycle $(1\,2\,3)$ and let $r = 2\sigma - 3\tau \in FS_3$. Then

$$\begin{aligned} r \cdot (\alpha e_1 + \beta e_2 + \gamma e_3) &= 2(\alpha e_{\sigma(1)} + \beta e_{\sigma(2)} + \gamma e_{\sigma(3)}) - 3(\alpha e_{\tau(1)} + \beta e_{\tau(2)} + \gamma e_{\tau(3)}) \\ &= 2(\alpha e_2 + \beta e_1 + \gamma e_3) - 3(\alpha e_2 + \beta e_3 + \gamma e_1) \\ &= (2\beta - 3\gamma)e_1 - \alpha e_2 + (2\gamma - 3\beta)e_3. \end{aligned}$$

(4) If $\psi : H \to GL(V)$ is any representation of $H$ and $\varphi : G \to H$ is any group homomorphism, then the composition $\psi \circ \varphi$ is a representation of $G$. For example, let $V$ be the $FS_n$-module of dimension $n$ described in the preceding example. If $\pi : G \to S_n$ is any permutation representation of $G$, the composition of $\pi$ with the representation above gives a linear representation of $G$. In other words, $V$ becomes an $FG$-module under the action

$$g \cdot e_i = e_{\pi(g)(i)}, \qquad \text{for all } g \in G.$$

Note that the regular representation, (2), is just the special case of this where $n = |G|$ and $\pi$ is the left regular permutation representation of $G$.

(5) Any homomorphism of $G$ into the multiplicative group $F^\times = GL_1(F)$ is a degree 1 (matrix) representation. For example, suppose $G = \langle g \rangle \cong Z_n$ is the cyclic group of order $n$ and $\zeta$ is a fixed $n^{\text{th}}$ root of 1 in $F$. Let $g^i \mapsto \zeta^i$, for all $i \in \mathbb{Z}$. This representation of $\langle g \rangle$ is a faithful representation if and only if $\zeta$ is a primitive $n^{\text{th}}$ root of 1.