

Theorem 21. (*Long Exact Sequence in Group Cohomology*) Suppose

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is a short exact sequence of G -modules. Then there is a long exact sequence:

$$\begin{aligned} 0 \longrightarrow A^G \longrightarrow B^G \longrightarrow C^G \xrightarrow{\delta_0} H^1(G, A) \longrightarrow H^1(G, B) \longrightarrow H^1(G, C) \xrightarrow{\delta_1} \cdots \\ \cdots \xrightarrow{\delta_{n-1}} H^n(G, A) \longrightarrow H^n(G, B) \longrightarrow H^n(G, C) \xrightarrow{\delta_n} H^{n+1}(G, A) \longrightarrow \cdots \end{aligned}$$

of abelian groups.

Among many other uses of the long exact sequence in Theorem 21 is a technique called *dimension shifting* which makes it possible to analyze the cohomology group $H^{n+1}(G, A)$ of dimension $n + 1$ for A by instead considering a cohomology group of dimension n for a different G -module. The technique is based on finding a G -module almost all of whose cohomology groups are zero. Such modules are given a name:

Definition. A G -module M is called *cohomologically trivial for G* if $H^n(G, M) = 0$ for all $n \geq 1$.

Corollary 22. (*Dimension Shifting*) Suppose $0 \rightarrow A \rightarrow M \rightarrow C \rightarrow 0$ is a short exact sequence of G -modules and that M is cohomologically trivial for G . Then there is an exact sequence

$$0 \longrightarrow A^G \longrightarrow M^G \longrightarrow C^G \longrightarrow H^1(G, A) \longrightarrow 0$$

and

$$H^{n+1}(G, A) \cong H^n(G, C) \text{ for all } n \geq 1.$$

Proof: Since M is cohomologically trivial for G , the portion

$$H^n(G, M) \longrightarrow H^n(G, C) \longrightarrow H^{n+1}(G, A) \longrightarrow H^{n+1}(G, M)$$

of the long exact sequence in Theorem 21 reduces to

$$0 \longrightarrow H^n(G, C) \longrightarrow H^{n+1}(G, A) \longrightarrow 0$$

which shows that $H^n(G, C) \cong H^{n+1}(G, A)$ for $n \geq 1$. Similarly, the first portion of the long exact sequence in Theorem 21 gives the first statement in the corollary.

We now indicate a natural construction that produces a G -module given a module over a subgroup H of G . When $H = 1$ is the trivial group this construction produces a cohomologically trivial module M and an exact sequence as in Corollary 22 for any G -module A .

Definition. If H is a subgroup of G and A is an H -module, define the *induced G -module* $M_H^G(A)$ to be $\text{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, A)$. In other words, $M_H^G(A)$ is the set of maps f from G to A satisfying $f(hx) = hf(x)$ for every $x \in G$ and $h \in H$.

The action of an element $g \in G$ on $f \in M_H^G(A)$ is given by $(g \cdot f)(x) = f(xg)$ for $x \in G$ (cf. Exercise 10 in Section 10.5).

Recall that if H is a subgroup of G and A is an H -module, then the module $\mathbb{Z}G \otimes_{\mathbb{Z}H} A$ obtained by extension of scalars from $\mathbb{Z}H$ to $\mathbb{Z}G$ is a G -module. For a finite group G , or more generally if H has finite index in G , we have $M_H^G(A) \cong \mathbb{Z}G \otimes_{\mathbb{Z}H} A$ (cf. Exercise 10). When G is infinite this need no longer be the case (cf. Exercise 11). The module $\mathbb{Z}G \otimes_{\mathbb{Z}H} A$ is sometimes called the *induced G -module* and the module $M_H^G(A)$ is sometimes referred to as the *coinduced G -module*. For finite groups, associativity of the tensor product shows that $M_H^G(M_K^H(A)) = M_K^G(A)$ for subgroups $K \leq H \leq G$, and the same result holds in general (this follows from the definition using Exercise 7).

Examples

- (1) If H is a subgroup of G and $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is a short exact sequence of H -modules then $0 \rightarrow M_H^G(A) \rightarrow M_H^G(B) \rightarrow M_H^G(C) \rightarrow 0$ is a short exact sequence of G -modules, since $M_H^G(A) \cong \mathbb{Z}G \otimes_{\mathbb{Z}H} A$ and $\mathbb{Z}G$ is free, hence flat, over $\mathbb{Z}H$.
- (2) When G is finite and A is the trivial H -module \mathbb{Z} , the module $M_H^G(\mathbb{Z})$ is a free \mathbb{Z} -module of rank $m = |G : H|$. There is a basis b_1, \dots, b_m such that G permutes these basis elements in the same way it permutes the left cosets of H in G by left multiplication, i.e., if we let $b_i \leftrightarrow g_i H$ then $gb_i = b_j$ if and only if $gg_i H = g_j H$. The module $M_H^G(\mathbb{Z})$ is the *permutation module* over \mathbb{Z} for G with stabilizer H . A special case of interest is when $G = S_m$ and $H = S_{m-1}$ where S_m permutes $\{1, 2, \dots, m\}$ as usual. Permutation modules and induced modules over fields are studied in Part VI.
- (3) Any abelian group A is an H -module when $H = 1$ is the trivial group. The corresponding induced G -module $M_1^G(A)$ is just the collection of all maps f from G into A . For $g \in G$ the map $g \cdot f \in M_1^G(A)$ satisfies $(g \cdot f)(x) = f(xg)$ for $x \in G$.
- (4) Suppose A is a G -module. Then there is a natural map

$$\varphi : A \longrightarrow M_1^G(A)$$

from A into the induced G -module $M_1^G(A)$ in the previous example defined by mapping $a \in A$ to the function f_a with $f_a(x) = xa$ for all $x \in G$. It is clear that φ is a group homomorphism, and $f_{ga}(x) = x(ga) = (xg)a = f_a(xg) = (g \cdot f_a)(x)$ shows that φ is a G -module homomorphism as well. Since $f_a(1) = a$, it follows that f_a is the zero function on G if and only if $a = 0$ in A , so that φ is an injection. Hence we may identify A as a G -submodule of the induced module $M_1^G(A)$.

- (5) More generally, if A is a G -module and H is any subgroup of G then the function $f_a(x)$ in the previous example is an element in the subgroup $M_H^G(A)$ since we have $f_a(hx) = (hx)(a) = h(xa) = hf_a(x)$ for all $h \in H$. The associated map from A to $M_H^G(A)$ is an injective G -module homomorphism.
- (6) The fixed points $(M_H^G(A))^G$ are maps f from G to A with $gf = f$ for all $g \in G$, i.e., with $(gf)(x) = f(x)$ for all $g, x \in G$. By definition of the G -action on $M_H^G(A)$, this is the equation $f(xg) = f(x)$ for all $g, x \in G$. Taking $x = 1$ shows that f is constant on all of G : $f(g) = f(1) = a \in A$. The constant function $f = a$ is an element of $M_H^G(A)$ if and only if $a = f(hx) = hf(x) = ha$ for all $h \in H$, so $(M_H^G(A))^G \cong A^H$.

An element $f_a(x)$ in the previous example is contained in the subgroup $(M_H^G(A))^G$ if and only if xa is constant for $x \in G$, i.e., if and only if $a \in A^G$.

One of the important properties of the G -module $M_H^G(A)$ induced from the H -module A is that its cohomology with respect to G is the same as the cohomology of A with respect to H :

Proposition 23. (*Shapiro's Lemma*) For any subgroup H of G and any H -module A we have $H^n(G, M_H^G(A)) \cong H^n(H, A)$ for $n \geq 0$.

Proof: Let $\cdots \rightarrow P_n \rightarrow \cdots \rightarrow P_0 \rightarrow \mathbb{Z} \rightarrow 0$ be a resolution of \mathbb{Z} by projective G -modules (for example, the standard resolution). The cohomology groups $H^n(G, M_H^G(A))$ are computed by taking homomorphisms from this resolution into $M_H^G(A) = \text{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, A)$. Since $\mathbb{Z}G$ is a free $\mathbb{Z}H$ -module it follows that this G -module resolution is also a resolution of \mathbb{Z} by projective H -modules, hence by taking homomorphisms into A the same resolution may be used to compute the cohomology groups $H^n(H, A)$. To see that these two collections of cohomology groups are isomorphic, we use the natural isomorphism of abelian groups

$$\Phi : \text{Hom}_{\mathbb{Z}G}(P_n, \text{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, A)) \cong \text{Hom}_{\mathbb{Z}H}(P_n, A)$$

given by $\Phi(f)(p) = f(p)(1)$, for all $f \in \text{Hom}_{\mathbb{Z}G}(P_n, \text{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, A))$ and $p \in P_n$. The inverse isomorphism is defined by taking $\Psi(f')(p)$ to be the map from $\mathbb{Z}G$ to A that takes $g \in G$ to the element $f'(gp)$ in A for all $f' \in \text{Hom}_{\mathbb{Z}H}(P_n, A)$ and $p \in P_n$, i.e., $(\Psi(f'))(p)(g) = f'(gp)$. Note this is well defined because P_n is a G -module. (These maps are a special case of an Adjoint Associativity Theorem, cf. Exercise 7.) Since these isomorphisms commute with the cochain maps, they induce isomorphisms on the corresponding cohomology groups, i.e., $H^n(G, M_H^G(A)) \cong H^n(H, A)$, as required.

Corollary 24. For any G -module A the module $M_1^G(A)$ is cohomologically trivial for G , i.e., $H^n(G, M_1^G(A)) = 0$ for all $n \geq 1$.

Proof: This follows immediately from the proposition applied with $H = 1$ together with the computation of the cohomology of the trivial group in Example 2 preceding Proposition 20.

By the corollary, the fourth example above gives us a short exact sequence of G -modules

$$0 \longrightarrow A \xrightarrow{\varphi} M \longrightarrow C \longrightarrow 0$$

where $M = M_1^G(A)$ is cohomologically trivial for G and where C is the quotient of $M_1^G(A)$ by the image of A . The dimension shifting result in Corollary 22 then becomes:

Corollary 25. For any G -module A we have $H^{n+1}(G, A) \cong H^n(G, M_1^G(A)/A)$ for all $n \geq 1$.

We next consider several important maps relating various cohomology groups. Some applications of the use of these homomorphisms appear in the following two sections.

In general, suppose we have two groups G and G' and that A is a G -module and A' is a G' -module. If $\varphi : G' \rightarrow G$ is a group homomorphism then A becomes a G' -module by defining $g' \cdot a = \varphi(g')a$ for $g' \in G'$ and $a \in A$. If now $\psi : A \rightarrow A'$ is a homomorphism of abelian groups then we consider whether ψ is a G' -module homomorphism:

Definition. Suppose A is a G -module and A' is a G' -module. The group homomorphisms $\varphi : G' \rightarrow G$ and $\psi : A \rightarrow A'$ are said to be *compatible* if ψ is a G' -module homomorphism when A is made into a G' -module by means of φ , i.e., if $\psi(\varphi(g')a) = g'\psi(a)$ for all $g' \in G'$ and $a \in A$.

The point of compatible homomorphisms is that they induce group homomorphisms on associated cohomology groups, as follows.

If $\varphi : G' \rightarrow G$ and $\psi : A \rightarrow A'$ are homomorphisms, then φ induces a homomorphism $\varphi^n : (G')^n \rightarrow G^n$, and so a homomorphism from $C^n(G, A)$ to $C^n(G', A)$ that maps f to $f \circ \varphi^n$. The map ψ induces a homomorphism from $C^n(G', A)$ to $C^n(G', A')$ that maps f to $\psi \circ f$. Taken together we obtain an induced homomorphism

$$\begin{aligned}\lambda_n : C^n(G, A) &\longrightarrow C^n(G', A') \\ f &\longmapsto \psi \circ f \circ \varphi^n.\end{aligned}$$

If in addition φ and ψ are *compatible* homomorphisms, then it is easy to check that the induced maps λ_n commute with the coboundary operator:

$$\lambda_{n+1} \circ d_n = d_n \circ \lambda_n$$

for all $n \geq 0$. It follows that λ_n maps cocycles to cocycles and coboundaries to coboundaries, hence induces a group homomorphism on cohomology:

$$\lambda_n : H^n(G, A) \longrightarrow H^n(G', A')$$

for $n \geq 0$.

We consider several instances of such maps:

Examples

- (1) Suppose $G = G'$ and φ is the identity map. Then to say that the group homomorphism $\psi : A \rightarrow A'$ is compatible with φ is simply the statement that ψ is a G -module homomorphism. Hence any G -module homomorphism from A to A' induces a group homomorphism

$$H^n(G, A) \longrightarrow H^n(G, A') \quad \text{for } n \geq 0.$$

In particular, if $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is a short exact sequence of G -modules we obtain induced homomorphisms from $H^n(G, A)$ to $H^n(G, B)$ and from $H^n(G, B)$ to $H^n(G, C)$ for $n \geq 0$. These are simply the homomorphisms in the long exact sequence of Theorem 21.

- (2) (*The Restriction Homomorphism*) If A is a G -module, then A is also an H -module for any subgroup H of G . The inclusion map $\varphi : H \rightarrow G$ of H into G and the identity