# 5 CHAPTER Trigonometry

## 5.1 Angle Measure

The word *trigonometry* comes from the Greek for "triangle measurement." More specifically, it means the study of relationships between the size of sides and the size of angles in triangles. Euclid says very little about this. He has theorems about equal angles and the sum of angles, and one angle being twice another or simply larger than another, but he never actually *measures* angles. He does not represent angles by numbers, nor does he represent them by lengths or areas. This suggests that angle measure may be a deep concept, perhaps beyond the scope of traditional geometry. The Greeks had some inkling of this when they tried unsuccessfully to construct the area bounded by the unit circle, the problem they called *squaring the circle*. In modern terms, squaring the circle amounts to constructing the number $\pi$, which is both the area of the unit circle and half its circumference. It is also the natural measure of the straight angle, formed by two right angles, so constructing $\pi$ is in fact a fundamental question about the measurement of angles.

Probably the only way to understand $\pi$ well enough to know whether it is constructible is to use advanced calculus, which is beyond the scope of this book. However, we can understand angle



J. Stillwell, *Numbers and Geometry*
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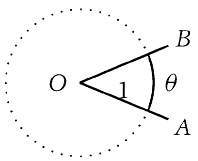


**FIGURE 5.1** Representing an angle by an arc.

measure with less; the concept can be made clear with the help of analytic geometry and the theory of real numbers developed in Chapter 3. We shall also find that this is enough to capture the elusive number $\pi$ in the form of an infinite sum or product.

Let us begin with Euclid's idea of angle, as a pair of rays $OA$ and $OB$ we call angle $AOB$. The implication of this notation is that $OA$ is the first ray in the angle and $OB$ is the second. But there is still the problem of explaining which way to travel from $OA$ to $OB$—clockwise or counterclockwise (recall the discussion of orientation in Chapter 3). The easiest way out of this problem and several others, is to draw a unit circle centered on $O$ and to choose one of the arcs between the rays to mark the intended angle (Figure 5.1, with the chosen arc drawn heavily).

The *measure* of the angle $AOB$ can then be defined as the *length* $\theta$ of the arc $AB$. We have not yet defined length of arcs, admittedly, but this is not hard. The length of any arc between points $P$ and $Q$ on the circle may be defined as the least upper bound of the length of polygons $P_1P_2\ldots P_{n-1}P_n$ joining points $P_1, P_2, \ldots P_{n-1}, P_n$ that lie in that order on the arc between $P$ and $Q$ (Figure 5.2). As implied in the exercises in Section 3.7, this least upper bound exists because a polygon joining points on the circle is shorter than a square enclosing the circle, so there is an upper bound to the set of polygonal

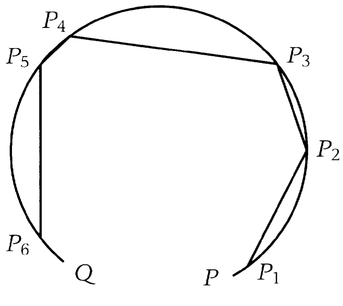


**FIGURE 5.2** Arc and polygon.

lengths and hence a least upper bound by the completeness of the real numbers.

Finally, we define the number $\pi$ to be the length of a semicircle of radius 1. It will be some time before we can give a precise value of $\pi$, but in the meantime we need to know what we are talking about when we say that the length of the whole unit circle is $2\pi$ and the like.

# Exercises

In the early history of $\pi$, some very rough estimates were used. For example, there is a verse in the Bible (Kings 7:23) about a "molten sea, ten cubits from the one brim to the other: it was round all about ... and a line of thirty cubits did compass it round about." If the sea was circular, this assumes $\pi = 3$. This value is easily seen to be too small.

5.1.1. By inscribing a regular hexagon in a circle, show that $\pi > 3$.

The idea of approximating the circle by polygons dominated the study of $\pi$ from the time of Archimedes (around 250 B.C.) until about 1500 A.D. Polygons inside and outside the circle were used to narrow the interval in which $\pi$ was known to lie.

5.1.2. Use the squares in Figure 5.3 to show that $2\sqrt{2} < \pi < 4$ and the octagons in Figure 5.3 to show that $4\sqrt{2-\sqrt{2}} < \pi < 8(\sqrt{2}-1)$. (The latter approximations give $3.06 < \pi < 3.32$.)

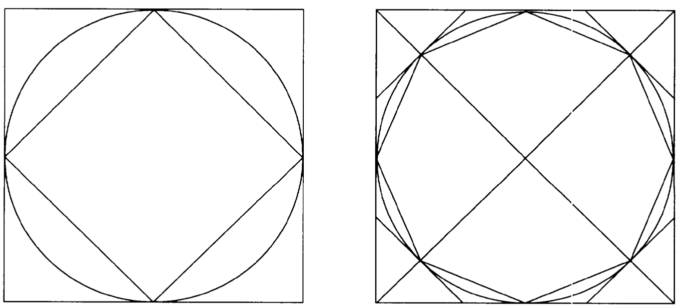

**FIGURE 5.3** Approximating the circle by squares and octagons.

The first accurate bounds on $\pi$ were found by Archimedes, who used inner and outer polygons with 96 sides to show that

$$3\frac{10}{71} < \pi < 3\frac{1}{7}.$$

This neat improvement on the school value of $3\frac{1}{7}$ gives the decimal estimates $3.140 < \pi < 3.143$, and hence gives $\pi$ correct to two decimal places.

Outer polygons are not needed to *define* the length of the circle, except to ensure that there is an upper bound to the length of polygons inside the circle, because the least upper bound of the lengths of inner polygons exists by the completeness of the real numbers. However, they are useful for finding how close a given inner polygon comes to the circle; it is closer to the circle than to any outer polygon.

5.1.3. Use the triangle inequality to show that *any* polygon inside the circle is shorter than every polygon outside the circle.

Outer polygons also assure us that it is sensible to define the length of the circle as the least upper bound of lengths of inner polygons, because we can show that the difference in length between inner and outer polygons can be made as small as we please. This means it is equivalent to use the (equally natural) definition that the length of the circle is the greatest lower bound of the lengths of outer polygons.

Figure 5.4 helps to explain why the difference in length can be made as small as we please. It shows a sector of the unit circle, the half-side $y$ of an inner polygon, and the half-side $y + \varepsilon$ of an outer polygon.

5.1.4. Show that $\frac{y}{y+\varepsilon} = 1 - \delta$. Hence conclude that, by suitable choice of $\delta$, we can make the ratio of lengths of an inner and outer polygon as close to 1 as we please. This implies that the difference between their lengths can be made as close to 0 as we please. Why?

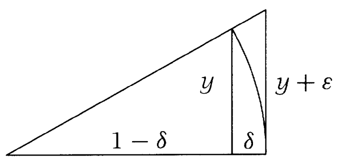


**FIGURE 5.4** Sides of inner and outer polygons.