(a) R *is row-reduced;*

(b) *every row of* R *which has all its entries* 0 *occurs below every row which has a non-zero entry;*

(c) *if rows* 1, . . . , r *are the non-zero rows of* R, *and if the leading non-zero entry of row* i *occurs in column* k_i, $i = 1, \ldots, r$, *then* $k_1 < k_2 < \cdots < k_r$.

One can also describe an $m \times n$ row-reduced echelon matrix R as follows. Either every entry in R is 0, or there exists a positive integer r, $1 \leq r \leq m$, and r positive integers $k_1, \ldots, k_r$ with $1 \leq k_i \leq n$ and

(a) $R_{ij} = 0$ for $i > r$, and $R_{ij} = 0$ if $j < k_i$.

(b) $R_{ik_j} = \delta_{ij}$, $1 \leq i \leq r$, $1 \leq j \leq r$.

(c) $k_1 < \cdots < k_r$.

EXAMPLE 8. Two examples of row-reduced echelon matrices are the $n \times n$ identity matrix, and the $m \times n$ **zero matrix** $0^{m,n}$, in which all entries are 0. The reader should have no difficulty in making other examples, but we should like to give one non-trivial one:

$$\begin{bmatrix} 0 & 1 & -3 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Theorem 5. *Every* $m \times n$ *matrix* A *is row-equivalent to a row-reduced echelon matrix.*

Proof. We know that A is row-equivalent to a row-reduced matrix. All that we need observe is that by performing a finite number of row interchanges on a row-reduced matrix we can bring it to row-reduced echelon form. ∎

In Examples 5 and 6, we saw the significance of row-reduced matrices in solving homogeneous systems of linear equations. Let us now discuss briefly the system $RX = 0$, when R is a row-reduced echelon matrix. Let rows $1, \ldots, r$ be the non-zero rows of R, and suppose that the leading non-zero entry of row i occurs in column k_i. The system $RX = 0$ then consists of r non-trivial equations. Also the unknown x_{k_i} will occur (with non-zero coefficient) only in the ith equation. If we let $u_1, \ldots, u_{n-r}$ denote the $(n - r)$ unknowns which are different from $x_{k_1}, \ldots, x_{k_r}$, then the r non-trivial equations in $RX = 0$ are of the form

$$\begin{aligned} x_{k_1} + \sum_{j=1}^{n-r} C_{1j}u_j &= 0 \\ \vdots \qquad & \quad \vdots \\ x_{k_r} + \sum_{j=1}^{n-r} C_{rj}u_j &= 0. \end{aligned} \tag{1-3}$$

All the solutions to the system of equations $RX = 0$ are obtained by assigning any values whatsoever to $u_1, \ldots, u_{n-r}$ and then computing the corresponding values of $x_{k_1}, \ldots, x_{k_r}$ from (1-3). For example, if R is the matrix displayed in Example 8, then $r = 2$, $k_1 = 2$, $k_2 = 4$, and the two non-trivial equations in the system $RX = 0$ are

$$\begin{aligned} x_2 - 3x_3 \quad + \tfrac{1}{2}x_5 &= 0 \quad \text{or} \quad x_2 = 3x_3 - \tfrac{1}{2}x_5 \\ x_4 + 2x_5 &= 0 \quad \text{or} \quad x_4 = -2x_5. \end{aligned}$$

So we may assign any values to x_1, x_3, and x_5, say $x_1 = a$, $x_3 = b$, $x_5 = c$, and obtain the solution $(a, 3b - \frac{1}{2}c, b, -2c, c)$.

Let us observe one thing more in connection with the system of equations $RX = 0$. If the number r of non-zero rows in R is less than n, then the system $RX = 0$ has a non-trivial solution, that is, a solution $(x_1, \ldots, x_n)$ in which not every x_j is 0. For, since $r < n$, we can choose some x_j which is not among the r unknowns $x_{k_1}, \ldots, x_{k_r}$, and we can then construct a solution as above in which this x_j is 1. This observation leads us to one of the most fundamental facts concerning systems of homogeneous linear equations.

Theorem 6. *If A is an m $\times$ n matrix and m $<$ n, then the homogeneous system of linear equations AX $=$ 0 has a non-trivial solution.*

Proof. Let R be a row-reduced echelon matrix which is row-equivalent to A. Then the systems $AX = 0$ and $RX = 0$ have the same solutions by Theorem 3. If r is the number of non-zero rows in R, then certainly $r \leq m$, and since $m < n$, we have $r < n$. It follows immediately from our remarks above that $AX = 0$ has a non-trivial solution. ∎

Theorem 7. *If A is an n $\times$ n (square) matrix, then A is row-equivalent to the n $\times$ n identity matrix if and only if the system of equations AX $=$ 0 has only the trivial solution.*

Proof. If A is row-equivalent to I, then $AX = 0$ and $IX = 0$ have the same solutions. Conversely, suppose $AX = 0$ has only the trivial solution $X = 0$. Let R be an $n \times n$ row-reduced echelon matrix which is row-equivalent to A, and let r be the number of non-zero rows of R. Then $RX = 0$ has no non-trivial solution. Thus $r \geq n$. But since R has n rows, certainly $r \leq n$, and we have $r = n$. Since this means that R actually has a leading non-zero entry of 1 in each of its n rows, and since these 1's occur each in a different one of the n columns, R must be the $n \times n$ identity matrix. ∎

Let us now ask what elementary row operations do toward solving a system of linear equations $AX = Y$ which is not homogeneous. At the outset, one must observe one basic difference between this and the homogeneous case, namely, that while the homogeneous system always has the trivial solution $x_1 = \cdots = x_n = 0$, an inhomogeneous system need have no solution at all.

We form the **augmented matrix** A' of the system $AX = Y$. This is the $m \times (n + 1)$ matrix whose first n columns are the columns of A and whose last column is Y. More precisely,

$$\begin{aligned} A'_{ij} &= A_{ij}, \quad \text{if} \quad j \le n \\ A'_{i(n+1)} &= y_i. \end{aligned}$$

Suppose we perform a sequence of elementary row operations on A, arriving at a row-reduced echelon matrix R. If we perform this same sequence of row operations on the augmented matrix A', we will arrive at a matrix R' whose first n columns are the columns of R and whose last column contains certain scalars $z_1, \ldots, z_m$. The scalars z_i are the entries of the $m \times 1$ matrix

$$Z = \begin{bmatrix} z_1 \\ \vdots \\ z_m \end{bmatrix}$$

which results from applying the sequence of row operations to the matrix Y. It should be clear to the reader that, just as in the proof of Theorem 3, the systems $AX = Y$ and $RX = Z$ are equivalent and hence have the same solutions. It is very easy to determine whether the system $RX = Z$ has any solutions and to determine all the solutions if any exist. For, if R has r non-zero rows, with the leading non-zero entry of row i occurring in column k_i, $i = 1, \ldots, r$, then the first r equations of $RX = Z$ effectively express $x_{k_1}, \ldots, x_{k_r}$ in terms of the $(n - r)$ remaining x_j and the scalars $z_1, \ldots, z_r$. The last $(m - r)$ equations are

$$\begin{aligned} 0 &= z_{r+1} \\ \vdots & \quad\; \vdots \\ 0 &= z_m \end{aligned}$$

and accordingly the condition for the system to have a solution is $z_i = 0$ for $i > r$. If this condition is satisfied, all solutions to the system are found just as in the homogeneous case, by assigning arbitrary values to $(n - r)$ of the x_j and then computing x_{k_i} from the ith equation.

EXAMPLE 9. Let F be the field of rational numbers and

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 1 \\ 0 & 5 & -1 \end{bmatrix}$$

and suppose that we wish to solve the system $AX = Y$ for some y_1, y_2, and y_3. Let us perform a sequence of row operations on the augmented matrix A' which row-reduces A:

$$\begin{bmatrix} 1 & -2 & 1 & y_1 \\ 2 & 1 & 1 & y_2 \\ 0 & 5 & -1 & y_3 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & -2 & 1 & y_1 \\ 0 & 5 & -1 & (y_2 - 2y_1) \\ 0 & 5 & -1 & y_3 \end{bmatrix} \xrightarrow{(2)}$$

$$\begin{bmatrix} 1 & -2 & 1 & y_1 \\ 0 & 5 & -1 & (y_2 - 2y_1) \\ 0 & 0 & 0 & (y_3 - y_2 + 2y_1) \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & -2 & 1 & y_1 \\ 0 & 1 & -\frac{1}{5} & \frac{1}{5}(y_2 - 2y_1) \\ 0 & 0 & 0 & (y_3 - y_2 + 2y_1) \end{bmatrix} \xrightarrow{(2)}$$

$$\begin{bmatrix} 1 & 0 & \frac{3}{5} & \frac{1}{5}(y_1 + 2y_2) \\ 0 & 1 & -\frac{1}{5} & \frac{1}{5}(y_2 - 2y_1) \\ 0 & 0 & 0 & (y_3 - y_2 + 2y_1) \end{bmatrix}.$$

The condition that the system $AX = Y$ have a solution is thus

$$2y_1 - y_2 + y_3 = 0$$

and if the given scalars y_i satisfy this condition, all solutions are obtained by assigning a value c to x_3 and then computing

$$\begin{aligned} x_1 &= -\tfrac{3}{5}c + \tfrac{1}{5}(y_1 + 2y_2) \\ x_2 &= \tfrac{1}{5}c + \tfrac{1}{5}(y_2 - 2y_1). \end{aligned}$$

Let us observe one final thing about the system $AX = Y$. Suppose the entries of the matrix A and the scalars $y_1, \ldots, y_m$ happen to lie in a subfield F_1 of the field F. If the system of equations $AX = Y$ has a solution with $x_1, \ldots, x_n$ in F, it has a solution with $x_1, \ldots, x_n$ in F_1. For, over either field, the condition for the system to have a solution is that certain relations hold between $y_1, \ldots, y_m$ in F_1 (the relations $z_i = 0$ for $i > r$, above). For example, if $AX = Y$ is a system of linear equations in which the scalars y_k and A_{ij} are real numbers, and if there is a solution in which $x_1, \ldots, x_n$ are complex numbers, then there is a solution with $x_1, \ldots, x_n$ real numbers.

Exercises

1. Find all solutions to the following system of equations by row-reducing the coefficient matrix:

$$\begin{aligned} \tfrac{1}{3}x_1 + 2x_2 - 6x_3 &= 0 \\ -4x_1 + 5x_3 &= 0 \\ -3x_1 + 6x_2 - 13x_3 &= 0 \\ -\tfrac{7}{3}x_1 + 2x_2 - \tfrac{8}{3}x_3 &= 0 \end{aligned}$$

2. Find a row-reduced echelon matrix which is row-equivalent to

$$A = \begin{bmatrix} 1 & -i \\ 2 & 2 \\ i & 1 + i \end{bmatrix}.$$

What are the solutions of $AX = 0$?

3. Describe explicitly all 2×2 row-reduced echelon matrices.

4. Consider the system of equations

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 1 \\ 2x_1 + 2x_3 &= 1 \\ x_1 - 3x_2 + 4x_3 &= 2. \end{aligned}$$

Does this system have a solution? If so, describe explicitly all solutions.

5. Give an example of a system of two linear equations in two unknowns which has no solution.

6. Show that the system

$$\begin{aligned} x_1 - 2x_2 + x_3 + 2x_4 &= 1 \\ x_1 + x_2 - x_3 + x_4 &= 2 \\ x_1 + 7x_2 - 5x_3 - x_4 &= 3 \end{aligned}$$

has no solution.

7. Find all solutions of

$$\begin{aligned} 2x_1 - 3x_2 - 7x_3 + 5x_4 + 2x_5 &= -2 \\ x_1 - 2x_2 - 4x_3 + 3x_4 + x_5 &= -2 \\ 2x_1 - 4x_3 + 2x_4 + x_5 &= 3 \\ x_1 - 5x_2 - 7x_3 + 6x_4 + 2x_5 &= -7. \end{aligned}$$

8. Let

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{bmatrix}.$$

For which triples (y_1, y_2, y_3) does the system $AX = Y$ have a solution?

9. Let

$$A = \begin{bmatrix} 3 & -6 & 2 & -1 \\ -2 & 4 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 1 & -2 & 1 & 0 \end{bmatrix}.$$

For which (y_1, y_2, y_3, y_4) does the system of equations $AX = Y$ have a solution?

10. Suppose R and R' are 2×3 row-reduced echelon matrices and that the systems $RX = 0$ and $R'X = 0$ have exactly the same solutions. Prove that $R = R'$.

1.5. Matrix Multiplication

It is apparent (or should be, at any rate) that the process of forming linear combinations of the rows of a matrix is a fundamental one. For this reason it is advantageous to introduce a systematic scheme for indicating just what operations are to be performed. More specifically, suppose B is an $n \times p$ matrix over a field F with rows $\beta_1, \dots, \beta_n$ and that from B we construct a matrix C with rows $\gamma_1, \dots, \gamma_m$ by forming certain linear combinations

$$\gamma_i = A_{i1}\beta_1 + A_{i2}\beta_2 + \cdots + A_{in}\beta_n. \tag{1-4}$$

The rows of C are determined by the mn scalars A_{ij} which are themselves the entries of an $m \times n$ matrix A. If (1-4) is expanded to

$$(C_{i1} \cdots C_{ip}) = \sum_{r=1}^{n} (A_{ir}B_{r1} \cdots A_{ir}B_{rp})$$

we see that the entries of C are given by

$$C_{ij} = \sum_{r=1}^{n} A_{ir}B_{rj}.$$

Definition. *Let* A *be an* m $\times$ n *matrix over the field* F *and let* B *be an* n $\times$ p *matrix over* F. *The* **product** AB *is the* m $\times$ p *matrix* C *whose* i, j *entry is*

$$\mathrm{C}_{ij} = \sum_{r=1}^{n} \mathrm{A}_{ir}\mathrm{B}_{rj}.$$

EXAMPLE 10. Here are some products of matrices with rational entries.

(a)
$$\begin{bmatrix} 5 & -1 & 2 \\ 0 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 5 & -1 & 2 \\ 15 & 4 & 8 \end{bmatrix}$$

Here

$$\begin{aligned} \gamma_1 &= (5 \quad -1 \quad 2) = 1 \cdot (5 \quad -1 \quad 2) + 0 \cdot (15 \quad 4 \quad 8) \\ \gamma_2 &= (0 \quad 7 \quad 2) = -3(5 \quad -1 \quad 2) + 1 \cdot (15 \quad 4 \quad 8) \end{aligned}$$

(b)
$$\begin{bmatrix} 0 & 6 & 1 \\ 9 & 12 & -8 \\ 12 & 62 & -3 \\ 3 & 8 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 3 \\ 5 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 6 & 1 \\ 3 & 8 & -2 \end{bmatrix}$$

Here

$$\begin{aligned} \gamma_2 &= (9 \quad 12 \quad -8) = -2(0 \quad 6 \quad 1) + 3(3 \quad 8 \quad -2) \\ \gamma_3 &= (12 \quad 62 \quad -3) = 5(0 \quad 6 \quad 1) + 4(3 \quad 8 \quad -2) \end{aligned}$$

(c)
$$\begin{bmatrix} 8 \\ 29 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$

(d)
$$\begin{bmatrix} -2 & -4 \\ 6 & 12 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} [2 \quad 4]$$

Here

$$\gamma_2 = (6 \quad 12) = 3(2 \quad 4)$$

(e)
$$[2 \quad 4] \begin{bmatrix} -1 \\ 3 \end{bmatrix} = [10]$$

(f)
$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -5 & 2 \\ 2 & 3 & 4 \\ 9 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(g)
$$\begin{bmatrix} 1 & -5 & 2 \\ 2 & 3 & 4 \\ 9 & -1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 9 & 0 \end{bmatrix}$$