Applying the mean value theorem to the integrand we have

$$\psi(x + he_k, t) - \psi(x, t) = h\, D_k\psi(z, t),$$

where z lies on the line segment joining x and x + he_k. Hence (10.17) becomes

$$\frac{\varphi(x + he_k) - \varphi(x)}{h} = \int_a^b D_k\psi(z, t)\, dt .$$

Therefore

$$\frac{\varphi(x + he_k) - \varphi(x)}{h} - \int_a^b D_k\psi(x, t)\, dt = \int_a^b \{D_k\psi(z, t) - D_k\psi(x, t)\}\, dt.$$

The last integral has absolute value not exceeding

$$\int_a^b |D_k\psi(z, t) - D_k\psi(x, t)|\, dt \le (b - a) \max |D_k\psi(z, t) - D_k\psi(x, t)|$$

where the maximum is taken for all z on the segment joining x to x + he_k, and all t in [a, b]. Now we invoke the uniform continuity of D_k on S $\times$ J (Theorem 9.10) to conclude that for every $\epsilon > 0$ there is a $\delta > 0$ such that this maximum is $< \epsilon/(b - a)$, whenever $0 < |h| < \delta$. Therefore

$$\left|\frac{\varphi(x + he_k) - \varphi(x)}{h} - \int_a^b D_k\psi(x, t)\, dt\right| < \epsilon \qquad \text{whenever } 0 < |h| < \delta.$$

This proves that $D_k\varphi(x)$ exists and equals $\int_a^b D_k\psi(x, t)\, dt$, as required.

Now we shall use this theorem to give the following necessary and sufficient condition for a vector field to be a gradient on a convex set.

THEOREM 10.9. *Let $f = (f_1, \ldots, f_n)$ be a continuously differentiable vector field on an open convex set S in $\mathbf{R}^n$. Then f is a gradient on S if and only if we have*

$$D_k f_j(x) = D_j f_k(x) \tag{10.18}$$

for each x in S and all $k, j = 1, 2, \ldots, n$.

Proof. We know, from Theorem 10.6, that the conditions are necessary. To prove sufficiency, we assume (10.18) and construct a potential φ on S.

For simplicity, we assume that S contains the origin. Let $\varphi(x)$ be the integral of f along the line segment from 0 to an arbitrary point x in S. As shown earlier in Equation (10.10) we have

$$\varphi(x) = \int_0^1 f(tx) \cdot x\, dt = \int_0^1 \psi(x, t)\, dt,$$

where $\psi(\boldsymbol{x}, t) = \boldsymbol{f}(t\boldsymbol{x}) \cdot \boldsymbol{x}$. There is a closed n-dimensional subinterval T of S with non-empty interior such that ψ satisfies the hypotheses of Theorem 10.8 in $T \times J$, where $J = [0, 1]$. Therefore the partial derivative $D_k\varphi(\boldsymbol{x})$ exists for each $k = 1, 2, \ldots, n$ and can be computed by differentiating under the integral sign,

$$D_k\varphi(\boldsymbol{x}) = \int_0^1 D_k\psi(\boldsymbol{x}, t)\, dt .$$

To compute $D_k\psi(\boldsymbol{x}, t)$, we differentiate the dot product $\boldsymbol{f}(t\boldsymbol{x}) \cdot \boldsymbol{x}$ and obtain

$$\begin{aligned} D_k\psi(\boldsymbol{x}, t) &= \boldsymbol{f}(t\boldsymbol{x}) \cdot D_k\boldsymbol{x} + D_k\{\boldsymbol{f}(t\boldsymbol{x})\} \cdot \boldsymbol{x} \\ &= \boldsymbol{f}(t\boldsymbol{x}) \cdot \boldsymbol{e}_k + t(D_k f_1(t\boldsymbol{x}), \ldots, D_k f_n(t\boldsymbol{x})) \cdot \boldsymbol{x} \\ &= f_k(t\boldsymbol{x}) + t(D_1 f_k(t\boldsymbol{x}), \ldots, D_n f_k(t\boldsymbol{x})) \cdot \boldsymbol{x} , \end{aligned}$$

where in the last step we used the relation (10.18). Therefore we have

$$D_k\psi(\boldsymbol{x}, t) = f_k(t\boldsymbol{x}) + t\nabla f_k(t\boldsymbol{x}) \cdot \boldsymbol{x} .$$

Now let $g(t) = f_k(t\boldsymbol{x})$. By the chain rule we have

$$g'(t) = \nabla f_k(t\boldsymbol{x}) \cdot \boldsymbol{x}$$

so the last formula for $D_k\psi(\boldsymbol{x}, t)$ becomes

$$D_k\psi(\boldsymbol{x}, t) = g(t) + tg'(t) .$$

Integrating this from 0 to 1 we find

(10.19) $$D_k\varphi(\boldsymbol{x}) = \int_0^1 D_k\psi(\boldsymbol{x}, t) = \int_0^1 g(t)\, dt + \int_0^1 tg'(t)\, dt .$$

We integrate by parts in the last integral to obtain

$$\int_0^1 tg'(t)\, dt = tg(t) \Big|_0^1 - \int_0^1 g(t)\, dt = g(1) - \int_0^1 g(t)\, dt.$$

Therefore (10.19) becomes

$$D_k\varphi(\boldsymbol{x}) = g(1) = f_k(\boldsymbol{x}) .$$

This shows that $\nabla\varphi = \boldsymbol{f}$ on S, which completes the proof.

11

MULTIPLE INTEGRALS

11.1 Introduction

Volume I discussed integrals $\int_a^b f(x)\,dx$, first for functions defined and bounded on finite intervals, and later for unbounded functions and infinite intervals. Chapter 10 of Volume II generalized the concept by introducing line integrals. This chapter extends the concept in yet another direction. The one-dimensional interval [a, b] is replaced by a two-dimensional set Q, called the *region of integration.* First we consider rectangular regions; later we consider more general regions with curvilinear boundaries. The integrand is a scalar field f defined and bounded on Q. The resulting integral is called a *double integral* and is denoted by the symbol

$$\iint_Q f, \qquad \text{or by} \qquad \iint_Q f(x, y)\,dx\,dy.$$

As in the one-dimensional case, the symbols dx and dy play no role in the definition of the double integral; however, they are useful in computations and transformations of integrals.

The program in this chapter consists of several stages. First we discuss the definition of the double integral. The approach here is analogous to the one-dimensional case treated in Volume I. The integral is defined first for step functions and then for more general functions. As in the one-dimensional case, the definition does not provide a useful procedure for actual computation of integrals. We shall find that most double integrals occurring in practice can be computed by repeated one-dimensional integration. We shall also find a connection between double integrals and line integrals. Applications of double integrals to problems involving area, volume, mass, center of mass, and related concepts are also given. Finally, we indicate how the concepts can be extended to n-space.

11.2 Partitions of rectangles. Step functions

Let Q be a rectangle, the Cartesian product of two closed intervals [a, b] and [c, d],

$$Q = [a, b] \times [c, d] = \{(x, y) \mid x \in [a, b] \quad \text{and} \quad y \in [c, d]\}.$$

An example is shown in Figure 11.1. Consider two partitions P_1 and P_2 of [a, b] and [c, d],

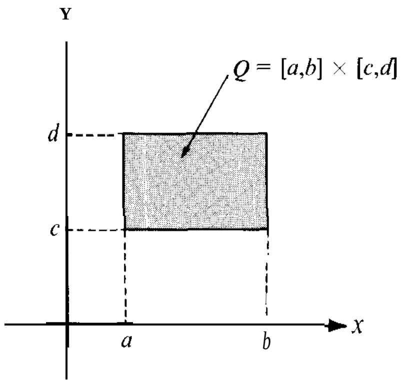

FIGURE 11.1 A rectangle Q, the Cartesian product of two intervals.

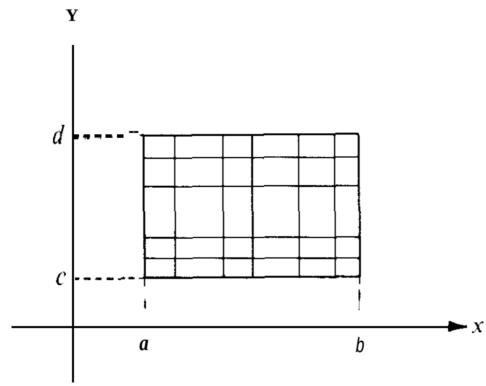

FIGURE 11.2 A partition of a rectangle Q.

respectively, say

$$P_1 = \{x_0, x_1, \ldots, x_{n-1}, x_n\} \quad \text{and} \quad P_2 = \{y_0, y_1, \ldots, y_{m-1}, y_m\},$$

where $x_0 = a$, $x_n = b$, $y_0 = c$, $y_m = d$. The Cartesian product $P_1 \times P_2$ is said to be a partition of Q. Since P_1 decomposes $[a, b]$ into n subintervals and P_2 decomposes $[c, d]$ into m subintervals, the partition $P = P_1 \times P_2$ decomposes Q into mn subrectangles. Figure 11.2 illustrates an example of a partition of Q into 30 subrectangles. A partition P' of Q is said to be finer than P if $P \subseteq P'$, that is, if every point in P is also in P'.

The Cartesian product of two open subintervals of P_1 and P_2 is a subrectangle with its edges missing. This is called an open subrectangle of P or of Q.

DEFINITION OF STEP FUNCTION. *A function f dejined on a rectangle Q is said to be a step function if a partition P of Q exists such that f is constant on each of the open subrectangles of P.*

The graph of an example is shown in Figure 11.3. Most of the graph consists of horizontal rectangular patches. A step function also has well-defined values at each of the boundary

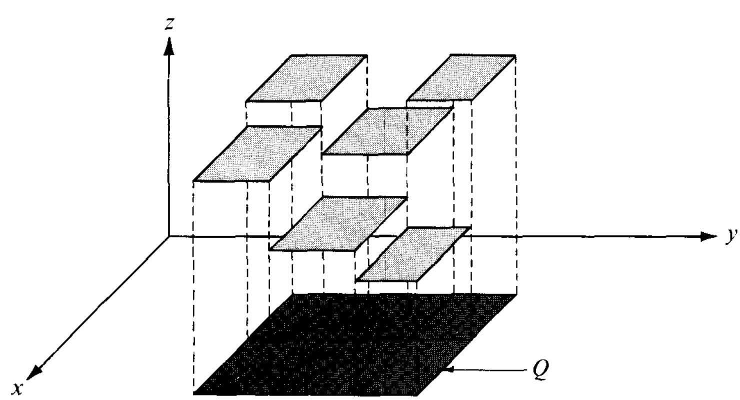

FIGURE 11.3 The graph of a step function defined on a rectangle Q.

points of the subrectangles, but the actual values at these points are not relevant to integration theory.

If f and g are two step functions defined on a given rectangle Q, then the linear combination $c_1 f + c_2 g$ is also a step function. In fact, if P and P' are partitions of Q such that f is constant on the open subrectangles of P and g is constant on the open subrectangles of P', then $c_1 f + c_2 g$ is constant on the open subrectangles of the union $P \cup P'$ (which we may call a common refinement of P and P'). Thus, the set of step functions defined on Q forms a linear space.

11.3 The double integral of a step function

Let $P = P_1 \times P_2$ be a partition of a rectangle Q into mn subrectangles and let f be a step function that is constant on the open subrectangles of Q. Let the subrectangle determined by $[x_{i-1}, x_i]$ and $[y_{j-1}, y_j]$ be denoted by Q_{ij} and let c_{ij} denote the constant value that f takes at the interior points of Q_{ij}. If f is positive, the volume of the rectangular box with base Q_{ij} and altitude c_{ij} is the product

$$c_{ij} \cdot (x_i - x_{i-1})(y_j - y_{j-1}).$$

For any step function f, positive or not, the sum of all these products is defined to be the double integral of f over Q. Thus, we have the following definition.

DEFINITION OF THE DOUBLE INTEGRAL OF A STEP FUNCTION. *Let f be a step function which takes the constant value c_{ij} on the open subrectangle $(x_{i-1}, x_i) \times (y_{j-1}, y_j)$ of a rectangle Q. The double integral of f over Q is defined by the formula*

$$\iint_Q f = \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} \cdot (x_i - x_{i-1})(y_j - y_{j-1}). \tag{11.1}$$

As in the one-dimensional case, the value of the integral does not change if the partition P is replaced by any finer partition P'. Thus, the value of the integral is independent of the choice of P so long as f is constant on the open subrectangles of Q.

For brevity, we sometimes write Δx_i instead of $(x_i - x_{i-1})$ and Δy_j instead of $(y_j - y_{j-1})$, and the sum in (11.1) becomes

$$\sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij}\, \Delta x_i\, \Delta y_j .$$

To remind ourselves how this sum is formed, we write the symbol for the integral as

$$\iint_Q f(x, y)\, dx\, dy .$$

This symbol is merely an alternative notation for $\iint_Q f$.

Note that if f is constant on the interior of Q, say $f(x, y) = k$ when $a < x < b$ and $c < y < d$, we have

$$\iint_Q f = k(b - a)(d - c), \tag{11.2}$$

regardless of the values off on the edges of Q. Since we have

$$b - a = \int_a^b dx \qquad \text{and} \qquad d - c = \int_c^d dy,$$

formula (11.2) can also be written as

$$\iint_Q f = \int_c^d \left[\int_a^b f(x, y)\, dx\right] dy = \int_a^b \left[\int_c^d f(x, y)\, dy\right] dx . \tag{11.3}$$

The integrals which appear on the right are one-dimensional integrals, and the formula is said to provide an evaluation of the double integral by *repeated* or *iterated* integration. In particular, when f is a step function of the type described above, we can write

$$\iint_{Q_{ij}} f = \int_{y_{j-1}}^{y_j} \left[\int_{x_{i-1}}^{x_i} f(x, y)\, dx\right] dy = \int_{x_{i-1}}^{x_i} \left[\int_{y_{j-1}}^{y_j} f(x, y)\, dy\right] dx.$$

Summing on i and j and using (11. 1), we find that (11.3) holds for step functions.

The following further properties of the double integral of a step function are generalizations of the corresponding one-dimensional theorems. They may be proved as direct consequences of the definition in (11.1) or by use of formula (11.3) and the companion theorems for one-dimensional integrals. In the following theorems the symbols s and t denote step functions defined on a rectangle Q. To avoid trivial special cases we assume that Q is a nondegenerate rectangle; in other words, that Q is not merely a single point or a line segment.

THEOREM 11.1. LINEARITY PROPERTY. *For every real c_1 and c_2 we have*

$$\iint_Q [c_1 s(x, y) + c_2 t(x, y)]\, dx\, dy = c_1 \iint_Q s(x, y)\, dx\, dy + c_2 \iint_Q t(x, y)\, dx\, dy .$$

THEOREM 11.2. ADDITIVE PROPERTY. *If Q is subdivided into two rectangles Q_1 and Q_2, then*

$$\iint_Q s(x, y)\, dx\, dy = \iint_{Q_1} s(x, y)\, dx\, dy + \iint_{Q_2} s(x, y)\, dx\, dy .$$

THEOREM 11.3 COMPARISON THEOREM. *If $s(x, y) \le t(x, y)$ for every (x, y) in Q, we have*

$$\iint_Q s(x, y)\, dx\, dy \le \iint_Q t(x, y)\, dx\, dy.$$

In particular, if $t(x, y) \ge 0$ for every (x, y) in Q, then

$$\iint_Q t(x, y)\, dx\, dy \ge 0 .$$

The proofs of these theorems are left as exercises.

11.4 The definition of the double integral of a function defined and bounded on a rectangle

Let f be a function that is defined and bounded on a rectangle Q; specifically, suppose that

$$|f(x, y)| \leq M \qquad \text{if} \quad (x, y) \in Q.$$

Then f may be surrounded from above and from below by two constant step functions s and t, where $s(x, y) = -M$ and $t(x, y) = M$ for all (x, y) in Q. Now consider *any* two step functions s and t, defined on Q, such that

$$s(x, y) \leq f(x, y) \leq t(x, y) \qquad \text{for every point } (x, y) \text{ in } Q. \tag{11.4}$$

DEFINITION OF THE INTEGRAL OF A BOUNDED FUNCTION OVER A RECTANGLE. *If there is one and only one number I such that*

$$\iint_Q s \leq I \leq \iint_Q t \tag{11.5}$$

for every pair of step functions satisfying the inequalities in (11.4), this number I is called the double integral of f over Q and is denoted by the symbol

$$\iint_Q f \qquad \text{or} \qquad \iint_Q f(x, y)\, dx\, dy.$$

When such an I exists the function f is said to be integrable on Q.

11.5 Upper and lower double integrals

The definition of the double integral is entirely analogous to the one-dimensional case. Upper and lower double integrals can also be defined as was done in the one-dimensional case.

Assume f is bounded on a rectangle Q and let s and t be step functions satisfying (11.4). We say that s is *below* f, and t is *above* f, and we write $s \leq f \leq t$. Let S denote the set of all numbers $\iint_Q s$ obtained as s runs through all step functions below f, and let T be the set of all numbers $\iint_Q t$ obtained as t runs through all step functions above f. Both sets S and T are nonempty since f is bounded. Also, $\iint_Q s \leq \iint_Q t$ if $s \leq f \leq t$, so every number in S is less than every number in T. Therefore S has a supremum, and T has an infimum, and they satisfy the inequalities

$$\iint_Q s \leq \sup S \leq \inf T \leq \iint_Q t$$

for all s and t satisfying $s \leq f \leq t$. This shows that both numbers $\sup S$ and $\inf T$ satisfy (11.5). Therefore, f is integrable on Q if and only if $\sup S = \inf T$, in which case we have

$$\iint_Q f = \sup S = \inf T.$$

The number sup S is called the *lower integral* of f and is denoted by $\underline{I}(f)$. The number inf T is called the *upper integral* of f and is denoted by $\bar{I}(f)$. Thus, we have

$$\underline{I}(f) = \sup\left\{\iint_Q s \mid s \le f\right\}, \qquad \bar{I}(f) = \inf\left\{\iint_Q t \mid f \le t\right\}.$$

The foregoing argument proves the following theorem.

THEOREM 11.4. *Every function f which is bounded on a rectangle Q has a lower integral $\underline{I}(f)$ and an upper integral $\bar{I}(f)$ satisfying the inequalities*

$$\iint_Q s \le \underline{I}(f) \le \bar{I}(f) \le \iint_Q t$$

for all step functions s and t with $s \le f \le t$. The function f is integrable on Q if and only if its upper and lower integrals are equal, in which case we have

$$\iint_Q f = \underline{I}(f) = \bar{I}(f).$$

Since the foregoing definitions are entirely analogous to the one-dimensional case, it is not surprising to learn that the linearity property, the additive property, and the comparison theorem as stated for step functions in Section 11.3, also hold for double integrals in general. The proofs of these statements are analogous to those in the one-dimensional case and will be omitted.

11.6 Evaluation of a double integral by repeated one-dimensional integration

In one-dimensional integration theory, the second fundamental theorem of calculus provides a practical method for calculating integrals. The next theorem accomplishes the same result in the two-dimensional theory; it enables us to evaluate certain double integrals by means of two successive one-dimensional integrations. The result is an extension of formula (11.3), which we have already proved for step functions.

THEOREM 11.5. *Let f be defined and bounded on a rectangle $Q = [a, b] \times [c, d]$, and assume that f is integrable on Q. For each fixed y in $[c, d]$ assume that the one-dimensional integral $\int_a^b f(x, y)\,dx$ exists, and denote its value by $A(y)$. If the integral $\int_c^d A(y)\,dy$ exists it is equal to the double integral $\iint_Q f$. In other words, we have the formula*

$$\iint_Q f(x, y)\,dx\,dy = \int_c^d \left[\int_a^b f(x, y)\,dx\right] dy. \tag{11.6}$$

Proof. Choose any two step functions s and t satisfying $s \le f \le t$ on Q. Integrating with respect to x over the interval $[a, b]$ we have

$$\int_a^b s(x, y)\,dx \le A(y) \le \int_a^b t(x, y)\,dx.$$

Since the integral $\int_c^d A(y)\, dy$ exists, we can integrate both these inequalities with respect to y over $[c, d]$ and use Equation (11.3) to obtain

$$\iint_Q s \le \int_c^d A(y)\, dy \le \iint_Q t .$$

Therefore $\int_c^d A(y)\, dy$ is a number which lies between $\iint_Q s$ and $\iint_Q t$ for all step functions s and t approximating f from below and from above, respectively. Since f is integrable on Q, the only number with this property is the double integral of f over Q. Therefore $\int_c^d A(y)\, dy = \iint_Q f$, which proves Equation (11.6).

Formula (11.6) is said to provide an evaluation of the double integral by repeated or iterated integration. The process is described by saying that first we integrate f with respect to x from a to b (holding y fixed), and then we integrate the result with respect to y from c to d. If we interchange the order of integration, we have a similar formula, namely,

$$\iint_Q f(x, y)\, dx\, dy = \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx , \tag{11.7}$$

which holds if we assume that $\int_c^d f(x, y)\, dy$ exists for each fixed x in $[a, b]$ and is integrable on $[a, b]$.

11.7 Geometric interpretation of the double integral as a volume

Theorem 11.5 has a simple geometric interpretation, illustrated in Figure 11.4. If f is nonnegative, the set S of points (x, y, z) in 3-space with (x, y) in Q and $0 \le z \le f(x, y)$ is

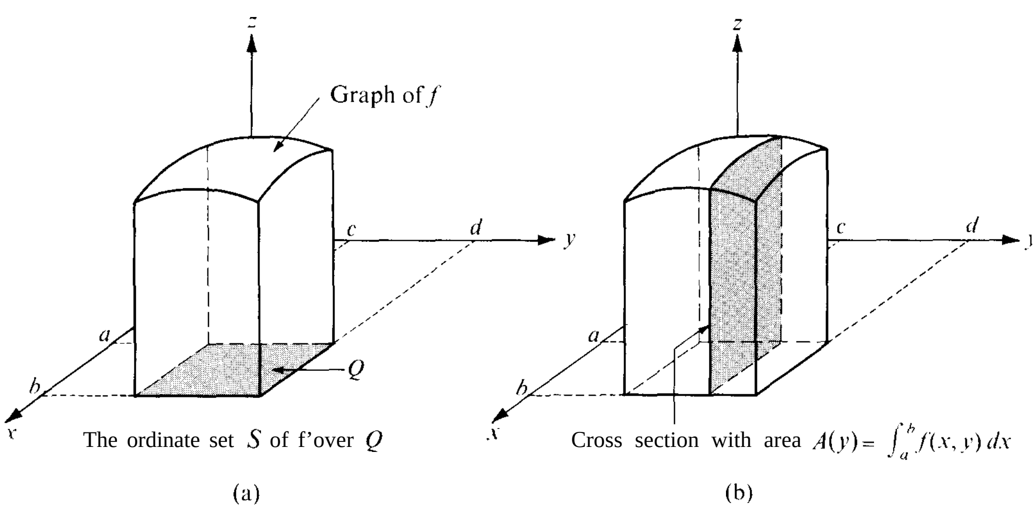

FIGURE 11.4 The volume of S is the integral of the cross-sectional area: $v(S) = \int_c^d A(y)\, dy$.

called the ***ordinate set off over*** Q. It consists of those points between the rectangle Q and the surface z = ***f*** (x, y) . (See Figure 11.4(a).) For each *y* in the interval [c, ***d],*** the integral

$$A(y) = \int_a^b f(x, y)\, dx$$

is the area of the cross section of S cut by a plane parallel to the xz-plane (the shaded region in Figure 11.4(b)). Since the cross-sectional area A(y) is integrable on [c, ***d],*** Theorem 2.7 of Volume I tells us that the integral $\int_c^d$ A(y) dy is equal to v(S), the volume of S. Thus, for nonnegative integrands, Theorem 11.5 shows that the volume of the ordinate set off over Q is equal to the double integral $\iint_Q f$.

Equation (11.7) gives another way of computing the volume of the ordinate set. This time we integrate the area of the cross sections cut by planes parallel to the yz-plane.

11.8 Worked examples

In this section we illustrate Theorem 11.5 with two numerical examples.

EXAMPLE 1. If Q = [-1, 1] x $[0, \pi/2]$, evaluate $\iint_Q$ (x sin *y* — ye”) *dx dy*, given that the integral exists. The region of integration is shown in Figure 11.5.

Solution. Integrating first with respect to x and calling the result A(y), we have

$$A(y) = \int_{-1}^{1} (x \sin y - ye”)\, dx = \left(\frac{x^2}{2} \sin y - y e\right)\Bigg|_{x=-1}^{x=1} = -ey + y/e .$$

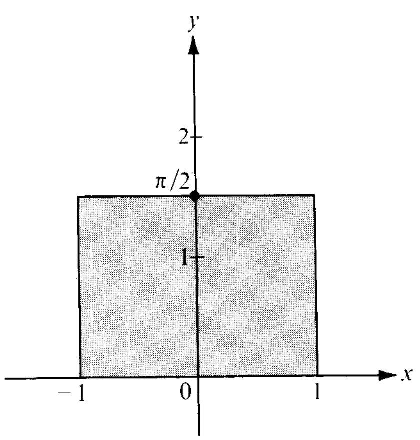

FIGURE 11.5 The region of integration for Example 1.

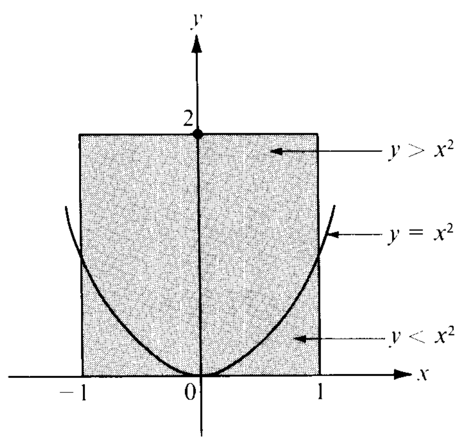

FIGURE 11.6 The region of integration for Example 2.