$$\text{(10-1)}\qquad \begin{aligned} f(c\alpha_1 + \alpha_2, \beta) &= cf(\alpha_1, \beta) + f(\alpha_2, \beta)\\ f(\alpha, c\beta_1 + \beta_2) &= cf(\alpha, \beta_1) + f(\alpha, \beta_2). \end{aligned}$$

If we let $V \times V$ denote the set of all ordered pairs of vectors in V, this definition can be rephrased as follows: A bilinear form on V is a function f from $V \times V$ into F which is linear as a function of either of its arguments when the other is fixed. The zero function from $V \times V$ into F is clearly a bilinear form. It is also true that any linear combination of bilinear forms on V is again a bilinear form. To prove this, it is sufficient to consider linear combinations of the type $cf + g$, where f and g are bilinear forms on V. The proof that $cf + g$ satisfies (10-1) is similar to many others we have given, and we shall thus omit it. All this may be summarized by saying that the set of all bilinear forms on V is a subspace of the space of all functions from $V \times V$ into F (Example 3, Chapter 2). We shall denote the space of bilinear forms on V by $L(V, V, F)$.

Example 1. Let V be a vector space over the field F and let L_1 and L_2 be linear functions on V. Define f by

$$f(\alpha, \beta) = L_1(\alpha)L_2(\beta).$$

If we fix β and regard f as a function of α, then we simply have a scalar multiple of the linear functional L_1. With α fixed, f is a scalar multiple of L_2. Thus it is clear that f is a bilinear form on V.

Example 2. Let m and n be positive integers and F a field. Let V be the vector space of all $m \times n$ matrices over F. Let A be a fixed $m \times m$ matrix over F. Define

$$f_A(X, Y) = \operatorname{tr}(X^tAY).$$

Then f_A is a bilinear form on V. For, if X, Y, and Z are $m \times n$ matrices over F,

$$\begin{aligned} f_A(cX + Z, Y) &= \operatorname{tr}[(cX + Z)^tAY]\\ &= \operatorname{tr}(cXtAY) + \operatorname{tr}(Z^tAY)\\ &= cf_A(X, Y) + f_A(Z, Y). \end{aligned}$$

Of course, we have used the fact that the transpose operation and the trace function are linear. It is even easier to show that f_A is linear as a function of its second argument. In the special case $n = 1$, the matrix X^tAY is 1×1, i.e., a scalar, and the bilinear form is simply

$$\begin{aligned} f_A(X, Y) &= X^tAY\\ &= \sum_i \sum_j A_{ij}x_iy_j. \end{aligned}$$

We shall presently show that every bilinear form on the space of $m \times 1$ matrices is of this type, i.e., is f_A for some $m \times m$ matrix A.

EXAMPLE 3. Let F be a field. Let us find all bilinear forms on the space F^2. Suppose f is such a bilinear form. If $\alpha = (x_1, x_2)$ and $\beta = (y_1, y_2)$ are vectors in F^2, then

$$\begin{aligned} f(\alpha, \beta) &= f(x_1\epsilon_1 + x_2\epsilon_2, \beta) \\ &= x_1 f(\epsilon_1, \beta) + x_2 f(\epsilon_2, \beta) \\ &= x_1 f(\epsilon_1, y_1\epsilon_1 + y_2\epsilon_2) + x_2 f(\epsilon_2, y_1\epsilon_1 + y_2\epsilon_2) \\ &= x_1y_1 f(\epsilon_1, \epsilon_1) + x_1y_2 f(\epsilon_1, \epsilon_2) + x_2y_1 f(\epsilon_2, \epsilon_1) + x_2y_2 f(\epsilon_2, \epsilon_2). \end{aligned}$$

Thus f is completely determined by the four scalars $A_{ij} = f(\epsilon_i, \epsilon_j)$ by

$$\begin{aligned} f(\alpha, \beta) &= A_{11}x_1y_1 + A_{12}x_1y_2 + A_{21}x_2y_1 + A_{22}x_2y_2 \\ &= \sum_{i,j} A_{ij}x_iy_j. \end{aligned}$$

If X and Y are the coordinate matrices of α and β, and if A is the 2×2 matrix with entries $A(i, j) = A_{ij} = f(\epsilon_i, \epsilon_j)$, then

$$f(\alpha, \beta) = X^tAY. \tag{10-2}$$

We observed in Example 2 that if A is any 2×2 matrix over F, then (10-2) defines a bilinear form on F^2. We see that the bilinear forms on F^2 are precisely those obtained from a 2×2 matrix as in (10-2).

The discussion in Example 3 can be generalized so as to describe all bilinear forms on a finite-dimensional vector space. Let V be a finite-dimensional vector space over the field F and let $\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}$ be an ordered basis for V. Suppose f is a bilinear form on V. If

$$\alpha = x_1\alpha_1 + \cdots + x_n\alpha_n \quad \text{and} \quad \beta = y_1\alpha_1 + \cdots + y_n\alpha_n$$

are vectors in V, then

$$\begin{aligned} f(\alpha, \beta) &= f\left(\sum_i x_i\alpha_i, \beta\right) \\ &= \sum_i x_i f(\alpha_i, \beta) \\ &= \sum_i x_i f\left(\alpha_i, \sum_j y_j\alpha_j\right) \\ &= \sum_i \sum_j x_iy_j f(\alpha_i, \alpha_j). \end{aligned}$$

If we let $A_{ij} = f(\alpha_i, \alpha_j)$, then

$$\begin{aligned} f(\alpha, \beta) &= \sum_i \sum_j A_{ij}x_iy_j \\ &= X^tAY \end{aligned}$$

where X and Y are the coordinate matrices of α and β in the ordered basis $\mathcal{B}$. Thus every bilinear form on V is of the type

$$f(\alpha, \beta) = [\alpha]^t_{\mathcal{B}} A [\beta]_{\mathcal{B}} \tag{10-3}$$

for some $n \times n$ matrix A over F. Conversely, if we are given any $n \times n$ matrix A, it is easy to see that (10-3) defines a bilinear form f on V, such that $A_{ij} = f(\alpha_i, \alpha_j)$.

Definition. *Let V be a finite-dimensional vector space, and let $\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}$ be an ordered basis for V. If f is a bilinear form on V, the* **matrix of f in the ordered basis** $\mathcal{B}$ *is the n × n matrix A with entries $A_{ij} = f(\alpha_i, \alpha_j)$. At times, we shall denote this matrix by $[f]_{\mathcal{B}}$.*

Theorem 1. *Let V be a finite-dimensional vector space over the field F. For each ordered basis $\mathcal{B}$ of V, the function which associates with each bilinear form on V its matrix in the ordered basis $\mathcal{B}$ is an isomorphism of the space L(V, V, F) onto the space of n × n matrices over the field F.*

Proof. We observed above that $f \rightarrow [f]_{\mathcal{B}}$ is a one-one correspondence between the set of bilinear forms on V and the set of all $n \times n$ matrices over F. That this is a linear transformation is easy to see, because

$$(cf + g)(\alpha_i, \alpha_j) = cf(\alpha_i, \alpha_j) + g(\alpha_i, \alpha_j)$$

for each i and j. This simply says that

$$[cf + g]_{\mathcal{B}} = c[f]_{\mathcal{B}} + [g]_{\mathcal{B}}. \quad \blacksquare$$

Corollary. *If $\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}$ is an ordered basis for V, and $\mathcal{B}^* = \{L_1, \ldots, L_n\}$ is the dual basis for V*, then the n^2 bilinear forms*

$$f_{ij}(\alpha, \beta) = L_i(\alpha)L_j(\beta), \qquad 1 \leq i \leq n, 1 \leq j \leq n$$

form a basis for the space L(V, V, F). In particular, the dimension of L(V, V, F) is n^2.

Proof. The dual basis $\{L_1, \ldots, L_n\}$ is essentially defined by the fact that $L_i(\alpha)$ is the ith coordinate of α in the ordered basis $\mathcal{B}$ (for any α in V). Now the functions f_{ij} defined by

$$f_{ij}(\alpha, \beta) = L_i(\alpha)L_j(\beta)$$

are bilinear forms of the type considered in Example 1. If

$$\alpha = x_1\alpha_1 + \cdots + x_n\alpha_n \quad \text{and} \quad \beta = y_1\alpha_1 + \cdots + y_n\alpha_n,$$

then

$$f_{ij}(\alpha, \beta) = x_iy_j.$$

Let f be any bilinear form on V and let A be the matrix of f in the ordered basis $\mathcal{B}$. Then

$$f(\alpha, \beta) = \sum_{i,j} A_{ij}x_iy_j$$

which simply says that

$$f = \sum_{i,j} A_{ij}f_{ij}.$$

It is now clear that the n^2 forms f_{ij} comprise a basis for $L(V, V, F)$. $\blacksquare$

One can rephrase the proof of the corollary as follows. The bilinear form f_{ij} has as its matrix in the ordered basis $\mathcal{B}$ the matrix 'unit' $E^{i,j}$, whose only non-zero entry is a 1 in row i and column j. Since these matrix units comprise a basis for the space of $n \times n$ matrices, the forms f_{ij} comprise a basis for the space of bilinear forms.

The concept of the matrix of a bilinear form in an ordered basis is similar to that of the matrix of a linear operator in an ordered basis. Just as for linear operators, we shall be interested in what happens to the matrix representing a bilinear form, as we change from one ordered basis to another. So, suppose $\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}$ and $\mathcal{B}' = \{\alpha_1', \ldots, \alpha_n'\}$ are two ordered bases for V and that f is a bilinear form on V. How are the matrices $[f]_{\mathcal{B}}$ and $[f]_{\mathcal{B}'}$ related? Well, let P be the (invertible) $n \times n$ matrix such that

$$[\alpha]_{\mathcal{B}} = P[\alpha]_{\mathcal{B}'}$$

for all α in V. In other words, define P by

$$\alpha_j' = \sum_{i=1}^{n} P_{ij}\alpha_i.$$

For any vectors α, β in V

$$\begin{aligned} f(\alpha, \beta) &= [\alpha]_{\mathcal{B}}^t[f]_{\mathcal{B}}[\beta]_{\mathcal{B}} \\ &= (P[\alpha]_{\mathcal{B}'})^t[f]_{\mathcal{B}}P[\beta]_{\mathcal{B}'} \\ &= [\alpha]_{\mathcal{B}'}^t(P^t[f]_{\mathcal{B}}P)[\beta]_{\mathcal{B}'}. \end{aligned}$$

By the definition and uniqueness of the matrix representing f in the ordered basis $\mathcal{B}'$, we must have

$$[f]_{\mathcal{B}'} = P^t[f]_{\mathcal{B}}P. \tag{10-4}$$

EXAMPLE 4. Let V be the vector space R^2. Let f be the bilinear form defined on $\alpha = (x_1, x_2)$ and $\beta = (y_1, y_2)$ by

$$f(\alpha, \beta) = x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2.$$

Now

$$f(\alpha, \beta) = [x_1, x_2]\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

and so the matrix of f in the standard ordered basis $\mathcal{B} = \{\epsilon_1, \epsilon_2\}$ is

$$[f]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Let $\mathcal{B}' = \{\epsilon_1', \epsilon_2'\}$ be the ordered basis defined by $\epsilon_1' = (1, -1)$, $\epsilon_2' = (1, 1)$. In this case, the matrix P which changes coordinates from $\mathcal{B}'$ to $\mathcal{B}$ is

$$P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Thus

$$\begin{aligned} [f]_{\mathcal{B}'} &= P^t[f]_{\mathcal{B}}P \\ &= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}.$$

What this means is that if we express the vectors α and β by means of their coordinates in the basis $\mathcal{B}'$, say

$$\alpha = x_1'\epsilon_1' + x_2'\epsilon_2', \qquad \beta = y_1'\epsilon_1' + y_2'\epsilon_2'$$

then

$$f(\alpha, \beta) = 4x_2'y_2'.$$

One consequence of the change of basis formula (10-4) is the following: If A and B are $n \times n$ matrices which represent the same bilinear form on V in (possibly) different ordered bases, then A and B have the same rank. For, if P is an invertible $n \times n$ matrix and $B = P^tAP$, it is evident that A and B have the same rank. This makes it possible to define the rank of a bilinear form on V as the rank of any matrix which represents the form in an ordered basis for V.

It is desirable to give a more intrinsic definition of the rank of a bilinear form. This can be done as follows: Suppose f is a bilinear form on the vector space V. If we fix a vector α in V, then $f(\alpha, \beta)$ is linear as a function of β. In this way, each fixed α determines a linear functional on V; let us denote this linear functional by $L_f(\alpha)$. To repeat, if α is a vector in V, then $L_f(\alpha)$ is the linear functional on V whose value on any vector β is $f(\alpha, \beta)$. This gives us a transformation $\alpha \to L_f(\alpha)$ from V into the dual space V^*. Since

$$f(c\alpha_1 + \alpha_2, \beta) = cf(\alpha_1, \beta) + f(\alpha_2, \beta)$$

we see that

$$L_f(c\alpha_1 + \alpha_2) = cL_f(\alpha_1) + L_f(\alpha_2)$$

that is, L_f is a linear transformation from V into V^*.

In a similar manner, f determines a linear transformation R_f from V into V^*. For each fixed β in V, $f(\alpha, \beta)$ is linear as a function of α. We define $R_f(\beta)$ to be the linear functional on V whose value on the vector α is $f(\alpha, \beta)$.

Theorem 2. *Let* f *be a bilinear form on the finite-dimensional vector space* V. *Let* L_f *and* R_f *be the linear transformations from* V *into* V^* *defined by* $(L_f\alpha)(\beta) = f(\alpha, \beta) = (R_f\beta)(\alpha)$. *Then rank* (L_f) = *rank* (R_f).

Proof. One can give a 'coordinate free' proof of this theorem. Such a proof is similar to the proof (in Section 3.7) that the row-rank of a matrix is equal to its column-rank. So, here we shall give a proof which proceeds by choosing a coordinate system (basis) and then using the 'row-rank equals column-rank' theorem.

To prove rank (L_f) = rank (R_f), it will suffice to prove that L_f and

R_f have the same nullity. Let $\mathcal{B}$ be an ordered basis for V, and let $A = [f]_{\mathcal{B}}$. If α and β are vectors in V, with coordinate matrices X and Y in the ordered basis $\mathcal{B}$, then $f(\alpha, \beta) = X^tAY$. Now $R_f(\beta) = 0$ means that $f(\alpha, \beta) = 0$ for every α in V, i.e., that $X^tAY = 0$ for every $n \times 1$ matrix X. The latter condition simply says that $AY = 0$. The nullity of R_f is therefore equal to the dimension of the space of solutions of $AY = 0$.

Similarly, $L_f(\alpha) = 0$ if and only if $X^tAY = 0$ for every $n \times 1$ matrix Y. Thus α is in the null space of L_f if and only if $X^tA = 0$, i.e., $A^tX = 0$. The nullity of L_f is therefore equal to the dimension of the space of solutions of $A^tX = 0$. Since the matrices A and A^t have the same column-rank, we see that

$$\text{nullity } (L_f) = \text{nullity } (R_f). \quad \blacksquare$$

Definition. *If* f *is a bilinear form on the finite-dimensional space* V, *the* **rank** *of* f *is the integer* r = *rank* (L_f) = *rank* (R_f).

Corollary 1. *The rank of a bilinear form is equal to the rank of the matrix of the form in any ordered basis.*

Corollary 2. *If* f *is a bilinear form on the* n*-dimensional vector space* V, *the following are equivalent:*

(a) *rank* (f) = n.
(b) *For each non-zero* α *in* V, *there is a* β *in* V *such that* $f(\alpha, \beta) \neq 0$.
(c) *For each non-zero* β *in* V, *there is an* α *in* V *such that* $f(\alpha, \beta) \neq 0$.

Proof. Statement (b) simply says that the null space of L_f is the zero subspace. Statement (c) says that the null space of R_f is the zero subspace. The linear transformations L_f and R_f have nullity 0 if and only if they have rank n, i.e., if and only if rank $(f) = n$. $\blacksquare$

Definition. *A bilinear form* f *on a vector space* V *is called* **non-degenerate** (*or* **non-singular**) *if it satisfies conditions* (b) *and* (c) *of Corollary* 2.

If V is finite-dimensional, then f is non-degenerate provided f satisfies any one of the three conditions of Corollary 2. In particular, f is non-degenerate (non-singular) if and only if its matrix in some (every) ordered basis for V is a non-singular matrix.

EXAMPLE 5. Let $V = R^n$, and let f be the bilinear form defined on $\alpha = (x_1, \ldots, x_n)$ and $\beta = (y_1, \ldots, y_n)$ by

$$f(\alpha, \beta) = x_1y_1 + \cdots + x_ny_n.$$