A general ring need not have maximal ideals. For example, take any abelian group which has no maximal subgroups (for example, $\mathbb{Q}$ — cf. Exercise 16, Section 6.1) and make it into a trivial ring by defining $ab = 0$ for all a, b. In such a ring the ideals are simply the subgroups and so there are no maximal ideals. The zero ring has no maximal ideals, hence any result involving maximal ideals forces a ring to be nonzero. The next proposition shows that rings with an identity $1 \neq 0$ always possess maximal ideals. Like many such general existence theorems (e.g., the result that a finitely generated group has maximal subgroups or that every vector space has a basis) the proof relies on Zorn's Lemma (see Appendix I). In many specific rings, however, the presence of maximal ideals is often obvious, independent of Zorn's Lemma.

Proposition 11. In a ring with identity every proper ideal is contained in a maximal ideal.

Proof: Let R be a ring with identity and let I be a proper ideal (so R cannot be the zero ring, i.e., $1 \neq 0$). Let $\mathcal{S}$ be the set of all proper ideals of R which contain I. Then $\mathcal{S}$ is nonempty ($I \in \mathcal{S}$) and is partially ordered by inclusion. If $\mathcal{C}$ is a chain in $\mathcal{S}$, define J to be the union of all ideals in $\mathcal{C}$:

$$J = \bigcup_{A \in \mathcal{C}} A.$$

We first show that J is an ideal. Certainly J is nonempty because $\mathcal{C}$ is nonempty — specifically, $0 \in J$ since 0 is in every ideal A. If $a, b \in J$, then there are ideals $A, B \in \mathcal{C}$ such that $a \in A$ and $b \in B$. By definition of a chain either $A \subseteq B$ or $B \subseteq A$. In either case $a - b \in J$, so J is closed under subtraction. Since each $A \in \mathcal{C}$ is closed under left and right multiplication by elements of R, so is J. This proves J is an ideal.

If J is not a proper ideal then $1 \in J$. In this case, by definition of J we must have $1 \in A$ for some $A \in \mathcal{C}$. This is a contradiction because each A is a proper ideal ($A \in \mathcal{C} \subseteq \mathcal{S}$). This proves that each chain has an upper bound in $\mathcal{S}$. By Zorn's Lemma $\mathcal{S}$ has a maximal element which is therefore a maximal (proper) ideal containing I.

For commutative rings the next result characterizes maximal ideals by the structure of their quotient rings.

Proposition 12. Assume R is commutative. The ideal M is a maximal ideal if and only if the quotient ring R/M is a field.

Proof: This follows from the Lattice Isomorphism Theorem together with Proposition 9(2). The ideal M is maximal if and only if there are no ideals I with $M \subset I \subset R$. By the Lattice Isomorphism Theorem the ideals of R containing M correspond bijectively with the ideals of R/M, so M is maximal if and only if the only ideals of R/M are 0 and R/M. By Proposition 9(2) we see that M is maximal if and only if R/M is a field.

The proposition above indicates how to *construct* some fields: take the quotient of any commutative ring R with identity by a maximal ideal in R. We shall use this in Part IV to construct all finite fields by taking quotients of the ring $\mathbb{Z}[x]$ by maximal ideals.

Examples

(1) Let n be a nonnegative integer. The ideal $n\mathbb{Z}$ of $\mathbb{Z}$ is a maximal ideal if and only if $\mathbb{Z}/n\mathbb{Z}$ is a field. We saw in Section 3 that this is the case if and only if n is a prime number. This also follows directly from the containment of ideals of $\mathbb{Z}$ described in Example 2 above.

(2) The ideal $(2, x)$ is a maximal ideal in $\mathbb{Z}[x]$ because its quotient ring is the field $\mathbb{Z}/2\mathbb{Z}$ — cf. Example 3 above and Example 5 at the end of Section 3.

(3) The ideal (x) in $\mathbb{Z}[x]$ is not a maximal ideal because $(x) \subset (2, x) \subset \mathbb{Z}[x]$. The quotient ring $\mathbb{Z}[x]/(x)$ is isomorphic to $\mathbb{Z}$ (the ideal (x) in $\mathbb{Z}[x]$ is the kernel of the surjective ring homomorphism from $\mathbb{Z}[x]$ to $\mathbb{Z}$ given by evaluation at 0). Since $\mathbb{Z}$ is not a field, we see again that (x) is not a maximal ideal in $\mathbb{Z}[x]$.

(4) Let R be the ring of all functions from [0,1] to $\mathbb{R}$ and for each $a \in [0, 1]$ let M_a be the kernel of evaluation at a. Since evaluation is a surjective homomorphism from R to $\mathbb{R}$, we see that $R/M_a \cong \mathbb{R}$ and hence M_a is a maximal ideal. Similarly, the kernel of evaluation at any fixed point is a maximal ideal in the ring of continuous real valued functions on $[0, 1]$.

(5) If F is a field and G is a finite group, then the augmentation ideal I is a maximal ideal of the group ring FG (cf. Example 7 at the end of the preceding section). The augmentation ideal is the kernel of the augmentation map which is a surjective homomorphism onto the field F (i.e., $FG/I \cong F$, a field). Note that Proposition 12 does not apply directly since FG need not be commutative, however, the implication in Proposition 12 that I is a maximal ideal if R/I is a field holds for arbitrary rings.

Definition. Assume R is commutative. An ideal P is called a *prime ideal* if $P \neq R$ and whenever the product ab of two elements $a, b \in R$ is an element of P, then at least one of a and b is an element of P.

The notion of a maximal ideal is fairly intuitive but the definition of a prime ideal may seem a little strange. It is, however, a natural generalization of the notion of a "prime" in the integers $\mathbb{Z}$. Let n be a nonnegative integer. According to the above definition the ideal $n\mathbb{Z}$ is a *prime* ideal provided $n \neq 1$ (to ensure that the ideal is proper) and provided every time the product ab of two integers is an element of $n\mathbb{Z}$, at least one of a, b is an element of $n\mathbb{Z}$. Put another way, if $n \neq 0$, it must have the property that whenever n divides ab, n must divide a or divide b. This is equivalent to the usual definition that n is a prime number. Thus *the prime ideals of $\mathbb{Z}$ are just the ideals $p\mathbb{Z}$ of $\mathbb{Z}$ generated by prime numbers p together with the ideal 0.*

For the integers $\mathbb{Z}$ there is no difference between the maximal ideals and the nonzero prime ideals. This is not true in general, but we shall see shortly that every maximal ideal is a prime ideal. First we translate the notion of prime ideals into properties of quotient rings as we did for maximal ideals in Proposition 12. Recall that an integral domain is a commutative ring with identity $1 \neq 0$ that has no zero divisors.

Proposition 13. Assume R is commutative. Then the ideal P is a prime ideal in R if and only if the quotient ring R/P is an integral domain.

Proof: This proof is simply a matter of translating the definition of a prime ideal into the language of quotients. The ideal P is prime if and only if $P \neq R$ and whenever

$ab \in P$, then either $a \in P$ or $b \in P$. Use the bar notation for elements of R/P: $\bar{r} = r + P$. Note that $r \in P$ if and only if the element $\bar{r}$ is zero in the quotient ring R/P. Thus in the terminology of quotients P is a prime ideal if and only if $\overline{R} \neq \bar{0}$ and whenever $\overline{ab} = \bar{a}\bar{b} = \bar{0}$, then either $\bar{a} = \bar{0}$ or $\bar{b} = \bar{0}$, i.e., R/P is an integral domain.

It follows in particular that a commutative ring with identity is an integral domain if and only if 0 is a prime ideal.

Corollary 14. Assume R is commutative. Every maximal ideal of R is a prime ideal.

Proof: If M is a maximal ideal then R/M is a field by Proposition 12. A field is an integral domain so the corollary follows from Proposition 13.

Examples

(1) The principal ideals generated by primes in $\mathbb{Z}$ are both prime and maximal ideals. The zero ideal in $\mathbb{Z}$ is prime but not maximal.

(2) The ideal (x) is a prime ideal in $\mathbb{Z}[x]$ since $\mathbb{Z}[x]/(x) \cong \mathbb{Z}$. This ideal is not a maximal ideal. The ideal 0 is a prime ideal in $\mathbb{Z}[x]$, but is not a maximal ideal.

EXERCISES

Let R be a ring with identity $1 \neq 0$.

1. Let L_j be the left ideal of $M_n(R)$ consisting of arbitrary entries in the j^{th} column and zero in all other entries and let E_{ij} be the element of $M_n(R)$ whose i, j entry is 1 and whose other entries are all 0. Prove that $L_j = M_n(R)E_{ij}$ for any i. [See Exercise 6, Section 2.]

2. Assume R is commutative. Prove that the augmentation ideal in the group ring RG is generated by $\{g - 1 \mid g \in G\}$. Prove that if $G = \langle \sigma \rangle$ is cyclic then the augmentation ideal is generated by $\sigma - 1$.

3. (a) Let p be a prime and let G be an abelian group of order p^n. Prove that the nilradical of the group ring $\mathbb{F}_pG$ is the augmentation ideal (cf. Exercise 29, Section 3). [Use the preceding exercise.]

(b) Let $G = \{g_1, \dots, g_n\}$ be a finite group and assume R is commutative. Prove that if r is any element of the augmentation ideal of RG then $r(g_1 + \cdots + g_n) = 0$. [Use the preceding exercise.]

4. Assume R is commutative. Prove that R is a field if and only if 0 is a maximal ideal.

5. Prove that if M is an ideal such that R/M is a field then M is a maximal ideal (do not assume R is commutative).

6. Prove that R is a division ring if and only if its only left ideals are (0) and R. (The analogous result holds when "left" is replaced by "right.")

7. Let R be a commutative ring with 1. Prove that the principal ideal generated by x in the polynomial ring $R[x]$ is a prime ideal if and only if R is an integral domain. Prove that (x) is a maximal ideal if and only if R is a field.

8. Let R be an integral domain. Prove that $(a) = (b)$ for some elements $a, b \in R$, if and only if $a = ub$ for some unit u of R.

9. Let R be the ring of all continuous functions on $[0, 1]$ and let I be the collection of functions $f(x)$ in R with $f(1/3) = f(1/2) = 0$. Prove that I is an ideal of R but is not a prime ideal.