Example 9. Let F be a field and V the vector space of all polynomial functions from F into F. Let D be the differentiation operator defined in Example 2, and let T be the linear operator 'multiplication by x':

$$(Tf)(x) = xf(x).$$

Then $DT \neq TD$. In fact, the reader should find it easy to verify that $DT - TD = I$, the identity operator.

Even though the 'multiplication' we have on $L(V, V)$ is not commutative, it is nicely related to the vector space operations of $L(V, V)$.

Example 10. Let $\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}$ be an ordered basis for a vector space V. Consider the linear operators $E^{p,q}$ which arose in the proof of Theorem 5:

$$E^{p,q}(\alpha_i) = \delta_{iq}\alpha_p.$$

These n^2 linear operators form a basis for the space of linear operators on V. What is $E^{p,q}E^{r,s}$? We have

$$\begin{aligned}(E^{p,q}E^{r,s})(\alpha_i) &= E^{p,q}(\delta_{is}\alpha_r)\\ &= \delta_{is}E^{p,q}(\alpha_r)\\ &= \delta_{is}\delta_{rq}\alpha_p.\end{aligned}$$

Therefore,

$$E^{p,q}E^{r,s} = \begin{cases}0, & \text{if } r \neq q\\ E^{p,s}, & \text{if } q = r.\end{cases}$$

Let T be a linear operator on V. We showed in the proof of Theorem 5 that if

$$\begin{aligned}A_j &= [T\alpha_j]_{\mathcal{B}}\\ A &= [A_1, \ldots, A_n]\end{aligned}$$

then

$$T = \sum_p \sum_q A_{pq}E^{p,q}.$$

If

$$U = \sum_r \sum_s B_{rs}E^{r,s}$$

is another linear operator on V, then the last lemma tells us that

$$\begin{aligned}TU &= (\sum_p \sum_q A_{pq}E^{p,q})(\sum_r \sum_s B_{rs}E^{r,s})\\ &= \sum_p \sum_q \sum_r \sum_s A_{pq}B_{rs}E^{p,q}E^{r,s}.\end{aligned}$$

As we have noted, the only terms which survive in this huge sum are the terms where $q = r$, and since $E^{p,r}E^{r,s} = E^{p,s}$, we have

$$\begin{aligned}TU &= \sum_p \sum_s (\sum_r A_{pr}B_{rs})E^{p,s}\\ &= \sum_p \sum_s (AB)_{ps}E^{p,s}.\end{aligned}$$

Thus, the effect of composing T and U is to multiply the matrices A and B.

In our discussion of algebraic operations with linear transformations we have not yet said anything about invertibility. One specific question of interest is this. For which linear operators T on the space V does there exist a linear operator T^{-1} such that $TT^{-1} = T^{-1}T = I$?

The function T from V into W is called **invertible** if there exists a function U from W into V such that UT is the identity function on V and TU is the identity function on W. If T is invertible, the function U is unique and is denoted by T^{-1}. (See Appendix.) Furthermore, T is invertible if and only if

1. T is 1:1, that is, $T\alpha = T\beta$ implies $\alpha = \beta$;
2. T is onto, that is, the range of T is (all of) W.

Theorem 7. *Let* V *and* W *be vector spaces over the field* F *and let* T *be a linear transformation from* V *into* W. *If* T *is invertible, then the inverse function* T^{-1} *is a linear transformation from* W *onto* V.

Proof. We repeat ourselves in order to underscore a point. When T is one-one and onto, there is a uniquely determined inverse function T^{-1} which maps W onto V such that $T^{-1}T$ is the identity function on V, and TT^{-1} is the identity function on W. What we are proving here is that if a linear function T is invertible, then the inverse T^{-1} is also linear.

Let β_1 and β_2 be vectors in W and let c be a scalar. We wish to show that

$$T^{-1}(c\beta_1 + \beta_2) = cT^{-1}\beta_1 + T^{-1}\beta_2.$$

Let $\alpha_i = T^{-1}\beta_i$, $i = 1, 2$, that is, let α_i be the unique vector in V such that $T\alpha_i = \beta_i$. Since T is linear,

$$\begin{aligned} T(c\alpha_1 + \alpha_2) &= cT\alpha_1 + T\alpha_2 \\ &= c\beta_1 + \beta_2. \end{aligned}$$

Thus $c\alpha_1 + \alpha_2$ is the unique vector in V which is sent by T into $c\beta_1 + \beta_2$, and so

$$\begin{aligned} T^{-1}(c\beta_1 + \beta_2) &= c\alpha_1 + \alpha_2 \\ &= c(T^{-1}\beta_1) + T^{-1}\beta_2 \end{aligned}$$

and T^{-1} is linear. ▌

Suppose that we have an invertible linear transformation T from V onto W and an invertible linear transformation U from W onto Z. Then UT is invertible and $(UT)^{-1} = T^{-1}U^{-1}$. That conclusion does not require the linearity nor does it involve checking separately that UT is 1:1 and onto. All it involves is verifying that $T^{-1}U^{-1}$ is both a left and a right inverse for UT.

If T is linear, then $T(\alpha - \beta) = T\alpha - T\beta$; hence, $T\alpha = T\beta$ if and only if $T(\alpha - \beta) = 0$. This simplifies enormously the verification that T is 1:1. Let us call a linear transformation T **non-singular** if $T\gamma = 0$ implies

$\gamma = 0$, i.e., if the null space of T is $\{0\}$. Evidently, T is 1:1 if and only if T is non-singular. The extension of this remark is that non-singular linear transformations are those which preserve linear independence.

Theorem 8. *Let* T *be a linear transformation from* V *into* W. *Then* T *is non-singular if and only if* T *carries each linearly independent subset of* V *onto a linearly independent subset of* W.

Proof. First suppose that T is non-singular. Let S be a linearly independent subset of V. If $\alpha_1, \ldots, \alpha_k$ are vectors in S, then the vectors $T\alpha_1, \ldots, T\alpha_k$ are linearly independent; for if

$$c_1(T\alpha_1) + \cdots + c_k(T\alpha_k) = 0$$

then

$$T(c_1\alpha_1 + \cdots + c_k\alpha_k) = 0$$

and since T is non-singular

$$c_1\alpha_1 + \cdots + c_k\alpha_k = 0$$

from which it follows that each $c_i = 0$ because S is an independent set. This argument shows that the image of S under T is independent.

Suppose that T carries independent subsets onto independent subsets. Let α be a non-zero vector in V. Then the set S consisting of the one vector α is independent. The image of S is the set consisting of the one vector $T\alpha$, and this set is independent. Therefore $T\alpha \neq 0$, because the set consisting of the zero vector alone is dependent. This shows that the null space of T is the zero subspace, i.e., T is non-singular. ∎

EXAMPLE 11. Let F be a subfield of the complex numbers (or a field of characteristic zero) and let V be the space of polynomial functions over F. Consider the differentiation operator D and the 'multiplication by x' operator T, from Example 9. Since D sends all constants into 0, D is singular; however, V is not finite dimensional, the range of D is all of V, and it is possible to define a right inverse for D. For example, if E is the indefinite integral operator:

$$E(c_0 + c_1x + \cdots + c_nx^n) = c_0x + \frac{1}{2}c_1x^2 + \cdots + \frac{1}{n+1}c_nx^{n+1}$$

then E is a linear operator on V and $DE = I$. On the other hand, $ED \neq I$ because ED sends the constants into 0. The operator T is in what we might call the reverse situation. If $xf(x) = 0$ for all x, then $f = 0$. Thus T is non-singular and it is possible to find a left inverse for T. For example if U is the operation 'remove the constant term and divide by x':

$$U(c_0 + c_1x + \cdots + c_nx^n) = c_1 + c_2x + \cdots + c_nx^{n-1}$$

then U is a linear operator on V and $UT = I$. But $TU \neq I$ since every function in the range of TU is in the range of T, which is the space of polynomial functions f such that $f(0) = 0$.

EXAMPLE 12. Let F be a field and let T be the linear operator on F^2 defined by

$$T(x_1, x_2) = (x_1 + x_2, x_1).$$

Then T is non-singular, because if $T(x_1, x_2) = 0$ we have

$$\begin{aligned} x_1 + x_2 &= 0 \\ x_1 &= 0 \end{aligned}$$

so that $x_1 = x_2 = 0$. We also see that T is onto; for, let (z_1, z_2) be any vector in F^2. To show that (z_1, z_2) is in the range of T we must find scalars x_1 and x_2 such that

$$\begin{aligned} x_1 + x_2 &= z_1 \\ x_1 &= z_2 \end{aligned}$$

and the obvious solution is $x_1 = z_2$, $x_2 = z_1 - z_2$. This last computation gives us an explicit formula for T^{-1}, namely,

$$T^{-1}(z_1, z_2) = (z_2, z_1 - z_2).$$

We have seen in Example 11 that a linear transformation may be non-singular without being onto and may be onto without being non-singular. The present example illustrates an important case in which that cannot happen.

Theorem 9. *Let* V *and* W *be finite-dimensional vector spaces over the field* F *such that dim* V = *dim* W. *If* T *is a linear transformation from* V *into* W, *the following are equivalent:*

(i) T *is invertible.*
(ii) T *is non-singular.*
(iii) T *is onto, that is, the range of* T *is* W.

Proof. Let $n = \dim V = \dim W$. From Theorem 2 we know that

$$\text{rank } (T) + \text{nullity } (T) = n.$$

Now T is non-singular if and only if nullity $(T) = 0$, and (since $n = \dim W$) the range of T is W if and only if rank $(T) = n$. Since the rank plus the nullity is n, the nullity is 0 precisely when the rank is n. Therefore T is non-singular if and only if $T(V) = W$. So, if either condition (ii) or (iii) holds, the other is satisfied as well and T is invertible. ∎

We caution the reader not to apply Theorem 9 except in the presence of finite-dimensionality and with $\dim V = \dim W$. Under the hypotheses of Theorem 9, the conditions (i), (ii), and (iii) are also equivalent to these.

(iv) *If* $\{\alpha_1, \ldots, \alpha_n\}$ *is basis for* V, *then* $\{T\alpha_1, \ldots, T\alpha_n\}$ *is a basis for* W.

(v) *There is some basis* $\{\alpha_1, \ldots, \alpha_n\}$ *for* V *such that* $\{T\alpha_1, \ldots, T\alpha_n\}$ *is a basis for* W.

We shall give a proof of the equivalence of the five conditions which contains a different proof that (i), (ii), and (iii) are equivalent.

(i) $\rightarrow$ (ii). If T is invertible, T is non-singular. (ii) $\rightarrow$ (iii). Suppose T is non-singular. Let $\{\alpha_1, \ldots, \alpha_n\}$ be a basis for V. By Theorem 8, $\{T\alpha_1, \ldots, T\alpha_n\}$ is a linearly independent set of vectors in W, and since the dimension of W is also n, this set of vectors is a basis for W. Now let β be any vector in W. There are scalars $c_1, \ldots, c_n$ such that

$$\begin{aligned}\beta &= c_1(T\alpha_1) + \cdots + c_n(T\alpha_n)\\ &= T(c_1\alpha_1 + \cdots + c_n\alpha_n)\end{aligned}$$

which shows that β is in the range of T. (iii) $\rightarrow$ (iv). We now assume that T is onto. If $\{\alpha_1, \ldots, \alpha_n\}$ is any basis for V, the vectors $T\alpha_1, \ldots, T\alpha_n$ span the range of T, which is all of W by assumption. Since the dimension of W is n, these n vectors must be linearly independent, that is, must comprise a basis for W. (iv) $\rightarrow$ (v). This requires no comment. (v) $\rightarrow$ (i). Suppose there is some basis $\{\alpha_1, \ldots, \alpha_n\}$ for V such that $\{T\alpha_1, \ldots, T\alpha_n\}$ is a basis for W. Since the $T\alpha_i$ span W, it is clear that the range of T is all of W. If $\alpha = c_1\alpha_1 + \cdots + c_n\alpha_n$ is in the null space of T, then

$$T(c_1\alpha_1 + \cdots + c_n\alpha_n) = 0$$

or

$$c_1(T\alpha_1) + \cdots + c_n(T\alpha_n) = 0$$

and since the $T\alpha_i$ are independent each $c_i = 0$, and thus $\alpha = 0$. We have shown that the range of T is W, and that T is non-singular, hence T is invertible.

The set of invertible linear operators on a space V, with the operation of composition, provides a nice example of what is known in algebra as a 'group.' Although we shall not have time to discuss groups in any detail, we shall at least give the definition.

Definition. *A* **group** *consists of the following.*

1. *A set* G;

2. *A rule (or operation) which associates with each pair of elements* x, y *in* G *an element* xy *in* G *in such a way that*

(a) x(yz) = (xy)z, *for all* x, y, *and* z *in* G (*associativity*);

(b) *there is an element* e *in* G *such that* ex = xe = x, *for every* x *in* G;

(c) *to each element* x *in* G *there corresponds an element* x^{-1} *in* G *such that* $xx^{-1} = x^{-1}x = e$.

We have seen that composition $(U, T) \rightarrow UT$ associates with each pair of invertible linear operators on a space V another invertible operator on V. Composition is an associative operation. The identity operator I