12. Let W be a finite-dimensional subspace of an inner product space V, and let E be the orthogonal projection of V on W. Prove that $(E\alpha|\beta) = (\alpha|E\beta)$ for all α, β in V.

13. Let S be a subset of an inner product space V. Show that $(S^\perp)^\perp$ contains the subspace spanned by S. When V is finite-dimensional, show that $(S^\perp)^\perp$ is the subspace spanned by S.

14. Let V be a finite-dimensional inner product space, and let $\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}$ be an orthonormal basis for V. Let T be a linear operator on V and A the matrix of T in the ordered basis $\mathcal{B}$. Prove that

$$A_{ij} = (T\alpha_j|\alpha_i).$$

15. Suppose $V = W_1 \oplus W_2$ and that f_1 and f_2 are inner products on W_1 and W_2, respectively. Show that there is a unique inner product f on V such that

(a) $W_2 = W_1^\perp$;

(b) $f(\alpha, \beta) = f_k(\alpha, \beta)$, when α, β are in W_k, $k = 1, 2$.

16. Let V be an inner product space and W a finite-dimensional subspace of V. There are (in general) many projections which have W as their range. One of these, the orthogonal projection on W, has the property that $||E\alpha|| \leq ||\alpha||$ for every α in V. Prove that if E is a projection with range W, such that $||E\alpha|| \leq ||\alpha||$ for all α in V, then E is the orthogonal projection on W.

17. Let V be the real inner product space consisting of the space of real-valued continuous functions on the interval, $-1 \leq t \leq 1$, with the inner product

$$(f|g) = \int_{-1}^{1} f(t)g(t)\,dt.$$

Let W be the subspace of odd functions, i.e., functions satisfying $f(-t) = -f(t)$. Find the orthogonal complement of W.

8.3. Linear Functionals and Adjoints

The first portion of this section treats linear functionals on an inner product space and their relation to the inner product. The basic result is that any linear functional f on a finite-dimensional inner product space is 'inner product with a fixed vector in the space,' i.e., that such an f has the form $f(\alpha) = (\alpha|\beta)$ for some fixed β in V. We use this result to prove the existence of the 'adjoint' of a linear operator T on V, this being a linear operator T^* such that $(T\alpha|\beta) = (\alpha|T^*\beta)$ for all α and β in V. Through the use of an orthonormal basis, this adjoint operation on linear operators (passing from T to T^*) is identified with the operation of forming the conjugate transpose of a matrix. We explore slightly the analogy between the adjoint operation and conjugation on complex numbers.

Let V be any inner product space, and let β be some fixed vector in V. We define a function f_β from V into the scalar field by

$$f_\beta(\alpha) = (\alpha|\beta).$$

This function f_β is a linear functional on V, because, by its very definition, $(\alpha|\beta)$ is linear as a function of α. If V is finite-dimensional, every linear functional on V arises in this way from some β.

Theorem 6. *Let V be a finite-dimensional inner product space, and f a linear functional on V. Then there exists a unique vector β in V such that $f(\alpha) = (\alpha|\beta)$ for all α in V.*

Proof. Let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be an orthonormal basis for V. Put

$$\beta = \sum_{j=1}^{n} \overline{f(\alpha_j)}\alpha_j \tag{8-13}$$

and let f_β be the linear functional defined by

$$f_\beta(\alpha) = (\alpha|\beta).$$

Then

$$f_\beta(\alpha_k) = (\alpha_k|\sum_j \overline{f(\alpha_j)}\alpha_j) = f(\alpha_k).$$

Since this is true for each α_k, it follows that $f = f_\beta$. Now suppose γ is a vector in V such that $(\alpha|\beta) = (\alpha|\gamma)$ for all α. Then $(\beta - \gamma|\beta - \gamma) = 0$ and $\beta = \gamma$. Thus there is exactly one vector β determining the linear functional f in the stated manner. ∎

The proof of this theorem can be reworded slightly, in terms of the representation of linear functionals in a basis. If we choose an orthonormal basis $\{\alpha_1, \ldots, \alpha_n\}$ for V, the inner product of $\alpha = x_1\alpha_1 + \cdots + x_n\alpha_n$ and $\beta = y_1\alpha_1 + \cdots + y_n\alpha_n$ will be

$$(\alpha|\beta) = x_1\bar{y}_1 + \cdots + x_n\bar{y}_n.$$

If f is any linear functional on V, then f has the form

$$f(\alpha) = c_1x_1 + \cdots + c_nx_n$$

for some fixed scalars $c_1, \ldots, c_n$ determined by the basis. Of course $c_j = f(\alpha_j)$. If we wish to find a vector β in V such that $(\alpha|\beta) = f(\alpha)$ for all α, then clearly the coordinates y_j of β must satisfy $\bar{y}_j = c_j$ or $y_j = \overline{f(\alpha_j)}$. Accordingly,

$$\beta = \overline{f(\alpha_1)}\alpha_1 + \cdots + \overline{f(\alpha_n)}\alpha_n$$

is the desired vector.

Some further comments are in order. The proof of Theorem 6 that we have given is admirably brief, but it fails to emphasize the essential geometric fact that β lies in the orthogonal complement of the null space of f. Let W be the null space of f. Then $V = W + W^\perp$, and f is completely determined by its values on $W^\perp$. In fact, if P is the orthogonal projection of V on $W^\perp$, then

$$f(\alpha) = f(P\alpha)$$

for all α in V. Suppose $f \neq 0$. Then f is of rank 1 and $\dim (W^\perp) = 1$. If γ is any non-zero vector in $W^\perp$, it follows that

$$P\alpha = \frac{(\alpha|\gamma)}{||\gamma||^2}\,\gamma$$

for all α in V. Thus

$$f(\alpha) = (\alpha|\gamma) \cdot \frac{f(\gamma)}{||\gamma||^2}$$

for all α, and $\beta = [\overline{f(\gamma)}/||\gamma||^2]\,\gamma$.

Example 16. We should give one example showing that Theorem 6 is not true without the assumption that V is finite dimensional. Let V be the vector space of polynomials over the field of complex numbers, with the inner product

$$(f|g) = \int_0^1 f(t)\overline{g(t)}\,dt.$$

This inner product can also be defined algebraically. If $f = \Sigma\, a_k x^k$ and $g = \Sigma\, b_k x^k$, then

$$(f|g) = \sum_{j,k} \frac{1}{j+k+1}\, a_j \overline{b_k}.$$

Let z be a fixed complex number, and let L be the linear functional 'evaluation at z':

$$L(f) = f(z).$$

Is there a polynomial g such that $(f|g) = L(f)$ for every f? The answer is no; for suppose we have

$$f(z) = \int_0^1 f(t)\overline{g(t)}\,dt$$

for every f. Let $h = x - z$, so that for any f we have $(hf)(z) = 0$. Then

$$0 = \int_0^1 h(t)f(t)\overline{g(t)}\,dt$$

for all f. In particular this holds when $f = \bar{h}g$ so that

$$\int_0^1 |h(t)|^2|g(t)|^2\,dt = 0$$

and so $hg = 0$. Since $h \neq 0$, it must be that $g = 0$. But L is not the zero functional; hence, no such g exists.

One can generalize the example somewhat, to the case where L is a linear combination of point evaluations. Suppose we select fixed complex numbers $z_1, \ldots, z_n$ and scalars $c_1, \ldots, c_n$ and let

$$L(f) = c_1 f(z_1) + \cdots + c_n f(z_n).$$

Then L is a linear functional on V, but there is no g with $L(f) = (f|g)$, unless $c_1 = c_2 = \cdots = c_n = 0$. Just repeat the above argument with $h = (x - z_1) \cdots (x - z_n)$.

We turn now to the concept of the adjoint of a linear operator.

Theorem 7. *For any linear operator* T *on a finite-dimensional inner product space* V, *there exists a unique linear operator* T* *on* V *such that*

$$(\text{8-14}) \qquad (\mathrm{T}\alpha|\beta) = (\alpha|\mathrm{T}^*\beta)$$

for all α, β *in* V.

Proof. Let β be any vector in V. Then $\alpha \to (T\alpha|\beta)$ is a linear functional on V. By Theorem 6 there is a unique vector β' in V such that $(T\alpha|\beta) = (\alpha|\beta')$ for every α in V. Let T^* denote the mapping $\beta \to \beta'$:

$$\beta' = T^*\beta.$$

We have (8-14), but we must verify that T^* is a linear operator. Let β, γ be in V and let c be a scalar. Then for any α,

$$\begin{aligned}(\alpha|T^*(c\beta + \gamma)) &= (T\alpha|c\beta + \gamma)\\ &= (T\alpha|c\beta) + (T\alpha|\gamma)\\ &= \bar{c}(T\alpha|\beta) + (T\alpha|\gamma)\\ &= \bar{c}(\alpha|T^*\beta) + (\alpha|T^*\gamma)\\ &= (\alpha|cT^*\beta) + (\alpha|T^*\gamma)\\ &= (\alpha|cT^*\beta + T^*\gamma).\end{aligned}$$

Thus $T^*(c\beta + \gamma) = cT^*\beta + T^*\gamma$ and T^* is linear.

The uniqueness of T^* is clear. For any β in V, the vector $T^*\beta$ is uniquely determined as the vector β' such that $(T\alpha|\beta) = (\alpha|\beta')$ for every α. ▌

Theorem 8. *Let* V *be a finite-dimensional inner product space and let* $\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}$ *be an (ordered) orthonormal basis for* V. *Let* T *be a linear operator on* V *and let* A *be the matrix of* T *in the ordered basis* $\mathcal{B}$. *Then* $A_{kj} = (T\alpha_j|\alpha_k)$.

Proof. Since $\mathcal{B}$ is an orthonormal basis, we have

$$\alpha = \sum_{k=1}^{n} (\alpha|\alpha_k)\alpha_k.$$

The matrix A is defined by

$$T\alpha_j = \sum_{k=1}^{n} A_{kj}\alpha_k$$

and since

$$T\alpha_j = \sum_{k=1}^{n} (T\alpha_j|\alpha_k)\alpha_k$$

we have $A_{kj} = (T\alpha_j|\alpha_k)$. ▌

Corollary. *Let V be a finite-dimensional inner product space, and let T be a linear operator on V. In any orthonormal basis for V, the matrix of T* is the conjugate transpose of the matrix of T.*

Proof. Let $\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}$ be an orthonormal basis for V, let $A = [T]_{\mathcal{B}}$ and $B = [T^*]_{\mathcal{B}}$. According to Theorem 8,

$$A_{kj} = (T\alpha_j|\alpha_k)$$
$$B_{kj} = (T^*\alpha_j|\alpha_k).$$

By the definition of T^* we then have

$$\begin{aligned} B_{kj} &= (T^*\alpha_j|\alpha_k) \\ &= \overline{(\alpha_k|T^*\alpha_j)} \\ &= \overline{(T\alpha_k|\alpha_j)} \\ &= \overline{A}_{jk}. \quad \blacksquare \end{aligned}$$

EXAMPLE 17. Let V be a finite-dimensional inner product space and E the orthogonal projection of V on a subspace W. Then for any vectors α and β in V.

$$\begin{aligned} (E\alpha|\beta) &= (E\alpha|E\beta + (1 - E)\beta) \\ &= (E\alpha|E\beta) \\ &= (E\alpha + (1 - E)\alpha|E\beta) \\ &= (\alpha|E\beta). \end{aligned}$$

From the uniqueness of the operator E^* it follows that $E^* = E$. Now consider the projection E described in Example 14. Then

$$A = \frac{1}{154}\begin{bmatrix} 9 & 36 & -3 \\ 36 & 144 & -12 \\ -3 & -12 & 1 \end{bmatrix}$$

is the matrix of E in the standard orthonormal basis. Since $E = E^*$, A is also the matrix of E^*, and because $A = A^*$, this does not contradict the preceding corollary. On the other hand, suppose

$$\begin{aligned} \alpha_1 &= (154, 0, 0) \\ \alpha_2 &= (145, -36, 3) \\ \alpha_3 &= (-36, 10, 12). \end{aligned}$$

Then $\{\alpha_1, \alpha_2, \alpha_3\}$ is a basis, and

$$\begin{aligned} E\alpha_1 &= (9, 36, -3) \\ E\alpha_2 &= (0, 0, 0) \\ E\alpha_3 &= (0, 0, 0). \end{aligned}$$

Since $(9, 36, -3) = -(154, 0, 0) - (145, -36, 3)$, the matrix B of E in the basis $\{\alpha_1, \alpha_2, \alpha_3\}$ is defined by the equation

$$B = \begin{bmatrix} -1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In this case $B \neq B^*$, and B^* is not the matrix of $E^* = E$ in the basis $\{\alpha_1, \alpha_2, \alpha_3\}$. Applying the corollary, we conclude that $\{\alpha_1, \alpha_2, \alpha_3\}$ is not an orthonormal basis. Of course this is quite obvious anyway.

Definition. *Let* T *be a linear operator on an inner product space* V. *Then we say that* T **has an adjoint on** V *if there exists a linear operator* T* *on* V *such that* $(T\alpha|\beta) = (\alpha|T^*\beta)$ *for all* α *and* β *in* V.

By Theorem 7 every linear operator on a finite-dimensional inner product space V has an adjoint on V. In the infinite-dimensional case this is not always true. But in any case there is at most one such operator T^*; when it exists, we call it the **adjoint** of T.

Two comments should be made about the finite-dimensional case.

1. The adjoint of T depends not only on T but on the inner product as well.

2. As shown by Example 17, in an arbitrary ordered basis $\mathcal{B}$, the relation between $[T]_{\mathcal{B}}$ and $[T^*]_{\mathcal{B}}$ is more complicated than that given in the corollary above.

Example 18. Let V be $C^{n\times 1}$, the space of complex $n \times 1$ matrices, with inner product $(X|Y) = Y^*X$. If A is an $n \times n$ matrix with complex entries, the adjoint of the linear operator $X \to AX$ is the operator $X \to A^*X$. For

$$(AX|Y) = Y^*AX = (A^*Y)^*X = (X|A^*Y).$$

The reader should convince himself that this is really a special case of the last corollary.

Example 19. This is similar to Example 18. Let V be $C^{n\times n}$ with the inner product $(A|B) = \operatorname{tr}(B^*A)$. Let M be a fixed $n \times n$ matrix over C. The adjoint of left multiplication by M is left multiplication by M^*. Of course, 'left multiplication by M' is the linear operator L_M defined by $L_M(A) = MA$.

$$\begin{aligned}(L_M(A)|B) &= \operatorname{tr}(B^*(MA)) \\ &= \operatorname{tr}(MAB^*) \\ &= \operatorname{tr}(AB^*M) \\ &= \operatorname{tr}(A(M^*B)^*) \\ &= (A|L_M{}^*(B)).\end{aligned}$$