Since $\mathbf{F}'_m(\mathbf{0})$ is invertible, the left side of (19) is not $\mathbf{0}$, and therefore there is a k such that $m \le k \le n$ and $(D_m \alpha_k)(\mathbf{0}) \ne 0$.

Let B_m be the flip that interchanges m and this k (if $k = m$, B_m is the identity) and define

$$\mathbf{G}_m(\mathbf{x}) = \mathbf{x} + [\alpha_k(\mathbf{x}) - x_m]\mathbf{e}_m \qquad (\mathbf{x} \in V_m). \tag{20}$$

Then $\mathbf{G}_m \in \mathscr{C}'(V_m)$, $\mathbf{G}_m$ is primitive, and $\mathbf{G}'_m(\mathbf{0})$ is invertible, since $(D_m \alpha_k)(\mathbf{0}) \ne 0$.

The inverse function theorem shows therefore that there is an open set U_m, with $\mathbf{0} \in U_m \subset V_m$, such that $\mathbf{G}_m$ is a 1-1 mapping of U_m onto a neighborhood V_{m+1} of $\mathbf{0}$, in which $\mathbf{G}_m^{-1}$ is continuously differentiable. Define $\mathbf{F}_{m+1}$ by

$$\mathbf{F}_{m+1}(\mathbf{y}) = B_m \mathbf{F}_m \circ \mathbf{G}_m^{-1}(\mathbf{y}) \qquad (\mathbf{y} \in V_{m+1}). \tag{21}$$

Then $\mathbf{F}_{m+1} \in \mathscr{C}'(V_{m+1})$, $\mathbf{F}_{m+1}(\mathbf{0}) = \mathbf{0}$, and $\mathbf{F}'_{m+1}(\mathbf{0})$ is invertible (by the chain rule). Also, for $\mathbf{x} \in U_m$,

$$\begin{aligned} P_m \mathbf{F}_{m+1}(\mathbf{G}_m(\mathbf{x})) &= P_m B_m \mathbf{F}_m(\mathbf{x}) \\ &= P_m[P_{m-1}\mathbf{x} + \alpha_k(\mathbf{x})\mathbf{e}_m + \cdots] \\ &= P_{m-1}\mathbf{x} + \alpha_k(\mathbf{x})\mathbf{e}_m \\ &= P_m \mathbf{G}_m(\mathbf{x}) \end{aligned} \tag{22}$$

so that

$$P_m \mathbf{F}_{m+1}(\mathbf{y}) = P_m \mathbf{y} \qquad (\mathbf{y} \in V_{m+1}). \tag{23}$$

Our induction hypothesis holds therefore with $m + 1$ in place of m.

[In (22), we first used (21), then (18) and the definition of B_m, then the definition of P_m, and finally (20).]

Since $B_m B_m = I$, (21), with $\mathbf{y} = \mathbf{G}_m(\mathbf{x})$, is equivalent to

$$\mathbf{F}_m(\mathbf{x}) = B_m \mathbf{F}_{m+1}(\mathbf{G}_m(\mathbf{x})) \qquad (\mathbf{x} \in U_m). \tag{24}$$

If we apply this with $m = 1, \ldots, n - 1$, we successively obtain

$$\begin{aligned} \mathbf{F} = \mathbf{F}_1 &= B_1 \mathbf{F}_2 \circ \mathbf{G}_1 \\ &= B_1 B_2 \mathbf{F}_3 \circ \mathbf{G}_2 \circ \mathbf{G}_1 = \cdots \\ &= B_1 \cdots B_{n-1} \mathbf{F}_n \circ \mathbf{G}_{n-1} \circ \cdots \circ \mathbf{G}_1 \end{aligned}$$

in some neighborhood of $\mathbf{0}$. By (17), $\mathbf{F}_n$ is primitive. This completes the proof.

PARTITIONS OF UNITY

10.8 Theorem *Suppose K is a compact subset of R^n, and $\{V_\alpha\}$ is an open cover of K. Then there exist functions $\psi_1, \ldots, \psi_s \in \mathscr{C}(R^n)$* such that

(*a*) $0 \leq \psi_i \leq 1$ *for* $1 \leq i \leq s$;
(*b*) *each ψ_i has its support in some V_α, and*
(*c*) $\psi_1(\mathbf{x}) + \cdots + \psi_s(\mathbf{x}) = 1$ *for every* $\mathbf{x} \in K$.

Because of (*c*), $\{\psi_i\}$ is called a *partition of unity*, and (*b*) is sometimes expressed by saying that $\{\psi_i\}$ is *subordinate* to the cover $\{V_\alpha\}$.

Corollary *If $f \in \mathscr{C}(R^n)$ and the support of f lies in K, then*

$$f = \sum_{i=1}^{s} \psi_i f. \tag{25}$$

Each $\psi_i f$ has its support in some V_α.

The point of (25) is that it furnishes a representation of f as a sum of continuous functions $\psi_i f$ with "small" supports.

Proof Associate with each $\mathbf{x} \in K$ an index $\alpha(\mathbf{x})$ so that $\mathbf{x} \in V_{\alpha(\mathbf{x})}$. Then there are open balls $B(\mathbf{x})$ and $W(\mathbf{x})$, centered at $\mathbf{x}$, with

$$\overline{B(\mathbf{x})} \subset W(\mathbf{x}) \subset \overline{W(\mathbf{x})} \subset V_{\alpha(\mathbf{x})}. \tag{26}$$

Since K is compact, there are points $\mathbf{x}_1, \ldots, \mathbf{x}_s$ in K such that

$$K \subset B(\mathbf{x}_1) \cup \cdots \cup B(\mathbf{x}_s). \tag{27}$$

By (26), there are functions $\varphi_1, \ldots, \varphi_s \in \mathscr{C}(R^n)$, such that $\varphi_i(\mathbf{x}) = 1$ on $B(\mathbf{x}_i)$, $\varphi_i(\mathbf{x}) = 0$ outside $W(\mathbf{x}_i)$, and $0 \leq \varphi_i(\mathbf{x}) \leq 1$ on R^n. Define $\psi_1 = \varphi_1$ and

$$\psi_{i+1} = (1 - \varphi_1) \cdots (1 - \varphi_i)\varphi_{i+1} \tag{28}$$

for $i = 1, \ldots, s - 1$.

Properties (*a*) and (*b*) are clear. The relation

$$\psi_1 + \cdots + \psi_i = 1 - (1 - \varphi_1) \cdots (1 - \varphi_i) \tag{29}$$

is trivial for $i = 1$. If (29) holds for some $i < s$, addition of (28) and (29) yields (29) with $i + 1$ in place of i. It follows that

$$\sum_{i=1}^{s} \psi_i(\mathbf{x}) = 1 - \prod_{i=1}^{s} [1 - \varphi_i(\mathbf{x})] \qquad (\mathbf{x} \in R^n). \tag{30}$$

If $\mathbf{x} \in K$, then $\mathbf{x} \in B(\mathbf{x}_i)$ for some i, hence $\varphi_i(\mathbf{x}) = 1$, and the product in (30) is 0. This proves (*c*).

CHANGE OF VARIABLES

We can now describe the effect of a change of variables on a multiple integral. For simplicity, we confine ourselves here to continuous functions with compact support, although this is too restrictive for many applications. This is illustrated by Exercises 9 to 13.

10.9 Theorem *Suppose T is a 1-1 $\mathscr{C}'$-mapping of an open set $E \subset R^k$ into R^k such that $J_T(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in E$. If f is a continuous function on R^k whose support is compact and lies in $T(E)$, then*

$$\int_{R^k} f(\mathbf{y})\, d\mathbf{y} = \int_{R^k} f(T(\mathbf{x}))\,|J_T(\mathbf{x})|\, d\mathbf{x}. \tag{31}$$

We recall that J_T is the Jacobian of T. The assumption $J_T(\mathbf{x}) \neq 0$ implies, by the inverse function theorem, that T^{-1} is continuous on $T(E)$, and this ensures that the integrand on the right of (31) has compact support in E (Theorem 4.14).

The appearance of the *absolute value* of $J_T(\mathbf{x})$ in (31) may call for a comment. Take the case $k = 1$, and suppose T is a 1-1 $\mathscr{C}'$-mapping of R^1 onto R^1. Then $J_T(x) = T'(x)$; and if T is *increasing*, we have

$$\int_{R^1} f(y)\, dy = \int_{R^1} f(T(x))T'(x)\, dx, \tag{32}$$

by Theorems 6.19 and 6.17, for all continuous f with compact support. But if T decreases, then $T'(x) < 0$; and if f is positive in the interior of its support, the left side of (32) is positive and the right side is negative. A correct equation is obtained if T' is replaced by $|T'|$ in (32).

The point is that the integrals we are now considering are integrals of functions over subsets of R^k, and we associate no direction or orientation with these subsets. We shall adopt a different point of view when we come to integration of differential forms over surfaces.

Proof It follows from the remarks just made that (31) is true if T is a primitive $\mathscr{C}'$-mapping (see Definition 10.5), and Theorem 10.2 shows that (31) is true if T is a linear mapping which merely interchanges two coordinates.

If the theorem is true for transformations P, Q, and if $S(\mathbf{x}) = P(Q(\mathbf{x}))$, then

$$\begin{aligned}\int f(\mathbf{z})\, d\mathbf{z} &= \int f(P(\mathbf{y}))\,|J_P(\mathbf{y})|\, d\mathbf{y}\\ &= \int f(P(Q(\mathbf{x})))\,|J_P(Q(\mathbf{x}))|\,|J_Q(\mathbf{x})|\, d\mathbf{x}\\ &= \int f(S(\mathbf{x}))\,|J_S(\mathbf{x})|\, d\mathbf{x},\end{aligned}$$

since

$$J_P(Q(\mathbf{x}))J_Q(\mathbf{x}) = \det P'(Q(\mathbf{x})) \det Q'(\mathbf{x})$$
$$= \det P'(Q(\mathbf{x}))Q'(\mathbf{x}) = \det S'(\mathbf{x}) = J_S(\mathbf{x}),$$

by the multiplication theorem for determinants and the chain rule. Thus the theorem is also true for S.

Each point $\mathbf{a} \in E$ has a neighborhood $U \subset E$ in which

$$T(\mathbf{x}) = T(\mathbf{a}) + B_1 \cdots B_{k-1}\mathbf{G}_k \circ \mathbf{G}_{k-1} \circ \cdots \circ \mathbf{G}_1(\mathbf{x}-\mathbf{a}), \tag{33}$$

where $\mathbf{G}_i$ and B_i are as in Theorem 10.7. Setting $V = T(U)$, it follows that (31) holds if the support of f lies in V. Thus:

Each point $\mathbf{y} \in T(E)$ *lies in an open set* $V_{\mathbf{y}} \subset T(E)$ *such that* (31) *holds for all continuous functions whose support lies in* $V_{\mathbf{y}}$.

Now let f be a continuous function with compact support $K \subset T(E)$. Since $\{V_{\mathbf{y}}\}$ covers K, the Corollary to Theorem 10.8 shows that $f = \Sigma\psi_i f$, where each ψ_i is continuous, and each ψ_i has its support in some $V_{\mathbf{y}}$. Thus (31) holds for each $\psi_i f$, and hence also for their sum f.

DIFFERENTIAL FORMS

We shall now develop some of the machinery that is needed for the n-dimensional version of the fundamental theorem of calculus which is usually called *Stokes' theorem*. The original form of Stokes' theorem arose in applications of vector analysis to electromagnetism and was stated in terms of the curl of a vector field. Green's theorem and the divergence theorem are other special cases. These topics are briefly discussed at the end of the chapter.

It is a curious feature of Stokes' theorem that the only thing that is difficult about it is the elaborate structure of definitions that are needed for its statement. These definitions concern differential forms, their derivatives, boundaries, and orientation. Once these concepts are understood, the statement of the theorem is very brief and succinct, and its proof presents little difficulty.

Up to now we have considered derivatives of functions of several variables only for functions defined in *open* sets. This was done to avoid difficulties that can occur at boundary points. It will now be convenient, however, to discuss differentiable functions on *compact* sets. We therefore adopt the following convention:

To say that $\mathbf{f}$ is a $\mathscr{C}'$-mapping (or a $\mathscr{C}''$-mapping) of a compact set $D \subset R^k$ into R^n means that there is a $\mathscr{C}'$-mapping (or a $\mathscr{C}''$-mapping) $\mathbf{g}$ of an open set $W \subset R^k$ into R^n such that $D \subset W$ and such that $\mathbf{g}(\mathbf{x}) = \mathbf{f}(\mathbf{x})$ for all $\mathbf{x} \in D$.

10.10 Definition Suppose E is an open set in R^n. A *k-surface* in E is a $\mathscr{C}'$-mapping Φ from a compact set $D \subset R^k$ into E.

D is called the *parameter domain* of Φ. Points of D will be denoted by $\mathbf{u} = (u_1, \ldots, u_k)$.

We shall confine ourselves to the simple situation in which D is either a k-cell or the k-simplex Q^k described in Example 10.4. The reason for this is that we shall have to integrate over D, and we have not yet discussed integration over more complicated subsets of R^k. It will be seen that this restriction on D (which will be tacitly made from now on) entails no significant loss of generality in the resulting theory of differential forms.

We stress that k-surfaces in E are defined to be *mappings* into E, not subsets of E. This agrees with our earlier definition of curves (Definition 6.26). In fact, 1-surfaces are precisely the same as continuously differentiable curves.

10.11 Definition Suppose E is an open set in R^n. A *differential form of order* $k \geq 1$ *in* E (briefly, a *k-form in* E) is a function ω, symbolically represented by the sum

$$\omega = \sum a_{i_1 \cdots i_k}(\mathbf{x})\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \tag{34}$$

(the indices $i_1, \ldots, i_k$ range independently from 1 to n), which assigns to each k-surface Φ in E a number $\omega(\Phi) = \int_\Phi \omega$, according to the rule

$$\int_\Phi \omega = \int_D \sum a_{i_1 \cdots i_k}(\Phi(\mathbf{u})) \frac{\partial(x_{i_1}, \ldots, x_{i_k})}{\partial(u_1, \ldots, u_k)}\, d\mathbf{u}, \tag{35}$$

where D is the parameter domain of Φ.

The functions $a_{i_1 \cdots i_k}$ are assumed to be real and continuous in E. If $\phi_1, \ldots, \phi_n$ are the components of Φ, the Jacobian in (35) is the one determined by the mapping

$$(u_1, \ldots, u_k) \to (\phi_{i_1}(\mathbf{u}), \ldots, \phi_{i_k}(\mathbf{u})).$$

Note that the right side of (35) is an integral over D, as defined in Definition 10.1 (or Example 10.4) and that (35) is the *definition* of the symbol $\int_\Phi \omega$.

A k-form ω is said to be of class $\mathscr{C}'$ or $\mathscr{C}''$ if the functions $a_{i_1 \cdots i_k}$ in (34) are all of class $\mathscr{C}'$ or $\mathscr{C}''$.

A 0-form in E is defined to be a continuous function in E.

10.12 Examples

(*a*) Let γ be a 1-surface (a curve of class $\mathscr{C}'$) in R^3, with parameter domain $[0, 1]$.

Write (x, y, z) in place of (x_1, x_2, x_3), and put

$$\omega = x\, dy + y\, dx.$$

Then

$$\int_\gamma \omega = \int_0^1 [\gamma_1(t)\gamma_2'(t) + \gamma_2(t)\gamma_1'(t)]\, dt = \gamma_1(1)\gamma_2(1) - \gamma_1(0)\gamma_2(0).$$

Note that in this example $\int_\gamma \omega$ depends only on the initial point $\gamma(0)$ and on the end point $\gamma(1)$ of γ. In particular, $\int_\gamma \omega = 0$ for every closed curve γ. (As we shall see later, this is true for every 1-form ω which is *exact*.)

Integrals of 1-forms are often called *line integrals*.

(*b*) Fix $a > 0$, $b > 0$, and define

$$\gamma(t) = (a \cos t, b \sin t) \qquad (0 \le t \le 2\pi),$$

so that γ is a closed curve in R^2. (Its range is an ellipse.) Then

$$\int_\gamma x\, dy = \int_0^{2\pi} ab \cos^2 t\, dt = \pi ab,$$

whereas

$$\int_\gamma y\, dx = -\int_0^{2\pi} ab \sin^2 t\, dt = -\pi ab.$$

Note that $\int_\gamma x\, dy$ is the area of the region bounded by γ. This is a special case of Green's theorem.

(*c*) Let D be the 3-cell defined by

$$0 \le r \le 1, \qquad 0 \le \theta \le \pi, \qquad 0 \le \varphi \le 2\pi.$$

Define $\Phi(r, \theta, \varphi) = (x, y, z)$, where

$$\begin{aligned} x &= r \sin\theta \cos\varphi \\ y &= r \sin\theta \sin\varphi \\ z &= r \cos\theta. \end{aligned}$$

Then

$$J_\Phi(r, \theta, \varphi) = \frac{\partial(x, y, z)}{\partial(r, \theta, \varphi)} = r^2 \sin\theta.$$

Hence

(36)
$$\int_\Phi dx \wedge dy \wedge dz = \int_D J_\Phi = \frac{4\pi}{3}.$$

Note that Φ maps D onto the closed unit ball of R^3, that the mapping is 1-1 in the interior of D (but certain boundary points are identified by Φ), and that the integral (36) is equal to the volume of $\Phi(D)$.

10.13 Elementary properties Let $\omega, \omega_1, \omega_2$ be k-forms in E. We write $\omega_1 = \omega_2$ if and only if $\omega_1(\Phi) = \omega_2(\Phi)$ for every k-surface Φ in E. In particular, $\omega = 0$ means that $\omega(\Phi) = 0$ for every k-surface Φ in E. If c is a real number, then $c\omega$ is the k-form defined by

$$\int_\Phi c\omega = c \int_\Phi \omega, \tag{37}$$

and $\omega = \omega_1 + \omega_2$ means that

$$\int_\Phi \omega = \int_\Phi \omega_1 + \int_\Phi \omega_2 \tag{38}$$

for every k-surface Φ in E. As a special case of (37), note that $-\omega$ is defined so that

$$\int_\Phi (-\omega) = -\int_\Phi d\omega. \tag{39}$$

Consider a k-form

$$\omega = a(\mathbf{x})\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \tag{40}$$

and let $\bar{\omega}$ be the k-form obtained by interchanging some pair of subscripts in (40). If (35) and (39) are combined with the fact that a determinant changes sign if two of its rows are interchanged, we see that

$$\bar{\omega} = -\omega. \tag{41}$$

As a special case of this, note that the *anticommutative relation*

$$dx_i \wedge dx_j = -dx_j \wedge dx_i \tag{42}$$

holds for all i and j. In particular,

$$dx_i \wedge dx_i = 0 \qquad (i = 1, \ldots, n). \tag{43}$$

More generally, let us return to (40), and assume that $i_r = i_s$ for some $r \neq s$. If these two subscripts are interchanged, then $\bar{\omega} = \omega$, hence $\omega = 0$, by (41).

In other words, *if ω is given by* (40), *then $\omega = 0$ unless the subscripts $i_1, \ldots, i_k$ are all distinct.*

If ω is as in (34), the summands with repeated subscripts can therefore be omitted without changing ω.

It follows that 0 is the only k-form in any open subset of R^n, if $k > n$.

The anticommutativity expressed by (42) is the reason for the inordinate amount of attention that has to be paid to minus signs when studying differential forms.