1. Prove that if G is an abelian simple group then $G \cong Z_p$ for some prime p (do not assume G is a finite group).
2. Exhibit all 3 composition series for Q_8 and all 7 composition series for D_8. List the composition factors in each case.
3. Find a composition series for the quasidihedral group of order 16 (cf. Exercise 11, Section 2.5). Deduce that QD_{16} is solvable.
4. Use Cauchy's Theorem and induction to show that a finite abelian group has a subgroup of order n for each positive divisor n of its order.
5. Prove that subgroups and quotient groups of a solvable group are solvable.
6. Prove part (1) of the Jordan–Hölder Theorem by induction on $|G|$.
7. If G is a finite group and $H \trianglelefteq G$ prove that there is a composition series of G, one of whose terms is H.
8. Let G be a *finite* group. Prove that the following are equivalent:
   (i) G is solvable
   (ii) G has a chain of subgroups: $1 = H_0 \trianglelefteq H_1 \trianglelefteq H_2 \trianglelefteq \ldots \trianglelefteq H_s = G$ such that H_{i+1}/H_i is cyclic, $0 \le i \le s-1$
   (iii) all composition factors of G are of prime order
   (iv) G has a chain of subgroups: $1 = N_0 \trianglelefteq N_1 \trianglelefteq N_2 \trianglelefteq \ldots \trianglelefteq N_t = G$ such that each N_i is a normal subgroup of G and N_{i+1}/N_i is abelian, $0 \le i \le t-1$.

   [For (iv), prove that a minimal nontrivial normal subgroup M of G is necessarily abelian and then use induction. To see that M is abelian, let $N \trianglelefteq M$ be of prime index (by (iii)) and show that $x^{-1}y^{-1}xy \in N$ for all $x, y \in M$ (cf. Exercise 40, Section 1). Apply the same argument to gNg^{-1} to show that $x^{-1}y^{-1}xy$ lies in the intersection of all G-conjugates of N, and use the minimality of M to conclude that $x^{-1}y^{-1}xy = 1$.]
9. Prove the following special case of part (2) of the Jordan–Hölder Theorem: assume the finite group G has two composition series
$$1 = N_0 \trianglelefteq N_1 \trianglelefteq \ldots \trianglelefteq N_r = G \qquad \text{and} \qquad 1 = M_0 \trianglelefteq M_1 \trianglelefteq M_2 = G.$$
   Show that $r = 2$ and that the list of composition factors is the same. [Use the Second Isomorphism Theorem.]
10. Prove part (2) of the Jordan–Hölder Theorem by induction on $\min\{r, s\}$. [Apply the inductive hypothesis to $H = N_{r-1} \cap M_{s-1}$ and use the preceding exercises.]
11. Prove that if H is a nontrivial normal subgroup of the solvable group G then there is a nontrivial subgroup A of H with $A \trianglelefteq G$ and A abelian.
12. Prove (without using the Feit–Thompson Theorem) that the following are equivalent:
    (i) every group of odd order is solvable
    (ii) the only simple groups of odd order are those of prime order.

3.5 TRANSPOSITIONS AND THE ALTERNATING GROUP

Transpositions and Generation of S_n

As we saw in Section 1.3 (and will prove in the next chapter) every element of S_n can be written as a product of disjoint cycles in an essentially unique fashion. In contrast, every element of S_n can be written in many different ways as a (nondisjoint) product of cycles. For example, even in S_3 the element $\sigma = (1\,2\,3)$ may be written

$$\sigma = (1\,2\,3) = (1\,3)(1\,2) = (1\,2)(1\,3)(1\,2)(1\,3) = (1\,2)(2\,3)$$

and, in fact, there are an infinite number of different ways to write σ. Not requiring the cycles to be disjoint totally destroys the uniqueness of a representation of a permutation as a product of cycles. We can, however, obtain a sort of "parity check" from writing permutations (nonuniquely) as products of 2-cycles.

Definition. A 2-cycle is called a *transposition*.

Intuitively, every permutation of $\{1, 2, \ldots, n\}$ can be realized by a succession of transpositions or simple interchanges of pairs of elements (try this on a small deck of cards sometime!). We illustrate how this may be done. First observe that

$$(a_1\, a_2 \ldots a_m) = (a_1\, a_m)(a_1\, a_{m-1})(a_1\, a_{m-2}) \ldots (a_1\, a_2)$$

for any m-cycle. Now any permutation in S_n may be written as a product of cycles (for instance, its cycle decomposition). Writing each of these cycles in turn as a product of transpositions by the above procedure we see that

every element of S_n may be written as a product of transpositions

or, equivalently,

$$S_n = \langle T \rangle \quad \text{where} \quad T = \{(i\;j) \mid 1 \le i < j \le n\}.$$

For example, the permutation σ in Section 1.3 may be written

$$\begin{aligned}\sigma &= (1\,12\,8\,10\,4)(2\,13)(5\,11\,7)(6\,9)\\ &= (1\,4)(1\,10)(1\,8)(1\,12)(2\,13)(5\,7)(5\,11)(6\,9).\end{aligned}$$

The Alternating Group

Again we emphasize that for any $\sigma \in S_n$ there may be many ways of writing σ as a product of transpositions. For fixed σ we now show that the parity (i.e., an odd or even number of terms) is the same for any product of transpositions equaling σ.

Let $x_1, \ldots, x_n$ be independent variables and let Δ be the polynomial

$$\Delta = \prod_{1 \le i < j \le n} (x_i - x_j),$$

i.e., the product of all the terms $x_i - x_j$ for $i < j$. For example, when $n = 4$,

$$\Delta = (x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)(x_3 - x_4).$$

For each $\sigma \in S_n$ let σ act on Δ by permuting the variables in the same way it permutes their indices:

$$\sigma(\Delta) = \prod_{1 \le i < j \le n} (x_{\sigma(i)} - x_{\sigma(j)}).$$

For example, if $n = 4$ and $\sigma = (1\ 2\ 3\ 4)$ then

$$\sigma(\Delta) = (x_2 - x_3)(x_2 - x_4)(x_2 - x_1)(x_3 - x_4)(x_3 - x_1)(x_4 - x_1)$$

(we have written the factors in the same order as above and applied σ to each factor to get $\sigma(\Delta)$). Note (in general) that Δ contains one factor $x_i - x_j$ for all $i < j$, and since σ is a bijection of the indices, $\sigma(\Delta)$ must contain either $x_i - x_j$ or $x_j - x_i$, but not both (and certainly no $x_i - x_i$ terms), for all $i < j$. If $\sigma(\Delta)$ has a factor $x_j - x_i$ where $j > i$, write this term as $-(x_i - x_j)$. Collecting all the changes in sign together we see that Δ and $\sigma(\Delta)$ have the same factors up to a product of -1's, i.e.,

$$\sigma(\Delta) = \pm\Delta, \qquad \text{for all } \sigma \in S_n.$$

For each $\sigma \in S_n$ let

$$\epsilon(\sigma) = \begin{cases} +1, & \text{if } \sigma(\Delta) = \Delta \\ -1, & \text{if } \sigma(\Delta) = -\Delta. \end{cases}$$

In the example above with $n = 4$ and $\sigma = (1\ 2\ 3\ 4)$, there are exactly 3 factors of the form $x_j - x_i$ where $j > i$ in $\sigma(\Delta)$, each of which contributes a factor of -1. Hence

$$(1\ 2\ 3\ 4)(\Delta) = (-1)^3(\Delta) = -\Delta,$$

so

$$\epsilon((1\ 2\ 3\ 4)) = -1.$$

Definition.

(1) $\epsilon(\sigma)$ is called the *sign* of σ.

(2) σ is called an *even permutation* if $\epsilon(\sigma) = 1$ and an *odd permutation* if $\epsilon(\sigma) = -1$

The next result shows that the sign of a permutation defines a homomorphism.

Proposition 23. The map $\epsilon : S_n \to \{\pm 1\}$ is a homomorphism (where $\{\pm 1\}$ is a multiplicative version of the cyclic group of order 2).

Proof: By definition,

$$(\tau\sigma)(\Delta) = \prod_{1 \le i < j \le n} (x_{\tau\sigma(i)} - x_{\tau\sigma(j)}).$$

Suppose that $\sigma(\Delta)$ has exactly k factors of the form $x_j - x_i$ with $j > i$, that is $\epsilon(\sigma) = (-1)^k$. When calculating $(\tau\sigma)(\Delta)$, after first applying σ to the indices we see that $(\tau\sigma)(\Delta)$ has exactly k factors of the form $x_{\tau(j)} - x_{\tau(i)}$ with $j > i$. Interchanging the order of the terms in these k factors introduces the sign change $(-1)^k = \epsilon(\sigma)$, and now all factors of $(\tau\sigma)(\Delta)$ are of the form $x_{\tau(p)} - x_{\tau(q)}$, with $p < q$. Thus

$$(\tau\sigma)(\Delta) = \epsilon(\sigma) \prod_{1 \le p < q \le n} (x_{\tau(p)} - x_{\tau(q)}).$$

Since by definition of ϵ

$$\prod_{1 \le p < q \le n} (x_{\tau(p)} - x_{\tau(q)}) = \epsilon(\tau)\Delta$$

we have $(\tau\sigma)(\Delta) = \epsilon(\sigma)\epsilon(\tau)\Delta$. Thus $\epsilon(\tau\sigma) = \epsilon(\sigma)\epsilon(\tau) = \epsilon(\tau)\epsilon(\sigma)$, as claimed.

To see the proof in action, let $n = 4$, $\sigma = (1\,2\,3\,4)$, $\tau = (4\,2\,3)$ so $\tau\sigma = (1\,3\,2\,4)$. By definition (using the explicit Δ in this case),

$$\begin{aligned}(\tau\sigma)(\Delta) &= (1\,3\,2\,4)(\Delta)\\ &= (x_3 - x_4)(x_3 - x_2)(x_3 - x_1)(x_4 - x_2)(x_4 - x_1)(x_2 - x_1)\\ &= (-1)^5\Delta\end{aligned}$$

where all factors except the first one are flipped to recover Δ. This shows $\epsilon(\tau\sigma) = -1$. On the other hand, since we already computed $\sigma(\Delta)$

$$\begin{aligned}(\tau\sigma)(\Delta) &= \tau(\sigma(\Delta))\\ &= (x_{\tau(2)} - x_{\tau(3)})(x_{\tau(2)} - x_{\tau(4)})(x_{\tau(2)} - x_{\tau(1)})(x_{\tau(3)} - x_{\tau(4)}) \times\\ &\qquad \times (x_{\tau(3)} - x_{\tau(1)})(x_{\tau(4)} - x_{\tau(1)})\\ &= (-1)^3 \prod_{1\le p<q\le 4} (x_{\tau(p)} - x_{\tau(q)}) = (-1)^3\tau(\Delta)\end{aligned}$$

where here the third, fifth, and sixth factors need to have their terms interchanged in order to put all factors in the form $x_{\tau(p)} - x_{\tau(q)}$ with $p < q$. We already calculated that $\epsilon(\sigma) = (-1)^3 = -1$ and, by the same method, it is easy to see that $\epsilon(\tau) = (-1)^2 = 1$ so $\epsilon(\tau\sigma) = -1 = \epsilon(\tau)\epsilon(\sigma)$.

The next step is to compute $\epsilon((i\ j))$, for any transposition $(i\ j)$. Rather than compute this directly for arbitrary i and j we do it first for $i = 1$ and $j = 2$ and reduce the general case to this. It is clear that applying $(1\ 2)$ to Δ (regardless of what n is) will flip exactly one factor, namely $x_1 - x_2$; thus $\epsilon((1\ 2)) = -1$. Now for any transposition $(i\ j)$ let λ be the permutation which interchanges 1 and i, interchanges 2 and j, and leaves all other numbers fixed (if $i = 1$ or $j = 2$, λ fixes i or j, respectively). Then it is easy to see that $(i\ j) = \lambda(1\ 2)\lambda$ (compute what the right hand side does to any $k \in \{1, 2, \ldots, n\}$). Since ϵ is a homomorphism we obtain

$$\begin{aligned}\epsilon((i\ j)) &= \epsilon(\lambda(1\ 2)\lambda)\\ &= \epsilon(\lambda)\epsilon((1\ 2))\epsilon(\lambda)\\ &= (-1)\epsilon(\lambda)^2\\ &= -1.\end{aligned}$$

This proves

Proposition 24. Transpositions are all odd permutations and ϵ is a surjective homomorphism.

Definition. The *alternating group of degree n*, denoted by A_n, is the kernel of the homomorphism ϵ (i.e., the set of even permutations).

Note that by the First Isomorphism Theorem $S_n/A_n \cong \epsilon(S_n) = \{\pm 1\}$, so that the order of A_n is easily determined: $|A_n| = \frac{1}{2}|S_n| = \frac{1}{2}(n!)$. Also, $S_n - A_n$ is the coset of