14. For any group G define the *dual group* of G (denoted $\widehat{G}$) to be the set of all homomorphisms from G into the multiplicative group of roots of unity in $\mathbb{C}$. Define a group operation in $\widehat{G}$ by pointwise multiplication of functions: if χ, ψ are homomorphisms from G into the group of roots of unity then $\chi\psi$ is the homomorphism given by $(\chi\psi)(g) = \chi(g)\psi(g)$ for all $g \in G$, where the latter multiplication takes place in $\mathbb{C}$.
   (a) Show that this operation on $\widehat{G}$ makes $\widehat{G}$ into an abelian group. [Show that the identity is the map $g \mapsto 1$ for all $g \in G$ and the inverse of $\chi \in \widehat{G}$ is the map $g \mapsto \chi(g)^{-1}$.]
   (b) If G is a finite abelian group, prove that $\widehat{G} \cong G$. [Write G as $\langle x_1 \rangle \times \cdots \times \langle x_r \rangle$ and if $n_i = |x_i|$ define χ_i to be the homomorphism which sends x_i to $e^{2\pi i/n_i}$ and sends x_j to 1, for all $j \neq i$. Prove χ_i has order n_i in $\widehat{G}$ and $\widehat{G} = \langle \chi_1 \rangle \times \cdots \times \langle \chi_r \rangle$.]

   (This result is often phrased: a finite abelian group is self-dual. It implies that the lattice diagram of a finite abelian group is the same when it is turned upside down. Note however that there is no *natural* isomorphism between G and its dual (the isomorphism depends on a choice of a set of generators for G). This is frequently stated in the form: a finite abelian group is *noncanonically* isomorphic to its dual.)
15. Let $G = \langle x \rangle \times \langle y \rangle$ where $|x| = 8$ and $|y| = 4$.
   (a) Find all pairs a, b in G such that $G = \langle a \rangle \times \langle b \rangle$ (where a and b are expressed in terms of x and y).
   (b) Let $H = \langle x^2y, y^2 \rangle \cong Z_4 \times Z_2$. Prove that there are no elements a, b of G such that $G = \langle a \rangle \times \langle b \rangle$ and $H = \langle a^2 \rangle \times \langle b^2 \rangle$ (i.e., one cannot pick direct product generators for G in such a way that some powers of these are direct product generators for H).
16. Prove that no finitely generated abelian group is divisible (cf. Exercise 19, Section 2.4).

5.3 TABLE OF GROUPS OF SMALL ORDER

At this point we can give a table of the isomorphism types for most of the groups of small order.

Each of the unfamiliar non-abelian groups in the table on the following page will be constructed in Section 5 on semidirect products (which will also explain the notation used for them). For the present we give generators and relations for each of them (i.e., presentations of them).

The group $Z_3 \rtimes Z_4$ of order 12 can be described by the generators and relations:

$$\langle x, y \mid x^4 = y^3 = 1,\ x^{-1}yx = y^{-1} \rangle,$$

namely, it has a normal Sylow 3-subgroup ($\langle y \rangle$) which is inverted by an element of order 4 (x) acting by conjugation (x^2 centralizes y).

The group $(Z_3 \times Z_3) \rtimes Z_2$ has generators and relations:

$$\langle x, y, z \mid x^2 = y^3 = z^3 = 1,\ yz = zy,\ x^{-1}yx = y^{-1},\ x^{-1}zx = z^{-1} \rangle,$$

namely, it has a normal Sylow 3-subgroup isomorphic to $Z_3 \times Z_3$ ($\langle y, z \rangle$) inverted by an element of order 2 (x) acting by conjugation.

The group $Z_5 \rtimes Z_4$ of order 20 has generators and relations:

$$\langle x, y \mid x^4 = y^5 = 1,\ x^{-1}yx = y^{-1} \rangle,$$

namely, it has a normal Sylow 5-subgroup ($\langle y \rangle$) which is inverted by an element of order 4 (x) acting by conjugation (x^2 centralizes y).

Order	No. of Isomorphism Types	Abelian Groups	Non-abelian Groups
1	1	Z_1	none
2	1	Z_2	none
3	1	Z_3	none
4	2	$Z_4,\ Z_2 \times Z_2$	none
5	1	Z_5	none
6	2	Z_6	S_3
7	1	Z_7	none
8	5	$Z_8,\ Z_4 \times Z_2,$ $Z_2 \times Z_2 \times Z_2$	$D_8,\ Q_8$
9	2	$Z_9,\ Z_3 \times Z_3$	none
10	2	Z_{10}	D_{10}
11	1	Z_{11}	none
12	5	$Z_{12},\ Z_6 \times Z_2$	$A_4,\ D_{12},\ Z_3 \rtimes Z_4$
13	1	Z_{13}	none
14	2	Z_{14}	D_{14}
15	1	Z_{15}	none
16	14	$Z_{16},\ Z_8 \times Z_2,$ $Z_4 \times Z_4,\ Z_4 \times Z_2 \times Z_2,$ $Z_2 \times Z_2 \times Z_2 \times Z_2$	not listed
17	1	Z_{17}	none
18	5	$Z_{18},\ Z_6 \times Z_3$	$D_{18},\ S_3 \times Z_3,$ $(Z_3 \times Z_3) \rtimes Z_2$
19	1	Z_{19}	none
20	5	$Z_{20},\ Z_{10} \times Z_2$	D_{20} $Z_5 \rtimes Z_4,\ F_{20}$

The group F_{20} of order 20 has generators and relations:

$$\langle x, y \mid x^4 = y^5 = 1,\ xyx^{-1} = y^2 \rangle,$$

namely, it has a normal Sylow 5-subgroup ($\langle y \rangle$) which is squared by an element of order 4 (x) acting by conjugation. One can check that this group occurs as the normalizer of a Sylow 5-subgroup in S_5, e.g.,

$$F_{20} = \langle (2\,3\,5\,4),\ (1\,2\,3\,4\,5) \rangle.$$

This group is called the *Frobenius group* of order 20.

EXERCISE

1. Prove that D_{16}, $Z_2 \times D_8$, $Z_2 \times Q_8$, $Z_4 * D_8$, QD_{16} and M are nonisomorphic non-abelian groups of order 16 (where $Z_4 * D_8$ is described in Exercise 12, Section 1 and QD_{16} and M are described in the exercises in Section 2.5).

5.4 RECOGNIZING DIRECT PRODUCTS

So far we have seen that direct products may be used to both construct "larger" groups from "smaller" ones and to decompose finitely generated abelian groups into cyclic factors. Even certain non-abelian groups, which may be given in some other form, may be decomposed as direct products of smaller groups. The purpose of this section is to indicate a criterion to recognize when a group is the direct product of some of its subgroups and to illustrate the criterion with some examples.

Before doing so we introduce some standard notation and elementary results on commutators which will streamline the presentation and which will be used again in Chapter 6 when we consider nilpotent groups.

Definition. Let G be a group, let $x, y \in G$ and let A, B be nonempty subsets of G.

(1) Define $[x, y] = x^{-1}y^{-1}xy$, called the *commutator* of x and y.

(2) Define $[A, B] = \langle\, [a, b] \mid a \in A,\ b \in B \,\rangle$, the group generated by commutators of elements from A and from B.

(3) Define $G' = \langle\, [x, y] \mid x, y \in G \,\rangle$, the subgroup of G generated by commutators of elements from G, called the *commutator subgroup* of G.

The commutator of x and y is 1 if and only if x and y commute, which explains the terminology. The following proposition shows how commutators measure the "difference" in G between xy and yx.

Proposition 7. Let G be a group, let $x, y \in G$ and let $H \leq G$. Then

(1) $xy = yx[x, y]$ (in particular, $xy = yx$ if and only if $[x, y] = 1$).

(2) $H \trianglelefteq G$ if and only if $[H, G] \leq H$.

(3) $\sigma[x, y] = [\sigma(x), \sigma(y)]$ for any automorphism σ of G, G' char G and G/G' is abelian.

(4) G/G' is the largest abelian quotient of G in the sense that if $H \trianglelefteq G$ and G/H is abelian, then $G' \leq H$. Conversely, if $G' \leq H$, then $H \trianglelefteq G$ and G/H is abelian.

(5) If $\varphi : G \to A$ is any homomorphism of G into an abelian group A, then φ factors through G' i.e., $G' \leq \ker\varphi$ and the following diagram commutes:

$$\begin{tikzcd} G \arrow[r] \arrow[dr, "\varphi"'] & G/G' \arrow[d] \\ & A \end{tikzcd}$$

Proof: (1) This is immediate from the definition of $[x, y]$.

(2) By definition, $H \trianglelefteq G$ if and only if $g^{-1}hg \in H$ for all $g \in G$ and all $h \in H$. For $h \in H$, $g^{-1}hg \in H$ if and only if $h^{-1}g^{-1}hg \in H$, so that $H \trianglelefteq G$ if and only if $[h, g] \in H$ for all $h \in H$ and all $g \in G$. Thus $H \trianglelefteq G$ if and only if $[H, G] \leq H$, which is (2).

(3) Let $\sigma \in \mathrm{Aut}(G)$ be an automorphism of G and let $x, y \in G$. Then

$$\begin{aligned}\sigma([x, y]) &= \sigma(x^{-1}y^{-1}xy)\\ &= \sigma(x)^{-1}\sigma(y)^{-1}\sigma(x)\sigma(y)\\ &= [\sigma(x), \sigma(y)].\end{aligned}$$

Thus for every commutator $[x, y]$ of G', $\sigma([x, y])$ is again a commutator. Since σ has a 2-sided inverse, it follows that it maps the set of commutators bijectively onto itself. Since the commutators are a generating set for G', $\sigma(G') = G'$, that is, G' char G.

To see that G/G' is abelian, let xG' and yG' be arbitrary elements of G/G'. By definition of the group operation in G/G' and since $[x, y] \in G'$ we have

$$\begin{aligned}(xG')(yG') &= (xy)G'\\ &= (yx[x, y])G'\\ &= (yx)G' = (yG')(xG'),\end{aligned}$$

which completes the proof of (3).

(4) Suppose $H \trianglelefteq G$ and G/H is abelian. Then for all $x, y \in G$ we have $(xH)(yH) = (yH)(xH)$, so

$$\begin{aligned}1H &= (xH)^{-1}(yH)^{-1}(xH)(yH)\\ &= x^{-1}y^{-1}xyH\\ &= [x, y]H.\end{aligned}$$

Thus $[x, y] \in H$ for all $x, y \in G$, so that $G' \leq H$.

Conversely, if $G' \leq H$, then since G/G' is abelian by (3), every subgroup of G/G' is normal. In particular, $H/G' \trianglelefteq G/G'$. By the Lattice Isomorphism Theorem $H \trianglelefteq G$. By the Third Isomorphism Theorem

$$G/H \cong (G/G')/(H/G')$$

hence G/H is abelian (being isomorphic to a quotient of the abelian group G/G'). This proves (4).

(5) This is (4) phrased in terms of homomorphisms.

Passing to the quotient by the commutator subgroup of G collapses all commutators to the identity so that all elements in the quotient group commute. As (4) indicates, a strong converse to this also holds: a quotient of G by H is abelian if and only if the commutator subgroup is contained in H (i.e., if and only if G' is mapped to the identity in the quotient G/H).

We shall exhibit a group (of order 96) in the next section with the property that one of the elements of its commutator subgroup *cannot* be written as a single commutator $[x, y]$ for any x and y. Thus G' does not necessarily consist only of the set of (single) commutators (but is the group *generated* by these elements).