p	q	$1/p + 1/q$	E	F	V	Polyhedron
3	3	2/3	6	4	4	tetrahedron
3	4	7/12	12	8	6	octahedron
4	3	7/12	12	6	8	cube
3	5	8/15	30	20	12	icosahedron
5	3	8/15	30	12	20	dodecahedron

TABLE 9.1. The regular polyhedra

der because, having discovered the dodecahedron, he failed to ascribe his discovery to Pythagoras.

When Plato proposed that the creator, whom he called the 'demiurge', used the regular polyhedra when forming the universe, he may not have been that far off. The tetrahedron, cube and octahedron can be found in nature as crystals. The octahedron, icosahedron and dodecahedron occur as the skeletons of certain radiolarians (a type of microscopic sea animal).

Are there only five regular polyhedra? Yes. In fact, a proof of this is found at the end of Euclid's Elements (300 BC). This proof is based on the fact that if q regular p-gons meet at a vertex, then the sum of the q angles in the q faces is less than 360°. This is proved rigorously in Proposition 21 of Book XI of the *Elements*, but it can be seen intuitively by imagining someone cutting the q edges and flattening the angle. For example, the three angles at the vertex of a cube clearly add up to 270°, which is less than 360°.

In general, for a regular polyhedron whose faces are regular p-gons, with q faces meeting at each vertex,

$$q(p-2)180^\circ/p < 360^\circ \, ,$$

which may be simplified to yield $1/2 < 1/p + 1/q$. We can easily see that there are just five possibilities for p and q, as in Table 9.1. (In the table, E is the number of edges the polyhedron has, F the number of faces, and V the number of vertices.)

We have not as yet explained how the numbers E, F and V in our table are calculated. It so happens that these numbers are related by a simple formula, even when we consider an arbitrary polyhedron, regular or not. Here are some examples:

	F	V	E
cube	6	8	12
tetrahedron	4	4	6
pyramid	5	5	8
prism	5	6	9

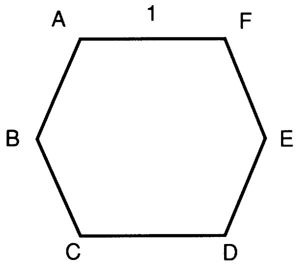

FIGURE 9.2. Cross-section of an icosahedron

We note that in each example

$$F + V - E = 2.$$

This is in fact a general rule, valid for all polyhedra. While it may first have been noted by Descartes, it was only proved by Euler and is known as *Euler's formula.* We shall give a proof in Chapter 30 of Part I.

Now suppose we are looking at a regular solid in which each face has p edges and in which q edges meet at each vertex. It follows immediately that

$$pF = 2E, \qquad qV = 2E.$$

Substituting $F = 2E/p$ and $V = 2E/q$ into Euler's formula, we obtain

$$2E/p + 2E/q - E = 2,$$

and, after dividing by $2E$,

$$1/p + 1/q - 1/2 = 1/E.$$

This allows us to calculate E from p and q, and then F and V.

The ancient Greeks were fascinated by the five regular solids. Without the help of trigonometry or calculus, they managed to prove all their basic properties. Book XIII of the *Elements* (300 BC) is devoted to showing that, for each of these five solids, there is a sphere passing through all its vertices. In each of the five cases, Book XIII calculates the ratio of the side of the regular polyhedron to the radius of this 'circumscribing' sphere.

For example, if one cuts an icosahedron in half, cutting along a side, the resulting cross-section is as in Figure 9.2. AF and CD are edges; AC and DF are diagonals in the regular pentagons formed by the sides of the icosahedron. (You may have to construct an icosahedron to see this.) Thus, if AF and CD each have unit length, AC and DF each have length $\frac{1}{2}(1+\sqrt{5})$. The diameter of the circumscribing sphere is CF, which is the hypotenuse of the right triangle with sides CD and DF. Thus $CF^2 = 1^2 + (\frac{1}{2}(1+\sqrt{5}))^2$, and hence the radius of the sphere is $\frac{1}{2}\sqrt{\frac{1}{2}(5+\sqrt{5})}$.

Exercises

1. Construct a dodecahedron, e.g., by taping together 12 identical regular pentagons cut out of cardboard.

2. Show that the radius of a sphere passing through the vertices of a dodecahedron with side 1 is $(\sqrt{3}+\sqrt{15})/4$.

3. Show that the volume of a dodecahedron of side 1 is $(15+7\sqrt{5})/4$.

4. Given a polyhedron, not necessarily regular, in which exactly 3 edges meet at each vertex. Show that $V = 2K$, $E = 3K$ and $F = K+2$ for some positive integer K.

5. Under the condition of the previous exercise, if F_p is the number of faces with p sides, show that
$$\sum_p (6-p)F_p = 12.$$

6. If all faces of a polygon are hexagons or pentagons and if three edges meet at each vertex, prove that the number of pentagons is twelve. (There are molecules of such a shape with twenty hexagons, called 'buckyballs', a form of carbon called 'buckminster fullerene'. See Chung and Sternberg [1993].)