subgroup which cannot be extended to the whole group take G to be any simple group and let φ be any representation of H with the property that $\ker\varphi$ is a proper, nontrivial normal subgroup of H. If φ extended to a representation Φ of G then the kernel of Φ would be a proper, nontrivial subgroup of G, contrary to G being a simple group. We shall see that the method of induced characters produces a representation Φ of G from a given representation φ of its subgroup H but that $\Phi|_H \neq \varphi$ in general (indeed, unless $H = G$ the degree of Φ will be greater than the degree of φ).

We saw in Example 5 following Corollary 9 in Section 10.4 that because FH is a subring of FG, the ring FG is an (FG, FH)-bimodule; and so for any left FH-module V, the abelian group $FG \otimes_{FH} V$ is a left FG-module (called the extension of scalars from FH to FG for V). In the representation theory of finite groups this extension is given a special name.

Definition. Let H be a subgroup of the finite group G and let V be an FH-module affording the representation φ of H. The FG-module $FG \otimes_{FH} V$ is called the *induced module* of V and the representation of G it affords is called the *induced representation* of φ. If ψ is the character of φ then the character of the induced representation is called the *induced character* and is denoted by $\mathrm{Ind}_H^G(\psi)$.

Theorem 11. Let H be a subgroup of the finite group G and let $g_1, \dots, g_m$ be representatives for the distinct left cosets of H in G. Let V be an FH-module affording the matrix representation φ of H of degree n. The FG-module $W = FG \otimes_{FH} V$ has dimension nm over F and there is a basis of W such that W affords the matrix representation Φ defined for each $g \in G$ by

$$\Phi(g) = \begin{pmatrix} \varphi(g_1^{-1}gg_1) & \cdots & \varphi(g_1^{-1}gg_m) \\ \vdots & \vdots & \vdots \\ \varphi(g_m^{-1}gg_1) & \cdots & \varphi(g_m^{-1}gg_m) \end{pmatrix}$$

where each $\varphi(g_i^{-1}gg_j)$ is an $n \times n$ block appearing in the i, j block position of $\Phi(g)$, and where $\varphi(g_i^{-1}gg_j)$ is defined to be the zero block whenever $g_i^{-1}gg_j \notin H$.

Proof: First note that FG is a free right FH-module:

$$FG = g_1FH \oplus g_2FH \oplus \cdots \oplus g_mFH.$$

Since tensor products commute with direct sums (Theorem 17, Section 10.4), as abelian groups we have

$$W = FG \otimes_{FH} V \cong (g_1 \otimes V) \oplus (g_2 \otimes V) \oplus \cdots \oplus (g_m \otimes V).$$

Since F is in the center of FG it follows that this is an F-vector space isomorphism as well. Thus if $v_1, v_2, \dots, v_n$ is a basis of V affording the matrix representation φ, then $\{g_i \otimes v_j \mid 1 \le i \le m,\ 1 \le j \le n\}$ is a basis of W. This shows the dimension of W is mn. Order the basis into m sets, each of size n as

$$g_1 \otimes v_1, g_1 \otimes v_2, \dots, g_1 \otimes v_n, g_2 \otimes v_1, \dots, g_2 \otimes v_n, \dots\dots, g_m \otimes v_n.$$

We compute the matrix representation $\Phi(g)$ of each g acting on W with respect to this basis. Fix j and g, and let $gg_j = g_i h$ for some index i and some $h \in H$. Then for every k

$$\begin{aligned} g(g_j \otimes v_k) = (gg_j) \otimes v_k &= g_i \otimes hv_k \\ &= \sum_{t=1}^{n} a_{tk}(h)(g_i \otimes v_t) \end{aligned}$$

where a_{tk} is the t, k coefficient of the matrix of h acting on V with respect to the basis $\{v_1, \dots, v_n\}$. In other words, the action of g on W maps the j^{th} block of n basis vectors of W to the i^{th} block of basis vectors, and then has the matrix $\varphi(h)$ on that block. Since $h = g_i^{-1} g g_j$, this describes the block matrix $\Phi(g)$ of the theorem, as needed.

Corollary 12. In the notation of Theorem 11

(1) if ψ is the character afforded by V then the induced character is given by

$$\mathrm{Ind}_H^G(\psi)(g) = \sum_{i=1}^{m} \psi(g_i^{-1} g g_i)$$

where $\psi(g_i^{-1} g g_i)$ is defined to be 0 if $g_i^{-1} g g_i \notin H$, and

(2) $\mathrm{Ind}_H^G(\psi)(g) = 0$ if g is not conjugate in G to some element of H. In particular, if H is a normal subgroup of G then $\mathrm{Ind}_H^G(\psi)$ is zero on all elements of $G - H$.

Remark: Since the character ψ of H is constant on the conjugacy classes of H we have $\psi(g) = \psi(h^{-1} g h)$ for all $h \in H$. As h runs over all elements of H, xh runs over all elements of the coset xH. Thus the formula for the induced character may also be written

$$\mathrm{Ind}_H^G(\psi)(g) = \frac{1}{|H|} \sum_{x \in G} \psi(x^{-1} g x)$$

where the elements x in each fixed coset give the same character value $|H|$ times (which accounts for the factor of $1/|H|$), and again $\psi(x^{-1} g x) = 0$ if $x^{-1} g x \notin H$.

Proof: From the matrix of g computed above, the blocks $\varphi(g_i^{-1} g g_i)$ down the diagonal of $\Phi(g)$ are zero except when $g_i^{-1} g g_i \in H$. Thus the trace of the block matrix $\Phi(g)$ is the sum of the traces of the matrices $\varphi(g_i^{-1} g g_i)$ for which $g_i^{-1} g g_i \in H$. Since the trace of $\varphi(g_i^{-1} g g_i)$ is $\psi(g_i^{-1} g g_i)$, part (1) holds.

If $g_i^{-1} g g_i \notin H$ for all coset representatives g_i then each term in the sum for $\mathrm{Ind}_H^G(\psi)(g)$ is zero. In particular, if g is not in the normal subgroup H then neither is any conjugate of g, so $\mathrm{Ind}_H^G(\psi)$ is zero on g.

Examples

(1) Let $G = D_{12} = \langle r, s \mid r^6 = s^2 = 1,\ rs = sr^{-1} \rangle$ be the dihedral group of order 12 and let $H = \{1, s, r^3, sr^3\}$, so that H is isomorphic to the Klein 4-group and $|G : H| = 3$. Following the notation of Theorem 11 we exhibit the matrices for r and s of the induced representation of a specific representation φ of H. Let the representation of H on a 2-dimensional vector space over $\mathbb{Q}$ with respect to some basis v_1, v_2 be given by

$$\varphi(s) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = A, \quad \varphi(r^3) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = B, \quad \varphi(sr^3) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = C,$$

so $n = 2$, $m = 3$ and the induced representation Φ has degree $nm = 6$. Fix representatives $g_1 = 1$, $g_2 = r$, and $g_3 = r^2$ for the left cosets of H in G, so that $g_k = r^{k-1}$. Then

$$g_i^{-1} r g_j = r^{-(i-1)+1+(j-1)} = r^{j-i+1}, \text{ and}$$
$$g_i^{-1} s g_j = s r^{(i-1)+(j-1)} = s r^{i+j-2}.$$

Thus the 6×6 matrices for the induced representation are seen to be

$$\Phi(r) = \begin{pmatrix} 0 & 0 & B \\ I & 0 & 0 \\ 0 & I & 0 \end{pmatrix} \qquad \Phi(s) = \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & C \\ 0 & C & 0 \end{pmatrix}$$

where the 2×2 matrices A, B and C are given above, I is the 2×2 identity matrix and 0 denotes the 2×2 zero matrix.

(2) If H is any subgroup of G and ψ_1 is the principal character of H, then $\mathrm{Ind}_H^G(\psi_1)(g)$ counts 1 for each coset representative g_i such that $g_i^{-1} g g_i \in H$. Since $g_i^{-1} g g_i \in H$ if and only if g fixes the left coset $g_i H$ under left multiplication, $\mathrm{Ind}_H^G(\psi_1)(g)$ is the number of points fixed by g in the permutation representation of g on the left cosets of H. Thus by Example 3 of Section 18.3 we see that: *if ψ_1 is the principal character of H then $\mathrm{Ind}_H^G(\psi_1)$ is the permutation character on the left cosets of H in G.* In the special case when $H = 1$, this implies *if χ_1 is the principal character of the trivial subgroup $H = 1$ then $\mathrm{Ind}_1^G(\chi_1)$ is the regular character of G.* This also shows that an induced character is not, in general, irreducible even if the character from which it is induced is irreducible.

(3) Let $G = S_3$ and let ψ be a nonprincipal linear character of $A_3 = \langle x \rangle$, so that $\psi(x) = \zeta$, for some primitive cube root of unity ζ (the character tables of $A_3 = Z_3$ and S_3 appear in Section 1). Let $\Psi = \mathrm{Ind}_{A_3}^{S_3}(\psi)$. Thus Ψ has degree $1 \cdot |S_3 : A_3| = 2$ and, by the corollary, Ψ is zero on all transpositions. If y is any transposition then $1, y$ is a set of left coset representatives of A_3 in S_3 and $y^{-1}xy = x^2$. Thus $\Psi(x) = \psi(x) + \psi(x^2)$ equals $\zeta + \zeta^2 = -1$. This shows that if ψ is either of the two nonprincipal irreducible characters of A_3 then the induced character of ψ is the (unique) irreducible character of S_3 of degree 2. In particular, different characters of a subgroup may induce the same character of the whole group.

(4) Let $G = D_8$ have its usual generators and relations and let $H = \langle s \rangle$. Let ψ be the nonprincipal irreducible character of H and let $\Psi = \mathrm{Ind}_H^G(\psi)$. Pick left coset representatives $1, r, r^2, r^3$ for H. By Theorem 11, $\Psi(1) = 4$. Since $\psi(s) = -1$, one computes directly that $\Psi(s) = -2$. By Corollary 12(2) we obtain $\Psi(r) = \Psi(r^2) = \Psi(sr) = 0$. In the notation of the character table of D_8 in Section 1, by the orthogonality relations we obtain $\Psi = \chi_2 + \chi_4 + \chi_5$ (which may be checked by inspection).

For the remainder of this section the field F is taken to be the complex numbers: $F = \mathbb{C}$.

Before concluding with an application of induced characters to simple groups we compute the characters of an important class of groups.