a segment CE (with E on CD produced) which is a times as long as CD. One often writes $m(a,b)$ as ab.

How can we make an M-set into a functor? Let F map the category M to the category of sets so that $F(*)$ is X and, for any $a \in M$. $F(a) : X \to X$ is the function which sends $b \in X$ onto $F(a)(b) = ab$. Thus $F(1)$ is the identity function on X and $F(a \cdot a') = F(a) \circ F(a')$. For, if $b \in X$ then $F(a \cdot a')(b) = (aa')b = a(a'b) = F(a)(F(a')(b))$.

Call a category *small* if its class of objects and its class of arrows are both *sets*. (The distinction between 'set' and 'class' is made clear in the set theory of Gödel and Bernays.) *Cat* is the category whose objects are small categories and whose arrows are the functors from one small category to another. The fact that the small categories themselves form a category again illustrates the slogan that 'interesting objects congregate in categories'.

Exercises

1. Complete the details of Example 33.3 on pre-ordered sets.

2. Show in detail that *Cat* is indeed a category.

3. A graph is *small* if its class of objects and its class of arrows are both sets. Show that there is a category *Grph* whose objects are small graphs, and whose arrows are like functors, except that they need not satisfy the equations (2) and (3) in the definition of a functor. (*Grph* will be discussed in some detail in the following section.)

34

Natural Transformations

Category theory began in 1945 with Eilenberg and Mac Lane's article 'General Theory of Natural Equivalences'. In this chapter we will investigate the notion of a 'natural equivalence'.

Let A and B be categories and let F and G be functors from A to B. A *natural transformation* t from F to G is a mapping that assigns to every object a of A an arrow $t(a)$ in B from $F(a)$ to $G(a)$, such that, for any arrow f in A from a to b, $G(f) \circ t(a) = t(b) \circ F(f)$. This can be pictured as follows:

$$\begin{tikzcd}
\mathrm{F(a)} \arrow[r, "\mathrm{t(a)}"] \arrow[d, "\mathrm{F(f)}"'] & \mathrm{G(a)} \arrow[d, "\mathrm{G(f)}"] \\
\mathrm{F(b)} \arrow[r, "\mathrm{t(b)}"] & \mathrm{G(b)}
\end{tikzcd}$$

Note that a and b are objects in category A, whereas all of the objects and arrows in the picture are in B.

EXAMPLE 34.1

We saw in Example 33.4 that a set S can be viewed as a functor F from the discrete one-object category to the category of sets, in such a way that F takes its object to S and its arrow to the identity function on S. Let S and S' be two sets, and F and F' the corresponding functors. Given a function f from S to S', let t be a mapping that assigns to the object $*$ of the discrete one-object category the arrow f from S to S'. Then $F'(1_*) \circ t(*) = 1_{S'} \circ f = f = f \circ 1_S = t(*) \circ F(1_*)$. Hence t is a natural transformation from F to F'. Conversely, it is easy to show that every natural transformation from F to F' must have this form.

EXAMPLE 34.2

A *small graph* consists of a set of objects, a set of arrows, and two mappings (*source* and *target*) from the set of arrows to the set of objects. A *morphism* F between small graphs is a mapping which sends the objects of the first graph to the objects of the second, and the arrows of the first graph to the arrows of the second. Moreover, if $f : a \to b$ in the first graph, we require that $F(f) : F(a) \to F(b)$. That is, graph morphisms preserve source and target. In Example 33.5 we saw that a graph may be viewed as a functor. Now we shall show that a graph morphism can be viewed as a natural transformation between functors which represent graphs.

Let A be the two-object category of Example 33.5 and F any functor from A to the category of sets. We saw that $(Y, X, F(s), F(t))$ forms a small graph, call it G.

Suppose h is a graph morphism from G to $G' = (Y', X', F'(s), F'(t))$, where F' is a second functor from A to the category of sets and $F'(\mathbf{a}) = X'$, $F'(\mathbf{o}) = Y'$. Let τ be a function from the set $\{\mathbf{a}, \mathbf{o}\}$ to the class of arrows in *Set*, which assigns to $\mathbf{a}$ the map $h : X \to X'$ and to $\mathbf{o}$ the map $h : Y \to Y'$. To show that τ is a natural transformation, we must show that, if f is either s or t, then $F'(f) \circ \tau(\mathbf{a}) = \tau(\mathbf{o}) \circ F(f)$. (The equations $F'(1_{\mathbf{a}}) \circ \tau(\mathbf{a}) = \tau(\mathbf{a}) \circ F'(1_{\mathbf{a}})$ and $F'(1_{\mathbf{o}}) \circ \tau(\mathbf{o}) = \tau(\mathbf{o}) \circ F'(1_{\mathbf{o}})$ follow at once.)

Both $F'(s) \circ \tau(\mathbf{a})$ and $\tau(\mathbf{o}) \circ F(s)$ map X to Y'. If $a \in X$, then $(F'(s) \circ \tau(\mathbf{a}))(a) = F'(s)(h(a))$, which is the source of $h(a)$ in Y'. Since h is a graph morphism, $F'(s)(h(a)) = h(F(s)(a))$. But this equals $\tau(\mathbf{o}) \circ F(s)(a)$. Similarly, one can show that $F'(t) \circ \tau(\mathbf{a}) = \tau(\mathbf{o}) \circ F(t)$. Thus τ is a natural transformation.

EXAMPLE 34.3

We saw in Chapter 33 that an *M-set* can also be regarded as a functor. If (M, X, m) and (M, X', m') are two M-sets, an M*-homomorphism* is a function f from X to X' such that, if $a \in M$ and $b \in X$, $f(m(a, b)) = m'(a, f(b))$. We usually write this equation as $f(ab) = af(b)$. We will let the reader show that such an M-homomorphism may be viewed as a natural transformation between the functors that represent the M-sets.

It may seem from the above examples that category theory is merely a complicated way of expressing simpler ideas. However, one should bear in mind that the abstract definitions of the theory embody the ideas and methods of many branches of mathematics at once, and thus may serve to unify their separate proofs and results. Chapter 35 should help to illustrate this fact.

Exercises

1. Show that every M-homomorphism can be viewed as a natural transformation.

2. Write an essay supporting one of the following views:

   (a) Category theory is a perfect example of useless abstraction. Instead of giving us something new in mathematics, it merely burdens us with a new jargon.

   (b) Category theory is the crown of contemporary mathematics. It combines insights from different branches of mathematics and provides a common language for discussing them.