means geometry with notions of length, angle, and curvature, not necessarily Euclidean geometry. In fact, the natural geometric models of most surfaces are hyperbolic.

It remains to be seen whether topology as a whole will ever be subordinate to ordinary geometry. This is conjectured to be the case in three dimensions though the situation has so far been too complicated to resolve completely [see Thurston (1997) or Weeks (1985)]. It does appear that here, too, hyperbolic geometry is the most important geometry. In four or more dimensions it would be rash to speculate, though geometric methods have been important in recent breakthroughs [for example Donaldson (1983)]. In this chapter we make a virtue of a necessity by confining our discussion to the topology of surfaces. This is the only area that is sufficiently understandable and relevant when set against the background of the rest of this book. Fortunately, this area is also rich enough to illustrate some important topological ideas, while still being mathematically tractable and visual.

We begin the discussion of surface topology at its historical starting point, the theory of polyhedra.

22.2 Polyhedron Formulas of Descartes and Euler

The first topological property of polyhedra seems to have been discovered by Descartes around 1630. Descartes' short paper on the subject is lost, but its contents are known from a copy made by Leibniz in 1676, discovered among Leibniz' papers in 1860 and published in Prouhet (1860). A detailed study of this paper, including a translation and facsimile of the Leibniz manuscript, has been published by Federico (1982).

The same property was rediscovered by Euler (1752), and it is now known as the *Euler characteristic.* If a polyhedron has V vertices, E edges, and F faces, then its Euler characteristic is $V-E+F$. Euler showed that this quantity has certain invariance by showing

$$V-E+F=2$$

for all convex polyhedra, a result now known as the *Euler polyhedron formula.* Descartes already had the same result implicitly in the pair of formulas

$$P=2F+2V-4, \quad P=2E,$$

proach, which also yields the value of $V-E+F$ for *nonspherical* surfaces, is explained in the next section.

EXERCISES

Here is the proof of the Euler polyhedron formula by Legendre (1794).

22.2.1 Consider the projection of a convex polyhedron onto a sphere, whose faces are therefore spherical polygons. Use the fact that

$$\text{area of a spherical } n\text{-gon} = \text{angle sum} - (n-2)\pi$$

to conclude that

$$\text{total area} = 4\pi = (\textstyle\sum \text{all angles}) - \pi(\sum \text{all } n) + 2\pi F.$$

22.2.2 Show also that

$$\textstyle\sum \text{all } n = 2E, \quad \sum \text{all angles} = 2\pi V$$

whence

$$V-E+F=2.$$

The invariance of the Euler characteristic gives a simple topological proof that there are only five regular polyhedra. In fact, it shows that only five polyhedra are *topologically regular* in the following sense: for some $m,n>2$ their "faces" are topological m-gons on a topological sphere, n of which meet at each vertex. We show as follows that $V-E+F=2$ allows only the pairs

$$(m,n) = (3,3),(3,4),(3,5),(4,3),(5,3),$$

corresponding to the known regular polyhedra (Section 2.2).

22.2.3 Given that there are F faces, deduce that $E=mF/2$ and $V=mF/n$.

22.2.4 Apply the formula $V-E+F=2$ to conclude that $4n/(2m+2n-mn)$ is a positive integer.

22.2.5 Show that $2m+2n-mn>0$, that is $2\frac{m}{n}+2>m$, only for the above pairs (m,n).

22.2.6 Also check that $2m+2n-mn$ divides $4n$ for these pairs.

22.3 The Classification of Surfaces

Between the 1850s and the 1880s, several different lines of research led to the demand for a topological classification of surfaces. One line, descending from Euler, was the classification of polyhedra. Another was the

Riemann surface representation of algebraic curves, coming from Riemann (1851, 1857). Related to this was the problem of classifying symmetry groups of tessellations, considered by Poincaré (1882) and Klein (1882b) (see Section 22.6). Finally, there was the problem of classifying smooth closed surfaces in ordinary space [Möbius (1863)]. These different lines of research converged when it was realized that in each case the surface could be subdivided into faces by edges (not necessarily straight, of course) so that it became a generalized polyhedron. The generalized polyhedra are what were traditionally called *closed* surfaces, now described by topologists as *compact and without boundary.*

The subdivision argument for the invariance of the Euler characteristic $V - E + F$ applies to any such polyhedron, not just those homeomorphic to the sphere and not just those with straight edges and flat faces. Various mathematicians [Riemann (1851), Jordan (1866)] came to the conclusion that any closed surface is determined, up to homeomorphism, by its Euler characteristic. It also emerged that the different possible Euler characteristics were represented by the "normal form" surfaces seen in Figure 22.1, which were discovered by Möbius (1863). It is certainly plausible that these forms are distinct, topologically, because of their different numbers of "holes." The main part of the proof is to show that any closed surface is homeomorphic to one of them.

The assumptions of Riemann (that the surface is a Riemann surface) and Möbius (that the surface is smoothly embedded in $\mathbb{R}^3$) were a little too special to yield a purely topological proof, and in addition they contained a hidden assumption of *orientability* ("two-sidedness"). A rigorous proof, from an axiomatic definition of generalized polyhedron, was given by Dehn and Heegaard (1907). The closed orientable surfaces indeed turn out to be those pictured in Figure 22.1, but in addition there are *nonorientable* surfaces, which are not homeomorphic to orientable surfaces.

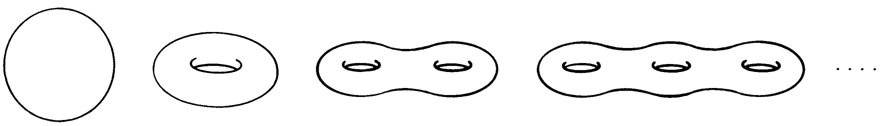

Figure 22.1: Closed orientable surfaces

A nonorientable surface may be defined as one that contains a *Möbius band*, a nonclosed surface discovered independently by Möbius and Listing in 1858 (Figure 22.2). Closed nonorientable surfaces cannot occur as Riemann surfaces, nor can they lie in $\mathbb{R}^3$ without crossing themselves; nevertheless, they include some important surfaces, such as the projective plane (Exercise 8.5.5). The nonorientable surfaces are also determined, up to homeomorphism, by the Euler characteristic.

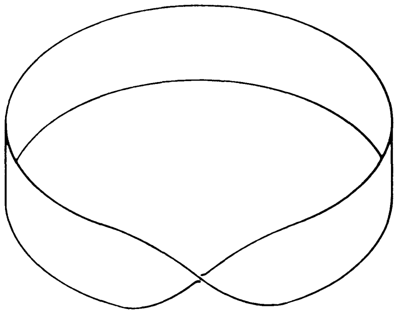

Figure 22.2: The Möbius band

The Möbius forms of closed orientable surfaces were given standard polyhedral structures by Klein (1882b). These are "minimal" subdivisions with just one face and, except for the sphere, with just one vertex. When the Klein subdivision of a surface is cut along its edges, one obtains a *fundamental polygon*, from which the surface may be reconstructed by identifying like-labeled edges [Figure 22.3, which is taken from Hilbert and Cohn-Vossen (1932)].

It is often more convenient to work with the polygon rather than the surface or its polyhedral structure. For example, since Brahana (1921), most proofs of the classification theorem have used polygons rather than polyhedra, "cutting and pasting" them (instead of subdividing and amalgamating) until Klein's fundamental polygons are obtained. The fundamental polygon gives a very easy calculation of the Euler characteristic χ and shows it to be related to the *genus* g (number of "holes") by

$$\chi = 2 - 2g$$

(Exercise 22.3.1). Of course, the genus determines the surface more simply than the Euler characteristic, but we shall see that the Euler characteristic is a better reflection of geometric properties.

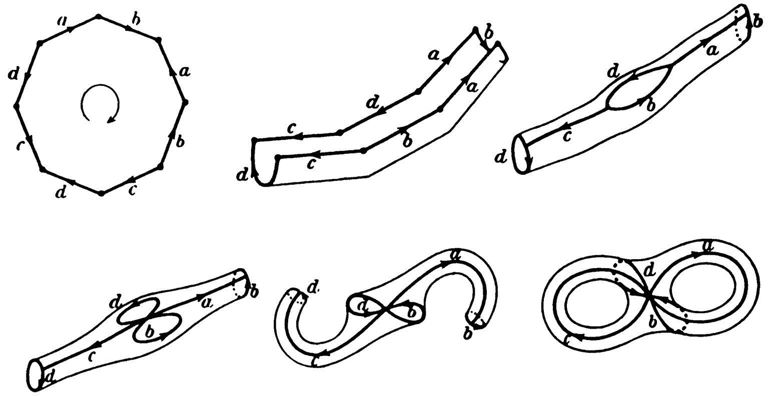

Figure 22.3: Constructing a surface by edge pasting

EXERCISES

22.3.1 Show that the standard polyhedron for a surface of genus $g \geq 1$ has $V = 1$, $E = 2g$, $F = 1$, whence $\chi = 2 - 2g$.

The standard polygon for the genus g surface has a boundary path of the form $a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$, where successive letters denote successive edges and those with exponents -1 have oppositely directed arrows. Edges with the same letter are pasted together, with arrows matching.

22.3.2 Each sequence $a_i b_i a_i^{-1} b_i^{-1}$ is called a *handle*. Justify this term by drawing the surface that results from pasting together the matching edges of the polygon bounded by $a_i b_i a_i^{-1} b_i^{-1} c$. The result should be a "handle-shaped" surface with boundary curve c.

Another simple fundamental polygon is the "$2n$-gon with opposite edges pasted together," in other words, the polygon with boundary path of the form

$$a_1 a_2 \cdots a_n a_1^{-1} a_2^{-1} \cdots a_n^{-1}.$$

22.3.3 Show that for both $n = 2$ and $n = 3$ the surface obtained from the polygon $a_1 a_2 \cdots a_n a_1^{-1} a_2^{-1} \cdots a_n^{-1}$ is a torus.

22.3.4 Show that if n is even, the vertices of the polygon $a_1 a_2 \cdots a_n a_1^{-1} a_2^{-1} \cdots a_n^{-1}$ become a single vertex after pasting, and if n is odd they become two. Hence find the Euler characteristic of the surface for any n.

22.4 Descartes and Gauss–Bonnet

The first theorem in the Descartes manuscript is a remarkable statement about the total "curvature" of a convex polyhedron, not at first appearing to have any topological content. It is a spatial analogue of the obvious theorem that the sum of the external angles of a convex polygon is 2π. The latter theorem can be seen intuitively by considering the total turn of a line that is transported around the polygon (Figure 22.4).

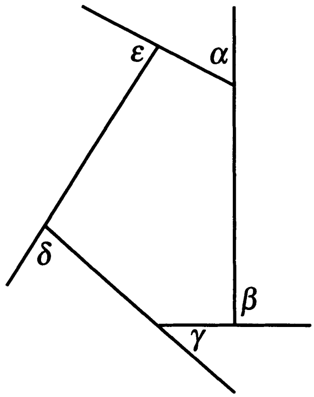

Figure 22.4: Total turn around a polygon

Figure 22.5 shows a different proof, which generalizes to polyhedra.

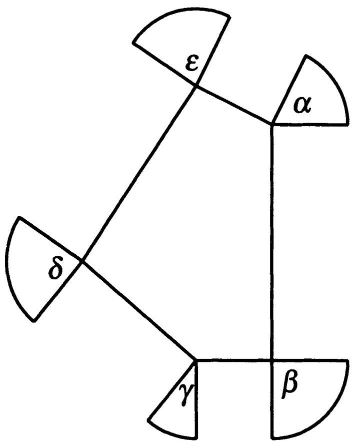

Figure 22.5: Adding the sectors bounded by normals

At each vertex, construct a sector of a unit circle, bounded by normals to the two edges at that vertex. Clearly, the angle of the sector equals the external angle at that vertex. Also, adjacent sides of adjacent sectors are perpendicular to the same edge, hence parallel, so the sectors can be fitted together to form a complete disk, of total angle (circumference) 2π.

To generalize this to polyhedra, define the *exterior solid angle* at each vertex P to be the (area of the) sector of a unit ball bounded by planes normal to the edges at P (Figure 22.6). As before, adjacent sides of adjacent sectors are parallel, hence the sectors can be fitted together to form a complete ball, of total solid angle (area) 4π. Descartes only stated that the total exterior solid angle is 4π, without even defining exterior solid angle. The foregoing proof is based on the reconstruction by Pólya (1954a).

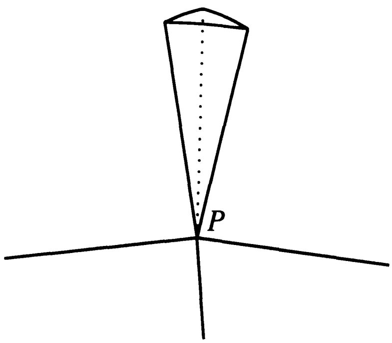

Figure 22.6: The exterior solid angle

The theorem about polygons has an analogue for simple closed smooth curves $\mathscr{C}$, namely, $\int_{\mathscr{C}} \kappa \, ds = 2\pi$ (Section 17.2). This leads us to wonder whether the Descartes theorem has an analogue for smooth closed convex surfaces $\mathscr{S}$, say, $\iint_{\mathscr{S}} \kappa_1 \kappa_2 \, dA = 4\pi$, where $\kappa_1 \kappa_2$ is the Gaussian curvature. This is so, and in fact there is a proof like the polyhedron proof using yet another characterization of Gaussian curvature due to Gauss (1827).

If we take a small geodesic polygon $\mathscr{P}$ on the surface $\mathscr{S}$, then the "total curvature" of the portion $\mathscr{P}$ can be represented by an "exterior solid angle" $\mathscr{A}$ bounded by parallels to the normals to $\mathscr{S}$ along the sides of $\mathscr{P}$ (Figure 22.7). Gauss showed that the measure of $\mathscr{A}$—the area it cuts out of the unit sphere—is $\iint_{\mathscr{P}} \kappa_1 \kappa_2 \, dA$. But it is also clear, by the parallelism of adjacent sides of adjacent exterior solid angles $\mathscr{A}$, that the $\mathscr{A}$'s corresponding to a partition of $\mathscr{S}$ by geodesic polygons $\mathscr{P}$ fit together to form a complete ball. Hence $\iint_{\mathscr{S}} \kappa_1 \kappa_2 \, dA = 4\pi$.

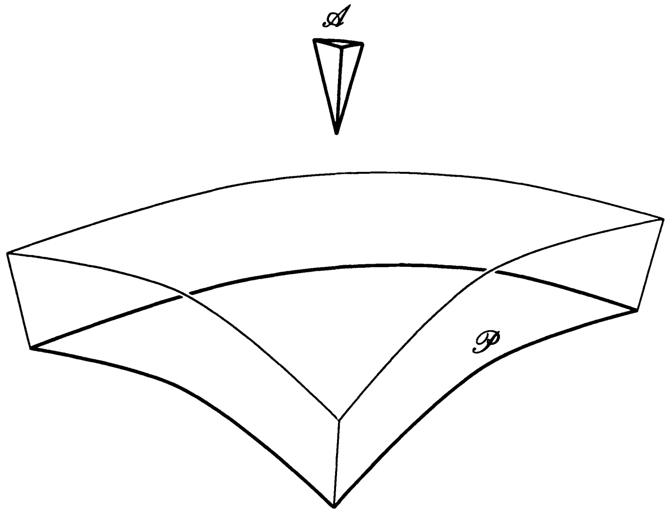

Figure 22.7: The solid angle of total curvature

This is a "global" form of the Gauss–Bonnet theorem. When Descartes' theorem was first published in 1860, the Gauss–Bonnet theorem was already known, and the analogy between the two was noted by Bertrand (1860). Bertrand, however, made the qualification that "the beautiful conception of Gauss could not in any manner be considered as a corollary to that of Descartes." This may be true in a narrow sense; nevertheless, the Descartes and Gauss–Bonnet theorems can be viewed as limiting cases of each other. Gauss–Bonnet $\Rightarrow$ Descartes by concentrating the curvature of a surface at vertices until it becomes a polyhedron, while Descartes $\Rightarrow$ Gauss–Bonnet by increasing the number of vertices of a polyhedron until it becomes a smooth surface. It is interesting, though probably accidental, that Descartes actually uses the word "curvatura" to describe the exterior solid angle.

22.5 Euler Characteristic and Curvature

There is another, more "intrinsic" proof of Descartes' theorem that reveals the fact that total exterior solid angle is really $2\pi \times$ Euler characteristic. In fact, knowledge of the total exterior angle yields a proof of the Euler characteristic polyhedron formula. This seems to have been the way in which Descartes discovered his version of the formula.