We have to check that this is a legitimate notion of equality. We need to verify the reflexivity, symmetry, transitivity, and substitution axioms (see Section A.7). We leave reflexivity and symmetry to Exercise 4.1.1 and instead verify the transitivity axiom. Suppose we know that $a—b = c—d$ and $c—d = e—f$. Then we have $a + d = c + b$ and $c + f = d + e$. Adding the two equations together we obtain $a+d+c+f = c+b+d+e$. By Proposition 2.2.6 we can cancel the c and d, obtaining $a+f = b+e$, i.e., $a—b = e—f$. Thus the cancellation law was needed to make sure that our notion of equality is sound. As for the substitution axiom, we cannot verify it at this stage because we have not yet defined any operations on the integers. However, when we do define our basic operations on the integers, such as addition, multiplication, and order, we will have to verify the substitution axiom at that time in order to ensure that the definition is valid. (We will only need to do this for the basic operations; more advanced operations on the integers, such as exponentiation, will be defined in terms of the basic ones, and so we do not need to re-verify the substitution axiom for the advanced operations.)

Now we define two basic arithmetic operations on integers: addition and multiplication.

Definition 4.1.2. The sum of two integers, $(a—b) + (c—d)$, is defined by the formula

$$(a—b) + (c—d) := (a + c)—(b + d).$$

The product of two integers, $(a—b) \times (c—d)$, is defined by

$$(a—b) \times (c—d) := (ac + bd)—(ad + bc).$$

Thus for instance, $(3—5) + (1—4)$ is equal to $(4—9)$. There is however one thing we have to check before we can accept these definitions - we have to check that if we replace one of the integers by an equal integer, that the sum or product does not change. For instance, $(3—5)$ is equal to $(2—4)$, so $(3—5) + (1—4)$ ought to have the same value as $(2—4) + (1—4)$, otherwise this would not give a consistent definition of addition. Fortunately, this is the case:

Lemma 4.1.3 (Addition and multiplication are well-defined). *Let a, b, a', b', c, d be natural numbers. If $(a—b) = (a'—b')$, then $(a—b) + (c—d) = (a'—b') + (c—d)$ and $(a—b) \times (c—d) = (a'—b') \times (c—d)$, and also $(c—d) + (a—b) = (c—d) + (a'—b')$ and $(c—d) \times (a—b) = (c—d) \times (a'—b')$. Thus addition and multiplication are well-defined operations (equal inputs give equal outputs).*

Proof. To prove that $(a—b) + (c—d) = (a'—b') + (c—d)$, we evaluate both sides as $(a + c)—(b + d)$ and $(a' + c)—(b' + d)$. Thus we need to show that $a + c + b' + d = a' + c + b + d$. But since $(a—b) = (a'—b')$, we have $a + b' = a' + b$, and so by adding $c + d$ to both sides we obtain the claim. Now we show that $(a—b) \times (c—d) = (a'—b') \times (c—d)$. Both sides evaluate to $(ac + bd)—(ad + bc)$ and $(a'c + b'd)—(a'd + b'c)$, so we have to show that $ac + bd + a'd + b'c = a'c + b'd + ad + bc$. But the left-hand side factors as $c(a + b') + d(a' + b)$, while the right factors as $c(a' + b) + d(a + b')$. Since $a + b' = a' + b$, the two sides are equal. The other two identities are proven similarly. $\square$

The integers $n—0$ behave in the same way as the natural numbers n; indeed one can check that $(n—0) + (m—0) = (n + m)—0$ and $(n—0) \times (m—0) = nm—0$. Furthermore, $(n—0)$ is equal to $(m—0)$ if and only if $n = m$. (The mathematical term for this is that there is an *isomorphism* between the natural numbers n and those integers of the form $n—0$.) Thus we may *identify* the natural numbers with integers by setting $n \equiv n—0$; this does not affect our definitions of addition or multiplication or equality since they are consistent with each other. For instance the natural number 3 is now considered to be the same as the integer $3—0$, thus $3 = 3—0$. In particular 0 is equal to $0—0$ and 1 is equal to $1—0$. Of course, if we set n equal to $n—0$, then it will also be equal to any other integer which is equal to $n—0$, for instance 3 is equal not only to $3—0$, but also to $4—1$, $5—2$, etc.

We can now define incrementation on the integers by defining $x{+}{+} := x + 1$ for any integer x; this is of course consistent with our definition of the increment operation for natural numbers. However, this is no longer an important operation for us, as it has been now superceded by the more general notion of addition.

Now we consider some other basic operations on the integers.

Definition 4.1.4 (Negation of integers). If $(a—b)$ is an integer, we define the negation $-(a—b)$ to be the integer $(b—a)$. In particular if $n = n—0$ is a positive natural number, we can define its negation $-n = 0—n$.

For instance $-(3—5) = (5—3)$. One can check this definition is well-defined (Exercise 4.1.2).

We can now show that the integers correspond exactly to what we expect.

Lemma 4.1.5 (Trichotomy of integers). *Let x be an integer. Then exactly one of the following three statements is true: (a) x is zero; (b) x is equal to a positive natural number n; or (c) x is the negation $-n$ of a positive natural number n.*

Proof. We first show that at least one of (a), (b), (c) is true. By definition, $x = a—b$ for some natural numbers a, b. We have three cases: $a > b$, $a = b$, or $a < b$. If $a > b$ then $a = b + c$ for some positive natural number c, which means that $a—b = c—0 = c$, which is (b). If $a = b$, then $a—b = a—a = 0—0 = 0$, which is (a). If $a < b$, then $b > a$, so that $b—a = n$ for some natural number n by the previous reasoning, and thus $a—b = -n$, which is (c).

Now we show that no more than one of (a), (b), (c) can hold at a time. By definition, a positive natural number is non-zero, so (a) and (b) cannot simultaneously be true. If (a) and (c) were simultaneously true, then $0 = -n$ for some positive natural n; thus $(0—0) = (0—n)$, so that $0 + n = 0 + 0$, so that $n = 0$, a contradiction. If (b) and (c) were simultaneously true, then $n = -m$ for some positive n, m, so that $(n—0) = (0—m)$, so that $n + m = 0 + 0$, which contradicts Proposition 2.2.8. Thus exactly one of (a), (b), (c) is true for any integer x. $\square$

If n is a positive natural number, we call $-n$ a *negative integer*. Thus every integer is positive, zero, or negative, but not more than one of these at a time.

One could well ask why we don't use Lemma 4.1.5 to *define* the integers; i.e., why didn't we just say an integer is anything which is either a positive natural number, zero, or the negative of a natural number. The reason is that if we did so, the rules for adding and multiplying integers would split into many different cases (e.g., negative times positive equals positive; negative plus positive is either negative, positive, or zero, depending on which term is larger, etc.) and to verify all the properties would end up being much messier.

We now summarize the algebraic properties of the integers.

Proposition 4.1.6 (Laws of algebra for integers). *Let x, y, z be integers. Then we have*

$$x + y = y + x$$
$$(x + y) + z = x + (y + z)$$
$$x + 0 = 0 + x = x$$
$$x + (-x) = (-x) + x = 0$$
$$xy = yx$$
$$(xy)z = x(yz)$$
$$x1 = 1x = x$$
$$x(y + z) = xy + xz$$
$$(y + z)x = yx + zx.$$

Remark 4.1.7. The above set of nine identities have a name; they are asserting that the integers form a *commutative ring*. (If one deleted the identity $xy = yx$, then they would only assert that the integers form a *ring*). Note that some of these identities were already proven for the natural numbers, but this does not automatically mean that they also hold for the integers because the integers are a larger set than the natural numbers. On the other hand, this proposition supercedes many of the propositions derived earlier for natural numbers.

Proof. There are two ways to prove these identities. One is to use Lemma 4.1.5 and split into a lot of cases depending on whether x, y, z are zero, positive, or negative. This becomes very messy. A shorter way is to write $x = (a—b)$, $y = (c—d)$, and $z = (e—f)$ for some natural numbers a, b, c, d, e, f, and expand these identities in terms of a, b, c, d, e, f and use the algebra of the natural numbers. This allows each identity to be proven in a few lines. We shall just prove the longest one, namely $(xy)z = x(yz)$:

$$\begin{aligned}
(xy)z &= ((a—b)(c—d))\,(e—f) \\
&= ((ac+bd)—(ad+bc))\,(e—f) \\
&= ((ace+bde+adf+bcf)—(acf+bdf+ade+bce));\\
x(yz) &= (a—b)\,((c—d)(e—f)) \\
&= (a—b)\,((ce+df)—(cf+de)) \\
&= ((ace+adf+bcf+bde)—(acf+ade+bce+bdf))
\end{aligned}$$

and so one can see that $(xy)z$ and $x(yz)$ are equal. The other identities are proven in a similar fashion; see Exercise 4.1.4. □

We now define the operation of *subtraction* $x-y$ of two integers by the formula

$$x - y := x + (-y).$$

We do not need to verify the substitution axiom for this operation, since we have defined subtraction in terms of two other operations on integers, namely addition and negation, and we have already verified that those operations are well-defined.

One can easily check now that if a and b are natural numbers, then

$$a - b = a + -b = (a—0) + (0—b) = a—b,$$

and so $a—b$ is just the same thing as $a - b$. Because of this we can now discard the — notation, and use the familiar operation of subtraction instead. (As remarked before, we could not use subtraction immediately because it would be circular.)

We can now generalize Lemma 2.3.3 and Corollary 2.3.7 from the natural numbers to the integers:

Proposition 4.1.8 (Integers have no zero divisors). *Let a and b be integers such that $ab = 0$. Then either $a = 0$ or $b = 0$ (or both).*

Proof. See Exercise 4.1.5. □

Corollary 4.1.9 (Cancellation law for integers). *If a, b, c are integers such that $ac = bc$ and c is non-zero, then $a = b$.*

Proof. See Exercise 4.1.6. □

We now extend the notion of order, which was defined on the natural numbers, to the integers by repeating the definition verbatim:

Definition 4.1.10 (Ordering of the integers). Let n and m be integers. We say that n is *greater than or equal to* m, and write $n \geq m$ or $m \leq n$, iff we have $n = m + a$ for some natural number a. We say that n is *strictly greater than* m, and write $n > m$ or $m < n$, iff $n \geq m$ and $n \neq m$.

Thus for instance $5 > -3$, because $5 = -3 + 8$ and $5 \neq -3$. Clearly this definition is consistent with the notion of order on the natural numbers, since we are using the same definition.

Using the laws of algebra in Proposition 4.1.6 it is not hard to show the following properties of order:

Lemma 4.1.11 (Properties of order). *Let a, b, c be integers.*

(a) *$a > b$ if and only if $a - b$ is a positive natural number.*

(b) *(Addition preserves order) If $a > b$, then $a + c > b + c$.*

(c) *(Positive multiplication preserves order) If $a > b$ and c is positive, then $ac > bc$.*

(d) *(Negation reverses order) If $a > b$, then $-a < -b$.*

(e) *(Order is transitive) If $a > b$ and $b > c$, then $a > c$.*

(f) *(Order trichotomy) Exactly one of the statements $a > b$, $a < b$, or $a = b$ is true.*

Proof. See Exercise 4.1.7. □

Exercise 4.1.1. Verify that the definition of equality on the integers is both reflexive and symmetric.

Exercise 4.1.2. Show that the definition of negation on the integers is well-defined in the sense that if $(a—b) = (a'—b')$, then $-(a—b) = -(a'—b')$ (so equal integers have equal negations).

Exercise 4.1.3. Show that $(-1) \times a = -a$ for every integer a.

Exercise 4.1.4. Prove the remaining identities in Proposition 4.1.6. (Hint: one can save some work by using some identities to prove others. For instance, once you know that $xy = yx$, you get for free that $x1 = 1x$, and once you also prove $x(y + z) = xy + xz$, you automatically get $(y + z)x = yx + zx$ for free.)

Exercise 4.1.5. Prove Proposition 4.1.8. (Hint: while this proposition is not quite the same as Lemma 2.3.3, it is certainly legitimate to use Lemma 2.3.3 in the course of proving Proposition 4.1.8.)

Exercise 4.1.6. Prove Corollary 4.1.9. (Hint: there are two ways to do this. One is to use Proposition 4.1.8 to conclude that $a - b$ must be zero. Another way is to combine Corollary 2.3.7 with Lemma 4.1.5.)

Exercise 4.1.7. Prove Lemma 4.1.11. (Hint: use the first part of this lemma to prove all the others.)

Exercise 4.1.8. Show that the principle of induction (Axiom 2.5) does not apply directly to the integers. More precisely, give an example of a property $P(n)$ pertaining to an integer n such that $P(0)$ is true, and that $P(n)$ implies $P(n{+}{+})$ for all integers n, but that $P(n)$ is not true for all integers n. Thus induction is not as useful a tool for dealing with the integers as it is with the natural numbers. (The situation becomes even worse with the rational and real numbers, which we shall define shortly.)

4.2 The rationals

We have now constructed the integers, with the operations of addition, subtraction, multiplication, and order and verified all the expected algebraic and order-theoretic properties. Now we will use a similar construction to build the rationals, adding division to our mix of operations.

Just like the integers were constructed by subtracting two natural numbers, the rationals can be constructed by dividing two integers, though of course we have to make the usual caveat that the denominator should be non-zero[2]. Of course, just as two differences $a-b$ and $c-d$ can be equal if $a+d = c+b$, we know (from more advanced knowledge) that two quotients a/b and c/d can be equal if $ad = bc$. Thus, in analogy with the integers, we create a new meaningless symbol // (which will eventually be superceded by division), and define

Definition 4.2.1. A *rational number* is an expression of the form $a//b$, where a and b are integers and b is non-zero; $a//0$ is not considered to be a rational number. Two rational numbers are considered to be equal, $a//b = c//d$, if and only if $ad = cb$. The set of all rational numbers is denoted $\mathbf{Q}$.

Thus for instance $3//4 = 6//8 = -3//-4$, but $3//4 \neq 4//3$. This is a valid definition of equality (Exercise 4.2.1). Now we need a notion of addition, multiplication, and negation. Again, we will take advantage of our pre-existing knowledge, which tells us that $a/b + c/d$ should equal $(ad+bc)/(bd)$ and that $a/b * c/d$ should equal ac/bd, while $-(a/b)$ equals $(-a)/b$. Motivated by this foreknowledge, we define

Definition 4.2.2. If $a//b$ and $c//d$ are rational numbers, we define their sum

$$(a//b) + (c//d) := (ad+bc)//(bd)$$

their product

$$(a//b) * (c//d) := (ac)//(bd)$$

and the negation

$$-(a//b) := (-a)//b.$$

[2]There is no reasonable way we can divide by zero, since one cannot have both the identities $(a/b)*b = a$ and $c*0 = 0$ hold simultaneously if b is allowed to be zero. However, we can eventually get a reasonable notion of dividing by a quantity which *approaches* zero - think of L'Hôpital's rule (see Section 10.5), which suffices for doing things like defining differentiation.

Note that if b and d are non-zero, then bd is also non-zero, by Proposition 4.1.8, so the sum or product of a rational number remains a rational number.

Lemma 4.2.3. *The sum, product, and negation operations on rational numbers are well-defined, in the sense that if one replaces $a//b$ with another rational number $a'//b'$ which is equal to $a//b$, then the output of the above operations remains unchanged, and similarly for $c//d$.*

Proof. We just verify this for addition; we leave the remaining claims to Exercise 4.2.2. Suppose $a//b = a'//b'$, so that b and b' are non-zero and $ab' = a'b$. We now show that $a//b + c//d = a'//b' + c//d$. By definition, the left-hand side is $(ad+bc)//bd$ and the right-hand side is $(a'd + b'c)//b'd$, so we have to show that

$$(ad + bc)b'd = (a'd + b'c)bd,$$

which expands to

$$ab'd^2 + bb'cd = a'bd^2 + bb'cd.$$

But since $ab' = a'b$, the claim follows. Similarly if one replaces $c//d$ by $c'//d'$. $\square$

We note that the rational numbers $a//1$ behave in a manner identical to the integers a:

$$\begin{aligned}(a//1) + (b//1) &= (a + b)//1;\\ (a//1) \times (b//1) &= (ab//1);\\ -(a//1) &= (-a)//1.\end{aligned}$$

Also, $a//1$ and $b//1$ are only equal when a and b are equal. Because of this, we will identify a with $a//1$ for each integer a: $a \equiv a//1$; the above identities then guarantee that the arithmetic of the integers is consistent with the arithmetic of the rationals. Thus just as we embedded the natural numbers inside the integers, we embed the integers inside the rational numbers. In particular, all natural numbers are rational numbers, for instance 0 is equal to $0//1$ and 1 is equal to $1//1$.

Observe that a rational number $a//b$ is equal to $0 = 0//1$ if and only if $a \times 1 = b \times 0$, i.e., if the numerator a is equal to 0. Thus if a and b are non-zero then so is $a//b$.

We now define a new operation on the rationals: reciprocal. If $x = a//b$ is a non-zero rational (so that $a, b \neq 0$) then we define the *reciprocal* x^{-1} of x to be the rational number $x^{-1} := b//a$. It is easy to check that this operation is consistent with our notion of equality: if two rational numbers $a//b$, $a'//b'$ are equal, then their reciprocals are also equal. (In contrast, an operation such as "numerator" is not well-defined: the rationals $3//4$ and $6//8$ are equal, but have unequal numerators, so we have to be careful when referring to such terms as "the numerator of x".) We however leave the reciprocal of 0 undefined.

We now summarize the algebraic properties of the rationals.

Proposition 4.2.4 (Laws of algebra for rationals). *Let x, y, z be rationals. Then the following laws of algebra hold:*

$$x + y = y + x$$
$$(x + y) + z = x + (y + z)$$
$$x + 0 = 0 + x = x$$
$$x + (-x) = (-x) + x = 0$$
$$xy = yx$$
$$(xy)z = x(yz)$$
$$x1 = 1x = x$$
$$x(y + z) = xy + xz$$
$$(y + z)x = yx + zx.$$

If x is non-zero, we also have

$$xx^{-1} = x^{-1}x = 1.$$

Remark 4.2.5. The above set of ten identities have a name; they are asserting that the rationals $\mathbf{Q}$ form a *field*. This is better than being a commutative ring because of the tenth identity

$xx^{-1} = x^{-1}x = 1$. Note that this proposition supercedes Proposition 4.1.6.

Proof. To prove this identity, one writes $x = a//b$, $y = c//d$, $z = e//f$ for some integers a, c, e and non-zero integers b, d, f, and verifies each identity in turn using the algebra of the integers. We shall just prove the longest one, namely $(x+y)+z = x+(y+z)$:

$$\begin{aligned}
(x+y)+z &= ((a//b) + (c//d)) + (e//f) \\
&= ((ad+bc)//bd) + (e//f) \\
&= (adf + bcf + bde)//bdf; \\
x + (y+z) &= (a//b) + ((c//d) + (e//f)) \\
&= (a//b) + ((cf + de)//df) \\
&= (adf + bcf + bde)//bdf
\end{aligned}$$

and so one can see that $(x+y)+z$ and $x+(y+z)$ are equal. The other identities are proven in a similar fashion and are left to Exercise 4.2.3. □

We can now define the *quotient* x/y of two rational numbers x and y, *provided that* y is non-zero, by the formula

$$x/y := x \times y^{-1}.$$

Thus, for instance

$$(3//4)/(5//6) = (3//4) \times (6//5) = (18//20) = (9//10).$$

Using this formula, it is easy to see that $a/b = a//b$ for every integer a and every non-zero integer b. Thus we can now discard the $//$ notation, and use the more customary a/b instead of $a//b$.

Proposition 4.2.4allows us to use all the normal rules of algebra; we will now proceed to do so without further comment.

In the previous section we organized the integers into positive, zero, and negative numbers. We now do the same for the rationals.

Definition 4.2.6. A rational number x is said to be *positive* iff we have $x = a/b$ for some positive integers a and b. It is said to be *negative* iff we have $x = -y$ for some positive rational y (i.e., $x = (-a)/b$ for some positive integers a and b).

Thus for instance, every positive integer is a positive rational number, and every negative integer is a negative rational number, so our new definition is consistent with our old one.

Lemma 4.2.7 (Trichotomy of rationals). *Let x be a rational number. Then exactly one of the following three statements is true: (a) x is equal to 0. (b) x is a positive rational number. (c) x is a negative rational number.*

Proof. See Exercise 4.2.4. □

Definition 4.2.8 (Ordering of the rationals). Let x and y be rational numbers. We say that $x > y$ iff $x - y$ is a positive rational number, and $x < y$ iff $x - y$ is a negative rational number. We write $x \geq y$ iff either $x > y$ or $x = y$, and similarly define $x \leq y$.

Proposition 4.2.9 (Basic properties of order on the rationals). *Let x, y, z be rational numbers. Then the following properties hold.*

(a) *(Order trichotomy) Exactly one of the three statements $x = y$, $x < y$, or $x > y$ is true.*

(b) *(Order is anti-symmetric) One has $x < y$ if and only if $y > x$.*

(c) *(Order is transitive) If $x < y$ and $y < z$, then $x < z$.*

(d) *(Addition preserves order) If $x < y$, then $x + z < y + z$.*

(e) *(Positive multiplication preserves order) If $x < y$ and z is positive, then $xz < yz$.*

Proof. See Exercise 4.2.5. □

Remark 4.2.10. The above five properties in Proposition 4.2.9, combined with the field axioms in Proposition 4.2.4, have a name: they assert that the rationals **Q** form an *ordered field*. It is important to keep in mind that Proposition 4.2.9(e) only works when z is positive, see Exercise 4.2.6.

Exercise 4.2.1. Show that the definition of equality for the rational numbers is reflexive, symmetric, and transitive. (Hint: for transitivity, use Corollary 2.3.7.)

Exercise 4.2.2. Prove the remaining components of Lemma 4.2.3.

Exercise 4.2.3. Prove the remaining components of Proposition 4.2.4. (Hint: as with Proposition 4.1.6, you can save some work by using some identities to prove others.)

Exercise 4.2.4. Prove Lemma 4.2.7. (Note that, as in Proposition 2.2.13, you have to prove two different things: firstly, that *at least* one of (a), (b), (c) is true; and secondly, that *at most* one of (a), (b), (c) is true.)

Exercise 4.2.5. Prove Proposition 4.2.9.

Exercise 4.2.6. Show that if x, y, z are real numbers such that $x < y$ and z is *negative*, then $xz > yz$.

4.3 Absolute value and exponentiation

We have already introduced the four basic arithmetic operations of addition, subtraction, multiplication, and division on the rationals. (Recall that subtraction and division came from the more primitive notions of negation and reciprocal by the formulae $x - y := x + (-y)$ and $x/y := x \times y^{-1}$.) We also have a notion of order $<$, and have organized the rationals into the positive rationals, the negative rationals, and zero. In short, we have shown that the rationals $\mathbf{Q}$ form an *ordered field*.

One can now use these basic operations to construct more operations. There are many such operations we can construct, but we shall just introduce two particularly useful ones: absolute value and exponentiation.

Definition 4.3.1 (Absolute value). If x is a rational number, the *absolute value* $|x|$ of x is defined as follows. If x is positive, then $|x| := x$. If x is negative, then $|x| := -x$. If x is zero, then $|x| := 0$.

Definition 4.3.2 (Distance). Let x and y be real numbers. The quantity $|x - y|$ is called the *distance between x and y* and is sometimes denoted $d(x, y)$, thus $d(x, y) := |x - y|$. For instance, $d(3, 5) = 2$.

Proposition 4.3.3 (Basic properties of absolute value and distance). *Let x, y, z be rational numbers.*

(a) *(Non-degeneracy of absolute value) We have $|x| \geq 0$. Also, $|x| = 0$ if and only if x is 0.*

(b) *(Triangle inequality for absolute value) We have $|x + y| \leq |x| + |y|$.*

(c) *We have the inequalities $-y \leq x \leq y$ if and only if $y \geq |x|$. In particular, we have $-|x| \leq x \leq |x|$.*

(d) *(Multiplicativity of absolute value) We have $|xy| = |x|\ |y|$. In particular, $|-x| = |x|$.*

(e) *(Non-degeneracy of distance) We have $d(x, y) \geq 0$. Also, $d(x, y) = 0$ if and only if $x = y$.*

(f) *(Symmetry of distance) $d(x, y) = d(y, x)$.*

(g) *(Triangle inequality for distance) $d(x, z) \leq d(x, y) + d(y, z)$.*

Proof. See Exercise 4.3.1. □

Absolute value is useful for measuring how "close" two numbers are. Let us make a somewhat artificial definition:

Definition 4.3.4 (ε-closeness). Let $\varepsilon > 0$, and x, y be rational numbers. We say that y is *ε-close* to x iff we have $d(y, x) \leq \varepsilon$.

Remark 4.3.5. This definition is not standard in mathematics textbooks; we will use it as "scaffolding" to construct the more important notions of limits (and of Cauchy sequences) later on, and once we have those more advanced notions we will discard the notion of ε-close.

Examples 4.3.6. The numbers 0.99 and 1.01 are 0.1-close, but they are not 0.01 close, because $d(0.99, 1.01) = |0.99 - 1.01| = 0.02$ is larger than 0.01. The numbers 2 and 2 are ε-close for every positive ε.

We do not bother defining a notion of ε-close when ε is zero or negative, because if ε is zero then x and y are only ε-close when they are equal, and when ε is negative then x and y are never ε-close. (In any event it is a long-standing tradition in analysis that the Greek letters ε, δ should only denote small positive numbers.)

Some basic properties of ε-closeness are the following.

Proposition 4.3.7. *Let x, y, z, w be rational numbers.*

(a) *If $x = y$, then x is ε-close to y for every $\varepsilon > 0$. Conversely, if x is ε-close to y for every $\varepsilon > 0$, then we have $x = y$.*

(b) *Let $\varepsilon > 0$. If x is ε-close to y, then y is ε-close to x.*

(c) *Let $\varepsilon, \delta > 0$. If x is ε-close to y, and y is δ-close to z, then x and z are $(\varepsilon + \delta)$-close.*

(d) *Let $\varepsilon, \delta > 0$. If x and y are ε-close, and z and w are δ-close, then $x + z$ and $y + w$ are $(\varepsilon + \delta)$-close, and $x - z$ and $y - w$ are also $(\varepsilon + \delta)$-close.*

(e) *Let $\varepsilon > 0$. If x and y are ε-close, they are also ε'-close for every $\varepsilon' > \varepsilon$.*

(f) *Let $\varepsilon > 0$. If y and z are both ε-close to x, and w is between y and z (i.e., $y \le w \le z$ or $z \le w \le y$), then w is also ε-close to x.*

(g) *Let $\varepsilon > 0$. If x and y are ε-close, and z is non-zero, then xz and yz are $\varepsilon|z|$-close.*

(h) *Let $\varepsilon, \delta > 0$. If x and y are ε-close, and z and w are δ-close, then xz and yw are $(\varepsilon|z| + \delta|x| + \varepsilon\delta)$-close.*

Proof. We only prove the most difficult one, (h); we leave (a)-(g) to Exercise 4.3.2. Let $\varepsilon, \delta > 0$, and suppose that x and y are ε-close. If we write $a := y - x$, then we have $y = x + a$ and that $|a| \le \varepsilon$. Similarly, if z and w are δ-close, and we define $b := w - z$, then $w = z + b$ and $|b| \le \delta$.