No segment of the form

$$\left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right), \tag{24}$$

where k and m are positive integers, has a point in common with P. Since every segment (α, β) contains a segment of the form (24), if

$$3^{-m} < \frac{\beta - \alpha}{6},$$

P contains no segment.

To show that P is perfect, it is enough to show that P contains no isolated point. Let $x \in P$, and let S be any segment containing x. Let I_n be that interval of E_n which contains x. Choose n large enough, so that $I_n \subset S$. Let x_n be an endpoint of I_n, such that $x_n \neq x$.

It follows from the construction of P that $x_n \in P$. Hence x is a limit point of P, and P is perfect.

One of the most interesting properties of the Cantor set is that it provides us with an example of an uncountable set of measure zero (the concept of measure will be discussed in Chap. 11).

CONNECTED SETS

2.45 Definition Two subsets A and B of a metric space X are said to be *separated* if both $A \cap \bar{B}$ and $\bar{A} \cap B$ are empty, i.e., if no point of A lies in the closure of B and no point of B lies in the closure of A.

A set $E \subset X$ is said to be *connected* if E is *not* a union of two nonempty separated sets.

2.46 Remark Separated sets are of course disjoint, but disjoint sets need not be separated. For example, the interval $[0, 1]$ and the segment $(1, 2)$ are *not* separated, since 1 is a limit point of $(1, 2)$. However, the segments $(0, 1)$ and $(1, 2)$ *are* separated.

The connected subsets of the line have a particularly simple structure:

2.47 Theorem *A subset E of the real line R^1 is connected if and only if it has the following property: If $x \in E$, $y \in E$, and $x < z < y$, then $z \in E$.*

Proof If there exist $x \in E$, $y \in E$, and some $z \in (x, y)$ such that $z \notin E$, then $E = A_z \cup B_z$ where

$$A_z = E \cap (-\infty, z), \qquad B_z = E \cap (z, \infty).$$

Since $x \in A_z$ and $y \in B_z$, A and B are nonempty. Since $A_z \subset (-\infty, z)$ and $B_z \subset (z, \infty)$, they are separated. Hence E is not connected.

To prove the converse, suppose E is not connected. Then there are nonempty separated sets A and B such that $A \cup B = E$. Pick $x \in A$, $y \in B$, and assume (without loss of generality) that $x < y$. Define

$$z = \sup (A \cap [x, y]).$$

By Theorem 2.28, $z \in \bar{A}$; hence $z \notin B$. In particular, $x \le z < y$.

If $z \notin A$, it follows that $x < z < y$ and $z \notin E$.

If $z \in A$, then $z \notin \bar{B}$, hence there exists z_1 such that $z < z_1 < y$ and $z_1 \notin B$. Then $x < z_1 < y$ and $z_1 \notin E$.

EXERCISES

1. Prove that the empty set is a subset of every set.
2. A complex number z is said to be *algebraic* if there are integers $a_0, \ldots, a_n$, not all zero, such that
$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0.$$
Prove that the set of all algebraic numbers is countable. *Hint:* For every positive integer N there are only finitely many equations with
$$n + |a_0| + |a_1| + \cdots + |a_n| = N.$$
3. Prove that there exist real numbers which are not algebraic.
4. Is the set of all irrational real numbers countable?
5. Construct a bounded set of real numbers with exactly three limit points.
6. Let E' be the set of all limit points of a set E. Prove that E' is closed. Prove that E and $\bar{E}$ have the same limit points. (Recall that $\bar{E} = E \cup E'$.) Do E and E' always have the same limit points?
7. Let $A_1, A_2, A_3, \ldots$ be subsets of a metric space.
(*a*) If $B_n = \bigcup_{i=1}^{n} A_i$, prove that $\bar{B}_n = \bigcup_{i=1}^{n} \bar{A}_i$, for $n = 1, 2, 3, \ldots$.
(*b*) If $B = \bigcup_{i=1}^{\infty} A_i$, prove that $\bar{B} \supset \bigcup_{i=1}^{\infty} \bar{A}_i$.
Show, by an example, that this inclusion can be proper.
8. Is every point of every open set $E \subset R^2$ a limit point of E? Answer the same question for closed sets in R^2.
9. Let E° denote the set of all interior points of a set E. [See Definition 2.18(*e*); E° is called the *interior* of E.]
(*a*) Prove that E° is always open.
(*b*) Prove that E is open if and only if $E^\circ = E$.
(*c*) If $G \subset E$ and G is open, prove that $G \subset E^\circ$.
(*d*) Prove that the complement of E° is the closure of the complement of E.
(*e*) Do E and $\bar{E}$ always have the same interiors?
(*f*) Do E and E° always have the same closures?

10. Let X be an infinite set. For $p \in X$ and $q \in X$, define

$$d(p, q) = \begin{cases} 1 & (\text{if } p \neq q) \\ 0 & (\text{if } p = q). \end{cases}$$

Prove that this is a metric. Which subsets of the resulting metric space are open? Which are closed? Which are compact?

11. For $x \in R^1$ and $y \in R^1$, define

$$\begin{aligned} d_1(x, y) &= (x - y)^2, \\ d_2(x, y) &= \sqrt{|x - y|}, \\ d_3(x, y) &= |x^2 - y^2|, \\ d_4(x, y) &= |x - 2y|, \\ d_5(x, y) &= \frac{|x - y|}{1 + |x - y|}. \end{aligned}$$

Determine, for each of these, whether it is a metric or not.

12. Let $K \subset R^1$ consist of 0 and the numbers $1/n$, for $n = 1, 2, 3, \ldots$. Prove that K is compact directly from the definition (without using the Heine-Borel theorem).

13. Construct a compact set of real numbers whose limit points form a countable set.

14. Give an example of an open cover of the segment $(0, 1)$ which has no finite subcover.

15. Show that Theorem 2.36 and its Corollary become false (in R^1, for example) if the word "compact" is replaced by "closed" or by "bounded."

16. Regard Q, the set of all rational numbers, as a metric space, with $d(p, q) = |p - q|$. Let E be the set of all $p \in Q$ such that $2 < p^2 < 3$. Show that E is closed and bounded in Q, but that E is not compact. Is E open in Q?

17. Let E be the set of all $x \in [0, 1]$ whose decimal expansion contains only the digits 4 and 7. Is E countable? Is E dense in $[0, 1]$? Is E compact? Is E perfect?

18. Is there a nonempty perfect set in R^1 which contains no rational number?

19. (*a*) If A and B are disjoint closed sets in some metric space X, prove that they are separated.

(*b*) Prove the same for disjoint open sets.

(*c*) Fix $p \in X$, $\delta > 0$, define A to be the set of all $q \in X$ for which $d(p, q) < \delta$, define B similarly, with $>$ in place of $<$. Prove that A and B are separated.

(*d*) Prove that every connected metric space with at least two points is uncountable. *Hint*: Use (*c*).

20. Are closures and interiors of connected sets always connected? (Look at subsets of R^2.)

21. Let A and B be separated subsets of some R^k, suppose $\mathbf{a} \in A$, $\mathbf{b} \in B$, and define

$$\mathbf{p}(t) = (1 - t)\mathbf{a} + t\mathbf{b}$$

for $t \in R^1$. Put $A_0 = \mathbf{p}^{-1}(A)$, $B_0 = \mathbf{p}^{-1}(B)$. [Thus $t \in A_0$ if and only if $\mathbf{p}(t) \in A$.]

(*a*) Prove that A_0 and B_0 are separated subsets of R^1.
(*b*) Prove that there exists $t_0 \in (0, 1)$ such that $\mathbf{p}(t_0) \notin A \cup B$.
(*c*) Prove that every convex subset of R^k is connected.

22. A metric space is called *separable* if it contains a countable dense subset. Show that R^k is separable. *Hint:* Consider the set of points which have only rational coordinates.

23. A collection $\{V_\alpha\}$ of open subsets of X is said to be a *base* for X if the following is true: For every $x \in X$ and every open set $G \subset X$ such that $x \in G$, we have $x \in V_\alpha \subset G$ for some α. In other words, every open set in X is the union of a subcollection of $\{V_\alpha\}$.

Prove that every separable metric space has a *countable* base. *Hint:* Take all neighborhoods with rational radius and center in some countable dense subset of X.

24. Let X be a metric space in which every infinite subset has a limit point. Prove that X is separable. *Hint:* Fix $\delta > 0$, and pick $x_1 \in X$. Having chosen $x_1, \ldots, x_j \in X$, choose $x_{j+1} \in X$, if possible, so that $d(x_i, x_{j+1}) \geq \delta$ for $i = 1, \ldots, j$. Show that this process must stop after a finite number of steps, and that X can therefore be covered by finitely many neighborhoods of radius δ. Take $\delta = 1/n$ ($n = 1, 2, 3, \ldots$), and consider the centers of the corresponding neighborhoods.

25. Prove that every compact metric space K has a countable base, and that K is therefore separable. *Hint:* For every positive integer n, there are finitely many neighborhoods of radius $1/n$ whose union covers K.

26. Let X be a metric space in which every infinite subset has a limit point. Prove that X is compact. *Hint:* By Exercises 23 and 24, X has a countable base. It follows that every open cover of X has a *countable* subcover $\{G_n\}$, $n = 1, 2, 3, \ldots$. If no finite subcollection of $\{G_n\}$ covers X, then the complement F_n of $G_1 \cup \cdots \cup G_n$ is nonempty for each n, but $\bigcap F_n$ is empty. If E is a set which contains a point from each F_n, consider a limit point of E, and obtain a contradiction.

27. Define a point p in a metric space X to be a *condensation point* of a set $E \subset X$ if every neighborhood of p contains uncountably many points of E.

Suppose $E \subset R^k$, E is uncountable, and let P be the set of all condensation points of E. Prove that P is perfect and that at most countably many points of E are not in P. In other words, show that $P^c \cap E$ is at most countable. *Hint:* Let $\{V_n\}$ be a countable base of R^k, let W be the union of those V_n for which $E \cap V_n$ is at most countable, and show that $P = W^c$.

28. Prove that every closed set in a separable metric space is the union of a (possibly empty) perfect set and a set which is at most countable. (*Corollary:* Every countable closed set in R^k has isolated points.) *Hint*: Use Exercise 27.

29. Prove that every open set in R^1 is the union of an at most countable collection of disjoint segments. *Hint:* Use Exercise 22.

30. Imitate the proof of Theorem 2.43 to obtain the following result:

> If $R^k = \bigcup_1^\infty F_n$, where each F_n is a closed subset of R^k, then at least one F_n has a nonempty interior.
>
> *Equivalent statement:* If G_n is a dense open subset of R^k, for $n = 1, 2, 3, \ldots,$ then $\bigcap_1^\infty G_n$ is not empty (in fact, it is dense in R^k).

(This is a special case of Baire's theorem; see Exercise 22, Chap. 3, for the general case.)

3

NUMERICAL SEQUENCES AND SERIES

As the title indicates, this chapter will deal primarily with sequences and series of complex numbers. The basic facts about convergence, however, are just as easily explained in a more general setting. The first three sections will therefore be concerned with sequences in euclidean spaces, or even in metric spaces.

CONVERGENT SEQUENCES

3.1 Definition A sequence $\{p_n\}$ in a metric space X is said to *converge* if there is a point $p \in X$ with the following property: For every $\varepsilon > 0$ there is an integer N such that $n \geq N$ implies that $d(p_n, p) < \varepsilon$. (Here d denotes the distance in X.)

In this case we also say that $\{p_n\}$ converges to p, or that p is the limit of $\{p_n\}$ [see Theorem 3.2(*b*)], and we write $p_n \to p$, or

$$\lim_{n \to \infty} p_n = p.$$

If $\{p_n\}$ does not converge, it is said to *diverge*.

It might be well to point out that our definition of "convergent sequence" depends not only on $\{p_n\}$ but also on X; for instance, the sequence $\{1/n\}$ converges in R^1 (to 0), but fails to converge in the set of all positive real numbers [with $d(x, y) = |x - y|$]. In cases of possible ambiguity, we can be more precise and specify "convergent in X" rather than "convergent."

We recall that the set of all points p_n ($n = 1, 2, 3, \ldots$) is the *range* of $\{p_n\}$. The range of a sequence may be a finite set, or it may be infinite. The sequence $\{p_n\}$ is said to be *bounded* if its range is bounded.

As examples, consider the following sequences of complex numbers (that is, $X = R^2$):

(*a*) If $s_n = 1/n$, then $\lim_{n\to\infty} s_n = 0$; the range is infinite, and the sequence is bounded.
(*b*) If $s_n = n^2$, the sequence $\{s_n\}$ is unbounded, is divergent, and has infinite range.
(*c*) If $s_n = 1 + [(-1)^n/n]$, the sequence $\{s_n\}$ converges to 1, is bounded, and has infinite range.
(*d*) If $s_n = i^n$, the sequence $\{s_n\}$ is divergent, is bounded, and has finite range.
(*e*) If $s_n = 1$ ($n = 1, 2, 3, \ldots$), then $\{s_n\}$ converges to 1, is bounded, and has finite range.

We now summarize some important properties of convergent sequences in metric spaces.

3.2 Theorem *Let $\{p_n\}$ be a sequence in a metric space X.*

(*a*) *$\{p_n\}$ converges to $p \in X$ if and only if every neighborhood of p contains p_n* for *all but finitely many n.*
(*b*) *If $p \in X$, $p' \in X$, and if $\{p_n\}$ converges to p and to p', then $p' = p$.*
(*c*) *If $\{p_n\}$ converges, then $\{p_n\}$ is bounded.*
(*d*) *If $E \subset X$ and if p is a limit point of E, then there is a sequence $\{p_n\}$ in E such that $p = \lim_{n\to\infty} p_n$.*

Proof (*a*) Suppose $p_n \to p$ and let V be a neighborhood of p. For some $\varepsilon > 0$, the conditions $d(q, p) < \varepsilon$, $q \in X$ imply $q \in V$. Corresponding to this ε, there exists N such that $n \geq N$ implies $d(p_n, p) < \varepsilon$. Thus $n \geq N$ implies $p_n \in V$.

Conversely, suppose every neighborhood of p contains all but finitely many of the p_n. Fix $\varepsilon > 0$, and let V be the set of all $q \in X$ such that $d(p, q) < \varepsilon$. By assumption, there exists N (corresponding to this V) such that $p_n \in V$ if $n \geq N$. Thus $d(p_n, p) < \varepsilon$ if $n \geq N$; hence $p_n \to p$.