Computing $\rho(zg^{-1})$ using (12) then gives

$$\sum_{j=1}^{r} \chi_j(1)\chi_j(zg^{-1}) = \alpha_g|G|. \tag{18.13}$$

Let φ_j be the irreducible representation afforded by M_j, $1 \leq j \leq r$. Since we may consider φ_j as an algebra homomorphism from $\mathbb{C}G$ into $\mathrm{End}(M_j)$, we obtain $\varphi_j(zg^{-1}) = \varphi_j(z)\varphi_j(g^{-1})$. Also, we have already observed that $\varphi_j(z)$ is 0 if $j \neq i$ and $\varphi_i(z)$ is the identity endomorphism on M_i. Thus

$$\varphi_j(zg^{-1}) = \begin{cases} 0 & \text{if } j \neq i \\ \varphi_i(g^{-1}) & \text{if } j = i. \end{cases}$$

This proves $\chi_j(zg^{-1}) = \chi_i(g^{-1})\delta_{ij}$, where δ_{ij} is zero if $i \neq j$ and is 1 if $i = j$ (called the Kronecker delta). Substituting this into equation (13) gives $\alpha_g = \frac{1}{|G|}\chi_i(1)\chi_i(g^{-1})$. This is the coefficient of g in the statement of the proposition, completing the proof.

The orthonormality of the irreducible characters will follow directly from the orthogonality of the central primitive idempotents via the following calculation:

$$\begin{aligned} z_i\delta_{ij} &= z_i z_j \\ &= \frac{\chi_i(1)}{|G|}\frac{\chi_j(1)}{|G|}\sum_{g,h\in G}\chi_i(g^{-1})\chi_j(h^{-1})gh \\ &= \frac{\chi_i(1)}{|G|}\frac{\chi_j(1)}{|G|}\sum_{y\in G}\left[\sum_{x\in G}\chi_i(xy^{-1})\chi_j(x^{-1})\right]y \end{aligned}$$

(to get the latter sum from the former substitute y for gh and x for h). Since the elements of G are a basis of $\mathbb{C}G$ we may equate coefficients with those of z_i found in Proposition 13 to get (the coefficient of g)

$$\delta_{ij}\frac{\chi_i(1)}{|G|}\chi_i(g^{-1}) = \frac{\chi_i(1)\chi_j(1)}{|G|^2}\sum_{x\in G}\chi_i(xg^{-1})\chi_j(x^{-1}).$$

Simplifying (and replacing g by g^{-1}) gives

$$\delta_{ij}\frac{\chi_i(g)}{\chi_j(1)} = \frac{1}{|G|}\sum_{x\in G}\chi_i(xg)\chi_j(x^{-1}) \quad \text{for all } g \in G. \tag{18.14}$$

Taking $g = 1$ in (14) gives

$$\delta_{ij} = \frac{1}{|G|}\sum_{x\in G}\chi_i(x)\chi_j(x^{-1}). \tag{18.15}$$

The sum on the right side would be precisely the inner product (χ_i, χ_j) if $\chi_j(x^{-1})$ were equal to $\overline{\chi_j(x)}$; this is the content of the next proposition.

Proposition 14. If ψ is any character of G then $\psi(x)$ is a sum of roots of 1 in $\mathbb{C}$ and $\psi(x^{-1}) = \overline{\psi(x)}$ for all $x \in G$.

Proof: Let φ be a representation whose character is ψ, fix an element $x \in G$ and let $|x| = k$. Since the minimal polynomial of $\varphi(x)$ divides $X^k - 1$ (hence has distinct roots), there is a basis of the underlying vector space such that the matrix of $\varphi(x)$ with respect to this basis is a diagonal matrix with k^{th} roots of 1 on the diagonal. Since $\psi(x)$ is the sum of the diagonal entries (and does not depend on the choice of basis), $\psi(x)$ is a sum of roots of 1. Moreover, if ϵ is a root of 1, $\epsilon^{-1} = \overline{\epsilon}$. Thus the inverse of a diagonal matrix with roots of 1 on the diagonal is the diagonal matrix with the complex conjugates of those roots of 1 on the diagonal. Since the complex conjugate of a sum is the sum of the complex conjugates, $\psi(x^{-1}) = \operatorname{tr}\varphi(x^{-1}) = \overline{\operatorname{tr}\varphi(x)} = \overline{\psi(x)}$.

Keep in mind that in the proof of Proposition 14 we first fixed a group element x and then chose a basis of the representation space so that $\varphi(x)$ was a diagonal matrix. It is always possible to diagonalize a single element but it is possible to *simultaneously* diagonalize all $\varphi(x)$'s if and only if φ is similar to a sum of degree 1 representations.

Combining the above proposition with equation (15) proves:

Theorem 15. *(The First Orthogonality Relation for Group Characters)* Let G be a finite group and let $\chi_1, \dots, \chi_r$ be the irreducible characters of G over $\mathbb{C}$. Then with respect to the inner product $(\ ,\ )$ above we have

$$(\chi_i, \chi_j) = \delta_{ij}$$

and the irreducible characters are an orthonormal basis for the space of class functions. In particular, if θ is any class function then

$$\theta = \sum_{i=1}^{r} (\theta, \chi_i)\chi_i.$$

Proof: We have just established that the irreducible characters form an orthonormal basis for the space of class functions. If θ is any class function, write $\theta = \sum_{i=1}^{r} a_i \chi_i$, for some $a_i \in \mathbb{C}$. It follows from linearity of the Hermitian product that $a_i = (\theta, \chi_i)$, as stated.

We list without proof the Second Orthogonality Relation; we shall not require it for the applications in this book.

Theorem 16. *(The Second Orthogonality Relation for Group Characters)* Under the notation above, for any $x, y \in G$

$$\sum_{i=1}^{r} \chi_i(x)\overline{\chi_i(y)} = \begin{cases} |C_G(x)| & \text{if } x \text{ and } y \text{ are conjugate in } G \\ 0 & \text{otherwise.} \end{cases}$$

Definition. For θ any class function on G the *norm* of θ is $(\theta, \theta)^{1/2}$ and will be denoted by $\|\theta\|$.

When a class function is written in terms of the irreducible characters, $\theta = \sum \alpha_i \chi_i$, its norm is easily calculated as $\|\theta\| = (\sum \alpha_i^2)^{1/2}$. It follows that

a character has norm 1 if and only if it is irreducible.

Finally, observe that computations of the inner product of characters θ and ψ may be simplified as follows. If $\mathcal{K}_1, \ldots, \mathcal{K}_r$ are the conjugacy classes of G with sizes $d_1, \ldots, d_r$ and representatives $g_1, \ldots, g_r$ respectively, then the value $\theta(g_i)\overline{\psi(g_i)}$ appears d_i times in the sum for (θ, ψ), once for each element of $\mathcal{K}_i$. Collecting these terms gives

$$(\theta, \psi) = \frac{1}{|G|} \sum_{i=1}^{r} d_i \theta(g_i)\overline{\psi(g_i)},$$

a sum only over representatives of the conjugacy classes of G. In particular, the norm of θ is given by

$$\|\theta\|^2 = (\theta, \theta) = \frac{1}{|G|} \sum_{i=1}^{r} d_i |\theta(g_i)|^2.$$

Examples

(1) Let $G = S_3$ and let π be the permutation character of degree 3 described in the examples at the beginning of this section. Recall that $\pi(\sigma)$ equals the number of elements in $\{1, 2, 3\}$ fixed by σ. The conjugacy classes of S_3 are represented by 1, (1 2) and (1 2 3) of sizes 1, 3 and 2 respectively, and $\pi(1) = 3$, $\pi((1\ 2)) = 1$, $\pi((1\ 2\ 3)) = 0$. Hence

$$\begin{aligned}\|\pi\|^2 &= \frac{1}{6}\left[1\,\pi(1)^2 + 3\,\pi((1\ 2))^2 + 2\,\pi((1\ 2\ 3))^2\right] \\ &= \frac{1}{6}(9 + 3 + 0) = 2\end{aligned}$$

This implies that π is a sum of two distinct irreducible characters, each appearing with multiplicity 1. Let χ_1 be the principal character of S_3, so that $\chi_1(\sigma) = \overline{\chi_1(\sigma)} = 1$ for all $\sigma \in S_3$. Then

$$\begin{aligned}(\pi, \chi_1) &= \frac{1}{6}\left[1\,\pi(1)\overline{\chi_1(1)} + 3\,\pi((1\ 2))\overline{\chi_1((1\ 2))} + 2\,\pi((1\ 2\ 3))\overline{\chi_1((1\ 2\ 3))}\,\right] \\ &= \frac{1}{6}(3 + 3 + 0) = 1\end{aligned}$$

so the principal character appears as a constituent of π with multiplicity 1. This proves $\pi = \chi_1 + \chi_2$ for some irreducible character χ_2 of S_3 of degree 2 (and agrees with our earlier decomposition of this representation). This also shows that the value of χ_2 on $\sigma \in S_3$ is the number of fixed points of σ minus 1.

(2) Let $G = S_4$ and let π be the natural permutation character of degree 4 (so again $\pi(\sigma)$ is the number of fixed points of σ). The conjugacy classes of S_4 are represented by 1, (1 2), (1 2 3), (1 2 3 4) and (1 2)(3 4) of sizes 1, 6, 8, 6 and 3 respectively. Again we compute:

$$\begin{aligned}\|\pi\|^2 &= \frac{1}{24}\left[1\,\pi(1)^2 + 6\,\pi((1\ 2))^2 + 8\,\pi((1\ 2\ 3))^2 + 6\,\pi((1\ 2\ 3\ 4))^2\right. \\ &\qquad \left. + 3\,\pi((1\ 2)(3\ 4))^2\right] \\ &= \frac{1}{24}(16 + 24 + 8 + 0 + 0) = 2\,.\end{aligned}$$

so π has two distinct irreducible constituents. If χ_1 is the principal character of S_4, then

$$\begin{aligned}(\pi, \chi_1) &= \frac{1}{24}\left[1\,\pi(1) + 6\,\pi((1\,2)) + 8\,\pi((1\,2\,3))\right.\\ &\qquad\qquad \left. + 6\,\pi((1\,2\,3\,4)) + 3\,\pi((1\,2)(3\,4))\right]\\ &= \frac{1}{24}(4 + 12 + 8 + 0 + 0) = 1.\end{aligned}$$

This proves that the degree 4 permutation character is the sum of the principal character and an irreducible character of degree 3.

(3) Let $G = D_8$, where

$$D_8 = \langle\, r, s \mid s^2 = r^4 = 1,\ rs = sr^{-1}\,\rangle.$$

The conjugacy classes of D_8 are represented by $1, s, r, r^2$ and sr and have sizes 1, 2, 2, 1 and 2, respectively. Let φ be the degree 2 matrix representation of D_8 obtained as in Example 6 in Section 1 from embedding a square in $\mathbb{R}^2$:

$$\varphi(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \varphi(r) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ \varphi(r^2) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},\ \varphi(sr) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let ψ be the character of this representation (where we consider the real matrices as a subset of the complex matrices). Again, since ψ is real valued one computes

$$\begin{aligned}\|\,\psi\,\|^2 &= \frac{1}{8}\left[1\psi(1)^2 + 2\psi(s)^2 + 2\psi(r)^2 + 1\psi(r^2)^2 + 2\psi(sr)^2\right]\\ &= \frac{1}{8}(4 + 0 + 0 + 4 + 0) = 1.\end{aligned}$$

This proves the representation φ is irreducible (even if we allow similarity transformations by complex matrices).

We have seen that the sum of two characters is again a character. Specifically, if ψ_1 and ψ_2 are characters of representations φ_1 and φ_2, then $\psi_1 + \psi_2$ is the character of $\varphi_1 + \varphi_2$.

Proposition 17. If ψ_1 and ψ_2 are characters, then so is their product $\psi_1\psi_2$.

Proof: Let V_1 and V_2 be $\mathbb{C}G$-modules affording characters ψ_1 and ψ_2 and define $W = V_1 \otimes_{\mathbb{C}} V_2$. Since each $g \in G$ acts as a linear transformation on V_1 and V_2, the action of g on simple tensors by $g(v_1 \otimes v_2) = (gv_1) \otimes (gv_2)$ extends by linearity to a well defined linear transformation on W by Proposition 17 in Section 11.2. One easily checks that this action also makes W into a $\mathbb{C}G$-module. By Exercise 38 in Section 11.2 the character afforded by W is $\psi_1\psi_2$.

The next chapter will contain further explicit character computations as well as some applications of group characters to proving theorems about certain classes of groups.