Then f is a non-degenerate bilinear form on R^n. The matrix of f in the standard ordered basis is the $n \times n$ identity matrix:

$$f(X, Y) = X^t Y.$$

This f is usually called the dot (or scalar) product. The reader is probably familiar with this bilinear form, at least in the case $n = 3$. Geometrically, the number $f(\alpha, \beta)$ is the product of the length of α, the length of β, and the cosine of the angle between α and β. In particular, $f(\alpha, \beta) = 0$ if and only if the vectors α and β are orthogonal (perpendicular).

Exercises

1. Which of the following functions f, defined on vectors $\alpha = (x_1, x_2)$ and $\beta = (y_1, y_2)$ in R^2, are bilinear forms?

(a) $f(\alpha, \beta) = 1$.
(b) $f(\alpha, \beta) = (x_1 - y_1)^2 + x_2 y_2$.
(c) $f(\alpha, \beta) = (x_1 + y_1)^2 - (x_1 - y_1)^2$.
(d) $f(\alpha, \beta) = x_1 y_2 - x_2 y_1$.

2. Let f be the bilinear form on R^2 defined by

$$f((x_1, y_1), (x_2, y_2)) = x_1 y_1 + x_2 y_2.$$

Find the matrix of f in each of the following bases:

$$\{(1, 0), (0, 1)\}, \qquad \{(1, -1), (1, 1)\}, \qquad \{(1, 2), (3, 4)\}.$$

3. Let V be the space of all 2×3 matrices over R, and let f be the bilinear form on V defined by $f(X, Y) = \operatorname{trace}(X^t A Y)$, where

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Find the matrix of f in the ordered basis

$$\{E^{11}, E^{12}, E^{13}, E^{21}, E^{22}, E^{23}\}$$

where E^{ij} is the matrix whose only non-zero entry is a 1 in row i and column j.

4. Describe explicitly all bilinear forms f on R^3 with the property that $f(\alpha, \beta) = f(\beta, \alpha)$ for all α, β.

5. Describe the bilinear forms on R^3 which satisfy $f(\alpha, \beta) = -f(\beta, \alpha)$ for all α, β.

6. Let n be a positive integer, and let V be the space of all $n \times n$ matrices over the field of complex numbers. Show that the equation

$$f(A, B) = n \operatorname{tr}(AB) - \operatorname{tr}(A) \operatorname{tr}(B)$$

defines a bilinear form f on V. Is it true that $f(A, B) = f(B, A)$ for all A, B?

7. Let f be the bilinear form defined in Exercise 6. Show that f is degenerate (not non-degenerate). Let V_1 be the subspace of V consisting of the matrices of trace 0, and let f_1 be the restriction of f to V_1. Show that f_1 is non-degenerate.

8. Let f be the bilinear form defined in Exercise 6, and let V_2 be the subspace of V consisting of all matrices A such that trace $(A) = 0$ and $A^* = -A$ (A^* is the conjugate transpose of A). Denote by f_2 the restriction of f to V_2. Show that f_2 is negative definite, i.e., that $f_2(A, A) < 0$ for each non-zero A in V_2.

9. Let f be the bilinear form defined in Exercise 6. Let W be the set of all matrices A in V such that $f(A, B) = 0$ for all B. Show that W is a subspace of V. Describe W explicitly and find its dimension.

10. Let f be any bilinear form on a finite-dimensional vector space V. Let W be the subspace of all β such that $f(\alpha, \beta) = 0$ for every α. Show that

$$\text{rank } f = \dim V - \dim W.$$

Use this result and the result of Exercise 9 to compute the rank of the bilinear form defined in Exercise 6.

11. Let f be a bilinear form on a finite-dimensional vector space V. Suppose V_1 is a subspace of V with the property that the restriction of f to V_1 is non-degenerate. Show that rank $f \geq \dim V_1$.

12. Let f, g be bilinear forms on a finite-dimensional vector space V. Suppose g is non-singular. Show that there exist unique linear operators T_1, T_2 on V such that

$$f(\alpha, \beta) = g(T_1\alpha, \beta) = g(\alpha, T_2\beta)$$

for all α, β.

13. Show that the result given in Exercise 12 need not be true if g is singular.

14. Let f be a bilinear form on a finite-dimensional vector space V. Show that f can be expressed as a product of two linear functionals (i.e., $f(\alpha, \beta) = L_1(\alpha)L_2(\beta)$ for L_1, L_2 in V^*) if and only if f has rank 1.

10.2. Symmetric Bilinear Forms

The main purpose of this section is to answer the following question: If f is a bilinear form on the finite-dimensional vector space V, when is there an ordered basis $\mathcal{B}$ for V in which f is represented by a diagonal matrix? We prove that this is possible if and only if f is a symmetric bilinear form, i.e., $f(\alpha, \beta) = f(\beta, \alpha)$. The theorem is proved only when the scalar field has characteristic zero, that is, that if n is a positive integer the sum $1 + \cdots + 1$ (n times) in F is not 0.

Definition. *Let* f *be a bilinear form on the vector space* V. *We say that* f *is* **symmetric** *if* $f(\alpha, \beta) = f(\beta, \alpha)$ *for all vectors* α, β *in* V.

If V is a finite-dimensional, the bilinear form f is symmetric if and only if its matrix A in some (or every) ordered basis is symmetric, $A^t = A$. To see this, one inquires when the bilinear form

$$f(X, Y) = X^tAY$$

is symmetric. This happens if and only if $X^tAY = Y^tAX$ for all column matrices X and Y. Since X^tAY is a 1×1 matrix, we have $X^tAY = Y^tA^tX$. Thus f is symmetric if and only if $Y^tA^tX = Y^tAX$ for all X, Y. Clearly this just means that $A = A^t$. In particular, one should note that if there is an ordered basis for V in which f is represented by a diagonal matrix, then f is symmetric, for any diagonal matrix is a symmetric matrix.

If f is a symmetric bilinear form, the **quadratic form associated with** f is the function q from V into F defined by

$$q(\alpha) = f(\alpha, \alpha).$$

If F is a subfield of the complex numbers, the symmetric bilinear form f is completely determined by its associated quadratic form, according to the **polarization identity**

$$f(\alpha, \beta) = \tfrac{1}{4}q(\alpha + \beta) - \tfrac{1}{4}q(\alpha - \beta). \tag{10-5}$$

The establishment of (10-5) is a routine computation, which we omit. If f is the bilinear form of Example 5, the dot product, the associated quadratic form is

$$q(x_1, \ldots, x_n) = x_1^2 + \cdots + x_n^2.$$

In other words, $q(\alpha)$ is the square of the length of α. For the bilinear form $f_A(X, Y) = X^tAY$, the associated quadratic form is

$$q_A(X) = X^tAX = \sum_{i,j} A_{ij}x_ix_j.$$

One important class of symmetric bilinear forms consists of the inner products on real vector spaces, discussed in Chapter 8. If V is a *real* vector space, an **inner product** on V is a symmetric bilinear form f on V which satisfies

$$f(\alpha, \alpha) > 0 \quad \text{if} \quad \alpha \neq 0. \tag{10-6}$$

A bilinear form satisfying (10-6) is called **positive definite.** Thus, an inner product on a real vector space is a positive definite, symmetric bilinear form on that space. Note that an inner product is non-degenerate. Two vectors α, β are called **orthogonal** with respect to the inner product f if $f(\alpha, \beta) = 0$. The quadratic form $q(\alpha) = f(\alpha, \alpha)$ takes only non-negative values, and $q(\alpha)$ is usually thought of as the square of the length of α. Of course, these concepts of length and orthogonality stem from the most important example of an inner product—the dot product of Example 5.

If f is any symmetric bilinear form on a vector space V, it is convenient to apply some of the terminology of inner products to f. It is especially convenient to say that α and β are orthogonal with respect to f if $f(\alpha, \beta) = 0$. It is not advisable to think of $f(\alpha, \alpha)$ as the square of the length of α; for example, if V is a complex vector space, we may have $f(\alpha, \alpha) = \sqrt{-1}$, or on a real vector space, $f(\alpha, \alpha) = -2$.

We turn now to the basic theorem of this section. In reading the proof, the reader should find it helpful to think of the special case in which V is a real vector space and f is an inner product on V.

Theorem 3. *Let* V *be a finite-dimensional vector space over a field of characteristic zero, and let* f *be a symmetric bilinear form on* V. *Then there is an ordered basis for* V *in which* f *is represented by a diagonal matrix.*

Proof. What we must find is an ordered basis

$$\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}$$

such that $f(\alpha_i, \alpha_j) = 0$ for $i \neq j$. If $f = 0$ or $n = 1$, the theorem is obviously true. Thus we may suppose $f \neq 0$ and $n > 1$. If $f(\alpha, \alpha) = 0$ for every α in V, the associated quadratic form q is identically 0, and the polarization identity (10-5) shows that $f = 0$. Thus there is a vector α in V such that $f(\alpha, \alpha) = q(\alpha) \neq 0$. Let W be the one-dimensional subspace of V which is spanned by α, and let $W^\perp$ be the set of all vectors β in V such that $f(\alpha, \beta) = 0$. Now we claim that $V = W \oplus W^\perp$. Certainly the subspaces W and $W^\perp$ are independent. A typical vector in W is $c\alpha$, where c is a scalar. If $c\alpha$ is also in $W^\perp$, then $f(c\alpha, c\alpha) = c^2 f(\alpha, \alpha) = 0$. But $f(\alpha, \alpha) \neq 0$, thus $c = 0$. Also, each vector in V is the sum of a vector in W and a vector in $W^\perp$. For, let γ be any vector in V, and put

$$\beta = \gamma - \frac{f(\gamma, \alpha)}{f(\alpha, \alpha)}\,\alpha.$$

Then

$$f(\alpha, \beta) = f(\alpha, \gamma) - \frac{f(\gamma, \alpha)}{f(\alpha, \alpha)} f(\alpha, \alpha)$$

and since f is symmetric, $f(\alpha, \beta) = 0$. Thus β is in the subspace $W^\perp$. The expression

$$\gamma = \frac{f(\gamma, \alpha)}{f(\alpha, \alpha)}\,\alpha + \beta$$

shows us that $V = W + W^\perp$.

The restriction of f to $W^\perp$ is a symmetric bilinear form on $W^\perp$. Since $W^\perp$ has dimension $(n - 1)$, we may assume by induction that $W^\perp$ has a basis $\{\alpha_2, \ldots, \alpha_n\}$ such that

$$f(\alpha_i, \alpha_j) = 0, \qquad i \neq j\ (i \geq 2, j \geq 2).$$

Putting $\alpha_1 = \alpha$, we obtain a basis $\{\alpha_1, \ldots, \alpha_n\}$ for V such that $f(\alpha_i, \alpha_j) = 0$ for $i \neq j$. ∎

Corollary. *Let* F *be a subfield of the complex numbers, and let* A *be a symmetric* n × n *matrix over* F. *Then there is an invertible* n × n *matrix* P *over* F *such that* P^tAP *is diagonal.*

In case F is the field of real numbers, the invertible matrix P in this corollary can be chosen to be an *orthogonal* matrix, i.e., $P^t = P^{-1}$. In other words, if A is a real symmetric $n \times n$ matrix, there is a real orthogonal matrix P such that P^tAP is diagonal; however, this is not at all apparent from what we did above (see Chapter 8).

Theorem 4. *Let* V *be a finite-dimensional vector space over the field of complex numbers. Let* f *be a symmetric bilinear form on* V *which has rank* r. *Then there is an ordered basis* $\mathcal{B} = \{\beta_1, \ldots, \beta_n\}$ *for* V *such that*

(i) *the matrix of* f *in the ordered basis* $\mathcal{B}$ *is diagonal;*

(ii) $f(\beta_j, \beta_j) = \begin{cases} 1, & j = 1, \ldots, r \\ 0, & j > r. \end{cases}$

Proof. By Theorem 3, there is an ordered basis $\{\alpha_1, \ldots, \alpha_n\}$ for V such that

$$f(\alpha_i, \alpha_j) = 0 \quad \text{for} \quad i \neq j.$$

Since f has rank r, so does its matrix in the ordered basis $\{\alpha_1, \ldots, \alpha_n\}$. Thus we must have $f(\alpha_j, \alpha_j) \neq 0$ for precisely r values of j. By reordering the vectors α_j, we may assume that

$$f(\alpha_j, \alpha_j) \neq 0, \qquad j = 1, \ldots, r.$$

Now we use the fact that the scalar field is the field of complex numbers. If $\sqrt{f(\alpha_j, \alpha_j)}$ denotes any complex square root of $f(\alpha_j, \alpha_j)$, and if we put

$$\beta_j = \begin{cases} \dfrac{1}{\sqrt{f(\alpha_j, \alpha_j)}}\, \alpha_j, & j = 1, \ldots, r \\ \alpha_j, & j > r \end{cases}$$

the basis $\{\beta_1, \ldots, \beta_n\}$ satisfies conditions (i) and (ii). ∎

Of course, Theorem 4 is valid if the scalar field is any subfield of the complex numbers in which each element has a square root. It is not valid, for example, when the scalar field is the field of real numbers. Over the field of real numbers, we have the following substitute for Theorem 4.

Theorem 5. *Let* V *be an* n*-dimensional vector space over the field of real numbers, and let* f *be a symmetric bilinear form on* V *which has rank* r. *Then there is an ordered basis* $\{\beta_1, \beta_2, \ldots, \beta_n\}$ *for* V *in which the matrix of* f *is diagonal and such that*

$$f(\beta_j, \beta_j) = \pm 1, \qquad j = 1, \ldots, r.$$

Furthermore, the number of basis vectors β_j *for which* $f(\beta_j, \beta_j) = 1$ *is independent of the choice of basis.*

Proof. There is a basis $\{\alpha_1, \ldots, \alpha_n\}$ for V such that

$$\begin{aligned} f(\alpha_i, \alpha_j) &= 0, & i &\neq j \\ f(\alpha_j, \alpha_j) &\neq 0, & 1 &\leq j \leq r \\ f(\alpha_j, \alpha_j) &= 0, & j &> r. \end{aligned}$$