Note that if q is a positive rational number, then the Cauchy sequence $q, q, q, \ldots$ is positively bounded away from zero, and hence $\mathrm{LIM}_{n\to\infty} q = q$ is a positive real number. Thus the notion of positivity for rationals is consistent with that for reals. Similarly, the notion of negativity for rationals is consistent with that for reals.

Once we have defined positive and negative numbers, we can define absolute value and order.

Definition 5.4.5 (Absolute value). Let x be a real number. We define the *absolute value* $|x|$ of x to equal x if x is positive, $-x$ when x is negative, and 0 when x is zero.

Definition 5.4.6 (Ordering of the real numbers). Let x and y be real numbers. We say that x is *greater than* y, and write $x > y$, if $x - y$ is a positive real number, and $x < y$ iff $x - y$ is a negative real number. We define $x \geq y$ iff $x > y$ or $x = y$, and similarly define $x \leq y$.

Comparing this with the definition of order on the rationals from Definition 4.2.8 we see that order on the reals is consistent with order on the rationals, i.e., if two rational numbers q, q' are such that q is less than q' in the rational number system, then q is still less than q' in the real number system, and similarly for "greater than". In the same way we see that the definition of absolute value given here is consistent with that in Definition 4.3.1.

Proposition 5.4.7. *All the claims in Proposition 4.2.9 which held for rationals, continue to hold for real numbers.*

Proof. We just prove one of the claims and leave the rest to Exercise 5.4.2. Suppose we have $x < y$ and z a positive real, and want to conclude that $xz < yz$. Since $x < y$, $y - x$ is positive, hence by Proposition 5.4.4 we have $(y - x)z = yz - xz$ is positive, hence $xz < yz$. $\square$

As an application of these propositions, we prove

Proposition 5.4.8. *Let x be a positive real number. Then x^{-1} is also positive. Also, if y is another positive number and $x > y$, then $x^{-1} < y^{-1}$.*

Proof. Let x be positive. Since $xx^{-1} = 1$, the real number x^{-1} cannot be zero (since $x0 = 0 \neq 1$). Also, from Proposition 5.4.4 it is easy to see that a positive number times a negative number is negative; this shows that x^{-1} cannot be negative, since this would imply that $xx^{-1} = 1$ is negative, a contradiction. Thus, by Proposition 5.4.4, the only possibility left is that x^{-1} is positive.

Now let y be positive as well, so x^{-1} and y^{-1} are also positive. If $x^{-1} \geq y^{-1}$, then by Proposition 5.4.7 we have $xx^{-1} > yx^{-1} \geq yy^{-1}$, thus $1 > 1$, which is a contradiction. Thus we must have $x^{-1} < y^{-1}$. □

Another application is that the laws of exponentiation (Proposition 4.3.12) that were previously proven for rationals, are also true for reals; see Section 5.6.

We have already seen that the formal limit of positive rationals need not be positive; it could be zero, as the example $0.1, 0.01, 0.001, \ldots$ showed. However, the formal limit of *non-negative* rationals (i.e., rationals that are either positive or zero) is non-negative.

Proposition 5.4.9 (The non-negative reals are closed). *Let a_1, a_2, $a_3, \ldots$ be a Cauchy sequence of non-negative rational numbers. Then $\mathrm{LIM}_{n\to\infty} a_n$ is a non-negative real number.*

Eventually, we will see a better explanation of this fact: the set of non-negative reals is *closed*, whereas the set of positive reals is *open*. See Section 12.2.

Proof. We argue by contradiction, and suppose that the real number $x := \mathrm{LIM}_{n\to\infty} a_n$ is a negative number. Then by definition of negative real number, we have $x = \mathrm{LIM}_{n\to\infty} b_n$ for some sequence b_n which is negatively bounded away from zero, i.e., there is a negative rational $-c < 0$ such that $b_n \leq -c$ for all $n \geq 1$. On the other hand, we have $a_n \geq 0$ for all $n \geq 1$, by hypothesis. Thus the numbers a_n and b_n are never $c/2$-close, since $c/2 < c$. Thus the sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are not eventually $c/2$-close. Since $c/2 > 0$, this implies that $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are not equivalent. But this contradicts the fact that both these sequences have x as their formal limit. □

Corollary 5.4.10. *Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be Cauchy sequences of rationals such that $a_n \geq b_n$ for all $n \geq 1$. Then* $\mathrm{LIM}_{n\to\infty}a_n \geq \mathrm{LIM}_{n\to\infty}b_n$.

Proof. Apply Proposition 5.4.9 to the sequence $a_n - b_n$. □

Remark 5.4.11. Note that the above Corollary does not work if the $\geq$ signs are replaced by $>$: for instance if $a_n := 1 + 1/n$ and $b_n := 1 - 1/n$, then a_n is always strictly greater than b_n, but the formal limit of a_n is not greater than the formal limit of b_n, instead they are equal.

We now define distance $d(x, y) := |x - y|$ just as we did for the rationals. In fact, Propositions 4.3.3 and 4.3.7 hold not only for the rationals, but for the reals; the proof is identical, since the real numbers obey all the laws of algebra and order that the rationals do.

We now observe that while positive real numbers can be arbitrarily large or small, they cannot be than all of the positive integers, or smaller in magnitude than all of the positive rationals:

Proposition 5.4.12 (Bounding of reals by rationals). *Let x be a positive real number. Then there exists a positive rational number q such that $q \leq x$, and there exists a positive integer N such that $x \leq N$.*

Proof. Since x is a positive real, it is the formal limit of some Cauchy sequence $(a_n)_{n=1}^{\infty}$ which is positively bounded away from zero. Also, by Lemma 5.1.15, this sequence is bounded. Thus we have rationals $q > 0$ and r such that $q \leq a_n \leq r$ for all $n \geq 1$. But by Proposition 4.4.1 we know that there is some integer N

such that $r \leq N$; since q is positive and $q \leq r \leq N$, we see that N is positive. Thus $q \leq a_n \leq N$ for all $n \geq 1$. Applying Corollary 5.4.10 we obtain that $q \leq x \leq N$, as desired. □

Corollary 5.4.13 (Archimedean property). *Let x and ε be any positive real numbers. Then there exists a positive integer M such that $M\varepsilon > x$.*

Proof. The number x/ε is positive, and hence by Proposition 5.4.12 there exists a positive integer N such that $x/\varepsilon \leq N$. If we set $M := N + 1$, then $x/\varepsilon < M$. Now multiply by ε. □

This property is quite important; it says that no matter how large x is and how small ε is, if one keeps adding ε to itself, one will eventually overtake x.

Proposition 5.4.14. *Given any two real numbers $x < y$, we can find a rational number q such that $x < q < y$.*

Proof. See Exercise 5.4.5. □

We have now completed our construction of the real numbers. This number system contains the rationals, and has almost everything that the rational number system has: the arithmetic operations, the laws of algebra, the laws of order. However, we have not yet demonstrated any *advantages* that the real numbers have over the rationals; so far, even after much effort, all we have done is shown that they are *at least as good as* the rational number system. But in the next few sections we show that the real numbers can do more things than rationals: for example, we can take square roots in a real number system.

Remark 5.4.15. Up until now, we have not addressed the fact that real numbers can be expressed using the decimal system. For instance, the formal limit of

$$1.4, 1.41, 1.414, 1.4142, 1.41421, \ldots$$

is more conventionally represented as the decimal $1.41421\ldots$. We will address this in an Appendix (§B), but for now let us just remark that there are some subtleties in the decimal system, for instance $0.9999\ldots$ and $1.000\ldots$ are in fact the same real number.

Exercise 5.4.1. Prove Proposition 5.4.4. (Hint: if x is not zero, and x is the formal limit of some sequence $(a_n)_{n=1}^\infty$, then this sequence cannot be eventually ε-close to the zero sequence $(0)_{n=1}^\infty$ for every single $\varepsilon > 0$. Use this to show that the sequence $(a_n)_{n=1}^\infty$ is eventually either positively bounded away from zero or negatively bounded away from zero.)

Exercise 5.4.2. Prove the remaining claims in Proposition 5.4.7.

Exercise 5.4.3. Show that for every real number x there is exactly one integer N such that $N \leq x < N+1$. (This integer N is called the *integer part* of x, and is sometimes denoted $N = \lfloor x \rfloor$.)

Exercise 5.4.4. Show that for any positive real number $x > 0$ there exists a positive integer N such that $x > 1/N > 0$.

Exercise 5.4.5. Prove Proposition 5.4.14. (Hint: use Exercise 5.4.4. You may also need to argue by contradiction.)

Exercise 5.4.6. Let x, y be real numbers and let $\varepsilon > 0$ be a positive real. Show that $|x-y| < \varepsilon$ if and only if $y-\varepsilon < x < y+\varepsilon$, and that $|x-y| \leq \varepsilon$ if and only if $y - \varepsilon \leq x \leq y + \varepsilon$.

Exercise 5.4.7. Let x and y be real numbers. Show that $x \leq y + \varepsilon$ for all real numbers $\varepsilon > 0$ if and only if $x \leq y$. Show that $|x - y| \leq \varepsilon$ for all real numbers $\varepsilon > 0$ if and only if $x = y$.

Exercise 5.4.8. Let $(a_n)_{n=1}^\infty$ be a Cauchy sequence of rationals, and let x be a real number. Show that if $a_n \leq x$ for all $n \geq 1$, then $\mathrm{LIM}_{n\to\infty} a_n \leq x$. Similarly, show that if $a_n \geq x$ for all $n \geq 1$, then $\mathrm{LIM}_{n\to\infty} a_n \geq x$. (Hint: prove by contradiction. Use Proposition 5.4.14 to find a rational between $\mathrm{LIM}_{n\to\infty} a_n$ and x, and then use Proposition 5.4.9.)

5.5 The least upper bound property

We now give one of the most basic advantages of the real numbers over the rationals; one can take the *least upper bound* $\sup(E)$ of any subset E of the real numbers $\mathbf{R}$.

Definition 5.5.1 (Upper bound). Let E be a subset of $\mathbf{R}$, and let M be a real number. We say that M is an *upper bound* for E, iff we have $x \leq M$ for every element x in E.

Example 5.5.2. Let E be the interval $E := \{x \in \mathbf{R} : 0 \leq x \leq 1\}$. Then 1 is an upper bound for E, since every element of E is less than or equal to 1. It is also true that 2 is an upper bound for E, and indeed every number greater or equal to 1 is an upper bound for E. On the other hand, any other number, such as 0.5, is not an upper bound, because 0.5 is not larger than *every* element in E. (Merely being larger than *some* elements of E is not necessarily enough to make 0.5 an upper bound.)

Example 5.5.3. Let $\mathbf{R}^+$ be the set of positive reals: $\mathbf{R}^+ := \{x \in \mathbf{R} : x > 0\}$. Then $\mathbf{R}^+$ does not have any upper bounds[3] at all (why?).

Example 5.5.4. Let $\emptyset$ be the empty set. Then every number M is an upper bound for $\emptyset$, because M is greater than every element of the empty set (this is a vacuously true statement, but still true).

It is clear that if M is an upper bound of E, then any larger number $M' \geq M$ is also an upper bound of E. On the other hand, it is not so clear whether it is also possible for any number smaller than M to also be an upper bound of E. This motivates the following definition:

Definition 5.5.5 (Least upper bound). Let E be a subset of $\mathbf{R}$, and M be a real number. We say that M is a *least upper bound* for E iff (a) M is an upper bound for E, and also (b) any other upper bound M' for E must be larger than or equal to M.

Example 5.5.6. Let E be the interval $E := \{x \in \mathbf{R} : 0 \leq x \leq 1\}$. Then, as noted before, E has many upper bounds, indeed every number greater than or equal to 1 is an upper bound. But only 1 is the *least* upper bound; all other upper bounds are larger than 1.

Example 5.5.7. The empty set does not have a least upper bound (why?).

---

[3]More precisely, $\mathbf{R}^+$ has no upper bounds which are real numbers. In Section 6.2 we shall introduce the *extended real number system* $\mathbf{R}^*$, which allows one to give the upper bound of $+\infty$ for sets such as $\mathbf{R}^+$.

Proposition 5.5.8 (Uniqueness of least upper bound). *Let E be a subset of* $\mathbf{R}$*. Then E can have at most one least upper bound.*

Proof. Let M_1 and M_2 be two least upper bounds, say M_1 and M_2. Since M_1 is a least upper bound and M_2 is an upper bound, then by definition of least upper bound we have $M_2 \geq M_1$. Since M_2 is a least upper bound and M_1 is an upper bound, we similarly have $M_1 \geq M_2$. Thus $M_1 = M_2$. Thus there is at most one least upper bound. □

Now we come to an important property of the real numbers:

Theorem 5.5.9 (Existence of least upper bound). *Let E be a non-empty subset of* $\mathbf{R}$*. If E has an upper bound, (i.e., E has some upper bound M), then it must have exactly one least upper bound.*

Proof. This theorem will take quite a bit of effort to prove, and many of the steps will be left as exercises.

Let E be a non-empty subset of $\mathbf{R}$ with an upper bound M. By Proposition 5.5.8, we know that E has at most one least upper bound; we have to show that E has at least one least upper bound. Since E is non-empty, we can choose some element x_0 in E.

Let $n \geq 1$ be a positive integer. We know that E has an upper bound M. By the Archimedean property (Corollary 5.4.13), we can find an integer K such that $K/n \geq M$, and hence K/n is also an upper bound for E. By the Archimedean property again, there exists another integer L such that $L/n < x_0$. Since x_0 lies in E, we see that L/n is *not* an upper bound for E. Since K/n is an upper bound but L/n is not, we see that $K \geq L$.

Since K/n is an upper bound for E and L/n is not, we can find an integer $L < m_n \leq K$ with the property that m_n/n is an upper bound for E, but $(m_n - 1)/n$ is not (see Exercise 5.5.2). In fact, this integer m_n is unique (Exercise 5.5.3). We subscript m_n by n to emphasize the fact that this integer m depends on the choice of n. This gives a well-defined (and unique) sequence $m_1, m_2, m_3, \ldots$ of integers, with each of the m_n/n being upper bounds and each of the $(m_n - 1)/n$ not being upper bounds.

Now let $N \geq 1$ be a positive integer, and let $n, n' \geq N$ be integers larger than or equal to N. Since m_n/n is an upper bound for E and $(m_{n'} - 1)/n'$ is not, we must have $m_n/n > (m_{n'} - 1)/n'$ (why?). After a little algebra, this implies that

$$\frac{m_n}{n} - \frac{m_{n'}}{n'} > -\frac{1}{n'} \geq -\frac{1}{N}.$$

Similarly, since $m_{n'}/n'$ is an upper bound for E and $(m_n - 1)/n$ is not, we have $m_{n'}/n' > (m_n - 1)/n$, and hence

$$\frac{m_n}{n} - \frac{m_{n'}}{n'} \leq \frac{1}{n} \leq \frac{1}{N}.$$

Putting these two bounds together, we see that

$$|\frac{m_n}{n} - \frac{m_{n'}}{n'}| \leq \frac{1}{N} \text{ for all } n, n' \geq N \geq 1.$$

This implies that $\frac{m_n}{n}$ is a Cauchy sequence (Exercise 5.5.4). Since the $\frac{m_n}{n}$ are rational numbers, we can now define the real number S as

$$S := \mathrm{LIM}_{n\to\infty}\frac{m_n}{n}.$$

From Exercise 5.3.5 we conclude that

$$S = \mathrm{LIM}_{n\to\infty}\frac{m_n - 1}{n}.$$

To finish the proof of the theorem, we need to show that S is the least upper bound for E. First we show that it is an upper bound. Let x be any element of E. Then, since m_n/n is an upper bound for E, we have $x \leq m_n/n$ for all $n \geq 1$. Applying Exercise 5.4.8, we conclude that $x \leq \mathrm{LIM}_{n\to\infty} m_n/n = S$. Thus S is indeed an upper bound for E.

Now we show it is a least upper bound. Suppose y is an upper bound for E. Since $(m_n - 1)/n$ is not an upper bound, we conclude that $y \geq (m_n - 1)/n$ for all $n \geq 1$. Applying Exercise 5.4.8, we conclude that $y \geq \mathrm{LIM}_{n\to\infty}(m_n - 1)/n = S$. Thus the upper bound S is less than or equal to every upper bound of E, and S is thus a least upper bound of E. □

Definition 5.5.10 (Supremum). Let E be a subset of the real numbers. If E is non-empty and has some upper bound, we define $\sup(E)$ to be the least upper bound of E (this is well-defined by Theorem 5.5.9). We introduce two additional symbols, $+\infty$ and $-\infty$. If E is non-empty and has no upper bound, we set $\sup(E) := +\infty$; if E is empty, we set $\sup(E) := -\infty$. We refer to $\sup(E)$ as the *supremum* of E, and also denote it by $\sup E$.

Remark 5.5.11. At present, $+\infty$ and $-\infty$ are meaningless symbols; we have no operations on them at present, and none of our results involving real numbers apply to $+\infty$ and $-\infty$, because these are not real numbers. In Section 6.2 we add $+\infty$ and $-\infty$ to the reals to form the *extended real number system*, but this system is not as convenient to work with as the real number system, because many of the laws of algebra break down. For instance, it is not a good idea to try to define $+\infty + -\infty$; setting this equal to 0 causes some problems.

Now we give an example of how the least upper bound property is useful.

Proposition 5.5.12. *There exists a positive real number x such that $x^2 = 2$.*

Remark 5.5.13. Comparing this result with Proposition 4.4.4, we see that certain numbers are real but not rational. The proof of this proposition also shows that the rationals $\mathbf{Q}$ do not obey the least upper bound property, otherwise one could use that property to construct a square root of 2, which by Proposition 4.4.4 is not possible.

Proof. Let E be the set $\{y \in \mathbf{R} : y \geq 0 \text{ and } y^2 < 2\}$; thus E is the set of all non-negative real numbers whose square is less than 2. Observe that E has an upper bound of 2 (because if $y > 2$, then $y^2 > 4 > 2$ and hence $y \notin E$). Also, E is non-empty (for instance, 1 is an element of E). Thus by the least upper bound property, we have a real number $x := \sup(E)$ which is the least upper bound of E. Then x is greater than or equal to 1 (since

$1 \in E$) and less than or equal to 2 (since 2 is an upper bound for E). So x is positive. Now we show that $x^2 = 2$.

We argue this by contradiction. We show that both $x^2 < 2$ and $x^2 > 2$ lead to contradictions. First suppose that $x^2 < 2$. Let $0 < \varepsilon < 1$ be a small number; then we have

$$(x+\varepsilon)^2 = x^2 + 2\varepsilon x + \varepsilon^2 \leq x^2 + 4\varepsilon + \varepsilon = x^2 + 5\varepsilon$$

since $x \leq 2$ and $\varepsilon^2 \leq \varepsilon$. Since $x^2 < 2$, we see that we can choose an $0 < \varepsilon < 1$ such that $x^2 + 5\varepsilon < 2$, thus $(x+\varepsilon)^2 < 2$. By construction of E, this means that $x+\varepsilon \in E$; but this contradicts the fact that x is an upper bound of E.

Now suppose that $x^2 > 2$. Let $0 < \varepsilon < 1$ be a small number; then we have

$$(x-\varepsilon)^2 = x^2 - 2\varepsilon x + \varepsilon^2 \geq x^2 - 2\varepsilon x \geq x^2 - 4\varepsilon$$

since $x \leq 2$ and $\varepsilon^2 \geq 0$. Since $x^2 > 2$, we can choose $0 < \varepsilon < 1$ such that $x^2 - 4\varepsilon > 2$, and thus $(x-\varepsilon)^2 > 2$. But then this implies that $x - \varepsilon \geq y$ for all $y \in E$. (Why? If $x - \varepsilon < y$ then $(x-\varepsilon)^2 < y^2 \leq 2$, a contradiction.) Thus $x - \varepsilon$ is an upper bound for E, which contradicts the fact that x is the *least* upper bound of E. From these two contradictions we see that $x^2 = 2$, as desired. □

Remark 5.5.14. In Chapter 6 we will use the least upper bound property to develop the theory of limits, which allows us to do many more things than just take square roots.

Remark 5.5.15. We can of course talk about lower bounds, and greatest lower bounds, of sets E; the greatest lower bound of a set E is also known as the *infimum*[4] of E and is denoted $\inf(E)$ or $\inf E$. Everything we say about suprema has a counterpart for

[4]Supremum means "highest" and infimum means "lowest", and the plurals are suprema and infima. Supremum is to superior, and infimum to inferior, as maximum is to major, and minimum to minor. The root words are "super", which means "above", and "infer", which means "below" (this usage only survives in a few rare English words such as "infernal", with the Latin prefix "sub" having mostly replaced "infer" in English).

infima; we will usually leave such statements to the reader. A precise relationship between the two notions is given by Exercise 5.5.1. See also Section 6.2.

Exercise 5.5.1. Let E be a subset of the real numbers $\mathbf{R}$, and suppose that E has a least upper bound M which is a real number, i.e., $M = \sup(E)$. Let $-E$ be the set

$$-E := \{-x : x \in E\}.$$

Show that $-M$ is the greatest lower bound of $-E$, i.e., $-M = \inf(-E)$.

Exercise 5.5.2. Let E be a non-empty subset of $\mathbf{R}$, let $n \geq 1$ be an integer, and let $L < K$ be integers. Suppose that K/n is an upper bound for E, but that L/n is not an upper bound for E. Without using Theorem 5.5.9, show that there exists an integer $L < m \leq K$ such that m/n is an upper bound for E, but that $(m-1)/n$ is not an upper bound for E. (Hint: prove by contradiction, and use induction. It may also help to draw a picture of the situation.)

Exercise 5.5.3. Let E be a non-empty subset of $\mathbf{R}$, let $n \geq 1$ be an integer, and let m, m' be integers with the properties that m/n and m'/n are upper bounds for E, but $(m-1)/n$ and $(m'-1)/n$ are not upper bounds for E. Show that $m = m'$. This shows that the integer m constructed in Exercise 5.5.2 is unique. (Hint: again, drawing a picture will be helpful.)

Exercise 5.5.4. Let $q_1, q_2, q_3, \ldots$ be a sequence of rational numbers with the property that $|q_n - q_{n'}| \leq \frac{1}{M}$ whenever $M \geq 1$ is an integer and $n, n' \geq M$. Show that $q_1, q_2, q_3, \ldots$ is a Cauchy sequence. Furthermore, if $S := \mathrm{LIM}_{n\to\infty} q_n$, show that $|q_M - S| \leq \frac{1}{M}$ for every $M \geq 1$. (Hint: use Exercise 5.4.8.)

5.6 Real exponentiation, part I

In Section 4.3 we defined exponentiation x^n when x is rational and n is a natural number, or when x is a non-zero rational and n is an integer. Now that we have all the arithmetic operations on the reals (and Proposition 5.4.7 assures us that the arithmetic properties of the rationals that we are used to, continue to hold for the reals) we can similarly define exponentiation of the reals.

Definition 5.6.1 (Exponentiating a real by a natural number). Let x be a real number. To raise x to the power 0, we define $x^0 := 1$. Now suppose recursively that x^n has been defined for some natural number n, then we define $x^{n+1} := x^n \times x$.

Definition 5.6.2 (Exponentiating a real by an integer). Let x be a non-zero real number. Then for any negative integer $-n$, we define $x^{-n} := 1/x^n$.

Clearly these definitions are consistent with the definition of rational exponentiation given earlier. We can then assert

Proposition 5.6.3. *All the properties in Propositions 4.3.10 and 4.3.12 remain valid if x and y are assumed to be real numbers instead of rational numbers.*

Instead of giving an actual proof of this proposition, we shall give a meta-proof (an argument appealing to the nature of proofs, rather than the nature of real and rational numbers).

Meta-proof. If one inspects the proof of Propositions 4.3.10 and 4.3.12 we see that they rely on the laws of algebra and the laws of order for the rationals (Propositions 4.2.4 and 4.2.9). But by Propositions 5.3.11, 5.4.7, and the identity $xx^{-1} = x^{-1}x = 1$ we know that all these laws of algebra and order continue to hold for real numbers as well as rationals. Thus we can modify the proof of Proposition 4.3.10 and 4.3.12 to hold in the case when x and y are real. □

Now we consider exponentiation to exponents which are not integers. We begin with the notion of an n^{th} root, which we can define using our notion of supremum.

Definition 5.6.4. Let $x > 0$ be a positive real, and let $n \geq 1$ be a positive integer. We define $x^{1/n}$, also known as the n^{th} *root of* x, by the formula

$$x^{1/n} := \sup\{y \in \mathbf{R} : y \geq 0 \text{ and } y^n \leq x\}.$$

We often write $\sqrt{x}$ for $x^{1/2}$.

Note we do not define the n^{th} root of zero at this point, nor do we define the n^{th} root of a negative number. The former issue will be addressed presently; as for the latter, we will leave the n^{th} roots of negative numbers undefined for the rest of the text (one can define these n^{th} roots once one defines the complex numbers, but we shall refrain from doing so).

Lemma 5.6.5 (Existence of n^{th} roots). *Let $x > 0$ be a positive real, and let $n \geq 1$ be a positive integer. Then the set $E := \{y \in \mathbf{R} : y \geq 0 \text{ and } y^n \leq x\}$ is non-empty and is also bounded above. In particular, $x^{1/n}$ is a real number.*

Proof. The set E contains 0 (why?), so it is certainly not empty. Now we show it has an upper bound. We divide into two cases: $x \leq 1$ and $x > 1$. First suppose that we are in the case where $x \leq 1$. Then we claim that the set E is bounded above by 1. To see this, suppose for sake of contradiction that there was an element $y \in E$ for which $y > 1$. But then $y^n > 1$ (why?), and hence $y^n > x$, a contradiction. Thus E has an upper bound. Now suppose that we are in the case where $x > 1$. Then we claim that the set E is bounded above by x. To see this, suppose for contradiction that there was an element $y \in E$ for which $y > x$. Since $x > 1$, we thus have $y > 1$. Since $y > x$ and $y > 1$, we have $y^n > x$ (why?), a contradiction. Thus in both cases E has an upper bound, and so $x^{1/n}$ is finite. $\square$

We list some basic properties of n^{th} root below.

Lemma 5.6.6. *Let $x, y > 0$ be positive reals, and let $n, m \geq 1$ be positive integers.*

(a) *If $y = x^{1/n}$, then $y^n = x$.*

(b) *Conversely, if $y^n = x$, then $y = x^{1/n}$.*

(c) *$x^{1/n}$ is a positive real number.*

(d) *We have $x > y$ if and only if $x^{1/n} > y^{1/n}$.*