smooth if the interval [a, b] can be partitioned into a finite number of subintervals in each of which the path is smooth.

Figure 10.1 shows the graph of a piecewise smooth path. In this example the curve has a tangent line at all but a finite number of its points. These exceptional points subdivide the curve into arcs, along each of which the tangent line turns continuously.

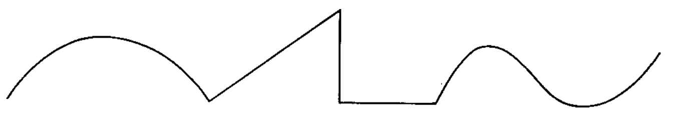


FIGURE 10.1 The graph of a piecewise smooth path in the plane.

DEFINITION OF LINE INTEGRAL. *Let a be a piecewise smooth path in n-space defined on an interval [a, b], and let* $\boldsymbol{f}$ *be a vector field defined and bounded on the graph of a. The line integral of* $\boldsymbol{f}$ *along a is denoted by the symbol* $\int \boldsymbol{f} \cdot da$ *and is defined by the equation*

$$\int \boldsymbol{f} \cdot da = \int_a^b \boldsymbol{f}[\boldsymbol{\alpha}(t)] \,.\, a'(t)\ dt, \tag{10.1}$$

whenever the integral on the right exists, either as a proper or improper integral.

Note: In most examples that occur in practice the dot product $\boldsymbol{f}[\boldsymbol{\alpha}(t)] \cdot \boldsymbol{\alpha}'(t)$ is bounded on [a, b] and continuous except possibly at a finite number of points, in which case the integral exists as a proper integral.

10.3 Other notations for line integrals

If C denotes the graph of **a,** the line integral $\int \boldsymbol{f} \cdot da$ is also written as $\int_C \boldsymbol{f} \cdot da$ and is called *the integral off along C.*

If a = *a(a)* and $\boldsymbol{b}$ = *a(b)* denote the end points of C, the line integral is sometimes written as $\int_a^b$ *for* as $\int_a^b \boldsymbol{f} \cdot da$ and is called the line integral off from $\boldsymbol{a}$ to $\boldsymbol{b}$ along **a.** When the notation $\int_a^b \boldsymbol{f}$ is used it should be kept in mind that the integral depends not only on the end points a and $\boldsymbol{b}$ but also on the path **a** joining them.

When $\boldsymbol{a} = \boldsymbol{b}$ the path is said to be *closed.* The symbol $\oint$ is often used to indicate integration along a closed path.

When $\boldsymbol{f}$ and *a* are expressed in terms of their components, say

$$\boldsymbol{f} = (f_1, \ldots, f_n) \qquad \text{and} \qquad \boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n),$$

the integral on the right of (10.1) becomes a sum of integrals,

$$\sum_{k=1}^{n} \int_a^b f_k[\boldsymbol{\alpha}(t)]\alpha_k'(t)\ dt.$$

In this case the line integral is also written as $\int f_1\ da, + \cdots + f_n\ da,.$

In the two-dimensional case the path α is usually described by a pair of parametric equations,

$$x = \alpha_1(t), \qquad y = \alpha_2(t),$$

and the line integral $\int_C f \cdot da$ is written as $\int_C f_1\, dx + f_2\, dy$, or as $\int_C f_1(x, y)\, dx + f_2(x, y)\, dy$.

In the three-dimensional case we use three parametric equations,

$$x = \alpha_1(t), \qquad y = \alpha_2(t), \qquad z = \alpha_3(t)$$

and write the line integral $\int_C f \cdot da$ as $\int_C f_1\, dx + f_2\, dy + f_3\, dz$, or as

$$\mathrm{I}_C\, f_1(x, y, z)\, dx + f_2(x, y, z)\, dy + f_3(x, y, z)\, dz.$$

EXAMPLE. Let f be a two-dimensional vector field given by

$$f(x, y) = \mathrm{Jy}\, \boldsymbol{i} + (x^3 + y)\boldsymbol{j}$$

for all (x, y) withy $\geq$ 0. Calculate the line integral of f from (0, 0) to (1, 1) along each of the following paths :

(a) the line with parametric equations $x = t$, $y = t$, $0 \leq t \leq 1$;
(b) the path with parametric equations $x = t^2$, $y = t^3$, $0 \leq t \leq 1$.

Solution. For the path in part (a) we take $a(t) = ti + t\boldsymbol{j}$. Then $a'(t) = i + \boldsymbol{j}$ and $f[\alpha(t)] = \sqrt{t}\, \boldsymbol{i} + (t^3 + t)j$. Therefore the dot product of $f[\alpha(t)]$ and $a'(t)$ is equal to $\sqrt{t} + t^3 + t$ and we **find**

$$\int_{(0,0)}^{(1,1)} f \cdot da = \int_0^1 (Ji + t^3 + t)\, dt = \frac{17}{12}.$$

For the path in part (b) we take $a(t) = t^2\boldsymbol{i} + t^3\boldsymbol{j}$. Then $a'(t) = 2ti + 3t^2\boldsymbol{j}$ and $f[\alpha(t)] = t^{3/2}\boldsymbol{i} + (t^6 + t^3)\boldsymbol{j}$. Therefore

$$f[\alpha(t)] \cdot a'(t) = 2t^{5/2} + 3t^8 + 3t^5,$$

so

$$\int_{(0,0)}^{(1,1)} f \cdot da = \int_0^1 (2t^{5/2} + 3t^8 + 3t^5)\, dt = \frac{59}{42}.$$

This example shows that the integral from one point to another may depend on the path joining the two points.

Now let us carry out the calculation for part (b) once more, using the same curve but with a different parametric representation. The same curve can be described by the function

$$\beta(t) = ti + t^{3/2}\boldsymbol{j}, \qquad \text{where } 0 \leq t \leq 1.$$

This leads to the relation

$$\boldsymbol{f}[\boldsymbol{\beta}(t)] \cdot \boldsymbol{\beta}'(t) = (t^{3/4}\boldsymbol{i} + (t^3 + t^{3/2})\boldsymbol{j}) \cdot (\boldsymbol{i} + \tfrac{3}{2}t^{1/2}\boldsymbol{j}) = t^{3/4} + \tfrac{3}{2}t^{7/2} + \tfrac{3}{2}t^2,$$

the integral of which from 0 to 1 is $59/42$, as before. This calculation illustrates that the value of the integral is independent of the parametric representation used to describe the curve. This is a general property of line integrals which is proved in the next section.

10.4 Basic properties of line integrals

Since line integrals are defined in terms of ordinary integrals, it is not surprising to find that they share many of the properties of ordinary integrals. For example, they have a *linearity property* with respect to the integrand,

$$\int (a\boldsymbol{f} + b\boldsymbol{g}) \cdot d\boldsymbol{\alpha} = a \int \boldsymbol{f} \cdot d\boldsymbol{\alpha} + b \int \boldsymbol{g} \cdot d\boldsymbol{\alpha},$$

and an *additiveproperty* with respect to the path of integration:

$$\int_C \boldsymbol{f} \cdot d\boldsymbol{\alpha} = \int_{C_1} \boldsymbol{f} \cdot d\boldsymbol{\alpha} + \int_{C_2} \boldsymbol{f} \cdot d\boldsymbol{\alpha},$$

where the two curves C_1 and C_2 make up the curve C. That is, C is described by a function a defined on an interval $[a, b]$, and the curves C_1 and C_2 are those traced out by $a(t)$ as t varies over subintervals $[a, c]$ and. $[c, b]$, respectively, for some c satisfying $a < c < b$. The proofs of these properties follow immediately from the definition of the line integral; they are left as exercises for the reader.

Next we examine the behavior of line integrals under a change of parameter. Let a be a continuous path defined on an interval $[a, b]$, let u be a real-valued function that is differentiable, with u' never zero on an interval $[c, d]$, and such that the range of u is $[a, b]$. Then the function $\boldsymbol{\beta}$ defined on $[c, d]$ by the equation

$$\boldsymbol{\beta}(t) = \boldsymbol{\alpha}[u(t)]$$

is a continuous path having the same graph as a. Two paths a and $\boldsymbol{\beta}$ so related are called *equivalent.* They are said to provide different parametric representations of the same curve. The function u is said to define a change of parameter.

Let C denote the common graph of two equivalent paths a and $\boldsymbol{\beta}$. If the derivative of u is always positive on $[c, d]$ the function u is increasing and we say that the two paths a and $\boldsymbol{\beta}$ trace out C in the *same direction.* If the derivative of u is always negative we say that a and $\boldsymbol{\beta}$ trace out C in *opposite directions.* In the first case the function u is said to be *orientation-preserving;* in the second case u is said to be *orientation-reversing.* An example is shown in Figure 10.2.

The next theorem shows that a line integral remains unchanged under a change of parameter that preserves orientation; it reverses its sign if the change of parameter reverses orientation. We assume both intergals $\int \boldsymbol{f} \cdot da$ and $\int \boldsymbol{f} \cdot d\boldsymbol{\beta}$ exist.

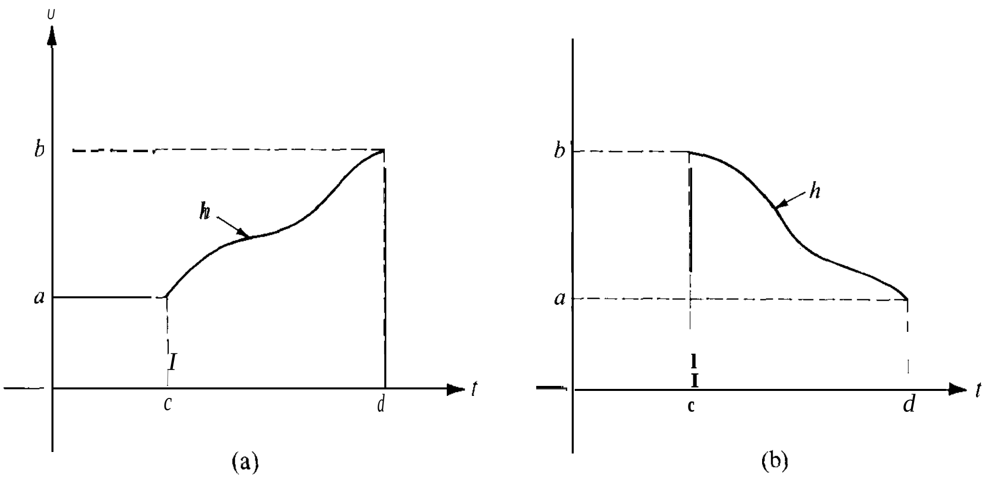


FIGURE 10.2 A change of parameter defined by $u = h(t)$. In (a), the function h preserves orientation. In (b), the function h reverses the orientation.

THEOREM 10.1. BEHAVIOR OF A LINE INTEGRAL UNDER A CHANGE OF PARAMETER. *Let* a *and* $\boldsymbol{\beta}$ *be equivalent piecewise smooth paths. Then we have*

$$\int_C \boldsymbol{f} \cdot d\boldsymbol{\alpha} = \int_C \boldsymbol{f} \cdot d\boldsymbol{\beta}$$

if a *and* $\boldsymbol{\beta}$ *trace out C in the same direction; and*

$$\int_C \boldsymbol{f} \cdot d\boldsymbol{\alpha} = -\int_C \boldsymbol{f} \cdot d\boldsymbol{\beta}$$

if a *and* $\boldsymbol{\beta}$ *trace out C in opposite directions.*

Proof, It suffices to prove the theorem for smooth paths; then we invoke the additive property with respect to the path of integration to deduce the result for piecewise smooth paths.

The proof is a simple application of the chain rule. The paths a and $\boldsymbol{\beta}$ are related by an equation of the form $\boldsymbol{\beta}(t) = \boldsymbol{\alpha}[u(t)]$, where u is defined on an interval [c, d] and a is defined on an interval $[a, b]$. From the chain rule we have

$$\boldsymbol{\beta}'(t) = \boldsymbol{\alpha}'[u(t)]u'(t).$$

Therefore we find

$$\int_C \boldsymbol{f} \cdot d\boldsymbol{\beta} = \int_c^d \boldsymbol{f}\,[\boldsymbol{\beta}(t)] \cdot \boldsymbol{\beta}'(t)\,dt = \int_c^d \boldsymbol{f}(\boldsymbol{\alpha}[u(t)]) \cdot \boldsymbol{\alpha}'[u(t)]u'(t)\,dt.$$

In the last integral we introduce the substitution v = u(t), $dv = u'(t)\,dt$ to obtain

$$\int_C \boldsymbol{f} \cdot d\boldsymbol{\beta} = \int_{u(c)}^{u(d)} \boldsymbol{f}(\boldsymbol{\alpha}(v)) \cdot a'(v)\,dv = \pm \int_a^b \boldsymbol{f}(\boldsymbol{\alpha}(v)) \cdot a'(v)\,dv = \pm \int_C \boldsymbol{f} \cdot d\boldsymbol{\alpha},$$

where the + sign is used if $a = u(c)$ and $b = u(d)$, and the $-$ sign is used if $a = u(d)$ and $b = u(c)$. The first case occurs if α and β trace out C in the same direction, the second if they trace out C in opposite directions.

10.5 Exercises

In each of Exercises 1 through 8 calculate the line integral of the vector field f along the path described.

1. $f(x, y) = (x^2 - 2xy)i + (y^2 - 2xy)j$, from $(-1, 1)$ to $(1, 1)$ along the parabola $y = x^2$.
2. $f(x, y) = (2a - y)i + xj$, along the path described by $\alpha(t) = a(t - \sin t)i + a(1 - \cos t)j$, $0 \le t \le 2\pi$.
3. $f(x, y, z) = (y^2 - z^2)i + 2yzj - x^2k$, along the path described by $\alpha(t) = ti + t^2j + t^3k$, $0 \le t \le 1$.
4. $f(x, y) = (x^2 + y^2)i + (x^2 - y^2)j$, from $(0, 0)$ to $(2,0)$ along the curve $y = 1 - |1 - x|$.
5. $f(x, y) = (x + y)i + (x - y)j$, once around the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ in a counterclockwise direction.
6. $f(x, y, z) = 2xyi + (x^2 + z)j + yk$, from $(1, 0, 2)$ to $(3, 4, 1)$ along a line segment.
7. $f(x, y, z) = xi + yj + (xz - y)k$, from $(0, 0, 0)$ to $(1, 2, 4)$ along a line segment.
8. $f(x, y, z) = xi + yj + (xz - y)k$, along the path described by $\alpha(t) = t^2i + 2tj + 4t^3k$, $0 \le t \le 1$.

In each of Exercises 9 through 12, compute the value of the given line integral.

9. $\int_C (x^2 - 2xy)\,dx + (y^2 - 2xy)\,dy$, where C is a path from $(-2, 4)$ to $(1, 1)$ along the parabola $y = x^2$.
10. $\displaystyle\int_C \frac{(x + y)\,dx - (x - y)\,dy}{x^2 + y^2}$, where C is the circle $x^2 + y^2 = a^2$, traversed once in a counterclockwise direction.
11. $\displaystyle\int_C \frac{dx + dy}{|x| + |y|}$, where C is the square with vertices $(1, 0)$, $(0, 1)$, $(-1, 0)$, and $(0, -1)$, traversed once in a counterclockwise direction.
12. $\int_C y\,dx + z\,dy + x\,dz$, where
    (a) C is the curve of intersection of the two surfaces $x + y = 2$ and $x^2 + y^2 + z^2 = 2(x + y)$. The curve is to be traversed once in a direction that appears clockwise when viewed from the origin.
    (b) C is the intersection of the two surfaces $z = xy$ and $x^2 + y^2 = 1$, traversed once in a direction that appears counterclockwise when viewed from high above the xy-plane.

10.6 The concept of work as a line intregal

Consider a particle which moves along a curve under the action of a force field f. If the curve is the graph of a piecewise smooth path α, the *work* done by f is defined to be the line integral $\int f \cdot d\alpha$. The following examples illustrate some fundamental properties of work.

EXAMPLE 1. *Work done by a constant force.* If f is a constant force, say $f = c$, it can be shown that the work done by f in moving a particle from a point a to a point b along any piecewise smooth path joining a and b is $c \cdot (b - a)$, the dot product of the force and the displacement vector $b - a$. We shall prove this in a special case.

We let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a path joining a and b, say $\alpha(a) = a$ and $\alpha(b) = b$, and we write $\boldsymbol{c} = (c_1, \ldots, c_n)$. Assume $\boldsymbol{\alpha}'$ is continuous on $[a, b]$. Then the work done by $\boldsymbol{f}$ is equal to

$$\int \boldsymbol{f} \cdot d\boldsymbol{\alpha} = \sum_{k=1}^{n} c_k \int_a^b \alpha_k'(t)\, dt = \sum_{k=1}^{n} c_k[\alpha_k(b) - \alpha_k(a)] = \boldsymbol{c} \cdot [\boldsymbol{\alpha}(b) - \boldsymbol{\alpha}(a)] = \boldsymbol{c} \cdot (\boldsymbol{b} - \boldsymbol{a}).$$

For this force field the work depends only on the end points $\boldsymbol{a}$ and $\boldsymbol{b}$ and not on the curve joining them. Not all force fields have this property. Those which do are called *conservative.* The example on p. 325 is a nonconservative force field. In a later section we shall determine all conservative force fields.

EXAMPLE 2. *The principle of work and energy.* A particle of mass m moves along a curve under the action of a force field $\boldsymbol{f}$. If the speed of the particle at time t is $v(t)$, its kinetic energy is defined to be $\frac{1}{2}mv^2(t)$. Prove that *the change in kinetic energy in any time interval is equal to the work done by $\boldsymbol{f}$ during this time interval.*

Solution. Let $\boldsymbol{r}(t)$ denote the position of the particle at time t. The work done by $\boldsymbol{f}$ during a time interval $[a, b]$ is $\int_{\boldsymbol{r}(a)}^{\boldsymbol{r}(b)} \boldsymbol{f} \cdot d\boldsymbol{r}$. We wish to prove that

$$\int_{\boldsymbol{r}(a)}^{\boldsymbol{r}(b)} \boldsymbol{f} \cdot d\boldsymbol{r} = \tfrac{1}{2}mv^2(b) - \tfrac{1}{2}mv^2(a).$$

From Newton's second law of motion we have

$$\boldsymbol{f}[\boldsymbol{r}(t)] = m\boldsymbol{r}''(t) = m\boldsymbol{v}'(t),$$

where $\boldsymbol{v}(t)$ denotes the velocity vector at time t. The speed is the length of the velocity vector, $v(t) = \|\boldsymbol{v}(t)\|$. Therefore

$$\boldsymbol{f}[\boldsymbol{r}(t)] \cdot \boldsymbol{r}'(t) = \boldsymbol{f}[\boldsymbol{r}(t)] \cdot \boldsymbol{v}(t) = m\boldsymbol{v}'(t) \cdot \boldsymbol{v}(t) = \tfrac{1}{2}m \frac{d}{dt}(\boldsymbol{v}(t) \cdot \boldsymbol{v}(t)) = \tfrac{1}{2}m \frac{d}{dt}(v^2(t)).$$

Integrating from a to b we obtain

$$\int_{\boldsymbol{r}(a)}^{\boldsymbol{r}(b)} \boldsymbol{f} \cdot d\boldsymbol{r} = \int_a^b \boldsymbol{f}[\boldsymbol{r}(t)] \cdot \boldsymbol{r}'(t)\, dt = \tfrac{1}{2}mv^2(t)\Big|_a^b = \tfrac{1}{2}mv^2(b) - \tfrac{1}{2}mv^2(a),$$

as required.

10.7 Line integrals with respect to arc length

Let $\boldsymbol{\alpha}$ be a path with $\boldsymbol{\alpha}'$ continuous on an interval $[a, b]$. The graph of $\boldsymbol{\alpha}$ is a rectifiable curve. In Volume I we proved that the corresponding arc-length function s is given by the integral

$$s(t) = \int_a^t \|\boldsymbol{\alpha}'(u)\|\, du.$$

The derivative of arc length is given by

$$s'(t) = \|\boldsymbol{\alpha}'(t)\|.$$

Let φ be a scalar field defined and bounded on C, the graph of a. The *line integral of φ with respect to arc length along* C is denoted by the symbol $\int_C \varphi\, ds$ and is defined by the equation

$$\int_C \varphi\, ds = \int_a^b \varphi[\boldsymbol{\alpha}(t)]s'(t)\, dt,$$

whenever the integral on the right exists.

Now consider a scalar field φ given by $\varphi[\boldsymbol{\alpha}(t)] = \boldsymbol{f}[\boldsymbol{\alpha}(t)] \cdot \boldsymbol{T}(t)$, the dot product of a vector fieldfdefined on C and the unit tangent vector $T(t) = (d\boldsymbol{\alpha}/ds)$. In this case the line integral $\int_C \varphi\, ds$ is the same as the line integral $\int_C \boldsymbol{f} \cdot da$ because

$$\boldsymbol{f}[\boldsymbol{\alpha}(t)] \cdot \text{a'(t)} = \boldsymbol{f}[\boldsymbol{\alpha}(t)] \cdot \frac{d\boldsymbol{\alpha}}{ds}\frac{ds}{dt} = \boldsymbol{f}[\boldsymbol{\alpha}(t)] \cdot \boldsymbol{T}(t)s'(t) = \varphi[\boldsymbol{\alpha}(t)]s'(t).$$

When $\boldsymbol{f}$ denotes a velocity field, the dot product $\boldsymbol{f} \cdot T$ is the tangential component of velocity, and the line integral $\int_C \boldsymbol{f} \cdot T\, ds$ is called the *flow integral* off along C. When C is a closed curve the flow integral is called the *circulation* off along C. These terms are commonly used in the theory of fluid flow.

10.8 Further applications of line integrals

Line integrals with respect to arc length also occur in problems concerned with mass distribution along a curve. For example, think of a curve C in 3-space as a wire made of a thin material of varying density. Assume the density is described by a scalar field φ, where $\varphi(x, y, z)$ is the *mass per unit length* at the point (x, y, z) of C. Then the total mass M of the wire is defined to be the line integral of φ with respect to arc length:

$$M = \int_C \varphi(x, y, z)\, ds.$$

The *center of mass* of the wire is defined to be the point $(\bar{x}, \bar{y}, \bar{z})$ whose coordinates are determined by the equations

$$\bar{x}M = \int_C x\varphi(x, y, z)\, ds, \qquad \bar{y}M = \int_C y\varphi(x, y, z)\, ds, \qquad \bar{z}M = \int_C z\varphi(x, y, z)\, ds.$$

A wire of constant density is called *uniform.* In this case the center of mass is also called the *centroid.*

EXAMPLE 1. Compute the mass M of one coil of a spring having the shape of the helix whose vector equation is

$$\boldsymbol{\alpha}(t) = a\cos t\,\boldsymbol{i} + a\sin t\,\boldsymbol{j} + bt\boldsymbol{k}$$

if the density at (x, y, z) is $x^2 + y^2 + z^2$.

Solution. The integral for A4 is

$$M = \int_C (x^2 + y^2 + z^2)\, ds = \int_0^{2\pi} (a" \cos^2 t + a^2 \sin^2 t + b^2t^2)s'(t)\, dt.$$

Since s'(t) = $\|\alpha'(t)\|$ and **a'(f)** = $-a$ sin t i + a cos $tj + bk$, **we** have s'(t) $= \sqrt{a^2 + b^2}$ and hence

$$M = \sqrt{a^2 + b^2} \int_0^{2\pi} (a' + b^2t^2)\, dt = \sqrt{a^2 + b^2} \left(2\pi a^2 + \frac{8}{3}\pi^3 b^2\right).$$

In this example the z-coordinate $\bar{z}$ of the center of mass is given by

$$\bar{z}M = \int_C z(x^2 + y^2 + z")\, ds = \sqrt{a^2 + b^2} \int_0^{2\pi} bt(a^2 + b^2t^2)\, dt$$
$$= b\sqrt{a^2 + b^2}\,(2\pi^2 a^2 + 4\pi^4 b^2).$$

The coordinates $\bar{x}$ and $\bar{y}$ are requested in Exercise 15 of Section 10.9.

Line integrals can be used to define the moment of inertia of a wire with respect to an axis. If $\delta(x, y, z)$ represents the perpendicular distance from a point (x, y, z) of C to an axis **L,** the moment of inertia I_L is defined to be the line integral

$$I_L = \int_C \delta^2(x, y, z)\varphi(x, y, z)\, ds,$$

where $\varphi(x, y, z)$ is the density at (x, y, z). The moments of inertia about the coordinate axes are denoted by I,, ***I,,*** and ***I,.***

EXAMPLE 2. Compute the moment of inertia I_z of the spring coil in Example 1.

Solution. Here $\delta^2(x, y, z) = x^2 + y^2 = a^2$ and $\varphi(x, y, z) = x^2 + y^2 + z^2$, so we have

$$I_z = \int_C (x^2 + y^2)(x^2 + y^2 + z^2)\, ds = a^2 \int_C (x^2 + y^2 + z")\, ds = Ma^2,$$

where M is the mass, as computed in Example 1.

10.9 Exercises

1. A force field f in 3-space is given by $f(x, y, z) = xi + yj + (xz - y)k$. Compute the work done by this force in moving a particle from (0, 0, 0) to (1, 2, 4) along the line segment joining these two points.
2. Find the amount of work done by the force $f(x, y) = (x^2 - y^2)i + 2xyj$ in moving a particle (in a counterclockwise direction) once around the square bounded by the coordinate axes and thelinesx =aandy $= a, a >$O.
3. A two-dimensional force fieldfis given by the equation $f(x, y) = cxyi + x^6y^2j$, where c is a positive constant. This force acts on a particle which must move from (0, 0) to the line x = 1 along a curve of the form

$$y = ax^b, \qquad \text{where } a > 0 \qquad \text{and} \qquad b > O.$$

Find a value of a (in terms of c) such that the work done by this force is independent of **b.**

4. A force field f in 3-space is given by the formula $f(x, y, z) = yzi + xzj + x(y + 1)k$. Calculate the work done by fin moving a particle once around the triangle with vertices (0, 0, 0), (1, 1, 1), (-1, 1, -1) in that order.

5. Calculate the work done by the force field $f(x, y, z) = (y - z)i + (z - x)j + (x - y)k$ along the curve of intersection of the sphere $x^2 + y^2 + z^2 = 4$ and the plane $z = y \tan \theta$, where $0 < \theta < \pi/2$. The path is transversed in a direction that appears counterclockwise when viewed from high above the xy-plane.
6. Calculate the work done by the force field $f(x, y, z) = y^2 i + z^2 j + x^2 k$ along the curve of intersection of the sphere $x^2 + y^2 + z^2 = a^2$ and the cylinder $x^2 + y^2 = ax$, where $z \geq 0$ and $a > 0$. The path is traversed in a direction that appears clockwise when viewed from high above the xy-plane.

Calculate the line integral with respect to arc length in each of Exercises 7 through 10.

7. $\int_C (x + y)\, ds$, where C is the triangle with vertices (0, 0), (1, 0), and (0, 1), traversed in a counterclockwise direction.
8. $\int_C y^2\, ds$, where C has the vector equation

$$a(t) = a(t - \sin t)i + a(1 - \cos t)j, \qquad 0 \leq t \leq 2\pi.$$

9. $\int_C (x^2 + y^2)\, ds$, where C has the vector equation

$$a(t) = a(\cos t + t \sin t)i + a(\sin t - t \cos t)j, \qquad 0 \leq t \leq 2\pi.$$

10. $\int_C z\, ds$, where C has the vector equation

$$a(t) = t \cos t\, i + t \sin t\, j + tk, \qquad 0 \leq t \leq t_0.$$

11. Consider a uniform semicircular wire of radius a.
(a) Show that the centroid lies on the axis of symmetry at a distance $2a/\pi$ from the center.
(b) Show that the moment of inertia about the diameter through the end points of the wire is $\frac{1}{2}Ma^2$, where M is the mass of the wire.
12. A wire has the shape of the circle $x^2 + y^2 = a^2$. Determine its mass and moment of inertia about a diameter if the density at (x, y) is $|x| + |y|$.
13. Find the mass of a wire whose shape is that of the curve of intersection of the sphere $x^2 + y^2 + z^2 = 1$ and the plane $x + y + z = 0$ if the density of the wire at (x, y, z) is x^2.
14. A uniform wire has the shape of that portion of the curve of intersection of the two surfaces $x^2 + y^2 = z^2$ and $y^2 = x$ connecting the points (0, 0, 0) and (1, 1, $\sqrt{2}$). Find the z-coordinate of its centroid.
15. Determine the coordinates $\bar{x}$ and $\bar{y}$ of the center of mass of the spring coil described in Example 1 of Section 10.8.
16. For the spring coil described in Example 1 of Section 10.8, compute the moments of inertia I_x and I_y.

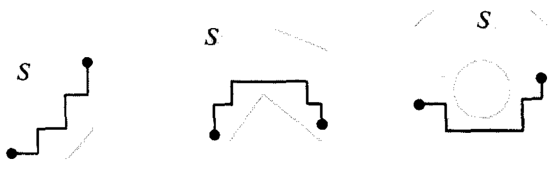
10.10 Open connected sets. Independence of the path

Let S be an open set in $\mathbf{R}^n$. The set S is called *connected* if every pair of points in S can be joined by a piecewise smooth path whose graph lies in S. That is, for every pair of points a and b in S there is a piecewise smooth path α defined on an interval $[a, b]$ such that $\alpha(t) \in S$ for each t in $[a, b]$, with $\alpha(a) = a$ and $\alpha(b) = b$.

Three examples of open connected sets in the plane are shown in Figure 10.3. Examples in 3-space analogous to these would be (a) a solid ellipsoid, (b) a solid polyhedron, and (c) a solid torus; in each case only the interior points are considered.

An open set S is said to be *disconnected* if S is the union of two or more disjoint non-empty open sets. An example is shown in Figure 10.4. It can be shown that the class of open connected sets is identical with the class of open sets that are not disconnected.?

Now let f be a vector field that is continuous on an open connected set S. Choose two points $\boldsymbol{a}$ and $\boldsymbol{b}$ in S and consider the line integral off from $\boldsymbol{a}$ to $\boldsymbol{b}$ along some piecewise smooth path in S. The value of the integral depends, in general, on the path joining a to $\boldsymbol{b}$. For some vector fields, the integral depends only on the end points a and $\boldsymbol{b}$ and not on the path which joins them. In this case we say the integral is *independent of the path from* $\boldsymbol{a}$ *to* $\boldsymbol{b}$. We say the line integral ***off*** is *independent of the path in* S if it is independent of the path from a to **b** for every pair of points a and $\boldsymbol{b}$ in S.



(a) (b) (c)

FIGURE 10.3 Examples of open connected sets.

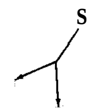


FIGURE 10.4 A disconnected set S, the union of two disjoint circular disks.

Which vector fields have line integrals independent of the path? To answer this question we extend the first and second fundamental theorems of calculus to line integrals.

10.11 The second fundamental theorem of calculus for line integrals

The second fundamental theorem for real functions, as proved in Volume I (Theorem 5.3), states that

$$\int_a^b \varphi'(t)\, dt = \varphi(b) - \varphi(a)\,,$$

provided that φ' is continuous on some open interval containing both a and b. To extend this result to line integrals we need a slightly stronger version of the theorem in which continuity of φ' is assumed only in the open interval *(a, b).*

THEOREM 10.2. *Let φ be a realfunction that is continuous on a closed interval [a, b] and assume that the integral $\int_a^b$ y'(t) dt exists.* ***If*** *φ' is continuous on the open interval (a, b), we have*

$$\int_a^b \varphi'(t)\, dt = \varphi(b) - \varphi(a)\,.$$

Proof. For each x in *[a, b]* define $f(x) = \int_a^x \varphi'(t)\, dt$. We wish to prove that

$$f(b) = \varphi(b) - \varphi(a)\,. \tag{10.2}$$

† For a further discussion of connectedness, see Chapter 8 of the author's *Mathematical Analysis,* Addison-Wesley, Reading, Mass., 1957.

By Theorem 3.4 of Volume I, f is continuous on the closed interval $[a, b]$. By Theorem 5.1 of Volume I, f is differentiable on the open interval (a, b), with $f'(x) = \varphi'(x)$ for each x in (a, b). Therefore, by the zero-derivative theorem (Theorem 5.2 of Volume I), the difference $f - \varphi$ is constant on the open interval (a, b). By continuity, $f - \varphi$ is also constant on the closed interval $[a, b]$. In particular, $f(b) - \varphi(b) = f(a) - \varphi(a)$. But since $f(a) = 0$, this proves (10.2).

THEOREM 10.3. SECOND FUNDAMENTAL THEOREM OF CALCULUS FOR LINE INTEGRALS. *Let φ be a differentiable scalar field with a continuous gradient $\nabla\varphi$ on an open connected set S in $\mathbf{R}^n$. Then for any two points $\boldsymbol{a}$ and $\boldsymbol{b}$ joined by a piecewise smooth path $\boldsymbol{\alpha}$ in S we have*

$$\int_{\boldsymbol{a}}^{\boldsymbol{b}} \nabla\varphi \cdot d\boldsymbol{\alpha} = \varphi(\boldsymbol{b}) - \varphi(\boldsymbol{a}).$$

Proof. Choose any two points $\boldsymbol{a}$ and $\boldsymbol{b}$ in S and join them by a piecewise smooth path $\boldsymbol{\alpha}$ in S defined on an interval $[a, b]$. Assume first that $\boldsymbol{\alpha}$ is *smooth* on $[a, b]$. Then the line integral of $\nabla\varphi$ from $\boldsymbol{a}$ to $\boldsymbol{b}$ along $\boldsymbol{\alpha}$ is given by

$$\int_{\boldsymbol{a}}^{\boldsymbol{b}} \nabla\varphi \cdot d\boldsymbol{\alpha} = \int_a^b \nabla\varphi[\boldsymbol{\alpha}(t)] \cdot \boldsymbol{\alpha}'(t)\, dt .$$

By the chain rule we have

$$\nabla\varphi[\boldsymbol{\alpha}(t)] \cdot \boldsymbol{\alpha}'(t) = g'(t),$$

where g is the composite function defined on $[a, b]$ by the formula

$$g(t) = \varphi[\boldsymbol{\alpha}(t)].$$

The derivative g' is continuous on the open interval (a, b) because $\nabla\varphi$ is continuous on S and $\boldsymbol{\alpha}$ is smooth. Therefore we can apply Theorem 10.2 to g to obtain

$$\int_{\boldsymbol{a}}^{\boldsymbol{b}} \nabla\varphi \cdot d\boldsymbol{\alpha} = \int_a^b g'(t)\, dt = g(b) - g(a) = \varphi[\boldsymbol{\alpha}(b)] - \varphi[\boldsymbol{\alpha}(a)] = \varphi(\boldsymbol{b}) - \varphi(\boldsymbol{a}).$$

This proves the theorem if $\boldsymbol{\alpha}$ is smooth.

When $\boldsymbol{\alpha}$ is piecewise smooth we partition the interval $[a, b]$ into a finite number (say r) of subintervals $[t_{k-1}, t_k]$, in each of which $\boldsymbol{\alpha}$ is smooth, and we apply the result just proved to each subinterval. This gives us

$$\int_{\boldsymbol{a}}^{\boldsymbol{b}} \nabla\varphi = \sum_{k=1}^{r} \int_{\boldsymbol{\alpha}(t_{k-1})}^{\boldsymbol{\alpha}(t_k)} \nabla\varphi = \sum_{k=1}^{r} \{\varphi[\boldsymbol{\alpha}(t_k)] - \varphi[\boldsymbol{\alpha}(t_{k-1})]\} = \varphi(\boldsymbol{b}) - \varphi(\boldsymbol{a}),$$

as required.

As a consequence of Theorem 10.3 we see that the line integral of a gradient is independent of the path in any open connected set S in which the gradient is continuous. For a closed path we have $\boldsymbol{b} = \boldsymbol{a}$, so $\varphi(\boldsymbol{b}) - \varphi(\boldsymbol{a}) = 0$. In other words, ***the line integral of a*** *continuous gradient is zero around* every *piecewise smooth closed path in S.* In Section 10.14 we shall prove (in Theorem 10.4) that gradients are the *only* continuous vector fields with this property.

10.12 Applications to mechanics

If a vector field $\boldsymbol{f}$ is the gradient of a scalar field φ, then φ is called a *potential function* for $\boldsymbol{f}$. In 3-space, the level sets of φ are called *equipotential surfaces;* in 2-space they are called *equipotential lines.* (If φ denotes temperature, the word "equipotential" is replaced by "isothermal"; if φ denotes pressure the word "isobaric" is used.)

EXAMPLE 1. In 3-space, let $\varphi(x, y, z) = r^n$, where $r = (x^2 + y^2 + z^2)^{1/2}$. For every integer n we have

$$\nabla(r^n) = nr^{n-2}\boldsymbol{r},$$

where $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$. (See Exercise 8 of Section 8.14.) Therefore φ is a potential of the vector field

$$\boldsymbol{f}(x, y, z) = nr^{n-2}\boldsymbol{r}.$$

The equipotential surfaces of φ are concentric spheres centered at the origin.

EXAMPLE 2. *The Newtonian potential.* Newton's gravitation law states that the force $\boldsymbol{f}$ which a particle of mass M exerts on another particle of mass m is a vector of length GmM/r^2, where G is a constant and r is the distance between the two particles. Place the origin at the particle of mass M, and let $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ be the position vector from the origin to the particle of mass m. Then $r = \|\boldsymbol{r}\|$ and $-\boldsymbol{r}/r$ is a unit vector with the same direction as $\boldsymbol{f}$, so Newton's law becomes

$$\boldsymbol{f} = -GmMr^{-3}\boldsymbol{r}.$$

Taking $n = -1$ in Example 1 we see that the gravitational force $\boldsymbol{f}$ is the gradient of the scalar field given by

$$\varphi(x, y, z) = GmMr^{-1}.$$

This is called the *Newtonian potential.*

The work done by the gravitational force in moving the particle of mass m from (x_1, y_1, z_1) to (x_2, y_2, z_2) is

$$\varphi(x_1, y_1, z_1) - \varphi(x_2, y_2, z_2) = GmM\left(\frac{1}{r_1} - \frac{1}{r_2}\right),$$

where $r_1 = (x_1^2 + y_1^2 + z_1^2)^{1/2}$ and $r_2 = (x_2^2 + y_2^2 + z_2^2)^{1/2}$. If the two points lie on the same equipotential surface then $r_1 = r_2$ and no work is done.

EXAMPLE 3. *The principle of conservation of mechanical energy.* Let $\boldsymbol{f}$ be a continuous force field having a potential φ in an open connected set S. Theorem 10.3 tells us that the work done by $\boldsymbol{f}$ in moving a particle from $\boldsymbol{a}$ to $\boldsymbol{x}$ along any piecewise smooth path in S is

$\varphi(x) - \varphi(a)$, the change in the potential function. In Example 2 of Section 10.6 we proved that this work is also equal to the change in kinetic energy of the particle, $k(x) - k(a)$ where $k(x)$ denotes the kinetic energy of the particle when it is located at x. Thus, we have

$$k(x) - k(a) = \varphi(x) - \varphi(a),$$

or

$$k(x) - \varphi(x) = k(a) - \varphi(a). \tag{10.3}$$

The scalar $-\varphi(x)$ is called the *potential energy*† of the particle.

If a is kept fixed and x is allowed to vary over the set S, Equation (10.3) tells us that the sum of $k(x)$ and $-\varphi(x)$ is constant. In other words, *if a force field is a gradient, the sum of the kinetic and potential energies of a particle moving in this field is constant.* In mechanics this is called the principle of conservation of (mechanical) energy. A force field with a potential function is said to be *conservative* because the total energy, kinetic plus potential, is conserved. In a conservative field, no net work is done in moving a particle around a closed curve back to its starting point. A force field will not be conservative if friction or viscosity exists in the system, since these tend to convert mechanical energy into heat energy.

10.13 Exercises

1. Determine which of the following open sets S in $\mathbf{R}^2$ are connected. For each connected set, choose two arbitrary distinct points in S and explain how you would find a piecewise smooth curve in S connecting the two points.
(a) $S = \{(x, y) \mid x^2 + y^2 \geq 0\}$. (c) $S = \{(x, y) \mid x^2 + y^2 < 1\}$.
(b) $S = \{(x, y) \mid x^2 + y^2 > 0\}$. (d) $S = \{(x, y) \mid 1 < x^2 + y^2 < 2\}$.
(e) $S = \{(x, y) \mid x^2 + y^2 > 1$ and $(x - 3)^2 + y^2 > 1\}$.
(f) $S = \{(x, y) \mid x^2 + y^2 < 1$ or $(x - 3)^2 + y^2 < 1\}$.
2. Given a two-dimensional vector field

$$f(x, y) = P(x, y)i + Q(x, y)j,$$

where the partial derivatives $\partial P/\partial y$ and $\partial Q/\partial x$ are continuous on an open set S. If f is the gradient of some potential φ, prove that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

at each point of S.
3. For each of the following vector fields, use the result of Exercise 2 to prove that f is *not* a gradient. Then find a closed path C such that $\oint_C f \neq 0$.
(a) $f(x, y) = yi - xj$.
(b) $f(x, y) = yi + (xy - x)j$.
4. Given a three-dimensional vector field

$$f(x, y, z) = P(x, y, z)i + Q(x, y, z)j + R(x, y, z)k,$$

† Some authors refer to $-\varphi$ as the potential function of f so that the potential energy at x will be equal to the value of the potential function φ at x.