Prove that $z^2 = w$ if $v \geq 0$ and that $(\bar{z})^2 = w$ if $v \leq 0$. Conclude that every complex number (with one exception!) has two complex square roots.

11. If z is a complex number, prove that there exists an $r \geq 0$ and a complex number w with $|w| = 1$ such that $z = rw$. Are w and r always uniquely determined by z?

12. If $z_1, \ldots, z_n$ are complex, prove that

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|.$$

13. If x, y are complex, prove that

$$||x| - |y|| \leq |x - y|.$$

14. If z is a complex number such that $|z| = 1$, that is, such that $z\bar{z} = 1$, compute

$$|1 + z|^2 + |1 - z|^2.$$

15. Under what conditions does equality hold in the Schwarz inequality?

16. Suppose $k \geq 3$, $\mathbf{x}, \mathbf{y} \in R^k$, $|\mathbf{x} - \mathbf{y}| = d > 0$, and $r > 0$. Prove:

(*a*) If $2r > d$, there are infinitely many $\mathbf{z} \in R^k$ such that

$$|\mathbf{z} - \mathbf{x}| = |\mathbf{z} - \mathbf{y}| = r.$$

(*b*) If $2r = d$, there is exactly one such $\mathbf{z}$.

(*c*) If $2r < d$, there is no such $\mathbf{z}$.

How must these statements be modified if k is 2 or 1?

17. Prove that

$$|\mathbf{x} + \mathbf{y}|^2 + |\mathbf{x} - \mathbf{y}|^2 = 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2$$

if $\mathbf{x} \in R^k$ and $\mathbf{y} \in R^k$. Interpret this geometrically, as a statement about parallelograms.

18. If $k \geq 2$ and $\mathbf{x} \in R^k$, prove that there exists $\mathbf{y} \in R^k$ such that $\mathbf{y} \neq \mathbf{0}$ but $\mathbf{x} \cdot \mathbf{y} = 0$. Is this also true if $k = 1$?

19. Suppose $\mathbf{a} \in R^k$, $\mathbf{b} \in R^k$. Find $\mathbf{c} \in R^k$ and $r > 0$ such that

$$|\mathbf{x} - \mathbf{a}| = 2|\mathbf{x} - \mathbf{b}|$$

if and only if $|\mathbf{x} - \mathbf{c}| = r$.

(*Solution:* $3\mathbf{c} = 4\mathbf{b} - \mathbf{a}$, $3r = 2|\mathbf{b} - \mathbf{a}|$.)

20. With reference to the Appendix, suppose that property (III) were omitted from the definition of a cut. Keep the same definitions of order and addition. Show that the resulting ordered set has the least-upper-bound property, that addition satisfies axioms (A1) to (A4) (with a slightly different zero-element!) but that (A5) fails.

2

BASIC TOPOLOGY

FINITE, COUNTABLE, AND UNCOUNTABLE SETS

We begin this section with a definition of the function concept.

2.1 Definition Consider two sets A and B, whose elements may be any objects whatsoever, and suppose that with each element x of A there is associated, in some manner, an element of B, which we denote by $f(x)$. Then f is said to be a *function* from A to B (or a *mapping* of A into B). The set A is called the *domain* of f (we also say f is defined on A), and the elements $f(x)$ are called the *values* of f. The set of all values of f is called the *range* of f.

2.2 Definition Let A and B be two sets and let f be a mapping of A into B. If $E \subset A, f(E)$ is defined to be the set of all elements $f(x)$, for $x \in E$. We call $f(E)$ the *image* of E under f. In this notation, $f(A)$ is the range of f. It is clear that $f(A) \subset B$. If $f(A) = B$, we say that f maps A *onto* B. (Note that, according to this usage, *onto* is more specific than *into*.)

If $E \subset B, f^{-1}(E)$ denotes the set of all $x \in A$ such that $f(x) \in E$. We call f^{-1} (E) the *inverse image* of E under f. If $y \in B, f^{-1}(y)$ is the set of all $x \in A$

such that $f(x) = y$. If, for each $y \in B, f^{-1}(y)$ consists of at most one element of A, then f is said to be a 1-1 (*one-to-one*) mapping of A into B. This may also be expressed as follows: f is a 1-1 mapping of A into B provided that $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$, $x_1 \in A$, $x_2 \in A$.

(The notation $x_1 \neq x_2$ means that x_1 and x_2 are distinct elements; otherwise we write $x_1 = x_2$.)

2.3 Definition If there exists a 1-1 mapping of A *onto* B, we say that A and B can be put in 1-1 *correspondence*, or that A and B have the same *cardinal number*, or, briefly, that A and B are *equivalent*, and we write $A \sim B$. This relation clearly has the following properties:

It is reflexive: $A \sim A$.
It is symmetric: If $A \sim B$, then $B \sim A$.
It is transitive: If $A \sim B$ and $B \sim C$, then $A \sim C$.

Any relation with these three properties is called an *equivalence relation*.

2.4 Definition For any positive integer n, let J_n be the set whose elements are the integers $1, 2, \ldots, n$; let J be the set consisting of all positive integers. For any set A, we say:

(*a*) A is *finite* if $A \sim J_n$ for some n (the empty set is also considered to be finite).
(*b*) A is *infinite* if A is not finite.
(*c*) A is *countable* if $A \sim J$.
(*d*) A is *uncountable* if A is neither finite nor countable.
(*e*) A is *at most countable* if A is finite or countable.

Countable sets are sometimes called *enumerable*, or *denumerable*.

For two finite sets A and B, we evidently have $A \sim B$ if and only if A and B contain the same number of elements. For infinite sets, however, the idea of "having the same number of elements" becomes quite vague, whereas the notion of 1-1 correspondence retains its clarity.

2.5 Example Let A be the set of all integers. Then A is countable. For, consider the following arrangement of the sets A and J:

$$\begin{array}{ll} A: & 0, 1, -1, 2, -2, 3, -3, \ldots \\ J: & 1, 2, 3, 4, 5, 6, 7, \ldots \end{array}$$

We can, in this example, even give an explicit formula for a function f from J to A which sets up a 1-1 correspondence:

$$f(n) = \begin{cases} \dfrac{n}{2} & (n \text{ even}), \\[2ex] -\dfrac{n-1}{2} & (n \text{ odd}). \end{cases}$$

2.6 Remark A finite set cannot be equivalent to one of its proper subsets. That this is, however, possible for infinite sets, is shown by Example 2.5, in which J is a proper subset of A.

In fact, we could replace Definition 2.4(b) by the statement: A is infinite if A is equivalent to one of its proper subsets.

2.7 Definition By a *sequence*, we mean a function f defined on the set J of all positive integers. If $f(n) = x_n$, for $n \in J$, it is customary to denote the sequence f by the symbol $\{x_n\}$, or sometimes by $x_1, x_2, x_3, \ldots$. The values of f, that is, the elements x_n, are called the *terms* of the sequence. If A is a set and if $x_n \in A$ for all $n \in J$, then $\{x_n\}$ is said to be a *sequence in* A, or a *sequence of elements of* A.

Note that the terms $x_1, x_2, x_3, \ldots$ of a sequence need not be distinct.

Since every countable set is the range of a 1-1 function defined on J, we may regard every countable set as the range of a sequence of distinct terms. Speaking more loosely, we may say that the elements of any countable set can be "arranged in a sequence."

Sometimes it is convenient to replace J in this definition by the set of all nonnegative integers, i.e., to start with 0 rather than with 1.

2.8 Theorem *Every infinite subset of a countable set A is countable.*

Proof Suppose $E \subset A$, and E is infinite. Arrange the elements x of A in a sequence $\{x_n\}$ of distinct elements. Construct a sequence $\{n_k\}$ as follows:

Let n_1 be the smallest positive integer such that $x_{n_1} \in E$. Having chosen $n_1, \ldots, n_{k-1}$ $(k = 2, 3, 4, \ldots)$, let n_k be the smallest integer greater than n_{k-1} such that $x_{n_k} \in E$.

Putting $f(k) = x_{n_k}$ $(k = 1, 2, 3, \ldots)$, we obtain a 1-1 correspondence between E and J.

The theorem shows that, roughly speaking, countable sets represent the "smallest" infinity: No uncountable set can be a subset of a countable set.

2.9 Definition Let A and Ω be sets, and suppose that with each element α of A there is associated a subset of Ω which we denote by E_α.

The set whose elements are the sets E_α will be denoted by $\{E_\alpha\}$. Instead of speaking of sets of sets, we shall sometimes speak of a collection of sets, or a family of sets.

The *union* of the sets E_α is defined to be the set S such that $x \in S$ if and only if $x \in E_\alpha$ for at least one $\alpha \in A$. We use the notation

$$S = \bigcup_{\alpha \in A} E_\alpha. \tag{1}$$

If A consists of the integers $1, 2, \ldots, n$, one usually writes

$$S = \bigcup_{m=1}^{n} E_m \tag{2}$$

or

$$S = E_1 \cup E_2 \cup \cdots \cup E_n. \tag{3}$$

If A is the set of all positive integers, the usual notation is

$$S = \bigcup_{m=1}^{\infty} E_m. \tag{4}$$

The symbol ∞ in (4) merely indicates that the union of a *countable* collection of sets is taken, and should not be confused with the symbols $+\infty$, $-\infty$, introduced in Definition 1.23.

The *intersection* of the sets E_α is defined to be the set P such that $x \in P$ if and only if $x \in E_\alpha$ for every $\alpha \in A$. We use the notation

$$P = \bigcap_{\alpha \in A} E_\alpha, \tag{5}$$

or

$$P = \bigcap_{m=1}^{n} E_m = E_1 \cap E_2 \cap \cdots \cap E_n, \tag{6}$$

or

$$P = \bigcap_{m=1}^{\infty} E_m, \tag{7}$$

as for unions. If $A \cap B$ is not empty, we say that A and B *intersect*; otherwise they are *disjoint*.

2.10 Examples

(*a*) Suppose E_1 consists of 1, 2, 3 and E_2 consists of 2, 3, 4. Then $E_1 \cup E_2$ consists of 1, 2, 3, 4, whereas $E_1 \cap E_2$ consists of 2, 3.

(*b*) Let A be the set of real numbers x such that $0 < x \leq 1$. For every $x \in A$, let E_x be the set of real numbers y such that $0 < y < x$. Then

(i) $E_x \subset E_z$ if and only if $0 < x \leq z \leq 1$;

(ii) $\bigcup_{x \in A} E_x = E_1$;

(iii) $\bigcap_{x \in A} E_x$ is empty;

(i) and (ii) are clear. To prove (iii), we note that for every $y > 0$, $y \notin E_x$ if $x < y$. Hence $y \notin \bigcap_{x \in A} E_x$.

2.11 Remarks Many properties of unions and intersections are quite similar to those of sums and products; in fact, the words sum and product were sometimes used in this connection, and the symbols Σ and Π were written in place of $\bigcup$ and $\bigcap$.

The commutative and associative laws are trivial:

$$A \cup B = B \cup A; \qquad A \cap B = B \cap A. \tag{8}$$

$$(A \cup B) \cup C = A \cup (B \cup C); \qquad (A \cap B) \cap C = A \cap (B \cap C). \tag{9}$$

Thus the omission of parentheses in (3) and (6) is justified.

The distributive law also holds:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C). \tag{10}$$

To prove this, let the left and right members of (10) be denoted by E and F, respectively.

Suppose $x \in E$. Then $x \in A$ and $x \in B \cup C$, that is, $x \in B$ or $x \in C$ (possibly both). Hence $x \in A \cap B$ or $x \in A \cap C$, so that $x \in F$. Thus $E \subset F$.

Next, suppose $x \in F$. Then $x \in A \cap B$ or $x \in A \cap C$. That is, $x \in A$, and $x \in B \cup C$. Hence $x \in A \cap (B \cup C)$, so that $F \subset E$.

It follows that $E = F$.

We list a few more relations which are easily verified:

$$A \subset A \cup B, \tag{11}$$

$$A \cap B \subset A. \tag{12}$$

If 0 denotes the empty set, then

$$A \cup 0 = A, \qquad A \cap 0 = 0. \tag{13}$$

If $A \subset B$, then

$$A \cup B = B, \qquad A \cap B = A. \tag{14}$$

2.12 Theorem *Let $\{E_n\}$, $n = 1, 2, 3, \ldots$, be a sequence of countable sets, and put*

$$S = \bigcup_{n=1}^{\infty} E_n. \tag{15}$$

Then S is countable.

Proof Let every set E_n be arranged in a sequence $\{x_{nk}\}$, $k = 1, 2, 3, \ldots$, and consider the infinite array

$$\begin{array}{ccccc} x_{11} & x_{12} & x_{13} & x_{14} & \cdots \\ x_{21} & x_{22} & x_{23} & x_{24} & \cdots \\ x_{31} & x_{32} & x_{33} & x_{34} & \cdots \\ x_{41} & x_{42} & x_{43} & x_{44} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{array} \tag{16}$$

in which the elements of E_n form the nth row. The array contains all elements of S. As indicated by the arrows, these elements can be arranged in a sequence

$$x_{11};\ x_{21}, x_{12};\ x_{31}, x_{22}, x_{13};\ x_{41}, x_{32}, x_{23}, x_{14};\ \ldots \tag{17}$$

If any two of the sets E_n have elements in common, these will appear more than once in (17). Hence there is a subset T of the set of all positive integers such that $S \sim T$, which shows that S is at most countable (Theorem 2.8). Since $E_1 \subset S$, and E_1 is infinite, S is infinite, and thus countable.

Corollary *Suppose A is at most countable, and, for every $\alpha \in A$, B_α is at most countable. Put*

$$T = \bigcup_{\alpha \in A} B_\alpha.$$

Then T is at most countable.

For T is equivalent to a subset of (15).

2.13 Theorem *Let A be a countable set, and let B_n be the set of all n-tuples $(a_1, \ldots, a_n)$, where $a_k \in A$ $(k = 1, \ldots, n)$, and the elements $a_1, \ldots, a_n$ need not be distinct. Then B_n is countable.*

Proof That B_1 is countable is evident, since $B_1 = A$. Suppose B_{n-1} is countable $(n = 2, 3, 4, \ldots)$. The elements of B_n are of the form

$$(b, a) \qquad (b \in B_{n-1},\ a \in A). \tag{18}$$

For every fixed b, the set of pairs (b, a) is equivalent to A, and hence countable. Thus B_n is the union of a countable set of countable sets. By Theorem 2.12, B_n is countable.

The theorem follows by induction.