N as a submodule of a $\mathbb{Q}$-module. The two $\mathbb{Z}$-modules $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ exhibit extremely different behaviors when we try to "extend scalars" from $\mathbb{Z}$ to $\mathbb{Q}$: the first module maps injectively into some $\mathbb{Q}$-module, the second always maps to 0 in a $\mathbb{Q}$-module.

We now construct for a general R-module N an S-module that is the "best possible" target in which to try to embed N. We shall also see that this module determines *all* of the possible R-module homomorphisms of N into S-modules, in particular determining when N is contained in some S-module (cf. Corollary 9). In the case of $R = \mathbb{Z}$ and $S = \mathbb{Q}$ this construction will give us $\mathbb{Q}$ when applied to the module $N = \mathbb{Z}$, and will give us 0 when applied to the module $N = \mathbb{Z}/2\mathbb{Z}$ (Examples 2 and 3 following Corollary 9).

If the R-module N were already an S-module then of course there is no difficulty in "extending" the scalars from R to S, so we begin the construction by returning to the basic module axioms in order to examine whether we can define "products" of the form sn, for $s \in S$ and $n \in N$. These axioms start with an abelian group N together with a map from $S \times N$ to N, where the image of the pair (s, n) is denoted by sn. It is therefore natural to consider the free $\mathbb{Z}$-module (i.e., , the free abelian group) on the set $S \times N$, i.e., the collection of all finite commuting sums of elements of the form (s_i, n_i) where $s_i \in S$ and $n_i \in N$. This is an abelian group where there are no relations between any distinct pairs (s, n) and (s', n'), i.e., no relations between the "formal products" sn, and in this abelian group the original module N has been thoroughly distinguished from the new "coefficients" from S. To satisfy the relations necessary for an S-module structure imposed in equation (1) and the compatibility relation with the action of R on N in (2′), we must take the quotient of this abelian group by the subgroup H generated by all elements of the form

$$\begin{gathered}(s_1 + s_2, n) - (s_1, n) - (s_2, n),\\ (s, n_1 + n_2) - (s, n_1) - (s, n_2), \text{ and}\\ (sr, n) - (s, rn),\end{gathered} \tag{10.3}$$

for $s, s_1, s_2 \in S$, $n, n_1, n_2 \in N$ and $r \in R$, where rn in the last element refers to the R-module structure already defined on N.

The resulting quotient group is denoted by $S \otimes_R N$ (or just $S \otimes N$ if R is clear from the context) and is called the *tensor product of S and N over R*. If $s \otimes n$ denotes the coset containing (s, n) in $S \otimes_R N$ then by definition of the quotient we have forced the relations

$$\begin{gathered}(s_1 + s_2) \otimes n = s_1 \otimes n + s_2 \otimes n,\\ s \otimes (n_1 + n_2) = s \otimes n_1 + s \otimes n_2, \text{ and}\\ sr \otimes n = s \otimes rn.\end{gathered} \tag{10.4}$$

The elements of $S \otimes_R N$ are called *tensors* and can be written (non-uniquely in general) as finite sums of "simple tensors" of the form $s \otimes n$ with $s \in S$, $n \in N$.

We now show that the tensor product $S \otimes_R N$ is naturally a left S-module under the action defined by

$$s\Big(\sum_{\text{finite}} s_i \otimes n_i\Big) = \sum_{\text{finite}} (ss_i) \otimes n_i. \tag{10.5}$$

We first check this is well defined, i.e., independent of the representation of the element of $S \otimes_R N$ as a sum of simple tensors. Note first that if s' is any element of S then

$$(s'(s_1 + s_2), n) - (s's_1, n) - (s's_2, n) \ \big(= (s's_1 + s's_2, n) - (s's_1, n) - (s's_2, n)\big),$$
$$(s's, n_1 + n_2) - (s's, n_1) - (s's, n_2), \text{ and}$$
$$(s'(sr), n) - (s's, rn) \ \big(= ((s's)r, n) - (s's, rn)\big)$$

each belongs to the set of generators in (3), so in particular each lies in the subgroup H. This shows that multiplying the first entries of the generators in (3) on the left by s' gives another element of H (in fact another generator). Since any element of H is a sum of elements as in (3), it follows that for any element $\sum(s_i, n_i)$ in H also $\sum(s's_i, n_i)$ lies in H. Suppose now that $\sum s_i \otimes n_i = \sum s_i' \otimes n_i'$ are two representations for the same element in $S \otimes_R N$. Then $\sum(s_i, n_i) - \sum(s_i', n_i')$ is an element of H, and by what we have just seen, for any $s \in S$ also $\sum(ss_i, n_i) - \sum(ss_i', n_i')$ is an element of H. But this means that $\sum ss_i \otimes n_i = \sum ss_i' \otimes n_i'$ in $S \otimes_R N$, so the expression in (5) is indeed well defined.

It is now straightforward using the relations in (4) to check that the action defined in (5) makes $S \otimes_R N$ into a left S-module. For example, on the simple tensor $s_i \otimes n_i$,

$$\begin{aligned}
(s + s')\,(s_i \otimes n_i) &= ((s + s')s_i) \otimes n_i && \text{by definition (5)}\\
&= (ss_i + s's_i) \otimes n_i &&\\
&= ss_i \otimes n_i + s's_i \otimes n_i && \text{by the first relation in (4)}\\
&= s\,(s_i \otimes n_i) + s'\,(s_i \otimes n_i) && \text{by definition (5)}\,.
\end{aligned}$$

The module $S \otimes_R N$ is called *the (left) S-module obtained by extension of scalars from the (left) R-module N.*

There is a natural map $\iota : N \to S \otimes_R N$ defined by $n \mapsto 1 \otimes n$ (i.e., first map $n \in N$ to the element $(1, n)$ in the free abelian group and then pass to the quotient group). Since $1 \otimes rn = r \otimes n = r(1 \otimes n)$ by (4) and (5), it is easy to check that ι is an R-module homomorphism from N to $S \otimes_R N$. Since we have passed to a quotient group, however, ι is not injective in general. Hence, while there is a natural R-module homomorphism from the original left R-module N to the left S-module $S \otimes_R N$, in general $S \otimes_R N$ need not contain (an isomorphic copy of) N. On the other hand, the relations in equation (3) were the *minimal* relations that we had to impose in order to obtain an S-module, so it is reasonable to expect that the tensor product $S \otimes_R N$ is the "best possible" S-module to serve as target for an R-module homomorphism from N. The next theorem makes this more precise by showing that any other R-module homomorphism from N factors through this one, and is referred to as the *universal property* for the tensor product $S \otimes_R N$. The analogous result for the general tensor product is given in Theorem 10.

Theorem 8. Let R be a subring of S, let N be a left R-module and let $\iota : N \to S \otimes_R N$ be the R-module homomorphism defined by $\iota(n) = 1 \otimes n$. Suppose that L is any left S-module (hence also an R-module) and that $\varphi : N \to L$ is an R-module homomorphism from N to L. Then there is a unique S-module homomorphism $\Phi : S \otimes_R N \to L$ such that φ factors through Φ, i.e., $\varphi = \Phi \circ \iota$ and the diagram

$$\begin{tikzcd} N \arrow[r, "\iota"] \arrow[dr, "\varphi"'] & S \otimes_R N \arrow[d, "\Phi"] \\ & L \end{tikzcd}$$

commutes. Conversely, if $\Phi : S \otimes_R N \to L$ is an S-module homomorphism then $\varphi = \Phi \circ \iota$ is an R-module homomorphism from N to L.

Proof: Suppose $\varphi : N \to L$ is an R-module homomorphism to the S-module L. By the universal property of free modules (Theorem 6 in Section 3) there is a $\mathbb{Z}$-module homomorphism from the free $\mathbb{Z}$-module F on the set $S \times N$ to L that sends each generator (s, n) to $s\varphi(n)$. Since φ is an R-module homomorphism, the generators of the subgroup H in equation (3) all map to zero in L. Hence this $\mathbb{Z}$-module homomorphism factors through H, i.e., there is a well defined $\mathbb{Z}$-module homomorphism Φ from $F/H = S \otimes_R N$ to L satisfying $\Phi(s \otimes n) = s\varphi(n)$. Moreover, on simple tensors we have

$$s'\Phi(s \otimes n) = s'(s\varphi(n)) = (s's)\varphi(n) = \Phi((s's) \otimes n) = \Phi(s'(s \otimes n)).$$

for any $s' \in S$. Since Φ is additive it follows that Φ is an S-module homomorphism, which proves the existence statement of the theorem. The module $S \otimes_R N$ is generated as an S-module by elements of the form $1 \otimes n$, so any S-module homomorphism is uniquely determined by its values on these elements. Since $\Phi(1 \otimes n) = \varphi(n)$, it follows that the S-module homomorphism Φ is uniquely determined by φ, which proves the uniqueness statement of the theorem. The converse statement is immediate.

The universal property of $S \otimes_R N$ in Theorem 8 shows that R-module homomorphisms of N into S-modules arise from S-module homomorphisms from $S \otimes_R N$. In particular this determines when it is possible to map N injectively into some S-module:

Corollary 9. Let $\iota : N \to S \otimes_R N$ be the R-module homomorphism in Theorem 8. Then $N/\ker \iota$ is the unique largest quotient of N that can be embedded in any S-module. In particular, N can be embedded as an R-submodule of some left S-module if and only if ι is injective (in which case N is isomorphic to the R-submodule $\iota(N)$ of the S-module $S \otimes_R N$).

Proof: The quotient $N/\ker \iota$ is mapped injectively (by ι) into the S-module $S \otimes_R N$. Suppose now that φ is an R-module homomorphism injecting the quotient $N/\ker \varphi$ of N into an S-module L. Then, by Theorem 8, $\ker \iota$ is mapped to 0 by φ, i.e., $\ker \iota \subseteq \ker \varphi$. Hence $N/\ker \varphi$ is a quotient of $N/\ker \iota$ (namely, the quotient by the submodule $\ker \varphi/\ker \iota$). It follows that $N/\ker \iota$ is the unique largest quotient of N that can be embedded in any S-module. The last statement in the corollary follows immediately.