ENCODINGS OF GRAPHS

We first consider parameters related to three types of graph encoding. Each model of encoding involves assigning vectors to vertices, and the parameter is the minimum length of vectors that suffice. We study the maximum of this parameter over n-vertex graphs. The parameters are intersection number, product dimension, and squashed-cube dimension.

8.4.1. Definition. An **intersection representation** of **length** t assigns each vertex a 0,1-vector of length t such that $u \leftrightarrow v$ if and only if their vectors have a 1 in a common position. Equivalently, it assigns each $x \in V(G)$ a set $S_x \subseteq [t]$ such that $u \leftrightarrow v$ if and only if $S_u \cap S_v \neq \emptyset$. The **intersection number** $\theta'(G)$ is the minimum length of an intersection representation of G.

The elements of $[t]$ in a representation correspond to complete subgraphs that cover $E(G)$. This motivates our use of θ' for intersection number: $\theta(G)$ is the minimum number of cliques needed to cover $V(G)$.

8.4.2. Proposition. (Erdős–Goodman–Pósa [1966]) The intersection number equals the minimum number of complete subgraphs needed to cover $E(G)$.

Proof: We define a natural correspondence between representations of length t and coverings of $E(G)$ by t complete subgraphs. Each $i \in [t]$ generates a clique $\{v \in V(G) : i \in S_v\}$. The resulting complete subgraphs cover $E(G)$, since $u \leftrightarrow v$ if and only if $S_u \cap S_v \neq \emptyset$.

Conversely, if complete subgraphs $Q_1, \ldots, Q_t$ cover $E(G)$, then assigning $\{i : v \in V(Q_i)\}$ to each vertex v yields an intersection representation. ■

Hence $\theta'(G) = e(G)$ if G is triangle-free, and $\theta'(K_{\lfloor n/2\rfloor, \lceil n/2\rceil}) = \lfloor n^2/4 \rfloor$. In fact, this is the unique n-vertex graph maximizing $\theta'(G)$. Exercise 1 suggests a direct proof of the bound; here we present a stronger result.

Let **F** be a family of graphs. For an input graph G, the **F-decomposition** problem is to decompose G into the minimum number of graphs in **F**. When **F** is not closed under taking subgraphs, **F**-decomposition may require more subgraphs than **F-covering**. For example, we can cover the kite with two complete subgraphs, but three complete subgraphs are needed to decompose it.

Proving $\theta'(G) \le \lfloor n^2/4 \rfloor$ for n-vertex graphs means showing that every n-vertex graph can be covered with $\lfloor n^2/4 \rfloor$ complete subgraphs; we prove the stronger result that there is always a decomposition using at most this many complete subgraphs. In fact, we can find such a decomposition greedily.

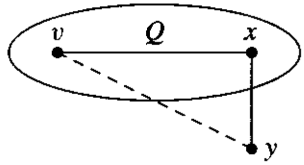
8.4.3. Theorem. (McGuinness [1994]) Every greedy clique decomposition of an n-vertex graph uses at most $\lfloor n^2/4 \rfloor$ cliques.

Proof: We use induction on n. The claim is obvious for $n \le 2$; consider $n > 2$. Let $\mathbf{Q} = Q_1, \ldots, Q_m$ be a greedy decomposition of G, meaning that each Q_i is a maximal complete subgraph in $G - \cup_{j<i} E(Q_j)$. Note that deleting Q_j from the list **Q** leaves a greedy decomposition of $G - E(Q_j)$.

If each Q_i has at least three edges, then $m < n^2/6$, so we may assume that some Q_j is an edge xy. Let R consist of the elements of $\mathbf{Q} - \{Q_j\}$ that are incident to x, and let S consist of those incident to y. The set $\mathbf{Q}' = \mathbf{Q} - (R \cup S \cup \{Q_j\})$ is a greedy decomposition of a subgraph of $G - x - y$. By the induction hypothesis, $|\mathbf{Q}'| \le (n-2)^2/4$. Hence it suffices to prove that $|R| + |S| \le n - 2$.

We prove this by choosing distinct vertices in $V(G) - \{x, y\}$ from the vertex sets of the elements of $R \cup S$. Since each edge is deleted exactly once, each $v \notin \{x, y\}$ appears once in R if $v \in N(x)$ and once in S if $v \in N(y)$. Consider $Q \in R$. If Q uses a vertex $v \notin N(y)$, then we choose such a v for Q. If $V(Q) \subseteq N(y)$, then we choose for Q a vertex $v \in Q$ such that vy belongs to the earliest element of $\mathbf{Q}$ that contains both y and some vertex of Q. Call this element Q'; note that Q' is the only element of S containing v. Since Q and xy are maximal when chosen, Q' precedes both of these in $\mathbf{Q}$. For elements of S, choose vertices by reversing the roles of x and y.

We have shown that if v belongs to some $Q \in R$ and to some $Q' \in S$, and v is chosen for one of them, then the one for which it is chosen occurs after the other one in the list $\mathbf{Q}$. Hence no vertex is chosen twice. We conclude that $|R| + |S| \le n - 2$ and $m \le n^2/4$. ■



Both Chung [1981] and Győri–Kostochka [1979] strengthened the decomposition bound, proving that every n-vertex graph has a decomposition into complete subgraphs whose orders sum to at most $\lfloor n(G)^2/2 \rfloor$.

Now we consider the second encoding model.

8.4.4. Definition. A **product representation** of **length** t assigns the vertices distinct vectors of length t so that $u \leftrightarrow v$ if and only if their vectors differ in every position. The **product dimension** $\operatorname{pdim} G$ is the minimum length of such a representation of G.

By devoting one coordinate to each $e \in E(\overline{G})$, in which the vertices of e have value 0 and other vertices have distinct positive values, we obtain $\operatorname{pdim} G \le e(\overline{G})$ (if G is not a complete graph).

8.4.5. Example. Every complete graph has product dimension 1. For $\overline{K}_n$, each pair of vertices must agree in some coordinate, but we cannot assign two vertices the same vector. Hence two coordinates are needed, and assigning $(0, j)$ to v_j for each j suffices.

For $K_1 + K_{n-1}$, the vectors for the clique must differ in each coordinate. The vector for the isolated vertex must agree with each of the others somewhere, but it cannot agree with more than one in a single coordinate. Hence at least $n-1$ coordinates are needed. This suffices, by using $(1, 2, \dots, n-1)$ for the isolated vertex and $(i, i, \dots, i)$ for the ith vertex of the clique. ■

Again we can describe the parameter using complete graphs.

8.4.6. Definition. An **equivalence** on G is a spanning subgraph of G whose components are complete graphs.

8.4.7. Proposition. The product dimension of G is the minimum number of equivalences $E_1, \dots, E_t$ such that $\bigcup E_i = \overline{G}$ and $\bigcap E_i = \emptyset$.

Proof: Again there is a natural bijection. Given a product representation, the ith coordinate generates E_i, with a component for each value used in the ith coordinate. Every nonadjacent pair agrees in some coordinate, so every edge of $\overline{G}$ is covered.

Conversely, given $E_1, \dots, E_t$, each component of E_i becomes a fixed value in the ith coordinate of a representation. The requirement $\cap E_i = \emptyset$ is the requirement of using distinct vectors in the product representation. ■

8.4.8. Lemma. If $\chi'(\overline{G}) > 1$, then pdim $G \le \chi'(\overline{G})$, with equality if $\overline{G}$ is triangle-free.

Proof: Every matching is a disjoint union of complete graphs and becomes an equivalence by the addition of isolated vertices; hence $\chi'(\overline{G})$ equivalences cover $\overline{G}$. If $\chi'(\overline{G}) > 1$, then these equivalences have no common edge.

If $\overline{G}$ is triangle-free, then every equivalence used in a cover of $\overline{G}$ is a matching plus isolated edges, and thus $\chi'(\overline{G}) \le$ pdim G. ■

8.4.9. Corollary. For $n \ge 3$, the maximum product dimension of an n-vertex graph is $n-1$.

Proof: Let G be an n-vertex graph. By Lemma 8.4.8 and Vizing's Theorem (Theorem 7.1.10), pdim $G \le \chi'(\overline{G}) \le \Delta(\overline{G}) + 1 \le n$. Furthermore, the bound improves to $n-1$ unless $\Delta(\overline{G}) = n-1$. Let S be the set of vertices of degree $n-1$ in $\overline{G}$; we may assume that $|S| = k \ge 1$.

By Lemma 8.4.8 and Vizing's Theorem again, pdim $(G-S) \le n-k$. By duplicating coordinates if needed, we obtain a product representation of $G-S$ of length $n-k$. Let x^i be the vector assigned to v_i in this representation.

Each vertex of S is isolated in G. We now assign to each $v \in S$ the vector whose ith coordinate, for $1 \le i \le n-k$, is the ith coordinate of x^i. If $k=1$, then this completes a representation of G with length $n-1$. If $k > 1$, then we have assigned the same vector to all of S; we add one coordinate using distinct values to complete a representation of length $n-k+1$, which is less than $n-1$.

Since pdim $(K_1 + K_{n-1}) = n-1$ (Example 8.4.5), the bound is sharp. ■

Lovász–Nešetřil–Pultr [1980] characterized the n-vertex graphs with product dimension $n-1$ (Exercise 4). They also proved a general lower bound using a dimension argument in linear algebra.

8.4.10. Theorem. (Lovász–Nešetřil–Pultr [1980]) Let $u_1, \ldots, u_r$ and $v_1, \ldots, v_r$ be two lists of vertices (not necessarily distinct) in a graph G. If $u_i \leftrightarrow v_j$ for $i = j$ and $u_i \not\leftrightarrow v_j$ for $i < j$, then pdim $G \ge \lceil \lg r \rceil$.

Proof: Let G have a representation of length d. Let $x^1, \ldots, x^r$ and $y^1, \ldots, y^r$ be the vectors for $u_1, \ldots, u_r$ and $v_1, \ldots, v_r$, respectively. The vectors x^i and y^i differ in every coordinate, but x^i and y^j agree in some coordinate if $i \ne j$. Hence $\prod_{k=1}^{d}(x_k^i - y_k^j)$ is nonzero if and only if $i = j$.

We use this product property to construct r linearly independent vectors in $\mathbb{R}^{2^d}$; this proves that $r \le 2^d$ and hence that pdim $G \ge \lceil \lg r \rceil$. Expansion of $\prod_{k=1}^{d}(w_k - z_k)$ for $w, z \in \mathbb{R}^d$ yields the sum $\sum_{S \subseteq [d]} \prod_{i \in S} w_i \prod_{j \in \overline{S}}(-z_j)$. To relate r to 2^d, we view this as a dot product in $\mathbb{R}^{2^d}$, with coordinates indexed by the subsets of $[d]$. For each $w \in \mathbb{R}^d$, define two vectors in $\mathbb{R}^{2^d}$ by setting $\overline{w}_S = \prod_{i \in S} w_i$ and $\hat{w}_S = \prod_{i \notin S}(-w_i)$ for the coordinate $S \subseteq [d]$. With this definition, the dot product $\overline{w} \cdot \hat{z}$ equals $\prod_{k=1}^{d}(w_k - z_k)$. The conditions on the x's and y's thus imply that $\overline{x}^i \cdot \hat{y}^j$ is nonzero if and only if $i = j$.

We claim that $\overline{x}^1, \ldots, \overline{x}^r$ are independent. Consider a linear dependence $\sum_{i=1}^{r} c_i \overline{x}^i = \mathbf{0}$. Taking the dot product of $\hat{y}^r$ with both sides kills all terms below $i = r$, yielding $c_r \overline{x}^r \cdot \hat{y}^r = 0$. Since $\overline{x}^r \cdot \hat{y}^r \ne 0$, we have $c_r = 0$. We can now apply the same argument using $\hat{y}^{r-1}$. Knowing that $c_r = 0$ yields $c_{r-1}\overline{x}^{r-1} \cdot \hat{y}^{r-1} = 0$. Successively decreasing the index yields $c_j = 0$ for all j. We conclude that $\overline{x}^1, \ldots, \overline{x}^r$ are independent, which requires $2^d \ge r$. ∎

8.4.11. Example. *Matchings:* pdim $(n/2)K_2 = \lceil \lg n \rceil$. Given k coordinates, the graph encoded by using all 2^k binary k-tuples as codes is $2^{k-1}K_2$, since only with its complement does a vector disagree in each position. If n is not a power of 2, then we can discard complementary pairs to obtain a construction. The lower bound follows from Theorem 8.4.10, using each vertex in each list (for example, set $u_i = v_{n+1-i}$). ∎

In our third encoding model, we want to recover more detailed information: distance between vertices. This arises from an addressing problem in communication networks. Each message should travel a shortest path to its destination. Without central control, a vertex receiving a message must determine where to send it using only the name of the destination. If the vectors for two vertices yield the distance between them in G, then a vertex can compare the destination vector with the vectors for its neighbors and send the message to a neighbor closest to the destination.

For a connected graph G, we want to assign vectors to vertices such that the distance between vertices is the number of positions where their vectors differ. This is an **isometric** or **"distance-preserving"** embedding of G into $H = K_{n_1} \square \cdots \square K_{n_t}$, meaning a mapping $f\colon V(G) \to V(H)$ such that $d_G(u, v) = d_H(f(u), f(v))$. However, many connected graphs have no isometric embedding in a cartesian product of cliques; C_{2k+1} for $k \ge 2$ is an example (Exercise 11).

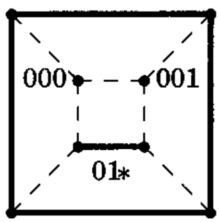
Hence we introduce a "don't care" symbol $*$. Let $S = \{0, 1, *\}$, and define a symmetric function d by $d(0, 1) = 1$ and $d(0, *) = 0 = d(1, *)$. Let S^N denote the set of N-tuples (vectors) with entries in S, and for $a, b \in S^N$ let $d_S(a, b) = \sum d(a_i, b_i)$. For each graph G, we obtain for some N an encoding $f\colon V(G) \to S^N$ so that $d_G(u, v) = d_S(f(u), f(v))$ for all $u, v \in V(G)$.

Each $a \in S^N$ corresponds to a subcube of Q_N, the N-dimensional cube; the dimension of the subcube is the number of $*$s in a. For $a, b \in S^N$, the minimum distance between vertices of the corresponding subcubes is $d_S(a, b)$. The vectors assigned to distinct vertices correspond to disjoint subcubes, else their distance would be 0. If we contract the edges of each assigned subcube, we obtain a "squashed cube" H. The distance-preserving map $f\colon V(G) \to S^N$ is an isometric embedding of G in H.

8.4.12. Definition. A **squashed-cube embedding** of **length** N is a map $f\colon V(G) \to S^N$ such that $d_G(u, v) = d_S(f(u), f(v))$. The **squashed-cube dimension** qdim G is the minimum length of such an embedding of G.

8.4.13. Example. The vectors 000, 001, 01$*$, and 1$**$ form a squashed-cube embedding of K_4 with length 3. Two adjacent vertices of the 3-cube remain unchanged, an edge adjacent to both collapses, and the entire opposite face collapses. The resulting graph is K_4. The image subcubes appear below in bold. The construction generalizes to embed K_n in a squashed $n-1$-dimensional cube.

The path P_n embeds isometrically in Q_{n-1} without squashings, using $00\cdots00$, $10\cdots00$, $11\cdots00, \ldots, 11\cdots10$, $11\cdots11$. No shorter embedding exists, because the distance between the endpoints of P_n is $n-1$, and each coordinate contributes at most 1 to the distance between vectors. ■



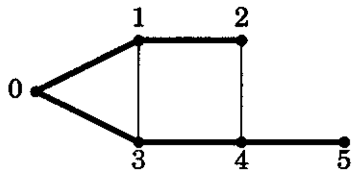
8.4.14. Proposition. For a graph G, qdim $(G) \le \sum_{i<j} d_G(v_i, v_j)$.

Proof: For each pair i, j with $i < j$, we dedicate a block of $d_G(v_i, v_j)$ coordinates. Set these coordinates to 0 for v_i, to 1 for v_j, and to $*$ for other vertices. Given two vertices, the only coordinates where neither contains $*$ are the coordinates dedicated to the pair, so $d_G(v_i, v_j) = d_S(f(v_i), f(v_j))$. ■

Using an eigenvalue technique (Exercise 8.6.14), Graham and Pollak [1971, 1973] proved a general lower bound on qdim (G) that yields qdim $K_n = n-1$. Hence K_n and P_n both have squashed-cube dimension $n-1$; Graham and Pollak conjectured that qdim $G \le n-1$ for every n-vertex connected graph. Graham offered \$100 for a proof, and Winkler found an encoding scheme to prove this "Squashed-Cube Conjecture".

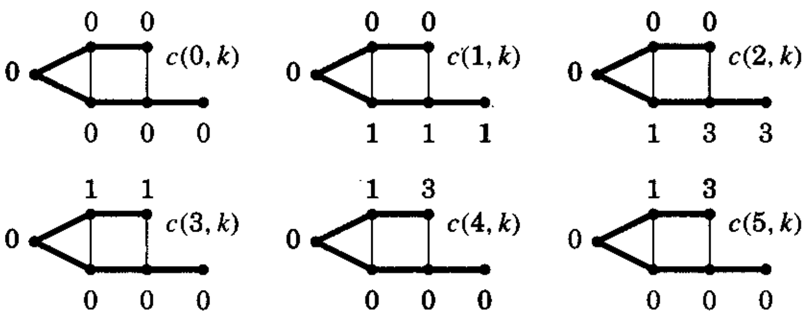
Winkler's proof generates an explicit $n-1$-dimensional squashed-cube encoding for each connected n-vertex graph G. We begin by indexing the vertices; choose v_0 arbitrarily. Next, find a spanning tree T such that $d_T(v, v_0) = d_G(v, v_0)$ for all $v \in V(G)$ (T can be generated by a breadth-first search from v_0). Now, number the vertices by a *depth*-first search in T. In other words, having chosen the indexing for $v_0, \dots, v_i$, let v_{i+1} be an unvisited child of v_i in T, if one exists; otherwise backtrack toward the root until a vertex with such a child is found. The resulting indices increase along every path from v_0 in T.

8.4.15. Example. *Depth-first numbering of a breadth-first spanning tree.* Below, the bold edges belong to T and the solid edges to $G - T$. We will use this example to illustrate several steps in the proof. ∎



We henceforth fix T and this ordering and refer to vertices by their index in this ordering. Let P_i be the vertex set of the $i,0$-path in T, let i' be the parent of i in T (the next vertex on the path from i to 0), and let $i \wedge j = \max(P_i \cap P_j)$ be the vertex at which the $i,0$-path and $j,0$-path meet. Given a depth-first numbering of a breadth-first tree T in G, let $c(i,j) = d_T(i,j) - d_G(i,j)$ be the **discrepancy** of two vertices i, j.

8.4.16. Example. In the graph marked $c(i,k)$ below, we record at each vertex k the discrepancy $c(i,k)$ for the tree T in Example 8.4.15. ∎



8.4.17. Lemma. (Winkler [1983]). Discrepancy has the following properties.

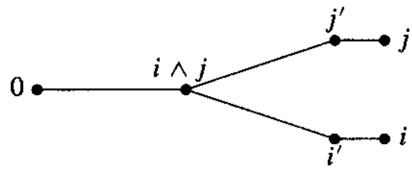
a) $c(i,j) = c(j,i) \geq 0$.

b) If $i \in P_j$, then $c(i,j) = 0$.

c) If neither $i \in P_j$ nor $j \in P_i$, then $c(i,j') \leq c(i,j) \leq c(i,j') + 2$.

Proof: (a) Distance in graphs is symmetric, and the shortest i,j-path in G is no longer than the path between them in T. (b) The preservation of distances to

v_0 implies that the i, j-path in T is a shortest i, j-path in G. (c) Since j' belongs to the i, j-path in T, we have $d_T(i, j) - d_T(i, j') = 1$. Since $jj' \in E(G)$, we have $|d_G(i, j) - d_G(i, j')| \le 1$. Thus $c(i, j) - c(i, j')$ is 0, 1, or 2. ∎



With this notion of discrepancy, we can give an overview of how Winkler's encoding works. We use a search tree because it gives us $n - 1$ natural coordinates. Distance in the tree is an "approximation" to distance in the graph; it needs to be adjusted (reduced) by the discrepancy. Winkler's encoding puts a 1 in coordinate k for exactly one of vertices i and j for exactly $d_T(i, j)$ values of k. We want the other code to have a 0 in exactly $d_G(i, j)$ of these coordinates, so we perform the adjustment by having $*$ in exactly $c(i, j)$ of the coordinates where one code has a 1. The problem is to design the encoding to achieve this simultaneously for all pairs of vertices.

8.4.18. Theorem. (Winkler [1983]) Every connected n-vertex graph G has squashed-cube dimension at most $n - 1$.

Proof: Choose a tree T and numbering $0, \dots, n-1$ as described above. We define an encoding $f(i) = (f_1(i), \dots, f_{n-1}(i))$ and verify that $d_G(i, j) = d_S(a_i, a_j)$. The encoding is

$$f_k(i) = \begin{cases} 1 & \text{if } k \in P_i \\ * & \text{if } c(i,k) - c(i,k') = 2 \\ * & \text{if } c(i,k) - c(i,k') = 1 \text{ and } i < k \text{ and } c(i,k) \text{ is even} \\ * & \text{if } c(i,k) - c(i,k') = 1 \text{ and } i > k \text{ and } c(i,k) \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

(The vectors in the encoding for Example 8.4.16 are $f(0) = 00000$, $f(1) = 10000$, $f(2) = 110{*}0$, $f(3) = {*}0100$, $f(4) = {*}{*}110$, and $f(5) = {*}{*}111$.)

To prove that $d_S(f(i), f(j)) = d_G(i, j)$, we count the coordinates where one of $f(i), f(j)$ has a 1 and the other has a 0. Such coordinates k belong to $P_i \cup P_j$, where all the 1's are located. By symmetry, we may assume that $i < j$. Hence $j \notin P_i$, and we consider the two cases $i \in P_j$ and $i \notin P_j$.

If $i \in P_j$, then $d_G(i, j) = d_T(i, j) = |P_j - P_i|$, and $f_k(i) = f_k(j) = 1$ if and only if $k \in P_i$. The coordinates where exactly one of $f(i), f(j)$ has a 1 all lie in $P_j - P_i$. For $k \in P_j - P_i$, we have $f_k(i) = 0$, and thus $d_G(i, j) = d_S(f(i), f(j))$.

If $i \notin P_j$, then exactly one of $\{f_i(k), f_j(k)\}$ equals 1 precisely when $k \in (P_j - P_i) \cup (P_i - P_j)$. We need to prove that the other vector has $*$ in exactly $c(i, j)$ of these coordinates. This will yield

$$d_S(f(i), f(j)) = |P_j - P_i| + |P_i - P_j| - c(i, j) = d_T(i, j) - c(i, j) = d_G(i, j).$$

In Example 8.4.16, $(P_5 - P_2) \cup (P_2 - P_5)$ is all five coordinates; since $f(2)$ and $f(5)$ together have $*$ in three of these coordinates, we have $d_S(f(2), f(5)) = d_G(2, 5) = 2$, as desired.

To locate the $*$'s in these positions, consider the change in discrepancies as we bring either of i, j to the point where P_i, P_j meet. Consider two lists:

$$0 = c(i, i \wedge j) \le \cdots \le c(i, j') \le c(i, j)$$
$$0 = c(i \wedge j, j) \le \cdots \le c(i', j) \le c(i, j).$$

We will obtain one $*$ in $f(i)$ for each even m with $0 < m \le c(i, j)$ and one $*$ in $f(j)$ for each odd m with $0 < m \le c(i, j)$.

For even m with $0 < m \le c(i, j)$, let j_m be the unique vertex such that $c(i, j_m) \ge m$ and $c(i, j_m') < m$. Even when the value m is not in the first list, j_m is well-defined. Because c changes by at most 2 with each step, the values of j_m are distinct. Furthermore, the depth-first ordering guarantees $i < k$ for all $k \in P_j - P_i$. Thus $f_k(i) = *$ for $k \in P_j - P_i$ if and only if $k = j_m$ for some even m. In Example 8.4.16, for $(i, j) = (2, 5)$ we have $j_2 = 4$ and $f_4(2) = *$.

Similarly, for odd m with $0 < m \le c(i, j)$, let i_m be the unique vertex such that $c(i_m, j) \ge m$ and $c(i_m', j) < m$. As before, the values of i_m are distinct and well-defined. The depth-first ordering guarantees $j > k$ for all $k \in P_i - P_j$, so $a_j(k) = *$ for $k \in P_i - P_j$ if and only if $k = i_m$ for some odd m. In Example 8.4.16, for $(i, j) = (2, 5)$ we have $i_1 = 1$, $i_3 = 2$, and $f_1(j) = f_3(j) = *$.

Thus, we have counted the $*$'s in $P_i - P_j \cup P_j - P_i$. Their number is the number of even integers between 1 and $c(i, j)$ plus the number of odd integers between 1 and $c(i, j)$, which together equals $c(i, j)$. ■

BRANCHINGS AND GOSSIP

We have studied the problem of finding the maximum number of pairwise edge-disjoint spanning trees in a graph; this equals the maximum k such that for every vertex partition P, there are at least $k(|P| - 1)$ edges crossing between sets of P (Corollary 8.2.59). Here we consider an analogous problem for digraphs that is related to Menger's Theorem (Exercise 14). Menger's Theorem is a min-max theorem that focuses on vertex pairs. We examine "connectedness" from a single vertex to the rest of the digraph.

8.4.19. Definition. An r-**branching** in a digraph is a rooted tree "branching out" from r. Vertex r has indegree 0, all other vertices have indegree 1, and all other vertices are reachable from r. Let $\kappa'(r; G)$ denote the minimum number of edges whose deletion makes some vertex unreachable from r.

Deleting the edges entering a set $X \subseteq V(G) - \{r\}$ makes each vertex of X unreachable from r. On the other hand, a minimal set whose deletion makes some vertex unreachable includes all edges leaving the set of reached vertices. Hence $\kappa'(r; G)$ equals the minimum, over nonempty $X \subseteq V(G) - \{r\}$, of the number of edges entering X.

In a set of pairwise edge-disjoint r-branchings, each must use at least one edge entering X. Thus there are at most $\kappa'(r; G)$ pairwise edge-disjoint r-branchings in G. Edmonds proved that this bound is achievable. Our discussion allows multiple edges.

8.4.20. Theorem. (Edmonds' Branching Theorem [1973]) For a vertex r in a digraph G, the maximum number of pairwise edge-disjoint r-branchings in G is $\kappa'(r; G)$.

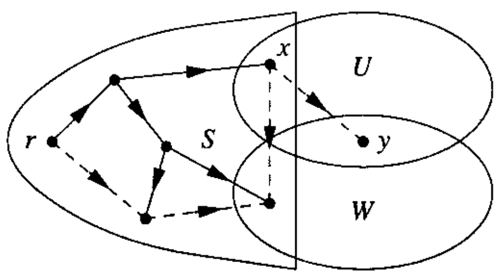
Proof: (Lovász [1976]) Let V be the vertex set of G. The upper bound holds since each subset of $V - r$ is entered by at least one edge in every r-branching. We prove the existence of $\kappa'(r; G)$ edge-disjoint r-branchings by induction on $k = \kappa'(r; G)$. For $k = 1$, a breadth-first search suffices to grow an r-branching, since every vertex is reachable. For $k > 1$, we seek an r-branching T such that $\kappa'(r; G - E(T)) = k - 1$; the induction hypothesis then supplies $k - 1$ additional r-branchings.

A *partial* r-branching is an r-branching of an induced subgraph of G. Let T be a partial r-branching of maximum order such that $\kappa'(r; G - E(T)) \ge k - 1$. The vertex r itself is such a branching, with $E(T) = \emptyset$. Let $S = V(T)$. If $S = V$, then we are done, so we may assume that $S \ne V$.

For $X \subseteq V - r$, let e_X denote the number of edges in $G - E(T)$ that enter X. If $e_X \ge k$ for every $X \subseteq V - r$ that intersects $V - S$, then we can extend T by adding any edge from S to $V - S$. Hence we can choose a smallest set $U \subseteq V - r$ that intersects $V - S$ and is entered by exactly $k - 1$ edges. (In the illustration, T consists of the solid edges.)

Because $\kappa'(r; G) = k$ and we have deleted no edge entering $U - S$, we still have $e_{U-S} \ge k$. However, $e_U = k - 1$, so there must be an edge xy from $S \cap U$ to $U - S$. We claim that xy can be added to enlarge T, contradicting the maximality of T. We need only verify that at least $k - 1$ edges still enter each $W \subseteq V - r$ when we delete xy from $G - E(T)$. This holds trivially unless $x \in V - W$ and $y \in W$. It suffices to prove that $e_W \ge k$ for such a W.

The quantity $e_W + e_U$ counts edges entering W and entering U. Except for the edges between $U - W$ and $W - U$, these enter $W \cup U$, and those entering $W \cap U$ are counted twice. Thus $e_W + e_U \ge e_{W \cup U} + e_{W \cap U}$. We have $e_{W \cup U} \ge k - 1$ by the defining property of T, $e_U = k - 1$ by construction, and $e_{W \cap U} \ge k$ by $x \in U - W$ and the minimality of U. Hence $e_W \ge k - 1 - (k - 1) + k = k$, as desired. ∎



Lovász's proof can be converted to an algorithm for finding the maximum number of pairwise disjoint r-branchings; Tarjan [1974/75] gave another algorithm. We might call $\kappa'(r; G)$ the **local-global edge-connectivity**. Theorem 8.4.20 has several equivalent forms:

8.4.21. Corollary. If G is a directed graph, r is a vertex of G, and $k \geq 0$, then the following statements are equivalent.
A) G has k pairwise edge-disjoint r-branchings.
B) $\kappa'(r; G) \geq k$; equivalently, $|[\overline{X}, X]| \geq k$ for all $X \subseteq V(G) - \{r\}$.
C) For each $s \neq r$ there exist k pairwise edge-disjoint r, s-paths.
D) There exist k pairwise edge-disjoint spanning trees of the underlying (undirected) graph that for each $s \neq r$ contain among them exactly k edges of the digraph G entering s.

Proof: A $\Leftrightarrow$ B is Edmonds' Theorem, B $\Leftrightarrow$ C is Menger's Theorem, and A $\Rightarrow$ D is immediate. For D $\Rightarrow$ B, assume that the trees exist and consider $U \subseteq V - r$. Each spanning tree has at most $|U| - 1$ edges within U, so the trees together have at most $k(|U| - 1)$ edges within U. By hypothesis, the edges of the digraph G corresponding to these trees contain exactly $k\,|U|$ edges with heads in U, so at least k edges enter U. ■

Schrijver observed that Edmonds' Branching Theorem can also be proved using matroid union and matroid intersection. Discard the edges entering the root r. Let M_1 be the union of k copies of the cycle matroid on the underlying undirected graph. Let M_2 be the matroid in which a set of edges is independent if and only if no $k + 1$ of them have the same head (this is the direct sum of uniform matroids of rank k). There exist k disjoint r-branchings if and only if these two matroids have a common independent set of size $k(n(G) - 1)$.

Pairwise edge-disjoint r-branchings provide a fault-tolerant static protocol for message transmissions from r; alternative trees are available. We next consider a static protocol for transmissions from each vertex to every other. Each transmission is two-way, but they are performed in a specified order.

The resulting question is the **gossip problem**. Consider n gossips, each having a tidbit of information. Being gossips, each wants to know all the information, and when two communicate they tell each other everything they know. How many telephone calls are needed to transmit all the information? Several solutions were published in the early 1970s.

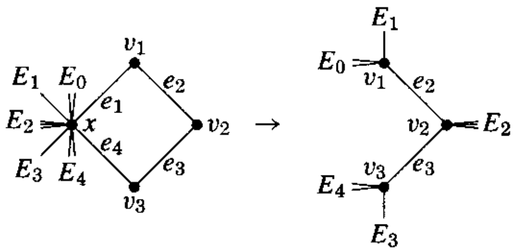
Succeeding with $2n - 3$ calls is easy: everyone calls x, and then x calls everyone back, saving one call by combining the last call in and first call out. When $n \geq 4$, $2n - 4$ calls suffice: first the others call in to a set S of four people, then S shares the information in two successive pairings, and finally the others receive calls back from S, using a total of $(n - 4) + 4 + (n - 4) = 2n - 4$ calls. Using a graph model, we show that this is optimal.

8.4.22. Definition. An **ordered graph** is a graph with an ordering of the edges (multiple edges allowed). An **increasing path** is a path via successively later edges. A **gossip scheme** is an ordered graph having an increasing path from each vertex to every other vertex. A gossip scheme **satisfies NOHO** ("No One Hears his or her Own information") if it has no increasing x, y-path plus a later edge between x and y.

8.4.23. Theorem. For $n \geq 4$, the minimum number of edges in a gossip scheme on n vertices is $2n - 4$.

Proof: (Baker–Shostak [1972]). We freely use "calls" in place of "edges" to emphasize the ordering and the possibility of repeated edges. The scheme described above uses $2n - 4$ calls, and case analysis shows that it is optimal for $n = 4$. This provides the basis for a proof by induction on n. For $n > 4$, we may assume that every gossip scheme with $n-1$ vertices uses at least $2n-6$ calls. If $2n-4$ is not optimal for n vertices, then we can add calls to the optimal scheme (if necessary) to obtain an n-vertex gossip scheme G with exactly $2n - 5$ calls.

Claim 1. G satisfies NOHO. Otherwise, G has an increasing path from x to v_k along edges $e_1, \dots, e_k$ followed by a call $e_{k+1} = v_k x$. Delete e_1 and e_{k+1}. Partition the other calls involving x into $k + 2$ sets: E_0 consists of those before e_1, E_i for $1 \leq i \leq k$ consists of those between e_i and e_{i+1}, and E_{k+1} consists of those after e_{k+1}. In each edge $e \in E_i$, replace x by v_1, v_i, or v_k in the cases $i = 0$, $1 \leq i \leq k$, or $i = k + 1$, respectively (see illustration). Now $E(G) - \{e_1, e_{k+1}\}$ is a gossip scheme on $V(G) - \{x\}$, because every increasing path through x is replaced by an increasing path that consists of the same edges and perhaps additional edges from $\{e_i\}$. The scheme has $2(n-1)-5$ edges, which contradicts the induction hypothesis.



Claim 2. $d(x)-3$ calls are useless to x, and hence $\delta(G) \geq 3$. Let $O(x)$ be the set of calls on which some vertex is reached for the first time by an increasing path "Out" from x; these calls form a tree. The tree $I(x)$ of edges useful "In" to x is $O(x)$ for the reverse order on $E(G)$. We show that $O(x) \cap I(x)$ is the set of edges incident to x. If an increasing x, y-path reaches $y \in N(x)$ before the edge xy, then x violates NOHO; hence $xy \in O(x)$. Similarly, $xy \in I(x)$. Conversely, if $O(x) \cap I(x)$ contains an edge e not incident to x, then an increasing path from x containing e and an increasing path to x containing e combine to violate NOHO for x. Hence $|O(x) \cap I(x)| = d(x)$. The edges "useless to x" are those not in $O(x) \cup I(x)$. We have

$$\left|\overline{O(x) \cup I(x)}\right| = 2n - 5 - (n-1) - (n-1) + d(x) = d(x) - 3.$$

Since this counts a set of edges, $\delta(G) \geq 3$.

Claim 3. The subgraph obtained by deleting the first call and the last call made by each vertex has at least five components and has no isolated vertex. Let xy be the first call involving x. If the first call involving y is yz with $z \neq x$, then by definition it occurs before xy, and these two calls do not communicate from x to z. After yz and xy, an increasing x,z-path violates NOHO at z. Hence the set F of first calls is a matching, and there are $n/2$ of them. Similarly, the set L of last calls is a matching of size $n/2$. The graph $G - F - L$ has $n - 5$ edges and hence at least five components, by Proposition 1.2.11. It has no isolated vertex, since $\delta(G) \geq 3$.

The contradiction. Since $e(G) = 2n - 5 < 2n$, some vertex x has degree at most 3. Let C_1, C_2, C_3 be the components of $G - F - L$ containing x, its first neighbor, and its last neighbor, respectively (its middle neighbor is in C_1). Edges of $G - F - L$ can belong to $O(x)$ only via paths that start with the first or middle edge involving x and avoid $F \cup L$, so they belong to C_1 or C_2. Similarly, edges of $G - F - L$ belonging to $I(x)$ appear only in C_1 or C_3. The edges of the remaining components, of which there are at least two, are useless to x ($G - F - L$ has no isolated vertex), but Claim 3 allows only $d(x) - 3 = 0$ edges useless to x. ■

In practical applications, we might wish to minimize the total length of the messages or the total time (assuming that each vertex participates in at most one call per time unit). We can also restrict the pairs that are allowed to call each other. Gossiping can be completed in $2n - 4$ if and only if the graph of allowable calls is connected and has a 4-cycle (Bumby [1981], Kleitman–Shearer [1980]). Other variations consider digraphs (Exercises 15–16), fault-tolerance, conference calls, etc.

LIST COLORING AND CHOOSABILITY

List coloring is a more general version of the vertex coloring problem. We still pick a single color for each vertex, but the set of colors available at each vertex may be restricted. This model was introduced independently in Vizing [1976] and Erdős–Rubin–Taylor [1979].

8.4.24. Definition. For each vertex v in a graph G, let $L(v)$ denote a list of colors available at v. A **list coloring** or **choice function** is a proper coloring f such that $f(v) \in L(v)$ for all v. A graph G is k**-choosable** or **list** k**-colorable** if every assignment of k-element lists to the vertices permits a proper list coloring. The **list chromatic number**, **choice number**, or **choosability** $\chi_l(G)$ is the minimum k such that G is k-choosable.

Since the lists could be identical, $\chi_l(G) \geq \chi(G)$. If the lists have size at least $1 + \Delta(G)$, then coloring the vertices in succession leaves an available color at each vertex. This argument is analogous to the greedy coloring algorithm and proves that $\chi_l(G) \leq 1 + \Delta(G)$ (see Exercise 22 for other analogues with

$\chi(G)$). It is not possible, however, to place an upper bound on $\chi_l(G)$ in terms of $\chi(G)$; there are bipartite graphs with arbitrarily large list chromatic number.

8.4.25. Proposition. (Erdős–Rubin–Taylor [1979]) If $m = \binom{2k-1}{k}$; then $K_{m,m}$ is not k-choosable.

Proof: Let X, Y be the bipartition of $G = K_{m,m}$. Assign the distinct k-subsets of $[2k-1]$ as the lists for the vertices of X, and do the same for Y. Consider a choice function f. If f uses fewer than k distinct choices in X, then there is a k-set $S \subseteq [2k-1]$ not used, which means that no color was chosen for the vertex of X having S as its list. If f uses at least k colors on vertices of X, then there is a k-set $S \subseteq [2k-1]$ of colors used in X, and no color can be properly chosen for the vertex of Y with list S. ∎

List chromatic number is more difficult to compute than chromatic number, because the statements of the upper bound and lower bound both involve universal quantifiers. Determining the 3-choosable complete bipartite graphs was difficult. For $3 \le m \le n$, $K_{m,n}$ is 3-choosable if and only if

$m = 3$ and $n \le 26$ (Erdős–Rubin–Taylor [1979]), or
$m = 4$ and $n \le 20$ (Mahadev–Roberts–Santhanakrishnan [1991]), or
$m = 5$ and $n \le 12$ (Shende–Tesman [1994]), or
$m = 6$ and $n \le 10$ (O'Donnell [1995]).

Alon and Tarsi [1992] used a polynomial associated with a graph to obtain upper bounds on $\chi_l(G)$ (see also Alon [1993]). Fleischner and Stiebitz [1992] used the technique to prove that adding n disjoint triangles to a $3n$-cycle yields a 3-colorable graph; they proved the stronger result that it is 3-choosable.

There is also an edge-coloring variant, where we assign lists to the edges and must choose a proper edge-coloring.

8.4.26. Definition. Let $L(e)$ denote the list of colors available for e. A **list edge-coloring** is a proper edge-coloring f with $f(e)$ chosen from $L(e)$ for each e. The **edge-choosability** $\chi_l'(G)$ is the minimum k such that every assignment of lists of size k yields a proper list edge-coloring. Equivalently, $\chi_l'(G) = \chi_l(L(G))$, where $L(G)$ is the line graph of G.

The argument for $\chi'(G) \le 2\Delta(G) - 1$ also yields $\chi_l'(G) \le 2\Delta(G) - 1$ (Exercise 22) and thus $\chi_l'(G) < 2\chi'(G)$. As in ordinary coloring, the use of line graphs expresses the edge version as a special case of the vertex version, and it behaves much better. Even so, the conjectured bound for edge-choosability is surprising. It was suggested independently by many researchers, including Vizing, Gupta, Albertson, Collins, and Tucker, and it seems to have been published first in Bollobás–Harris [1985] (see also Bollobás [1986]).

8.4.27. Conjecture. (List Coloring Conjecture) $\chi_l'(G) = \chi'(G)$ for all G. ∎

For simple graphs, this conjecture and Vizing's Theorem (Theorem 7.1.10) would yield $\chi_l'(G) \le \Delta(G) + 1$. Bollobás and Harris [1985] proved that $\chi_l'(G) <$

$c\Delta(G)$ when $c > 11/6$ for sufficiently large $\Delta(G)$. This and later improvements used probabilistic methods. Kahn [1996] proved the conjecture asymptotically: $\chi_l'(G) \le (1 + o(1))\Delta(G)$. Häggkvist and Janssen [1997] sharpened the error term: $\chi_l'(G) \le d + O(d^{2/3}\sqrt{\log d})$ when $d = \Delta(G)$. Molloy and Reed [1999] further sharpened (and generalized) the bound.

The special case of the List Coloring Conjecture for $G = K_{n,n}$ was posed by Dinitz in 1979. (Janssen [1993] proved it for $K_{n,n-1}$.) The Dinitz Conjecture became popular in its matrix formulation: If each position of an n by n grid contains a set of size n, then it is possible to choose one element from each set so that the elements chosen in each row are distinct and the elements chosen in each column are distinct.

Galvin [1995] proved the List Coloring Conjecture for bipartite graphs, which includes the Dinitz Conjecture (see also Slivnik [1996]). Here we prove only the Dinitz Conjecture, using the Stable Matching Problem (Section 3.2).

8.4.28. Definition. A **kernel** of a digraph is an independent set S having a successor of every vertex outside S. A digraph is **kernel-perfect** if every induced subdigraph has a kernel. Given a function $f\colon V(G) \to \mathbb{N}$, the graph G is **f-choosable** if a proper coloring can be chosen from the lists at the vertices whenever $|L(x)| = f(x)$ for each x.

We used the concept of "kernel" in Application 1.4.14 (digraphs without odd cycles, for example, have kernels). An f-choosable graph is k-choosable for $k = \max f(x)$, since adding colors to a list cannot make the choice more difficult.

8.4.29. Lemma. (Bondy–Boppana–Siegel) If D is a kernel-perfect orientation of G and $f(x) = 1 + d_D^+(x)$ for all $x \in V(G)$, then G is f-choosable.

Proof: We use induction on $n(G)$; the statement is trivial for $n(G) = 1$. For $n(G) > 1$, consider an assignment of lists, with the list $L(x)$ having size $f(x)$. Choose a color c appearing in some list. Let $U = \{v\colon c \in L(v)\}$. Let S be the kernel of the induced subdigraph $D[U]$. Assign color c to all of S, which is permissible since S is independent.

Delete c from $L(v)$ for each $v \in U - S$. Delete additional colors arbitrarily from other lists to reduce $L(x)$ for each $x \in V(D) - S$ to size $f'(x)$, where $f'(x) = 1 + d_{D-S}^+(x)$. Since each vertex not in S has a successor in S, we have $f'(x) < f(x)$ for $x \in V(D) - S$, which accommodates the deletion of c from the lists. By the induction hypothesis, D' is $f'(x)$-choosable, so we can complete a list coloring for G by adding a list coloring of D' to the use of c on S. ■

8.4.30. Theorem. (Galvin [1995]) $\chi_l'(K_{n,n}) = n$.

Proof: Since $\chi_l'(G) = \chi_l(L(G))$, it suffices by Lemma 8.4.29 to prove that $L(K_{n,n})$ has a kernel-perfect orientation with each vertex having indegree and outdegree $n - 1$. The graph $L(K_{n,n})$ is the cartesian product $K_n \,\square\, K_n$ (Exercise 7.1.8); placed in an n by n grid, vertices are adjacent if and only if they are in the same row or in the same column.