$\mathcal{B}^* = \{f_1, \ldots, f_n\}$ is the dual basis, then f_i is precisely the function which assigns to each vector α in V the ith coordinate of α relative to the ordered basis $\mathcal{B}$. Thus we may also call the f_i the coordinate functions for $\mathcal{B}$. The formula (3-13), when combined with (3-14) tells us the following: If f is in V^*, and we let $f(\alpha_i) = a_i$, then when

$$\alpha = x_1\alpha_1 + \cdots + x_n\alpha_n$$

we have

$$f(\alpha) = a_1x_1 + \cdots + a_nx_n. \tag{3-15}$$

In other words, if we choose an ordered basis $\mathcal{B}$ for V and describe each vector in V by its n-tuple of coordinates $(x_1, \ldots, x_n)$ relative to $\mathcal{B}$, then every linear functional on V has the form (3-15). This is the natural generalization of Example 18, which is the special case $V = F^n$ and $\mathcal{B} = \{\epsilon_1, \ldots, \epsilon_n\}$.

Example 22. Let V be the vector space of all polynomial functions from R into R which have degree less than or equal to 2. Let t_1, t_2, and t_3 be any three *distinct* real numbers, and let

$$L_i(p) = p(t_i).$$

Then L_1, L_2, and L_3 are linear functionals on V. These functionals are linearly independent; for, suppose

$$L = c_1L_1 + c_2L_2 + c_3L_3.$$

If $L = 0$, i.e., if $L(p) = 0$ for each p in V, then applying L to the particular polynomial 'functions' 1, x, x^2, we obtain

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ t_1c_1 + t_2c_2 + t_3c_3 &= 0 \\ t_1^2c_1 + t_2^2c_2 + t_3^2c_3 &= 0 \end{aligned}$$

From this it follows that $c_1 = c_2 = c_3 = 0$, because (as a short computation shows) the matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ t_1 & t_2 & t_3 \\ t_1^2 & t_2^2 & t_3^2 \end{bmatrix}$$

is invertible when t_1, t_2, and t_3 are distinct. Now the L_i are independent, and since V has dimension 3, these functionals form a basis for V^*. What is the basis for V, of which this is the dual? Such a basis $\{p_1, p_2, p_3\}$ for V must satisfy

$$L_i(p_j) = \delta_{ij}$$

or

$$p_j(t_i) = \delta_{ij}.$$

These polynomial functions are rather easily seen to be

$$p_1(x) = \frac{(x - t_2)(x - t_3)}{(t_1 - t_2)(t_1 - t_3)}$$
$$p_2(x) = \frac{(x - t_1)(x - t_3)}{(t_2 - t_1)(t_2 - t_3)}$$
$$p_3(x) = \frac{(x - t_1)(x - t_2)}{(t_3 - t_1)(t_3 - t_2)}.$$

The basis $\{p_1, p_2, p_3\}$ for V is interesting, because according to (3-14) we have for each p in V

$$p = p(t_1)p_1 + p(t_2)p_2 + p(t_3)p_3.$$

Thus, if c_1, c_2, and c_3 are any real numbers, there is exactly one polynomial function p over R which has degree at most 2 and satisfies $p(t_j) = c_j$, $j = 1, 2, 3$. This polynomial function is $p = c_1p_1 + c_2p_2 + c_3p_3$.

Now let us discuss the relationship between linear functionals and subspaces. If f is a non-zero linear functional, then the rank of f is 1 because the range of f is a non-zero subspace of the scalar field and must (therefore) be the scalar field. If the underlying space V is finite-dimensional, the rank plus nullity theorem (Theorem 2) tells us that the null space N_f has dimension

$$\dim N_f = \dim V - 1.$$

In a vector space of dimension n, a subspace of dimension $n - 1$ is called a **hyperspace.** Such spaces are sometimes called hyperplanes or subspaces of codimension 1. Is every hyperspace the null space of a linear functional? The answer is easily seen to be yes. It is not much more difficult to show that each d-dimensional subspace of an n-dimensional space is the intersection of the null spaces of $(n - d)$ linear functionals (Theorem 16 below).

Definition. *If* V *is a vector space over the field* F *and* S *is a subset of* V, *the* **annihilator** *of* S *is the set* S^0 *of linear functionals* f *on* V *such that* $f(\alpha) = 0$ *for every* α *in* S.

It should be clear to the reader that S^0 is a subspace of V^*, whether S is a subspace of V or not. If S is the set consisting of the zero vector alone, then $S^0 = V^*$. If $S = V$, then S^0 is the zero subspace of V^*. (This is easy to see when V is finite-dimensional.)

Theorem 16. *Let* V *be a finite-dimensional vector space over the field* F, *and let* W *be a subspace of* V. *Then*

$$\dim W + \dim W^0 = \dim V.$$

Proof. Let k be the dimension of W and $\{\alpha_1, \ldots, \alpha_k\}$ a basis for W. Choose vectors $\alpha_{k+1}, \ldots, \alpha_n$ in V such that $\{\alpha_1, \ldots, \alpha_n\}$ is a basis for V. Let $\{f_1, \ldots, f_n\}$ be the basis for V^* which is dual to this basis for V.

The claim is that $\{f_{k+1}, \ldots, f_n\}$ is a basis for the annihilator W^0. Certainly f_i belongs to W^0 for $i \geq k+1$, because

$$f_i(\alpha_j) = \delta_{ij}$$

and $\delta_{ij} = 0$ if $i \geq k+1$ and $j \leq k$; from this it follows that, for $i \geq k+1$, $f_i(\alpha) = 0$ whenever α is a linear combination of $\alpha_1, \ldots, \alpha_k$. The functionals $f_{k+1}, \ldots, f_n$ are independent, so all we must show is that they span W^0. Suppose f is in V^*. Now

$$f = \sum_{i=1}^{n} f(\alpha_i) f_i$$

so that if f is in W^0 we have $f(\alpha_i) = 0$ for $i \leq k$ and

$$f = \sum_{i=k+1}^{n} f(\alpha_i) f_i.$$

We have shown that if $\dim W = k$ and $\dim V = n$ then $\dim W^0 = n - k$. ∎

Corollary. *If* W *is a* k*-dimensional subspace of an* n*-dimensional vector space* V, *then* W *is the intersection of* (n − k) *hyperspaces in* V.

Proof. This is a corollary of the proof of Theorem 16 rather than its statement. In the notation of the proof, W is exactly the set of vectors α such that $f_i(\alpha) = 0$, $i = k+1, \ldots, n$. In case $k = n - 1$, W is the null space of f_n. ∎

Corollary. *If* W_1 *and* W_2 *are subspaces of a finite-dimensional vector space, then* $W_1 = W_2$ *if and only if* $W_1^0 = W_2^0$.

Proof. If $W_1 = W_2$, then of course $W_1^0 = W_2^0$. If $W_1 \neq W_2$, then one of the two subspaces contains a vector which is not in the other. Suppose there is a vector α which is in W_2 but not in W_1. By the previous corollaries (or the proof of Theorem 16) there is a linear functional f such that $f(\beta) = 0$ for all β in W, but $f(\alpha) \neq 0$. Then f is in W_1^0 but not in W_2^0 and $W_1^0 \neq W_2^0$. ∎

In the next section we shall give different proofs for these two corollaries. The first corollary says that, if we select some ordered basis for the space, each k-dimensional subspace can be described by specifying $(n - k)$ homogeneous linear conditions on the coordinates relative to that basis.

Let us look briefly at systems of homogeneous linear equations from the point of view of linear functionals. Suppose we have a system of linear equations,

$$\begin{array}{ccc} A_{11}x_1 + \cdots + A_{1n}x_n & = & 0 \\ \vdots \qquad\qquad \vdots & & \\ A_{m1}x_1 + \cdots + A_{mn}x_n & = & 0 \end{array}$$

for which we wish to find the solutions. If we let f_i, $i = 1, \ldots, m$, be the linear functional on F^n defined by

$$f_i(x_1, \ldots, x_n) = A_{i1}x_1 + \cdots + A_{in}x_n$$

then we are seeking the subspace of F^n of all α such that

$$f_i(\alpha) = 0, \qquad i = 1, \ldots, m.$$

In other words, we are seeking the subspace annihilated by $f_1, \ldots, f_m$. Row-reduction of the coefficient matrix provides us with a systematic method of finding this subspace. The n-tuple $(A_{i1}, \ldots, A_{in})$ gives the coordinates of the linear functional f_i relative to the basis which is dual to the standard basis for F^n. The row space of the coefficient matrix may thus be regarded as the space of linear functionals spanned by $f_1, \ldots, f_m$. The solution space is the subspace annihilated by this space of functionals.

Now one may look at the system of equations from the 'dual' point of view. That is, suppose that we are given m vectors in F^n

$$\alpha_i = (A_{i1}, \ldots, A_{in})$$

and we wish to find the annihilator of the subspace spanned by these vectors. Since a typical linear functional on F^n has the form

$$f(x_1, \ldots, x_n) = c_1x_1 + \cdots + c_nx_n$$

the condition that f be in this annihilator is that

$$\sum_{j=1}^{n} A_{ij}c_j = 0, \qquad i = 1, \ldots, m$$

that is, that $(c_1, \ldots, c_n)$ be a solution of the system $AX = 0$. From this point of view, row-reduction gives us a systematic method of finding the annihilator of the subspace spanned by a given finite set of vectors in F^n.

Example 23. Here are three linear functionals on R^4:

$$\begin{aligned} f_1(x_1, x_2, x_3, x_4) &= x_1 + 2x_2 + 2x_3 + x_4 \\ f_2(x_1, x_2, x_3, x_4) &= 2x_2 + x_4 \\ f_3(x_1, x_2, x_3, x_4) &= -2x_1 - 4x_3 + 3x_4. \end{aligned}$$

The subspace which they annihilate may be found explicitly by finding the row-reduced echelon form of the matrix

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 2 & 0 & 1 \\ -2 & 0 & -4 & 3 \end{bmatrix}.$$

A short calculation, or a peek at Example 21 of Chapter 2, shows that

$$R = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Therefore, the linear functionals

$$\begin{aligned} g_1(x_1, x_2, x_3, x_4) &= x_1 + 2x_3 \\ g_2(x_1, x_2, x_3, x_4) &= x_2 \\ g_3(x_1, x_2, x_3, x_4) &= x_4 \end{aligned}$$

span the same subspace of $(R^4)^*$ and annihilate the same subspace of R^4 as do f_1, f_2, f_3. The subspace annihilated consists of the vectors with

$$\begin{aligned} x_1 &= -2x_3 \\ x_2 &= x_4 = 0. \end{aligned}$$

Example 24. Let W be the subspace of R^5 which is spanned by the vectors

$$\begin{aligned} \alpha_1 &= (2, -2, 3, 4, -1), \qquad & \alpha_3 &= (0, 0, -1, -2, 3) \\ \alpha_2 &= (-1, 1, 2, 5, 2), \qquad & \alpha_4 &= (1, -1, 2, 3, 0). \end{aligned}$$

How does one describe W^0, the annihilator of W? Let us form the 4×5 matrix A with row vectors $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, and find the row-reduced echelon matrix R which is row-equivalent to A:

$$A = \begin{bmatrix} 2 & -2 & 3 & 4 & -1 \\ -1 & 1 & 2 & 5 & 2 \\ 0 & 0 & -1 & -2 & 3 \\ 1 & -1 & 2 & 3 & 0 \end{bmatrix} \longrightarrow R = \begin{bmatrix} 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

If f is a linear functional on R^5:

$$f(x_1, \ldots, x_5) = \sum_{j=1}^{5} c_j x_j$$

then f is in W^0 if and only if $f(\alpha_i) = 0$, $i = 1, 2, 3, 4$, i.e., if and only if

$$\sum_{j=1}^{5} A_{ij} c_j = 0, \qquad 1 \leq i \leq 4.$$

This is equivalent to

$$\sum_{j=1}^{5} R_{ij} c_j = 0, \qquad 1 \leq i \leq 3$$

or

$$\begin{aligned} c_1 - c_2 - c_4 &= 0 \\ c_3 + 2c_4 &= 0 \\ c_5 &= 0. \end{aligned}$$

We obtain all such linear functionals f by assigning arbitrary values to c_2 and c_4, say $c_2 = a$ and $c_4 = b$, and then finding the corresponding $c_1 = a + b$, $c_3 = -2b$, $c_5 = 0$. So W^0 consists of all linear functionals f of the form

$$f(x_1, x_2, x_3, x_4, x_5) = (a + b)x_1 + ax_2 - 2bx_3 + bx_4.$$

The dimension of W^0 is 2 and a basis $\{f_1, f_2\}$ for W^0 can be found by first taking $a = 1$, $b = 0$ and then $a = 0$, $b = 1$:

$$f_1(x_1, \ldots, x_5) = x_1 + x_2$$
$$f_2(x_1, \ldots, x_5) = x_1 - 2x_3 + x_4.$$

The above general f in W^0 is $f = af_1 + bf_2$.

Exercises

1. In R^3, let $\alpha_1 = (1, 0, 1)$, $\alpha_2 = (0, 1, -2)$, $\alpha_3 = (-1, -1, 0)$.
   (a) If f is a linear functional on R^3 such that

$$f(\alpha_1) = 1, \qquad f(\alpha_2) = -1, \qquad f(\alpha_3) = 3,$$

and if $\alpha = (a, b, c)$, find $f(\alpha)$.
   (b) Describe explicitly a linear functional f on R^3 such that

$$f(\alpha_1) = f(\alpha_2) = 0 \quad \text{but} \quad f(\alpha_3) \neq 0.$$

   (c) Let f be any linear functional such that

$$f(\alpha_1) = f(\alpha_2) = 0 \quad \text{and} \quad f(\alpha_3) \neq 0.$$

If $\alpha = (2, 3, -1)$, show that $f(\alpha) \neq 0$.

2. Let $\mathcal{B} = \{\alpha_1, \alpha_2, \alpha_3\}$ be the basis for C^3 defined by

$$\alpha_1 = (1, 0, -1), \qquad \alpha_2 = (1, 1, 1), \qquad \alpha_3 = (2, 2, 0).$$

Find the dual basis of $\mathcal{B}$.

3. If A and B are $n \times n$ matrices over the field F, show that trace (AB) = trace (BA). Now show that similar matrices have the same trace.

4. Let V be the vector space of all polynomial functions p from R into R which have degree 2 or less:

$$p(x) = c_0 + c_1x + c_2x^2.$$

Define three linear functionals on V by

$$f_1(p) = \int_0^1 p(x)\,dx, \qquad f_2(p) = \int_0^2 p(x)\,dx, \qquad f_3(p) = \int_0^{-1} p(x)\,dx.$$

Show that $\{f_1, f_2, f_3\}$ is a basis for V^* by exhibiting the basis for V of which it is the dual.

5. If A and B are $n \times n$ complex matrices, show that $AB - BA = I$ is impossible.

6. Let m and n be positive integers and F a field. Let $f_1, \ldots, f_m$ be linear functionals on F^n. For α in F^n define

$$T\alpha = (f_1(\alpha), \ldots, f_m(\alpha)).$$

Show that T is a linear transformation from F^n into F^m. Then show that every linear transformation from F^n into F^m is of the above form, for some $f_1, \ldots, f_m$.

7. Let $\alpha_1 = (1, 0, -1, 2)$ and $\alpha_2 = (2, 3, 1, 1)$, and let W be the subspace of R^4 spanned by α_1 and α_2. Which linear functionals f: