Conversely, suppose the Cauchy condition holds. By Theorem 3.11, the sequence $\{f_n(x)\}$ converges, for every x, to a limit which we may call $f(x)$. Thus the sequence $\{f_n\}$ converges on E, to f. We have to prove that the convergence is uniform.

Let $\varepsilon > 0$ be given, and choose N such that (13) holds. Fix n, and let $m \to \infty$ in (13). Since $f_m(x) \to f(x)$ as $m \to \infty$, this gives

$$\tag{14} |f_n(x) - f(x)| \leq \varepsilon$$

for every $n \geq N$ and every $x \in E$, which completes the proof.

The following criterion is sometimes useful.

7.9 Theorem *Suppose*

$$\lim_{n \to \infty} f_n(x) = f(x) \qquad (x \in E).$$

Put

$$M_n = \sup_{x \in E} |f_n(x) - f(x)|.$$

Then $f_n \to f$ *uniformly on* E *if and only if* $M_n \to 0$ *as* $n \to \infty$.

Since this is an immediate consequence of Definition 7.7, we omit the details of the proof.

For series, there is a very convenient test for uniform convergence, due to Weierstrass.

7.10 Theorem *Suppose* $\{f_n\}$ *is a sequence of functions defined on* E, *and suppose*

$$|f_n(x)| \leq M_n \qquad (x \in E, n = 1, 2, 3, \ldots).$$

Then Σf_n *converges uniformly on* E *if* ΣM_n *converges.*

Note that the converse is not asserted (and is, in fact, not true).

Proof If ΣM_n converges, then, for arbitrary $\varepsilon > 0$,

$$\left| \sum_{i=n}^{m} f_i(x) \right| \leq \sum_{i=n}^{m} M_i \leq \varepsilon \qquad (x \in E),$$

provided m and n are large enough. Uniform convergence now follows from Theorem 7.8.

UNIFORM CONVERGENCE AND CONTINUITY

7.11   Theorem   *Suppose $f_n \to f$ uniformly on a set E in a metric space. Let x be a limit point of E, and suppose that*

$$\text{(15)} \qquad \lim_{t \to x} f_n(t) = A_n \qquad (n = 1, 2, 3, \ldots).$$

Then $\{A_n\}$ converges, and

$$\text{(16)} \qquad \lim_{t \to x} f(t) = \lim_{n \to \infty} A_n.$$

In other words, the conclusion is that

$$\text{(17)} \qquad \lim_{t \to x} \lim_{n \to \infty} f_n(t) = \lim_{n \to \infty} \lim_{t \to x} f_n(t).$$

Proof   Let $\varepsilon > 0$ be given. By the uniform convergence of $\{f_n\}$, there exists N such that $n \geq N$, $m \geq N$, $t \in E$ imply

$$\text{(18)} \qquad |f_n(t) - f_m(t)| \leq \varepsilon.$$

Letting $t \to x$ in (18), we obtain

$$|A_n - A_m| \leq \varepsilon$$

for $n \geq N, m \geq N$, so that $\{A_n\}$ is a Cauchy sequence and therefore converges, say to A.

Next,

$$\text{(19)} \qquad |f(t) - A| \leq |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A|.$$

We first choose n such that

$$\text{(20)} \qquad |f(t) - f_n(t)| \leq \frac{\varepsilon}{3}$$

for all $t \in E$ (this is possible by the uniform convergence), and such that

$$\text{(21)} \qquad |A_n - A| \leq \frac{\varepsilon}{3}.$$

Then, for this n, we choose a neighborhood V of x such that

$$\text{(22)} \qquad |f_n(t) - A_n| \leq \frac{\varepsilon}{3}$$

if $t \in V \cap E$, $t \neq x$.

Substituting the inequalities (20) to (22) into (19), we see that

$$|f(t) - A| \leq \varepsilon,$$

provided $t \in V \cap E$, $t \neq x$. This is equivalent to (16).

7.12 Theorem *If $\{f_n\}$ is a sequence of continuous functions on E, and if $f_n \to f$ uniformly on E, then f is continuous on E.*

This very important result is an immediate corollary of Theorem 7.11.

The converse is not true; that is, a sequence of continuous functions may converge to a continuous function, although the convergence is not uniform. Example 7.6 is of this kind (to see this, apply Theorem 7.9). But there is a case in which we can assert the converse.

7.13 Theorem *Suppose K is compact, and*

(a) *$\{f_n\}$ is a sequence of continuous functions on K,*

(b) *$\{f_n\}$ converges pointwise to a continuous function f on K,*

(c) *$f_n(x) \geq f_{n+1}(x)$ for all $x \in K$, $n = 1, 2, 3, \ldots$.*

Then $f_n \to f$ uniformly on K.

Proof Put $g_n = f_n - f$. Then g_n is continuous, $g_n \to 0$ pointwise, and $g_n \geq g_{n+1}$. We have to prove that $g_n \to 0$ uniformly on K.

Let $\varepsilon > 0$ be given. Let K_n be the set of all $x \in K$ with $g_n(x) \geq \varepsilon$. Since g_n is continuous, K_n is closed (Theorem 4.8), hence compact (Theorem 2.35). Since $g_n \geq g_{n+1}$, we have $K_n \supset K_{n+1}$. Fix $x \in K$. Since $g_n(x) \to 0$, we see that $x \notin K_n$ if n is sufficiently large. Thus $x \notin \bigcap K_n$. In other words, $\bigcap K_n$ is empty. Hence K_N is empty for some N (Theorem 2.36). It follows that $0 \leq g_n(x) < \varepsilon$ for all $x \in K$ and for all $n \geq N$. This proves the theorem.

Let us note that compactness is really needed here. For instance, if

$$f_n(x) = \frac{1}{nx + 1} \qquad (0 < x < 1; n = 1, 2, 3, \ldots)$$

then $f_n(x) \to 0$ monotonically in $(0, 1)$, but the convergence is not uniform.

7.14 Definition If X is a metric space, $\mathscr{C}(X)$ will denote the set of all complex-valued, continuous, bounded functions with domain X.

[Note that boundedness is redundant if X is compact (Theorem 4.15). Thus $\mathscr{C}(X)$ consists of all complex continuous functions on X if X is compact.] We associate with each $f \in \mathscr{C}(X)$ its *supremum norm*

$$\|f\| = \sup_{x \in X} |f(x)|.$$

Since f is assumed to be bounded, $\|f\| < \infty$. It is obvious that $\|f\| = 0$ only if $f(x) = 0$ for every $x \in X$, that is, only if $f = 0$. If $h = f + g$, then

$$|h(x)| \leq |f(x)| + |g(x)| \leq \|f\| + \|g\|$$

for all $x \in X$; hence

$$\|f + g\| \leq \|f\| + \|g\|.$$

If we define the distance between $f \in \mathscr{C}(X)$ and $g \in \mathscr{C}(X)$ to be $\|f - g\|$, it follows that Axioms 2.15 for a metric are satisfied.

We have thus made $\mathscr{C}(X)$ into a metric space.

Theorem 7.9 can be rephrased as follows:

A sequence $\{f_n\}$ converges to f with respect to the metric of $\mathscr{C}(X)$ if and only if $f_n \to f$ uniformly on X.

Accordingly, closed subsets of $\mathscr{C}(X)$ are sometimes called *uniformly closed*, the closure of a set $\mathscr{A} \subset \mathscr{C}(X)$ is called its *uniform closure*, and so on.

7.15 Theorem *The above metric makes $\mathscr{C}(X)$ into a complete metric space.*

Proof Let $\{f_n\}$ be a Cauchy sequence in $\mathscr{C}(X)$. This means that to each $\varepsilon > 0$ corresponds an N such that $\|f_n - f_m\| < \varepsilon$ if $n \geq N$ and $m \geq N$. It follows (by Theorem 7.8) that there is a function f with domain X to which $\{f_n\}$ converges uniformly. By Theorem 7.12, f is continuous. Moreover, f is bounded, since there is an n such that $|f(x) - f_n(x)| < 1$ for all $x \in X$, and f_n is bounded.

Thus $f \in \mathscr{C}(X)$, and since $f_n \to f$ uniformly on X, we have $\|f - f_n\| \to 0$ as $n \to \infty$.

UNIFORM CONVERGENCE AND INTEGRATION

7.16 Theorem *Let α be monotonically increasing on $[a, b]$. Suppose $f_n \in \mathscr{R}(\alpha)$ on $[a, b]$, for $n = 1, 2, 3, \ldots$, and suppose $f_n \to f$ uniformly on $[a, b]$. Then $f \in \mathscr{R}(\alpha)$ on $[a, b]$, and*

$$(23) \qquad \int_a^b f \, d\alpha = \lim_{n \to \infty} \int_a^b f_n \, d\alpha.$$

(The existence of the limit is part of the conclusion.)

Proof It suffices to prove this for real f_n. Put

$$(24) \qquad \varepsilon_n = \sup |f_n(x) - f(x)|,$$

the supremum being taken over $a \leq x \leq b$. Then

$$f_n - \varepsilon_n \leq f \leq f_n + \varepsilon_n,$$

so that the upper and lower integrals of f (see Definition 6.2) satisfy

$$(25) \qquad \int_a^b (f_n - \varepsilon_n) \, d\alpha \leq \underline{\int} f \, d\alpha \leq \overline{\int} f \, d\alpha \leq \int_a^b (f_n + \varepsilon_n) \, d\alpha.$$

Hence

$$0 \leq \overline{\int} f \, d\alpha - \underline{\int} f \, d\alpha \leq 2\varepsilon_n [\alpha(b) - \alpha(a)].$$

Since $\varepsilon_n \to 0$ as $n \to \infty$ (Theorem 7.9), the upper and lower integrals of f are equal.

Thus $f \in \mathscr{R}(\alpha)$. Another application of (25) now yields

$$(26) \qquad \left| \int_a^b f \, d\alpha - \int_a^b f_n \, d\alpha \right| \le \varepsilon_n[\alpha(b) - \alpha(a)].$$

This implies (23).

Corollary If $f_n \in \mathscr{R}(\alpha)$ on $[a, b]$ and if

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \qquad (a \le x \le b),$$

the series converging uniformly on $[a, b]$, then

$$\int_a^b f \, d\alpha = \sum_{n=1}^{\infty} \int_a^b f_n \, d\alpha.$$

In other words, the series may be integrated term by term.

UNIFORM CONVERGENCE AND DIFFERENTIATION

We have already seen, in Example 7.5, that uniform convergence of $\{f_n\}$ implies nothing about the sequence $\{f_n'\}$. Thus stronger hypotheses are required for the assertion that $f_n' \to f'$ if $f_n \to f$.

7.17 Theorem *Suppose $\{f_n\}$ is a sequence of functions, differentiable on $[a, b]$ and such that $\{f_n(x_0)\}$ converges for some point x_0 on $[a, b]$. If $\{f_n'\}$ converges uniformly on $[a, b]$, then $\{f_n\}$ converges uniformly on $[a, b]$, to a function f, and*

$$(27) \qquad f'(x) = \lim_{n \to \infty} f_n'(x) \qquad (a \le x \le b).$$

Proof Let $\varepsilon > 0$ be given. Choose N such that $n \ge N$, $m \ge N$, implies

$$(28) \qquad |f_n(x_0) - f_m(x_0)| < \frac{\varepsilon}{2}$$

and

$$(29) \qquad |f_n'(t) - f_m'(t)| < \frac{\varepsilon}{2(b - a)} \qquad (a \le t \le b).$$

If we apply the mean value theorem 5.19 to the function $f_n - f_m$, (29) shows that

(30)
$$|f_n(x) - f_m(x) - f_n(t) + f_m(t)| \leq \frac{|x - t|\varepsilon}{2(b - a)} \leq \frac{\varepsilon}{2}$$

for any x and t on $[a, b]$, if $n \geq N, m \geq N$. The inequality

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f_m(x) - f_n(x_0) + f_m(x_0)| + |f_n(x_0) - f_m(x_0)|$$

implies, by (28) and (30), that

$$|f_n(x) - f_m(x)| < \varepsilon \qquad (a \leq x \leq b, n \geq N, m \geq N),$$

so that $\{f_n\}$ converges uniformly on $[a, b]$. Let

$$f(x) = \lim_{n \to \infty} f_n(x) \qquad (a \leq x \leq b).$$

Let us now fix a point x on $[a, b]$ and define

(31)
$$\phi_n(t) = \frac{f_n(t) - f_n(x)}{t - x}, \qquad \phi(t) = \frac{f(t) - f(x)}{t - x}$$

for $a \leq t \leq b, t \neq x$. Then

(32)
$$\lim_{t \to x} \phi_n(t) = f_n'(x) \qquad (n = 1, 2, 3, \ldots).$$

The first inequality in (30) shows that

$$|\phi_n(t) - \phi_m(t)| \leq \frac{\varepsilon}{2(b - a)} \qquad (n \geq N, m \geq N),$$

so that $\{\phi_n\}$ converges uniformly, for $t \neq x$. Since $\{f_n\}$ converges to f, we conclude from (31) that

(33)
$$\lim_{n \to \infty} \phi_n(t) = \phi(t)$$

uniformly for $a \leq t \leq b, t \neq x$.

If we now apply Theorem 7.11 to $\{\phi_n\}$, (32) and (33) show that

$$\lim_{t \to x} \phi(t) = \lim_{n \to \infty} f_n'(x);$$

and this is (27), by the definition of $\phi(t)$.

Remark: If the continuity of the functions f_n' is assumed in addition to the above hypotheses, then a much shorter proof of (27) can be based on Theorem 7.16 and the fundamental theorem of calculus.

7.18   Theorem   *There exists a real continuous function on the real line which is nowhere differentiable.*

Proof   Define

(34)
$$\varphi(x) = |x| \qquad (-1 \leq x \leq 1)$$

and extend the definition of $\varphi(x)$ to all real x by requiring that

(35)
$$\varphi(x + 2) = \varphi(x).$$

Then, for all s and t,

(36)
$$|\varphi(s) - \varphi(t)| \leq |s - t|.$$

In particular, φ is continuous on R^1. Define

(37)
$$f(x) = \sum_{n=0}^{\infty} (\tfrac{3}{4})^n \varphi(4^n x).$$

Since $0 \leq \varphi \leq 1$, Theorem 7.10 shows that the series (37) converges uniformly on R^1. By Theorem 7.12, f is continuous on R^1.

Now fix a real number x and a positive integer m. Put

(38)
$$\delta_m = \pm \tfrac{1}{2} \cdot 4^{-m}$$

where the sign is so chosen that no integer lies between $4^m x$ and $4^m(x + \delta_m)$. This can be done, since $4^m |\delta_m| = \tfrac{1}{2}$. Define

(39)
$$\gamma_n = \frac{\varphi(4^n(x + \delta_m)) - \varphi(4^n x)}{\delta_m}.$$

When $n > m$, then $4^n \delta_m$ is an even integer, so that $\gamma_n = 0$. When $0 \leq n \leq m$, (36) implies that $|\gamma_n| \leq 4^n$.

Since $|\gamma_m| = 4^m$, we conclude that

$$\left| \frac{f(x + \delta_m) - f(x)}{\delta_m} \right| = \left| \sum_{n=0}^{m} \left(\frac{3}{4}\right)^n \gamma_n \right|$$

$$\geq 3^m - \sum_{n=0}^{m-1} 3^n$$

$$= \tfrac{1}{2}(3^m + 1).$$

As $m \to \infty$, $\delta_m \to 0$. It follows that f is not differentiable at x.

EQUICONTINUOUS FAMILIES OF FUNCTIONS

In Theorem 3.6 we saw that every bounded sequence of complex numbers contains a convergent subsequence, and the question arises whether something similar is true for sequences of functions. To make the question more precise, we shall define two kinds of boundedness.

7.19 Definition Let $\{f_n\}$ be a sequence of functions defined on a set E.

We say that $\{f_n\}$ is *pointwise bounded* on E if the sequence $\{f_n(x)\}$ is bounded for every $x \in E$, that is, if there exists a finite-valued function ϕ defined on E such that

$$|f_n(x)| < \phi(x) \qquad (x \in E, n = 1, 2, 3, \ldots).$$

We say that $\{f_n\}$ is *uniformly bounded* on E if there exists a number M such that

$$|f_n(x)| < M \qquad (x \in E, n = 1, 2, 3, \ldots).$$

Now if $\{f_n\}$ is pointwise bounded on E and E_1 is a countable subset of E, it is always possible to find a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}(x)\}$ converges for every $x \in E_1$. This can be done by the diagonal process which is used in the proof of Theorem 7.23.

However, even if $\{f_n\}$ is a uniformly bounded sequence of continuous functions on a compact set E, there need not exist a subsequence which converges pointwise on E. In the following example, this would be quite troublesome to prove with the equipment which we have at hand so far, but the proof is quite simple if we appeal to a theorem from Chap. 11.

7.20 Example Let

$$f_n(x) = \sin nx \qquad (0 \le x \le 2\pi, n = 1, 2, 3, \ldots).$$

Suppose there exists a sequence $\{n_k\}$ such that $\{\sin n_k x\}$ converges, for every $x \in [0, 2\pi]$. In that case we must have

$$\lim_{k \to \infty} (\sin n_k x - \sin n_{k+1} x) = 0 \qquad (0 \le x \le 2\pi);$$

hence

(40) $$\lim_{k \to \infty} (\sin n_k x - \sin n_{k+1} x)^2 = 0 \qquad (0 \le x \le 2\pi).$$

By Lebesgue's theorem concerning integration of boundedly convergent sequences (Theorem 11.32), (40) implies

(41) $$\lim_{k \to \infty} \int_0^{2\pi} (\sin n_k x - \sin n_{k+1} x)^2 \, dx = 0.$$

But a simple calculation shows that

$$\int_0^{2\pi} (\sin n_k x - \sin n_{k+1} x)^2 \, dx = 2\pi,$$

which contradicts (41).