x_0 to simultaneously be an interior and an exterior point of E. If x_0 is a boundary point of E, then it could be an element of E, but it could also not lie in E; we give some examples below.

Example 12.2.7. We work on the real line $\mathbf{R}$ with the standard metric d. Let E be the half-open interval $E = [1, 2)$. The point 1.5 is an interior point of E, since one can find a ball (for instance $B(1.5, 0.1)$) centered at 1.5 which lies in E. The point 3 is an exterior point of E, since one can find a ball (for instance $B(3, 0.1)$) centered at 3 which is disjoint from E. The points 1 and 2 however, are neither interior points nor exterior points of E, and are thus boundary points of E. Thus in this case $int(E) = (1, 2)$, $ext(E) = (-\infty, 1) \cup (2, \infty)$, and $\partial E = \{1, 2\}$. Note that in this case one of the boundary points is an element of E, while the other is not.

Example 12.2.8. When we give a set X the discrete metric d_{disc}, and E is any subset of X, then every element of E is an interior point of E, every point not contained in E is an exterior point of E, and there are no boundary points; see Exercise 12.2.1.

Definition 12.2.9 (Closure). Let (X, d) be a metric space, let E be a subset of X, and let x_0 be a point in X. We say that x_0 is an *adherent point* of E if for every radius $r > 0$, the ball $B(x_0, r)$ has a non-empty intersection with E. The set of all adherent points of E is called the *closure* of E and is denoted $\overline{E}$.

Note that these notions are consistent with the corresponding notions on the real line defined in Definitions 9.1.8, 9.1.10 (why?).

The following proposition links the notions of adherent point with interior and boundary point, and also to that of convergence.

Proposition 12.2.10. *Let (X, d) be a metric space, let E be a subset of X, and let x_0 be a point in X. Then the following statements are logically equivalent.*

(a) *x_0 is an adherent point of E.*

(b) *x_0 is either an interior point or a boundary point of E.*

(c) *There exists a sequence* $(x_n)_{n=1}^{\infty}$ *in* E *which converges to* x_0 *with respect to the metric* d.

Proof. See Exercise 12.2.2. □

From the equivalence of Proposition 12.2.10(a) and (b) we obtain an immediate corollary:

Corollary 12.2.11. *Let* (X, d) *be a metric space, and let* E *be a subset of* X. *Then* $\overline{E} = \text{int}(E) \cup \partial E = X \backslash \text{ext}(E)$.

As remarked earlier, the boundary of a set E may or may not lie in E. Depending on how the boundary is situated, we may call a set open, closed, or neither:

Definition 12.2.12 (Open and closed sets). Let (X, d) be a metric space, and let E be a subset of X. We say that E is *closed* if it contains all of its boundary points, i.e., $\partial E \subseteq E$. We say that E is *open* if it contains none of its boundary points, i.e., $\partial E \cap E = \emptyset$. If E contains some of its boundary points but not others, then it is neither open nor closed.

Example 12.2.13. We work in the real line $\mathbf{R}$ with the standard metric d. The set $(1, 2)$ does not contain either of its boundary points 1, 2 and is hence open. The set $[1, 2]$ contains both of its boundary points 1, 2 and is hence closed. The set $[1, 2)$ contains one of its boundary points 1, but does not contain the other boundary point 2, but not the other, so is neither open nor closed.

Remark 12.2.14. It is possible for a set to be simultaneously open and closed, if it has no boundary. For instance, in a metric space (X, d), the whole space X has no boundary (every point in X is an interior point - why?), and so X is both open and closed. The empty set $\emptyset$ also has no boundary (every point in X is an exterior point - why?), and so is both open and closed. In many cases these are the only sets that are simultaneously open and closed, but there are exceptions. For instance, using the discrete metric d_{disc}, *every* set is both open and closed! (why?)

From the above two remarks we see that the notions of being open and being closed are *not* negations of each other; there are sets that are both open and closed, and there are sets which are neither open and closed. Thus, if one knew for instance that E was not an open set, it would be erroneous to conclude from this that E was a closed set, and similarly with the rôles of open and closed reversed. The correct relationship between open and closed sets is given by Proposition 12.2.15(e) below.

Now we list some more properties of open and closed sets.

Proposition 12.2.15 (Basic properties of open and closed sets). *Let (X, d) be a metric space.*

(a) *Let E be a subset of X. Then E is open if and only if $E = \operatorname{int}(E)$. In other words, E is open if and only if for every $x \in E$, there exists an $r > 0$ such that $B(x, r) \subseteq E$.*

(b) *Let E be a subset of X. Then E is closed if and only if E contains all its adherent points. In other words, E is closed if and only if for every convergent sequence $(x_n)_{n=m}^{\infty}$ in E, the limit $\lim_{n\to\infty} x_n$ of that sequence also lies in E.*

(c) *For any $x_0 \in X$ and $r > 0$, then the ball $B(x_0, r)$ is an open set. The set $\{x \in X : d(x, x_0) \leq r\}$ is a closed set. (This set is sometimes called the closed ball of radius r centered at x_0.)*

(d) *Any singleton set $\{x_0\}$, where $x_0 \in X$, is automatically closed.*

(e) *If E is a subset of X, then E is open if and only if the complement $X \backslash E := \{x \in X : x \notin E\}$ is closed.*

(f) *If $E_1, \ldots, E_n$ are a finite collection of open sets in X, then $E_1 \cap E_2 \cap \ldots \cap E_n$ is also open. If $F_1, \ldots, F_n$ is a finite collection of closed sets in X, then $F_1 \cup F_2 \cup \ldots \cup F_n$ is also closed.*

(g) *If* $\{E_\alpha\}_{\alpha \in I}$ *is a collection of open sets in* X *(where the index set* I *could be finite, countable, or uncountable), then the union* $\bigcup_{\alpha \in I} E_\alpha := \{x \in X : x \in E_\alpha$ *for some* $\alpha \in A\}$ *is also open. If* $\{F_\alpha\}_{\alpha \in I}$ *is a collection of closed sets in* X*, then the intersection* $\bigcap_{\alpha \in I} F_\alpha := \{x \in X : x \in F_\alpha$ *for all* $\alpha \in I\}$ *is also closed.*

(h) *If* E *is any subset of* X*, then* $\mathrm{int}(E)$ *is the largest open set which is contained in* E*; in other words,* $\mathrm{int}(E)$ *is open, and given any other open set* $V \subseteq E$*, we have* $V \subseteq int(E)$*. Similarly* $\overline{E}$ *is the smallest closed set which contains* E*; in other words,* $\overline{E}$ *is closed, and given any other closed set* $K \supset E$*,* $K \supset \overline{E}$*.*

Proof. See Exercise 12.2.3. □

Exercise 12.2.1. Verify the claims in Example 12.2.8.

Exercise 12.2.2. Prove Proposition 12.2.10. (Hint: for some of the implications one will need the axiom of choice, as in Lemma 8.4.5.)

Exercise 12.2.3. Prove Proposition 12.2.15. (Hint: you can use earlier parts of the proposition to prove later ones.)

Exercise 12.2.4. Let (X, d) be a metric space, x_0 be a point in X, and $r > 0$. Let B be the open ball $B := B(x_0, r) = \{x \in X : d(x, x_0) < r\}$, and let C be the closed ball $C := \{x \in X : d(x, x_0) \leq r\}$.

(a) Show that $\overline{B} \subseteq C$.

(b) Give an example of a metric space (X, d), a point x_0, and a radius $r > 0$ such that $\overline{B}$ is *not* equal to C.

12.3 Relative topology

When we defined notions such as open and closed sets, we mentioned that such concepts depended on the choice of metric one uses. For instance, on the real line $\mathbf{R}$, if one uses the usual metric $d(x, y) = |x - y|$, then the set $\{1\}$ is not open, however if instead one uses the discrete metric d_{disc}, then $\{1\}$ is now an open set (why?).

However, it is not just the choice of metric which determines what is open and what is not - it is also the choice of *ambient space* X. Here are some examples.

Example 12.3.1. Consider the plane $\mathbf{R}^2$ with the Euclidean metric d_{l^2}. Inside the plane, we can find the x-axis $X := \{(x, 0) : x \in \mathbf{R}\}$. The metric d_{l^2} can be restricted to X, creating a subspace $(X, d_{l^2}|_{X \times X})$ of $(\mathbf{R}^2, d_{l^2})$. (This subspace is essentially the same as the real line $(\mathbf{R}, d)$ with the usual metric; the precise way of stating this is that $(X, d_{l^2}|_{X \times X})$ is *isometric* to $(\mathbf{R}, d)$. We will not pursue this concept further in this text, however.) Now consider the set

$$E := \{(x, 0) : -1 < x < 1\}$$

which is both a subset of X and of $\mathbf{R}^2$. Viewed as a subset of $\mathbf{R}^2$, it is not open, because the point 0, for instance, lies in E but is not an interior point of E. (Any ball $B_{\mathbf{R}^2, d_{l^2}}(0, r)$ will contain at least one point that lies outside of the x-axis, and hence outside of E. On the other hand, if viewed as a subset of X, it is open; every point of E is an interior point of E *with respect to the metric space* $(X, d_{l^2}|_{X \times X})$. For instance, the point 0 is now an interior point of E, because the ball $B_{X, d_{l^2}|_{X \times X}}(0, 1)$ is contained in E (in fact, in this case it *is* E.)

Example 12.3.2. Consider the real line $\mathbf{R}$ with the standard metric d, and let X be the interval $X := (-1, 1)$ contained inside $\mathbf{R}$; we can then restrict the metric d to X, creating a subspace $(X, d|_{X \times X})$ of $(\mathbf{R}, d)$. Now consider the set $[0, 1)$. This set is not closed in $\mathbf{R}$, because the point 1 is adherent to $[0, 1)$ but is not contained in $[0, 1)$. However, when considered as a subset of X, the set $[0, 1)$ now becomes closed; the point 1 is not an element of X and so is no longer considered an adherent point of $[0, 1)$, and so now $[0, 1)$ contains all of its adherent points.

To clarify this distinction, we make a definition.

Definition 12.3.3 (Relative topology). Let (X, d) be a metric space, let Y be a subset of X, and let E be a subset of Y. We say that E is *relatively open with respect to* Y if it is open in the metric subspace $(Y, d|_{Y\times Y})$. Similarly, we say that E is *relatively closed with respect to* Y if it is closed in the metric space $(Y, d|_{Y\times Y})$.

The relationship between open (or closed) sets in X, and relatively open (or relatively closed) sets in Y, is the following.

Proposition 12.3.4. *Let (X, d) be a metric space, let Y be a subset of X, and let E be a subset of Y.*

(a) *E is relatively open with respect to Y if and only if $E = V \cap Y$ for some set $V \subseteq X$ which is open in X.*

(b) *E is relatively closed with respect to Y if and only if $E = K \cap Y$ for some set $K \subseteq X$ which is closed in X.*

Proof. We just prove (a), and leave (b) to Exercise 12.3.1. First suppose that E is relatively open with respect to Y. Then, E is open in the metric space $(Y, d|_{Y\times Y})$. Thus, for every $x \in E$, there exists a radius $r > 0$ such that the ball $B_{(Y,d|_{Y\times Y})}(x, r)$ is contained in E. This radius r depends on x; to emphasize this we write r_x instead of r, thus for every $x \in E$ the ball $B_{(Y,d|_{Y\times Y})}(x, r_x)$ is contained in E. (Note that we have used the axiom of choice, Proposition 8.4.7, to do this.)

Now consider the set

$$V := \bigcup_{x\in E} B_{(X,d)}(x, r_x).$$

This is a subset of X. By Proposition 12.2.15(c) and (g), V is open. Now we prove that $E = V \cap Y$. Certainly any point x in E lies in $V \cap Y$, since it lies in Y and it also lies in $B_{(X,d)}(x, r_x)$, and hence in V. Now suppose that y is a point in $V \cap Y$. Then $y \in V$, which implies that there exists an $x \in E$ such that $y \in B_{(X,d)}(x, r_x)$. But since y is also in Y, this implies that $y \in B_{(Y,d|_{Y\times Y})}(x, r_x)$. But by definition of r_x, this means that $y \in E$, as desired. Thus we have found an open set V for which $E = V \cap Y$ as desired.

Now we do the converse. Suppose that $E = V \cap Y$ for some open set V; we have to show that E is relatively open with respect to Y. Let x be any point in E; we have to show that x is an interior point of E in the metric space $(Y, d|_{Y\times Y})$. Since $x \in E$, we know $x \in V$. Since V is open in X, we know that there is a radius $r > 0$ such that $B_{(X,d)}(x, r)$ is contained in V. Strictly speaking, r depends on x, and so we could write r_x instead of r, but for this argument we will only use a single choice of x (as opposed to the argument in the previous paragraph) and so we will not bother to subscript r here. Since $E = V \cap Y$, this means that $B_{(X,d)}(x, r) \cap Y$ is contained in E. But $B_{(X,d)}(x, r) \cap Y$ is exactly the same as $B_{(Y,d|_{Y\times Y})}(x, r)$ (why?), and so $B_{(Y,d|_{Y\times Y})}(x, r)$ is contained in E. Thus x is an interior point of E in the metric space $(Y, d|_{Y\times Y})$, as desired. □

Exercise 12.3.1. Prove Proposition 12.3.4(b).

12.4 Cauchy sequences and complete metric spaces

We now generalize much of the theory of limits of sequences from Chapter 6 to the setting of general metric spaces. We begin by generalizing the notion of a *subsequence* from Definition 6.6.1:

Definition 12.4.1 (Subsequences). Suppose that $(x^{(n)})_{n=m}^{\infty}$ is a sequence of points in a metric space (X, d). Suppose that $n_1, n_2, n_3, \ldots$ is an increasing sequence of integers which are at least as large as m, thus

$$m \leq n_1 < n_2 < n_3 < \ldots.$$

Then we call the sequence $(x^{(n_j)})_{j=1}^{\infty}$ a *subsequence* of the original sequence $(x^{(n)})_{n=m}^{\infty}$.

Examples 12.4.2. the sequence $((\frac{1}{j^2}, \frac{1}{j^2}))_{j=1}^{\infty}$ in $\mathbf{R}^2$ is a subsequence of the sequence $((\frac{1}{n}, \frac{1}{n}))_{n=1}^{\infty}$ (in this case, $n_j := j^2$). The sequence $1, 1, 1, 1, \ldots$ is a subsequence of $1, 0, 1, 0, 1, \ldots$.

If a sequence converges, then so do all of its subsequences:

Lemma 12.4.3. *Let $(x^{(n)})_{n=m}^{\infty}$ be a sequence in (X,d) which converges to some limit x_0. Then every subsequence $(x^{(n_j)})_{j=1}^{\infty}$ of that sequence also converges to x_0.*

Proof. See Exercise 12.4.3. □

On the other hand, it is possible for a subsequence to be convergent without the sequence as a whole being convergent. For example, the sequence $1, 0, 1, 0, 1, \ldots$ is not convergent, even though certain subsequences of it (such as $1, 1, 1, \ldots$) converge. To quantify this phenomenon, we generalize Definition 6.4.1 as follows:

Definition 12.4.4 (Limit points). Suppose that $(x^{(n)})_{n=m}^{\infty}$ is a sequence of points in a metric space (X,d), and let $L \in X$. We say that L is a *limit point* of $(x^{(n)})_{n=m}^{\infty}$ iff for every $N \geq m$ and $\varepsilon > 0$ there exists an $n \geq N$ such that $d(x^{(n)}, L) \leq \varepsilon$.

Proposition 12.4.5. *Let $(x^{(n)})_{n=m}^{\infty}$ be a sequence of points in a metric space (X,d), and let $L \in X$. Then the following are equivalent:*

- *L is a limit point of $(x^{(n)})_{n=m}^{\infty}$.*
- *There exists a subsequence $(x^{(n_j)})_{j=1}^{\infty}$ of the original sequence $(x^{(n)})_{n=m}^{\infty}$ which converges to L.*

Proof. See Exercise 12.4.2. □

Next, we review the notion of a *Cauchy sequence* from Definition 6.1.3 (see also Definition 5.1.8).

Definition 12.4.6 (Cauchy sequences). Let $(x^{(n)})_{n=m}^{\infty}$ be a sequence of points in a metric space (X,d). We say that this sequence is a *Cauchy sequence* iff for every $\varepsilon > 0$, there exists an $N \geq m$ such that $d(x^{(j)}, x^{(k)}) < \varepsilon$ for all $j, k \geq N$.

Lemma 12.4.7 (Convergent sequences are Cauchy sequences). *Let $(x^{(n)})_{n=m}^{\infty}$ be a sequence in (X,d) which converges to some limit x_0. Then $(x^{(n)})_{n=m}^{\infty}$ is also a Cauchy sequence.*

Proof. See Exercise 12.4.3. □

It is also easy to check that subsequence of a Cauchy sequence is also a Cauchy sequence (why)? However, not every Cauchy sequence converges:

Example 12.4.8. (Informal) Consider the sequence

$$3, 3.1, 3.14, 3.141, 3.1415, \ldots$$

in the metric space $(\mathbf{Q}, d)$ (the rationals $\mathbf{Q}$ with the usual metric $d(x, y) := |x - y|$). While this sequence is convergent in $\mathbf{R}$ (it converges to π), it does not converge in $\mathbf{Q}$ (since $\pi \notin \mathbf{Q}$, and a sequence cannot converge to two different limits).

So in certain metric spaces, Cauchy sequences do not necessarily converge. However, if even part of a Cauchy sequence converges, then the entire Cauchy sequence must converge (to the same limit):

Lemma 12.4.9. *Let $(x^{(n)})_{n=m}^{\infty}$ be a Cauchy sequence in (X, d). Suppose that there is some subsequence $(x^{(n_j)})_{j=1}^{\infty}$ of this sequence which converges to a limit x_0 in X. Then the original sequence $(x^{(n)})_{n=m}^{\infty}$ also converges to x_0.*

Proof. See Exercise 12.4.4. □

In Example 12.4.8 we saw an example of a metric space which contained Cauchy sequences which did not converge. However, in Theorem 6.4.18 we saw that in the metric space $(\mathbf{R}, d)$, every Cauchy sequence did have a limit. This motivates the following definition.

Definition 12.4.10 (Complete metric spaces). A metric space (X, d) is said to be *complete* iff every Cauchy sequence in (X, d) is in fact convergent in (X, d).

Example 12.4.11. By Theorem 6.4.18, the reals $(\mathbf{R}, d)$ are complete; by Example 12.4.8, the rationals $(\mathbf{Q}, d)$, on the other hand, are not complete.

Complete metric spaces have some nice properties. For instance, they are *intrinsically closed*: no matter what space one places them in, they are always closed sets. More precisely:

Proposition 12.4.12. (a) *Let (X, d) be a metric space, and let $(Y, d|_{Y\times Y})$ be a subspace of (X, d). If $(Y, d|_{Y\times Y})$ is complete, then Y must be closed in X.*

(b) *Conversely, suppose that (X, d) is a complete metric space, and Y is a closed subset of X. Then the subspace $(Y, d|_{Y\times Y})$ is also complete.*

Proof. See Exercise 12.4.7. □

In contrast, an incomplete metric space such as $(\mathbf{Q}, d)$ may be considered closed in some spaces (for instance, $\mathbf{Q}$ is closed in $\mathbf{Q}$) but not in others (for instance, $\mathbf{Q}$ is not closed in $\mathbf{R}$). Indeed, it turns out that given any incomplete metric space (X, d), there exists a *completion* $(\overline{X}, \overline{d})$, which is a larger metric space containing (X, d) which is complete, and such that X is not closed in $\overline{X}$ (indeed, the closure of X in $(\overline{X}, \overline{d})$ will be all of $\overline{X}$); see Exercise 12.4.8. For instance, one possible completion of $\mathbf{Q}$ is $\mathbf{R}$.

Exercise 12.4.1. Prove Lemma 12.4.3. (Hint: review your proof of Proposition 6.6.5.)

Exercise 12.4.2. Prove Proposition 12.4.5. (Hint: review your proof of Proposition 6.6.6.)

Exercise 12.4.3. Prove Lemma 12.4.7. (Hint: review your proof of Proposition 6.1.12.)

Exercise 12.4.4. Prove Lemma 12.4.9.

Exercise 12.4.5. Let $(x^{(n)})_{n=m}^{\infty}$ be a sequence of points in a metric space (X, d), and let $L \in X$. Show that if L is a limit point of the sequence $(x^{(n)})_{n=m}^{\infty}$, then L is an adherent point of the set $\{x^{(n)} : n \geq m\}$. Is the converse true?

Exercise 12.4.6. Show that every Cauchy sequence can have at most one limit point.

Exercise 12.4.7. Prove Proposition 12.4.12.

Exercise 12.4.8. The following construction generalizes the construction of the reals from the rationals in Chapter 5, allowing one to view any metric space as a subspace of a complete metric space. In what follows we let (X, d) be a metric space.

(a) Given any Cauchy sequence $(x_n)_{n=1}^{\infty}$ in X, we introduce the *formal limit* $\mathrm{LIM}_{n\to\infty} x_n$. We say that two formal limits $\mathrm{LIM}_{n\to\infty} x_n$ and $\mathrm{LIM}_{n\to\infty} y_n$ are equal if $\lim_{n\to\infty} d(x_n, y_n)$ is equal to zero. Show that this equality relation obeys the reflexive, symmetry, and transitive axioms.

(b) Let $\overline{X}$ be the space of all formal limits of Cauchy sequences in X, with the above equality relation. Define a metric $d_{\overline{X}} : \overline{X} \times \overline{X} \to \mathbf{R}^+$ by setting

$$d_{\overline{X}}(\mathrm{LIM}_{n\to\infty} x_n, \mathrm{LIM}_{n\to\infty} y_n) := \lim_{n\to\infty} d(x_n, y_n).$$

Show that this function is well-defined (this means not only that the limit $\lim_{n\to\infty} d(x_n, y_n)$ exists, but also that the axiom of substitution is obeyed; cf. Lemma 5.3.7), and gives $\overline{X}$ the structure of a metric space.

(c) Show that the metric space $(\overline{X}, d_{\overline{X}})$ is complete.

(d) We identify an element $x \in X$ with the corresponding formal limit $\mathrm{LIM}_{n\to\infty} x$ in $\overline{X}$; show that this is legitimate by verifying that $x = y \iff \mathrm{LIM}_{n\to\infty} x = \mathrm{LIM}_{n\to\infty} y$. With this identification, show that $d(x, y) = d_{\overline{X}}(x, y)$, and thus (X, d) can now be thought of as a subspace of $(\overline{X}, d_{\overline{X}})$.

(e) Show that the closure of X in $\overline{X}$ is $\overline{X}$ (which explains the choice of notation $\overline{X}$).

(f) Show that the formal limit agrees with the actual limit, thus if $(x_n)_{n=1}^{\infty}$ is any Cauchy sequence in X, then we have $\lim_{n\to\infty} x_n = \mathrm{LIM}_{n\to\infty} x_n$ in $\overline{X}$.

12.5 Compact metric spaces

We now come to one of the most useful notions in point set topology, that of *compactness*. Recall the Heine-Borel theorem (Theorem 9.1.24), which asserted that every sequence in a closed and bounded subset X of the real line $\mathbf{R}$ had a convergent subsequence whose limit was also in X. Conversely, only the closed and bounded sets have this property. This property turns out to be so useful that we give it a name.

Definition 12.5.1 (Compactness). A metric space (X, d) is said to be *compact* iff every sequence in (X, d) has at least one convergent subsequence. A subset Y of a metric space X is said to be *compact* if the subspace $(Y, d|_{Y\times Y})$ is compact.

Remark 12.5.2. The notion of a set Y being compact is *intrinsic*, in the sense that it only depends on the metric function $d|_{Y\times Y}$ restricted to Y, and not on the choice of the ambient space X. The notions of completeness in Definition 12.4.10, and of boundedness below in Definition 12.5.3, are also intrinsic, but the notions of open and closed are not (see the discussion in Section 12.3).

Thus, Theorem 9.1.24 shows that in the real line $\mathbf{R}$ with the usual metric, every closed and bounded set is compact, and conversely every compact set is closed and bounded.

Now we investigate how the Heine-Borel extends to other metric spaces.

Definition 12.5.3 (Bounded sets). Let (X, d) be a metric space, and let Y be a subset of X. We say that Y is *bounded* iff there exists a ball $B(x, r)$ in X which contains Y.

Remark 12.5.4. This definition is compatible with the definition of a bounded set in Definition 9.1.22 (Exercise 12.5.1).

Proposition 12.5.5. *Let (X, d) be a compact metric space. Then (X, d) is both complete and bounded.*

Proof. See Exercise 12.5.2. □

From this proposition and Proposition 12.4.12(a) we obtain one half of the Heine-Borel theorem for general metric spaces:

Corollary 12.5.6 (Compact sets are closed and bounded). *Let (X, d) be a metric space, and let Y be a compact subset of X. Then Y is closed and bounded.*

The other half of the Heine-Borel theorem is true in Euclidean spaces:

Theorem 12.5.7 (Heine-Borel theorem). *Let $(\mathbf{R}^n, d)$ be a Euclidean space with either the Euclidean metric, the taxicab metric, or the sup norm metric. Let E be a subset of $\mathbf{R}^n$. Then E is compact if and only if it is closed and bounded.*

Proof. See Exercise 12.5.3. □

However, the Heine-Borel theorem is not true for more general metrics. For instance, the integers $\mathbf{Z}$ with the discrete metric is closed (indeed, it is complete) and bounded, but not compact, since the sequence $1, 2, 3, 4, \ldots$ is in $\mathbf{Z}$ but has no convergent subsequence (why?). Another example is in Exercise 12.5.8. However, a version of the Heine-Borel theorem is available if one is willing to replace closedness with the stronger notion of completeness, and boundedness with the stronger notion of *total boundedness*; see Exercise 12.5.10.

One can characterize compactness topologically via the following, rather strange-sounding statement: every open cover of a compact set has a finite subcover.

Theorem 12.5.8. *Let (X, d) be a metric space, and let Y be a compact subset of X. Let $(V_\alpha)_{\alpha \in I}$ be a collection of open sets in X, and suppose that*

$$Y \subseteq \bigcup_{\alpha \in I} V_\alpha.$$

(i.e., the collection $(V_\alpha)_{\alpha \in I}$ covers Y). Then there exists a finite subset F of I such that

$$Y \subseteq \bigcup_{\alpha \in F} V_\alpha.$$

Proof. We assume for sake of contradiction that there does not exist any finite subset F of A for which $Y \subset \bigcup_{\alpha \in F} V_\alpha$.

Let y be any element of Y. Then y must lie in at least one of the sets V_α. Since each V_α is open, there must therefore be an $r > 0$ such that $B_{(X,d)}(y,r) \subseteq V_\alpha$. Now let $r(y)$ denote the quantity

$$r(y) := \sup\{r \in (0,\infty) : B_{(X,d)}(y,r) \subseteq V_\alpha \text{ for some } \alpha \in A\}.$$

By the above discussion, we know that $r(y) > 0$ for all $y \in Y$. Now, let r_0 denote the quantity

$$r_0 := \inf\{r(y) : y \in Y\}.$$

Since $r(y) > 0$ for all $y \in Y$, we have $r_0 \geq 0$. There are two cases: $r_0 = 0$ and $r_0 > 0$.

- **Case 1:** $r_0 = 0$. Then for every integer $n \geq 1$, there is at least one point y in Y such that $r(y) < 1/n$ (why?). We thus choose, for each $n \geq 1$, a point $y^{(n)}$ in Y such that $r(y^{(n)}) < 1/n$ (we can do this because of the axiom of choice, see Proposition 8.4.7). In particular we have $\lim_{n\to\infty} r(y^{(n)}) = 0$, by the squeeze test. The sequence $(y^{(n)})_{n=1}^\infty$ is a sequence in Y; since Y is compact, we can thus find a subsequence $(y^{(n_j)})_{j=1}^\infty$ which converges to a point $y_0 \in Y$.

  As before, we know that there exists some $\alpha \in I$ such that $y_0 \in V_\alpha$, and hence (since V_α is open) there exists some $\varepsilon > 0$ such that $B(y_0, \varepsilon) \subseteq V_\alpha$. Since $y^{(n)}$ converges to y_0, there must exist an $N \geq 1$ such that $y^{(n)} \in B(y_0, \varepsilon/2)$ for all $n \geq N$. In particular, by the triangle inequality we have $B(y^{(n)}, \varepsilon/2) \subseteq B(y_0, \varepsilon)$, and thus $B(y^{(n)}, \varepsilon/2) \subseteq V_\alpha$. By definition of $r(y^{(n)})$, this implies that $r(y^{(n)}) \geq \varepsilon/2$ for all $n \geq N$. But this contradicts the fact that $\lim_{n\to\infty} r(y^{(n)}) = 0$.

- **Case 2:** $r_0 > 0$. In this case we now have $r(y) > r_0/2$ for all $y \in Y$. This implies that for every $y \in Y$ there exists an $\alpha \in A$ such that $B(y, r_0/2) \in V_\alpha$ (why?).

  We now construct a sequence $y^{(1)}, y^{(2)}, \ldots$ by the following recursive procedure. We let $y^{(1)}$ be any point in Y. The ball $B(y^{(1)}, r_0/2)$ is contained in one of the V_α and thus cannot cover all of Y, since we would then obtain a finite cover, a contradiction. Thus there exists a point $y^{(2)}$ which does not lie in $B(y^{(1)}, r_0/2)$, so in particular $d(y^{(2)}, y^{(1)}) \geq r_0/2$. Choose such a point $y^{(2)}$. The set $B(y^{(1)}, r_0/2) \cup B(y^{(2)}, r_0/2)$ cannot cover all of Y, since we would then obtain two sets V_{α_1} and V_{α_2} which covered Y, a contradiction again. So we can choose a point $y^{(3)}$ which does not lie in $B(y^{(1)}, r_0/2) \cup B(y^{(2)}, r_0/2)$, so in particular $d(y^{(3)}, y^{(1)}) \geq r_0/2$ and $d(y^{(3)}, y^{(2)}) \geq r_0/2$. Continuing in this fashion we obtain a sequence $(y^{(n)})_{n=1}^\infty$ in Y with the property that $d(y^{(k)}, y^{(j)}) \geq r_0/2$ for all $k \geq j$. In particular the sequence $(y^{(n)})_{n=1}^\infty$ is not a Cauchy sequence, and in fact no subsequence of $(y^{(n)})_{n=1}^\infty$ can be a Cauchy sequence either. But this contradicts the assumption that Y is compact (by Lemma 12.4.7).

□

It turns out that Theorem 12.5.8 has a converse: if Y has the property that every open cover has a finite sub-cover, then it is compact (Exercise 12.5.11). In fact, this property is often considered the more fundamental notion of compactness than the sequence-based one. (For metric spaces, the two notions, that of compactness and sequential compactness, are equivalent, but for more general *topological spaces*, the two notions are slightly different; see Exercise 13.5.8.)

Theorem 12.5.8 has an important corollary: that every nested sequence of non-empty compact sets is still non-empty.

Corollary 12.5.9. *Let (X, d) be a metric space, and let K_1, K_2, K_3 be a sequence of non-empty compact subsets of X such that*

$$K_1 \supset K_2 \supset K_3 \supset \dots.$$

Then the intersection $\bigcap_{n=1}^\infty K_n$ is non-empty.

Proof. See Exercise 12.5.6. □