Suppose now that ϕ has length greater than 1 and that every combinator ψ shorter than ϕ is such that $\lambda_x\psi$ is equal to a combinator. We claim that $\lambda_x\phi$ is equal to a combinator. Indeed, $\phi = \psi`\chi$, where ψ and χ are shorter than ϕ, hence

$$\lambda_x\phi = \lambda_x(\psi`\chi) = (S`(\lambda_x\psi))`(\lambda_x\chi).$$

This is a combinator because $\lambda_x\psi$ and $\lambda_x\chi$ are combinators by our inductional assumption.

Using the methods of this proof, we can express 2 as a combinator:

$$\begin{array}{rcll} 2 & = & \lambda_f\lambda_x(f`(f`x)) & \\ & = & \lambda_f((S`\lambda_x f)`\lambda_x(f`x)) & \\ & = & \lambda_f((S`\lambda_x f)`f) & \text{by } \mathbf{R2} \\ & = & (S`(\lambda_f(S`\lambda_x f)))`\lambda_f f & \\ & = & (S`(\lambda_f(S`(K`f))))`I & \\ & = & (S`((S`\lambda_f S)`\lambda_f(K`f)))`I & \\ & = & (S`((S`(K`S))`K))`I & \text{by } \mathbf{R2}. \end{array}$$

Exercises

1. Assuming that m, p and q are natural numbers expressed in the lambda calculus, show that $(m^p)^q = m^{(q\times p)}$.
2. Prove that $0^{\Sigma`n} = 0$.
3. Prove that $I = (S`K)`K$.
4. Express m^n, $m \times n$, and $m + n$ in terms of I, K, S, m and n.

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Logic from Aristotle to Russell

Logic was not always regarded as a branch of mathematics, certainly not by Aristotle (384–322 BC), who was the first to write about logic in the West. Among the principles which he recognized are the following:

$$\begin{array}{ll} \neg\neg p \Longleftrightarrow p & \text{(double negation),} \\ p \vee \neg p & \text{(excluded third),} \\ (p \Rightarrow q) \Longleftrightarrow (\neg q \Rightarrow \neg p) & \text{(contraposition).} \end{array}$$

He also looked at modal logic and showed how possibility can be defined in terms of necessity.

Aristotle's major concern was with a type of argument called the 'syllogism', which predominated in logical thinking for the next two thousand years. It dealt with four types of basic statements:

SaP meaning *all* S *are* P,
SeP meaning *no* S *are* P,
SiP meaning *some* S *are* P,
SoP meaning *some* S *are not* P.

He realized that PeS is equivalent to SeP and that PiS is equivalent to SiP and he adhered to a convention that SaP implies SiP. (Today we use words differently: we assert that *all* unicorns have horns, but deny that *some* unicorns have horns. Evidently Aristotle did not believe in the empty set.)

A *syllogism* is an argument which infers one such basic statement from two others. Here are the first four 'figures' of the syllogism:

$$\frac{\begin{array}{c}\text{MaP}\\ \text{SaM}\end{array}}{\text{SaP}} \qquad \frac{\begin{array}{c}\text{MeP}\\ \text{SaM}\end{array}}{\text{SeP}} \qquad \frac{\begin{array}{c}\text{MaP}\\ \text{SiM}\end{array}}{\text{SiP}} \qquad \frac{\begin{array}{c}\text{MeP}\\ \text{SiM}\end{array}}{\text{SoP}}$$

William of Shyreswood (1250 AD) gave these syllogisms the names

$$\text{b}\underline{\text{a}}\text{rb}\underline{\text{a}}\text{r}\underline{\text{a}}, \qquad \text{c}\underline{\text{e}}\text{l}\underline{\text{a}}\text{r}\underline{\text{e}}\text{nt}, \qquad \text{d}\underline{\text{a}}\text{r}\underline{\text{i}}\underline{\text{i}}, \qquad \text{f}\underline{\text{e}}\text{r}\underline{\text{i}}\underline{\text{o}},$$

— to make them easier to remember. There were more such figures, which we shall not discuss here. Here is a typical argument illustrating the 'ferio':

$$\frac{\begin{array}{c}\text{no minister is prudent}\\ \text{some socialists are ministers}\end{array}}{\text{some socialists are not prudent}}$$

The Stoics (200 BC), Philo of Megara in particular, essentially introduced truth tables into logic, thus anticipating Ludwig Wittgenstein (1889–1951). They discussed the problem of whether 'p or q' is true when both p and q are true and whether 'if p then q' is always true when p is false. They arrived at the modern conventions, expressed in the following *truth tables*:

$p \vee q$	$p \Rightarrow q$
T T T	T T T
T T F	T F F
F T T	F T T
F F F	F T F

The Stoics were commited to the view that there are only two truth values (T and F). In particular, they believed that a statement like 'there will be a battle tomorrow' is either true or false, although Aristotle seems to have had some second thoughts about this. This belief was associated in