Examples

(1) For any ring R and any left R-module N we have $R \otimes_R N \cong N$ (so "extending scalars from R to R" does not change the module). This follows by taking φ to be the identity map from N to itself (and $S = R$) in Theorem 8: ι is then an isomorphism with inverse isomorphism given by Φ. In particular, if A is any abelian group (i.e., a $\mathbb{Z}$-module), then $\mathbb{Z} \otimes_{\mathbb{Z}} A = A$.

(2) Let $R = \mathbb{Z}$, $S = \mathbb{Q}$ and let A be a finite abelian group of order n. In this case the $\mathbb{Q}$-module $\mathbb{Q} \otimes_{\mathbb{Z}} A$ obtained by extension of scalars from the $\mathbb{Z}$-module A is 0. To see this, observe first that in any tensor product $1 \otimes 0 = 1 \otimes (0+0) = 1 \otimes 0 + 1 \otimes 0$, by the second relation in (4), so

$$1 \otimes 0 = 0.$$

Now, for any simple tensor $q \otimes a$ we can write the rational number q as $(q/n)n$. Then since $na = 0$ in A by Lagrange's Theorem, we have

$$q \otimes a = (\frac{q}{n} \cdot n) \otimes a = \frac{q}{n} \otimes (na) = (q/n) \otimes 0 = (q/n)(1 \otimes 0) = 0.$$

It follows that $\mathbb{Q} \otimes_{\mathbb{Z}} A = 0$. In particular, the map $\iota : A \to S \otimes_R A$ is the zero map. By Theorem 8, we see again that any homomorphism of a finite abelian group into a rational vector space is the zero map. In particular, if A is nontrivial, then the original $\mathbb{Z}$-module A is not contained in the $\mathbb{Q}$-module obtained by extension of scalars.

(3) *Extension of scalars for free modules:* If $N \cong R^n$ is a free module of rank n over R then $S \otimes_R N \cong S^n$ is a free module of rank n over S. We shall prove this shortly (Corollary 18) when we discuss tensor products of direct sums. For example, $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}^n \cong \mathbb{Q}^n$. In this case the module obtained by extension of scalars contains (an isomorphic copy of) the original R-module N. For example, $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}^n \cong \mathbb{Q}^n$ and $\mathbb{Z}^n$ is a subgroup of the abelian group $\mathbb{Q}^n$.

(4) *Extension of scalars for vector spaces:* As a special case of the previous example, let F be a subfield of the field K and let V be an n-dimensional vector space over F (i.e., $V \cong F^n$). Then $K \otimes_F V \cong K^n$ is a vector space over the larger field K of the same dimension, and the original vector space V is contained in $K \otimes_F V$ as an F-vector subspace.

(5) *Induced modules for finite groups:* Let R be a commutative ring with 1, let G be a finite group and let H be a subgroup of G. As in Section 7.2 we may form the group ring RG and its subring RH. For any RH-module N define the *induced module* $RG \otimes_{RH} N$. In this way we obtain an RG-module for each RH-module N. We shall study properties of induced modules and some of their important applications to group theory in Chapters 17 and 19.

The general tensor product construction follows along the same lines as the extension of scalars above, but before describing it we make two observations from this special case. The first is that the construction of $S \otimes_R N$ as an *abelian group* involved only the elements in equation (3), which in turn only required S to be a *right* R-module and N to be a *left* R-module. In a similar way we shall construct an *abelian group* $M \otimes_R N$ for *any right* R-module M and *any left* R-module N. The second observation is that the S*-module* structure on $S \otimes_R N$ defined by equation (5) required only a *left* S-module structure on S together with a "compatibility relation"

$$s'(sr) = (s's)r \qquad \text{for } s, s' \in S, r \in R,$$

between this left S-module structure and the right R-module structure on S (this was needed in order to deduce that (5) was well defined). We first consider the general construction of $M \otimes_R N$ as an abelian group, after which we shall return to the question of when this abelian group can be given a module structure.

Suppose then that N is a left R-module and that M is a right R-module. The quotient of the free $\mathbb{Z}$-module on the set $M \times N$ by the subgroup generated by all elements of the form

$$\begin{gathered}(m_1+m_2,n)-(m_1,n)-(m_2,n),\\ (m,n_1+n_2)-(m,n_1)-(m,n_2), \text{ and}\\ (mr,n)-(m,rn),\end{gathered} \tag{10.6}$$

for $m, m_1, m_2 \in M, n, n_1, n_2 \in N$ and $r \in R$ is an abelian group, denoted by $M \otimes_R N$, or simply $M \otimes N$ if the ring R is clear from the context, and is called the *tensor product of M and N over R*. The elements of $M \otimes_R N$ are called *tensors*, and the coset, $m \otimes n$, of (m, n) in $M \otimes_R N$ is called a simple tensor. We have the relations

$$\begin{gathered}(m_1+m_2)\otimes n = m_1\otimes n + m_2\otimes n,\\ m\otimes(n_1+n_2) = m\otimes n_1 + m\otimes n_2, \text{ and}\\ mr\otimes n = m\otimes rn.\end{gathered} \tag{10.7}$$

Every tensor can be written (non-uniquely in general) as a finite sum of simple tensors.

Remark: We emphasize that care must be taken when working with tensors, since each $m \otimes n$ represents a *coset* in some quotient group, and so we may have $m \otimes n = m' \otimes n'$ where $m \neq m'$ or $n \neq n'$. More generally, an element of $M \otimes N$ may be expressible in many different ways as a sum of simple tensors. In particular, care must be taken when defining maps from $M \otimes_R N$ to another group or module, since a map from $M \otimes N$ which is described on the generators $m \otimes n$ in terms of m and n is not well defined unless it is shown to be independent of the particular choice of $m \otimes n$ as a coset representative.

Another point where care must be exercised is in reference to the element $m \otimes n$ when the modules M and N or the ring R are not clear from the context. The first two examples of extension of scalars give an instance where M is a submodule of a larger module M', and for some $m \in M$ and $n \in N$ we have $m \otimes n = 0$ in $M' \otimes_R N$ but $m \otimes n$ is *nonzero* in $M \otimes_R N$. This is possible because the symbol "$m \otimes n$" represents different cosets, hence possibly different elements, in the two tensor products. In particular, these two examples show that $M \otimes_R N$ need not be a subgroup of $M' \otimes_R N$ even when M is a submodule of M' (cf. also Exercise 2).

Mapping $M \times N$ to the free $\mathbb{Z}$-module on $M \times N$ and then passing to the quotient defines a map $\iota : M \times N \to M \otimes_R N$ with $\iota(m, n) = m \otimes n$. This map is in general not a group homomorphism, but it is additive in both m and n separately and satisfies $\iota(mr, n) = mr \otimes n = m \otimes rn = \iota(m, rn)$. Such maps are given a name:

Definition. Let M be a right R-module, let N be a left R-module and let L be an abelian group (written additively). A map $\varphi : M \times N \to L$ is called *R-balanced* or *middle linear with respect to R* if

$$\begin{aligned} \varphi(m_1 + m_2, n) &= \varphi(m_1, n) + \varphi(m_2, n) \\ \varphi(m, n_1 + n_2) &= \varphi(m, n_1) + \varphi(m, n_2) \\ \varphi(m, rn) &= \varphi(mr, n) \end{aligned}$$

for all $m, m_1, m_2 \in M, n, n_1, n_2 \in N$, and $r \in R$.

With this terminology, it follows immediately from the relations in (7) that the map $\iota : M \times N \to M \otimes_R N$ is R-balanced. The next theorem proves the extremely useful *universal property of the tensor product* with respect to balanced maps.

Theorem 10. Suppose R is a ring with 1, M is a right R-module, and N is a left R-module. Let $M \otimes_R N$ be the tensor product of M and N over R and let $\iota : M \times N \to M \otimes_R N$ be the R-balanced map defined above.

(1) If $\Phi : M \otimes_R N \to L$ is any group homomorphism from $M \otimes_R N$ to an abelian group L then the composite map $\varphi = \Phi \circ \iota$ is an R-balanced map from $M \times N$ to L.

(2) Conversely, suppose L is an abelian group and $\varphi : M \times N \to L$ is any R-balanced map. Then there is a unique group homomorphism $\Phi : M \otimes_R N \to L$ such that φ factors through ι, i.e., $\varphi = \Phi \circ \iota$ as in (1).

Equivalently, the correspondence $\varphi \leftrightarrow \Phi$ in the commutative diagram

$$\begin{tikzcd} M \times N \arrow[r, "\iota"] \arrow[dr, "\varphi"'] & M \otimes_R N \arrow[d, "\Phi"] \\ & L \end{tikzcd}$$

establishes a bijection

$$\left\{ \begin{array}{c} R\text{-balanced maps} \\ \varphi : M \times N \to L \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{group homomorphisms} \\ \Phi : M \otimes_R N \to L \end{array} \right\}.$$

Proof: The proof of (1) is immediate from the properties of ι above. For (2), the map φ defines a unique $\mathbb{Z}$-module homomorphism $\tilde{\varphi}$ from the free group on $M \times N$ to L (Theorem 6 in Section 3) such that $\tilde{\varphi}(m, n) = \varphi(m, n) \in L$. Since φ is R-balanced, $\tilde{\varphi}$ maps each of the elements in equation (6) to 0; for example

$$\tilde{\varphi}\left((mr, n) - (m, rn)\right) = \varphi(mr, n) - \varphi(m, rn) = 0.$$

It follows that the kernel of $\tilde{\varphi}$ contains the subgroup generated by these elements, hence $\tilde{\varphi}$ induces a homomorphism Φ on the quotient group $M \otimes_R N$ to L. By definition we then have

$$\Phi(m \otimes n) = \tilde{\varphi}(m, n) = \varphi(m, n),$$

i.e., $\varphi = \Phi \circ \iota$. The homomorphism Φ is uniquely determined by this equation since the elements $m \otimes n$ generate $M \otimes_R N$ as an abelian group. This completes the proof.