8(1957), pp. 686–695, which treats much more general groups. Because we are dealing with a specific group order, our arguments are simpler and numerically more explicit, yet they retain some of the key ideas of Suzuki's work. Moreover, Suzuki's paper and its successor, *Finite groups in which the centralizer of any non-identity element is nilpotent*, by W. Feit, M. Hall and J. Thompson, Math. Zeit., 74(1960), pp. 1–17, are prototypes for the lengthy and difficult Feit–Thompson Theorem (cf. Section 3.4). Our discussion also conveys some of the flavor of these fundamental papers. In particular, each of these papers follows the basic development in which the structure and embedding of the Sylow subgroups is first determined and then character theory (with heavy reliance on induced characters) is applied.

For the remainder of this section we assume G is a simple group of order $3^3 \cdot 7 \cdot 13 \cdot 409$. We list some properties of G which may be verified using the methods stemming from Sylow's Theorem discussed in Section 6.2. The details are left as exercises.

(1) Let $q_1 = 3$, let Q_1 be a Sylow 3-subgroup of G and let $N_1 = N_G(Q_1)$. Then Q_1 is an elementary abelian 3-group of order 3^3 and N_1 is a Frobenius group of order $3^3 \cdot 13$ with Frobenius kernel Q_1 and with N_1/Q_1 acting irreducibly by conjugation on Q_1.

(2) Let $q_2 = 7$, let Q_2 be a Sylow 7-subgroup of G and let $N_2 = N_G(Q_2)$. Then Q_2 is cyclic of order 7 and N_2 is the non-abelian group of order $7 \cdot 3$ (so N_2 is a Frobenius group with Frobenius kernel Q_2).

(3) Let $q_3 = 13$, let Q_3 be a Sylow 11-subgroup of G and let $N_3 = N_G(Q_3)$. Then Q_3 is cyclic of order 13 and N_3 is the non-abelian group of order $13 \cdot 3$ (so N_3 is a Frobenius group with Frobenius kernel Q_3).

(4) Let $q_4 = 409$, let Q_4 be a Sylow 409-subgroup of G and let $N_4 = N_G(Q_4)$. Then Q_4 is cyclic of order 409 and N_4 is the non-abelian group of order $409 \cdot 3$ (so N_4 is a Frobenius group with Frobenius kernel Q_4).

(5) Every nonidentity element of G has prime order and $Q_i \cap Q_i^g = 1$ for every $g \in G - N_i$, for each $i = 1, 2, 3, 4$. The nonidentity conjugacy classes of G are:
  (a) 2 classes of elements of order 3 (each of these classes has size $7 \cdot 13 \cdot 409$)
  (b) 2 classes of elements of order 7 (each of these classes has size $3^3 \cdot 13 \cdot 409$)
  (c) 4 classes of elements of order 13 (each of these classes has size $3^3 \cdot 7 \cdot 409$)
  (d) 136 classes of elements of order 409 (each of these classes has size $3^3 \cdot 7 \cdot 13$),
  and so there are 145 conjugacy classes in G.

Since each of the groups N_i is a Frobenius group satisfying the hypothesis of Proposition 13, the number of characters of N_i of degree > 1 may be read off from that proposition:

(i) N_1 has 2 irreducible characters of degree 13
(ii) N_2 has 2 irreducible characters of degree 3
(iii) N_3 has 4 irreducible characters of degree 3
(iv) N_4 has 136 irreducible characters of degree 3.

From now on, to simplify notation, for any subgroup H of G and any generalized character μ of H let

$$\mu^* = \mathrm{Ind}_H^G(\mu)$$

so a star will always denote induction from a subgroup to the whole group G and the subgroup will be clear from the context.

The following lemma is a key point in the proof. It shows how the vanishing of induced characters described in Corollary 12 (together with the *trivial intersection* property of the Sylow subgroups Q_i, namely the fact that $Q_i \cap Q_i^g = 1$ for all $g \in G - N_G(Q_i)$) may be used to relate inner products of certain generalized characters to the inner products of their induced generalized characters. For these computations it is important that the generalized characters are zero on the identity (which explains why we are considering *differences* of characters of the same degree).

Lemma 15. For any $i \in \{1, 2, 3, 4\}$ let $q = q_i$, let $Q = Q_i$, let $N = N_i$ and let $p = |N : Q|$. Let $\psi_1, \dots, \psi_4$ be any irreducible characters of N of degree p (not necessarily distinct) and let $\alpha = \psi_1 - \psi_2$ and $\beta = \psi_3 - \psi_4$. Then α and β are generalized characters of N which are zero on every element of N of order not equal to q. Furthermore, α^* and β^* are generalized characters of G which are zero on every element of G of order not equal to q and

$$(\alpha^*, \beta^*)_G = (\alpha, \beta)_N$$

(where $(\ , \ )_H$ denotes the usual Hermitian product of class functions computed in the group H). In other words, induction from N to G is an inner product preserving map on such generalized characters α, β of N.

Proof: By Proposition 13, there are nonprincipal characters $\lambda_1, \dots, \lambda_4$ of Q of degree 1 such that $\psi_j = \mathrm{Ind}_Q^N(\lambda_j)$ for $j = 1, \dots, 4$. By Corollary 12 therefore, each ψ_j vanishes on $N - Q$, hence so do α and β. Note that since $\psi_j(1) = p$ for all j we have $\alpha(1) = \beta(1) = 0$. By the transitivity of induction, $\psi_j^* = \mathrm{Ind}_N^G(\psi_j) = \mathrm{Ind}_Q^G(\lambda_j)$ for all j. Again by Corollary 12 applied to the latter induced character we see that ψ_j^* vanishes on all elements not conjugate in G to some element of Q, hence so do both α^* and β^*. Since the induced characters ψ_j^* all have degree $|G : Q|$, the generalized characters α^* and β^* are zero on the identity. Thus α^* and β^* vanish on all elements of G which are not of order q. Finally, if $g_1, \dots, g_m$ are representatives for the left cosets of N in G with $g_1 = 1$, then because $Q \cap Q^{g_k} = 1$ for all $k > 1$ (by (5) above), it follows immediately from the formula for induced (generalized) characters that $\alpha^*(x) = \alpha(x)$ and $\beta^*(x) = \beta(x)$ for all nonidentity elements $x \in Q$ (i.e., for all elements $x \in N$ of order q). Furthermore, by Sylow's Theorem every element of G of order q lies in a conjugate of Q, hence the collection of G-conjugates of the set $Q - \{1\}$ partition the elements of order q in G into $|G : N|$ disjoint subsets. Since α^* and β^* are class functions on G, the sum of $\alpha^*(x)\overline{\beta^*(x)}$ as x runs over any of these subsets is the same. These facts imply

$$\begin{aligned}
(\alpha^*, \beta^*)_G &= \frac{1}{|G|} \sum_{x \in G} \alpha^*(x)\overline{\beta^*(x)} \\
&= \frac{1}{|G|} \sum_{\substack{x \in G \\ |x| = q}} \alpha^*(x)\overline{\beta^*(x)} \\
&= \frac{1}{|G|} \sum_{\substack{x \in N \\ |x| = q}} |G : N| \alpha^*(x)\overline{\beta^*(x)}
\end{aligned}$$

$$= \frac{1}{|N|} \sum_{x \in N} \alpha(x)\overline{\beta(x)} = (\alpha, \beta)_N.$$

This completes the proof.

The next lemma sets up a correspondence between the irreducible characters of N_i of degree > 1 and some nonprincipal irreducible characters of G.

Lemma 16. For any $i \in \{1, 2, 3, 4\}$ let $q = q_i$, let $Q = Q_i$, let $N = N_i$ and let $p = |N : Q|$. Let $\psi_1, \dots, \psi_k$ be the distinct irreducible characters of N of degree p. Then there are distinct irreducible characters $\chi_1, \dots, \chi_k$ of G, all of which have the same degree, and a fixed sign $\epsilon = \pm 1$ such that $\psi_1^* - \psi_j^* = \epsilon(\chi_1 - \chi_j)$ for all $j = 2, 3, \dots, k$.

Proof: Let $\alpha_j = \psi_1 - \psi_j$ for $j = 2, 3, \dots, k$ so α_j satisfies the hypothesis of Lemma 15. Since $\psi_1 \neq \psi_j$, by Lemma 15

$$2 = ||\,\alpha_j\,||^2 = (\alpha_j, \alpha_j)_N = (\alpha_j^*, \alpha_j^*)_G = ||\,\alpha_j^*\,||^2$$

for all j. Thus α_j^* must have two distinct irreducible characters of G as its irreducible constituents. Since $\alpha_j^*(1) = 0$ it must be a difference of two distinct irreducible characters, both of which have the same degree. In particular, the lemma holds if $k = 2$ (which is the case for $q = 3$ and $q = 7$). Assume therefore that $k > 2$ and write

$$\alpha_2^* = \psi_1^* - \psi_2^* = \epsilon(\chi - \chi')$$
$$\alpha_3^* = \psi_1^* - \psi_3^* = \epsilon'(\theta - \theta')$$

for some irreducible characters $\chi, \chi', \theta, \theta'$ of G and some signs ϵ, ϵ'. As proved above, $\chi \neq \chi'$ and $\theta \neq \theta'$. Interchanging θ and θ' if necessary, we may assume $\epsilon = \epsilon'$. Thus

$$\alpha_3^* - \alpha_2^* = \psi_2^* - \psi_3^* = \epsilon(\theta - \theta' - \chi + \chi').$$

By Lemma 15, $\psi_2^* - \psi_3^* = (\psi_2 - \psi_3)^*$ also has exactly two distinct irreducible constituents, hence either $\theta = \chi$ or $\theta' = \chi'$. Replacing ϵ by $-\epsilon$ if necessary we may assume that $\theta = \chi$ so that now we have

$$\alpha_2^* = \psi_1^* - \psi_2^* = \epsilon(\chi - \chi')$$
$$\alpha_3^* = \psi_1^* - \psi_3^* = \epsilon(\chi - \theta')$$

where χ, χ' and θ are distinct irreducible characters of G and the sign ϵ is determined. Label $\chi = \chi_1$, $\chi' = \chi_2$ and $\theta = \chi_3$. Now one similarly checks that for each $j \geq 3$ there is an irreducible character χ_j of G such that

$$\alpha_j^* = \psi_1^* - \psi_j^* = \epsilon(\chi_1 - \chi_j)$$

and $\chi_1, \dots, \chi_k$ are distinct. Since all χ_j's have the same degree as χ_1, the proof is complete.

We remark that it need not be the case that $\chi_j = \psi_j^*$ for any j, but only that the differences of irreducible characters of N induce to differences of irreducible characters of G.

The irreducible characters χ_j of G obtained via Lemma 16 are called *exceptional characters* associated to Q.