Lemma 6.5.2. *Let x be a real number. Then the limit $\lim_{n\to\infty} x^n$ exists and is equal to zero when $|x| < 1$, exists and is equal to 1 when $x = 1$, and diverges when $x = -1$ or when $|x| > 1$.*

Proof. See Exercise 6.5.2. □

Lemma 6.5.3. *For any $x > 0$, we have $\lim_{n\to\infty} x^{1/n} = 1$.*

Proof. See Exercise 6.5.3. □

We will derive a few more standard limits later on, once we develop the root and ratio tests for series and for sequences.

Exercise 6.5.1. Show that $\lim_{n\to\infty} 1/n^q = 0$ for any rational $q > 0$. (Hint: use Corollary 6.5.1 and the limit laws, Theorem 6.1.19.) Conclude that the limit $\lim_{n\to\infty} n^q$ does not exist. (Hint: argue by contradiction using Theorem 6.1.19(e).)

Exercise 6.5.2. Prove Lemma 6.5.2. (Hint: use Proposition 6.3.10, Exercise 6.3.4, and the squeeze test.)

Exercise 6.5.3. Prove Lemma 6.5.3. (Hint: you may need to treat the cases $x \geq 1$ and $x < 1$ separately. You might wish to first use Lemma 6.5.2 to prove the preliminary result that for every $\varepsilon > 0$ and every real number $M > 0$, there exists an n such that $M^{1/n} \leq 1 + \varepsilon$.)

6.6 Subsequences

This chapter has been devoted to the study of sequences $(a_n)_{n=1}^\infty$ of real numbers, and their limits. Some sequences were convergent to a single limit, while others had multiple limit points. For instance, the sequence

$$1.1, 0.1, 1.01, 0.01, 1.001, 0.001, 1.0001, \ldots$$

has two limit points at 0 and 1 (which are incidentally also the lim inf and lim sup respectively), but is not actually convergent (since the lim sup and lim inf are not equal). However, while this sequence is not convergent, it does appear to contain convergent components; it seems to be a mixture of two convergent subsequences, namely

$$1.1, 1.01, 1.001, \ldots$$

and

$$0.1, 0.01, 0.001, \ldots.$$

To make this notion more precise, we need a notion of subsequence.

Definition 6.6.1 (Subsequences). Let $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ be sequences of real numbers. We say that $(b_n)_{n=0}^{\infty}$ is a *subsequence* of $(a_n)_{n=0}^{\infty}$ iff there exists a function $f : \mathbf{N} \to \mathbf{N}$ which is strictly increasing (i.e., $f(n+1) > f(n)$ for all $n \in \mathbf{N}$) such that

$$b_n = a_{f(n)} \text{ for all } n \in \mathbf{N}.$$

Example 6.6.2. If $(a_n)_{n=0}^{\infty}$ is a sequence, then $(a_{2n})_{n=0}^{\infty}$ is a subsequence of $(a_n)_{n=0}^{\infty}$, since the function $f : \mathbf{N} \to \mathbf{N}$ defined by $f(n) := 2n$ is a strictly increasing function from $\mathbf{N}$ to $\mathbf{N}$. Note that we do not assume f to be bijective, although it is necessarily injective (why?). More informally, the sequence

$$a_0, a_2, a_4, a_6, \ldots$$

is a subsequence of

$$a_0, a_1, a_2, a_3, a_4, \ldots.$$

Example 6.6.3. The two sequences

$$1.1, 1.01, 1.001, \ldots$$

and

$$0.1, 0.01, 0.001, \ldots$$

mentioned earlier are both subsequences of

$$1.1, 0.1, 1.01, 0.01, 1.001, 1.0001, \ldots$$

The property of being a subsequence is reflexive and transitive, though not symmetric:

Lemma 6.6.4. *Let $(a_n)_{n=0}^\infty$, $(b_n)_{n=0}^\infty$, and $(c_n)_{n=0}^\infty$ be sequences of real numbers. Then $(a_n)_{n=0}^\infty$ is a subsequence of $(a_n)_{n=0}^\infty$. Furthermore, if $(b_n)_{n=0}^\infty$ is a subsequence of $(a_n)_{n=0}^\infty$, and $(c_n)_{n=0}^\infty$ is a subsequence of $(b_n)_{n=0}^\infty$, then $(c_n)_{n=0}^\infty$ is a subsequence of $(a_n)_{n=0}^\infty$.*

Proof. See Exercise 6.6.1. □

We now relate the concept of subsequences to the concept of limits and limit points.

Proposition 6.6.5 (Subsequences related to limits). *Let $(a_n)_{n=0}^\infty$ be a sequence of real numbers, and let L be a real number. Then the following two statements are logically equivalent (each one implies the other):*

(a) *The sequence $(a_n)_{n=0}^\infty$ converges to L.*

(b) *Every subsequence of $(a_n)_{n=0}^\infty$ converges to L.*

Proof. See Exercise 6.6.4. □

Proposition 6.6.6 (Subsequences related to limit points). *Let $(a_n)_{n=0}^\infty$ be a sequence of real numbers, and let L be a real number. Then the following two statements are logically equivalent.*

(a) *L is a limit point of $(a_n)_{n=0}^\infty$.*

(b) *There exists a subsequence of $(a_n)_{n=0}^\infty$ which converges to L.*

Proof. See Exercise 6.6.5. □

Remark 6.6.7. The above two propositions give a sharp contrast between the notion of a limit, and that of a limit point. When a sequence has a limit L, then *all* subsequences also converge to L. But when a sequence has L as a limit point, then only *some* subsequences converge to L.

We can now prove an important theorem in real analysis, due to Bernard Bolzano (1781–1848) and Karl Weierstrass (1815–1897): every bounded sequence has a convergent subsequence.

Theorem 6.6.8 (Bolzano-Weierstrass theorem). *Let $(a_n)_{n=0}^\infty$ be a bounded sequence (i.e., there exists a real number $M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbf{N}$). Then there is at least one subsequence of $(a_n)_{n=0}^\infty$ which converges.*

Proof. Let L be the limit superior of the sequence $(a_n)_{n=0}^\infty$. Since we have $-M \leq a_n \leq M$ for all natural numbers n, it follows from the comparison principle (Lemma 6.4.13) that $-M \leq L \leq M$. In particular, L is a real number (not $+\infty$ or $-\infty$). By Proposition 6.4.12(e), L is thus a limit point of $(a_n)_{n=0}^\infty$. Thus by Proposition 6.6.6, there exists a subsequence of $(a_n)_{n=0}^\infty$ which converges (in fact, it converges to L). $\square$

Note that we could as well have used the limit inferior instead of the limit superior in the above argument.

Remark 6.6.9. The Bolzano-Weierstrass theorem says that if a sequence is bounded, then eventually it has no choice but to converge in some places; it has "no room" to spread out and stop itself from acquiring limit points. It is not true for unbounded sequences; for instance, the sequence $1, 2, 3, \ldots$ has no convergent subsequences whatsoever (why?). In the language of topology, this means that the interval $\{x \in \mathbf{R} : -M \leq x \leq M\}$ is *compact*, whereas an unbounded set such as the real line $\mathbf{R}$ is not compact. The distinction between compact sets and non-compact sets will be very important in later chapters - of similar importance to the distinction between finite sets and infinite sets.

Exercise 6.6.1. Prove Lemma 6.6.4.

Exercise 6.6.2. Can you find two sequences $(a_n)_{n=0}^\infty$ and $(b_n)_{n=0}^\infty$ which are not the same sequence, but such that each is a subsequence of the other?

Exercise 6.6.3. Let $(a_n)_{n=0}^\infty$ be a sequence which is not bounded. Show that there exists a subsequence $(b_n)_{n=0}^\infty$ of $(a_n)_{n=0}^\infty$ such that $\lim_{n\to\infty} 1/b_n$ exists and is equal to zero. (Hint: for each natural number j, introduce the quantity $n_j := \min\{n \in \mathbf{N} : |a_n| \geq j\}$, first explaining why the set $\{n \in \mathbf{N} : |a_n| \geq j\}$ is non-empty. Then set $b_j := a_{n_j}$.)

Exercise 6.6.4. Prove Proposition 6.6.5. (Note that one of the two implications has a very short proof.)

Exercise 6.6.5. Prove Proposition 6.6.6. (Hint: to show that (a) implies (b), define the numbers n_j for each natural numbers j by the formula $n_j := \min\{n \in \mathbf{N} : |a_n - L| \leq 1/j\}$, explaining why the set $\{n \in \mathbf{N} : |a_n - L| \leq 1/j\}$ is non-empty. Then consider the sequence a_{n_j}.)

6.7 Real exponentiation, part II

We finally return to the topic of exponentiation of real numbers that we started in Section 5.6. In that section we defined x^q for all rational q and positive real numbers x, but we have not yet defined x^α when α is real. We now rectify this situation using limits (in a similar way as to how we defined all the other standard operations on the real numbers). First, we need a lemma:

Lemma 6.7.1 (Continuity of exponentiation). *Let $x > 0$, and let α be a real number. Let $(q_n)_{n=1}^\infty$ be any sequence of rational numbers converging to α. Then $(x^{q_n})_{n=1}^\infty$ is also a convergent sequence. Furthermore, if $(q'_n)_{n=1}^\infty$ is any other sequence of rational numbers converging to α, then $(x^{q'_n})_{n=1}^\infty$ has the same limit as $(x^{q_n})_{n=1}^\infty$:*

$$\lim_{n\to\infty} x^{q_n} = \lim_{n\to\infty} x^{q'_n}.$$

Proof. There are three cases: $x < 1$, $x = 1$, and $x > 1$. The case $x = 1$ is rather easy (because then $x^q = 1$ for all rational q). We shall just do the case $x > 1$, and leave the case $x < 1$ (which is very similar) to the reader.

Let us first prove that $(x^{q_n})_{n=1}^\infty$ converges. By Proposition 6.4.18 it is enough to show that $(x^{q_n})_{n=1}^\infty$ is a Cauchy sequence.

To do this, we need to estimate the distance between x^{q_n} and x^{q_m}; let us say for the time being that $q_n \geq q_m$, so that $x^{q_n} \geq x^{q_m}$

(since $x > 1$). We have

$$d(x^{q_n}, x^{q_m}) = x^{q_n} - x^{q_m} = x^{q_m}(x^{q_n - q_m} - 1).$$

Since $(q_n)_{n=1}^{\infty}$ is a convergent sequence, it has some upper bound M; since $x > 1$, we have $x^{q_m} \leq x^M$. Thus

$$d(x^{q_n}, x^{q_m}) = |x^{q_n} - x^{q_m}| \leq x^M(x^{q_n - q_m} - 1).$$

Now let $\varepsilon > 0$. We know by Lemma 6.5.3 that the sequence $(x^{1/k})_{k=1}^{\infty}$ is eventually εx^{-M}-close to 1. Thus there exists some $K \geq 1$ such that

$$|x^{1/K} - 1| \leq \varepsilon x^{-M}.$$

Now since $(q_n)_{n=1}^{\infty}$ is convergent, it is a Cauchy sequence, and so there is an $N \geq 1$ such that q_n and q_m are $1/K$-close for all $n, m \geq N$. Thus we have

$$d(x^{q_n}, x^{q_m}) = x^M(x^{q_n - q_m} - 1) \leq x^M(x^{1/K} - 1) \leq x^M \varepsilon x^{-M} = \varepsilon$$

for every $n, m \geq N$ such that $q_n \geq q_m$. By symmetry we also have this bound when $n, m \geq N$ and $q_n \leq q_m$. Thus the sequence $(x^{q_n})_{n=N}^{\infty}$ is ε-steady. Thus the sequence $(x^{q_n})_{n=1}^{\infty}$ is eventually ε-steady for every $\varepsilon > 0$, and is thus a Cauchy sequence as desired. This proves the convergence of $(x^{q_n})_{n=1}^{\infty}$.

Now we prove the second claim. It will suffice to show that

$$\lim_{n\to\infty} x^{q_n - q'_n} = 1,$$

since the claim would then follow from limit laws (since $x^{q_n} = x^{q_n - q'_n} x^{q'_n}$).

Write $r_n := q_n - q'_n$; by limit laws we know that $(r_n)_{n=1}^{\infty}$ converges to 0. We have to show that for every $\varepsilon > 0$, the sequence $(x^{r_n})_{n=1}^{\infty}$ is eventually ε-close to 1. But from Lemma 6.5.3 we know that the sequence $(x^{1/k})_{k=1}^{\infty}$ is eventually ε-close to 1. Since $\lim_{k\to\infty} x^{-1/k}$ is also equal to 1 by Lemma 6.5.3, we know that $(x^{-1/k})_{k=1}^{\infty}$ is also eventually ε-close to 1. Thus we can find a K such that $x^{1/K}$ and $x^{-1/K}$ are both ε-close to 1. But since $(r_n)_{n=1}^{\infty}$ is convergent to 0, it is eventually $1/K$-close to 0, so that eventually $-1/K \leq r_n \leq 1/K$, and thus $x^{-1/K} \leq x^{r_n} \leq x^{1/K}$. In particular x^{r_n} is also eventually ε-close to 1 (see Proposition 4.3.7(f)), as desired. □

We may now make the following definition.

Definition 6.7.2 (Exponentiation to a real exponent). Let $x > 0$ be real, and let α be a real number. We define the quantity x^α by the formula $x^\alpha = \lim_{n\to\infty} x^{q_n}$, where $(q_n)_{n=1}^\infty$ is any sequence of rational numbers converging to α.

Let us check that this definition is well-defined. First of all, given any real number α we always have at least one sequence $(q_n)_{n=1}^\infty$ of rational numbers converging to α, by the definition of real numbers (and Proposition 6.1.15). Secondly, given any such sequence $(q_n)_{n=1}^\infty$, the limit $\lim_{n\to\infty} x^{q_n}$ exists by Lemma 6.7.1. Finally, even though there can be multiple choices for the sequence $(q_n)_{n=1}^\infty$, they all give the same limit by Lemma 6.7.1. Thus this definition is well-defined.

If α is not just real but rational, i.e., $\alpha = q$ for some rational q, then this definition could in principle be inconsistent with our earlier definition of exponentiation in Section 6.7. But in this case α is clearly the limit of the sequence $(q)_{n=1}^\infty$, so by definition $x^\alpha = \lim_{n\to\infty} x^q = x^q$. Thus the new definition of exponentiation is consistent with the old one.

Proposition 6.7.3. *All the results of Lemma 5.6.9, which held for rational numbers q and r, continue to hold for real numbers q and r.*

Proof. We demonstrate this for the identity $x^{q+r} = x^q x^r$ (i.e., the first part of Lemma 5.6.9(b)); the other parts are similar and are left to Exercise 6.7.1. The idea is to start with Lemma 5.6.9 for rationals and then take limits to obtain the corresponding results for reals.

Let q and r be real numbers. Then we can write $q = \lim_{n\to\infty} q_n$ and $r = \lim_{n\to\infty} r_n$ for some sequences $(q_n)_{n=1}^\infty$ and $(r_n)_{n=1}^\infty$ of rationals, by the definition of real numbers (and Proposition 6.1.15).

Then by the limit laws, $q + r$ is the limit of $(q_n + r_n)_{n=1}^{\infty}$. By definition of real exponentiation, we have

$$x^{q+r} = \lim_{n\to\infty} x^{q_n+r_n}; \quad x^q = \lim_{n\to\infty} x^{q_n}; \quad x^r = \lim_{n\to\infty} x^{r_n}.$$

But by Lemma 5.6.9(b) (applied to *rational* exponents) we have $x^{q_n+r_n} = x^{q_n}x^{r_n}$. Thus by limit laws we have $x^{q+r} = x^q x^r$, as desired. □

Exercise 6.7.1. Prove the remaining components of Proposition 6.7.3.

Chapter 7

Series

Now that we have developed a reasonable theory of limits of sequences, we will use that theory to develop a theory of infinite series

$$\sum_{n=m}^{\infty} a_n = a_m + a_{m+1} + a_{m+2} + \dots.$$

But before we develop infinite series, we must first develop the theory of finite series.

7.1 Finite series

Definition 7.1.1 (Finite series). Let m, n be integers, and let $(a_i)_{i=m}^n$ be a finite sequence of real numbers, assigning a real number a_i to each integer i between m and n inclusive (i.e., $m \leq i \leq n$). Then we define the finite sum (or finite series) $\sum_{i=m}^n a_i$ by the recursive formula

$$\sum_{i=m}^{n} a_i := 0 \text{ whenever } n < m;$$

$$\sum_{i=m}^{n+1} a_i := \left(\sum_{i=m}^{n} a_i\right) + a_{n+1} \text{ whenever } n \geq m - 1.$$

Thus for instance we have the identities

$$\sum_{i=m}^{m-2} a_i = 0; \quad \sum_{i=m}^{m-1} a_i = 0; \quad \sum_{i=m}^{m} a_i = a_m;$$

$$\sum_{i=m}^{m+1} a_i = a_m + a_{m+1}; \quad \sum_{i=m}^{m+2} a_i = a_m + a_{m+1} + a_{m+2}$$

(why?). Because of this, we sometimes express $\sum_{i=m}^{n} a_i$ less formally as

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \ldots + a_n.$$

Remark 7.1.2. The difference between "sum" and "series" is a subtle linguistic one. Strictly speaking, a series is an *expression* of the form $\sum_{i=m}^{n} a_i$; this series is mathematically (but not semantically) equal to a real number, which is then the *sum* of that series. For instance, $1 + 2 + 3 + 4 + 5$ is a series, whose sum is 15; if one were to be very picky about semantics, one would not consider 15 a series and one would not consider $1+2+3+4+5$ a sum, despite the two expressions having the same value. However, we will not be very careful about this distinction as it is purely linguistic and has no bearing on the mathematics; the expressions $1+2+3+4+5$ and 15 are the same number, and thus *mathematically* interchangeable, in the sense of the axiom of substitution (see Section A.7), even if they are not semantically interchangeable.

Remark 7.1.3. Note that the variable i (sometimes called the *index of summation*) is a *bound variable* (sometimes called a *dummy variable*); the expression $\sum_{i=m}^{n} a_i$ does not actually depend on any quantity named i. In particular, one can replace the index of summation i with any other symbol, and obtain the same sum:

$$\sum_{i=m}^{n} a_i = \sum_{j=m}^{n} a_j.$$

We list some basic properties of summation below.

Lemma 7.1.4.

(a) *Let $m \leq n < p$ be integers, and let a_i be a real number assigned to each integer $m \leq i \leq p$. Then we have*

$$\sum_{i=m}^{n} a_i + \sum_{i=n+1}^{p} a_i = \sum_{i=m}^{p} a_i.$$

(b) *Let $m \le n$ be integers, k be another integer, and let a_i be a real number assigned to each integer $m \le i \le n$. Then we have*

$$\sum_{i=m}^{n} a_i = \sum_{j=m+k}^{n+k} a_{j-k}.$$

(c) *Let $m \le n$ be integers, and let a_i, b_i be real numbers assigned to each integer $m \le i \le n$. Then we have*

$$\sum_{i=m}^{n}(a_i + b_i) = \left(\sum_{i=m}^{n} a_i\right) + \left(\sum_{i=m}^{n} b_i\right)$$

(d) *Let $m \le n$ be integers, and let a_i be a real number assigned to each integer $m \le i \le n$, and let c be another real number. Then we have*

$$\sum_{i=m}^{n}(ca_i) = c\left(\sum_{i=m}^{n} a_i\right).$$

(e) *(Triangle inequality for finite series) Let $m \le n$ be integers, and let a_i be a real number assigned to each integer $m \le i \le n$. Then we have*

$$|\sum_{i=m}^{n} a_i| \le \sum_{i=m}^{n} |a_i|.$$

(f) *(Comparison test for finite series) Let $m \le n$ be integers, and let a_i, b_i be real numbers assigned to each integer $m \le i \le n$. Suppose that $a_i \le b_i$ for all $m \le i \le n$. Then we have*

$$\sum_{i=m}^{n} a_i \le \sum_{i=m}^{n} b_i.$$

Proof. See Exercise 7.1.1. □

Remark 7.1.5. In the future we may omit some of the parentheses in series expressions, for instance we may write $\sum_{i=m}^{n}(a_i + b_i)$ simply as $\sum_{i=m}^{n} a_i + b_i$. This is reasonably safe from being misinterpreted, because the alternative interpretation $(\sum_{i=m}^{n} a_i) + b_i$ does not make any sense (the index i in b_i is meaningless outside of the summation, since i is only a dummy variable).

One can use finite series to also define summations over finite sets:

Definition 7.1.6 (Summations over finite sets). Let X be a finite set with n elements (where $n \in \mathbf{N}$), and let $f : X \to \mathbf{R}$ be a function from X to the real numbers (i.e., f assigns a real number $f(x)$ to each element x of X). Then we can define the finite sum $\sum_{x \in X} f(x)$ as follows. We first select any bijection g from $\{i \in \mathbf{N} : 1 \le i \le n\}$ to X; such a bijection exists since X is assumed to have n elements. We then define

$$\sum_{x \in X} f(x) := \sum_{i=1}^{n} f(g(i)).$$

Example 7.1.7. Let X be the three-element set $X := \{a, b, c\}$, where a, b, c are distinct objects, and let $f : X \to \mathbf{R}$ be the function $f(a) := 2$, $f(b) := 5$, $f(c) := -1$. In order to compute the sum $\sum_{x \in X} f(x)$, we select a bijection $g : \{1, 2, 3\} \to X$, e.g., $g(1) := a$, $g(2) := b$, $g(3) := c$. We then have

$$\sum_{x \in X} f(x) = \sum_{i=1}^{3} f(g(i)) = f(a) + f(b) + f(c) = 6.$$

One could pick another bijection from $\{1, 2, 3\}$ to X, e.g., $h(1) := c$, $h(2) := b$, $h(3) = c$, but the end result is still the same:

$$\sum_{x \in X} f(x) = \sum_{i=1}^{3} f(h(i)) = f(c) + f(b) + f(a) = 6.$$

To verify that this definition actually does give a single, well-defined value to $\sum_{x \in X} f(x)$, one has to check that different bijections g from $\{i \in \mathbf{N} : 1 \le i \le n\}$ to X give the same sum. In other words, we must prove

Proposition 7.1.8 (Finite summations are well-defined). *Let X be a finite set with n elements (where $n \in \mathbf{N}$), let $f : X \to \mathbf{R}$ be a function, and let $g : \{i \in \mathbf{N} : 1 \leq i \leq n\} \to X$ and $h : \{i \in \mathbf{N} : 1 \leq i \leq n\} \to X$ be bijections. Then we have*

$$\sum_{i=1}^{n} f(g(i)) = \sum_{i=1}^{n} f(h(i)).$$

Remark 7.1.9. The issue is somewhat more complicated when summing over infinite sets; see Section 8.2.

Proof. We use induction on n; more precisely, we let $P(n)$ be the assertion that "For any set X of n elements, any function $f : X \to \mathbf{R}$, and any two bijections g, h from $\{i \in \mathbf{N} : 1 \leq i \leq n\}$ to X, we have $\sum_{i=1}^{n} f(g(i)) = \sum_{i=1}^{n} f(h(i))$". (More informally, $P(n)$ is the assertion that Proposition 7.1.8 is true for that value of n.) We want to prove that $P(n)$ is true for all natural numbers n.

We first check the base case $P(0)$. In this case $\sum_{i=1}^{0} f(g(i))$ and $\sum_{i=1}^{0} f(h(i))$ both equal to 0, by definition of finite series, so we are done.

Now suppose inductively that $P(n)$ is true; we now prove that $P(n+1)$ is true. Thus, let X be a set with $n+1$ elements, let $f : X \to \mathbf{R}$ be a function, and let g and h be bijections from $\{i \in \mathbf{N} : 1 \leq i \leq n+1\}$ to X. We have to prove that

$$\sum_{i=1}^{n+1} f(g(i)) = \sum_{i=1}^{n+1} f(h(i)). \tag{7.1}$$

Let $x := g(n+1)$; thus x is an element of X. By definition of finite series, we can expand the left-hand side of (7.1) as

$$\sum_{i=1}^{n+1} f(g(i)) = \left(\sum_{i=1}^{n} f(g(i)) \right) + x.$$

Now let us look at the right-hand side of (7.1). Ideally we would like to have $h(n+1)$ also equal to x - this would allow us to use the inductive hypothesis $P(n)$ much more easily - but we cannot assume this. However, since h is a bijection, we do know that there is *some* index j, with $1 \leq j \leq n+1$, for which $h(j) = x$. We now use Lemma 7.1.4 and the definition of finite series to write

$$\begin{aligned}\sum_{i=1}^{n+1} f(h(i)) &= \left(\sum_{i=1}^{j} f(h(i))\right) + \left(\sum_{i=j+1}^{n+1} f(h(i))\right)\\ &= \left(\sum_{i=1}^{j-1} f(h(i))\right) + f(h(j)) + \left(\sum_{i=j+1}^{n+1} f(h(i))\right)\\ &= \left(\sum_{i=1}^{j-1} f(h(i))\right) + x + \left(\sum_{i=j}^{n} f(h(i+1))\right)\end{aligned}$$

We now define the function $\tilde{h} : \{i \in \mathbf{N} : 1 \leq i \leq n\} \to X - \{x\}$ by setting $\tilde{h}(i) := h(i)$ when $i < j$ and $\tilde{h}(i) := h(i+1)$ when $i \geq j$. We can thus write the right-hand side of (7.1) as

$$= \left(\sum_{i=1}^{j-1} f(\tilde{h}(i))\right) + x + \left(\sum_{i=j}^{n} f(\tilde{h}(i))\right) = \left(\sum_{i=1}^{n} f(\tilde{h}(i))\right) + x$$

where we have used Lemma 7.1.4 once again. Thus to finish the proof of (7.1) we have to show that

$$\sum_{i=1}^{n} f(g(i)) = \sum_{i=1}^{n} f(\tilde{h}(i)). \tag{7.2}$$

But the function g (when restricted to $\{i \in \mathbf{N} : 1 \leq i \leq n\}$) is a bijection from $\{i \in \mathbf{N} : 1 \leq i \leq n\} \to X - \{x\}$ (why?). The function $\tilde{h}$ is also a bijection from $\{i \in \mathbf{N} : 1 \leq i \leq n\} \to X - \{x\}$ (why? cf. Lemma 3.6.9). Since $X - \{x\}$ has n elements (by Lemma 3.6.9), the claim 7.2 then follows directly from the induction hypothesis $P(n)$. $\square$

Remark 7.1.10. Suppose that X is a set, that $P(x)$ is a property pertaining to an element x of X, and $f : \{y \in X : P(y) \text{ is true}\} \to$

$\mathbf{R}$ is a function. Then we will often abbreviate

$$\sum_{x \in \{y \in X : P(y) \text{ is true}\}} f(x)$$

as $\sum_{x \in X : P(x) \text{ is true}} f(x)$ or even as $\sum_{P(x) \text{ is true}} f(x)$ when there is no chance of confusion. For instance, $\sum_{n \in \mathbf{N}: 2 \leq n \leq 4} f(x)$ or $\sum_{2 \leq n \leq 4} f(x)$ is short-hand for $\sum_{n \in \{2,3,4\}} f(x) = f(2) + f(3) + f(4)$.

The following properties of summation on finite sets are fairly obvious, but do require a rigourous proof:

Proposition 7.1.11 (Basic properties of summation over finite sets).

(a) *If X is empty, and $f : X \to \mathbf{R}$ is a function (i.e., f is the empty function), we have*

$$\sum_{x \in X} f(x) = 0.$$

(b) *If X consists of a single element, $X = \{x_0\}$, and $f : X \to \mathbf{R}$ is a function, we have*

$$\sum_{x \in X} f(x) = f(x_0).$$

(c) *(Substitution, part I) If X is a finite set, $f : X \to \mathbf{R}$ is a function, and $g : Y \to X$ is a bijection, then*

$$\sum_{x \in X} f(x) = \sum_{y \in Y} f(g(y)).$$

(d) *(Substitution, part II) Let $n \leq m$ be integers, and let X be the set $X := \{i \in \mathbf{Z} : n \leq i \leq m\}$. If a_i is a real number assigned to each integer $i \in X$, then we have*

$$\sum_{i=n}^{m} a_i = \sum_{i \in X} a_i.$$