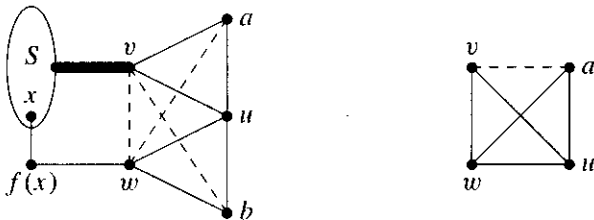


8.6.39. Theorem. (Friendship Theorem—Wilf [1971]). If G is a graph in which any two distinct vertices have exactly one common neighbor, then G has a vertex joined to all others.

Proof: The symmetry of the condition suggests that G might be regular. If G is regular, then it is strongly regular with $\lambda = \mu = 1$. By Theorem 8.6.36, $\frac{1}{2}(n-1 \pm k/\sqrt{k-1})$ now must be an integer. Hence $k/\sqrt{k-1}$ is an integer, which happens only when $k = 2$. However, K_3 is the only 2-regular graph satisfying the condition, and it does have vertices of degree $n - 1$.

Now suppose that G is not regular. We show that $v \not\leftrightarrow w$ requires $d(v) = d(w)$. Insistence on unique common neighbors forbids 4-cycles. Let u be the common neighbor of $\{v, w\}$. Let a be the common neighbor of $\{u, v\}$, and let b be the common neighbor of $\{u, w\}$. Every $x \in S = N(v) - \{u, a\}$ has a common neighbor $f(x)$ with w . If $f(x) = b$ for some $x \in S$, then x, b, u, v is a 4-cycle. If $f(x) = f(x')$ for distinct $x, x' \in S$, then $x, v, x', f(x)$ is a 4-cycle. We have thus shown that $d(w) \geq d(v)$. By symmetry, $d(v) \geq d(w)$.

Since G is not regular, it has two vertices v, w with $d(w) \neq d(v)$. By the preceding paragraph, $v \leftrightarrow w$. Let u be their common neighbor. Since u cannot have the same degree as each of them, we may assume that $d(u) \neq d(v)$. If G has a vertex $x \notin N(v)$, then $d(x) = d(v)$, but this requires $x \leftrightarrow w$ and $x \leftrightarrow u$. This creates the 4-cycle v, u, x, w . Hence $d(v) = n - 1$. ■



EXERCISES

- 8.6.1. Interpretation of cycle space and bond space.** Given a graph G , prove that
- The symmetric difference of two even subgraphs is an even subgraph.
 - The symmetric difference of two edge cuts is an edge cut, and
 - Every edge cut shares an even number of edges with every even subgraph.

8.6.2. Dimension of cycle space and bond space. By parts (a) and (b) of Exercise 8.6.1, the cycle space \mathbf{C} and bond space \mathbf{B} of a graph G are binary vector spaces. Prove that when G is connected, \mathbf{C} has dimension $e(G) - n(G) + 1$ and \mathbf{B} has dimension $n(G) - 1$. (Hint: Show that the cycles created by adding one edge to a particular spanning tree form a basis for the cycle space. Show that $n(G) - 1$ bonds that isolate single vertices form a basis for the bond space, or use orthogonality.)

8.6.3. Recall that the *closed neighborhood* of a vertex v is $N(v) \cup \{v\}$.

a) Let S be a set of vertices in a simple graph G whose neighborhoods are identical. Prove that some eigenvalue of G has multiplicity at least $|S| - 1$. What is it?

b) Let S be a set of vertices in a simple graph G whose closed neighborhoods are identical. Prove that some eigenvalue of G has multiplicity at least $|S| - 1$. What is it?

8.6.4. Let σ_k be the number of subgraphs of a graph G that are k -cycles. Let $L_k = \sum \lambda_i^k$ and $D_k = \sum d_i^k$ be the sums of the k th powers of the eigenvalues and the vertex degrees. Obtain formulas for σ_3 and σ_4 in terms of $\{L_k\}$ and $\{D_k\}$.

8.6.5. Deletion formulas for the characteristic polynomial. For clarity in this problem, we write $\phi(G; \lambda)$ as ϕ_G . Let $v [xy]$ be an arbitrary vertex [edge] of G , and let $Z(v) [Z(xy)]$ be the collection of cycles containing $v [xy]$. Prove that the characteristic polynomial satisfies the following recurrences.

$$a) \phi_G = \lambda \phi_{G-v} - \sum_{u \in N(v)} \phi_{G-v-u} - 2 \sum_{C \in Z(v)} \phi_{G-V(C)}.$$

$$b) \phi_G = \phi_{G-xy} - \phi_{G-x-y} - 2 \sum_{C \in Z(xy)} \phi_{G-V(C)}.$$

(Hint: Induction or Sachs's formula can be used. Also, the edge-deletion formula can be proved from the vertex-deletion formula. Comment: When G is a forest and v is a leaf with neighbor u , the formulas reduce to $\phi_G = \lambda \phi_{G-v} - \phi_{G-v-u}$ and $\phi_G = \phi_{G-xy} - \phi_{G-x-y}$.)

8.6.6. Characteristic polynomial for paths and cycles.

a) Use Exercise 8.6.5 to find recurrences for $\phi(P_n; \lambda)$ and for $\phi(C_n; \lambda)$.

b) Without solving the recurrence, prove that $\{2 \cos(2\pi j/n) : 0 \leq j \leq n-1\}$ are the eigenvalues of C_n .

c) Given $\text{Spec}(C_n)$, compute $\text{Spec } G$, where G is the graph obtained from C_n by adding edges joining vertices at distance 2 in C_n .

8.6.7. For a tree, prove that the coefficient of λ^{n-2k} in the characteristic polynomial is $(-1)^k \mu_k(G)$, where $\mu_k(G)$ is the number of matchings of size k . Use this to construct a pair of nonisomorphic "co-spectral" 8-vertex trees; both have characteristic polynomial $\lambda^8 - 7\lambda^6 + 9\lambda^4$. (Comment: As $n \rightarrow \infty$, almost no trees are uniquely determined by their spectra.) (Schwenk [1973])

8.6.8. (+) Let T be a tree. Prove that $\alpha(T)$ is the number of nonnegative eigenvalues of T . (Hint: See Theorem 8.6.20.) (Cvetković-Doob-Sachs [1979, p233])

8.6.9. Let λ be an eigenvalue of a graph G with n vertices and m edges. Prove that $|\lambda| \leq \sqrt{2m(n-1)/n}$.

8.6.10. Let $\lambda_1, \dots, \lambda_m$ and μ_1, \dots, μ_n be the eigenvalues of G and H , respectively. Show that the mn eigenvalues of $G \square H$ are $\{\lambda_i + \mu_j\}$. Use this to derive the spectrum of the k -cube. (Hint: Given an eigenvector of $A(G)$ associated with λ_i and an eigenvector of $A(H)$ associated with μ_j , construct an eigenvector for $A(G \square H)$ associated with $\lambda_i + \mu_j$.)

8.6.11. Compute the spectrum of the complete p -partite graph $K_{m, \dots, m}$. (Hint: Use the expression $A(\bar{G}) = J - I - A(G)$ for the adjacency matrix of the complement.)

8.6.12. Given $\phi(G; x) = x^8 - 24x^6 - 64x^5 - 48x^4$, determine G .

8.6.13. (!) Prove that G is bipartite if G is connected and $\lambda_{\max}(G) = -\lambda_{\min}(G)$.

8.6.14. (!) Given a graph G , let $R(G)$ be the matrix whose i, j th entry is $d_G(v_i, v_j)$. Prove that the squashed-cube dimension of a graph (Definition 8.4.12) is at least the maximum of the number of positive eigenvalues and the number of negative eigenvalues of $R(G)$. Conclude that the squashed cube dimension of K_n is $n-1$. (Hint: Rewrite the quadratic form $x^T R x$ as a sum of squares of linear functions, and apply Sylvester's Law of Inertia.)

8.6.15. (!) The **Laplacian matrix** Q of a graph G is $D - A$, where D is the diagonal matrix of degrees and A is the adjacency matrix. The **Laplacian spectrum** is the list of eigenvalues of Q .

a) Prove that the smallest eigenvalue of Q is 0.

b) Prove that if G is connected, then eigenvalue 0 has multiplicity 1.

c) Prove that if G is k -regular, then $k - \lambda$ is a Laplacian eigenvalue if and only if λ is an ordinary eigenvalue of G , with the same multiplicity.

8.6.16. Given a real symmetric matrix partitioned as $M = \begin{pmatrix} P & Q \\ Q^t & R \end{pmatrix}$ with P, R square, a lemma in linear algebra yields $\lambda_{\max}(M) + \lambda_{\min}(M) \leq \lambda_{\max}(P) + \lambda_{\max}(R)$.

a) Let A be a real symmetric matrix partitioned into t^2 submatrices $A_{i,j}$ such that the diagonal submatrices A_{ii} are square. Prove that

$$\lambda_{\max}(A) + (t-1)\lambda_{\min}(A) \leq \sum_{i=1}^m \lambda_{\max}(A_{ii}).$$

b) Prove that $\chi(G) \geq 1 + \lambda_{\max}(G)/(-\lambda_{\min}(G))$ when G is nontrivial. (Wilf)

c) Use the Four Color Theorem to prove that $\lambda_1(G) + 3\lambda_n(G) \leq 0$ for planar graphs.

8.6.17. (!) Use Theorem 8.6.28 to count the spanning trees in $K_{m,m}$. (Comment: See Exercise 2.2.11.)

8.6.18. (+) Given a matrix A , let $b_{i,j}$ equal $(-1)^{i+j}$ times the matrix obtained by deleting row i and column j of A . Let $\text{Adj } A$ be the matrix whose entry in position i, j is $b_{j,i}$. The definition of the determinant by expansion along rows of A yields $A(\text{Adj } A) = (\det A)I$. Use this formula to prove that if the sum of the columns of A is the vector 0, then $b_{i,j}$ is independent of j . (Comment: With the next exercise, this completes the proof of the Matrix Tree Theorem (Theorem 2.2.12).)

8.6.19. (+) Let $C = AB$, where A and B are $n \times m$ and $m \times n$ matrices. Given $S \subseteq [m]$, let A_S be the $n \times n$ matrix whose columns are the columns of A indexed by S , and let B_S be the $n \times n$ matrix whose rows are the rows of B indexed by S . Prove the Binet–Cauchy Formula: $\det C = \sum_S \det A_S \det B_S$, where the summation extends over all n -element subsets of $[m]$. (Hint: Consider the matrix equation $\begin{pmatrix} I_n & 0 \\ A & I_m \end{pmatrix} \begin{pmatrix} -I_m & B \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -I_m & B \\ 0 & AB \end{pmatrix}$.)

8.6.20. A matrix is **totally unimodular** if every square submatrix has determinant in $\{0, 1, -1\}$. Prove that the incidence matrix of a simple graph is totally unimodular if and only if the graph is bipartite. (Reminder: The incidence matrix of a simple graph has two $+1$'s in each column.)

8.6.21. (–) Let G be an (n, k, c) -magnifier with vertices v_1, \dots, v_n . Let H be the bipartite graph with partite sets $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_n\}$ such that $x_i y_j \in E(H)$ if and only if $i = j$ or $v_i v_j \in E(G)$. Prove that H is an $(n, k+1, c)$ -expander.

8.6.22. *Existence of expanders of linear size.*

a) Let X be a random variable giving the size of the union of k s -subsets of $[n]$ chosen at random from $\binom{[n]}{s}$. Prove that $P(X \leq l) \leq \binom{n}{l} (l/n)^{ks}$.

b) (+) For $\alpha\beta < 1$, prove that there is a constant k such that, when n is sufficiently large, there exists a subgraph of $K_{n,n}$ with maximum degree at most k such that $|N(S)| \geq \beta|S|$ whenever $|S| \leq \alpha n$. (Hint: Generate bipartite subgraphs of $K_{n,n}$ by taking the union of k random perfect matchings.)

c) Conclude the existence of k such that n, k, c -expanders exist for all sufficiently large n . An (n, α, β, d) -**expander** is a bipartite graph $G \subseteq K_{A,B}$ with $|A| = |B| = n$, $\Delta(G) \leq d$, and $|N(S)| \geq \beta|S|$ whenever $|S| \leq \alpha n$.

8.6.23. Let G be a triangle-free graph on n vertices in which every pair of nonadjacent vertices has exactly two common neighbors. Prove that G is regular and that $n = 1 +$

$\binom{k+1}{2}$, where k is the degree of the vertices in G . Prove that G is strongly regular. What constraints on k are implied by the integrality conditions? Construct examples for all $k \in \{1, 2, 5\}$. A realization for $k = 10$ is known using combinatorial designs.)

8.6.24. (+) Prove that the Petersen graph is strongly regular, and determine its spectrum (the spectrum is easy with properties of strongly regular graphs and not hard without them). Apply the spectrum to show that edges of the complete graph K_{10} cannot be partitioned into three disjoint copies of the Petersen graph. (Hint: Use the spectrum to prove that two copies of the Petersen matrix have a common eigenvector other than the constant vector.) (Schwenk [1983])

8.6.25. Let $F = G \square H$, where G and H are simple graphs. Prove that if every two non-adjacent vertices in F have exactly two common neighbors, then G and H are complete graphs.

8.6.26. The **subconstituents** of a graph are the induced subgraphs of the form $G[U]$, where $v \in V(G)$ and $U = N(v)$ or $U = N[v]$. Vince [1989] defined G to be **superregular** if G has no vertices or if G is regular and every subconstituent of G is superregular. Let \mathbf{S} be the class consisting of $\{aK_b: a, b \geq 0\}$ (disjoint unions of isomorphic cliques), $\{K_m \square K_m: m \geq 0\}$, C_5 , and the complements of these graphs.

a) Prove that every graph in \mathbf{S} is superregular and that every disconnected superregular graph is in \mathbf{S} . (Comment: In fact, every superregular graph is in \mathbf{S} , but the complete inductive proof of this requires several pages (Maddox [1996], West [1996])

b) Prove that every superregular graph is strongly regular.

8.6.27. (+) *Automorphisms and eigenvalues.*

a) Prove that σ is an automorphism of G if and only if the permutation matrix corresponding to σ commutes with the adjacency matrix of G ; that is, $PA = AP$.

b) Let x be an eigenvector of G for an eigenvalue of multiplicity 1, and let P be the permutation matrix for an automorphism of G . Prove that $Px = \pm x$.

c) Conclude that when every eigenvalue of G has multiplicity 1, every automorphism of G is an involution, meaning that repeating it yields the identity. (Mowshowitz [1969], Petersdorf-Sachs [1969])

8.6.28. (+) Light bulbs l_1, \dots, l_n are controlled by switches s_1, \dots, s_n . The i th switch changes the on/off status of the i th light and possibly others, but s_i changes the status of l_i if and only if s_i changes the status of l_i . Initially all the lights are off. Prove that it is possible to turn all the lights on. (Peled [1992]) (Hint: This uses vector spaces, not eigenvalues.)

Appendix A

Mathematical Background

This appendix summarizes aspects of language and mathematics that are not directly part of graph theory but provide useful background for learning graph theory. Where appropriate, we mention examples in the context of graphs, so it is best to read this appendix in conjunction with Chapter 1. This presentation is modeled on material in the first half of *Mathematical Thinking*, by John P. D'Angelo and Douglas B. West (Prentice–Hall, second edition, 2000).

SETS

Our most primitive mathematical notion is that of a **set**. It is so fundamental that we cannot define it in terms of simpler concepts. We think of a set as a collection of distinct objects with a precise description that provides a way of deciding (in principle) whether a given object is in it.

A.1. Definition. The objects in a set are its **elements** or **members**. When x is an element of A , we write $x \in A$ and say “ x **belongs to** A ”. When x is not in A , we write $x \notin A$. If every element of a set B belongs to A , then B is a **subset** of A , and A **contains** B ; we write $B \subseteq A$ or $A \supseteq B$.

For example, we may speak of the set A of graphs with n vertices. When we impose an additional restriction, such as requiring that the graphs also be connected, we obtain a subset of A .

When we list the elements of a set explicitly, we put braces around the list; “ $A = \{-1, 1\}$ ” specifies the set A consisting of the elements -1 and 1 . Writing the elements in a different order does not change a set. We write $x, y \in S$ to mean that both x and y are elements of S .

A.2. Example. We use the characters \mathbb{N} , \mathbb{Z} , \mathbb{Q} , and \mathbb{R} to name the sets of **natural numbers**, **integers**, **rational numbers**, and **real numbers**, respectively. Each set in this list is contained in the next, so we write $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$.

We treat these sets and their elements as familiar objects. By convention, 0 is not a natural number, so $\mathbb{N} = \{1, 2, 3, \dots\}$. The set of integers is $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$. The set \mathbb{Q} of rational numbers is the set of real numbers expressible as a/b with $a, b \in \mathbb{Z}$ and $b \neq 0$.

We also take as familiar the elementary arithmetic properties of these number systems. These include the rules that permit algebraic manipulation of expressions, equalities, and inequalities. They also include elementary properties about divisibility of integers. ■

A.3. Definition. Sets A and B are **equal**, written $A = B$, if they have the same elements. The **empty set**, written \emptyset , is the unique set with no elements. A **proper subset** of a set A is a subset that is not A itself.

The empty set is a subset of every set, and every set is a subset of itself. The definition of subgraph (Definition 1.1.16) is similar. Every graph is a subgraph of itself, but something must be discarded to obtain a proper subgraph.

“Solving a mathematical problem” often means describing a given set more simply. We must show that the set of objects satisfying the new description is equal to the given set.

A.4. Remark. *Equality of sets.* To prove that $A = B$, we prove that every element of A is in B and that every element of B is in A ; in other words, $A \subseteq B$ and $B \subseteq A$. It also suffices to turn the description of one set into the description of the other by operations that do not change membership.

This book proves many characterization theorems for classes of graphs. Such a theorem states that two sets are the same (example: the set of bipartite graphs is equal to the set of graphs without odd cycles—Theorem 1.2.18).

Often, a mathematical model defines a set S of *solutions*; these are the objects that satisfy the conditions of the problem. We want to list or describe the solutions explicitly; this specifies a set T . The problem is to show that $S = T$. Proving $S \subseteq T$ means showing that every solution belongs to T . Proving $T \subseteq S$ means showing that every member of T is a solution. ■

A.5. Remark. *Specifying a set.* Given a set A , we may want to specify a subset S consisting of the elements of A that satisfy a given condition. To do so, we write “ $S = \{x \in A : \text{condition}(x)\}$ ”. We read this as “ S is the set of elements x in A such that x satisfies ‘condition’”. For example, the expression $\{n \in \mathbb{N} : n^2 \leq 25\}$ is another way to name the set $\{1, 2, 3, 4, 5\}$.

In this format, the set A is the **universe** for x ; we can drop this part of the notation when the context makes it clear. For example, $\{n^2 : n \in \mathbb{N}\}$ is the set of positive integers squares. ■

Many special sets have common names and/or notation.

A.6. Definition. When $a, b \in \mathbb{Z}$, we write $\{a, \dots, b\}$ for $\{i \in \mathbb{Z} : a \leq i \leq b\}$. When $n \in \mathbb{N}$, we write $[n]$ for $\{1, \dots, n\}$; also $[0] = \emptyset$. The set of **even numbers**

is $\{2k: k \in \mathbb{Z}\}$. The set of **odd numbers** is $\{2k + 1: k \in \mathbb{Z}\}$. The **parity** of an integer states whether it is even or odd.

Note that 0 is an even number. We say “even” and “odd” for numbers *only* when discussing integers. Every integer is even or odd; none is both.

A.7. Definition. A **partition** of a set A is a list A_1, \dots, A_k of subsets of A such that each element of A appears in exactly one subset in the list.

The set of even numbers and the set of odd numbers partition \mathbb{Z} . In a partition of A into A_1, \dots, A_k , the sets A_1, \dots, A_k in the list are called “blocks” or “classes” or “parts” or “partite sets”. The use of “blocks” is common in combinatorics, but graph theory has another definition for the word “block”, so we usually use “classes” or “sets”. “Partite sets” is used only for the sets in a partition of the vertex set of a graph into independent sets.

A.8. Remark. *Conventions about universes.* When we write “[n]”, it is understood that n is a nonnegative integer. When we speak of n as the number of vertices in a graph, by context we know that n is a natural number. When we say only that a number is positive without specifying the number system containing it, we mean that it is a positive real number. Thus, “consider $x > 0$ ” means “let x be a positive real number”, but in “For $n \geq 2$, let G be a n -vertex graph” our convention is that $n \in \mathbb{N}$. ■

A.9. Definition. A set A is **finite** if there is a one-to-one correspondence between A and $[n]$ for some $n \in \mathbb{N} \cup \{0\}$. This n is the **size** of A , written $|A|$.

Another elementary property of number systems is that a set A cannot be in one-to-one correspondence with both $[m]$ and $[n]$ when $m \neq n$. Thus the size of a finite set is a well-defined integer. **Counting** a set means determining its size.

A.10. Remark. *“If” in definitions.* It is a common convention in defining mathematical properties to say that an object has a certain property **if** it satisfies a certain condition. Subsequently, the condition can be substituted for the property and vice versa, so the “if” really means “if and only if”. This conventional usage in definitions reflects the notion that the concept being defined does not exist until the definition is complete. ■

There are several natural ways to obtain new sets from old sets.

A.11. Definition. Let A and B be sets. Their **union** $A \cup B$ consists of all elements in A or in B (or both). Their **intersection** $A \cap B$ consists of all elements in both A and B . Their **difference** $A - B$ consists of the elements of A that are not in B . Their **symmetric difference** $A \triangle B$ is the set of elements belonging to exactly one of A and B .

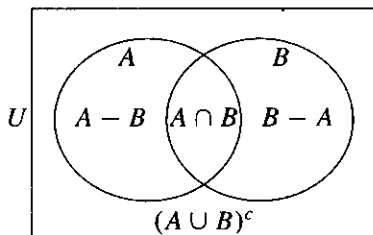
Two sets are **disjoint** if their intersection is the empty set \emptyset . If a set A

is contained in some universe U under discussion, then the **complement** \bar{A} of A is the set of elements of U not in A .

When we speak of taking the “complement” of a simple graph, we are keeping the vertex set unchanged and taking the complement of the edge set (viewed as pairs of vertices) within the universe of vertex pairs. Other times we speak of the complement \bar{S} of a set of vertices S in G ; in this case we mean $\bar{S} = V(G) - S$.

A.12. Remark. In a **Venn diagram**, an outer box represents the universe under consideration, and regions within the box correspond to sets. Non-overlapping regions correspond to disjoint sets. The four regions in the Venn diagram for two sets A and B represent $A \cap B$, $\overline{(A \cup B)}$, $A - B$, and $B - A$. Note that $A \Delta B = (A - B) \cup (B - A)$,

Since $A - B$ consists of the elements in A and not in B , we have $A - B = A \cap \bar{B}$. Similarly, the diagram suggests that \bar{B} is the union of $A - B$ and $\overline{(A \cup B)}$, which are disjoint. It also suggests that the symmetric difference $A \Delta B$ is obtained from the union by deleting the intersection. ■



A.13. Remark. When A and B are sets, $A \Delta B = (A \cup B) - (A \cap B)$. The union starts with all elements in at least one of A and B ; we delete those in both.

When A and B are finite sets, $|A \cup B| + |A \cap B| = |A| + |B|$. Each element of the intersection is counted twice on both sides, each element of the symmetric difference is counted once on both sides, and no other elements are counted. ■

A.14. Definition. A list with entries in A consists of elements of A in a specified order, with repetition allowed. A **k -tuple** is a list with k entries. We write A^k for the set of k -tuples with entries in A . When $A = \{0, 1\}$, A^k is the set of **binary k -tuples**.

An **ordered pair** (x, y) is a list with two entries. The **cartesian product** of sets S and T , written $S \times T$, is the set $\{(x, y): x \in S, y \in T\}$.

Note that $A^2 = A \times A$ and $A^k = \{(x_1, \dots, x_k): x_i \in A\}$. We read “ x_i ” as “ x sub i ”. When $S = T = \mathbb{Z}$, the cartesian product $S \times T$ is the **integer lattice**, the set of points in the plane with integer coordinates.

QUANTIFIERS AND PROOFS

Roughly speaking, a mathematical statement is a statement that can be determined to be true or false. This requires correct mathematical grammar, and it requires that variables be “quantified”.

For example, the sentence $x^2 - 4 = 0$ cannot be determined to be true or false because we do not know the value of x . It becomes a mathematical statement if we precede it with “When $x = 3$,” or “For $x \in \{2, -2\}$,” or “For some integer x ,”.

If a sentence $P(x)$ becomes a mathematical statement whenever the variable x takes a value in the set S , then the two sentences below are mathematical statements.

“For all x in S , the sentence $P(x)$ is true.”

“For some x in S , the sentence $P(x)$ is true.”

A.15. Definition. In the statement “For all x in S , $P(x)$ is true”, the variable x is **universally quantified**. We write this as $(\forall x \in S)P(x)$ and say that \forall is a **universal quantifier**. In “For some x in S , $P(x)$ is true”, the variable x is **existentially quantified**. We write this as $(\exists x \in S)P(x)$ and say that \exists is an **existential quantifier**. The set of allowed values for a variable is its **universe**.

A.16. Remark. *English words that express quantification.* Typically, “every” and “for all” represent universal quantifiers, while “some” and “there is” represent existential quantifiers. We can also express universal quantification by referring to an arbitrary element of the universe, as in “Let x be an integer”, or “A student failing the exam will fail the course”. Below we list common indicators of quantification.

Universal (\forall)	(helpers)	Existential (\exists)	(helpers)
for [all], for every		for some	
if	then	there exists	such that
whenever, for, given		at least one	for which
every, any	satisfies	some	satisfies
a, arbitrary	must, is	has a	such that
let	be		

The “helpers” may be absent. Consider “The square of a real number is non-negative”. This means $x^2 \geq 0$ for *every* $x \in \mathbb{R}$; it is not a statement about one real number and cannot be verified by an example. When we write “A bipartite graph has no odd cycle”, we mean “in every bipartite graph there is no odd cycle”. When we write “Let G be a bipartite graph”, we mean that every bipartite graph is under consideration. When we take an “arbitrary” vertex in a graph, we are considering each one individually. When we discuss an “arbitrary” pair of vertices in a graph, we are considering each pair, one at a time.

The difference between “for every G ” and “for every graph G ” is that the latter specifies the universe for the universally quantified variable G . ■

Existential quantifiers state lower bounds; “there is a” and “there are two” mean “at least one” and “at least two”. Phrases like “there is a unique” and “there are exactly two” indicate equality. Sometimes equality is clear from context, but it does not hurt to make it explicit when it is intended.

A statement may have more than one quantifier. Consider the sentence “There are triangle-free graphs with arbitrarily large chromatic number”. Phrased using explicit quantifiers, this means “For every $n \in \mathbb{N}$, there exists a triangle-free graph with chromatic number at least n ”. The expression “arbitrarily large” often conveys an implicit universal quantifier in this way.

In contrast, the expression “sufficiently large” imposes an implicit existential quantifier. The statement “ $2^n > n^{1000}$ when n is sufficiently large” means “There exists $N \in \mathbb{N}$ such that for all $n \geq N$, the inequality $2^n > n^{1000}$ holds”.

A.17. Remark. The meaning of a statement with more than one quantifier depends on their order. Compare these two sentences:

“For every graph G , there exists $m \in \mathbb{N}$ such that every $v \in V(G)$ has degree at most m ”
 “There exists $m \in \mathbb{N}$ such that for every graph G , every $v \in V(G)$ has degree at most m ”

The first statement is true; the second is false. Every (finite) graph has a maximum degree, but there is no maximum over all graphs. We write the two sentences in logical notation as

$$\begin{aligned} &(\forall G)(\exists m \in \mathbb{N})(\forall v \in V(G))(d_G(v) \leq m). \\ &(\exists m \in \mathbb{N})(\forall G)(\forall v \in V(G))(d_G(v) \leq m). \end{aligned}$$

In English, quantifiers often appear at the ends of sentence to enhance readability, as in “I feel happy every time I learn something new.” In sentences with abstract concepts and more than one quantifier, we adopt conventions about order to avoid confusion. Quantifiers apply in the order in which they are stated. In particular, a variable is chosen in terms of the preceding variables.

For example, in $(\forall G)(\exists m \in \mathbb{N})P(G, m)$, we have the freedom to choose m after knowing what G is. In $(\exists m \in \mathbb{N})(\forall G)P(G, m)$, we must choose a single m that works for all G . ■

A.18. Remark. *Negation of quantified statements.* The logical symbol for negation is \neg . If it is false that all $x \in S$ make $P(x)$ true, then there must be some $x \in S$ such that $P(x)$ is false. Similarly, negating an existentially quantified statement yields a universally quantified negation. In notation,

$$\begin{aligned} \neg[(\forall x \in S)P(x)] &\text{ has the same meaning as } (\exists x \in S)(\neg P(x)). \\ \neg[(\exists x \in S)P(x)] &\text{ has the same meaning as } (\forall x \in S)(\neg P(x)). \end{aligned}$$

The universe of quantification *does not change* when the statement is negated. For example, the false statement in Remark A.17 was

$$(\exists m \in \mathbb{N})(\forall G)(\forall v \in V(G))(d_G(v) \leq m).$$

Its negation is the same as $(\forall m \in \mathbb{N})(\exists G)[\neg((\forall v \in V(G))(d_G(v) \leq m))]$, which we further simplify to $(\forall m \in \mathbb{N})(\exists G)(\exists v \in V(G))(d_G(v) > m)$. This statement is

“for every natural number m , there is some graph having a vertex with degree greater than m ”, which is true. ■

Logical connectives permit us to build compound statements.

A.19. Definition. *Logical connectives.* In the following table, we define the operations named in the first column by the truth values specified in the last column.

Name	Symbol	Meaning	Condition for truth
Negation	$\neg P$	not P	P false
Conjunction	$P \wedge Q$	P and Q	both true
Disjunction	$P \vee Q$	P or Q	at least one true
Biconditional	$P \Leftrightarrow Q$	P if & only if Q	same truth value
Conditional	$P \Rightarrow Q$	P implies Q	Q true whenever P true

A.20. Remark. Conjunction and disjunction are quantifiers over the truth of their component statements. A conjunction (“and”) is true precisely when all of its component statements are true. A disjunction (“or”) is true precisely when there exists a true statement among its components. Our understanding of negation thus yields logical equivalence between $\neg(P \wedge Q)$ and $(\neg P) \vee (\neg Q)$ and between $\neg(P \vee Q)$ and $(\neg P) \wedge (\neg Q)$. ■

A.21. Definition. In the conditional statement $P \Rightarrow Q$, we call P the **hypothesis** and Q the **conclusion**. The statement $Q \Rightarrow P$ is the **converse** of $P \Rightarrow Q$.

A.22. Remark. *Conditionals.* Conditional statements are the only type in Definition A.19 whose meaning changes when P and Q are interchanged. There is no general implication between $P \Rightarrow Q$ and its converse $Q \Rightarrow P$. Consider these three statements about a graph G : P is “ G is a path”, Q is “ G is bipartite”, and R is “ G has no odd cycles”. Here $P \Rightarrow Q$ is true but $Q \Rightarrow P$ is false. On the other hand, both $Q \Rightarrow R$ and $R \Rightarrow Q$ are true.

Note that here G is a variable. We have dropped G from the notation for the statements because the context is clear. The precise meaning of $P \Rightarrow Q$ using G is $(\forall G)(P(G) \Rightarrow Q(G))$.

A conditional statement is false when and only when the hypothesis is true and the conclusion is false. Thus the meaning of $P \Rightarrow Q$ is $(\neg P) \vee Q$; the two are logically equivalent. Every conditional statement with a false hypothesis is true, regardless of whether the conclusion is true. The meaning of $\neg(P \Rightarrow Q)$ is $P \wedge (\neg Q)$.

Below we list ways to say $P \Rightarrow Q$ in English. ■

If P (is true), then Q (is true).	P is true only if Q is true.
Q is true whenever P is true.	P is a sufficient condition for Q .
Q is true if P is true.	Q is a necessary condition for P .

The business of mathematics is proving implications. Note that universally quantified statements can be interpreted as conditional statements. The statement “ $(\forall G \in \mathbf{G})(P(G))$ ” has the same meaning as “If $G \in \mathbf{G}$, then $P(G)$ ” (consider the two statements when \mathbf{G} is the family of bipartite graphs and $P(G)$ is the assertion that G has no odd cycles).

The basic proof methods come from the meaning of conditional statements.

A.23. Remark. Proving implications. The **direct method** of proving $P \Rightarrow Q$ is to assume that P is true and then to apply mathematical reasoning to deduce that Q is true. When P is “ $x \in A$ ” and Q is “ $Q(x)$ ”, the direct method considers an *arbitrary* $x \in A$ and deduces $Q(x)$. There is no “proof by example”. The proof must apply to every member of A as a possible instance of x .

The **contrapositive** of $P \Rightarrow Q$ is $\neg Q \Rightarrow \neg P$. Each of these statements fails only when P is true and Q is false. Thus they are equivalent; we can prove $P \Rightarrow Q$ by proving $\neg Q \Rightarrow \neg P$. This is the **contrapositive method**.

We have observed that $(P \Rightarrow Q) \Leftrightarrow \neg[P \wedge (\neg Q)]$. Hence we can prove $P \Rightarrow Q$ by proving that P and $\neg Q$ cannot both be true. We do this by obtaining a contradiction after assuming both P and $\neg Q$. This is the **method of contradiction**.

The two latter methods are **indirect proof**. When the direct method for $P \Rightarrow Q$ doesn't seem to work, we say “Well, suppose not”. At that point we are starting from the assumption $\neg Q$. We need not know in advance whether we are seeking to derive $\neg P$ (contrapositive method) or seeking to use P and $\neg Q$ to obtain a contradiction. ■

Examples of each of these methods appear in the text. Indirect proof is promising when the negation of the conclusion provides useful information. This approach may be easier than finding a direct proof, because both the hypothesis and the negation of the conclusion can be used. If the contradiction we obtain is the impossibility of our original assumption $\neg Q$, then usually we can rewrite the proof in simpler language as a direct proof. If instead we obtain $\neg P$, then we have proved the contrapositive.

A.24. Remark. Biconditional statements. The biconditional statement “ $P \Leftrightarrow Q$ ” has the same meaning as “ $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$ ”. We read it as “ P if and only if Q ”, where “ $Q \Rightarrow P$ ” is “ P if Q ”, and “ $P \Rightarrow Q$ ” is “ P only if Q ”.

Although sometimes we can prove a biconditional statement by a chain of equivalences, usually we prove a conditional statement and its converse; the latter is also a conditional statement. For each we have the three fundamental methods above. To prove $P \Leftrightarrow Q$, we must prove one statement in each column in the table below. The lines are the direct method, the contrapositive method, and the method of contradiction, respectively. Proving two statements in the same column would amount to proving the same statement twice. ■

$P \Rightarrow Q$	$Q \Rightarrow P$
$\neg Q \Rightarrow \neg P$	$\neg P \Rightarrow \neg Q$
$\neg(P \wedge \neg Q)$	$\neg(Q \wedge \neg P)$

Students sometimes wonder about the precise meanings of words like “theorem”, “lemma”, and “corollary” that are used to designate mathematical results. In Greek, *lemma* means “premise” and *theorema* means “thesis to be proved”. Thus a theorem is a major result requiring some effort. A lemma is a lesser statement, usually proved in order to help prove other statements. A proposition is something “proposed” to be proved; typically this takes less effort than a theorem. The word *corollary* comes from Latin, as a modification of a word meaning “gift”; a corollary follows easily from a theorem or proposition, without much additional work.

INDUCTION AND RECURRENCE

Many statements having a natural number as a variable can be proved using the technique of induction. In Theorem 1.2.1, we describe the strong version of induction. Here we review the ordinary version that most students learn when they first encounter induction. It involves the Well Ordering Property for the natural numbers, which states that every nonempty subset of \mathbb{N} has a least element. We take this as an axiom, as part of our intuitive understanding of what \mathbb{N} is. Although we then state the Principle of Induction as a Theorem, in reality it is equivalent to the Well Ordering Property for \mathbb{N} .

A.25. Theorem. (Principle of Induction) For each natural number n , let $P(n)$ be a mathematical statement. If properties (a) and (b) below hold, then for each $n \in \mathbb{N}$ the statement $P(n)$ is true.

a) $P(1)$ is true.

b) For $k \in \mathbb{N}$, if $P(k)$ is true, then $P(k + 1)$ is true.

Proof: If $P(n)$ is not true for all n , then the set of natural numbers where it fails is nonempty. By the Well Ordering Property, there is a least natural number in this set. By (a), this number cannot be 1. By (b), it cannot be bigger than 1. The contradiction implies that $P(n)$ is true for all n . ■

When applying the method of induction, we prove statement (a) in Theorem A.25 as the **basis step** and statement (b) as the **induction step**. Statement (b) is a conditional statement, and its hypothesis (“ $P(k)$ ” is true) is the **induction hypothesis**. We present one example in rather formal language.

A.26. Proposition. If S is a set of n lines in the plane such that every two have exactly one common point and no three have a common point, then S cuts the plane into $1 + n(n + 1)/2$ regions.

Proof: We use induction on n to prove the claim for all $n \in \mathbb{N}$. Let $P(n)$ be the statement that the claim holds for all such sets of n lines.

Basis step ($P(1)$). With one line the number of regions is 2, which equals $1 + 1(1 + 1)/2$.

Induction step ($P(k) \Rightarrow P(k + 1)$). The statement $P(k)$ is the induction hypothesis. Let S be a set of $k + 1$ lines meeting the conditions. Select a line L

in S (the dashed line in the figure), and let S' be the set of k lines obtained by deleting L from S .

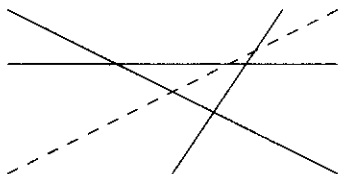
Since S' meets the conditions, the induction hypothesis states that S' cuts the plane into $1 + k(k + 1)/2$ regions. When we replace L , some regions are cut. The increase in the number of regions is the number of regions that L cuts. It moves from one of these regions to another each time it crosses a line in S' . Since L crosses each line in S' once, the lines in S' cut L into $k + 1$ pieces. Each piece corresponds to a region that L cuts.

Thus the number of regions formed by S is $k + 1$ more than the number of regions formed by S' . The number of regions formed by S is

$$1 + k(k + 1)/2 + (k + 1) = 1 + (k + 1)(k + 2)/2.$$

We have proved that $P(k)$ implies $P(k + 1)$.

By the principle of induction, the claim holds for every $n \in \mathbb{N}$. ■



A.27. Remark. The discussion of Proposition A.26 suggests several comments about proof by induction. Note first that we could also have used $n = 0$ as the basis step to prove the statement for all nonnegative n .

It is not immediately obvious from the statement of the problem that the number of regions is the same for all sets of n lines, but this follows because we proved a formula for this number that depends only on n .

In the proof of the induction step, we began with L , an instance of the larger-sized problem. This approach ensures that we have considered all such instances; we return to this point shortly.

We proved $P(k + 1)$ from $P(k)$ as suggested by statement (b) of Theorem A.25. In most examples in this book, we use a different phrasing that is more consistent with strong induction as introduced in Section 1.2. To prove $P(n)$ for all $n \in \mathbb{N}$, in this example we would write “Basis step: $n = 1, \dots$ ” and then “Induction step: $n > 1, \dots$ ”. In the proof of the induction step, we would consider an arbitrary set S of n lines and apply the induction hypothesis to the set S' obtained by deleting one line L .

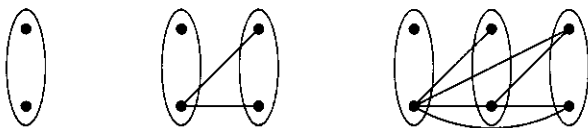
The content of the proof is the same in both phrasings. The phrasing that we have just described emphasizes the item about which the claim is proved. The basis step directly verifies the claim for the smallest value of the induction parameter. When the parameter has a larger value, the claim about the item is proved using the hypothesis that it holds for an earlier item; this is the induction step. Invoking it (repeatedly) yields the claim for each subsequent value of the parameter. ■

When learning to use induction in graph theory, many students have trouble with two particular aspects. One is when the statement $P(n)$ being proved by induction is itself a conditional statement $A(n) \Rightarrow B(n)$. The induction hypothesis is the statement $A(n-1) \Rightarrow B(n-1)$. A template for the induction step in this situation is presented in Remark 1.3.25, and there are examples of this throughout Chapter 1.

The other pitfall we call the “induction trap”, discussed at length in Example 1.3.26. Here we provide another example, using the language of proving $P(n+1)$ from $P(n)$ that sometimes leads students into the trap.

A.28. Example. The Handshake Problem. Let a **handshake party** of order n (henceforth “ n -party”) be a party with n married couples where no spouses shake hands with each other and the $2n-1$ people other than the host shake hands with different numbers of people. We use induction on n to prove that in every n -party, the hostess shakes with exactly $n-1$ people.

We model the party using a simple graph in which the vertices are the people at the party and the edges are the pairs who shake hands. The degree of a vertex is its number of handshaking partners. If no one shakes with his or her spouse, then each degree is between 0 and $2n-2$. The condition that the $2n-1$ numbers other than the host’s are distinct implies that the degrees are 0 through $2n-2$. The figure below shows for $n \in \{1, 2, 3\}$ the graph that is forced; each circled pair of vertices indicates a married couple, with host and hostess rightmost in each graph.



Basis step: If $n = 1$, then the hostess shakes with 0 (which equals $n-1$), because the host and hostess don’t shake.

Induction step (INVALID): The induction hypothesis is that the claim holds for n -parties. Consider such a party. By the induction hypothesis, the degree of the hostess is $n-1$. By our earlier discussion, the degrees of vertices other than the host are $0, \dots, 2n-2$. We form an $(n+1)$ -party by adding one more couple. Let one member of the new couple shake with everyone in the first n couples; the other shakes with no one. This increases the degree of each of the earlier vertices by 1, so those degrees other than the host are now $1, \dots, 2n-1$, and the new couple have degrees 0 and $2n$. Hence the larger configuration is an $(n+1)$ -party. The degree of the hostess has increased by 1, so it is n .

Induction step (VALID): The induction hypothesis is that the claim holds for n -parties. Consider an $(n+1)$ -party. By our earlier discussion, the degrees other than the host are $0, \dots, 2n$. Let p_i denote the person of degree i among these. Since p_{2n} shakes with all but one person, the person p_0 who shakes with no one must be the only person missed by p_{2n} . Hence p_0 is the spouse of