now in another fashion. We already saw that $Q_8/\langle -1 \rangle \cong V_4$. The discussion for D_8 in the next paragraph could be applied equally well to Q_8 to give an independent identification of the isomorphism type of the quotient.

Let $G = D_8$ and let $Z = \langle r^2 \rangle = Z(D_8)$. Since $Z = \{1, r^2\}$, each coset, gZ, consists of the two element set $\{g, gr^2\}$. Since these cosets partition the 8 elements of D_8 into pairs, there must be 4 (disjoint) left cosets of Z in D_8:

$$\bar{1} = 1Z, \quad \bar{r} = rZ, \quad \bar{s} = sZ, \quad \text{and} \quad \overline{rs} = rsZ.$$

Now by the classification of groups of order 4 (Exercise 10, Section 2.5) we know that $D_8/Z(D_8) \cong Z_4$ or V_4. To determine which of these two is correct (i.e., determine the isomorphism type of the quotient) simply observe that

$$(\bar{r})^2 = r^2Z = 1Z = \bar{1}$$
$$(\bar{s})^2 = s^2Z = 1Z = \bar{1}$$
$$(\overline{rs})^2 = (rs)^2Z = 1Z = \bar{1}$$

so every nonidentity element in D_8/Z has order 2. In particular there is no element of order 4 in the quotient, hence D_8/Z is not cyclic so $D_8/Z(D_8) \cong V_4$.

EXERCISES

Let G and H be groups.

1. Let $\varphi : G \to H$ be a homomorphism and let E be a subgroup of H. Prove that $\varphi^{-1}(E) \leq G$ (i.e., the preimage or pullback of a subgroup under a homomorphism is a subgroup). If $E \trianglelefteq H$ prove that $\varphi^{-1}(E) \trianglelefteq G$. Deduce that $\ker \varphi \trianglelefteq G$.
2. Let $\varphi : G \to H$ be a homomorphism of groups with kernel K and let $a, b \in \varphi(G)$. Let $X \in G/K$ be the fiber above a and let Y be the fiber above b, i.e., $X = \varphi^{-1}(a)$, $Y = \varphi^{-1}(b)$. Fix an element u of X (so $\varphi(u) = a$). Prove that if $XY = Z$ in the quotient group G/K and w is any member of Z, then there is some $v \in Y$ such that $uv = w$. [Show $u^{-1}w \in Y$.]
3. Let A be an abelian group and let B be a subgroup of A. Prove that A/B is abelian. Give an example of a non-abelian group G containing a proper normal subgroup N such that G/N is abelian.
4. Prove that in the quotient group G/N, $(gN)^\alpha = g^\alpha N$ for all $\alpha \in \mathbb{Z}$.
5. Use the preceding exercise to prove that the order of the element gN in G/N is n, where n is the smallest positive integer such that $g^n \in N$ (and gN has infinite order if no such positive integer exists). Give an example to show that the order of gN in G/N may be strictly smaller than the order of g in G.
6. Define $\varphi : \mathbb{R}^\times \to \{\pm 1\}$ by letting $\varphi(x)$ be x divided by the absolute value of x. Describe the fibers of φ and prove that φ is a homomorphism.
7. Define $\pi : \mathbb{R}^2 \to \mathbb{R}$ by $\pi((x, y)) = x + y$. Prove that π is a surjective homomorphism and describe the kernel and fibers of π geometrically.
8. Let $\varphi : \mathbb{R}^\times \to \mathbb{R}^\times$ be the map sending x to the absolute value of x. Prove that φ is a homomorphism and find the image of φ. Describe the kernel and the fibers of φ.
9. Define $\varphi : \mathbb{C}^\times \to \mathbb{R}^\times$ by $\varphi(a + bi) = a^2 + b^2$. Prove that φ is a homomorphism and find the image of φ. Describe the kernel and the fibers of φ geometrically (as subsets of the plane).

10. Let $\varphi : \mathbb{Z}/8\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$ by $\varphi(\bar{a}) = \bar{a}$. Show that this is a well defined, surjective homomorphism and describe its fibers and kernel explicitly (showing that φ is well defined involves the fact that $\bar{a}$ has a different meaning in the domain and range of φ).

11. Let F be a field and let $G = \{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in F,\ ac \neq 0\} \leq GL_2(F)$.

(a) Prove that the map $\varphi : \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mapsto a$ is a surjective homomorphism from G onto $F^\times$ (recall that $F^\times$ is the multiplicative group of nonzero elements in F). Describe the fibers and kernel of φ.

(b) Prove that the map $\psi : \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mapsto (a, c)$ is a surjective homomorphism from G onto $F^\times \times F^\times$. Describe the fibers and kernel of ψ.

(c) Let $H = \{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in F\}$. Prove that H is isomorphic to the additive group F.

12. Let G be the additive group of real numbers, let H be the multiplicative group of complex numbers of absolute value 1 (the unit circle S^1 in the complex plane) and let $\varphi : G \to H$ be the homomorphism $\varphi : r \mapsto e^{2\pi i r}$. Draw the points on a real line which lie in the kernel of φ. Describe similarly the elements in the fibers of φ above the points -1, i, and $e^{4\pi i/3}$ of H. (Figure 1 of the text for this homomorphism φ is usually depicted using the following diagram.)

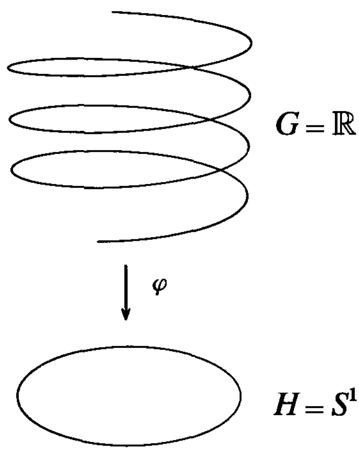

Fig. 5

13. Repeat the preceding exercise with the map φ replaced by the map $\varphi : r \mapsto e^{4\pi i r}$.

14. Consider the additive quotient group $\mathbb{Q}/\mathbb{Z}$.

(a) Show that every coset of $\mathbb{Z}$ in $\mathbb{Q}$ contains exactly one representative $q \in \mathbb{Q}$ in the range $0 \leq q < 1$.

(b) Show that every element of $\mathbb{Q}/\mathbb{Z}$ has finite order but that there are elements of arbitrarily large order.

(c) Show that $\mathbb{Q}/\mathbb{Z}$ is the torsion subgroup of $\mathbb{R}/\mathbb{Z}$ (cf. Exercise 6, Section 2.1).

(d) Prove that $\mathbb{Q}/\mathbb{Z}$ is isomorphic to the multiplicative group of root of unity in $\mathbb{C}^\times$.

15. Prove that a quotient of a divisible abelian group by any proper subgroup is also divisible. Deduce that $\mathbb{Q}/\mathbb{Z}$ is divisible (cf. Exercise 19, Section 2.4).

16. Let G be a group, let N be a normal subgroup of G and let $\overline{G} = G/N$. Prove that if

$G = \langle x, y \rangle$ then $\overline{G} = \langle \overline{x}, \overline{y} \rangle$. Prove more generally that if $G = \langle S \rangle$ for any subset S of G, then $\overline{G} = \langle \overline{S} \rangle$.

17. Let G be the dihedral group of order 16 (whose lattice appears in Section 2.5):

$$G = \langle r, s \mid r^8 = s^2 = 1,\ rs = sr^{-1} \rangle$$

and let $\overline{G} = G/\langle r^4 \rangle$ be the quotient of G by the subgroup generated by r^4 (this subgroup is the center of G, hence is normal).

(a) Show that the order of $\overline{G}$ is 8.

(b) Exhibit each element of $\overline{G}$ in the form $\overline{s}^a\overline{r}^b$, for some integers a and b.

(c) Find the order of each of the elements of $\overline{G}$ exhibited in (b).

(d) Write each of the following elements of $\overline{G}$ in the form $\overline{s}^a\overline{r}^b$, for some integers a and b as in (b): $\overline{rs}$, $\overline{sr^{-2}s}$, $\overline{s^{-1}r^{-1}sr}$.

(e) Prove that $\overline{H} = \langle \overline{s}, \overline{r}^2 \rangle$ is a normal subgroup of $\overline{G}$ and $\overline{H}$ is isomorphic to the Klein 4-group. Describe the isomorphism type of the complete preimage of $\overline{H}$ in G.

(f) Find the center of $\overline{G}$ and describe the isomorphism type of $\overline{G}/Z(\overline{G})$.

18. Let G be the quasidihedral group of order 16 (whose lattice was computed in Exercise 11 of Section 2.5):

$$G = \langle \sigma, \tau \mid \sigma^8 = \tau^2 = 1,\ \sigma\tau = \tau\sigma^3 \rangle$$

and let $\overline{G} = G/\langle \sigma^4 \rangle$ be the quotient of G by the subgroup generated by σ^4 (this subgroup is the center of G, hence is normal).

(a) Show that the order of $\overline{G}$ is 8.

(b) Exhibit each element of $\overline{G}$ in the form $\overline{\tau}^a\overline{\sigma}^b$, for some integers a and b.

(c) Find the order of each of the elements of $\overline{G}$ exhibited in (b).

(d) Write each of the following elements of $\overline{G}$ in the form $\overline{\tau}^a\overline{\sigma}^b$, for some integers a and b as in (b): $\overline{\sigma\tau}$, $\overline{\tau\sigma^{-2}\tau}$, $\overline{\tau^{-1}\sigma^{-1}\tau\sigma}$.

(e) Prove that $\overline{G} \cong D_8$.

19. Let G be the modular group of order 16 (whose lattice was computed in Exercise 14 of Section 2.5):

$$G = \langle u, v \mid u^2 = v^8 = 1,\ vu = uv^5 \rangle$$

and let $\overline{G} = G/\langle v^4 \rangle$ be the quotient of G by the subgroup generated by v^4 (this subgroup is contained in the center of G, hence is normal).

(a) Show that the order of $\overline{G}$ is 8.

(b) Exhibit each element of $\overline{G}$ in the form $\overline{u}^a\overline{v}^b$, for some integers a and b.

(c) Find the order of each of the elements of $\overline{G}$ exhibited in (b).

(d) Write each of the following elements of $\overline{G}$ in the form $\overline{u}^a\overline{v}^b$, for some integers a and b as in (b): $\overline{vu}$, $\overline{uv^{-2}u}$, $\overline{u^{-1}v^{-1}uv}$.

(e) Prove that $\overline{G}$ is abelian and is isomorphic to $Z_2 \times Z_4$.

20. Let $G = \mathbb{Z}/24\mathbb{Z}$ and let $\widetilde{G} = G/\langle \overline{12} \rangle$, where for each integer a we simplify notation by writing $\widetilde{\overline{a}}$ as $\widetilde{a}$.

(a) Show that $\widetilde{G} = \{\widetilde{0}, \widetilde{1}, \dots, \widetilde{11}\}$.

(b) Find the order of each element of $\widetilde{G}$.

(c) Prove that $\widetilde{G} \cong \mathbb{Z}/12\mathbb{Z}$. (Thus $(\mathbb{Z}/24\mathbb{Z})/(12\mathbb{Z}/24\mathbb{Z}) \cong \mathbb{Z}/12\mathbb{Z}$, just as if we inverted and cancelled the $24\mathbb{Z}$'s.)

21. Let $G = Z_4 \times Z_4$ be given in terms of the following generators and relations:

$$G = \langle x, y \mid x^4 = y^4 = 1,\ xy = yx \rangle.$$

Let $\overline{G} = G/\langle x^2y^2 \rangle$ (note that every subgroup of the abelian group G is normal).

(a) Show that the order of $\overline{G}$ is 8.

(b) Exhibit each element of $\overline{G}$ in the form $\overline{x}^a\overline{y}^b$, for some integers a and b.

(c) Find the order of each of the elements of $\overline{G}$ exhibited in (b).

(d) Prove that $\overline{G} \cong Z_4 \times Z_2$.

22. **(a)** Prove that if H and K are normal subgroups of a group G then their intersection $H \cap K$ is also a normal subgroup of G.

(b) Prove that the intersection of an arbitrary nonempty collection of normal subgroups of a group is a normal subgroup (do not assume the collection is countable).

23. Prove that the join (cf. Section 2.5) of any nonempty collection of normal subgroups of a group is a normal subgroup.

24. Prove that if $N \trianglelefteq G$ and H is any subgroup of G then $N \cap H \trianglelefteq H$.

25. **(a)** Prove that a subgroup N of G is normal if and only if $gNg^{-1} \subseteq N$ for *all* $g \in G$.

(b) Let $G = GL_2(\mathbb{Q})$, let N be the subgroup of upper triangular matrices with integer entries and 1's on the diagonal, and let g be the diagonal matrix with entries 2,1. Show that $gNg^{-1} \subseteq N$ but g does *not* normalize N.

26. Let $a, b \in G$.

(a) Prove that the conjugate of the product of a and b is the product of the conjugate of a and the conjugate of b. Prove that the order of a and the order of any conjugate of a are the same.

(b) Prove that the conjugate of a^{-1} is the inverse of the conjugate of a.

(c) Let $N = \langle S \rangle$ for some subset S of G. Prove that $N \trianglelefteq G$ if $gSg^{-1} \subseteq N$ for all $g \in G$.

(d) Deduce that if N is the cyclic group $\langle x \rangle$, then N is normal in G if and only if for each $g \in G$, $gxg^{-1} = x^k$ for some $k \in \mathbb{Z}$.

(e) Let n be a positive integer. Prove that the subgroup N of G generated by all the elements of G of order n is a normal subgroup of G.

27. Let N be a *finite* subgroup of a group G. Show that $gNg^{-1} \subseteq N$ if and only if $gNg^{-1} = N$. Deduce that $N_G(N) = \{g \in G \mid gNg^{-1} \subseteq N\}$.

28. Let N be a *finite* subgroup of a group G and assume $N = \langle S \rangle$ for some subset S of G. Prove that an element $g \in G$ normalizes N if and only if $gSg^{-1} \subseteq N$.

29. Let N be a *finite* subgroup of G and suppose $G = \langle T \rangle$ and $N = \langle S \rangle$ for some subsets S and T of G. Prove that N is normal in G if and only if $tSt^{-1} \subseteq N$ for all $t \in T$.

30. Let $N \leq G$ and let $g \in G$. Prove that $gN = Ng$ if and only if $g \in N_G(N)$.

31. Prove that if $H \leq G$ and N is a normal subgroup of H then $H \leq N_G(N)$. Deduce that $N_G(N)$ is the largest subgroup of G in which N is normal (i.e., is the join of all subgroups H for which $N \trianglelefteq H$).

32. Prove that every subgroup of Q_8 is normal. For each subgroup find the isomorphism type of its corresponding quotient. [You may use the lattice of subgroups for Q_8 in Section 2.5.]

33. Find all normal subgroups of D_8 and for each of these find the isomorphism type of its corresponding quotient. [You may use the lattice of subgroups for D_8 in Section 2.5.]

34. Let $D_{2n} = \langle r, s \mid r^n = s^2 = 1,\ rs = sr^{-1} \rangle$ be the usual presentation of the dihedral group of order $2n$ and let k be a positive integer dividing n.

(a) Prove that $\langle r^k \rangle$ is a normal subgroup of D_{2n}.

(b) Prove that $D_{2n}/\langle r^k \rangle \cong D_{2k}$.