Theorem 10 is extremely useful in defining homomorphisms on $M \otimes_R N$ since it replaces the often tedious check that maps defined on simple tensors $m \otimes n$ are well defined with a check that a related map defined on ordered pairs (m, n) is balanced.

The first consequence of the universal property in Theorem 10 is a characterization of the tensor product $M \otimes_R N$ as an abelian group:

Corollary 11. Suppose D is an abelian group and $\iota' : M \times N \to D$ is an R-balanced map such that

(i) the image of ι' generates D as an abelian group, and

(ii) every R-balanced map defined on $M \times N$ factors through ι' as in Theorem 10.

Then there is an isomorphism $f : M \otimes_R N \cong D$ of abelian groups with $\iota' = f \circ \iota$.

Proof: Since $\iota' : M \times N \to D$ is a balanced map, the universal property in (2) of Theorem 10 implies there is a (unique) homomorphism $f : M \otimes_R N \to D$ with $\iota' = f \circ \iota$. In particular $\iota'(m, n) = f(m \otimes n)$ for every $m \in M$, $n \in N$. By the first assumption on ι', these elements generate D as an abelian group, so f is a surjective map. Now, the balanced map $\iota : M \times N \to M \otimes_R N$ together with the second assumption on ι' implies there is a (unique) homomorphism $g : D \to M \otimes_R N$ with $\iota = g \circ \iota'$. Then $m \otimes n = (g \circ f)(m \otimes n)$. Since the simple tensors $m \otimes n$ generate $M \otimes_R N$, it follows that $g \circ f$ is the identity map on $M \otimes_R N$ and so f is injective, hence an isomorphism. This establishes the corollary.

We now return to the question of giving the abelian group $M \otimes_R N$ a *module* structure. As we observed in the special case of extending scalars from R to S for the R-module N, the S-module structure on $S \otimes_R N$ required only a left S-module structure on S together with the compatibility relation $s'(sr) = (s's)r$ for $s, s' \in S$ and $r \in R$. In this special case this relation was simply a consequence of the associative law in the ring S. To obtain an S-module structure on $M \otimes_R N$ more generally we impose a similar structure on M:

Definition. Let R and S be any rings with 1. An abelian group M is called an (S, R)-*bimodule* if M is a left S-module, a right R-module, and $s(mr) = (sm)r$ for all $s \in S$, $r \in R$ and $m \in M$.

Examples

(1) Any ring S is an (S, R)-bimodule for any subring R with $1_R = 1_S$ by the associativity of the multiplication in S. More generally, if $f : R \to S$ is any ring homomorphism with $f(1_R) = 1_S$ then S can be considered as a right R-module with the action $s \cdot r = sf(r)$, and with respect to this action S becomes an (S, R)-bimodule.

(2) Let I be an ideal (two-sided) in the ring R. Then the quotient ring R/I is an $(R/I, R)$-bimodule. This is easy to see directly and is also a special case of the previous example (with respect to the canonical projection homomorphism $R \to R/I$).

(3) Suppose that R is a commutative ring. Then a left (respectively, right) R-module M can always be given the structure of a right (respectively, left) R-module by defining $mr = rm$ (respectively, $rm = mr$), for all $m \in M$ and $r \in R$, and this makes M into

an (R, R)-bimodule. Hence every module (right or left) over a commutative ring R has at least one natural (R, R)-bimodule structure.

(4) Suppose that M is a left S-module and R is a subring contained in the *center* of S (for example, if S is commutative). Then in particular R is commutative so M can be given a right R-module structure as in the previous example. Then for any $s \in S$, $r \in R$ and $m \in M$ by definition of the right action of R we have

$$(sm)r = r(sm) = (rs)m = (sr)m = s(rm) = s(mr)$$

(note that we have used the fact that r commutes with s in the middle equality). Hence M is an (S, R)-bimodule with respect to this definition of the right action of R.

Since the situation in Example 3 occurs so frequently, we give this bimodule structure a name:

Definition. Suppose M is a left (or right) R-module over the commutative ring R. Then the (R, R)-bimodule structure on M defined by letting the left and right R-actions coincide, i.e., $mr = rm$ for all $m \in M$ and $r \in R$, will be called the *standard* R-module structure on M.

Suppose now that N is a left R-module and M is an (S, R)-bimodule. Then just as in the example of extension of scalars the (S, R)-bimodule structure on M implies that

$$s\left(\sum_{\text{finite}} m_i \otimes n_i\right) = \sum_{\text{finite}} (sm_i) \otimes n_i \tag{10.8}$$

gives a well defined action of S under which *$M \otimes_R N$ is a left S-module*. Note that Theorem 10 may be used to give an alternate proof that (8) is well defined, replacing the direct calculations on the relations defining the tensor product with the easier check that a map is R-balanced, as follows. It is very easy to see that for each fixed $s \in S$ the map $(m, n) \mapsto sm \otimes n$ is an R-balanced map from $M \times N$ to $M \otimes_R N$. By Theorem 10 there is a well defined group homomorphism λ_s from $M \otimes_R N$ to itself such that $\lambda_s(m \otimes n) = sm \otimes n$. Since the right side of (8) is then $\lambda_s(\sum m_i \otimes n_i)$, the fact that λ_s is well defined shows that this expression is indeed independent of the representation of the tensor $\sum m_i \otimes n_i$ as a sum of simple tensors. Because λ_s is additive, equation (8) holds.

By a completely parallel argument, if M is a right R-module and N is an (R, S)-bimodule then the tensor product $M \otimes_R N$ has the structure of a *right* S-module, where $(\sum m_i \otimes n_i)s = \sum m_i \otimes (n_i s)$.

Before giving some more examples of tensor products it is worthwhile to highlight one frequently encountered special case of the previous discussion, namely the case when M and N are two left modules over a *commutative* ring R and $S = R$ (in some works on tensor products this is the only case considered). Then the standard R-module structure on M defined previously gives M the structure of an (R, R)-bimodule, so in this case the tensor product $M \otimes_R N$ always has the structure of a left R-module.

The corresponding map $\iota : M \times N \to M \otimes_R N$ maps $M \times N$ into an R-module and is additive in each factor. Since $r(m \otimes n) = rm \otimes n = mr \otimes n = m \otimes rn$ it also satisfies

$$r\iota(m, n) = \iota(rm, n) = \iota(m, rn).$$

Such maps are given a name:

Definition. Let R be a commutative ring with 1 and let M, N, and L be left R-modules. The map $\varphi : M \times N \to L$ is called *R-bilinear* if it is R-linear in each factor, i.e., if

$$\varphi(r_1 m_1 + r_2 m_2, n) = r_1\varphi(m_1, n) + r_2\varphi(m_2, n), \quad \text{and}$$
$$\varphi(m, r_1 n_1 + r_2 n_2) = r_1\varphi(m, n_1) + r_2\varphi(m, n_2)$$

for all $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$ and $r_1, r_2 \in R$.

With this terminology Theorem 10 gives

Corollary 12. Suppose R is a commutative ring. Let M and N be two left R-modules and let $M \otimes_R N$ be the tensor product of M and N over R, where M is given the standard R-module structure. Then $M \otimes_R N$ is a left R-module with

$$r(m \otimes n) = (rm) \otimes n = (mr) \otimes n = m \otimes (rn),$$

and the map $\iota : M \times N \to M \otimes_R N$ with $\iota(m, n) = m \otimes n$ is an R-bilinear map. If L is any left R-module then there is a bijection

$$\left\{ \begin{array}{c} R\text{-bilinear maps} \\ \varphi : M \times N \to L \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} R\text{-module homomorphisms} \\ \Phi : M \otimes_R N \to L \end{array} \right\}$$

where the correspondence between φ and Φ is given by the commutative diagram

$$\begin{tikzcd} M \times N \arrow[r, "\iota"] \arrow[dr, "\varphi"'] & M \otimes_R N \arrow[d, "\Phi"] \\ & L \end{tikzcd}$$

Proof: We have shown $M \otimes_R N$ is an R-module and that ι is bilinear. It remains only to check that in the bijective correspondence in Theorem 10 the bilinear maps correspond with the R-module homomorphisms. If $\varphi : M \times N \to L$ is bilinear then it is an R-balanced map, so the corresponding $\Phi : M \otimes_R N$ is a group homomorphism. Moreover, on simple tensors $\Phi((rm) \otimes n) = \varphi(rm, n) = r\varphi(m, n) = r\Phi(m \otimes n)$, where the middle equality holds because φ is R-linear in the first variable. Since Φ is additive this extends to sums of simple tensors to show Φ is an R-module homomorphism. Conversely, if Φ is an R-module homomorphism it is an exercise to see that the corresponding balanced map φ is bilinear.

Examples

(1) In any tensor product $M \otimes_R N$ we have $m \otimes 0 = m \otimes (0 + 0) = (m \otimes 0) + (m \otimes 0)$, so $m \otimes 0 = 0$. Likewise $0 \otimes n = 0$.

(2) We have $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} = 0$, since $3a = a$ for $a \in \mathbb{Z}/2\mathbb{Z}$ so that

$$a \otimes b = 3a \otimes b = a \otimes 3b = a \otimes 0 = 0$$

and every simple tensor is reduced to 0. In particular $1 \otimes 1 = 0$. It follows that there are no nonzero balanced (or bilinear) maps from $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ to any abelian group.