

The Jacobson radical is analogous to the Frattini subgroup of a group, and it enjoys some corresponding properties (cf. Exercise 24 in Section 6.1):

Proposition 1. Let \mathcal{J} be the Jacobson radical of the commutative ring R .

- (1) If I is a proper ideal of R , then so is (I, \mathcal{J}) , the ideal generated by I and \mathcal{J} .
- (2) The Jacobson radical contains the nilradical of R : $\text{rad } 0 \subseteq \text{Jac } R$.
- (3) An element x belongs to \mathcal{J} if and only if $1 - rx$ is a unit for all $r \in R$.
- (4) (*Nakayama's Lemma*) If M is any finitely generated R -module and $\mathcal{J}M = M$, then $M = 0$.

Proof: If I is a proper ideal in R , then $I \subseteq M$ for some maximal ideal M . Since $\mathcal{J} \subseteq M$, also $(I, \mathcal{J}) \subseteq M$, which proves (1).

Part (2) follows from the definitions of the two radicals and Proposition 12 in Section 15.2 since maximal ideals are prime.

Suppose $1 - rx$ is not a unit and let M be a maximal ideal containing $1 - rx$. Since $1 \notin M$, $rx \notin M$, so x cannot belong to \mathcal{J} because $\mathcal{J} \subseteq M$. Conversely, suppose $x \notin \mathcal{J}$, i.e., there is a maximal ideal M with $x \notin M$. Then $R = (x, M)$, hence $1 = rx + y$ for some $y \in M$. Thus $1 - rx = y \in M$ and so $1 - rx$ is not a unit, which proves (3).

To prove (4), assume $M \neq 0$ and let n be the smallest integer such that M is generated by n elements, say m_1, \dots, m_n . Since $M = \mathcal{J}M$ we have

$$m_n = r_1 m_1 + r_2 m_2 + \cdots + r_n m_n \quad \text{for some } r_1, r_2, \dots, r_n \in \mathcal{J}.$$

Thus $(1 - r_n)m_n = r_1 m_1 + \cdots + r_{n-1} m_{n-1}$. By (3), $1 - r_n$ is a unit, so m_n lies in the module generated by m_1, \dots, m_{n-1} , contradicting the minimality of n . Hence $M = 0$, completing the proof.

Definition. A commutative ring R is said to be *Artinian* or to satisfy the *descending chain condition on ideals* (or *D.C.C. on ideals*) if there is no infinite decreasing chain of ideals in R , i.e., whenever $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ is a decreasing chain of ideals of R , then there is a positive integer m such that $I_k = I_m$ for all $k \geq m$. Similarly, an R -module M is said to be Artinian if it satisfies D.C.C. on submodules.

It is immediate from the Lattice Isomorphism Theorem that every quotient R/I of an Artinian ring R by an ideal I is again an Artinian ring.

The following result for Artinian rings is parallel to results in Theorem 15.2. The proof is completely analogous, and so is left as an exercise.

Proposition 2. The following are equivalent:

- (1) R is an Artinian ring.
- (2) Every nonempty set of ideals of R contains a minimal element under inclusion.

The next result gives the main structure theorem for Artinian rings.

Theorem 3. Let R be an Artinian ring.

- (1) There are only finitely many maximal ideals in R .
- (2) The quotient $R/(\text{Jac } R)$ is a direct product of a finite number of fields. More precisely, if M_1, \dots, M_n are the finitely many maximal ideals in R then

$$R/(\text{Jac } R) \cong k_1 \times \cdots \times k_n,$$

where k_i is the field R/M_i for $1 \leq i \leq n$.

- (3) Every prime ideal of R is maximal, i.e., R has Krull dimension 0. The Jacobson radical of R equals the nilradical of R and is a nilpotent ideal: $(\text{Jac } R)^m = 0$ for some $m \geq 1$.
- (4) The ring R is isomorphic to the direct product of a finite number of Artinian local rings.
- (5) Every Artinian ring is Noetherian.

Proof: To prove (1), let \mathcal{S} be the set of all ideals of R that are the intersection of a finite number of maximal ideals. By Proposition 2, \mathcal{S} has a minimal element, say $M_1 \cap M_2 \cap \cdots \cap M_n$. Then for any maximal ideal M we have

$$M \cap M_1 \cap M_2 \cap \cdots \cap M_n = M_1 \cap M_2 \cap \cdots \cap M_n,$$

so $M \supseteq M_1 \cap M_2 \cap \cdots \cap M_n$. By Exercise 11 in Section 7.4, $M \supseteq M_i$ for some i . Thus $M = M_i$ and so M_1, \dots, M_n are all the maximal ideals of R .

The proof of (2) is immediate from the Chinese Remainder Theorem (Section 7.6) applied to M_1, \dots, M_n , since these maximal ideals are clearly pairwise comaximal and their intersection is $\text{Jac } R$.

For (3), we first prove $\mathcal{J} = \text{Jac } R$ is nilpotent. By D.C.C. there is some $m > 0$ such that $\mathcal{J}^m = \mathcal{J}^{m+i}$ for all positive i . By way of contradiction assume $\mathcal{J}^m \neq 0$. Let \mathcal{S} be the set of proper ideals I such that $I\mathcal{J}^m \neq 0$, so $\mathcal{J} \in \mathcal{S}$. Let I_0 be a minimal element of \mathcal{S} . There is some $x \in I_0$ such that $x\mathcal{J}^m \neq 0$, so by minimality we must have $I_0 = (x)$. But now $((x)\mathcal{J})\mathcal{J}^m = x\mathcal{J}^{m+1} = x\mathcal{J}^m$, so it follows by minimality of (x) that $(x) = (x)\mathcal{J}$. By Nakayama's Lemma above, $(x) = 0$, a contradiction. This proves $\text{Jac } R$ is nilpotent.

Since $\text{Jac } R$ is nilpotent, in particular $\text{Jac } R \subseteq \text{rad } R$, so these two ideals are equal by the second statement in Proposition 1.

Every prime ideal P in R contains the nilradical of R , hence contains $\text{Jac } R$ by what has already been proved. The image of P is a prime ideal in the quotient ring $R/(\text{Jac } R) = k_1 \times \cdots \times k_n$. But in a direct product of rings $R_1 \times R_2$ (where each R_i has a 1) every ideal is of the form $I_1 \times I_2$, where I_j is an ideal of R_j for $j = 1, 2$ (cf. Exercise 3 in Section 7.6). It follows that a prime ideal in $k_1 \times \cdots \times k_n$ consists of the elements that are 0 in one of the components. In particular, such a prime ideal is also a maximal ideal in $k_1 \times \cdots \times k_n$ and it follows that P was a maximal ideal in R , which finishes the proof of (3).

Let M_1, \dots, M_n be all the distinct maximal ideals of R and let $(\text{Jac } R)^m = 0$ as in (3). Then

$$\prod_{i=1}^n M_i^m \subseteq \left(\prod_{i=1}^n M_i \right)^m \subseteq (\text{Jac } R)^m = 0.$$

By the Chinese Remainder Theorem it follows that

$$R \cong (R/M_1^m) \times (R/M_2^m) \times \cdots \times (R/M_n^m),$$

and each R/M_i^m is an Artinian ring with unique maximal ideal M_i/M_i^m , proving (4).

To prove (5), it suffices by (4) to prove that an Artinian local ring is Noetherian, so assume R is Artinian with unique maximal ideal M . In this case we have $M = \text{Jac } R$, so $M^m = (\text{Jac } R)^m = 0$ for some positive m . Then $R \cong R/M^m$, and in this case it is an exercise to see that R/M^m is Noetherian if and only if it is Artinian (cf. Exercise 8).

Corollary 4. The ring R is Artinian if and only if R is Noetherian and has Krull dimension 0.

Proof: The forward implication was proved in Theorem 3. Suppose now that R is Noetherian and that R has Krull dimension 0, i.e., that prime ideals of R are maximal. Since R is Noetherian, by Corollary 22(3) in Section 15.2, the ideal $(0) = P_1 \cdots P_n$ is the product of (not necessarily distinct) prime ideals, and these prime ideals are then maximal since R has dimension 0. By the Chinese Remainder Theorem, R is isomorphic to the direct product of a finite number of Noetherian rings of the form R/M^m where M is a maximal ideal in R . As in the proof of (5) of the theorem, R/M^m is Artinian, and it follows that R is Artinian.

Examples

- (1) Let $n > 1$ be an integer. Since the ring $R = \mathbb{Z}/n\mathbb{Z}$ is finite, it is Artinian. If $n = p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s}$ is the unique factorization of n into distinct prime powers, then

$$\mathbb{Z}/n\mathbb{Z} \cong (\mathbb{Z}/p_1^{a_1}\mathbb{Z}) \times (\mathbb{Z}/p_2^{a_2}\mathbb{Z}) \times \cdots \times (\mathbb{Z}/p_s^{a_s}\mathbb{Z}).$$

Each $\mathbb{Z}/p_i^{a_i}\mathbb{Z}$ is an Artinian local ring with unique maximal ideal $(p_i)/(p_i^{a_i})$, so this is the decomposition of $\mathbb{Z}/n\mathbb{Z}$ given by Theorem 3(4). The Jacobson radical of R is the ideal generated by $p_1 p_2 \cdots p_s$, the squarefree part of n and $R/(\text{Jac } R) \cong (\mathbb{Z}/p_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/p_s\mathbb{Z})$ is a direct product of fields. The ideals generated by p_i for $i = 1, \dots, s$ are the maximal ideals of R .

- (2) For any field k , a k -algebra R that is finite dimensional as a vector space over k is Artinian because ideals in R are in particular k -subspaces of R , hence the length of any chain of ideals in R is bounded by $\dim_k R$.
- (3) Suppose f is a nonzero polynomial in $k[x]$ where k is a field. Then the quotient ring $R = k[x]/(f(x))$ is Artinian by the previous example. The decomposition of R as a direct product of Artinian local rings is given by

$$k[x]/(f(x)) \cong k[x]/(f_1(x)^{a_1}) \times \cdots \times k[x]/(f_s(x)^{a_s})$$

where $f(x) = f_1(x)^{a_1} \cdots f_s(x)^{a_s}$ is the factorization of $f(x)$ into powers of distinct irreducibles in $k[x]$ (cf. Proposition 16 in Section 9.5). The Jacobson radical of R is the ideal generated by the squarefree part of $f(x)$ and the maximal ideals of R are the ideals generated by the irreducible factors $f_i(x)$ for $i = 1, \dots, s$ similar to Example 1.