Thus our proof by mathematical induction is complete.

This method, but not the above proof, is explained verbally by Proclus in commenting on a passage in Plato's *Republic*. Today, it is easy to obtain explicit formulas for the numbers a_n and b_n. First, one proves by mathematical induction that

$$a_n + b_n\sqrt{2} = (1+\sqrt{2})^n .$$

Replacing the square root by its negative, one obtains

$$a_n - b_n\sqrt{2} = (1-\sqrt{2})^n .$$

Therefore,

$$a_n = \frac{1}{2}((1+\sqrt{2})^n + (1-\sqrt{2})^n),$$

$$b_n = \frac{1}{2\sqrt{2}}((1+\sqrt{2})^n - (1-\sqrt{2})^n).$$

Although the Pythagoreans did not know it, they had actually found all solutions of the equations $x^2 - 2y^2 = \pm 1$ in positive integers. Suppose, for example, $x^2 - 2y^2 = 1$. Let n be the largest natural number such that $(1+\sqrt{2})^n \leq x + y\sqrt{2}$, then

$$(1+\sqrt{2})^n \leq x + y\sqrt{2} < (1+\sqrt{2})^{n+1}.$$

Multiplying this by $(1-\sqrt{2})^n = a_n - b_n\sqrt{2}$ and assuming that n is even, we obtain

$$(1) \qquad 1 \leq (x + y\sqrt{2})(a_n - b_n\sqrt{2}) < 1 + \sqrt{2}.$$

Taking negative reciprocals of this, we get

$$(2) \qquad -1 \leq (-x + y\sqrt{2})(a_n + b_n\sqrt{2}) < 1 - \sqrt{2}.$$

Adding (1) and (2) and dividing by $2\sqrt{2}$, we obtain

$$0 \leq ya_n - xb_n < 1/\sqrt{2}.$$

Since $ya_n - xb_n$ is a whole number, it must be 0, hence $ya_n = xb_n$. Now we know that x and y are relatively prime, and so are a_n and b_n . It easily follows that $x = a_n$ and $y = b_n$, where n is even.

If n is odd or if $x^2 - 2y^2 = -1$, we proceed similarly.

Exercises

1. Prove that the decimal expression of $\sqrt{2}$ is not ultimately periodic.

2. Prove that the following numbers are not rational: $\sqrt{3}$, $\sqrt[3]{2}$ and $\log_{10} 2$.

3. If a, b, c and d are integers and $a + b\sqrt{2} = c + d\sqrt{2}$, show that $a = c$ and $b = d$.

4. Solve the following equations for positive integers:

$$x^2 - 4y^2 = 1, \quad x^2 - 3y^2 = 1.$$

11

From Heraclitus to Democritus

Heraclitus of Ephesus (in western Turkey) flourished about 500 BC, Parmenides of Elea (in southern Italy) about 480 BC, Zeno of Elea about 460 BC, Empedocles in Sicily about 440 BC, Democritus of Abdera (in north-eastern Greece) about 420 BC.

In the *Metaphysics* (986b4-8), Aristotle tells us that the Pythagoreans had a list of opposites: one, many; finite, infinite; male, female; etc. It was perhaps this list which led Heraclitus to his view that everything that happens is the result of a struggle between opposites. He proclaimed that all change is the result of strife.

Heraclitus believed that everything is in flux. It was he who asserted that one cannot step into the same river twice. Not surprisingly, he thought the fundamental substance was fire, and declared that all matter can be transformed into fire (and vice versa), just as all goods can be exchanged for gold. Did he anticipate the modern discovery that mass can be transformed into energy?

Heraclitus has had a great deal of influence on the twentieth century, largely through the nineteenth century Prussian philosopher Hegel. Influenced by Heraclitus, Hegel taught that the universe is a sort of debating society in which 'thesis' and 'antithesis' are forever struggling to produce a 'synthesis'. Marx adopted this philosophy, giving it a materialistic slant, and the views of Heraclitus ended up forming part of the official doctrine of Marxist governments, now much in decline.

Heraclitus has had less influence on logic. On one occasion he expressed his doctrine of continual change by saying that the river we step into both is and is not the same. Yet, in most logical systems, any statement of the

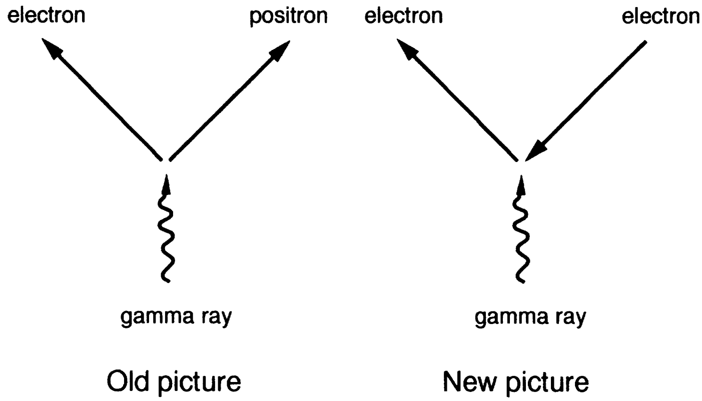

FIGURE 11.1. Positron as an electron travelling backwards in time

form 'p and not p' is regarded to be false. Hegelians sometimes adopt a similar mode of speech, claiming that 'a is not always equal to a'. Needless to say, this doctrine has not been applied to mathematics.

Yet Marxist philosophers try to understand not only history, but also mathematics in terms of a dialectic process. According to Lenin, subtraction is the antithesis of addition, yielding arithmetic as a synthesis, and integration is the antithesis of differentiation, the synthesis being calculus. Quite recently, the American mathematician Lawvere has suggested that a foundation of mathematics be built on a dialectic process in which the striving opposites are so-called 'adjoint functors', but this concept is too technical to be explained here.

Parmenides took the view opposite to that of Heraclitus, proclaiming that nothing changes, that change is an illusion: from the point of view of the 'goddess', the past and the future are all there at the same time. This is a bit like the view of the modern physicist and his four-dimensional space-time, in which the ever-changing events are replaced by unchanging world-lines.

Richard Feynman has recently shown that this way of viewing the universe allows one to give a more elegant and instructive explanation of certain fundamental processes. For example, to explain how an electron and a positron annihilate each other, giving rise to a γ ray, we may take it that the positron is an electron traveling backwards in time, having been deflected with a γ ray splitting off. Simultaneous pair creation is explained similarly. One may even speculate that there is only one electron in the universe. See Figure 11.1.

Zeno was a disciple of Parmenides. He produced four arguments attempting to prove that motion is impossible, his so-called 'paradoxes'. What he