defines an inner product on V. Furthermore, if we introduce the **conjugate transpose** matrix B^*, where $B^*_{kj} = \overline{B}_{jk}$, we may express this inner product on $F^{n\times n}$ in terms of the trace function:

$$(A|B) = \operatorname{tr}(AB^*) = \operatorname{tr}(B^*A).$$

For

$$\begin{aligned}\operatorname{tr}(AB^*) &= \sum_j (AB^*)_{jj}\\ &= \sum_j \sum_k A_{jk}B^*_{kj}\\ &= \sum_j \sum_k A_{jk}\overline{B}_{jk}.\end{aligned}$$

Example 4. Let $F^{n\times 1}$ be the space of $n \times 1$ (column) matrices over F, and let Q be an $n \times n$ invertible matrix over F. For X, Y in $F^{n\times 1}$ set

$$(X|Y) = Y^*Q^*QX.$$

We are identifying the 1×1 matrix on the right with its single entry. When Q is the identity matrix, this inner product is essentially the same as that in Example 1; we call it the **standard inner product** on $F^{n\times 1}$. The reader should note that the terminology 'standard inner product' is used in two special contexts. For a general finite-dimensional vector space over F, there is no obvious inner product that one may call standard.

Example 5. Let V be the vector space of all continuous complex-valued functions on the unit interval, $0 \leq t \leq 1$. Let

$$(f|g) = \int_0^1 f(t)\overline{g(t)}\, dt.$$

The reader is probably more familiar with the space of real-valued continuous functions on the unit interval, and for this space the complex conjugate on g may be omitted.

Example 6. This is really a whole class of examples. One may construct new inner products from a given one by the following method. Let V and W be vector spaces over F and suppose $(\ |\ )$ is an inner product on W. If T is a non-singular linear transformation from V into W, then the equation

$$p_T(\alpha, \beta) = (T\alpha|T\beta)$$

defines an inner product p_T on V. The inner product in Example 4 is a special case of this situation. The following are also special cases.

(a) Let V be a finite-dimensional vector space, and let

$$\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}$$

be an ordered basis for V. Let $\epsilon_1, \ldots, \epsilon_n$ be the standard basis vectors in F^n, and let T be the linear transformation from V into F^n such that $T\alpha_j = \epsilon_j$, $j = 1, \ldots, n$. In other words, let T be the 'natural' isomorphism of V onto F^n that is determined by $\mathcal{B}$. If we take the standard inner product on F^n, then

$$p_T(\sum_j x_j\alpha_j, \sum_k y_k\alpha_k) = \sum_{j=1}^{n} x_j\bar{y}_j.$$

Thus, for any basis for V there is an inner product on V with the property $(\alpha_j|\alpha_k) = \delta_{jk}$; in fact, it is easy to show that there is exactly one such inner product. Later we shall show that every inner product on V is determined by some basis $\mathcal{B}$ in the above manner.

(b) We look again at Example 5 and take $V = W$, the space of continuous functions on the unit interval. Let T be the linear operator 'multiplication by t,' that is, $(Tf)(t) = tf(t)$, $0 \leq t \leq 1$. It is easy to see that T is linear. Also T is non-singular; for suppose $Tf = 0$. Then $tf(t) = 0$ for $0 \leq t \leq 1$; hence $f(t) = 0$ for $t > 0$. Since f is continuous, we have $f(0) = 0$ as well, or $f = 0$. Now using the inner product of Example 5, we construct a new inner product on V by setting

$$\begin{aligned} p_T(f, g) &= \int_0^1 (Tf)(t)\overline{(Tg)(t)}\,dt \\ &= \int_0^1 f(t)\overline{g(t)}t^2\,dt. \end{aligned}$$

We turn now to some general observations about inner products. Suppose V is a complex vector space with an inner product. Then for all α, β in V

$$(\alpha|\beta) = \operatorname{Re}(\alpha|\beta) + i\operatorname{Im}(\alpha|\beta)$$

where $\operatorname{Re}(\alpha|\beta)$ and $\operatorname{Im}(\alpha|\beta)$ are the real and imaginary parts of the complex number $(\alpha|\beta)$. If z is a complex number, then $\operatorname{Im}(z) = \operatorname{Re}(-iz)$. It follows that

$$\operatorname{Im}(\alpha|\beta) = \operatorname{Re}[-i(\alpha|\beta)] = \operatorname{Re}(\alpha|i\beta).$$

Thus the inner product is completely determined by its 'real part' in accordance with

$$(\alpha|\beta) = \operatorname{Re}(\alpha|\beta) + i\operatorname{Re}(\alpha|i\beta). \tag{8-2}$$

Occasionally it is very useful to know that an inner product on a real or complex vector space is determined by another function, the so-called quadratic form determined by the inner product. To define it, we first denote the positive square root of $(\alpha|\alpha)$ by $||\alpha||$; $||\alpha||$ is called the **norm** of α with respect to the inner product. By looking at the standard inner products in R^1, C^1, R^2, and R^3, the reader should be able to convince himself that it is appropriate to think of the norm of α as the 'length' or 'magnitude' of α. The **quadratic form** determined by the inner product is the function that assigns to each vector α the scalar $||\alpha||^2$. It follows from the properties of the inner product that

$$||\alpha \pm \beta||^2 = ||\alpha||^2 \pm 2 \operatorname{Re} (\alpha|\beta) + ||\beta||^2$$

for all vectors α and β. Thus in the real case

$$(\alpha|\beta) = \frac{1}{4} ||\alpha + \beta||^2 - \frac{1}{4} ||\alpha - \beta||^2. \tag{8-3}$$

In the complex case we use (8-2) to obtain the more complicated expression

$$(\alpha|\beta) = \frac{1}{4} ||\alpha + \beta||^2 - \frac{1}{4} ||\alpha - \beta||^2 + \frac{i}{4} ||\alpha + i\beta||^2 - \frac{i}{4} ||\alpha - i\beta||^2. \tag{8-4}$$

Equations (8-3) and (8-4) are called the **polarization identities.** Note that (8-4) may also be written as follows:

$$(\alpha|\beta) = \frac{1}{4} \sum_{n=1}^{4} i^n ||\alpha + i^n\beta||^2.$$

The properties obtained above hold for any inner product on a real or complex vector space V, regardless of its dimension. We turn now to the case in which V is finite-dimensional. As one might guess, an inner product on a finite-dimensional space may always be described in terms of an ordered basis by means of a matrix.

Suppose that V is finite-dimensional, that

$$\mathfrak{B} = \{\alpha_1, \ldots, \alpha_n\}$$

is an ordered basis for V, and that we are given a particular inner product on V; we shall show that the inner product is completely determined by the values

$$G_{jk} = (\alpha_k|\alpha_j) \tag{8-5}$$

it assumes on pairs of vectors in $\mathfrak{B}$. If $\alpha = \sum_k x_k\alpha_k$ and $\beta = \sum_j y_j\alpha_j$, then

$$\begin{aligned}
(\alpha|\beta) &= (\sum_k x_n\alpha_k|\beta) \\
&= \sum_k x_k(\alpha_k|\beta) \\
&= \sum_k x_k \sum_j \bar{y}_j(\alpha_k|\alpha_j) \\
&= \sum_{j,k} \bar{y}_j G_{jk} x_k \\
&= Y^*GX
\end{aligned}$$

where X, Y are the coordinate matrices of α, β in the ordered basis $\mathfrak{B}$, and G is the matrix with entries $G_{jk} = (\alpha_k|\alpha_j)$. We call G the **matrix of the inner product in the ordered basis** $\mathfrak{B}$. It follows from (8-5) that G is hermitian, i.e., that $G = G^*$; however, G is a rather special kind of hermitian matrix. For G must satisfy the additional condition

$$X^*GX > 0, \qquad X \neq 0. \tag{8-6}$$

In particular, G must be invertible. For otherwise there exists an $X \neq 0$ such that $GX = 0$, and for any such X, (8-6) is impossible. More explicitly, (8-6) says that for any scalars $x_1, \ldots, x_n$ not all of which are 0

$$\sum_{j,k} \bar{x}_j G_{jk} x_k > 0. \tag{8-7}$$

From this we see immediately that each diagonal entry of G must be positive; however, this condition on the diagonal entries is by no means sufficient to insure the validity of (8-6). Sufficient conditions for the validity of (8-6) will be given later.

The above process is reversible; that is, if G is any $n \times n$ matrix over F which satisfies (8-6) and the condition $G = G^*$, then G is the matrix in the ordered basis $\mathcal{B}$ of an inner product on V. This inner product is given by the equation

$$(\alpha|\beta) = Y^*GX$$

where X and Y are the coordinate matrices of α and β in the ordered basis $\mathcal{B}$.

Exercises

1. Let V be a vector space and $(\ |\ )$ an inner product on V.

(a) Show that $(0|\beta) = 0$ for all β in V.

(b) Show that if $(\alpha|\beta) = 0$ for all β in V, then $\alpha = 0$.

2. Let V be a vector space over F. Show that the sum of two inner products on V is an inner product on V. Is the difference of two inner products an inner product? Show that a positive multiple of an inner product is an inner product.

3. Describe explicitly all inner products on R^1 and on C^1.

4. Verify that the standard inner product on F^n is an inner product.

5. Let $(\ |\ )$ be the standard inner product on R^2.

(a) Let $\alpha = (1, 2)$, $\beta = (-1, 1)$. If γ is a vector such that $(\alpha|\gamma) = -1$ and $(\beta|\gamma) = 3$, find γ.

(b) Show that for any α in R^2 we have $\alpha = (\alpha|\epsilon_1)\epsilon_1 + (\alpha|\epsilon_2)\epsilon_2$.

6. Let $(\ |\ )$ be the standard inner product on R^2, and let T be the linear operator $T(x_1, x_2) = (-x_2, x_1)$. Now T is 'rotation through 90°' and has the property that $(\alpha|T\alpha) = 0$ for all α in R^2. Find all inner products $[\ |\ ]$ on R^2 such that $[\alpha|T\alpha] = 0$ for each α.

7. Let $(\ |\ )$ be the standard inner product on C^2. Prove that there is no non-zero linear operator on C^2 such that $(\alpha|T\alpha) = 0$ for every α in C^2. Generalize.

8. Let A be a 2×2 matrix with real entries. For X, Y in $R^{2\times 1}$ let

$$f_A(X, Y) = Y^t A X.$$

Show that f_A is an inner product on $R^{2\times 1}$ if and only if $A = A^t$, $A_{11} > 0$, $A_{22} > 0$, and $\det A > 0$.

9. Let V be a real or complex vector space with an inner product. Show that the quadratic form determined by the inner product satisfies the **parallelogram law**

$$||\alpha + \beta||^2 + ||\alpha - \beta||^2 = 2||\alpha||^2 + 2||\beta||^2.$$

10. Let $(\ |\ )$ be the inner product on R^2 defined in Example 2, and let $\mathcal{B}$ be the standard ordered basis for R^2. Find the matrix of this inner product relative to $\mathcal{B}$.

11. Show that the formula

$$\left(\sum_j a_j x^j \Big| \sum_k b_k x^k\right) = \sum_{j,k} \frac{a_j b_k}{j + k + 1}$$

defines an inner product on the space $R[x]$ of polynomials over the field R. Let W be the subspace of polynomials of degree less than or equal to n. Restrict the above inner product to W, and find the matrix of this inner product on W, relative to the ordered basis $\{1, x, x^2, \ldots, x^n\}$. (*Hint:* To show that the formula defines an inner product, observe that

$$(f|g) = \int_0^1 f(t)g(t)\, dt$$

and work with the integral.)

12. Let V be a finite-dimensional vector space and let $\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}$ be a basis for V. Let $(\ |\ )$ be an inner product on V. If $c_1, \ldots, c_n$ are any n scalars, show that there is exactly one vector α in V such that $(\alpha|\alpha_j) = c_j$, $j = 1, \ldots, n$.

13. Let V be a complex vector space. A function J from V into V is called a **conjugation** if $J(\alpha + \beta) = J(\alpha) + J(\beta)$, $J(c\alpha) = \bar{c}J(\alpha)$, and $J(J(\alpha)) = \alpha$, for all scalars c and all α, β in V. If J is a conjugation show that:

(a) The set W of all α in V such that $J\alpha = \alpha$ is a vector space over R with respect to the operations defined in V.

(b) For each α in V there exist unique vectors β, γ in W such that $\alpha = \beta + i\gamma$.

14. Let V be a complex vector space and W a subset of V with the following properties:

(a) W is a real vector space with respect to the operations defined in V.

(b) For each α in V there exist unique vectors β, γ in W such that $\alpha = \beta + i\gamma$.

Show that the equation $J\alpha = \beta - i\gamma$ defines a conjugation on V such that $J\alpha = \alpha$ if and only if α belongs to W, and show also that J is the only conjugation on V with this property.

15. Find all conjugations on C^1 and C^2.

16. Let W be a finite-dimensional real subspace of a complex vector space V. Show that W satisfies condition (b) of Exercise 14 if and only if every basis of W is also a basis of V.

17. Let V be a complex vector space, J a conjugation on V, W the set of α in V such that $J\alpha = \alpha$, and f an inner product on W. Show that:

(a) There is a unique inner product g on V such that $g(\alpha, \beta) = f(\alpha, \beta)$ for all α, β in W,

(b) $g(J\alpha, J\beta) = g(\beta, \alpha)$ for all α, β in V.

What does part (a) say about the relation between the standard inner products on R^1 and C^1, or on R^n and C^n?

8.2. Inner Product Spaces

Now that we have some idea of what an inner product is, we shall turn our attention to what can be said about the combination of a vector space and some particular inner product on it. Specifically, we shall establish the basic properties of the concepts of 'length' and 'orthogonality' which are imposed on the space by the inner product.

Definition. *An* **inner product space** *is a real or complex vector space, together with a specified inner product on that space.*

A finite-dimensional real inner product space is often called a **Euclidean space.** A complex inner product space is often referred to as a **unitary space.**

Theorem 1. *If* V *is an inner product space, then for any vectors* α, β *in* V *and any scalar* c

(i) $||c\alpha|| = |c|\,||\alpha||$;
(ii) $||\alpha|| > 0$ *for* $\alpha \neq 0$;
(iii) $|(\alpha|\beta)| \leq ||\alpha||\,||\beta||$;
(iv) $||\alpha + \beta|| \leq ||\alpha|| + ||\beta||$.

Proof. Statements (i) and (ii) follow almost immediately from the various definitions involved. The inequality in (iii) is clearly valid when $\alpha = 0$. If $\alpha \neq 0$, put

$$\gamma = \beta - \frac{(\beta|\alpha)}{||\alpha||^2}\,\alpha.$$

Then $(\gamma|\alpha) = 0$ and

$$\begin{aligned} 0 \leq ||\gamma||^2 &= \left(\beta - \frac{(\beta|\alpha)}{||\alpha||^2}\,\alpha \,\middle|\, \beta - \frac{(\beta|\alpha)}{||\alpha||^2}\,\alpha\right) \\ &= (\beta|\beta) - \frac{(\beta|\alpha)(\alpha|\beta)}{||\alpha||^2} \\ &= ||\beta||^2 - \frac{|(\alpha|\beta)|^2}{||\alpha||^2}. \end{aligned}$$