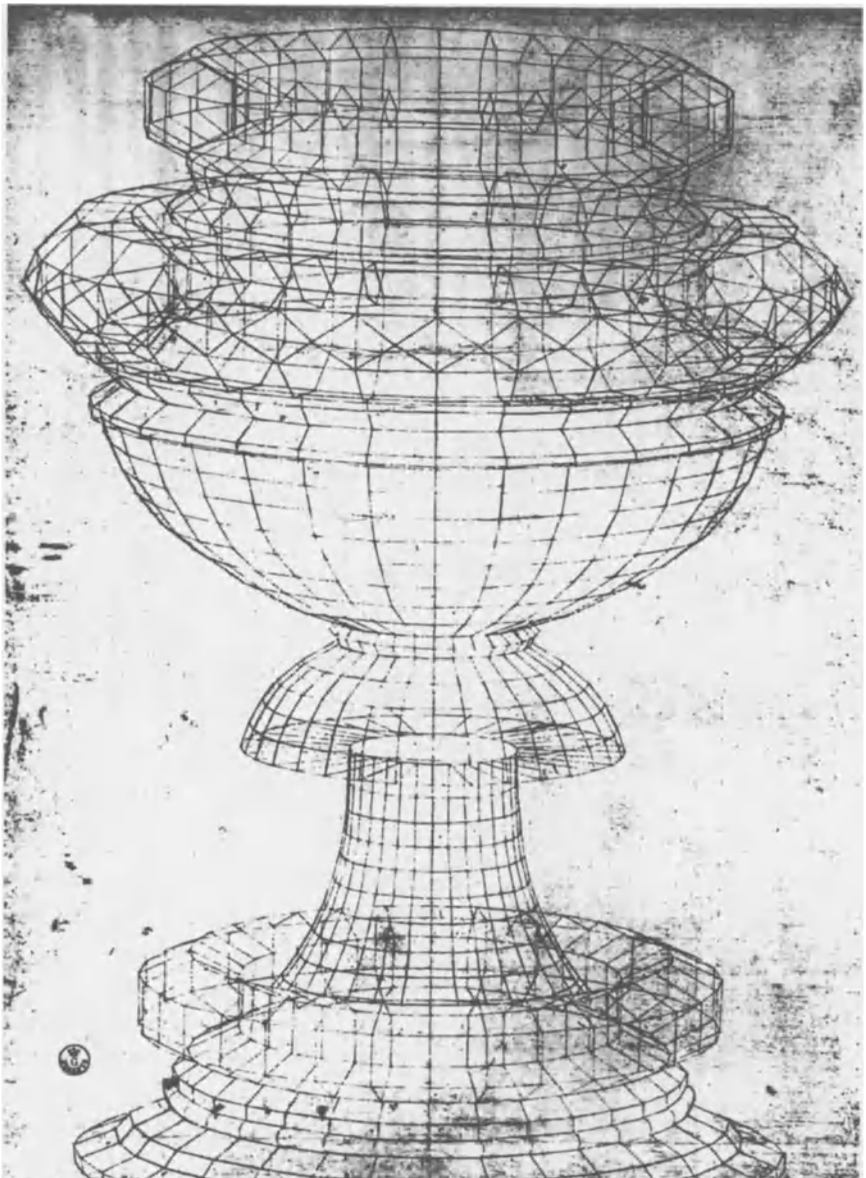

Figure 8.12: Drawing of a chalice by Uccello. (Uffizi, Florence)

Moreover, the projective viewpoint gave something else that would have been incomprehensible to the Greeks: a clear account of the behavior of curves at infinity.

For example, Desargues (1639) [in Taton (1951), p. 137] distinguished the ellipse, parabola, and hyperbola by their numbers of points at infinity, 0, 1, and 2, respectively. The points at infinity on the parabola and hyperbola can be seen quite plainly by tilting the ordinary views of them into perspective views (Figures 8.13 and 8.14). The parabola has just one point at infinity because it crosses each ray through 0, except the y-axis, at just one finite point. As for the hyperbola, its two points at infinity are where it touches its asymptotes, as seen in Figure 8.14. The continuation of the hyperbola above the horizon results from projecting the lower branch through the same center of projection (Figure 8.15).

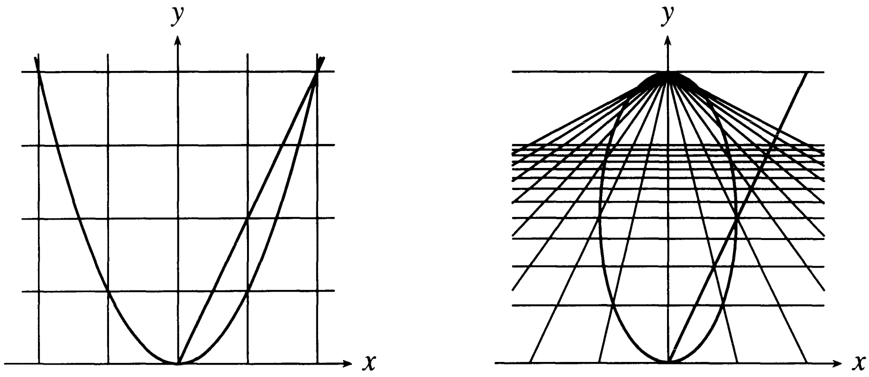

Figure 8.13: The parabola

Projective geometry goes beyond describing the behavior of curves at infinity. The line at infinity is no different from any other line and can be deprived of its special status. Then all projective views of a curve are equally valid, and one can say, for example, that all conic sections are ellipses when suitably viewed. This is no surprise if one remembers conic sections not as second-degree curves but as sections of the cone. Of course they all look the same from the vertex of the cone.

More surprisingly, a great simplification of cubic curves also occurs when they are viewed projectively. As mentioned in Section 7.4, Newton (1695) classified cubic curves into 72 types (and missed 6). However, in his Section 29, "On the Genesis of Curves by Shadows," Newton claimed that

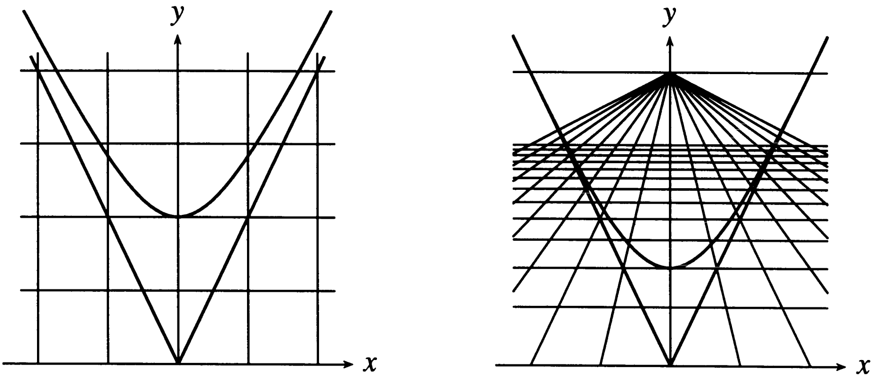

Figure 8.14: The hyperbola

each cubic curve can be projected onto one of just five types. As mentioned in Section 7.4, this includes the result that $y = x^3$ can be projected onto $y^2 = x^3$. The proof of this is an easy calculation when coordinates are introduced (see Exercise 8.5.3), but one already gets an inkling of it from the perspective view of $y = x^3$. (See Figure 8.16. The lower half of the cusp is the view of $y = x^3$ below the horizon; the upper half comes from projecting the view behind one's head through P to the picture plane in front.)

Conversely, $y^2 = x^3$ has an inflection at infinity. Newton's projective classification comes about by studying the behavior at infinity of all cubics and observing that each has characteristics already possessed, not necessarily at infinity, by curves of the form

$$y^2 = Ax^3 + Bx^2 + Cx + D.$$

Newton had already divided these into five types in his analytic classification (they are the five shown in Figure 7.3). Newton's result was improved only in the nineteenth century, when projective classification over the complex numbers reduced the number of types of cubics into just three. We discuss this later in connection with the development of complex numbers (Section 15.5).

Exercises

As suggested above, the points at infinity of a curve may be counted by considering intersections of the curve with lines through the origin, and observing where they tend to infinity.

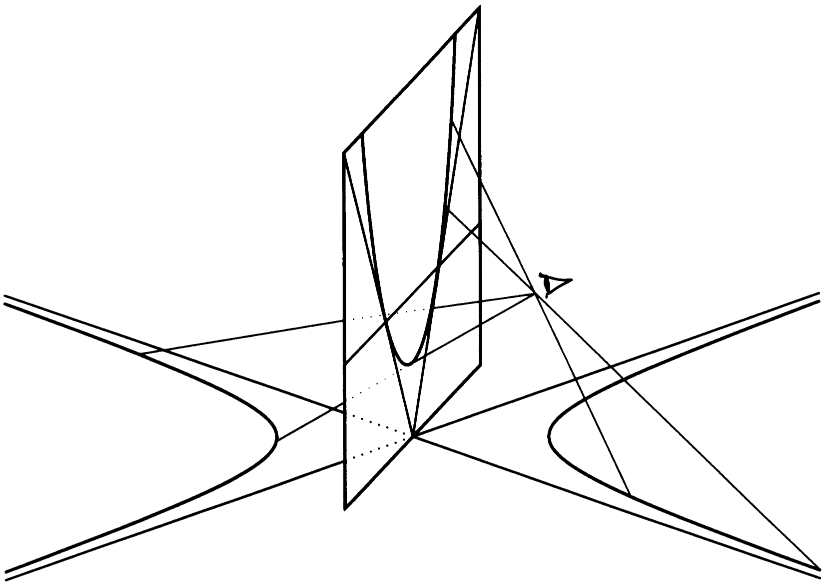

Figure 8.15: Branches of the hyperbola

8.4.1 Use this method to explain why

- the hyperbola $xy = 1$ has two points at infinity,
- the curve $y = x^3$ has one point at infinity.

Figures 8.13 and 8.14 were made by taking Alberti's veil to be the (x,z)-plane in (x,y,z)-space, with the "eye" at $(0,-4,4)$ viewing the (x,y)-plane.

8.4.2 Find the parametric equations of the line from $(0,-4,4)$ to $(x',y',0)$, and hence show that this line meets the veil where

$$x = \frac{4x'}{y'+4}, \quad z = \frac{4y'}{y'+4}.$$

8.4.3 Renaming the coordinates x, z in the veil as X, Y respectively, show that

$$x' = \frac{4X}{4-Y}, \quad y' = \frac{4Y}{4-Y}.$$

8.4.4 Deduce from Exercise 8.4.3 that the points (x',y') on the parabola $y = x^2$ have image on the veil

$$X^2 + \frac{(Y-2)^2}{4} = 1,$$

and check that this is the ellipse shown in Figure 8.13.

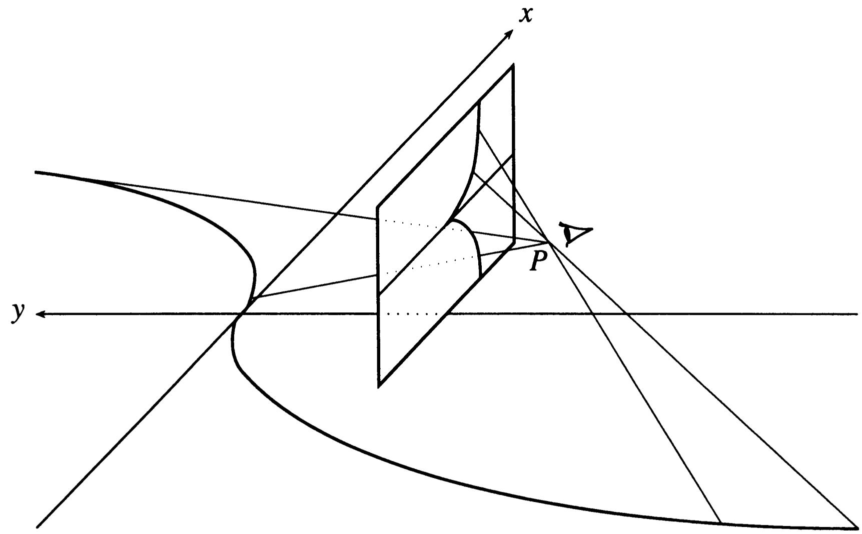

Figure 8.16: Perspective view of a cubic curve

8.5 Homogeneous Coordinates

The way in which projective geometry allows infinity to be put on the same footing as the finite points of the plane is intuitively clear when one thinks of the horizon in a picture, which is a line like any other. However, the most convenient way to formalize the idea is to introduce coordinates. This did not happen in Desargues' time, perhaps because of the resistance to coordinates in elementary geometry that was then prevalent (see Sections 7.4 and 7.5). Suitable coordinates, now known as *homogeneous coordinates*, were invented by Möbius (1827) and Plücker (1830). Homogeneous coordinates give a natural extension of the cartesian plane $\mathbb{R}^2$ by points at infinity by assigning new coordinates to the points already present and creating new points with the coordinates left over.

The homogeneous coordinates of a point $(X,Y) \in \mathbb{R}^2$ are all the real triples (Xz, Yz, z) with $z \neq 0$, that is, all real triples (x,y,z) with $x/z = X$, $y/z = Y$. If, following Klein (1925), we take X, Y to be the x, y coordinates in the plane $z = 1$, then these triples are just the coordinates of points $\neq O$ on the line in $\mathbb{R}^3$ from O to (X,Y) (Figure 8.17). Thus homogeneous coordinates give a one-to-one correspondence between points $(X,Y) \in \mathbb{R}^2$ and nonhorizontal lines through O in $\mathbb{R}^3$. The horizontal lines, whose points have coordinates $(x, y, 0)$, naturally correspond to points at infinity. Moreover, there is a natural way to determine which points at infinity "belong to" a given curve.

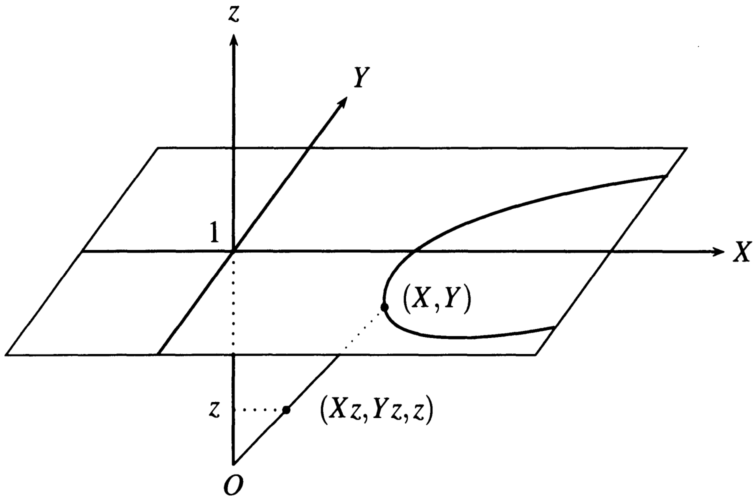

Figure 8.17: Constructing homogeneous coordinates

Each curve C in $\mathbb{R}^2$, expressed by an equation

$$p(X,Y) = 0 \tag{1}$$

say, can be reexpressed by the equation

$$p\left(\frac{x}{z}, \frac{y}{z}\right) = 0 \tag{2}$$

for $z \neq 0$. If p is a polynomial of degree n, we can extend (2) to all values of z by multiplying through by z^n, giving

$$z^n p\left(\frac{x}{z}, \frac{y}{z}\right) = \bar{p}(x,y,z) = 0, \tag{3}$$

where $\bar{p}$ is a *homogeneous polynomial* of degree n in x, y, z [that is, if (x, y, z) is a solution of (3), so is (tx, ty, tz)—as it should be, since these triples are coordinates of the same point]. For example, if the curve in $\mathbb{R}^2$ is the line $aX + bY + c = 0$, then the corresponding homogeneous equation (3) is $ax + by + cz = 0$.