Example

Let $R = \mathbb{Z}_{(2)}$ and $S = \mathbb{Q}$ as in the preceding set of examples. Define $\varphi^* : \operatorname{Spec}\mathbb{Q} \to \operatorname{Spec}\mathbb{Z}_{(2)}$ by $\varphi^*((0)) = (2)$ (which is Zariski continuous). Define $\varphi^{\#} : \mathcal{O}(\operatorname{Spec} R) \to \mathcal{O}(\operatorname{Spec} S)$ to be the inclusion map $\mathbb{Z}_{(2)} \hookrightarrow \mathbb{Q}$ and define $\varphi^{\#}$ for all other $U \subseteq \operatorname{Spec} R$ simply to be the zero map. It is straightforward to check that these homomorphisms commute with the restriction maps. This family of maps does *not* arise from a ring homomorphism, however, because on the stalks for $(0) \in \operatorname{Spec} S$ and $\varphi^*((0)) = (2) \in \operatorname{Spec} R$ the induced homomorphism

$$\varphi^{\#} : \mathcal{O}_{\operatorname{Spec} R,(2)} \hookrightarrow \mathcal{O}_{\operatorname{Spec} S,(0)}$$

is the injection $\mathbb{Z}_{(2)} \hookrightarrow \mathbb{Q}$, which is not a *local* homomorphism (the inverse image of (0) is (0) and not the maximal ideal $2\mathbb{Z}_{(2)}$).

The proof of Theorem 59 shows that a morphism $(\varphi^*, \varphi^{\#})$ of affine schemes necessarily comes from the ring homomorphism defined by $\varphi^{\#}$ on global sections. In this example, the homomorphism on global sections is the inclusion map of R into S. The inclusion map from R to S defines a map from $\operatorname{Spec} S$ to $\operatorname{Spec} R$ that maps $(0) \in \operatorname{Spec} S$ to $(0) \in \operatorname{Spec} R$ and not to $(2) \in \operatorname{Spec} R$, so this map does not agree with the original map φ^*.

The previous example shows that the converse in Theorem 59 would not be true without the third (local homomorphism) condition in the definition of a morphism of affine schemes. As a result, Theorem 59 shows that the appropriate place to view affine schemes is in the category of *locally ringed spaces*. Roughly speaking, a locally ringed space is a topological space X together with a collection of rings $\mathcal{O}(U)$ for each open subset of X (with a compatible set of homomorphisms from $\mathcal{O}(U)$ to $\mathcal{O}(U')$ if $U' \subseteq U$ and with some local conditions on the sections) such that the stalks $\mathcal{O}_P = \varinjlim \mathcal{O}(U)$ for $P \in U$ are local rings. The morphisms in this category are continuous maps between the topological spaces together with ring homomorphisms between corresponding $\mathcal{O}(U)$ with precisely the same conditions as imposed in the definition of a morphism of affine schemes.

A *scheme* is a locally ringed space in which each point lies in a neighborhood isomorphic to an affine scheme (with some compatibility conditions between such neighborhoods), and is a fundamental object of study in modern algebraic geometry. The affine schemes considered here form the building blocks that are "glued together" to define general schemes in the same way that ordinary Euclidean spaces form the building blocks that are "glued together" to define manifolds in analysis.

EXERCISES

All rings are assumed commutative with identity, and all ring homomorphisms are assumed to map identities to identities.

1. If N is the nilradical of R, prove that $\operatorname{Spec} R$ and $\operatorname{Spec} R/N$ are homeomorphic. [Show that the natural homomorphism from R to R/N induces a Zariski continuous isomorphism from $\operatorname{Spec} R/N$ to $\operatorname{Spec} R$.]

2. Let I be an ideal in the ring R. Prove that the continuous map from $\operatorname{Spec} R/I$ to $\operatorname{Spec} R$ induced by the canonical projection homomorphism $R \to R/I$ maps $\operatorname{Spec} R/I$ homeomorphically onto the closed set $\mathcal{Z}(I)$ in $\operatorname{Spec} R$.

3. Prove that two elements $f, g \in R$ have the same values at all elements P in $\operatorname{Spec} R$ if and only if $f - g$ is contained in the nilradical of R. In particular, prove that an element in an affine k-algebra is uniquely determined by its values.

4. Let k be an arbitrary field, not necessarily algebraically closed. Prove that the prime ideals in $k[x, y]$ (i.e., the elements of $\operatorname{Spec} k[x, y]$) are
   (i) (0),
   (ii) (f) where f is an irreducible polynomial in $k[x, y]$, and
   (iii) $(p(x), g(x, y))$ where $p(x)$ is an irreducible polynomial in $k[x]$ and $g(x, y)$ is an irreducible polynomial in $k[x, y]$ that is irreducible modulo $p(x)$, i.e., $g(x, y)$ remains irreducible in the quotient $k[x, y]/(p(x))$.

   Prove that $\operatorname{mSpec} k[x, y]$ consists of the primes in (iii). [Use Exercise 20 in Section 1.]

5. Let $\mathfrak{m} = (p(x), g(x, y))$ be a maximal ideal in $k[x, y]$ as in the previous exercise. Show that $K = k[x, y]/\mathfrak{m}$ is an algebraic field extension of k, so that $k[x, y]$ can also be viewed as a subring of $K[x, y]$. If x, y are mapped to $\alpha, \beta \in K$, respectively, under the canonical homomorphism $k[x, y] \to k[x, y]/\mathfrak{m}$, prove that $\mathfrak{m} = k[x, y] \cap (x - \alpha, y - \beta) \subseteq K[x, y]$.

6. Describe the elements in $\operatorname{Spec} \mathbb{R}[x]$ and $\operatorname{Spec} \mathbb{C}[x]$. Describe the elements in $\operatorname{Spec} \mathbb{Z}_{(2)}[x]$ where $\mathbb{Z}_{(2)} = \{a/b \in \mathbb{Q} \mid b \text{ is odd}\}$ is the localization of $\mathbb{Z}$ at the prime (2).

7. Let $(f) = (x^5 + x + 1)$ in $\operatorname{Spec} \mathbb{Z}[x]$ viewed as fibered over $\operatorname{Spec} \mathbb{Z}$ as in Example 3 following Proposition 55. Show that there are two closed points in the fiber over (2), three closed points in the fiber over (5), four closed points in the fiber over (19), and five closed points in the fiber over (211).

8. Let $(f) = (x^4 + 1)$ in $\operatorname{Spec} \mathbb{Z}[x]$ viewed as fibered over $\operatorname{Spec} \mathbb{Z}$ as in Example 3 following Proposition 55. Prove that there is one closed point in the fiber over (2), four closed points in the fiber over p for p odd, $p \equiv 1 \bmod 8$, and two closed points in the fiber over p for all other odd primes p (cf. Corollary 16 in Section 3 of Chapter 14).

9. Prove that the elements in the fiber over (p) of the Zariski continuous map from $\operatorname{Spec} \mathbb{Z}[x]$ to $\operatorname{Spec} \mathbb{Z}$ are homeomorphic with the elements in $\operatorname{Spec}(\mathbb{Z}[x] \otimes_{\mathbb{Z}} \mathbb{F}_p)$.

10. Let $X = \operatorname{Spec} R$ and let X_f be the principal open set corresponding to $f \in R$. Prove that $X_f \cap X_g = X_{fg}$. Prove that $X_f = X$ if and only if f is a unit in R, and that $X_f = \emptyset$ if and only if f is nilpotent.

11. If X_f and X_g are principal open sets in $X = \operatorname{Spec} R$, prove that the open set $X_f \cup X_g$ is the complement of the closed set $\mathcal{Z}(I)$ where $I = (f, g)$ is the ideal in R generated by f and g.

12. Prove that a Zariski open subset U of $X = \operatorname{Spec} R$ is quasicompact if and only if U is a finite union of principal open subsets. Give an example of a ring R, a Zariski open subset U of $\operatorname{Spec} R$, and a Zariski open covering of U that cannot be reduced to a finite subcovering.

13. Let $\varphi : R \to S$ be a homomorphism of rings. Prove that under the induced map φ^* from $Y = \operatorname{Spec} S$ to $X = \operatorname{Spec} R$ the full preimage of the principal open set X_f in X is the principal open set $Y_{\varphi(f)}$ in Y.

14. Suppose that $R = R_1 \times R_2$ is the direct product of the rings R_1 and R_2. Prove that $X = \operatorname{Spec} R$ is the disjoint union of open subspaces X_1, X_2 (which are therefore also closed), where X_1 is homeomorphic to $\operatorname{Spec} R_1$ and X_2 is homeomorphic to $\operatorname{Spec} R_2$.

15. Prove that $X = \operatorname{Spec} R$ is not connected if and only if R is the direct product of two nonzero rings if and only if R contains an idempotent e with $e \neq 0, 1$ (cf. the previous exercise).

16. Prove that $X = \operatorname{Spec} R$ is irreducible (i.e., any two nonempty open subsets have a nontrivial intersection) if and only if $X_f \cap X_g \neq \emptyset$ for any two nonempty principal open sets X_f and X_g. Deduce that $X = \operatorname{Spec} R$ is irreducible if and only if the nilradical of R is a prime ideal. [Use Exercise 10.]

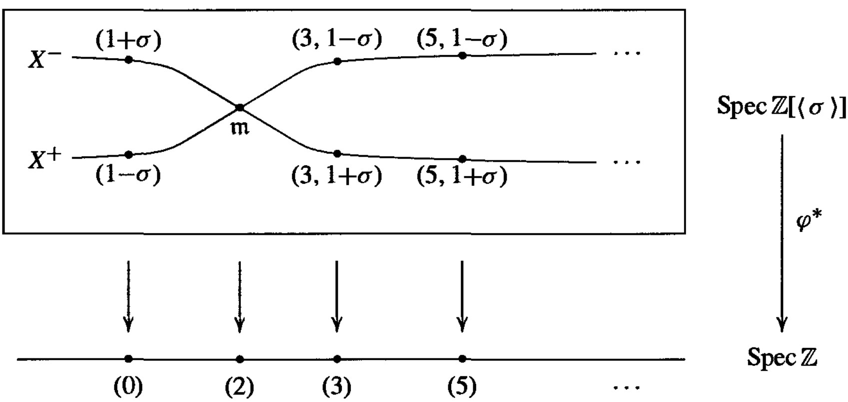
17. Let $G = \langle \sigma \rangle$ be a group of order 2, let $R = \mathbb{Z}[G] = \{a + b\sigma \mid a, b \in \mathbb{Z}\}$ be the corresponding group ring, and let $X = \operatorname{Spec} R$.

(a) Prove that the nilradical of R is (0) but is not a prime ideal. Prove that $X = X^+ \cup X^-$ where $X^+ = \mathcal{Z}(1-\sigma)$ and $X^- = \mathcal{Z}(1+\sigma)$. [Use $(1+\sigma)(1-\sigma) = 0$.]

(b) Prove that the homomorphism $\mathbb{Z}[G] \to \mathbb{Z}$ defined by mapping σ to 1 induces a homeomorphism of X^+ with $\operatorname{Spec} \mathbb{Z}$, and the homomorphism mapping σ to -1 induces a homeomorphism of X^- with $\operatorname{Spec} \mathbb{Z}$.

(c) Prove that $X^+ \cap X^-$ consists of the single element $\mathfrak{m} = (1+\sigma, 1-\sigma) = (2, 1-\sigma)$ and that this is a closed point in X.

(d) Show that $(1-\sigma)$ and $(1+\sigma)$ are the unique non-closed points in X, with closures X^+ and X^-, respectively. Describe the closed points, $\operatorname{mSpec} R$, in X and prove that $\operatorname{Spec} \mathbb{Z}[\langle \sigma \rangle]$ can be pictured as follows:



18. Let $\mathcal{O}$ be the structure sheaf on $X = \operatorname{Spec} R$, let U be an open set in X, and suppose $s, t \in \mathcal{O}(U)$. If $s = a/f_1^n$ on X_{f_1} and $t = b/f_2^m$ on X_{f_2}, show that

$$st = (abf_1^m f_2^n)/(f_1 f_2)^{n+m} \quad \text{and} \quad s + t = (af_1^m f_2^{m+n} + bf_1^{m+n} f_2^n)/(f_1 f_2)^{n+m}$$

on $X_{f_1 f_2}$. Deduce that $\mathcal{O}(U)$ is a commutative ring with identity.

19. Let $\mathcal{O}$ be the structure sheaf on $X = \operatorname{Spec} R$, let $V \subseteq U$ be open sets in X, and let $s \in \mathcal{O}(U)$. Suppose $P \in V$ and that $s = a/f^n$ on $X_f \subseteq U$.

(a) Show that there is a principal open set $X_{f'} \subseteq V \cap X_f$ containing P.

(b) Show that $(f')^m = bf$ for some $b \in R$.

(c) Show that $s = (ab^n)/(f')^{mn}$ on $X_{f'}$ and conclude that restricting s to V gives a well defined ring homomorphism from $\mathcal{O}(U)$ to $\mathcal{O}(V)$.

20. Let $\varphi : R \to S$ be a homomorphism of rings, let $X = \operatorname{Spec} R$, $Y = \operatorname{Spec} S$, and let $V \subseteq U$ be Zariski open subsets of X. Set $V' = (\varphi^*)^{-1}(V)$ and $U' = (\varphi^*)^{-1}(U)$, the corresponding Zariski open subsets of Y with respect to the continuous map $\varphi^* : Y \to X$ induced by φ. Prove that the induced map $\varphi^\# : \mathcal{O}_X(U) \to \mathcal{O}_Y(U')$ on sections is a ring homomorphism. Prove that $V' \subseteq U'$ and that $\varphi^\#$ is compatible with restriction i.e., that