$\mathrm{Hom}(A_1, A_2) = \{f\}$ and $\mathrm{Hom}(A_2, A_1) = \emptyset$. Note that the objects A_1 and A_2 and the morphism f are "primitives" in the sense that A_1 and A_2 are not defined to be sets and f is simply an arrow (literally) from A_1 to A_2; it is not defined as a set map on the elements of some set. One can continue this way and define **N** to be the category with N objects $A_1, A_2, \ldots, A_N$ with the only nonidentity morphisms being a unique arrow from A_i to A_j for every $j > i$ (so that composition of arrows is uniquely determined).

(6) If G is a group, form the category **G** as follows. The only object is G and $\mathrm{Hom}(G, G) = G$; the composition of two functions f and g is the product gf in the group G. Note that $\mathrm{Hom}(G, G)$ has an identity morphism: the identity of the group G.

Definition. Let **C** and **D** be categories.

(1) We say $\mathcal{F}$ is a *covariant functor* from **C** to **D** if
   (a) for every object A in **C**, $\mathcal{F}A$ is an object in **D**, and
   (b) for every $f \in \mathrm{Hom}_{\mathbf{C}}(A, B)$ we have $\mathcal{F}(f) \in \mathrm{Hom}_{\mathbf{D}}(\mathcal{F}A, \mathcal{F}B)$,

   such that the following axioms are satisfied:
   (i) if gf is a composition of morphisms in **C**, then $\mathcal{F}(gf) = \mathcal{F}(g)\mathcal{F}(f)$ in **D**, and
   (ii) $\mathcal{F}(1_A) = 1_{\mathcal{F}A}$.

(2) We say $\mathcal{F}$ is a *contravariant functor* from **C** to **D** if the conditions in (1) hold but property (b) and axiom (i) are replaced by:
   (b′) for every $f \in \mathrm{Hom}_{\mathbf{C}}(A, B)$, $\mathcal{F}(f) \in \mathrm{Hom}_{\mathbf{D}}(\mathcal{F}B, \mathcal{F}A)$,
   (i′) if gf is a composition of morphisms in **C**, then $\mathcal{F}(gf) = \mathcal{F}(f)\mathcal{F}(g)$ in **D**

   (i.e., contravariant functors reverse the arrows).

Examples

In each of these examples the verification of the axioms for a functor are left as exercises. Additional examples of functors appear in the exercises at the end of this section.

(1) The identity functor $\mathcal{I}_C$ maps any category **C** to itself by sending objects and morphisms to themselves. More generally, if **C** is a subcategory of **D**, the *inclusion functor* maps **C** into **D** by sending objects and morphisms to themselves.

(2) Let $\mathcal{F}$ be the functor from **Grp** to **Set** that maps any group G to the same set G and any group homomorphism φ to the same set map φ. This functor is called the *forgetful functor* since it "removes" or "forgets" the structure of the groups and the homomorphisms between them. Likewise there are forgetful functors from the categories **Ab**, R–**mod**, **Top**, etc., to **Set**.

(3) The *abelianizing* functor maps **Grp** to **Ab** by sending each group G to the abelian group $G^{\mathrm{ab}} = G/G'$, where G' is the commutator subgroup of G (cf. Section 5.4). Each group homomorphism $\varphi : G \to H$ is mapped to the induced homomorphism on quotient groups:

$$\overline{\varphi} : G^{\mathrm{ab}} \to H^{\mathrm{ab}} \qquad \text{by} \qquad \overline{\varphi}(xG') = \varphi(x)H'.$$

The definition of the commutator subgroup ensures that $\overline{\varphi}$ is well defined and the axioms for a functor are satisfied.

(4) Let R be a ring and let D be a left R-module. For each left R-module N the set $\mathrm{Hom}_R(D, N)$ is an abelian group, and is an R-module if R is commutative (cf. Proposition 2 in Section 10.2). If $\varphi : N_1 \to N_2$ is an R-module homomorphism, then for every $f \in \mathrm{Hom}_R(D, N_1)$ we have $\varphi \circ f \in \mathrm{Hom}_R(D, N_2)$. Thus

$\varphi' : \mathrm{Hom}_R(D, N_1) \to \mathrm{Hom}_R(D, N_2)$ by $\varphi'(f) = \varphi \circ f$. This shows the map

$$\mathcal{H}om(D, \_\_) : N \longrightarrow \mathrm{Hom}_R(D, N)$$
$$\mathcal{H}om(D, \_\_) : \varphi \longrightarrow \varphi'$$

is a covariant functor from R–**Mod** to **Grp**. If R is commutative, it maps R–**Mod** to itself.

(5) In the notation of the preceding example, we observe that if $\varphi : N_1 \to N_2$, then for every $g \in \mathrm{Hom}_R(N_2, D)$ we have $g \circ \varphi \in \mathrm{Hom}_R(N_1, D)$. Thus $\varphi' : \mathrm{Hom}_R(N_2, D) \to \mathrm{Hom}_R(N_1, D)$ by $\varphi'(g) = g \circ \varphi$. In this case the map

$$\mathcal{H}om(\_\_, D) : N \longrightarrow \mathrm{Hom}_R(N, D)$$
$$\mathcal{H}om(\_\_, D) : \varphi \longrightarrow \varphi'$$

defines a *contravariant* functor.

(6) When D is a right R-module the map $D \otimes_R \_\_ : N \to D \otimes_R N$ defines a covariant functor from R–**Mod** to **Ab** (or to R–**Mod** when R is commutative). Here the morphism $\varphi : N_1 \to N_2$ maps to the morphism $1 \otimes \varphi$.

Likewise when D is a left R-module $\_\_ \otimes_R D : N \to N \otimes_R D$ defines a covariant functor from the category of right R-modules to **Ab** (or to R–**Mod** when R is commutative), where the morphism φ maps to the morphism $\varphi \otimes 1$.

(7) Let K be a field and let K–**fdVec** be the category of all finite dimensional vector spaces over K, where morphisms in this category are K-linear transformations. We define the *double dual* functor $\mathcal{D}^2$ from K–**fdVec** to itself. Recall from Section 11.3 that the dual space, V^*, of V is defined as $V^* = \mathrm{Hom}_K(V, K)$; the double dual of V is $V^{**} = \mathrm{Hom}_K(V^*, K)$. Then $\mathcal{D}^2$ is defined on objects by mapping a vector space V to its double dual V^{**}. If $\varphi : V \to W$ is a linear transformation of finite dimensional spaces, then

$$\mathcal{D}^2(\varphi) : V^{**} \to W^{**} \qquad \text{by} \qquad \mathcal{D}^2(\varphi)(E_v) = E_{\varphi(v)},$$

where E_v denotes "evaluation at v" for each $v \in V$. By Theorem 19 in Section 11.3, $E_v \in V^{**}$, and each element of V^{**} is of the form E_v for a unique $v \in V$. Since $\varphi(v) \in W$ we have $E_{\varphi(v)} \in W^{**}$, so $\mathcal{D}^2(\varphi)$ is well defined.

The functor $\mathcal{F}$ from **C** to **D** is called *faithful* (or is called *full*) if for every pair of objects A and B in **C** the map $\mathcal{F} : \mathrm{Hom}(A, B) \to \mathrm{Hom}(\mathcal{F}A, \mathcal{F}B)$ is injective (or surjective, respectively). Thus, for example, the forgetful functor is faithful but not full.

EXERCISES

1. Let N be a group and let **Nor**–N be the collection of all groups that contain N as a normal subgroup. A morphism between objects A and B is any group homomorphism that maps N into N.

(a) Prove that **Nor**–N is a category.

(b) Show how the projection homomorphism $G \mapsto G/N$ may be used to define a functor from **Nor**–N to **Grp**.

2. Let H be a group. Define a map $\mathcal{H}\times$ from **Grp** to itself on objects and morphisms as follows:

$$\mathcal{H}\times : G \to H \times G, \text{ and}$$
$$\text{if } \varphi : G_1 \to G_2 \quad \text{then } \mathcal{H}\times(\varphi) : H \times G_1 \to H \times G_2 \quad \text{by} \quad (h, g) \mapsto (h, \varphi(g)).$$

Prove that $\mathcal{H}\times$ is a functor.

3. Show that the map **Ring** to **Grp** by mapping a ring to its group of units (i.e., $R \mapsto R^{\times}$) defines a functor. Show by explicit examples that this functor is neither faithful nor full.
4. Show that for each $n \geq 1$ the map $\mathcal{GL}_n : R \to GL_n(R)$ defines a functor from **CRing** to **Grp**. [Define $\mathcal{GL}_n$ on morphisms by applying each ring homomorphism to the entries of a matrix.]
5. Supply the details that show the double dual map described in Example 7 satisfies the axioms of a functor.

2. NATURAL TRANSFORMATIONS AND UNIVERSALS

As mentioned in the introduction to this appendix, one of the motivations for the inception of category theory was to give a precise definition of the notion of "natural" isomorphism. We now do so, and see how some natural maps mentioned in the text are instances of the categorical concept. We likewise give the categorical definition of "universal arrows" and view some occurrences of universal properties in the text in this light.

Definition. Let **C** and **D** be categories and let $\mathcal{F}$, $\mathcal{G}$ be covariant functors from **C** to **D**. A *natural transformation* or *morphism of functors* from $\mathcal{F}$ to $\mathcal{G}$ is a map η that assigns to each object A in **C** a morphism η_A in $\mathrm{Hom}_{\mathbf{D}}(\mathcal{F}A, \mathcal{G}A)$ with the following property: for every pair of objects A and B in **C** and every $f \in \mathrm{Hom}_{\mathbf{C}}(A, B)$ we have $\mathcal{G}(f)\eta_A = \eta_B\mathcal{F}(f)$, i.e., the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}A & \xrightarrow{\eta_A} & \mathcal{G}A \\ {\scriptstyle \mathcal{F}(f)}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{G}(f)} \\ \mathcal{F}B & \xrightarrow{\eta_B} & \mathcal{G}B \end{array}$$

If each η_A is an isomorphism, η is called a *natural isomorphism* of functors.

Consider the special case where $\mathbf{C} = \mathbf{D}$ and **C** is a subcategory of **Set**, and where $\mathcal{F}$ is the identity functor. There is a natural transformation η from the identity functor to $\mathcal{G}$ if whenever $\mathcal{G}$ maps the object A to the object $\mathcal{G}A$ there is a morphism η_A from A to $\mathcal{G}A$, and whenever there is a morphism f from A to B the morphism $\mathcal{G}(f)$ is compatible with f as a map from $\mathcal{G}A$ to $\mathcal{G}B$. In fact $\mathcal{G}(f)$ is uniquely determined by f as a map from the subset $\eta_A(A)$ in $\mathcal{G}A$ to the subset $\eta_B(B)$ of $\mathcal{G}B$. If η is a natural isomorphism, then the value of $\mathcal{G}$ on every morphism is completely determined by η, namely $\mathcal{G}(f) = \eta_B f \eta_A^{-1}$. In this case the functor $\mathcal{G}$ is entirely specified by η. We shall see that some of the examples of functors in the preceding section arise this way.

Examples

(1) For any categories **C** and **D** and any functor $\mathcal{F}$ from **C** to **D** the identity is a natural isomorphism from $\mathcal{F}$ to itself: $\eta_A = 1_{\mathcal{F}A}$ for every object A in **C**.