Proposition 17. Let $\varphi : V \to X$ and $\psi : W \to Y$ be linear transformations of finite dimensional vector spaces. Then the Kronecker product of matrices representing φ and ψ is a matrix representation of $\varphi \otimes \psi$.

Example

Let $V = X = \mathbb{R}^3$, both with basis v_1, v_2, v_3, and $W = Y = \mathbb{R}^2$, both with basis w_1, w_2. Suppose $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ is the linear transformation given by $\varphi(av_1 + bv_2 + cv_3) = cv_1 + 2av_2 - 3bv_3$ and $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation given by $\psi(aw_1 + bw_2) = (a + 3b)w_1 + (4b - 2a)w_2$. With respect to the chosen bases, the matrices for φ and ψ are

$$\begin{pmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & -3 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 3 \\ -2 & 4 \end{pmatrix},$$

respectively. Then with respect to the ordered basis

$$\mathcal{B} = \{v_1 \otimes w_1,\ v_1 \otimes w_2,\ v_2 \otimes w_1,\ v_2 \otimes w_2,\ v_3 \otimes w_1,\ v_3 \otimes w_2\}$$

we have

$$M_{\mathcal{B}}^{\mathcal{B}}(\varphi \otimes \psi) = \left(\begin{array}{cc:cc:cc} 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & -2 & 4 \\ \hdashline 2 & 6 & 0 & 0 & 0 & 0 \\ -4 & 8 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & -3 & -9 & 0 & 0 \\ 0 & 0 & 6 & -12 & 0 & 0 \end{array}\right),$$

obtained (as indicated by the dashed lines) by multiplying the 2×2 matrix for ψ successively by the entries in the matrix for φ.

EXERCISES

1. Let V be the collection of polynomials with coefficients in $\mathbb{Q}$ in the variable x of degree at most 5. Determine the transition matrix from the basis $1, x, x^2, \dots, x^5$ for V to the basis $1, 1 + x, 1 + x + x^2, \dots, 1 + x + x^2 + x^3 + x^4 + x^5$ for V.
2. Let V be the vector space of the preceding exercise. Let $\varphi = d/dx$ be the linear transformation of V to itself given by usual differentiation of a polynomial with respect to x. Determine the matrix of φ with respect to the two bases for V in the previous exercise.
3. Let V be the collection of polynomials with coefficients in F in the variable x of degree at most n. Determine the transition matrix from the basis $1, x, x^2, \dots, x^n$ for V to the elements
$$1,\ x - \lambda, \dots,\ (x - \lambda)^{n-1},\ (x - \lambda)^n$$
where λ is a fixed element of F. Conclude that these elements are a basis for V.
4. Let φ be the linear transformation of $\mathbb{R}^2$ to itself given by rotation counterclockwise around the origin through an angle θ. Show that the matrix of φ with respect to the standard basis for $\mathbb{R}^2$ is $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.
5. Show that the $m \times n$ matrix A is nonsingular if and only if the linear transformation φ is a nonsingular linear transformation from the n-dimensional space V to the m-dimensional space W, where $A = M_{\mathcal{B}}^{\mathcal{E}}(\varphi)$, regardless of the choice of bases $\mathcal{B}$ and $\mathcal{E}$.

6. Prove if $\varphi \in \mathrm{Hom}_F(F^n, F^m)$, and $\mathcal{B}, \mathcal{E}$ are the natural bases of F^n, F^m **respectively, then** the range of φ equals the span of the set of columns of $M_{\mathcal{B}}^{\mathcal{E}}(\varphi)$. Deduce that the **rank of φ** (as a linear transformation) equals the column rank of $M_{\mathcal{B}}^{\mathcal{E}}(\varphi)$.
7. Prove that any two similar matrices have the same row rank and the same column rank.
8. Let V be an n-dimensional vector space over F and let φ be a linear transformation of the vector space V to itself.
   (a) Prove that if V has a basis consisting of eigenvectors for φ (cf. Exercise 8 of Section 1) then the matrix representing φ with respect to this basis (for both domain and range) is diagonal with the eigenvalues as diagonal entries.
   (b) If A is the $n \times n$ matrix representing φ with respect to a given basis for V (for both domain and range) prove that A is similar to a diagonal matrix if and only if V has a basis of eigenvectors for φ.
9. If W is a subspace of the vector space V stable under the linear transformation φ (i.e., $\varphi(W) \subseteq W$), show that φ induces linear transformations $\varphi|_W$ on W and $\bar{\varphi}$ on the quotient vector space V/W. If $\varphi|_W$ and $\widetilde{\varphi}$ are nonsingular prove φ is nonsingular. Prove the converse holds if V has finite dimension and give a counterexample with V infinite dimensional.
10. Let V be an n-dimensional vector space and let φ be a linear transformation of V to itself. Suppose W is a subspace of V of dimension m that is stable under φ.
   (a) Prove that there is a basis for V with respect to which the matrix for φ is of the form
   $$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$
   where A is an $m \times m$ matrix, B is an $m \times (n-m)$ matrix and C is an $(n-m) \times (n-m)$ matrix (such a matrix is called *block upper triangular*).
   (b) Prove that if there is a subspace W' invariant under φ so that $V = W \oplus W'$ decomposes as a direct sum then the bases for W and W' give a basis for V with respect to which the matrix for φ is *block diagonal*:
   $$\begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}$$
   where A is an $m \times m$ matrix and C is an $(n-m) \times (n-m)$ matrix.
   (c) Prove conversely that if there is a basis for V with respect to which φ is block diagonal as in (b) then there are φ-invariant subspaces W and W' of dimensions m and $n-m$, respectively, with $V = W \oplus W'$.
11. Let φ be a linear transformation from the finite dimensional vector space V to itself such that $\varphi^2 = \varphi$.
   (a) Prove that image $\varphi \cap \ker \varphi = 0$.
   (b) Prove that $V = \text{image } \varphi \oplus \ker \varphi$.
   (c) Prove that there is a basis of V such that the matrix of φ with respect to this basis is a diagonal matrix whose entries are all 0 or 1.

   A linear transformation φ satisfying $\varphi^2 = \varphi$ is called an *idempotent* linear transformation. This exercise proves that idempotent linear transformations are simply projections onto some subspace.
12. Let $V = \mathbb{R}^2$, $v_1 = (1, 0)$, $v_2 = (0, 1)$, so that v_1, v_2 are a basis for V. Let φ be the linear transformation of V to itself whose matrix with respect to this basis is $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$. Prove that if W is the subspace generated by v_1 then W is stable under the action of φ. Prove that there is no subspace W' invariant under φ so that $V = W \oplus W'$.

13. Let V be a vector space of dimension n and let W be a vector space of dimension m over a field F. Suppose A is the $m \times n$ matrix representing a linear transformation φ from V to W with respect to the bases $\mathcal{B}_1$ for V and $\mathcal{E}_1$ for W. Suppose similarly that B is the $m \times n$ matrix representing φ with respect to the bases $\mathcal{B}_2$ for V and $\mathcal{E}_2$ for W. Let $P = M_{\mathcal{B}_2}^{\mathcal{B}_1}(I)$ where I denotes the identity map from V to V, and let $Q = M_{\mathcal{E}_2}^{\mathcal{E}_1}(I)$ where I denotes the identity map from W to W. Prove that $Q^{-1} = M_{\mathcal{E}_1}^{\mathcal{E}_2}(I)$ and that $Q^{-1}AP = B$, giving the general relation between matrices representing the same linear transformation but with respect to different choices of bases.

The following exercises recall the *Gauss–Jordan* elimination process. This is one of the fastest computational methods for the solution of a number of problems involving vector spaces — solving systems of linear equations, determining inverses of matrices, computing determinants, determining the span of a set of vectors, determining linear independence of a set of vectors etc.

Consider the system of m linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= c_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= c_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= c_m \end{aligned} \tag{11.4}$$

in the n unknowns $x_1, x_2, \ldots, x_n$ where $a_{ij}, c_i, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$ are elements of the field F. Associated to this system is the *coefficient matrix*:

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}$$

and the *augmented matrix*:

$$(A \mid C) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \ldots & a_{1n} & c_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & c_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & c_m \end{array}\right)$$

(the term *augmented* refers to the presence of the column matrix $C = (c_i)$ in addition to the coefficient matrix $A = (a_{ij})$). The set of solutions in F of this system of equations is not altered if we perform any of the following three operations:

(1) interchange any two equations
(2) add a multiple of one equation to another
(3) multiply any equation by a nonzero element from F,

which correspond to the following three *elementary row operations* on the augmented matrix:

(1) interchange any two rows
(2) add a multiple of one row to another
(3) multiply any row by a unit in F, i.e., by any nonzero element in F.

If a matrix A can be transformed into a matrix C by a series of elementary row operations then A is said to be *row reduced* to C.

14. Prove that if A can be row reduced to C then C can be row reduced to A. Prove that the relation "$A \sim C$ if and only if A can be row reduced to C" is an equivalence relation. [Observe that the elementary row operations are reversible.]

Matrices lying in the same equivalence class under this equivalence relation are said to be *row equivalent.*

15. Prove that the row rank of two row equivalent matrices is the same. [It suffices to prove this for two matrices differing by an elementary row operation.]

An $m \times n$ matrix is said to be in *reduced row echelon form* if

(a) the first nonzero entry a_{ij_i} in row i is 1 and all other entries in the corresponding j_i^{th} column are zero, and

(b) $j_1 < j_2 < \ldots < j_r$ where r is the number of nonzero rows, i.e., the number of initial zeros in each row is strictly increasing (hence the term *echelon*).

An augmented matrix $(A \mid C)$ is said to be in reduced row echelon form if its coefficient matrix A is in reduced row echelon form. For example, the following two matrices are in reduced row echelon form:

$$\left(\begin{array}{cccccc|c} 1 & 0 & 5 & 7 & 0 & 3 & 0 \\ 0 & 1 & -1 & 1 & 0 & -4 & -1 \\ 0 & 0 & 0 & 0 & 1 & 6 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \qquad \left(\begin{array}{cccc|c} 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & -3 \end{array}\right)$$

(with $j_1 = 1$, $j_2 = 2$, $j_3 = 5$ for the first matrix and $j_1 = 2$, $j_2 = 4$ for the second matrix). The first nonzero entry in any given row of the coefficient matrix of a reduced row echelon augmented matrix (in position (i, j_i) by definition) is sometimes referred to as a *pivotal* element (so the pivotal elements in the first matrix are in positions (1,1), (2,2) and (3,5) and the pivotal elements in the second matrix are in positions (1,2) and (2,4)). The columns containing pivotal elements will be called *pivotal* columns and the columns of the coefficient matrix not containing pivotal elements will be called *nonpivotal.*

16. Prove by induction that any augmented matrix can be put in reduced row echelon form by a series of elementary row operations.

17. Let A and C be two matrices in reduced row echelon form. Prove that if A and C are row equivalent then $A = C$.

18. Prove that the row rank of a matrix in reduced row echelon form is the number of nonzero rows.

19. Prove that the reduced row echelon forms of the matrices

$$\left(\begin{array}{cccccc|c} 1 & 1 & 4 & 8 & 0 & -1 & -1 \\ 1 & 2 & 3 & 9 & 0 & -5 & -2 \\ 0 & -2 & 2 & -2 & 1 & 14 & 3 \\ 1 & 4 & 1 & 11 & 0 & -13 & -4 \end{array}\right) \qquad \left(\begin{array}{cccc|c} 0 & -3 & 3 & 1 & 5 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 2 & -2 & 0 & -3 \end{array}\right)$$

are the two matrices preceding Exercise 16.

The point of the reduced row echelon form is that the corresponding system of linear equations is in a particularly simple form, from which the solutions to the system $AX = C$ in (4) can be determined immediately:

20. (*Solving Systems of Linear Equations*) Let $(A' \mid C')$ be the reduced row echelon form of the augmented matrix $(A \mid C)$. The number of zero rows of A' is clearly at least as great as the number of zero rows of $(A' \mid C')$.