Therefore the required polynomial is

$$\begin{aligned} P(x) &= y_0L_0(x) + y_1L_1(x) + y_2L_2(x) + y_3L_3(x) \\ &= -\frac{y_0}{12}(x+1)(x-1)(x-2) + \frac{y_1}{6}(x+2)(x-1)(x-2) \\ &\quad - \frac{y_2}{6}(x+2)(x+1)(x-2) + \frac{y_3}{12}(x+2)(x+1)(x-1). \end{aligned}$$

To compute the value of $P(x)$ for a specific x it is usually better to leave the polynomial in this form rather than to rewrite it in increasing powers of x. For example, if $y_0 = -5$, $y_1 = 1$, $y_2 = 1$, and $y_3 = 7$, the value of $P(x)$ for $x = \frac{3}{2}$ is given by

$$\begin{aligned} P(\tfrac{3}{2}) &= \tfrac{5}{12}(\tfrac{5}{2})(\tfrac{1}{2})(-\tfrac{1}{2}) + \tfrac{1}{6}(\tfrac{7}{2})(\tfrac{1}{2})(-\tfrac{1}{2}) - \tfrac{1}{6}(\tfrac{7}{2})(\tfrac{5}{2})(-\tfrac{1}{2}) + \tfrac{7}{12}(\tfrac{7}{2})(\tfrac{5}{2})(\tfrac{1}{2}) \\ &= -\tfrac{25}{96} - \tfrac{7}{48} + \tfrac{35}{48} + \tfrac{245}{96} = \tfrac{276}{96} = 2\tfrac{7}{8}. \end{aligned}$$

15.7 Equally spaced interpolation points

In the foregoing discussion the interpolation points $x_0, x_1, \ldots, x_n$ were assumed to be distinct but otherwise arbitrary. Now we assume they are equally spaced and show that the Lagrange coefficients $L_k(x)$ can be considerably simplified. Suppose $x_0 < x_1 < x_2 < \cdots < x_n$, and let h denote the distance between adjacent points. Then we can write

$$x_j = x_0 + jh$$

for $j = 0, 1, 2, \ldots, n$. Since $x_k - x_j = (k - j)h$, Equation (15.10) becomes

$$L_k(x) = \prod_{\substack{j=0 \\ j\neq k}}^{n} \frac{x - x_0 - jh}{(k-j)h} = \prod_{\substack{j=0 \\ j\neq k}}^{n} \frac{t-j}{k-j}, \tag{15.11}$$

where

$$t = \frac{x - x_0}{h}.$$

In the last term on the right of (15.11) the product of the factors independent of t is

$$\begin{aligned} \prod_{\substack{j=0 \\ j\neq k}}^{n} \frac{1}{k-j} &= \left(\prod_{j=0}^{k-1} \frac{1}{k-j}\right)\left(\prod_{j=k+1}^{n} \frac{1}{k-j}\right) = \frac{1}{k!}\prod_{j=k+1}^{n} \frac{(-1)}{j-k} \\ &= \frac{(-1)^{n-k}}{k!\,(n-k)!} = \frac{(-1)^{n-k}}{n!}\binom{n}{k}, \end{aligned} \tag{15.12}$$

where $\binom{n}{k}$ is the binomial coefficient. Since $x = x_0 + th$, Equation (15.11) now becomes

$$L_k(x_0 + th) = \frac{(-1)^{n-k}}{n!}\binom{n}{k}\prod_{\substack{j=0 \\ j\neq k}}^{n}(t-j). \tag{15.13}$$

For each fixed n, the right member of (15.13) is a function of k and t that can be tabulated. Extensive tables of the Lagrangian coefficients for equally spaced interpolation points have been prepared by the National Bureau of Standards. (See Reference 13 in the bibliography at the end of this chapter.) If x and h are chosen so that the number $t = (x - x_0)/h$ is one for which the Lagrangian coefficients $L_k(x_0 + th)$ are tabulated, the actual calculation of $P(x_0 + th)$ is reduced to a multiplication of the $f(x_k)$ by the tabulated $L_k(x_0 + th)$, followed by addition.

15.8 Error analysis in polynomial interpolation

Let f be a function defined on an interval $[a, b]$ containing the $n + 1$ distinct points $x_0, x_1, \ldots, x_n$, and let P be the interpolation polynomial of degree $\leq n$ which agrees with f at these points. If we alter the values of f at points other than the interpolation points we do not alter the polynomial P. This shows that the function f and the polynomial P may differ considerably at points other than the interpolation points. If the given function f has certain qualities of "smoothness" throughout the interval $[a, b]$ we can expect that the interpolating polynomial P will be a good approximation to f at points other than the x_k. The next theorem gives a useful expression that enables us to study the error in polynomial interpolation when the given function has a derivative of order $n + 1$ throughout $[a, b]$.

THEOREM 15.3. *Let $x_0, x_1, \ldots, x_n$ be $n + 1$ distinct points in the domain of a function f, and let P be the interpolation polynomial of degree $\leq n$ that agrees with f at these points. Choose a point x in the domain of f and let $[\alpha, \beta]$ be any closed interval containing the points $x_0, x_1, \ldots, x_n$, and x. If f has a derivative of order $n + 1$ in the interval $[\alpha, \beta]$ there is at least one point c in the open interval (α, β) such that*

$$f(x) - P(x) = \frac{A(x)}{(n+1)!} f^{(n+1)}(c), \tag{15.14}$$

where

$$A(x) = (x - x_0)(x - x_1) \cdots (x - x_n).$$

Note: Point c depends on both x and n.

Proof. If x is one of the interpolation points x_k, then $A(x_k) = 0$ and Equation (15.14) is trivially satisfied for any choice of c in (α, β). Suppose, then, that x is not one of the interpolation points. Keep x fixed and define a new function F on $[\alpha, \beta]$ by the equation

$$F(t) = A(x)[f(t) - P(t)] - A(t)[f(x) - P(x)]. \tag{15.15}$$

The right-hand side of this equation, as a function of t, has a derivative of order $n + 1$; hence the same is true of the left-hand side. Since $P(t)$ is a polynomial in t of degree $\leq n$, its $(n + 1)$st derivative is identically zero. The polynomial $A(t)$ has degree $n + 1$, the term of highest degree being t^{n+1}, and we have $A^{(n+1)}(t) = (n + 1)!$. Therefore, if we differentiate Equation (15.15) $n + 1$ times with respect to t we obtain the formula

$$F^{(n+1)}(t) = A(x)f^{(n+1)}(t) - (n+1)!\,[f(x) - P(x)]. \tag{15.16}$$

From the definition in Equation (15.15) we see that F has the value zero at the $n + 1$ interpolation points $x_0, x_1, \ldots, x_n$ and *also* at the point x. Therefore $F(t) = 0$ at $n + 2$ distinct points in the interval $[\alpha, \beta]$. These points determine $n + 1$ adjacent subintervals of $[\alpha, \beta]$ and the function F vanishes at both endpoints of each of these subintervals. By Rolle's theorem, the derivative $F'(t)$ must be zero for at least one t interior to each subinterval. If we choose exactly one such t from each subinterval we obtain $n + 1$ distinct points in the open interval (α, β) at which $F'(t) = 0$. These points, in turn, determine n subintervals at whose endpoints we have $F'(t) = 0$. Applying Rolle's theorem to F' we find that the second derivative $F''(t)$ is zero for at least n distinct points in (α, β). After applying Rolle's theorem $n + 1$ times in this manner we finally find that there is at least one point c in (α, β) at which $F^{(n+1)}(c) = 0$. Substituting this value of c in Equation (15.16) we obtain

$$(n + 1)!\,[f(x) - P(x)] = A(x)f^{(n+1)}(c),$$

which is the same as (15.14). This completes the proof.

It should be noted that, as with approximation by Taylor polynomials, the error term involves the $(n + 1)$st derivative $f^{(n+1)}(c)$ evaluated at an unknown point c. If the extreme values of $f^{(n+1)}$ in $[\alpha, \beta]$ are known, useful upper and lower bounds for the error can be obtained.

Suppose now that the interpolation points are equally spaced and that $x_0 < x_1 < x_2 < \cdots < x_n$. If h denotes the spacing we can write

$$x_j = x_0 + jh \qquad \text{and} \qquad x = x_0 + th,$$

where $t = (x - x_0)/h$. Since $x - x_j = (t - j)h$, the polynomial $A(x)$ can be written as

$$A(x) = \prod_{j=0}^{n} (x - x_j) = h^{n+1} \prod_{j=0}^{n} (t - j).$$

Formula (15.14) now becomes

$$f(x) - P(x) = \frac{f^{(n+1)}(c)}{(n + 1)!}\, h^{n+1} \prod_{j=0}^{n} (t - j), \tag{15.17}$$

with $t = (x - x_0)/h$.

EXAMPLE. *Error in linear interpolation.* Suppose a function f with a second derivative is tabulated and we wish to estimate its value at a point x intermediate to two consecutive entries x_0 and $x_0 + h$. If we use linear interpolation we approximate the graph off over the interval $[x_0, x_0 + h]$ by a straight line, as shown in Figure 15.1. If P denotes the linear interpolating polynomial, the error estimate in (15.17) becomes

$$f(x) - P(x) = \frac{f''(c)}{2!}\, h^2 t(t - 1), \tag{15.18}$$

where $t = (x - x_0)/h$. When x lies between x_0 and $x_0 + h$ we have $0 < t < 1$ and the maximum value of $|t(t - 1)|$ in this interval is $\frac{1}{4}$. Therefore (15.18) gives us the estimate

$$|f(x) - P(x)| \le \frac{|f''(c)|\, h^2}{8}.$$

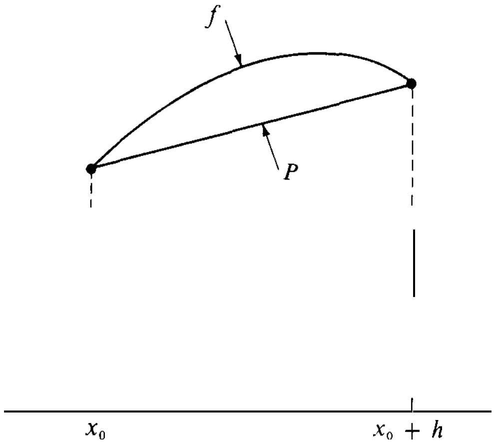

FIGURE 15.1 Linear interpolation.

The point c is an unknown point in the interval $(x_0, x_0 + h)$. If the second derivative f'' is bounded in this interval, say $|f''(x)| \le M$, the error estimate becomes

$$|f(x) - P(x)| \le \frac{Mh^2}{8}.$$

In particular, if f is a sine or cosine, then $|f''(x)| \le 1$ for all x and we have $|f(x) - P(x)| \le h^2/8$. If a table of sines or cosines has entries for every degree (one degree $= \pi/180$ radians) we have $h = \pi/180$, so

$$\frac{h^2}{8} = \frac{\pi^2}{8(180)^2} < \frac{10}{259{,}200} < \frac{1}{25{,}000} = 0.00004.$$

Since this error does not exceed $\frac{1}{2}$ in the fourth decimal place, linear interpolation would be satisfactory in a four-place table. The error estimate can be improved in portions of the table where $|f''(c)|$ is considerably less than 1.

15.9 Exercises

1. In each case find the polynomial P of lowest possible degree satisfying the given conditions.
   (a) $P(-1) = 0,\ P(0) = 2,\ P(2) = 7.$
   (b) $P(1) = 1,\ P(2) = 0,\ P(3) = 0,\ P(4) = 1.$
   (c) $P(1) = 1,\ P(2) = 2,\ P(3) = 3,\ P(0) = 1.$
   (d) $P(0) = -2,\ P(1) = 0,\ P(-1) = -2,\ P(2) = 16.$
   (e) $P(-2) = 11,\ P(-1) = -11,\ P(0) = -5,\ P(1) = -1.$
2. Let $f(x) = \cos(\pi x/4)$. Find the polynomial of smallest possible degree that takes the same values as f at the points $-2, -\frac{4}{3}, 0, \frac{4}{3}, 2$.
3. Let P be a polynomial of degree $\le n$ and let $A(x) = (x - x_0)(x - x_1) \cdots (x - x_n)$, where $x_0, x_1, \ldots, x_n$ are $n + 1$ distinct points.
   (a) Show that for any polynomial B the polynomial Q given by $Q(x) = P(x) + A(x)B(x)$ agrees with P at the points $x_0, x_1, \ldots, x_n$.

(b) Prove also the converse. That is, if Q is any polynomial that agrees with P at the points $x_0, x_1, \ldots, x_n$, then $Q(x) = P(x) + A(x)B(x)$ for some polynomial B.

4. (a) Find the polynomial Q of lowest possible degree that satisfies the conditions

$$Q(-2) = -5, \qquad Q(-1) = -1, \qquad Q(1) = 1, \qquad Q'(0) = -1.$$

[Hint: First find a polynomial P that takes the prescribed values at $-2, -1, 1$, and then use Exercise 3 to determine Q.]

(b) Find the polynomial Q of lowest possible degree that satisfies the conditions in part (a) with $Q'(0) = -3$ instead of $Q'(0) = -1$.

5. Let $f(x) = \log_4 x$ for $x > 0$. Compute $P(32)$, where P is the polynomial of lowest possible degree that agrees with f at the points :

(a) $x = 1, 64$.
(b) $x = 1, 16, 256$.
(c) $x = 4, 16, 64$.
(d) $x = 1, 4, 16, 64, 256$.

In each case compute the difference $f(32) - P(32)$. These examples show that the accuracy in polynomial interpolation is not necessarily improved by increasing the number of interpolation points.

6. The Lagrange interpolation coefficients $L_k(x)$ given by Equation (15.10) depend not only on x but also on the interpolation points $x_0, x_1, \ldots, x_n$. We can indicate this dependence by writing $L_k(x) = L_k(x; X)$, where X denotes the vector in $(n + 1)$-space given by $X = (x_0, x_1, \ldots, x_n)$. For a given real number b, let b denote the vector in $(n + 1)$-space all of whose components are equal to b. If $a \neq 0$, show that

$$L_k(ax + b\,;\, aX + b) = L_k(x;\, X).$$

This is called the invariance *property* of the Lagrange interpolation coefficients. The next exercise shows how this property can be used to help simplify calculations in practice.

7. Let P denote the polynomial of degree ≤ 4 that has the values

$$P(2.4) = 72, \qquad P(2.5) = 30, \qquad P(2.7) = 18, \qquad P(2.8) = 24, \qquad P(3.0) = 180.$$

(a) Introduce new interpolation points u_j related to the given points x by the equation $u_j = 10x_j - 24$. The u_j are integers. For each $k = 0, 1, 2, 3, 4$, determine the Lagrange interpolation coefficients $L_k(x)$ in terms of u, where $u = 10x - 24$.

(b) Use the invariance property of Exercise 6 to compute $P(2.6)$.

8. A table of the function $f(x) = \log x$ contains entries for $x = 1$ to $x = 10$ at intervals of 0.001. Values intermediate to each pair of consecutive entries are to be computed by linear interpolation. Assume the entries in the table are exact.

(a) Show that the error in linear interpolation will not exceed $\frac{1}{8}$ in the sixth decimal place.
(b) For what values of x will linear interpolation be satisfactory for a seven-place table?
(c) What should be the spacing of the entries in the interval $1 \leq x \leq 2$ so that linear interpolation will be satisfactory in a seven-place table?

In Exercises 9 through 15, $x_0, x_1, \ldots, x_n$ are distinct points and

$$A(x) = \prod_{j=0}^{n} (x - x_j), \qquad A_k(x) = \prod_{\substack{j=0 \\ j\neq k}}^{n} (x - x_j), \qquad L_k(x) = \frac{A_k(x)}{A_k(x_k)}.$$

9. Derive the formula $A'(x) = \sum_{k=0}^{n} A_k(x)$ by use of (a) logarithmic differentiation; (b) Lagrange's interpolation formula.
10. Prove each of the following formulas:

(a) $\displaystyle\sum_{k=0}^{n} L_k(x) = 1$ and $\displaystyle\sum_{k=0}^{n} \frac{A_k'(x)}{A'(x_k)} = 0$ for all x.

(b) $\displaystyle\sum_{k=0}^{n} \frac{1}{A'(x_k)} = 0.$ *[Hint:* Use part (a) with suitable values of x.]

11. Let P be any polynomial of degree $\le n$. Show that the coefficient of x^n is equal to

$$\sum_{k=0}^{n} \frac{P(x_k)}{A'(x_k)}.$$

12. (a) Determine a and b so that the polynomial

$$P_k(x) = \{a + b(x - x_k)\}L_k(x)^2$$

will have the following properties :

$$P_k(x_i) = 0 \quad \text{for all } i, \qquad P_k'(x_k) = 1, \qquad \text{and} \qquad P_k'(x_i) = 0 \quad \text{for } i \ne k.$$

(b) Determine c and d so that the polynomial

$$Q_k(x) = \{c + d(x - x_k)\}L_k(x)^2$$

will have the following properties:

$$Q_k(x_k) = 1, \qquad Q_k(x_i) = 0 \quad \text{for } i \ne k, \qquad \text{and} \qquad Q_k'(x_i) = 0 \quad \text{for all } i.$$

(c) Let $H(x) = \sum_{k=0}^{n} f(x_k)Q_k(x) + \sum_{k=0}^{n} f'(x_k)P_k(x)$, where f is a given function that is differentiable at $x_0, x_1, \ldots, x_n$. Prove that

$$H(x_i) = f(x_i) \qquad \text{and} \qquad H'(x_i) = f'(x_i) \qquad \text{for all } i.$$

Prove also that there is at most one polynomial $H(x)$ of degree $\le 2n + 1$ with this property.

13. (a) Let P and Q be two polynomials of degree $\le n$ satisfying the $n + 1$ conditions

$$P(x_0) = Q(x_0), \qquad P'(x_1) = Q'(x_1), \qquad P''(x_2) = Q''(x_2), \qquad \ldots, \qquad P^{(n)}(x_n) = Q^{(n)}(x_n).$$

Prove that $P(x) = Q(x)$ for all x.

(b) Let $B_0(x) = 1$, and for $n \ge 1$ define

$$B_n(x) = \frac{x(x - n)^{n-1}}{n!}.$$

Show that $B_n'(x) = B_{n-1}(x - 1)$ for $n \ge 1$ and deduce that

$$B_n(0) = B_n'(1) = B_n''(2) = \cdots = B_n^{(n-1)}(n - 1) = 0 \qquad \text{and} \qquad B_n^{(n)}(n) = 1.$$

(c) Show that the one and only polynomial of degree $\le n$ satisfying the conditions

$$P(0) = c_0, \qquad P'(1) = c_1, \qquad P''(2) = c_2, \qquad \ldots, \qquad P^{(n)}(n) = c_n$$

is given by

$$P(x) = \sum_{k=0}^{n} c_k B_k(x) .$$

(d) If $x_k = x_0 + kh$ for $k = 0, 1, 2, \ldots, n$, where $h > 0$, generalize the results in (b) and (c).

14. Assume $x_0, x_1, \ldots, x_n$ are integers satisfying $x_0 < x_1 < \cdots < x_n$.

(a) Prove that $|A'(x_k)| \ge k!\,(n - k)!$ and deduce that

$$\sum_{k=0}^{n} \frac{1}{|A'(x_k)|} \le \frac{2^n}{n!} .$$

(b) Let P be any polynomial of degree n, with the term of highest degree equal to x^n. Let M denote the largest of the numbers $|P(x_0)|, |P(x_1)|, \ldots, |P(x_n)|$. Prove that $M \ge n!/2^n$. *[Hint: Use part (a) and Exercise 11.]*

15. Prove the following formulas. In parts (a) and (b), x is any point different from $x_0, x_1, \ldots, x_n$.

(a) $$\frac{A'(x)}{A(x)} = \sum_{j=0}^{n} \frac{1}{x - x_j} .$$

(b) $$\frac{A''(x)}{A'(x)} = \frac{A_k(x)}{A'(x)} \sum_{\substack{j=0 \\ j \ne k}}^{n} \frac{1}{x - x_j} + \sum_{\substack{j=0 \\ j \ne k}}^{n} \frac{1}{x - x_j} - \frac{A(x)}{A'(x)} \sum_{\substack{j=0 \\ j \ne k}}^{n} \frac{1}{(x - x_j)^2} .$$

(c) $$\frac{A''(x_k)}{A'(x_k)} = 2 \sum_{\substack{j=0 \\ j \ne k}}^{n} \frac{1}{x_k - x_j} .$$

16. Let $P_n(x)$ be the polynomial of degree $\le n$ that agrees with the function $f(x) = e^{ax}$ at the $n + 1$ integers $x = 0, 1, \ldots, n$. Since this polynomial depends on a we denote it by $P_n(x; a)$. Prove that the limit

$$\lim_{a \to 0} \frac{P_n(x; a) - 1}{a}$$

exists and is a polynomial in x. Determine this polynomial explicitly.

15.10 Newton's interpolation formula

Let P_n denote the interpolation polynomial of degree $\le n$ that agrees with a given function f at $n + 1$ distinct points $x_0, x_1, \ldots, x_n$. Lagrange's interpolation formula tells us that

$$P_n(x) = \sum_{k=0}^{n} L_k(x) f(x_k) ,$$

where $L_k(x)$ is a polynomial of degree n (the Lagrange interpolation coefficient) given by the product formula

$$L_k(x) = \prod_{\substack{j=0 \\ j \ne k}}^{n} \frac{x - x_j}{x_k - x_j}, \qquad \text{for} \quad k = 0, 1, 2, \ldots, n. \tag{15.19}$$

Suppose we adjoin a new interpolation point x_{n+1} to the given points $x_0, x_1, \ldots, x_n$. To determine the corresponding polynomial P_{n+1} by Lagrange's formula it is necessary to compute a new interpolation coefficient L_{n+1} and to recompute all the earlier coefficients $L_0, L_1, \ldots, L_n$, each of which is now a polynomial of degree $n + 1$. In practice this involves considerable labor. Therefore it is desirable to have another formula for determining P_n that provides an easier transition from P_n to P_{n+1}. One such formula was discovered by Newton; we shall derive it from the following theorem.

THEOREM 15.4. *Given $n + 2$ distinct points $x_0, x_1, \ldots, x_n, x_{n+1}$. Let P_n be the polynomial of degree $\leq n$ that agrees with a given function f at $x_0, \ldots, x_n$, and let P_{n+1} be the polynomial of degree $\leq n + 1$ that agrees with f at $x_0, x_1, \ldots, x_n, x_{n+1}$. Then there is a constant c_{n+1}, uniquely determined by f and by the interpolation points $x_0, \ldots, x_{n+1}$, such that*

$$P_{n+1}(x) = P_n(x) + c_{n+1}(x - x_0) \cdots (x - x_n). \tag{15.20}$$

Proof. Let $Q(x) = P_n(x) + c(x - x_0) \cdots (x - x_n)$, where c is an unspecified constant. Then Q is a polynomial of degree $\leq n + 1$ that agrees with P_n and hence with f at each of the $n + 1$ points $x_0, \ldots, x_n$. Now we choose c to make Q agree with f also at x_{n+1}. This requires

$$f(x_{n+1}) = P_n(x_{n+1}) + c(x_{n+1} - x_0) \cdots (x_{n+1} - x_n).$$

Since the coefficient of c is nonzero, this equation has a unique solution which we call c_{n+1}. Taking $c = c_{n+1}$ we see that $Q = P_{n+1}$.

The next theorem expresses $P_n(x)$ in terms of the numbers $c_1, \ldots, c_n$.

THEOREM 15.5. NEWTON'S INTERPOLATION FORMULA. *If $x_0, \ldots, x_n$ are distinct, we have*

$$P_n(x) = f(x_0) + \sum_{k=1}^{n} c_k(x - x_0) \cdots (x - x_{k-1}). \tag{15.21}$$

Proof. We define $P_0(x) = f(x_0)$ and take $n = 0$ in (15.20) to obtain

$$P_1(x) = f(x_0) + c_1(x - x_0).$$

Now take $n = 1$ in (15.20) to get

$$P_2(x) = P_1(x) + c_2(x - x_0)(x - x_1) = f(x_0) + c_1(x - x_0) + c_2(x - x_0)(x - x_1).$$

By induction, we obtain (15.21).

The property of Newton's formula expressed in Equation (15.20) enables us to calculate P_{n+1} simply by adding one new term to P_n. This property is not possessed by Lagrange's formula.

The usefulness of Newton's formula depends, of course, on the ease with which the coefficients $c_1, c_2, \ldots, c_n$ can be computed. The next theorem shows that c_n is a linear combination of the function values $f(x_0), \ldots, f(x_n)$.

THEOREM 15.6. *The coefficients in Newton's interpolation formula are given by*

$$c_n = \sum_{k=0}^{n} \frac{f(x_k)}{A_k(x_k)}, \qquad \text{where} \quad A_k(x_k) = \prod_{\substack{j=0 \\ j\neq k}}^{n} (x_k - x_j). \tag{15.22}$$

Proof. By Lagrange's formula we have

$$P_n(x) = \sum_{k=0}^{n} L_k(x) f(x_k),$$

where $L_k(x)$ is the polynomial of degree n given by (15.19). Since the coefficient of x^n in $L_k(x)$ is $1/A_k(x_k)$, the coefficient of x^n in *P,(x)* is the sum appearing in (15.22). On the other hand, Newton's formula shows that the coefficient of x^n in *P,(x)* is equal to c,. This completes the proof.

Equation (15.22) provides a straightforward way for calculating the coefficients in Newton's formula. The numbers $A_k(x_k)$ also occur as factors in the denominator of the Lagrange interpolation coefficient $L_k(x)$. The next section describes an alternate method for computing the coefficients when the interpolation points are equally spaced.

15.11 Equally spaced interpolation points. The forward difference operator

In the case of equally spaced interpolation points with $x_k = x_0 + kh$ for $k = 0, 1, \ldots, n$ we can use Equation (15.12) to obtain

$$\frac{1}{A_k(x_k)} = \prod_{\substack{j=0 \\ j\neq k}}^{n} \frac{1}{x_k - x_j} = \frac{1}{h^n} \prod_{\substack{j=0 \\ j\neq k}}^{n} \frac{1}{k-j} = \frac{(-1)^{n-k}}{n!\,h^{n}} \binom{n}{k}.$$

In this case the formula for c_n in Theorem 15.6 becomes

$$c_n = \frac{1}{n!\,h^n} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x_k), \tag{15.23}$$

The sum on the right can be calculated in another way in terms of a linear operator A called the *forward difference operator.*

DEFINITION. *Let h be a fixed real number and let* f *be a given function. The function* Af *defined by the equation*

$$\Delta f(x) = f(x+h) - f(x)$$

is called the first forward difference off. It is dejined at those points x for which both x and $x + h$ *are in the domain off. Higher order differences* $\Delta^2 f$, $\Delta^3 f$, ... *are defined inductively as follows:*

$$\Delta^{k+1} f = \Delta(\Delta^k f) \qquad \text{for } k = 1, 2, 3, \ldots.$$

Note: The notations $\Delta_h f(x)$ and Af (x; *h)* are also used for $\Delta f(x)$ when it is desirable to indicate the dependence on h. It is convenient to define $\Delta^0 f = f$.

The nth difference $\Delta^n f(x)$ is a linear combination of the function values $f(x), f(x + h), \ldots, f(x + nh)$. For example, we have

$$\begin{aligned}\Delta^2 f(x) &= \{f(x + 2h) - f(x + h)\} - \{f(x + h) - f(x)\}\\ &= f(x + 2h) - 2f(x + h) + f(x).\end{aligned}$$

In general, we have

$$\Delta^n f(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x + kh). \tag{15.24}$$

This is easily proved by induction on n, using the law of Pascal's triangle for binomial coefficients :

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}.$$

Now suppose f is defined at $n + 1$ equally spaced points $x_k = x_0 + kh$ for $k = 0, 1, \ldots, n$. Then from (15.23) and (15.24) we obtain the formula

$$c_n = \frac{1}{h^n n!} \Delta^n f(x_0) .$$

This provides a rapid method for calculating the coefficients in Newton's interpolation formula. The diagram in Table 15.1, called a *difference* ***table,*** shows how the successive differences can be systematically calculated from a tabulation of the values of f at equally spaced points. In the table we have written f_k for $f(x_k)$.

Newton's interpolation formula (15.21) now becomes

$$P_n(x) = f(x_0) + \sum_{k=1}^{n} \frac{\Delta^k f(x_0)}{k!\, h^k} \prod_{j=0}^{k-1} (x - x_j). \tag{15.25}$$

TABLE 15.1

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
x_0	f_0			
		$f_1 - f_0 = \Delta f(x_0)$		
x_1	f_1		$\Delta f(x_1) - \Delta f(x_0) = \Delta^2 f(x_0)$	
		$f_2 - f_1 = \Delta f(x_1)$		$\Delta^2 f(x_1) - \Delta^2 f(x_0) = \Delta^3 f(x_0)$
x_2	f_2		$\Delta f(x_2) - \Delta f(x_1) = \Delta^2 f(x_1)$	
		$f_3 - f_2 = \Delta f(x_2)$		
x_3	f_3			

If we write

$$\prod_{j=0}^{k-1}(x - x_j) = \prod_{j=0}^{k-1}(x - x_0 - jh) = h^k \prod_{j=0}^{k-1}\left(\frac{x - x_0}{h} - j\right) = h^k \prod_{j=0}^{k-1}(t - j),$$

where $t = (x - x_0)/h$, Equation (15.25) becomes

$$P_n(x) = f(x_0) + \sum_{k=1}^{n} \frac{\Delta^k f(x_0)}{k!} \prod_{j=0}^{k-1}(t - j) . \tag{15.26}$$

15.12 Factorial polynomials

The product $t(t - 1) \cdots (t - k + 1)$ which appears in the sum in (15.26) is a polynomial in t of degree k called a *factorial polynomial,* or the *factorial kth power* of t. It is denoted by the symbol $t^{(k)}$. Thus, by definition,

$$t^{(k)} = \prod_{j=0}^{k-1}(t - j) .$$

We also define $t^{(0)} = 1$. If we consider the forward difference operator Δ with $h = 1$, that is, $\Delta f(x) = f(x + 1) - f(x)$, we find that

$$\Delta t^{(n)} = n t^{(n-1)} \qquad \text{for } n \geq 1 .$$

This is analogous to the differentiation formula $Dt^n = nt^{n-1}$ for ordinary powers. Thus, the factorial power $t^{(n)}$ is related to differences much in the same way that the ordinary power t^n is related to derivatives.

With the use of factorial polynomials, Newton's interpolation formula (15.26) becomes

$$P_n(x_0 + th) = \sum_{k=0}^{n} \frac{\Delta^k f(x_0)}{k!} t^{(k)} .$$

Expressed in this form, Newton's formula resembles the Taylor formula for the polynomial of degree $\leq n$ that agrees with f and its first n derivatives at x_0. If we write

$$\binom{t}{k} = \frac{t^{(k)}}{k!} = \frac{t(t - 1) \cdots (t - k + 1)}{k!} ,$$

Newton's formula takes the form

$$P_n(x_0 + th) = \sum_{k=0}^{n} \binom{t}{k} \Delta^k f(x_0) .$$

Further properties of factorial polynomials are developed in the following exercises.

15.13 Exercises

1. Let $\Delta f(x) = f(x + h)$ -f(x). Iffis a polynomial of degree n, say

$$f(x) = \sum_{r=0}^{n} a_r x^r$$

with $a_n \neq 0$, show that (a) $\Delta^k f(x)$ is a polynomial of degree $n - k$ if $k \le n$; (b) $\Delta^n f(x) = n!\, h^n a_n$; (c) $\Delta^k f(x) = 0$ for $k > n$.

2. Let $\Delta f(x) = f(x + h)$ -f(x). If $f(x) = \sin(ax + b)$, prove that

$$\Delta^n f(x) = \left(2 \sin \frac{ah}{2}\right)^n \sin\left(ax + b + \frac{nah + n\pi}{2}\right).$$

3. Let $\Delta f(x) = f(x + h)$ -f(x).
(a) If $f(x) = a^x$, where $a > 0$, show that $\Delta^k f(x) = (a^h - 1)^k a^x$.
(b) If $g(x) = (1 + a)^{x/h}$, where $a > 0$, show that $\Delta^k g(x) = a^k g(x)$.
(c) Show that the polynomial P_n of degree n that takes: the values $P_n(k) = (1 + a)^k$ for $k = 0, 1, 2, \ldots, n$ is given by

$$P_n(x) = \sum_{k=0}^{n} \frac{a^k}{k!} x^{(k)}.$$

4. Let $x^{(n)}$ be the factorial nth power of x. Since $x^{(n)}$ is a polynomial in x of degree n with the value 0 when $x = 0$, we can write

$$x^{(n)} = \sum_{k=1}^{n} S_{k,n} x^k.$$

The numbers $S_{k,n}$ are called *Stirling numbers of the first kind.* From the definition of $x^{(n)}$ it is clear that $S_{n,n} = 1$ for $n \ge 0$.
(a) Show that $S_{n-1,n} = -n(n - 1)/2$ and that $S_{1,n} = (-1)^{n-1}(n - 1)!$ for $n \ge 1$.
(b) Prove that $S_{k,n+1} = S_{k-1,n} - nS_{k,n}$. Use this relation to verify the entries in Table 15.2, a table of Stirling numbers of the first'kind, and construct the next three rows of the table.

TABLE 15.2

n	$S_{1,n}$	$S_{2,n}$	$S_{3,n}$	$S_{4,n}$	$S_{5,n}$	$S_{6,n}$	$S_{7,n}$
1	1						
2	-1	1					
3	2	-3	1				
4	-6	11	-6	1			
5	24	-50	35	-10	1		
6	-120	274	-225	85	-15	1	
7	720	-1764	1624	-735	175	-21	1

(c) Express the polynomial $x^{(4)} + 3x^{(3)} + 2x^{(1)} + 1$ as a. linear combination of powers of x.

5. (a) Prove that

$$x = x^{(1)}, \qquad x^2 = x^{(1)} + x^{(2)}, \qquad x^3 = x^{(1)} + 3x^{(2)} + x^{(3)},$$