The key step is to show that the exterior solid angle at a vertex P is expressible intrinsically as $2\pi - (\alpha_1 + \alpha_2 + \cdots + \alpha_n)$, where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the face angles that meet at P. These are *not* the angles $\alpha_1', \alpha_2', \ldots, \alpha_n'$ between the planes that bound the exterior solid angle, but it turns out (Exercise 22.5.1) that

$$\alpha_i + \alpha_i' = \pi$$

for each i, whence the measure of the exterior solid angle, which comes from $\alpha_i' + \alpha_2' + \cdots + \alpha_n'$ by Harriot's theorem (Section 17.6), also comes from $\alpha_1 + \alpha_2 + \cdots + \alpha_n$.

Knowing now that the exterior solid angle at P equals $2\pi - \sum$ face angles at P, we get

$$\text{total exterior solid angle} = 2\pi V - \sum \text{all face angles},$$

where V is the total number of vertices. By grouping the face angles according to the types of faces, we also find (Exercise 22.5.2) that

$$\sum \text{all face angles} = \pi(2E - 2F),$$

whence

$$\begin{aligned} \text{total exterior solid angle} &= 2\pi(V - E + F) \\ &= 2\pi \times \text{Euler characteristic}. \end{aligned}$$

In the case of convex polyhedra, where we already know that total exterior solid angle $= 4\pi$, this gives Euler characteristic $= 2$. More important, the derivation is valid for polyhedra of arbitrary Euler characteristic, showing that the total exterior solid angle is really the *same* as the Euler characteristic, up to a constant multiple.

There is a similar intrinsic proof of the Gauss–Bonnet theorem, again valid for arbitrary Euler characteristic, which shows that

$$\text{total curvature} = \iint_{\mathcal{S}} \kappa_1 \kappa_2 \, dA = 2\pi \times \text{Euler characteristic}$$

(Exercise 22.5.3). Legendre's (1794) proof of the Euler polyhedron formula is the special case of the argument for constant curvature.

Thus the Euler characteristic regulates the total curvature of a surface. In particular, if the curvature is constant, it must have the same sign as the Euler characteristic. This in turn has implications for the geometry of the surface. As we saw in Section 17.4, surfaces of constant positive curvature have spherical geometry, those of zero curvature have Euclidean geometry, and those of negative curvature have hyperbolic geometry. In the next section we shall see that there is a natural way to impose constant curvature on surfaces of arbitrary Euler characteristic. It will then follow that the natural geometry of a surface is spherical, Euclidean, or hyperbolic according as its Euler characteristic is positive, zero, or negative. Moreover, if the absolute value of the curvature is taken to be 1, then the Gauss–Bonnet theorem gives

$$\text{area} = |2\pi \times \text{Euler characteristic}|.$$

This makes surface topology completely subordinate to geometry, at least for orientable surfaces, because it says that the topology of a surface is completely determined by the sign of its curvature and its area.

These results were implicit in the work of Poincaré and Klein in the 1880s. Perhaps Klein was the first to see clearly how the geometry of a surface determines its topology [see, for example, Klein (1928), p. 264].

EXERCISES

Figure 22.8 shows the region around a vertex P of a polyhedron and the exterior solid angle of P centered at O and bounded by the planes OAB, OBC, OCA perpendicular to the edges through P.

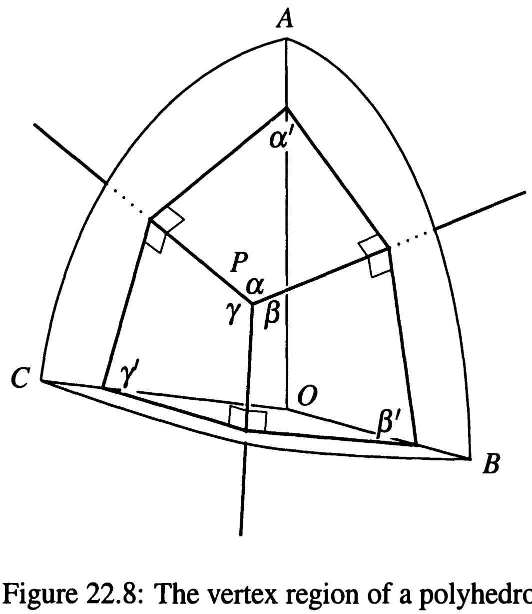

Figure 22.8: The vertex region of a polyhedron

22.5.1 Show that there are right angles where indicated, and hence that

$$\alpha+\alpha'=\pi, \quad \beta+\beta'=\pi, \quad \gamma+\gamma'=\pi.$$

Now to relate face angles to E and F, it helps to write

$$F=F_3+F_4+F_5+\cdots,$$

where $F_3=$ number of 3-gon faces, $F_4=$ number of 4-gon faces, and so on.

22.5.2 Show that

$$E=\frac{1}{2}(3F_3+4F_4+5F_5+\cdots),$$

and deduce that in an ordinary polyhedron (that is, one with flat faces)

$$\sum \text{all face angles} = \pi(2E-2F)$$

using the fact the angle sum of an n-gon is $(n-2)\pi$.

22.5.3 Prove the global form of the Gauss–Bonnet theorem,

$$\iint_{\mathcal{S}} \kappa_1\kappa_2\, dA = 2\pi \times \text{Euler characteristic},$$

by partitioning the closed surface $\mathcal{S}$ into geodesic polygons and applying the ordinary form of the Gauss–Bonnet theorem (Section 17.6).

22.6 Surfaces and Planes

In Section 16.5 we noticed that an elliptic function defines a mapping of a plane onto a torus. Such mappings are also interesting in the topological context, where they are called *universal coverings*. In general, a mapping $\varphi:\tilde{S}\to S$ of a surface $\tilde{S}$ onto a surface S is called a *covering* if it is a homeomorphism locally, that is, when restricted to sufficiently small pieces of $\tilde{S}$. The mapping of the plane onto the torus in Section 16.5 is a covering because it is a homeomorphism when restricted to any region smaller than a period parallelogram.

Another interesting example of a covering we have already met is the mapping of the sphere onto the projective plane given by Klein (1874) (Section 8.5). This map sends each pair of antipodal points of the sphere to the same point of the projective plane, and hence is a homeomorphism when restricted to any part of the sphere smaller than a hemisphere.

One more example is Beltrami's (1868a) covering of the pseudosphere by a horocyclic sector (Section 18.4). Topologically, this covering is the

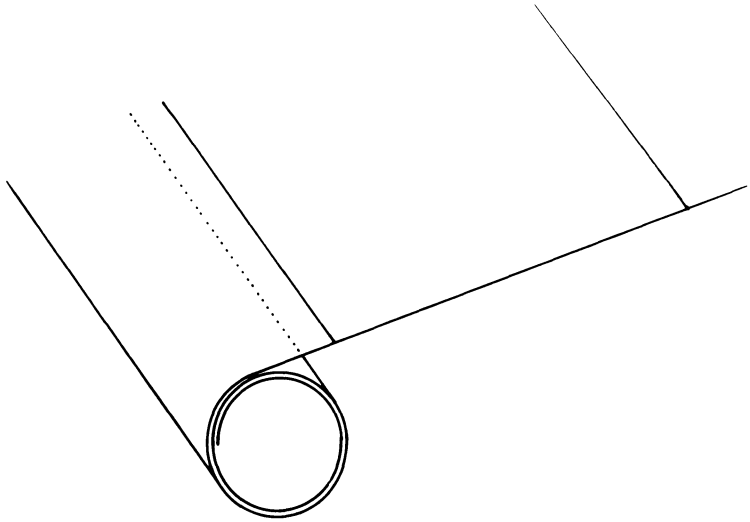

Figure 22.9: Covering a cylinder

same as the covering of a half-cylinder by a half-plane (Figure 22.9). All these coverings are *universal* in the sense that the covering surface $\tilde{S}$ (sphere or plane) can be covered only by $\tilde{S}$ itself.

An example of a *nonuniversal* covering is the covering of the torus by the cylinder, intuitively like an infinite snake swallowing its own tail (Figure 22.10). This is nonuniversal because the cylinder can in turn be covered by the plane, just as the half-cylinder is covered by the half-plane in Figure 22.9. In fact, by composing the coverings plane $\rightarrow$ cylinder $\rightarrow$ torus, we recover our first example, the plane $\rightarrow$ torus covering.

Since the sphere can be covered only by itself, the first interesting examples of coverings are those of orientable surfaces of genus ≥ 1 (that is, Euler characteristic ≤ 0). All of these surfaces can be covered by planes. Moreover, each nonorientable surface can be doubly covered by an orientable surface in the same way that the projective plane is covered by the sphere, so the main thing we need to understand is the universal covering of orientable surfaces of genus ≥ 1 by planes.

The basic idea is due to Schwarz, and it became generally known through a letter from Klein to Poincaré [Klein (1882a)]. To construct the universal covering of a surface S one takes infinitely many copies of a fun-

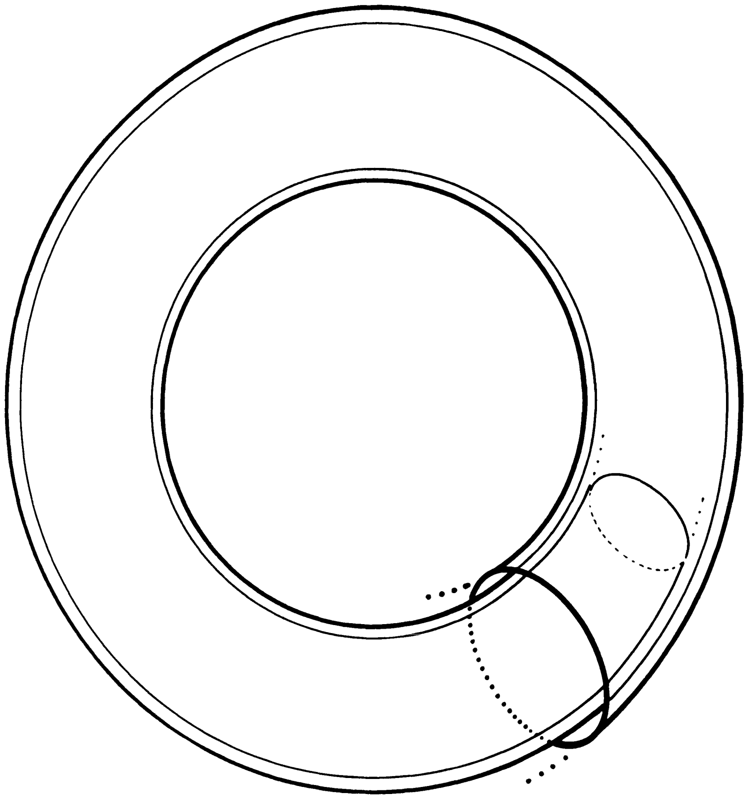

Figure 22.10: Covering a torus

damental polygon F for S and arranges them in the plane so that *adjacent* copies of F meet in the same way that F meets *itself* on S. For example, the torus T in Figure 22.11 has the square fundamental polygon F shown, which meets itself along $\vec{a}$ and $\vec{b}$ in S (where the arrows indicate that edges must agree in direction as well as label).

If instead we take infinitely many separate copies of F and join adjacent copies $\vec{a}$ to $\vec{a}$ and $\vec{b}$ to $\vec{b}$, then we obtain a plane $\tilde{T}$, tessellated as in Figure 22.12. The universal covering $\tilde{T} \to T$ is then defined by mapping each copy of F in $\tilde{T}$ in the natural way onto the F in T.

The tessellation of Figure 22.12 can of course be realized by squares in the Euclidean plane. We can therefore impose a Euclidean geometry on the torus by defining the distance between (sufficiently close) points on the torus to be the Euclidean distance between appropriate preimage points in

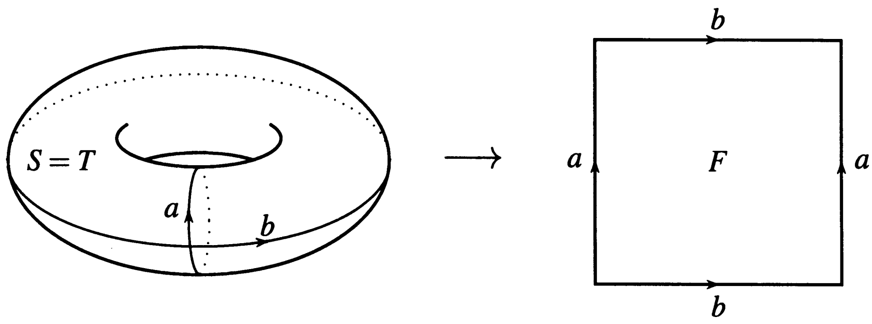

Figure 22.11: The torus and its fundamental polygon

the plane. In particular, the "straight lines" (geodesics) on the torus are the images of straight lines in the Euclidean plane. The torus geometry is not quite the geometry of the plane, of course, since there are closed geodesics, such as the images of the line segments a and b. However, it is Euclidean when restricted to sufficiently small regions. For example, the angle sum of each triangle on the torus is π.

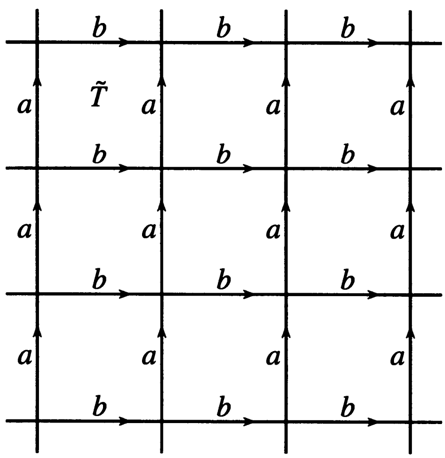

Figure 22.12: Tessellation of the torus cover

For surfaces of genus >1—that is, of negative Euler characteristic—the Gauss–Bonnet theorem predicts negative curvature and hence the natural covering plane should be hyperbolic. This can also be seen directly from the combinatorial nature of the tessellation on the universal cover. For example, the fundamental polygon F of the surface S of genus 2 is an octagon (Figure 22.13).

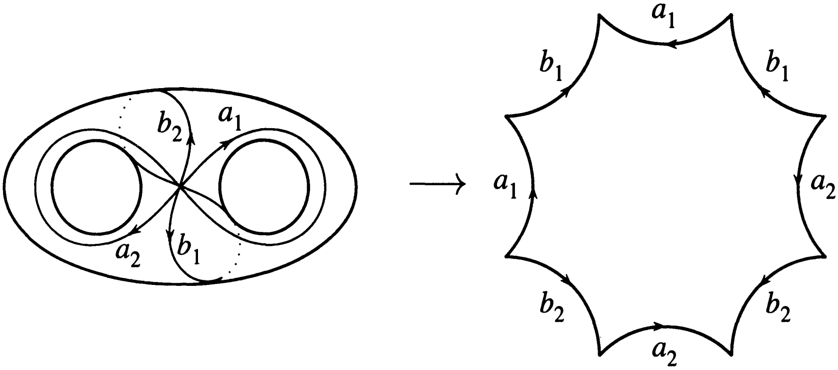

Figure 22.13: Genus 2 surface and its fundamental polygon

In the universal covering, eight of these octagons have to meet at each vertex, as the eight corners of the single F meet on S. Such a tessellation is impossible, by regular octagons, in the Euclidean plane, but it exists in the hyperbolic plane, as Figure 22.14 shows.

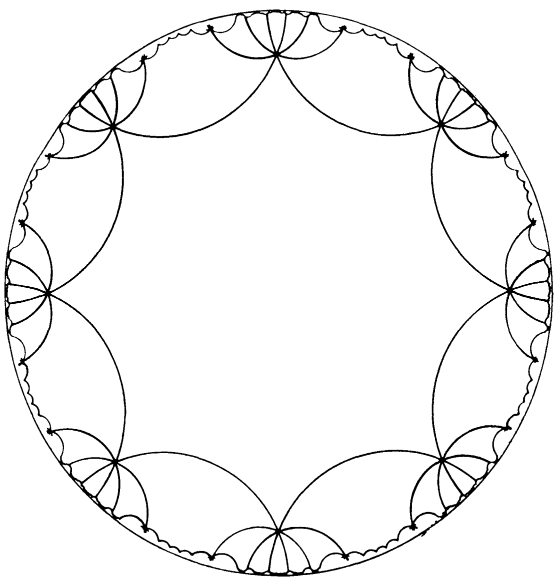

Figure 22.14: Tessellation of the genus 2 covering

In fact, this tessellation is obtained by amalgamating triangles in the Gauss tessellation (Figure 18.15). The tessellations for general genus >1 can similarly be realized geometrically in the hyperbolic plane and were among the hyperbolic tessellations considered by Poincaré (1882) and Klein (1882b). The distance function, hence the curvature and local geometry, can be transported from the covering plane to the surface as we did above for the torus.

EXERCISES

When surfaces of genus >1 are realized as surfaces of constant negative curvature, their genus can be read off from their area.

22.6.1 Show that the fundamental polygon for an orientable surface of genus p is a $4p$-gon with angle sum 2π.

22.6.2 Deduce that its Euler characteristic is proportional to its angular defect and hence to its area.

22.6.3 Conclude, using Exercise 22.3.1, that the area determines the genus.

22.7 The Fundamental Group

Another way to explore the meaning of the universal cover $\tilde{S}$ is to use it to plot paths on the surface S. As a point P moves on S, each preimage $\tilde{P}$ of P moves analogously on $\tilde{S}$. The only difference is that as P crosses an edge of the fundamental polygon on S, $\tilde{P}$ crosses from one fundamental polygon to another on $\tilde{S}$. Thus $\tilde{P}$ will not necessarily return to its starting point, even when P does. In fact, we can see that the displacement of $\tilde{P}$ in some way measures the extent to which P winds around the surface S. Figure 22.15 shows an example. As P winds once around the torus, more or less in the direction of $\vec{a}$, $\tilde{P}$ wanders from one end to the other of a segment $\vec{a}$ on $\tilde{S}$.

We say that closed paths p, p' with initial point O on S "wind in the same way," or are *homotopic*, if p can be deformed into p' with O fixed and without leaving the surface. Now if the path p of P is deformed into p', with O fixed, then the path $\tilde{p}$ of $\tilde{P}$ is deformed into a $\tilde{p}'$ with the same initial and final points, $\tilde{O}^{(1)}$ and $\tilde{O}^{(2)}$, as $\tilde{p}$. Hence each homotopy class corresponds simply to a *displacement* of the universal cover $\tilde{S}$ which moves $\tilde{O}^{(1)}$ to $\tilde{O}^{(2)}$. The different preimages $\tilde{P}$ will of course start at different preimages $\tilde{O}^{(1)}$ of O, but a single displacement of $\tilde{S}$ moves them all to their final positions $\tilde{O}^{(2)}$. Moreover, the displacement moves the whole tessellation of $\tilde{S}$ onto itself: it is a rigid motion of the tessellation.