Given the usual definition of the 'circle', it turns out, somewhat surprisingly, that 'circles' in the Poincaré sense are ordinary circles. It is just that their Poincaré centers are not where you would expect.

Using Steiner's geometry of 'inversion', one can prove that, under this interpretation, postulates I to IV are theorems in Euclidean geometry. However, postulate V, so interpreted, is not; through the center of the unit circle, for example, one can draw many 'lines' which do not meet a given orthogonal arc.

If V followed from I to IV, then V interpreted in the Poincaré sense, would both hold and not hold. We would have a statement about Euclidean lines and circles (related to the unit circle) which was both provable and disprovable relative to I to IV. Thus, assuming that I to IV are consistent, so are I to IV together with (H).

The reader can find the details of the 'proof by inversion' on pages 402 to 407 of volume 1 of Eves [1963]. Note that, although Eves uses logarithms in his proof, the reasoning is just the same if one drops them. (Eves uses logarithms because he wants the smallest 'distance' to be 0, not 1, and because of some results he aims to derive in a later section of his book.)

Nowadays people consider not only hyperbolic, but also 'elliptic geometry'. This was developed by Riemann in 1854, but is not to be confused with the more general 'Riemannian geometry', which we shall discuss below. In elliptic geometry, the straight lines are finite, and there are *no* parallels. A 'point' is like a pair of points on a sphere, and a 'line' is like a great circle on that sphere. Unfortunately, elliptic geometry does not satisfy postulate II, according to the way we interpreted 'continuously'.

The attitude of modern mathematicians is that one can vary the postulates of Euclid at will, constructing as many different geometries as one wishes. In the 19th century, this was a radical idea. People thought of Euclid's axioms as necessary truths about space, and hence truths which underlay the whole of astronomy and physics.

A modern physicist uses whichever geometry suits his purposes. According to the general theory of relativity, space-time is a four-dimensional Riemannian geometry, but with its curvature varying from place to place, depending on the local density of matter. The sum of the angles of a triangle might be two right angles (as in Euclidean geometry) if one was in a vacuum; however, if matter were present, the angle sum would differ from two right angles (as in non-Euclidean geometry), on account of the bending of light rays under the gravitational influence of that matter. Ideas such as these would have amazed mathematicians living in the early part of the 19th century.

We have seen in the last chapter that Euclid's postulates were not really adequate to describe the system he had in mind. Surprisingly, it was only in 1899 that Hilbert gave a completely adequate axiomatic description of three-dimensional Euclidean space. Since Hilbert required 21 postulates, or 'axioms' as he preferred to call them, we shall only state some of them here to give the flavour of his work.

Hilbert deals with the following undefined concepts: point, line, plane, incidence (between points and lines, between points and planes, between lines and planes), order (a ternary relation of 'betweenness' for three collinear points) and congruence (a binary relation between 'segments', which are themselves defined in terms of betweenness).

He lists seven axioms of incidence. For example, the first two can be combined to say this:

'Given two distinct points A and B there is a unique line a such that A lies on a and B lies on a.'

He lists five axioms of order. For example, the first of these says this:

'If B is between A and C then B is between C and A.'

More significant is his fifth axiom of order:

'If A, B, and C are three non-collinear points, and if a is a line which meets the segment AB, then a also meets the segment AC or the segment BC.'

He lists six axioms of congruence; for example the second one says

'If $AB \equiv A'B'$ and $AB \equiv A''B''$ then $A'B' \equiv A''B''$.'

He lists two axioms of continuity. The first of these is the so-called axiom of Archimedes: 'If e and f are geometric quantities and $e \neq 0$, then there is a natural number n such that $ne > f$.'

The second is his controversial axiom of completeness. He only reluctantly added it to the French translation of his lecture notes when it became apparent that otherwise one still could not deduce Euclid's Proposition 1. It was later simplified by Bernays as follows:

'No points can be added to a straight line so that all other postulates remain valid.'

Exercises

1. How does one use the parallel postulate to show, in Euclidean geometry, that every triangle has a circumcircle?

2. Show that, in the Poincaré model, there is exactly one line through any two distinct points. That is, prove, in Euclidean geometry, that, given any two points in the interior of a circle, there is exactly one other circle which goes through those points and is orthogonal to the first circle.

3. 'The true geometry is the one which is the simplest and most beautiful.' Write a short essay on this statement, saying something about the relation between the simple, the beautiful and the true.