34. Let φ be the linear transformation $\varphi : \mathbb{R}^4 \to \mathbb{R}^2$ such that

$$\varphi((1,0,0,0)) = (1,-1) \qquad \varphi((1,-1,0,0)) = (0,0)$$
$$\varphi((1,-1,1,0)) = (1,-1) \qquad \varphi((1,-1,1,-1)) = (0,0).$$

Determine a basis for the image and for the kernel of φ.

35. Let V be the set of all 2×2 matrices with real entries and let $\varphi : V \to \mathbb{R}$ be the map defined by sending a matrix $A \in V$ to the sum of the diagonal entries of A (the *trace* of A).

(a) Show that

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

is a basis for V.

(b) Prove that φ is a linear transformation and determine the matrix of φ with respect to the basis in (a) for V. Determine the dimension of and a basis for the kernel of φ.

36. Let V be the 6-dimensional vector space over $\mathbb{Q}$ consisting of the polynomials in the variable x of degree at most 5. Let φ be the map of V to itself defined by $\varphi(f) = x^2 f'' - 6xf' + 12f$, where f'' denotes the usual second derivative (with respect to x) of the polynomial $f \in V$ and f' similarly denotes the usual first derivative.

(a) Prove that φ is a linear transformation of V to itself.

(b) Determine a basis for the image and for the kernel of φ.

37. Let V be the 7-dimensional vector space over the field F consisting of the polynomials in the variable x of degree at most 6. Let φ be the linear transformation of V to itself defined by $\varphi(f) = f'$, where f' denotes the usual derivative (with respect to x) of the polynomial $f \in V$. For each of the fields below, determine a basis for the image and for the kernel of φ:

(a) $F = \mathbb{R}$

(b) $F = \mathbb{F}_2$, the finite field of 2 elements (note that, for example, $(x^2)' = 2x = 0$ over this field)

(c) $F = \mathbb{F}_3$

(d) $F = \mathbb{F}_5$.

38. Let A and B be square matrices. Prove that the trace of their Kronecker product is the product of their traces: $\mathrm{tr}\,(A \otimes B) = \mathrm{tr}\,(A)\,\mathrm{tr}\,(B)$. (Recall that the trace of a square matrix is the sum of its diagonal entries.)

39. Let F be a subfield of K and let $\psi : V \to W$ be a linear transformation of finite dimensional vector spaces over F.

(a) Prove that $1 \otimes \psi$ is a K–linear transformation from the vector spaces $K \otimes_F V$ to $K \otimes_F W$ over K. (Here 1 denotes the identity map from K to itself.)

(b) Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ and $\mathcal{E} = \{w_1, \ldots, w_m\}$ be bases of V and W respectively. Prove that the matrix of $1 \otimes \psi$ with respect to the bases $\{1 \otimes v_1, \ldots, 1 \otimes v_n\}$ and $\{1 \otimes w_1, \ldots, 1 \otimes w_m\}$ is the same as the matrix of ψ with respect to $\mathcal{B}$ and $\mathcal{E}$.

11.3 DUAL VECTOR SPACES

Definition.

(1) For V any vector space over F let $V^* = \mathrm{Hom}_F(V, F)$ be the space of linear transformations from V to F, called the *dual space* of V. Elements of V^* are called *linear functionals*.

(2) If $\mathcal{B} = \{v_1, v_2, \dots, v_n\}$ is a basis of the finite dimensional space V, define $v_i^* \in V^*$ for each $i \in \{1, 2, \dots, n\}$ by its action on the basis $\mathcal{B}$:

$$v_i^*(v_j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases} \qquad 1 \le j \le n. \tag{11.6}$$

Proposition 18. With notations as above, $\{v_1^*, v_2^*, \dots, v_n^*\}$ is a basis of V^*. In particular, if V is finite dimensional then V^* has the same dimension as V.

Proof: Observe that since V is finite dimensional, $\dim V^* = \dim \mathrm{Hom}_F(V, F) = \dim V = n$ (Corollary 11), so since there are n of the v_i^*'s it suffices to prove that they are linearly independent. If

$$\alpha_1 v_1^* + \alpha_2 v_2^* + \cdots + \alpha_n v_n^* = 0 \quad \text{in } \mathrm{Hom}_F(V, F),$$

then applying this element to v_i and using equation (6) above we obtain $\alpha_i = 0$. Since i is arbitrary these elements are linearly independent.

Definition. The basis $\{v_1^*, v_2^*, \dots, v_n^*\}$ of V^* is called the *dual basis* to $\{v_1, v_2, \dots, v_n\}$.

The exercises later show that if V is infinite dimensional it is always true that $\dim V < \dim V^*$. For spaces of arbitrary dimension the space V^* is the "algebraic" dual space to V. If V has some additional structure, for example a continuous structure (i.e., a topology), then one may define other types of dual spaces (e.g., the continuous dual of V, defined by requiring the linear functionals to be *continuous* maps). One has to be careful when reading other works (particularly analysis books) to ascertain what qualifiers are implicit in the use of the terms "dual space" and "linear functional."

Example

Let $[a, b]$ be a closed interval in $\mathbb{R}$ and let V be the real vector space of all continuous functions $f : [a, b] \to \mathbb{R}$. If $a < b$, V is infinite dimensional. For each $g \in V$ the function $\varphi_g : V \to \mathbb{R}$ defined by $\varphi_g(f) = \int_a^b f(t)g(t)dt$ is a linear functional on V.

Definition. The dual of V^*, namely V^{**}, is called the *double dual* or *second dual* of V.

Note that for a finite dimensional space V, $\dim V = \dim V^*$ and also $\dim V^* = \dim V^{**}$, hence V and V^{**} are isomorphic vector spaces. For infinite dimensional spaces $\dim V < \dim V^{**}$ (cf. the exercises) so V and V^{**} cannot be isomorphic. In the case of finite dimensional spaces there is a *natural*, i.e., basis independent or coordinate free way of exhibiting the isomorphism between a vector space and its second dual. The basic idea, in a more general setting, is as follows: if X is any set and S is any set of functions of X into the field F, we normally think of choosing or fixing an $f \in S$ and computing $f(x)$ as x ranges over all of X. Alternatively, we could think of fixing a point x in X and computing $f(x)$ as f ranges over all of S. The latter process, called *evaluation at* x shows that for each $x \in X$ there is a function $E_x : S \to F$ defined by

$E_x(f) = f(x)$ (i.e., evaluate f at x). This gives a map $x \mapsto E_x$ of X into the set of F-valued functions on S. If S "separates points" in the sense that for distinct points x and y of X there is some $f \in S$ such that $f(x) \neq f(y)$, then the map $x \mapsto E_x$ is injective. The proof of the next lemma applies this "role reversal" process to the situation where $X = V$ and $S = V^*$, proves E_x is a linear F-valued function on S, that is, E_x belongs to the dual space of V^*, and proves the map $x \mapsto E_x$ is a linear transformation from V into V^{**}. Note that throughout this process there is no mention of the word "basis" (although it is convenient to know the dimension of V^{**} — a fact we established by picking bases). In particular, the proof does not start with the familiar phrase "pick a basis of V ...."

Theorem 19. There is a natural injective linear transformation from V to V^{**}. If V is finite dimensional then this linear transformation is an isomorphism.

Proof: Let $v \in V$. Define the map (*evaluation at* v)

$$E_v : V^* \to F \qquad \text{by} \qquad E_v(f) = f(v).$$

Then $E_v(f + \alpha g) = (f + \alpha g)(v) = f(v) + \alpha g(v) = E_v(f) + \alpha E_g(v)$, so that E_v is a linear transformation from V^* to F. Hence E_v is an element of $\mathrm{Hom}_F(V^*, F) = V^{**}$. This defines a natural map

$$\varphi : V \to V^{**} \qquad \text{by} \qquad \varphi(v) = E_v.$$

The map φ is a *linear* map, as follows: for $v, w \in V$ and $\alpha \in F$,

$$E_{v+\alpha w}(f) = f(v + \alpha w) = f(v) + \alpha f(w) = E_v(f) + \alpha E_w(f)$$

for every $f \in V^*$, and so

$$\varphi(v + \alpha w) = E_{v+\alpha w} = E_v + \alpha E_w = \varphi(v) + \alpha\varphi(w).$$

To see that φ is injective let v be any nonzero vector in V. By the Building Up Lemma there is a basis $\mathcal{B}$ containing v. Let f be the linear transformation from V to F defined by sending v to 1 and every element of $\mathcal{B} - \{v\}$ to zero. Then $f \in V^*$ and $E_v(f) = f(v) = 1$. Thus $\varphi(v) = E_v$ is not zero in V^{**}. This proves $\ker\varphi = 0$, i.e., φ is injective.

If V has finite dimension n then by Proposition 18, V^* and hence also V^{**} has dimension n. In this case φ is an injective linear transformation from V to a finite dimensional vector space of the same dimension, hence is an isomorphism.

Let V, W be finite dimensional vector spaces over F with bases $\mathcal{B}, \mathcal{E}$, respectively and let $\mathcal{B}^*, \mathcal{E}^*$ be the dual bases. Fix some $\varphi \in \mathrm{Hom}_F(V, W)$. Then for each $f \in W^*$, the composite $f \circ \varphi$ is a linear transformation from V to F, that is $f \circ \varphi \in V^*$. Thus the map $f \mapsto f \circ \varphi$ defines a function from W^* to V^*. We denote this induced function on dual spaces by φ^*.

Theorem 20. With notations as above, φ^* is a linear transformation from W^* to V^* and $M_{\mathcal{E}^*}^{\mathcal{B}^*}(\varphi^*)$ is the transpose of the matrix $M_{\mathcal{B}}^{\mathcal{E}}(\varphi)$ (recall that the transpose of the matrix (a_{ij}) is the matrix (a_{ji})).

Proof: The map φ^* is linear because $(f + \alpha g) \circ \varphi = (f \circ \varphi) + \alpha(g \circ \varphi)$. The equations which define φ are (from its matrix)

$$\varphi(v_j) = \sum_{i=1}^{m} \alpha_{ij} w_i \qquad 1 \leq j \leq n.$$

To compute the matrix for φ^*, observe that by the definitions of φ^* and w_k^*

$$\varphi^*(w_k^*)(v_j) = (w_k^* \circ \varphi)(v_j) = w_k^*\left(\sum_{i=1}^{m} \alpha_{ij} w_i\right) = \alpha_{kj}.$$

Also

$$(\sum_{i=1}^{n} \alpha_{ki} v_i^*)(v_j) = \alpha_{kj}$$

for all j. This shows that the two linear functionals below agree on a basis of V, hence they are the same element of V^*:

$$\varphi^*(w_k^*) = \sum_{i=1}^{n} \alpha_{ki} v_i^*.$$

This determines the matrix for φ^* with respect to the bases $\mathcal{E}^*$ and $\mathcal{B}^*$ as the transpose of the matrix for φ.

Corollary 21. For any matrix A, the row rank of A equals the column rank of A.

Proof: Let $\varphi : V \to W$ be a linear transformation whose matrix with respect to some fixed bases of V and W is A. By Theorem 20 the matrix of $\varphi^* : W^* \to V^*$ with respect to the dual bases is the transpose of A. The column rank of A is the rank of φ and the row rank of A (= the column rank of the transpose of A) is the rank of φ^* (cf. Exercise 6 of Section 2). It therefore suffices to show that φ and φ^* have the same rank. Now

$$\begin{aligned} f \in \ker\varphi^* &\Leftrightarrow \varphi^*(f) = 0 \Leftrightarrow f \circ \varphi(v) = 0, \quad \text{for all } v \in V \\ &\Leftrightarrow \varphi(V) \subseteq \ker f \Leftrightarrow f \in \operatorname{Ann}(\varphi(V)), \end{aligned}$$

where $\operatorname{Ann}(S)$ is the annihilator of S described in Exercise 3 below. Thus $\operatorname{Ann}(\varphi(V)) = \ker\varphi^*$. By Exercise 3, $\dim \operatorname{Ann}(\varphi(V)) = \dim W - \dim \varphi(V)$. By Corollary 8, $\dim \ker\varphi^* = \dim W^* - \dim \varphi^*(W^*)$. Since W and W^* have the same dimension, $\dim \varphi(V) = \dim \varphi^*(W^*)$ as needed.