of the History of Mathematics, p. 225, Ferrari was murdered by his sister or by her boyfriend.

Rafael Bombelli of Bologna (1526–1572) published an algebra book in 1572, in which he traced the history of the subject back to Diophantus. He discussed complex radicals at length and showed that the irreducible case of the cubic equation leads to three real roots (see Case 3 Chapter 25). He also pointed out that the ancient Greek problem of trisecting an angle was equivalent to solving a cubic equation. He wrote $\overset{n}{\smile}$ for x^n.

Francois Viète, also called Vieta, (1540–1603) was a French lawyer and member of parliament, but his avocation was mathematics. He wrote *In artem analyticem isagoge* in 1591, in which he applied algebra to geometry. (Hitherto people had applied geometry to algebra.) He was challenged by King Henry IV to solve a special equation of the 45th degree, and managed to give the answer in a few minutes, having noticed that the equation was satisfied by the chord of an angle of $360°/45$. He constructed the circles touching three given circles, using only Euclidean geometry, and thus recaptured an ancient construction that was probably contained in a lost book by Apollonius. Viète deciphered a Spanish code for the French. His solutions of cubic and quadratic equations were just like ours.

We close this chapter with a question about astrology. Why did many mathematicians, such as Ptolemy, Cardano and later, Kepler, waste their time and talent on astrology when a little reflection reveals that it seems unlikely to have any truth in it? Astrology is based on the unproven and implausible assumption that there is a correlation between the constellations of the stars at the time of a person's birth and his or her character and ultimate life history. Did Ptolemy, et al., only espouse the practice of astrology because it helped to supplement their incomes, or did they genuinely believe in it, as perhaps a majority of people still do today? Kepler, for one, had a cynical view of astrology, as we shall see.

Exercises

1. Professor Smith learns a secret mathematical technique from Professor Brown only because he solemnly promises Brown that he will never publish it. Later Smith discovers that this technique was long ago published, by Professor Jones, in an obscure little journal that no one ever reads. Is it morally permissible for Brown to publish this technique (giving due credit, of course, to Smith and Jones)? Support your answer with reasons.

2. Solve Pacioli's problem, given above.

3. Solve Tartaglia's puzzle about the three couples.

4. Solve Tartaglia's puzzle about the oil.

5. Let $a_1 = 1/\sqrt{2}$, $a_{n+1} = \sqrt{\frac{1}{2} + \frac{1}{2}a_n}$. Viète proved that

$$2/\pi = a_1 a_2 \cdots a_n \cdots$$

Prove this formula. (Hint: first show

$$(\sin x)/(2^n \sin(x/2^n)) = (\cos x/2)(\cos x/2^2)(\cos x/2^3)\cdots(\cos x/2^n),$$

whence

$$(\sin x)/x = (\cos x/2)(\cos x/2^2)(\cos x/2^3)\cdots$$

and, finally, let $x = \pi/2$.)

25

The Cubic and Quartic Equations

Cardano was the first person to use imaginary numbers in print. In this chapter, we shall use imaginary numbers to present what is essentially Cardano's solution to the cubic equation. We shall also give Ludovico Ferrari's solution to the fourth degree polynomial equation.

Recall that $i = \sqrt{-1}$. Recall that, if a, a', b and b' are real numbers and $a + bi = a' + b'i$, then $a = a'$ and $b = b'$ (lest $i = (a - a')/(b - b')$, a real number). Let $\omega = \frac{-1}{2} + \frac{1}{2}\sqrt{3}i$. A quick calculation shows that $\omega^2 = \frac{-1}{2} - \frac{1}{2}\sqrt{3}i$, and hence $\omega^3 = 1$.

Lemma 25.1. *Any complex number $x+iy$ can be written in the* polar form *$r(\cos A + i\sin A)$, where r is a non-negative real number and A is a real number.*

Lemma 25.2. $(r(\cos A + i\sin A))^3 = r^3(\cos 3A + i\sin 3A)$.

Proof:

$$\cos 3A = \cos^3 A - 3\cos A\sin^2 A,$$
$$\sin 3A = 3\cos^2 A\sin A - \sin^3 A.$$

Lemma 25.3. *The equation $z^3 = 1$ has exactly three complex solutions: $1, \omega, \omega^2$.*

Lemma 25.4. *If b is any given complex number $\neq 0$, the equation $z^3 = b$ has exactly three complex solutions. If z_1 is one solution, the others are $z_1\omega$ and $z_1\omega^2$.*

Lemma 25.5. *If y and p are any complex numbers, there are complex numbers u and v such that $u + v = y$ and $uv = p$.* (This is the old Babylonian