The second consequence of the Desargues theorem is called the "scissors theorem." I do not know how common this name is, but it is used on p. 69 of the book *Fundamentals of Mathematics II. Geometry*, edited by Behnke, Bachmann, Fladt, and Kunle. In any case, it is an apt name, as you will see from Figure 6.13.

Scissors theorem. *If $ABCD$ and $A'B'C'D'$ are quadrilaterals with vertices alternately on two lines, and if AB is parallel to $A'B'$, BC to $B'C'$, and AD to $A'D'$, then also CD is parallel to $C'D'$.*

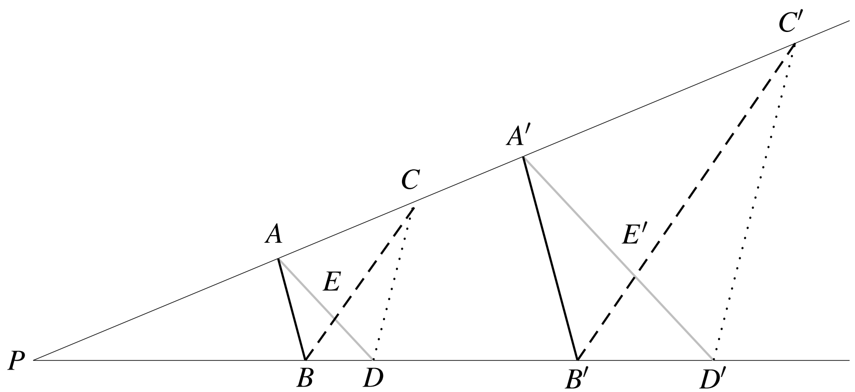

Figure 6.13: The scissors theorem

To prove this theorem, let E be the intersection of AD and BC and let E' be the intersection of $A'D'$ and $B'C'$, as shown in Figure 6.13. Then the triangles ABE and $A'B'E'$ have corresponding sides parallel. Hence, they are in perspective from the intersection P of AA' and BB', by the converse Desargues theorem.

But then the triangles CDE and $C'D'E'$ are also in perspective from P. Because their sides CE and $C'E'$, DE and $D'E'$, are parallel by assumption, it follows from the Desargues theorem that CD and $C'D'$ are also parallel, as required. □

The scissors theorem just proved says that if the black, gray, and dashed lines in Figure 6.13 are parallel, then so are the dotted lines. *What if the black, gray, and dotted lines are parallel: Are the dashed lines again parallel?* The answer is yes, and the proof is similar, but with a slightly different picture (Figure 6.14).

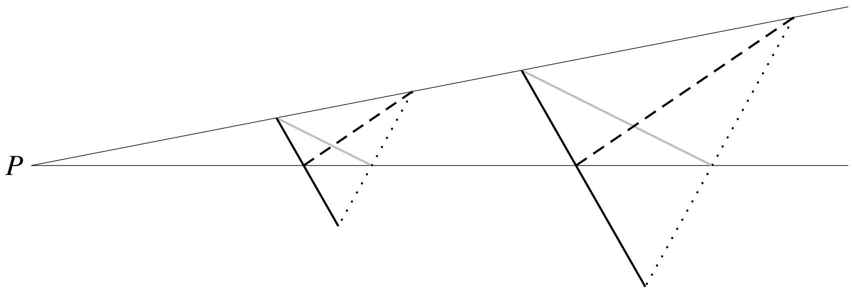

Figure 6.14: Second case of the scissors theorem

We have extended the black and dotted lines until they meet, forming triangles with their corresponding black, gray, and dotted sides parallel. Then it follows from the converse Desargues theorem that these triangles are in perspective from P. But then so are the triangles with dashed, black, and dotted sides. Hence their dashed sides are parallel by the Desargues theorem. □

Remark. In practice, the scissors theorem is often used in the following way. We have a pair of scissors $ABCD$ and another figure $D'A'B'C'F'$ with parallel pairs of black, gray, dashed, and dotted lines as shown in Figure 6.15. We want to prove that $D' = F'$ (so the ends of the gray and dotted lines *coincide*, and the second figure is also a pair of scissors).

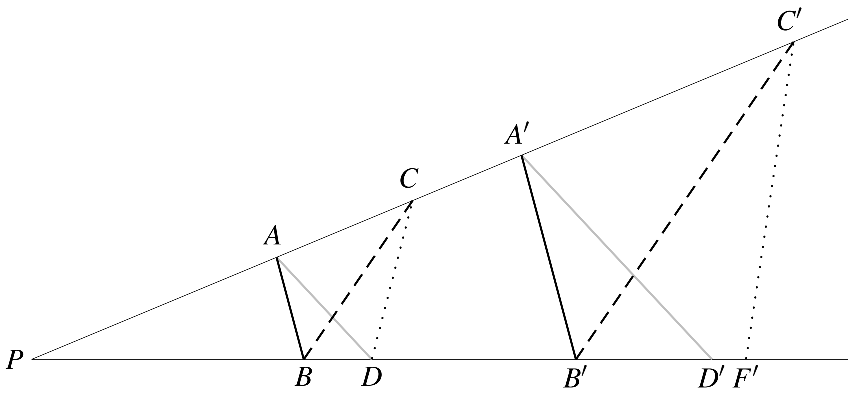

Figure 6.15: Applying the scissors theorem

This coincidence happens because the line $C'D'$ is parallel to CD by the scissors theorem, so $C'D'$ is the *same* line as $C'F'$, and hence $D' = F'$.

Exercises

Because the Desargues theorem implies its converse, another way to show that the Desargues theorem fails in the Moulton plane is to show that its converse fails. This plan is easily implemented with the help of Figure 6.16. (Moulton himself used this figure when he introduced the Moulton plane in 1902.)

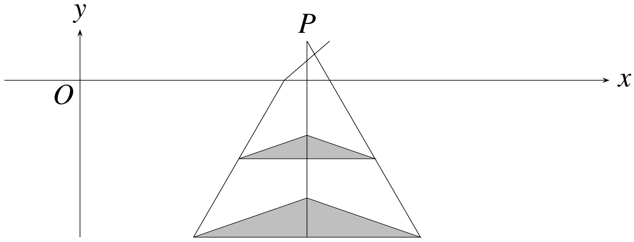

Figure 6.16: The converse Desargues theorem fails in the Moulton plane

6.3.1 Explain why Figure 6.16 shows the failure of the converse Desargues theorem in the Moulton plane.

6.3.2 Formulate a converse to the little Desargues theorem, and show that it follows from the little Desargues theorem.

6.3.3 Show that the converse little Desargues theorem implies a "little scissors theorem" in which the quadrilaterals have their vertices on parallel lines.

6.3.4 Design a figure that directly shows the failure of the little scissors theorem in the Moulton plane.

6.4 Projective arithmetic

If we choose any two lines in a projective plane as the x- and y-axes, we can add and multiply any points on the x-axis by certain constructions. The constructions resemble constructions of Euclidean geometry, but they use straightedge only, so they make sense in projective geometry. To keep them simple, we use lines we call "parallel," but this merely means lines meeting on a designated "line at infinity." The real difficulty is that the construction of $a+b$, for example, is different from the construction of $b+a$, so it is a "coincidence" if $a+b=b+a$. Similarly, it is a "coincidence" if $ab=ba$, or if any other law of algebra holds. Fortunately, we can show that the required coincidences actually occur, because they are implied by certain geometric coincidences, namely, the Pappus and Desargues theorems.

Addition

To construct the sum $a+b$ of points a and b on the x-axis, we take any line $\mathscr{L}$ parallel to the x-axis and construct the lines shown in Figure 6.17:

1. A line from a to the point where $\mathscr{L}$ meets the y-axis.
2. A line from b parallel to the y-axis.
3. A parallel to the first line through the intersection of the second line and $\mathscr{L}$.

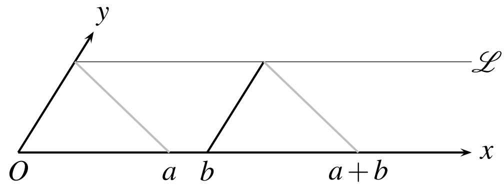

Figure 6.17: Construction of the sum

This construction is similar in spirit to the construction of the sum in Section 1.1. There we "copied a length" by moving it from one place to another by a compass. The spirit of the compass remains in the projective construction: the black line and the gray line form a "compass" that "copies" the length Oa to the point b.

We need the line $\mathscr{L}$ to construct $a+b$, but we get the same point $a+b$ from any other line $\mathscr{L}'$ parallel to the x-axis. This coincidence follows from the little Desargues theorem as shown in Figure 6.18.

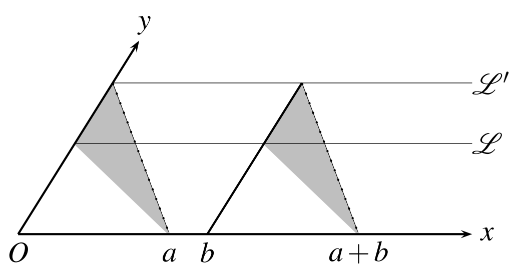

Figure 6.18: Why the sum is independent of the choice of $\mathscr{L}$

The black sides of the solid triangles are parallel by construction, as are the gray sides, one of which ends at the point $a+b$ constructed from $\mathscr{L}$. Then it follows from the little Desargues theorem that the dotted sides are also parallel, and one of them ends at the point $a+b$ constructed from $\mathscr{L}'$. Hence, the same point $a+b$ is constructed from both $\mathscr{L}$ and $\mathscr{L}'$.

Multiplication

To construct the product ab of two points a and b on the x-axis, we first need to choose a point $\neq O$ on the x-axis to be 1. We also choose a point $\neq O$ to be the 1 on the y-axis. The point ab is constructed by drawing the black and gray lines from 1 and a on the x-axis to 1 on the y-axis, and then drawing their parallels as shown in Figure 6.19. This construction is the projective version of “multiplication by a” done in Section 1.4.

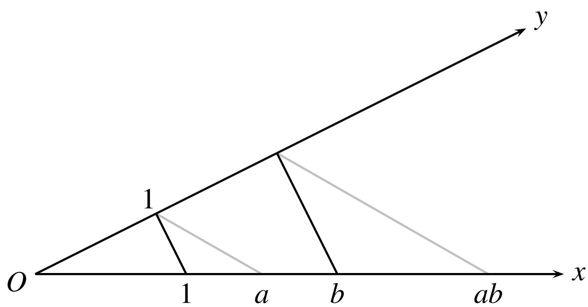

Figure 6.19: Construction of the product ab

Choosing the 1 on the x-axis means choosing a unit of length on the x-axis, so the position of ab definitely depends on it. For example, $ab = b$ if $a = 1$ but $ab \neq b$ if $a \neq 1$. However, the position of ab does not depend on the choice of 1 on the y-axis, as the scissors theorem shows (Figure 6.20).

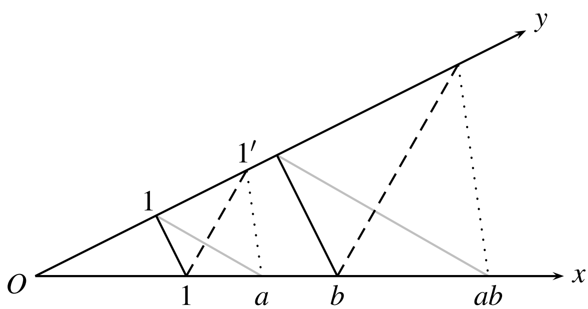

Figure 6.20: Why the product is independent of the 1 on the y-axis

If we choose $1'$ instead of 1 to construct ab, the path from b to ab follows the dashed and the dotted line instead of the black and the gray line. But it ends in the same place, because the dotted line to ab is parallel to the dotted line to a, by the scissors theorem.

Interchangeability of the axes

Once we have chosen points called 1 on both the x- and y-axes, it is natural to let each point a on the x-axis correspond to the point on the y-axis obtained by drawing the line through a parallel to the line through the points 1 on both axes (Figure 6.21).

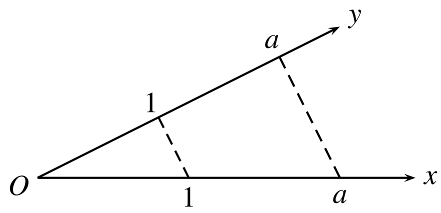

Figure 6.21: Corresponding points

It is also natural to define sum and product on the y-axis by constructions like those on the x-axis. But then the question arises: Do the y-axis sum and product correspond to the x-axis sum and product?

To show that *sums correspond*, we need to construct $a+b$ on the x-axis, and then show that the corresponding point $a+b$ on the y-axis is the y-axis sum of the y-axis a and b. Figure 6.22 shows how this construction is done.

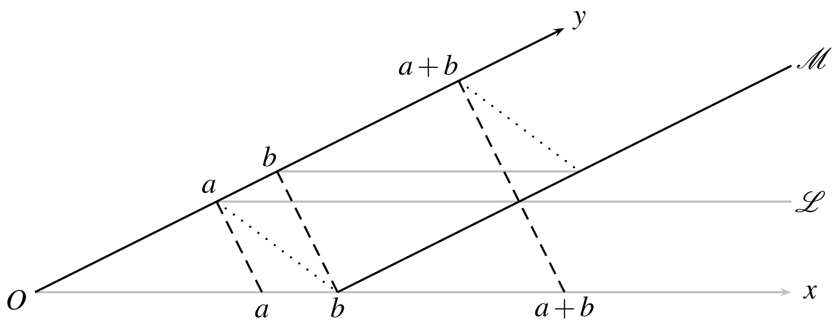

Figure 6.22: Corresponding sums

We construct $a+b$ on the x-axis using the line $\mathscr{L}$ through a on the y-axis. That is, draw the line $\mathscr{M}$ through b on the x-axis and parallel to the y-axis, and then draw the line (dashed) from the intersection of $\mathscr{M}$ and $\mathscr{L}$ parallel to the line from a on the x-axis to the intersection of $\mathscr{L}$ and the y-axis. This dashed line meets the x-axis at $a+b$, and (because it is parallel to the line from a to a) it also meets the y-axis at $a+b$.

Now we construct $a+b$ on the y-axis using the line $\mathscr{M}$ (as on the x-axis, the sum does not depend on line chosen, as long as it is parallel to the y-axis). That is, draw the line through b on the y-axis parallel to the x-axis, and then draw the line (dotted) parallel to the line from a on the y-axis to b on the x-axis (because this b is the intersection of the x-axis with $\mathscr{M}$).

But then, as is clear from Figure 6.22, we have a Pappus configuration of gray, dashed, and dotted lines between the y-axis and $\mathscr{M}$, hence the dotted line (leading to the y-axis sum) and the dashed line (leading to the point corresponding to the x-axis sum) end at the same point, as required. $\square$

To show that *products correspond*, we use the scissors theorem from Section 6.3. Figure 6.23 shows the corresponding points 1, a, b, and ab on both axes. The gray lines construct ab on the x-axis, and the dotted lines construct ab on the y-axis.

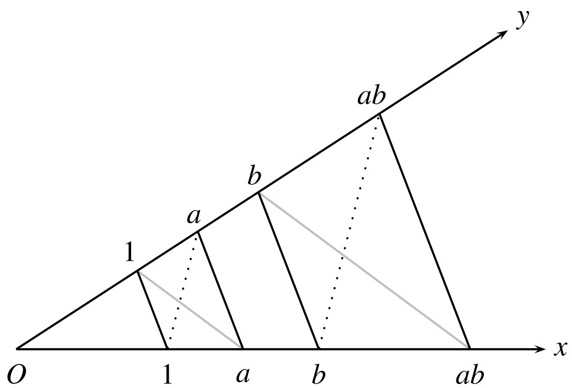

Figure 6.23: Construction of the product ab on both axes

It follows from the scissors theorem that the dotted line on the right ends at the same point as the black line from ab on the x-axis parallel to the lines connecting the corresponding points a and the corresponding points b. Hence, the product of a and b on the y-axis (at the end of the dotted line) is indeed the point corresponding to ab on the x-axis. $\square$

Exercises

These definitions of sum and product lead immediately to some of the simpler laws of algebra, namely, those concerned with the behavior of 0 and 1. The complete list of algebraic laws is given in the Section 6.5.

6.4.1 Show that $a+O=a$ for any a, so O functions as the zero on the x-axis.

6.4.2 Show that, for any a, there is a point b that serves as $-a$; that is, $a+b=O$. (Warning: Do not be tempted to use measurement to find b. Work backward from $O=a+b$, reversing the construction of the sum.)

6.4.3 Show that $a1=a$ for any a.

6.4.4 Show that, for any $a \neq O$, there is a b that serves as a^{-1}; that is, $ab=1$. (Again, do the construction of the product in reverse.)

You will notice that we have not attempted to define sums or products involving the point at infinity ∞ on the x-axis.

6.4.5 What happens when we try to construct $a+\infty$?

6.4.6 What is $-\infty$?

You should find that the answers to Exercises 6.4.5 and 6.4.6 are incompatible with ordinary arithmetic. This is why we do not include ∞ among the points we add and multiply.

6.5 The field axioms

In calculating with numbers, and particularly in calculating with symbols ("algebra"), we assume several things: that there are particular numbers 0 and 1; that each number a has a *additive inverse*, $-a$; that each number $a \neq 0$ has a *reciprocal*, a^{-1}; and that the following *field axioms* hold. (We introduced these in the discussion of vector spaces in Section 4.8.)

$$
\begin{array}{ccr}
a+b=b+a, & ab=ba & \text{(commutative laws)}\\
a+(b+c)=(a+b)+c, & a(bc)=(ab)c & \text{(associative laws)}\\
a+0=a, & a1=a & \text{(identity laws)}\\
a+(-a)=0, & aa^{-1}=1 & \text{(inverse laws)}\\
\multicolumn{2}{c}{a(b+c)=ab+ac} & \text{(distributive law)}
\end{array}
$$

We generally use these laws unconsciously. They are used so often, and they are so obviously true of numbers, that we do not notice them. But for the projective sum and product of points, they are *not* obviously true. It is not even clear that $a+b=b+a$, because the construction of $a+b$ is different from the construction of $b+a$. It is truly a *coincidence* that $a+b=b+a$ in projective geometry, the result of a geometric coincidence of the type discussed in Section 6.2.

In this chapter, we show that just two coincidences—the theorems of Pappus and Desargues—imply all nine field axioms. In fact, it is known that Pappus alone is sufficient, because it implies Desargues. We do not prove this fact here, partly because it is difficult, and partly because the Desargues theorem itself is interesting: It implies all the field axioms except $ab = ba$. Thus, the theorems of Pappus and Desargues have *algebraic content* that can be measured accurately by the field axioms they imply. Pappus implies all nine, and Desargues only eight—all but $ab = ba$.

Proof of the commutative laws

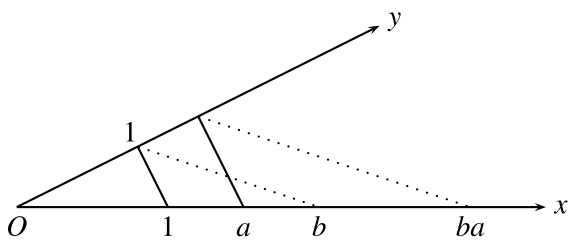

Figure 6.24: Construction of the product ba

We begin with the law $ab = ba$, which is the most important consequence of the Pappus theorem. Figure 6.24 shows the construction of ba from a and b, lying at the end of the second dotted line. It is different from the construction of ab, and Figure 6.25 shows the constructions of both ab and ba on the same diagram.

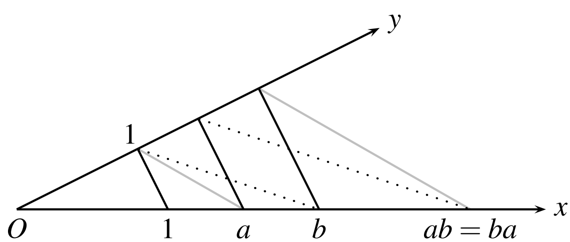

Figure 6.25: Construction of both ab and ba

Then $ab = ba$ because the gray and dotted lines end at the same place, by the Pappus theorem. The Pappus configuration in Figure 6.25 consists of all the lines except the line joining 1 on the x-axis to 1 on the y-axis. □

There is a similar proof that $a+b=b+a$. Remember from Section 6.4 that $a+b$ is the result of attaching the segment Oa at b. Thus, $b+a$ is the result of attaching Ob at a, which is different from the construction of $a+b$. Looking at both constructions together (Figure 6.26), we see that the gray line leads to $a+b$ and the dotted line leads to $b+a$. However, both of these lines end at the same point, thanks to the Pappus theorem.

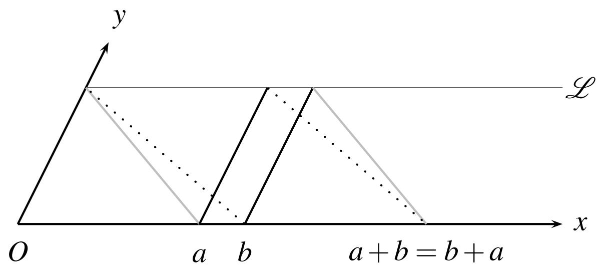

Figure 6.26: Construction of both $a+b$ and $b+a$

Exercises

6.5.1 Look back to the vector proof of the Pappus theorem in Section 4.2, and point out where it uses the assumption $ab=ba$.

The Pappus configuration that proves $a+b=b+a$ is actually a special one, because the vertices of the hexagon lie on parallel lines. The same special configuration also occurs in the proof in Section 6.4 that sums correspond on the x- and y-axes.

The special configuration corresponds to a special Pappus theorem, sometimes called the "little Pappus theorem." It is usually stated without mention of parallel lines; in which case, one has to talk about opposite sides of the hexagon meeting on a line $\mathscr{L}$.

6.5.2 Given that the assumptions of the little Pappus theorem are a hexagon with vertices on two lines that meet at a point P, and two pairs of opposite sides meeting on a line $\mathscr{L}$ that goes through P, what is the conclusion?

It is known that the little Desargues theorem implies the little Pappus theorem; a proof is in *Fundamentals of Mathematics, II* by Behnke *et al.*, p. 70. Thus, the results deduced here from the little Pappus theorem can also be deduced from the little Desargues theorem (although generally not as easily).

6.6 The associative laws

First we look at the associative law of addition, $a+(b+c)=(a+b)+c$. Figure 6.27 shows the construction of $a+(b+c)$. We have to construct $b+c$ from b and c first, and then add a as was done in Figure 6.17. Next we have to construct $(a+b)+c$, which means constructing $a+b$ first and then adding it to c as shown in Figure 6.28.

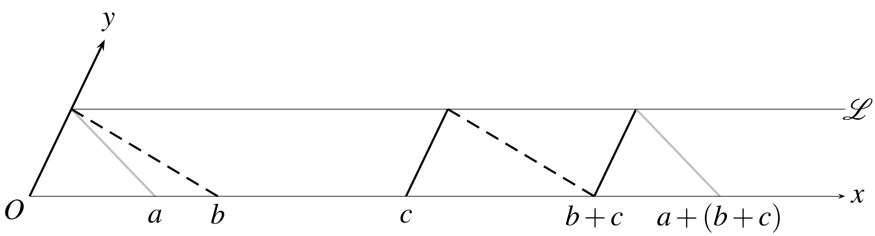

Figure 6.27: Construction of $a+(b+c)$

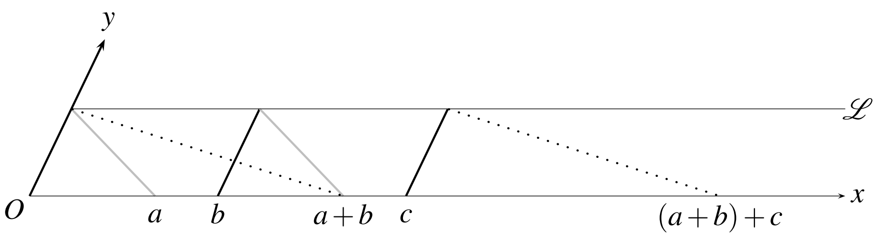

Figure 6.28: Construction of $(a+b)+c$

Figure 6.29 shows both Figures 6.27 and 6.28 on the same diagram. Here we need Desargues or, more precisely, the scissors theorem.

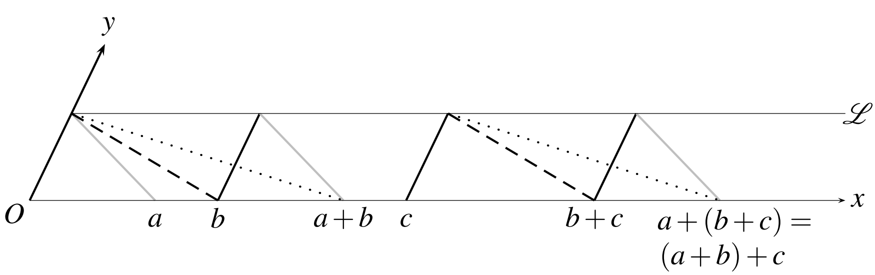

Figure 6.29: Why $a+(b+c)$ and $(a+b)+c$ coincide

One can clearly see two pairs of scissors, each consisting of a dashed line, a dotted line, a black line, and a gray line. In the scissors on the right, the gray line ends at $a+(b+c)$ and the dotted line at $(a+b)+c$. But the ends of these lines coincide, by the scissors theorem. Hence $a+(b+c) = (a+b)+c$. □

Because the scissors in this proof lie between parallel lines, we need only the little scissors theorem (and hence only the little Desargues theorem, by the remark in the previous exercise set).

Next we consider the associative law of multiplication, $a(bc) = (ab)c$. The diagram (Figure 6.30) is similar, except that the pairs of scissors lie between nonparallel lines (the x- and y-axes), so now we need the full Desargues theorem.

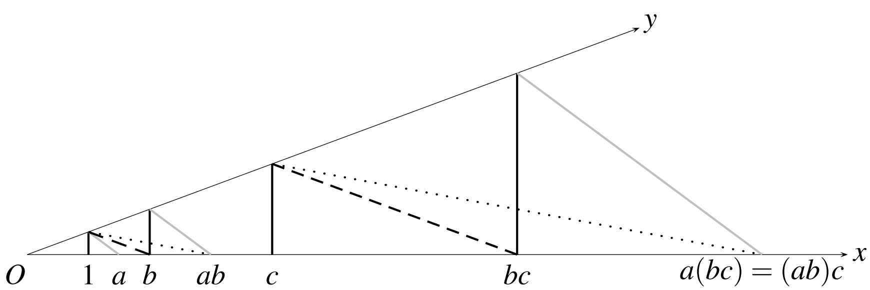

Figure 6.30: Why $a(bc)$ and $(ab)c$ coincide

The gray line ends at $a(bc)$ and the dotted line ends at $(ab)c$. But the ends of these lines coincide, by the scissors theorem, so $a(bc) = (ab)c$. □

Exercises

There is an algebraic system that satisfies all of the field axioms except the commutative law of multiplication. It is called the *quaternions* and is denoted by $\mathbb{H}$, after Sir William Rowan Hamilton, who discovered the quaternions in 1843.

In 1845, Arthur Cayley showed that the quaternions could be defined as 2×2 complex matrices of the form

$$\mathbf{q} = \begin{pmatrix} a+ib & c+id \\ -c+id & a-ib \end{pmatrix}.$$

Most of their properties follow from general properties of matrices. In fact, all the laws of algebra are immediate except the existence of $\mathbf{q}^{-1}$ and commutative multiplication.

6.6.1 Show that $\mathbf{q}$ has determinant $a^2+b^2+c^2+d^2$ and hence that $\mathbf{q}^{-1}$ exists for any nonzero quaternion $\mathbf{q}$.

6.6.2 Find specific quaternions $\mathbf{s}$ and $\mathbf{t}$ such that $\mathbf{st} \neq \mathbf{ts}$.

We can write any quaternion as $\mathbf{q} = a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, where

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

6.6.3 Verify that $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -\mathbf{1}$ (This is Hamilton's description of the quaternions).

It is possible to define the *quaternion projective plane* $\mathbb{HP}^2$ using quaternion coordinates. $\mathbb{HP}^2$ satisfies the Desargues theorem because it is possible to do the necessary calculations without using commutative multiplication. But it does *not* satisfy the Pappus theorem, because this implies commutative multiplication for $\mathbb{H}$. $\mathbb{HP}^2$ is therefore a *non-Pappian* plane—probably the most natural example.

6.7 The distributive law

To prove the distributive law $a(b+c) = ab+ac$, we take advantage of the ability to do addition and multiplication on both axes. We construct $b+c$ from b and c on the x-axis, and then map b, c, and $b+c$ to ab, ac, and $a(b+c)$ on the y-axis by lines parallel to the line from 1 on the x-axis to a on the y-axis. Then we use addition on the y-axis to construct $ab+ac$ there, and finally, use the Pappus theorem to show that $ab+ac$ and $a(b+c)$ are the same point. Here are the details.

First, observe that we can map any b on the x-axis to ab on the y-axis by sending it along a line parallel to the line from 1 on the x-axis to a on the y-axis (Figure 6.31).

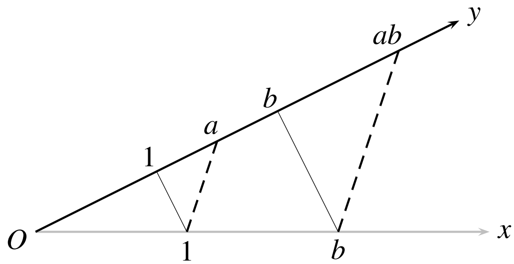

Figure 6.31: Multiplication via parallels

This is the same as constructing ab from a and b on the y-axis, because the line from b to b is parallel to the line from 1 to 1, as required by the definition of multiplication.

Next we add b and c on the x-axis, using a special choice of line $\mathscr{L}$: the parallel through ab on the y-axis. We also connect b, c, and $b+c$, respectively, to ab, ac, and $a(b+c)$ on the y-axis by parallel lines, shown dashed in Figure 6.32. The line through c that constructs $b+c$, namely the parallel $\mathscr{M}$ to the y-axis, is used in turn to add ab and ac on the y-axis.

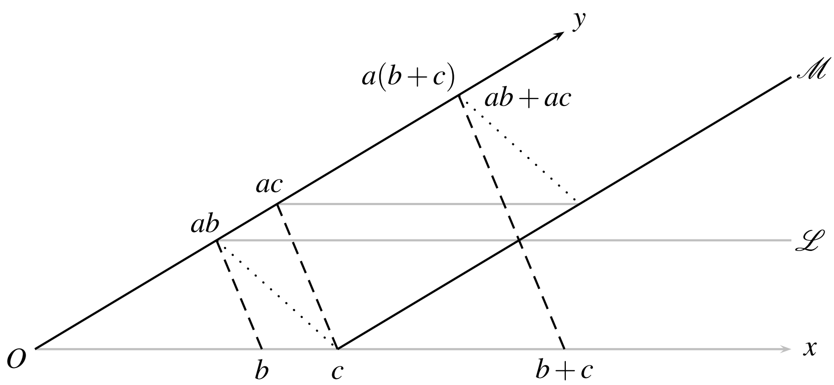

Figure 6.32: Why $a(b+c) = ab+ac$

This figure has the same structure as Figure 6.22; only the labels have changed. Now the dashed line ends at $a(b+c)$, and the dotted line ends at $ab+ac$. But again the endpoints coincide by the theorem of Pappus, and so $a(b+c) = ab+ac$. $\square$

Exercises

We need not prove the other distributive law, $(b+c)a = ba+bc$, because we are assuming Pappus, so multiplication is commutative.

6.7.1 Explain in this case why $a(b+c) = ab+ac$ implies $(b+c)a = ba+bc$.

However, in some important systems with noncommutative multiplication, both distributive laws remain valid.

6.7.2 Explain why both distributive laws are valid for the quaternions.

6.7.3 More generally, show that both distributive laws are valid for matrices.

6.8 Discussion

The idea of developing projective geometry without the use of numbers comes from the German mathematician Christian von Staudt in 1847. His compatriots Hermann Wiener and David Hilbert took the idea further in the 1890s, and it reached a high point with the publication of Hilbert's book, *Grundlagen der Geometrie* (Foundations of geometry), in 1899. It was Hilbert who first established a clear correlation between geometric and algebraic structure:

- Pappus with commutative multiplication,
- Desargues with associative multiplication.

The correlation is significant because some important algebraic systems satisfy all the field axioms except commutative multiplication. The best-known example is the quaternions, which has been known since 1843, but, for some reason, Hilbert did not mention it. To construct a non-Pappian plane, he created a rather artificial noncommutative coordinate system.

It is perhaps a lucky accident of history that Hilbert discovered the role of the Desargues theorem at all. He was forced to use it because, in 1899, it was still not known that Pappus implies Desargues. This implication was first proved by Gerhard Hessenberg in 1904. Even then the proof was faulty, and the mistake was not corrected until years later.

The whole circle of ideas was neatly tied up by yet another German mathematician, Ruth Moufang, in 1930. She found that the *little* Desargues theorem also has algebraic significance. In a projective plane satisfying the little Desargues theorem, with addition and multiplication defined as in Section 6.4, one can prove all the field axioms *except* commutativity and associativity. One can even prove a partial associativity law called *cancellation* or *alternativity*:

$$a^{-1}(ab) = b = (ba)a^{-1} \qquad \text{(alternativity)}$$

The commutative, associative, and alternative laws are beautifully exemplified by the possible multiplication operations that can be defined "reasonably" on the Euclidean spaces $\mathbb{R}^n$. ("Reasonably" means respecting at least the dimension of the space. For more on the problem of generalizing the idea of number to n dimensions, see the book *Numbers* by D. Ebbinghaus *et al.*)

- Commutative multiplication is possible only on $\mathbb{R}^1$ and $\mathbb{R}^2$, and it yields the number systems $\mathbb{R}$ and $\mathbb{C}$.
- Associative, but noncommutative, multiplication is possible only on $\mathbb{R}^4$, and it yields the quaternions $\mathbb{H}$.
- Alternative, but nonassociative, multiplication is possible only on $\mathbb{R}^8$, and it yields a system called the *octonions* $\mathbb{O}$. The octonions were discovered by a friend of Hamilton called John Graves, in 1843, and they were discovered independently by Cayley in 1845.

Ruth Moufang was the first to recognize the importance of quaternions and octonions in projective geometry. She pointed out the quaternion projective plane, as a natural example of a non-Pappian plane, and was the first to discuss the *octonion projective plane* $\mathbb{OP}^2$. $\mathbb{OP}^2$ is the most natural example of a plane that satisfies little Desargues but not Desargues.

In Section 5.4, we sketched the construction of the *real projective space* $\mathbb{RP}^3$ by means of homogeneous coordinates. This idea is easily generalized to obtain the n-dimensional real projective space $\mathbb{RP}^n$, and one can obtain $\mathbb{CP}^n$ and $\mathbb{HP}^n$ in precisely the same way. Surprisingly, the idea does *not* work for the octonions. The only octonion projective spaces are the octonion projective line $\mathbb{OP}^1 = \mathbb{O} \cup \{\infty\}$ and the octonion projective plane $\mathbb{OP}^2$ discovered by Moufang.

The reason for the nonexistence of $\mathbb{OP}^3$ is extremely interesting and has to do with the nature of the Desargues theorem in three dimensions. Remember that the Desargues theorem assumes a pair of triangles in perspective and concludes that the intersections of corresponding sides lie on a line. We know (because of the example of the Moulton plane) that the conclusion does not follow by basic incidence properties of points and lines. But *if the triangles lie in three-dimensional space, the conclusion follows by basic incidence properties of points, lines, and planes.*

The spatial Desargues theorem is clear from a picture that emphasizes the placement of the triangles in three dimensions, such as Figure 6.33. The planes containing the two triangles meet in a line $\mathscr{L}$, where the pairs of corresponding sides necessarily meet also.

The argument is a little trickier if the two triangles lie in the same plane. But, *provided the plane lies in a projective space*, it can be carried out (one shows that the planar configuration is a "shadow" of a spatial configuration).

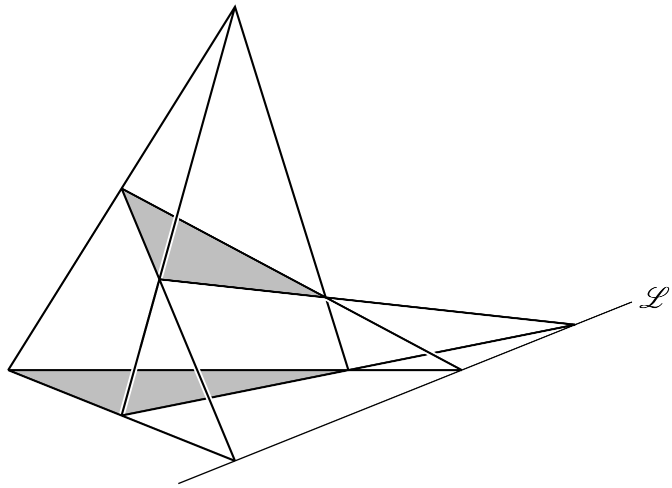

Figure 6.33: The spatial Desargues configuration

Thus, *the Desargues theorem holds in any projective space of at least three dimensions.* This is why $\mathbb{OP}^3$ cannot exist. If it did, the Desargues theorem would hold in it, and we could then show that $\mathbb{O}$ is associative—which it is not. Q. E. D.

7

Transformations

PREVIEW

In this book, we have seen at least three geometries: Euclidean, vector, and projective. In Euclidean geometry, the basic concept is length, but angle and straightness derive from it. In vector geometry, the basic concepts are vector sums and scalar multiples, but from these we derive others, such as midpoints of line segments. Finally, in projective geometry, the basic concept is straightness. Length and angle have no meaning, but a certain combination of lengths—the *cross-ratio*—is meaningful because it is unchanged by projection.

We found the cross-ratio as an *invariant of projective transformations*. The concept of length was not discovered this way, but nevertheless, it *is* an invariant of certain transformations. It is an invariant of the *isometries*, for the simple reason that isometries are *defined* to be transformations of the plane that preserve length.

These examples are two among many that suggest *geometry is the study of invariants of groups of transformations*. This definition of geometry was first proposed by the German mathematician Felix Klein in 1872. Klein's concept of geometry is perhaps still not broad enough, but it does cover the geometry in this book.

In this chapter we look again at Euclidean, vector, and projective geometry from Klein's viewpoint, first explaining precisely what "transformation" and "group" mean. It turns out that the appropriate transformations for projective geometry are *linear*, and that linear transformations also play an important role in Euclidean and vector geometry.

Linear transformations also pave the way for *hyperbolic geometry*, a new "non-Euclidean" geometry that we study in Chapter 8.