(a) Prove that $U^{-1}N$ is an abelian group under the addition defined by $\overline{(u_1, n_1)} + \overline{(u_2, n_2)} = \overline{(u_1u_2, u_2n_1 + u_1n_2)}$. Prove that $r\overline{(u,n)} = \overline{(u, rn)}$ defines an action of R on $U^{-1}N$ making it into an R-module. [This is an example of *localization* considered in general in Section 4 of Chapter 15, cf. also Section 5 in Chapter 7.]

(b) Show that the map from $Q \times N$ to $U^{-1}N$ defined by sending $(a/b, n)$ to $\overline{(b, an)}$ for $a \in R$, $b \in U$, $n \in N$, is an R-balanced map, so induces a homomorphism f from $Q \otimes_R N$ to $U^{-1}N$. Show that the map g from $U^{-1}N$ to $Q \otimes_R N$ defined by $g(\overline{(u,n)}) = (1/u) \otimes n$ is well defined and is an inverse homomorphism to f. Conclude that $Q \otimes_R N \cong U^{-1}N$ as R-modules.

(c) Conclude from (b) that $(1/d) \otimes n$ is 0 in $Q \otimes_R N$ if and only if $rn = 0$ for some nonzero $r \in R$.

(d) If A is an abelian group, show that $\mathbb{Q} \otimes_{\mathbb{Z}} A = 0$ if and only if A is a torsion abelian group (i.e., every element of A has finite order).

9. Suppose R is an integral domain with quotient field Q and let N be any R-module. Let $Q \otimes_R N$ be the module obtained from N by extension of scalars from R to Q. Prove that the kernel of the R-module homomorphism $\iota : N \to Q \otimes_R N$ is the torsion submodule of N (cf. Exercise 8 in Section 1). [Use the previous exercise.]

10. Suppose R is commutative and $N \cong R^n$ is a free R-module of rank n with R-module basis $e_1, \ldots, e_n$.

(a) For any nonzero R-module M show that every element of $M \otimes N$ can be written uniquely in the form $\sum_{i=1}^n m_i \otimes e_i$ where $m_i \in M$. Deduce that if $\sum_{i=1}^n m_i \otimes e_i = 0$ in $M \otimes N$ then $m_i = 0$ for $i = 1, \ldots, n$.

(b) Show that if $\sum m_i \otimes n_i = 0$ in $M \otimes N$ where the n_i are merely assumed to be R-linearly independent then it is not necessarily true that all the m_i are 0. [Consider $R = \mathbb{Z}$, $n = 1$, $M = \mathbb{Z}/2\mathbb{Z}$, and the element $1 \otimes 2$.]

11. Let $\{e_1, e_2\}$ be a basis of $V = \mathbb{R}^2$. Show that the element $e_1 \otimes e_2 + e_2 \otimes e_1$ in $V \otimes_{\mathbb{R}} V$ cannot be written as a simple tensor $v \otimes w$ for any $v, w \in \mathbb{R}^2$.

12. Let V be a vector space over the field F and let v, v' be nonzero elements of V. Prove that $v \otimes v' = v' \otimes v$ in $V \otimes_F V$ if and only if $v = av'$ for some $a \in F$.

13. Prove that the usual dot product of vectors defined by letting $(a_1, \ldots, a_n) \cdot (b_1, \ldots, b_n)$ be $a_1b_1 + \cdots + a_nb_n$ is a bilinear map from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}$.

14. Let I be an arbitrary nonempty index set and for each $i \in I$ let N_i be a left R-module. Let M be a right R-module. Prove the group isomorphism: $M \otimes (\oplus_{i \in I} N_i) \cong \oplus_{i \in I}(M \otimes N_i)$, where the direct sum of an arbitrary collection of modules is defined in Exercise 20, Section 3. [Use the same argument as for the direct sum of two modules, taking care to note where the direct *sum* hypothesis is needed — cf. the next exercise.]

15. Show that tensor products do not commute with direct products in general. [Consider the extension of scalars from $\mathbb{Z}$ to $\mathbb{Q}$ of the direct product of the modules $M_i = \mathbb{Z}/2^i\mathbb{Z}$, $i = 1, 2, \ldots$]

16. Suppose R is commutative and let I and J be ideals of R, so R/I and R/J are naturally R-modules.

(a) Prove that every element of $R/I \otimes_R R/J$ can be written as a simple tensor of the form $(1 \bmod I) \otimes (r \bmod J)$.

(b) Prove that there is an R-module isomorphism $R/I \otimes_R R/J \cong R/(I + J)$ mapping $(r \bmod I) \otimes (r' \bmod J)$ to $rr' \bmod (I + J)$.

17. Let $I = (2, x)$ be the ideal generated by 2 and x in the ring $R = \mathbb{Z}[x]$. The ring $\mathbb{Z}/2\mathbb{Z} = R/I$ is naturally an R-module annihilated by both 2 and x.

(a) Show that the map $\varphi : I \times I \to \mathbb{Z}/2\mathbb{Z}$ defined by

$$\varphi(a_0 + a_1x + \cdots + a_nx^n, b_0 + b_1x + \cdots + b_mx^m) = \frac{a_0}{2}b_1 \bmod 2$$

is R-bilinear.

(b) Show that there is an R-module homomorphism from $I \otimes_R I \to \mathbb{Z}/2\mathbb{Z}$ mapping $p(x) \otimes q(x)$ to $\frac{p(0)}{2}q'(0)$ where q' denotes the usual polynomial derivative of q.

(c) Show that $2 \otimes x \neq x \otimes 2$ in $I \otimes_R I$.

18. Suppose I is a principal ideal in the integral domain R. Prove that the R-module $I \otimes_R I$ has no nonzero torsion elements (i.e., $rm = 0$ with $0 \neq r \in R$ and $m \in I \otimes_R I$ implies that $m = 0$).

19. Let $I = (2, x)$ be the ideal generated by 2 and x in the ring $R = \mathbb{Z}[x]$ as in Exercise 17. Show that the nonzero element $2 \otimes x - x \otimes 2$ in $I \otimes_R I$ is a torsion element. Show in fact that $2 \otimes x - x \otimes 2$ is annihilated by both 2 and x and that the submodule of $I \otimes_R I$ generated by $2 \otimes x - x \otimes 2$ is isomorphic to R/I.

20. Let $I = (2, x)$ be the ideal generated by 2 and x in the ring $R = \mathbb{Z}[x]$. Show that the element $2 \otimes 2 + x \otimes x$ in $I \otimes_R I$ is not a simple tensor, i.e., cannot be written as $a \otimes b$ for some $a, b \in I$.

21. Suppose R is commutative and let I and J be ideals of R.

(a) Show there is a surjective R-module homomorphism from $I \otimes_R J$ to the product ideal IJ mapping $i \otimes j$ to the element ij.

(b) Give an example to show that the map in (a) need not be injective (cf. Exercise 17).

22. Suppose that M is a left and a right R-module such that $rm = mr$ for all $r \in R$ and $m \in M$. Show that the elements r_1r_2 and r_2r_1 act the same on M for every $r_1, r_2 \in R$. (This explains why the assumption that R is commutative in the definition of an R-algebra is a fairly natural one.)

23. Verify the details that the multiplication in Proposition 19 makes $A \otimes_R B$ into an R-algebra.

24. Prove that the extension of scalars from $\mathbb{Z}$ to the Gaussian integers $\mathbb{Z}[i]$ of the ring $\mathbb{R}$ is isomorphic to $\mathbb{C}$ as a ring: $\mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{C}$ as rings.

25. Let R be a subring of the commutative ring S and let x be an indeterminate over S. Prove that $S[x]$ and $S \otimes_R R[x]$ are isomorphic as S-algebras.

26. Let S be a commutative ring containing R (with $1_S = 1_R$) and let $x_1, \ldots, x_n$ be independent indeterminates over the ring S. Show that for every ideal I in the polynomial ring $R[x_1, \ldots, x_n]$ that $S \otimes_R (R[x_1, \ldots, x_n]/I) \cong S[x_1, \ldots, x_n]/IS[x_1, \ldots, x_n]$ as S-algebras.

The next exercise shows the ring $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ introduced at the end of this section is isomorphic to $\mathbb{C} \times \mathbb{C}$. One may also prove this via Exercise 26 and Proposition 16 in Section 9.5, since $\mathbb{C} \cong \mathbb{R}[x]/(x^2 + 1)$. The ring $\mathbb{C} \times \mathbb{C}$ is also discussed in Exercise 23 of Section 1.

27. (a) Write down a formula for the multiplication of two elements $a \cdot 1 + b \cdot e_2 + c \cdot e_3 + d \cdot e_4$ and $a' \cdot 1 + b' \cdot e_2 + c' \cdot e_3 + d' \cdot e_4$ in the example $A = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ following Proposition 21 (where $1 = 1 \otimes 1$ is the identity of A).

(b) Let $\epsilon_1 = \frac{1}{2}(1 \otimes 1 + i \otimes i)$ and $\epsilon_2 = \frac{1}{2}(1 \otimes 1 - i \otimes i)$. Show that $\epsilon_1\epsilon_2 = 0$, $\epsilon_1 + \epsilon_2 = 1$, and $\epsilon_j^2 = \epsilon_j$ for $j = 1, 2$ (ϵ_1 and ϵ_2 are called *orthogonal idempotents* in A). Deduce that A is isomorphic as a ring to the direct product of two principal ideals: $A \cong A\epsilon_1 \times A\epsilon_2$ (cf. Exercise 1, Section 7.6).

(c) Prove that the map $\varphi : \mathbb{C} \times \mathbb{C} \to \mathbb{C} \times \mathbb{C}$ by $\varphi(z_1, z_2) = (z_1z_2, z_1\overline{z_2})$, where $\overline{z_2}$ denotes the complex conjugate of z_2, is an $\mathbb{R}$-bilinear map.

(d) Let Φ be the $\mathbb{R}$-module homomorphism from A to $\mathbb{C} \times \mathbb{C}$ obtained from φ in (c). Show that $\Phi(\epsilon_1) = (0, 1)$ and $\Phi(\epsilon_2) = (1, 0)$. Show also that Φ is $\mathbb{C}$-linear, where the action of $\mathbb{C}$ is on the left tensor factor in A and on both factors in $\mathbb{C} \times \mathbb{C}$. Deduce that Φ is surjective. Show that Φ is a $\mathbb{C}$-algebra isomorphism.

10.5 EXACT SEQUENCES—PROJECTIVE, INJECTIVE, AND FLAT MODULES

One of the fundamental results for studying the structure of an algebraic object B (e.g., a group, a ring, or a module) is the First Isomorphism Theorem, which relates the subobjects of B (the normal subgroups, the ideals, or the submodules, respectively) with the possible homomorphic images of B. We have already seen many examples applying this theorem to understand the structure of B from an understanding of its "smaller" constituents—for example in analyzing the structure of the dihedral group D_8 by determining its center and the resulting quotient by the center.

In most of these examples we began *first* with a given B and then determined some of its basic properties by constructing a homomorphism φ (often given implicitly by the specification of $\ker\varphi \subseteq B$) and examining both $\ker\varphi$ and the resulting quotient $B/\ker\varphi$. We now consider in some greater detail the reverse situation, namely whether we may *first* specify the "smaller constituents." More precisely, we consider whether, given two modules A and C, there exists a module B containing (an isomorphic copy of) A such that the resulting quotient module B/A is isomorphic to C—in which case B is said to be an *extension of C by A*. It is then natural to ask how many such B exist for a given A and C, and the extent to which properties of B are determined by the corresponding properties of A and C. There are, of course, analogous problems in the contexts of groups and rings. This is the *extension problem* first discussed (for groups) in Section 3.4; in this section we shall be primarily concerned with left modules over a ring R, making note where necessary of the modifications required for some other structures, notably noncommutative groups. As in the previous section, throughout this section all rings contain a 1.

We first introduce a very convenient notation. To say that A is isomorphic to a submodule of B, is to say that there is an injective homomorphism $\psi : A \to B$ (so then $A \cong \psi(A) \subseteq B$). To say that C is isomorphic to the resulting quotient is to say that there is a surjective homomorphism $\varphi : B \to C$ with $\ker\varphi = \psi(A)$. In particular this gives us a pair of homomorphisms:

$$A \xrightarrow{\psi} B \xrightarrow{\varphi} C$$

with image $\psi = \ker\varphi$. A pair of homomorphisms with this property is given a name:

Definition.

(1) The pair of homomorphisms $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ is said to be *exact* (at Y) if $\text{image}\,\alpha = \ker\beta$.

(2) A sequence $\cdots \to X_{n-1} \to X_n \to X_{n+1} \to \cdots$ of homomorphisms is said to be an *exact sequence* if it is exact at every X_n between a pair of homomorphisms.