

must map  $y$  to a power of  $\gamma$ . There are therefore  $p$  homomorphisms  $\varphi_i : P \rightarrow \text{Aut}(Q)$  given by  $\varphi_i(y) = \gamma^i$ ,  $0 \leq i \leq p - 1$ . Since  $\varphi_0$  is the trivial homomorphism,  $Q \rtimes_{\varphi_0} P \cong Q \times P$  as before. Each  $\varphi_i$  for  $i \neq 0$  gives rise to a non-abelian group,  $G_i$ , of order  $pq$ . It is straightforward to check that these groups are all isomorphic because for each  $\varphi_i$ ,  $i > 0$ , there is some generator  $y_i$  of  $P$  such that  $\varphi_i(y_i) = \gamma$ . Thus, up to a choice for the (arbitrary) generator of  $P$ , these semidirect products are all the same (see Exercise 6. See also Exercise 28 of Section 4.3).

### Example: (Groups of Order 30)

By the examples following Sylow's Theorem every group  $G$  of order 30 contains a subgroup  $H$  of order 15. By the preceding example  $H$  is cyclic and  $H$  is normal in  $G$  (index 2). By Sylow's Theorem there is a subgroup  $K$  of  $G$  of order 2. Thus  $G = HK$  and  $H \cap K = 1$  so  $G \cong H \rtimes K$ , for some  $\varphi : K \rightarrow \text{Aut}(H)$ . By Proposition 4.16,

$$\text{Aut}(Z_{15}) \cong (\mathbb{Z}/15\mathbb{Z})^\times \cong Z_4 \times Z_2.$$

The latter isomorphism can be computed directly, or one can use Exercise 11 of the preceding section: writing  $H$  as  $\langle a \rangle \times \langle b \rangle \cong Z_5 \times Z_3$ , we have (since these two subgroups are characteristic in  $H$ )

$$\text{Aut}(H) \cong \text{Aut}(Z_5) \times \text{Aut}(Z_3).$$

In particular,  $\text{Aut}(H)$  contains precisely three elements of order 2, whose actions on the group  $H = \langle a \rangle \times \langle b \rangle$  are the following:

$$\begin{cases} a &\mapsto &a \\ b &\mapsto &b^{-1} \end{cases} \quad \begin{cases} a &\mapsto &a^{-1} \\ b &\mapsto &b \end{cases} \quad \begin{cases} a &\mapsto &a^{-1} \\ b &\mapsto &b^{-1} \end{cases}.$$

Thus there are three nontrivial homomorphisms from  $K$  into  $\text{Aut}(H)$  given by sending the generator of  $K$  into one of these three elements of order 2 (as usual, the trivial homomorphism gives the direct product:  $H \times K \cong Z_{30}$ ).

Let  $K = \langle k \rangle$ . If the homomorphism  $\varphi_1 : K \rightarrow \text{Aut}(H)$  is defined by mapping  $k$  to the first automorphism above (so that  $k \cdot a = a$  and  $k \cdot b = b^{-1}$  gives the action of  $k$  on  $H$ ) then  $G_1 = H \rtimes_{\varphi_1} K$  is easily seen to be isomorphic to  $Z_5 \times D_6$  (note that in this semidirect product  $k$  centralizes the element  $a$  of  $H$  of order 5, so the factorization as a direct product is  $\langle a \rangle \times \langle b, k \rangle$ ).

If  $\varphi_2$  is defined by mapping  $k$  to the second automorphism above, then  $G_2 = H \rtimes_{\varphi_2} K$  is easily seen to be isomorphic to  $Z_3 \times D_{10}$  (note that in this semidirect product  $k$  centralizes the element  $b$  of  $H$  of order 3, so the factorization as a direct product is  $\langle b \rangle \times \langle a, k \rangle$ ).

If  $\varphi_3$  is defined by mapping  $k$  to the third automorphism above then  $G_3 = H \rtimes_{\varphi_3} K$  is easily seen to be isomorphic to  $D_{30}$ .

Note that these groups are all nonisomorphic since their centers have orders 30 (in the abelian case), 5 (for  $G_1$ ), 3 (for  $G_2$ ), and 1 (for  $G_3$ ).

We emphasize that although (in hindsight) this procedure does not give rise to any groups we could not already have constructed using only direct products, the argument proves that this is the *complete* list of isomorphism types of groups of order 30.

### Example: (Groups of Order 12)

Let  $G$  be a group of order 12, let  $V \in \text{Syl}_2(G)$  and let  $T \in \text{Syl}_3(G)$ . By the discussion of groups of order 12 in Section 4.5 we know that either  $V$  or  $T$  is normal in  $G$  (for purposes of illustration we shall not invoke the full force of our results from Chapter 4, namely that either  $T \trianglelefteq G$  or  $G \cong A_4$ ). By Lagrange's Theorem  $V \cap T = 1$ . Thus  $G$  is a semidirect product. Note that  $V \cong Z_4$  or  $Z_2 \times Z_2$  and  $T \cong Z_3$ .

### Case 1: $V \trianglelefteq G$

We must determine all possible homomorphisms from  $T$  into  $\text{Aut}(V)$ . If  $V \cong Z_4$ , then  $\text{Aut}(V) \cong Z_2$  and there are no nontrivial homomorphisms from  $T$  into  $\text{Aut}(V)$ . Thus the only group of order 12 with a normal cyclic Sylow 2-subgroup is  $Z_{12}$ .

Assume therefore that  $V \cong Z_2 \times Z_2$ . In this case  $\text{Aut}(V) \cong S_3$  and there is a unique subgroup of  $\text{Aut}(V)$  of order 3, say  $\langle \gamma \rangle$ . Thus if  $T = \langle y \rangle$ , there are three possible homomorphisms from  $T$  into  $\text{Aut}(V)$ :

$$\varphi_i : T \rightarrow \text{Aut}(V) \quad \text{defined by} \quad \varphi_i(y) = \gamma^i, \quad i = 0, 1, 2.$$

As usual,  $\varphi_0$  is the trivial homomorphism, which gives rise to the direct product  $Z_2 \times Z_2 \times Z_3$ . Homomorphisms  $\varphi_1$  and  $\varphi_2$  give rise to isomorphic semidirect products because they differ only in the choice of a generator for  $T$  (i.e.,  $\varphi_1(y) = \gamma$  and  $\varphi_2(y') = \gamma$ , where  $y' = y^2$  and  $y'$  is another choice of generator for  $T$  — see also Exercise 6). The unique non-abelian group in this case is  $A_4$ .

### Case 2: $T \trianglelefteq G$

We must determine all possible homomorphisms from  $V$  into  $\text{Aut}(T)$ . Note that  $\text{Aut}(T) = \langle \lambda \rangle \cong Z_2$ , where  $\lambda$  inverts  $T$ . If  $V = \langle x \rangle \cong Z_4$ , there are precisely two homomorphisms from  $V$  into  $\text{Aut}(T)$ : the trivial homomorphism and the homomorphism which sends  $x$  to  $\lambda$ . As usual, the trivial homomorphism gives rise to the direct product:  $Z_3 \times Z_4 \cong Z_{12}$ . The nontrivial homomorphism gives the semidirect product which was discussed in Example 2 following Proposition 11 of this section.

Finally, assume  $V = \langle a \rangle \times \langle b \rangle \cong Z_2 \times Z_2$ . There are precisely three nontrivial homomorphisms from  $V$  into  $\text{Aut}(T)$  determined by specifying their kernels as one of the three subgroups of order 2 in  $V$ . For example,  $\varphi_1(a) = \lambda$  and  $\varphi_1(b) = \lambda$  has kernel  $\langle ab \rangle$ , that is, in this semidirect product both  $a$  and  $b$  act by inverting  $T$  and  $ab$  centralizes  $T$ . If  $\varphi_2$  and  $\varphi_3$  have kernels  $\langle a \rangle$  and  $\langle b \rangle$ , respectively, then one easily checks that the resulting three semidirect products are all isomorphic to  $S_3 \times Z_2$ , where the  $Z_2$  direct factor is the kernel of  $\varphi_i$ . For example,

$$T \rtimes_{\varphi_1} V = \langle a, T \rangle \times \langle ab \rangle.$$

In summary, there are precisely 5 groups of order 12, three of which are non-abelian.

### Example: (Groups of Order $p^3$ , $p$ an odd prime)

Let  $G$  be a group of order  $p^3$ ,  $p$  an odd prime, and assume  $G$  is not cyclic. By Exercise 9 of the previous section the map  $x \mapsto x^p$  is a homomorphism from  $G$  into  $Z(G)$  and the kernel of this homomorphism has order  $p^2$  or  $p^3$ . In the former case  $G$  must contain an element of order  $p^2$  and in the latter case every nonidentity element of  $G$  has order  $p$ .

#### Case 1: $G$ has an element of order $p^2$

Let  $x$  be an element of order  $p^2$  and let  $H = \langle x \rangle$ . Note that since  $H$  has index  $p$ ,  $H$  is normal in  $G$  by Corollary 4.5. If  $E$  is the kernel of the  $p^{\text{th}}$  power map, then in this case  $E \cong Z_p \times Z_p$  and  $E \cap H = \langle x^p \rangle$ . Let  $y$  be any element of  $E - H$  and let  $K = \langle y \rangle$ . By construction,  $H \cap K = 1$  and so  $G$  is isomorphic to  $Z_{p^2} \rtimes Z_p$ , for some  $\varphi : K \rightarrow \text{Aut}(H)$ . If  $\varphi$  is the trivial homomorphism,  $G \cong Z_{p^2} \times Z_p$ , so we need only consider the nontrivial homomorphisms. By Proposition 4.17  $\text{Aut}(H) \cong Z_{p(p-1)}$  is cyclic and so contains a unique subgroup of order  $p$ , explicitly given by  $\langle \gamma \rangle$  where

$$\gamma(x) = x^{1+p}.$$

As usual, up to choice of a generator for the cyclic group  $K$ , there is only one nontrivial homomorphism,  $\varphi$ , from  $K$  into  $\text{Aut}(H)$ , given by  $\varphi(y) = \gamma$ ; hence up to isomorphism

there is a unique non-abelian group  $H \rtimes K$  in this case. This group is described in Example 7 above.

**Case 2:** every nonidentity element of  $G$  has order  $p$

In this case let  $H$  be any subgroup of  $G$  of order  $p^2$  (see Exercise 29, Section 4.3). Necessarily  $H \cong Z_p \times Z_p$ . Let  $K = \langle y \rangle$  for any element  $y$  of  $G - H$ . Since  $H$  has index  $p$ ,  $H \trianglelefteq G$  and since  $K$  has order  $p$  but is not contained in  $H$ ,  $H \cap K = 1$ . Then  $G$  is isomorphic to  $(Z_p \times Z_p) \rtimes Z_p$ , for some  $\varphi : K \rightarrow \text{Aut}(H)$ . If  $\varphi$  is trivial,  $G \cong Z_p \times Z_p \times Z_p$  (the elementary abelian group), so we may assume  $\varphi$  is nontrivial. By Proposition 4.17,

$$\text{Aut}(H) \cong GL_2(\mathbb{F}_p)$$

so  $|\text{Aut}(H)| = (p^2 - 1)(p^2 - p)$ . Note that a Sylow  $p$ -subgroup of  $\text{Aut}(H)$  has order  $p$  so all subgroups of order  $p$  in  $\text{Aut}(H)$  are conjugate in  $\text{Aut}(H)$  by Sylow's Theorem. Explicitly, (as discussed in Example 7 above) every subgroup of order  $p$  in  $\text{Aut}(H)$  is conjugate to  $\langle \gamma \rangle$ , where if  $H = \langle a \rangle \times \langle b \rangle$ , the automorphism  $\gamma$  is defined by

$$\gamma(a) = ab \quad \text{and} \quad \gamma(b) = b.$$

With respect to the  $\mathbb{F}_p$ -basis  $a, b$  of the 2-dimensional vector space  $H$  the automorphism has matrix

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \in GL_2(\mathbb{F}_p).$$

Thus (again quoting Exercise 6) there is a unique isomorphism type of semidirect product in this case.

Finally, since the two non-abelian groups have different orders for the kernels of the  $p^{\text{th}}$  power maps, they are not isomorphic. A presentation for this group is also given in Example 7 above.

## EXERCISES

Let  $H$  and  $K$  be groups, let  $\varphi$  be a homomorphism from  $K$  into  $\text{Aut}(H)$  and, as usual, identify  $H$  and  $K$  as subgroups of  $G = H \rtimes_{\varphi} K$ .

1. Prove that  $C_K(H) = \ker \varphi$  (recall that  $C_K(H) = C_G(H) \cap K$ ).
2. Prove that  $C_H(K) = N_H(K)$ .
3. In Example 1 following the proof of Proposition 11 prove that every element of  $G - H$  has order 2. Prove that  $G$  is abelian if and only if  $h^2 = 1$  for all  $h \in H$ .
4. Let  $p = 2$  and check that the construction of the two non-abelian groups of order  $p^3$  is valid in this case. Prove that *both* resulting groups are isomorphic to  $D_8$ .
5. Let  $G = \text{Hol}(Z_2 \times Z_2)$ .
  - (a) Prove that  $G = H \rtimes K$  where  $H = Z_2 \times Z_2$  and  $K \cong S_3$ . Deduce that  $|G| = 24$ .
  - (b) Prove that  $G$  is isomorphic to  $S_4$ . [Obtain a homomorphism from  $G$  into  $S_4$  by letting  $G$  act on the left cosets of  $K$ . Use Exercise 1 to show this representation is faithful.]
6. Assume that  $K$  is a cyclic group,  $H$  is an arbitrary group and  $\varphi_1$  and  $\varphi_2$  are homomorphisms from  $K$  into  $\text{Aut}(H)$  such that  $\varphi_1(K)$  and  $\varphi_2(K)$  are conjugate subgroups of  $\text{Aut}(H)$ . If  $K$  is infinite assume  $\varphi_1$  and  $\varphi_2$  are injective. Prove by constructing an explicit isomorphism that  $H \rtimes_{\varphi_1} K \cong H \rtimes_{\varphi_2} K$  (in particular, if the subgroups  $\varphi_1(K)$  and  $\varphi_2(K)$  are equal in  $\text{Aut}(H)$ , then the resulting semidirect products are isomorphic). [Suppose  $\sigma \varphi_1(K)\sigma^{-1} = \varphi_2(K)$  so that for some  $a \in \mathbb{Z}$  we have  $\sigma \varphi_1(k)\sigma^{-1} = \varphi_2(k)^a$  for all  $k \in K$ . Show that the map