1. Linear Equations

1.1. Fields

We assume that the reader is familiar with the elementary algebra of real and complex numbers. For a large portion of this book the algebraic properties of numbers which we shall use are easily deduced from the following brief list of properties of addition and multiplication. We let F denote either the set of real numbers or the set of complex numbers.

1. Addition is commutative,

$$x + y = y + x$$

for all x and y in F.

2. Addition is associative,

$$x + (y + z) = (x + y) + z$$

for all x, y, and z in F.

3. There is a unique element 0 (zero) in F such that $x + 0 = x$, for every x in F.

4. To each x in F there corresponds a unique element $(-x)$ in F such that $x + (-x) = 0$.

5. Multiplication is commutative,

$$xy = yx$$

for all x and y in F.

6. Multiplication is associative,

$$x(yz) = (xy)z$$

for all x, y, and z in F.

7. There is a unique non-zero element 1 (one) in F such that $x1 = x$, for every x in F.

8. To each non-zero x in F there corresponds a unique element x^{-1} (or $1/x$) in F such that $xx^{-1} = 1$.

9. Multiplication distributes over addition; that is, $x(y + z) = xy + xz$, for all x, y, and z in F.

Suppose one has a set F of objects $x, y, z, \ldots$ and two operations on the elements of F as follows. The first operation, called addition, associates with each pair of elements x, y in F an element $(x + y)$ in F; the second operation, called multiplication, associates with each pair x, y an element xy in F; and these two operations satisfy conditions (1)–(9) above. The set F, together with these two operations, is then called a **field.** Roughly speaking, a field is a set together with some operations on the objects in that set which behave like ordinary addition, subtraction, multiplication, and division of numbers in the sense that they obey the nine rules of algebra listed above. With the usual operations of addition and multiplication, the set C of complex numbers is a field, as is the set R of real numbers.

For most of this book the 'numbers' we use may as well be the elements from any field F. To allow for this generality, we shall use the word 'scalar' rather than 'number.' Not much will be lost to the reader if he always assumes that the field of scalars is a subfield of the field of complex numbers. A **subfield** of the field C is a set F of complex numbers which is itself a field under the usual operations of addition and multiplication of complex numbers. This means that 0 and 1 are in the set F, and that if x and y are elements of F, so are $(x + y)$, $-x$, xy, and x^{-1} (if $x \neq 0$). An example of such a subfield is the field R of real numbers; for, if we identify the real numbers with the complex numbers $(a + ib)$ for which $b = 0$, the 0 and 1 of the complex field are real numbers, and if x and y are real, so are $(x + y)$, $-x$, xy, and x^{-1} (if $x \neq 0$). We shall give other examples below. The point of our discussing subfields is essentially this: If we are working with scalars from a certain subfield of C, then the performance of the operations of addition, subtraction, multiplication, or division on these scalars does not take us out of the given subfield.

EXAMPLE 1. The set of **positive integers:** 1, 2, 3, ..., is not a subfield of C, for a variety of reasons. For example, 0 is not a positive integer; for no positive integer n is $-n$ a positive integer; for no positive integer n except 1 is $1/n$ a positive integer.

EXAMPLE 2. The set of **integers:** ..., -2, -1, 0, 1, 2, ..., is not a subfield of C, because for an integer n, $1/n$ is not an integer unless n is 1 or

-1. With the usual operations of addition and multiplication, the set of integers satisfies all of the conditions (1)–(9) except condition (8).

EXAMPLE 3. The set of **rational numbers,** that is, numbers of the form p/q, where p and q are integers and $q \neq 0$, is a subfield of the field of complex numbers. The division which is not possible within the set of integers is possible within the set of rational numbers. The interested reader should verify that any subfield of C must contain every rational number.

EXAMPLE 4. The set of all complex numbers of the form $x + y\sqrt{2}$, where x and y are rational, is a subfield of C. We leave it to the reader to verify this.

In the examples and exercises of this book, the reader should assume that the field involved is a subfield of the complex numbers, unless it is expressly stated that the field is more general. We do not want to dwell on this point; however, we should indicate why we adopt such a convention. If F is a field, it may be possible to add the unit 1 to itself a finite number of times and obtain 0 (see Exercise 5 following Section 1.2):

$$1 + 1 + \cdots + 1 = 0.$$

That does not happen in the complex number field (or in any subfield thereof). If it does happen in F, then the least n such that the sum of n 1's is 0 is called the **characteristic** of the field F. If it does not happen in F, then (for some strange reason) F is called a field of **characteristic zero.** Often, when we assume F is a subfield of C, what we want to guarantee is that F is a field of characteristic zero; but, in a first exposure to linear algebra, it is usually better not to worry too much about characteristics of fields.

1.2. Systems of Linear Equations

Suppose F is a field. We consider the problem of finding n scalars (elements of F) $x_1, \ldots, x_n$ which satisfy the conditions

$$\begin{aligned} A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n &= y_1 \\ A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n &= y_2 \\ \vdots \qquad \vdots \qquad\quad\;\; \vdots \quad\;\; & \;\;\; \vdots \\ A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n &= y_m \end{aligned} \tag{1-1}$$

where $y_1, \ldots, y_m$ and A_{ij}, $1 \leq i \leq m$, $1 \leq j \leq n$, are given elements of F. We call (1-1) a **system of m linear equations in n unknowns.** Any n-tuple $(x_1, \ldots, x_n)$ of elements of F which satisfies each of the equations in (1-1) is called a **solution** of the system. If $y_1 = y_2 = \cdots = y_m = 0$, we say that the system is **homogeneous,** or that each of the equations is homogeneous.

Perhaps the most fundamental technique for finding the solutions of a system of linear equations is the technique of elimination. We can illustrate this technique on the homogeneous system

$$\begin{aligned} 2x_1 - x_2 + x_3 &= 0 \\ x_1 + 3x_2 + 4x_3 &= 0. \end{aligned}$$

If we add (-2) times the second equation to the first equation, we obtain

$$-7x_2 - 7x_3 = 0$$

or, $x_2 = -x_3$. If we add 3 times the first equation to the second equation, we obtain

$$7x_1 + 7x_3 = 0$$

or, $x_1 = -x_3$. So we conclude that if (x_1, x_2, x_3) is a solution then $x_1 = x_2 = -x_3$. Conversely, one can readily verify that any such triple is a solution. Thus the set of solutions consists of all triples $(-a, -a, a)$.

We found the solutions to this system of equations by 'eliminating unknowns,' that is, by multiplying equations by scalars and then adding to produce equations in which some of the x_j were not present. We wish to formalize this process slightly so that we may understand why it works, and so that we may carry out the computations necessary to solve a system in an organized manner.

For the general system (1-1), suppose we select m scalars $c_1, \ldots, c_m$, multiply the jth equation by c_j and then add. We obtain the equation

$$\begin{aligned} (c_1A_{11} + \cdots + c_mA_{m1})x_1 + \cdots + (c_1A_{1n} + \cdots + c_mA_{mn})x_n \\ = c_1y_1 + \cdots + c_my_m. \end{aligned}$$

Such an equation we shall call a **linear combination** of the equations in (1-1). Evidently, any solution of the entire system of equations (1-1) will also be a solution of this new equation. This is the fundamental idea of the elimination process. If we have another system of linear equations

$$\begin{aligned} B_{11}x_1 + \cdots + B_{1n}x_n &= z_1 \\ \vdots \qquad\qquad\quad \vdots \quad &\quad \vdots \\ B_{k1}x_1 + \cdots + B_{kn}x_n &= z_k \end{aligned} \tag{1-2}$$

in which each of the k equations is a linear combination of the equations in (1-1), then every solution of (1-1) is a solution of this new system. Of course it may happen that some solutions of (1-2) are not solutions of (1-1). This clearly does not happen if each equation in the original system is a linear combination of the equations in the new system. Let us say that two systems of linear equations are **equivalent** if each equation in each system is a linear combination of the equations in the other system. We can then formally state our observations as follows.

Theorem 1. *Equivalent systems of linear equations have exactly the same solutions.*

If the elimination process is to be effective in finding the solutions of a system like (1-1), then one must see how, by forming linear combinations of the given equations, to produce an equivalent system of equations which is easier to solve. In the next section we shall discuss one method of doing this.

Exercises

1. Verify that the set of complex numbers described in Example 4 is a subfield of C.

2. Let F be the field of complex numbers. Are the following two systems of linear equations equivalent? If so, express each equation in each system as a linear combination of the equations in the other system.

$$\begin{aligned} x_1 - x_2 &= 0 \\ 2x_1 + x_2 &= 0 \end{aligned} \qquad \begin{aligned} 3x_1 + x_2 &= 0 \\ x_1 + x_2 &= 0 \end{aligned}$$

3. Test the following systems of equations as in Exercise 2.

$$\begin{aligned} -x_1 + x_2 + 4x_3 &= 0 \\ x_1 + 3x_2 + 8x_3 &= 0 \\ \tfrac{1}{2}x_1 + x_2 + \tfrac{5}{2}x_3 &= 0 \end{aligned} \qquad \begin{aligned} x_1 \qquad - x_3 &= 0 \\ x_2 + 3x_3 &= 0 \end{aligned}$$

4. Test the following systems as in Exercise 2.

$$\begin{aligned} 2x_1 + (-1 + i)x_2 \qquad + x_4 &= 0 \\ 3x_2 - 2ix_3 + 5x_4 &= 0 \end{aligned} \qquad \begin{aligned} \left(1 + \frac{i}{2}\right)x_1 + 8x_2 - ix_3 - x_4 &= 0 \\ \tfrac{2}{3}x_1 - \tfrac{1}{2}x_2 + x_3 + 7x_4 &= 0 \end{aligned}$$

5. Let F be a set which contains exactly two elements, 0 and 1. Define an addition and multiplication by the tables:

$+$	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

Verify that the set F, together with these two operations, is a field.

6. Prove that if two homogeneous systems of linear equations in two unknowns have the same solutions, then they are equivalent.

7. Prove that each subfield of the field of complex numbers contains every rational number.

8. Prove that each field of characteristic zero contains a copy of the rational number field.