Proof: The only item remaining to be proved in the first statement is the exactness of (12) at $\mathrm{Hom}_R(M, D)$. The proof of this statement is very similar to the proof of the corresponding result in Theorem 28 and is left as an exercise. Note also that the injectivity of ψ is not required, which proves the "if" portion of the final statement of the theorem.

Suppose now that the sequence (12) is exact for all R-modules D. We first show that $\varphi : M \to N$ is a surjection. Take $D = N/\varphi(M)$. If $\pi_1 : N \to N/\varphi(M)$ is the natural projection homomorphism, then $\pi_1 \circ \varphi(M) = 0$ by definition of π_1. Since $\pi_1 \circ \varphi = \varphi'(\pi_1)$, this means that the element $\pi_1 \in \mathrm{Hom}_R(N, N/\varphi(M))$ is mapped to 0 by φ'. Since φ' is assumed to be injective for all modules D, this means π_1 is the zero map, i.e., $N = \varphi(M)$ and so φ is a surjection. We next show that $\varphi \circ \psi = 0$, which will imply that image $\psi \subseteq \ker \varphi$. For this we take $D = N$ and observe that the identity map id_N on N is contained in $\mathrm{Hom}_R(N, N)$, hence $\varphi'(id_N) \in \mathrm{Hom}_R(M, N)$. Then the exactness of (12) for $D = N$ implies that $\varphi'(id_N) \in \ker \psi'$, so $\psi'(\varphi'(id_N)) = 0$. Then $id_N \circ \psi \circ \varphi = 0$, i.e., $\psi \circ \varphi = 0$, as claimed. Finally, we show that $\ker \varphi \subseteq$ image ψ. Let $D = M/\psi(L)$ and let $\pi_2 : M \to M/\psi(L)$ be the natural projection. Then $\psi'(\pi_2) = 0$ since $\pi_2(\psi(L)) = 0$ by definition of π_2. The exactness of (12) for this D then implies that π_2 is in the image of φ', say $\pi_2 = \varphi'(f)$ for some homomorphism $f \in \mathrm{Hom}_R(N, M/\psi(L))$, i.e., $\pi_2 = f \circ \varphi$. If $m \in \ker \varphi$ then $\pi_2(m) = f(\varphi(m)) = 0$, which means that $m \in \psi(L)$ since π_2 is just the projection from M into the quotient $M/\psi(L)$. Hence $\ker \varphi \subseteq$ image ψ, completing the proof.

By Theorem 33, the sequence

$$0 \longrightarrow \mathrm{Hom}_R(N, D) \xrightarrow{\varphi'} \mathrm{Hom}_R(M, D) \xrightarrow{\psi'} \mathrm{Hom}_R(L, D) \longrightarrow 0$$

is in general *not* a short exact sequence since ψ' need not be surjective, and the question of whether this sequence is exact precisely measures the extent to which homomorphisms from M to D are uniquely determined by pairs of homomorphisms from L and N to D.

The second statement in Proposition 29 shows that this sequence is exact when the original exact sequence $0 \to L \to M \to N \to 0$ is a *split* exact sequence. In fact in this case the sequence $0 \to \mathrm{Hom}_R(N, D) \xrightarrow{\varphi'} \mathrm{Hom}_R(M, D) \xrightarrow{\psi'} \mathrm{Hom}_R(L, D) \to 0$ is also a split exact sequence of abelian groups for every R-module D. Exercise 14 shows that a converse holds: if $0 \to \mathrm{Hom}_R(N, D) \xrightarrow{\varphi'} \mathrm{Hom}_R(M, D) \xrightarrow{\psi'} \mathrm{Hom}_R(L, D) \to 0$ is exact for every R-module D then $0 \to L \xrightarrow{\psi} M \xrightarrow{\varphi} N \to 0$ is a split short exact sequence (which then implies that if the Hom sequence is exact for every D, then in fact it is split exact for every D).

There is also a dual version of the first three parts of Proposition 30, which describes the R-modules D having the property that the sequence (12) in Theorem 33 can *always* be extended to a short exact sequence:

Proposition 34. Let Q be an R-module. Then the following are equivalent:

(1) For any R-modules L, M, and N, if

$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow 0$$

is a short exact sequence, then

$$0 \longrightarrow \mathrm{Hom}_R(N, Q) \xrightarrow{\varphi'} \mathrm{Hom}_R(M, Q) \xrightarrow{\psi'} \mathrm{Hom}_R(L, Q) \longrightarrow 0$$

is also a short exact sequence.

(2) For any R-modules L and M, if $0 \to L \xrightarrow{\psi} M$ is exact, then every R-module homomorphism from L into Q lifts to an R-module homomorphism of M into Q, i.e., given $f \in \mathrm{Hom}_R(L, Q)$ there is a lift $F \in \mathrm{Hom}_R(M, Q)$ making the following diagram commute:

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "\psi"] \arrow[d, "f"'] & M \arrow[dl, dashed, "F"] \\ & Q & \end{tikzcd}$$

(3) If Q is a submodule of the R-module M then Q is a direct summand of M, i.e., every short exact sequence $0 \to Q \to M \to N \to 0$ splits.

Proof: The equivalence of (1) and (2) is part of Theorem 33. Suppose now that (2) is satisfied and let $0 \to Q \xrightarrow{\psi} M \xrightarrow{\varphi} N \to 0$ be exact. Taking $L = Q$ and f the identity map from Q to itself, it follows by (2) that there is a homomorphism $F : M \to Q$ with $F \circ \psi = 1$, so F is a splitting homomorphism for the sequence, which proves (3). The proof that (3) implies (2) is outlined in the exercises.

Definition. An R-module Q is called *injective* if it satisfies any of the equivalent conditions of Proposition 34.

The third statement in Proposition 34 can be rephrased as saying that any module M into which Q injects has (an isomorphic copy of) Q as a direct summand, which explains the terminology.

If D is fixed, then given any R-module X we have an associated abelian group $\mathrm{Hom}_R(X, D)$. Further, an R-module homomorphism $\alpha : X \to Y$ induces an abelian group homomorphism $\alpha' : \mathrm{Hom}_R(Y, D) \to \mathrm{Hom}_R(X, D)$, defined by $\alpha'(f) = f \circ \alpha$, that "reverses" the direction of the arrow. Put another way, the map $\mathrm{Hom}_R(D, \underline{\ \ })$ is a *contravariant functor* from the category of R-modules to the category of abelian groups (cf. Appendix II). Theorem 33 shows that applying this functor to the terms in the exact sequence

$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow 0$$

produces an exact sequence

$$0 \to \mathrm{Hom}_R(N, D) \xrightarrow{\varphi'} \mathrm{Hom}_R(M, D) \xrightarrow{\psi'} \mathrm{Hom}_R(L, D).$$

This is referred to by saying that $\mathrm{Hom}_R(\underline{\ \ }, D)$ is a *left exact* (contravariant) functor. Note that the functor $\mathrm{Hom}_R(\underline{\ \ }, D)$ and the functor $\mathrm{Hom}_R(D, \underline{\ \ })$ considered earlier are both left exact; the former reverses the directions of the maps in the original short exact sequence, the latter maintains the directions of the maps.

By Proposition 34, the functor $\operatorname{Hom}_R(\_\_, D)$ is *exact*, i.e., always takes short exact sequences to short exact sequences (and hence exact sequences of any length to exact sequences), if and only if D is injective. We summarize this in the following proposition, which is dual to the covariant result of Corollary 32.

Corollary 35. If D is an R-module, then the functor $\operatorname{Hom}_R(\_\_, D)$ from the category of R-modules to the category of abelian groups is left exact. It is exact if and only if D is an injective R-module.

We have seen that an R-module is projective if and only if it is a direct summand of a free R-module. Providing such a simple characterization of injective R-modules is not so easy. The next result gives a criterion for Q to be an injective R-module (a result due to Baer, who introduced the notion of injective modules around 1940), and using it we can give a characterization of injective modules when $R = \mathbb{Z}$ (or, more generally, when R is a P.I.D.). Recall that a $\mathbb{Z}$-module A (i.e., an abelian group, written additively) is said to be *divisible* if $A = nA$ for all nonzero integers n. For example, both $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are divisible (cf. Exercises 18 and 19 in Section 2.4 and Exercise 15 in Section 3.1).

Proposition 36. Let Q be an R-module.

(1) *(Baer's Criterion)* The module Q is injective if and only if for every left ideal I of R any R-module homomorphism $g : I \to Q$ can be extended to an R-module homomorphism $G : R \to Q$.

(2) If R is a P.I.D. then Q is injective if and only if $rQ = Q$ for every nonzero $r \in R$. In particular, a $\mathbb{Z}$-module is injective if and only if it is divisible. When R is a P.I.D., quotient modules of injective R-modules are again injective.

Proof: If Q is injective and $g : I \to Q$ is an R-module homomorphism from the nonzero ideal I of R into Q, then g can be extended to an R-module homomorphism from R into Q by Proposition 34(2) applied to the exact sequence $0 \to I \to R$, which proves the "only if" portion of (1). Suppose conversely that every homomorphism $g : I \to Q$ can be lifted to a homomorphism $G : R \to Q$. To show that Q is injective we must show that if $0 \to L \to M$ is exact and $f : L \to Q$ is an R-module homomorphism then there is a lift $F : M \to Q$ extending f. If $\mathcal{S}$ is the collection (f', L') of lifts $f' : L' \to Q$ of f to a submodule L' of M containing L, then the ordering $(f', L') \le (f'', L'')$ if $L' \subseteq L''$ and $f'' = f'$ on L' partially orders $\mathcal{S}$. Since $\mathcal{S} \neq \emptyset$, by Zorn's Lemma there is a maximal element (F, M') in $\mathcal{S}$. The map $F : M' \to Q$ is a lift of f and it suffices to show that $M' = M$. Suppose that there is some element $m \in M$ not contained in M' and let $I = \{r \in R \mid rm \in M'\}$. It is easy to check that I is a left ideal in R, and the map $g : I \to Q$ defined by $g(x) = F(xm)$ is an R-module homomorphism from I to Q. By hypothesis, there is a lift $G : R \to Q$ of g. Consider the submodule $M' + Rm$ of M, and define the map $F' : M' + Rm \to Q$ by $F'(m' + rm) = F(m') + G(r)$. If $m_1 + r_1 m = m_2 + r_2 m$ then $(r_1 - r_2)m = m_2 - m_1$