In $D(\epsilon_j, \epsilon_k, \ldots, \alpha_n)$ we next replace $\alpha_3$ by $\sum A(3, l)\epsilon_l$ and so on. We finally obtain a complicated but theoretically important expression for $D(A)$, namely

$$D(A) = \sum_{k_1, k_2, \ldots, k_n} A(1, k_1)A(2, k_2) \cdots A(n, k_n)D(\epsilon_{k_1}, \epsilon_{k_2}, \ldots, \epsilon_{k_n}). \tag{5-10}$$

In (5-10) the sum is extended over all sequences $(k_1, k_2, \ldots, k_n)$ of positive integers not exceeding $n$. This shows that $D$ is a finite sum of functions of the type described by (5-2). It should be noted that (5-10) is a consequence just of assumption that $D$ is $n$-linear, and that a special case of (5-10) was obtained in Example 2. Since $D$ is alternating,

$$D(\epsilon_{k_1}, \epsilon_{k_2}, \ldots, \epsilon_{k_n}) = 0$$

whenever two of the indices $k_i$ are equal. A sequence $(k_1, k_2, \ldots, k_n)$ of positive integers not exceeding $n$, with the property that no two of the $k_i$ are equal, is called a **permutation of degree $n$.** In (5-10) we need therefore sum only over those sequences which are permutations of degree $n$.

Since a finite sequence, or $n$-tuple, is a function defined on the first $n$ positive integers, a permutation of degree $n$ may be defined as a one-one function from the set $\{1, 2, \ldots, n\}$ onto itself. Such a function $\sigma$ corresponds to the $n$-tuple $(\sigma 1, \sigma 2, \ldots, \sigma n)$ and is thus simply a rule for ordering $1, 2, \ldots, n$ in some well-defined way.

If $D$ is an alternating $n$-linear function and $A$ is an $n \times n$ matrix over $K$, we then have

$$D(A) = \sum_{\sigma} A(1, \sigma 1) \cdots A(n, \sigma n)D(\epsilon_{\sigma 1}, \ldots, \epsilon_{\sigma n}) \tag{5-11}$$

where the sum is extended over the distinct permutations $\sigma$ of degree $n$.

Next we shall show that

$$D(\epsilon_{\sigma 1}, \ldots, \epsilon_{\sigma n}) = \pm D(\epsilon_1, \ldots, \epsilon_n) \tag{5-12}$$

where the sign $\pm$ depends only on the permutation $\sigma$. The reason for this is as follows. The sequence $(\sigma 1, \sigma 2, \ldots, \sigma n)$ can be obtained from the sequence $(1, 2, \ldots, n)$ by a finite number of interchanges of pairs of elements. For example, if $\sigma 1 \neq 1$, we can transpose 1 and $\sigma 1$, obtaining $(\sigma 1, \ldots, 1, \ldots)$. Proceeding in this way we shall arrive at the sequence $(\sigma 1, \ldots, \sigma n)$ after $n$ or less such interchanges of pairs. Since $D$ is alternating, the sign of its value changes each time that we interchange two of the rows $\epsilon_i$ and $\epsilon_j$. Thus, if we pass from $(1, 2, \ldots, n)$ to $(\sigma 1, \sigma 2, \ldots, \sigma n)$ by means of $m$ interchanges of pairs $(i, j)$, we shall have

$$D(\epsilon_{\sigma 1}, \ldots, \epsilon_{\sigma n}) = (-1)^m D(\epsilon_1, \ldots, \epsilon_n).$$

In particular, if $D$ is a determinant function

$$D(\epsilon_{\sigma 1}, \ldots, \epsilon_{\sigma n}) = (-1)^m \tag{5-13}$$

where $m$ depends only upon $\sigma$, not upon $D$. Thus all determinant functions assign the same value to the matrix with rows $\epsilon_{\sigma 1}, \ldots, \epsilon_{\sigma n}$, and this value is either 1 or $-1$.

Now a basic fact about permutations is the following. If $\sigma$ is a permutation of degree $n$, one can pass from the sequence $(1, 2, \ldots, n)$ to the sequence $(\sigma 1, \sigma 2, \ldots, \sigma n)$ by a succession of interchanges of pairs, and this can be done in a variety of ways; however, no matter how it is done, the number of interchanges used is either always even or always odd. The permutation is then called **even** or **odd,** respectively. One defines the **sign** of a permutation by

$$\operatorname{sgn} \sigma = \begin{cases} 1, & \text{if } \sigma \text{ is even} \\ -1, & \text{if } \sigma \text{ is odd} \end{cases}$$

the symbol '1' denoting here the integer 1.

We shall show below that this basic property of permutations can be deduced from what we already know about determinant functions. Let us assume this for the time being. Then the integer $m$ occurring in (5-13) is always even if $\sigma$ is an even permutation, and is always odd if $\sigma$ is an odd permutation. For any alternating $n$-linear function $D$ we then have

$$D(\epsilon_{\sigma 1}, \ldots, \epsilon_{\sigma n}) = (\operatorname{sgn} \sigma) D(\epsilon_1, \ldots, \epsilon_n)$$

and using (5-11)

$$D(A) = \left[\sum_\sigma (\operatorname{sgn} \sigma) A(1, \sigma 1) \cdots A(n, \sigma n)\right] D(I). \tag{5-14}$$

Of course $I$ denotes the $n \times n$ identity matrix.

From (5-14) we see that there is precisely one determinant function on $n \times n$ matrices over $K$. If we denote this function by det, it is given by

$$\det (A) = \sum_\sigma (\operatorname{sgn} \sigma) A(1, \sigma 1) \cdots A(n, \sigma n) \tag{5-15}$$

the sum being extended over the distinct permutations $\sigma$ of degree $n$. We can formally summarize as follows.

***Theorem 2.*** *Let* K *be a commutative ring with identity and let* n *be a positive integer. There is precisely one determinant function on the set of* $n \times n$ *matrices over* K, *and it is the function* det *defined by (5-15). If* D *is any alternating* n*-linear function on* $K^{n \times n}$, *then for each* $n \times n$ *matrix* A

$$D(A) = (\det A) D(I).$$

This is the theorem we have been seeking, but we have left a gap in the proof. That gap is the proof that for a given permutation $\sigma$, when we pass from $(1, 2, \ldots, n)$ to $(\sigma 1, \sigma 2, \ldots, \sigma n)$ by interchanging pairs, the number of interchanges is always even or always odd. This basic combinatorial fact can be proved without any reference to determinants;

however, we should like to point out how it follows from the *existence* of a determinant function on $n \times n$ matrices.

Let us take $K$ to be the ring of integers. Let $D$ be a determinant function on $n \times n$ matrices over $K$. Let $\sigma$ be a permutation of degree $n$, and suppose we pass from $(1, 2, \ldots, n)$ to $(\sigma 1, \sigma 2, \ldots, \sigma n)$ by $m$ interchanges of pairs $(i, j)$, $i \neq j$. As we showed in (5-13)

$$(-1)^m = D(\epsilon_{\sigma 1}, \ldots, \epsilon_{\sigma n})$$

that is, the number $(-1)^m$ must be the value of $D$ on the matrix with rows $\epsilon_{\sigma 1}, \ldots, \epsilon_{\sigma n}$. If

$$D(\epsilon_{\sigma 1}, \ldots, \epsilon_{\sigma n}) = 1,$$

then $m$ must be even. If

$$D(\epsilon_{\sigma 1}, \ldots, \epsilon_{\sigma n}) = -1,$$

then $m$ must be odd.

Since we have an explicit formula for the determinant of an $n \times n$ matrix and this formula involves the permutations of degree $n$, let us conclude this section by making a few more observations about permutations. First, let us note that there are precisely $n! = 1 \cdot 2 \cdots n$ permutations of degree $n$. For, if $\sigma$ is such a permutation, there are $n$ possible choices for $\sigma 1$; when this choice has been made, there are $(n - 1)$ choices for $\sigma 2$, then $(n - 2)$ choices for $\sigma 3$, and so on. So there are

$$n(n-1)(n-2) \cdots 2 \cdot 1 = n!$$

permutations $\sigma$. The formula (5-15) for det $(A)$ thus gives det $(A)$ as a sum of $n!$ terms, one for each permutation of degree $n$. A given term is a product

$$A(1, \sigma 1) \cdots A(n, \sigma n)$$

of $n$ entries of $A$, one entry from each row and one from each column, and is prefixed by a '$+$' or '$-$' sign according as $\sigma$ is an even or odd permutation.

When permutations are regarded as one-one functions from the set $\{1, 2, \ldots, n\}$ onto itself, one can define a product of permutations. The product of $\sigma$ and $\tau$ will simply be the composed function $\sigma\tau$ defined by

$$(\sigma\tau)(i) = \sigma(\tau(i)).$$

If $\epsilon$ denotes the identity permutation, $\epsilon(i) = i$, then each $\sigma$ has an inverse $\sigma^{-1}$ such that

$$\sigma\sigma^{-1} = \sigma^{-1}\sigma = \epsilon.$$

One can summarize these observations by saying that, under the operation of composition, the set of permutations of degree $n$ is a group. This group is usually called the **symmetric group of degree *n*.**

From the point of view of products of permutations, the basic property of the sign of a permutation is that

$$\text{(5-16)} \qquad \operatorname{sgn}(\sigma\tau) = (\operatorname{sgn}\sigma)(\operatorname{sgn}\tau).$$

In other words, $\sigma\tau$ is an even permutation if $\sigma$ and $\tau$ are either both even or both odd, while $\sigma\tau$ is odd if one of the two permutations is odd and the other is even. One can see this from the definition of the sign in terms of successive interchanges of pairs $(i, j)$. It may also be instructive if we point out how $\operatorname{sgn}(\sigma\tau) = (\operatorname{sgn}\sigma)(\operatorname{sgn}\tau)$ follows from a fundamental property of determinants.

Let $K$ be the ring of integers and let $\sigma$ and $\tau$ be permutations of degree $n$. Let $\epsilon_1, \ldots, \epsilon_n$ be the rows of the $n \times n$ identity matrix over $K$, let $A$ be the matrix with rows $\epsilon_{\tau 1}, \ldots, \epsilon_{\tau n}$, and let $B$ be the matrix with rows $\epsilon_{\sigma 1}, \ldots, \epsilon_{\sigma n}$. The $i$th row of $A$ contains exactly one non-zero entry, namely the 1 in column $\tau i$. From this it is easy to see that $\epsilon_{\sigma\tau i}$ is the $i$th row of the product matrix $AB$. Now

$$\det(A) = \operatorname{sgn}\tau, \qquad \det(B) = \operatorname{sgn}\sigma, \qquad \text{and} \quad \det(AB) = \operatorname{sgn}(\sigma\tau).$$

So we shall have $\operatorname{sgn}(\sigma\tau) = (\operatorname{sgn}\sigma)(\operatorname{sgn}\tau)$ as soon as we prove the following.

***Theorem 3.*** *Let K be a commutative ring with identity, and let A and B be n $\times$ n matrices over K. Then*

$$\det(AB) = (\det A)(\det B).$$

*Proof.* Let $B$ be a fixed $n \times n$ matrix over $K$, and for each $n \times n$ matrix $A$ define $D(A) = \det(AB)$. If we denote the rows of $A$ by $\alpha_1, \ldots, \alpha_n$, then

$$D(\alpha_1, \ldots, \alpha_n) = \det(\alpha_1 B, \ldots, \alpha_n B).$$

Here $\alpha_j B$ denotes the $1 \times n$ matrix which is the product of the $1 \times n$ matrix $\alpha_j$ and the $n \times n$ matrix $B$. Since

$$(c\alpha_i + \alpha_i')B = c\alpha_i B + \alpha_i' B$$

and det is $n$-linear, it is easy to see that $D$ is $n$-linear. If $\alpha_i = \alpha_j$, then $\alpha_i B = \alpha_j B$, and since det is alternating,

$$D(\alpha_1, \ldots, \alpha_n) = 0.$$

Hence, $D$ is alternating. Now $D$ is an alternating $n$-linear function, and by Theorem 2

$$D(A) = (\det A)D(I).$$

But $D(I) = \det(IB) = \det B$, so

$$\det(AB) = D(A) = (\det A)(\det B). \quad \blacksquare$$

The fact that $\operatorname{sgn}(\sigma\tau) = (\operatorname{sgn}\sigma)(\operatorname{sgn}\tau)$ is only one of many corollaries to Theorem 3. We shall consider some of these corollaries in the next section.

## *Exercises*

**1.** If $K$ is a commutative ring with identity and $A$ is the matrix over $K$ given by

$$A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$$

show that $\det A = 0$.

**2.** Prove that the determinant of the Vandermonde matrix

$$\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$$

is $(b - a)(c - a)(c - b)$.

**3.** List explicitly the six permutations of degree 3, state which are odd and which are even, and use this to give the complete formula (5-15) for the determinant of a $3 \times 3$ matrix.

**4.** Let $\sigma$ and $\tau$ be the permutations of degree 4 defined by $\sigma 1 = 2$, $\sigma 2 = 3$, $\sigma 3 = 4$, $\sigma 4 = 1$, $\tau 1 = 3$, $\tau 2 = 1$, $\tau 3 = 2$, $\tau 4 = 4$.

(a) Is $\sigma$ odd or even? Is $\tau$ odd or even?

(b) Find $\sigma\tau$ and $\tau\sigma$.

**5.** If $A$ is an invertible $n \times n$ matrix over a field, show that $\det A \neq 0$.

**6.** Let $A$ be a $2 \times 2$ matrix over a field. Prove that $\det (I + A) = 1 + \det A$ if and only if $\operatorname{trace} (A) = 0$.

**7.** An $n \times n$ matrix $A$ is called **triangular** if $A_{ij} = 0$ whenever $i > j$ or if $A_{ij} = 0$ whenever $i < j$. Prove that the determinant of a triangular matrix is the product $A_{11}A_{22} \cdots A_{nn}$ of its diagonal entries.

**8.** Let $A$ be a $3 \times 3$ matrix over the field of complex numbers. We form the matrix $xI - A$ with polynomial entries, the $i, j$ entry of this matrix being the polynomial $\delta_{ij}x - A_{ij}$. If $f = \det (xI - A)$, show that $f$ is a monic polynomial of degree 3. If we write

$$f = (x - c_1)(x - c_2)(x - c_3)$$

with complex numbers $c_1$, $c_2$, and $c_3$, prove that

$$c_1 + c_2 + c_3 = \operatorname{trace} (A) \quad \text{and} \quad c_1c_2c_3 = \det A.$$

**9.** Let $n$ be a positive integer and $F$ a field. If $\sigma$ is a permutation of degree $n$, prove that the function

$$T(x_1, \ldots, x_n) = (x_{\sigma 1}, \ldots, x_{\sigma n})$$

is an invertible linear operator on $F^n$.

**10.** Let $F$ be a field, $n$ a positive integer, and $S$ the set of $n \times n$ matrices over $F$. Let $V$ be the vector space of all functions from $S$ into $F$. Let $W$ be the set of alternating $n$-linear functions on $S$. Prove that $W$ is a subspace of $V$. What is the dimension of $W$?