Homomorphisms of Tensor Algebras

If $\varphi : M \to N$ is any R-module homomorphism, then there is an induced map on the k^{th} tensor power:

$$\mathcal{T}^k(\varphi) : m_1 \otimes m_2 \otimes \cdots \otimes m_k \longmapsto \varphi(m_1) \otimes \varphi(m_2) \otimes \cdots \otimes \varphi(m_k).$$

It follows directly that this map sends generators of each of the homogeneous components of the ideals $\mathcal{C}(M)$ and $\mathcal{A}(M)$ to themselves. Thus φ induces R-module homomorphisms on the quotients:

$$\mathcal{S}^k(\varphi) : \mathcal{S}^k(M) \longrightarrow \mathcal{S}^k(N) \qquad \text{and} \qquad \bigwedge^k(\varphi) : \bigwedge^k(M) \longrightarrow \bigwedge^k(N).$$

Moreover, each of these three maps is a ring homomorphism (hence they are graded R-algebra homomorphisms).

Of particular interest is the case when $M = V$ is an n-dimensional vector space over the field F and $\varphi : V \to V$ is an endomorphism. In this case by Corollary 37, $\bigwedge^n(\varphi)$ maps the 1-dimensional space $\bigwedge^n(V)$ to itself. Let $v_1, \dots, v_n$ be a basis of V, so that $v_1 \wedge \cdots \wedge v_n$ is a basis of $\bigwedge^n(V)$. Then

$$\bigwedge^n(\varphi)(v_1 \wedge \cdots \wedge v_n) = \varphi(v_1) \wedge \cdots \wedge \varphi(v_n) = D(\varphi) v_1 \wedge \cdots \wedge v_n$$

for some scalar $D(\varphi) \in F$.

For any $n \times n$ matrix A over F we can define the associated endomorphism φ (with respect to the given basis $v_1, \dots, v_n$), which gives a map $D : M_{n \times n}(F) \to F$ where $D(A) = D(\varphi)$. It is easy to check that this map D satisfies the three axioms for a determinant function in Section 4. Then the uniqueness statement of Theorem 24 gives:

Proposition 38. If φ is an endomorphism on a n-dimensional vector space V, then $\bigwedge^n(\varphi)(w) = \det(\varphi) w$ for all $w \in \bigwedge^n(V)$.

Note that Proposition 38 characterizes the determinant of the endomorphism φ as a certain naturally induced *linear* map on $\bigwedge^n(V)$. The fact that the determinant arises naturally when considering alternating multilinear maps also explains the source of the map φ in the example above.

As with the tensor product, the maps $\mathcal{S}^k(\varphi)$ and $\bigwedge^k(\varphi)$ induced from an injective map from M to N need not remain injective (so $\bigwedge^2(M)$ need not be a submodule of $\bigwedge^2(N)$ when M is a submodule of N, for example).

Example

The inclusion $\varphi : I \hookrightarrow R$ of the ideal (x, y) into the ring $R = \mathbb{Z}[x, y]$, both considered as R-modules, induces a map

$$\bigwedge^2(\varphi) : \bigwedge^2(I) \to \bigwedge^2(R).$$

Since $\bigwedge^2(R) = 0$ and $\bigwedge^2(I) \neq 0$, the map cannot be injective.

One can show that if M is an R-module *direct summand* of N, then $\mathcal{T}(M)$ (respectively, $\mathcal{S}(M)$ and $\bigwedge(M)$) is an R-subalgebra of $\mathcal{T}(N)$ (respectively, $\mathcal{S}(N)$ and $\bigwedge(N)$) (cf. the exercises). When $R = F$ is a field then *every* subspace M of N is a direct summand of N and so the corresponding algebra for M is a subalgebra of the algebra for N.

Symmetric and Alternating Tensors

The symmetric and exterior algebras can in some instances also be defined in terms of *symmetric* and *alternating* tensors (defined below), which identify these algebras as *sub*algebras of the tensor algebra rather than as quotient algebras.

For any R-module M there is a natural left group action of the symmetric group S_k on $M \times M \times \cdots \times M$ (k factors) given by permuting the factors:

$$\sigma(m_1, m_2, \dots, m_k) = (m_{\sigma^{-1}(1)}, m_{\sigma^{-1}(2)}, \dots, m_{\sigma^{-1}(k)}) \qquad \text{for each } \sigma \in S_k$$

(the reason for σ^{-1} is to make this a *left* group action, cf. Exercise 8 of Section 5.1). This map is clearly R-multilinear, so there is a well defined R-linear left group action of S_k on $\mathcal{T}^k(M)$ which is defined on simple tensors by

$$\sigma(m_1 \otimes m_2 \otimes \cdots \otimes m_k) = m_{\sigma^{-1}(1)} \otimes m_{\sigma^{-1}(2)} \otimes \cdots \otimes m_{\sigma^{-1}(k)} \qquad \text{for each } \sigma \in S_k.$$

Definition.

(1) An element $z \in \mathcal{T}^k(M)$ is called a *symmetric* k-tensor if $\sigma z = z$ for all σ in the symmetric group S_k.

(2) An element $z \in \mathcal{T}^k(M)$ is called an *alternating* k-tensor if $\sigma z = \epsilon(\sigma) z$ for all σ in the symmetric group S_k, where $\epsilon(\sigma)$ is the sign, ± 1, of the permutation σ.

It is immediate from the definition that the collection of symmetric (respectively, alternating) k-tensors is an R-submodule of the module of all k-tensors.

Example

The elements $m \otimes m$ and $m_1 \otimes m_2 + m_2 \otimes m_1$ are symmetric 2-tensors. The element $m_1 \otimes m_2 - m_2 \otimes m_1$ is an alternating 2-tensor.

It is also clear from the definition that both $\mathcal{C}^k(M)$ and $\mathcal{A}^k(M)$ are stable under the action of S_k, hence there is an induced action on the quotients $\mathcal{S}^k(M)$ and $\bigwedge^k(M)$.

Proposition 39. Let σ be an element in the symmetric group S_k and let $\epsilon(\sigma)$ be the sign of the permutation σ. Then

(1) for every $w \in \mathcal{S}^k(M)$ we have $\sigma w = w$, and

(2) for every $w \in \bigwedge^k(M)$ we have $\sigma w = \epsilon(\sigma) w$.

Proof: The first statement is immediate from (1) in Theorem 34. We showed in the course of the proof of Theorem 36 that

$$m_1 \wedge \cdots \wedge m_i \wedge m_{i+1} \wedge \cdots \wedge m_k = -m_1 \wedge \cdots \wedge m_{i+1} \wedge m_i \wedge \cdots \wedge m_k,$$

which shows that the formula in (2) is valid on simple products for the transposition $\sigma = (i\, i{+}1)$. Since these transpositions generate S_k and ϵ is a group homomorphism it follows that (2) is valid for any $\sigma \in S_k$ on simple products w. Since both sides are R-linear in w, it follows that (2) holds for all $w \in \bigwedge^k(M)$.

By Proposition 39, the symmetric group S_k acts trivially on both the *sub*module of symmetric k-tensors and the *quotient* module $\mathcal{S}^k(M)$, the k^{th} symmetric power of M. Similarly, S_k acts the same way on the submodule of alternating k-tensors as on $\bigwedge^k(M)$, the k^{th} exterior power of M. We now show that when $k!$ is a unit in R that these respective submodules and quotient modules are isomorphic (where $k!$ is the sum of the 1 of R with itself $k!$ times).

For any k-tensor $z \in \mathcal{T}^k(M)$ define

$$Sym(z) = \sum_{\sigma \in S_k} \sigma z$$
$$Alt(z) = \sum_{\sigma \in S_k} \epsilon(\sigma)\, \sigma z.$$

For any k-tensor z, the k-tensor $Sym(z)$ is symmetric and the k-tensor $Alt(z)$ is alternating. For example, for any $\tau \in S_k$

$$\begin{aligned}
\tau\, Alt(z) &= \sum_{\sigma \in S_k} \epsilon(\sigma)\, \tau\sigma\, z \\
&= \sum_{\sigma' \in S_k} \epsilon(\tau^{-1}\sigma')\, \sigma' z \quad \text{(letting } \sigma' = \tau\sigma) \\
&= \epsilon(\tau^{-1}) \sum_{\sigma' \in S_k} \epsilon(\sigma')\, \sigma' z = \epsilon(\tau) Alt(z).
\end{aligned}$$

The tensor $Sym(z)$ is sometimes called the *symmetrization* of z and $Alt(z)$ the *skew-symmetrization* of z.

If z is already a symmetric (respectively, alternating) tensor then $Sym(z)$ (respectively, $Alt(z)$) is just $k!z$. It follows that Sym (respectively, Alt) is an R-module endomorphism of $\mathcal{T}^k(M)$ whose image lies in the submodule of symmetric (respectively, alternating) tensors. In general these maps are not surjective, but if $k!$ is a unit in R then

$$\begin{aligned}
\frac{1}{k!} Sym(z) &= z \quad \text{for any symmetric tensor } z, \text{ and} \\
\frac{1}{k!} Alt(z) &= z \quad \text{for any alternating tensor } z
\end{aligned}$$

so that in this case the maps $(1/k!)Sym$ and $(1/k!)Alt$ give surjective R-module homomorphisms from $\mathcal{T}^k(M)$ to the submodule of symmetric (respectively, alternating) tensors.

Proposition 40. Suppose $k!$ is a unit in the ring R and M is an R-module. Then

(1) The map $(1/k!)Sym$ induces an R-module isomorphism between the k^{th} symmetric power of M and the R-submodule of symmetric k-tensors:
$$\frac{1}{k!}Sym : S^k(M) \cong \{\text{symmetric } k\text{-tensors}\}.$$

(2) The map $(1/k!)Alt$ induces an R-module isomorphism between the k^{th} exterior power of M and the R-submodule of alternating k-tensors:
$$\frac{1}{k!}Alt : \bigwedge^k(M) \cong \{\text{alternating } k\text{-tensors}\}.$$

Proof: We have seen that the respective maps are surjective R-homomorphisms from $\mathcal{T}^k(M)$ so to prove the proposition it suffices to check that their kernels are $\mathcal{C}^k(M)$ and $\mathcal{A}^k(M)$, respectively. We show the first and leave the second to the exercises. It is clear that Sym is 0 on any difference of two k-tensors which differ only in the order of their factors, so $\mathcal{C}^k(M)$ is contained in the kernel of $(1/k!)Sym$ by (1) of Theorem 34. For the reverse inclusion, observe that
$$z - \frac{1}{k!}Sym(z) = \frac{1}{k!}\sum_{\sigma\in S_k}(z - \sigma z)$$
for any k-tensor z. If z is in the kernel of Sym then the left hand side of this equality is just z; and since $z - \sigma z \in \mathcal{C}^k(M)$ for every $\sigma \in S_k$ (again by (1) of Theorem 34), it follows that $z \in \mathcal{C}^k(M)$, completing the proof.

The maps $(1/k!)Sym$ and $(1/k!)Alt$ are *projections* (cf. Exercise 11 in Section 2) onto the submodules of symmetric and antisymmetric tensors, respectively. Equivalently, if $k!$ is a unit in R, we have R-module direct sums
$$\mathcal{T}^k(M) = \ker(\pi) \oplus \text{image}(\pi)$$
for $\pi = (1/k!)Sym$ or $\pi = (1/k!)Alt$. In the former case the kernel consists of $\mathcal{C}^k(M)$ and the image is the collection of symmetric tensors (in which case $\mathcal{C}^k(M)$ is said to form an R-module *complement* to the symmetric tensors). In the latter case the kernel is $\mathcal{A}^k(M)$ and the image consists of the alternating tensors.

The R-linear left group action of S_k on $\mathcal{T}^k(M)$ makes $\mathcal{T}^k(M)$ into a module over the group ring RS_k (analogous to the formation of $F[x]$-modules described in Section 10.1). In terms of this module structure these projections give RS_k-submodule complements to the RS_k-submodules $\mathcal{C}^k(M)$ and $\mathcal{A}^k(M)$. The "averaging" technique used to construct these maps can be used to prove a very general result (Maschke's Theorem in Section 18.1) related to actions of finite groups on vector spaces (which is the subject of the "representation theory" of finite groups in Part VI).

If $k!$ is not invertible in R then in general we do not have such S_k-invariant direct sum decompositions so it is not in general possible to identify, for example, the k^{th} exterior power of M with the alternating k-tensors of M.

Note also that when $k!$ is invertible it is possible to *define* the k^{th} exterior power of M as the collection of alternating k-tensors (this equivalent approach is sometimes found