2.3.3. Assuming that any polygon can be cut into triangles, show that the angle sum of any n-gon is $(n-2)\pi$, where π denotes two right angles.

2.3.4. Deduce from Exercise 2.3.3 that the only ways to tile the plane with copies of a single regular n-gon (that is, an n-gon with equal sides and equal angles) are by equilateral triangles, squares, and regular hexagons.

2.3.5. Show that the plane can be tiled with copies of any single triangle.

2.4 Angles and Circles

One of the first theorems in Euclid's *Elements* says that *the base angles of an isosceles ("equal sides") triangle are equal.* The most elegant proof of this theorem was found by another Greek mathematician, Pappus, around 300 A.D. It goes like this. Suppose ABC is a triangle with $AB = BC$ (Figure 2.13). Because $AB = BC$, this triangle can be turned over and placed so that BC replaces AB, and AB replaces BC. In other words, the triangle exactly fills the space it filled in its old position. In particular, the base angle BAC fills the space previously filled by angle BCA, so these two angles are equal.

Triangles that occupy the same space were called *congruent* by Euclid. He used the idea of moving one triangle to coincide with another to prove the two triangles congruent when they agree in certain angles and sides. The preceding argument uses "side-angle-side" agreement: if two triangles agree in two sides and the included angle, then one can be moved to coincide with the other. Congru-

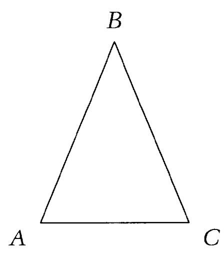

FIGURE 2.13 An isosceles triangle.

ence also occurs in the angle-side-angle and side-side-side cases. Later mathematicians felt that the idea of motion did not belong in synthetic geometry, and instead stated the congruence of triangles with side-angle-side, angle-side-angle, or side-side-side agreement as axioms. This was done in Hilbert's *Foundations of Geometry* (1899), for example. The idea of motion came back in Felix Klein's definition of geometry, which we shall discuss in Chapter 3.

Whichever approach is adopted, the theorem on the base angles of an isosceles triangle is the key to many other results. Perhaps the most important is the theorem relating angles in a circle: *an arc of a circle subtends twice the angle at the center as it does at the circumference*. Figure 2.14 shows the situation in question—the arc AB and the angles AOB and APB it subtends at the center and circumference, respectively—together with a construction line PQ, which gives away the plot.

Because the lines OA and OP are radii of the circle, they are equal. Therefore, triangle POA is isosceles, with equal base angles α as shown. The external angle QOA is therefore 2α because it, like the interior angles α, forms a sum of two right angles with the interior angle AOP. Similarly, the triangle POB has equal angles β as shown and an exterior angle 2β. Thus the angle $2(\alpha + \beta)$ at the center is twice the angle $\alpha + \beta$ at the circumference.

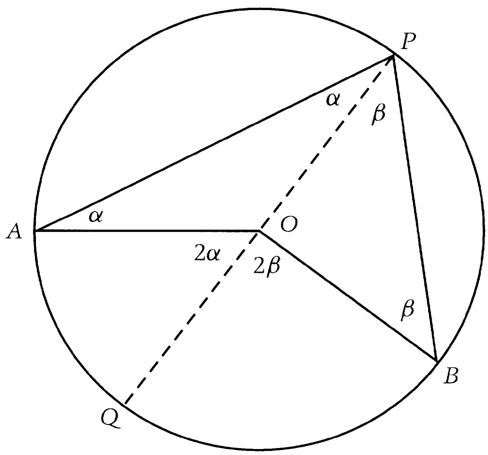

FIGURE 2.14 Angles subtended by an arc.

In the special case where the arc AB is half the circumference so that the angle at the center is straight, we find that the angle at the circumference is a right angle. This should remind you of the theorem of Thales mentioned in the exercises for Section 2.1.

Exercises

We originally stated the theorem of Thales by saying that any right-angled triangle fits in a semicircle. The special case of the theorem about angles in a circle says, rather, that any triangle in a semicircle is right-angled. The two theorems are actually converses of each other. However, they are both true, and the relationship between them can be traced back to converse theorems about isosceles triangles.

2.4.1. Prove that a triangle with two equal angles is isosceles.

2.4.2. What form of congruence axiom is involved in Exercise 2.4.1?

From Exercise 2.4.1, which is the converse theorem about isosceles triangles, we deduce the converse Thales' theorem. It is based on Figure 2.15.

2.4.3. If triangle PAB has a right angle at P and PO is drawn to make the equal angles marked α, show that this also results in equal angles marked β.

2.4.4. Deduce from Exercise 2.4.3 that each right-angled triangle fits in a semicircle.

The fact the angle at the circumference is half the angle at the center implies that the angle at the circumference is *constant*. This means that

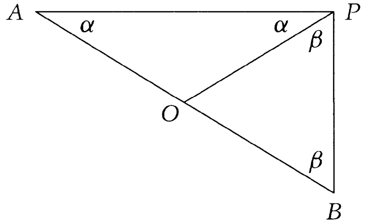

FIGURE 2.15 A right-angled triangle.

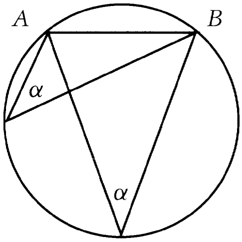

FIGURE 2.16 Apparent size of a chord of a circle.

the chord AB looks the same size, viewed from any point on the circle (Figure 2.16).

Now suppose that we vary the circle through A and B, and consider the effect on the apparent size of AB (a problem of practical importance if, say, you are trying to score a goal between goalposts A and B).

2.4.5. Show that the maximum apparent size of AB, viewed from a line CD, occurs at the point where CD is tangential to a circle through A and B.

2.5 Length and Area

Arithmetic and geometry come together in the idea of *measurement*, first for lengths, but more interestingly for areas. In fact, the very word "geometry" comes from the Greek for "land measurement." To measure lengths, we choose a fixed line segment as the unit of length and attempt to express other lengths as multiples of it. By joining copies of the unit end to end we can obtain any natural number multiple of the unit, and by dividing these into equal parts, we also obtain any rational multiple of the unit. For most practical purposes, this is sufficient, because rational multiples of the unit can be as small as we please. However, we know from Section 1.1 that $\sqrt{2}$ is not rational, and mathematicians would like to be able to speak of a length $\sqrt{2}$, even though the similar length 1.414 might be near enough for surveying or carpentry.

The fundamental problem in measurement is to find enough numbers to represent all possible lengths. This problem is more difficult than it looks, and we shall postpone it until the next chapter.

For the time being we shall just assume that every length is a number. The Greeks did not believe such an assumption was necessary, but this caused difficulties with their theory of area, as we shall see now and in the next section.

Just as length is measured by counting unit lengths, area is measured by counting unit *squares*, that is, squares whose sides are of unit length. For example, a rectangle of height 3 and width 5 can be cut into $3 \times 5 = 15$ unit squares, as Figure 2.17 clearly shows; hence it has area 15.

How convenient that we call it a 3×5 rectangle! Multiplication is the natural symbol to describe rectangles, because it gives the number of unit squares in them. And not only when the sides are integer multiples of the unit. A rectangle of height $3/2$ and width $5/2$ can similarly be cut into 15 squares of side $1/2$, each of which has area $1/4$ (because four of them make a unit square). Hence the area of the $3/2 \times 5/2$ rectangle is $3/2 \times 5/2 = 15/4$.

The same idea, cutting into little fractional squares, shows that the area of an $r \times s$ rectangle is rs for any rational multiples r and s of the unit. But what about, say, a square with side $\sqrt{2}$? Is its area $\sqrt{2} \times \sqrt{2} = 2$? Well, the area of an $r \times r$ square should be close to the area of a $\sqrt{2} \times \sqrt{2}$ square when r is a rational number close to $\sqrt{2}$. If so, 2 is the area of the $\sqrt{2} \times \sqrt{2}$ square, because 2 is the number approached by the values r^2 as r approaches $\sqrt{2}$.

For the Greeks, the area of the $\sqrt{2} \times \sqrt{2}$ square was not a problem, because they *defined* $\sqrt{2}$ to be the side of a square of area 2. The price they paid for this was having to develop a separate arithmetic of lengths and areas, since they did not regard $\sqrt{2}$ as a number. If one wants all lengths to be numbers, defining the area of a rectangle is the same as defining the product of irrational lengths, and it can only be done by comparing the rectangle with arbitrarily close rational rectangles. Once the area of a rectangle is known to be

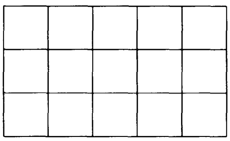

FIGURE 2.17 A 3×5 rectangle.

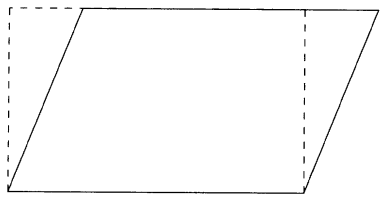

FIGURE 2.18 Area of a parallelogram.

height × width, however, there is a simple way to find the area of other polygons: by *cutting and pasting*.

For example, the standard proof that the area of a triangle is $\frac{1}{2}$base×height is achieved by cutting and pasting. We first argue that the area of the triangle is half the area of the parallelogram obtained by pasting two copies of the triangle together, then that

$$\text{area of parallelogram} = \text{base} \times \text{height}$$

by cutting a triangle off one end of the parallelogram and pasting it on to the other to make a rectangle with the same base and height (Figure 2.18). After this, the area of any other polygon follows, because any polygon can be cut into triangles.

Exercises

It is not quite obvious that any polygon can be cut into triangles, so we should check that this is true, because it is the only way we know to define the area of a polygon.

2.5.1. A polygon Π is *convex* if the line segment connecting any two points of Π is contained in Π. Show that a convex polygon with n sides can be cut into n triangles.

Thus it now suffices to prove that any polygon can be cut into a finite number of convex polygons. This can be done in two easy steps.

2.5.2. Show that any finite set of lines divides the plane into convex polygons.

2.5.3. Deduce from Exercise 2.5.2 that any polygon is cut into convex pieces by the lines that extend its own edges.

2.6 The Pythagorean Theorem

Having seen the main ideas of Greek geometry, it is worth looking again at the Pythagorean theorem, to see where it fits into the big picture. Logically, it comes after the basic theory of area, and in fact Euclid uses the fact that a triangle has half the area of a parallelogram with the same base and height. His proof goes as follows (referring to Figure 2.19).

square $ABFG$ on one side of the triangle

$= 2 \times$ triangle CFB

(same base and height),

$= 2 \times$ triangle ABD

(because the triangles are congruent by agreement of side-angle-side),

$=$ rectangle $BMLD$

(same base and height).

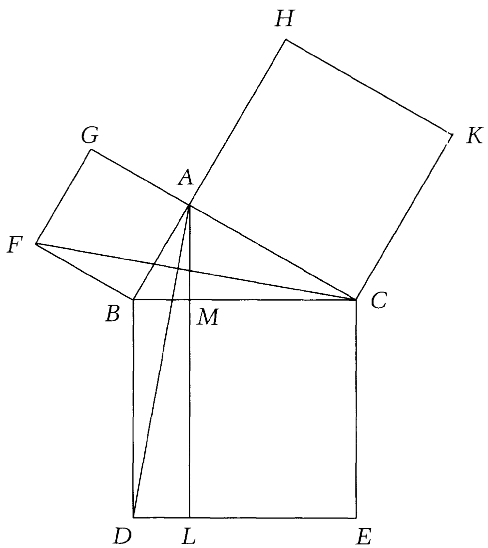

FIGURE 2.19 Areas related to the right-angled triangle.

Similarly, square $ACKH$ on the other side of the triangle equals rectangle $MCEL$, so the squares on the two sides sum to the square on the hypotenuse. □

As mentioned in Section 2.1, the Pythagorean theorem was discovered in several different cultures, and in fact some of them discovered it long before the time of Pythagoras. However, Pythagoras and his followers (the *Pythagoreans*) deserve special mention because they also discovered that $\sqrt{2}$ is irrational. According to legend, this discovery caused great dismay because it conflicted with the Pythagorean philosophy that "all is number." The Pythagoreans initially believed that all things, including lengths, could be measured by natural numbers or their ratios. Yet they could not deny that the diagonal of the unit square was a length, and according to the Pythagorean theorem its square was equal to 2, hence the side and diagonal of the square were *not* natural number multiples of a common unit.

The first fruits of the conflict were bitter, to our taste, but they had a huge influence on the development of mathematics.

- Separation of arithmetic and geometry.
- Development of a separate arithmetic of lengths and areas.
- Preference for the latter "geometric" arithmetic, and the development of a corresponding "geometric algebra."

In geometric algebra, lengths are added by joining them end to end and multiplied by forming the rectangle with them as adjacent sides, the product being interpreted as its area. Areas are added by pasting. From the Pythagorean viewpoint, it is natural to relate lengths via areas. The basic example is the Pythagorean theorem itself, which says that the sides and hypotenuse of a right-angled triangle are simply related via their squares, even though they are *not* simply related as lengths.

The sweeter fruits of the conflict grew from the eventual reconciliation of arithmetic and geometry. This began only in the 17th century, when Fermat and Descartes introduced analytic geometry, and it was not completed until the late 19th century. It was difficult because:

- Arithmetic and geometry involve very different styles of thought, and it was not clearly possible or desirable to do geometry "arithmetically."
- Defects in Euclid's geometry were very deep and subtle. It was not clear that they had anything to do with arithmetic—or the lack of it.
- Arithmetic was in no position to mend the subtle defects of geometry until its own foundations were sound. In particular, a clear concept of *number* was needed.

But the greater the difficulties, the greater the creativity needed to overcome them. The process of reconciliation began with the help of new developments in algebra in the 16th century. This made the methods of arithmetic competitive in geometry for the first time. The process was accelerated by calculus, which gave answers to previously inaccessible questions about lengths and areas of curves. But it was also calculus, with its focus on the "infinitesimal," that most needed a clear concept of number. In 1858, Dedekind realized that calculus, geometry, and the concept of *irrational* could all be clarified in one fell swoop. He defined the concept of *real number* to capture all possible lengths, and thus completed the reconciliation of arithmetic and geometry. The details may be found in Chapter 3.

Exercises

In geometric algebra, a product of three lengths was interpreted as a volume, and there was no interpretation of products of four or more lengths.

2.6.1. With these definitions of addition and multiplication, show that the associative, commutative and distributive laws (Section 1.4) are valid.

2.6.2. Show that the formula $(a + b)^2 = a^2 + 2ab + b^2$ has a natural interpretation in terms of addition and multiplication of lengths.

2.6.3. Also give a geometric interpretation of the identity $b^2 - a^2 = (b - a)(b + a)$.