and positive rationals and reals were studied before negative integers were considered.

Each of the sets after **N** 'extends' the preceding one and, with the exception of **H**, is motivated by our desire to solve equations which are otherwise unsolvable. Thus, **Z, Q, R**, and **C** are needed to solve the equations $x+2=0$, $2x=3$, $x^2=2$ and $x^2+1=0$, respectively.

Mathematicians often 'construct' a set on this list from the set just above it. For example, they build the rationals out of the integers by equating rationals with certain classes of pairs of integers: the fraction $2/3$ is equated with $\{(2,3),(4,6),\ldots,(-2,-3),\ldots\}$. However, the list begins with the natural numbers. How were they constructed? According to Kronecker, 'God created the natural numbers, all the other numbers were made by man'. Still, as we shall see, some people have tried to construct the natural numbers also. This has led one wit to suggest that 'man created the natural numbers, all the others were Dieudonné'.

We should note that there are other kinds of numbers that do not appear on our list, such as the transfinite numbers and the infinitesimals.

What are the natural numbers? In particular, what is the number 2? We know it is not just the sign or numeral '2' (or 'II'). It is not something perishable or changing. But what is it? There is more than one answer.

The *Platonists* hold that numbers are abstract, necessarily existing objects . The number two is that Platonic 'form' or 'idea' in virtue of which things have the property of two-ness.

For the *logicists*, numbers are things which can be defined in terms of logic. For example, for Bertrand Russell, 2 is

$$\{x \mid \exists_y \exists_z (y \neq z \wedge x = \{y,z\})\}.$$

In other words, 2 is the set of all unordered pairs. On the other hand, von Neumann claimed that 2 is the set $\{0,1\}$, where $1=\{0\}$, and 0 is just the empty set. Pursuing yet another approach, Church held that 2 is the process of iteration

$$\lambda_f(f \circ f),$$

more precisely, the mapping which assigns to every function f its iterate $f \circ f$, defined by $(f \circ f)(x) = f(f(x))$.

For the *formalists*, 2 is just a class of expressions manipulated according to certain rules. Often they do not define numbers, but rather give axioms that characterize them. For example, Peano sees 2 as $SS0$, where '$SS0$' is a string of symbols which are manipulated according to certain axioms (see below).

The *intuitionists* hold that numbers are mental entities which would not exist unless people thought about them. For Brouwer, 2 is the concept which expresses the principle of 'two-ity'.

We see that each of these schools has a different view of the matter. What is the true answer? Fortunately, Professor Tournesol was able to discover this when attending a conference in France. The number 2 is a pair of platinum balls kept at room temperature in the second drawer at the Bureau of Standards in Paris.

2

Natural Numbers (Peano's Approach)

For Peano, the natural number system is a triple $(N, 0, S)$, where N is a set, 0 is an element of that set, and S is a function whose domain and codomain are that set, such that the following axioms hold for all elements x and y of $\mathbf{N}$:

1. $Sx \neq 0$,
2. $Sx = Sy \Rightarrow x = y$,
3. when $\phi(x)$ is any proposition involving the natural number x, if
   (a) $\phi(0)$,
   (b) $\forall_{x \in N} \phi(x) \Rightarrow \phi(Sx)$,

   then $\phi(x)$ is true for any natural number x.

We define 1 as $S0$ and 2 as $SS0 = S1$, etc. (Normally, we leave out the parentheses.)

Note that (3) above is not a single axiom, but a whole 'scheme' or class of axioms, one for each ϕ. This scheme is called *mathematical induction.*

In Peano's system we define addition as follows:

$$\begin{aligned} x + 0 &= x, \\ x + Sy &= S(x + y). \end{aligned}$$

Such definitions are called *recursive* definitions. We add 3 and 2 thus: $3 + 2 = 3 + S1 = S(3 + 1) = S(3 + S0) = S(S(3 + 0)) = SS(SSS0) = 5$.

Multiplication is defined recursively as follows:

$$\begin{aligned} x \cdot 0 &= 0, \\ x \cdot Sy &= (x \cdot y) + x. \end{aligned}$$

We sometimes write $(x \cdot y) + x$ as $xy + x$. Assuming that we already know how to add, we can now mutiply. Thus $3 \cdot 2 = 3 \cdot S1 = (3 \cdot 1) + 3 = (3 \cdot S0) + 3 = ((3 \cdot 0) + 3) + 3 = (0 + 3) + 3 = 3 + 3 = 6$.

Exponentiation is defined recursively as follows:

$$\begin{aligned} x^0 &= 1, \\ x^{Sy} &= (x^y) \cdot x. \end{aligned}$$

Here even $0^0 = 1$. The reader who has already learned how to multiply will now have no difficulty in calculating 3^2. All the usual laws of arithmetic follow from the above axioms and definitions; for example:

commutativity: $x + y = y + x, \quad xy = yx$;

associativity: $x + (y + z) = (x + y) + z, \quad x(yz) = (xy)z$;

distributivity: $x(y + z) = xy + xz, \quad (x + y)z = xz + yz$;

laws of indices: $(x^y)^z = x^{zy}, \quad x^y x^z = x^{y+z}, \quad x^z y^z = (xy)^z$.

We give some examples of how the above laws (and others) follow from Peano's axioms and definitions.

Lemma 2.1. $0 + x = x$.

Proof: We prove this using mathematical induction. First we show that it is true when $x = 0$: $0 + 0 = 0$ by the definition of addition (df+). Second we assume it holds for x and prove it for Sx. Our assumption that it is true for x is called the *induction hypothesis* (hyp). We have

$$\begin{aligned} 0 + Sx &= S(0 + x) && \text{(df+)} \\ &= S(x + 0) && \text{(hyp)} \\ &= Sx && \text{(df+).} \end{aligned}$$

Since the assumption that the result holds for x implies that it holds for Sx , and since it is true for 0, we may conclude, by mathematical induction, that it is true for all natural numbers.

Lemma 2.2. $Sx + y = S(x + y)$.

Proof: Use induction on y. The result is true when $y = 0$: $Sx + 0 = Sx$ (df +) and $Sx = S(x + 0)$ (df +). If the result holds for y,

$$\begin{aligned} Sx + Sy &= S(Sx + y) && \text{(df+)} \\ &= SS(x + y) && \text{(hyp)} \\ &= S(x + Sy) && \text{(df+).} \end{aligned}$$