i-th convergent, and we prove the claim for the $(i+1)$-th convergent. Note that we obtain the $(i+1)$-th convergent by replacing a_i by $a_i + 1/a_{i+1}$ in the formula that expresses the numerator and denominator of the i-th convergent in terms of the $(i-1)$-th and $(i-2)$-th. That is, the $(i+1)$-th convergent is

$$\frac{(a_i + \frac{1}{a_{i+1}})b_{i-1} + b_{i-2}}{(a_i + \frac{1}{a_{i+1}})c_{i-1} + c_{i-2}} = \frac{a_{i+1}(a_i b_{i-1} + b_{i-2}) + b_{i-1}}{a_{i+1}(a_i c_{i-1} + c_{i-2}) + c_{i-1}} = \frac{a_{i+1}b_i + b_{i-1}}{a_{i+1}c_i + c_{i-1}},$$

by the induction assumption. This completes the induction, and proves part (a).

Part (c) is also easy to prove by induction. The induction step goes as follows:

$$\begin{aligned} b_{i+1}c_i - b_i c_{i+1} &= (a_{i+1}b_i + b_{i-1})c_i - b_i(a_{i+1}c_i + c_{i-1}) = b_{i-1}c_i - b_i c_{i-1} \\ &= -(-1)^{i-1} = (-1)^i, \end{aligned}$$

so part (c) for i implies part (c) for $i+1$. Finally, part (b) follows from part (c), because any common divisor of b_i and c_i must divide $(-1)^{i-1}$, which is ± 1. This proves the proposition.

If we divide the equation in Proposition V.4.1(c) by $c_i c_{i-1}$, we find that

$$\frac{b_i}{c_i} - \frac{b_{i-1}}{c_{i-1}} = \frac{(-1)^{i-1}}{c_i c_{i-1}}.$$

Since the c_i clearly form a strictly increasing sequence of positive integers, this equality shows that the sequence of convergents behaves like an alternating series, i.e., it oscillates back and forth with shrinking amplitude; thus, the sequence of convergents converges to a limit.

Finally, it is not hard to see that the limit of the convergents is the number x which was expanded in the first place. To see that, notice that x can be obtained by forming the $(i+1)$-th convergent with a_{i+1} replaced by $1/x_i$. Thus, by Proposition V.4.1(a) (with i replaced by $i+1$ and a_{i+1} replaced by $1/x_i$), we have

$$x = \frac{b_i/x_i + b_{i-1}}{c_i/x_i + c_{i-1}} = \frac{b_i + x_i b_{i-1}}{c_i + x_i c_{i-1}},$$

and this is strictly between b_{i-1}/c_{i-1} and b_i/c_i. (To see this, consider the two vectors $\mathbf{u} = (b_i, c_i)$ and $\mathbf{v} = (b_{i-1}, c_{i-1})$ in the plane, both in the same quadrant; note that the slope of the vector $\mathbf{u} + x_i\mathbf{v}$ is intermediate between the slopes of $\mathbf{u}$ and $\mathbf{v}$.) Thus, the sequence b_i/c_i oscillates around x and converges to x.

Continued fractions have many special properties that cause them to come up in several different branches of mathematics. For example, they provide a way of generating "best possible" rational approximations to real numbers (in the sense that any rational number that is closer to x than b_i/c_i