Exercise 16.1.2. Prove Lemma 16.1.5. (Hint: for (a), first show that f is bounded on $[0, 1]$.)

Exercise 16.1.3. Show that $C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$ with the sup-norm metric d_∞ is a metric space. Furthermore, show that this metric space is complete.

16.2 Inner products on periodic functions

From Lemma 16.1.5 we know that we can add, subtract, multiply, and take limits of continuous periodic functions. We will need a couple more operations on the space $C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$, though. The first one is that of *inner product.*

Definition 16.2.1 (Inner product). If $f, g \in C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$, we define the *inner product* $\langle f, g\rangle$ to be the quantity

$$\langle f, g\rangle = \int_{[0,1]} f(x)\overline{g(x)}\, dx.$$

Remark 16.2.2. In order to integrate a complex-valued function, $f(x) = g(x) + ih(x)$, we use the definition that $\int_{[a,b]} f := \int_{[a,b]} g + i \int_{[a,b]} h$; i.e., we integrate the real and imaginary parts of the function separately. Thus for instance $\int_{[1,2]}(1 + ix)\, dx = \int_{[1,2]} 1\, dx + i \int_{[1,2]} x\, dx = 1 + \frac{3}{2}i$. It is easy to verify that all the standard rules of calculus (integration by parts, fundamental theorem of calculus, substitution, etc.) still hold when the functions are complex-valued instead of real-valued.

Example 16.2.3. Let f be the constant function $f(x) := 1$, and let $g(x)$ be the function $g(x) := e^{2\pi i x}$. Then we have

$$\begin{aligned}
\langle f, g \rangle &= \int_{[0,1]} 1\overline{e^{2\pi i x}}\, dx \\
&= \int_{[0,1]} e^{-2\pi i x}\, dx \\
&= \frac{e^{-2\pi i x}}{-2\pi i}\Big|_{x=0}^{x=1} \\
&= \frac{e^{-2\pi i} - e^{0}}{-2\pi i} \\
&= \frac{1-1}{-2\pi i} \\
&= 0.
\end{aligned}$$

Remark 16.2.4. In general, the inner product $\langle f, g \rangle$ will be a complex number. (Note that $f(x)\overline{g(x)}$ will be Riemann integrable since both functions are bounded and continuous.)

Roughly speaking, the inner product $\langle f, g \rangle$ is to the space $C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$ what the dot product $x \cdot y$ is to Euclidean spaces such as $\mathbf{R}^n$. We list some basic properties of the inner product below; a more in-depth study of inner products on vector spaces can be found in any linear algebra text but is beyond the scope of this text.

Lemma 16.2.5. *Let $f, g, h \in C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$.*

(a) *(Hermitian property) We have $\langle g, f \rangle = \overline{\langle f, g \rangle}$.*

(b) *(Positivity) We have $\langle f, f \rangle \geq 0$. Furthermore, we have $\langle f, f \rangle = 0$ if and only if $f = 0$ (i.e., $f(x) = 0$ for all $x \in \mathbf{R}$).*

(c) *(Linearity in the first variable) We have $\langle f+g, h \rangle = \langle f, h \rangle + \langle g, h \rangle$. For any complex number c, we have $\langle cf, g \rangle = c\langle f, g \rangle$.*

(d) *(Antilinearity in the second variable) We have $\langle f, g+h \rangle = \langle f, g \rangle + \langle f, h \rangle$. For any complex number c, we have $\langle f, cg \rangle = \overline{c}\langle f, g \rangle$.*

Proof. See Exercise 16.2.1. □

From the positivity property, it makes sense to define the L^2 *norm* $\|f\|_2$ of a function $f \in C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$ by the formula

$$\|f\|_2 := \sqrt{\langle f, f\rangle} = \left(\int_{[0,1]} f(x)\overline{f(x)}\, dx\right)^{1/2} = \left(\int_{[0,1]} |f(x)|^2\, dx\right)^{1/2}.$$

Thus $\|f\|_2 \geq 0$ for all f. The norm $\|f\|_2$ is sometimes called the *root mean square* of f.

Example 16.2.6. If $f(x)$ is the function $e^{2\pi i x}$, then

$$\|f\|_2 = \left(\int_{[0,1]} e^{2\pi i x} e^{-2\pi i x}\, dx\right)^{1/2} = \left(\int_{[0,1]} 1\, dx\right)^{1/2} = 1^{1/2} = 1.$$

This L^2 norm is related to, but is distinct from, the L^∞ norm $\|f\|_\infty := \sup_{x\in\mathbf{R}} |f(x)|$. For instance, if $f(x) = \sin(x)$, then $\|f\|_\infty = 1$ but $\|f\|_2 = \frac{1}{\sqrt{2}}$. In general, the best one can say is that $0 \leq \|f\|_2 \leq \|f\|_\infty$; see Exercise 16.2.3.

Some basic properties of the L^2 norm are given below.

Lemma 16.2.7. *Let $f, g \in C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$.*

(a) *(Non-degeneracy) We have $\|f\|_2 = 0$ if and only if $f = 0$.*

(b) *(Cauchy-Schwarz inequality) We have $|\langle f, g\rangle| \leq \|f\|_2\|g\|_2$.*

(c) *(Triangle inequality) We have $\|f + g\|_2 \leq \|f\|_2 + \|g\|_2$.*

(d) *(Pythagoras' theorem) If $\langle f, g\rangle = 0$, then $\|f+g\|_2^2 = \|f\|_2^2 + \|g\|_2^2$.*

(e) *(Homogeneity) We have $\|cf\|_2 = |c|\|f\|_2$ for all $c \in \mathbf{C}$.*

Proof. See Exercise 16.2.4. □

In light of Pythagoras' theorem, we sometimes say that f and g are *orthogonal* iff $\langle f, g\rangle = 0$.

We can now define the L^2 *metric* d_{L^2} on $C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$ by defining

$$d_{L^2}(f, g) := \|f - g\|_2 = \left(\int_{[0,1]} |f(x) - g(x)|^2\, dx\right)^{1/2}.$$

Remark 16.2.8. One can verify that d_{L^2} is indeed a metric (Exercise 16.2.2). Indeed, the L^2 metric is very similar to the l^2 metric on Euclidean spaces $\mathbf{R}^n$, which is why the notation is deliberately chosen to be similar; you should compare the two metrics yourself to see the analogy.

Note that a sequence f_n of functions in $C(\mathbf{R}/\mathbf{Z};\mathbf{C})$ will *converge in the L^2 metric* to $f \in C(\mathbf{R}/\mathbf{Z};\mathbf{C})$ if $d_{L^2}(f_n, f) \to 0$ as $n \to \infty$, or in other words that

$$\lim_{n\to\infty} \int_{[0,1]} |f_n(x) - f(x)|^2\, dx = 0.$$

Remark 16.2.9. The notion of convergence in L^2 metric is different from that of uniform or pointwise convergence; see Exercise 16.2.6.

Remark 16.2.10. The L^2 metric is not as well-behaved as the L^∞ metric. For instance, it turns out the space $C(\mathbf{R}/\mathbf{Z};\mathbf{C})$ is not complete in the L^2 metric, despite being complete in the L^∞ metric; see Exercise 16.2.5.

Exercise 16.2.1. Prove Lemma 16.2.5. (Hint: the last part of (b) is a little tricky. You may need to prove by contradiction, assuming that f is not the zero function, and then show that $\int_{[0,1]} |f(x)|^2$ is strictly positive. You will need to use the fact that f, and hence $|f|$, is continuous, to do this.)

Exercise 16.2.2. Prove that the L^2 metric d_{L^2} on $C(\mathbf{R}/\mathbf{Z};\mathbf{C})$ does indeed turn $C(\mathbf{R}/\mathbf{Z};\mathbf{C})$ into a metric space. (cf. Exercise 12.1.6).

Exercise 16.2.3. If $f \in C(\mathbf{R}/\mathbf{Z};\mathbf{C})$ is a non-zero function, show that $0 < \|f\|_2 \le \|f\|_{L^\infty}$. Conversely, if $0 < A \le B$ are real numbers, so that there exists a non-zero function $f \in C(\mathbf{R}/\mathbf{Z};\mathbf{C})$ such that $\|f\|_2 = A$ and $\|f\|_\infty = B$. (Hint: let g be a non-constant non-negative real-valued function in $C(\mathbf{R}/\mathbf{Z};\mathbf{C})$, and consider functions f of the form $f = (c + dg)^{1/2}$ for some constant real numbers $c, d > 0$.)

Exercise 16.2.4. Prove Lemma 16.2.7. (Hint: use Lemma 16.2.5 frequently. For the Cauchy-Schwarz inequality, begin with the positivity property $\langle f, f\rangle \geq 0$, but with f replaced by the function $f\|g\|_2^2 - \langle f, g\rangle g$, and then simplify using Lemma 16.2.5. You may have to treat the case $\|g\|_2 = 0$ separately. Use the Cauchy-Schwarz inequality to prove the triangle inequality.)

Exercise 16.2.5. Find a sequence of continuous periodic functions which converge in L^2 to a discontinuous periodic function. (Hint: try converging to the square wave function.)

Exercise 16.2.6. Let $f \in C(\mathbf{R}/\mathbf{Z}, \mathbf{C})$, and let $(f_n)_{n=1}^\infty$ be a sequence of functions in $C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$.

(a) Show that if f_n converges uniformly to f, then f_n also converges to f in the L^2 metric.

(b) Give an example where f_n converges to f in the L^2 metric, but does not converge to f uniformly. (Hint: take $f = 0$. Try to make the functions f_n large in sup norm.)

(c) Give an example where f_n converges to f in the L^2 metric, but does not converge to f pointwise. (Hint: take $f = 0$. Try to make the functions f_n large at one point.)

(d) Give an example where f_n converges to f pointwise, but does not converge to f in the L^2 metric. (Hint: take $f = 0$. Try to make the functions f_n large in L^2 norm.)

16.3 Trigonometric polynomials

We now define the concept of a *trigonometric polynomial*. Just as polynomials are combinations of the functions x^n (sometimes called *monomials*), trigonometric polynomials are combinations of the functions $e^{2\pi inx}$ (sometimes called *characters*).

Definition 16.3.1 (Characters). For every integer n, we let $e_n \in C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$ denote the function

$$e_n(x) := e^{2\pi inx}.$$

This is sometimes referred to as the *character with frequency n*.

Definition 16.3.2 (Trigonometric polynomials). A function f in $C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$ is said to be a *trigonometric polynomial* if we can write $f = \sum_{n=-N}^{N} c_n e_n$ for some integer $N \geq 0$ and some complex numbers $(c_n)_{n=-N}^{N}$.

Example 16.3.3. The function $f = 4e_{-2} + ie_{-1} - 2e_0 + 0e_1 - 3e_2$ is a trigonometric polynomial; it can be written more explicitly as

$$f(x) = 4e^{-4\pi ix} + ie^{-2\pi ix} - 2 - 3e^{4\pi ix}.$$

Example 16.3.4. For any integer n, the function $\cos(2\pi nx)$ is a trigonometric polynomial, since

$$\cos(2\pi nx) = \frac{e^{2\pi inx} + e^{-2\pi inx}}{2} = \frac{1}{2}e_{-n} + \frac{1}{2}e_n.$$

Similarly the function $\sin(2\pi nx) = \frac{-1}{2i}e_{-n} + \frac{1}{2i}e_n$ is a trigonometric polynomial. In fact, any linear combination of sines and cosines is also a trigonometric polynomial, for instance $3 + i\cos(2\pi x) + 4i\sin(4\pi x)$ is a trigonometric polynomial.

The Fourier theorem will allow us to write any function in $C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$ as a Fourier series, which is to trigonometric polynomials what power series is to polynomials. To do this we will use the inner product structure from the previous section. The key computation is

Lemma 16.3.5 (Characters are an orthonormal system). *For any integers n and m, we have $\langle e_n, e_m \rangle = 1$ when $n = m$ and $\langle e_n, e_m \rangle = 0$ when $n \neq m$. Also, we have $\|e_n\| = 1$.*

Proof. See Exercise 16.3.2. □

As a consequence, we have a formula for the co-efficients of a trigonometric polynomial.

Corollary 16.3.6. *Let $f = \sum_{n=-N}^{N} c_n e_n$ be a trigonometric polynomial. Then we have the formula*

$$c_n = \langle f, e_n \rangle$$

for all integers $-N \leq n \leq N$. Also, we have $0 = \langle f, e_n \rangle$ whenever $n > N$ or $n < -N$. Also, we have the identity

$$\|f\|_2^2 = \sum_{n=-N}^{N} |c_n|^2.$$

Proof. See Exercise 16.3.3. □

We rewrite the conclusion of this corollary in a different way.

Definition 16.3.7 (Fourier transform). For any function $f \in C(\mathbf{R}/\mathbf{Z};\mathbf{R})$, and any integer $n \in \mathbf{Z}$, we define the n^{th} *Fourier coefficient of* f, denoted $\hat{f}(n)$, by the formula

$$\hat{f}(n) := \langle f, e_n \rangle = \int_{[0,1]} f(x)e^{-2\pi i n x}\, dx.$$

The function $\hat{f} : \mathbf{Z} \to \mathbf{C}$ is called the *Fourier transform* of f.

From Corollary 16.3.6, we see that whenever $f = \sum_{n=-N}^{N} c_n e_n$ is a trigonometric polynomial, we have

$$f = \sum_{n=-N}^{N} \langle f, e_n \rangle e_n = \sum_{n=-\infty}^{\infty} \langle f, e_n \rangle e_n$$

and in particular we have the *Fourier inversion formula*

$$f = \sum_{n=-\infty}^{\infty} \hat{f}(n) e_n$$

or in other words

$$f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x}.$$

The right-hand side is referred to as the *Fourier series* of f. Also, from the second identity of Corollary 16.3.6 we have the *Plancherel formula*

$$\|f\|_2^2 = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2.$$

Remark 16.3.8. We stress that at present we have only proven the Fourier inversion and Plancherel formulae in the case when f is a trigonometric polynomial. Note that in this case that the Fourier coefficients $\hat{f}(n)$ are mostly zero (indeed, they can only be non-zero when $-N \leq n \leq N$), and so this infinite sum is really just a finite sum in disguise. In particular there are no issues about what sense the above series converge in; they both converge pointwise, uniformly, and in L^2 metric, since they are just finite sums.

In the next few sections we will extend the Fourier inversion and Plancherel formulae to general functions in $C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$, not just trigonometric polynomials. (It is also possible to extend the formula to discontinuous functions such as the square wave, but we will not do so here). To do this we will need a version of the Weierstrass approximation theorem, this time requiring that a continuous periodic function be approximated uniformly by *trigonometric* polynomials. Just as convolutions were used in the proof of the polynomial Weierstrass approximation theorem, we will also need a notion of convolution tailored for periodic functions.

Exercise 16.3.1. Show that the sum or product of any two trigonometric polynomials is again a trigonometric polynomial.

Exercise 16.3.2. Prove Lemma 16.3.5.

Exercise 16.3.3. Prove Corollary 16.3.6. (Hint: use Lemma 16.3.5. For the second identity, either use Pythagoras' theorem and induction, or substitute $f = \sum_{n=-N}^{N} c_n e_n$ and expand everything out.)

16.4 Periodic convolutions

The goal of this section is to prove the Weierstrass approximation theorem for trigonometric polynomials:

Theorem 16.4.1. *Let $f \in C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$, and let $\varepsilon > 0$. Then there exists a trignometric polynomial P such that $\|f - P\|_\infty \leq \varepsilon$.*

This theorem asserts that any continuous periodic function can be uniformly approximated by trigonometric polynomials. To put it another way, if we let $P(\mathbf{R}/\mathbf{Z}; \mathbf{C})$ denote the space of all trigonometric polynomials, then the closure of $P(\mathbf{R}/\mathbf{Z}; \mathbf{C})$ in the L^∞ metric is $C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$.

It is possible to prove this theorem directly from the Weierstrass approximation theorem for polynomials (Theorem 14.8.3), and both theorems are a special case of a much more general theorem known as the *Stone-Weierstrass theorem*, which we will not discuss here. However we shall instead prove this theorem from scratch, in order to introduce a couple of interesting notions, notably that of periodic convolution. The proof here, though, should strongly remind you of the arguments used to prove Theorem 14.8.3.

Definition 16.4.2 (Periodic convolution). Let $f, g \in C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$. Then we define the *periodic convolution* $f * g : \mathbf{R} \to \mathbf{C}$ of f and g by the formula

$$f * g(x) := \int_{[0,1]} f(y)g(x-y)\, dy.$$

Remark 16.4.3. Note that this formula is slightly different from the convolution for compactly supported functions defined in Definition 14.8.9, because we are only integrating over $[0, 1]$ and not on all of $\mathbf{R}$. Thus, in principle we have given the symbol $f * g$ two conflicting meanings. However, in practice there will be no confusion, because it is not possible for a non-zero function to both be periodic and compactly supported (Exercise 16.4.1).

Lemma 16.4.4 (Basic properties of periodic convolution). *Let $f, g, h \in C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$.*

(a) *(Closure) The convolution $f*g$ is continuous and $\mathbf{Z}$-periodic. In other words, $f * g \in C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$.*

(b) *(Commutativity) We have $f * g = g * f$.*

(c) *(Bilinearity) We have $f * (g + h) = f * g + f * h$ and $(f + g) * h = f * h + g * h$. For any complex number c, we have $c(f * g) = (cf) * g = f * (cg)$.*

Proof. See Exercise 16.4.2. □

Now we observe an interesting identity: for any $f \in C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$ and any integer n, we have

$$f * e_n = \hat{f}(n) e_n.$$

To prove this, we compute

$$f * e_n(x) = \int_{[0,1]} f(y) e^{2\pi i n(x-y)}\, dy$$

$$= e^{2\pi i n x} \int_{[0,1]} f(y) e^{-2\pi i n y}\, dy = \hat{f}(n) e^{2\pi i n x} = \hat{f}(n) e_n$$

as desired.

More generally, we see from Lemma 16.4.4(iii) that for any trigonometric polynomial $P = \sum_{n=-N}^{n=N} c_n e_n$, we have

$$f * P = \sum_{n=-N}^{n=N} c_n (f * e_n) = \sum_{n=-N}^{n=N} \hat{f}(n) c_n e_n.$$

Thus the periodic convolution of any function in $C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$ with a trigonometric polynomial, is again a trigonometric polynomial. (Compare with Lemma 14.8.13.)

Next, we introduce the periodic analogue of an approximation to the identity.

Definition 16.4.5 (Periodic approximation to the identity). Let $\varepsilon > 0$ and $0 < \delta < 1/2$. A function $f \in C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$ is said to be a *periodic* (ε, δ) *approximation to the identity* if the following properties are true:

(a) $f(x) \geq 0$ for all $x \in \mathbf{R}$, and $\int_{[0,1]} f = 1$.

(b) We have $f(x) < \varepsilon$ for all $\delta \leq |x| \leq 1 - \delta$.

Now we have an analogue of Lemma 14.8.8:

Lemma 16.4.6. *For every $\varepsilon > 0$ and $0 < \delta < 1/2$, there exists a trigonometric polynomial P which is an (ε, δ) approximation to the identity.*

Proof. We sketch the proof of this lemma here, and leave the completion of it to Exercise 16.4.3. Let $N \geq 1$ be an integer. We define the *Fejér kernel* F_N to be the function

$$F_N = \sum_{n=-N}^{N} (1 - \frac{|n|}{N}) e_n.$$

Clearly F_N is a trigonometric polynomial. We observe the identity

$$F_N = \frac{1}{N} |\sum_{n=0}^{N-1} e_n|^2$$

(why?). But from the geometric series formula (Lemma 7.3.3) we have

$$\sum_{n=0}^{N-1} e_n(x) = \frac{e_N - e_0}{e_1 - e_0} = \frac{e^{\pi i (N-1)x} \sin(\pi N x)}{\sin(\pi x)}$$

when x is not an integer, (why?) and hence we have the formula

$$F_N(x) = \frac{\sin(\pi N x)^2}{N \sin(\pi x)^2}.$$

When x is an integer, the geometric series formula does not apply, but one has $F_N(x) = N$ in that case, as one can see by direct computation. In either case we see that $F_N(x) \geq 0$ for any x. Also, we have

$$\int_{[0,1]} F_N(x)\, dx = \sum_{n=-N}^{N} (1 - \frac{|n|}{N}) \int_{[0,1]} e_n = (1 - \frac{|0|}{N})1 = 1$$

(why?). Finally, since $\sin(\pi N x) \leq 1$, we have

$$F_N(x) \leq \frac{1}{N \sin(\pi x)^2} \leq \frac{1}{N \sin(\pi \delta)^2}$$

whenever $\delta < |x| < 1 - \delta$ (this is because sin is increasing on $[0, \pi/2]$ and decreasing on $[\pi/2, \pi]$). Thus by choosing N large enough, we can make $F_N(x) \leq \varepsilon$ for all $\delta < |x| < 1 - \delta$. $\square$

Proof of Theorem 16.4.1. Let f be any element of $C(\mathbf{R}/\mathbf{Z};\mathbf{C})$; we know that f is bounded, so that we have some $M > 0$ such that $|f(x)| \leq M$ for all $x \in \mathbf{R}$.

Let $\varepsilon > 0$ be arbitrary. Since f is uniformly continuous, there exists a $\delta > 0$ such that $|f(x) - f(y)| \leq \varepsilon$ whenever $|x - y| \leq \delta$. Now use Lemma 16.4.6 to find a trigonometric polynomial P which is a (ε, δ) approximation to the identity. Then $f * P$ is also a trigonometric polynomial. We now estimate $\|f - f * P\|_\infty$.

Let x be any real number. We have

$$\begin{aligned}
|f(x) - f * P(x)| &= |f(x) - P * f(x)| \\
&= |f(x) - \int_{[0,1]} f(x-y)P(y)\, dy| \\
&= |\int_{[0,1]} f(x)P(y)\, dy - \int_{[0,1]} f(x-y)P(y)\, dy| \\
&= |\int_{[0,1]} (f(x) - f(x-y))P(y)\, dy| \\
&\leq \int_{[0,1]} |f(x) - f(x-y)|P(y)\, dy.
\end{aligned}$$

The right-hand side can be split as

$$\begin{aligned}
\int_{[0,\delta]} |f(x) - f(x-y)|P(y)\, dy + \int_{[\delta,1-\delta]} |f(x) - f(x-y)|P(y)\, dy \\
+ \int_{[1-\delta,1]} |f(x) - f(x-y)|P(y)\, dy
\end{aligned}$$

which we can bound from above by

$$\begin{aligned}
&\leq \int_{[0,\delta]} \varepsilon P(y)\, dy + \int_{[\delta,1-\delta]} 2M\varepsilon\, dy \\
&\quad + \int_{[1-\delta,1]} |f(x-1) - f(x-y)|P(y)\, dy \\
&\leq \int_{[0,\delta]} \varepsilon P(y)\, dy + \int_{[\delta,1-\delta]} 2M\varepsilon\, dy + \int_{[1-\delta,1]} \varepsilon P(y)\, dy \\
&\leq \varepsilon + 2M\varepsilon + \varepsilon \\
&= (2M + 2)\varepsilon.
\end{aligned}$$

Thus we have $\|f - f * P\|_\infty \leq (2M + 2)\varepsilon$. Since M is fixed and ε is arbitrary, we can thus make $f * P$ arbitrarily close to f in sup norm, which proves the periodic Weierstrass approximation theorem. □

Exercise 16.4.1. Show that if $f : \mathbf{R} \to \mathbf{C}$ is both compactly supported and $\mathbf{Z}$-periodic, then it is identically zero.

Exercise 16.4.2. Prove Lemma 16.4.4. (Hint: to prove that $f * g$ is continuous, you will have to do something like use the fact that f is bounded, and g is uniformly continuous, or vice versa. To prove that $f * g = g * f$, you will need to use the periodicity to "cut and paste" the interval $[0, 1]$.)

Exercise 16.4.3. Fill in the gaps marked (why?) in Lemma 16.4.6. (Hint: for the first identity, use the identities $|z|^2 = z\overline{z}$, $\overline{e_n} = e_{-n}$, and $e_n e_m = e_{n+m}$.)

16.5 The Fourier and Plancherel theorems

Using the Weierstrass approximation theorem (Theorem 16.4.1), we can now generalize the Fourier and Plancherel identities to arbitrary continuous periodic functions.

Theorem 16.5.1 (Fourier theorem). *For any $f \in C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$, the series $\sum_{n=-\infty}^{\infty} \hat{f}(n) e_n$ converges in L^2 metric to f. In other words, we have*

$$\lim_{N \to \infty} \|f - \sum_{n=-N}^{N} \hat{f}(n) e_n\|_2 = 0.$$

Proof. Let $\varepsilon > 0$. We have to show that there exists an N_0 such that $\|f - \sum_{n=-N}^{N} \hat{f}(n) e_n\|_2 \leq \varepsilon$ for all sufficiently large N.

By the Weierstrass approximation theorem (Theorem 16.4.1), we can find a trigonometric polynomial $P = \sum_{n=-N_0}^{N_0} c_n e_n$ such that $\|f - P\|_\infty \leq \varepsilon$, for some $N_0 > 0$. In particular we have $\|f - P\|_2 \leq \varepsilon$.

Now let $N > N_0$, and let $F_N := \sum_{n=-N}^{n=N} \hat{f}(n)e_n$. We claim that $\|f - F_N\|_2 \leq \varepsilon$. First observe that for any $|m| \leq N$, we have

$$\langle f - F_N, e_m \rangle = \langle f, e_m \rangle - \sum_{n=-N}^{N} \hat{f}(n)\langle e_n, e_m \rangle = \hat{f}(m) - \hat{f}(m) = 0,$$

where we have used Lemma 16.3.5. In particular we have

$$\langle f - F_N, F_N - P \rangle = 0$$

since we can write $F_N - P$ as a linear combination of the e_m for which $|m| \leq N$. By Pythagoras' theorem we therefore have

$$\|f - P\|_2^2 = \|f - F_N\|_2^2 + \|F_N - P\|_2^2$$

and in particular

$$\|f - F_N\|_2 \leq \|f - P\|_2 \leq \varepsilon$$

as desired. □

Remark 16.5.2. Note that we have only obtained convergence of the Fourier series $\sum_{n=-\infty}^{\infty} \hat{f}(n)e_n$ to f in the L^2 metric. One may ask whether one has convergence in the uniform or pointwise sense as well, but it turns out (perhaps somewhat surprisingly) that the answer is no to both of those questions. However, if one assumes that the function f is not only continuous, but is also continuously differentiable, then one can recover pointwise convergence; if one assumes continuously twice differentiable, then one gets uniform convergence as well. These results are beyond the scope of this text and will not be proven here. However, we will prove one theorem about when one can improve the L^2 convergence to uniform convergence:

Theorem 16.5.3. *Let $f \in C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$, and suppose that the series $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|$ is absolutely convergent. Then the series $\sum_{n=-\infty}^{\infty} \hat{f}(n)e_n$ converges uniformly to f. In other words, we have*

$$\lim_{N\to\infty} \|f - \sum_{n=-N}^{N} \hat{f}(n)e_n\|_\infty = 0.$$

Proof. By the Weierstrass M-test (Theorem 14.5.7), we see that $\sum_{n=-\infty}^{\infty} \hat{f}(n)e_n$ converges to *some* function F, which by Lemma 16.1.5(iii) is also continuous and $\mathbf{Z}$-periodic. (Strictly speaking, the Weierstrass M test was phrased for series from $n = 1$ to $n = \infty$, but also works for series from $n = -\infty$ to $n = +\infty$; this can be seen by splitting the doubly infinite series into two pieces.) Thus

$$\lim_{N\to\infty} \|F - \sum_{n=-N}^{N} \hat{f}(n)e_n\|_\infty = 0$$

which implies that

$$\lim_{N\to\infty} \|F - \sum_{n=-N}^{N} \hat{f}(n)e_n\|_2 = 0$$

since the L^2 norm is always less than or equal to the L^∞ norm. But the sequence $\sum_{n=-N}^{N} \hat{f}(n)e_n$ is already converging in L^2 metric to f by the Fourier theorem, so can only converge in L^2 metric to F if $F = f$ (cf. Proposition 12.1.20). Thus $F = f$, and so we have

$$\lim_{N\to\infty} \|f - \sum_{n=-N}^{N} \hat{f}(n)e_n\|_\infty = 0$$

as desired. □

As a corollary of the Fourier theorem, we obtain

Theorem 16.5.4 (Plancherel theorem). *For any $f \in C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$, the series $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2$ is absolutely convergent, and*

$$\|f\|_2^2 = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2.$$

This theorem is also known as *Parseval's theorem*.