# 7

# Pythagoras and His School

Pythagoras (570–500 BC) was born in Samos, a Greek island off the coast of what is now Turkey. According to ancient sources (Iamblichus, Porphyry and Diogenes Laërtius), he traveled and studied in the Persian empire, which extended then from northern Greece to the Indus Valley and included ancient Mesopotamia. We know (Plimpton 322) that the Babylonians understood what is now called the 'theorem of Pythagoras', although the latter may have given the first proof. Pythagoras may have learned the theory of 'Pythagorean triangles' from the Babylonians.

According to the above mentioned sources, Pythagoras also studied under the Zoroastrian priests, the so-called 'Magi'. However, judging from his belief in reincarnation and his vegetarianism, it is more likely that he was influenced by Hindu tradition. Even his mathematics has an Indian flavour.

About 525 BC, Pythagoras emigrated to Croton (modern Crotone) in southern Italy, where he founded a society, half-way between a political party and a religious cult, which came to be known as the 'Pythagorean Brotherhood.' Some members of this society were admitted to an inner circle consisting of the so-called 'mathematicians'.

The word 'mathematics' was in fact introduced by Pythagoras. The first part of this word is an old Indo-European root, related to the English word 'mind'. The modern meaning of 'mathematics' is due to Aristotle.

Whereas Thales had claimed that 'all is water', Pythagoras taught that 'all is number'. For Pythagoras this implied that everything could be understood in terms of whole numbers and their ratios. In particular, he implicitly expected that every line segment was a whole number or a ratio of whole numbers (in terms of a given unit length). It seems that the dis-

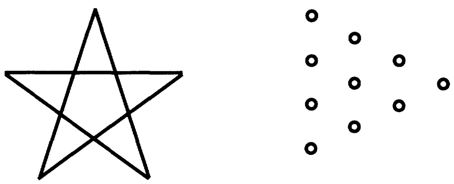

FIGURE 7.1. Pythagorean star and the fourth triangular number

covery of the irrationality of the diagonal of the square of side 1 was made by his followers and that Pythagoras himself was not aware of this.

In his philosophy, Pythagoras reserved a special place for the number 10. He called it the 'divine number', noting that 10 is a triangular number and realizing that the five-pointed 'Pythagorean star' (Figure 7.1) has 10 vertices.

The Pythagoreans ascribed all their mathematical discoveries to Pythagoras, but there is not, in fact, a single theorem which we can safely credit to him. For example, in his preface to the *Introductio Arithmetica*, written by a Pythagorean, Nichomachus of Gerasa (100 AD), Iamblichus (300 AD) credits Pythagoras with a knowledge of the amicable pair 220 and 284. (Two natural numbers are *amicable* if each is the sum of the proper divisors of the other.) However, we have no way of knowing for certain whether amicable numbers had been recognized as early as 500 BC. Yet, according to a famous anecdote, when someone challenged his slogan 'all is number' by asking 'then what is friendship?', Pythagoras replied that friendship is as 220 is to 284.

Leaving behind the shadowy figure of the Master, let us review the accomplishments of his followers. Although they were primarily a religious and political group, they did a fair amount of work in arithmetic, geometry, astronomy and music – the four subjects later forming the medieval *quadrivium*. (In the university curriculum of the Middle Ages, these subjects were meant to follow the 'trivial' subjects: grammar, rhetoric and logic.)

## Theorem of Pythagoras

The Pythagoreans are probably responsible for the proof found in Euclid's *Elements*, Book I, Proposition 47. They also found a proof of the converse of the theorem of Pythagoras.

## Means

They examined the relationships between the following means:
arithmetic ($\frac{1}{2}(a+b)$ ), geometric ($\sqrt{ab}$) and harmonic ($2ab/(a+b)$).

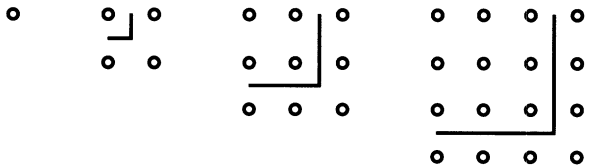

FIGURE 7.2. The sequence of squares

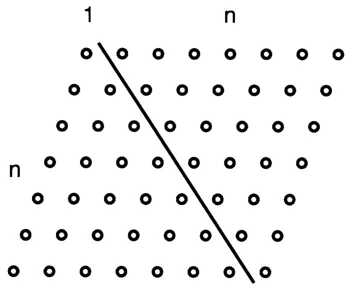


FIGURE 7.3. Triangular numbers

## Perfect Numbers

They found a formula giving perfect numbers. See Chapter 8.

## Regular Solids

They discovered the dodecahedron. See Chapter 9.

## Irrationality of $\sqrt{2}$

They discovered that the square root of 2 is not rational. They used the integer solutions of $x^2 - 2y^2 = \pm 1$ to find approximations to it. See Chapter 10.

## Figurative Numbers

They found proofs for several algebraic relations by means of studying figurative numbers. For example, looking at the sequence of squares, expressed in terms of 'arrays of pebbles' (Figure 7.2), they noticed that $n^2 + (2n+1) = (n+1)^2$ and hence $1 + 3 + 5 + \cdots + (2n-1) = n^2$:

Fitting two 'triangular numbers' into a parallelogram, they noticed that the $n$th triangular number is $\frac{1}{2}n(n+1)$.

Looking at the sequence of triangular numbers, expressed in terms of pebble arrays, they realized that the difference between the $(n+1)$th and $n$th triangular number is just $n+1$. From this they concluded that $1 + 2 + 3 + \cdots + n =$ the $n$th triangular number $= \frac{1}{2}n(n+1)$. See Figure 7.3.

The study of figurative numbers is alive and well today. For example, recently some very advanced mathematics was used to show, for the first