

Corollary 35. Let V be an n -dimensional vector space over the field F . Then $S(V)$ is isomorphic as a graded F -algebra to the ring of polynomials in n variables over F (i.e., the isomorphism is also a vector space isomorphism from $S^k(V)$ onto the space of all homogeneous polynomials of degree k). In particular, $\dim_F(S^k(V)) = \binom{k+n-1}{n-1}$.

Proof: Let $\mathcal{B} = \{v_1, \dots, v_n\}$ be a basis of V . By Proposition 32 there is a bijection between a basis of $\mathcal{T}^k(V)$ and the set \mathcal{B}^k of ordered k -tuples of elements from \mathcal{B} . Define two k -tuples in \mathcal{B}^k to be equivalent if there is some permutation of the entries of one that gives the other — this is easily seen to be an equivalence relation on \mathcal{B}^k . Let $S(\mathcal{B}^k)$ denote the corresponding set of equivalence classes. Any symmetric k -multilinear function from V^k to a vector space over F will be constant on all of the basis tensors whose corresponding k -tuples lie in the same equivalence class; conversely, any function from $S(\mathcal{B}^k)$ can be uniquely extended to a symmetric k -multilinear function on V^k . It follows that the vector space over F with basis $S(\mathcal{B}^k)$ satisfies the universal property of $S^k(V)$ in Theorem 34(2), hence is isomorphic to $S^k(V)$. Each equivalence class has a unique representative of the form $(v_1^{a_1}, v_2^{a_2}, \dots, v_n^{a_n})$, where $v_i^{a_i}$ denotes the sequence v_i, v_i, \dots, v_i taken a_i times, each $a_i \geq 0$, and $a_1 + \dots + a_n = k$. Thus there is a bijection between the basis $S^k(\mathcal{B})$ and the set $x_1^{a_1} \dots x_n^{a_n}$ of monic monomials of degree k in the polynomial ring $F[x_1, \dots, x_n]$. This bijection extends to an isomorphism of graded F -algebras, proving the first part of the corollary. The computation of the dimension of $S^k(V)$ (i.e., the number of monic monomials of degree k) is left as an exercise.

Exterior Algebras

Recall from Section 4 that a multilinear map $\varphi : M \times \dots \times M \rightarrow N$ is called *alternating* if $\varphi(m_1, \dots, m_k) = 0$ whenever $m_i = m_{i+1}$ for some i . (The definition is the same for any R -module as for vector spaces.) We saw that the determinant map was alternating, and was uniquely determined by some additional constraints. We can apply Proposition 33 to construct an algebra through which alternating multilinear maps must factor in a manner similar to the construction of the symmetric algebra (through which symmetric multilinear maps factor).

Definition. The *exterior algebra* of an R -module M is the R -algebra obtained by taking the quotient of the tensor algebra $\mathcal{T}(M)$ by the ideal $\mathcal{A}(M)$ generated by all elements of the form $m \otimes m$, for $m \in M$. The exterior algebra $\mathcal{T}(M)/\mathcal{A}(M)$ is denoted by $\bigwedge(M)$ and the image of $m_1 \otimes m_2 \otimes \dots \otimes m_k$ in $\bigwedge(M)$ is denoted by $m_1 \wedge m_2 \wedge \dots \wedge m_k$.

As with the symmetric algebra, the ideal $\mathcal{A}(M)$ is generated by homogeneous elements hence is a graded ideal. By Proposition 33 the exterior algebra is graded, with k^{th} homogeneous component $\bigwedge^k(M) = \mathcal{T}^k(M)/\mathcal{A}^k(M)$. We can again identify R with $\bigwedge^0(M)$ and M with $\bigwedge^1(M)$ and so consider M as an R -submodule of the R -algebra $\bigwedge(M)$. The R -module $\bigwedge^k(M)$ is called the k^{th} *exterior power* of M .

The multiplication

$$(m_1 \wedge \dots \wedge m_i) \wedge (m'_1 \wedge \dots \wedge m'_j) = m_1 \wedge \dots \wedge m_i \wedge m'_1 \wedge \dots \wedge m'_j$$

in the exterior algebra is called the *wedge* (or *exterior*) *product*. By definition of the quotient, this multiplication is alternating in the sense that the product $m_1 \wedge \cdots \wedge m_k$ is 0 in $\bigwedge(M)$ if $m_i = m_{i+1}$ for any $1 \leq i < k$. Then

$$\begin{aligned} 0 &= (m + m') \wedge (m + m') \\ &= (m \wedge m) + (m \wedge m') + (m' \wedge m) + (m' \wedge m') \\ &= (m \wedge m') + (m' \wedge m) \end{aligned}$$

shows that the multiplication is also anticommutative on simple tensors:

$$m \wedge m' = -m' \wedge m \quad \text{for all } m, m' \in M.$$

This anticommutativity does not extend to arbitrary products, however, i.e., we need not have $ab = -ba$ for all $a, b \in \bigwedge(M)$ (cf. Exercise 4).

Theorem 36. Let M be an R -module over the commutative ring R and let $\bigwedge(M)$ be its exterior algebra.

- (1) The k^{th} exterior power, $\bigwedge^k(M)$, of M is equal to $M \otimes \cdots \otimes M$ (k factors) modulo the submodule generated by all elements of the form

$$m_1 \otimes m_2 \otimes \cdots \otimes m_k \quad \text{where } m_i = m_j \text{ for some } i \neq j.$$

In particular,

$$m_1 \wedge m_2 \wedge \cdots \wedge m_k = 0 \quad \text{if } m_i = m_j \text{ for some } i \neq j.$$

- (2) (*Universal Property for Alternating Multilinear Maps*) If $\varphi : M \times \cdots \times M \rightarrow N$ is an alternating k -multilinear map then there is a unique R -module homomorphism $\Phi : \bigwedge^k(M) \rightarrow N$ such that $\varphi = \Phi \circ \iota$, where

$$\iota : M \times \cdots \times M \rightarrow \bigwedge^k(M)$$

is the map defined by

$$\iota(m_1, \dots, m_k) = m_1 \wedge \cdots \wedge m_k.$$

Remark: The exterior algebra also satisfies a universal property similar to (3) of Theorem 34, namely with respect to R -module homomorphisms from M to R -algebras A satisfying $a^2 = 0$ for all $a \in A$ (cf. Exercise 6).

Proof: The k -tensors $\mathcal{A}^k(M)$ in the ideal $\mathcal{A}(M)$ are finite sums of elements of the form

$$m_1 \otimes \cdots \otimes m_{i-1} \otimes (m \otimes m) \otimes m_{i+2} \otimes \cdots \otimes m_k$$

with $m_1, \dots, m_k, m \in M$ (where $k \geq 2$ and $1 \leq i < k$), which is a k -tensor with two equal entries (in positions i and $i+1$), so is of the form in (1). For the reverse inclusion, note that since

$$\begin{aligned} m' \otimes m &= -m \otimes m' + [(m + m') \otimes (m + m') - m \otimes m - m' \otimes m'] \\ &\equiv -m \otimes m' \pmod{\mathcal{A}(M)}, \end{aligned}$$

interchanging any two consecutive entries and multiplying by -1 in a simple k -tensor gives an equivalent tensor modulo $\mathcal{A}^k(M)$. Using such a sequence of interchanges and sign changes we can arrange for the equal entries m_i and m_j of a simple tensor as in (1) to be adjacent, which gives an element of $\mathcal{A}^k(M)$. It follows that the generators in (1) are contained in $\mathcal{A}^k(M)$, which proves the first part of the theorem.

As in Theorem 34, the proof of (2) follows easily from the corresponding result for the tensor algebra in Theorem 31 since $\mathcal{A}^k(M)$ is contained in the kernel of any alternating map from $\mathcal{T}^k(M)$ to N .

Examples

- (1) Suppose V is a one-dimensional vector space over F with basis element v . Then $\bigwedge^k(V)$ consists of finite sums of elements of the form $\alpha_1 v \wedge \alpha_2 v \wedge \cdots \wedge \alpha_k v$, i.e., $\alpha_1 \alpha_2 \cdots \alpha_k (v \wedge v \wedge \cdots \wedge v)$ for $\alpha_1, \dots, \alpha_k \in F$. Since $v \wedge v = 0$, it follows that $\bigwedge^0(V) = F$, $\bigwedge^1(V) = V$, and $\bigwedge^i(V) = 0$ for $i \geq 2$, so as a graded F -algebra we have

$$\bigwedge(V) = F \oplus V \oplus 0 \oplus 0 \oplus \dots$$

- (2) Suppose now that V is a two-dimensional vector space over F with basis v, v' . Here $\bigwedge^k(V)$ consists of finite sums of elements of the form $(\alpha_1 v + \alpha'_1 v') \wedge \cdots \wedge (\alpha_k v + \alpha'_k v')$. Such an element is a sum of elements that are simple wedge products involving only v and v' . For example, an element in $\bigwedge^2(V)$ is a sum of elements of the form

$$\begin{aligned} (av + bv') \wedge (cv + dv') &= ac(v \wedge v) + ad(v \wedge v') + bc(v' \wedge v) \\ &\quad + bd(v' \wedge v') \\ &= (ad - bc)v \wedge v'. \end{aligned}$$

It follows that $\bigwedge^i(V) = 0$ for $i \geq 3$ since then at least one of v, v' appears twice in such simple products.

We can see directly from $\bigwedge^2(V) = \mathcal{T}^2(V)/\mathcal{A}^2(V)$ that $v \wedge v' \neq 0$, as follows. The vector space $\mathcal{T}^2(V)$ is 4-dimensional with $v \otimes v, v \otimes v', v' \otimes v, v' \otimes v'$ as basis (Proposition 16). The elements $v \otimes v, v \otimes v' + v' \otimes v, v' \otimes v'$ and $v \otimes v'$ are therefore also a basis for $\mathcal{T}^2(V)$. The subspace $\mathcal{A}^2(V)$ consists of all the 2-tensors in the ideal generated by the tensors

$$(av + bv') \otimes (av + bv') = a^2(v \otimes v) + ab(v \otimes v' + v' \otimes v) + b^2(v' \otimes v'),$$

from which it is clear that $\mathcal{A}^2(V)$ is contained in the 3-dimensional subspace having $v \otimes v, v \otimes v' + v' \otimes v$, and $v' \otimes v'$ as basis. In particular, the basis element $v \otimes v'$ of $\mathcal{T}^2(V)$ is not contained in $\mathcal{A}^2(V)$, i.e., $v \wedge v' \neq 0$ in $\bigwedge^2(V)$.

It follows that $\bigwedge^0(V) = F$, $\bigwedge^1(V) = V$, $\bigwedge^2(V) = F(v \wedge v')$, and $\bigwedge^i(V) = 0$ for $i \geq 3$, so as a graded F -algebra we have

$$\bigwedge(V) = F \oplus V \oplus F(v \wedge v') \oplus 0 \oplus \dots$$

As the previous examples illustrate, unlike the tensor and symmetric algebras, for finite dimensional vector spaces the exterior algebra is finite dimensional:

Corollary 37. Let V be a finite dimensional vector space over the field F with basis $B = \{v_1, \dots, v_n\}$. Then the vectors

$$v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_k} \quad \text{for } 1 \leq i_1 < i_2 < \cdots < i_k \leq n$$

are a basis of $\bigwedge^k(V)$, and $\bigwedge^k(V) = 0$ when $k > n$ (when $k = 0$ the basis vector is the element $1 \in F$). In particular, $\dim_F(\bigwedge^k(V)) = \binom{n}{k}$.

Proof: As the proof of Theorem 36 shows, modulo $\mathcal{A}^k(M)$, the order of the terms in any simple k -tensor can be rearranged up to introducing a sign change. It follows that the k -tensors in the corollary (which have been arranged with increasing subscripts on the v_i and with no repeated entries) are generators for $\bigwedge^k(V)$. To show these vectors are linearly independent it suffices to exhibit an alternating k -multilinear function from V^k to F which is 1 on a given $v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_k}$ and zero on all other generators. Such a function f is defined on the basis of $\mathcal{T}^k(V)$ in Proposition 32 by $f(v_{j_1} \otimes v_{j_2} \otimes \cdots \otimes v_{j_k}) = \epsilon(\sigma)$ if σ is the unique permutation of (j_1, j_2, \dots, j_k) into (i_1, i_2, \dots, i_k) , and f is zero on every basis tensor whose k -tuple of indices cannot be permuted to (i_1, i_2, \dots, i_k) (where $\epsilon(\sigma)$ is the sign of σ). Note that f is zero on any basis tensor with repeated entries. The value $\epsilon(\sigma)$ ensures that when f is extended to all elements of $\mathcal{T}^k(V)$ it gives an alternating map, i.e., f factors through $\mathcal{A}^k(V)$. Hence f is the desired function. The computation of the dimension of $\bigwedge^k(V)$ (i.e., of the number of increasing sequences of k -tuples of indices) is left to the exercises.

The results in Corollary 37 are true for any *free* R -module of rank n . In particular if $M \cong R^n$ with R -module basis m_1, \dots, m_n then

$$\bigwedge^n(M) = R(m_1 \wedge \cdots \wedge m_n)$$

is a free (rank 1) R -module with generator $m_1 \wedge \cdots \wedge m_n$ and

$$\bigwedge^{n+1}(M) = \bigwedge^{n+2}(M) = \cdots = 0.$$

Example

Let R be the polynomial ring $\mathbb{Z}[x, y]$ in the variables x and y . If $M = R$, then $\bigwedge^2(M) = 0$ so, for example, there are no nontrivial alternating bilinear maps on $R \times R$ by the universal property of $\bigwedge^2(R)$ with respect to such maps (Theorem 36).

Suppose now that $M = I$ is the ideal (x, y) generated by x and y in R . Then $I \wedge I \neq 0$. Perhaps the easiest way to see this is to construct a nontrivial alternating bilinear map on $I \times I$. The map

$$\varphi(ax + by, cx + dy) = (ad - bc) \bmod (x, y)$$

is a well defined alternating R -bilinear map from $I \times I$ to $\mathbb{Z} = R/I$ (cf. Exercise 7). Since $\varphi(x, y) = 1$, it follows that $x \wedge y \in \bigwedge^2(I)$ is nonzero. Unlike the situation of free modules as in the examples following Theorem 36 (where arguments involving *bases* could be used), in this case it is not at all a trivial matter to give a direct verification that $x \wedge y \neq 0$ in $\bigwedge^2(I)$.

Remark: The ideal I is an example of a rank 1 (but *not* free) R -module (the rank of a module over an integral domain is defined in Section 12.1), and this example shows that the results of Corollary 37 are not true in general if the R -module is not free over R .