Likewise, an intersection of multiplicity 3 can be explained as the limit of three distinct intersections, for example, of $y = \varepsilon x$ with $y = x^3$ (Figure 15.2)

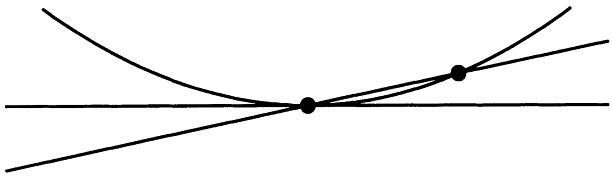

Figure 15.1: Intersection of multiplicity 2

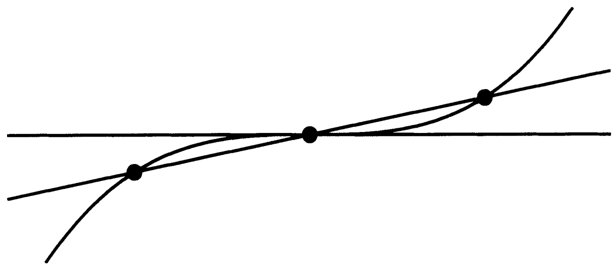

Figure 15.2: Intersection of multiplicity 3

At first glance this idea seems to break down with multiplicity 4, since $y = \varepsilon x$ meets $y = x^4$ at only two points, $x = 0$ and $x = \sqrt[3]{\varepsilon}$. The explanation is that there are also two complex roots in this case ($\sqrt[3]{\varepsilon}$ times the two complex cube roots of 1), hence we cannot neglect complex roots if we want to get the geometrically "correct" number of intersections.

The fundamental theorem of algebra (Section 14.6) gives us n roots of an nth-degree equation (2) and consequently n intersections of the polynomial curve (1) with the axis $y = 0$. To get n roots, however, we have to admit complex values of x, hence we have to consider "curves" for which x and y are complex in order to obtain n intersections. This, and other tidy consequences of the fundamental theorem of algebra (for example, the "coincident point" interpretation of multiplicity; see Exercise 15.1.1), persuaded eighteenth-century mathematicians to admit complex numbers into the theory of curves before complex numbers themselves were understood—and even before the fundamental theorem of algebra was proved.

The most elegant consequence was Bézout's theorem that a curve C_m of degree m meets a curve C_n of degree n at mn points. As we saw in Section 8.6, if homogeneous coordinates are used to take account of points at infinity, then the intersections of C_m and C_n correspond to the solutions of an equation $r_{mn}(x,y) = 0$, which is homogeneous of degree mn. We can now use the fundamental theorem of algebra to show that $r_{mn}(x,y)$ is the product of mn linear factors as follows:

$$\begin{aligned} r_{mn}(x,y) &= y^{mn} r_{mn}\left(\frac{x}{y}, 1\right) \\ &= y^{mn} \prod_{i=1}^{p} \left(b_i \frac{x}{y} - a_i\right) \qquad \text{for some } p \leq mn \end{aligned}$$

by the fundamental theorem, since $r_{mn}(x/y, 1)$ is a polynomial of degree $p \leq mn$ in the single variable x/y. But then

$$\begin{aligned} r_{mn}(x,y) &= y^{mn-p} \prod_{i=1}^{p} (b_i x - a_i y) \\ &= \prod_{i=1}^{mn} (b_i x - a_i y) \end{aligned}$$

since each factor y in front (if any) is trivially of the form $b_i x - a_i y$.

It follows that the equation $r_{mn}(x,y) = 0$ has mn solutions, and hence there are mn intersections of C_m and C_n, counting multiplicities.

EXERCISES

15.1.1 Show that $y = \varepsilon x$ meets $y = x^n$ in n distinct points when $\varepsilon \neq 0$, and list them (for example, with the help of de Moivre's theorem).

If a curve K has a double point at O, then a line $y = tx$ may have double contact with K at O even though nearby lines $y = (t+\varepsilon)x$ do not meet K at nearby points other than O. In this case the double contact may be explained as contact with the two branches of the curve at O.

15.1.2 Consider the lines $y = tx$ through the double point O of $y^2 = x^2(x+1)$. Show that each such line has double contact with the curve at O, except when $t = \pm 1$. How do you account for the multiplicities when $t = \pm 1$?

15.1.3 Show that $y = tx$ also has double contact with $y^2 = x^3$ at its cusp point O. Try to explain this by viewing $y^2 = x^3$ as the result of "shrinking the loop" of $y^2 = x^2(x+\varepsilon)$ (letting $\varepsilon \to 0$).

15.1.4 Show that the line $y = tx$ has double contact at O with the lemniscate $(x^2+y^2)^2 = x^2 - y^2$ except for two values of t, for which it has quadruple contact.

15.1.5 Explain the multiplicities found in Exercise 15.1.4 with the help of the known shape of the lemniscate (Figure 12.1).

15.2 The Complex Projective Line

We saw in Section 8.5 that adding a point at infinity to the real line $\mathbb{R}$ in $\mathbb{R} \times \mathbb{R}$ forms a closed curve that is qualitatively like a circle. Indeed, a real projective line in the sphere model of the real projective plane $\mathbb{R}P^2$ has much the same geometric properties as a great circle on a sphere, after one allows for the fact that antipodal points on the sphere are the same point on $\mathbb{R}P^2$. The situation with the complex "line" $\mathbb{C}$ is similar but more difficult to visualize. $\mathbb{C}$ is already two-dimensional, as we saw in Gauss' proof of the fundamental theorem of algebra, hence the complex "plane" $\mathbb{C} \times \mathbb{C}$ is four-dimensional and virtually impossible to visualize.

To avoid an excursion into four-dimensional space, we first revise our approach to the real projective line. In Section 8.5 we considered ordinary lines L, in a horizontal plane not passing through the origin, and extended each to a projective line whose "points" are the lines through the origin O, in the plane through O and L. The nonhorizontal lines in this family correspond to points of L, and the horizontal line in the family to the point at infinity of L. We now use this construction again to demonstrate directly the qualitative, or more precisely *topological*, equivalence between a projective line and a circle (Figure 15.3).

The origin N is taken to be the top point of a circle that, at its bottom point, touches our line $L = \mathbb{R}$. There is a continuous one-to-one correspondence between lines through N and points of the circle. Each nonhorizontal line corresponds to its intersection $x' \neq N$ with the circle, while the horizontal line corresponds to N itself. Thus the projective completion of $\mathbb{R}$, which we now call $\mathbb{R}P^1$, is *topologically the same* as the circle, in the sense that there is a continuous one-to-one correspondence between them. Moreover, we can understand projective completion of $\mathbb{R}$ topologically as a process of adding one "point" that is "approached" as one tends to infinity, in either direction, along $\mathbb{R}$, for as x tends to infinity in either direction, x' tends to the same point, N, on the circle.

We can now view projective completion of $\mathbb{C}$ in the same way using

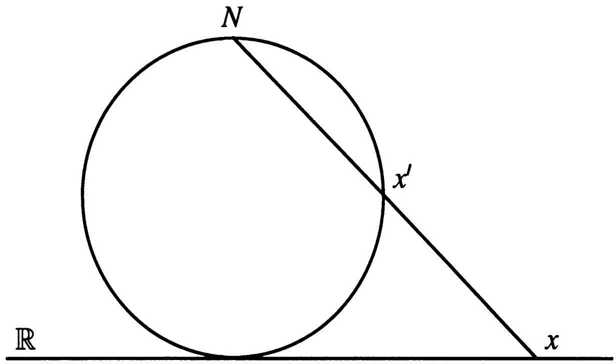

Figure 15.3: The real projective line

Figure 15.4, which shows the so-called *stereographic projection* of the plane $\mathbb{C}$ into a sphere. Each point $z \in \mathbb{C}$ is projected to a point z' on the tangential sphere S by the ray through z and the north pole N of S. This establishes a continuous one-to-one correspondence between points z of $\mathbb{C}$ and points $z' \neq N$ on S. Moreover, as z tends to infinity in any direction, z' tends to N, hence the projective completion of $\mathbb{C}P^1$ of $\mathbb{C}$ is topologically the same as the complete sphere S, with the point at ∞ of $\mathbb{C}$ corresponding to N.

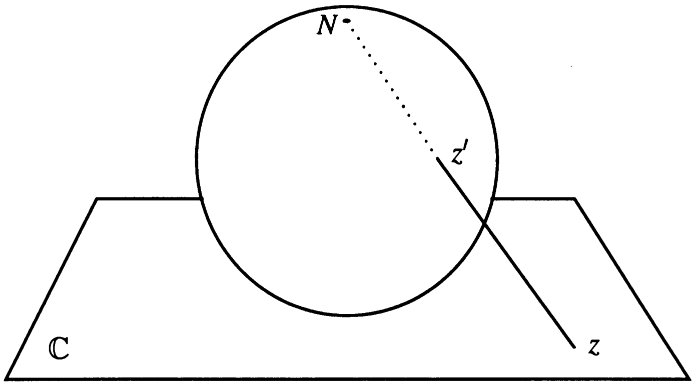

Figure 15.4: The complex projective line

Since one also wants to complete $\mathbb{C}$ by a point ∞ in this way for complex analysis, geometry and analysis are both served by passing from $\mathbb{C}$ to $\mathbb{C}P^1$. Gauss seems to have been the first to appreciate the advantages of

$\mathbb{C} \cup \{\infty\}$ over $\mathbb{C}$, hence one often calls $\mathbb{C}P^1$ the *Gauss sphere* in analysis. [Unfortunately, only a few unpublished, undated fragments of Gauss' work on this topic seem to have survived; see Gauss (1819).] Algebraic geometers call $\mathbb{C}P^1$ the (complex) projective line, since it is the formal equivalent of a real line, even though it is topologically a surface. Similarly, complex curves are topologically surfaces, known to analysts as *Riemann surfaces*, though algebraic geometers prefer to call them "curves."

The "surface" viewpoint is helpful when studying intrinsic properties of complex curves. For example, *genus* (introduced in connection with parameterization in Sections 11.3 to 11.5) turns out to have a very simple meaning in the topology of surfaces (see Section 15.4). On the other hand, the "curve" viewpoint is helpful when studying intersections of curves and their embedding in $\mathbb{C} \times \mathbb{C}$ or its projective completion $\mathbb{C}P^2$. Instead of trying to imagine two planes meeting in a single point of $\mathbb{C} \times \mathbb{C}$, for example, it is better to imagine the intersection as analogous to that of real lines in a real plane—as the single solution of two linear equations. After all, we are working with $\mathbb{C}$ to remove anomalies that occur with $\mathbb{R}$, not for the sake of doing something different, and we expect that much of the behavior of real curves will recur with complex ones.

EXERCISES

Since addition and multiplication are continuous functions, it is quite easy to find one-to-one continuous maps between certain complex algebraic curves and the sphere.

15.2.1 Show that the projective completion of the curve $Y = X^2$ is topologically a sphere by considering its parameterization

$$X = t, \quad Y = t^2,$$

where t ranges over the sphere $\mathbb{C} \cup \{\infty\}$. Namely, show that the mapping $t \mapsto (t, t^2)$ is one-to-one and continuous.

15.2.2 Similarly show that the projective completion of $Y^2 = X^3$ is topologically a sphere by considering its parameterization

$$X = t^2, \quad Y = t^3$$

and the continuous mapping $t \mapsto (t^2, t^3)$.

15.2.3 Consider the mapping of the t sphere onto the projective completion of $Y^2 = X^2(X+1)$ defined by $t \mapsto P(t)$, where $P(t)$ is the third intersection of the curve with the line $Y = tX$ through the double point (found in Exercise 7.4.2).

Show that this mapping is continuous and that it is one-to-one except at the points $t = \pm 1$, which are both mapped to the point O on the curve. Conclude that the curve is topologically the same as a sphere with two points identified (Figure 15.5).

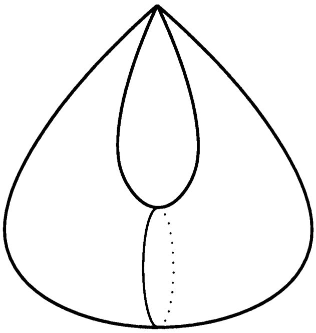

Figure 15.5: The singular sphere

15.3 Branch Points

The key to the topological form of a complex curve $p(x,y) = 0$ lies in its *branch points*, the points α where the Newton–Puiseux expansion of y begins with a fractional power of $(x - \alpha)$ (see Section 10.5). The nature of branch points was first described by Riemann (1851) as part of a revolutionary new geometric theory of complex functions. Riemann's idea, one of the most illuminating in the history of mathematics, was to represent a relation $p(x,y) = 0$ between complex x and complex y by covering a plane (or sphere) representing the x variable by a surface representing the y variable, the point or points of the y surface over a given point $x = \alpha$ being those values of y that satisfy $p(\alpha, y) = 0$.

If the equation $p(\alpha, y) = 0$ is of degree n in y, there will in general be n distinct y values for a given α, consequently n "sheets" of the y surface lying over the x-plane in the neighborhood of $x = \alpha$. At finitely many exceptional values of x, sheets merge due to concidence of roots, and the Newton–Puiseux theory says that at such a point y behaves like a fractional power of x at 0. Our main problem, therefore, is to understand the behavior of the Riemann surface for $y = x^{m/n}$ in the neighborhood of 0.

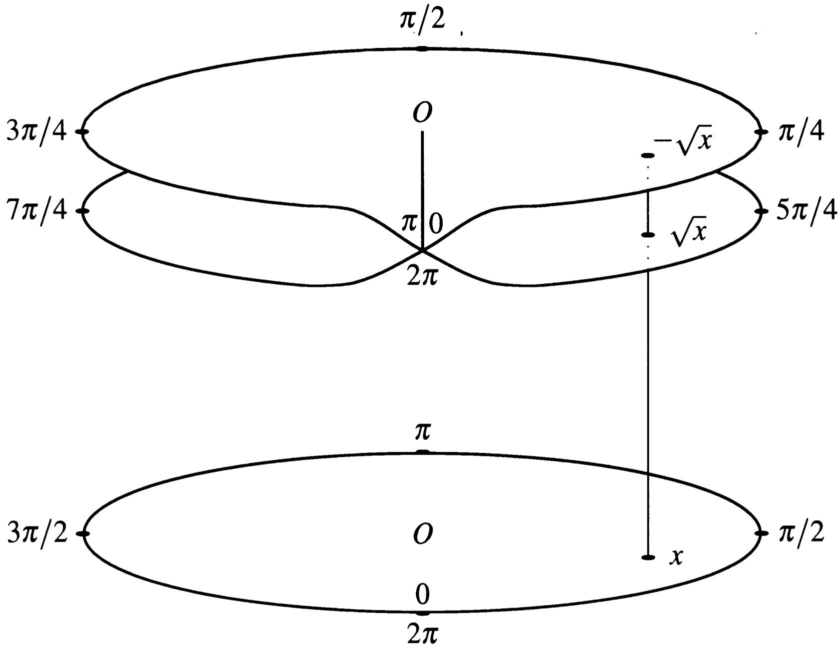

Figure 15.6: Branch point for the square root

The idea can be grasped sufficiently well from seeing the special case $y = x^{1/2}$. If we consider the unit disk in the y-plane and try to deform it so that the points $y = \pm\sqrt{x}$ lie above the point x in the unit disk of x-plane, then the result is something like Figure 15.6.

The angles θ on the disk boundaries are the arguments of the corresponding points $e^{i\theta}$. If

$$x = e^{i\theta} = e^{i(\theta+2\pi)}$$

then

$$y = e^{i\theta/2}, \qquad e^{i(\theta/2+\pi)},$$

giving the values shown. A more graphic depiction of the branch point is seen in Figure 15.7, taken from an early textbook on Riemann's theory [Neumann (1865), endpaper].

It should be noted that the awkward appearance of the branch point, in particular the line of self-intersection, is a consequence of representing the relation $y^2 = x$ in fewer dimensions than the four it really requires. If we similarly attempt to represent the relation $y^2 = x$ between real x and y by

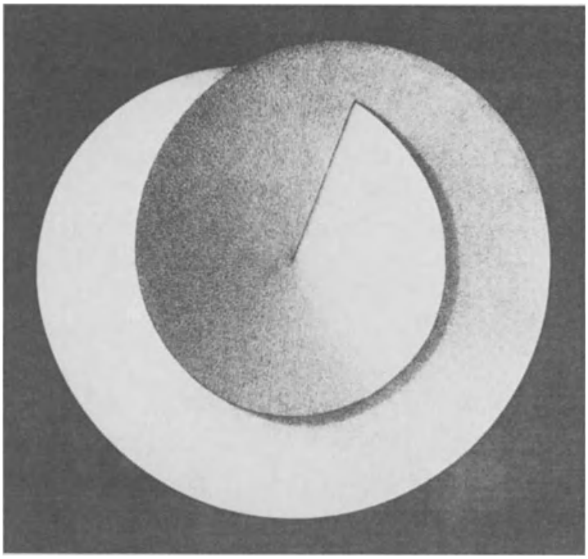

Figure 15.7: Neumann's picture of a branch point

laying the y-axis along the x-axis so that $y = \pm\sqrt{x}$ are on top of x, then the result is an awkward folded "branch point" at 0 (Figure 15.8). This is a consequence of trying to represent the relation in one dimension. In reality, as the second part of the figure shows, when viewed as a curve in the plane the relation is just as smooth at 0 as anywhere else. (Notice, incidentally, that the folded line in Figure 15.8, the real y-axis, corresponds to the self-intersection line in Figure 15.7.)

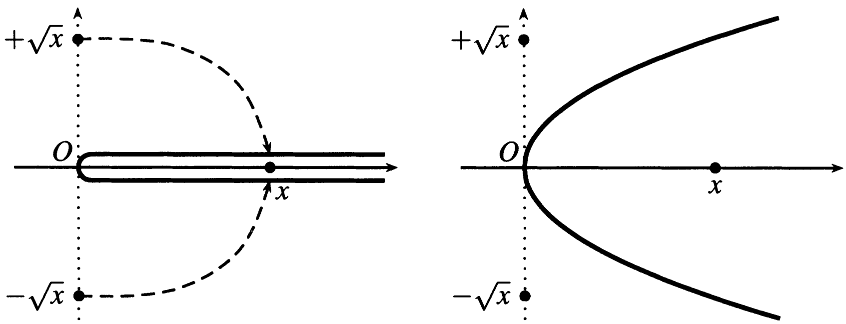

Figure 15.8: A one-dimensional branch point