10

The Crisis of Incommensurables

Two lengths a and b are said to be *commensurable* if there exist positive integers p and q such that $a/b = p/q$. When the Pythagoreans claimed that all things are numbers, they probably meant to imply that all pairs of lengths are commensurable. They were aware of the fact that, if a vibrating string is divided into two parts, of lengths a and b, so that a melodious tone is produced, then a and b are commensurable.

Unfortunately for the Pythagoreans, they soon discovered that the diagonal of a square is not commensurable with its side. A simple proof of this is found in Aristotle's *Prior Analytics* 41a23-30. Let $ABCD$ be a square, say of side $AB = 1$. By the Theorem of Pythagoras, the diagonal AC measures $\sqrt{2}$. Suppose $\sqrt{2} = AC/AB = p/q$, where p and q are positive integers. We may assume, without loss of generality, that p and q are relatively prime. In particular, they are not both even. Now $p^2 = 2q^2$, so that p^2 is even. As the Pythagoreans well knew, the square of an odd number is odd and the square of an even number is even. Thus, from the fact that p^2 is even, it follows that p is even. Putting $p = 2r$, we have $2q^2 = (2r)^2$, hence $q = 2r^2$. But this means that q is even as well, contradicting the fact that p and q are relatively prime. Thus, the assumption that AC and AB are commensurable must be false. Today we would express this result by saying that $\sqrt{2}$ is irrational.

The Pythagoreans tried to keep this discovery a secret, as it seemed to undermine their whole philosophy. Some say that it was Hippasus, whom we met before, who leaked the secret, and that he drowned in a shipwreck as a punishment for having done so. It seems that Hippasus was the Trotsky of the Pythagorean society.

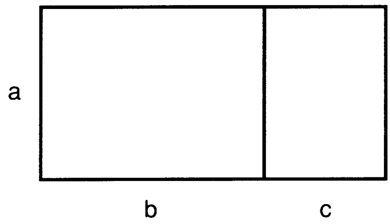

FIGURE 10.1. The distributive law

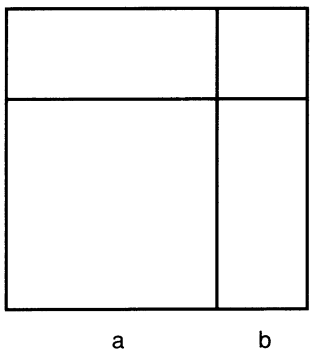

FIGURE 10.2. Binomial expansion $(a+b)^2 = a^2 + 2ab + b^2$

The Greeks did not have infinite decimals. They did not know how to handle a number like $\sqrt{2}$ in an arithmetical or algebraic fashion the way we do now, although it has recently been claimed that they could represent real numbers by continued fractions. They did, however, know that $\sqrt{2}$ was a length, and they turned to geometry for an understanding of it. The problem of incommensurables was one of the reasons that they preferred to do what we would call algebra in a geometric manner.

For example, the distributive law $a(b+c) = ab + ac$ was thought of as an addition rule for areas of rectangles, as in Figure 10.1.

Euclid put it thus:

> If there are two straight lines, and one of them be cut into any number of segments whatever, the rectangle contained by the two straight lines is equal to the rectangles contained by the uncut straight line and each of the segments (*Elements* II 1).

The law $(a+b)^2 = a^2 + 2ab + b^2$ is illustrated in Figure 10.2. We shall refrain from putting this law into words.

However, as a third example, we again quote Euclid:

> If a straight line be cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole to-

gether with the square on the straight line between the points of section is equal to the square on the half (*Elements* II 5).

This is equivalent to the identity $(a+b)(a-b) = a^2 - b^2$.

The Pythagoreans found ways of approximating $\sqrt{2}$ as closely as could be desired by rational numbers. Using our modern algebraic notation, we can express their method as follows.

If $x^2 - 2y^2 = \pm 1$, with x and y positive integers, then x^2 is approximately equal to $2y^2$, so that x/y is approximately equal to $\sqrt{2}$. More precisely,

$$(x/y - \sqrt{2})(x/y + \sqrt{2}) = x^2/y^2 - 2 = \pm 1/y^2$$

so that

$$x/y - \sqrt{2} = \pm 1/(y^2(x/y + \sqrt{2})).$$

Since $x/y + \sqrt{2} > 1$, it follows that

$$|x/y - \sqrt{2}| < 1/y^2 .$$

Thus, if we can find positive integer solutions of $x^2 - 2y^2 = \pm 1$ with y sufficiently large, then we can find rational approximations to $\sqrt{2}$ as close as we please.

To find positive integers x and y such that $x^2 - 2y^2 = \pm 1$, the Pythagoreans proceeded as follows. Putting

$$a_1 = 1, \qquad b_1 = 1$$

and defining inductively

$$a_{n+1} = a_n + 2b_n, \quad b_{n+1} = a_n + b_n,$$

they obtained the following table:

n	a_n	b_n	a_n/b_n
1	1	1	1
2	3	2	3/2
3	7	5	7/5
4	17	12	17/12

etc., in which the last column contains successive approximations to $\sqrt{2}$.

Indeed, it is not difficult to prove by mathematical induction that

$$a_n^2 - 2b_n^2 = (-1)^n .$$

This is surely true when $n = 1$, so suppose it holds for n. Then

$$\begin{aligned} a_{n+1}^2 - 2b_{n+1}^2 &= (a_n + 2b_n)^2 - 2(a_n + b_n)^2 \\ &= a_n^2 + 4a_nb_n + 4b_n^2 - 2a_n^2 - 4a_nb_n - 2b_n^2 \\ &= -a_n^2 + 2b_n^2 \\ &= -(-1)^n = (-1)^{n+1}. \end{aligned}$$