Examples 11.1.12. The set $\mathbf{P} = \{\{1\}, (1,3), [3,5), \{5\}, (5,8], \emptyset\}$ of bounded intervals is a partition of $[1,8]$, because all the intervals in $\mathbf{P}$ lie in $[1,8]$, and each element of $[1,8]$ lies in exactly one interval in $\mathbf{P}$. Note that one could have removed the empty set from $\mathbf{P}$ and still obtain a partition. However, the set $\{[1,4],[3,5]\}$ is not a partition of $[1,5]$ because some elements of $[1,5]$ are included in more than one interval in the set. The set $\{(1,3),(3,5)\}$ is not a partition of $(1,5)$ because some elements of $(1,5)$ are not included in any interval in the set. The set $\{(0,3),[3,5)\}$ is not a partition of $(1,5)$ because some intervals in the set are not contained in $(1,5)$.

Now we come to a basic property about length:

Theorem 11.1.13 (Length is finitely additive). *Let I be a bounded interval, n be a natural number, and let $\mathbf{P}$ be a partition of I of cardinality n. Then*

$$|I| = \sum_{J \in \mathbf{P}} |J|.$$

Proof. We prove this by induction on n. More precisely, we let $P(n)$ be the property that whenever I is a bounded interval, and whenever $\mathbf{P}$ is a partition of I with cardinality n, that $|I| = \sum_{J \in \mathbf{P}} |J|$.

The base case $P(0)$ is trivial; the only way that I can be partitioned into an empty partition is if I is itself empty (why?), at which point the claim is easy. The case $P(1)$ is also very easy; the only way that I can be partitioned into a singleton set $\{J\}$ is if $J = I$ (why?), at which point the claim is again very easy.

Now suppose inductively that $P(n)$ is true for some $n \geq 1$, and now we prove $P(n+1)$. Let I be a bounded interval, and let $\mathbf{P}$ be a partition of I of cardinality $n+1$.

If I is the empty set or a point, then all the intervals in $\mathbf{P}$ must also be either the empty set or a point (why?), and so every interval has length zero and the claim is trivial. Thus we will assume that I is an interval of the form (a,b), $(a,b]$, $[a,b)$, or $[a,b]$.

Let us first suppose that $b \in I$, i.e., I is either $(a, b]$ or $[a, b]$. Since $b \in I$, we know that one of the intervals K in $\mathbf{P}$ contains b. Since K is contained in I, it must therefore be of the form $(c, b]$, $[c, b]$, or $\{b\}$ for some real number c, with $a \leq c \leq b$ (in the latter case of $K = \{b\}$, we set $c := b$). In particular, this means that the set $I - K$ is also an interval of the form $[a, c]$, (a, c), $(a, c]$, $[a, c)$ when $c > a$, or a point or empty set when $a = c$. Either way, we easily see that

$$|I| = |K| + |I - K|.$$

On the other hand, since $\mathbf{P}$ forms a partition of I, we see that $\mathbf{P} - \{K\}$ forms a partition of $I - K$ (why?). By the induction hypothesis, we thus have

$$|I - K| = \sum_{J \in \mathbf{P} - \{K\}} |J|.$$

Combining these two identities (and using the laws of addition for finite sets, see Proposition 7.1.11) we obtain

$$|I| = \sum_{J \in \mathbf{P}} |J|$$

as desired.

Now suppose that $b \notin I$, i.e., I is either (a, b) or $[a, b)$. Then one of the intervals K also is of the form (c, b) or $[c, b)$ (see Exercise 11.1.3). In particular, this means that the set $I - K$ is also an interval of the form $[a, c]$, (a, c), $(a, c]$, $[a, c)$ when $c > a$, or a point or empty set when $a = c$. The rest of the argument then proceeds as above. □

There are two more things we need to do with partitions. One is to say when one partition is finer than another, and the other is to talk about the common refinement of two partitions.

Definition 11.1.14 (Finer and coarser partitions). Let I be a bounded interval, and let $\mathbf{P}$ and $\mathbf{P}'$ be two partitions of I. We say that $\mathbf{P}'$ is *finer* than $\mathbf{P}$ (or equivalently, that $\mathbf{P}$ is *coarser* than $\mathbf{P}'$) if for every J in $\mathbf{P}'$, there exists a K in $\mathbf{P}$ such that $J \subseteq K$.

Example 11.1.15. The partition $\{[1, 2), \{2\}, (2, 3), [3, 4]\}$ is finer than $\{[1, 2], (2, 4]\}$ (why?). Both partitions are finer than $\{[1, 4]\}$, which is the coarsest possible partition of $[1, 4]$. Note that there is no such thing as a "finest" partition of $[1, 4]$. (Why? recall all partitions are assumed to be finite.) We do not compare partitions of different intervals, for instance if $\mathbf{P}$ is a partition of $[1, 4]$ and $\mathbf{P}'$ is a partition of $[2, 5]$ then we would not say that $\mathbf{P}$ is coarser or finer than $\mathbf{P}'$.

Definition 11.1.16 (Common refinement). Let I be a bounded interval, and let $\mathbf{P}$ and $\mathbf{P}'$ be two partitions of I. We define the *common refinement* $\mathbf{P}\#\mathbf{P}'$ of $\mathbf{P}$ and $\mathbf{P}'$ to be the set

$$\mathbf{P}\#\mathbf{P}' := \{K \cap J : K \in \mathbf{P} \text{ and } J \in \mathbf{P}'\}.$$

Example 11.1.17. Let $\mathbf{P} := \{[1, 3), [3, 4]\}$ and $\mathbf{P}' := \{[1, 2], (2, 4]\}$ be two partitions of $[1, 4]$. Then $\mathbf{P}\#\mathbf{P}'$ is the set $\{[1, 2], (2, 3), [3, 4], \emptyset\}$ (why?).

Lemma 11.1.18. *Let I be a bounded interval, and let $\mathbf{P}$ and $\mathbf{P}'$ be two partitions of I. Then $\mathbf{P}\#\mathbf{P}'$ is also a partition of I, and is both finer than $\mathbf{P}$ and finer than $\mathbf{P}'$.*

Proof. See Exercise 11.1.4. □

Exercise 11.1.1. Prove Lemma 11.1.4. (Hint: in order to show that (a) implies (b) in the case when X is non-empty, consider the supremum and infimum of X.)

Exercise 11.1.2. Prove Corollary 11.1.6. (Hint: use Lemma 11.1.4, and explain why the intersection of two bounded sets is automatically bounded, and why the intersection of two connected sets is automatically connected.)

Exercise 11.1.3. Let I be a bounded interval of the form $I = (a, b)$ or $I = [a, b)$ for some real numbers $a < b$. Let $I_1, \ldots, I_n$ be a partition of I. Prove that one of the intervals I_j in this partition is of the form $I_j = (c, b)$ or $I_j = [c, b)$ for some $a \leq c \leq b$. (Hint: prove by contradiction. First show that if I_j is *not* of the form (c, b) or $[c, b)$ for any $a \leq c \leq b$, then $\sup I_j$ is *strictly* less than b.)

Exercise 11.1.4. Prove Lemma 11.1.18.

11.2 Piecewise constant functions

We can now describe the class of "simple" functions which we can integrate very easily.

Definition 11.2.1 (Constant functions). Let X be a subset of $\mathbf{R}$, and let $f : X \to \mathbf{R}$ be a function. We say that f is *constant* iff there exists a real number c such that $f(x) = c$ for all $x \in X$. If E is a subset of X, we say that f is *constant on* E if the restriction $f|_E$ of f to E is constant, in other words there exists a real number c such that $f(x) = c$ for all $x \in E$. We refer to c as the *constant value* of f on E.

Remark 11.2.2. If E is a non-empty set, then a function f which is constant on f can have only one constant value; it is not possible for a function to always equal 3 on E while simultaneously always equalling 4. However, if E is empty, every real number c is a constant value for f on E (why?).

Definition 11.2.3 (Piecewise constant functions I). Let I be a bounded interval, let $f : I \to \mathbf{R}$ be a function, and let $\mathbf{P}$ be a partition of I. We say that f is *piecewise constant with respect to* $\mathbf{P}$ if for every $J \in \mathbf{P}$, f is constant on J.

Example 11.2.4. The function $f : [1, 6] \to \mathbf{R}$ defined by

$$f(x) = \begin{cases} 7 & \text{if } 1 \leq x < 3 \\ 4 & \text{if } x = 3 \\ 5 & \text{if } 3 < x < 6 \\ 2 & \text{if } x = 6 \end{cases}$$

is piecewise constant with respect to the partition $\{[1,3), \{3\}, (3,6), \{6\}\}$ of $[1,6]$. Note that it is also piecewise constant with respect to some other partitions as well; for instance, it is piecewise constant with respect to the partition $\{[1,2), \{2\}, (2,3), \{3\}, (3,5), [5,6), \{6\}, \emptyset\}$.

Definition 11.2.5 (Piecewise constant functions II). Let I be a bounded interval, and let $f : I \to \mathbf{R}$ be a function. We say that f is *piecewise constant on* I if there exists a partition $\mathbf{P}$ of I such that f is piecewise constant with respect to $\mathbf{P}$.

Example 11.2.6. The function used in the previous example is piecewise constant on $[1, 6]$. Also, every constant function on a bounded interval I is automatically piecewise constant (why?).

Lemma 11.2.7. *Let I be a bounded interval, let $\mathbf{P}$ be a partition of I, and let $f : I \to \mathbf{R}$ be a function which is piecewise constant with respect to $\mathbf{P}$. Let $\mathbf{P}'$ be a partition of I which is finer than $\mathbf{P}$. Then f is also piecewise constant with respect to $\mathbf{P}'$.*

Proof. See Exercise 11.2.1. □

The space of piecewise constant functions is closed under algebraic operations:

Lemma 11.2.8. *Let I be a bounded interval, and let $f : I \to \mathbf{R}$ and $g : I \to \mathbf{R}$ be piecewise constant functions on I. Then the functions $f+g$, $f-g$, $\max(f, g)$ and fg are also piecewise constant functions on I. Here of course $\max(f, g) : I \to \mathbf{R}$ is the function $\max(f, g)(x) := \max(f(x), g(x))$. If g does not vanish anywhere on I (i.e., $g(x) \neq 0$ for all $x \in I$) then f/g is also a piecewise constant function on I.*

Proof. See Exercise 11.2.2. □

We are now ready to integrate piecewise constant functions. We begin with a temporary definition of an integral with respect to a partition.

Definition 11.2.9 (Piecewise constant integral I). Let I be a bounded interval, let $\mathbf{P}$ be a partition of I. Let $f : I \to \mathbf{R}$ be a function which is piecewise constant with respect to $\mathbf{P}$. Then we define the *piecewise constant integral* $p.c. \int_{[\mathbf{P}]} f$ of f with respect to the partition $\mathbf{P}$ by the formula

$$p.c. \int_{[\mathbf{P}]} f := \sum_{J \in \mathbf{P}} c_J |J|,$$

where for each J in $\mathbf{P}$, we let c_J be the constant value of f on J.

Remark 11.2.10. This definition seems like it could be ill-defined, because if J is empty then every number c_J can be the constant value of f on J, but fortunately in such cases $|J|$ is zero and so the choice of c_J is irrelevant. The notation $p.c. \int_{[\mathbf{P}]} f$ is rather artificial, but we shall only need it temporarily, en route to a more useful definition. Note that since $\mathbf{P}$ is finite, the sum $\sum_{J \in \mathbf{P}} c_J |J|$ is always well-defined (it is never divergent or infinite).

Remark 11.2.11. The piecewise constant integral corresponds intuitively to one's notion of area, given that the area of a rectangle ought to be the product of the lengths of the sides. (Of course, if f is negative somewhere, then the "area" $c_J|J|$ would also be negative.)

Example 11.2.12. Let $f : [1, 4] \to \mathbf{R}$ be the function

$$f(x) = \begin{cases} 2 & \text{if } 1 \le x < 3 \\ 4 & \text{if } x = 3 \\ 6 & \text{if } 3 < x \le 4 \end{cases}$$

and let $\mathbf{P} := \{[1, 3), \{3\}, (3, 4]\}$. Then

$$\begin{aligned} p.c. \int_{[\mathbf{P}]} f &= c_{[1,3)}|[1, 3)| + c_{\{3\}}|\{3\}| + c_{(3,4]}|(3, 4]| \\ &= 2 \times 2 + 4 \times 0 + 6 \times 1 \\ &= 10. \end{aligned}$$

Alternatively, if we let $\mathbf{P}' := \{[1, 2), [2, 3), \{3\}, (3, 4], \emptyset\}$ then

$$\begin{aligned} p.c. \int_{[\mathbf{P}']} f &= c_{[1,2)}|[1, 2)| + c_{[2,3)}|[2, 3)| + c_{\{3\}}|\{3\}| \\ &\quad + c_{(3,4]}|(3, 4]| + c_{\emptyset}|\emptyset| \\ &= 2 \times 1 + 2 \times 1 + 4 \times 0 + 6 \times 1 + c_{\emptyset} \times 0 \\ &= 10. \end{aligned}$$

This example suggests that this integral does not really depend on what partition you pick, so long as your function is piecewise constant with respect to that partition. That is indeed true:

Proposition 11.2.13 (Piecewise constant integral is independent of partition). *Let I be a bounded interval, and let $f : I \to \mathbf{R}$ be a function. Suppose that $\mathbf{P}$ and $\mathbf{P}'$ are partitions of I such that f is piecewise constant both with respect to $\mathbf{P}$ and with respect to $\mathbf{P}'$. Then p.c. $\int_{[\mathbf{P}]} f =$ p.c. $\int_{[\mathbf{P}']} f$.*

Proof. See Exercise 11.2.3. □

Because of this proposition, we can now make the following definition:

Definition 11.2.14 (Piecewise constant integral II). Let I be a bounded interval, and let $f : I \to \mathbf{R}$ be a piecewise constant function on I. We define the *piecewise constant integral p.c.* $\int_I f$ by the formula

$$p.c. \int_I f := p.c. \int_{[\mathbf{P}]} f,$$

where $\mathbf{P}$ is any partition of I with respect to which f is piecewise constant. (Note that Proposition 11.2.13 tells us that the precise choice of this partition is irrelevant.)

Example 11.2.15. If f is the function given in Example 11.2.12, then *p.c.* $\int_{[1,4]} f = 10$.

We now give some basic properties of the piecewise constant integral. These laws will eventually be superceded by the corresponding laws for the Riemann integral (Theorem 11.4.1).

Theorem 11.2.16 (Laws of integration). *Let I be a bounded interval, and let $f : I \to \mathbf{R}$ and $g : I \to \mathbf{R}$ be piecewise constant functions on I.*

(a) *We have p.c. $\int_I (f + g) =$ p.c. $\int_I f +$ p.c. $\int_I g$.*

(b) *For any real number c, we have p.c. $\int_I (cf) = c($p.c. $\int_I f)$.*

(c) *We have p.c. $\int_I (f - g) =$ p.c. $\int_I f -$ p.c. $\int_I g$.*

(d) *If $f(x) \geq 0$ for all $x \in I$, then p.c. $\int_I f \geq 0$.*

(e) *If* $f(x) \geq g(x)$ *for all* $x \in I$, *then* p.c. $\int_I f \geq$ p.c. $\int_I g$.

(f) *If* f *is the constant function* $f(x) = c$ *for all* x *in* I, *then* p.c. $\int_I f = c|I|$.

(g) *Let* J *be a bounded interval containing* I *(i.e.,* $I \subseteq J$*), and let* $F : J \to \mathbf{R}$ *be the function*

$$F(x) := \begin{cases} f(x) & \text{if } x \in I \\ 0 & \text{if } x \notin I \end{cases}$$

Then F *is piecewise constant on* J, *and* p.c. $\int_J F =$ p.c. $\int_I f$.

(h) *Suppose that* $\{J, K\}$ *is a partition of* I *into two intervals* J *and* K. *Then the functions* $f|_J : J \to \mathbf{R}$ *and* $f|_K : K \to \mathbf{R}$ *are piecewise constant on* J *and* K *respectively, and we have*

$$\text{p.c.} \int_I f = \text{p.c.} \int_J f|_J + \text{p.c.} \int_K f|_K.$$

Proof. See Exercise 11.2.4. □

This concludes our integration of piecewise constant functions. We now turn to the question of how to integrate bounded functions.

Exercise 11.2.1. Prove Lemma 11.2.7.

Exercise 11.2.2. Prove Lemma 11.2.8. (Hint: use Lemmas 11.1.18 and 11.2.7 to make f and g piecewise constant with respect to the *same* partition of I.)

Exercise 11.2.3. Prove Proposition 11.2.13. (Hint: first use Theorem 11.1.13 to show that both integrals are equal to p.c. $\int_{[\mathbf{P} \# \mathbf{P}']} f$.)

Exercise 11.2.4. Prove Theorem 11.2.16. (Hint: you can use earlier parts of the theorem to prove some of the later parts of the theorem. See also the hint to Exercise 11.2.2.)

11.3 Upper and lower Riemann integrals

Now let $f : I \to \mathbf{R}$ be a bounded function defined on a bounded interval I. We want to define the Riemann integral $\int_I f$. To do this we first need to define the notion of upper and lower Riemann integrals $\overline{\int}_I f$ and $\underline{\int}_I f$. These notions are related to the Riemann integral in much the same way that the lim sup and lim inf of a sequence are related to the limit of that sequence.

Definition 11.3.1 (Majorization of functions). Let $f : I \to \mathbf{R}$ and $g : I \to \mathbf{R}$. We say that g *majorizes* f *on* I if we have $g(x) \geq f(x)$ for all $x \in I$, and that g *minorizes* f *on* I if $g(x) \leq f(x)$ for all $x \in I$.

The idea of the Riemann integral is to try to integrate a function by first majorizing or minorizing that function by a piecewise constant function (which we already know how to integrate).

Definition 11.3.2 (Upper and lower Riemann integrals). Let $f : I \to \mathbf{R}$ be a bounded function defined on a bounded interval I. We define the *upper Riemann integral* $\overline{\int}_I f$ by the formula

$$\overline{\int}_I f := \inf\{p.c. \int_I g : g \text{ is a p.c. function on } I \text{ which majorizes } f\}$$

and the *lower Riemann integral* $\underline{\int}_I f$ by the formula

$$\underline{\int}_I f := \sup\{p.c. \int_I g : g \text{ is a p.c. function on } I \text{ which minorizes } f\}$$

We give a crude but useful bound on the lower and upper integral:

Lemma 11.3.3. *Let $f : I \to \mathbf{R}$ be a function on a bounded interval I which is bounded by some real number M, i.e., $-M \leq f(x) \leq M$ for all $x \in I$. Then we have*

$$-M|I| \leq \underline{\int}_I f \leq \overline{\int}_I f \leq M|I|.$$

In particular, both the lower and upper Riemann integrals are real numbers (i.e., they are not infinite).

Proof. The function $g : I \to \mathbf{R}$ defined by $g(x) = M$ is constant, hence piecewise constant, and majorizes f; thus $\overline{\int}_I f \leq p.c. \int_I g = M|I|$ by definition of the upper Riemann integral. A similar argument gives $-M|I| \leq \underline{\int}_I f$. Finally, we have to show that $\underline{\int}_I f \leq \overline{\int}_I f$. Let g be any piecewise constant function majorizing f, and let h be any piecewise constant function minorizing f. Then g majorizes h, and hence $p.c. \int_I h \leq p.c. \int_I g$. Taking suprema in h, we obtain that $\underline{\int}_I f \leq p.c. \int_I g$. Taking infima in g, we thus obtain $\underline{\int}_I f \leq \overline{\int}_I g$, as desired. □

We now know that the upper Riemann integral is always at least as large as the lower Riemann integral. If the two integrals match, then we can define the Riemann integral:

Definition 11.3.4 (Riemann integral). Let $f : I \to \mathbf{R}$ be a bounded function on a bounded interval I. If $\underline{\int}_I f = \overline{\int}_I f$, then we say that f is *Riemann integrable on I* and define

$$\int_I f := \underline{\int}_I f = \overline{\int}_I f.$$

If the upper and lower Riemann integrals are unequal, we say that f is not Riemann integrable.

Remark 11.3.5. Compare this definition to the relationship between the lim sup, lim inf, and limit of a sequence a_n that was established in Proposition 6.4.12(f); the lim sup is always greater than or equal to the lim inf, but they are only equal when the sequence converges, and in this case they are both equal to the limit of the sequence. The definition given above may differ from the definition you may have encountered in your calculus courses, based on Riemann sums. However, the two definitions turn out to be equivalent; this is the purpose of the next section.

Remark 11.3.6. Note that we do not consider unbounded functions to be Riemann integrable; an integral involving such functions is known as an *improper integral.* It is possible to still evaluate such integrals using more sophisticated integration methods (such as the Lebesgue integral); we shall do this in Chapter 19.

The Riemann integral is consistent with (and supercedes) the piecewise constant integral:

Lemma 11.3.7. *Let $f : I \to \mathbf{R}$ be a piecewise constant function on a bounded interval I. Then f is Riemann integrable, and $\int_I f = p.c. \int_I f$.*

Proof. See Exercise 11.3.3. □

Remark 11.3.8. Because of this lemma, we will not refer to the piecewise constant integral $p.c. \int_I$ again, and just use the Riemann integral $\int_I$ throughout (until this integral is itself superceded by the Lebesgue integral in Chapter 19). We observe one special case of Lemma 11.3.7: if I is a point or the empty set, then $\int_I f = 0$ for all functions $f : I \to \mathbf{R}$. (Note that all such functions are automatically constant.)

We have just shown that every piecewise constant function is Riemann integrable. However, the Riemann integral is more general, and can integrate a wider class of functions; we shall see this shortly. For now, we connect the Riemann integral we have just defined to the concept of a *Riemann sum*, which you may have seen in other treatments of the Riemann integral.

Definition 11.3.9 (Riemann sums). Let $f : I \to \mathbf{R}$ be a bounded function on a bounded interval I, and let $\mathbf{P}$ be a partition of I. We define the *upper Riemann sum* $U(f, \mathbf{P})$ and the *lower Riemann sum* $L(f, \mathbf{P})$ by

$$U(f, \mathbf{P}) := \sum_{J \in \mathbf{P} : J \neq \emptyset} (\sup_{x \in J} f(x))|J|$$

and

$$L(f, \mathbf{P}) := \sum_{J \in \mathbf{P} : J \neq \emptyset} (\inf_{x \in J} f(x))|J|.$$

Remark 11.3.10. The restriction $J \neq \emptyset$ is required because the quantities $\inf_{x \in J} f(x)$ and $\sup_{x \in J} f(x)$ are infinite (or negative infinite) if J is empty.

We now connect these Riemann sums to the upper and lower Riemann integral.

Lemma 11.3.11. *Let $f : I \to \mathbf{R}$ be a bounded function on a bounded interval I, and let g be a function which majorizes f and which is piecewise constant with respect to some partition $\mathbf{P}$ of I. Then*

$$p.c. \int_I g \geq U(f, \mathbf{P}).$$

Similarly, if h is a function which minorizes f and is piecewise constant with respect to $\mathbf{P}$, then

$$p.c. \int_I h \leq L(f, \mathbf{P}).$$

Proof. See Exercise 11.3.4. □

Proposition 11.3.12. *Let $f : I \to \mathbf{R}$ be a bounded function on a bounded interval I. Then*

$$\overline{\int}_I f = \inf\{U(f, \mathbf{P}) : \mathbf{P} \text{ is a partition of } I\}$$

and

$$\underline{\int}_I f = \sup\{L(f, \mathbf{P}) : \mathbf{P} \text{ is a partition of } I\}$$

Proof. See Exercise 11.3.5. □

Exercise 11.3.1. Let $f : I \to \mathbf{R}$, $g : I \to \mathbf{R}$, and $h : I \to \mathbf{R}$ be functions. Show that if f majorizes g and g majorizes h, then f majorizes h. Show that if f and g majorize each other, then they must be equal.

Exercise 11.3.2. Let $f : I \to \mathbf{R}$, $g : I \to \mathbf{R}$, and $h : I \to \mathbf{R}$ be functions. If f majorizes g, is it true that $f + h$ majorizes $g + h$? Is it true that $f \cdot h$ majorizes $g \cdot h$? If c is a real number, is it true that cf majorizes cg?

Exercise 11.3.3. Prove Lemma 11.3.7.

Exercise 11.3.4. Prove Lemma 11.3.11.

Exercise 11.3.5. Prove Proposition 11.3.12. (Hint: you will need Lemma 11.3.11, even though this Lemma will only do half of the job.)

11.4 Basic properties of the Riemann integral

Just as we did with limits, series, and derivatives, we now give the basic laws for manipulating the Riemann integral. These laws will eventually be superceded by the corresponding laws for the Lebesgue integral (Proposition 19.3.3).

Theorem 11.4.1 (Laws of Riemann integration). *Let I be a bounded interval, and let $f : I \to \mathbf{R}$ and $g : I \to \mathbf{R}$ be Riemann integrable functions on I.*

(a) *The function $f+g$ is Riemann integrable, and we have $\int_I (f+g) = \int_I f + \int_I g$.*

(b) *For any real number c, the function cf is Riemann integrable, and we have $\int_I (cf) = c(\int_I f)$.*

(c) *The function $f-g$ is Riemann integrable, and we have $\int_I (f-g) = \int_I f - \int_I g$.*

(d) *If $f(x) \geq 0$ for all $x \in I$, then $\int_I f \geq 0$.*

(e) *If $f(x) \geq g(x)$ for all $x \in I$, then $\int_I f \geq \int_I g$.*

(f) *If f is the constant function $f(x) = c$ for all x in I, then $\int_I f = c|I|$.*

(g) *Let J be a bounded interval containing I (i.e., $I \subseteq J$), and let $F : J \to \mathbf{R}$ be the function*

$$F(x) := \begin{cases} f(x) & \text{if } x \in I \\ 0 & \text{if } x \notin I \end{cases}$$

Then F is Riemann integrable on J, and $\int_J F = \int_I f$.

(h) *Suppose that $\{J, K\}$ is a partition of I into two intervals J and K. Then the functions $f|_J : J \to \mathbf{R}$ and $f|_K : K \to \mathbf{R}$ are Riemann integrable on J and K respectively, and we have*

$$\int_I f = \int_J f|_J + \int_K f|_K.$$

Proof. See Exercise 11.4.1. □

Remark 11.4.2. We often abbreviate $\int_J f|_J$ as $\int_J f$, even though f is really defined on a larger domain than just J.

Theorem 11.4.1 asserts that the sum or difference of any two Riemann integrable functions is Riemann integrable, as is any scalar multiple cf of a Riemann integrable function f. We now give some further ways to create Riemann integrable functions.

Theorem 11.4.3 (Max and min preserve integrability). *Let I be a bounded interval, and let $f : I \to \mathbf{R}$ and $g : I \to \mathbf{R}$ be a Riemann integrable function. Then the functions* $\max(f, g) : I \to \mathbf{R}$ *and* $\min(f, g) : I \to \mathbf{R}$ *defined by* $\max(f, g)(x) := \max(f(x), g(x))$ *and* $\min(f, g)(x) := \min(f(x), g(x))$ *are also Riemann integrable.*

Proof. We shall just prove the claim for $\max(f, g)$, the case of $\min(f, g)$ being similar. First note that since f and g are bounded, then $\max(f, g)$ is also bounded.

Let $\varepsilon > 0$. Since $\int_I f = \underline{\int}_I f$, there exists a piecewise constant function $\underline{f} : I \to \mathbf{R}$ which minorizes f on I such that

$$\int_I \underline{f} \geq \int_I f - \varepsilon.$$

Similarly we can find a piecewise constant $\underline{g} : I \to \mathbf{R}$ which minorizes g on I such that

$$\int_I \underline{g} \geq \int_I g - \varepsilon,$$

and we can find piecewise functions $\overline{f}$, $\overline{g}$ which majorize f, g respectively on I such that

$$\int_I \overline{f} \leq \int_I f + \varepsilon$$

and

$$\int_I \overline{g} \leq \int_I g + \varepsilon.$$

In particular, if $h : I \to \mathbf{R}$ denotes the function

$$h := (\overline{f} - \underline{f}) + (\overline{g} - \underline{g})$$

we have

$$\int_I h \leq 4\varepsilon.$$

On the other hand, $\max(\underline{f}, \underline{g})$ is a piecewise constant function on I (why?) which minorizes $\max(f, g)$ (why?), while $\max(\overline{f}, \overline{g})$ is similarly a piecewise constant function on I which majorizes $\max(f, g)$. Thus

$$\int_I \max(\underline{f}, \underline{g}) \leq \underline{\int}_I \max(f, g) \leq \overline{\int}_I \max(f, g) \leq \int_I \max(\overline{f}, \overline{g}),$$

and so

$$0 \leq \overline{\int}_I \max(f, g) - \underline{\int}_I \max(f, g) \leq \int_I \max(\overline{f}, \overline{g}) - \max(\underline{f}, \underline{g}).$$

But we have

$$\overline{f}(x) = \underline{f}(x) + (\overline{f} - \underline{f})(x) \leq \underline{f}(x) + h(x)$$

and similarly

$$\overline{g}(x) = \underline{g}(x) + (\overline{g} - \underline{g})(x) \leq \underline{g}(x) + h(x)$$

and thus

$$\max(\overline{f}(x), \overline{g}(x)) \leq \max(\underline{f}(x), \underline{g}(x)) + h(x).$$

Inserting this into the previous inequality, we obtain

$$0 \leq \overline{\int}_I \max(f, g) - \underline{\int}_I \max(f, g) \leq \int_I h \leq 4\varepsilon.$$

To summarize, we have shown that

$$0 \leq \overline{\int}_I \max(f, g) - \underline{\int}_I \max(f, g) \leq 4\varepsilon$$