2. Given β in F^n, how does one determine whether β is a linear combination of $\alpha_1, \ldots, \alpha_m$, i.e., whether β is in the subspace W?

3. How can one give an explicit description of the subspace W?

The third question is a little vague, since it does not specify what is meant by an 'explicit description'; however, we shall clear up this point by giving the sort of description we have in mind. With this description, questions (1) and (2) can be answered immediately.

Let A be the $m \times n$ matrix with row vectors α_i:

$$\alpha_i = (A_{i1}, \ldots, A_{in}).$$

Perform a sequence of elementary row operations, starting with A and terminating with a row-reduced echelon matrix R. We have previously described how to do this. At this point, the dimension of W (the row space of A) is apparent, since this dimension is simply the number of non-zero row vectors of R. If $\rho_1, \ldots, \rho_r$ are the non-zero row vectors of R, then $\mathcal{B} = \{\rho_1, \ldots, \rho_r\}$ is a basis for W. If the first non-zero coordinate of ρ_i is the k_ith one, then we have for $i \leq r$

(a) $\qquad R(i, j) = 0, \quad \text{if} \quad j < k_i$

(b) $\qquad R(i, k_j) = \delta_{ij}$

(c) $\qquad k_1 < \cdots < k_r.$

The subspace W consists of all vectors

$$\begin{aligned}\beta &= c_1\rho_1 + \cdots + c_r\rho_r \\ &= \sum_{i=1}^{r} c_i(R_{i1}, \ldots, R_{in}).\end{aligned}$$

The coordinates $b_1, \ldots, b_n$ of such a vector β are then

$$b_j = \sum_{i=1}^{r} c_i R_{ij}. \tag{2-23}$$

In particular, $b_{k_j} = c_j$, and so if $\beta = (b_1, \ldots, b_n)$ is a linear combination of the ρ_i, it must be the particular linear combination

$$\beta = \sum_{i=1}^{r} b_{k_i}\rho_i. \tag{2-24}$$

The conditions on β that (2-24) should hold are

$$b_j = \sum_{i=1}^{r} b_{k_i}R_{ij}, \qquad j = 1, \ldots, n. \tag{2-25}$$

Now (2-25) is the explicit description of the subspace W spanned by $\alpha_1, \ldots, \alpha_m$, that is, the subspace consists of all vectors β in F^n whose coordinates satisfy (2-25). What kind of description is (2-25)? In the first place it describes W as all solutions $\beta = (b_1, \ldots, b_n)$ of the system of homogeneous linear equations (2-25). This system of equations is of a very special nature, because it expresses $(n - r)$ of the coordinates as linear combinations of the r distinguished coordinates $b_{k_1}, \ldots, b_{k_r}$. One has complete freedom of choice in the coordinates b_{k_i}, that is, if $c_1, \ldots, c_r$ are *any* r scalars, there is one and only one vector β in W which has c_i as its k_ith coordinate.

The significant point here is this: Given the vectors α_i, row-reduction is a straightforward method of determining the integers $r, k_1, \ldots, k_r$ and the scalars R_{ij} which give the description (2-25) of the subspace spanned by $\alpha_1, \ldots, \alpha_m$. One should observe as we did in Theorem 11 that every subspace W of F^n has a description of the type (2-25). We should also point out some things about question (2). We have already stated how one can find an invertible $m \times m$ matrix P such that $R = PA$, in Section 1.4. The knowledge of P enables one to find the scalars $x_1, \ldots, x_m$ such that

$$\beta = x_1\alpha_1 + \cdots + x_m\alpha_m$$

when this is possible. For the row vectors of R are given by

$$\rho_i = \sum_{j=1}^{m} P_{ij}\alpha_j$$

so that if β is a linear combination of the α_j, we have

$$\begin{aligned}\beta &= \sum_{i=1}^{r} b_{k_i}\rho_i \\ &= \sum_{i=1}^{r} b_{k_i} \sum_{j=1}^{m} P_{ij}\alpha_j \\ &= \sum_{j=1}^{m} \sum_{i=1}^{r} b_{k_i}P_{ij}\alpha_j\end{aligned}$$

and thus

$$x_j = \sum_{i=1}^{r} b_{k_i}P_{ij}$$

is one possible choice for the x_j (there may be many).

The question of whether $\beta = (b_1, \ldots, b_n)$ is a linear combination of the α_i, and if so, what the scalars x_i are, can also be looked at by asking whether the system of equations

$$\sum_{i=1}^{m} A_{ij}x_i = b_j, \qquad j = 1, \ldots, n$$

has a solution and what the solutions are. The coefficient matrix of this system of equations is the $n \times m$ matrix B with *column* vectors $\alpha_1, \ldots, \alpha_m$. In Chapter 1 we discussed the use of elementary row operations in solving a system of equations $BX = Y$. Let us consider one example in which we adopt both points of view in answering questions about subspaces of F^n.

EXAMPLE 21. Let us pose the following problem. Let W be the subspace of R^4 spanned by the vectors

$$\begin{aligned}\alpha_1 &= (1, 2, 2, 1)\\ \alpha_2 &= (0, 2, 0, 1)\\ \alpha_3 &= (-2, 0, -4, 3).\end{aligned}$$

(a) Prove that α_1, α_2, α_3 form a basis for W, i.e., that these vectors are linearly independent.

(b) Let $\beta = (b_1, b_2, b_3, b_4)$ be a vector in W. What are the coordinates of β relative to the ordered basis $\{\alpha_1, \alpha_2, \alpha_3\}$?

(c) Let

$$\begin{aligned}\alpha_1' &= (1, 0, 2, 0)\\ \alpha_2' &= (0, 2, 0, 1)\\ \alpha_3' &= (0, 0, 0, 3).\end{aligned}$$

Show that α_1', α_2', α_3' form a basis for W.

(d) If β is in W, let X denote the coordinate matrix of β relative to the α-basis and X' the coordinate matrix of β relative to the α'-basis. Find the 3×3 matrix P such that $X = PX'$ for every such β.

To answer these questions by the first method we form the matrix A with row vectors α_1, α_2, α_3, find the row-reduced echelon matrix R which is row-equivalent to A and simultaneously perform the same operations on the identity to obtain the invertible matrix Q such that $R = QA$:

$$\begin{bmatrix} 1 & 2 & 2 & 1\\ 0 & 2 & 0 & 1\\ -2 & 0 & -4 & 3 \end{bmatrix} \rightarrow R = \begin{bmatrix} 1 & 0 & 2 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1 \end{bmatrix} \rightarrow Q = \frac{1}{6}\begin{bmatrix} 6 & -6 & 0\\ -2 & 5 & -1\\ 4 & -4 & 2 \end{bmatrix}$$

(a) Clearly R has rank 3, so α_1, α_2 and α_3 are independent.

(b) Which vectors $\beta = (b_1, b_2, b_3, b_4)$ are in W? We have the basis for W given by ρ_1, ρ_2, ρ_3, the row vectors of R. One can see at a glance that the span of ρ_1, ρ_2, ρ_3 consists of the vectors β for which $b_3 = 2b_1$. For such a β we have

$$\begin{aligned}\beta &= b_1\rho_1 + b_2\rho_2 + b_4\rho_3\\ &= [b_1, b_2, b_4]R\\ &= [b_1 \quad b_2 \quad b_4]QA\\ &= x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3\end{aligned}$$

where $x_i = [b_1 \quad b_2 \quad b_4]Q_i$:

$$\begin{aligned} x_1 &= b_1 - \tfrac{1}{3}b_2 + \tfrac{2}{3}b_4\\ x_2 &= -b_1 + \tfrac{5}{6}b_2 - \tfrac{2}{3}b_4\\ x_3 &= - \tfrac{1}{6}b_2 + \tfrac{1}{3}b_4. \end{aligned} \tag{2-26}$$

(c) The vectors α_1', α_2', α_3' are all of the form (y_1, y_2, y_3, y_4) with $y_3 = 2y_1$ and thus they are in W. One can see at a glance that they are independent.

(d) The matrix P has for its columns

$$P_j = [\alpha_j']_{\mathcal{B}}$$

where $\mathcal{B} = \{\alpha_1, \alpha_2, \alpha_3\}$. The equations (2-26) tell us how to find the coordinate matrices for α_1', α_2', α_3'. For example with $\beta = \alpha_1'$ we have $b_1 = 1$, $b_2 = 0$, $b_3 = 2$, $b_4 = 0$, and

$$\begin{aligned} x_1 &= 1 - \tfrac{1}{3}(0) + \tfrac{2}{3}(0) = 1 \\ x_2 &= -1 + \tfrac{5}{6}(0) - \tfrac{2}{3}(0) = -1 \\ x_3 &= - \tfrac{1}{6}(0) + \tfrac{1}{3}(0) = 0. \end{aligned}$$

Thus $\alpha_1' = \alpha_1 - \alpha_2$. Similarly we obtain $\alpha_2' = \alpha_2$ and $\alpha_3' = 2\alpha_1 - 2\alpha_2 + \alpha_3$. Hence

$$P = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

Now let us see how we would answer the questions by the second method which we described. We form the 4×3 matrix B with column vectors α_1, α_2, α_3:

$$B = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 0 \\ 2 & 0 & -4 \\ 1 & 1 & 3 \end{bmatrix}$$

We inquire for which y_1, y_2, y_3, y_4 the system $BX = Y$ has a solution.

$$\begin{bmatrix} 1 & 0 & -2 & y_1 \\ 2 & 2 & 0 & y_2 \\ 2 & 0 & -4 & y_3 \\ 1 & 1 & 3 & y_4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -2 & y_1 \\ 0 & 2 & 4 & y_2 - 2y_1 \\ 0 & 0 & 0 & y_3 - 2y_1 \\ 0 & 1 & 5 & y_4 - y_1 \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} 1 & 0 & -2 & y_1 \\ 0 & 0 & -6 & y_2 - 2y_4 \\ 0 & 1 & 5 & y_4 - y_1 \\ 0 & 0 & 0 & y_3 - 2y_1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & y_1 - \frac{1}{3}y_2 + \frac{2}{3}y_4 \\ 0 & 0 & 1 & \frac{1}{6}(2y_4 - y_2) \\ 0 & 1 & 0 & -y_1 + \frac{5}{6}y_2 - \frac{2}{3}y_4 \\ 0 & 0 & 0 & y_3 - 2y_1 \end{bmatrix}$$

Thus the condition that the system $BX = Y$ have a solution is $y_3 = 2y_1$. So $\beta = (b_1, b_2, b_3, b_4)$ is in W if and only if $b_3 = 2b_1$. If β is in W, then the coordinates (x_1, x_2, x_3) in the ordered basis $\{\alpha_1, \alpha_2, \alpha_3\}$ can be read off from the last matrix above. We obtain once again the formulas (2-26) for those coordinates.

The questions (c) and (d) are now answered as before.

EXAMPLE 22. We consider the 5×5 matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 2 & 4 & 1 & 10 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and the following problems concerning A

(a) Find an invertible matrix P such that PA is a row-reduced echelon matrix R.
(b) Find a basis for the row space W of A.
(c) Say which vectors $(b_1, b_2, b_3, b_4, b_5)$ are in W.
(d) Find the coordinate matrix of each vector $(b_1, b_2, b_3, b_4, b_5)$ in W in the ordered basis chosen in (b).
(e) Write each vector $(b_1, b_2, b_3, b_4, b_5)$ in W as a linear combination of the rows of A.
(f) Give an explicit description of the vector space V of all 5×1 column matrices X such that $AX = 0$.
(g) Find a basis for V.
(h) For what 5×1 column matrices Y does the equation $AX = Y$ have solutions X?

To solve these problems we form the augmented matrix A' of the system $AX = Y$ and apply an appropriate sequence of row operations to A'.

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & y_1 \\ 1 & 2 & -1 & -1 & 0 & y_2 \\ 0 & 0 & 1 & 4 & 0 & y_3 \\ 2 & 4 & 1 & 10 & 1 & y_4 \\ 0 & 0 & 0 & 0 & 1 & y_5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & 3 & 0 & y_1 \\ 0 & 0 & -1 & -4 & 0 & -y_1 + y_2 \\ 0 & 0 & 1 & 4 & 0 & y_3 \\ 0 & 0 & 1 & 4 & 1 & -2y_1 + y_4 \\ 0 & 0 & 0 & 0 & 1 & y_5 \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & y_1 \\ 0 & 0 & 1 & 4 & 0 & y_1 - y_2 \\ 0 & 0 & 0 & 0 & 0 & -y_1 + y_2 + y_3 \\ 0 & 0 & 0 & 0 & 1 & -3y_1 + y_2 + y_4 \\ 0 & 0 & 0 & 0 & 1 & y_5 \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & y_1 \\ 0 & 0 & 1 & 4 & 0 & y_1 - y_2 \\ 0 & 0 & 0 & 0 & 1 & y_5 \\ 0 & 0 & 0 & 0 & 0 & -y_1 + y_2 + y_3 \\ 0 & 0 & 0 & 0 & 0 & -3y_1 + y_2 + y_4 - y_5 \end{bmatrix}$$

(a) If

$$PY = \begin{bmatrix} y_1 \\ y_1 - y_2 \\ y_5 \\ -y_1 + y_2 + y_3 \\ -3y_1 + y_2 + y_4 - y_5 \end{bmatrix}$$

for all Y, then

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & 1 & 1 & 0 & 0 \\ -3 & 1 & 0 & 1 & -1 \end{bmatrix}$$

hence PA is the row-reduced echelon matrix

$$R = \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

It should be stressed that the matrix P is not unique. There are, in fact, many invertible matrices P (which arise from different choices for the operations used to reduce A') such that $PA = R$.

(b) As a basis for W we may take the non-zero rows

$$\begin{aligned} \rho_1 &= (1 \quad 2 \quad 0 \quad 3 \quad 0) \\ \rho_2 &= (0 \quad 0 \quad 1 \quad 4 \quad 0) \\ \rho_3 &= (0 \quad 0 \quad 0 \quad 0 \quad 1) \end{aligned}$$

of R.

(c) The row-space W consists of all vectors of the form

$$\begin{aligned} \beta &= c_1\rho_1 + c_2\rho_2 + c_3\rho_3 \\ &= (c_1, 2c_1, c_2, 3c_1 + 4c_2, c_3) \end{aligned}$$

where c_1, c_2, c_3 are arbitrary scalars. Thus $(b_1, b_2, b_3, b_4, b_5)$ is in W if and only if

$$(b_1, b_2, b_3, b_4, b_5) = b_1\rho_1 + b_3\rho_2 + b_5\rho_3$$

which is true if and only if

$$\begin{aligned} b_2 &= 2b_1 \\ b_4 &= 3b_1 + 4b_3. \end{aligned}$$

These equations are instances of the general system (2-25), and using them we may tell at a glance whether a given vector lies in W. Thus $(-5, -10, 1, -11, 20)$ is a linear combination of the rows of A, but $(1, 2, 3, 4, 5)$ is not.

(d) The coordinate matrix of the vector $(b_1, 2b_1, b_3, 3b_1 + 4b_3, b_5)$ in the basis $\{\rho_1, \rho_2, \rho_3\}$ is evidently

$$\begin{bmatrix} b_1 \\ b_3 \\ b_5 \end{bmatrix}.$$

(e) There are many ways to write the vectors in W as linear combinations of the rows of A. Perhaps the easiest method is to follow the first procedure indicated before Example 21:

$$\begin{aligned} \beta &= (b_1, 2b_1, b_3, 3b_1 + 4b_3, b_5) \\ &= [b_1, b_3, b_5, 0, 0] \cdot R \\ &= [b_1, b_3, b_5, 0, 0] \cdot PA \\ &= [b_1, b_3, b_5, 0, 0] \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & 1 & 1 & 0 & 0 \\ -3 & 1 & 0 & 1 & -1 \end{bmatrix} \cdot A \\ &= [b_1 + b_3, -b_3, 0, 0, b_5] \cdot A. \end{aligned}$$

In particular, with $\beta = (-5, -10, 1, -11, 20)$ we have

$$\beta = (-4, -1, 0, 0, 20) \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 2 & 4 & 1 & 10 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

(f) The equations in the system $RX = 0$ are

$$\begin{aligned} x_1 + 2x_2 + 3x_4 &= 0 \\ x_3 + 4x_4 &= 0 \\ x_5 &= 0. \end{aligned}$$

Thus V consists of all columns of the form

$$X = \begin{bmatrix} -2x_2 - 3x_4 \\ x_2 \\ -4x_4 \\ x_4 \\ 0 \end{bmatrix}$$

where x_2 and x_4 are arbitrary.

(g) The columns

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -3 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}$$

form a basis of V. This is an example of the basis described in Example 15.