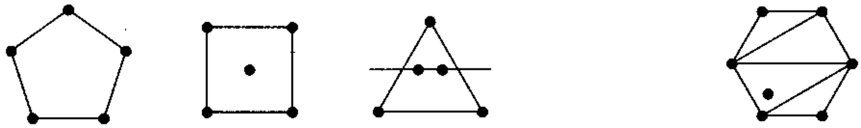
immediately. If it is a triangle, then the other two points lie inside. By the pigeonhole principle(!), two of the vertices of the triangle are on one side of the line through the two inside points. These two vertices together with the two points inside form a convex quadrilateral, as illustrated below.



In a convex m-gon, any four corners determine a convex quadrilateral. We need the converse: *(2) If every 4-subset of m points in the plane forms a convex quadrilateral, then the m points form a convex m-gon.* If the claim fails, then the convex hull of the m points consists of t points, for some $t < m$. The remaining points lie inside the t-gon. When we triangulate the t-gon, as illustrated on the right above, a point inside lies in one of the triangles. With the vertices of that triangle, it forms a 4-set that does not determine a convex quadrilateral.

To prove the theorem, let $N = R(m, 5; 4)$. Given N points in a plane with no three on a line, color each 4-set by convexity: red if it determines a convex quadrilateral, blue if it does not. By fact (1), there cannot be five points whose 4-subsets are all blue. By Ramsey's Theorem, this means there are m points whose 4-subsets are all red. By fact (2), they form a convex m-gon. Hence $N(m)$ exists and is at most $R(m, 5; 4)$. ∎

The bound $R(m, 5; 4)$ is very loose. It is exact for $m = 4$, where fact (1) implies that $N(4) = 5 = R(4, 5; 4)$. In contrast, $N(5) = 9$ (Exercise 10), but $R(5, 5; 4)$ is enormous. Erdős and Szekeres conjectured that $N(m) = 2^{m-2} + 1$ and proved that $2^{m-2} \le N(m) \le \binom{2m-4}{m-2} + 1$.

Another application concerns search strategies for numbers stored in tables. From a set U, a subset of size n is stored in a table of size n according to some rule for storing n-sets. Yao [1981] used Ramsey's Theorem to prove that when U is large, the strategy minimizing the worst-case number of probes required to test whether some element of U is in the table is to store the chosen set in sorted order and test membership by binary search. (For small U, this strategy is not best!) The value that Ramsey's Theorem yields for "large" is probably much larger than needed.

RAMSEY NUMBERS

Ramsey's Theorem defines the Ramsey numbers $R(p_1, \ldots, p_k; r)$. No exact formula is known, and few Ramsey numbers have been computed. To prove that $R(p_1, \ldots, p_k; r) = N$, we must exhibit a k-coloring of the r-sets among $N-1$ points that meets no quota (or show that one exists without constructing it), *and* we must show that *every* coloring on N points meets some quota.

In principle, we could use a computer to examine all k-colorings of $\binom{[n]}{r}$ for successive n until we find the first N such that every such coloring meets a quota p_i for some i. Even for 2-color Ramsey numbers, $2^{\binom{n}{2}}$ rapidly becomes too large to contemplate. Erdős joked that if an alien being threatened to destroy us unless we told it the exact value of $R(5, 5)$, then we should set all the computers in the world to work on an exhaustive solution. If we were asked for $R(6, 6)$, then his advice was to try to destroy the alien.

When $r = 2$, we abbreviate the notation $R(p_1, \ldots, p_k; r)$ to $R(p_1, \ldots, p_k)$. When $p = p_1 = \cdots = p_k$, we abbreviate it to $R_k(p; r)$. For $r > 2$, little is known other than $R(4, 4; 3) = 13$ (McKay–Radziszowski [1991]). Even for $r = 2$, only one Ramsey number is known exactly when $k > 2$, which is $R(3, 3, 3) = 17$. The table below contains the known values of $R(p, q)$ and the best known upper and lower bounds for several other values as of July 1999. Several of these bounds have improved slightly since the first edition of this book. The current bounds are maintained in Radziszowski [1995], which is periodically updated.

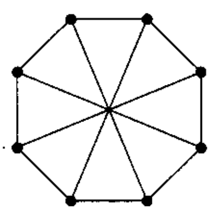
	3	4	5	6	7	8	9
3	6	9	14	18	23	28	36
4		18	25	35/41	49/61	55/84	69/115
5			43/49	58/87	80/143	95/216	116/316
6				102/165	109/298	122/495	153/780

The computations of $R(3, 9)$ (Grinstead–Roberts [1982]), $R(3, 8)$ (McKay–Zhang [1992]), and $R(4, 5)$ (McKay–Radziszowski [1995]) are recent; the others are much older (due primarily to Greenwood–Gleason [1955], Kalbfleisch [1967], and Graver–Yackel [1968]).

We prove only the first two of these results (see Exercise 16 for $R(3, 5)$). When $r = k = 2$, we simplify terminology by using two colors called "in" and "out". Ramsey's Theorem for this case then becomes: "There exists a minimum integer $R(p, q)$ such that every graph on $R(p, q)$ vertices has a clique of size p or an independent set of size q".

8.3.9. Example. $R(3, 3) = 6$. We showed earlier that $R(3, 3) \leq 6$. Since the 5-cycle has no triangle and no independent 3-set, $R(3, 3) \geq 6$. ■

8.3.10. Example. $R(3, 4) = 9$. The graph below has no K_3 and no $\overline{K}_4$, since four independent vertices on an 8-cycle include pairs of opposite vertices on the cycle. Hence $R(3, 4) \geq 9$.



Given a vertex x in a graph G, we can add x to two adjacent neighbors to form a triangle or add x to an independent 3-set of nonneighbors to form an independent 4-set. Since $R(2, 4) = 4$ and $R(3, 3) = 6$, we conclude that if x has four neighbors or has six nonneighbors, then G has a triangle or an independent 4-set. Avoiding both possibilities limits x to at most three neighbors and at most five nonneighbors, which yields $n(G) \le 9$. If this occurs for a 9-vertex graph, then *every* vertex has exactly three neighbors. Since the degree-sum formula forbids 3-regular graphs of order 9, we obtain $R(3,4) = 9$. ■

The proof of Ramsey's Theorem yields a (very large) recursive upper bound on $R(p,q;r)$. Graham–Rothschild–Spencer [1980, 1990] explains how large.

8.3.11. Theorem. $R(p,q) \le R(p-1,q) + R(p,q-1)$. If both summands on the right are even, then the inequality is strict.

Proof: If a vertex in an arbitrary graph has $R(p-1,q)$ neighbors or $R(p,q-1)$ nonneighbors, then the graph has a p-clique or an independent q-set. With $R(p-1,q)+R(p,q-1)$ points altogether in the graph, the pigeonhole principle guarantees that one of these possibilities occurs. Equality in the bound requires a regular graph with $R(p-1,q)+R(p,q-1)-1$ vertices. If both summands are even, this requires a regular graph of odd degree on an odd number of vertices, which is impossible. ■

Since $R(p,2) = R(2,p) = p$, Theorem 8.3.11 yields $R(p,q) \le \binom{p+q-2}{p-1}$ (Exercise 15). The lack of exact answers has led to study of asymptotics. For fixed q and large p, $R(p,q) \le cp^{q-1}\log\log p/\log p$ (Graver–Yackel [1968], Chung–Grinstead [1983]). For $q = 3$, the answer is known within a constant factor:

$$c'p^2/\log p \le R(p,3) \le cp^2/\log p.$$

The upper bound is due to Ajtai–Komlós–Szemerédi [1980]; the lower to Kim [1995]. All these bounds use probabilistic methods (Section 8.5).

Ramsey numbers for equal quotas are called **diagonal Ramsey numbers**. Asymptotically, the upper bound of $\binom{2p-2}{p-1}$ for $R(p,p)$ is $c4^p/\sqrt{p}$. Exercise 14 presents a constructive lower bound that is polynomial in p. The best known constructive lower bound grows faster than every polynomial in p but slower than every exponential in p (Frankl–Wilson [1981], Exercise 29).

An exponential lower bound can be proved by counting methods. It yields

$$\sqrt{2} \le \liminf R(p,p)^{1/p} \le \limsup R(p,p)^{1/p} \le 4.$$

Determination of this limit (and whether it exists) is the foremost open problem about Ramsey numbers.

8.3.12. Theorem. (Erdős [1947]). $R(p,p) > (e\sqrt{2})^{-1}p2^{p/2}(1+o(1))$.

Proof: Consider the graphs with vertex set $[n]$. Each possible p-clique occurs in $2^{\binom{n}{2}-\binom{p}{2}}$ of these $2^{\binom{n}{2}}$ graphs. Similarly, each p-set occurs as an independent set in $2^{\binom{n}{2}-\binom{p}{2}}$ of these graphs. Discarding this amount for each possible p-clique and each possible independent p-set leaves a lower bound on the number of graphs having no p-clique or independent p-set.

Since there are $\binom{n}{p}$ ways to choose p vertices, the inequality $2\binom{n}{p}2^{-\binom{p}{2}} < 1$ thus implies $R(p, p) > n$. Rough approximations yield $\binom{n}{p}2^{1-\binom{p}{2}} < 1$ whenever $n < 2^{p/2}$. More careful approximations (using Stirling's formula to approximate the factorials) lead to the result claimed. ■

GRAPH RAMSEY THEORY

Ramsey's Theorem for $r = 2$ says that k-coloring the edges of a large enough complete graph forces a monochromatic complete subgraph. A monochromatic p-clique contains a monochromatic copy of every p-vertex graph. Perhaps monochromatic copies of graphs with fewer edges can be forced by coloring a smaller graph than needed to force K_p. For example, 2-coloring the edges of K_3 always yields a monochromatic P_3, although six points are needed to force a monochromatic triangle. This suggests many Ramsey number questions, some easier to answer than the questions for cliques.

8.3.13. Definition. Given simple graphs $G_1, \dots, G_k$, the **(graph) Ramsey number** $R(G_1, \dots, G_k)$ is the smallest integer n such that every k-coloring of $E(K_n)$ contains a copy of G_i in color i for some i. When $G_i = G$ for all i, we write $R_k(G)$ for $R(G_1, \dots, G_k)$.

Burr [1983] determined $R(G, G)$, called the "Ramsey number of G", for all 113 graphs with at most six edges and no isolated vertices. Nice formulas are known for $R(G_1, G_2)$ in some cases. Again our two colors are red and blue.

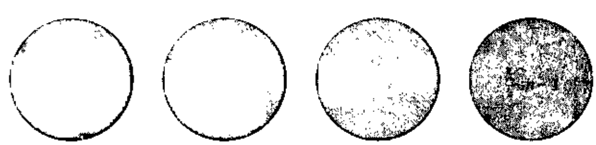
8.3.14. Theorem. (Chvátal [1977]) If T is an m-vertex tree, then $R(T, K_n) = (m-1)(n-1)+1$.

Proof: For the lower bound, color $K_{(m-1)(n-1)}$ by letting the red graph be $(n-1)K_{m-1}$. With red components of order $m-1$, there is no red m-vertex tree. The blue edges form an $n-1$-partite graph and hence cannot contain K_n.

The proof of the upper bound uses induction on each parameter, focusing on the neighbors of one vertex. Our presentation uses induction on n, invoking a property of trees that we proved in Chapter 2 by induction on m. The basis step is $n = 1$; no edges are needed to obtain K_1.

Given a 2-coloring of $E(K_{(m-1)(n-1)+1})$, consider a vertex x. If x has more than $(m-1)(n-2)$ neighbors along blue edges, then the induction hypothesis yields a red T or a blue K_{n-1} among them. This yields a red T or a blue K_n (with x) in the full coloring.

Otherwise, every vertex has at most $(m-1)(n-2)$ incident blue edges and thus at least $m-1$ incident red edges. This yields a red T, because every graph with minimum degree at least $m-1$ contains T (Proposition 2.1.8). ■

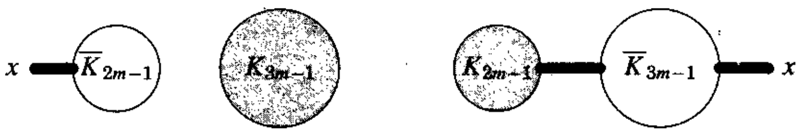


Whenever the largest component of G has m vertices and $\chi(H) = n$, the construction in Theorem 8.3.14 yields $R(G, H) \geq (m-1)(n-1)+1$ (Chvátal–Harary [1972]). Burr and Erdős [1983] conjectured that $R(G, K_n) = (m-1)(n-1)+1$ when m is sufficiently large relative to $n(H)$ and $\max_{F\subseteq G} \frac{e(F)}{n(F)}$. Although this holds (Burr [1981]) when G has many vertices of degree 2 and in some other cases, Brandt [2000] showed that for every nonbipartite graph H (such as K_n) and every $h \in \mathbb{R}$, there is a threshold d_0 such that $R(G, H) > hn(G)$ for almost every d-regular graph G with $d > d_0$.

In the upper bound for Theorem 8.3.14, it is crucial that the color classes in H are single vertices. When this fails, the lower bound can be very weak. When $G = H = mK_3$, for example, the Chvátal–Harary result yields $R(G, H) \geq (3-1)(3-1)+1 = 5$, but the correct value is $5m$. Here the coloring for the lower bound is surprisingly asymmetric, considering the symmetry of the inputs.

8.3.15. Theorem. (Burr–Erdős–Spencer [1975]) $R(mK_3, mK_3) = 5m$ for $m \geq 3$.

Proof: Let the red graph be $K_{3m-1} + K_{1,2m-1}$, as shown below. Every triangle in this graph uses three vertices from the $3m-1$-clique, but the clique does not have enough vertices to make m disjoint triangles. The complementary blue graph is $(K_{2m-1} + K_1) \vee \overline{K}_{3m-1}$. Every blue triangle has at least 2 vertices in the copy of K_{2m-1}, so there cannot be m disjoint blue triangles.



For the upper bound, we use induction on m. Basis step: $m = 2$. This requires a case analysis that is fairly short if phrased carefully (Exercise 26).

Induction step: $m \geq 3$. Since $5m > R(3,3) = 6$, we know that every 2-coloring contains a monochromatic triangle. Discarding vertices of triangles as we find them, we can continue to find monochromatic triangles while at least six vertices remain. Since $5m - 3m \geq 6$ for $m \geq 3$, we find m disjoint monochromatic triangles. If these all have the same color, then we are done.

Otherwise, we have at least one triangle in each color. Let abc be a red triangle, and let def be a blue triangle disjoint from it. Of the nine edges between them, we may assume by symmetry that at least five are red. Some pair of these must have a common endpoint in def.

Now we have a red triangle and a blue triangle with a common vertex; together they have five vertices. Since $m > 2$, the induction hypothesis for the coloring on the remaining $5m-5$ vertices yields $(m-1)K_3$ in one color. We add the appropriately colored triangle from the five special vertices. ■

Readers worried about the omission of the basis step in Theorem 8.3.15 may consider coloring K_{11}. Avoiding $2K_3$ forces a bowtie (monochromatic triangles with a common vertex) as above, but then we find another monochromatic triangle among the remaining six points. This completes a proof that $R(mK_3, mK_3) \leq 5m + 1$. Related results appear in Exercises 27–28.

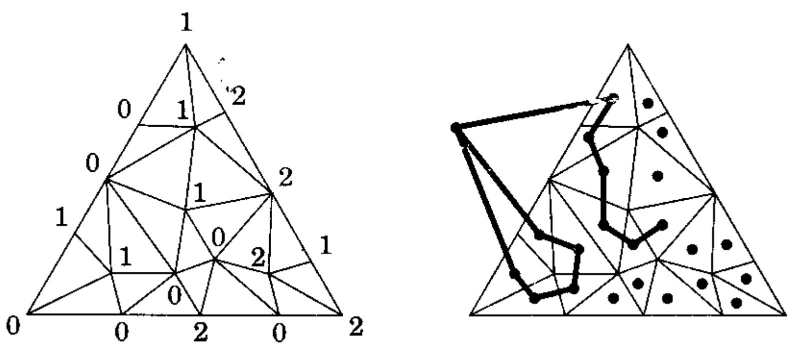
We mention one remarkable result. The Ramsey number of an arbitrary graph may be exponential in the number of vertices, such as for K_n. Chvátal, Rödl, Szemerédi, and Trotter [1983] proved that for the class of graphs with maximum degree d, the Ramsey number grows at most linearly in the number of vertices! In other words, $R(G, G) \le cn(G)$, where c is a constant depending only on d. Of course, the constant is a fast-growing function of d, but it does not depend on $n(G)$. The proof uses the Szemerédi Regularity Lemma [1978], itself a difficult result with many applications.

SPERNER'S LEMMA AND BANDWIDTH

Although Sperner's Lemma is not generally considered part of Ramsey theory, we include this material in this section because Sperner's Lemma has the flavor of Ramsey theory: every labeling of a triangulation that satisfies certain boundary conditions contains a piece with a special labeling (one element from *each* class). Like Ramsey's Theorem, Sperner's Lemma uses very simple ideas but has subtle applications; Ramsey's Theorem relies on the pigeonhole principle and induction, while Sperner's Lemma uses only a parity argument (and induction for a generalization to higher dimensions).

8.3.16. Definition. A **simplicial subdivision** of a large triangle T is a partition of T into triangular **cells** such that every intersection of two cells is a common edge or corner. We call the corners of cells **nodes**. A **proper labeling** of a simplicial subdivison of T assigns labels from $\{0, 1, 2\}$ to the nodes, avoiding label i on the ith edge of T, for $i \in \{0, 1, 2\}$. A **completely labeled cell** is a cell having all three labels on its corners.

In a proper labeling, each label appears at one corner of T, and label i avoids the edge of T joining the corners not labeled i. The figure below illustrates a simplicial subdivision and the graph we will obtain from it to prove that it has a completely labeled cell.



8.3.17. Theorem. (Sperner's Lemma [1928]) Every properly labeled simplicial subdivision has a completely labeled cell.

Proof: We prove the stronger result that there are an odd number of completely labeled cells. We seek such a cell by beginning outside T and entering a cell by crossing an edge with labels 0 and 1. If we reach a cell whose third label is 2, we are finished. If not, then the third label is 0 or 1, and the cell has another 0,1-edge. By crossing it, we enter a new cell and can continue looking for a cell with the third label.

This suggests defining a graph G encoding the possible steps. We include a vertex for each cell plus one vertex for the outside region. Two vertices of G are adjacent if those regions share a boundary edge with endpoints labeled 0 and 1. The graph on the right above results from the proper labeling on the left.

A completely labeled cell becomes a vertex of degree 1 in G. A cell with no 0 or no 1 becomes a vertex of degree 0. The remaining cells have corners labeled $0, 0, 1$ or $0, 1, 1$ and become vertices of degree 2. Hence the desired cells become vertices of degree 1 in G, and these are the only cells that become vertices of odd degree. We have transformed the original problem into the problem of showing that G has such a vertex of degree 1.

The vertex v for the outside region also has odd degree. As we travel from the 0-corner to the 1-corner along the edge of T that avoids label 2, we cross an edge of G involving v every time we switch from a 0 to a 1 or back again. Since we start with 0 and end with 1, we switch an odd number of times. Hence v has odd degree. Since the number of vertices of odd degree in every graph is even, the number of vertices other than v having odd degree is odd, so there are an odd number of completely labeled cells. ■

8.3.18. Application. *Brouwer Fixed-Point Theorem.* Brouwer's Theorem (for two dimensions) can be interpreted as saying that a continuous mapping from a triangular region T to itself must have a fixed point. Suppose that the corners of T are the points (vectors) v_0, v_1, v_2. Just as we can express a point on a segment uniquely as a weighted average of its endpoints, so we can express each $v \in T$ uniquely as a weighted average of the corners: $v = a_0v_0 + a_1v_1 + a_2v_2$, where $\sum a_i = 1$ and each $a_i \geq 0$ (Exercise 37). We can specify v by its vector of coefficients $a = (a_0, a_1, a_2)$.

Define sets S_0, S_1, S_2 from the mapping f by placing $a \in S_i$ if $a_i' \leq a_i$, where $f(a) = a'$. Since the coefficients of each point sum to 1, every point in T belongs to some S_i, and a point belongs to all three sets if and only if it is a fixed point for f. We want to show that the three sets have a common point.

Given a simplicial subdivision of T, for each node a choose a label i such that $a \in S_i$. Points on the edge of T opposite v_i have ith coordinate 0. Their ith coordinate cannot decrease under f, so we can choose a label other than i for each point on that edge. The resulting labeling is proper, and Sperner's Lemma guarantees a completely labeled cell. Repeating the process using triangulations with successively smaller cells yields a sequence of successively smaller completely labeled triangles. Let the jth triangle have corners x_j, y_j, z_j with labels 0,1,2, respectively. In each S_i, we obtain an infinite sequence of points.

The remaining details are topological; we merely suggest the steps. Since f is continuous, each S_i is closed and bounded. Every infinite sequence of points in a closed and bounded set has a convergent subsequence. Hence $\{x_1, x_2, \ldots\}$ has a convergent subsequence; let x_{i_k} be its kth entry. Because the distance from x_{i_k} to y_{i_k} and z_{i_k} approaches 0, these subsequences also converge to the same point. Since S_0, S_1, S_2 are closed and bounded, this limit point belongs to all three of them and is a fixed point of f. ∎

We also apply Sperner's Lemma to solve a problem on the "triangular grid".

8.3.19. Definition. When the vertices of G are numbered with distinct integers, the **dilation** is the maximum difference between integers assigned to adjacent vertices. The **bandwidth** $B(G)$ of a graph G is the minimum dilation of a numbering of G.

Dilation is always minimized when there are no gaps in the numbering, but it can be convenient to allow gaps (Exercise 42). The name "bandwidth" comes from matrix theory; the optimal numbering describes a permutation of the rows and columns of the adjacency matrix so that the 1's appear only in diagonal bands close to the main diagonal; arranging the matrix in this order can speed up computation of the inverse. Another motivation is to minimize the delay between adjacent vertices when the vertices must be processed in a linear order. Computation of bandwidth is NP-hard even for trees with maximum degree 3 (Garey–Graham–Johnson–Knuth [1978]).

We present two fundamental lower bounds on bandwidth.

8.3.20. Lemma. $B(G) \ge \max_{H\subseteq G} \frac{n(H)-1}{\operatorname{diam} H}$.

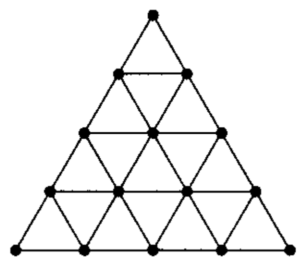
Proof: Every numbering of G contains a numbering of each subgraph of G. On every subgraph H, two numbers differing by at least $n(H) - 1$ are used. By the pigeonhole principle, some edge on a path between the two corresponding vertices has dilation at least $n(H) - 1$ divided by the distance between them. ∎

8.3.21. Lemma. (Harper [1966]) $B(G) \ge \max_k \min\{|\partial S| \colon |S| = k\}$, where ∂S denotes the subset of vertices in a set $S \subseteq V(G)$ that have at least one neighbor outside S.

Proof: For every value of k, some set S of k vertices must be the first k vertices in the optimal numbering of G. The bandwidth of G must be at least $|\partial S|$, because the vertex among ∂S that has the least label has an edge of dilation at least $|\partial S|$ to its neighborhood above S. ∎

Chung [1988] named the first bound the **local density** bound. The computation of Harper's bound is usually difficult. For the cube Q_k, the value is $\sum_{i=0}^{n-1} \binom{i}{\lfloor i/2 \rfloor}$. For the grid $P_m \square P_n$, the value of Harper's lower bound is $\min\{m, n\}$, which can be achieved (Exercise 43).

8.3.22. Example. *The triangular grid.* The triangular grid T_l consists of vertices (i, j, k) such that i, j, k are nonnegative integers summing to l, with two vertices adjacent if the total of the absolute differences in corresponding coordinates is 2. Below we show T_4. Numbering the vertices by rows produces an upper bound of $l + 1$ for $B(T_l)$. This is optimal, but the local density bound is only about $l/2$, and Harper's bound is about $l/\sqrt{2}$. Sperner's Lemma can be used to prove that $l + 1$ is optimal. ∎



Let G be the graph formed by a simplicial subdivision of a triangle. The outer boundary of G is a cycle, the bounded regions are triangles, and the cycle is partitioned into three paths by the corners of the large triangle. We say that a **connector** is a vertex set inducing a connected subgraph that contains a vertex of each boundary path.

8.3.23. Lemma. (Hochberg–McDiarmid–Saks [1995]) Let T be a simplicial subdivision in which each vertex is assigned red or blue. Let R, B be the subgraphs induced by the red and by the blue vertices, respectively. For each such coloring, exactly one of R, B contains a connector.

Proof: For each vertex v, consider the vertices reachable from v using vertices with the same color as v. If the three sides are not all reachable, label v with the smallest index of a side not reachable from v. For the vertices on the ith side, the label i does not appear. If there is no connector, then each node has a label, and this is a proper labeling of T.

By Sperner's Lemma, there is a completely labeled cell. Since the cell has three corners and we only used two colors R, B, two of the corners of this cell have the same color. Since they are adjacent, they can reach the same set of vertices in their color. Hence the least side unreachable from them cannot be different. This contradiction means that we could not have constructed the specified labeling. Hence there is a vertex from which every side is reachable.

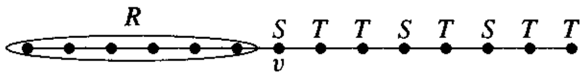
If one color has a connector, it partitions the remaining vertices into sets such that from each set at least one side is unreachable. Hence there cannot be connectors in both colors. ∎

8.3.24. Theorem. (Hochberg–McDiarmid–Saks [1995]) Let G be a graph that triangulates a region bounded by a cycle C partitioned into three paths. If k is the minimum over $v \in V(G)$ of the sum of the distances from v to each of the three paths, then $B(G) \geq k + 1$.

Proof: Let f be a numbering of G. Let t be the maximum index such that the subgraph induced by the vertices numbered $1, \ldots, t$ does not have a component meeting all three paths. Let R be this vertex set, let S be the set of vertices outside R having neighbors in R, and let T be the remaining vertices.

By construction, the vertex v with $f(v) = t + 1$ belongs to S. Since $R \cup \{v\}$ contains a connector, $R \cup S$ contains a connector, and T does not. Since there is no edge between R and T and R contains no connector, $R \cup T$ contains no connector. Now Lemma 8.3.23 implies that S contains a connector. The set S equals $\partial(S \cup T)$ for the terminal segment $S \cup T$ in the numbering. Therefore, the numbering has difference at least $|S|$ on some edge from S to R.

A connector contains walks from each of its vertices to each of the three boundary paths. By hypothesis, the sum of the lengths of these walks from any fixed vertex is at least k. There exists a vertex in S for which these walks in S are disjoint paths. Hence $|S| \geq k + 1$. ■



8.3.25. Corollary. The triangular grid T_l has bandwidth $l + 1$.

Proof: For each vertex (i, j, k) in T_l, the distances to the three sides are i, j, k, respectively, so the sum of the distances is l. By Theorem 8.3.24, the bandwidth is at least $l + 1$, which we have observed is achievable. ■

EXERCISES

8.3.1. (−) Each of two concentric discs has 20 radial sections of equal size. For each disc, 10 sections are painted red and 10 blue, in some arrangement. Prove that the two discs can be aligned so that at least 10 sections on the inner disc match colors with the corresponding sections on the outer disc.

8.3.2. For $n \in \mathbb{N}$, let S be a set of $n + 1$ elements in $\{1, \ldots, 2n\}$. Prove that S has two elements with greatest common factor 1 and has two elements such that one divides the other. For each conclusion, exhibit a subset of size n where it does not hold; hence these conclusions are best possible.

8.3.3. Use partial sums and the pigeonhole principle to prove the following statements.

a) Every set of n integers contains a nonempty subset whose sum is divisible by n. (Also exhibit a collection of $n - 1$ integers with no such subset.)

b) Given $x \in \mathbb{R}$, prove that at least one of $\{x, 2x, \ldots, (n-1)x\}$ differs by at most $1/n$ from an integer.

8.3.4. (!) A private club has 90 rooms and 100 members. Keys are given to the members so that any 90 members have access to the rooms in the sense that each of these 90 members will have a key to a different room. (They do not share their keys.) Prove that at least 990 keys are needed and that 990 suffice.

8.3.5. Let T be a tree. Use the technique of Theorem 8.3.2 to prove that the center of T consists of one vertex or two adjacent vertices (this proves Theorem 2.1.13 again). (Jordan [1869], Graham–Entringer–Székely [1994])

8.3.6. Prove that every set of $2^m + 1$ integer lattice points in $\mathbb{R}^m$ contains a pair of points whose centroid (mean vector) is also an integer lattice point.

8.3.7. Prove that every 2-coloring of the integer lattice points in $\mathbb{R}^m$ has a collection of n points with the same color whose centroid (mean vector) is an integer lattice point also having that color. (Hint: Ramsey's Theorem is not needed; there is a short proof using only the pigeonhole principle.) (Bòna [1990])

8.3.8. Let S be a collection of $n+1$ positive integers summing to k. For $k \le 2n+1$, prove that S has a subset with sum i for each $i \in [k]$. For each n, exhibit a collection for which this fails when $k = 2n+2$.

8.3.9. For even n, construct an ordering of $E(K_n)$ so that the maximum length of an increasing trail is $n-1$. (Comment: This proves that Theorem 8.3.4 is best possible when n is even. It also is best possible when n is odd and at least 9, but the construction is much more difficult.) (Graham–Kleitman [1973])

8.3.10. Let S be a set of nine points in the plane (no three collinear). Prove that S contains the vertex set of a convex 5-gon. Exhibit a set of eight points without this property.

8.3.11. (!) Let S be a set of $R(m, m; 3)$ points in the plane no three of which are collinear. Prove that S contains m points that form a convex m-gon. (Tarsi)

8.3.12. Recall that a digraph is *simple* if no two edges have the same ordered pair of endpoints. A **monotone tournament** is a tournament in which the orientation of the edges always agrees with the order of the indices on the vertices or always disagrees with that order. A **complete loopless digraph** has one copy of each ordered pair of distinct vertices as an edge. Given m, prove that if N is sufficiently large, then every simple loopless digraph with vertex set $[N]$ has an independent set of order m or a monotone tournament of order m or a complete loopless digraph of order m.

8.3.13. (!) *Schur's Theorem.* (Schur [1916])

a) Given $k > 0$, prove that there exists a least integer s_k such that every k-coloring of the integers $1, \dots, s_k$ yields a monochromatic x, y, z (not necessarily distinct) such that $x + y = z$. (Hint: Apply Ramsey's Theorem for $r = 2$.)

b) Prove constructively that $s_k \ge 3s_{k-1} - 1$ and hence that $s_k \ge (3^k - 1)/2$.

8.3.14. (!) The **composition** or **lexicographic product** of two simple graphs G and H is the simple graph $G[H]$ whose vertex set is $V(G) \times V(H)$, with edges given by $(u, v) \leftrightarrow (u', v')$ if and only if (1) uu' is an edge of G, or (2) $u = u'$ and vv' is an edge of H.

a) Prove that $\alpha(G[H]) = \alpha(G)\alpha(H)$.

b) Prove that the complement of $G[H]$ is $\overline{G}[\overline{H}]$.

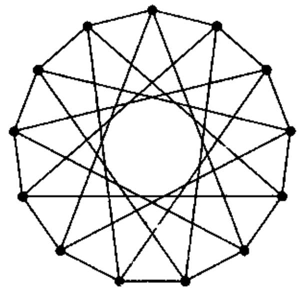
c) Use parts (a) and (b) to prove by construction that

$$R(pq+1, pq+1) - 1 \ge [R(p+1, p+1) - 1] \times [R(q+1, q+1) - 1].$$

d) Deduce that $R(2^n + 1, 2^n + 1) \ge 5^n + 1$ for $n \ge 0$ and compare this lower bound to the nonconstructive lower bound for $R(k, k)$. (Abbott [1972])

8.3.15. (−) Verify that $R(p, 2) = R(2, p) = p$. Use this and Theorem 8.3.11 to prove that $R(p, q) \le \binom{p+q-2}{p-1}$.

8.3.16. (−) Use the graph below to prove that $R(3,5) = 14$.



8.3.17. *Ramsey numbers for $r = 2$ and multiple colors.*

a) Let $p = (p_1, \ldots, p_k)$, and let q_i be obtained from p by subtracting 1 from p_i but leaving the other coordinates unchanged. Prove that $R(p) \le \sum_{i=1}^{k} R(q_i) - k + 2$.

b) Prove that $R(p_1 + 1, \ldots, p_k + 1) \le \frac{(p_1+\cdots+p_k)!}{p_1!\cdots p_k!}$.

8.3.18. Let $r_k = R_k(3; 2)$ (this is the value of n such that k-coloring $E(K_n)$ forces a monochromatic triangle).

a) Show that $r_k \le k(r_{k-1} - 1) + 2$.

b) Use part (a) to show that $r_k \le \lfloor k!e \rfloor + 1$, so that $r_3 \le 17$. (Comment: $r_3 = 17$, but the lower bound requires a clever 3-coloring of K_{16} that arises from the finite field $GF(2^4)$).

8.3.19. Prove that $R_k(p; r+1) \le r + k^M$, where $M = \binom{R_k(p;r)}{r}$.

8.3.20. (+) *Off-diagonal Ramsey numbers.*

a) Prove that $R(k, l) > n$ if $\binom{n}{k}p^{\binom{k}{2}} + \binom{n}{l}(1-p)^{\binom{l}{2}} < 1$ for some $p \in (0, 1)$. Prove that $R(k, l) > n - \binom{n}{k}p^{\binom{k}{2}} - \binom{n}{l}(1-p)^{\binom{l}{2}}$ for all $n \in \mathbb{N}$ and $p \in (0, 1)$.

b) Use part (a) to prove $R(3, k) > k^{3/2+o(1)}$. What lower bound on $R(3, k)$ can be obtained from the first part of (a)? (Spencer [1977])

c) Use part (a) to obtain a lower bound for $R_k(q)$.

8.3.21. (!) Determine the Ramsey number $R(K_{1,m}, K_{1,n})$. (Hint: The answer depends on whether m and n are even or odd.)

8.3.22. (!) Let T be a tree with m vertices. Given that $m - 1$ divides $n - 1$, determine the Ramsey number $R(T, K_{1,n})$. (Burr [1974])

8.3.23. If $p > (m-1)(n-1)$, prove that every 2-coloring of $E(K_p)$ in which the red graph is transitively orientable contains a red m-clique or a blue n-clique, and prove that this is best possible. (Brozinsky–Nishiura) (Hint. Use perfect graphs.)

8.3.24. Show that $R(T, K_{n_1}, \ldots, K_{n_k}) = (m-1)(R(n_1, \ldots, n_k) - 1) + 1$ when T is a tree with m vertices. (Burr)

8.3.25. Prove that $R(C_4, C_4) = 6$. (Comment: There are many proofs.)

8.3.26. Prove that $R(2K_3, 2K_3) = 10$. (Hint: Reduce to the case of a bowtie with triangles of both colors plus monochromatic 5-cycles; then use symmetry.)

8.3.27. (!) Prove that $R(mK_2, mK_2) = 3m - 1$.

8.3.28. (!) For $1 \le i \le k$, let G_i be a graph on p_i vertices, and fix a multiplicity m_i. Prove that $R(m_1G_1, \dots, m_kG_k) \le \sum(m_i - 1)p_i + R(G_1, \dots, G_k)$.

8.3.29. Frankl and Wilson [1981] explicitly constructed graphs with n vertices that have no clique or independent set with size exceeding $2^{c\sqrt{\log n \log\log n}}$, where c is a particular constant. Prove that this gives a lower bound for $R(p, p)$ that grows faster than every polynomial in p but slower than every exponential in p.

8.3.30. (!) For every simple graph G, determine $R(P_3, G)$ as a function only of the number of vertices of G and the maximum size of a matching in $\overline{G}$.

8.3.31. (!) Let r and s be natural numbers with $r + s \not\equiv 0 \bmod 4$. Prove that every 2-coloring of $E(K_{r,s})$ has a monochromatic connected graph with at least $\lceil r/2 \rceil + \lceil s/2 \rceil$ vertices. Conclude that every 3-coloring of $E(K_{r+s})$ contains a monochromatic connected subgraph with more than $(r + s)/2$ vertices. Show that this fails when 4 divides $r + s$.

8.3.32. *Forcing 4-cycles.*

a) Prove that if $\sum_{v \in V(G)} \binom{d(v)}{2} > \binom{n(G)}{2}$, then G contains a 4-cycle.

b) Prove that if $e(G) > \frac{n(G)}{4}(1 + \sqrt{4n(G) - 3})$, then G contains a 4-cycle.

c) Prove that $R_k(C_4) \le k^2 + k + 2$. (Chung–Graham [1975])

8.3.33. (!) Bondy [1971a] proved that $x \not\leftrightarrow y$ implies $d(x) + d(y) \ge n(G)$, then $G = K_{t,t}$ or G has a cycle of each length from 3 to n. Use this to prove that $R(C_m, K_{1,n}) = \max\{m, 2n + 1\}$, except possibly if m is even and at most $2n$. (Lawrence [1973])

8.3.34. (!) Prove that every 2-coloring of $E(K_n)$ has a Hamiltonian cycle that is monochromatic or consists of two monochromatic paths. (Hint: Use induction on n.) (Lovász [1979, p85, p482 - attributed to H. Raynaud])

8.3.35. (+) Let f be a 2-coloring of $E(K_n)$, and suppose that $k \ge 3$. Prove the following:

a) If f has a monochromatic C_{2k+1}, then f also has a monochromatic C_{2k}.

b) If f has a monochromatic C_{2k}, then f also has a monochromatic C_{2k-1} or $2K_k$.

c) If $m \ge 5$, then $R(C_m, C_m) \le 2m - 1$ (see Exercise 8.3.25 for $m = 4$). (Hint: Use parts (a) and (b) and the result of Erdős–Gallai [1959] (Theorem 8.4.35) that $e(G) > (m-1)(n(G) - 1)/2$ forces a cycle of length at least m in G. There remains one difficult case).

8.3.36. The **Ramsey multiplicity** of G is the minimum number of monochromatic copies of G in a 2-coloring of the edges of a clique on $R(G, G)$ vertices. Show that the Ramsey multiplicity of K_3 is 2.

8.3.37. Prove that each point in a triangular region has a unique expression as a convex combination of the vertices of the triangle (convex combinations are linear combinations where the coefficients are nonnegative and sum to 1).

8.3.38. *Sperner's Lemma in higher dimensions.* A **k-dimensional simplex** consists of the convex combinations of $k + 1$ points in $\mathbb{R}^k$ not lying in a hyperplane. A **simplicial subdivision** expresses a k-dimensional simplex as a union of k-dimensional simplices (cells) such that any two cells intersect in the simplex determined by their common corners. A **completely labeled** cell has $\{0, \dots, k\}$ at its corners.

State a general definition of "proper labeling" so that every proper labeling of a simplicial subdivision of a k-simplex contains a completely labeled cell. Prove this theorem. (Hint: The proof of Sperner's Lemma in two dimensions (Theorem 8.3.17) is an instance of the induction step for a proof by induction on k.)

8.3.39. (−) Compute the bandwidths of P_n, K_n, and C_n.

8.3.40. Compute the bandwidth of $K_{n_1,\ldots,n_k}$. (Eitner [1979])

8.3.41. (!) Prove that every tree with k leaves is the union of $\lceil k/2 \rceil$ pairwise intersecting paths (Exercise 2.1.37). Use this to prove that the bandwidth of a tree with k leaves is at most $\lceil k/2 \rceil$. (Ando–Kaneko–Gervacio [1996])

8.3.42. (+) Let G be a caterpillar (Definition 2.2.17), and let m be an integer such that $\left\lceil \frac{n(H)-1}{\operatorname{diam} H} \right\rceil \le m$ for all $H \subseteq G$. Prove that $B(G) \le m$. (Hint: Prove that G has a numbering f in which $f(v)$ is a multiple of m whenever v is on the spine and $|f(u) - f(v)| \le m$ for all $u \leftrightarrow v$.) (Sysło–Zak [1982], Miller [1981])

8.3.43. *Bandwidth of grids.*

a) Compute the local density lower bound for the bandwidth of $P_m \square P_n$.

b) Let S be a k-set of vertices in $P_n \square P_n$ with a_i vertices in the ith row and b_j vertices in the jth column. Prove that $|\partial T| \le |\partial S|$ if T is the set consisting of the first a_i vertices in the ith row for each i.

c) Prove that $|\partial S|$ is minimized over k-sets in $V(P_n \square P_n)$ by some S such that $a_1 \ge \cdots \ge a_n$ and $b_1 \ge \cdots \ge b_n$. Conclude that Harper's lower bound for $B(P_n \square P_n)$ is n.

d) Conclude that $B(P_m \square P_n) = \min\{m, n\}$. (Chvátalová [1975])

8.3.44. (+) Let G be a simple graph with order n and bandwidth b.

a) For $e \in \overline{G}$, prove that $B(G + e) \le 2b$.

b) Prove that if $n \ge 6b$, then $B(G + e)$ can be as large as $2b$.

(Comment: The maximum of $B(G + e)$ is $b + 1$ if $n \le 3b + 4$ and is $\lceil (n-1)/3 \rceil$ if $3b + 5 \le n \le 6b - 2$.) (Wang–West–Yao [1995])

8.4. More Extremal Problems

Extremal graph theory is a huge area. In Section 1.3 we described the distinction between optimization problems (find an extremal structure in the input graph) and extremal problems (find an extremal instance over a class of graphs), and we have studied both types of problems throughout this book. In this section we study the latter type. The archetypal example is the Turán problem: find the maximum number of edges in a graph not containing H as a subgraph. We list one additional example from each chapter.

Objective	*Class of graphs*	*Answer*	*Reference*
max $e(G)$	n vertices and k components	$\binom{n-k+1}{2}$	Exercise 1.3.40
max girth	diameter k and not a tree	$2k + 1$	Exercise 2.1.61
max $\beta(G)$	$\alpha'(G) \le k$	$2k$	Exercise 3.3.10
min $\alpha(G)$	$\kappa(G) = k$ and diameter d	$\lceil (d+1)/2 \rceil$	Exercise 4.2.22
max $\chi(G)$	$2K_2$-free and $\omega(G) = k$	$\binom{k+1}{2}$	Exercise 5.2.11
max $\chi(G)$	outerplanar	3	Exercise 6.3.12
max $e(G)$	$n(G) = n$ and non-Hamiltonian	$\binom{n-1}{2} + 1$	Exercise 7.2.26
max $n(G)$	$\omega(G) < p$ and $\alpha(G) < q$	$R(p, q) - 1$	Section 8.3

With such enormous variety of extremal problems, we can only hope in this section to exhibit a small sample of interesting results.