13

Mechanics

13.1 Mechanics before Calculus

The ambiguous title reflects the dual purpose of this section: to give a brief survey of the mechanics that came before calculus and to introduce the thesis that mechanics was psychologically, if not logically, a prerequisite for calculus itself. The remainder of the chapter expands on this thesis, demonstrating how several important fields in calculus (and beyond) originated in the study of mechanical problems. Lack of space, not to mention lack of expertise, prevents my venturing far into the history of mechanical concepts, so I shall assume some understanding of time, velocity, acceleration, force, and the like, and concentrate on the mathematics that emerged from reflection on these notions. These mathematical developments will be pursued as far as the nineteenth century. More details may be found in Dugas (1957, 1958) and Truesdell (1954, 1960). In the last century, mathematics seems to have been the motivation for mechanics rather than the other way round. The outstanding mechanical concepts of the twentieth century—relativity and quantum mechanics—would not have been conceivable without nineteenth-century advances in pure mathematics, some of which we discuss later.

It is mentioned in Section 4.5 that Archimedes made the only substantial contribution to mechanics in antiquity by introducing the basics of statics (balance of a lever requires equality of moments on the two sides) and hydrostatics (a body immersed in a fluid experiences an upward force equal to the weight of fluid displaced). Archimedes' famous results on areas and volumes were in fact discovered, as he revealed in his *Method*, by hypothetically balancing thin slices of different figures. Thus the earliest nontrivial results in calculus, if by calculus one means a method for discovering results about limits, relied on concepts from mechanics.

The medieval mathematician Oresme also was mentioned (Section 7.1) for his use of coordinates to give a geometric representation of functions. The relationship Oresme represented was in fact velocity v as a function of time t. He understood that displacement is then represented by the area under the curve, and hence in the case of constant acceleration (or "uniformly deformed velocity," as he called it) the displacement equals total time $\times$ velocity at the middle instant (Figure 13.1). This result is known as the "Merton acceleration theorem" [see, for example, Clagett (1959), p. 355] because it originated in the work of a group of mathematicians at Merton College, Oxford, in the 1330s. The first proofs were arithmetical and far less transparent than Oresme's figure.

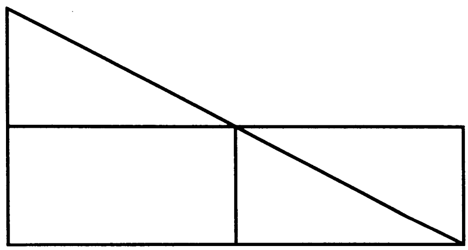

Figure 13.1: The Merton acceleration theorem

While constant acceleration was understood theoretically in the 1330s, it was not clear that it was actually a natural occurrence—namely, with falling bodies—until the time of Galileo (1564–1642). Galileo announced the equivalent result, that displacement of a body falling from rest at time $t = 0$ is proportional to t^2, in a letter [Galileo (1604)]. At first he was uncertain whether this derived from a velocity proportional to time $v = kt$ (that is, constant acceleration) or proportional to distance $v = ks$, but he resolved the question correctly in favor of $v = kt$ later [Galileo (1638)]. By composing the uniformly increasing vertical velocity with constant horizontal velocity, Galileo derived for the first time the correct trajectory of a projectile: the parabola.

The motion of projectiles was a matter of weighty importance in the Renaissance, and presumably observed often, yet the trajectories suggested before Galileo were quite preposterous (see, for example, Figure 6.3). The belief, deriving from Aristotle, that motion could be sustained only by continued application of a force led mathematicians to ignore the evidence and to draw trajectories in which the horizontal velocity dwindled to zero. Galileo overthrew this mistaken belief by affirming the *principle of inertia*: a body not subject to external forces travels with constant velocity.

The principle of inertia was Newton's starting point in mechanics; indeed, it is often called Newton's first law. It is a special case of his second law, that force is proportional to mass $\times$ acceleration [Newton (1687), p. 13]. Under this law, the motion of a body is determined by composition of the forces acting on it. The correct law for the composition of forces, that forces add vectorially, had been discovered in the case of perpendicular forces by Stevin (1586) and in the general case by Roberval [published in Mersenne (1636)]. The motion is thus determined by vector addition of the corresponding accelerations, the method Galileo used in the case of the projectile.

The determination of velocity and displacement from acceleration are of course problems of integration, so mechanics contributed a natural class of problems to calculus just at the time the subject was emerging. But more than this was true. The early practitioners of calculus believed that continuity was an essential attribute of functions, and the only way they were able to define continuity was ultimately by falling back on the dependence of a velocity or displacement on time. From this viewpoint, *all* problems of integration and differentiation were problems of mechanics, and Newton described them as such when explaining how his calculus of infinite series could be applied:

> It now remains, in illustration of this analytical art, to deliver some typical problems and such especially as the nature of curves will present. But first of all I would observe that difficulties of this sort may all be reduced to these two problems alone, which I may be permitted to propose with regard to the space traversed by any local motion however accelerated or retarded:
>
> 1. Given the length of space continuously (that is, at every time), to find the speed of motion at any time proposed.
> 2. Given the speed of motion continuously, to find the length of space described at any time proposed
>
> [Newton (1671), p. 71].

Of course we now know that the first problem requires differentiability rather than continuity for its solution, but the pioneers of calculus thought that differentiability was implied by continuity, and hence did not recognize it as a distinct notion. In fact it was a mechanical question—the problem of the vibrating string—whose investigation brought the distinction to light (see Section 13.4).

13.2 Celestial Mechanics

Astronomy has been a powerful stimulus to mathematics since ancient times. The epicyclic theory of Apollonius and Ptolemy introduced an interesting family of algebraic and transcendental curves, as we saw in Section 2.5, and the theory itself ruled Western astronomy until the seventeenth century. Even Copernicus (1472–1543), when he overthrew Ptolemy's earth-centered system with a sun-centered system in his *De revolutionibus orbium coelestium* [Copernicus (1543)], was unwilling to give up epicycles. Taking the sun as the center of the system simplifies the orbits of the planets but does not make them circular, so Copernicus, accepting the Ptolemaic philosophy that orbits must be generated by circular motions, modeled them by epicycles. In fact he used more epicycles than Ptolemy.

A more important advance, from the mathematical point of view, was Kepler's introduction of elliptical orbits in his *Astronomia nova* [Kepler (1609)]. When Newton explained these orbits as a consequence of the inverse square law of gravitation in the *Principia* [Newton (1687), p. 56] he showed that there was a deeper level of explanation—the infinitesimal level—where simplicity could be attained even when it was not possible at the global level. The force on a given body B_1 is simply the vector sum of the forces due to the other bodies $B_2, \ldots, B_n$ in the system, determined by their masses and distances from B_1 by the inverse square law and, by Newton's second law, this determines the acceleration of B_1. The accelerations of $B_2, \ldots, B_n$ are similarly determined, hence the behavior of the system is completely determined by the inverse square law, once initial positions and velocities are given. The inverse square law is an infinitesimal law in the sense that it describes the limiting behavior of a body—its acceleration—and not its global behavior such as the shape or period of its orbit.

As we now know, it is rarely possible to describe the global behavior of a dynamical system explicitly, so Newton found the only viable basis for dynamics in directing attention to infinitesimal behavior. Unfortunately, he communicated this insight poorly by expressing it in geometrical terms, in the belief that calculus, which he had used to discover his results, was inappropriate in a serious publication. By the eighteenth century this belief had been dispelled by Leibniz and his followers, and definitive formulations of dynamics in terms of calculus were given by Euler and Lagrange. They recognized that the infinitesimal behavior of a dynamical system was typically described by a system of *differential equations* and that the global behavior was derivable from these equations, in principle, by integration.

The question remained, however, whether the inverse square law did indeed account for the observed global behavior of the solar system. In a system with only two bodies, Newton showed [Newton (1687), p. 166] that each describes a conic section relative to the other, in normal cases an ellipse as stated by Kepler. With a three-body system, such as the earth–moon–sun, no simple global description was possible, and Newton could obtain only qualitative results through approximations. With the many bodies in the solar system, extremely complex behavior was possible, and for 100 years mathematicians were unable to account for some of the phenomena actually observed.

A famous example was the so-called secular variation of Jupiter and Saturn, which was detected by Halley in 1695 from the observations then available. For several centuries Jupiter had been speeding up (spiraling toward the sun) and Saturn had been slowing down (spiraling outward). The problem was to explain this behavior and to determine whether it would continue, with the eventual destruction of Jupiter and disappearance of Saturn. Euler and Lagrange worked on the problem without success; then, in the centenary year of *Principia*, Laplace (1787) succeeded in explaining the phenomenon. He showed that the secular variation was actually periodic, with Jupiter and Saturn returning to their initial positions every 929 years. Laplace viewed this as confirmation not only of the Newtonian theory but also of the stability of the solar system, though it seems that the latter is still an open question.

Laplace introduced the term "celestial mechanics" and left no doubt that the theory had arrived with his monumental *Mécanique céleste*, a work of five volumes that appeared between 1799 and 1825. In astronomy, the theory had its finest hour in 1846, with the discovery of Neptune, whose position had been computed by Adams and Leverrier from observed perturbations in the orbit of Uranus. The difficult question of stability was taken up again in the three volume *Les méthodes nouvelles de la mécanique*