EXAMPLE 4. In the space $C(a,\ b)$, define

$$(f,\ g) = \int_a^b w(t)f(t)g(t)\ dt\,,$$

where w is a fixed positive function in $C(a,$ b). The function w is called a *weightfunction.*
In Example 3 we have w(t) = 1 for all t.

EXAMPLE 5. In the linear space of all real polynomials, define

$$(f,\ g) = \int_0^\infty e^{-t}f(t)g(t)\ dt\,.$$

Because of the exponential factor, this improper integral converges for every choice of
polynomials f and g.

THEOREM   *1.8.   In a Euclidean space V, every inner product satisfies the Cauchy-Schwarz
inequality:*

$$|(x, y)|^2 \leq (x,\ x)(y, y) \qquad \text{for all } x \text{ andy in } V.$$

Moreover, the equality sign holds if and only **if x** *and y are dependent.*

Proof.   If either x = 0 or y = 0 the result holds trivially, so we can assume that both
x and y are nonzero. Let z = $ax + by$, where a and b are scalars to be specified later. We
have the inequality (z, z) $\geq$ 0 for all a and b. When we express this inequality in terms of x
and y with an appropriate choice of a and b we will obtain the Cauchy-Schwarz inequality.
To express $(z,$ z) in terms of x and y we use properties $(1')$, (2) and (3') to obtain

$$(z, z) = (ax + \mathbf{by},\quad ax + \mathbf{by})\ \ = (\mathbf{ax}, ax) + (\mathbf{ax}, by) + (by, ax) + (by, by)$$
$$= a\bar{a}(x, x) + a\bar{b}(x, y)\ + b\bar{a}(y, x) + b\bar{b}(y, y) \geq 0.$$

Taking a = (y, y) and cancelling the positive factor $(y,$ y) in the inequality we obtain

$$(y, y)(x, x) + \bar{b}(x, y)\ + b(y,\ x) + b\bar{b} \geq 0.$$

Now we take $b = -(x, y)$. Then 6 = $-$ (y, x) and the last inequality simplifies to

$$(y, y)(x, x) \geq (x, y)(y, x) = |(x, y)|^2.$$

This proves the Cauchy-Schwarz inequality. The equality sign holds throughout the proof
if and only if z = 0. This holds, in turn, if and only if x and y are dependent.

EXAMPLE.   Applying Theorem 1.8 to the space $C(a,\ b)$ with the inner product $(f,\ g) =$
$\int_a^b f(t)g(t)\ dt$, we find that the Cauchy-Schwarz inequality becomes

$$\left(\int_a^b f(t)g(t)\ dt\right)^2 \leq \left(\int_a^b f^2(t)\ dt\right)\left(\int_a^b g^2(t)\ dt\right).$$

The inner product can be used to introduce the metric concept of length in any Euclidean space.

DEFINITION. *In a Euclidean space V, the nonnegative number* $\|x\|$ *defined by the equation*

$$\|x\| = (x, x)^{1/2}$$

is called the norm of the element x.

When the Cauchy-Schwarz inequality is expressed in terms of norms, it becomes

$$|(x, y)| \leq \|x\| \, \|y\| \, .$$

Since it may be possible to define an inner product in many different ways, the norm of an element will depend on the choice of inner product. This lack of uniqueness is to be expected. It is analogous to the fact that we can assign different numbers to measure the length of a given line segment, depending on the choice of scale or unit of measurement. The next theorem gives fundamental properties of norms that do not depend on the choice of inner product.

THEOREM 1.9. *In a Euclidean space, every norm has the following properties for all elements x and y and all scalars c:*
   (a) $\|x\| = 0$ *if* $x = O$.
   (b) $\|x\| > 0$ *if* $x \neq O$ *(positivity).*
   (c) $\|cx\| = |c| \|x\|$ *(homogeneity).*
   (d) $\|x + y\| \leq \|x\| + \|y\|$ *(triangle inequality).*
   The equality sign holds in (d) *if* x = O, *if* y = O, *or if* y = cx *for some c > 0.*

Proof. Properties (a), (b) and (c) follow at once from the axioms for an inner product. To prove (d), we note that

$$\|x + y\|^2 = (x + y, x + y) = (x, x) + (y, y) + (x, y) + (y, x)$$
$$= \|x\|^2 + \|y\|^2 + (x, y) + (x, y).$$

The sum $(x, y) + (x, y)$ is real. The Cauchy-Schwarz inequality shows that $|(x, y)| \leq \|x\| \|y\|$ and $|(x, y)| \leq \|x\| \, \|y\|$, so we have

$$\|x + y\|^2 \leq \|x\|^2 + \|y\|^2 + 2\|x\| \, \|y\| = (\|x\| + \|y\|)^2.$$

This proves (d). When y = cx, where c > 0, we have

$$\|x + y\| = \|x + cx\| = (1 + c) \|x\| = \|x\| + \|cx\| = \|x\| + \|y\| \, .$$

DEFINITION. *In a real Euclidean space V, the angle between two nonzero elements x and y is defined to be that number θ in the interval $0 \le \theta \le \pi$ which satisfies the equation*

$$(1.7) \qquad \cos \theta = \frac{(x, y)}{\|x\| \, \|y\|} \, .$$

Note: The Cauchy-Schwarz inequality shows that the quotient on the right of (1.7) lies in the interval $[-1, 1]$, so there is exactly one θ in $[0, \pi]$ whose cosine is equal to this quotient.

1.12 Orthogonality in a Euclidean space

DEFINITION. *In a Euclidean space V, two elements x and y are called orthogonal if their inner product is zero. A subset S of V is called an orthogonal set if (x, y) = 0 for every pair of distinct elements x and y in S. An orthogonal set is called orthonormal if each of its elements has norm 1.*

The zero element is orthogonal to every element of *V;* it is the only element orthogonal to itself. The next theorem shows a relation between orthogonality and independence.

THEOREM 1.10. *In a Euclidean space V, every orthogonal set of nonzero elements is independent. In particular, in a jinite-dimensional Euclidean space with* dim $V = n$, *every orthogonal set consisting of n nonzero elements is a basis for V.*

Proof. Let S be an orthogonal set of nonzero elements in *V,* and suppose some finite linear combination of elements of S is zero, say

$$\sum_{i=1}^{k} c_i x_i = O \, ,$$

where each $x_i \in S$. Taking the inner product of each member with x_1 and using the fact that $(x_i, x_i) = 0$ if $i \ne 1$, we find that $c_1(x_1, x_1) = 0$. But $(x_1, x_1) \ne 0$ since $x_1 \ne 0$ so $c_1 = 0$. Repeating the argument with x_1 replaced by x_j, we find that each $c_j = 0$. This proves that S is independent. If dim $V = n$ and if S consists of n elements, Theorem 1.7(b) shows that S is a basis for *V.*

EXAMPLE. In the real linear space $C(0, 2\pi)$ with the inner product $(f, g) = \int_0^{2\pi} f(x)g(x) \, dx$, let S be the set of trigonometric functions $\{u_0, u_1, u_2, \ldots\}$ given by

$$u_0(x) = 1, \qquad u_{2n-1}(x) = \cos nx, \qquad u_{2n}(x) = \sin nx, \qquad \text{for } n = 1, 2, \ldots.$$

If $m \ne n$, we have the orthogonality relations

$$\int_0^{2\pi} u_n(x)u_m(x) \, dx = 0,$$

so S is an orthogonal set. Since no member of S is the zero element, S is independent. The norm of each element of S is easily calculated. We have $(u_0, u_0) = \int_0^{2\pi} dx = 2\pi$ and, for $n \geq 1$, we have

$$(u_{2n-1}, u_{2n-1}) = \int_0^{2\pi} \cos^2 nx \ dx = \pi, \qquad (u_{2n}, u_{2n}) = \int_0^{2\pi} \sin^2 nx \ dx = \pi.$$

Therefore, $\|u_0\| = \sqrt{2\pi}$ and $\|u_n\| = \sqrt{\pi}$ for $n \geq 1$. Dividing each u_n by its norm, we obtain an orthonormal set $\{\varphi_0, \varphi_1, \varphi_2, \ldots\}$ where $\varphi_n = u_n/\|u_n\|$ Thus, we have

$$\varphi_0(x) = \frac{1}{\sqrt{2\pi}}, \qquad \varphi_{2n-1}(x) = \frac{\cos nx}{\sqrt{\pi}}, \qquad \varphi_{2n}(x) = \frac{\sin nx}{\sqrt{\pi}}, \qquad \text{for} \quad n \geq 1$$

   In Section 1.14 we shall prove that every finite-dimensional Euclidean space has an orthogonal basis. The next theorem shows how to compute the components of an element relative to such a basis.

   THEOREM *I.11. Let V he a finite-dimensional Euclidean space with dimension n, and assume that $S = \{e_1, \ldots, e_n\}$s an orthogonal basis for V.* **[fan** *element x is expressed as a linear combination of the basis elements, say*

(1.8) $$x = \sum_{i=1}^n c_i e_i,$$

then its components relative to the ordered basis (e, , ... , e_n) *are given by the formulas*

(1.9) $$\frac{(x, e_j)}{(e_j, e_j)} \quad \text{for} \quad j = 1, 2, \ldots, n.$$

In particular, if S is an orthonormal basis, each c_j is given by

(1.10) $$c_j = (x, e_j).$$

   Proof. Taking the inner product of each member of (1.8) with e_j, we obtain

$$(x, e_j) = \sum_{i=1}^n c_i(e_i, e_j) = c_j(e_j, e_j)$$

since $(e_i, e_j) = 0$ if $i \neq j$. This implies (1.9), and when $(e_j, e_j) = 1$, we obtain (1.10).

   If $\{e_1, \ldots, e_n\}$ is an orthonormal basis, Equation (1.9) can be written in the form

(1.11) $$x = \sum_{i=1}^n (x, e_i)e_i.$$

   The next theorem shows that in a finite-dimensional Euclidean space with an orthonormal basis the inner product of two elements can be computed in terms of their components.

THEOREM 1.12. *Let V be a finite-dimensional Euclidean space of dimension n, and assume that $\{e_1, \ldots, e_n\}$ is an orthonormal basis for V. Then for every pair of elements x and y in V, we have*

(1.12)
$$(x, y) = \sum_{i=1}^{n} (x, e_i)\overline{(y, e_i)} \qquad \textbf{(Parseval's formula)}.$$

In particular, when x = y , we have

(1.13)
$$\|x\|^2 = \sum_{i=1}^{n} |(x, e_i)|^2.$$

Proof, Taking the inner product of both members of Equation (1.11) withy and using the linearity property of the inner product, we obtain (1.12). When $x = y$, Equation (1.12) reduces to (1.13).

Note: Equation (1.12) is named in honor of M. A. Parseval (circa 1776–1836), who obtained this type of formula in a special function space. Equation (1.13) is a generalization of the theorem of Pythagoras.

1.13 Exercises

1. Let $x = (x_1, \ldots, x_n)$ andy $= (y_1, \ldots, y_n)$ be arbitrary vectors in V_n . In each case, determine whether (x, y) is an inner product for V_n if (x, y) is defined by the formula given. In case (x, y) is not an inner product, tell which axioms are not satisfied.

(a) $(x, y) = \sum_{i=1}^{n} x_i |y_i|$

(b) $(x, y) = \left| \sum_{i=1}^{n} x_i y_i \right|.$

(c) $(x, y) = \sum_{i=1}^{n} x_i \sum_{j=1}^{n} y_j.$

(d) $(x, y) = \left(\sum_{i=1}^{n} x_i^2 y_i^2 \right)^{1/2},$

(e) $(x, y) = \sum_{i=1}^{n} (x_i + y_i)^2 - \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} y_i^2.$

2. Suppose we retain the first three axioms for a real inner product (symmetry, linearity, and homogeneity but replace the fourth axiom by a new axiom (4'): (x, x) = 0 if and only if x = 0. Prove that either (x, x) > 0 for all $x \neq 0$ or else (x, x) < 0 for all $x \neq 0$.

[*Hint:* Assume (x, x) > 0 for some $x \neq 0$ and (y, y) < 0 for some $y \neq 0$. In the space spanned by {x, y), find an element $z \neq 0$ with (z, z) = 0.]

Prove that each of the statements in Exercises 3 through 7 is valid for all elements x and y in a real Euclidean space.

3. (x, y) = 0 if and only if $\|x + y\| = \|x - y\|$.
4. (x, y) = 0 if and only if $\|x + y\|^2 = \|x\|^2 + \|y\|^2$.
5. (x, y) = 0 if and only if $\|x + cy\| \geq \|x\|$ for all real c.
6. (x + y, x - y) = 0 if and only if $\|x\| = \|y\|$.
7. If x and y are nonzero elements making an angle θ with each other, then

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2 \|x\| \|y\| \cos \theta.$$

8. In the real linear space C(l, e), define an inner product by the equation

$$(f, g) = \int_1^e (\log x) f(x) g(x)\, dx \, ,$$

 (a) If $f(x) = \sqrt{x}$, compute $\|f\|$.
 (b) Find a linear polynomial $g(x) = a + bx$ that is orthogonal to the constant function $f(x) = 1$.

9. In the real linear space C($-1, 1$), let $(f, g) = \int_{-1}^1 f(t)g(t)\, dt$. Consider the three functions u_1, u_2, u_3 given by

$$u_1(t) = 1 \, , \qquad u_2(t) = t \, , \qquad u_3(t) = 1 + t \, .$$

Prove that two of them are orthogonal, two make an angle $\pi/3$ with each other, and two make an angle $\pi/6$ with each other.

10. In the linear space P_n of all real polynomials of degree $\leq n$, define

$$(f, g) = \sum_{k=0}^n f\left(\frac{k}{n}\right) g\left(\frac{k}{n}\right).$$

 (a) Prove that (f, g) is an inner product for P_n .
 (b) Compute (f, g) when $f(t) = t$ and $g(t) = at + b$.
 (c) If $f(t) = t$, find all linear polynomials g orthogonal to f.

11. In the linear space of all real polynomials, define $(f, g) = \int_0^\infty e^{-t} f(t) g(t)\, dt$.
 (a) Prove that this improper integral converges absolutely for all polynomials f and g.
 (b) If $x_n(t) = t^n$ for $n = 0, 1, 2, \ldots$, prove that $(x_n, x_m) = (m + n)!$.
 (c) Compute (f, g) when $f(t) = (t + 1)^2$ and $g(t) = t^2 + 1$.
 (d) Find all linear polynomials $g(t) = a + bt$ orthogonal to $f(t) = 1 + t$.

12. In the linear space of all real polynomials, determine whether or not (f, g) is an inner product if (f, g) is defined by the formula given. In case (f, g) is not an inner product, indicate which axioms are violated. In (c), f' and g' denote derivatives.

 (a) $(f, g) = f(1) g(1)$. (c) $(f, g) = \int_0^1 f'(t) g'(t)\, dt$.

 (b) $(f, g) = \left| \int_0^1 f(t) g(t)\, dt \right|$. (d) $(f, g) = \left(\int_0^1 f(t)\, dt \right) \left(\int_0^1 g(t)\, dt \right)$.

13. Let V consist of all infinite sequences {x,} of real numbers for which the series Σx_n^2 converges. If x = $\{x_n\}$ and $y = \{y_n\}$ are two elements of V, define

$$(x, y) = \sum_{n=1}^\infty x_n y_n.$$

 (a) Prove that this series converges absolutely.

 [*Hint:* Use the Cauchy-Schwarz inequality to estimate the sum $\sum_{n=1}^M |x_n y_n|$.]

 (b) Prove that V is a linear space with (x, y) as an inner product.
 (c) Compute (x, y) if $x_n = 1/n$ and $y_n = 1/(n + 1)$ for $n \geq 1$.
 (d) Compute (x, y) if $x_n = 2^n$ and $y_n = 1/n!$ for $n \geq 1$.

14. Let V be the set of all real functions f continuous on $[0, +\infty)$ and such that the integral $\int_0^\infty e^{-t} f^2(t)\, dt$ converges. Define $(f, g) = \int_0^\infty e^{-t} f(t) g(t)\, dt$.

(a) Prove that the integral for (f, g) converges absolutely for each pair of functions f and g in V.

[Hint: Use the Cauchy-Schwarz inequality to estimate the integral $\int_0^M e^{-t} \mid f(t)g(t) \mid dt$.]

(b) Prove that V is a linear space with (f, g) as an inner product.

(c) Compute (f, g) if $f(t) = e^{-t}$ and $g(t) = t^n$, where $n = 0, 1, 2, \ldots$.

15. In a complex Euclidean space, prove that the inner product has the following properties for all elements x, y and z, and all complex a and b.

    (a) $(ax, by) = a\bar{b}(x, y)$.   (b) $(x, ay + bz) = \bar{a}(x, y) + \bar{b}(x, z)$.

16. Prove that the following identities are valid in every Euclidean space.

    (a) $\|x + y\|^2 = \|x\|^2 + \|y\|^2 + (x, y) + (y, x)$.

    (b) $\|x + y\|^2 - \|x - y\|^2 = 2(x, y) + 2(y, x)$.

    (c) $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$.

17. Prove that the space of all complex-valued functions continuous on an interval *[a, b]* becomes a unitary space if we define an inner product by the formula

$$(f, g) = \int_a^b w(t)f(t)\overline{g(t)}\, dt,$$

where w is a fixed positive function, continuous on *[a, b]*.

1.14 Construction of orthogonal sets. The Gram-Scltmidt process

Every finite-dimensional linear space has a finite basis. If the space is Euclidean, we can always construct an **orthogonal** basis. This result will be deduced as a consequence of a general theorem whose proof shows how to construct orthogonal sets in any Euclidean space, finite or infinite dimensional. The construction is called the **Gram-Schmidt** *orthog-* **onalizationprocess,** in honor of J. P. Gram (1850-1916) and E. Schmidt (18451921).

    THEOREM   1.13. ORTHOGONALIZATION   THEOREM.   *Let $x_1, x_2, \ldots$, be ajinite or* infinite *sequence of elements in a Euclidean space V, and let $L(x_1, \ldots, x_k)$ denote the subspace spanned by the first k of these elements. Then there is a corresponding sequence of elements $y_1, y_2, \ldots$, in V which has the following properties for each integer k:*

    (a) **The element y_k is orthogonal to every element in the subspace $L(y_1, \ldots, y_{k-1})$.**

    (b) **The subspace spanned by $y_1, \ldots, y_k$ is the same as that spanned by $x_1, \ldots, x_k$:**

$$L(y_1, \ldots, y_k) = L(x_1, \ldots, x_k).$$

    *(c) **The sequence $y_1, y_2, \ldots$, is unique, except for scalar factors. That is, if $y_1', y_2', \ldots$, is** **another sequence of elements in V satisfying properties** (a) **and** (b) **for all k, then for each k** **there is a scalar c_k such that $y_k' = c_k y_k$.***

    Proof.   We construct the elements $y_1, y_2, \ldots$, by induction. To start the process, we take $y_1 = x_1$. Now assume we have constructed $y_1, \ldots, y_r$ so that (a) and (b) are satisfied when $k = r$. Then we define y_{r+1} by the equation

(1.14) $$y_{r+1} = x_{r+1} - \sum_{i=1}^{r} a_i y_i,$$

where the scalars $a_1, \ldots, a_r$ are to be determined. For $j \leq r$, the inner product of y_{r+1} with y_j is given by

$$(y_{r+1}, y_j) = (x_{r+1}, y_j) - \sum_{i=1}^{r} a_i(y_i, y_j) = (x_{r+1}, y_j) - a_j(y_j, y_j),$$

since $(y_i, y_j) = 0$ if $i \neq j$. If $y_j \neq O$, we can make y_{r+1} orthogonal to y_j by taking

(1.15) $$a_j = \frac{(x_{r+1}, y_j)}{(y_j, y_j)}.$$

If $y_j = O$, then y_{r+1} is orthogonal to y_j for any choice of a_j, and in this case we choose $a_j = 0$. Thus, the element y_{r+1} is well defined and is orthogonal to each of the earlier elements $y_1, \ldots, y_r$. Therefore, it is orthogonal to every element in the subspace

$$L(y_1, \ldots, y_r).$$

This proves (a) when $k = r + 1$.

To prove (b) when $k = r + 1$, we must show that $L(y_1, \ldots, y_{r+1}) = L(x_1, \ldots, x_{r+1})$, given that $L(y_1, \ldots, y_r) = L(x_1, \ldots, x_r)$. The first r elements $y_1, \ldots, y_r$ are in

$$L(x_1, \ldots, x_r)$$

and hence they are in the larger subspace $L(x_1, \ldots, x_{r+1})$. The new element y_{r+1} given by (1.14) is a difference of two elements in $L(x_1, \ldots, x_{r+1})$ so it, too, is in $L(x_1, \ldots, x_{r+1})$. This proves that

$$L(y_1, \ldots, y_{r+1}) \subseteq L(x_1, \ldots, x_{r+1}).$$

Equation (1.14) shows that x_{r+1} is the sum of two elements in $L(y_1, \ldots, y_{r+1})$ so a similar argument gives the inclusion in the other direction:

$$L(x_1, \ldots, x_{r+1}) \subseteq L(y_1, \ldots, y_{r+1}).$$

This proves (b) when $k = r + 1$. Therefore both (a) and (b) are proved by induction on k.

Finally we prove (c) by induction on k. The case $k = 1$ is trivial. Therefore, assume (c) is true for $k = r$ and consider the element y'_{r+1}. Because of (b), this element is in

$$L(y_1, \ldots, y_{r+1}),$$

so we can write

$$y'_{r+1} = \sum_{i=1}^{r+1} c_i y_i = z_r + c_{r+1} y_{r+1},$$

where $z_r \in L(y_1, \ldots, y_r)$. We wish to prove that $z_r = 0$. By property (a), both y'_{r+1} and $c_{r+1} y_{r+1}$ are orthogonal to z_r. Therefore, their difference, z_r, is orthogonal to z_r. In other words, z_r is orthogonal to itself, so $z_r = 0$. This completes the proof of the orthogonalization theorem.

In the foregoing construction, suppose we have $y_{r+1} = 0$ for some r. Then (1.14) shows that x_{r+1} is a linear combination of $y_1, \ldots, y_r$, and hence of $x_1, \ldots, x_r$, so the elements $x_1, \ldots, x_{r+1}$ are dependent. In other words, if the first k elements $x_1, \ldots, x_k$ are independent, then the corresponding elements $y_1, \ldots, y_k$ are *nonzero*. In this case the coefficients a_i in (1.14) are given by (1.15), and the formulas defining $y_1, \ldots, y_k$ become

$$(1.16) \quad y_1 = x_1, \qquad y_{r+1} = x_{r+1} - \sum_{i=1}^{r} \frac{(x_{r+1}, y_i)}{(y_i, y_i)} y_i \qquad \text{for} \quad r = 1, 2, \ldots, k-1.$$

These formulas describe the Gram-Schmidt process for constructing an orthogonal set of nonzero elements $y_1, \ldots, y_k$ which spans the same subspace as a given independent set $x_1, \ldots, x_k$. In particular, if $x_1, \ldots, x_k$ is a basis for a finite-dimensional Euclidean space, then $y_1, \ldots, y_k$ is an orthogonal basis for the same space. We can also convert this to an orthonormal basis by **normalizing** each element y_i, that is, by dividing it by its norm. Therefore, as a corollary of Theorem 1.13 we have the following.

THEOREM 1.14. *Every finite-dimensional Euclidean space has an orthonormal basis.*

If x and y are elements in a Euclidean space, with $y \neq O$, the element

$$\frac{(x, y)}{(y, y)} y$$

is called the **projection** of x **along** y. In the Gram-Schmidt process (1.16), we construct the element y_{r+1} by subtracting from x_{r+1} the projection of x_{r+1} along each of the earlier elements $y_1, \ldots, y_r$. Figure 1.1 illustrates the construction geometrically in the vector space V_3.
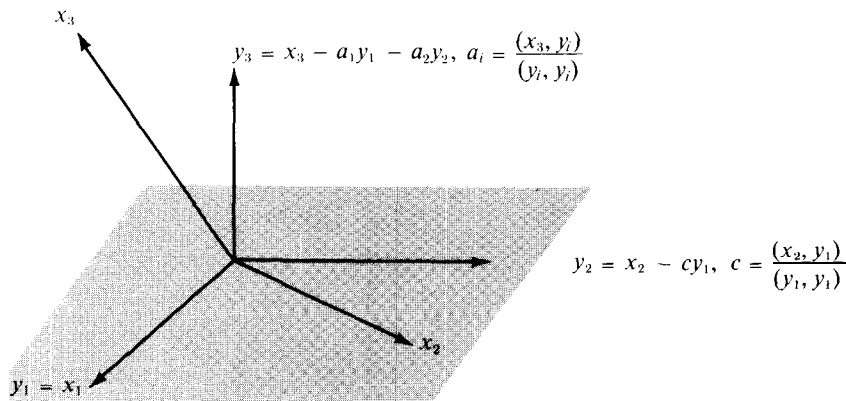

FIGURE 1.1 The Gram-Schmidt process in V_3. An orthogonal set $\{y_1, y_2, y_3\}$ is constructed from a given independent set $\{x_1, x_2, x_3\}$.