satisfies $IT = TI$ for each T, and for an invertible T there is (by Theorem 7) an invertible linear operator T^{-1} such that $TT^{-1} = T^{-1}T = I$. Thus the set of invertible linear operators on V, together with this operation, is a group. The set of invertible $n \times n$ matrices with matrix multiplication as the operation is another example of a group. A group is called **commutative** if it satisfies the condition $xy = yx$ for each x and y. The two examples we gave above are not commutative groups, in general. One often writes the operation in a commutative group as $(x, y) \to x + y$, rather than $(x, y) \to xy$, and then uses the symbol 0 for the 'identity' element e. The set of vectors in a vector space, together with the operation of vector addition, is a commutative group. A field can be described as a set with two operations, called addition and multiplication, which is a commutative group under addition, and in which the non-zero elements form a commutative group under multiplication, with the distributive law $x(y + z) = xy + xz$ holding.

Exercises

1. Let T and U be the linear operators on R^2 defined by

$$T(x_1, x_2) = (x_2, x_1) \quad \text{and} \quad U(x_1, x_2) = (x_1, 0).$$

(a) How would you describe T and U geometrically?

(b) Give rules like the ones defining T and U for each of the transformations $(U + T)$, UT, TU, T^2, U^2.

2. Let T be the (unique) linear operator on C^3 for which

$$T\epsilon_1 = (1, 0, i), \qquad T\epsilon_2 = (0, 1, 1), \qquad T\epsilon_3 = (i, 1, 0).$$

Is T invertible?

3. Let T be the linear operator on R^3 defined by

$$T(x_1, x_2, x_3) = (3x_1, x_1 - x_2, 2x_1 + x_2 + x_3).$$

Is T invertible? If so, find a rule for T^{-1} like the one which defines T.

4. For the linear operator T of Exercise 3, prove that

$$(T^2 - I)(T - 3I) = 0.$$

5. Let $C^{2\times 2}$ be the complex vector space of 2×2 matrices with complex entries. Let

$$B = \begin{bmatrix} 1 & -1 \\ -4 & 4 \end{bmatrix}$$

and let T be the linear operator on $C^{2\times 2}$ defined by $T(A) = BA$. What is the rank of T? Can you describe T^2?

6. Let T be a linear transformation from R^3 into R^2, and let U be a linear transformation from R^2 into R^3. Prove that the transformation UT is not invertible. Generalize the theorem.

7. Find two linear operators T and U on R^2 such that $TU = 0$ but $UT \neq 0$.

8. Let V be a vector space over the field F and T a linear operator on V. If $T^2 = 0$, what can you say about the relation of the range of T to the null space of T? Give an example of a linear operator T on R^2 such that $T^2 = 0$ but $T \neq 0$.

9. Let T be a linear operator on the finite-dimensional space V. Suppose there is a linear operator U on V such that $TU = I$. Prove that T is invertible and $U = T^{-1}$. Give an example which shows that this is false when V is not finite-dimensional. (*Hint:* Let $T = D$, the differentiation operator on the space of polynomial functions.)

10. Let A be an $m \times n$ matrix with entries in F and let T be the linear transformation from $F^{n\times 1}$ into $F^{m\times 1}$ defined by $T(X) = AX$. Show that if $m < n$ it may happen that T is onto without being non-singular. Similarly, show that if $m > n$ we may have T non-singular but not onto.

11. Let V be a finite-dimensional vector space and let T be a linear operator on V. Suppose that rank (T^2) = rank (T). Prove that the range and null space of T are disjoint, i.e., have only the zero vector in common.

12. Let p, m, and n be positive integers and F a field. Let V be the space of $m \times n$ matrices over F and W the space of $p \times n$ matrices over F. Let B be a fixed $p \times m$ matrix and let T be the linear transformation from V into W defined by $T(A) = BA$. Prove that T is invertible if and only if $p = m$ and B is an invertible $m \times m$ matrix.

3.3. Isomorphism

If V and W are vector spaces over the field F, any one-one linear transformation T of V onto W is called an **isomorphism of V onto W.** If there exists an isomorphism of V onto W, we say that V is **isomorphic** to W.

Note that V is trivially isomorphic to V, the identity operator being an isomorphism of V onto V. Also, if V is isomorphic to W via an isomorphism T, then W is isomorphic to V, because T^{-1} is an isomorphism of W onto V. The reader should find it easy to verify that if V is isomorphic to W and W is isomorphic to Z, then V is isomorphic to Z. Briefly, isomorphism is an equivalence relation on the class of vector spaces. If there exists an isomorphism of V onto W, we may sometimes say that V and W are isomorphic, rather than V is isomorphic to W. This will cause no confusion because V is isomorphic to W if and only if W is isomorphic to V.

Theorem 10. *Every* n*-dimensional vector space over the field* F *is isomorphic to the space* F^n.

Proof. Let V be an n-dimensional space over the field F and let $\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}$ be an ordered basis for V. We define a function T

from V into F^n, as follows: If α is in V, let $T\alpha$ be the n-tuple $(x_1, \ldots, x_n)$ of coordinates of α relative to the ordered basis $\mathcal{B}$, i.e., the n-tuple such that

$$\alpha = x_1\alpha_1 + \cdots + x_n\alpha_n.$$

In our discussion of coordinates in Chapter 2, we verified that this T is linear, one-one, and maps V onto F^n. ▮

For many purposes one often regards isomorphic vector spaces as being 'the same,' although the vectors and operations in the spaces may be quite different, that is, one often identifies isomorphic spaces. We shall not attempt a lengthy discussion of this idea at present but shall let the understanding of isomorphism and the sense in which isomorphic spaces are 'the same' grow as we continue our study of vector spaces.

We shall make a few brief comments. Suppose T is an isomorphism of V onto W. If S is a subset of V, then Theorem 8 tells us that S is linearly independent if and only if the set $T(S)$ in W is independent. Thus in deciding whether S is independent it doesn't matter whether we look at S or $T(S)$. From this one sees that an isomorphism is 'dimension preserving,' that is, any finite-dimensional subspace of V has the same dimension as its image under T. Here is a very simple illustration of this idea. Suppose A is an $m \times n$ matrix over the field F. We have really given two definitions of the solution space of the matrix A. The first is the set of all n-tuples $(x_1, \ldots, x_n)$ in F^n which satisfy each of the equations in the system $AX = 0$. The second is the set of all $n \times 1$ column matrices X such that $AX = 0$. The first solution space is thus a subspace of F^n and the second is a subspace of the space of all $n \times 1$ matrices over F. Now there is a completely obvious isomorphism between F^n and $F^{n\times 1}$, namely,

$$(x_1, \ldots, x_n) \to \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

Under this isomorphism, the first solution space of A is carried onto the second solution space. These spaces have the same dimension, and so if we want to prove a theorem about the dimension of the solution space, it is immaterial which space we choose to discuss. In fact, the reader would probably not balk if we chose to identify F^n and the space of $n \times 1$ matrices. We may do this when it is convenient, and when it is not convenient we shall not.

Exercises

1. Let V be the set of complex numbers and let F be the field of real numbers. With the usual operations, V is a vector space over F. Describe explicitly an isomorphism of this space onto R^2.

2. Let V be a vector space over the field of complex numbers, and suppose there is an isomorphism T of V onto C^3. Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be vectors in V such that

$$T\alpha_1 = (1, 0, i), \qquad T\alpha_2 = (-2, 1 + i, 0),$$
$$T\alpha_3 = (-1, 1, 1), \qquad T\alpha_4 = (\sqrt{2}, i, 3).$$

(a) Is α_1 in the subspace spanned by α_2 and α_3?

(b) Let W_1 be the subspace spanned by α_1 and α_2, and let W_2 be the subspace spanned by α_3 and α_4. What is the intersection of W_1 and W_2?

(c) Find a basis for the subspace of V spanned by the four vectors α_j.

3. Let W be the set of all 2×2 complex Hermitian matrices, that is, the set of 2×2 complex matrices A such that $A_{ij} = \overline{A_{ji}}$ (the bar denoting complex conjugation). As we pointed out in Example 6 of Chapter 2, W is a vector space over the field of *real* numbers, under the usual operations. Verify that

$$(x, y, z, t) \rightarrow \begin{bmatrix} t + x & y + iz \\ y - iz & t - x \end{bmatrix}$$

is an isomorphism of R^4 onto W.

4. Show that $F^{m \times n}$ is isomorphic to F^{mn}.

5. Let V be the set of complex numbers regarded as a vector space over the field of real numbers (Exercise 1). We define a function T from V into the space of 2×2 real matrices, as follows. If $z = x + iy$ with x and y real numbers, then

$$T(z) = \begin{bmatrix} x + 7y & 5y \\ -10y & x - 7y \end{bmatrix}.$$

(a) Verify that T is a one-one (real) linear transformation of V into the space of 2×2 real matrices.

(b) Verify that $T(z_1 z_2) = T(z_1)T(z_2)$.

(c) How would you describe the range of T?

6. Let V and W be finite-dimensional vector spaces over the field F. Prove that V and W are isomorphic if and only if $\dim V = \dim W$.

7. Let V and W be vector spaces over the field F and let U be an isomorphism of V onto W. Prove that $T \rightarrow UTU^{-1}$ is an isomorphism of $L(V, V)$ onto $L(W, W)$.

3.4. Representation of Transformations by Matrices

Let V be an n-dimensional vector space over the field F and let W be an m-dimensional vector space over F. Let $\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}$ be an ordered basis for V and $\mathcal{B}' = \{\beta_1, \ldots, \beta_m\}$ an ordered basis for W. If T is any linear transformation from V into W, then T is determined by its action on the vectors α_j. Each of the n vectors $T\alpha_j$ is uniquely expressible as a linear combination

$$T\alpha_j = \sum_{i=1}^{m} A_{ij}\beta_i \tag{3-3}$$

of the β_i, the scalars $A_{1j}, \ldots, A_{mj}$ being the coordinates of $T\alpha_j$ in the ordered basis $\mathcal{B}'$. Accordingly, the transformation T is determined by the mn scalars A_{ij} via the formulas (3-3). The $m \times n$ matrix A defined by $A(i, j) = A_{ij}$ is called **the matrix of T relative to the pair of ordered bases $\mathcal{B}$ and $\mathcal{B}'$.** Our immediate task is to understand explicitly how the matrix A determines the linear transformation T.

If $\alpha = x_1\alpha_1 + \cdots + x_n\alpha_n$ is a vector in V, then

$$\begin{aligned} T\alpha &= T\left(\sum_{j=1}^{n} x_j\alpha_j\right) \\ &= \sum_{j=1}^{n} x_j(T\alpha_j) \\ &= \sum_{j=1}^{n} x_j \sum_{i=1}^{m} A_{ij}\beta_i \\ &= \sum_{i=1}^{m}\left(\sum_{j=1}^{n} A_{ij}x_j\right)\beta_i. \end{aligned}$$

If X is the coordinate matrix of α in the ordered basis $\mathcal{B}$, then the computation above shows that AX is the coordinate matrix of the vector $T\alpha$ in the ordered basis $\mathcal{B}'$, because the scalar

$$\sum_{j=1}^{n} A_{ij}x_j$$

is the entry in the ith row of the column matrix AX. Let us also observe that if A is any $m \times n$ matrix over the field F, then

$$T\left(\sum_{j=1}^{n} x_j\alpha_j\right) = \sum_{i=1}^{m}\left(\sum_{j=1}^{n} A_{ij}x_j\right)\beta_i \tag{3-4}$$

defines a linear transformation T from V into W, the matrix of which is A, relative to $\mathcal{B}$, $\mathcal{B}'$. We summarize formally:

Theorem 11. *Let* V *be an* n*-dimensional vector space over the field* F *and* W *an* m*-dimensional vector space over* F*. Let* $\mathcal{B}$ *be an ordered basis for* V *and* $\mathcal{B}'$ *an ordered basis for* W*. For each linear transformation* T *from* V *into* W*, there is an* $m \times n$ *matrix* A *with entries in* F *such that*

$$[T\alpha]_{\mathcal{B}'} = A[\alpha]_{\mathcal{B}}$$

for every vector α *in* V*. Furthermore,* $T \to A$ *is a one-one correspondence between the set of all linear transformations from* V *into* W *and the set of all* $m \times n$ *matrices over the field* F*.*

The matrix A which is associated with T in Theorem 11 is called the **matrix of T relative to the ordered bases** $\mathcal{B}$, $\mathcal{B}'$. Note that Equation (3-3) says that A is the matrix whose columns $A_1, \ldots, A_n$ are given by

$$A_j = [T\alpha_j]_{\mathcal{B}'}, \qquad j = 1, \ldots, n.$$