

and let  $\varepsilon > 0$  be given. By Corollary 7.27 there exist real numbers  $c_1, \dots, c_n$  such that

$$(53) \quad \left| \sum_{i=1}^n c_i y^i - |y| \right| < \varepsilon \quad (-a \leq y \leq a).$$

Since  $\mathcal{B}$  is an algebra, the function

$$g = \sum_{i=1}^n c_i f^i$$

is a member of  $\mathcal{B}$ . By (52) and (53), we have

$$|g(x) - |f(x)|| < \varepsilon \quad (x \in K).$$

Since  $\mathcal{B}$  is uniformly closed, this shows that  $|f| \in \mathcal{B}$ .

**STEP 2** If  $f \in \mathcal{B}$  and  $g \in \mathcal{B}$ , then  $\max(f, g) \in \mathcal{B}$  and  $\min(f, g) \in \mathcal{B}$ .

By  $\max(f, g)$  we mean the function  $h$  defined by

$$h(x) = \begin{cases} f(x) & \text{if } f(x) \geq g(x), \\ g(x) & \text{if } f(x) < g(x), \end{cases}$$

and  $\min(f, g)$  is defined likewise.

**Proof** Step 2 follows from step 1 and the identities

$$\begin{aligned} \max(f, g) &= \frac{f+g}{2} + \frac{|f-g|}{2}, \\ \min(f, g) &= \frac{f+g}{2} - \frac{|f-g|}{2}. \end{aligned}$$

By iteration, the result can of course be extended to any finite set of functions: If  $f_1, \dots, f_n \in \mathcal{B}$ , then  $\max(f_1, \dots, f_n) \in \mathcal{B}$ , and

$$\min(f_1, \dots, f_n) \in \mathcal{B}.$$

**STEP 3** Given a real function  $f$ , continuous on  $K$ , a point  $x \in K$ , and  $\varepsilon > 0$ , there exists a function  $g_x \in \mathcal{B}$  such that  $g_x(x) = f(x)$  and

$$(54) \quad g_x(t) > f(t) - \varepsilon \quad (t \in K).$$

**Proof** Since  $\mathcal{A} \subset \mathcal{B}$  and  $\mathcal{A}$  satisfies the hypotheses of Theorem 7.31 so does  $\mathcal{B}$ . Hence, for every  $y \in K$ , we can find a function  $h_y \in \mathcal{B}$  such that

$$(55) \quad h_y(x) = f(x), \quad h_y(y) = f(y).$$

By the continuity of  $h_y$ , there exists an open set  $J_y$ , containing  $y$ , such that

$$(56) \quad h_y(t) > f(t) - \varepsilon \quad (t \in J_y).$$

Since  $K$  is compact, there is a finite set of points  $y_1, \dots, y_n$  such that

$$(57) \quad K \subset J_{y_1} \cup \dots \cup J_{y_n}.$$

Put

$$g_x = \max (h_{y_1}, \dots, h_{y_n}).$$

By step 2,  $g_x \in \mathcal{B}$ , and the relations (55) to (57) show that  $g_x$  has the other required properties.

**STEP 4** *Given a real function  $f$ , continuous on  $K$ , and  $\varepsilon > 0$ , there exists a function  $h \in \mathcal{B}$  such that*

$$(58) \quad |h(x) - f(x)| < \varepsilon \quad (x \in K).$$

Since  $\mathcal{B}$  is uniformly closed, this statement is equivalent to the conclusion of the theorem.

**Proof** Let us consider the functions  $g_x$ , for each  $x \in K$ , constructed in step 3. By the continuity of  $g_x$ , there exist open sets  $V_x$  containing  $x$ , such that

$$(59) \quad g_x(t) < f(t) + \varepsilon \quad (t \in V_x).$$

Since  $K$  is compact, there exists a finite set of points  $x_1, \dots, x_m$  such that

$$(60) \quad K \subset V_{x_1} \cup \dots \cup V_{x_m}.$$

Put

$$h = \min (g_{x_1}, \dots, g_{x_m}).$$

By step 2,  $h \in \mathcal{B}$ , and (54) implies

$$(61) \quad h(t) > f(t) - \varepsilon \quad (t \in K),$$

whereas (59) and (60) imply

$$(62) \quad h(t) < f(t) + \varepsilon \quad (t \in K).$$

Finally, (58) follows from (61) and (62).

Theorem 7.32 does not hold for complex algebras. A counterexample is given in Exercise 21. However, the conclusion of the theorem does hold, even for complex algebras, if an extra condition is imposed on  $\mathcal{A}$ , namely, that  $\mathcal{A}$  be *self-adjoint*. This means that for every  $f \in \mathcal{A}$  its complex conjugate  $\bar{f}$  must also belong to  $\mathcal{A}$ ;  $\bar{f}$  is defined by  $\bar{f}(x) = \overline{f(x)}$ .

**7.33 Theorem** Suppose  $\mathcal{A}$  is a self-adjoint algebra of complex continuous functions on a compact set  $K$ ,  $\mathcal{A}$  separates points on  $K$ , and  $\mathcal{A}$  vanishes at no point of  $K$ . Then the uniform closure  $\mathcal{B}$  of  $\mathcal{A}$  consists of all complex continuous functions on  $K$ . In other words,  $\mathcal{A}$  is dense  $\mathcal{C}(K)$ .

**Proof** Let  $\mathcal{A}_R$  be the set of all real functions on  $K$  which belong to  $\mathcal{A}$ .

If  $f \in \mathcal{A}$  and  $f = u + iv$ , with  $u, v$  real, then  $2u = f + \bar{f}$ , and since  $\mathcal{A}$  is self-adjoint, we see that  $u \in \mathcal{A}_R$ . If  $x_1 \neq x_2$ , there exists  $f \in \mathcal{A}$  such that  $f(x_1) = 1, f(x_2) = 0$ ; hence  $0 = u(x_2) \neq u(x_1) = 1$ , which shows that  $\mathcal{A}_R$  separates points on  $K$ . If  $x \in K$ , then  $g(x) \neq 0$  for some  $g \in \mathcal{A}$ , and there is a complex number  $\lambda$  such that  $\lambda g(x) > 0$ ; if  $f = \lambda g, f = u + iv$ , it follows that  $u(x) > 0$ ; hence  $\mathcal{A}_R$  vanishes at no point of  $K$ .

Thus  $\mathcal{A}_R$  satisfies the hypotheses of Theorem 7.32. It follows that every real continuous function on  $K$  lies in the uniform closure of  $\mathcal{A}_R$ , hence lies in  $\mathcal{B}$ . If  $f$  is a complex continuous function on  $K, f = u + iv$ , then  $u \in \mathcal{B}, v \in \mathcal{B}$ , hence  $f \in \mathcal{B}$ . This completes the proof.

## EXERCISES

1. Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.
2. If  $\{f_n\}$  and  $\{g_n\}$  converge uniformly on a set  $E$ , prove that  $\{f_n + g_n\}$  converges uniformly on  $E$ . If, in addition,  $\{f_n\}$  and  $\{g_n\}$  are sequences of bounded functions, prove that  $\{f_n g_n\}$  converges uniformly on  $E$ .
3. Construct sequences  $\{f_n\}, \{g_n\}$  which converge uniformly on some set  $E$ , but such that  $\{f_n g_n\}$  does not converge uniformly on  $E$  (of course,  $\{f_n g_n\}$  must converge on  $E$ ).
4. Consider

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1+n^2 x}.$$

For what values of  $x$  does the series converge absolutely? On what intervals does it converge uniformly? On what intervals does it fail to converge uniformly? Is  $f$  continuous wherever the series converges? Is  $f$  bounded?

5. Let

$$f_n(x) = \begin{cases} 0 & \left(x < \frac{1}{n+1}\right), \\ \sin^2 \frac{\pi}{x} & \left(\frac{1}{n+1} \leq x \leq \frac{1}{n}\right), \\ 0 & \left(\frac{1}{n} < x\right). \end{cases}$$

Show that  $\{f_n\}$  converges to a continuous function, but not uniformly. Use the series  $\sum f_n$  to show that absolute convergence, even for all  $x$ , does not imply uniform convergence.

6. Prove that the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^2 + n}{n^2}$$

converges uniformly in every bounded interval, but does not converge absolutely for any value of  $x$ .

7. For  $n = 1, 2, 3, \dots$ ,  $x$  real, put

$$f_n(x) = \frac{x}{1 + nx^2}.$$

Show that  $\{f_n\}$  converges uniformly to a function  $f$ , and that the equation

$$f'(x) = \lim_{n \rightarrow \infty} f'_n(x)$$

is correct if  $x \neq 0$ , but false if  $x = 0$ .

8. If

$$I(x) = \begin{cases} 0 & (x \leq 0), \\ 1 & (x > 0), \end{cases}$$

if  $\{x_n\}$  is a sequence of distinct points of  $(a, b)$ , and if  $\sum |c_n|$  converges, prove that the series

$$f(x) = \sum_{n=1}^{\infty} c_n I(x - x_n) \quad (a \leq x \leq b)$$

converges uniformly, and that  $f$  is continuous for every  $x \neq x_n$ .

9. Let  $\{f_n\}$  be a sequence of continuous functions which converges uniformly to a function  $f$  on a set  $E$ . Prove that

$$\lim_{n \rightarrow \infty} f_n(x_n) = f(x)$$

for every sequence of points  $x_n \in E$  such that  $x_n \rightarrow x$ , and  $x \in E$ . Is the converse of this true?

10. Letting  $(x)$  denote the fractional part of the real number  $x$  (see Exercise 16, Chap. 4, for the definition), consider the function

$$f(x) = \sum_{n=1}^{\infty} \frac{(nx)}{n^2} \quad (x \text{ real}).$$

Find all discontinuities of  $f$ , and show that they form a countable dense set. Show that  $f$  is nevertheless Riemann-integrable on every bounded interval.

11. Suppose  $\{f_n\}, \{g_n\}$  are defined on  $E$ , and
- (a)  $\Sigma f_n$  has uniformly bounded partial sums;
  - (b)  $g_n \rightarrow 0$  uniformly on  $E$ ;
  - (c)  $g_1(x) \geq g_2(x) \geq g_3(x) \geq \cdots$  for every  $x \in E$ .

Prove that  $\Sigma f_n g_n$  converges uniformly on  $E$ . *Hint:* Compare with Theorem 3.42.

12. Suppose  $g$  and  $f_n (n = 1, 2, 3, \dots)$  are defined on  $(0, \infty)$ , are Riemann-integrable on  $[t, T]$  whenever  $0 < t < T < \infty$ ,  $|f_n| \leq g$ ,  $f_n \rightarrow f$  uniformly on every compact subset of  $(0, \infty)$ , and

$$\int_0^{\infty} g(x) dx < \infty.$$

Prove that

$$\lim_{n \rightarrow \infty} \int_0^{\infty} f_n(x) dx = \int_0^{\infty} f(x) dx.$$

(See Exercises 7 and 8 of Chap. 6 for the relevant definitions.)

This is a rather weak form of Lebesgue's dominated convergence theorem (Theorem 11.32). Even in the context of the Riemann integral, uniform convergence can be replaced by pointwise convergence if it is assumed that  $f \in \mathcal{R}$ . (See the articles by F. Cunningham in *Math. Mag.*, vol. 40, 1967, pp. 179–186, and by H. Kestelman in *Amer. Math. Monthly*, vol. 77, 1970, pp. 182–187.)

13. Assume that  $\{f_n\}$  is a sequence of monotonically increasing functions on  $R^1$  with  $0 \leq f_n(x) \leq 1$  for all  $x$  and all  $n$ .
- (a) Prove that there is a function  $f$  and a sequence  $\{n_k\}$  such that

$$f(x) = \lim_{k \rightarrow \infty} f_{n_k}(x)$$

for every  $x \in R^1$ . (The existence of such a pointwise convergent subsequence is usually called *Helly's selection theorem*.)

- (b) If, moreover,  $f$  is continuous, prove that  $f_{n_k} \rightarrow f$  uniformly on compact sets.

*Hint:* (i) Some subsequence  $\{f_{n_i}\}$  converges at all rational points  $r$ , say, to  $f(r)$ . (ii) Define  $f(x)$ , for any  $x \in R^1$ , to be  $\sup f(r)$ , the sup being taken over all  $r \leq x$ . (iii) Show that  $f_{n_i}(x) \rightarrow f(x)$  at every  $x$  at which  $f$  is continuous. (This is where monotonicity is strongly used.) (iv) A subsequence of  $\{f_{n_i}\}$  converges at every point of discontinuity of  $f$  since there are at most countably many such points. This proves (a). To prove (b), modify your proof of (iii) appropriately.

14. Let  $f$  be a continuous real function on  $R^1$  with the following properties:  
 $0 \leq f(t) \leq 1$ ,  $f(t+2) = f(t)$  for every  $t$ , and

$$f(t) = \begin{cases} 0 & (0 \leq t \leq \frac{1}{3}) \\ 1 & (\frac{2}{3} \leq t \leq 1). \end{cases}$$

Put  $\Phi(t) = (x(t), y(t))$ , where

$$x(t) = \sum_{n=1}^{\infty} 2^{-n} f(3^{2n-1}t), \quad y(t) = \sum_{n=1}^{\infty} 2^{-n} f(3^{2n}t).$$

Prove that  $\Phi$  is continuous and that  $\Phi$  maps  $I = [0, 1]$  onto the unit square  $I^2 \subset R^2$ .  
 If fact, show that  $\Phi$  maps the Cantor set onto  $I^2$ .

*Hint:* Each  $(x_0, y_0) \in I^2$  has the form

$$x_0 = \sum_{n=1}^{\infty} 2^{-n} a_{2n-1}, \quad y_0 = \sum_{n=1}^{\infty} 2^{-n} a_{2n}$$

where each  $a_i$  is 0 or 1. If

$$t_0 = \sum_{i=1}^{\infty} 3^{-i-1} (2a_i)$$

show that  $f(3^k t_0) = a_k$ , and hence that  $x(t_0) = x_0$ ,  $y(t_0) = y_0$ .

(This simple example of a so-called "space-filling curve" is due to I. J. Schoenberg, *Bull. A.M.S.*, vol. 44, 1938, pp. 519.)

15. Suppose  $f$  is a real continuous function on  $R^1$ ,  $f_n(t) = f(nt)$  for  $n = 1, 2, 3, \dots$ , and  $\{f_n\}$  is equicontinuous on  $[0, 1]$ . What conclusion can you draw about  $f$ ?
16. Suppose  $\{f_n\}$  is an equicontinuous sequence of functions on a compact set  $K$ , and  $\{f_n\}$  converges pointwise on  $K$ . Prove that  $\{f_n\}$  converges uniformly on  $K$ .
17. Define the notions of uniform convergence and equicontinuity for mappings into any metric space. Show that Theorems 7.9 and 7.12 are valid for mappings into any metric space, that Theorems 7.8 and 7.11 are valid for mappings into any complete metric space, and that Theorems 7.10, 7.16, 7.17, 7.24, and 7.25 hold for vector-valued functions, that is, for mappings into any  $R^k$ .
18. Let  $\{f_n\}$  be a uniformly bounded sequence of functions which are Riemann-integrable on  $[a, b]$ , and put

$$F_n(x) = \int_a^x f_n(t) dt \quad (a \leq x \leq b).$$

Prove that there exists a subsequence  $\{F_{n_k}\}$  which converges uniformly on  $[a, b]$ .

19. Let  $K$  be a compact metric space, let  $S$  be a subset of  $\mathcal{C}(K)$ . Prove that  $S$  is compact (with respect to the metric defined in Section 7.14) if and only if  $S$  is uniformly closed, pointwise bounded, and equicontinuous. (If  $S$  is not equicontinuous, then  $S$  contains a sequence which has no equicontinuous subsequence, hence has no subsequence that converges uniformly on  $K$ .)

20. If  $f$  is continuous on  $[0, 1]$  and if

$$\int_0^1 f(x)x^n dx = 0 \quad (n = 0, 1, 2, \dots),$$

prove that  $f(x) = 0$  on  $[0, 1]$ . *Hint:* The integral of the product of  $f$  with any polynomial is zero. Use the Weierstrass theorem to show that  $\int_0^1 f^2(x) dx = 0$ .

21. Let  $K$  be the unit circle in the complex plane (i.e., the set of all  $z$  with  $|z| = 1$ ), and let  $\mathcal{A}$  be the algebra of all functions of the form

$$f(e^{i\theta}) = \sum_{n=0}^N c_n e^{in\theta} \quad (\theta \text{ real}).$$

Then  $\mathcal{A}$  separates points on  $K$  and  $\mathcal{A}$  vanishes at no point of  $K$ , but nevertheless there are continuous functions on  $K$  which are not in the uniform closure of  $\mathcal{A}$ .

*Hint:* For every  $f \in \mathcal{A}$

$$\int_0^{2\pi} f(e^{i\theta}) e^{i\theta} d\theta = 0,$$

and this is also true for every  $f$  in the closure of  $\mathcal{A}$ .

22. Assume  $f \in \mathcal{R}(a, b)$  on  $[a, b]$ , and prove that there are polynomials  $P_n$  such that

$$\lim_{n \rightarrow \infty} \int_a^b |f - P_n|^2 dx = 0.$$

(Compare with Exercise 12, Chap. 6.)

23. Put  $P_0 = 0$ , and define, for  $n = 0, 1, 2, \dots$ ,

$$P_{n+1}(x) = P_n(x) + \frac{x^2 - P_n^2(x)}{2}.$$

Prove that

$$\lim_{n \rightarrow \infty} P_n(x) = |x|,$$

uniformly on  $[-1, 1]$ .

(This makes it possible to prove the Stone-Weierstrass theorem without first proving Theorem 7.26.)

*Hint:* Use the identity

$$|x| - P_{n+1}(x) = [|x| - P_n(x)] \left[ 1 - \frac{|x| + P_n(x)}{2} \right]$$

to prove that  $0 \leq P_n(x) \leq P_{n+1}(x) \leq |x|$  if  $|x| \leq 1$ , and that

$$|x| - P_n(x) \leq |x| \left( 1 - \frac{|x|}{2} \right)^n < \frac{2}{n+1}$$

if  $|x| \leq 1$ .

24. Let  $X$  be a metric space, with metric  $d$ . Fix a point  $a \in X$ . Assign to each  $p \in X$  the function  $f_p$  defined by

$$f_p(x) = d(x, p) - d(x, a) \quad (x \in X).$$

Prove that  $|f_p(x)| \leq d(a, p)$  for all  $x \in X$ , and that therefore  $f_p \in \mathcal{C}(X)$ .  
Prove that

$$\|f_p - f_q\| = d(p, q)$$

for all  $p, q \in X$ .

If  $\Phi(p) = f_p$  it follows that  $\Phi$  is an *isometry* (a distance-preserving mapping) of  $X$  onto  $\Phi(X) \subset \mathcal{C}(X)$ .

Let  $Y$  be the closure of  $\Phi(X)$  in  $\mathcal{C}(X)$ . Show that  $Y$  is complete.

*Conclusion:*  $X$  is isometric to a dense subset of a complete metric space  $Y$ . (Exercise 24, Chap. 3 contains a different proof of this.)

25. Suppose  $\phi$  is a continuous bounded real function in the strip defined by  $0 \leq x \leq 1, -\infty < y < \infty$ . Prove that the initial-value problem

$$y' = \phi(x, y), \quad y(0) = c$$

has a solution. (Note that the hypotheses of this existence theorem are less stringent than those of the corresponding uniqueness theorem; see Exercise 27, Chap. 5.)

*Hint:* Fix  $n$ . For  $i = 0, \dots, n$ , put  $x_i = i/n$ . Let  $f_n$  be a continuous function on  $[0, 1]$  such that  $f_n(0) = c$ ,

$$f'_n(t) = \phi(x_i, f_n(x_i)) \quad \text{if } x_i < t < x_{i+1},$$

and put

$$\Delta_n(t) = f'_n(t) - \phi(t, f_n(t)),$$

except at the points  $x_i$ , where  $\Delta_n(t) = 0$ . Then

$$f_n(x) = c + \int_0^x [\phi(t, f_n(t)) + \Delta_n(t)] dt.$$

Choose  $M < \infty$  so that  $|\phi| \leq M$ . Verify the following assertions.

- (a)  $|f'_n| \leq M$ ,  $|\Delta_n| \leq 2M$ ,  $\Delta_n \in \mathcal{R}$ , and  $|f_n| \leq |c| + M = M_1$ , say, on  $[0, 1]$ , for all  $n$ .  
(b)  $\{f_n\}$  is equicontinuous on  $[0, 1]$ , since  $|f'_n| \leq M$ .  
(c) Some  $\{f_{n_k}\}$  converges to some  $f$ , uniformly on  $[0, 1]$ .  
(d) Since  $\phi$  is uniformly continuous on the rectangle  $0 \leq x \leq 1, |y| \leq M_1$ ,

$$\phi(t, f_{n_k}(t)) \rightarrow \phi(t, f(t))$$

uniformly on  $[0, 1]$ .

- (e)  $\Delta_n(t) \rightarrow 0$  uniformly on  $[0, 1]$ , since

$$\Delta_n(t) = \phi(x_i, f_n(x_i)) - \phi(t, f_n(t))$$

in  $(x_i, x_{i+1})$ .