When Lagrange wrote to Euler about his results in the calculus of variations, Euler was so impressed that he withheld his own results from publication so that Lagrange could publish first. Sad to say, such unselfish acts are rare.

After his wife died in 1783, Lagrange wore himself out publishing the *Mécanique.* The excesses of the Revolution upset him and he became subject to fits of depression. From these the lonely genius was rescued by the love of a teenaged girl, Renée Le Monnier, who insisted on marrying him in 1792. For the remaining twenty years of his life, Lagrange was both happy and mathematically productive.

Laplace was the son of humble parents but ended up as a marquis under the restored Bourbons. Politically, he was an opportunist, but occasionally he stood up for his principles. Napoleon once told him, 'you have written a big book on the universe without mentioning its creator', to which Laplace replied: 'I don't need that hypothesis'.

Laplace was more of a mathematical physicist than a pure mathematician. He introduced the potential V and showed that it satisfied

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

His greatest contribution to mathematics was the useful phrase 'it is easy to see', which peppers his *Mécanique Céleste.* In *The History of Mathematics*, David Burton reports:

> The American astronomer Nathaniel Bowditch (1773-1838), who translated four of the five volumes into English, observed, "I never came across one of Laplace's 'Thus it plainly appears' without feeling sure that I had hours of hard work before me to fill up the chasm and find out and show how it plainly appears."

Legendre was a great promoter of Euclid. He showed that the Parallel Postulate follows from the assumption that the plane contains real squares (i.e., quadrilaterals with four equal sides, each of whose angles is a right angle.) He also did work on the method of least squares.

Legendre is best known for his work in number theory. He was the first to prove that the Diophantine equation

$$x^5 + y^5 = z^5$$

has no nonzero integer solutions. He introduced the Legendre symbol $\left(\frac{n}{p}\right)$, where p is a prime and n an integer not divisible by p. He wrote $\left(\frac{n}{p}\right) = 1$ when n has the form $kp + r^2$ (with k and r integers) and $\left(\frac{n}{p}\right) = -1$ when n does not have this form.

Euler had conjectured a theorem, called the Law of Quadratic Reciprocity. Using the the Legendre symbol, it can be expressed as follows: if p and q are distinct odd primes, $\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{(p-1)(q-1)/4}$. Euler could not prove this, but Legendre found proofs for some special cases. It was Gauss who published the first complete proof, in 1801. Gauss later gave five other proofs of the same result.

The Legendre symbol $\left(\frac{n}{p}\right)$ can be calculated quite easily, in view of the following observation due to Euler:

$$n^{\frac{p-1}{2}} = \left(\frac{n}{p}\right) + \text{a multiple of } p.$$

Indeed, it follows from Fermat's Little Theorem that

$$(n^{\frac{p-1}{2}} - 1)(n^{\frac{p-1}{2}} + 1) = n^{p-1} - 1$$

is a multiple of p, so p divides either $n^{\frac{p-1}{2}} - 1$ or $n^{\frac{p-1}{2}} + 1$ (but not both, since it does not divide their difference). We claim that p divides the former if and only if $\left(\frac{n}{p}\right) = 1$, hence p divides the latter if and only if $\left(\frac{n}{p}\right) = -1$, from which facts the observation follows.

To see this, at least in one direction, suppose $\left(\frac{n}{p}\right) = 1$, that is, $n = r^2 + kp$ for some integers r and k. Then

$$n^{\frac{p-1}{2}} = (r^2 + kp)^{\frac{p-1}{2}} = r^{p-1} + k'p = 1 + k''p$$

for some integers k' and k'', hence p divides $n^{\frac{p-1}{2}} - 1$. The converse implication, though not difficult, is a little tricky, and we shall omit its proof.

Exercises

1. Give conditions sufficient for the convergence of the Maclaurin series.
2. Prove de Moivre's formula for positive integers n.
3. Show that e, as defined above, is bound below by 2 and above by 3.
4. Prove that the circumcenter, orthocenter and centroid of any triangle are collinear.
5. Give an example of a formula with exponents which is true when the exponents are natural numbers but not always true when the exponents are rationals.
6. Prove Wilson's Theorem.
7. Check the Law of Quadratic Reciprocity for $p = 5$ and $q = 13$.

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The Law of Quadratic Reciprocity

In this chapter we single out a great 19th century mathematician and present one of his most elegant proofs. The reader should be warned, however, that the 19th century was so rich in mathematics that it really deserves a book of its own.

Carl Friedrich Gauss (1777–1855) was inspired to become a mathematician by his discovery of a ruler and compass construction for the regular polygon with 17 sides — this when he was only a teenager — but his gift had revealed itself much earlier: as a three year old, he had pointed out an error in his father's payroll accounts!

Gauss's first major contribution was his Number Theory book *Disquisitiones Arithmeticae*, which appeared in 1801. As well as the first construction for the regular 17-gon, it included the first proof of the Fundamental Theorem of Arithmetic (that every integer can be uniquely written as a product of primes), the first proof that every prime p has a primitive root g (meaning that no two of the numbers $1, g, g^2, \ldots, g^{p-2}$ differ by a multiple of p), the first proof that every natural number is a sum of three triangular numbers, and the first proof of the theorem featured below, namely, the Law of Quadratic Reciprocity.

Gauss was also an astronomer and a physicist. In 1807, subsequent to his calculation of the position of the asteroid Ceres, he was appointed director of the observatory at Göttingen and, in 1809, he published a book on planetary astronomy. In physics, Gauss did pioneering work in electromagnetism; the *gauss* is a unit of measure denoting magnetic intensity.

We conclude this chapter with a short proof of the quadratic reciprocity law, adapted from one of the proofs given by Gauss.