The first bound $m - k$ in Proposition 6.3.13 is useful when G has few edges: the crossing number of a simple graph G is at least $e(G) - 3n + 6$, and when G is bipartite it is at least $e(G) - 2n + 4$. Iterating the argument improves the bound when $e(G)$ is larger, but for dense graphs this lower bound is weak.

Consider K_n, for example. Lacking an exact answer, we hope at least to determine the leading term in a polynomial expression for $\nu(K_n)$. To indicate a polynomial of degree k in n with leading term an^k, we often write $an^k + O(n^{k-1})$. This is consistent with the definition of "Big Oh" notation in Definition 3.2.3.

Proposition 6.3.13 yields $\nu(K_n) \geq \frac{1}{24}n^3 + O(n^2)$, but actually $\nu(K_n)$ grows like a polynomial of degree 4. The crossing number cannot exceed $\binom{n}{4}$, since we can place the vertices on the circumference of a circle and draw chords. For K_n, each set of four vertices contributes exactly one crossing. Actually, this is the worst possible straight-line drawing of K_n, since in every straight-line drawing, each set of four vertices contributes at most one crossing, depending on whether one vertex is inside the triangle formed by the other three. How many crossings can be saved by a better drawing?

6.3.14. Theorem. (R. Guy [1972]) $\frac{1}{80}n^4 + O(n^3) \leq \nu(K_n) \leq \frac{1}{64}n^4 + O(n^3)$.

Proof: A counting argument yields a recursive lower bound. A drawing of K_n with fewest crossings contains n drawings of K_{n-1}, each obtained by deleting one vertex. Each subdrawing has at least $\nu(K_{n-1}$ crossings. The total count is at least $n\nu(K_{n-1})$, but each crossing in the full drawing has been counted $(n-4)$ times. We conclude that $(n-4)\nu(K_n) \geq n\nu(K_{n-1})$.

From this inequality, we prove by induction on n that $\nu(K_n) \geq \frac{1}{5}\binom{n}{4}$ when $n \geq 5$. Basis step: $n = 5$. The crossing number of K_5 is 1. Induction step: $n > 5$. Using the induction hypothesis, we compute

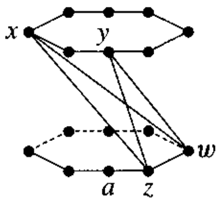
$$\nu(K_n) \geq \frac{n}{n-4}\nu(K_{n-1}) \geq \frac{n}{n-4}\frac{1}{5}\frac{(n-1)(n-2)(n-3)(n-4)}{24} = \frac{1}{5}\binom{n}{4}.$$

The denominator of the quartic term in the lower bound can be improved from 120 to 80 by considering copies of $K_{6,n-6}$, which has crossing number $6\left\lfloor\frac{n-6}{2}\right\rfloor\left\lfloor\frac{n-7}{2}\right\rfloor$ (Exercise 26b).

A better drawing lowers the upper bound from $\binom{n}{4}$ to $\frac{1}{64}n^4 + O(n^3)$. Consider $n = 2k$. Drawing K_n in the plane is equivalent to drawing it on a sphere or on the surface of a can. Place k vertices on the top rim of the can and k vertices on the bottom rim, drawing chords on the top and bottom for those k-cliques.

The edges from top to bottom fall into k natural classes. The "class number" is the circular separation between the top and bottom endpoints, ranging from $\left\lceil\frac{-k+1}{2}\right\rceil$ to $\left\lceil\frac{k-1}{2}\right\rceil$. We draw these edges to wind around the can as little as possible in passing from top to bottom, so edges in the same class don't cross. We now twist the can to make the class displacements run from 1 to k. This makes them easier to count but doesn't change the pairs of edges that cross.

Crossings on the side of the can involve two vertices on the top and two on the bottom. For top vertices x, y and bottom vertices z, w, where xz has smaller positive displacement than xw, we have a crossing for x, y, z, w if and only if the displacements to y, z, w are distinct positive values in increasing order. (For example, this holds for x, y, z, w in the illustration, but not for x, y, z, a; the edge ya winds around the can.) Hence there are $k\binom{k}{3}$ crossings on the side of the twisted can, and $\nu(K_n) \le 2\binom{k}{4} + k\binom{k}{3} = \frac{1}{64}n^4 + O(n^3)$. ■



6.3.15. Example. $\nu(K_{m,n})$. The most naive drawing puts the vertices of one partite set on one side of a channel and the vertices of the other partite set on the other side, with all edges drawn straight across. This has $\binom{n}{2}\binom{m}{2}$ crossings, but it is easy to reduce this by a factor of 4. Place the vertices of $K_{m,n}$ along two perpendicular axes. Put $\lceil n/2 \rceil$ vertices along the positive y-axis and $\lfloor n/2 \rfloor$ along the negative y-axis; similarly split the m vertices along the positive and negative x-axis. Adding up the four types of crossings generated when we join every vertex on the x-axis to every vertex on the y-axis yields $\nu(K_{m,n}) \le \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ (Zarankiewicz [1954]).

This bound is conjectured to be optimal (Guy [1969] tells the history). Kleitman [1970] proved it for $\min\{n, m\} \le 6$. Aided by a computer search, Woodall [1993] extended this so that the smallest unknown cases are $K_{7,11}$ and $K_{9,9}$. From Kleitman's result, Guy [1970] proved that $\nu(K_{m,n}) \ge \frac{m(m-1)}{5} \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$, which is not far from the upper bound (Exercise 26). ■

Another general lower bound for crossing number, conjectured in Erdős-Guy [1973], has an appealing geometric application. Our proof is inductive, generalizing the lower bound argument in Theorem 6.3.14. There is an elegant probabilistic proof in Exercise 8.5.11and stronger results in Pach–Tóth [1997].

6.3.16.* Theorem. (Ajtai-Chvátal-Newborn-Szemerédi [1982],Leighton[1983]) Let G be a simple graph. If $e(G) \ge 4n(G)$, then $\nu(G) \ge \frac{1}{64}e(G)^3/n(G)^2$.

Proof: Let $m = e(G)$ and $n = n(G)$. We use induction on n.

Basis step: $m \le 5n$ (this includes all simple graphs with at most 11 vertices). Note that $(\alpha - 3) \ge \frac{1}{64}\alpha^3$ when $4 \le \alpha \le 5$. Letting $m = \alpha n$ for $4 \le \alpha \le 5$, we obtain $\nu(G) \ge m - 3n \ge \frac{1}{64}m^3/n^2$, as desired.

Induction step: $n > 11$. Given an optimal drawing of G, each crossing appears in $n-4$ of the drawings obtained by deleting a single vertex. By the induction hypothesis, $\nu(G - v) \ge \frac{1}{64}\frac{(m-d(v))^3}{(n-1)^2}$. Thus $(n-4)\nu(G) \ge \sum_{v \in V(G)} \frac{1}{64}\frac{(m-d(v))^3}{(n-1)^2}$.

By convexity, the lower bound is always at least what results when the vertex degrees are all replaced by the average degree. In other words, $\sum(m - d(v))^3 \ge n(m - 2m/n)^3$. Also $(n-1)^2(n-4) \le (n-2)^3$. Thus

$$\nu(G) \ge \frac{1}{64}n\frac{(n-2)^3 m^3}{n^3(n-1)^2(n-4)} \ge \frac{1}{64}\frac{m^3}{n^2}\ .$$ ■

6.3.17.* Example. *Achieving the bound.* The order of magnitude in Theorem 6.3.16 is best possible. Consider $G = \frac{n}{2m} K_{2m/n}$, where $2m$ is a multiple of n. The total number of vertices is n, and the total number of edges is asymptotic to $\frac{n^2}{2m}\frac{1}{2}(\frac{2m}{n})^2 = m$. Since $\nu(K_r) \le \frac{1}{64}r^4$, we have $\nu(G) \le \frac{n^2}{2m}\frac{1}{64}(\frac{2m}{n})^4 = \frac{1}{8}\frac{m^3}{n^2}$. This is within a constant factor of the lower bound from Theorem 6.3.16. ■

We apply Theorem 6.3.16 to a problem in combinatorial geometry. Erdős [1946] asked how many unit distances can occur among a set of n points in the plane. If the points occur in a unit grid, then the graph of unit distances is the cartesian product of two paths, and this produces about $n - O(\sqrt{n})$ edges. By taking all the points of a refined grid that lie within an appropriate distance from the origin, Erdős obtained about $n^{1+c/\log\log n}$ unit distances. This growth rate is superlinear, but it is slower than $n^{1+\epsilon}$ for each positive ϵ.

Erdős also proved an upper bound of $O(n^{3/2})$. Since two circles of radius 1 intersect in at most two points, the graph G of unit distances cannot contain $K_{2,3}$. Thus each pair of points has at most two common neighbors. Since each vertex v is a common neighbor for its $\binom{d(v)}{2}$ pairs of neighbors, $\sum\binom{d(v)}{2} \le 2\binom{n}{2}$. Since $2e(G)/n$ is the average vertex degree, convexity yields $\sum\binom{d(v)}{2} \ge n\binom{2e(G)/n}{2}$. Together, these inequalities yield the desired bound (Exercise 5.2.25 considers the edge-maximization problem in general when a biclique is forbidden).

Using number-theoretic arguments about incidences between lines and points in a point set, Spencer–Szemerédi–Trotter [1984] improved the upper bound to $O(n^{4/3})$. Székely applied Theorem 6.3.16 to give an elegant and short graph-theoretic proof of this bound.

6.3.18.* Theorem. (Spencer–Szemerédi–Trotter [1984]) There are at most $4n^{4/3}$ pairs of points at distance 1 among a set of n points in the plane.

Proof: (Székely [1997]) By moving points or pairs of points without reducing the number of pairs at distance 1, we can ensure that each point is involved in such a pair and that no two points have distance 1 only from each other. If any point now is involved in only one unit distance pair, we can rotate it around its mate until it is distance 1 from another point. This reduces the problem to the case that every point is involved in at least two such pairs.

Let P be an optimal n-point configuration, with q unit distance pairs. We obtain a graph from P, not by using the unit distance pairs as edges, but rather by drawing a unit circle around each point. If a point in P is at distance 1 from k other points in P, then these points partition the circle into k arcs. Altogether we obtain $2q$ arcs. These are the edges of a loopless graph G.

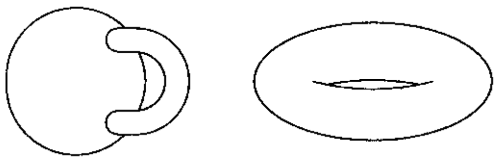
Since two points can appear on two (but not three) unit circles, G may have edges of multiplicity 2 but no larger multiplicity. We delete one copy of each duplicated edge to obtain a simple graph G' with at least q edges. We may assume that $q \ge 4n$; otherwise the bound already holds.

Because these arcs lie on n circles, they cannot produce many crossings; each pair of circles crosses at most twice. Thus our layout of G' has at most $2\binom{n}{2}$ crossings. By Theorem 6.3.16, G' has at least $\frac{1}{64}q^3/n^2$ crossings. Together, these inequalities yield $q \le 4n^{4/3}$. ■

SURFACES OF HIGHER GENUS (optional)

Instead of minimizing crossings in the plane, we could change the surface to avoid crossings. This is the effect of building overpasses and cloverleafs instead of installing traffic lights. The surface of the earth is a sphere, and for this discussion it is convenient to consider drawings on the sphere instead of in the plane. As observed in Remark 6.1.27, these settings are equivalent.

To avoid creating boundaries in the surface, we add an overpass by cutting two holes in the sphere and joining the edges of the holes by a tube. By stretching the tube and squeezing the rest of the sphere, we obtain a doughnut.



6.3.19. Definition. A **handle** is a tube joining two holes cut in a surface. The **torus** is the surface obtained by adding one handle to a sphere.

The torus is topologically the same as the sphere with one handle, in the sense that one surface can be continuously transformed into the other.†

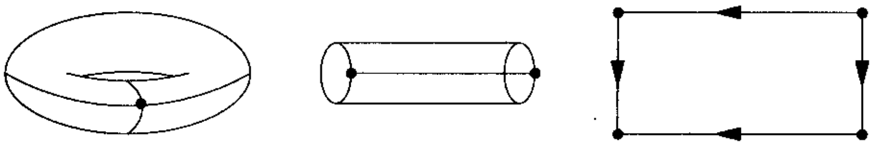
A large graph may have many crossings and need more handles. For any graph, adding enough handles to a drawing on the sphere will eliminate all crossings and produce an embedding. When we add some number of handles, it doesn't matter how we do it, because a fundamental result of topology says that two surfaces obtained by adding the same number of handles to a sphere can be continuously deformed into each other.

6.3.20. Definition. The **genus** of a surface obtained by adding handles to a sphere is the number of handles added; we use S_γ for the surface of genus γ. The **genus** of a graph G is the minimum γ such that G embeds on S_γ. The graphs embeddable on the surfaces of genus 0, 1, 2 are the **planar**, **toroidal**, and **double-toroidal** graphs, respectively (the surface with two handles is the **double-torus**).

The theory of planar graphs extends in some ways to graphs embeddable on higher surfaces; we discuss this only briefly, for cultural interest. Drawings of large graphs on surfaces of large genus are hard to follow, even on the **pretzel** (S_3). Locally, the surface looks like a plane sheet of paper. To draw the graph we want to lay the entire surface flat; to do this we must cut the surface. If we keep track of how the edges should be pasted back together to get the surface, we can describe the surface on a flat piece of paper. Consider first the torus.

†This is the source of the joke that a topologist is a person who can't tell the difference between a doughnut and a coffee cup.

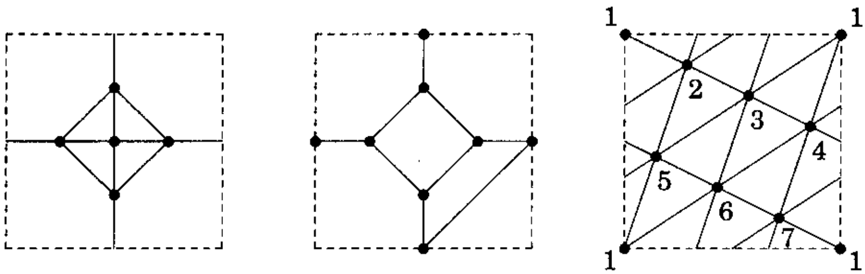
6.3.21. Example. *Combinatorial description of the torus.*



Cutting the closed tube once turns it into a cylinder, and then slitting the length of the cylinder allows us to lay it flat as a rectangle. Labeling the edges of the rectangle indicates how to paste it back together. The two sides of a cut labeled with the same letter are "identified".

Keeping track of the identifications is important because edges of an embedding on a surface may cross such a cut. When the edge reaches one border of the rectangle, it is reaching one side of the imagined cut. When it crosses the cut, it emerges from the identical point on the other copy of this border. The four "corners" of the rectangle correspond to the single point on the surface through which both cuts pass.

These ideas lead to nice toroidal embeddings of K_5, $K_{3,3}$, and K_7. ■



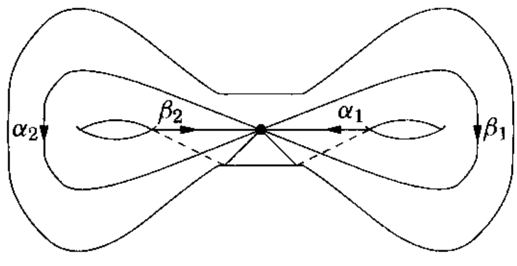
For surfaces of higher genus, there is some flexibility in making the cuts, but each way takes two cuts per handle before we can lay the surface flat. The usual representation comes from expressing the handles as "lobes" of the surface, with the cuts having a common point on the hub.

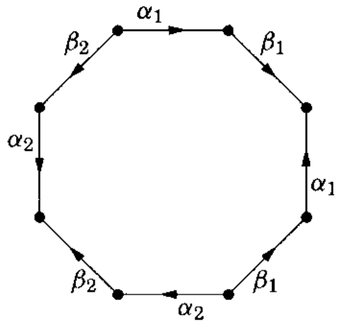
6.3.22. Example. *Laying the double torus flat.* Below is a polygonal representation for the double torus. Making the cuts is equivalent to adding loops at a single vertex until we have a one-face embedding of a bouquet of loops. In general, we make 2γ cuts through a single point to lay S_γ flat.

Keeping track of the borders from each cut leads to representing S_γ by a 4γ-gon in which a clockwise traversal of the boundary can be described by reading out the cuts as we traverse them. We record a cut using the notation of inverses when we traverse it in the opposite order.

Since we are following the boundary of a single face, with our left hand always on the wall, each edge will be followed once forward and once backward. For the example here, the traversal is $\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\alpha_2\beta_2\alpha_2^{-1}\beta_2^{-1}$.

Each surface S_γ has a layout of the form $\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\cdots\alpha_\gamma\beta_\gamma\alpha_\gamma^{-1}\beta_\gamma^{-1}$. Other layouts result from other ways of making the cuts – different ways of embedding a bouquet of 2γ loops. For example, the double torus can also be represented by an octagon with boundary $\alpha\beta\gamma\delta\alpha^{-1}\beta^{-1}\gamma^{-1}\delta^{-1}$. ■





6.3.23. Remark. *Euler's Formula for S_γ.* A **2-cell** is a region such that every closed curve in the interior can be continuously contracted to a point. A **2-cell embedding** is an embedding where every region is a 2-cell. Euler's Formula generalizes for 2-cell embeddings of connected graphs on S_γ (Exercise 35) as

$$n - e + f = 2 - 2\gamma.$$

For example, our embedding of K_7 on the torus ($\gamma = 1$) has 7 vertices, 21 edges, 14 faces, and $7 - 21 + 14 = 0$. The proof of Euler's Formula for S_γ is like the proof in the plane, except that the basis case of 1-vertex graphs needs more care. It requires showing that it takes 2γ cuts to lay the surface flat (that is, to obtain a 2-cell embedding of a graph with one vertex and one face). ■

6.3.24. Lemma. Every simple n-vertex graph embedded on S_γ has at most $3(n - 2 + 2\gamma)$ edges.

Proof: Exercise 35. ■

Note that K_7 satisfies Lemma 6.3.24 with equality on the torus ($\gamma = 1$), as every face in the toroidal embedding of K_7 is a 3-gon. Hence K_7 is a maximal toroidal graph. Rewriting $e \le 3(n - 2 + 2\gamma)$ yields a lower bound on the number of handles we must add to obtain a surface on which G is embeddable; thus $\gamma(G) \ge 1 + (e - 3n)/6$.

Lemma 6.3.24 leads to an analogue of the Four Color Theorem for S_γ.

6.3.25. Theorem. (Heawood's Formula—Heawood [1890]) If G is embeddable on S_γ with $\gamma > 0$, then $\chi(G) \le \left\lfloor (7 + \sqrt{1 + 48\gamma})/2 \right\rfloor$.

Proof: Let $c = (7 + \sqrt{1 + 48\gamma})/2$. It suffices to prove that every simple graph embeddable on S_γ has a vertex of degree at most $c - 1$; the bound on $\chi(G)$ then follows by induction on $n(G)$. Since $\chi(G) \le c$ for all graphs with at most c vertices, so need only consider $n(G) > c$.

We use Lemma 6.3.24 to show that the average (and hence minimum) degree is at most $c - 1$. The second inequality below follows from $\gamma > 0$ and $n > c$.

Since c satisfies $c^2 - 7c + (12 - 12\gamma) = 0$, we have $c - 1 = 6 - (12 - 12\gamma)/c$, so the average degree satisfies the desired bound.

$$\frac{2e}{n} \le \frac{6(n-2+2\gamma)}{n} \le 6 - \frac{12-12\gamma}{c} = c - 1. \qquad \blacksquare$$

The key inequality here fails when $\gamma = 0$. Thus the argument is invalid for planar graphs, even though the formula reduces to $\chi(G) \le 4$ when $\gamma = 0$. Proving that the Heawood bound is sharp involves embedding K_n on S_γ with $\gamma = \lceil (n-3)(n-4)/12 \rceil$. The proof breaks into cases by the congruence class of n modulo 12 (K_7 is the first example in the easy class). Completed in Ringel–Youngs [1968], it comprises the book *Map Color Theorem* (Ringel [1974]).

Having considered the coloring problem on S_γ, one naturally wonders which graphs embed on S_γ. Planar graphs have many characterizations, beginning with Kuratowski's Theorem (Theorem 6.2.2) and Wagner's Theorem (Exercise 6.2.12). On any surface, embeddability is preserved by deleting or contracting an edge. Thus every surface has a list of "minor-minimal" obstructions to embeddability. Wagner's Theorem states that the list for the plane is $\{K_{3,3}, K_5\}$; every nonplanar graph has one of these as a minor.

More than 800 minimal forbidden minors are known for the torus. For each surface, the list is finite; this follows from the much more general statement below (the *subdivision* relation in Kuratowski's Theorem leads to infinite lists).

6.3.26. Theorem. (The Graph Minor Theorem—Robertson–Seymour [1985]) In any infinite list of graphs, some graph is a minor of another. $\blacksquare$

This is perhaps the most difficult theorem known in graph theory. The complete proof takes well over 500 pages (without computer assistance) in a series of 20 papers stretching beyond the year 2000. It has many ramifications about structure of graphs and complexity of computation. The techniques involved in the proof have spawned new areas of graph theory. Some aspects of these techniques and their relation to the proof of the Graph Minor Theorem are presented in the final chapter of the text by Diestel [1997].

EXERCISES

6.3.1. (−) State a polynomial-time algorithm that takes an arbitrary planar graph as input and produces a proper 5-coloring of the graph.

6.3.2. (−) A graph G is k-**degenerate** if every subgraph of G has a vertex of degree at most k. Prove that every k-degenerate graph is $k+1$-colorable.

6.3.3. (−) Use the Four Color Theorem to prove that every outerplanar graph is 3-colorable.

6.3.4. (−) Determine the crossing numbers of $K_{2,2,2,2}$, $K_{4,4}$, and the Petersen graph.

• • • • •

6.3.5. Prove that every planar graph decomposes into two bipartite graphs. (Hedetniemi [1969], Mabry [1995])

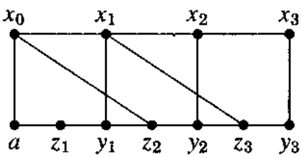
6.3.6. Without using the Four Color Theorem, prove that every planar graph with at most 12 vertices is 4-colorable. Use this to prove that every planar graph with at most 32 edges is 4-colorable.

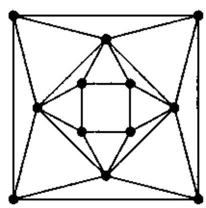
6.3.7. (!) Let H be a configuration in a planar triangulation (Definition 6.3.2). Let H' be obtained by labeling the neighbors of the ring vertices with their degrees and then deleting the ring vertices. Prove that H can be retrieved from H'.

6.3.8. Create a configuration with ring size 5 in a planar triangulation such that every internal vertex has degree at least five.

6.3.9. (+) Prove that every planar configuration having ring size at most four is reducible. (Hint: The ring is a separating cycle C. Prove that if smaller triangulations are 4-colorable, then the C-lobes of G have 4-colorings that agree on C.) (Birkhoff [1913])

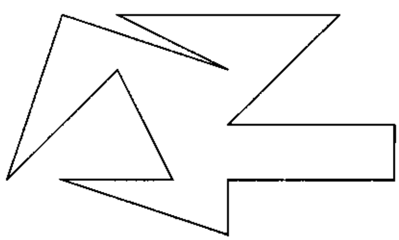
6.3.10. Grötzsch's Theorem [1959] (see Steinberg [1993], Thomassen [1994a]) states that a triangle-free planar graph G is 3-colorable. Hence $\alpha(G) \geq n(G)/3$. Tovey–Steinberg [1993] proved that $\alpha(G) > n(G)/3$ always. Prove that this is best possible by considering the family of graphs G_k defined as follows: G_1 is the 5-cycle, with vertices a, x_0, x_1, y_1, z_1 in order. For $k > 1$, G_k is obtained from G_{k-1} by adding the three vertices x_k, y_k, z_k and the five edges $x_{k-1}x_k, x_ky_k, y_kz_k, z_ky_{k-1}, z_kx_{k-2}$. The graph G_3 is shown on the left below. (Fraughnaugh [1985])





6.3.11. Define a sequence of plane graphs as follows. Let G_1 be C_4. For $n > 1$, obtain G_n from G_{n-1} by adding a new 4-cycle surrounding G_{n-1}, making each vertex of the new cycle also adjacent to two consecutive vertices of the previous outside face. The graph G_3 is shown on the right above. Prove that if n is even, then every proper 4-coloring of G_n uses each color on exactly n vertices. (Albertson)

6.3.12. (!) Without using the Four Color Theorem, prove that every outerplanar graph is 3-colorable. Apply this to prove the Art Gallery Theorem: If an art gallery is laid out as a simple polygon with n sides, then it is possible to place $\lfloor n/3 \rfloor$ guards such that every point of the interior is visible to some guard. Construct a polygon that requires $\lfloor n/3 \rfloor$ guards. (Chvátal [1975], Fisk [1978])



6.3.13. An *art gallery with walls* is a polygon plus some nonintersecting chords called "walls" that join vertices. Each interior wall has a tiny opening called a "doorway". A guard in a doorway can see everything in the two neighboring rooms, but a guard not in a doorway cannot see past a wall. Determine the minimum number t such that for every walled art gallery with n vertices, it is possible to place t guards so that every interior point is visible to some guard. (Hutchinson [1995], Kündgen [1999])

6.3.14. (+) Prove that a maximal planar graph is 3-colorable if and only if it is Eulerian. (Hint: For sufficiency, use induction on $n(G)$. Choose an appropriate pair or triple of adjacent vertices to replace with appropriate edges.) (Heawood [1898])

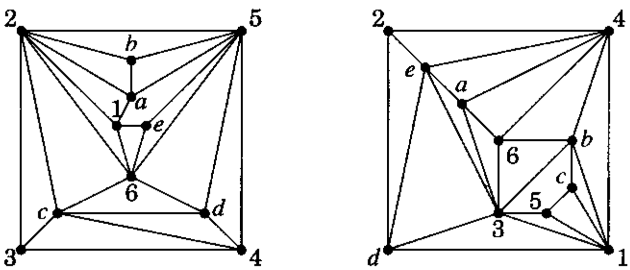
6.3.15. (!) Prove that the vertices of an outerplanar graph can be partitioned into two sets so that the subgraph induced by each set is a disjoint union of paths. (Hint: Define the partition using the parity of the distance from a fixed vertex.) (Akiyama–Era–Gervacio [1989], Goddard [1991])

6.3.16. (−) Prove that the 4-dimensional cube Q_4 is nonplanar. Decompose it into two isomorphic planar graphs; hence Q_4 has thickness 2.

6.3.17. Prove that K_n has thickness at least $\lfloor \frac{n+7}{6} \rfloor$. (Hint: $\lceil \frac{x}{r} \rceil = \lfloor \frac{x+r-1}{r} \rfloor$.) Show that equality holds for K_8 by finding a self-complementary planar graph with 8 vertices. (Comment: The thickness equals $\lfloor \frac{n+7}{6} \rfloor$ except that K_9 and K_{10} have thickness 3; Beineke–Harary [1965] for $n \not\equiv 4 \bmod 6$, and Alekseev–Gončakov [1976] for $n \equiv 4 \bmod 6$.)

6.3.18. Decompose K_9 into three pairwise-isomorphic planar graphs.

6.3.19. Prove that if G has thickness 2, then $\chi(G) \le 12$. Use the two graphs below to show that $\chi(G)$ may be as large as 9 when G has thickness 2. (Sulanke)



6.3.20. (!) When r is even and s is greater than $(r-2)^2/2$, prove that the thickness of $K_{r,s}$ is $r/2$. (Beineke–Harary–Moon [1964])

6.3.21. Determine $\nu(K_{1,2,2,2})$ and use it to compute $\nu(K_{2,2,2,2})$.

6.3.22. Prove that $K_{3,2,2}$ has no planar subgraph with 15 edges. Use this to give another proof that $\nu(K_{3,2,2}) \ge 2$.

6.3.23. Let M_n be the graph obtained from the cycle C_n by adding chords between vertices that are opposite (if n is even) or nearly opposite (if n is odd). The graph M_n is 3-regular if n is even, 4-regular if n is odd. Determine $\nu(M_n)$. (Guy–Harary [1967])

6.3.24. The graph P_n^k has vertex set $[n]$ and edge set $\{ij\colon |i-j| \le k\}$. Prove that P_n^3 is a maximal planar graph. Use a planar embedding of P_n^3 to prove that $\nu(P_n^4) = n - 4$. (Harary–Kainen [1993])

6.3.25. For every positive integer k, construct a graph that embeds on the torus but requires at least k crossings when drawn in the plane. (Hint: A single easily described toroidal family suffices; use Proposition 6.3.13.)

6.3.26. (!) Use Kleitman's computation that $\nu(K_{6,n}) = 6\left\lfloor\frac{n-6}{2}\right\rfloor\left\lfloor\frac{n-7}{2}\right\rfloor$ to give counting arguments for the following lower bounds.

a) $\nu(K_{m,n}) \ge m\frac{m-1}{5}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor$. (Guy [1970])

b) $\nu(K_p) \ge \frac{1}{80}p^4 + O(p^3)$.

6.3.27. (!) It is conjectured that $\nu(K_{m,n}) = \left\lfloor\frac{m}{2}\right\rfloor\left\lfloor\frac{m-1}{2}\right\rfloor\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor$. Suppose that this conjecture holds for $K_{m,n}$ and that m is odd. Prove that the conjecture then holds also for $K_{m+1,n}$. (Kleitman [1970])

6.3.28. (!) Suppose that m and n are odd. Prove that in all drawings of $K_{m,n}$, the parity of the number of pairs of edges that cross is the same. (We consider only drawings where edges cross at most once and edges sharing an endpoint do not cross.) Conclude that $\nu(K_{m,m})$ is odd when $m-3$ and $n-3$ are divisible by 3 and even otherwise.

6.3.29. Suppose that n is odd. Prove that in all drawings of K_n, the parity of the number of pairs of edges that cross is the same. Conclude that $\nu(K_n)$ is even when n is congruent to 1 or 3 modulo 8 and is odd when n is congruent to 5 or 7 modulo 8.

6.3.30. (!) It is known that $\nu(C_m \square C_n) = (m-2)n$ if $m \le \min\{5, n\}$. Also $\nu(K_4 \square C_n) = 3n$.

a) Find drawings in the plane to establish the upper bounds.

b) Prove that $\nu(C_3 \square C_3) \ge 2$. (Hint: Find three subdivisions of $K_{3,3}$ that together use each edge exactly twice.)

6.3.31. Let $f(n) = \nu(K_{n,n,n})$.

a) Show that $3\nu(K_{n,n}) \le f(n) \le 3\binom{n}{2}^2$.

b) Show that $\nu(K_{3,2,2}) = 2$ and $\nu(K_{3,3,1}) = 3$. Show that $5 \le \nu(K_{3,3,2}) \le 7$ and $9 \le \nu(K_{3,3,3}) \le 15$.

c) Exercise 6.3.26a shows that the lower bound in part (a) is at least $(3/20)n^4 + O(n^3)$. Improve it by using a recurrence to show that $f(n) \ge n^3(n-1)/6$.

d) The upper bound in part (a) is $\frac{3}{4}n^4 + O(n^3)$. Improve it to $f(n) \le \frac{9}{16}n^4 + O(n^3)$. (Hint: One construction embeds the graph on a tetrahedron and generalizes to a construction for $K_{l,m,n}$; another uses K_n and generalizes to a construction for $K_{n,\dots,n}$.)

6.3.32. (∗) Construct an embedding of a 3-regular nonbipartite simple graph on the torus so that every face has even length.

6.3.33. (∗) Suppose that n is at least 9 and is not a prime or twice a prime. Construct a 6-regular toroidal graph with n vertices.

6.3.34. (∗) An embedding of a graph on a surface is **regular** if its faces all have the same length. Construct regular embeddings of $K_{4,4}$, $K_{3,6}$, and $K_{3,3}$ on the torus.

6.3.35. (∗) Prove Euler's Formula for genus γ: For every 2-cell embedding of a graph on the surface S_γ, the numbers of vertices, edges, and faces satisfy $n - e + f = 2 - 2\gamma$. Conclude that an n-vertex graph embeddable on S_γ has at most $3(n - 2 + 2\gamma)$ edges.

6.3.36. (∗) Use Euler's Formula for S_γ to prove that $\gamma(K_{3,3,n}) \ge n - 2$, and determine the value exactly for $n \le 3$.

6.3.37. (∗) For every positive integer k, use Euler's Formula for higher surfaces to prove that there exists a planar graph G such that $\gamma(G \square K_2) \ge k$.

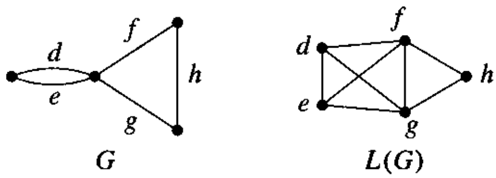
Chapter 7

Edges and Cycles

7.1. Line Graphs and Edge-coloring

Many questions about vertices have natural analogues for edges. Independent sets have no adjacent vertices; matchings have no "adjacent" edges. Vertex colorings partition vertices into independent sets; we can instead partition edges into matchings. These pairs of problems are related via line graphs (Definition 4.2.18). Here we repeat the definition, emphasizing our return to the context in which a graph may have multiple edges. We use "line graph" and $L(G)$ instead of "edge graph" because $E(G)$ already denotes the edge set.

7.1.1. Definition. The **line graph** of G, written $L(G)$, is the simple graph whose vertices are the edges of G, with $ef \in E(L(G))$ when e and f have a common endpoint in G.



Some questions about edges in a graph G can be phrased as questions about vertices in $L(G)$. When extended to all simple graphs, the vertex question may be more difficult. If we can solve it, then we can answer the original question about edges in G by applying the vertex result to $L(G)$.

In Chapter 1, we studied Eulerian circuits. An Eulerian circuit in G yields a spanning cycle in the line graph $L(G)$. (Exercise 7.2.10 shows that the converse need not hold!) In Section 7.2, we study spanning cycles for graphs in general. As discussed in Appendix B, this problem is computationally difficult.

In Chapter 3, we studied matchings. A matching in G becomes an independent set in $L(G)$. Thus $\alpha'(G) = \alpha(L(G))$, and the study of α' for graphs is the study of α for line graphs. Computing α is harder for general graphs than for line graphs. Section 3.1 considers this for bipartite graphs, and we describe the general case briefly in Appendix B.

In Chapter 4, we studied connectivity. Menger's Theorem gave a min-max relation for connectivity and internally disjoint paths in all graphs. By applying this theorem to an appropriate line graph, we proved the analogous min-max relation for edge-connectivity and edge-disjoint paths in all graphs.

In Chapter 5, we studied vertex coloring. Coloring edges so that each color class is a matching amounts to proper vertex coloring of the line graph. Thus edge-coloring is a special case of vertex coloring and therefore potentially easier. We discuss edge-coloring in this section. Our main result, when stated in terms of vertex coloring of line graphs, is an algorithm to compute $\chi(H)$ within 1 when H is the line graph of a simple graph.

Thus line graphs suggest the problems of edge-coloring and spanning cycles that are discussed in this chapter. We first study these separately. In Section 7.3, we study their connections to each other and to planar graphs.

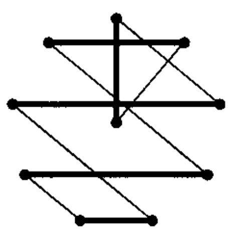
In applying algorithms for line graphs, we may need to know whether G is a line graph. There are good algorithms to check this; they use characterizations of line graphs, which we postpone to the end of this section.

EDGE-COLORINGS

In Example 1.1.11 that introduced vertex coloring, we needed to schedule Senate committees. Edge-coloring problems arise when the objects being scheduled are pairs of underlying elements.

7.1.2. Example. *Edge-coloring of K_{2n}.* In a league with $2n$ teams, we want to schedule games so that each pair of teams plays a game, but each team plays at most once a week. Since each team must play $2n-1$ others, the season lasts at least $2n-1$ weeks. The games of each week must form a matching. We can schedule the season in $2n-1$ weeks if and only if we can partition $E(K_{2n})$ into $2n-1$ matchings. Since K_{2n} is $2n-1$-regular, these must be perfect matchings.

The figure below describes the solution. Put one vertex in the center. Arrange the other $2n-1$ vertices cyclically, viewed as congruence classes modulo $2n-1$. As in Theorem 2.2.16, the *difference* between two congruence classes is 1 if they are consecutive, 2 if there is one class between them, and so on up to difference $n-1$. There are $2n-1$ edges with each difference i, for $1 \le i \le n-1$.



Each matching consists of one edge from each difference class plus one edge involving the center vertex. We show one such matching in bold. Rotating the picture (to obtain the solid matching) yields n new edges; again they are one of each length plus one to the center. The $2n-1$ rotations of the figure yield the desired matchings, since these matchings take distinct edges from each difference class and distinct edges involving the center vertex. ■

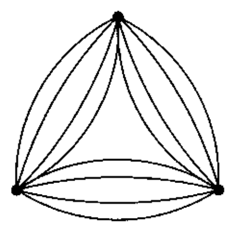
7.1.3. Definition. A **k-edge-coloring** of G is a labeling $f\colon E(G) \to S$, where $|S| = k$ (often we use $S = [k]$). The labels are **colors**; the edges of one color form a **color class**. A k-edge-coloring is **proper** if incident edges have different labels; that is, if each color class is a matching. A graph is **k-edge-colorable** if it has a proper k-edge-coloring. The **edge-chromatic number** $\chi'(G)$ of a loopless graph G is the least k such that G is k-edge-colorable.

Chromatic index is another name for $\chi'(G)$. Since edges sharing a vertex need different colors, $\chi'(G) \geq \Delta(G)$. Vizing [1964] and Gupta [1966] independently proved that $\Delta(G)+1$ colors suffice when G is simple; this is our main objective. A clique in $L(G)$ is a set of pairwise-intersecting edges of G. When G is simple, such edges form a star or a triangle in G (Exercise 9). For the hereditary class of line graphs of simple graphs, Vizing's Theorem thus states that $\chi(H) \leq \omega(H)+1$; thus line graphs are "almost" perfect.

In contrast to $\chi(G)$ in Chapter 5, multiple edges greatly affect $\chi'(G)$. A graph with a loop has no proper edge-coloring; the adjective "loopless" excludes loops but allows multiple edges.

7.1.4. Definition. In a graph G with multiple edges, we say that a vertex pair x, y is an edge of **multiplicity** m if there are m edges with endpoints x, y. We write $\mu(xy)$ for the multiplicity of the pair, and we write $\mu(G)$ for the maximum of the edge multiplicities in G.

7.1.5. Example. *The "Fat Triangle".* For loopless graphs with multiple edges, $\chi'(G)$ may exceed $\Delta(G)+1$. Shannon [1949] proved that the maximum of $\chi'(G)$ in terms of $\Delta(G)$ alone is $3\Delta(G)/2$ (see Theorem 7.1.13). Vizing and Gupta proved that $\chi'(G) \leq \Delta(G)+\mu(G)$, where $\mu(G)$ is the maximum edge multiplicity. The graph below achieves both bounds. The edges are pairwise intersecting and hence require distinct colors. Thus $\chi'(G) = 3\Delta(G)/2 = \Delta(G)+\mu(G)$. ■



7.1.6. Remark. We have observed that always $\chi'(G) \geq \Delta(G)$. The upper bound $\chi'(G) \leq 2\Delta(G)-1$ also follows easily. Color the edges in some order, always assigning the current edge the least-indexed color different from those already appearing on edges incident to it. Since no edge is incident to more than $2(\Delta(G)-1)$ other edges, this never uses more than $2\Delta(G)-1$ colors. The procedure is precisely greedy coloring for vertices of $L(G)$.

$$\chi'(G)=\chi(L(G))\leq\Delta(L(G))+1\leq 2\Delta(G)-1.$$ ■

For bipartite graphs, the results of Chapter 3 improve the upper bound of Remark 7.1.6, achieving the trivial lower bound even when multiple edges are allowed. Furthermore, there is a good algorithm to produce a proper $\Delta(G)$-edge-coloring in a bipartite graph G.

7.1.7. Theorem. (König [1916]) If G is bipartite, then $\chi'(G)=\Delta(G)$.

Proof: Corollary 3.1.13 states that every regular bipartite graph H has a 1-factor. By induction on $\Delta(H)$, this yields a proper $\Delta(H)$-edge-coloring. It therefore suffices to show that for every bipartite graph G with maximum degree k, there is a k-regular bipartite graph H containing G.

To construct such a graph, first add vertices to the smaller partite set of G, if necessary, to equalize the sizes. If the resulting G' is not regular, then each partite set has a vertex with degree less than k. Add an edge with these two vertices as endpoints. Continue adding such edges until the graph becomes k-regular; the resulting graph is H. ■

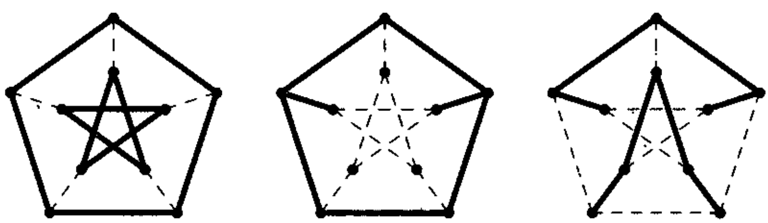
For a regular graph G, proper edge-coloring with $\Delta(G)$ colors is equivalent to decomposition into 1-factors.

7.1.8. Definition. A decomposition of a regular graph G into 1-factors is a **1-factorization** of G. A graph with a 1-factorization is **1-factorable**.

An odd cycle is not 1-factorable; $\chi'(C_{2m+1})=3>\Delta(C_{2m+1})$. The Petersen graph also requires an extra color, but only one extra color.

7.1.9. Example. *The Petersen graph is 4-edge-chromatic* (Petersen [1898]). The Petersen graph is 3-regular; 3-edge-colorability requires a 1-factorization. Deleting a perfect matching leaves a 2-factor; all components are cycles. The 1-factorization can be completed only if these are all even cycles.

Thus it suffices to show that every 2-factor is isomorphic to $2C_5$. Consider the drawing consisting of two 5-cycles and a matching (the **cross edges**) between them. We consider cases by the number of cross edges used.



Every cycle uses an even number of cross edges, so a 2-factor H has an even number m of cross edges. If $m = 0$ (left figure), then $H = 2C_5$.

If $m = 2$ (central figure), then the two cross edges have nonadjacent endpoints on the inner cycle or the outer cycle. On the cycle where their endpoints are nonadjacent, the remaining three vertices force all five edges of that cycle into H, which violates the 2-factor requirement.

If $m = 4$ (right figure), then the cycle edges forced into H by the unused cross edges form a $2P_5$ whose only completion to a 2-factor in H is $2C_5$.

Note that since C_5 is 3-edge-colorable, the graph is 4-edge-colorable. ■

Now we consider all simple graphs. We make $\Delta(G) + 1$ colors available and build a proper edge-coloring, incorporating edges one by one until we have a proper $\Delta(G) + 1$-edge-coloring of G. The algorithm runs surprisingly quickly.

7.1.10. Theorem. (Vizing [1964, 1965], Gupta [1966]) If G is a simple graph, then $\chi'(G) \le \Delta(G) + 1$.

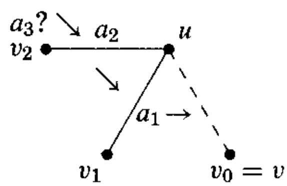
Proof: Let f be a proper $\Delta(G) + 1$-edge-coloring of a subgraph G' of G. If $G' \ne G$, then some edge uv is uncolored by f. After possibly recoloring some edges, we extend the coloring to include uv; call this an *augmentation*. After $e(G)$ augmentations, we obtain a proper $\Delta(G) + 1$-edge-coloring of G.

Since the number of colors exceeds $\Delta(G)$, every vertex has some color *not* appearing on its incident edges. Let a_0 be a color missing at u. We generate a list of neighbors of u and a corresponding list of colors. Begin with $v_0 = v$.

Let a_1 be a color missing at v_0. We may assume that a_1 appears at u on some edge uv_1; otherwise, we would use a_1 on uv_0.

Let a_2 be a color missing at v_1. We may assume that a_2 appears at u on some edge uv_2; otherwise, we would replace color a_1 with a_2 on uv_1 and then use a_1 on uv_0 to augment the coloring.

Having selected uv_{i-1} with color a_{i-1}, let a_i be a color missing at v_{i-1}. If a_i is missing at u, then we use a_i on uv_{i-1} and shift color a_j from uv_j to uv_{j-1} for $1 \le j \le i-1$ to complete the augmentation. We call this *downshifting from i*. If a_i appears at u (on some edge uv_i), then the process continues.



Since we have only $\Delta(G) + 1$ colors to choose from, the list of selected colors eventually repeats (or we complete the augmentation by downshifting). Let l be the smallest index such that a color missing at v_l is in the list $a_1, \dots, a_l$; let this color be a_k. Instead of extending the list, we use this repetition to perform the augmentation in one of several ways.

The color a_k missing at v_l is also missing at v_{k-1} and appears on uv_k. If a_0 does not appear at v_l, then we downshift from v_l and use color a_0 on uv_l to complete the augmentation. Hence we may assume that a_0 appears at v_l.