Let $\boldsymbol{x} = (x_1, \ldots, x_n)$, $\boldsymbol{u} = (u_1, \ldots, u_n)$, and $\boldsymbol{X} = (X_1, \ldots, X_n)$. Then these equations define a vector-valued mapping

$$\boldsymbol{X}: T \to S$$

from a set T in n-space to another set S in n-space. We assume the mapping $\boldsymbol{X}$ is one-to-one and continuously differentiable on T. The transformation formula for n-fold integrals assumes the form

$$\int_S f(\boldsymbol{x})\, d\boldsymbol{x} = \int_T f[\boldsymbol{X}(\boldsymbol{u})]\, |\det D\boldsymbol{X}(\boldsymbol{u})|\, d\boldsymbol{u}, \tag{11.47}$$

where $D\boldsymbol{X}(\boldsymbol{u}) = [D_j X_k(\boldsymbol{u})]$ is the Jacobian matrix of the vector field $\boldsymbol{X}$. In terms of components we have

$$D\boldsymbol{X}(\boldsymbol{u}) = \begin{bmatrix} D_1X_1(\boldsymbol{u}) & D_2X_1(\boldsymbol{u}) & \cdots & D_nX_1(\boldsymbol{u}) \\ \vdots & & & \vdots \\ D_1X_n(\boldsymbol{u}) & D_2X_n(\boldsymbol{u}) & \cdots & D_nX_n(\boldsymbol{u}) \end{bmatrix}.$$

As in the two-dimensional case, the transformation formula is valid if $\boldsymbol{X}$ is one-to-one on T and if the Jacobian determinant $J(\boldsymbol{u}) = \det D\boldsymbol{X}(\boldsymbol{u})$ is never zero on T. It is also valid if the mapping fails to be one-to-one on a subset of T having n-dimensional content zero, or if the Jacobian determinant vanishes on such a subset.

For the three-dimensional case we write (x, y, z) for (x_1, x_2, x_3), (u, v, w) for (u_1, u_2, u_3), and (X, Y, Z) for (X_1, X_2, X_3). The transformation formula for triple integrals takes the form

$$\iiint\limits_S f(x, y, z)\, dx\, dy\, dz = \iiint\limits_T f[X(u, v, w), Y(u, v, w), Z(u, v, w)]\, |J(u, v, w)|\, du\, dv\, dw, \tag{11.48}$$

where $J(u, v, w)$ is the Jacobian determinant,

$$J(u, v, w) = \begin{vmatrix} \dfrac{\partial X}{\partial u} & \dfrac{\partial Y}{\partial u} & \dfrac{\partial Z}{\partial u} \\[2ex] \dfrac{\partial X}{\partial v} & \dfrac{\partial Y}{\partial v} & \dfrac{\partial Z}{\partial v} \\[2ex] \dfrac{\partial X}{\partial w} & \dfrac{\partial Y}{\partial w} & \dfrac{\partial Z}{\partial w} \end{vmatrix}$$

In 3-space the Jacobian determinant can be thought of as a magnification factor for volumes. In fact, if we introduce the vector-valued function $\boldsymbol{r}$ defined by the equation

$$\boldsymbol{r}(u, v, w) = X(u, v, w)\boldsymbol{i} + Y(u, v, w)\boldsymbol{j} + Z(u, v, w)\boldsymbol{k},$$

and the vectors

$$V_1 = \frac{\partial r}{\partial u} = \frac{\partial X}{\partial u} i + \frac{\partial Y}{\partial u} j + \frac{\partial Z}{\partial u} k,$$

$$V_2 = \frac{\partial r}{\partial v} = \frac{\partial X}{\partial v} i + \frac{\partial Y}{\partial v} j + \frac{\partial Z}{\partial v} k,$$

$$V_3 = \frac{\partial r}{\partial w} = \frac{\partial X}{\partial w} i + \frac{\partial Y}{\partial w} j + \frac{\partial Z}{\partial w} k,$$

an argument similar to that given in Section 11.26 suggests that a rectangular parallelepiped of dimensions Au, Δv, Aw in uvw-space is carried onto a solid which is nearly a curvilinear "parallelepiped" in xyz-space determined by the three vectors V_1 Au, V_2 Δv, and V_3 Aw. (See Figure 11.30.) The boundaries of this solid are surfaces obtained by setting u = constant, v = constant, and w = constant, respectively. The volume of a parallelepiped is equal to the absolute value of the scalar triple product of the three vectors which determine it, so the volume of the curvilinear parallelepiped is nearly equal to

$$|(V_1\,\Delta u)\cdot(V_2\,\Delta v) \times (V_3\,\Delta w)| = |V_1 \cdot V_2 \times V_3|\ \text{Au}\ \Delta v\ \text{Aw} = |J(u, v, w)|\ \text{Au Au Aw}.$$

11.33 Worked examples

Two important special cases of (11.48) are discussed in the next two examples.

EXAMPLE 1. *Cylindrical coordinates.* Here we write r, θ, z for u, v, w and define the mapping by the equations

(11.49) $$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z.$$

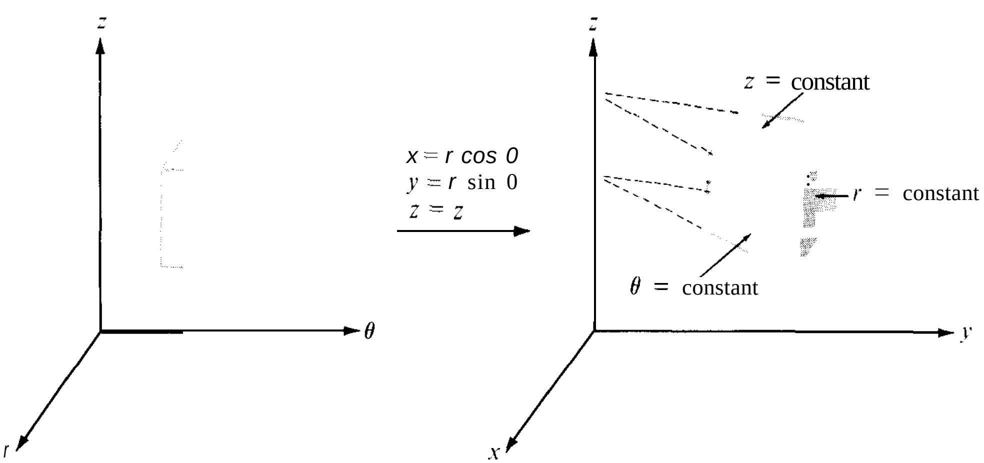

FIGURE 11.30 Transformation by cylindrical coordinates.

In other words, we replace x and y by their polar coordinates in the xy-plane and leave z unchanged. Again, to get a one-to-one mapping we must keep $r > 0$ and restrict θ to be in an interval of the form $\theta_0 \leq \theta < \theta_0 + 2\pi$. Figure 11.30 shows what happens to a rectangular parallelepiped in the r&-space.

The Jacobian determinant of the mapping in (11.49) is

$$J(r, \theta, z) = \begin{vmatrix} \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r,$$

and therefore the transformation formula in (11.48) becomes

$$\iiint_S f(x, y, z)\, dx\, dy\, dz = \iiint_T f(r\cos\theta, r\sin\theta, z) r\, dr\, d\theta\, dz\,.$$

The Jacobian determinant vanishes when $r = 0$, but this does not affect the validity of the transformation formula because the set of points with $r = 0$ has 3-dimensional content 0.

EXAMPLE **2. *Spherical coordinates.*** In this case the symbols ρ, θ, φ are used instead of u, v, w and the mapping is defined by the equations

$$x = \rho\cos\theta\sin\varphi, \qquad y = \rho\sin\theta\sin\varphi, \qquad z = \rho\cos\varphi\,.$$

The geometric meanings of ρ, θ, and φ are shown in Figure 11.31. To get a one-to-one mapping we keep $p > 0$, $0 \leq \theta < 2\pi$, and $0 \leq \varphi < \pi$. The surfaces p = constant are spheres centered at the origin, the surfaces θ = constant are planes passing through the z-axis, and the surfaces φ = constant are circular cones with their axes along the z-axis. Therefore a rectangular box in $\rho\theta\varphi$-space is mapped onto a solid of the type shown in Figure 11.31.

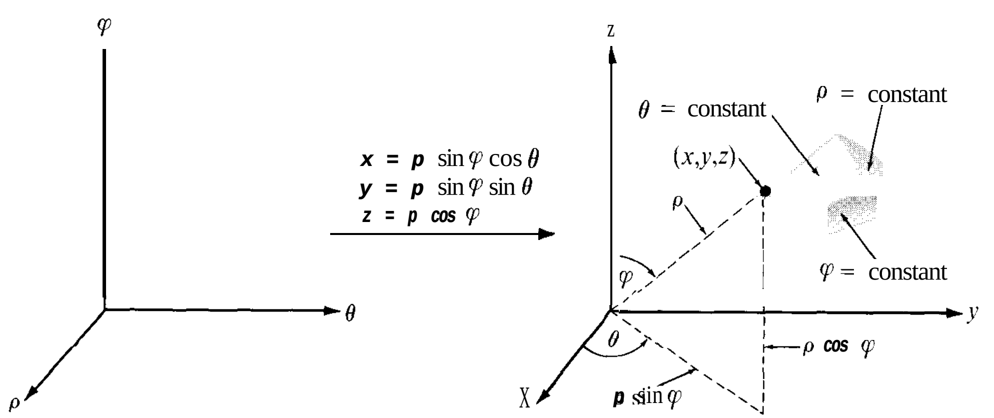

FIGURE 11.31 Transformation by spherical coordinates.

The Jacobian determinant of the mapping is

$$J(\rho, \theta, \varphi) = \begin{vmatrix} \cos\theta \sin\varphi & \sin\theta \sin\varphi & \cos\varphi \\ -\rho \sin\theta \sin\varphi & \rho\cos\theta\sin\varphi & 0 \\ \rho\cos\theta\cos\varphi & \rho\sin\theta\cos\varphi & -\rho\sin\varphi \end{vmatrix} = -\rho^2 \sin\varphi .$$

Since $\sin\varphi \geq 0$ if $0 \leq \varphi < \pi$, we have $|J(\rho, \theta, \varphi)| = \rho^2 \sin\varphi$ and the formula for transforming triple integrals becomes

$$\iiint_S f(x, y, z)\, dx\, dy\, dz = \iiint_T F(\rho, \theta, \varphi)\rho^2 \sin\varphi \, d\rho\, d\theta\, d\varphi ,$$

where $F(\rho, \theta, \varphi) = f(\rho\cos\theta\sin\varphi, \rho\sin\theta\sin\varphi, \rho\cos\varphi)$. Although the Jacobian determinant vanishes when $\varphi = 0$ the transformation formula is still valid because the set of points with $\varphi = 0$ has 3-dimensional content 0.

The concept of volume can be extended to certain classes of sets (called measurable sets) in n-space in such a way that if S is measurable then its volume is equal to the integral of the constant function 1 over S. That is, if $v(S)$ denotes the volume of S, we have

$$v(S) = \int \cdots \int_S dx_1 \cdots dx_n .$$

We shall not attempt to describe the class of sets for which this formula is valid. Instead, we illustrate how the integral can be calculated in some special cases.

EXAMPLE 3. ***Volume of an n-dimensional interval.*** If S is an n-dimensional interval, say $S = [a_1, b_1] \times \cdots \times [a_n, b_n]$, the multiple integral for $v(S)$ is the product of n one-dimensional integrals,

$$v(S) = \int_{a_1}^{b_1} dx_1 \cdots \int_{a_n}^{b_n} dx_n = (b_1 - a_1) \cdots (b_n - a_n).$$

This agrees with the formula given earlier for the volume of an n-dimensional interval.

EXAMPLE 4. ***The volume of an n-dimensional sphere.*** Let $S_n(a)$ denote the n-dimensional solid sphere (or n-ball) of radius a given by

$$S_n(a) = \{(x_1, \ldots, x_n) \mid x_1^2 + \cdots + x_n^2 \leq a^2\},$$

and let

$$V_n(a) = \int \cdots \int_{S_n(a)} dx_1 \cdots dx_n ,$$

the volume of $S_n(a)$. We shall prove that

(11.50) $$V_n(a) = \frac{\pi^{n/2}}{\Gamma(\frac{1}{2}n + 1)} a^n ,$$

where Γ is the gamma function. For $n = 1$ the formula gives $V_1(a) = 2a$, the length of the interval [-a, *a].* For n = 2 it gives $V,(a) = \pi a^2$, the area of a circular disk of radius *a.* We shall prove (11.50) for $n \geq 3$.

First we prove that for every $a > 0$ we have

$$V_n(a) = a^n V_n(1). \tag{11.51}$$

In other words, the volume of a sphere of radius a is a^n times the volume of a sphere of radius 1. To prove this we use the linear change of variable x = au to map S,(l) onto S,(a). The mapping has Jacobian determinant a^n. Hence

$$V_n(a) = \int_{S_n(a)} \cdots \int dx_1 \cdots dx, = j \cdots j_{S_n(1)} \; a^n \, du_1 \cdots du, = a^n V_n(1).$$

This proves (11.51). Therefore, to prove (11.50) it suffices to prove that

$$V_n(1) = \frac{\pi^{n/2}}{\Gamma(\frac{1}{2}n + 1)}. \tag{11.52}$$

First we note that $x_1^2 + \cdots + x_n^2 \leq 1$ if and only if

$$x_1^2 + \cdots + x_{n-2}^2 \leq 1 - x_{n-1}^2 - x_n^2 \qquad \text{and} \qquad x_{n-1}^2 + x_n^2 \leq 1.$$

Therefore we can write the integral for $V_n(1)$ as the iteration of an $(n - 2)$-fold integral and a double integral, as follows:

$$V_n(1) = \underset{x_{n-1}^2+x_n^2\leq 1}{j \; j} \left[\underset{x_1^2+\cdots+x_{n-2}^2 \leq 1-x_{n-1}^2-x_n^2}{j \cdots j} dx, \ldots dx_{n-2} \right] dx_{n-1} \, dx, . \tag{11.53}$$

The inner integral is extended over the (n − 2)-dimensional sphere S,-,(R), where $R = \sqrt{1 - x_{n-1}^2 - x_n^2}$, so it is equal to

$$V_{n-2}(R) = R^{n-2} V_{n-2}(1) = (1 - x_{n-1}^2 - x_n^2)^{n/2-1} V_{n-2}(1).$$

Now we write x for x_{n-1} and y for x_n . Then (11.53) becomes

$$V_n(1) = V_{n-2}(1) \iint_{x^2+y^2 \leq 1} (1 - x^2 - y^2)^{n/2-1} \, dx \, dy .$$

We evaluate the double integral by transforming it to polar coordinates and obtain

$$V_n(1) = V_{n-2}(1) \int_0^{2\pi} \int_0^1 (1 - r^2)^{n/2-1} r \, dr \, d\theta = V_{n-2}(1) \frac{2\pi}{n} .$$

In other words, the numbers $V_n(1)$ satisfy the recursion formula

$$V_n(1) = \frac{2\pi}{n} V_{n-2}(1) \qquad \text{if } n \geq 3.$$

But the sequence of numbers $\{f(n)\}$ defined by

$$f(n) = \frac{\pi^{n/2}}{\Gamma(\frac{1}{2}n + 1)}$$

satisfies the same recursion formula because $\Gamma(s+1) = s\Gamma(s)$. Also, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ (see Exercise 16, Section 11.28), so $\Gamma(\frac{3}{2}) = \frac{1}{2}\sqrt{\pi}$ and $f(1) = V_1(1) = 2$. Also, $f(2) = V_2(1) = \pi$, hence we have $f(n) = V_n(1)$ for all $n \geq 1$. This proves (11.52).

11.34 Exercises

Evaluate each of the triple integrals in Exercises 1 through 5. Make a sketch of the region of integration in each case. You may assume the existence of all the integrals encountered.

1. $\iiint_S xy^2z^3\,dx\,dy\,dz$, where S is the solid bounded by the surface $z = xy$ and the planes $y = x, x = 1$, and $z = 0$.
2. $\iiint_S (1 + x + y + z)^{-3}\,dx\,dy\,dz$, where S is the solid bounded by the three coordinate planes and the plane $x + y + z = 1$.
3. $\iiint_S xyz\,dx\,dy\,dz$, where $S = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1, x \geq 0, y \geq 0, z \geq 0\}$.
4. $\iiint_S \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) dx\,dy\,dz$, where S is the solid bounded by the ellipsoid
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$
5. $\iiint_S \sqrt{x^2 + y^2}\,dx\,dy\,dz$, where S is the solid formed by the upper nappe of the cone $z^2 = x^2 + y^2$ and the plane $z = 1$.

In Exercises 6, 7, and 8, a triple integral $\iiint_S f(x, y, z)\,dx\,dy\,dz$ of a positive function reduces to the iterated integral given. In each case describe the region of integration S by means of a sketch, showing its projection on the xy-plane. Then express the triple integral as one or more iterated integrals in which the first integration is with respect to y.

6. $\int_0^1 \left(\int_0^{1-x} \left[\int_0^{x+y} f(x, y, z)\,dz \right] dy \right) dx.$
7. $\int_{-1}^1 \left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left[\int_{\sqrt{x^2-y^2}}^1 f(x, y, z)\,dz \right] dy \right) dx.$
8. $\int_0^1 \left(\int_0^1 \left[\int_0^{x^2+y^2} f(x, y, z)\,dz \right] dy \right) dx.$
9. Show that:
$$\int_0^x \left(\int_0^v \left[\int_0^u f(t)\,dt \right] du \right) dv = \frac{1}{2} \int_0^x (x - t)^2 f(t)\,dt.$$

Evaluate the integrals in Exercises 10, 11, and 12 by transforming to cylindrical coordinates. You may assume the existence of all integrals encountered.

10. $\iiint_S (x^2 + y^2)\,dx\,dy\,dz$, where S is the solid bounded by the surface $x^2 + y^2 = 2z$ and the plane $z = 2$.

11. $\iiint_S dx\,dy\,dz$, where S is the solid bounded by the three coordinate planes, the surface $z = x^2 + y^2$, and the plane $x + y = 1$.
12. $\iiint_S (y^2 + z^2)\,dx\,dy\,dz$, where S is a right circular cone of altitude h with its base, of radius a, in the xy-plane and its axis along the z-axis.

Evaluate the integrals in Exercises 13, 14, and 15 by transforming to spherical coordinates.

13. $\iiint_S dx\,dy\,dz$, where S is a solid sphere of radius a and center at the origin.
14. $\iiint_S dx\,dy\,dz$, where S is the solid bounded by two concentric spheres of radii a and b, where $0 < a < b$, and the center is at the origin.
15. $\iiint_S [(x - a)^2 + (y - b)^2 + (z - c)^2]^{-1/2}\,dx\,dy\,dz$, where S is a solid sphere of radius R and center at the origin, and (a, b, c) is a fixed point outside this sphere.
16. Generalized spherical coordinates may be defined by the following mapping:

$$x = a\rho \cos^m \theta \sin^n \varphi, \qquad y = b\rho \sin^m \theta \sin^n \varphi, \qquad z = c\rho \cos^n \varphi,$$

where a, b, c, m, and n are positive constants. Show that the Jacobian determinant is equal to

$$-abcmn\rho^2 \cos^{m-1} \theta \sin^{m-1} \theta \cos^{n-1} \varphi \sin^{2n-1} \varphi.$$

Triple integrals can be used to compute volume, mass, center of mass, moment of inertia, and other physical concepts associated with solids. If S is a solid, its volume V is given by the triple integral

$$V = \iiint_S dx\,dy\,dz.$$

If the solid is assigned a density $f(x, y, z)$ at each of its points (x, y, z) (mass per unit volume), its mass M is defined to be

$$M = \iiint_S f(x, y, z)\,dx\,dy\,dz,$$

and its center of mass the point $(\bar{x}, \bar{y}, \bar{z})$ with coordinates

$$\bar{x} = \frac{1}{M} \iiint_S x f(x, y, z)\,dx\,dy\,dz,$$

and so on. The moment of inertia I_{xy} about the xy-plane is detined by the equation

$$I_{xy} = \iiint_S z^2 f(x, y, z)\,dx\,dy\,dz$$

and similar formulas are used to define I_{yz} and I_{zx}. The moment of inertia I_L about a line L is detined to be

$$I_L = \iiint_S \delta^2(x, y, z) f(x, y, z)\,dx\,dy\,dz,$$

where 6(x, y, z) denotes the perpendicular distance from a general point (x, y, z) of S to the line L.

17. Show that the moments of inertia about the coordinate axes are

$$I_x = I_{xy} + I_{xz}, \qquad I_y = I_{yx} + I_{yz}, \qquad I_z = I_{zx} + I_{zy}.$$

18. Find the volume of the solid bounded above by the sphere $x^2 + y^2 + z^2 = 5$ and below by the paraboloid $x^2 + y^2 = 4z$.
19. Find the volume of the solid bounded by the xy-plane, the cylinder $x^2 + y^2 = 2x$, and the cone $z = \sqrt{x^2 + y^2}$.
20. Compute the mass of the solid lying between two concentric spheres of radii a and b, where $0 < a < b$, if the density at each point is equal to the square of the distance of this point from the center.
21. A homogeneous solid right circular cone has altitude h. Prove that the distance of its centroid from the base is h/4.
22. Determine the center of mass of a right circular cone of altitude h if its density at each point is proportional to the distance of this point from the base.
23. Determine the center of mass of a right circular cone of altitude h if i.ts density at each point is proportional to the distance of this point from the axis of the cone.
24. A solid is bounded by two concentric hemispheres of radii a and b, where $0 < a < b$. If the density is constant, find the center of mass.
25. Find the center of mass of a cube of side h if its density at each point is proportional to the square of the distance of this point from one corner of the base. (Take the base in the xy-plane and place the edges on the coordinate axes.)
26. A right circular cone has altitude h, radius of base a, constant density, and mass M. Find its moment of inertia about an axis through the vertex parallel to the base.
27. Find the moment of inertia of a sphere of radius R and mass M about a diameter if the density is constant.
28. Find the moment of inertia of a cylinder of radius a and mass M if its density at each point is proportional to the distance of this point from the axis of the cylinder.
29. The stem of a mushroom is a right circular cylinder of diameter 1 and length 2, and its cap is a hemisphere of radius R. If the mushroom is a homogeneous solid with axial symmetry, and if its center of mass lies in the plane where the stem joins the cap, find R.
30. A new space satellite has a smooth unbroken skin made up of portions of two circular cylinders of equal diameters D whose axes meet at right angles. It is proposed to ship the satellite to Cape Kennedy in a cubical packing box of inner dimension D. Prove that one-third of the box will be waste space.
31. Let S,(u) denote the following set in n-space, where $a > 0$:

$$S_n(a) = \{(x_1, \ldots, x_n) \mid |x_1| + \cdots + |x_n| \le a\}.$$

When $n = 2$ the set is a square with vertices at $(0, \pm a)$ and $(\pm a, 0)$. When $n = 3$ it is an octahedron with vertices at $(0, 0, \pm a)$, $(0, \pm a, 0)$, and (fu, $0, 0)$. Let $V_n(a)$ denote the volume of S,(u), given by

$$V_n(a) = \int \cdots \int_{S_n(a)} dx_1 \cdots dx_n.$$

(a) Prove that $V_n(a) = a^n V_n(1)$.

(b) For $n \ge 2$, express the integral for $V_n(1)$ as an iteration of a one-dimensional integral and an $(n - 1)$-fold integral and show that

$$V_n(1) = V_{n-1}(1) \int_{-1}^{1} (1 - |x|)^{n-1}\, dx = \frac{2}{n} V_{n-1}(1).$$

(c) Use parts (a) and (b) to deduce that $V_n(a) = \dfrac{2^n a^n}{n!}$.

32. Let $S_n(a)$ denote the following set in n-space, where a > 0 and $n \geq 2$:

$$S_n(a) = \{(x_1, \ldots, x_n) \mid |x_i| + |x_n| \leq a \quad \text{for each } i = 1, \ldots, n-1\}.$$

(a) Make a sketch of $S_n(1)$ when $n = 2$ and when $n = 3$.
(b) Let $V_n(a) = \int \cdots \int_{S_n(a)} dx_1 \ldots dx_n$, and show that $V_n(a) = a^n V_n(1)$.
(c) Express the integral for $V_n(1)$ as an iteration of a one-dimensional integral and an $(n - 1)$-fold integral and deduce that $V_n(a) = 2^n a^n/n$.

33. (a) Refer to Example 4, p. 411. Express the integral for $V_n(1)$, the volume of the n-dimensional unit sphere, as the iteration of an $(n - 1)$-fold integral and a one-dimensional integral and thereby prove that

$$V_n(1) = 2\, V_{n-1}(1) \int_0^1 (1 - x^2)^{(n-1)/2}\, dx.$$

(b) Use part (a) and Equation (11.52) to deduce that

$$\int_0^{\pi/2} \cos^n t\, dt = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\dfrac{n+1}{2}\right)}{\Gamma\left(\dfrac{n}{2} + 1\right)}.$$

12

SURFACE INTEGRALS

12.1 Parametric representation of a surface

This chapter discusses surface integrals and their applications. A surface integral can be thought of as a two-dimensional analog of a line integral where the region of integration is a surface rather than a curve. Before we can discuss surface integrals intelligently, we must agree on what we shall mean by a surface.

Roughly speaking, a surface is the locus of a point moving in space with two degrees of freedom. In our study of analytic geometry in Volume I we discussed two methods for describing such a locus by mathematical formulas. One is the *implicit representation* in which we describe a surface as a set of points (x, y, z) satisfying an equation of the form $F(x, y, z) = 0$. Sometimes we can solve such an equation for one of the coordinates in terms of the other two, say for z in terms of x and y. When this is possible we obtain an *explicit representation* given by one or more equations of the form $z = f(x, y)$. For example, a sphere of radius 1 and center at the origin has the implicit representation $x^2 + y^2 + z^2 - 1 = 0$. When this equation is solved for z it leads to two solutions, $z = \sqrt{1 - x^2 - y^2}$ and $z = -\sqrt{1 - x^2 - y^2}$. The first gives an explicit representation of the upper hemisphere and the second of the lower hemisphere.

A third method for describing surfaces is more useful in the study of surface integrals; this is the *parametric* or *vector* representation in which we have three equations expressing x, y, and z in terms of two parameters u and v:

$$x = X(u, v), \qquad y = Y(u, v), \qquad z = Z(u, v). \tag{12.1}$$

Here the point (u, v) is allowed to vary over some two-dimensional connected set T in the uv-plane, and the corresponding points (x, y, z) trace out a surface in xyz-space. This method for describing a surface is analogous to the representation of a space curve by three parametric equations involving one parameter. The presence of the two parameters in (12.1) makes it possible to transmit two degrees of freedom to the point (x, y, z), as suggested by Figure 12.1. Another way of describing the same idea is to say that a surface is the image of a plane region T under the mapping defined by (12.1).

If we introduce the radius vector $\mathbf{r}$ from the origin to a general point (x, y, z) of the surface, we may combine the three parametric equations in (12.1) into one vector equation

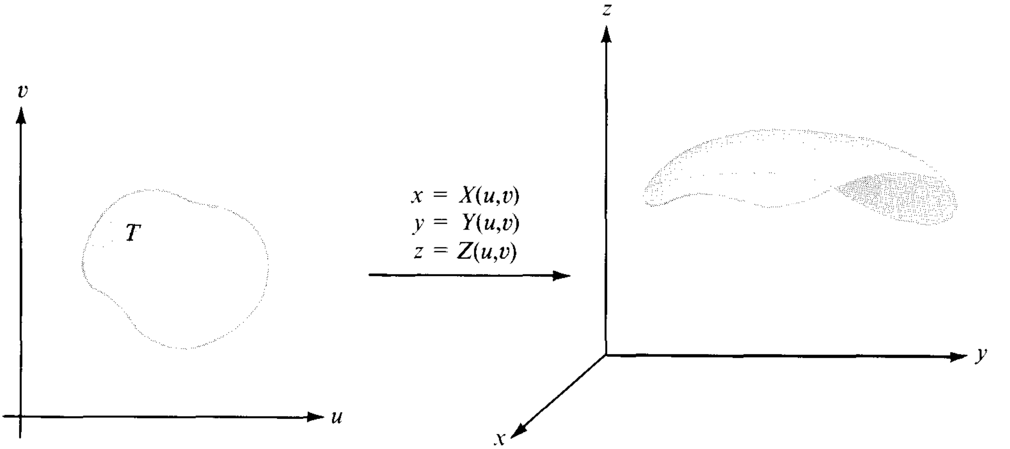

FIGURE 12.1 Parametric representation of a surface.

of the form

(12.2) $$\mathbf{r}(u, v) = X(u, v)\mathbf{i} + Y(u, v)\mathbf{j} + Z(u, v)\mathbf{k}, \qquad \text{where } (u, v) \in T.$$

This is called a ***vector equation*** for the surface.

There are, of course, many parametric representations for the same surface. One of these can always be obtained from an explicit form $z = f(x, y)$ by taking $X(u, v) = u$, $Y(u, v) = v$, $Z(u, v) = f(u, v)$. On the other hand, if we can solve the first two equations in (12.1) for u and v in terms of x and y and substitute in the third-we obtain an explicit representation $z = f(x, y)$.

EXAMPLE 1. ***A parametric representation of a sphere.*** The three equations

(12.3) $$x = a\cos u\cos v, \qquad y = a\sin u\cos v, \qquad z = a\sin u$$

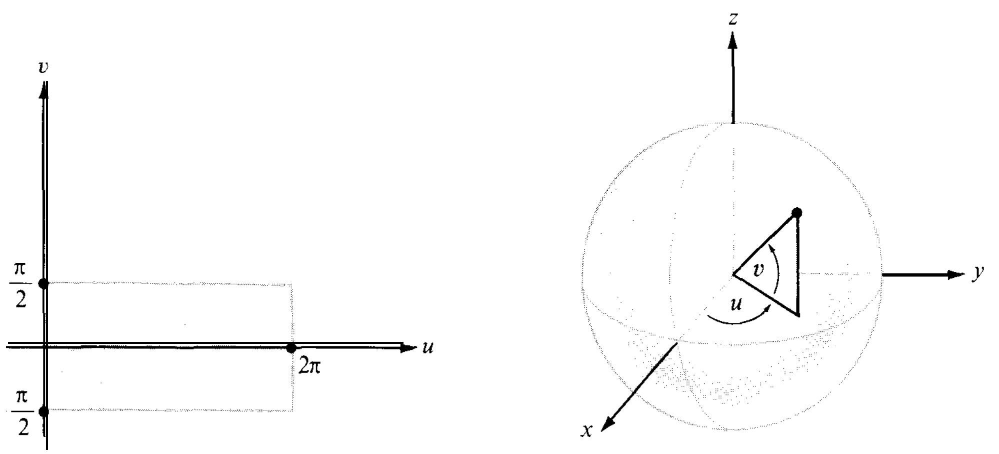

FIGURE 12.2 Parametric representation of a sphere.

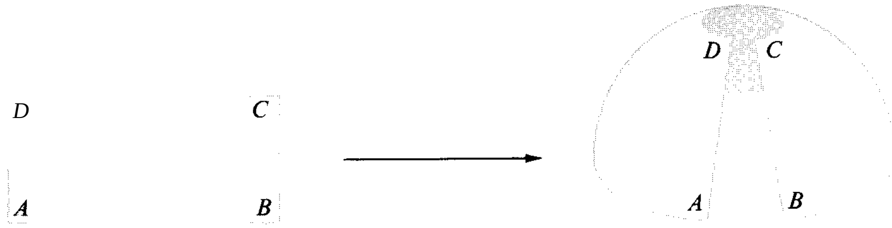

FIGURE 12.3 Deformation of a rectangle into a hemisphere.

serve as parametric equations for a sphere of radius a and center at the origin. If we square and add the three equations in (12.3) we find $x^2 + y^2 + z^2 = a^2$, and we see that every point (x, y, z) satisfying (12.3) lies on the sphere. The parameters u and v in this example may be interpreted geometrically as the angles shown in Figure 12.2. If we let the point (u, v) vary over the rectangle $T = [0, 2\pi] \times [-\frac{1}{2}\pi, \frac{1}{2}\pi]$, the points determined by (12.3) trace out the whole sphere. The upper hemisphere is the image of the rectangle $[0, 2\pi] \times [0, \frac{1}{2}\pi]$ and the lower hemisphere is the image of $[0, 2\pi] \times [-\frac{1}{2}\pi, 0]$. Figure 12.3 gives a concrete idea of how the rectangle $[0, 2\pi] \times [0, \frac{1}{2}\pi]$ is mapped onto the upper hemisphere. Imagine that the rectangle is made of a flexible plastic material capable of being stretched or shrunk. Figure 12.3 shows the rectangle being deformed into a hemisphere. The base AB eventually becomes the equator, the opposite edges AD and BC are brought into coincidence, and the upper edge DC shrinks to a point (the North Pole).

EXAMPLE **2. *A parametric representation of a cone.*** The vector equation

$$\boldsymbol{r}(u, v) = v \sin \alpha \cos u\, \boldsymbol{i} + v \sin \alpha \sin u\, \boldsymbol{j} + v \cos \alpha\, \boldsymbol{k}$$

represents the right circular cone shown in Figure 12.4, where α denotes half the vertex angle. Again, the parameters u and v may be given geometric interpretations; v is the distance from the vertex to the point (x, y, z) on the cone, and u is the polar-coordinate angle. When (u, v) is allowed to vary over a rectangle of the form $[0, 2\pi] \times [0, h]$, the corresponding points (x, y, z) trace out a cone of altitude $h \cos \alpha$. A plastic rectangle

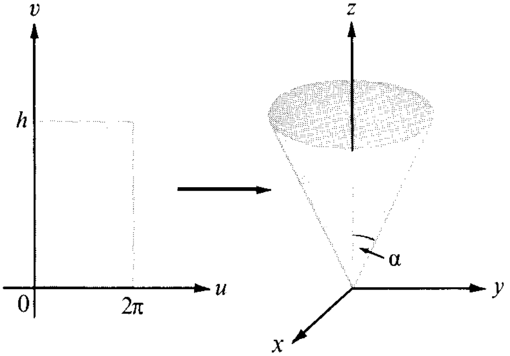

FIGURE 12.4 Parametric representation of a cone.

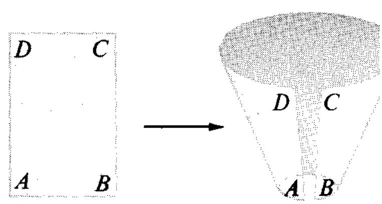

FIGURE 12.5 Deformation of a rectangle into a cone.

may be physically deformed into the cone by bringing the edges ***AD*** and ***BC*** into coincidence, as suggested by Figure 12.5, and letting the edge ***AB*** shrink to a point (the vertex of the cone). The surface in Figure 12.5 shows an intermediate stage of the deformation.

In the general study of surfaces, the functions X, Y, and Z that occur in the parametric equations (12.1) or in the vector equation (12.2) are assumed to be continuous on T. The image of T under the mapping r is called a ***parametric surface*** and will be denoted by the symbol $r(T)$. In many of the examples we shall discuss, T will be a rectangle, a circular disk, or some other simply connected set bounded by a simple closed curve. If the function r is one-to-one on T, the image $r(T)$ will be called a ***simple parametric surface.*** In such a case, distinct points of T map onto distinct points of the surface. In particular, every simple closed curve in T maps onto a simple closed curve lying on the surface.

A parametric surface $r(T)$ may degenerate to a point or to a curve. For example, if all three functions X, Y, and Z are constant, the image $r(T)$ is a single point. If X, Y, and Z are independent of v, the image $r(T)$ is a curve. Another example of a degenerate surface occurs when $X(u, v) = u + v$, $Y(u, v) = (u + v)^2$, and $Z(u, v) = (u + v)^3$, where $T = [0, 1] \times [0, 1]$. If we write $t = u + v$ we see that the surface degenerates to the space curve having parametric equations $x = t$, $y = t^2$, and $z = t^3$, where $0 \le t \le 2$. Such degeneracies can be avoided by placing further restrictions on the mapping function r, as described in the next section.

12.2 The fundamental vector product

Consider a surface described by the vector equation

$$\boldsymbol{r}(u, v) = X(u, v)\boldsymbol{i} + Y(u, v)\boldsymbol{j} + Z(u, v)\boldsymbol{k}, \qquad \text{where } (u, v) \in T.$$

If X, Y, and Z are differentiable on T we consider the two vectors

$$\frac{\partial \boldsymbol{r}}{\partial u} = \frac{\partial X}{\partial u}\boldsymbol{i} + \frac{\partial Y}{\partial u}\boldsymbol{j} + \frac{\partial Z}{\partial u}\boldsymbol{k}$$

and

$$\frac{\partial \boldsymbol{r}}{\partial v} = \frac{\partial X}{\partial v}\boldsymbol{i} + \frac{\partial Y}{\partial v}\boldsymbol{j} + \frac{\partial Z}{\partial v}\boldsymbol{k}.$$

The cross product of these two vectors $\partial \boldsymbol{r}/\partial u \times \partial \boldsymbol{r}/\partial v$ will be referred to as the ***fundamental vector product*** of the representation $\boldsymbol{r}$. Its components can be expressed as Jacobian determinants. In fact, we have

$$(12.4)\qquad \frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial X}{\partial u} & \frac{\partial Y}{\partial u} & \frac{\partial Z}{\partial u} \\ \frac{\partial X}{\partial v} & \frac{\partial Y}{\partial v} & \frac{\partial Z}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{\partial Y}{\partial u} & \frac{\partial Z}{\partial u} \\ \frac{\partial Y}{\partial v} & \frac{\partial Z}{\partial v} \end{vmatrix}\boldsymbol{i} + \begin{vmatrix} \frac{\partial Z}{\partial u} & \frac{\partial X}{\partial u} \\ \frac{\partial Z}{\partial v} & \frac{\partial X}{\partial v} \end{vmatrix}\boldsymbol{j} + \begin{vmatrix} \frac{\partial X}{\partial u} & \frac{\partial Y}{\partial u} \\ \frac{\partial X}{\partial v} & \frac{\partial Y}{\partial v} \end{vmatrix}\boldsymbol{k}$$

$$= \frac{\partial(Y, Z)}{\partial(u, v)}\boldsymbol{i} + \frac{\partial(Z, X)}{\partial(u, v)}\boldsymbol{j} + \frac{\partial(X, Y)}{\partial(u, v)}\boldsymbol{k}.$$