is replaced by

$$\frac{(\frac{1}{r}-\frac{1}{p})(\frac{1}{s}-\frac{1}{q})}{(\frac{1}{r}-\frac{1}{q})(\frac{1}{s}-\frac{1}{p})} = \frac{\frac{p-r}{pr}\cdot\frac{q-s}{qs}}{\frac{q-r}{qr}\cdot\frac{p-s}{ps}} \quad \text{taking common denominators,}$$

$$= \frac{(p-r)(q-s)}{(q-r)(p-s)} \quad \text{multiplying through by } pqrs,$$

$$= \frac{(r-p)(s-q)}{(r-q)(s-p)} \quad \text{changing the sign in all factors,}$$

and thus, the cross-ratio is unchanged in this case also. □

Is the cross-ratio visible?

If we take the four equally spaced points $p = 0$, $q = 1$, $r = 2$, and $s = 3$ on the line, then their cross-ratio is

$$\frac{(r-p)(s-q)}{(r-q)(s-p)} = \frac{2\times 2}{1\times 3} = \frac{4}{3}.$$

It follows that any projective image of these points also has cross-ratio $4/3$. Do four points on a line *look* equally spaced if their cross-ratio is 4/3? Test your eye on the quadruples of points in Figure 5.18, and then do Exercise 5.7.2 to find the correct answer.

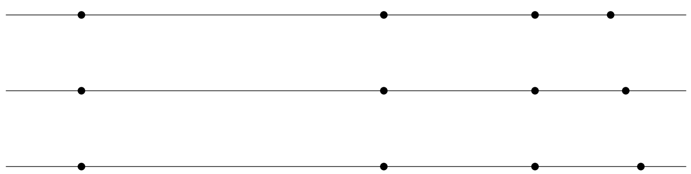


Figure 5.18: Which is a projective image of equally spaced points?

Exercises

We will see in the next section that any three points on $\mathbb{RP}^1$ can be projected to any three points. Hence, there cannot be an invariant involving just three points.

However, the invariance of the cross-ratio tells us that, once the images of three points are known, the whole projection map is known (compare with the “three-point determination” of isometries of the plane in Section 3.7).

5.7.1 Show that there is only one point s that has a given cross-ratio with given points p, q, and r.

In particular, if we have the points $p = 0$, $q = 2$, $r = 3$ (which we do in the three quadruples in Figure 5.18), there is exactly one s that gives the cross-ratio 4/3 required for "equally spaced" points.

5.7.2 Find the value of s that gives the cross-ratio 4/3, and hence find the "equally spaced" quadruple in Figure 5.18.

Before the discovery of perspective, artists sometimes attempted to draw a tiled floor by making the width of each row of tiles a constant fraction e of the one before.

5.7.3 Show that this method is not correct by computing the cross-ratio of four points separated by the distances 1, e, and e^2.

5.8 What is special about the cross-ratio?

In the remainder of this book, we use the abbreviation

$$[p,q;r,s] = \frac{(r-p)(s-q)}{(r-q)(s-p)}$$

for the cross-ratio of the four points p,q,r,s, taken in that order.

We have shown that the cross-ratio is an invariant of linear fractional transformations, but it is obviously not the only one. Examples of other invariants are (cross-ratio)^2 and cross-ratio $+1$. The cross-ratio is special because it is the *defining invariant* of linear fractional transformations. That is, *the linear fractional transformations are precisely the transformations of* $\mathbb{RP}^1$ *that preserve the cross-ratio.* (Thus, the cross-ratio defines linear fractional transformations the way that length defines isometries.)

We prove this fact among several others about linear fractional transformations and the cross-ratio.

Fourth point determination. *Given any three points* $p,q,r \in \mathbb{RP}^1$, *any other point* $x \in \mathbb{RP}^1$ *is uniquely determined by its cross-ratio* $[p,q;r,x] = y$ *with* p,q,r.

This statement holds because we can solve the equation

$$y = \frac{(r-p)(x-q)}{(r-q)(x-p)}$$

uniquely for x in terms of p,q,r, and y. $\square$

Existence of three-point maps. *Given three points $p, q, r \in \mathbb{RP}^1$ and three points $p', q', r' \in \mathbb{RP}^1$, there is a linear fractional transformation f sending p, q, r to p', q', r', respectively.*

This statement holds because there is a projection sending any three points p, q, r to any three points p', q', r', and any projection is linear fractional by Section 5.6. The way to project is shown in Figure 5.19.

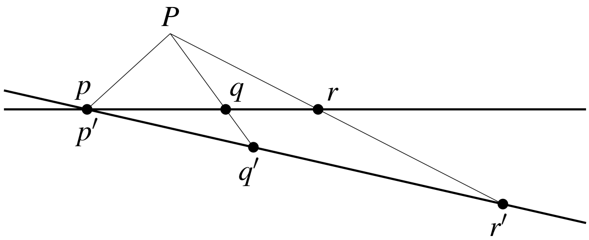


Figure 5.19: Projecting three points to three points

Without loss of generality, we can place the two copies of $\mathbb{RP}^1$ so that $p = p'$. Then the required projection is from the point P where the lines qq' and rr' meet. $\square$

Uniqueness of three-point maps. *Exactly one linear fractional function sends three points p, q, r to three points p', q', r', respectively.*

A linear fractional f sending p, q, r to p', q', r', respectively, must send any $x \neq p, q, r$ to x' satisfying $[p, q; r, x] = [p', q'; r', x']$, because f preserves the cross-ratio by Section 5.7. But x' is unique by fourth point determination, so there is exactly one such function f. $\square$

Characterization of linear fractional maps. *These are precisely the maps of $\mathbb{RP}^1$ that preserve the cross-ratio.*

By Section 5.7, any linear fractional map f preserves the cross-ratio. That is, $[f(p, f(q); f(r), f(s)] = [p, q; r, s]$ for any four points p, q, r, s.

Conversely, suppose that f is a map of $\mathbb{RP}^1$ with

$$[f(p), f(q); f(r), f(s)] = [p, q; r, s] \quad \text{for any four points } p, q, r, s.$$

By the existence of three-point maps, we can find a linear fractional g that agrees with f on p, q, r. But then, because f preserves the cross-ratio, g agrees with f on s also, by unique fourth point determination.

Thus, g agrees with f everywhere, so f is a linear fractional map. $\square$

The existence of three-point maps says that any three points on $\mathbb{RP}^1$ can be sent to any three points by a linear fractional transformation. Thus, any invariant of triples of points must have the *same value* for any triple, and so it is trivial. A nontrivial invariant must involve at least four points, and the cross-ratio is an example. It is in fact the *fundamental* example, in the following sense.

The fundamental invariant. *Any invariant of four points is a function of the cross-ratio.*

To see why, suppose $I(p,q,r,s)$ is a function, defined on quadruples of distinct points, that is invariant under linear fractional transformations. Thus,

$$I(f(p), f(q), f(r), f(s)) = I(p,q,r,s) \quad \text{for any linear fractional } f.$$

In other words, I has the same value on all quadruples (p',q',r',s') that result from (p,q,r,s) by a linear fractional transformation. But more is true: *I has the same value on all quadruples (p',q',r',s') with the same cross-ratio as (p,q,r,s)*, because such a quadruple (p',q',r',s') results from (p,q,r,s) by a linear fractional transformation. This follows from the existence and uniqueness of three-point maps:

- by existence, we can send p,q,r to p',q',r', respectively, by a linear fractional transformation f, and
- by uniqueness, f also sends s to s', the unique point that makes $[p,q;r,s] = [p',q';r',s']$.

Because I has the same value on all quadruples with the same cross-ratio, it is meaningful to view I as a function J *of* the cross-ratio, defined by

$$J([p,q;r,s]) = I(p,q,r,s). \qquad \square$$

Exercises

The following exercises illustrate the result above about invariant functions of quadruples. They show that the invariants obtained by *permuting the variables* in the cross-ratio $y = [p,q;r,s]$ are simple functions of y, such as $1/y$ and $y-1$.

5.8.1 If $[p,q;r,s] = y$, show that $[p,q;s,r] = 1/y$.

5.8.2 If $[p,q;r,s] = y$, show that $[q,p;r,s] = 1/y$.

5.8.3 Prove that $[p,q;r,s]+[p,r;q,s]=1$, so it follows that if $[p,q;r,s]=y$, then $[p,r;q,s]=1-y$.

The transformations $y \mapsto 1/y$ and $y \mapsto 1-y$ obtained in this way generate all transformations of the cross-ratio obtained by permuting its variables. There are six such transformations (even though there are 24 permutations of four variables).

5.8.4 Show that the functions of y obtained by combining $1/y$ and $1-y$ in all ways are

$$y, \quad \frac{1}{y}, \quad 1-y, \quad 1-\frac{1}{y}, \quad \frac{1}{1-y}, \quad \frac{y}{y-1}.$$

5.8.5 Explain why any permutation of four variables may be obtained by exchanges: either of the first two, the middle two, or the last two variables.

5.8.6 Deduce from Exercises 5.8.1–5.8.5 that the invariants obtained from the cross-ratio y by permuting its variables are precisely the six listed in Exercise 5.8.4.

The six linear fractional functions of y obtained in Exercise 5.8.4 constitute what is sometimes called the *cross-ratio group*. It is an example of a concept we will study in Chapter 7: the concept of a *group of transformations*. Unlike most of the groups studied there, this group is finite.

5.9 Discussion

The plane $\mathbb{RP}^2$ studied in this chapter is the most important projective plane, but it is far from being the only one. Many other projective planes can be constructed by imitating the construction of $\mathbb{RP}^2$, which is based on ordered triples (x,y,z) and linear equations $ax+by+cz=0$. It is not essential for x,y,z to be real numbers. As noted earlier, they could be complex numbers, but more generally they could be elements of any *field*. A field is any set with $+$ and $\times$ operations satisfying the nine field axioms listed in Section 4.8.

If $\mathbb{F}$ is any field, we can consider the space $\mathbb{F}^3$ of ordered triples (x,y,z) with $x,y,z \in \mathbb{F}$. Then the projective plane $\mathbb{FP}^2$ has

- "points," each of which is a set of triples (kx,ky,kz), where $x,y,z \in \mathbb{F}$ are fixed and k runs through the elements of $\mathbb{F}$,
- "lines," each of which consists of the "points" satisfying an equation of the form $ax+by+cz=0$ for some fixed $a,b,c \in \mathbb{F}$.

The projective plane axioms can be checked for $\mathbb{FP}^2$ just as they were for $\mathbb{RP}^2$. The same calculations apply, because the field axioms ensure that the same algebraic operations work in $\mathbb{F}$ (solving equations, for example). This gives a great variety of planes $\mathbb{FP}^2$, because there are a great variety of fields $\mathbb{F}$.

Perhaps the most familiar field, after $\mathbb{R}$ and $\mathbb{C}$, is the set $\mathbb{Q}$ of rational numbers. $\mathbb{QP}^2$ is not unlike $\mathbb{RP}^2$, except that all of its points have rational coordinates, and all of its lines are full of gaps, because they contain only rational points.

More surprising examples arise from taking $\mathbb{F}$ to be a *finite* field, of which there is one with p^n elements for each power p^n of each prime p. The simplest example is the field $\mathbb{F}_2$, whose members are the elements 0 and 1, with the following addition and multiplication tables.

$+$	0	1
0	0	1
1	1	0

$\times$	0	1
0	0	0
1	0	1

The projective plane $\mathbb{F}_2\mathbb{P}^2$ has seven points, corresponding to the seven nonzero points in $\mathbb{F}_2^3$:

$$(1,0,0),\ (0,1,0),\ (0,0,1),\ (0,1,1),\ (1,0,1),\ (1,1,0),\ (1,1,1).$$

These points are arranged in threes along the seven lines in Figure 5.20, one of which is drawn as a circle so as to connect its three points.

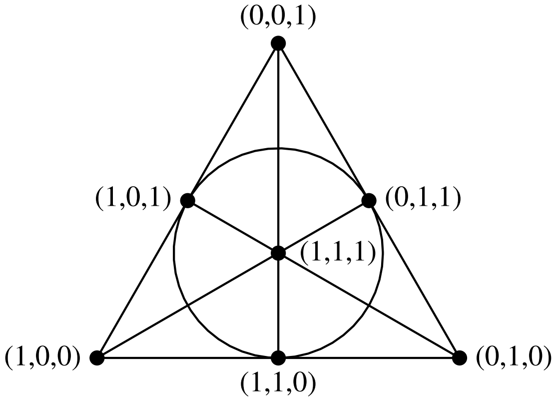


Figure 5.20: The smallest projective plane

Notice that the lines satisfy the linear equations

$$x = 0, \quad y = 0, \quad z = 0,$$
$$x + y = 0, \quad y + z = 0, \quad z + x = 0,$$
$$x + y + z = 0.$$

For example, the points on the circle satisfy $x + y + z = 0$. (Of course, the coordinates have nothing to do with position in the plane of the diagram. The figure is mainly symbolic, while attempting to show "points" collected into "lines.")

This structure is called the *Fano plane*, and it is the smallest projective plane. Despite being small, it is well-behaved, because its "lines" satisfy linear equations, just as lines do in the traditional geometric world. Thanks to finite fields, linear algebra works well in many finite structures. It has led to the wholesale development of finite geometries, many of which have applications in the mathematics of information and communication.

However, the three axioms for a finite projective plane do not ensure that the plane is of the form $\mathbb{F}\mathbb{P}^2$, with coordinates for points and linear equations for lines. They can be satisfied by bizarre "nonlinear" structures, as we will see in the next chapter. A fourth axiom is needed to engender a field $\mathbb{F}$ of coordinates, and the axiom is none other than the theorem of Pappus that we met briefly in Chapters 1 and 4. This state of affairs will be explained in Chapter 6.

The invariance of the cross-ratio

The invariance of the cross-ratio was discovered by Pappus around 300 CE and rediscovered by Desargues around 1640. It appears (not very clearly) as Proposition 129 in Book VII of Pappus' *Mathematical Collection* and again in *Manière universelle de Mr Desargues* in 1648. The latter is a pamphlet on perspective by written by Abraham Bosse, a disciple of Desargues. It also contains the first published statement of the Desargues theorem mentioned in Chapters 1 and 4. Because of this, and the fact that he wrote the first book on projective geometry, Desargues is considered to be the founder of the subject. Nevertheless, projective geometry was little known until the 19th century, when geometry expanded in all directions. In the more general 19th century geometry (which often included use of complex numbers), the cross-ratio continued to be a central concept.

One of the reasons we now consider the appropriate generalization of classical projective geometry to be projective geometry with coordinates in a field is that *the cross-ratio continues to make sense in this setting*.

Linear fractional transformations and the cross-ratio make sense when $\mathbb{R}$ is replaced by any field $\mathbb{F}$. The transformations $x \mapsto x+l$ and $x \mapsto kx$ make sense on $\mathbb{F}$, and $x \mapsto 1/x$ makes sense on $\mathbb{F} \cup \{\infty\}$ if we set $1/0 = \infty$ and $1/\infty = 0$. Then the transformations

$$x \mapsto \frac{ax+b}{cx+d}, \quad \text{where } a,b,c,d \in \mathbb{F} \text{ and } ad-bc \neq 0,$$

make sense on the "$\mathbb{F}$ *projective line*" $\mathbb{FP}^1 = \mathbb{F} \cup \{\infty\}$. The cross-ratio is invariant by the same calculation as in Section 5.7, thanks to the field axioms, because the usual calculations with fractions are valid in a field.

6

Projective planes

PREVIEW

In this chapter, geometry fights back against the forces of arithmetization. We show that coordinates need not be brought into geometry from outside—they can be defined by purely geometric means. Moreover, the geometry required to define coordinates and their arithmetic is *simpler* than Euclid's geometry. It is the *projective geometry* introduced in the previous chapter, but we have to build it from scratch using properties of straight lines alone.

We started this project in Section 5.3 by stating the three axioms for a projective plane. However, these axioms are satisfied by many structures, some of which have no reasonable system of coordinates. To build coordinates, we need at least one additional axiom, but for convenience we take two: the *Pappus* and *Desargues* properties that were proved with the help of coordinates in Chapter 4.

Here we proceed in the direction opposite to Chapter 4: Take Pappus and Desargues as axioms, and use them to define coordinates. The coordinates are points on a projective line, and we add and multiply them by constructions like those in Chapter 1. But instead of using parallel lines as we did there, we call lines "parallel" if they meet on a designated line called the "horizon" or the "line at infinity."

The main problem is to prove that our addition and multiplication operations satisfy the field axioms. This is where the theorems of Pappus and Desargues are crucial. Pappus is needed to prove the *commutative law* of multiplication, $ab = ba$, whereas Desargues is needed to prove the *associative law*, $a(bc) = (ab)c$.

6.1 Pappus and Desargues revisited

The theorems of Pappus and Desargues stated in Chapters 1 and 4 had a similar form: If two particular pairs of lines are parallel, then a third pair is parallel. Because parallel lines meet on the horizon, the Pappus and Desargues theorems also say that if two particular pairs of lines meet on the horizon, then so does a third pair. And because the horizon is not different from any other line, these theorems are really about three pairs of lines having their intersections on the *same* line.

In this projective setting, the Pappus theorem takes the form shown in Figure 6.1. The six vertices of the hexagon are shown as dots, and the opposite sides are shown as a black pair, a gray pair, and a dotted pair. The line on which each of the three pairs meet is labeled $\mathscr{L}$, and we have oriented the figure so that $\mathscr{L}$ is horizontal (but this is not at all necessary).

Projective Pappus theorem. *Six points, lying alternately on two straight lines, form a hexagon whose three pairs of opposite sides meet on a line.*

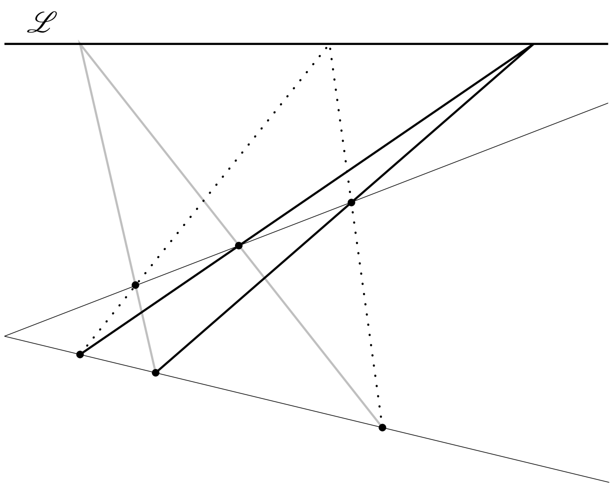


Figure 6.1: The projective Pappus configuration

This statement of the Pappus theorem is called *projective* because it involves only the concepts of points, lines, and meetings between them. Meetings between geometric objects are called *incidences*, and, for this reason, the Pappus theorem is also called an *incidence theorem*. The three axioms of a projective plane, given in Section 5.3, are the simplest examples of incidence theorems.

The projective Desargues theorem is another incidence theorem. It concerns the pairs of corresponding sides of two triangles, shown in solid gray in Figure 6.2. The triangles are in *perspective from a point P*, which means that each pair of corresponding vertices lies on a line through P. The three corresponding pairs of sides are again shown as black, gray, and dotted, and each pair meets on a line labeled $\mathscr{L}$.

Projective Desargues theorem. *If two triangles are in perspective from a point, then their pairs of corresponding sides meet on a line.*

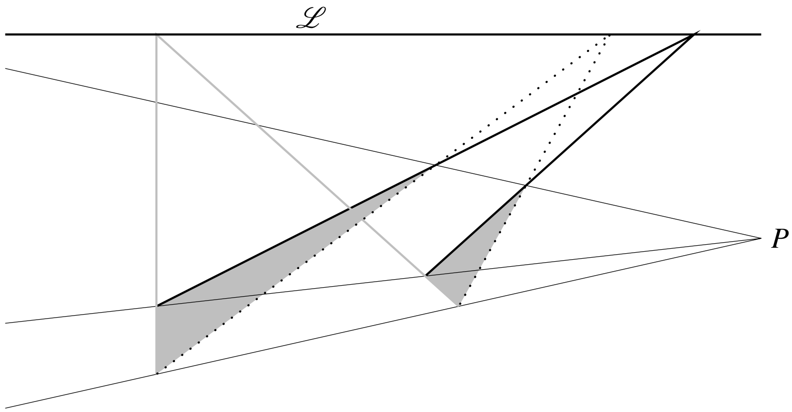


Figure 6.2: The projective Desargues configuration

An important special case of the Desargues theorem has the center of projection P on the line $\mathscr{L}$ where the corresponding sides of the triangles meet. This special case is called the *little Desargues theorem*, and it is shown in Figure 6.3.

Little Desargues theorem. *If two triangles are in perspective from a point P, and if two pairs of corresponding sides meet on a line $\mathscr{L}$ through P, then the third pair of corresponding sides also meets on $\mathscr{L}$.*

Because the projective Pappus and Desargues theorems involve only incidence concepts, one would like proofs of them that involve only the three axioms for a projective plane given in Section 5.3. Unfortunately, this is not possible, because there are examples of projective planes *not* satisfying the Pappus and Desargues theorems. What we can do, however, is take the Pappus and Desargues theorems as *new axioms*. Together with the original three axioms for projective planes, these two new axioms apply to a broad class of projective planes called *Pappian planes*.

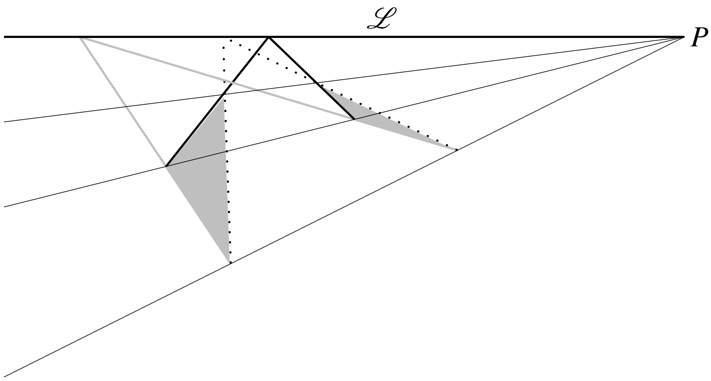


Figure 6.3: The little Desargues configuration

The Pappian planes include $\mathbb{RP}^2$ and many other planes, but not all. They turn out to be the planes with *coordinates* satisfying the same *laws of algebra* as the real numbers—the field axioms. The object of this chapter is to show how coordinates arise when the Pappus and Desargues theorems hold, and why they satisfy the field axioms. In doing so, we will see that projective geometry is *simpler* than algebra in a certain sense, because we use only five geometric axioms to derive the nine field axioms.

Exercises

In some projective planes, the Desargues theorem is false. Here is one example, which is called the *Moulton plane*. Its "points" are ordinary points of $\mathbb{R}^2$, together with a point at infinity for each family of parallel "lines." However, the "lines" of the Moulton plane are not all ordinary lines. They include the ordinary lines of negative, horizontal, or vertical slope, but each other "line" is a *broken line* consisting of a half line of slope $k > 0$ below the x-axis, joined to a half line of slope $k/2$ above the x-axis. Figure 6.4 shows some of the "lines."

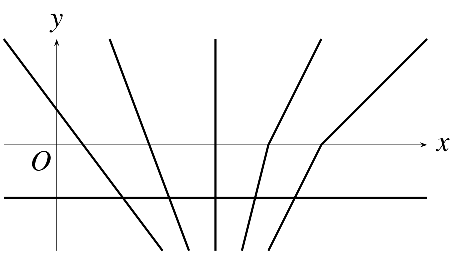


Figure 6.4: Lines of the Moulton plane

6.1.1 Find where the "line" from $(0,-1)$ to $(2,1/2)$ meets the x-axis.

6.1.2 Explain why any two "points" of the Moulton plane lie on a unique "line."

6.1.3 Explain why any two "lines" of the Moulton plane meet in a unique "point." (Parallel "lines" have a common "point at infinity" by definition, so do not worry about them.)

6.1.4 Give four "points," no three of which lie on the same "line."

6.1.5 Thus, the Moulton plane satisfies the three axioms of a projective plane. But it does not satisfy even the little Desargues theorem, as Figure 6.5 shows. Explain.

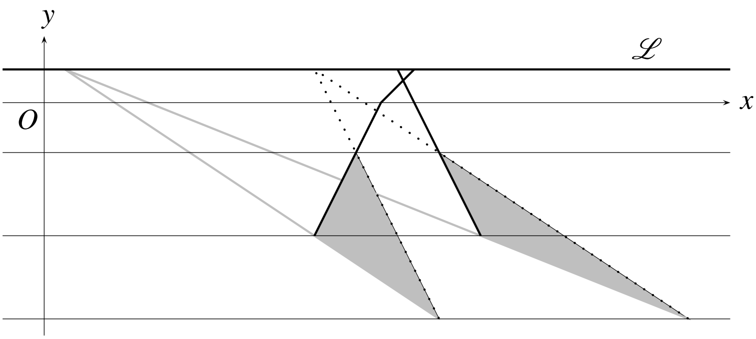
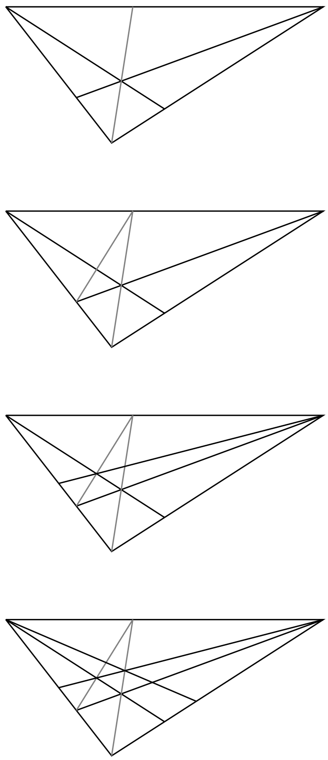


Figure 6.5: Failure of the little Desargues theorem in the Moulton plane

6.2 Coincidences

Two points A, B always lie on a line. But it is accidental, so to speak, if a third point C lies on the line through A and B. Such an accidental meeting is called a "coincidence" in everyday life, and this is a good name for it in projective geometry too: *co*incidence = two incidences together—in this case the incidence of A and B with a line, and the incidence of C with the same line.

The theorems of Pappus and Desargues state that certain coincidences occur. In fact, they are coincidences of the type just described, in which two points lie on a line and a third point lies on the same line. The perspective picture of the tiled floor also involves certain coincidences, as becomes clear when we look again at the first few steps in its construction (Figure 6.6).



Draw diagonal of first tile, extended to the horizon

Extend diagonal of second tile to the horizon

Draw side of second tile, through the new intersection

Draw second column of tiles, through the new intersection

Figure 6.6: Constructing the tiled floor

At this step, a coincidence occurs. Three of the points we have constructed lie on a straight line, which is shown dashed in Figure 6.7.

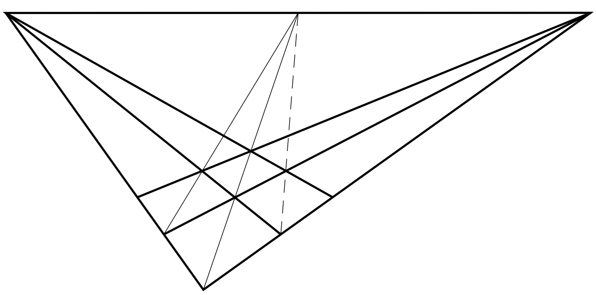


Figure 6.7: A coincidence in the tiled floor

This coincidence can be traced to a special case of the little Desargues theorem, which involves the two shaded triangles shown in Figure 6.8.

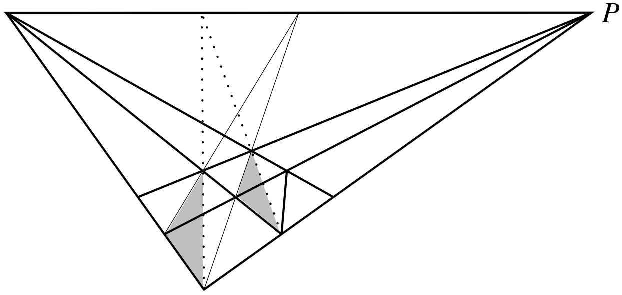


Figure 6.8: A little Desargues configuration in the tiled floor

This case of little Desargues says that the two dotted lines (diagonals of "double tiles") meet on the horizon. These lines give us a second little Desargues configuration, shown in Figure 6.9, from which we conclude that the dashed diagonals also meet on the horizon, as required to explain the coincidence in Figure 6.7.

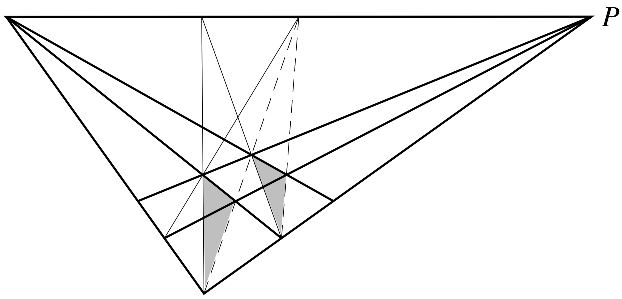


Figure 6.9: A second little Desargues configuration

Exercises

The occurrence of the little Desargues configuration in the tiled floor may be easier to see if we draw the lines meeting on the horizon as actual parallels. The little Desargues theorem itself is easier to state in terms of actual parallels (Figure 6.10).

6.2.1 Formulate an appropriate statement of the little Desargues theorem when one has parallels instead of lines meeting on $\mathscr{L}$.

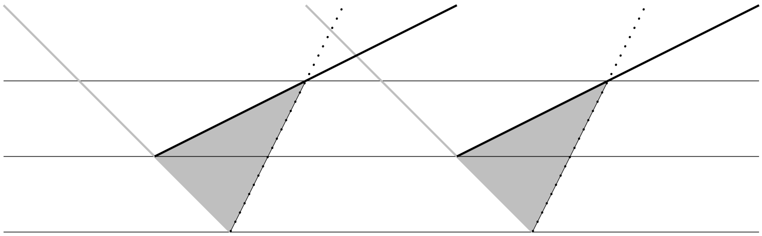


Figure 6.10: The parallel little Desargues configuration

6.2.2 Now redraw Figures 6.7, 6.8, and 6.9 so that the lines meeting on $\mathscr{L}$ are shown as actual parallels.

6.2.3 What is the nature of the "coincidence" in Figure 6.7 now?

6.2.4 Find occurrences of the little Desargues configuration in your diagrams. Hence, explain why the "coincidence" in Exercise 6.2.3 follows from your statement of the little Desargues theorem in Exercise 6.2.1.

The theorem that proves the coincidence in the drawing of the tiled floor is actually a special case of the little Desargues theorem: the case in which a vertex of one triangle lies on a side of the other. Thus, it is not clear that the coincidence is false in the Moulton plane, where we know only that the general little Desargues theorem is false by the exercises in Section 6.1.

6.2.5 By placing an x-axis in a suitable position on Figure 6.11, show that the tiled floor coincidence fails in the Moulton plane.

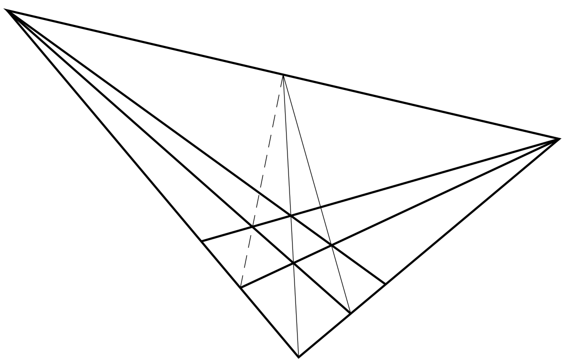


Figure 6.11: A coincidence that fails in the Moulton plane

6.3 Variations on the Desargues theorem

In Section 6.1, we stated the Desargues theorem in the form: *If two triangles are in perspective from a point, then their three pairs of corresponding sides meet on a line.* The Desargues theorem is a very flexible theorem, which appears in many forms, and two that we need later are the following. (We need these theorems only as consequences of the Desargues theorem, but they are actually equivalent to it.)

Converse Desargues theorem. *If corresponding sides of two triangles meet on a line, then the two triangles are in perspective from a point.*

To deduce this result from the Desargues theorem, let ABC and $A'B'C'$ be two triangles whose corresponding sides meet on the line $\mathscr{L}$. Let P be the intersection of AA' and BB', so we want to prove that P lies on CC' as well. Suppose that PC meets the line $B'C'$ at C'' (Figure 6.12 shows C'', hypothetically, unequal to C').

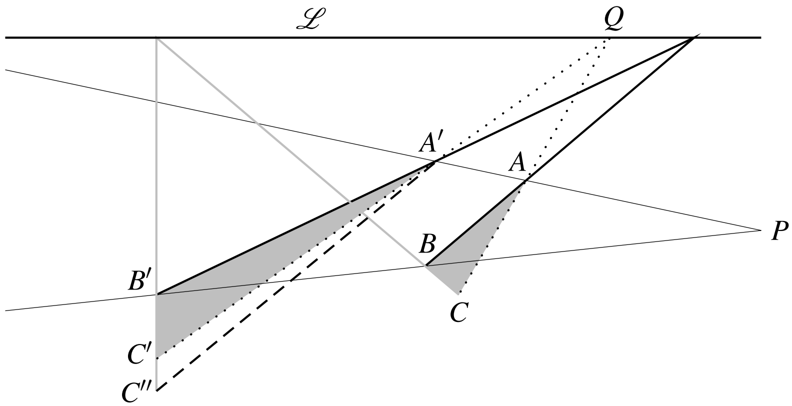


Figure 6.12: The converse Desargues theorem

Then the triangles ABC and $A'B'C''$ are in perspective from P and therefore, by the Desargues theorem, their corresponding sides meet on a line. We already know that AB meets $A'B'$ on $\mathscr{L}$, and that BC meets $B'C'$ on $\mathscr{L}$. Hence, AC meets $A'C''$ on $\mathscr{L}$, necessarily at the point Q where AC meets $\mathscr{L}$. It follows that QA' goes through C''. But we also know that QA' meets $B'C'$ at C'. Hence, $C'' = C'$.

Thus, C' is indeed on the line PC, so ABC and $A'B'C'$ are in perspective from P, as required. $\square$