To complete the above analysis we shall also prove the following result.

Theorem 8. *Suppose* P *is an* n × n *invertible matrix over* F. *Let* V *be an* n*-dimensional vector space over* F, *and let* $\mathcal{B}$ *be an ordered basis of* V. *Then there is a unique ordered basis* $\mathcal{B}'$ *of* V *such that*

$$\text{(i)} \qquad [\alpha]_{\mathcal{B}} = P[\alpha]_{\mathcal{B}'}$$
$$\text{(ii)} \qquad [\alpha]_{\mathcal{B}'} = P^{-1}[\alpha]_{\mathcal{B}}$$

for every vector α *in* V.

Proof. Let $\mathcal{B}$ consist of the vectors $\alpha_1, \ldots, \alpha_n$. If $\mathcal{B}' = \{\alpha'_1, \ldots, \alpha'_n\}$ is an ordered basis of V for which (i) is valid, it is clear that

$$\alpha'_j = \sum_{i=1}^{n} P_{ij}\alpha_i.$$

Thus we need only show that the vectors α'_j, defined by these equations, form a basis. Let $Q = P^{-1}$. Then

$$\begin{aligned} \sum_j Q_{jk}\alpha'_j &= \sum_j Q_{jk} \sum_i P_{ij}\alpha_i \\ &= \sum_j \sum_i P_{ij}Q_{jk}\, \alpha_i \\ &= \sum_i \Big(\sum_j P_{ij}Q_{jk}\Big)\, \alpha_i \\ &= \alpha_k. \end{aligned}$$

Thus the subspace spanned by the set

$$\mathcal{B}' = \{\alpha'_1, \ldots, \alpha'_n\}$$

contains $\mathcal{B}$ and hence equals V. Thus $\mathcal{B}'$ is a basis, and from its definition and Theorem 7, it is clear that (i) is valid and hence also (ii). ∎

Example 18. Let F be a field and let

$$\alpha = (x_1, x_2, \ldots, x_n)$$

be a vector in F^n. If $\mathcal{B}$ is the standard ordered basis of F^n,

$$\mathcal{B} = \{\epsilon_1, \ldots, \epsilon_n\}$$

the coordinate matrix of the vector α in the basis $\mathcal{B}$ is given by

$$[\alpha]_{\mathcal{B}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Example 19. Let R be the field of the real numbers and let θ be a fixed real number. The matrix

$$P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

is invertible with inverse,

$$P^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

Thus for each θ the set $\mathcal{B}'$ consisting of the vectors $(\cos\theta, \sin\theta)$, $(-\sin\theta, \cos\theta)$ is a basis for R^2; intuitively this basis may be described as the one obtained by rotating the standard basis through the angle θ. If α is the vector (x_1, x_2), then

$$[\alpha]_{\mathcal{B}'} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

or

$$\begin{aligned} x_1' &= x_1 \cos\theta + x_2 \sin\theta \\ x_2' &= -x_1 \sin\theta + x_2 \cos\theta. \end{aligned}$$

Example 20. Let F be a subfield of the complex numbers. The matrix

$$P = \begin{bmatrix} -1 & 4 & 5 \\ 0 & 2 & -3 \\ 0 & 0 & 8 \end{bmatrix}$$

is invertible with inverse

$$P^{-1} = \begin{bmatrix} -1 & 2 & \frac{11}{8} \\ 0 & \frac{1}{2} & \frac{3}{16} \\ 0 & 0 & \frac{1}{8} \end{bmatrix}.$$

Thus the vectors

$$\begin{aligned} \alpha_1' &= (-1, 0, 0) \\ \alpha_2' &= (4, 2, 0) \\ \alpha_3' &= (5, -3, 8) \end{aligned}$$

form a basis $\mathcal{B}'$ of F^3. The coordinates x_1', x_2', x_3' of the vector $\alpha = (x_1, x_2, x_3)$ in the basis $\mathcal{B}'$ are given by

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} -x_1 + 2x_2 + \frac{11}{8}x_3 \\ \frac{1}{2}x_2 + \frac{3}{16}x_3 \\ \frac{1}{8}x_3 \end{bmatrix} = \begin{bmatrix} -1 & 2 & \frac{11}{8} \\ 0 & \frac{1}{2} & \frac{3}{16} \\ 0 & 0 & \frac{1}{8} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

In particular,

$$(3, 2, -8) = -10\alpha_1' - \tfrac{1}{2}\alpha_2' - \alpha_3'.$$

Exercises

1. Show that the vectors

$$\begin{aligned} \alpha_1 &= (1, 1, 0, 0), & \alpha_2 &= (0, 0, 1, 1) \\ \alpha_3 &= (1, 0, 0, 4), & \alpha_4 &= (0, 0, 0, 2) \end{aligned}$$

form a basis for R^4. Find the coordinates of each of the standard basis vectors in the ordered basis $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$.

2. Find the coordinate matrix of the vector $(1, 0, 1)$ in the basis of C^3 consisting of the vectors $(2i, 1, 0)$, $(2, -1, 1)$, $(0, 1 + i, 1 - i)$, in that order.

3. Let $\mathfrak{B} = \{\alpha_1, \alpha_2, \alpha_3\}$ be the ordered basis for R^3 consisting of

$$\alpha_1 = (1, 0, -1), \qquad \alpha_2 = (1, 1, 1), \qquad \alpha_3 = (1, 0, 0).$$

What are the coordinates of the vector (a, b, c) in the ordered basis $\mathfrak{B}$?

4. Let W be the subspace of C^3 spanned by $\alpha_1 = (1, 0, i)$ and $\alpha_2 = (1 + i, 1, -1)$.

(a) Show that α_1 and α_2 form a basis for W.

(b) Show that the vectors $\beta_1 = (1, 1, 0)$ and $\beta_2 = (1, i, 1 + i)$ are in W and form another basis for W.

(c) What are the coordinates of α_1 and α_2 in the ordered basis $\{\beta_1, \beta_2\}$ for W?

5. Let $\alpha = (x_1, x_2)$ and $\beta = (y_1, y_2)$ be vectors in R^2 such that

$$x_1y_1 + x_2y_2 = 0, \qquad x_1^2 + x_2^2 = y_1^2 + y_2^2 = 1.$$

Prove that $\mathfrak{B} = \{\alpha, \beta\}$ is a basis for R^2. Find the coordinates of the vector (a, b) in the ordered basis $\mathfrak{B} = \{\alpha, \beta\}$. (The conditions on α and β say, geometrically, that α and β are perpendicular and each has length 1.)

6. Let V be the vector space over the complex numbers of all functions from R into C, i.e., the space of all complex-valued functions on the real line. Let $f_1(x) = 1$, $f_2(x) = e^{ix}$, $f_3(x) = e^{-ix}$.

(a) Prove that f_1, f_2, and f_3 are linearly independent.

(b) Let $g_1(x) = 1$, $g_2(x) = \cos x$, $g_3(x) = \sin x$. Find an invertible 3×3 matrix P such that

$$g_j = \sum_{i=1}^{3} P_{ij}f_i.$$

7. Let V be the (real) vector space of all polynomial functions from R into R of degree 2 or less, i.e., the space of all functions f of the form

$$f(x) = c_0 + c_1x + c_2x^2.$$

Let t be a fixed real number and define

$$g_1(x) = 1, \qquad g_2(x) = x + t, \qquad g_3(x) = (x + t)^2.$$

Prove that $\mathfrak{B} = \{g_1, g_2, g_3\}$ is a basis for V. If

$$f(x) = c_0 + c_1x + c_2x^2$$

what are the coordinates of f in this ordered basis $\mathfrak{B}$?

2.5. Summary of Row-Equivalence

In this section we shall utilize some elementary facts on bases and dimension in finite-dimensional vector spaces to complete our discussion of row-equivalence of matrices. We recall that if A is an $m \times n$ matrix over the field F the row vectors of A are the vectors $\alpha_1, \ldots, \alpha_m$ in F^n defined by

$$\alpha_i = (A_{i1}, \ldots, A_{in})$$

and that the row space of A is the subspace of F^n spanned by these vectors. The **row rank** of A is the dimension of the row space of A.

If P is a $k \times m$ matrix over F, then the product $B = PA$ is a $k \times n$ matrix whose row vectors $\beta_1, \ldots, \beta_k$ are linear combinations

$$\beta_i = P_{i1}\alpha_1 + \cdots + P_{im}\alpha_m$$

of the row vectors of A. Thus the row space of B is a subspace of the row space of A. If P is an $m \times m$ invertible matrix, then B is row-equivalent to A so that the symmetry of row-equivalence, or the equation $A = P^{-1}B$, implies that the row space of A is also a subspace of the row space of B.

Theorem 9. *Row-equivalent matrices have the same row space.*

Thus we see that to study the row space of A we may as well study the row space of a row-reduced echelon matrix which is row-equivalent to A. This we proceed to do.

Theorem 10. *Let* R *be a non-zero row-reduced echelon matrix. Then the non-zero row vectors of* R *form a basis for the row space of* R.

Proof. Let $\rho_1, \ldots, \rho_r$ be the non-zero row vectors of R:

$$\rho_i = (R_{i1}, \ldots, R_{in}).$$

Certainly these vectors span the row space of R; we need only prove they are linearly independent. Since R is a row-reduced echelon matrix, there are positive integers $k_1, \ldots, k_r$ such that, for $i \leq r$

$$\text{(2-18)} \qquad \begin{aligned} &\text{(a)}\ R(i, j) = 0 \quad \text{if} \quad j < k_i \\ &\text{(b)}\ R(i, k_j) = \delta_{ij} \\ &\text{(c)}\ k_1 < \cdots < k_r. \end{aligned}$$

Suppose $\beta = (b_1, \ldots, b_n)$ is a vector in the row space of R:

$$\text{(2-19)} \qquad \beta = c_1\rho_1 + \cdots + c_r\rho_r.$$

Then we claim that $c_j = b_{k_j}$. For, by (2-18)

$$\text{(2-20)} \qquad \begin{aligned} b_{k_j} &= \sum_{i=1}^{r} c_i R(i, k_j) \\ &= \sum_{i=1}^{r} c_i \delta_{ij} \\ &= c_j. \end{aligned}$$

In particular, if $\beta = 0$, i.e., if $c_1\rho_1 + \cdots + c_r\rho_r = 0$, then c_j must be the k_jth coordinate of the zero vector so that $c_j = 0$, $j = 1, \ldots, r$. Thus $\rho_1, \ldots, \rho_r$ are linearly independent. ▌

Theorem 11. *Let* m *and* n *be positive integers and let* F *be a field. Suppose* W *is a subspace of* F^n *and dim* W $\leq$ m. *Then there is precisely one* m $\times$ n *row-reduced echelon matrix over* F *which has* W *as its row space.*

Proof. There is at least one $m \times n$ row-reduced echelon matrix with row space W. Since $\dim W \leq m$, we can select some m vectors $\alpha_1, \ldots, \alpha_m$ in W which span W. Let A be the $m \times n$ matrix with row vectors $\alpha_1, \ldots, \alpha_m$ and let R be a row-reduced echelon matrix which is row-equivalent to A. Then the row space of R is W.

Now let R be any row-reduced echelon matrix which has W as its row space. Let $\rho_1, \ldots, \rho_r$ be the non-zero row vectors of R and suppose that the leading non-zero entry of ρ_i occurs in column k_i, $i = 1, \ldots, r$. The vectors $\rho_1, \ldots, \rho_r$ form a basis for W. In the proof of Theorem 10, we observed that if $\beta = (b_1, \ldots, b_n)$ is in W, then

$$\beta = c_1\rho_1 + \cdots + c_r\rho_r,$$

and $c_i = b_{k_i}$; in other words, the unique expression for β as a linear combination of $\rho_1, \ldots, \rho_r$ is

$$\beta = \sum_{i=1}^{r} b_{k_i}\rho_i. \tag{2-21}$$

Thus any vector β is determined if one knows the coordinates b_{k_i}, $i = 1, \ldots, r$. For example, ρ_s is the unique vector in W which has k_sth coordinate 1 and k_ith coordinate 0 for $i \neq s$.

Suppose β is in W and $\beta \neq 0$. We claim the first non-zero coordinate of β occurs in one of the columns k_s. Since

$$\beta = \sum_{i=1}^{r} b_{k_i}\rho_i$$

and $\beta \neq 0$, we can write

$$\beta = \sum_{i=s}^{r} b_{k_i}\rho_i, \qquad b_{k_s} \neq 0. \tag{2-22}$$

From the conditions (2-18) one has $R_{ij} = 0$ if $i > s$ and $j \leq k_s$. Thus

$$\beta = (0, \ldots, 0, \quad b_{k_s}, \ldots, b_n), \qquad b_{k_s} \neq 0$$

and the first non-zero coordinate of β occurs in column k_s. Note also that for each k_s, $s = 1, \ldots, r$, there exists a vector in W which has a non-zero k_sth coordinate, namely ρ_s.

It is now clear that R is uniquely determined by W. The description of R in terms of W is as follows. We consider all vectors $\beta = (b_1, \ldots, b_n)$ in W. If $\beta \neq 0$, then the first non-zero coordinate of β must occur in some column t:

$$\beta = (0, \ldots, 0, \quad b_t, \ldots, b_n), \qquad b_t \neq 0.$$

Let $k_1, \ldots, k_r$ be those positive integers t such that there is some $\beta \neq 0$ in W, the first non-zero coordinate of which occurs in column t. Arrange $k_1, \ldots, k_r$ in the order $k_1 < k_2 < \cdots < k_r$. For each of the positive integers k_s there will be one and only one vector ρ_s in W such that the k_sth coordinate of ρ_s is 1 and the k_ith coordinate of ρ_s is 0 for $i \neq s$. Then R is the $m \times n$ matrix which has row vectors $\rho_1, \ldots, \rho_r, 0, \ldots, 0$. ▮

Corollary. *Each* m × n *matrix* A *is row-equivalent to one and only one row-reduced echelon matrix.*

Proof. We know that A is row-equivalent to at least one row-reduced echelon matrix R. If A is row-equivalent to another such matrix R', then R is row-equivalent to R'; hence, R and R' have the same row space and must be identical. ∎

Corollary. *Let* A *and* B *be* m × n *matrices over the field* F. *Then* A *and* B *are row-equivalent if and only if they have the same row space.*

Proof. We know that if A and B are row-equivalent, then they have the same row space. So suppose that A and B have the same row space. Now A is row-equivalent to a row-reduced echelon matrix R and B is row-equivalent to a row-reduced echelon matrix R'. Since A and B have the same row space, R and R' have the same row space. Thus $R = R'$ and A is row-equivalent to B. ∎

To summarize—if A and B are $m \times n$ matrices over the field F, the following statements are equivalent:

1. A and B are row-equivalent.
2. A and B have the same row space.
3. $B = PA$, where P is an invertible $m \times m$ matrix.

A fourth equivalent statement is that the homogeneous systems $AX = 0$ and $BX = 0$ have the same solutions; however, although we know that the row-equivalence of A and B implies that these systems have the same solutions, it seems best to leave the proof of the converse until later.

2.6. Computations Concerning Subspaces

We should like now to show how elementary row operations provide a standardized method of answering certain concrete questions concerning subspaces of F^n. We have already derived the facts we shall need. They are gathered here for the convenience of the reader. The discussion applies to any n-dimensional vector space over the field F, if one selects a fixed ordered basis $\mathcal{B}$ and describes each vector α in V by the n-tuple $(x_1, \ldots, x_n)$ which gives the coordinates of α in the ordered basis $\mathcal{B}$.

Suppose we are given m vectors $\alpha_1, \ldots, \alpha_m$ in F^n. We consider the following questions.

1. How does one determine if the vectors $\alpha_1, \ldots, \alpha_m$ are linearly independent? More generally, how does one find the dimension of the subspace W spanned by these vectors?