Chapter 18

Lebesgue measure

In the previous chapter we discussed differentiation in several variable calculus. It is now only natural to consider the question of integration in several variable calculus. The general question we wish to answer is this: given some subset Ω of $\mathbf{R}^n$, and some real-valued function $f : \Omega \to \mathbf{R}$, is it possible to integrate f on Ω to obtain some number $\int_\Omega f$? (It is possible to consider other types of functions, such as complex-valued or vector-valued functions, but this turns out not to be too difficult once one knows how to integrate real-valued functions, since one can integrate a complex or vector valued function, by integrating each real-valued component of that function separately.)

In one dimension we already have developed (in Chapter 11) the notion of a *Riemann integral* $\int_{[a,b]} f$, which answers this question when Ω is an interval $\Omega = [a, b]$, and f is *Riemann integrable.* Exactly what Riemann integrability means is not important here, but let us just remark that every piecewise continuous function is Riemann integrable, and in particular every piecewise constant function is Riemann integrable. However, not all functions are Riemann integrable. It is possible to extend this notion of a Riemann integral to higher dimensions, but it requires quite a bit of effort and one can still only integrate "Riemann integrable" functions, which turn out to be a rather unsatisfactorily small class of functions. (For instance, the pointwise limit of Riemann integrable functions need not be Riemann integrable, and the same goes for an L^2 limit, although we have already seen that uniform limits of Riemann integrable functions remain Riemann integrable.)

Because of this, we must look beyond the Riemann integral to obtain a truly satisfactory notion of integration, one that can handle even very discontinuous functions. This leads to the notion of the *Lebesgue integral*, which we shall spend this chapter and the next constructing. The Lebesgue integral can handle a very large class of functions, including all the Riemann integrable functions but also many others as well; in fact, it is safe to say that it can integrate virtually any function that one actually needs in mathematics, at least if one works on Euclidean spaces and everything is absolutely integrable. (If one assumes the axiom of choice, then there are still some pathological functions one can construct which cannot be integrated by the Lebesgue integral, but these functions will not come up in real-life applications.)

Before we turn to the details, we begin with an informal discussion. In order to understand how to compute an integral $\int_\Omega f$, we must first understand a more basic and fundamental question: how does one compute the *length/area/volume* of Ω? To see why this question is connected to that of integration, observe that if one integrates the function 1 on the set Ω, then one should obtain the length of Ω (if Ω is one-dimensional), the area of Ω (if Ω is two-dimensional), or the volume of Ω (if Ω is three-dimensional). To avoid splitting into cases depending on the dimension, we shall refer to the *measure* of Ω as either the length, area, volume, (or hypervolume, etc.) of Ω, depending on what Euclidean space $\mathbf{R}^n$ we are working in.

Ideally, to every subset Ω of $\mathbf{R}^n$ we would like to associate a non-negative number $m(\Omega)$, which will be the measure of Ω (i.e., the length, area, volume, etc.). We allow the possibility for $m(\Omega)$ to be zero (e.g., if Ω is just a single point or the empty set) or for $m(\Omega)$ to be infinite (e.g., if Ω is all of $\mathbf{R}^n$). This measure should obey certain reasonable properties; for instance, the measure of the unit cube $(0,1)^n := \{(x_1, \ldots, x_n) : 0 < x_i < 1\}$ should equal 1, we should have $m(A \cup B) = m(A) + m(B)$ if A and B are disjoint, we should have $m(A) \leq m(B)$ whenever $A \subseteq B$, and we should have $m(x + A) = m(A)$ for any $x \in \mathbf{R}^n$ (i.e., if we shift A by the vector x the measure should be the same).

Remarkably, it turns out that such a measure *does not exist*; one cannot assign a non-negative number to *every* subset of $\mathbf{R}^n$ which has the above properties. This is quite a surprising fact, as it goes against one's intuitive concept of volume; we shall prove it later in these notes. (An even more dramatic example of this failure of intuition is the *Banach-Tarski paradox*, in which a unit ball in $\mathbf{R}^3$ is decomposed into five pieces, and then the five pieces are reassembled via translations and rotations to form two complete and disjoint unit balls, thus violating any concept of conservation of volume; however we will not discuss this paradox here.)

What these paradoxes mean is that it is impossible to find a reasonable way to assign a measure to every single subset of $\mathbf{R}^n$. However, we can salvage matters by only measuring a certain class of sets in $\mathbf{R}^n$ - the *measurable sets*. These are the only sets Ω for which we will define the measure $m(\Omega)$, and once one restricts one's attention to measurable sets, one recovers all the above properties again. Furthermore, almost all the sets one encounters in real life are measurable (e.g., all open and closed sets will be measurable), and so this turns out to be good enough to do analysis.

18.1 The goal: Lebesgue measure

Let $\mathbf{R}^n$ be a Euclidean space. Our goal in this chapter is to define a concept of *measurable set*, which will be a special kind of subset of $\mathbf{R}^n$, and for every such measurable set $\Omega \subset \mathbf{R}^n$, we will define the *Lebesgue measure* $m(\Omega)$ to be a certain number in $[0, \infty]$. The concept of measurable set will obey the following properties:

(i) (Borel property) Every open set in $\mathbf{R}^n$ is measurable, as is every closed set.

(ii) (Complementarity) If Ω is measurable, then $\mathbf{R}^n \backslash \Omega$ is also measurable.

(iii) (Boolean algebra property) If $(\Omega_j)_{j\in J}$ is any finite collection of measurable sets (so J is finite), then the union $\bigcup_{j\in J}\Omega_j$ and intersection $\bigcap_{j\in J}\Omega_j$ are also measurable.

(iv) (σ-algebra property) If $(\Omega_j)_{j\in J}$ are any countable collection of measurable sets (so J is countable), then the union $\bigcup_{j\in J}\Omega_j$ and intersection $\bigcap_{j\in J}\Omega_j$ are also measurable.

Note that some of these properties are redundant; for instance, (iv) will imply (iii), and once one knows all open sets are measurable, (ii) will imply that all closed sets are measurable also. The properties (i-iv) will ensure that virtually every set one cares about is measurable; though as indicated in the introduction, there do exist non-measurable sets.

To every measurable set Ω, we associate the *Lebesgue measure* $m(\Omega)$ of Ω, which will obey the following properties:

(v) (Empty set) The empty set $\emptyset$ has measure $m(\emptyset) = 0$.

(vi) (Positivity) We have $0 \leq m(\Omega) \leq +\infty$ for every measurable set Ω.

(vii) (Monotonicity) If $A \subseteq B$, and A and B are both measurable, then $m(A) \leq m(B)$.

(viii) (Finite sub-additivity) If $(A_j)_{j\in J}$ are a finite collection of measurable sets, then $m(\bigcup_{j\in J} A_j) \leq \sum_{j\in J} m(A_j)$.

(ix) (Finite additivity) If $(A_j)_{j\in J}$ are a finite collection of *disjoint* measurable sets, then $m(\bigcup_{j\in J} A_j) = \sum_{j\in J} m(A_j)$.

(x) (Countable sub-additivity) If $(A_j)_{j\in J}$ are a countable collection of measurable sets, then $m(\bigcup_{j\in J} A_j) \leq \sum_{j\in J} m(A_j)$.

(xi) (Countable additivity) If $(A_j)_{j\in J}$ are a countable collection of *disjoint* measurable sets, then $m(\bigcup_{j\in J} A_j) = \sum_{j\in J} m(A_j)$.

(xii) (Normalization) The unit cube $[0,1]^n = \{(x_1,\ldots,x_n) \in \mathbf{R}^n : 0 \leq x_j \leq 1 \text{ for all } 1 \leq j \leq n\}$ has measure $m([0,1]^n) = 1$.

(xiii) (Translation invariance) If Ω is a measurable set, and $x \in \mathbf{R}^n$, then $x + \Omega := \{x + y : y \in \Omega\}$ is also measurable, and $m(x + \Omega) = m(\Omega)$.

Again, many of these properties are redundant; for instance the countable additivity property can be used to deduce the finite additivity property, which in turn can be used to derive monotonicity (when combined with the positivity property). One can also obtain the sub-additivity properties from the additivity ones. Note that $m(\Omega)$ can be $+\infty$, and so in particular some of the sums in the above properties may also equal $+\infty$. (Since everything is positive we will never have to deal with indeterminate forms such as $-\infty + +\infty$.)

Our goal for this chapter can then be stated thus:

Theorem 18.1.1 (Existence of Lebesgue measure). . *There exists a concept of a measurable set, and a way to assign a number $m(\Omega)$ to every measurable subset $\Omega \subseteq \mathbf{R}^n$, which obeys all of the properties (i)-(xiii).*

It turns out that Lebesgue measure is pretty much unique; any other concept of measurability and measure which obeys axioms (i)-(xiii) will largely coincide with the construction we give. However there are other measures which obey only some of the above axioms; also, we may be interested in concepts of measure for other domains than Euclidean spaces $\mathbf{R}^n$. This leads to *measure theory*, which is an entire subject in itself and will not be pursued here; however we do remark that the concept of measures is very important in modern probability, and in the finer points of analysis (e.g., in the theory of distributions).

18.2 First attempt: Outer measure

Before we construct Lebesgue measure, we first discuss a somewhat naive approach to finding the measure of a set - namely, we try to cover the set by boxes, and then add up the volume of each box. This approach will almost work, giving us a concept called *outer measure* which can be applied to every set and obeys all of the properties (v)-(xiii) except for the additivity properties (ix), (xi). Later we will have to modify outer measure slightly to recover the additivity property.

We begin by starting with the notion of an open box.

Definition 18.2.1 (Open box). An *open box* (or *box* for short) B in $\mathbf{R}^n$ is any set of the form

$$B = \prod_{i=1}^{n}(a_i, b_i) := \{(x_1, \ldots, x_n) \in \mathbf{R}^n : x_i \in (a_i, b_i) \text{ for all } 1 \leq i \leq n\}$$

where $b_i \geq a_i$ are real numbers. We define the *volume* $\text{vol}(B)$ of this box to be the number

$$\text{vol}(B) := \prod_{i=1}^{n}(b_i - a_i) = (b_1 - a_1)(b_2 - a_2)\ldots(b_n - a_n).$$

For instance, the unit cube $(0,1)^n$ is a box, and has volume 1. In one dimension $n = 1$, boxes are the same as open intervals. One can easily check that in general dimension that open boxes are indeed open. Note that if we have $b_i = a_i$ for some i, then the box becomes empty, and has volume 0, but we still consider this to be a box (albeit a rather silly one). Sometimes we will use $\text{vol}_n(B)$ instead of $\text{vol}(B)$ to emphasize that we are dealing with n-dimensional volume, thus for instance $\text{vol}_1(B)$ would be the length of a one-dimensional box B, $\text{vol}_2(B)$ would be the area of a two-dimensional box B, etc.

Remark 18.2.2. We of course expect the measure $m(B)$ of a box to be the same as the volume $\text{vol}(B)$ of that box. This is in fact an inevitable consequence of the axioms (i)-(xiii) (see Exercise 18.2.5).

Definition 18.2.3 (Covering by boxes). Let $\Omega \subseteq \mathbf{R}^n$ be a subset of $\mathbf{R}^n$. We say that a collection $(B_j)_{j \in J}$ of boxes *cover* Ω iff $\Omega \subseteq \bigcup_{j \in J} B_j$.

Suppose $\Omega \subseteq \mathbf{R}^n$ can be covered by a finite or countable collection of boxes $(B_j)_{j \in J}$. If we wish Ω to be measurable, and if we wish to have a measure obeying the monotonicity and sub-additivity properties (vii), (viii), (x) and if we wish $m(B_j) = \mathrm{vol}(B_j)$ for every box j, then we must have

$$m(\Omega) \leq m(\bigcup_{j \in J} B_j) \leq \sum_{j \in J} m(B_j) = \sum_{j \in J} \mathrm{vol}(B_j).$$

We thus conclude

$$m(\Omega) \leq \inf\{\sum_{j = \in J} \mathrm{vol}(B_j) : (B_j)_{j \in J} \text{ covers } \Omega; J \text{ at most countable}\}.$$

Inspired by this, we define

Definition 18.2.4 (Outer measure). If Ω is a set, we define the *outer measure* $m^*(\Omega)$ of Ω to be the quantity

$$m^*(\Omega) := \inf\{\sum_{j=1}^{\infty} \mathrm{vol}(B_j) : (B_j)_{j \in J} \text{ covers } \Omega; J \text{ at most countable}\}.$$

Since $\sum_{j=1}^{\infty} \mathrm{vol}(B_j)$ is non-negative, we know that $m^*(\Omega) \geq 0$ for all Ω. However, it is quite possible that $m^*(\Omega)$ could equal $+\infty$. Note that because we are allowing ourselves to use a countable number of boxes, that every subset of $\mathbf{R}^n$ has at least one countable cover by boxes; in fact $\mathbf{R}^n$ itself can be covered by countably many translates of the unit cube $(0,1)^n$ (how?). We will sometimes write $m_n^*(\Omega)$ instead of $m^*(\Omega)$ to emphasize the fact that we are using n-dimensional outer measure.

Note that outer measure can be defined for every single set (not just the measurable ones), because we can take the infimum of any non-empty set. It obeys several of the desired properties of a measure:

Lemma 18.2.5 (Properties of outer measure). *Outer measure has the following six properties:*

(v) (Empty set) The empty set $\emptyset$ has outer measure $m^*(\emptyset) = 0$.

(vi) *(Positivity) We have* $0 \le m^*(\Omega) \le +\infty$ *for every measurable set* Ω.

(vii) *(Monotonicity) If* $A \subseteq B \subseteq \mathbf{R}^n$, *then* $m^*(A) \le m^*(B)$.

(viii) *(Finite sub-additivity) If* $(A_j)_{j \in J}$ *are a finite collection of subsets of* $\mathbf{R}^n$, *then* $m^*(\bigcup_{j \in J} A_j) \le \sum_{j \in J} m^*(A_j)$.

(x) *(Countable sub-additivity) If* $(A_j)_{j \in J}$ *are a countable collection of subsets of* $\mathbf{R}^n$, *then* $m^*(\bigcup_{j \in J} A_j) \le \sum_{j \in J} m^*(A_j)$.

(xiii) *(Translation invariance) If* Ω *is a subset of* $\mathbf{R}^n$, *and* $x \in \mathbf{R}^n$, *then* $m^*(x + \Omega) = m^*(\Omega)$.

Proof. See Exercise 18.2.1. □

The outer measure of a closed box is also what we expect:

Proposition 18.2.6 (Outer measure of closed box). *For any closed box*

$$B = \prod_{i=1}^{n} [a_i, b_i] := \{(x_1, \ldots, x_n) \in \mathbf{R}^n : x_i \in [a_i, b_i] \text{ for all } 1 \le i \le n\}$$

we have

$$m^*(B) = \prod_{i=1}^{n} (b_i - a_i).$$

Proof. Clearly, we can cover the closed box $B = \prod_{i=1}^{n} [a_i, b_i]$ by the open box $\prod_{i=1}^{n} (a_i - \varepsilon, b_i + \varepsilon)$ for every $\varepsilon > 0$. Thus we have

$$m^*(B) \le \mathrm{vol}\Big(\prod_{i=1}^{n} (a_i - \varepsilon, b_i + \varepsilon)\Big) = \prod_{i=1}^{n} (b_i - a_i + 2\varepsilon)$$

for every $\varepsilon > 0$. Taking limits as $\varepsilon \to 0$, we obtain

$$m^*(B) \le \prod_{i=1}^{n} (b_i - a_i).$$

To finish the proof, we need to show that

$$m^*(B) \geq \prod_{i=1}^{n}(b_i - a_i).$$

By the definition of $m^*(B)$, it suffices to show that

$$\sum_{j \in J} \mathrm{vol}(B_j) \geq \prod_{i=1}^{n}(b_i - a_i)$$

whenever $(B_j)_{j \in J}$ is a finite or countable cover of B.

Since B is closed and bounded, it is compact (by the Heine-Borel theorem, Theorem 12.5.7), and in particular every open cover has a finite subcover (Theorem 12.5.8). Thus to prove the above inequality for countable covers, it suffices to do it for finite covers (since if $(B_j)_{j \in J'}$ is a finite subcover of $(B_j)_{j \in J}$ then $\sum_{j \in J} \mathrm{vol}(B_j)$ will be greater than or equal to $\sum_{j \in J'} \mathrm{vol}(B_j)$).

To summarize, our goal is now to prove that

$$\sum_{j \in J} \mathrm{vol}(B^{(j)}) \geq \prod_{i=1}^{n}(b_i - a_i) \qquad (18.1)$$

whenever $(B^{(j)})_{j \in J}$ is a finite cover of $\prod_{j=1}^{n}[a_i, b_i]$; we have changed the subscript B_j to superscript $B^{(j)}$ because we will need the subscripts to denote components.

To prove the inequality (18.1), we shall use induction on the dimension n. First we consider the base case $n = 1$. Here B is just a closed interval $B = [a, b]$, and each box $B^{(j)}$ is just an open interval $B^{(j)} = (a_j, b_j)$. We have to show that

$$\sum_{j \in J}(b_j - a_j) \geq (b - a).$$

To do this we use the Riemann integral. For each $j \in J$, let $f^{(j)} : \mathbf{R} \to \mathbf{R}$ be the function such that $f^{(j)}(x) = 1$ when $x \in (a_j, b_j)$ and $f^{(j)}(x) = 0$ otherwise. Then we have that $f^{(j)}$ is Riemann integrable (because it is piecewise constant, and compactly supported) and

$$\int_{-\infty}^{\infty} f^{(j)} = b_j - a_j.$$

Summing this over all $j \in J$, and interchanging the integral with the finite sum, we have

$$\int_{-\infty}^{\infty} \sum_{j \in J} f^{(j)} = \sum_{j \in J} b_j - a_j.$$

But since the intervals (a_j, b_j) cover $[a, b]$, we have $\sum_{j \in J} f^{(j)}(x) \geq 1$ for all $x \in [a, b]$ (why?). For all other values if x, we have $\sum_{j \in J} f^{(j)}(x) \geq 0$. Thus

$$\int_{-\infty}^{\infty} \sum_{j \in J} f^{(j)} \geq \int_{[a,b]} 1 = b - a$$

and the claim follows by combining this inequality with the previous equality. This proves (18.1) when $n = 1$.

Now assume inductively that $n > 1$, and we have already proven the inequality (18.1) for dimensions $n - 1$. We shall use a similar argument to the preceding one. Each box $B^{(j)}$ is now of the form

$$B^{(j)} = \prod_{i=1}^{n} (a_i^{(j)}, b_i^{(j)}).$$

We can write this as

$$B^{(j)} = A^{(j)} \times (a_n^{(j)}, b_n^{(j)})$$

where $A^{(j)}$ is the $n - 1$-dimensional box $A^{(j)} := \prod_{i=1}^{n-1} (a_i^{(j)}, b_i^{(j)})$. Note that

$$\mathrm{vol}(B^{(j)}) = \mathrm{vol}_{n-1}(A^{(j)})(b_n^{(j)} - a_n^{(j)})$$

where we have subscripted vol_{n-1} by $n-1$ to emphasize that this is $n - 1$-dimensional volume being referred to here. We similarly write

$$B = A \times [a_n, b_n]$$

where $A := \prod_{i=1}^{n-1}[a_i, b_i]$, and again note that

$$\mathrm{vol}(B) = \mathrm{vol}_{n-1}(A)(b_n - a_n).$$

For each $j \in J$, let $f^{(j)}$ be the function such that $f^{(j)}(x_n) = \mathrm{vol}_{n-1}(A^{(j)})$ for all $x_n \in (a_n^{(j)}, b_n^{(j)})$, and $f^{(j)}(x_n) = 0$ for all other x_n. Then $f^{(j)}$ is Riemann integrable and

$$\int_{-\infty}^{\infty} f^{(j)} = \mathrm{vol}_{n-1}(A^{(j)})(b_n^{(j)} - a_n^{(j)}) = \mathrm{vol}(B^{(j)})$$

and hence

$$\sum_{j\in J} \mathrm{vol}(B^{(j)}) = \int_{-\infty}^{\infty} \sum_{j\in J} f^{(j)}.$$

Now let $x_n \in [a_n, b_n]$ and $(x_1, \ldots, x_{n-1}) \in A$. Then $(x_1, \ldots, x_n)$ lies in B, and hence lies in one of the $B^{(j)}$. Clearly we have $x_n \in (a_n^{(j)}, b_n^{(j)})$, and $(x_1, \ldots, x_{n-1}) \in A^{(j)}$. In particular, we see that for each $x_n \in [a_n, b_n]$, the set

$$\{A^{(j)} : j \in J; x_n \in (a_n^{(j)}, b_n^{(j)})\}$$

of $n-1$-dimensional boxes covers A. Applying the inductive hypothesis (18.1) at dimension $n-1$ we thus see that

$$\sum_{j\in J: x_n \in (a_n^{(j)}, b_n^{(j)})} \mathrm{vol}_{n-1}(A^{(j)}) \geq \mathrm{vol}_{n-1}(A),$$

or in other words

$$\sum_{j\in J} f^{(j)}(x_n) \geq \mathrm{vol}_{n-1}(A).$$

Integrating this over $[a_n, b_n]$, we obtain

$$\int_{[a_n,b_n]} \sum_{j\in J} f^{(j)} \geq \mathrm{vol}_{n-1}(A)(b_n - a_n) = \mathrm{vol}(B)$$

and in particular

$$\int_{-\infty}^{\infty} \sum_{j\in J} f^{(j)} \geq \mathrm{vol}_{n-1}(A)(b_n - a_n) = \mathrm{vol}(B)$$

since $\sum_{j \in J} f^{(j)}$ is always non-negative. Combining this with our previous identity for $\int_{-\infty}^{\infty} \sum_{j \in J} f^{(j)}$ we obtain (18.1), and the induction is complete. □

Once we obtain the measure of a closed box, the corresponding result for an open box is easy:

Corollary 18.2.7. *For any open box*

$$B = \prod_{i=1}^{n} (a_i, b_i) := \{(x_1, \ldots, x_n) \in \mathbf{R}^n : x_i \in [a_i, b_i] \text{ for all } 1 \leq i \leq n\},$$

we have

$$m^*(B) = \prod_{i=1}^{n} (b_i - a_i).$$

In particular, outer measure obeys the normalization (xii).

Proof. We may assume that $b_i > a_i$ for all i, since if $b_i = a_i$ this follows from Lemma 18.2.5(v). Now observe that

$$\prod_{i=1}^{n} [a_i + \varepsilon, b_i - \varepsilon] \subset \prod_{i=1}^{n} (a_i, b_i) \subset \prod_{i=1}^{n} [a_i, b_i]$$

for all $\varepsilon > 0$, assuming that ε is small enough that $b_i - \varepsilon > a_i + \varepsilon$ for all i. Applying Proposition 18.2.6 and Lemma 18.2.5(vii) we obtain

$$\prod_{i=1}^{n} (b_i - a_i - 2\varepsilon) \leq m^*(\prod_{i=1}^{n} (a_i, b_i)) \leq \prod_{i=1}^{n} (b_i - a_i).$$

Sending $\varepsilon \to 0$ and using the squeeze test (Corollary 6.4.14), one obtains the result. □

We now compute some examples of outer measure on the real line $\mathbf{R}$.

Example 18.2.8. Let us compute the one-dimensional measure of $\mathbf{R}$. Since $(-R, R) \subset \mathbf{R}$ for all $R > 0$, we have

$$m^*(\mathbf{R}) \geq m^*((-R, R)) = 2R$$

by Corollary 18.2.7. Letting $R \to +\infty$ we thus see that $m^*(\mathbf{R}) = +\infty$.

Example 18.2.9. Now let us compute the one-dimensional measure of $\mathbf{Q}$. From Proposition 18.2.6 we see that for each rational number $\mathbf{Q}$, the point $\{q\}$ has outer measure $m^*(\{q\}) = 0$. Since $\mathbf{Q}$ is clearly the union $\mathbf{Q} = \bigcup_{q \in \mathbf{Q}} \{q\}$ of all these rational points q, and $\mathbf{Q}$ is countable, we have

$$m^*(\mathbf{Q}) \leq \sum_{q \in \mathbf{Q}} m^*(\mathbf{Q}) = \sum_{q \in \mathbf{Q}} 0 = 0,$$

and so $m^*(Q)$ must equal zero. In fact, the same argument shows that every countable set has measure zero. (This, incidentally, gives another proof that the real numbers are uncountable, Corollary 8.3.4.)

Remark 18.2.10. One consequence of the fact that $m^*(\mathbf{Q}) = 0$ is that given any $\varepsilon > 0$, it is possible to cover the rationals $\mathbf{Q}$ by a countable number of intervals whose total length is less than ε. This fact is somewhat un-intuitive; can you find a more explicit way to construct such a countable covering of $\mathbf{Q}$ by short intervals?

Example 18.2.11. Now let us compute the one-dimensional measure of the irrationals $\mathbf{R}\backslash\mathbf{Q}$. From finite sub-additivity we have

$$m^*(\mathbf{R}) \leq m^*(\mathbf{R}\backslash\mathbf{Q}) + m^*(\mathbf{Q}).$$

Since $\mathbf{Q}$ has outer measure 0, and $m^*(\mathbf{R})$ has outer measure $+\infty$, we thus see that the irrationals $\mathbf{R}\backslash\mathbf{Q}$ have outer measure $+\infty$. A similar argument shows that $[0, 1]\backslash\mathbf{Q}$, the irrationals in $[0, 1]$, have outer measure 1 (why?).

Example 18.2.12. By Proposition 18.2.6, the unit interval $[0, 1]$ in $\mathbf{R}$ has one-dimensional outer measure 1, but the unit interval $\{(x, 0) : 0 \leq x \leq 1\}$ in $\mathbf{R}^2$ has two-dimensional outer measure 0. Thus one-dimensional outer measure and two-dimensional outer measure are quite different. Note that the above remarks and countable additivity imply that the entire x-axis of $\mathbf{R}^2$ has two-dimensional outer measure 0, despite the fact that $\mathbf{R}$ has infinite one-dimensional measure.

Exercise 18.2.1. Prove Lemma 18.2.5. (Hint: you will have to use the definition of inf, and probably introduce a parameter ε. You may have to treat separately the cases when certain outer measures are equal to $+\infty$. (viii) can be deduced from (x) and (v). For (x), label the index set J as $J = \{j_1, j_2, j_3, \ldots\}$, and for each A_j, pick a covering of A_j by boxes whose total volume is no larger than $m^*(A_j) + \varepsilon/2^j$.)

Exercise 18.2.2. Let A be a subset of $\mathbf{R}^n$, and let B be a subset of $\mathbf{R}^m$. Note that the Cartesian product $\{(a, b) : a \in A, b \in B\}$ is then a subset of $\mathbf{R}^{n+m}$. Show that $m^*_{n+m}(A \times B) \leq m^*_n(A)m^*_m(B)$. (It is in fact true that $m^*_{n+m}(A \times B) = m^*_n(A)m^*_m(B)$, but this is substantially harder to prove).

In Exercises 18.2.3-18.2.5, we assume that $\mathbf{R}^n$ is a Euclidean space, and we have a notion of measurable set in $\mathbf{R}^n$ (which may or may not coincide with the notion of Lebesgue measurable set) and a notion of measure (which may or may not co-incide with Lebesgue measure) which obeys axioms (i)-(xiii).

Exercise 18.2.3.

(a) Show that if $A_1 \subseteq A_2 \subseteq A_3 \ldots$ is an increasing sequence of measurable sets (so $A_j \subseteq A_{j+1}$ for every positive integer j), then we have $m(\bigcup_{j=1}^\infty A_j) = \lim_{j\to\infty} m(A_j)$.

(b) Show that if $A_1 \supseteq A_2 \supseteq A_3 \ldots$ is a decreasing sequence of measurable sets (so $A_j \supseteq A_{j+1}$ for every positive integer j), and $m(A_1) < +\infty$, then we have $m(\bigcap_{j=1}^\infty A_j) = \lim_{j\to\infty} m(A_j)$.

Exercise 18.2.4. Show that for any positive integer $q > 1$, that the open box

$$(0, 1/q)^n := \{(x_1, \ldots, x_n) \in \mathbf{R}^n : 0 < x_j < 1/q \text{ for all } 1 \leq j \leq n\}$$

and the closed box

$$[0,1/q]^n := \{(x_1,\ldots,x_n) \in \mathbf{R}^n : 0 \le x_j \le 1/q \text{ for all } 1 \le j \le n\}$$

both measure q^{-n}. (Hint: first show that $m((0,1/q)^n) \le q^{-n}$ for every $q \ge 1$ by covering $(0,1)^n$ by some translates of $(0,1/q)^n$. Using a similar argument, show that $m([0,1/q]^n) \ge q^{-n}$. Then show that $m([0,1/q]^n \backslash (0,1/q)^n) \le \varepsilon$ for every $\varepsilon > 0$, by covering the boundary of $[0,1/q]^n$ with some very small boxes.)

Exercise 18.2.5. Show that for any box B, that $m(B) = \text{vol}(B)$. (Hint: first prove this when the co-ordinates a_j, b_j are rational, using Exercise 18.2.4. Then take limits somehow (perhaps using Q1) to obtain the general case when the co-ordinates are real.)

Exercise 18.2.6. Use Lemma 18.2.5 and Proposition 18.2.6 to furnish another proof that the reals are uncountable (i.e., reprove Corollary 8.3.4).

18.3 Outer measure is not additive

In light of Lemma 18.2.5, it would seem now that all we need to do is to verify the additivity properties (ix), (xi), and we have everything we need to have a usable measure. Unfortunately, these properties fail for outer measure, even in one dimension.

Proposition 18.3.1 (Failure of countable additivity). *There exists a countable collection $(A_j)_{j\in J}$ of disjoint subsets of $\mathbf{R}$, such that $m^*(\bigcup_{j\in J} A_j) \neq \sum_{j\in J} m^*(A_j)$.*

Proof. We shall need some notation. Let $\mathbf{Q}$ be the rationals, and $\mathbf{R}$ be the reals. We say that a set $A \subset \mathbf{R}$ is a *coset* of $\mathbf{Q}$ if it is of the form $A = x + \mathbf{Q}$ for some real number x. For instance, $\sqrt{2} + \mathbf{Q}$ is a coset of $\mathbf{R}$, as is $\mathbf{Q}$ itself, since $\mathbf{Q} = 0 + \mathbf{Q}$. Note that a coset A can correspond to several values of x; for instance $2 + \mathbf{Q}$ is exactly the same coset as $0 + \mathbf{Q}$. Also observe that it is not possible for two cosets to partially overlap; if $x + \mathbf{Q}$ intersects $y + \mathbf{Q}$ in even just a single point z, then $x - y$ must be rational (why? use the identity $x - y = (x - z) - (y - z)$), and thus $x + \mathbf{Q}$ and $y + \mathbf{Q}$ must be equal (why?). So any two cosets are either identical or distinct.