This proof depends on the Stoic idea, also endorsed by Aristotle, that every proposition, here the proposition that $\sqrt{2}^{\sqrt{2}}$ is rational, is either true or false. What Kronecker would have objected to is that, at the end of the proof, we don't know whether $\alpha = \sqrt{2}^{\sqrt{2}}$ or $\alpha = \sqrt{2}$.

It is known, using some deep mathematics, that $\sqrt{2}^{\sqrt{2}}$ is irrational, but that is beside the point here. It is also beside the point that the stated theorem has an easy constructive proof: take $\alpha = \sqrt{2}$ and $\beta = 2\log_2 3$. It is possible to exhibit mathematical theorems for which no constructive proof exists, for instance the theorem which asserts that, if the axiom of choice is true, then there exist nonprincipal ultrafilters on the set of natural numbers. (The reader unfamiliar with ultrafilters may ignore this example.)

Poincaré, on the other hand, objected to *impredicative* definitions or constructions, essentially those which define or construct an entity in terms of entities of a higher type. For example, consider the usual proof that every nonempty set of real numbers that is bounded above has a least upper bound. According to Dedekind, a *real number* is a set α of rational numbers such that

1. both α and its complement α^c are nonempty;
2. every element of α is less than every element of α^c;
3. α has no greatest element.

Now let m be a nonempty set of real numbers which is bounded above. Put

$$\alpha \equiv \{x \in \mathbf{Q} \mid \exists_{y \in m} x \in y\}.$$

It is not hard to show that α satisfies 1 to 3 and that it is the least upper bound of m. However, being a *set* of reals, m has a higher type than α. The construction of α is thus impredicative in Poincaré's sense.

Most mathematicians feel that Poincaré was too sceptical here. If we disallowed constructions such as that of the least upper bound of m above, most of analysis would have to be abandoned.

Exercises

1. Suppose the barber in a certain village shaves all and only those men of the village who do not shave themselves. Prove that the barber is a woman.

2. Prove that, in the set theory of Zermelo–Fraenkel, there does not exist a 'universal' set y, such that $\forall_x (x \in y \Leftrightarrow x = x)$.

3. Prove that the impredicatively defined set α above is a real number according to Dedekind's definition as given in the text.

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Intuitionistic Propositional Calculus

It seems that, over many centuries, no philosopher or mathematician ever seriously questioned Aristotle's *law of the excluded third:* for every proposition p, either p or not p, symbolically $p \vee \neg p$. In retrospect, it appears that Aristotle himself had some doubts about applying this law when talking about events in time, e.g., when p was the proposition: there will be a sea battle tomorrow. But in mathematics, which deals with unchanging entities, the law of the excluded third was accepted as gospel truth, as was the equivalent assertion: for every proposition p, $\neg\neg p \Rightarrow p$; two negations make an affirmation.

It was the topologist Luitzen Egbertus Jan Brouwer (1882–1966) who observed that all nonconstructive arguments in mathematics depend on Aristotle's law and he proposed that we simply drop it, together with all its consequences, at least when talking about infinite collections. Surprisingly, it turns out that, if one follows Brouwer's suggestion, one is still left with a rich logical system adequate for all constructive mathematics. We shall present the outlines of such a system here, starting with the propositional calculus.

We consider the logical symbols $\top$, $\perp$, $\wedge$, $\vee$ and $\Rightarrow$, the first two counting as formulas, the last two being binary connectives between formulas, so that $A \wedge B$, $A \vee B$, and $A \Rightarrow B$ are formulas if A and B are. The usual reading of these formulas is as follows:

$$\begin{aligned} \top &\equiv \text{true}, \\ \perp &\equiv \text{false}, \\ A \wedge B &\equiv \text{(both) } A \text{ and } B, \end{aligned}$$

$$\begin{aligned} A \vee B &\equiv \text{(either) } A \text{ or } B, \\ A \Rightarrow B &\equiv \text{if } A \text{ then } B. \end{aligned}$$

However, *intuitionists*, as the followers of Brouwer are called, understand these words in a way subtly different from that of classical mathematicians, as we hope to make clear in the next chapter.

It is customary to make use of the *entailment* symbol $\vdash$, where

$$A_1, \ldots, A_n \vdash B$$

means that the assumptions $A_1, \ldots, A_n$ entail the conclusion B, or that B may be deduced from $A_1 \ldots, A_n$, n being any natural number. We often denote strings of formulas by capital Greek letters; thus $\Gamma \vdash B$ means that B may be inferred from the formulas in Γ. Note that Γ may be empty ($n = 0$), or consist of a single formula ($n = 1$).

We shall adopt the following *axioms:*

$$\vdash \top; \qquad \bot \vdash A; \qquad A \wedge B \vdash A; \qquad A \wedge B \vdash B;$$

$$A, B \vdash A \wedge B; \qquad A \vdash A \vee B; \qquad B \vdash A \vee B; \qquad A, A \Rightarrow B \vdash B.$$

The last of these has the Latin name 'modus ponens', which we shall abbreviate as MP. We also adopt two *rules of inference:*

$$\frac{\Gamma, A \vdash C, \quad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C}; \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}.$$

These are called 'argument by cases', abbreviated AC, and 'deduction rule', abbreviated DR, respectively.

In classical logic, one would be quite happy to establish these axioms and rules of inference with the help of truth tables. In intuitionistic logic, we are not allowed to use truth tables, as they would also establish Aristotle's $A \vee \neg A$ and $\neg\neg A \Rightarrow A$, which we have discarded, provided we define $\neg A \equiv A \Rightarrow \bot$, as we shall do from now on.

In addition to the above axioms and rules of inference, which describe the logical connectives, we also have the following, which describe the entailment symbol: