Proposition 42. Let R be a commutative ring with 1 and let $D^{-1}R$ be its localization with respect to the multiplicatively closed subset D of R containing 1.

(1) Localization commutes with finite sums and intersections of ideals: If I and J are ideals of R, then

$$D^{-1}(I+J) = D^{-1}(I) + D^{-1}(J) \quad \text{and} \quad D^{-1}(I \cap J) = D^{-1}(I) \cap D^{-1}(J).$$

Localization commutes with quotients:

$$D^{-1}R \,/\, D^{-1}I \cong D^{-1}(R/I),$$

(where the localization on the right is with respect to the image of D in the quotient R/I).

(2) Localization commutes with taking radicals: If N is the nilradical of R, then $D^{-1}N$ is the nilradical of $D^{-1}R$. If I is an ideal in R, then $\operatorname{rad}(D^{-1}I)$ is $D^{-1}(\operatorname{rad} I)$.

(3) Primary ideals correspond to primary ideals in the correspondence (3) of Proposition 38. More precisely, suppose Q is a P-primary ideal in R. If $D \cap P \neq \emptyset$ then $D^{-1}Q = D^{-1}R$. If $D \cap P = \emptyset$ then $D^{-1}P$ is a prime ideal, the extension $D^{-1}Q$ of Q is a $D^{-1}P$-primary ideal in $D^{-1}R$, and the contraction back to R of $D^{-1}Q$ is Q.

(4) Localization commutes with finite sums, intersections and quotients of modules: If L and N are submodules of the R-module M, then
(a) $D^{-1}(L+N) = D^{-1}L + D^{-1}N$ and $D^{-1}(L \cap N) = D^{-1}L \cap D^{-1}N$,
(b) $D^{-1}N$ is a submodule of $D^{-1}M$ and $D^{-1}M \,/\, D^{-1}N = D^{-1}(M/N)$.

(5) Localization commutes with finite direct sums of modules: If M and N are R-modules, then $D^{-1}(M \oplus N) \cong D^{-1}M \oplus D^{-1}N$.

(6) Localization is exact (i.e., $D^{-1}R$ is a flat R-module): If $0 \to L \xrightarrow{\psi} M \xrightarrow{\varphi} N \to 0$ is a short exact sequence of R-modules, then the induced sequence $0 \to D^{-1}L \xrightarrow{\psi'} D^{-1}M \xrightarrow{\varphi'} D^{-1}N \to 0$ of $D^{-1}R$-modules is also exact.

Proof: We first prove (6). Suppose that $0 \to L \xrightarrow{\psi} M \xrightarrow{\varphi} N \to 0$ is a short exact sequence of R-modules. Every element of $D^{-1}N$ is of the form n/d for some $n \in N$ and $d \in D$. Since φ is surjective, $n = \varphi(m)$ for some $m \in M$, so $\varphi'(m/d) = \varphi(m)/d = n/d$ and $\varphi' : D^{-1}M \to D^{-1}N$ is surjective. If m/d is in the kernel of φ' then $d_1\varphi(m) = 0$ for some $d_1 \in D$. Then $\varphi(d_1 m) = 0$ implies $d_1 m = \psi(l)$ for some $l \in L$ by the exactness of the original sequence at M, so $m/d = d_1 m/(d_1 d) = \psi(l)/(d_1 d) = \psi'(l/(d_1 d))$ and $\ker(\varphi') \subseteq \operatorname{image}(\psi')$. If $\psi(l)/d \in \operatorname{image}(\psi')$ then $\varphi'(\psi(l)/d) = \varphi(\psi(l))/d = 0$, which shows the reverse inclusion $\operatorname{image}(\psi') \subseteq \ker(\varphi')$, and we have exactness of the induced sequence at $D^{-1}M$. Finally, suppose $\psi'(l/d) = 0$. Then $d_2\psi(l) = 0$ for some $d_2 \in D$, i.e., $\psi(d_2 l) = 0$, so $d_2 l = 0$ by the injectivity of ψ. Hence $l/d = d_2 l/(d_2 d) = 0$ and ψ' is injective. This proves that the sequence $0 \to D^{-1}L \xrightarrow{\psi'} D^{-1}M \xrightarrow{\varphi'} D^{-1}N \to 0$ is exact.

To prove the first statement in (1), note that $(i+j)/d = i/d + j/d$ for $i \in I$, $j \in J$ and $d \in D$ shows $D^{-1}(I+J) \subseteq D^{-1}(I) + D^{-1}(J)$; and $i/d_1 + j/d_2 = (d_2 i + d_1 j)/(d_1 d_2)$ for $i \in I$, $j \in J$ and $d_1, d_2 \in D$ shows $D^{-1}(I) + D^{-1}(J) \subseteq D^{-1}(I+J)$. For the second statement, the inclusion $D^{-1}(I \cap J) \subseteq D^{-1}(I) \cap D^{-1}(J)$ is immediate. If

$a/d \in D^{-1}(I) \cap D^{-1}(J)$ then $d_1 a \in I$ and $d_2 a \in J$ for some $d_1, d_2 \in D$. Then $d_1 d_2 a \in I \cap J$ and $a/d = (d_1 d_2 a)/(d_1 d_2 d)$ gives the inclusion $D^{-1}(I) \cap D^{-1}(J) \subseteq D^{-1}(I \cap J)$. The last statement in (1) follows by applying (6) to the exact sequence $0 \to I \xrightarrow{\psi} R \xrightarrow{\varphi} R/I \to 0$.

To prove (2), suppose first that $a \in \operatorname{rad} I$, so that $a^n \in I$ for some $n \geq 1$. Then $(a/d)^n = a^n/d^n \in D^{-1}I$ so $D^{-1}(\operatorname{rad} I) \subseteq \operatorname{rad}(D^{-1}I)$. Conversely, if $a/d \in \operatorname{rad}(D^{-1}I)$ then $(a/d)^n \in D^{-1}I$ for some $n \geq 1$, i.e., $d_1 a^n \in I$ for some $d_1 \in D$. Hence $(d_1 a)^n = d_1^{n-1}(d_1 a^n) \in I$, so $d_1 a \in \operatorname{rad} I$ and then $a/d = d_1 a/(d_1 d) \in D^{-1}(\operatorname{rad} I)$ shows that $\operatorname{rad}(D^{-1}I) \subseteq D^{-1}(\operatorname{rad} I)$. This proves the second statement in (2), and the first statement follows by applying this to the ideal $I = (0)$.

For (3), note first that $D \cap P = \emptyset$ if and only if $D \cap Q = \emptyset$ (one inclusion is obvious and the other follows since $d \in D \cap P$ implies $d^n \in D \cap Q$ for some n). The statement for $D \cap P \neq \emptyset$ and the fact that $D^{-1}P$ is a prime ideal for $D \cap P = \emptyset$ were proved in Proposition 38. To see that $D^{-1}Q$ is a primary ideal in $D^{-1}R$, suppose that $(a/d_1)(b/d_2) \in D^{-1}Q$ and $a/d_1 \notin D^{-1}Q$. Then there is some element $d \in D$ so that $dab \in Q$, and since $a \notin Q$ and Q is primary, we have $(db)^n \in Q$ for some $n \geq 1$. Then $(b/d_2)^n = d^n b^n/(d^n d_2^n) \in D^{-1}Q$, so that $D^{-1}Q$ is primary. The radical of $D^{-1}Q$ is $D^{-1}P$ by (2). Finally, by (2) of Proposition 38, the contraction of $D^{-1}Q$ is an ideal of R containing Q and consists precisely of the elements $r \in R$ with $dr \in Q$ for some $d \in D$. Since Q is P-primary, the definition of primary implies that if $dr \in Q$ and $d \notin P$, then $r \in Q$, hence the contraction of $D^{-1}Q$ is Q.

The proof of (4) is essentially the same as the proof of (1) and is left as an exercise.

It is easy to see that if the exact sequence $0 \to L \xrightarrow{\psi} M \xrightarrow{\varphi} N \to 0$ of R-modules splits, then the exact sequence $0 \to D^{-1}L \xrightarrow{\psi'} D^{-1}M \xrightarrow{\varphi'} D^{-1}N \to 0$ of $D^{-1}R$-modules also splits, which gives (5).

Proposition 38 shows that localizing at the multiplicatively closed set D emphasizes the ideals of R not containing any elements of D since the other ideals of R become trivial when extended to $D^{-1}R$. The following proposition provides a more precise statement in terms of the effect of localization on primary decomposition of ideals.

Proposition 43. Let R be a Noetherian ring and let

$$I = Q_1 \cap \cdots \cap Q_m$$

be a minimal primary decomposition of the proper ideal I, where Q_i is a P_i-primary ideal. Suppose D is a multiplicatively closed set of R containing 1 and the primary ideals $Q_1, \dots, Q_m$ are numbered so that $D \cap P_i = \emptyset$ for $1 \leq i \leq t$ and $D \cap P_i \neq \emptyset$ for $t+1 \leq i \leq m$. Then

$$D^{-1}I = D^{-1}Q_1 \cap \cdots \cap D^{-1}Q_t$$

is a minimal primary decomposition of $D^{-1}I$ in $D^{-1}R$ and $D^{-1}Q_i$ is a $D^{-1}P_i$-primary ideal. Further, the contraction of $D^{-1}Q_i$ back to R is Q_i for $1 \leq i \leq t$ and

$$^c(D^{-1}I) = Q_1 \cap \cdots \cap Q_t$$

is a minimal primary decomposition of the contraction of $D^{-1}I$ back to R.

Proof: By (3) of Proposition 42, $D^{-1}Q_i = D^{-1}R$ for $t+1 \le i \le m$, and $D^{-1}Q_i$ is a $D^{-1}P_i$-primary ideal with pullback Q_i for $1 \le i \le t$. By (1) of the same proposition, $D^{-1}I = D^{-1}Q_1 \cap \cdots \cap D^{-1}Q_t$, and (3) shows that this is a primary decomposition. Contracting to R shows that $\mathcal{C}(D^{-1}I) = Q_1 \cap \cdots \cap Q_t$, which also implies that the decompositions are minimal.

In particular we can finish the proof of Theorem 21:

Corollary 44. The primary ideals belonging to the isolated primes in a minimal primary decomposition of I are uniquely defined by I.

Proof: Let P be a minimal element in the set $\{P_1, \ldots, P_m\}$ of primes belonging to I, and take $D = R - P$ in Proposition 43. Then $D \cap P_i = \emptyset$ only for $P = P_i$, so the contraction of the localization of I at D is precisely the primary ideal Q belonging to the minimal prime P. Since the prime ideals $\{P_1, \ldots, P_m\}$ of primes belonging to I are uniquely determined by I, it follows that the primary ideals Q belonging to the isolated primes of I are also uniquely determined by I.

The effect of isolating in on certain prime ideals by localization is particularly precise in the case of localizing at a prime P (considered in Example 3 following Corollary 37 above). We first recall the definition of an important type of ring (cf. Exercises 37–39 in Section 7.4).

Definition. A commutative ring with 1 that has a unique maximal ideal is called a *local ring.*

Proposition 45. Let R be a commutative ring with 1. Then the following are equivalent:

(1) R is a local ring with unique maximal ideal M

(2) if M is the set of elements of R that are not units, then M is an ideal

(3) there is a maximal ideal M of R such that every element $1+m$ with $m \in M$ is a unit in R.

Proof: If $a \in R$ then the ideal (a) is either R, in which case a is a unit, or is a proper ideal, in which case (a) is contained in a maximal ideal (Proposition 11 of Section 7.4). It follows that if R is a local ring and M is its unique maximal ideal then every $a \notin M$ is a unit, so M consists precisely of the set of nonunits in R, showing that (1) implies (2). It also follows that if the set M of nonunits in R is an ideal then this ideal must be the unique maximal ideal in R, so that (2) implies (1).

Suppose now that (3) is satisfied. If a is an element of R not contained in the maximal ideal M, then $(a) + M = R$, so that $ab + m = 1$ for some $b \in R$ and $m \in M$. Then $ab = 1 - m$ is a unit by assumption, so a is also a unit. This shows that M is the unique maximal ideal in R, so (3) implies (1). Conversely, if R is a local ring, then $1 + m \notin M$ for any $m \in M$, so $1 + m$ is a unit, so (1) implies (3).