

The second two statements in the proposition show that the collection

$$\mathcal{T} = \{\mathcal{Z}(I) \mid I \text{ is an ideal of } R\}$$

satisfies the three axioms for the closed sets of a topology on $\text{Spec } R$ as in Section 2.

Definition. The topology on $\text{Spec } R$ defined by the closed sets $\mathcal{Z}(I)$ for the ideals I of R is called the *Zariski topology* on $\text{Spec } R$.

By definition, the closure in the Zariski topology of the singleton set $\{P\}$ in $\text{Spec } R$ consists of all the prime ideals of R that contain P . In particular, a point P in $\text{Spec } R$ is closed in the Zariski topology if and only if the prime ideal P is not contained in any other prime ideals of R , i.e., if and only if P is a maximal ideal (so the Zariski topology on $\text{Spec } R$ is not generally Hausdorff). These points are given a name:

Definition. The maximal ideals of R are called the *closed points* in $\text{Spec } R$.

In terms of the terminology above, the points in $\text{Spec } R$ that are closed in the Zariski topology are precisely the points in $\text{mSpec } R$.

A closed subset of a topological space is *irreducible* if it is not the union of two proper closed subsets, or, equivalently, if every nonempty open set is dense. Arguments similar to those used to prove Proposition 17 show that the closed subset $Y = \mathcal{Z}(I)$ in $\text{Spec } R$ is irreducible if and only if $\mathcal{I}(Y) = \text{rad } I$ is prime (cf. Exercise 16).

The following proposition summarizes some of these results:

Proposition 54. The maps \mathcal{Z} and \mathcal{I} define inverse bijections

$$\{\text{Zariski closed subsets of } \text{Spec } R\} \xrightleftharpoons[\mathcal{Z}]{\mathcal{I}} \{\text{radical ideals of } R\}.$$

Under this correspondence the closed points in $\text{Spec } R$ correspond to the maximal ideals in R , and the irreducible subsets of $\text{Spec } R$ correspond to the prime ideals in R .

Examples

- (1) If $X = \text{Spec } \mathbb{Z}$ then X is irreducible and the nonzero primes give closed points in X . The point (0) is not a closed point, in fact the closure of (0) is all of X , i.e., (0) is *dense* in $\text{Spec } \mathbb{Z}$. For this reason the element (0) is called a *generic point* in $\text{Spec } \mathbb{Z}$. Since every ideal of \mathbb{Z} is principal, the Zariski closed sets in $\text{Spec } \mathbb{Z}$ are \emptyset , $\text{Spec } \mathbb{Z}$ and any finite set of nonzero prime ideals in \mathbb{Z} .
- (2) Suppose $X = \text{Spec } \mathbb{Z}[x]$ as in Example 3 previously. For each integer prime p the Zariski closure of the element $(p) \in X$ consists of the maximal ideals (p, g) of type (d). Likewise for each \mathbb{Q} -irreducible polynomial f of type (c), the Zariski closure of the element (f) is the collection of prime ideals of type (d) where g is some divisor of f in $\mathbb{Z}/p\mathbb{Z}[x]$.

Example: (Affine k -algebras)

Suppose $R = k[V]$ is the coordinate ring of some affine algebraic set $V \subseteq \mathbb{A}^n$ over an algebraically closed field k . Then $R = k[x_1, \dots, x_n]/\mathcal{I}(V)$ where $\mathcal{I}(V)$ is a radical ideal in $k[x_1, \dots, x_n]$. In particular R is a finitely generated k -algebra and since $\mathcal{I}(V)$ is radical, R contains no nonzero nilpotent elements.

Definition. A finitely generated algebra over an algebraically closed field k having no nonzero nilpotent elements is called an *affine k -algebra*.

If R is an affine k -algebra, then by Corollary 5 there is a surjective k -algebra homomorphism $\pi : k[x_1, \dots, x_n] \rightarrow R$ whose kernel $I = \ker \pi$ must be a radical ideal since R has no nonzero nilpotent elements. Let $V = Z(I) \subseteq \mathbb{A}^n$. Then $R \cong k[x_1, \dots, x_n]/I = k[V]$ is the coordinate ring of an affine algebraic set over k . Hence *affine k -algebras are precisely the rings arising as the rings of functions on affine algebraic sets over algebraically closed fields*.

By the Nullstellensatz, the points of $\text{mSpec } R$ are in bijective correspondence with V , and the points of $\text{Spec } R$ are in bijective correspondence with the subvarieties of V . By Theorem 6, morphisms between two affine algebraic sets correspond bijectively with (k -algebra) homomorphisms of affine k -algebras. In the language of categories these results show that over an algebraically closed field k there is an equivalence of categories

$$\left\{ \begin{array}{l} \text{affine algebraic sets} \\ \text{morphisms of algebraic sets} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{affine } k\text{-algebras} \\ k\text{-algebra homomorphisms} \end{array} \right\}.$$

The map from left to right sends the affine algebraic set V to its coordinate ring $k[V]$. The map from right to left sends the affine k -algebra R to $\text{mSpec } R$. The pair $(\text{mSpec } R, R)$ is sometimes called the *canonical model* of the affine k -algebra R .

Over an algebraically closed field k , a k -algebra homomorphism $\varphi : R \rightarrow S$ between two affine k -algebras as in the previous example has the property (by the Nullstellensatz) that the inverse image of a maximal ideal in S is a maximal ideal in R . As previously mentioned, one reason for considering $\text{Spec } R$ rather than just $\text{mSpec } R$ for more general rings is that inverse images of maximal ideals under ring homomorphisms are not in general maximal ideals. When R is an affine k -algebra corresponding to an affine algebraic set V , the space $\text{Spec } R$ contains not only the “geometric points” of V (in the form of the closed points in $\text{Spec } R$), but also the non-closed points corresponding to all of the subvarieties of V (in the form of the non-closed points in $\text{Spec } R$, i.e., the prime ideals P of R that are not maximal).

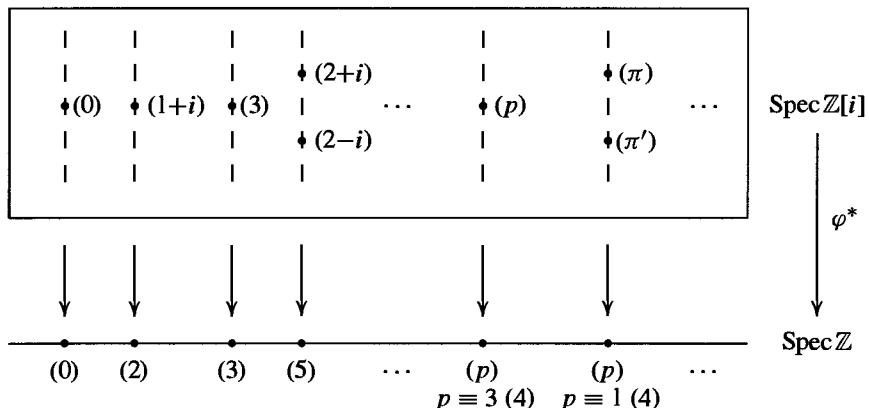
In general, if $\varphi : R \rightarrow S$ is a ring homomorphism mapping 1_R to 1_S and P is a prime ideal in S then $\varphi^{-1}(P)$ is a prime ideal in R . This defines a map $\varphi^* : \text{Spec } S \rightarrow \text{Spec } R$ with $\varphi^*(P) = \varphi^{-1}(P)$. If $Z(I) \subseteq \text{Spec } R$ is a Zariski closed subset of $\text{Spec } R$, then it is easy to show that $(\varphi^*)^{-1}(Z(I))$ is the Zariski closed subset $Z(\varphi(I)S)$ defined by the ideal generated by $\varphi(I)$ in S . Since the inverse image of a closed subset in $\text{Spec } R$ is a closed subset in $\text{Spec } S$, the induced map φ^* is continuous in the Zariski topology. This proves the following proposition.

Proposition 55. Every ring homomorphism $\varphi : R \rightarrow S$ mapping 1_R to 1_S induces a map $\varphi^* : \text{Spec } S \rightarrow \text{Spec } R$ that is continuous with respect to the Zariski topologies on $\text{Spec } R$ and $\text{Spec } S$.

While the generalization from affine algebraic sets to $\text{Spec } R$ for general rings R has made matters slightly more complicated, there are (at least) two very important benefits gained by this more general setting. The first is that $\text{Spec } R$ can be considered even for commutative rings R containing nilpotent elements; the second is that $\text{Spec } R$ need not be a k -algebra for any field k , and even when it is, the field k need not be algebraically closed. The fact that many of the properties found in the situation of affine k -algebras hold in more general settings then allows the application of “geometric” ideas to these situations (for example, to $\text{Spec } R$ when R is finite).

Examples

- (1) The natural inclusion $\varphi : \mathbb{Z} \rightarrow \mathbb{Z}[i]$ induces a map $\varphi^* : \text{Spec } \mathbb{Z}[i] \rightarrow \text{Spec } \mathbb{Z}$. The fiber of φ^* over the nonzero prime P in \mathbb{Z} consists of the prime ideals of $\mathbb{Z}[i]$ containing P . If $P = (p)$ where $p = 2$ or p is a prime congruent to 3 mod 4, then there is only one element in this fiber; if p is a prime congruent to 1 mod 4, then there are two elements in the fiber: the primes (π) and (π') where $p = \pi\pi'$ in $\mathbb{Z}[i]$, cf. Proposition 18 in Section 8.3. This can be represented pictorially in the following figure:



- (2) If k is an algebraically closed field then $\text{Spec } k[x]$ consists of (0) and the ideals $(x - a)$ for $a \in k$; the natural inclusion $\varphi : k[x] \rightarrow k[x, y]$ induces the Zariski continuous map $\varphi^* : \text{Spec } k[x, y] \rightarrow \text{Spec } k[x]$. The elements of $\text{Spec } k[x, y]$ are
- (0) ,
 - (f) where f is an irreducible polynomial in $k[x, y]$, and
 - $(x - a, y - b)$ with $a, b \in k$

(cf. Exercise 4). The prime (0) is Zariski dense in $\text{Spec } k[x, y]$; the Zariski closure of the primes in (b) consists of the primes $(x - a, y - b)$ in (c) with $f(a, b) = 0$; the closed points, i.e., the elements of $\text{mSpec } k[x, y]$, are the primes in (c).

By the Nullstellensatz, each prime ideal P in $\text{Spec } k[x, y]$ is uniquely determined by the corresponding zero set $\mathcal{Z}(P)$. The prime $(0) \in k[x, y]$ corresponds to \mathbb{A}^2 . The prime (f) corresponds to the points where $f(x, y) = 0$, and $P = (f)$ is the intersection of all the maximal ideals containing P . The maximal ideal $(x - a, y - b)$ corresponds to the point $(a, b) \in \mathbb{A}^2$. Fibered over $\text{Spec } k[x]$ by the map φ^* these primes can be pictured geometrically as in the diagram on the following page.

In this diagram, the prime $(x - a)$ in $\text{Spec } k[x]$ is identified with the element $a \in k$. The prime $(x) \in \text{Spec } k[x, y]$ corresponds to the points in \mathbb{A}^2 with $x = 0$, i.e.,