where the partial derivatives

$$\frac{\partial P}{\partial y}, \frac{\partial P}{\partial z}, \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial z}, \frac{\partial R}{\partial x}, \frac{\partial R}{\partial y},$$

are continuous on an open set S. Iffis the gradient of some potential function φ, prove that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$

at each point of S.

5. For each of the following vector fields, use the result of Exercise 4 to prove that f is not a gradient. Then find a closed path C such that $\oint_C f \neq 0$.
(a) $f(x, y, z) = yi + xj + xk$.
(b) $f(x, y, z) = xyi + (x^2 + 1)j + z^2k$.
6. A force field f is defined in 3-space by the equation

$$f(x, y, z) = yi + zj + yzk.$$

(a) Determine whether or not fis conservative.
(b) Calculate the work done in moving a particle along the curve described by

$$a(t) = \cos ti + \sin tj + e^t k$$

as t runs from 0 to π.
7. A two-dimensional force field F is described by the equation

$$F(x, y) = (x + y)i + (x - y)j.$$

(a) Show that the work done by this force in moving a particle along a curve

$$\alpha(t) = f(t)i + g(t)j, \qquad a \le t \le b,$$

depends only on $f(a)$, $f(b)$, $g(a)$, $g(b)$.
(b) Find the amount of work done when $f(a) = 1$, $f(b) = 2$, $g(a) = 3$, $g(b) = 4$.
8. A force field is given in polar coordinates by the equation

$$F(r,\theta) = -4 \sin \theta \; i + 4 \sin \theta j.$$

Compute the work done in moving a particle from the point $(1, 0)$ to the origin along the spiral whose polar equation is $r = e^{-\theta}$.
9. A radial or "central" force field F in the plane can be written in the form $F(x, y) = f(r)r$, where $r = xi + yj$ and $r = \|r\|$. Show that such a force field is conservative.
10. Find the work done by force $F(x, y) = (3y^2 + 2)i + 16xj$. in moving a particle from $(-1, 0)$ to $(1, 0)$ along the upper half of the ellipse $b^2x^2 + y^2 = b^2$. Which ellipse (that is, which value of b) makes the work a minimum?

10.14 The first fundamental theorem of calculus for line integrals

Section 10.11 extended the second fundamental theorem of calculus to line integrals. This section extends the first fundamental theorem. We recall that the first fundamental theorem states that every indefinite integral of a continuous function f has a derivative equal to f. That is, if

$$\varphi(x) = \int_a^x f(t)\, dt\,,$$

then at the points of continuity of f we have

$$\varphi'(x) = f(x).$$

To extend this theorem to line integrals we begin with a vector field $\boldsymbol{f}$, continuous on an open connected set S, and integrate it along a piecewise smooth curve C from a fixed point a in S to an arbitrary point x. Then we let φ denote the scalar field defined by the line integral

$$\varphi(\boldsymbol{x}) = \int_a^{\boldsymbol{x}} \boldsymbol{f} \cdot d\boldsymbol{\alpha}\,,$$

where $\boldsymbol{\alpha}$ describes C. Since S is connected, each point $\boldsymbol{x}$ in S can be reached by such a curve. For this definition of $\varphi(\boldsymbol{x})$ to be unambiguous, we need to know that the integral depends only on $\boldsymbol{x}$ and not on the particular path used to join a to x. Therefore, it is natural to require the line integral of f to be independent of the path in S. Under these conditions, the extension of the first fundamental theorem takes the following form:

THEOREM 10.4. FIRST FUNDAMENTAL THEOREM FOR LINE INTEGRALS. *Let $\boldsymbol{f}$ be a vector field that is continuous on an open connected set S in $\mathbf{R}^n$, and assume that the line integral of f is independent of the path in S. Let a be a fixed point of S and define a scalar field φ on S by the equation*

$$\varphi(\boldsymbol{x}) = \int_a^{\boldsymbol{x}} \boldsymbol{f} \cdot d\boldsymbol{\alpha}\,,$$

where $\boldsymbol{\alpha}$ is any piecewise smooth path in S joining a to x. Then the gradient of φ exists and is equal to $\boldsymbol{f}$; that is,

$$\nabla\varphi(\boldsymbol{x}) = f(x) \qquad \textit{for every } \boldsymbol{x} \textit{ in } S.$$

Proof, We shall prove that the partial derivative $D_k\varphi(\boldsymbol{x})$ exists and is equal to $f_k(\boldsymbol{x})$, the kth component of $f(x)$, for each $k = 1, 2, \ldots, n$ and each $\boldsymbol{x}$ in S.

Let $B(\boldsymbol{x}; r)$ be an n-ball with center at $\boldsymbol{x}$ and radius r lying in S. If $\boldsymbol{y}$ is a unit vector, the point $\boldsymbol{x} + h\boldsymbol{y}$ also lies in S for every real h satisfying $0 < |h| < r$, and we can form the difference quotient

$$\frac{\varphi(\boldsymbol{x} + h\boldsymbol{y}) - \varphi(\boldsymbol{x})}{h}\,.$$

Because of the additive property of line integrals, the numerator of this quotient can be written as

$$\varphi(\boldsymbol{x} + h\boldsymbol{y}) - \varphi(\boldsymbol{x}) = \int_{\boldsymbol{x}}^{\boldsymbol{x}+h\boldsymbol{y}} \boldsymbol{f} \cdot d\alpha,$$

and the path joining $\boldsymbol{x}$ to $\boldsymbol{x} + h\boldsymbol{y}$ can be any piecewise smooth path lying in S. In particular, we can use the line segment described by

$$a(t) = \boldsymbol{x} + thy, \qquad \text{where } 0 \le t \le 1.$$

Since $a'(t) = \boldsymbol{hy}$, the difference quotient becomes

$$\textbf{(10.4)} \qquad \frac{\varphi(\boldsymbol{x} + h\boldsymbol{y}) - \varphi(\boldsymbol{x})}{h} = \int_0^1 \boldsymbol{f}(\boldsymbol{x} + th\boldsymbol{y}) \cdot \boldsymbol{y}\, dt\,.$$

Now we take y = $\boldsymbol{e}_k$, the kth unit coordinate vector, and note that the integrand becomes $\boldsymbol{f}(\boldsymbol{x} + thy) \,.\, y = f_k(\boldsymbol{x} + the,)$. Then we make the change of variable $u = ht$, $du = h\, dt$, and we write (10.4) in the form

$$(10.5) \qquad \frac{\varphi(\boldsymbol{x} + h\boldsymbol{e}_k) - \varphi(\boldsymbol{x})}{h} = \frac{1}{h}\int_0^h f_k(\boldsymbol{x} + u\boldsymbol{e}_k)\, du = \frac{g(h) - g(0)}{h},$$

where g is the function defined on the open interval $(-r, r)$ by the equation

$$g(t) = \int_0^t f_k(\boldsymbol{x} + u\boldsymbol{e}_k)\, du\,.$$

Since each component f_k is continuous on S, the first fundamental theorem for ordinary integrals tells us that $g'(t)$ exists for each t in $(-r, r)$ and that

$$g'(t) = f_k(\boldsymbol{x} + t\boldsymbol{e}_k)\,.$$

In particular, $g'(0) = f_k(\boldsymbol{x})$. Therefore, if we let $h \to 0$ in (10.5) we find that

$$\lim_{h\to 0} \frac{\varphi(\boldsymbol{x} + h\boldsymbol{e}_k) - \varphi(\boldsymbol{x})}{h} = \lim_{h\to 0} \frac{g(h) - g(0)}{h} = g'(0) = f_k(\boldsymbol{x})\,.$$

This proves that the partial derivative $D_k\varphi(\boldsymbol{x})$ exists and equals $f_k(\boldsymbol{x})$, as asserted.

10.15 Necessary and sufficient conditions for a vector field to be a gradient

The first and second fundamental theorems for line integrals together tell us that a necessary and sufficient condition for a continuous vector field to be a gradient on an open connected set is for its line integral between any two points to be independent of the path. We shall prove now that this condition is equivalent to the statement that the line integral is zero around every piecewise smooth *closed* path. All these conditions are summarized in the following theorem.

THEOREM 10.5. NECESSARY AND SUFFICIENT CONDITIONS FOR A VECTOR FIELD TO BE A GRADIENT. *Let $\boldsymbol{f}$ be a vector field continuous on an open connected set S in $\mathbf{R}^n$. Then the following three statements are equivalent.*

(a) *$\boldsymbol{f}$ is the gradient of somepotentialfinction in S.*

(b) *The line integral off is independent of the path in S.*

(c) *The line integral off is zero around every piecewise smooth closedpath in S.*

Proof. We shall prove that (b) implies (a), (a) implies (c), and (c) implies (b). Statement (b) implies (a) because of the first fundamental theorem. The second fundamental theorem shows that (a) implies (c).

To complete the proof we show that (c) implies (b). Assume (c) holds and let C_1 and C_2 be any two piecewise smooth curves in S with the same end points. Let C_1 be the graph of a function $\boldsymbol{\alpha}$ defined on an interval $[a, b]$, and let C_2 be the graph of a function $\boldsymbol{\beta}$ defined on $[c, d]$.

Define a new function y as follows:

$$\boldsymbol{\gamma}(t) = \begin{cases} \boldsymbol{\alpha}(t) & \text{if } a \le t \le b, \\ \boldsymbol{\beta}(b + d - t) & \text{if } b \le t \le b + d - c. \end{cases}$$

Then y describes a closed curve C such that

$$\oint_C \boldsymbol{f} \cdot d\boldsymbol{\gamma} = \int_{C_1} \boldsymbol{f} \cdot d\boldsymbol{\alpha} - \int_{C_2} \boldsymbol{f} \cdot d\boldsymbol{\beta}.$$

Since $\oint_C \boldsymbol{f} \cdot dy = 0$ because of (c), we have $\int_{C_1} \boldsymbol{f} \cdot \boldsymbol{da} = \int_{C_2} \boldsymbol{f} \cdot d\boldsymbol{\beta}$, so the integral off is independent of the path. This proves (b). Thus, (a), (b), and (c) are equivalent.

Note: If $\oint_C \boldsymbol{f} \neq 0$ for a particular closed curve C, then $\boldsymbol{f}$ is *not* a gradient. However, if a line integral $\oint_C \boldsymbol{f}$ is zero for a particular closed curve C or even for infinitely many closed curves, it does not necessarily follow that $\boldsymbol{f}$ *is* a gradient. For example, the reader can easily verify that the line integral of the vector field $\boldsymbol{f}(x, y) = x\boldsymbol{i} + xy\boldsymbol{j}$ is zero for every circle C with center at the origin. Nevertheless, this particular vector field is not a gradient.

10.16 Necessary conditions for a vector field to be a gradient

The first fundamental theorem can be used to determine whether or not a given vector field is a gradient on an open connected set S. If the line integral off is independent of the path in S, we simply define a scalar field φ by integrating $\boldsymbol{f}$ from some fixed point to an arbitrary point $\boldsymbol{x}$ in S along a convenient path in S. Then we compute the partial derivatives of φ and compare $D_k\varphi$ with f_k, the kth component off. If $D_k\varphi(\boldsymbol{x}) = f_k(\boldsymbol{x})$ for every $\boldsymbol{x}$ in S and every k, then $\boldsymbol{f}$ is a gradient on S and φ is a potential. If $D_k\varphi(\boldsymbol{x}) \neq f_k(\boldsymbol{x})$ for some k and some x, then $\boldsymbol{f}$ is *not* a gradient on S.

The next theorem gives another test for determining when a vector field $\boldsymbol{f}$ is *not* a gradient. This test is especially useful in practice because it does not require any integration.

THEOREM 10.6. NECESSARY CONDITIONS FOR A VECTOR FIELD TO BE A GRADIENT. *Let $\boldsymbol{f} = (f_1, \ldots, f_n)$ be a continuously differentiable vector field on an open set S in* **R”**. *Iff $\boldsymbol{f}$ is a gradient on S, then the partial derivatives of the components off are related by the equations*

(10.6) $$D_i f_j(\boldsymbol{x}) = D_j f_i(\boldsymbol{x})$$

for $i, j = 1, 2, \ldots, n$ and every $\boldsymbol{x}$ in S.

Proof. If f is a gradient, then $f = \nabla\varphi$ for some potential function φ. This means that

$$f_j = D_j\varphi$$

for each $j = 1, 2, \ldots, n$. Differentiating both members of this equation with respect to x_i we find

$$D_i f_j = D_i D_j \varphi .$$

Similarly, we have

$$D_j f_i = D_j D_i \varphi .$$

Since the partial derivatives $D_i f_j$ and $D_j f_i$ are continuous on S, the two mixed partial derivatives $D_i D_j \varphi$ and $D_j D_i \varphi$ must be equal on S. This proves (10.6).

EXAMPLE 1. Determine whether or not the vector field

$$f(x, y) = 3x^2 y\boldsymbol{i} + x^3 y\boldsymbol{j}$$

is a gradient on any open subset of $\mathbf{R}^2$.

Solution. Here we have

$$f_1(x, y) = 3x^2 y, \qquad f_2(x, y) = x^3 y.$$

The partial derivatives $D_2 f_1$ and $D_1 f_2$ are given by

$$D_2 f_1(x, y) = 3x^2, \qquad D_1 f_2(x, y) = 3x^2 y.$$

Since $D_2 f_1(x, y) \neq D_1 f_2(x, y)$ except when $x = 0$ or $y = 1$, this vector field is not a gradient on any open subset of $\mathbf{R}^2$.

The next example shows that the conditions of Theorem 10.6 are not always sufficient for a vector field to be a gradient.

EXAMPLE 2. Let S be the set of all $(x, y) \neq (0, 0)$ in $\mathbf{R}^2$, and let f be the vector field defined on S by the equation

$$f(x, y) = \frac{-y}{x^2 + y^2}\boldsymbol{i} + \frac{x}{x^2 + y^2}\boldsymbol{j}.$$

Show that $D_1 f_2 = D_2 f_1$ everywhere on S but that, nevertheless, f is not a gradient on S.

Solution. The reader can easily verify that $D_1 f_2(x, y) = D_2 f_1(x, y)$ for all (x, y) in S. (See Exercise 17 in Section 10.18.)

To prove that f is not a gradient on S we compute the line integral of f around the unit circle given by

$$\boldsymbol{\alpha}(t) = \cos t\,\boldsymbol{i} + \sin t\,\boldsymbol{j}, \qquad 0 \leq t \leq 2\pi .$$

We obtain

$$\oint \boldsymbol{f} \cdot d\mathbf{a} = \int_0^{2\pi} \boldsymbol{f}[\boldsymbol{\alpha}(t)] \cdot a'(t)\, dt = \int_0^{2\pi} (\sin^2 t + \cos^2 t)\, dt = 2\pi .$$

Since the line integral around this closed path is not zero, $\boldsymbol{f}$ is not a gradient on S. Further properties of this vector field are discussed in Exercise 18 of Section 10.18.

At the end of this chapter we shall prove that the necessary conditions of Theorem 10.6 are also sufficient if they are satisfied on an open *convex* set. (See Theorem 10.9.)

10.17 Special methods for constructing potential functions

The first fundamental theorem for line integrals also gives us a method for constructing potential functions. If $\boldsymbol{f}$ is a continuous gradient on an open connected set S, the line integral off is independent of the path in S. Therefore we can find a potential φ simply by integrating $\boldsymbol{f}$ from some fixed point a to an arbitrary point $\boldsymbol{x}$ in S, using any piecewise smooth path lying in S. The scalar field so obtained depends on the choice of the initial point a. If we start from another initial point, say $\boldsymbol{b}$, we obtain a new potential ψ. But, because of the additive property of line integrals, φ and ψ can differ only by a constant, this constant being the integral off from a to $\boldsymbol{b}$.

The following examples illustrate the use of different choices of the path of integration.

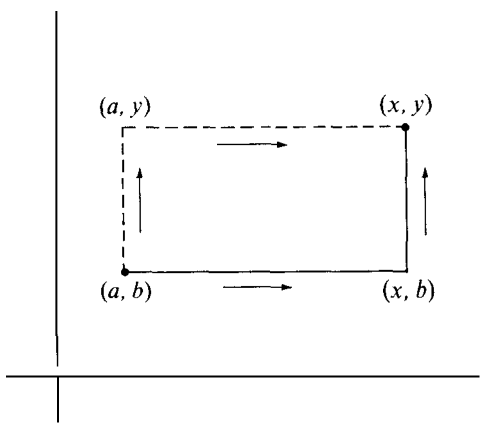

FIGURE 10.5 Two polygonal paths from (a, b) to (x, y).

EXAMPLE 1. ***Construction of a potential on an open rectangle.*** If $\boldsymbol{f}$ is a continuous gradient on an open rectangle in $\mathbf{R}^n$, a potential φ can be constructed by integrating from a fixed point to an arbitrary point along a set of line segments parallel to the coordinate axes. A two-dimensional example is shown in Figure 10.5. We can integrate first from ***(a, b)*** to (x, b) along a horizontal segment, then from (x, b) to (x, y) along a vertical segment. Along the horizontal segment we use the parametric representation

$$a(t) = ti + bj, \qquad a \le t \le x,$$

and along the vertical segment we use the representation

$$a(t) = xi + tj, \qquad b \le t \le y .$$

If $\boldsymbol{f}(x, y) = f_1(x, y)\boldsymbol{i} + f_2(x, y)\boldsymbol{j}$, the resulting formula for a potential $\varphi(x, y)$ is

$$\varphi(x, y) = \int_a^x f_1(t, b)\, dt + \int_b^y f_2(x, t)\, dt . \tag{10.7}$$

We could also integrate first from (a, b) to (a, y) along a vertical segment and then from (a, y) to (x, y) along a horizontal segment as indicated by the dotted lines in Figure 10.5. This gives us another formula for $\varphi(x, y)$,

$$\varphi(x, y) = \int_b^y f_2(a, t)\, dt + \int_a^x f_1(t, y)\, dt . \tag{10.8}$$

Both formulas (10.7) and (10.8) give the same value for $\varphi(x, y)$ because the line integral of a gradient is independent of the path.

EXAMPLE 2. *Construction of a potential function by the use of indefinite integrals.* The use of indefinite integrals often helps to simplify the calculation of potential functions. For example, suppose a three-dimensional vector field $\boldsymbol{f} = (f_1, f_2, f_3)$ is the gradient of a potential function φ on an open set S in $\mathbf{R}^3$. Then we have

$$\frac{\partial \varphi}{\partial x} = f_1, \qquad \frac{\partial \varphi}{\partial y} = f_2, \qquad \frac{\partial \varphi}{\partial z} = f_3$$

everywhere on S. Using indefinite integrals and integrating the first of these equations with respect to x (holding y and z constant) we find

$$\varphi(x, y, z) = \int f_1(x, y, z)\, dx + A(y, z) ,$$

where $A(y, z)$ is a "constant of integration" to be determined. Similarly, if we integrate the equation $\partial\varphi/\partial y = f_2$ with respect to y and the equation $\partial\varphi/\partial z = f_3$ with respect to z we obtain the further relations

$$\varphi(x, y, z) = \int f_2(x, y, z)\, dy + B(x, z)$$

and

$$\varphi(x, y, z) = \int f_3(x, y, z)\, dz + C(x, y) ,$$

where $B(x, z)$ and $C(x, y)$ are functions to be determined. Finding φ means finding three functions $A(y, z)$, $B(x, z)$, and $C(x, y)$ such that all three equations for $\varphi(x, y, z)$ agree in their right-hand members. In many cases this can be done by inspection, as illustrated by the following example.

EXAMPLE 3. Find a potential function φ for the vector field defined on $\mathbf{R}^3$ by the equation

$$\boldsymbol{f}(x, y, z) = (2xyz + z^2 - 2y^2 + 1)\boldsymbol{i} + (x^2z - 4xy)\boldsymbol{j} + (x^2y + 2xz - 2)\boldsymbol{k}.$$

Solution. Without knowing in advance whether or not $\boldsymbol{f}$ has a potential function φ, we try to construct a potential as outlined in Example 2, assuming that a potential φ exists.

Integrating the component f_1 with respect to x we find

$$\varphi(x, y, z) = \int (2xyz + z^2 - 2y^2 + 1)\, dx + A(y, z) = x^2yz + xz^2 - 2xy^2 + x + A(y, z).$$

Integrating f_2 with respect to y, and then f_3 with respect to z, we find

$$\varphi(x, y, z) = \int (x^2z - 4xy)\, dy + B(x, z) = x^2yz - 2xy^2 + B(x, z),$$

$$\varphi(x, y, z) = \int (x^2y + 2xz - 2)\, dz + C(x, y) = x^2yz + xz^2 - 2z + C(x, y).$$

By inspection we see that the choices $A(y, z) = -2z$, $B(x, z) = xz^2 + x - 2z$, and $C(x, y) = x - 2xy^2$ will make all three equations agree; hence the function φ given by the equation

$$\varphi(x, y, z) = x^2yz + xz^2 - 2xy^2 + x - 2z$$

is a potential for $\boldsymbol{f}$ on $\mathbf{R}^3$.

EXAMPLE ***4. Construction of a potential on a convex set.*** A set S in $\mathbf{R}^n$ is called ***convex*** if every pair of points in S can be joined by a line segment, all of whose points lie in S. An example is shown in Figure 10.6. Every open convex set is connected.

If $\boldsymbol{f}$ is a continuous gradient on an open convex set, then a potential φ can be constructed by integrating $\boldsymbol{f}$ from a fixed point $\boldsymbol{a}$ in S to an arbitrary point $\boldsymbol{x}$ along the line segment joining $\boldsymbol{a}$ to $\boldsymbol{x}$. The line segment can be parametrized by the function

$$\boldsymbol{\alpha}(t) = \boldsymbol{a} + t(\boldsymbol{x} - \boldsymbol{a}), \qquad \text{where } 0 \le t \le 1.$$

This gives us $\boldsymbol{\alpha}'(t) = \boldsymbol{x} - \boldsymbol{a}$, so the corresponding potential is given by the integral

$$\varphi(\boldsymbol{x}) = \int_0^1 \boldsymbol{f}(\boldsymbol{a} + t(\boldsymbol{x} - \boldsymbol{a})) \cdot (\boldsymbol{x} - \boldsymbol{a})\, dt. \tag{10.9}$$

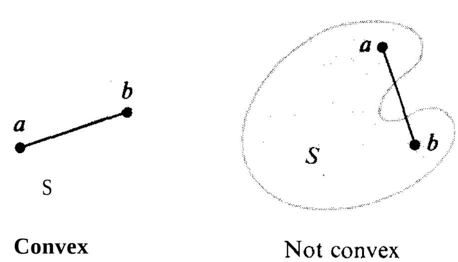

FIGURE 10.6 In a convex set S, the segment joining $\boldsymbol{a}$ and $\boldsymbol{b}$ is in S for all points $\boldsymbol{a}$ and $\boldsymbol{b}$ in S.

If S contains the origin we can take a = 0 and write (10.9) more simply as

$$\varphi(\boldsymbol{x}) = \int_0^1 \boldsymbol{f}(t\boldsymbol{x}) \cdot \boldsymbol{x}\, dt. \tag{10.10}$$

10.18 Exercises

In each of Exercises 1 through 12, a vector field $\boldsymbol{f}$ is defined by the formulas given. In each case determine whether or not $\boldsymbol{f}$ is the gradient of a scalar field. When $\boldsymbol{f}$ is a gradient, find a corresponding potential function φ.

1. $\boldsymbol{f}(x, y) = x\boldsymbol{i} + y\boldsymbol{j}$.
2. $\boldsymbol{f}(x, y) = 3x^2y\boldsymbol{i} + x^3\boldsymbol{j}$.
3. $\boldsymbol{f}(x, y) = (2xe^y + y)\boldsymbol{i} + (x^2e^y + x - 2y)\boldsymbol{j}$.
4. $\boldsymbol{f}(x, y) = (\sin y - y \sin x + x)\boldsymbol{i} + (\cos x + x\cos y + y)\boldsymbol{j}$.
5. $\boldsymbol{f}(x, y) = [\sin (xy) + xy \cos (xy)]\boldsymbol{i} + x^2 \cos (xy)\boldsymbol{j}$.
6. $\boldsymbol{f}(x, y, z) = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$.
7. $\boldsymbol{f}(x, y, z) = (x + z)\boldsymbol{i} - (y + z)\boldsymbol{j} + (x - y)\boldsymbol{k}$.
8. $\boldsymbol{f}(x, y, z) = 2xy^3\boldsymbol{i} + x^2z^3\boldsymbol{j} + 3x^2yz^2\boldsymbol{k}$.
9. $\boldsymbol{f}(x, y, z) = 3y^4z^2\boldsymbol{i} + 4x^3z^2\boldsymbol{j} - 3x^2y^2\boldsymbol{k}$.
10. $\boldsymbol{f}(x, y, z) = (2x^2 + 8xy^2)\boldsymbol{i} + (3x^3y - 3xy)\boldsymbol{j} - (4y^2z^2 + 2x^3z)\boldsymbol{k}$.
11. $\boldsymbol{f}(x, y, z) = (y^2 \cos x + z^3)\boldsymbol{i} - (4 - 2y \sin x)\boldsymbol{j} + (3xz^2 + 2)\boldsymbol{k}$.
12. $\boldsymbol{f}(x, y, z) = (4xy - 3x^2z^2 + 1)\boldsymbol{i} + 2(x^2 + 1)\boldsymbol{j} - (2x^3z + 3z^2)\boldsymbol{k}$.

13. A fluid flows in the xy-plane, each particle moving directly away from the origin. If a particle is at a distance r from the origin its speed is ar^n, where a and n are constants.
(a) Determine those values of a and n for which the velocity vector field is the gradient of some scalar field.
(b) Find a potential function of the velocity whenever the velocity is a gradient. The case $n = -1$ should be treated separately.
14. If both φ and ψ are potential functions for a continuous vector field $\boldsymbol{f}$ on an open connected set S in $\mathbf{R}^n$, prove that $\varphi - \psi$ is constant on S.
15. Let S be the set of all $\boldsymbol{x} \neq \boldsymbol{0}$ in $\mathbf{R}^n$. Let $r = \|\boldsymbol{x}\|$, and let $\boldsymbol{f}$ be the vector field defined on S by the equation
$$\boldsymbol{f}(\boldsymbol{x}) = r^p\boldsymbol{x},$$
where p is a real constant. Find a potential function for $\boldsymbol{f}$ on S. The case $p = -2$ should be treated separately.
16. Solve Exercise 15 for the vector field defined by the equation
$$\boldsymbol{f}(\boldsymbol{x}) = \frac{g'(r)}{r}\boldsymbol{x},$$
where g is a real function with a continuous derivative everywhere on $\mathbf{R}^1$.

The following exercises deal with the vector field $\boldsymbol{f}$ defined on the set S of all points $(x, y) \neq (0, 0)$ in $\mathbf{R}^2$ by the equation

$$\boldsymbol{f}(x, y) = -\frac{y}{x^2 + y^2}\boldsymbol{i} + \frac{x}{x^2 + y^2}\boldsymbol{j}.$$

In Example 2 of Section 10.16 we showed that f is not a gradient on S, even though $D_1 f_2 = D_2 f_1$ everywhere on S.

17. Verify that for every point (x, y) in S we have

$$D_1 f_2(x, y) = D_2 f_1(x, y) = \frac{y^2 - x^2}{(x^2 + y^2)^2}.$$

18. This exercise shows that f *is* a gradient on the set

$$T = \mathbf{R}^2 - \{(x, y) \mid y = 0, \quad x \le 0\},$$

consisting of all points-in the xy-plane except those on the nonpositive x-axis.
(a) If $(x, y) \in T$, express x and y in polar coordinates,

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

where $r > 0$ and $-\pi < \theta < \pi$. Prove that θ is given by the formulas

$$\theta = \begin{cases} \arctan\dfrac{y}{x} & \text{if } x > 0, \\ \dfrac{\pi}{2} & \text{if } x = 0, \\ \arctan\dfrac{y}{x} + \pi & \text{if } x < 0. \end{cases}$$

[Recall the definition of the arc tangent function: For any real t, $\arctan t$ is the unique real number φ which satisfies the two conditions $\tan\varphi = t$ and $-\pi/2 < \varphi < \pi/2$.]
(b) Deduce that

$$\frac{\partial\theta}{\partial x} = -\frac{y}{x^2 + y^2}, \qquad \frac{\partial\theta}{\partial y} = \frac{x}{x^2 + y^2}$$

for all (x, y) in T. This proves that θ is a potential function for f on the set T.
This exercise illustrates that the existence of a potential function depends not only on the vector field f but also on the set in which the vector field is defined.

10.19 Applications to exact differential equations of first order

Some differential equations of the first order can be solved with the aid of potential functions. Suppose we have a first-order differential equation of the form

$$y' = f(x, y).$$

If we multiply both sides by a nonvanishing factor $Q(x, y)$ we transform this equation to the form $Q(x, y)y' - f(x, y)Q(x, y) = 0$. If we then write $P(x, y)$ for $-f(x, y)Q(x, y)$ and use the Leibniz notation for derivatives, writing dy/dx for y', the differential equation takes the form

$$P(x, y)\, dx + Q(x, y)\, dy = 0. \tag{10.11}$$

We assume that P and Q are continuous on some open connected set S in the plane. With each such differential equation we can associate a vector field $\boldsymbol{V}$, where

$$\boldsymbol{V}(x, y) = P(x, y)\boldsymbol{i} + Q(x, y)\boldsymbol{j}.$$

The components P and Q are the coefficients of dx and dy in Equation (10.11). The differential equation in (10.11) is said to be ***exact*** in S if the vector field $\boldsymbol{V}$ is the gradient of a potential; that is, if $\boldsymbol{V}(x, y) = \nabla\varphi(x, y)$ for each point (x, y) in S, where φ is some scalar field. When such a φ exists we have $\partial\varphi/\partial x = P$ and $\partial\varphi/\partial y = Q$, and the differential equation in (10.11) becomes

$$\frac{\partial\varphi}{\partial x}\,dx + \frac{\partial\varphi}{\partial y}\,dy = 0.$$

We shall prove now that each solution of this differential equation satisfies the relation $\varphi(x, y) = C$, where C is a constant. More precisely, assume there is a solution Y of the differential equation (10.11) defined on an open interval (a, b) such that the point $(x, Y(x))$ is in S for each x in (a, b). We shall prove that

$$\varphi[x, Y(x)] = C$$

for some constant C. For this purpose we introduce the composite function g defined on (a, b) by the equation

$$g(x) = \varphi[x, Y(x)].$$

By the chain rule, the derivative of g is given by

$$(10.12)\qquad g'(x) = D_1\varphi[x, Y(x)] + D_2\varphi[x, Y(x)]\,Y'(x) = P(x, y) + Q(x, y)y',$$

where $y = Y(x)$ and $y' = Y'(x)$. If y satisfies (10.11), then $P(x, y) + Q(x, y)y' = 0$, so $g'(x) = 0$ for each x in (a, b) and, therefore, g is constant on (a, b). This proves that every solution y satisfies the equation $\varphi(x, y) = C$.

Now we may turn this argument around to find a solution of the differential equation. Suppose the equation

$$(10.13)\qquad \varphi(x, y) = C$$

defines y as a differentiable function of x, say $y = Y(x)$ for x in an interval (a, b), and let $g(x) = \varphi[x, Y(x)]$. Equation (10.13) implies that g is constant on (a, b). Hence, by (10.12), $P(x, y) + Q(x, y)y' = 0$, so y is a solution. Therefore, we have proved the following theorem :

THEOREM 10.7. *Assume that the differential equation*

$$(10.14)\qquad P(x, y)\,dx + Q(x, y)\,dy = 0$$

is exact in an open connected set S, and let φ be a scalar field satisfying

$$\frac{\partial \varphi}{\partial x} = P \quad \text{and} \quad \frac{\partial \varphi}{\partial y} = Q$$

everywhere in S. Then every solution $y = Y(x)$ of (10.14) whose graph lies in S satisfies the equation $\varphi[x, Y(x)] = C$ for some constant C. Conversely, if the equation

$$\varphi(x, y) = C$$

defines y implicitly as a differentiable function of x, then this function is a solution of the differential equation (10.14).

The foregoing theorem provides a straightforward method for solving exact differential equations of the first order. We simply construct a potential function φ and then write the equation. $\varphi(x, y) = C$, where C is a constant. Whenever this equation defines y implicitly as a function of x, the corresponding y satisfies (10.14). Therefore we can use Equation (10.13) as a representation of a one-parameter family of integral curves. Of course, the only admissible values of C are those for which $\varphi(x_0, y_0) = C$ for some (x_0, y_0) in S.

EXAMPLE 1. Consider the differential equation

$$\frac{dy}{dx} = -\frac{3x^2 + 6xy^2}{6x^2y + 4y^3}.$$

Clearing the fractions we may write the equation as

$$(3x^2 + 6xy^2)\, dx + (6x^2y + 4y^3)\, dy = 0.$$

This is now a special case of (10.14) with $P(x, y) = 3x^2 + 6xy^2$ and $Q(x, y) = 6x^2y + 4y^3$. Integrating P with respect to x and Q with respect toy and comparing results, we find that a potential function φ is given by the formula

$$\varphi(x, y) = x^3 + 3x^2y^2 + y^4.$$

By Theorem 10.7, each solution of the differential equation satisfies

$$x^3 + 3x^2y^2 + y^4 = C$$

for some C. This provides an implicit representation of a family of integral curves. In this particular case the equation is quadratic in y^2 and can be solved to give an explicit formula for y in terms of x and C.

EXAMPLE 2. Consider the first-order differential equation

$$y\, dx + 2x\, dy = 0. \tag{10.15}$$

Here $P(x, y) = y$ and $Q(x, y) = 2x$. Since $\partial P/\partial y = 1$ and $\partial Q/\partial x = 2$, this differential equation is not exact. However, if we multiply both sides by y we obtain an equation that *is* exact:

$$y^2\,dx + 2xy\,dy = 0. \tag{10.16}$$

A potential of the vector field $y^2\boldsymbol{i} + 2xy\boldsymbol{j}$ is $\varphi(x, y) = xy^2$, and every solution of (10.16) satisfies the relation $xy^2 = C$. This relation also represents a family of integral curves for Equation (10.15).

The multiplier y which converted (10.15) into an exact equation is called an ***integrating factor.*** In general, if multiplication of a first-order linear equation by a nonzero factor $\mu(x, y)$ results in an exact equation, the multiplier $\mu(x, y)$ is called an integrating factor of the original equation. A differential equation may have more than one integrating factor. For example, $\mu(x, y) = 2xy^3$ is another integrating factor of (10.15). Some special differential equations for which integrating factors can easily be found are discussed in the following set of exercises.

10.20 Exercises

Show that the differential equations in Exercises 1 through 5 are exact, and in each case find a one-parameter family of integral curves.

1. $(x + 2y)\,dx + (2x + y)\,dy = 0.$
2. $2xy\,dx + x^2\,dy = 0.$
3. $(x^2 - y)\,dx - (x + \sin^2 y)\,dy = 0.$
4. $4\sin x \sin 3y \cos x\,dx - 3\cos 3y \cos 2x\,dy = 0.$
5. $(3x^2y + 8xy^2)\,dx + (x^3 + 8x^2y + 12ye^y)\,dy = 0.$

6. Show that a linear first-order equation, $y' + P(x)y = Q(x)$, has the integrating factor $\mu(x) = e^{\int P(x)\,dx}$. Use this to solve the equation.
7. Let $\mu(x, y)$ be an integrating factor of the differential equation $P(x, y)\,dx + Q(x, y)\,dy = 0.$ Show that
$$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = Q\frac{\partial}{\partial x}\log|\mu| - P\frac{\partial}{\partial y}\log|\mu|.$$

   Use this equation to deduce the following rules for finding integrating factors:
   (a) If $(\partial P/\partial y - \partial Q/\partial x)/Q$ is a function of x alone, say $f(x)$, then $e^{\int f(x)\,dx}$ is an integrating factor.
   (b) If $(\partial Q/\partial x - \partial P/\partial y)/P$ is a function of y alone, say $g(y)$, then $e^{\int g(y)\,dy}$ is an integrating factor.
8. Use Exercise 7 to find integrating factors for the following equations, and determine a one-parameter family of integral curves.
   (a) $y\,dx - (2x + y)\,dy = 0.$
   (b) $(x^3 + y^3)\,dx - xy^2\,dy = 0.$
9. If $\partial P/\partial y - \partial Q/\partial x = f(x)Q(x, y) - g(y)P(x, y)$, show that $\exp\{\int f(x)\,dx + \int g(y)\,dy\}$ is an integrating factor of the differential equation $P(x, y)\,dx + Q(x, y)\,dy = 0$. Find such an integrating factor for each of the following equations and obtain a one-parameter family of integral curves.
   (a) $(2x^2y + y^2)\,dx + (2x^3 - xy)\,dy = 0.$
   (b) $(e^x \sec y - \tan y)\,dx + dy = 0.$

10. The following differential equations have an integrating factor in common. Find such an integrating factor and obtain a one-parameter family of integral curves for each equation.

$$(3y + 4xy^2)\,dx + (4x + 5x^2y)\,dy = 0,$$
$$(6y + x^2y^2)\,dx + (8x + x^3y)\,dy = 0.$$

10.21 Potential functions on convex sets

In Theorem 10.6 we proved that the conditions

$$D_i f_j(\boldsymbol{x}) = D_j f_i(\boldsymbol{x})$$

are *necessary* for a continuously differentiable vector field $\boldsymbol{f} = (f_1, \ldots, f_n)$ to be a gradient on an open set S in $\mathbf{R}^n$. We then showed, by an example, that these conditions are not always sufficient. In this section we prove that the conditions are sufficient whenever the set S is convex. The proof will make use of the following theorem concerning differentiation under the integral sign.

THEOREM 10.8. *Let S be a closed interval in $\mathbf{R}^n$ with nonempty interior and let $J = [a, b]$ be a closed interval in $\mathbf{R}^1$. Let J_{n+1} be the closed interval $S \times J$ in $\mathbf{R}^{n+1}$. Write each point in J_{n+1} as $(\boldsymbol{x}, t)$, where $\boldsymbol{x} \in S$ and $t \in J$,*

$$(\boldsymbol{x}, t) = (x_1, \ldots, x_n, t).$$

Assume that ψ is a scalar field defined on J_{n+1} such that the partial derivative $D_k\psi$ is continuous on J_{n+1}, where k is one of $1, 2, \ldots, n$. Define a scalar field φ on S by the equation

$$\varphi(\boldsymbol{x}) = \int_a^b \psi(\boldsymbol{x}, t)\,dt.$$

Then the partial derivative $D_k\varphi$ exists at each interior point of S and is given by the formula

$$D_k\varphi(\boldsymbol{x}) = \int_a^b D_k\psi(\boldsymbol{x}, t)\,dt.$$

In other words, we have

$$D_k \int_a^b \psi(\boldsymbol{x}, t)\,dt = \int_a^b D_k\psi(\boldsymbol{x}, t)\,dt.$$

Note: This theorem is usually described by saying that we can differentiate under the integral sign.

Proof. Choose any $\boldsymbol{x}$ in the interior of S. Since int S is open, there is an $r > 0$ such that $\boldsymbol{x} + h\boldsymbol{e}_k \in$ int S for all h satisfying $0 < |h| < r$. Here $\boldsymbol{e}_k$ is the kth unit coordinate vector in $\mathbf{R}^n$. For such h we have

$$\varphi(\boldsymbol{x} + h\boldsymbol{e}_k) - \varphi(\boldsymbol{x}) = \int_a^b \{\psi(\boldsymbol{x} + h\boldsymbol{e}_k, t) - \psi(\boldsymbol{x}, t)\}\,dt. \tag{10.17}$$