

The right side of (143) should be regarded as just an abbreviation for the last integral in (142). The point is that  $\mathbf{F}$  is defined on the range of  $\gamma$ , but  $\mathbf{t}$  is defined on  $[0, 1]$ ; thus  $\mathbf{F} \cdot \mathbf{t}$  has to be properly interpreted. Of course, when  $\gamma$  is one-to-one, then  $\mathbf{t}(u)$  can be replaced by  $\mathbf{t}(\gamma(u))$ , and this difficulty disappears.

**10.49 Integrals of 2-forms in  $R^3$**  Let  $\Phi$  be a 2-surface in an open set  $E \subset R^3$ , of class  $\mathcal{C}'$ , with parameter domain  $D \subset R^2$ . Let  $\mathbf{F}$  be a vector field in  $E$ , and define  $\omega_{\mathbf{F}}$  by (125). As in the preceding section, we shall obtain a different representation of the integral of  $\omega_{\mathbf{F}}$  over  $\Phi$ .

By (35) and (129),

$$\begin{aligned}\int_{\Phi} \omega_{\mathbf{F}} &= \int_{\Phi} (F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy) \\ &= \int_D \left\{ (F_1 \circ \Phi) \frac{\partial(y, z)}{\partial(u, v)} + (F_2 \circ \Phi) \frac{\partial(z, x)}{\partial(u, v)} + (F_3 \circ \Phi) \frac{\partial(x, y)}{\partial(u, v)} \right\} du dv \\ &= \int_D \mathbf{F}(\Phi(u, v)) \cdot \mathbf{N}(u, v) du dv.\end{aligned}$$

Now let  $\mathbf{n} = \mathbf{n}(u, v)$  be the unit vector in the direction of  $\mathbf{N}(u, v)$ . [If  $\mathbf{N}(u, v) = \mathbf{0}$  for some  $(u, v) \in D$ , take  $\mathbf{n}(u, v) = \mathbf{e}_1$ .] Then  $\mathbf{N} = |\mathbf{N}|\mathbf{n}$ , and therefore the last integral becomes

$$\int_D \mathbf{F}(\Phi(u, v)) \cdot \mathbf{n}(u, v) |\mathbf{N}(u, v)| du dv.$$

By (131), we can finally write this in the form

$$(144) \quad \int_{\Phi} \omega_{\mathbf{F}} = \int_{\Phi} (\mathbf{F} \cdot \mathbf{n}) dA.$$

With regard to the meaning of  $\mathbf{F} \cdot \mathbf{n}$ , the remark made at the end of Sec. 10.48 applies here as well.

We can now state the original form of Stokes' theorem.

**10.50 Stokes' formula** *If  $\mathbf{F}$  is a vector field of class  $\mathcal{C}'$  in an open set  $E \subset R^3$ , and if  $\Phi$  is a 2-surface of class  $\mathcal{C}''$  in  $E$ , then*

$$(145) \quad \int_{\Phi} (\nabla \times \mathbf{F}) \cdot \mathbf{n} dA = \int_{\partial\Phi} (\mathbf{F} \cdot \mathbf{t}) ds.$$

**Proof** Put  $\mathbf{H} = \nabla \times \mathbf{F}$ . Then, as in the proof of Theorem 10.43, we have

$$(146) \quad \omega_{\mathbf{H}} = d\lambda_{\mathbf{F}}.$$

Hence

$$\begin{aligned}\int_{\Phi} (\nabla \times \mathbf{F}) \cdot \mathbf{n} dA &= \int_{\Phi} (\mathbf{H} \cdot \mathbf{n}) dA = \int_{\Phi} \omega_{\mathbf{H}} \\ &= \int_{\Phi} d\lambda_{\mathbf{F}} = \int_{\partial\Phi} \lambda_{\mathbf{F}} = \int_{\partial\Phi} (\mathbf{F} \cdot \mathbf{t}) ds.\end{aligned}$$

Here we used the definition of  $\mathbf{H}$ , then (144) with  $\mathbf{H}$  in place of  $\mathbf{F}$ , then (146), then—the main step—Theorem 10.33, and finally (143), extended in the obvious way from curves to 1-chains.

**10.51 The divergence theorem** *If  $\mathbf{F}$  is a vector field of class  $\mathcal{C}'$  in an open set  $E \subset R^3$ , and if  $\Omega$  is a closed subset of  $E$  with positively oriented boundary  $\partial\Omega$  (as described in Sec. 10.31) then*

$$(147) \quad \int_{\Omega} (\nabla \cdot \mathbf{F}) dV = \int_{\partial\Omega} (\mathbf{F} \cdot \mathbf{n}) dA.$$

**Proof** By (125),

$$d\omega_{\mathbf{F}} = (\nabla \cdot \mathbf{F}) dx \wedge dy \wedge dz = (\nabla \cdot \mathbf{F}) dV.$$

Hence

$$\int_{\Omega} (\nabla \cdot \mathbf{F}) dV = \int_{\Omega} d\omega_{\mathbf{F}} = \int_{\partial\Omega} \omega_{\mathbf{F}} = \int_{\partial\Omega} (\mathbf{F} \cdot \mathbf{n}) dA,$$

by Theorem 10.33, applied to the 2-form  $\omega_{\mathbf{F}}$ , and (144).

## EXERCISES

1. Let  $H$  be a compact convex set in  $R^k$ , with nonempty interior. Let  $f \in \mathcal{C}(H)$ , put  $f(\mathbf{x}) = 0$  in the complement of  $H$ , and define  $\int_H f$  as in Definition 10.3.

Prove that  $\int_H f$  is independent of the order in which the  $k$  integrations are carried out.

*Hint:* Approximate  $f$  by functions that are continuous on  $R^k$  and whose supports are in  $H$ , as was done in Example 10.4.

2. For  $i = 1, 2, 3, \dots$ , let  $\varphi_i \in \mathcal{C}(R^1)$  have support in  $(2^{-i}, 2^{1-i})$ , such that  $\int \varphi_i = 1$ .

Put

$$f(x, y) = \sum_{i=1}^{\infty} [\varphi_i(x) - \varphi_{i+1}(x)] \varphi_i(y)$$

Then  $f$  has compact support in  $R^2$ ,  $f$  is continuous except at  $(0, 0)$ , and

$$\int dy \int f(x, y) dx = 0 \quad \text{but} \quad \int dx \int f(x, y) dy = 1.$$

Observe that  $f$  is unbounded in every neighborhood of  $(0, 0)$ .

3. (a) If  $F$  is as in Theorem 10.7, put  $A = F'(0)$ ,  $F_1(x) = A^{-1}F(x)$ . Then  $F'_1(0) = I$ . Show that

$$F_1(x) = G_n \circ G_{n-1} \circ \cdots \circ G_1(x)$$

in some neighborhood of  $0$ , for certain primitive mappings  $G_1, \dots, G_n$ . This gives another version of Theorem 10.7:

$$F(x) = F'(0)G_n \circ G_{n-1} \circ \cdots \circ G_1(x).$$

(b) Prove that the mapping  $(x, y) \rightarrow (y, x)$  of  $R^2$  onto  $R^2$  is not the composition of any two primitive mappings, in any neighborhood of the origin. (This shows that the flips  $B_i$  cannot be omitted from the statement of Theorem 10.7.)

4. For  $(x, y) \in R^2$ , define

$$F(x, y) = (e^x \cos y - 1, e^x \sin y).$$

Prove that  $F = G_2 \circ G_1$ , where

$$G_1(x, y) = (e^x \cos y - 1, y)$$

$$G_2(u, v) = (u, (1+u)\tan v)$$

are primitive in some neighborhood of  $(0, 0)$ .

Compute the Jacobians of  $G_1$ ,  $G_2$ ,  $F$  at  $(0, 0)$ . Define

$$H_2(x, y) = (x, e^x \sin y)$$

and find

$$H_1(u, v) = (h(u, v), v)$$

so that  $F = H_1 \circ H_2$  is some neighborhood of  $(0, 0)$ .

5. Formulate and prove an analogue of Theorem 10.8, in which  $K$  is a compact subset of an arbitrary metric space. (Replace the functions  $\varphi_i$  that occur in the proof of Theorem 10.8 by functions of the type constructed in Exercise 22 of Chap. 4.)
6. Strengthen the conclusion of Theorem 10.8 by showing that the functions  $\psi_i$  can be made differentiable, and even infinitely differentiable. (Use Exercise 1 of Chap. 8 in the construction of the auxiliary functions  $\varphi_i$ .)
7. (a) Show that the simplex  $Q^k$  is the smallest convex subset of  $R^k$  that contains  $0, e_1, \dots, e_k$ .  
(b) Show that affine mappings take convex sets to convex sets.
8. Let  $H$  be the parallelogram in  $R^2$  whose vertices are  $(1, 1), (3, 2), (4, 5), (2, 4)$ . Find the affine map  $T$  which sends  $(0, 0)$  to  $(1, 1)$ ,  $(1, 0)$  to  $(3, 2)$ ,  $(0, 1)$  to  $(2, 4)$ . Show that  $J_T = 5$ . Use  $T$  to convert the integral

$$\alpha = \int_H e^{x-y} dx dy$$

to an integral over  $I^2$  and thus compute  $\alpha$ .

9. Define  $(x, y) = T(r, \theta)$  on the rectangle

$$0 \leq r \leq a, \quad 0 \leq \theta \leq 2\pi$$

by the equations

$$x = r \cos \theta, \quad y = r \sin \theta.$$

Show that  $T$  maps this rectangle onto the closed disc  $D$  with center at  $(0, 0)$  and radius  $a$ , that  $T$  is one-to-one in the interior of the rectangle, and that  $J_T(r, \theta) = r$ . If  $f \in C(D)$ , prove the formula for integration in polar coordinates:

$$\int_D f(x, y) dx dy = \int_0^a \int_0^{2\pi} f(T(r, \theta))r dr d\theta.$$

*Hint:* Let  $D_0$  be the interior of  $D$ , minus the interval from  $(0, 0)$  to  $(0, a)$ . As it stands, Theorem 10.9 applies to continuous functions  $f$  whose support lies in  $D_0$ . To remove this restriction, proceed as in Example 10.4.

10. Let  $a \rightarrow \infty$  in Exercise 9 and prove that

$$\int_{R^2} f(x, y) dx dy = \int_0^\infty \int_0^{2\pi} f(T(r, \theta))r dr d\theta,$$

for continuous functions  $f$  that decrease sufficiently rapidly as  $|x| + |y| \rightarrow \infty$ . (Find a more precise formulation.) Apply this to

$$f(x, y) = \exp(-x^2 - y^2)$$

to derive formula (101) of Chap. 8.

11. Define  $(u, v) = T(s, t)$  on the strip

$$0 < s < \infty, \quad 0 < t < 1$$

by setting  $u = s - st$ ,  $v = st$ . Show that  $T$  is a 1-1 mapping of the strip onto the positive quadrant  $Q$  in  $R^2$ . Show that  $J_T(s, t) = s$ .

For  $x > 0$ ,  $y > 0$ , integrate

$$u^{x-1} e^{-u} v^{y-1} e^{-v}$$

over  $Q$ , use Theorem 10.9 to convert the integral to one over the strip, and derive formula (96) of Chap. 8 in this way.

(For this application, Theorem 10.9 has to be extended so as to cover certain improper integrals. Provide this extension.)

12. Let  $I^k$  be the set of all  $\mathbf{u} = (u_1, \dots, u_k) \in R^k$  with  $0 \leq u_i \leq 1$  for all  $i$ ; let  $Q^k$  be the set of all  $\mathbf{x} = (x_1, \dots, x_k) \in R^k$  with  $x_i \geq 0$ ,  $\sum x_i \leq 1$ . ( $I^k$  is the unit cube;  $Q^k$  is the standard simplex in  $R^k$ .) Define  $\mathbf{x} = T(\mathbf{u})$  by

$$\begin{aligned} x_1 &= u_1 \\ x_2 &= (1 - u_1)u_2 \\ &\dots \\ x_k &= (1 - u_1) \cdots (1 - u_{k-1})u_k. \end{aligned}$$

Show that

$$\sum_{i=1}^k x_i = 1 - \prod_{i=1}^k (1 - u_i).$$

Show that  $T$  maps  $I^k$  onto  $Q^k$ , that  $T$  is 1-1 in the interior of  $I^k$ , and that its inverse  $S$  is defined in the interior of  $Q^k$  by  $u_1 = x_1$  and

$$u_i = \frac{x_i}{1 - x_1 - \cdots - x_{i-1}}$$

for  $i = 2, \dots, k$ . Show that

$$J_T(\mathbf{u}) = (1 - u_1)^{k-1} (1 - u_2)^{k-2} \cdots (1 - u_{k-1}),$$

and

$$J_S(\mathbf{x}) = [(1 - x_1)(1 - x_1 - x_2) \cdots (1 - x_1 - \cdots - x_{k-1})]^{-1}.$$

**13.** Let  $r_1, \dots, r_k$  be nonnegative integers, and prove that

$$\int_{Q^k} x_1^{r_1} \cdots x_k^{r_k} dx = \frac{r_1! \cdots r_k!}{(k + r_1 + \cdots + r_k)!}$$

*Hint:* Use Exercise 12, Theorems 10.9 and 8.20.

Note that the special case  $r_1 = \cdots = r_k = 0$  shows that the volume of  $Q^k$  is  $1/k!$ .

**14.** Prove formula (46).

**15.** If  $\omega$  and  $\lambda$  are  $k$ - and  $m$ -forms, respectively, prove that

$$\omega \wedge \lambda = (-1)^{km} \lambda \wedge \omega.$$

**16.** If  $k \geq 2$  and  $\sigma = [\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_k]$  is an oriented affine  $k$ -simplex, prove that  $\partial^2 \sigma = 0$ , directly from the definition of the boundary operator  $\partial$ . Deduce from this that  $\partial^2 \Psi = 0$  for every chain  $\Psi$ .

*Hint:* For orientation, do it first for  $k = 2, k = 3$ . In general, if  $i < j$ , let  $\sigma_{ij}$  be the  $(k-2)$ -simplex obtained by deleting  $\mathbf{p}_i$  and  $\mathbf{p}_j$  from  $\sigma$ . Show that each  $\sigma_{ij}$  occurs twice in  $\partial^2 \sigma$ , with opposite sign.

**17.** Put  $J^2 = \tau_1 + \tau_2$ , where

$$\tau_1 = [0, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2], \quad \tau_2 = -[0, \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_1].$$

Explain why it is reasonable to call  $J^2$  the positively oriented unit square in  $R^2$ . Show that  $\partial J^2$  is the sum of 4 oriented affine 1-simplexes. Find these. What is  $\partial(\tau_1 - \tau_2)$ ?

**18.** Consider the oriented affine 3-simplex

$$\sigma_1 = [0, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3]$$

in  $R^3$ . Show that  $\sigma_1$  (regarded as a linear transformation) has determinant 1. Thus  $\sigma_1$  is positively oriented.

Let  $\sigma_2, \dots, \sigma_6$  be five other oriented 3-simplexes, obtained as follows: There are five permutations  $(i_1, i_2, i_3)$  of  $(1, 2, 3)$ , distinct from  $(1, 2, 3)$ . Associate with each  $(i_1, i_2, i_3)$  the simplex

$$s(i_1, i_2, i_3)[\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3}]$$

where  $s$  is the sign that occurs in the definition of the determinant. (This is how  $\tau_2$  was obtained from  $\tau_1$  in Exercise 17.)

Show that  $\sigma_2, \dots, \sigma_6$  are positively oriented.

Put  $J^3 = \sigma_1 + \dots + \sigma_6$ . Then  $J^3$  may be called the positively oriented unit cube in  $R^3$ .

Show that  $\partial J^3$  is the sum of 12 oriented affine 2-simplexes. (These 12 triangles cover the surface of the unit cube  $I^3$ .)

Show that  $\mathbf{x} = (x_1, x_2, x_3)$  is in the range of  $\sigma_1$  if and only if  $0 \leq x_3 \leq x_2 \leq x_1 \leq 1$ .

Show that the ranges of  $\sigma_1, \dots, \sigma_6$  have disjoint interiors, and that their union covers  $I^3$ . (Compare with Exercise 13; note that  $3! = 6$ .)

**19.** Let  $J^2$  and  $J^3$  be as in Exercise 17 and 18. Define

$$\begin{aligned} B_{01}(u, v) &= (0, u, v), & B_{11}(u, v) &= (1, u, v), \\ B_{02}(u, v) &= (u, 0, v), & B_{12}(u, v) &= (u, 1, v), \\ B_{03}(u, v) &= (u, v, 0), & B_{13}(u, v) &= (u, v, 1). \end{aligned}$$

These are affine, and map  $R^2$  into  $R^3$ .

Put  $\beta_{ri} = B_{ri}(J^2)$ , for  $r = 0, 1$ ,  $i = 1, 2, 3$ . Each  $\beta_{ri}$  is an affine-oriented 2-chain. (See Sec. 10.30.) Verify that

$$\partial J^3 = \sum_{i=1}^3 (-1)^i (\beta_{0i} - \beta_{1i}),$$

in agreement with Exercise 18.

**20.** State conditions under which the formula

$$\int_{\Phi} f d\omega = \int_{S\Phi} f \omega - \int_{\Phi} (df) \wedge \omega$$

is valid, and show that it generalizes the formula for integration by parts.

*Hint:*  $d(f\omega) = (df) \wedge \omega + f d\omega$ .

**21.** As in Example 10.36, consider the 1-form

$$\eta = \frac{x dy - y dx}{x^2 + y^2}$$

in  $R^2 - \{\mathbf{0}\}$ .

(a) Carry out the computation that leads to formula (113), and prove that  $d\eta = 0$ .

(b) Let  $\gamma(t) = (r \cos t, r \sin t)$ , for some  $r > 0$ , and let  $\Gamma$  be a  $C''$ -curve in  $R^2 - \{\mathbf{0}\}$ ,

with parameter interval  $[0, 2\pi]$ , with  $\Gamma(0) = \Gamma(2\pi)$ , such that the intervals  $[\gamma(t), \Gamma(t)]$  do not contain  $\mathbf{0}$  for any  $t \in [0, 2\pi]$ . Prove that

$$\int_{\Gamma} \eta = 2\pi.$$

*Hint:* For  $0 \leq t \leq 2\pi$ ,  $0 \leq u \leq 1$ , define

$$\Phi(t, u) = (1 - u)\Gamma(t) + u\gamma(t).$$

Then  $\Phi$  is a 2-surface in  $R^2 - \{\mathbf{0}\}$  whose parameter domain is the indicated rectangle. Because of cancellations (as in Example 10.32),

$$\partial\Phi = \Gamma - \gamma.$$

Use Stokes' theorem to deduce that

$$\int_{\Gamma} \eta = \int_{\gamma} \eta$$

because  $d\eta = 0$ .

(c) Take  $\Gamma(t) = (a \cos t, b \sin t)$  where  $a > 0, b > 0$  are fixed. Use part (b) to show that

$$\int_0^{2\pi} \frac{ab}{a^2 \cos^2 t + b^2 \sin^2 t} dt = 2\pi.$$

(d) Show that

$$\eta = d\left(\arctan \frac{y}{x}\right)$$

in any convex open set in which  $x \neq 0$ , and that

$$\eta = d\left(-\arctan \frac{x}{y}\right)$$

in any convex open set in which  $y \neq 0$ .

Explain why this justifies the notation  $\eta = d\theta$ , in spite of the fact that  $\eta$  is not exact in  $R^2 - \{\mathbf{0}\}$ .

(e) Show that (b) can be derived from (d).

(f) If  $\Gamma$  is any closed  $C'$ -curve in  $R^2 - \{\mathbf{0}\}$ , prove that

$$\frac{1}{2\pi} \int_{\Gamma} \eta = \text{Ind}(\Gamma).$$

(See Exercise 23 of Chap. 8 for the definition of the index of a curve.)

22. As in Example 10.37, define  $\zeta$  in  $R^3 - \{\mathbf{0}\}$  by

$$\zeta = \frac{x \, dy \wedge dz + y \, dz \wedge dx + z \, dx \wedge dy}{r^3}$$

where  $r = (x^2 + y^2 + z^2)^{1/2}$ , let  $D$  be the rectangle given by  $0 \leq u \leq \pi$ ,  $0 \leq v \leq 2\pi$ , and let  $\Sigma$  be the 2-surface in  $R^3$ , with parameter domain  $D$ , given by

$$x = \sin u \cos v, \quad y = \sin u \sin v, \quad z = \cos u.$$

(a) Prove that  $d\zeta = 0$  in  $R^3 - \{\mathbf{0}\}$ .

(b) Let  $S$  denote the restriction of  $\Sigma$  to a parameter domain  $E \subset D$ . Prove that

$$\int_S \zeta = \int_E \sin u \, du \, dv = A(S),$$

where  $A$  denotes area, as in Sec. 10.43. Note that this contains (115) as a special case.

(c) Suppose  $g, h_1, h_2, h_3$ , are  $C''$ -functions on  $[0, 1]$ ,  $g > 0$ . Let  $(x, y, z) = \Phi(s, t)$  define a 2-surface  $\Phi$ , with parameter domain  $I^2$ , by

$$x = g(t)h_1(s), \quad y = g(t)h_2(s), \quad z = g(t)h_3(s).$$

Prove that

$$\int_{\Phi} \zeta = 0,$$

directly from (35).

Note the shape of the range of  $\Phi$ : For fixed  $s$ ,  $\Phi(s, t)$  runs over an interval on a line through  $\mathbf{0}$ . The range of  $\Phi$  thus lies in a “cone” with vertex at the origin.

(d) Let  $E$  be a closed rectangle in  $D$ , with edges parallel to those of  $D$ . Suppose  $f \in C''(D)$ ,  $f > 0$ . Let  $\Omega$  be the 2-surface with parameter domain  $E$ , defined by

$$\Omega(u, v) = f(u, v) \Sigma(u, v).$$

Define  $S$  as in (b) and prove that

$$\int_{\Omega} \zeta = \int_S \zeta = A(S).$$

(Since  $S$  is the “radial projection” of  $\Omega$  into the unit sphere, this result makes it reasonable to call  $\int_{\Omega} \zeta$  the “solid angle” subtended by the range of  $\Omega$  at the origin.)

*Hint:* Consider the 3-surface  $\Psi$  given by

$$\Psi(t, u, v) = [1 - t + tf(u, v)] \Sigma(u, v),$$

where  $(u, v) \in E$ ,  $0 \leq t \leq 1$ . For fixed  $v$ , the mapping  $(t, u) \rightarrow \Psi(t, u, v)$  is a 2-sur-