in the study of numerical series. Most of the interesting numerical series turned out to be instances of power series for particular values of x, for example, the series for $\pi/4$ is the $x=1$ instance of the series for $\tan^{-1}x$.

The theory of power series began with the publication of the series

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

by Mercator (1668). As we have seen, this was obtained by integrating the geometric series

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

term by term. Now the most important transcendental functions—logs, exponentials, and the related circular and hyperbolic functions—are obtained by integration and inversion from algebraic functions, and fairly simple algebraic functions at that. For example, e^y is the inverse function of $y = \log x$, and

$$\log(1+x) = \int_0^x \frac{dt}{1+t},$$

$\sin y$ is the inverse function of $y = \sin^{-1}x$ and

$$\sin^{-1}x = \int_0^x \frac{dt}{\sqrt{1-t^2}},$$

$$\tan^{-1}x = \int_0^x \frac{dt}{1+t^2},$$

and so on. Thus the key to finding power series is finding series expansions of simple algebraic functions. Once this is done, term-by-term integration and Newton's method of series inversion (Section 9.5) yield power series for all the common functions.

Rational functions, such as $1/(1+t^2)$, can be expanded using geometric series; the crucial step was accomplished by Newton (1665a) when he discovered the general binomial theorem,

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots,$$

yielding the expansion of functions such as $1/\sqrt{1-t^2} = (1-t^2)^{-1/2}$. This theorem was also discovered independently by Gregory (1670). Both Newton and Gregory were inspired by the loose heuristic method of interpolation used by Wallis (1655a), but they refined it into a result now known as the *Gregory–Newton interpolation formula*:

$$f(a+h) = f(a) + \frac{h}{b}\Delta f(a) + \frac{(h/b)(h/b-1)}{2!}\Delta^2 f(a) + \cdots, \tag{1}$$

where

$$\begin{aligned}
\Delta f(a) &= f(a+b) - f(a),\\
\Delta^2 f(a) &= \Delta f(a+b) - \Delta f(a) = f(a+2b) - 2f(a+b) + f(a),\\
\Delta^3 f(a) &= \Delta^2 f(a+b) - \Delta^2 f(a) = f(a+2b) - 3f(a+2b) + 3f(a+b) - f(a),\\
&\ \vdots
\end{aligned}$$

This wonderful formula determines the value of f at an arbitrary point $a+h$ from the values at an infinite arithmetic sequence of points a, $a+b$, $a+2b$, ... The first n terms give an nth-degree polynomial in h taking the same values as f at $a, a+b, \ldots, a+nb$. Hence the formula is valid for any f that is the limit of its own approximating polynomials. This means all functions representable by power series, provided that the points $a, a+b, a+2b, \ldots$, are sensibly chosen (the points $\pi, 2\pi, 3\pi, \ldots$, are a bad choice for $\sin x$, since the x-axis is a polynomial curve through all of them).

Newton discovered the formula (1) after his special investigations on interpolation that led to the binomial theorem. Gregory discovered the general formula first and then used it to derive the binomial theorem (see exercises below), all independently of Newton. It even appears that Gregory used the interpolation theorem to discover Taylor's theorem 44 years before Brook Taylor. There is strong evidence that Gregory used Taylor's series for other results [Gregory (1671)], and Taylor's series

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots \tag{2}$$

is just the limiting case of (1) as $b \to 0$. Indeed, this is how it was derived by Taylor (1715). The passage from (1) and (2) is simple if one assumes plausible limiting behavior for the infinite sum. Notice that

$$\frac{\Delta f(a)}{b} = \frac{f(a+b) - f(a)}{b} \to f'(a) \quad \text{as} \quad b \to 0$$

and similarly

$$\frac{\Delta^2 f(a)}{b^2} \to f''(a), \qquad \frac{\Delta^3 f(a)}{b^3} \to f'''(a),$$

and so on. We write (1) as

$$f(a+h) = f(a) + h\frac{\Delta f(a)}{b} + \frac{h(h-b)}{2!}\frac{\Delta^2 f(a)}{b^2} + \cdots$$

and observe that the nth term $\to (h^n/n!)f^{(n)}(a)$ as $n \to \infty$. Assuming that the limit of the infinite sum is the sum of these limits, we then get Taylor's series (2) as the limit of (1) as $b \to 0$.

EXERCISES

10.2.1 Show that

$$\Delta^n f(a) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(a+ib),$$

where $\binom{n}{i}$ is the ordinary binomial coefficient.

10.2.2 If $a = 0$, $b = 1$, and $f(x) = (1+k)^x$, show that $\Delta^n f(0) = k^n$ using the finite binomial series

$$(1+h)^n = \sum_{i=0}^{n} \binom{n}{i} h^i.$$

10.3.3 Deduce the general binomial series

$$(1+k)^x = 1 + xk + \frac{x(x-1)}{2!}k^2 + \frac{x(x-1)(x-2)}{3!}k^3 + \cdots$$

using the Gregory–Newton interpolation formula.

10.3 An Interpolation on Interpolation

The importance of interpolation in the development of calculus seems to have been greatly underestimated. The topic rarely appears in calculus books today, and then only as a numerical method. Yet three of the most important founders of calculus, Newton, Gregory, and Leibniz, began their work with interpolation, and we have seen how this led to two of their most important results, the binomial theorem and Taylor's theorem. [For Leibniz's work, see Hofmann (1974).] With the relegation of interpolation to numerical methods, this connection has been lost. Of course, interpolation *is* a numerical method in practice, when one uses only a few terms of the Gregory–Newton series, but the full series is exact and hence of much greater interest. It was this interest in infinite expansions per se that set off Newton, Gregory, and Leibniz (as well as Wallis) from their predecessors in interpolation.

Interpolation goes back to ancient times as a method for estimating the values of functions between known values. But perhaps the first to glimpse the possibility of exact interpolation were Thomas Harriot (1560–1621) and Henry Briggs (1556–1630). The bulk of Harriot's work has not been published or even put in proper order, but a formula has been found in his papers which is equivalent to the first terms of the Gregory–Newton series [see Lohne (1965)]. Lohne dates this work of Harriot at 1611. Briggs may have learned something about interpolation from Harriot when the two were at Oxford around 1620. Briggs' *Arithmetica logarithmica* [Briggs (1624)], which is concerned with the calculation of logarithms, uses series for interpolation, and in the process gives the first instance of the binomial theorem for a fractional exponent

$$(1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1\cdot 1}{2\cdot 4}x^2 + \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}x^3 - \frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}x^4 + \cdots.$$

Gregory knew of Briggs' work, and Newton certainly *could* have known of it, though no strong evidence that he did has yet been found. For more information on the history of interpolation, see Whiteside (1961) and Goldstine (1977).

10.4 Summation of Series

The results on infinite series that we have seen so far are mostly decompositions or expansions rather than summations. That is, one begins with a "known" quantity or function and decomposes it into an infinite series. Solutions of the converse problem, summation of a given series, were comparatively rare. Archimedes' summation of $1+1/4+1/4^2+\cdots$ was one. Perhaps the next were of series such as $1/1\cdot 2+1/2\cdot 3+\cdots+1/n(n+1)+\cdots$, given by Mengoli (1650). The series $\sum 1/n(n+1)$ is easily summed because of the happy accident that

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1},$$

hence

$$\begin{aligned}\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1}.\end{aligned}$$

By letting $n \to \infty$ we then obtain the sum 1 for the infinite series.

The first really tough summation problem was $1 + 1/2^2 + 1/3^2 + \cdots$. Mengoli tackled this without success, as did the brothers Jakob and Johann Bernoulli in a series of papers [Bernoulli, Jakob and Johann (1704)]. The Bernoulli brothers were able to sum similar series, rediscovering Mengoli's $\sum 1/n(n+1)$ and also summing $\sum 1/(n^2-1)$, but for $\sum 1/n^2$ itself they could obtain only trivial results such as

$$\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots = \frac{1}{4}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right).$$

The solution was finally obtained by Euler (1734), long after the death of Jakob Bernoulli, and Johann Bernoulli exclaimed: "In this way my brother's most ardent wish is satisfied ... if only my brother were still alive!" (Johann Bernoulli, *Opera*, Vol. 4, p. 22). In fact, after hearing that the sum was $\pi^2/6$, Johann Bernoulli himself discovered a proof, which turned out to be the same as Euler's.

Euler (1707–1783) was probably the greatest virtuoso of series manipulation, and his first summation of $1 + 1/2^2 + 1/3^2 + \cdots$ was one of his most audacious arguments. (Later he gave more rigorous proofs.) Consider the equation

$$\frac{\sin\sqrt{x}}{\sqrt{x}} = 1 - \frac{x}{3!} + \frac{x^2}{5!} - \frac{x^3}{7!} + \cdots = 0 \tag{1}$$

easily obtained from the sine series of Section 8.5. This equation has roots $x_1 = \pi^2, x_2 = (2\pi)^2, x_3 = (3\pi)^2, \ldots$, but *not* 0, because $\sin\sqrt{x}/\sqrt{x} \to 1$ as $x \to 0$. Now if a *polynomial* equation

$$1 + a_1x + a_2x^2 + \cdots + a_nx^n = 0$$

has roots $x = x_1, x_2, \ldots, x_n$, then

$$1 + a_1x + \cdots + a_nx^n = \left(1 - \frac{x}{x_1}\right)\left(1 - \frac{x}{x_2}\right)\cdots\left(1 - \frac{x}{x_n}\right) \tag{2}$$

by Descartes' factor theorem (6.7). Also

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} = -\text{coefficient of } x = -a_1$$

since each x term in the expansion of the right-hand side of (2) comes from a term $-x/x_i$ in one factor multiplied by 1's in all the other factors. Assuming this is also true of the "infinite polynomial" equation (1), we get

$$\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\cdots = -\text{coefficient of } x = -\left(-\frac{1}{3!}\right),$$

that is,

$$\frac{1}{\pi^2}+\frac{1}{(2\pi)^2}+\frac{1}{(3\pi)^2}+\cdots = \frac{1}{6}.$$

Hence

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots = \frac{\pi^2}{6}.$$

Q.E.D.!

EXERCISES

Euler's reasoning also leads to a correct infinite product formula for $\sin x$, which in turn gives the Wallis product for $\pi/4$ (Section 9.4).

10.4.1 Deduce an infinite product for $\frac{\sin\sqrt{x}}{\sqrt{x}}$ from Euler's reasoning, and hence show that

$$\sin x = x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{2^2\pi^2}\right)\left(1-\frac{x^2}{3^2\pi^2}\right)\cdots.$$

10.4.2 By substituting $x=\pi/2$ in the infinite product for $\sin x$, show that

$$\frac{2}{\pi}=\frac{1\cdot 3}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{5\cdot 7}{6\cdot 6}\cdots,$$

and hence obtain Wallis' product for $\pi/4$.

10.5 Fractional Power Series

The introduction of power series helped to make mathematicians conscious of the function concept (see also Section 13.4) by drawing attention to the generality of the expression $a_0+a_1x+a_2x^2+\cdots$. However, not every function $f(x)$ is expressible as a power series $a_0+a_1x+a_2x^2+\cdots$. This is obvious in the case of functions that tend to infinity as $x\to 0$, since the power series has value a_0 when $x\to 0$. For other functions, such as $f(x)=x^{1/2}$, the behavior at 0 disallows a power-series expansion for a more subtle reason. These functions have *branching behavior* at 0; they are *many-valued*, and hence they are not functions in the strict sense. The function $x^{1/2}$, for example, is two-valued because each number has two square roots, one the negative of the other.

Such behavior is not reflected in a power series $a_0 + a_1x + a_2x^2 + \cdots$, which can be assigned only one value for each value of x. All fractional powers of x are many-valued—$x^{1/3}$ is three-valued, $x^{1/4}$ is four-valued, and so on—and many-valued behavior is typical of algebraic functions in general. We say that y is an *algebraic function* of x if x and y satisfy a polynomial equation $p(x,y) = 0$. It follows from the impossibility of solving most polynomial equations by radicals (Section 6.7) that algebraic functions are not generally expressible by radicals, that is, by finite expressions built from $+, -, \times, \div$, and fractional powers.

Nevertheless, it was the remarkable discovery of Newton (1671) that any algebraic function y can be expressed as a *fractional power series* in x:

$$y = a_0 + a_1x^{r_1} + a_2x^{r_2} + a_3x^{r_3} + \cdots,$$

where $r_1, r_2, r_3, \ldots$, are rational numbers. Furthermore, the series can be rewritten in the form

$$\begin{aligned}
a_0 &+ b_1x^{s_1}(c_{00} + c_{01}x + c_{02}x^2 + \cdots) \\
&+ b_2x^{s_2}(c_{10} + c_{11}x + c_{12}x^2 + \cdots) \\
&\vdots \\
&+ b_nx^{s_n}(c_{n0} + c_{n1}x + c_{n2}x^2 + \cdots)
\end{aligned}$$

that is, as a finite sum of ordinary power series with fractional powers of x as multipliers. This means that in the neighborhood of $x = 0$, the behavior of y is like that of a finite sum of fractional powers.

For example, if $y^2(1+x)^2 = x$, we have

$$\begin{aligned}
y &= \frac{x^{1/2}}{1+x} \\
&= x^{1/2}(1 - x + x^2 - x^3 + \cdots),
\end{aligned}$$

and near the origin, y has behavior similar to $x^{1/2}$; in particular there are two values of y for each x. Newton's contribution was an ingenious algorithm for obtaining the successive powers of x. The fractional powers themselves were not properly understood until the variables x and y were