Examples

Every Principal Ideal Domain is Noetherian since it satisfies condition (3) of Theorem 2. In particular, $\mathbb{Z}$, the polynomial ring $k[x]$ where k is a field, and the Gaussian integers $\mathbb{Z}[i]$, are Noetherian rings. The ring $\mathbb{Z}[x_1, x_2, \ldots]$ is not Noetherian since the ideal $(x_1, x_2, \ldots)$ cannot be generated by any finite set (any finite set of generators involves only finitely many of the x_i). Exercise 33(d) in Section 7.4 shows that the ring of continuous real valued functions on $[0, 1]$ is not Noetherian.

A Noetherian ring may have arbitrarily long ascending chains of ideals and may have infinitely long descending chains of ideals. For example, $\mathbb{Z}$ has the infinite descending chain

$$(2) \supset (4) \supset (8) \supset \cdots$$

i.e., a Noetherian ring need not satisfy the *descending chain condition on ideals (D.C.C.)*. We shall see, however, that a commutative ring satisfying D.C.C. on ideals necessarily also satisfies A.C.C., i.e., is Noetherian; such rings are called *Artinian* and are studied in Chapter 16.

The following theorem and its corollary, which we record here for completeness, were proved in Section 9.6 (Theorem 21 and Corollary 22, respectively).

Theorem 3. *(Hilbert's Basis Theorem)* If R is a Noetherian ring then so is the polynomial ring $R[x]$.

Note that Hilbert's Basis Theorem shows how larger Noetherian rings may be built from existing ones in a manner analogous to Theorem 7 of Section 9.3 (which proved that if R is a U.F.D., then so is $R[x]$).

Corollary 4. The polynomial ring $k[x_1, x_2, \ldots, x_n]$ with coefficients from a field k is a Noetherian ring.

Let k be a field. Recall that a ring R is a *k-algebra* if k is contained in the center of R and the identity of k is the identity of R.

Definition.

(1) The ring R is a *finitely generated k-algebra* if R is generated as a ring by k together with some finite set $r_1, r_2, \ldots, r_n$ of elements of R.

(2) Let R and S be k-algebras. A map $\psi : R \to S$ is a *k-algebra homomorphism* if ψ is a ring homomorphism that is the identity on k.

If R is a k-algebra then R is both a ring and a vector space over k, and it is important to distinguish the sense in which elements of R are generators for R. For example, the polynomial ring $k[x_1, \ldots, x_n]$ in a finite number of variables over k is a finitely generated k-algebra since $x_1, \ldots, x_n$ are ring generators, but for $n > 0$ this ring is an *infinite* dimensional vector space over k.

Corollary 5. The ring R is a finitely generated k-algebra if and only if there is some surjective k-algebra homomorphism

$$\varphi : k[x_1, x_2, \ldots, x_n] \to R$$

from the polynomial ring in a finite number of variables onto R that is the identity map on k. Any finitely generated k-algebra is therefore Noetherian.

Proof: If R is generated as a k-algebra by $r_1, \ldots, r_n$, then we may define the map $\varphi : k[x_1, \ldots, x_n] \to R$ by $\varphi(x_i) = r_i$ for all i and $\varphi(a) = a$ for all $a \in k$. Then φ extends uniquely to a surjective ring homomorphism. Conversely, given a surjective homomorphism φ, the images of $x_1, \ldots, x_n$ under φ then generate R as a k-algebra, proving that R is finitely generated. Since $k[x_1, \ldots, x_n]$ is Noetherian by the previous corollary, any finitely generated k-algebra is therefore the quotient of a Noetherian ring, hence also Noetherian by Proposition 1.

Example

Suppose the k-algebra R is finite dimensional as a vector space over k, for example when $R = k[x]/(f(x))$, where f is any nonzero polynomial in $k[x]$. Then in particular R is a finitely generated k-algebra since a vector space basis also generates R as a ring. In this case since ideals are also k-subspaces any ascending or descending chain of ideals has at most $\dim_k R + 1$ distinct terms, hence R satisfies both A.C.C. and D.C.C. on ideals.

The basic idea behind "algebraic geometry" is to equate geometric questions with algebraic questions involving ideals in rings such as $k[x_1, \ldots, x_n]$. The Noetherian nature of these rings reduces many questions to consideration of finitely many algebraic equations (and this was in turn one of the main original motivations for Hilbert's Basis Theorem). We first consider the principal geometric object, the notion of an "algebraic set" of points.

Affine Algebraic Sets

Recall that the set $\mathbb{A}^n$ of n-tuples of elements of the field k is called *affine n-space over k* (cf. Section 10.1). If $x_1, x_2, \ldots, x_n$ are independent variables over k, then the polynomials f in $k[x_1, x_2, \ldots, x_n]$ can be viewed as k-valued functions $f : \mathbb{A}^n \to k$ on $\mathbb{A}^n$ by evaluating f at the points in $\mathbb{A}^n$:

$$f : (a_1, a_2, \ldots, a_n) \mapsto f(a_1, a_2, \ldots, a_n) \in k.$$

This gives a ring of k-valued functions on $\mathbb{A}^n$, denoted by $k[\mathbb{A}^n]$ and called the *coordinate ring of* $\mathbb{A}^n$. For instance, when $k = \mathbb{R}$ and $n = 2$, the coordinate ring of Euclidean 2-space $\mathbb{R}^2$ is denoted by $\mathbb{R}[\mathbb{A}^2]$ and is the ring of polynomials in two variables, say x and y, acting as real valued functions on $\mathbb{R}^2$ (the usual "coordinate functions").

Each subset S of functions in the coordinate ring $k[\mathbb{A}^n]$ determines a subset $\mathcal{Z}(S)$ of affine space, namely the set of points where all functions in S are simultaneously zero:

$$\mathcal{Z}(S) = \{(a_1, a_2, \ldots, a_n) \in \mathbb{A}^n \mid f(a_1, a_2, \ldots, a_n) = 0 \text{ for all } f \in S\},$$

where $\mathcal{Z}(\emptyset) = \mathbb{A}^n$.

Definition. A subset V of $\mathbb{A}^n$ is called an *affine algebraic set* (or just an algebraic set) if V is the set of common zeros of some set S of polynomials, i.e., if $V = \mathcal{Z}(S)$ for some $S \subseteq k[\mathbb{A}^n]$. In this case $V = \mathcal{Z}(S)$ is called the *locus of S* in $\mathbb{A}^n$.

If $S = \{f\}$ or $\{f_1, \dots, f_m\}$ we shall simply write $\mathcal{Z}(f)$ or $\mathcal{Z}(f_1, \dots, f_m)$ for $\mathcal{Z}(S)$ and call it the locus of f or $f_1, \dots, f_m$, respectively. Note that the locus of a single polynomial of the form $f - g$ is the same as the solutions in affine n-space of the equation $f = g$, so affine algebraic sets are the solution sets to systems of polynomial equations, and as a result occur frequently in mathematics.

Examples

(1) If $n = 1$ then the locus of a single polynomial $f \in k[x]$ is the set of roots of f in k. The algebraic sets in $\mathbb{A}^1$ are $\emptyset$, any finite set, and k (cf. the exercises).

(2) The one point subsets of $\mathbb{A}^n$ for any n are affine algebraic since $\{(a_1, a_2, \dots, a_n)\}$ is $\mathcal{Z}(x_1 - a_1,\ x_2 - a_2,\ \dots,\ x_n - a_n)$. More generally, any finite subset of $\mathbb{A}^n$ is an affine algebraic set.

(3) One may define lines, planes, etc. in $\mathbb{A}^n$ — these are *linear algebraic sets*, the loci of sets of linear (degree 1) polynomials of $k[x_1, \dots, x_n]$. For example, a line in $\mathbb{A}^2$ is defined by an equation $ax + by = c$ (which is the locus of the polynomial $f(x, y) = ax + by - c \in k[x, y]$). A line in $\mathbb{A}^3$ is the locus of two linear polynomials of $k[x, y, z]$ that are not multiples of each other. In particular, the coordinate axes, coordinate planes, etc. in $\mathbb{A}^n$ are all affine algebraic sets. For instance, the x-axis in $\mathbb{A}^3$ is the zero set $\mathcal{Z}(y, z)$ and the x,y plane is the zero set $\mathcal{Z}(z)$.

(4) In general the algebraic set $\mathcal{Z}(f)$ of a nonconstant polynomial f is called a *hypersurface* in $\mathbb{A}^n$. Conic sections are familiar algebraic sets in the Euclidean plane $\mathbb{R}^2$. For example, the locus of $y - x^2$ is the parabola $y = x^2$, the locus of $x^2 + y^2 - 1$ is the unit circle, and $\mathcal{Z}(xy - 1)$ is the hyperbola $y = 1/x$. The x- and y-axes are the algebraic sets $\mathcal{Z}(y)$ and $\mathcal{Z}(x)$ respectively. Likewise, quadric surfaces such as the ellipsoid defined by the equation $x^2 + \dfrac{y^2}{4} + \dfrac{z^2}{9} = 1$ are affine algebraic sets in $\mathbb{R}^3$.

We leave as exercises the straightforward verification of the following properties of affine algebraic sets. Let S and T be subsets of $k[\mathbb{A}^n]$.

(1) If $S \subseteq T$ then $\mathcal{Z}(T) \subseteq \mathcal{Z}(S)$ (i.e., $\mathcal{Z}$ is inclusion reversing or *contravariant*).

(2) $\mathcal{Z}(S) = \mathcal{Z}(I)$, where $I = (S)$ is the ideal in $k[\mathbb{A}^n]$ generated by the subset S.

(3) The intersection of two affine algebraic sets is again an affine algebraic set, in fact $\mathcal{Z}(S) \cap \mathcal{Z}(T) = \mathcal{Z}(S \cup T)$. More generally an arbitrary intersection of affine algebraic sets is an algebraic set: if $\{S_j\}$ is any collection of subsets of $k[\mathbb{A}^n]$, then
$$\cap \mathcal{Z}(S_j) = \mathcal{Z}(\cup S_j).$$

(4) The union of two affine algebraic sets is again an affine algebraic set, in fact $\mathcal{Z}(I) \cup \mathcal{Z}(J) = \mathcal{Z}(IJ)$, where I and J are ideals and IJ is their product.

(5) $\mathcal{Z}(0) = \mathbb{A}^n$ and $\mathcal{Z}(1) = \emptyset$ (here 0 and 1 denote constant functions).

By (2), every affine algebraic set is the algebraic set corresponding to an *ideal* of the coordinate ring. Thus we may consider
$$\mathcal{Z} : \{\text{ideals of } k[\mathbb{A}^n]\} \to \{\text{affine algebraic sets in } \mathbb{A}^n\}.$$