is a dependence relation on the columns of A with, say, $\beta_i \neq 0$. By Cramer's Rule, $\beta_i \det A = 0$. Since R is an integral domain and $\beta_i \neq 0$, $\det A = 0$.

Conversely, assume the columns of A are independent. Consider the integral domain R as embedded in its quotient field F so that $M_{n\times n}(R)$ may be considered as a subring of $M_{n\times n}(F)$ (and note that the determinant function on the subring is the restriction of the determinant function from $M_{n\times n}(F)$). The columns of A in this way become elements of F^n. Any nonzero F-linear combination of the columns of A which is zero in F^n gives, by multiplying the coefficients by a common denominator, a nonzero R-linear dependence relation. The columns of A must therefore be independent vectors in F^n. Since A has n columns, these form a basis of F^n. Thus there are elements β_{ij} of F such that for each i, the i^{th} basis vector e_i in F^n may be expressed as

$$e_i = \beta_{1i}A_1 + \beta_{2i}A_2 + \cdots + \beta_{ni}A_n.$$

The $n \times n$ identity matrix is the one whose columns are $e_1, e_2, \ldots, e_n$. By Proposition 23 (with $\varphi = \det$), the determinant of the identity matrix is some F-multiple of $\det A$. Since the determinant of the identity matrix is 1, $\det A$ cannot be zero. This completes the proof.

Theorem 28. For matrices $A, B \in M_{n\times n}(R)$, $\det AB = (\det A)(\det B)$.

Proof: Let $B = (\beta_{ij})$ and let $A_1, A_2, \ldots, A_n$ be the columns of A. Then $C = AB$ is the $n \times n$ matrix whose j^{th} column is $C_j = \beta_{1j}A_1 + \beta_{2j}A_2 + \cdots + \beta_{nj}A_n$. By Proposition 23 applied to the multilinear function det we obtain

$$\det C = \det(C_1, \ldots, C_n) = \left[\sum_{\sigma\in S_n} \epsilon(\sigma)\beta_{\sigma(1)\,1}\beta_{\sigma(2)\,2}\ldots\beta_{\sigma(n)\,n}\right]\det(A_1, \ldots, A_n).$$

The sum inside the brackets is the formula for $\det B$, hence $\det C = (\det B)(\det A)$, as required ($R$ is commutative).

Definition. Let $A = (\alpha_{ij})$ be an $n \times n$ matrix. For each i, j, let A_{ij} be the $n-1 \times n-1$ matrix obtained from A by deleting its i^{th} row and j^{th} column (an $n-1 \times n-1$ *minor* of A). Then $(-1)^{i+j}\det(A_{ij})$ is called the *ij cofactor of A*.

Theorem 29. *(The Cofactor Expansion Formula along the i^{th} row)* If $A = (\alpha_{ij})$ is an $n \times n$ matrix, then for each fixed $i \in \{1, 2, \ldots, n\}$ the determinant of A can be computed from the formula

$$\det A = (-1)^{i+1}\alpha_{i1}\det A_{i1} + (-1)^{i+2}\alpha_{i2}\det A_{i2} + \cdots + (-1)^{i+n}\alpha_{in}\det A_{in}.$$

Proof: For each A let $D(A)$ be the element of R obtained from the cofactor expansion formula described above. We prove that D satisfies the axioms of a determinant function, hence is *the* determinant function. Proceed by induction on n. If $n = 1$, $D((\alpha)) = \alpha$, for all 1×1 matrices (α) and the result holds. Assume therefore that $n \geq 2$. To show that D is an alternating multilinear function of the columns, fix an index k and consider the k^{th} column as varying and all other columns as fixed. If $j \neq k$,

α_{ij} does not depend on k and $D(A_{ij})$ is linear in the k^{th} column by induction. Also, as the k^{th} column varies linearly so does α_{ik}, whereas $D(A_{ik})$ remains unchanged (the k^{th} column has been deleted from A_{ik}). Thus each term in the formula for D varies linearly in the k^{th} column. This proves D is multilinear in the columns.

To prove D is alternating assume columns k and $k+1$ of A are equal. If $j \neq k$ or $k+1$, the two equal columns of A become two equal columns in the matrix A_{ij}. By induction $D(A_{ij}) = 0$. The formula for D therefore has at most two nonzero terms: when $j = k$ and when $j = k+1$. The minor matrices A_{ik} and $A_{i\,k+1}$ are identical and $\alpha_{ik} = \alpha_{i\,k+1}$. Then the two remaining terms in the expansion for D, $(-1)^{i+k}\alpha_{ik}D(A_{ik})$ and $(-1)^{i+k+1}\alpha_{i\,k+1}D(A_{i\,k+1})$ are equal and appear with opposite signs, hence they cancel. Thus $D(A) = 0$ if A has two adjacent columns which are equal, i.e., D is alternating.

Finally, it follows easily from the formula and induction that $D(I) = 1$, where I is the identity matrix. This completes the induction.

Theorem 30. *(Cofactor Formula for the Inverse of a Matrix)* Let $A = (\alpha_{ij})$ be an $n \times n$ matrix and let B be the transpose of its matrix of cofactors, i.e., $B = (\beta_{ij})$, where $\beta_{ij} = (-1)^{i+j} \det A_{ji}$, $1 \leq i, j \leq n$. Then $AB = BA = (\det A)I$. Moreover, $\det A$ is a unit in R if and only if A is a unit in $M_{n\times n}(R)$; in this case the matrix $\dfrac{1}{\det A}B$ is the inverse of A.

Proof: The i, j entry of AB is $\alpha_{i1}\beta_{1j} + \alpha_{i2}\beta_{2j} + \cdots + \alpha_{in}\beta_{nj}$. By definition of the entries of B this equals

$$\alpha_{i1}(-1)^{j+1}D(A_{j1}) + \alpha_{i2}(-1)^{j+2}D(A_{j2}) + \cdots + \alpha_{in}(-1)^{j+n}D(A_{jn}). \tag{11.7}$$

If $i = j$, this is the cofactor expansion for $\det A$ along the i^{th} row. The diagonal entries of AB are thus all equal to $\det A$. If $i \neq j$, let $\overline{A}$ be the matrix A with the j^{th} row replaced by the i^{th} row, so $\det \overline{A} = 0$. By inspection $\overline{A}_{jk} = A_{jk}$ and $\alpha_{ik} = \overline{\alpha}_{jk}$ for every $k \in \{1, 2, \ldots, n\}$. By making these substitutions in equation (7) for each $k = 1, 2, \ldots, n$ one sees that the i, j entry in AB equals $\overline{\alpha}_{j1}(-1)^{1+j}D(\overline{A}_{j1}) + \cdots + \overline{\alpha}_{jn}(-1)^{n+j}D(\overline{A}_{jn})$. This expression is the cofactor expansion for $\det \overline{A}$ along the j^{th} row. Since, as noted above, $\det \overline{A} = 0$, this proves that all off diagonal terms of AB are zero, which proves that $AB = (\det A)I$.

It follows directly from the definition of B that the pair (A^t, B^t) satisfies the same hypotheses as the pair (A, B). By what has already been shown it follows that $(BA)^t = A^tB^t = (\det A^t)I$. Since $\det A^t = \det A$ and the transpose of a diagonal matrix is itself, we obtain $BA = (\det A)I$ as well.

If $d = \det A$ is a unit in R, then $d^{-1}B$ is a matrix with entries in R whose product with A (on either side) is the identity, i.e., A is a unit in $M_{n\times n}(R)$. Conversely, assume that A is a unit in R with (2-sided) inverse matrix C. Since $\det C \in R$ and

$$1 = \det I = \det AC = (\det A)(\det C) = (\det C)(\det A),$$

it follows that $\det A$ has a 2-sided inverse in R, as needed. This completes all parts of the proof.

EXERCISES

1. Formulate and prove the cofactor expansion formula along the j^{th} column of a square matrix A.
2. Let F be a field and let $A_1, A_2, \ldots, A_n$ be (column) vectors in F^n. Form the matrix A whose i^{th} column is A_i. Prove that these vectors form a basis of F^n if and only if $\det A \neq 0$.
3. Let R be any commutative ring with 1, let V be an R-module and let $x_1, x_2, \ldots, x_n \in V$. Assume that for some $A \in M_{n\times n}(R)$,
$$A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = 0.$$
Prove that $(\det A)x_i = 0$, for all $i \in \{1, 2, \ldots, n\}$.
4. (*Computing Determinants of Matrices*) This exercise outlines the use of Gauss–Jordan elimination (cf. the exercises in Section 2) to compute determinants. This is the most efficient general procedure for computing large determinants. Let A be an $n \times n$ matrix.
   (a) Prove that the elementary row operations have the following effect on determinants:
      (i) interchanging two rows changes the sign of the determinant
      (ii) adding a multiple of one row to another does not alter the determinant
      (iii) multiplying any row by a nonzero element u from F multiplies the determinant by u.
   (b) Prove that $\det A$ is nonzero if and only if A is row equivalent to the $n \times n$ identity matrix. Suppose A can be row reduced to the identity matrix using a total of s row interchanges as in (i) and by multiplying rows by the nonzero elements $u_1, u_2, \ldots, u_t$ as in (iii). Prove that $\det A = (-1)^s(u_1u_2\ldots u_t)^{-1}$.
5. Compute the determinants of the following matrices using row reduction:
$$A = \begin{pmatrix} 5 & 4 & -6 \\ -2 & 0 & 2 \\ 3 & 4 & -2 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & -4 & 4 \\ 2 & -1 & 4 & -8 \\ 1 & 0 & 1 & -2 \\ 0 & 1 & -2 & 3 \end{pmatrix}.$$
6. (*Minkowski's Criterion*) Suppose A is an $n \times n$ matrix with real entries such that the diagonal elements are all positive, the off-diagonal elements are all negative and the row sums are all positive. Prove that $\det A \neq 0$. [Consider the corresponding system of equations $AX = 0$ and suppose there is a nontrivial solution $(x_1, \ldots, x_n)$. If x_i has the largest absolute value show that the i^{th} equation leads to a contradiction.]

11.5 TENSOR ALGEBRAS, SYMMETRIC AND EXTERIOR ALGEBRAS

In this section R is any commutative ring with 1, and we assume the left and right actions of R on each R-module are the same. We shall primarily be interested in the special case when $R = F$ is a field, but the basic constructions hold in general.

Suppose M is an R-module. When tensor products were first introduced in Section 10.4 we spoke heuristically of forming "products" m_1m_2 of elements of M, and we constructed a new module $M \otimes M$ generated by such "products" $m_1 \otimes m_2$. The "value" of this product is not in M, so this does not give a ring structure on M itself. If, however, we iterate this by taking the "products" $m_1m_2m_3$ and $m_1m_2m_3m_4$, and all finite sums of such products, we can construct a ring containing M that is "universal" with respect to rings containing M (and, more generally, with respect to homomorphic images of M), as we now show.

For each integer $k \geq 1$, define

$$\mathcal{T}^k(M) = M \otimes_R M \otimes_R \cdots \otimes_R M \quad (k \text{ factors}),$$

and set $\mathcal{T}^0(M) = R$. The elements of $\mathcal{T}^k(M)$ are called *k-tensors*. Define

$$\mathcal{T}(M) = R \oplus \mathcal{T}^1(M) \oplus \mathcal{T}^2(M) \oplus \mathcal{T}^3(M) \cdots = \bigoplus_{k=0}^{\infty} \mathcal{T}^k(M).$$

Every element of $\mathcal{T}(M)$ is a finite linear combination of k-tensors for various $k \geq 0$. We identify M with $\mathcal{T}^1(M)$, so that M is an R-submodule of $\mathcal{T}(M)$.

Theorem 31. If M is any R-module over the commutative ring R then

(1) $\mathcal{T}(M)$ is an R-algebra containing M with multiplication defined by mapping

$$(m_1 \otimes \cdots \otimes m_i)(m_1' \otimes \cdots \otimes m_j') = m_1 \otimes \cdots \otimes m_i \otimes m_1' \otimes \cdots \otimes m_j'$$

and extended to sums via the distributive laws. With respect to this multiplication $\mathcal{T}^i(M)\mathcal{T}^j(M) \subseteq \mathcal{T}^{i+j}(M)$.

(2) *(Universal Property)* If A is any R-algebra and $\varphi : M \to A$ is an R-module homomorphism, then there is a unique R-algebra homomorphism $\Phi : \mathcal{T}(M) \to A$ such that $\Phi|_M = \varphi$.

Proof: The map

$$\underbrace{M \times M \times \cdots \times M}_{i \text{ factors}} \times \underbrace{M \times M \times \cdots \times M}_{j \text{ factors}} \to \mathcal{T}^{i+j}(M)$$

defined by

$$(m_1, \ldots, m_i, m_1', \ldots, m_j') \mapsto m_1 \otimes \ldots \otimes m_i \otimes m_1' \otimes \ldots \otimes m_j'$$

is R-multilinear, so induces a bilinear map $\mathcal{T}^i(M) \times \mathcal{T}^j(M)$ to $\mathcal{T}^{i+j}(M)$ which is easily checked to give a well defined multiplication satisfying (1) (cf. the proof of Proposition 21 in Section 10.4). To prove (2), assume that $\varphi : M \to A$ is an R-algebra homomorphism. Then

$$(m_1, m_2, \ldots, m_k) \mapsto \varphi(m_1)\varphi(m_2)\ldots\varphi(m_k)$$

defines an R-multilinear map from $M \times \cdots \times M$ (k times) to A. This in turn induces a unique R-module homomorphism Φ from $\mathcal{T}^k(M)$ to A (Corollary 16 of Section 10.4) mapping $m_1 \otimes \ldots \otimes m_k$ to the element on the right hand side above. It is easy to check from the definition of the multiplication in (1) that the resulting uniquely defined map $\Phi : \mathcal{T}(M) \to A$ is an R-algebra homomorphism.