Nonsingularity and Local Rings of Affine Plane Curves

Let k be an algebraically closed field and let C be an irreducible affine *curve* over k. In other words, C is an affine algebraic set whose coordinate ring $k[C]$ is an integral domain and whose field of rational functions $k(C)$ has transcendence degree 1 over k (cf. Section 15.4).

Recall that, by definition, the point v on C is nonsingular if $\mathfrak{m}_{v,C}/\mathfrak{m}_{v,C}^2$ is a 1-dimensional vector space over k, where $\mathfrak{m}_{v,C}$ is the unique maximal ideal in the local ring $\mathcal{O}_{v,C}$ of rational functions on C defined at v.

Proposition 12. Let v be a point on the irreducible affine curve C over k. Then C is nonsingular at v if and only if the local ring $\mathcal{O}_{v,C}$ is a Discrete Valuation Ring.

Proof: Suppose first that v is nonsingular. Then $\dim_k(\mathfrak{m}_{v,C}/\mathfrak{m}_{v,C}^2) = 1$, and since $\mathcal{O}_{v,C}$ is Noetherian, it follows from Exercise 12 in Section 1 that $\mathfrak{m}_{v,C}$ is principal. Hence $\mathcal{O}_{v,C}$ is a D.V.R. by Theorem 7(4). Conversely, suppose $\mathcal{O}_{v,C}$ is a D.V.R. and t is a uniformizing element for $\mathcal{O}_{v,C}$. Then every element in $\mathfrak{m}_{v,C}$ can be written uniquely in the form at for some a in $\mathcal{O}_{v,C}$. The map from $\mathfrak{m}_{v,C}$ to $\mathcal{O}_{v,C}/\mathfrak{m}_{v,C}$ defined by mapping at to $a \bmod \mathfrak{m}_{v,C}$ is easily checked to be a surjective $\mathcal{O}_{v,C}$-module homomorphism with kernel $\mathfrak{m}_{v,C}^2$. Hence $\mathfrak{m}_{v,C}/\mathfrak{m}_{v,C}^2$ is isomorphic as an $\mathcal{O}_{v,C}/\mathfrak{m}_{v,C}$-module to $\mathcal{O}_{v,C}/\mathfrak{m}_{v,C}$. Since $\mathcal{O}_{v,C}/\mathfrak{m}_{v,C} \cong k$ (Proposition 46(5) in Section 15.4), it follows that $\dim_k(\mathfrak{m}_{v,C}/\mathfrak{m}_{v,C}^2) = 1$, and so v is a nonsingular point on C.

Definition. If v is a nonsingular point on C with corresponding discrete valuation ν_v defined on $k(C)$, then $\nu_v(f) = n$ for $f \in k(V)$ is the *order of zero of f at v* (if $n \geq 0$) or the *order of the pole of f at v* (if $n < 0$).

Using the criterion for nonsingularity for points on curves in Proposition 12 we can prove a result first mentioned in Section 15.4:

Corollary 13. An irreducible affine curve C over an algebraically closed field k is smooth if and only if its coordinate ring $k[C]$ is integrally closed.

Proof: The curve C is smooth if and only if every localization $\mathcal{O}_{v,C}$ is a D.V.R. Since $k[C]$ has Krull dimension 1 (Exercise 11 in Section 1), the same is true for each $\mathcal{O}_{v,C}$. It then follows by Theorem 7(5) that every localization $\mathcal{O}_{v,C}$ is a D.V.R. if and only if $\mathcal{O}_{v,C}$ is integrally closed. By Proposition 49 in Section 15.4, this in turn is equivalent to the statement that $k[C]$ is integrally closed, which proves the corollary.

EXERCISES

1. Suppose R is a Discrete Valuation Ring with respect to the valuation ν on the fraction field K of R. If $x, y \in K$ with $\nu(x) < \nu(y)$ prove that $\nu(x + y) = \min(\nu(x), \nu(y))$. [Note that $x + y = x(1 + y/x)$.]
2. Suppose R is a Discrete Valuation Ring with unique maximal ideal M and quotient $F = R/M$. For any $n \geq 0$ show that M^n/M^{n+1} is a vector space over F and that $\dim_F(M^n/M^{n+1}) = 1$.

3. Suppose R is an integral domain that is also a local ring whose unique maximal ideal $M = (t)$ is nonzero and principal, and suppose that $\cap_{n\geq 1}(t^n) = 0$. Prove that R is a Discrete Valuation Ring. [Show that every nonzero element in R can be written in the form ut^n for some unit $u \in R$ and some $n \geq 0$.]
4. Suppose R is a Noetherian local ring whose unique maximal ideal $M = (t)$ is principal. Prove that either R is a Discrete Valuation Ring or $t^n = 0$ for some $n \geq 0$. In the latter case show that R is Artinian.
5. Suppose that R is a Noetherian integral domain that is also a local ring of Krull dimension 1. Let M be the unique maximal ideal of R and let $F = R/M$, so that M/M^2 is a vector space over F.
   (a) Prove that if $\dim_F(M/M^2) = 1$ then R is a Discrete Valuation Ring.
   (b) If every nonzero ideal of R is a power of M prove that R is a Discrete Valuation Ring.
6. Let R be an integral domain with fraction field K. Prove that every finitely generated R-submodule of K is a fractional ideal of R. If R is Noetherian, prove that A is a fractional ideal of R if and only if R is a finitely generated R-submodule of K.
7. If R is an integral domain and A is a fractional ideal of R, prove that if A is projective then A is finitely generated. Conclude that every integral domain that is not Noetherian contains an ideal that is not projective.
8. Suppose R is a Noetherian integral domain that is also a local ring with nonzero maximal ideal M. Prove that R is a D.V.R. if and only if the only M-primary ideals in R are the powers of M.
9. Let $C = \mathcal{Z}(xz - y^2, yz - x^3, z^2 - x^2y) \subset \mathbb{A}^3$ over the algebraically closed field k. If $v = (0, 0, 0) \in C$, prove that $\dim_k(\mathfrak{m}_{v,C}/\mathfrak{m}^2_{v,C}) = 3$ so that v is singular on C. Conclude that $k[C]$ is not integrally closed in $k(C)$ and determine its integral closure. [cf. Exercise 27, Section 15.4.]

16.3 DEDEKIND DOMAINS

In the previous section we showed that Discrete Valuation Rings are the local rings that are integrally closed Noetherian integral domains of Krull dimension 1. In this section we consider the effect of relaxing the condition that the ring be a local ring:

Definition. A *Dedekind Domain* is a Noetherian, integrally closed, integral domain of Krull dimension 1.

Equivalently, R is a Dedekind Domain if R is a Noetherian, integrally closed, integral domain that is not a field in which every nonzero prime ideal is maximal.

The first result shows that Dedekind Domains are a generalization of the class of Principal Ideal Domains. We shall see later (Theorem 22) that there is a structure theorem for finitely generated modules over a Dedekind Domain extending the corresponding result for P.I.D.s proved in Section 12.1.

Proposition 14.
(1) Every Principal Ideal Domain is a Dedekind Domain.
(2) The ring of integers in an algebraic number field is a Dedekind Domain.

Proof: A P.I.D. is clearly Noetherian, is integrally closed since it is a U.F.D. (Example 3, Section 15.3), and nonzero prime ideals are maximal (Proposition 7 in Section 8.2), which proves (1). Let $\mathcal{O}_K$ be the ring of integers in the number field K, i.e., the integral closure of $\mathbb{Z}$ in K. Then Corollary 25 in Section 15.3 shows that $\mathcal{O}_K$ is integrally closed, $\mathcal{O}_K$ is Noetherian by Theorem 29 in Section 15.3, and the fact that nonzero prime ideals in $\mathcal{O}_K$ are maximal was proved in the discussion following the same theorem. This proves (2).

The following theorem gives a number of important equivalent characterizations of Dedekind Domains. Recall that the basic properties of fractional ideals were developed in the previous section.

Theorem 15. Suppose R is an integral domain with fraction field $K \neq R$. The following are equivalent conditions for R to be a Dedekind Domain:

(1) The ring R is Noetherian, integrally closed, and every nonzero prime ideal is maximal.

(2) The ring R is Noetherian and for each nonzero prime P of R the localization R_P is a Discrete Valuation Ring.

(3) Every nonzero fractional ideal of R in K is invertible.

(4) Every nonzero fractional ideal of R in K is a projective R-module.

(5) Every nonzero proper ideal I of R can be written as a finite product of prime ideals: $I = P_1 P_2 \cdots P_n$ (not necessarily distinct).

When the condition in (5) holds, the set of primes $\{P_1, \ldots, P_n\}$ is uniquely determined and so every nonzero proper ideal I of R can be written uniquely (up to order) as a product of powers of prime ideals.

Proof: If R satisfies (1), then R_P is a D.V.R. by Corollary 8, so (1) implies (2). Conversely, assume each R_P is a D.V.R. Then R is integrally closed by Proposition 49 in Section 15.4 and every nonzero prime ideal is maximal by Proposition 46(3) in Section 15.4, so (2) implies (1).

Suppose now that (1) is satisfied and that A is a nonzero fractional ideal of R. Let $A' = \{x \in K \mid xA \subseteq R\}$ as in Proposition 9. For any prime ideal P of R the behavior of R-modules under localization shows that $(AA')_P = A_P(A')_P = A_P(A_P)'$ (cf. Exercise 4). Since R_P is a D.V.R. by what has already been shown, $A_P(A_P)' = R_P$ by Proposition 11. Hence $(AA')_P = R_P$ for all nonzero primes P of R, so $AA' = R$ (Exercise 13 in Section 15.4), and A is invertible, showing (1) implies (3). Conversely, suppose every nonzero fractional ideal of R is invertible. Then every ideal in R is finitely generated by Proposition 9(3), so R is Noetherian. Every localization R_P of R at a nonzero prime P is a local ring in which the nonzero fractional ideals are invertible (cf. Exercise 4), hence is a D.V.R. by Proposition 11. Hence (3) implies (2) and so (1), (2) and (3) are equivalent. The equivalence of these with (4) is given by Proposition 10.

Suppose now that (1) is satisfied, and let I be any nonzero proper ideal in R. Since R is Noetherian, I has a minimal primary decomposition $I = Q_1 \cap \cdots \cap Q_n$ as in Theorem 21 of Section 15.2. The associated primes $P_i = \text{rad } Q_i$ for $i = 1, \ldots, n$ are all distinct, and since primes are maximal in R by hypothesis, the associated primes are all pairwise comaximal, and it follows easily that the same is true for the Q_i (Exercise