A simple and useful test for convergence of a series of matrices can be given in terms of the *norm* of a matrix, a generalization of the absolute value of a number.

DEFINITION OF NORM OF A MATRIX. *If $A = [a_{ij}]$ is an $m \times n$ matrix of real or complex entries, the norm of A, denoted by $\|A\|$, is defined to be the nonnegative number given by the formula*

$$\|A\| = \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}| . \tag{7.11}$$

In other words, the norm of A is the sum of the absolute values of all its entries. There are other definitions of norms that are sometimes used, but we have chosen this one because of the ease with which we can prove the following properties.

THEOREM 7.1. FUNDAMENTAL PROPERTIES OF NORMS. *For rectangular matrices A and B, and all real or complex scalars c we have*

$$\|A + B\| \le \|A\| + \|B\| , \|AB\| \le \|A\| \, \|B\| , \|cA\| = |c| \, \|A\| .$$

Proof. We prove only the result for $\|AB\|$, assuming that A is $m \times n$ and B is $n \times p$. The proofs of the others are simpler and are left as exercises.

Writing $A = [a_{ik}]$, $B = [b_{kj}]$, we have $AB = [\sum_{k=1}^{n} a_{ik} b_{kj}]$, so from (7.11) we obtain

$$\|AB\| = \sum_{i=1}^{m} \sum_{j=1}^{p} \left| \sum_{k=1}^{n} a_{ik} b_{kj} \right| \le \sum_{i=1}^{m} \sum_{k=1}^{n} |a_{ik}| \sum_{j=1}^{p} |b_{kj}| \le \sum_{i=1}^{m} \sum_{k=1}^{n} |a_{ik}| \, \|B\| = \|A\| \, \|B\| .$$

Note that in the special case $B = A$ the inequality for $\|AB\|$ becomes $\|A^2\| \le \|A\|^2$. By induction we also have

$$\|A^k\| \le \|A\|^k \qquad \text{for } k = 1, 2, 3, \ldots .$$

These inequalities will be useful in the discussion of the exponential matrix.

The next theorem gives a useful sufficient condition for convergence of a series of matrices.

THEOREM 7.2. TEST FOR CONVERGENCE OF A MATRIX SERIES. *If $\{C_k\}$ is a sequence of $m \times n$ matrices such that $\sum_{k=1}^{\infty} \|C_k\|$ converges, then the matrix series $\sum_{k=1}^{\infty} C_k$ also converges.*

Proof. Let the ij-entry of C_k be denoted by $c_{ij}^{(k)}$. Since $|c_{ij}^{(k)}| \le \|C_k\|$, convergence of $\sum_{k=1}^{\infty} \|C_k\|$ implies absolute convergence of each series $\sum_{k=1}^{\infty} c_{ij}^{(k)}$. Hence each series $\sum_{k=1}^{\infty} c_{ij}^{(k)}$ is convergent, so the matrix series $\sum_{k=1}^{\infty} C_k$ is convergent.

7.4 Exercises

1. Verify that the linearity property of integrals also holds for integrals of matrix functions.
2. Verify each of the following differentiation rules for matrix functions, assuming P and Q are differentiable. In (a), P and Q must be of the same size so that $P + Q$ is meaningful. In

(b) and (d) they need not be of the same size provided the products are meaningful. In (c) and (d), Q is assumed to be nonsingular.

(a) $(P + Q)' = P' + Q'$. (c) $(Q^{-1})' = -Q^{-1}Q'Q^{-1}$.

(b) $(PQ)' = PQ' + P'Q$. (d) $(PQ^{-1})' = -PQ^{-1}Q'Q^{-1} + P'Q^{-1}$.

3. (a) Let P be a differentiable matrix function. Prove that the derivatives of P^2 and P^3 are given by the formulas

$$(P^2)' = PP' + P'P, \qquad (P^3)' = P^2P' + PP'P + P'P^2.$$

(b) Guess a general formula for the derivative of P^k and prove it by induction.

4. Let P be a differentiable matrix function and let g be a differentiable scalar function whose range is a subset of the domain of P. Define the composite function $F(t) = P[g(t)]$ and prove the chain rule, $F'(t) = g'(t)P'[g(t)]$.
5. Prove the zero-derivative theorem for matrix functions: ***If*** $P'(t) = 0$ ***for every*** t ***in an open interval*** (a, b), *then the matrix function* P *is constant on* (a, b).
6. State and prove generalizations of the first and second fundamental theorems of calculus for matrix functions.
7. State and prove a formula for integration by parts in which the integrands are matrix functions.
8. Prove the following properties of matrix norms :

$$\|A + B\| \le \|A\| + \|B\|, \qquad \|cA\| = |c| \, \|A\|.$$

9. If a matrix function P is integrable on an interval $[a, b]$ prove that

$$\left\| \int_a^b P(t)\, dt \right\| \le \int_a^b \|P(t)\|\, dt.$$

10. Let D be an $n \times n$ diagonal matrix, say $D = \text{diag}\,(\lambda_1, \ldots, \lambda_n)$. Prove that the matrix series $\sum_{k=0}^{\infty} D^k/k!$ converges and is also a diagonal matrix,

$$\sum_{k=0}^{\infty} \frac{D^k}{k!} = \text{diag}\,(e^{\lambda_1}, \ldots, e^{\lambda_n}).$$

(The term corresponding to $k = 0$ is understood to be the identity matrix I.)

11. Let D be an $n \times n$ diagonal matrix, $D = \text{diag}\,(\lambda_1, \ldots, \lambda_n)$. If the matrix series $\sum_{k=0}^{\infty} c_k D^k$ converges, prove that

$$\sum_{k=0}^{\infty} c_k D^k = \text{diag}\left(\sum_{k=0}^{\infty} c_k \lambda_1^k, \ldots, \sum_{k=0}^{\infty} c_k \lambda_n^k \right).$$

12. Assume that the matrix series $\sum_{k=1}^{\infty} C_k$ converges, where each C_k is an $n \times n$ matrix. Prove that the matrix series $\sum_{k=1}^{\infty} (AC_kB)$ also converges and that its sum is the matrix

$$A\left(\sum_{k=1}^{\infty} C_k \right) B.$$

Here A and B are matrices such that the products AC_kB are meaningful.

7.5 The exponential matrix

Using Theorem 7.2 it is easy to prove that the matrix series

$$(7.12)\qquad \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

converges for every square matrix A with real or complex entries. (The term corresponding to $k = 0$ is understood to be the identity matrix Z.) The norm of each term satisfies the inequality

$$\left\| \frac{A^k}{k!} \right\| \le \frac{\|A\|^k}{k!}.$$

Since the series $\sum a^k/k!$ converges for every real a, Theorem 7.2 implies that the series in (7.12) converges for every square matrix A.

DEFINITION OF THE EXPONENTIAL MATRIX. *For any $n \times n$ matrix A with real or complex entries we define the exponential e^A to be the n x n matrix given by the convergent series in* (7.12). *That is,*

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

Note that this definition implies $e^O = I$, where 0 is the zero matrix. Further properties of the exponential will be developed with the help of differential equations.

7.6 The differential equation satisfied by e^{tA}

Let t be a real number, let A be an n x n matrix, and let $E(t)$ be the n x n matrix given by

$$E(t) = e^{tA}.$$

We shall keep A fixed and study this matrix as a function oft. First we obtain a differential equation satisfied by E.

THEOREM 7.3. *For every real t the matrix function E defined by E(t) = e^{tA} satisfies the matrix differential equation*

$$E'(t) = E(t)A = AE(t).$$

Proof. From the definition of the exponential matrix we have

$$E(t) = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}.$$

Let $c_{ij}^{(k)}$ denote the ij-entry of A^k. Then the ij-entry of $t^kA^k/k!$ is $t^kc_{ij}^{(k)}/k!$. Hence, from the definition of a matrix series, we have

$$\sum_{k=0}^{\infty}\frac{t^kA^k}{k!} = \left[\sum_{k=0}^{\infty}\frac{t^k}{k!}c_{ij}^{(k)}\right]. \tag{7.13}$$

Each entry on the right of (7.13) is a power series in t, convergent for all t. Therefore its derivative exists for all t and is given by the differentiated series

$$\sum_{k=1}^{\infty}\frac{kt^{k-1}}{k!}c_{ij}^{(k)} = \sum_{k=0}^{\infty}\frac{t^k}{k!}c_{ij}^{(k+1)}.$$

This shows that the derivative $E'(t)$ exists and is given by the matrix series

$$E'(t) = \sum_{k=0}^{\infty}\frac{t^kA^{k+1}}{k!} = \left(\sum_{k=0}^{\infty}\frac{t^kA^k}{k!}\right)A = E(t)A.$$

In the last equation we used the property $A^{k+1} = A^kA$. Since A commutes with A^k *we* could also have written $A^{k+1} = AA^k$ to obtain the relation $E'(t) = AE(t)$. This completes the proof.

Note: The foregoing proof also shows that A commutes with e^{tA} .

7.7 Uniqueness theorem for the matrix differential equation $F'(t) = AF(t)$

In this section we prove a uniqueness theorem which characterizes all solutions of the matrix differential equation $F'(t) = AF(t)$. The proof makes use of the following theorem.

THEOREM *7.4.* NONSINGULARITY OF e^{tA}. *For any $n \times n$ matrix A and any scalar t we have*

$$e^{tA}e^{-tA} = I. \tag{7.14}$$

Hence e^{tA} is nonsingular, and its inverse is e^{-tA}.

Proof. Let F be the matrix function defined for all real t by the equation

$$F(t) = e^{tA}e^{-tA}.$$

We shall prove that $F(t)$ is the identity matrix I by showing that the derivative $F'(t)$ is the zero matrix. Differentiating F as a product, using the result of Theorem 7.3, we find

$$\begin{aligned}F'(t) &= e^{tA}(e^{-tA})' + (e^{tA})'e^{-tA} = e^{tA}(-Ae^{-tA}) + Ae^{tA}e^{-tA}\\ &= -Ae^{tA}e^{-tA} + Ae^{tA}e^{-tA} = O,\end{aligned}$$

since A commutes with e^{tA}. Therefore, by the zero-derivative theorem, F is a constant matrix. But $F(0) = e^{0A}e^{0A} = I$, so $F(t) = I$ for all t. This proves (7.14).

THEOREM 7.5. UNIQUENESS THEOREM. *Let A and B be given $n \times n$ constant matrices. Then the only $n \times n$ matrix function F satisfying the initial-value problem*

$$F'(t) = AF(t), \qquad F(0) = B$$

for $-\infty < t < +\infty$ *is*

$$F(t) = e^{tA}B. \tag{7.15}$$

Proof. First we note that $e^{tA}B$ is a solution. Now let F be any solution and consider the matrix function

$$G(t) = e^{-tA}F(t).$$

Differentiating this product we obtain

$$G'(t) = e^{-tA}F'(t) - Ae^{-tA}F(t) = e^{-tA}AF(t) - e^{-tA}AF(t) = 0.$$

Therefore $G(t)$ is a constant matrix,

$$G(t) = G(0) = F(0) = B.$$

In other words, $e^{-tA}F(t) = B$. Multiplying by e^{tA} and using (7.14) we obtain (7.15).

Note: The same type of proof shows that $F(t) = Be^{tA}$ is the only solution of the initial-value problem

$$F'(t) = F(t)A, \qquad F(0) = B.$$

7.8 The law of exponents for exponential matrices

The law of exponents $e^Ae^B = e^{A+B}$ is not always true for matrix exponentials. A counter example is given in Exercise 13 of Section 7.12. However, it is not difficult to prove that the formula is true for matrices A and B which commute.

THEOREM 7.6. *Let A and B be two $n \times n$ matrices which commute, $AB = BA$. Then we have*

$$e^{A+B} = e^Ae^B. \tag{7.16}$$

Proof. From the equation $AB = BA$ we find that

$$A^2B = A(BA) = (AB)A = (BA)A = BA^2,$$

so B commutes with A^2. By induction, B commutes with every power of A. By writing e^{tA} as a power series we find that B also commutes with e^{tA} for every real t.

Now let F be the matrix function defined by the equation

$$F(t) = e^{t(A+B)} - e^{tA}e^{tB}.$$

Differentiating $F(t)$ and using the fact that B commutes with e^{tA} we find

$$F'(t) = (A + B)e^{t(A+B)} - Ae^{tA}e^{tB} - e^{tA}Be^{tB}$$
$$= (A + B)e^{t(A+B)} - (A + B)e^{tA}e^{tB} = (A + B)F(t) .$$

By the uniqueness theorem we have

$$F(t) = e^{t(A+B)}F(0) .$$

But $F(0) = O$, so $F(t) = 0$ for all t. Hence

$$e^{t(A+B)} = e^{tA}e^{tB}$$

When $t = 1$ we obtain (7.16).

EXAMPLE. The matrices sA and tA commute for all scalars s and t. Hence we have

$$e^{sA}e^{tA} = e^{(s+t)A} .$$

7.9 Existence and uniqueness theorems for homogeneous linear systems with constant coefficients

The vector differential equation $Y'(t) = A\,Y(t)$, where A is an $n \times n$ constant matrix and Y is an n-dimensional vector function (regarded as an $n \times 1$ column matrix) is called a *homogeneous linear system with constant coefficients.* We shall use the exponential matrix to give an explicit formula for the solution of such a system.

THEOREM 7.7. *Let A be a given $n \times n$ constant matrix and let B be a given n-dimensional vector. Then the initial-value problem*

(7.17) $$Y'(t) = AY(t), \qquad Y(0) = B,$$

has a unique solution on the interval $-\infty < t < +\infty$. This solution is given by the formula

(7.18) $$Y(t) = e^{tA}B .$$

More generally, the unique solution of the initial value problem

$$Y'(t) = A\,Y(t), \qquad Y(a) = B,$$

is $Y(t) = e^{(t-a)A}B$.

Proof. Differentiation of (7.18) gives us $Y'(t) = Ae^{tA}B = A\,Y(t)$. Since $Y(0) = B$, this is a solution of the initial-value problem (7.17).

To prove that it is the only solution we argue as in the proof of Theorem 7.5. Let $Z(t)$ be another vector function satisfying $Z'(t) = AZ(t)$ with $Z(0) = B$, and let $G(t) = e^{-tA}Z(t)$. Then we easily verify that $G'(t) = O$, so $G(t) = G(0) = Z(0) = B$. In other words,

$e^{-tA}Z(t) = B$, so $Z(t) = e^{tA}B = Y(t)$. The more general case with initial value $Y(a) = B$ is treated in exactly the same way.

7.10 The problem of calculating e^{tA}

Although Theorem 7.7 gives an explicit formula for the solution of a homogeneous system with constant coefficients, there still remains the problem of actually computing the exponential matrix e^{tA}. If we were to calculate e^{tA} directly from the series definition we would have to compute all the powers A^k for $k = 0, 1, 2, \ldots$, and then compute the sum of each series $\sum_{k=0}^{\infty} t^k c_{ij}^{(k)}/k!$, where $c_{ij}^{(k)}$ is the ij-entry of A^k. In general this is a hopeless task unless A is a matrix whose powers may be readily calculated. For example, if A is a diagonal matrix, say

$$A = \operatorname{diag}(\lambda_1, \ldots, \lambda_n),$$

then every power of A is also a diagonal matrix, in fact,

$$A^k = \operatorname{diag}(\lambda_1^k, \ldots, \lambda_n^k).$$

Therefore in this case e^{tA} is a diagonal matrix given by

$$e^{tA} = \operatorname{diag}\left(\sum_{k=0}^{\infty} \frac{t^k}{k!}\lambda_1^k, \ldots, \sum_{k=0}^{\infty} \frac{t^k}{k!}\lambda_n^k\right) = \operatorname{diag}(e^{t\lambda_1}, \ldots, e^{t\lambda_n}).$$

Another easy case to handle is when A is a matrix which can be diagonalized. For example, if there is a nonsingular matrix C such that $C^{-1}AC$ is a diagonal matrix, say $C^{-1}AC = D$, then we have $A = CDC^{-1}$, from which we find

$$A^2 = (CDC^{-1})(CDC^{-1}) = CD^2C^{-1},$$

and, more generally,

$$A^k = CD^kC^{-1}.$$

Therefore in this case we have

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^kA^k}{k!} = \sum_{k=0}^{\infty} \frac{t^k}{k!} CD^kC^{-1} = C\left(\sum_{k=0}^{\infty} \frac{t^kD^k}{k!}\right)C^{-1} = Ce^{tD}C^{-1}.$$

Here the difficulty lies in determining C and its inverse. Once these are known, e^{tA} is easily calculated. Of course, not every matrix can be diagonalized so the usefulness of the foregoing remarks is limited.

EXAMPLE 1. Calculate e^{tA} for the 2×2 matrix $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$.

Solution. This matrix has distinct eigenvalues $\lambda_1 = 6$, $\lambda_2 = 1$, so there is a nonsingular matrix $C = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $C^{-1}AC = D$, where $D = \operatorname{diag}(\lambda_1, \lambda_2) = \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix}$. To determine C we can write $AC = CD$, or

$$\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix}.$$

Multiplying the matrices, we find that this equation is satisfied for any scalars a, b, c, d with $a = 4c$, $b = -d$. Taking $c = d = 1$ we choose

$$C = \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix}, \qquad C^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 1 \\ -1 & 4 \end{bmatrix}.$$

Therefore

$$e^{tA} = Ce^{tD}C^{-1} = \frac{1}{5}\begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{6t} & 0 \\ 0 & e^{t} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 4 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{6t} & e^{6t} \\ -e^{t} & 4e^{t} \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 4e^{6t} + e^{t} & 4e^{6t} - 4e^{t} \\ e^{6t} - e^{t} & e^{6t} + 4e^{t} \end{bmatrix}.$$

EXAMPLE 2. Solve the linear system

$$\begin{aligned} y_1' &= 5y_1 + 4y_2 \\ y_2' &= y_1 + 2y_2 \end{aligned}$$

subject to the initial conditions $y_1(0) = 2$, $y_2(0) = 3$.

Solution. In matrix form the system can be written as

$$Y'(t) = A\,Y(t), \qquad Y(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \qquad \text{where } A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}.$$

By Theorem 7.7 the solution is $Y(t) = e^{tA}\,Y(0)$. Using the matrix e^{tA} calculated in Example 1 we find

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 4e^{6t} + e^{t} & 4e^{6t} - 4e^{t} \\ e^{6t} - e^{t} & e^{6t} + 4e^{t} \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

from which we obtain

$$y_1 = 4e^{6t} - 2e^{t}, \qquad y_2 = e^{6t} + 2e^{t}.$$

There are many methods known for calculating e^{tA} when A cannot be diagonalized. Most of these methods are rather complicated and require preliminary matrix transformations, the nature of which depends on the multiplicities of the eigenvalues of A. In a later section we shall discuss a practical and straightforward method for calculating e^{tA} which can be used whether or not A can be diagonalized. It is valid for *all* matrices A and requires no preliminary transformations of any kind. This method was developed by E. J. Putzer in a paper in the *American Mathematical Monthly*, Vol. 73 (1966), pp. 2-7. It is based on a famous theorem attributed to Arthur Cayley (1821-1895) and William Rowan Hamilton

(1805–1865) which states that every square matrix satisfies its characteristic equation. First we shall prove the Cayley-Hamilton theorem and then we shall use it to obtain Putzer's formulas for calculating e^{tA}.

7.11 The Cayley-Hamilton theorem

THEOREM 7.8. CAYLEY-HAMILTON THEOREM. *Let A be an $n \times n$ matrix and let*

$$f(\lambda) = \det(\lambda I - A) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0 \tag{7.19}$$

be its characteristic polynomial. Then $f(A) = 0$. In other words, A satisfies the equation

$$A^n + c_{n-1}A^{n-1} + \cdots + c_1 A + c_0 I = O. \tag{7.20}$$

Proof. The proof is based on Theorem 3.12 which states that for any square matrix A we have

$$A(\operatorname{cof} A)^t = (\det A)I. \tag{7.21}$$

We apply this formula with A replaced by $\lambda I - A$. Since $\det(\lambda I - A) = f(\lambda)$, Equation (7.21) becomes

$$(\lambda I - A)\{\operatorname{cof}(\lambda I - A)\}^t = f(\lambda)I. \tag{7.22}$$

This equation is valid for all real λ. The idea of the proof is to show that it is also valid when λ is replaced by A.

The entries of the matrix $\operatorname{cof}(\lambda I - A)$ are the cofactors of $\lambda I - A$. Except for a factor ± 1, each such cofactor is the determinant of a minor of $\lambda I - A$ of order $n - 1$. Therefore each entry of $\operatorname{cof}(\lambda I - A)$, and hence of $\{\operatorname{cof}(\lambda I - A)\}^t$, is a polynomial in λ of degree $\le n - 1$. Therefore

$$\{\operatorname{cof}(\lambda I - A)\}^t = \sum_{k=0}^{n-1} \lambda^k B_k,$$

where each coefficient B_k is an $n \times n$ matrix with scalar entries. Using this in (7.22) we obtain the relation

$$(\lambda I - A)\sum_{k=0}^{n-1} \lambda^k B_k = f(\lambda)I \tag{7.23}$$

which can be rewritten in the form

$$\lambda^n B_{n-1} + \sum_{k=1}^{n-1} \lambda^k (B_{k-1} - AB_k) - AB_0 = \lambda^n I + \sum_{k=1}^{n-1} \lambda^k c_k I + c_0 I. \tag{7.24}$$

At this stage we equate coefficients of like powers of λ in (7.24) to obtain the equations

$$
\begin{aligned}
B_{n-1} &= I \\
B_{n-2} - AB_{n-1} &= c_{n-1}I \\
&\;\;\vdots \\
B_0 - AB, &= c_1 I \\
-AB_0 &= c_0 I.
\end{aligned}
\tag{7.25}
$$

Equating coefficients is permissible because (7.24) is equivalent to n^2 scalar equations, in each of which we may equate coefficients of like powers of il. Now we multiply the equations in (7.25) in succession by A”, A^{n-1}, . . . , A, I and add the results. The terms on the left cancel and we obtain

$$O = A\text{”} + c_{n-1}A^{n-1} + \cdots + c_1 A + c_0 I.$$

This proves the Cayley-Hamilton theorem.

Note: Hamilton proved the theorem in 1853 for a special class of matrices. A few years later, Cayley announced that the theorem is true for all matrices, but gave no proof.

EXAMPLE. The matrix $A = \begin{bmatrix} 5 & 4 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 2 \end{bmatrix}$ has characteristic polynomial

$$f(\lambda) = (\lambda - 1)(\lambda - 2)(\lambda - 6) = \lambda^3 - 9\lambda^2 + 20\lambda - 12.$$

The Cayley-Hamilton theorem states that A satisfies the equation

$$A^3 - 9A^2 + 20A - 12I = O. \tag{7.26}$$

This equation can be used to express A^3 and all higher powers of A in terms of I, A, and A^2. For example, we have

$$
\begin{aligned}
A^3 &= 9A^2 - 20A + 121, \\
A^4 &= 9A^3 - 20A^2 + 12A = 9(9A^2 - 20A + 12I) - 20A^2 + 12A \\
&= 61A^2 - 168A + 1081.
\end{aligned}
$$

It can also be used to express A^{-1} as a polynomial in A. From (7.26) we write $A(A^2 - 9A + 201) = 12I$, and we obtain

$$A^{-1} = \tfrac{1}{12}(A^2 - 9A + 201).$$

7.12 Exercises

In each of Exercises 1 through 4, (a) express A^{-1}, A^2 and all higher powers of A as a linear combination of Z and A. (The Cayley-Hamilton theorem can be of help.) (b) Calculate e^{tA}.

1. $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. 2. $A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$. 3. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. 4. $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

5. (a) If $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, prove that $e^{tA} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$.

(b) Find a corresponding formula for e^{tA} when $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, a, b real.

6. If $F(t) = \begin{bmatrix} t & t-1 \\ 0 & 1 \end{bmatrix}$, prove that $e^{F(t)} = eF(e^{t-1})$.

7. If A(t) is a scalar function of t, the derivative of $e^{A(t)}$ is $e^{A(t)}A'(t)$. Compute the derivative of $e^{A(t)}$ when $A(t) = \begin{bmatrix} 1 & t \\ 0 & 0 \end{bmatrix}$ and show that the result is not equal to either of the two products $e^{A(t)}A'(t)$ or $A'(t)e^{A(t)}$.

In each of Exercises 8, 9, 10, (a) calculate A^n, and express A^3 in terms of I, A, A^2. (b) Calculate e^{tA}.

8. $A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. 9. $A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. 10. $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

11. If $A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, express e^{tA} as a linear combination of I, A, A^2.

12. If $A = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}$, prove that $e^A = \begin{bmatrix} x^2 & xy & y^2 \\ 2xy & x^2 + y^2 & 2xy \\ y^2 & xy & x^2 \end{bmatrix}$, where $x = \cosh 1$ and $y = \sinh 1$.

13. This example shows that the equation $e^{A+B} = e^A e^B$ is not always true for matrix exponentials. Compute each of the matrices $e^A e^B$, $e^B e^A$, e^{A+B} when $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$, and note that the three results are distinct.

7.13 Putzer's method for calculating e^{tA}

The Cayley-Hamilton theorem shows that the nth power of any $n \times n$ matrix A can be expressed as a linear combination of the lower powers $I, A, A^2, \ldots, A^{n-1}$. It follows that each of the higher powers $A^{n+1}, A^{n+2}, \ldots$, can also be expressed as a linear combination of

$I, A, A^2, \ldots, A^{n-1}$. Therefore, in the infinite series defining e^{tA}, each term $t^kA^k/k!$ with $k \geq n$ is a linear combination of $t^kI, t^kA, t^kA^2, \ldots, t^kA^{n-1}$. Hence we can expect that e^{tA} should be expressible as a polynomial in A of the form

$$e^{tA} = \sum_{k=0}^{n-1} q_k(t)A^k , \tag{7.27}$$

where the scalar coefficients $q_k(t)$ depend on t. Putzer developed two useful methods for expressing e^{tA} as a polynomial in A. The next theorem describes the simpler of the two methods.

THEOREM 7.9. *Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of an $n \times n$ matrix A, and define a sequence of polynomials in A as follows:*

$$P_0(A) = I, \qquad P_k(A) = \prod_{m=1}^{k} (A - \lambda_m I), \qquad \textit{for} \quad k = 1, 2, \ldots, n. \tag{7.28}$$

Then we have

$$e^{tA} = \sum_{k=0}^{n-1} r_{k+1}(t)P_k(A) , \tag{7.29}$$

where the scalar coefficients $r_1(t), \ldots, r_n(t)$ are determined recursively from the system of linear differential equations

$$\begin{aligned} r_1'(t) &= \lambda_1 r_1(t) , & r_1(0) &= 1 , \\ r_{k+1}'(t) &= \lambda_{k+1} r_{k+1}(t) + r_k(t), & r_{k+1}(0) &= 0 , \qquad (k = 1, 2, \ldots, n-1) . \end{aligned} \tag{7.30}$$

Note: Equation (7.29) does not express e^{tA} directly in powers of A as indicated in (7.27), but as a linear combination of the polynomials $P_0(A), P_1(A), \ldots, P_{n-1}(A)$. These polynomials are easily calculated once the eigenvalues of A are determined. Also the multipliers $r_1(t), \ldots, r_n(t)$ in (7.30) are easily calculated. Although this requires solving a system of linear differential equations, this particular system has a triangular matrix and the solutions can be determined in succession.

Proof. Let $r_1(t), \ldots, r_n(t)$ be the scalar functions determined by (7.30) and define a matrix function F by the equation

$$F(t) = \sum_{k=0}^{n-1} r_{k+1}(t)P_k(A) . \tag{7.31}$$

Note that $F(0) = r_1(0)P_0(A) = I$. We will prove that $F(t) = e^{tA}$ by showing that F satisfies the same differential equation as e^{tA}, namely, $F'(t) = AF(t)$.

Differentiating (7.31) and using the recursion formulas (7.30) we obtain

$$F'(t) = \sum_{k=0}^{n-1} r_{k+1}'(t)P_k(A) = \sum_{k=0}^{n-1} \{r_k(t) + \lambda_{k+1} r_{k+1}(t)\} P_k(A) ,$$

where $r_n(t)$ is defined to be 0. We rewrite this in the form

$$F'(t) = \sum_{k=0}^{n-2} r_{k+1}(t)P_{k+1}(A) + \sum_{k=0}^{n-1} \lambda_{k+1}r_{k+1}(t)P_k(A),$$

then subtract $\lambda_n F(t) = \sum_{k=0}^{n-1} \lambda_n r_{k+1}(t)P_k(A)$ to obtain the relation

(7.32) $$F'(t) - \lambda_n F(t) = \sum_{k=0}^{n-2} r_{k+1}(t)\{P_{k+1}(A) + (\lambda_{k+1} - \lambda_n)P_k(A)\}.$$

But from (7.28) we see that $P_{k+1}(A) = (A - \lambda_{k+1}I)P_k(A)$, so

$$\begin{aligned} P_{k+1}(A) + (\lambda_{k+1} - \lambda_n)P_k(A) &= (A - \lambda_{k+1}I)P_k(A) + (\lambda_{k+1} - \lambda_n)P_k(A) \\ &= (A - \lambda_n I)P_k(A). \end{aligned}$$

Therefore Equation (7.32) becomes

$$\begin{aligned} F'(t) - \lambda_n F(t) = (A - \lambda_n I)\sum_{k=0}^{n-2} r_{k+1}(t)P_k(A) &= (A - \lambda_n I)\{F(t) - r_n(t)P_{n-1}(A)\} \\ &= (A - \lambda_n I)F(t) - r_n(t)P_n(A). \end{aligned}$$

The Cayley-Hamilton theorem implies that $P_n(A) = 0$, so the last equation becomes

$$F'(t) - \lambda_n F(t) = (A - \lambda_n I)F(t) = AF(t) - \lambda_n F(t),$$

from which we find $F'(t) = AF(t)$. Since $F(0) = I$, the uniqueness theorem (Theorem 7.7) shows that $F(t) = e^{tA}$.

EXAMPLE I. Express e^{tA} as a linear combination of Zand A if A is a 2 x 2 matrix with both its eigenvalues equal to il.

Solution. Writing $\lambda_1 = \lambda_2 = \lambda$, we are to solve the system of differential equations

$$\begin{aligned} r_1'(t) &= \lambda r_1(t), & r_1(0) &= 1, \\ r_2'(t) &= \lambda r_2(t) + r_1(t), & r_2(0) &= 0. \end{aligned}$$

Solving these first-order equations in succession we find

$$r_1(t) = e^{\lambda t}, \qquad r_2(t) = te^{\lambda t}.$$

Since $P_0(A) = I$ and $P_1(A) = A - \lambda I$, the required formula for e^{tA} is

(7.33) $$e^{tA} = e^{\lambda t}I + te^{\lambda t}(A - \lambda I) = e^{\lambda t}(1 - \lambda t)I + te^{\lambda t}A.$$

EXAMPLE 2. Solve Example 1 if the eigenvalues of A are λ and μ, where $\lambda \neq \mu$.

Solution. In this case the system of differential equations is

$$r_1'(t) = \lambda r_1(t), \qquad r_1(0) = 1,$$
$$r_2'(t) = \mu r_2(t) + r_1(t), \qquad r,(O) = 0.$$

Its solutions are given by

$$r_1(t) = e^{\lambda t}, \qquad r_2(t) = \frac{e^{\lambda t} - e^{\mu t}}{\lambda - \mu}.$$

Since $P,(A) = Z$ and $P,(A) = A - \lambda I$ the required formula for e^{tA} is

$$e^{tA} = e^{\lambda t} I + \frac{e^{\lambda t} - e^{\mu t}}{\lambda - \mu}(A - \lambda I) = \frac{\lambda e^{\mu t} - \mu e^{\lambda t}}{\lambda - \mu} I + \frac{e^{\lambda t} - e^{\mu t}}{\lambda - \mu} A. \tag{7.34}$$

If the eigenvalues λ, μ are complex numbers, the exponentials $e^{\lambda t}$ and $e^{\mu t}$ will also be complex numbers. But if λ and μ are complex conjugates, the scalars multiplying Z and A in (7.34) will be real. For example, suppose

$$\lambda = \alpha + i\beta, \qquad \mu = \alpha - i\beta, \qquad \beta \neq 0.$$

Then $\lambda - \mu = 2i\beta$ so Equation (7.34) becomes

$$\begin{aligned} e^{tA} &= e^{(\alpha+i\beta)t} I + \frac{e^{(\alpha+i\beta)t} - e^{(\alpha-i\beta)t}}{2i\beta}[A - (\alpha + i\beta)I] \\ &= e^{\alpha t}\left\{e^{i\beta t} I + \frac{e^{i\beta t} - e^{-i\beta t}}{2i\beta}(A - \alpha I - i\beta I)\right\} \\ &= e^{\alpha t}\left\{(\cos \beta t + i \sin \beta t) I + \frac{\sin \beta t}{\beta}(A - \alpha I - i\beta I)\right\}. \end{aligned}$$

The terms involving i cancel and we get

$$e^{tA} = \frac{e^{\alpha t}}{\beta}\{(\beta \cos \beta t - a \sin \beta t) I + \sin \beta t\, A\}. \tag{7.35}$$

7.14 Alternate methods for calculating e^{tA} in special cases

Putzer's method for expressing e^{tA} as a polynomial in A is completely general because it is valid for all square matrices A. A general method is not always the simplest method to use in certain special cases. In this section we give simpler methods for computing e^{tA} in three special cases: (a) When all the eigenvalues of A are equal, (b) when all the eigenvalues of A are distinct, and (c) when A has two distinct eigenvalues, exactly one of which has multiplicity 1.