Corollary 14.8.15 (Weierstrass approximation theorem I). *Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function supported on $[0, 1]$. Then for every $\varepsilon > 0$, there exists a function $P : \mathbf{R} \to \mathbf{R}$ which is polynomial on $[0, 1]$ and such that $|P(x) - f(x)| \leq \varepsilon$ for all $x \in [0, 1]$.*

Proof. See Exercise 14.8.15. □

Now we perform a series of modifications to convert Corollary 14.8.15 into the actual Weierstrass approximation theorem. We first need a simple lemma.

Lemma 14.8.16. *Let $f : [0, 1] \to \mathbf{R}$ be a continuous function which equals 0 on the boundary of $[0, 1]$, i.e., $f(0) = f(1) = 0$. Let $F : \mathbf{R} \to \mathbf{R}$ be the function defined by setting $F(x) := f(x)$ for $x \in [0, 1]$ and $F(x) := 0$ for $x \notin [0, 1]$. Then F is also continuous.*

Proof. See Exercise 14.8.16. □

Remark 14.8.17. The function F obtained in Lemma 14.8.16 is sometimes known as the *extension of f by zero.*

From Corollary 14.8.15 and Lemma 14.8.16 we immediately obtain

Corollary 14.8.18 (Weierstrass approximation theorem II). *Let $f : [0, 1] \to \mathbf{R}$ be a continuous function supported on $[0, 1]$ such that $f(0) = f(1) = 0$. Then for every $\varepsilon > 0$ there exists a polynomial $P : [0, 1] \to \mathbf{R}$ such that $|P(x) - f(x)| \leq \varepsilon$ for all $x \in [0, 1]$.*

Now we strengthen Corollary 14.8.18 by removing the assumption that $f(0) = f(1) = 0$.

Corollary 14.8.19 (Weierstrass approximation theorem III). *Let $f : [0, 1] \to \mathbf{R}$ be a continuous function supported on $[0, 1]$. Then for every $\varepsilon > 0$ there exists a polynomial $P : [0, 1] \to \mathbf{R}$ such that $|P(x) - f(x)| \leq \varepsilon$ for all $x \in [0, 1]$.*

Proof. Let $F : [0,1] \to \mathbf{R}$ denote the function

$$F(x) := f(x) - f(0) - x(f(1) - f(0)).$$

Observe that F is also continuous (why?), and that $F(0) = F(1) = 0$. By Corollary 14.8.18, we can thus find a polynomial $Q : [0,1] \to \mathbf{R}$ such that $|Q(x) - F(x)| \leq \varepsilon$ for all $x \in [0,1]$. But

$$Q(x) - F(x) = Q(x) + f(0) + x(f(1) - f(0)) - f(x),$$

so the claim follows by setting P to be the polynomial $P(x) := Q(x) + f(0) + x(f(1) - f(0))$. □

Finally, we can prove the full Weierstrass approximation theorem.

Proof of Theorem 14.8.3. Let $f : [a,b] \to \mathbf{R}$ be a continuous function on $[a,b]$. Let $g : [0,1] \to \mathbf{R}$ denote the function

$$g(x) := f(a + (b-a)x) \text{ for all } x \in [0,1]$$

Observe then that

$$f(y) = g((y-a)/(b-a)) \text{ for all } y \in [a,b].$$

The function g is continuous on $[0,1]$ (why?), and so by Corollary 14.8.19 we may find a polynomial $Q : [0,1] \to \mathbf{R}$ such that $|Q(x) - g(x)| \leq \varepsilon$ for all $x \in [0,1]$. In particular, for any $y \in [a,b]$, we have

$$|Q((y-a)/(b-a)) - g((y-a)/(b-a))| \leq \varepsilon.$$

If we thus set $P(y) := Q((y-a)/(b-a))$, then we observe that P is also a polynomial (why?), and so we have $|P(y) - g(y)| \leq \varepsilon$ for all $y \in [a,b]$, as desired. □

Remark 14.8.20. Note that the Weierstrass approximation theorem only works on bounded intervals $[a,b]$; continuous functions on $\mathbf{R}$ cannot be uniformly approximated by polynomials. For instance, the exponential function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) := e^x$ (which we shall study rigourously in Section 15.5) cannot be approximated by any polynomial, because exponential functions grow faster than any polynomial (Exercise 15.5.9) and so there is no way one can even make the sup metric between f and a polynomial finite.

Remark 14.8.21. There is a generalization of the Weierstrass approximation theorem to higher dimensions: if K is any compact subset of $\mathbf{R}^n$ (with the Euclidean metric d_{l^2}), and $f : K \to \mathbf{R}$ is a continuous function, then for every $\varepsilon > 0$ there exists a polynomial $P : K \to \mathbf{R}$ of n variables $x_1, \ldots, x_n$ such that $d_\infty(f, P) < \varepsilon$. This general theorem can be proven by a more complicated variant of the arguments here, but we will not do so. (There is in fact an even more general version of this theorem applicable to an arbitrary metric space, known as the *Stone-Weierstrass theorem*, but this is beyond the scope of this text.)

Exercise 14.8.1. Prove Lemma 14.8.5.

Exercise 14.8.2. (a) Prove that for any real number $0 \leq y \leq 1$ and any natural number $n \geq 0$, that $(1-y)^n \geq 1 - ny$. (Hint: induct on n. Alternatively, differentiate with respect to y.)

(b) Show that $\int_{-1}^{1} (1-x^2)^n \, dx \geq \frac{1}{\sqrt{n}}$. (Hint: for $|x| \leq 1/\sqrt{n}$, use part (a); for $|x| \geq 1/\sqrt{n}$, just use the fact that $(1-x^2)$ is positive. It is also possible to proceed via trigonometric substitution, but I would not recommend this unless you know what you are doing.)

(c) Prove Lemma 14.8.2. (Hint: choose $f(x)$ to equal $c(1-x^2)^N$ for $x \in [-1, 1]$ and to equal zero for $x \notin [-1, 1]$, where N is a large number N, where c is chosen so that f has integral 1, and use (b).)

Exercise 14.8.3. Let $f : \mathbf{R} \to \mathbf{R}$ be a compactly supported, continuous function. Show that f is bounded and uniformly continuous. (Hint: the idea is to use Proposition 13.3.2 and Theorem 13.3.5, but one must first deal with the issue that the domain $\mathbf{R}$ of f is non-compact.)

Exercise 14.8.4. Prove Proposition 14.8.11. (Hint: to show that $f * g$ is continuous, use Exercise 14.8.3.)

Exercise 14.8.5. Let $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ be continuous, compactly supported functions. Suppose that f is supported on the interval

$[0,1]$, and g is constant on the interval $[0,2]$ (i.e., there is a real number c such that $g(x) = c$ for all $x \in [0,2]$). Show that the convolution $f * g$ is constant on the interval $[1,2]$.

Exercise 14.8.6. (a) Let g be an (ε, δ) approximation to the identity. Show that $1 - 2\varepsilon \leq \int_{[-\delta,\delta]} g \leq 1$.

(b) Prove Lemma 14.8.14. (Hint: begin with the identity

$$f * g(x) = \int f(x-y)g(y)\ dy = \int_{[-\delta,\delta]} f(x-y)g(y)\ dy$$

$$+ \int_{[\delta,1]} f(x-y)g(y)\ dy + \int_{[-1,-\delta]} f(x-y)g(y)\ dy.$$

The idea is to show that the first integral is close to $f(x)$, and that the second and third integrals are very small. To achieve the former task, use (a) and the fact that $f(x)$ and $f(x-y)$ are within ε of each other; to achieve the latter task, use property (c) of the approximation to the identity and the fact that f is bounded.)

Exercise 14.8.7. Prove Corollary 14.8.15. (Hint: combine Exercise 14.8.3, Lemma 14.8.8, Lemma 14.8.13, and Lemma 14.8.14.)

Exercise 14.8.8. Let $f : [0,1] \to \mathbf{R}$ be a continuous function, and suppose that $\int_{[0,1]} f(x)x^n\ dx = 0$ for all non-negative integers $n = 0, 1, 2, \ldots$. Show that f must be the zero function $f \equiv 0$. (Hint: first show that $\int_{[0,1]} f(x)P(x)\ dx = 0$ for all polynomials P. Then, using the Weierstrass approximation theorem, show that $\int_{[0,1]} f(x)f(x)\ dx = 0$.)

Chapter 15

Power series

15.1 Formal power series

We now discuss an important subclass of series of functions, that of *power series*. As in earlier chapters, we begin by introducing the notion of a formal power series, and then focus in later sections on when the series converges to a meaningful function, and what one can say about the function obtained in this manner.

Definition 15.1.1 (Formal power series). Let a be a real number. A *formal power series centered at a* is any series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

where $c_0, c_1, \ldots$ is a sequence of real numbers (not depending on x); we refer to c_n as the n^{th} *coefficient* of this series. Note that each term $c_n(x-a)^n$ in this series is a function of a real variable x.

Example 15.1.2. The series $\sum_{n=0}^{\infty} n!(x-2)^n$ is a formal power series centered at 2. The series $\sum_{n=0}^{\infty} 2^x (x-3)^n$ is not a formal power series, since the coefficients 2^x depend on x.

We call these power series *formal* because we do not yet assume that these series converge for any x. However, these series are automatically guaranteed to converge when $x = a$ (why?).

In general, the closer x gets to a, the easier it is for this series to converge. To make this more precise, we need the following definition.

Definition 15.1.3 (Radius of convergence). Let $\sum_{n=0}^{\infty} c_n(x-a)^n$ be a formal power series. We define the *radius of convergence* R of this series to be the quantity

$$R := \frac{1}{\limsup_{n\to\infty} |c_n|^{1/n}}$$

where we adopt the convention that $\frac{1}{0} = +\infty$ and $\frac{1}{+\infty} = 0$.

Remark 15.1.4. Each number $|c_n|^{1/n}$ is non-negative, so the limit $\limsup_{n\to\infty} |c_n|^{1/n}$ can take on any value from 0 to $+\infty$, inclusive. Thus R can also take on any value between 0 and $+\infty$ inclusive (in particular it is not necessarily a real number). Note that the radius of convergence always exists, even if the sequence $|c_n|^{1/n}$ is not convergent, because the lim sup of any sequence always exists (though it might be $+\infty$ or $-\infty$).

Example 15.1.5. The series $\sum_{n=0}^{\infty} n(-2)^n(x-3)^n$ has radius of convergence

$$\frac{1}{\limsup_{n\to\infty} |n(-2^n)|^{1/n}} = \frac{1}{\limsup_{n\to\infty} 2n^{1/n}} = \frac{1}{2}.$$

The series $\sum_{n=0}^{\infty} 2^{n^2}(x+2)^n$ has radius of convergence

$$\frac{1}{\limsup_{n\to\infty} |2^{n^2}|^{1/n}} = \frac{1}{\limsup_{n\to\infty} 2^n} = \frac{1}{+\infty} = 0.$$

The series $\sum_{n=0}^{\infty} 2^{-n^2}(x+2)^n$ has radius of convergence

$$\frac{1}{\limsup_{n\to\infty} |2^{-n^2}|^{1/n}} = \frac{1}{\limsup_{n\to\infty} 2^{-n}} = \frac{1}{0} = +\infty.$$

The significance of the radius of convergence is the following.

Theorem 15.1.6. *Let $\sum_{n=0}^{\infty} c_n(x-a)^n$ be a formal power series, and let R be its radius of convergence.*

(a) *(Divergence outside of the radius of convergence) If $x \in \mathbf{R}$ is such that $|x-a| > R$, then the series $\sum_{n=0}^{\infty} c_n(x-a)^n$ is divergent for that value of x.*

(b) *(Convergence inside the radius of convergence) If $x \in \mathbf{R}$ is such that $|x-a| < R$, then the series $\sum_{n=0}^{\infty} c_n(x-a)^n$ is absolutely convergent for that value of x.*

For parts (c)-(e) we assume that $R > 0$ (i.e., the series converges at at least one other point than $x = a$). Let $f : (a-R, a+R) \to \mathbf{R}$ be the function $f(x) := \sum_{n=0}^{\infty} c_n(x-a)^n$; this function is guaranteed to exist by (b).

(c) *(Uniform convergence on compact sets) For any $0 < r < R$, the series $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges uniformly to f on the compact interval $[a-r, a+r]$. In particular, f is continuous on $(a-R, a+R)$.*

(d) *(Differentiation of power series) The function f is differentiable on $(a-R, a+R)$, and for any $0 < r < R$, the series $\sum_{n=0}^{\infty} nc_n(x-a)^{n-1}$ converges uniformly to f' on the interval $[a-r, a+r]$.*

(e) *(Integration of power series) For any closed interval $[y, z]$ contained in $(a-R, a+R)$, we have*

$$\int_{[y,z]} f = \sum_{n=0}^{\infty} c_n \frac{(z-a)^{n+1} - (y-a)^{n+1}}{n+1}.$$

Proof. See Exercise 15.1.1. □

Theorem 15.1.6 (a) and (b) of the above theorem give another way to find the radius of convergence, by using your favorite convergence test to work out the range of x for which the power series converges:

Example 15.1.7. Consider the power series $\sum_{n=0}^{\infty} n(x-1)^n$. The ratio test shows that this series converges when $|x-1| < 1$ and diverges when $|x-1| > 1$ (why?). Thus the only possible value for the radius of convergence is $R = 1$ (if $R < 1$, then we have contradicted Theorem 15.1.6(a); if $R > 1$, then we have contradicted Theorem 15.1.6(b)).

Remark 15.1.8. Theorem 15.1.6 is silent on what happens when $|x - a| = R$, i.e., at the points $a - R$ and $a + R$. Indeed, one can have either convergence or divergence at those points; see Exercise 15.1.2.

Remark 15.1.9. Note that while Theorem 15.1.6 assures us that the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ will converge pointwise on the interval $(a - R, a + R)$, it need not converge uniformly on that interval (see Exercise 15.1.2(e)). On the other hand, Theorem 15.1.6(c) assures us that the power series will converge on any smaller interval $[a - r, a + r]$. In particular, being uniformly convergent on every closed subinterval of $(a - R, a + R)$ is not enough to guarantee being uniformly convergent on all of $(a - R, a + R)$.

Exercise 15.1.1. Prove Theorem 15.1.6. (Hints: for (a) and (b), use the root test (Theorem 7.5.1). For (c), use the Weierstrass M-test (Theorem 14.5.7). For (d), use Theorem 14.7.1. For (e), use Corollary 14.8.18.

Exercise 15.1.2. Give examples of a formal power series $\sum_{n=0}^{\infty} c_n x^n$ centered at 0 with radius of convergence 1, which

(a) diverges at both $x = 1$ and $x = -1$;

(b) diverges at $x = 1$ but converges at $x = -1$;

(c) converges at $x = 1$ but diverges at $x = -1$;

(d) converges at both $x = 1$ and $x = -1$.

(e) converges pointwise on $(-1, 1)$, but does not converge uniformly on $(-1, 1)$.

15.2 Real analytic functions

A function $f(x)$ which is lucky enough to be representable as a power series has a special name; it is a *real analytic* function.

Definition 15.2.1 (Real analytic functions). Let E be a subset of $\mathbf{R}$, and let $f : E \to \mathbf{R}$ be a function. If a is an interior point of E, we say that f is *real analytic at a* if there exists an open interval $(a - r, a + r)$ in E for some $r > 0$ such that there exists a power series $\sum_{n=0}^{\infty} c_n(x - a)^n$ centered at a which has a radius of convergence greater than or equal to r, and which converges to f on $(a - r, a + r)$. If E is an open set, and f is real analytic at every point a of E, we say that f is *real analytic on E*.

Example 15.2.2. Consider the function $f : \mathbf{R}\backslash\{1\} \to \mathbf{R}$ defined by $f(x) := 1/(1 - x)$. This function is real analytic at 0 because we have a power series $\sum_{n=0}^{\infty} x^n$ centred at 0 which converges to $1/(1-x) = f(x)$ on the interval $(-1, 1)$. This function is also real analytic at 2 because we have a power series $\sum_{n=0}^{\infty}(-1)^{n+1}(x-2)^n$ which converges to $\frac{-1}{1-(-(x-2))} = \frac{1}{x-1} = f(x)$ on the interval $(1, 3)$ (why? use Lemma 7.3.3). In fact this function is real analytic on all of $\mathbf{R}\backslash\{1\}$; see Exercise 15.2.2.

Remark 15.2.3. The notion of being real analytic is closely related to another notion, that of being *complex analytic*, but this is a topic for complex analysis, and will not be discussed here.

We now discuss which functions are real analytic. From Theorem 15.1.6(c) and (d) we see that if f is real analytic at a point a, then f is both continuous and differentiable on $(a-r, a+r)$ for some $a \in \mathbf{R}$. We can in fact say more:

Definition 15.2.4 (k-times differentiability). Let E be a subset of $\mathbf{R}$. We say a function $f : E \to \mathbf{R}$ is *once differentiable on E* iff it is differentiable. More generally, for any $k \geq 2$ we say that $f : E \to \mathbf{R}$ is *k times differentiable on E*, or just *k times differentiable*, iff f is differentiable, and f' is $k - 1$ times differentiable. If f is k times differentiable, we define the k^{th} derivative $f^{(k)} : E \to \mathbf{R}$ by the recursive rule $f^{(1)} := f'$, and $f^{(k)} = (f^{(k-1)})'$ for all $k \geq 2$. We also define $f^{(0)} := f$ (this is f differentiated 0 times), and we allow every function to be zero times differentiable (since clearly $f^{(0)}$ exists for every f). A function is said to be *infinitely differentiable* (or *smooth*) iff it is k times differentiable for every $k \geq 0$.

Example 15.2.5. The function $f(x) := |x|^3$ is twice differentiable on $\mathbf{R}$, but not three times differentiable (why?). Indeed, $f^{(2)} = f'' = 6|x|$, which is not differentiable, at 0.

Proposition 15.2.6 (Real analytic functions are k-times differentiable). *Let E be a subset of $\mathbf{R}$, let a be an interior point of E, and and let f be a function which is real analytic at a, thus there is an $r > 0$ for which we have the power series expansion*

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

for all $x \in (a-r, a+r)$. Then for every $k \geq 0$, the function f is k-times differentiable on $(a-r, a+r)$, and for each $k \geq 0$ the k^{th} derivative is given by

$$\begin{aligned} f^{(k)}(x) &= \sum_{n=0}^{\infty} c_{n+k}(n+1)(n+2)\ldots(n+k)(x-a)^n \\ &= \sum_{n=0}^{\infty} c_{n+k} \frac{(n+k)!}{n!}(x-a)^n \end{aligned}$$

for all $x \in (a-r, a+r)$.

Proof. See Exercise 15.2.3. □

Corollary 15.2.7 (Real analytic functions are infinitely differentiable). *Let E be an open subset of $\mathbf{R}$, and let $f : E \to \mathbf{R}$ be a real analytic function on E. Then f is infinitely differentiable on E. Also, all derivatives of f are also real analytic on E.*

Proof. For every point $a \in E$ and $k \geq 0$, we know from Proposition 15.2.6 that f is k-times differentiable at a (we will have to apply Exercise 10.1.1 k times here, why?). Thus f is k-times differentiable on E for every $k \geq 0$ and is hence infinitely differentiable. Also, from Proposition 15.2.6 we see that each derivative $f^{(k)}$ of f has a convergent power series expansion at every $x \in E$ and thus $f^{(k)}$ is real analytic. □

Example 15.2.8. Consider the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) := |x|$. This function is not differentiable at $x = 0$, and hence cannot be real analytic at $x = 0$. It is however real analytic at every other point $x \in \mathbf{R}\backslash\{0\}$ (why?).

Remark 15.2.9. The converse statement to Corollary 15.2.7 is not true; there are infinitely differentiable functions which are not real analytic. See Exercise 15.5.4.

Proposition 15.2.6 has an important corollary, due to Brook Taylor (1685–1731).

Corollary 15.2.10 (Taylor's formula). *Let E be a subset of $\mathbf{R}$, let a be an interior point of E, and let $f : E \to \mathbf{R}$ be a function which is real analytic at a and has the power series expansion*

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

for all $x \in (a - r, a + r)$ and some $r > 0$. Then for any integer $k \geq 0$, we have

$$f^{(k)}(a) = k! c_k,$$

where $k! := 1 \times 2 \times \ldots \times k$ (and we adopt the convention that $0! = 1$). In particular, we have Taylor's formula

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

for all x in $(a - r, a + r)$.

Proof. See Exercise 15.2.4. □

The power series $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ is sometimes called the *Taylor series* of f around a. Taylor's formula thus asserts that if a function is real analytic, then it is equal to its Taylor series.

Remark 15.2.11. Note that Taylor's formula only works for functions which are real analytic; there are examples of functions which are infinitely differentiable but for which Taylor's theorem fails (see Exercise 15.5.4).

Another important corollary of Taylor's formula is that a real analytic function can have at most one power series at a point:

Corollary 15.2.12 (Uniqueness of power series). *Let E be a subset of $\mathbf{R}$, let a be an interior point of E, and let $f : E \to \mathbf{R}$ be a function which is real analytic at a. Suppose that f has two power series expansions*

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

and

$$f(x) = \sum_{n=0}^{\infty} d_n (x-a)^n$$

centered at a, ecah with a non-zero radius of convergence. Then $c_n = d_n$ for all $n \geq 0$.

Proof. By Corollary 15.2.10, we have $f^{(k)}(a) = k! c_k$ for all $k \geq 0$. But we also have $f^{(k)}(a) = k! d_k$, by similar reasoning. Since $k!$ is never zero, we can cancel it and obtain $c_k = d_k$ for all $k \geq 0$, as desired. □

Remark 15.2.13. While a real analytic function has a unique power series around any given point, it can certainly have different power series at different points. For instance, the function $f(x) := \frac{1}{1-x}$, defined on $\mathbf{R} - \{1\}$, has the power series

$$f(x) := \sum_{n=0}^{\infty} x^n$$

around 0, on the interval $(-1, 1)$, but also has the power series

$$f(x) = \frac{1}{1-x} = \frac{2}{1 - 2(x - \frac{1}{2})} = \sum_{n=0}^{\infty} 2(2(x - \frac{1}{2}))^n = \sum_{n=0}^{\infty} 2^{n+1}(x - \frac{1}{2})^n$$

around $1/2$, on the interval $(0, 1)$ (note that the above power series has a radius of convergence of $1/2$, thanks to the root test. See also Exercise 15.2.8.

Exercise 15.2.1. Let $n \geq 0$ be an integer, let c, a be real numbers, and let f be the function $f(x) := c(x-a)^n$. Show that f is infinitely differentiable, and that $f^{(k)}(x) = c\frac{n!}{(n-k)!}(x-a)^{n-k}$ for all integers $0 \leq k \leq n$. What happens when $k > n$?

Exercise 15.2.2. Show that the function f defined in Example 15.2.2 is real analytic on all of $\mathbf{R}\backslash\{1\}$.

Exercise 15.2.3. Prove Proposition 15.2.6. (Hint: induct on k and use Theorem 15.1.6(d)).

Exercise 15.2.4. Use Proposition 15.2.6 and Exercise 15.2.1 to prove Corollary 15.2.10.

Exercise 15.2.5. Let a, b be real numbers, and let $n \geq 0$ be an integer. Prove the identity

$$(x-a)^n = \sum_{m=0}^{n} \frac{n!}{m!(n-m)!}(b-a)^{n-m}(x-b)^n$$

for any real number x. (Hint: use the binomial formula, Exercise 7.1.4.) Explain why this identity is consistent with Taylor's theorem and Exercise 15.2.1. (Note however that Taylor's theorem cannot be rigourously applied until one verifies Exercise 15.2.6 below.)

Exercise 15.2.6. Using Exercise 15.2.5, show that every polynomial $P(x)$ of one variable is real analytic on $\mathbf{R}$.

Exercise 15.2.7. Let $m \geq 0$ be a positive integer, and let $0 < x < r$ be real numbers. Use Lemma 7.3.3 to establish the identity

$$\frac{r}{r-x} = \sum_{n=0}^{\infty} x^n r^{-n}$$

for all $x \in (-r, r)$. Using Proposition 15.2.6, conclude the identity

$$\frac{r}{(r-x)^{m+1}} = \sum_{n=m}^{\infty} \frac{n!}{m!(n-m)!} x^{n-m} r^{-n}$$

for all integers $m \geq 0$ and $x \in (-r, r)$. Also explain why the series on the right-hand side is absolutely convergent.

Exercise 15.2.8. Let E be a subset of $\mathbf{R}$, let a be an interior point of E, and let $f : E \to \mathbf{R}$ be a function which is real analytic in a, and has a power series expansion

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

at a which converges on the interval $(a-r, a+r)$. Let $(b-s, b+s)$ be any sub-interval of $(a-r, a+r)$ for some $s > 0$.

(a) Prove that $|a-b| \leq r-s$, so in particular $|a-b| < r$.

(b) Show that for every $0 < \varepsilon < r$, there exists a $C > 0$ such that $|c_n| \leq C(r-\varepsilon)^{-n}$ for all integers $n \geq 0$. (Hint: what do we know about the radius of convergence of the series $\sum_{n=0}^{\infty} c_n(x-a)^n$?)

(c) Show that the numbers $d_0, d_1, \ldots$ given by the formula

$$d_m := \sum_{n=m}^{\infty} \frac{n!}{m!(n-m)!}(b-a)^{n-m} c_n \text{ for all integers } m \geq 0$$

are well-defined, in the sense that the above series is absolutely convergent. (Hint: use (b) and the comparison test, Corollary 7.3.2, followed by Exercise 15.2.7.)

(d) Show that for every $0 < \varepsilon < s$ there exists a $C > 0$ such that

$$|d_m| \leq C(s-\varepsilon)^m$$

for all integers $m \geq 0$. (Hint: use the comparison test, and Exercise 15.2.7.)

(e) Show that the power series $\sum_{m=0}^{\infty} d_m(x-b)^n$ is absolutely convergent for $x \in (b-s, b+s)$ and converges to $f(x)$. (You may need Fubini's theorem for infinite series, Theorem 8.2.2, as well as Exercise 15.2.5).

(f) Conclude that f is real analytic at every point in $(a-r, a+r)$.

15.3 Abel's theorem

Let $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ be a power series centered at a with a radius of convergence $0 < R < \infty$ strictly between 0 and infinity.

From Theorem 15.1.6 we know that the power series converges absolutely whenever $|x-a| < R$, and diverges when $|x-a| > R$. However, at the boundary $|x-a| = R$ the situation is more complicated; the series may either converge or diverge (see Exercise 15.1.2). However, if the series does converge at the boundary point, then it is reasonably well behaved; in particular, it is continuous at that boundary point.

Theorem 15.3.1 (Abel's theorem). *Let $f(x) = \sum_{n=0}^\infty c_n(x-a)^n$ be a power series centered at a with radius of convergence $0 < R < \infty$. If the power series converges at $a+R$, then f is continuous at $a+R$, i.e.*

$$\lim_{x\to a+R: x\in(a-R,a+R)} \sum_{n=0}^\infty c_n(x-a)^n = \sum_{n=0}^\infty c_n R^n.$$

Similarly, if the power series converges at $a-R$, then f is continuous at $a-R$, i.e.

$$\lim_{x\to a-R: x\in(a-R,a+R)} \sum_{n=0}^\infty c_n(x-a)^n = \sum_{n=0}^\infty c_n(-R)^n.$$

Before we prove Abel's theorem, we need the following lemma.

Lemma 15.3.2 (Summation by parts formula). *Let $(a_n)_{n=0}^\infty$ and $(b_n)_{n=0}^\infty$ be sequences of real numbers which converge to limits A and B respectively, i.e., $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$. Suppose that the sum $\sum_{n=0}^\infty (a_{n+1}-a_n)b_n$ is convergent. Then the sum $\sum_{n=0}^\infty a_{n+1}(b_{n+1}-b_n)$ is also convergent, and*

$$\sum_{n=0}^\infty (a_{n+1}-a_n)b_n = AB - a_0b_0 - \sum_{n=0}^\infty a_{n+1}(b_{n+1}-b_n).$$

Proof. See Exercise 15.3.1. □

Remark 15.3.3. One should compare this formula with the more well-known *integration by parts formula*

$$\int_0^\infty f'(x)g(x)\,dx = f(x)g(x)|_0^\infty - \int_0^\infty f(x)g'(x)\,dx,$$

see Proposition 11.10.1.