Let p and q be distinct odd primes and write $f(x,y) = py - qx$. Consider the region of the Cartesian plane consisting of all pairs of real numbers (x,y) such that

$$1/2 < x < p/2, \quad 1/2 < y < q/2 \ .$$

This region is subdivided into four mutually disjoint parts

$$A: \ f(x,y) < -q/2, \quad B: \ -q/2 < f(x,y) < 0 \ ,$$

$$C: \ 0 < f(x,y) < p/2, \quad D: \ p/2 < f(x,y) \ .$$

Replacing x by $\frac{p+1}{2} - x$ and y by $\frac{q+1}{2} - y$, we establish a one-to-one correspondence between A and D. For any subset S of the Cartesian plane, let $L(S)$ denote the number of pairs of integers (x,y) in S; then clearly $L(A) = L(D)$. Now

$$L(A) + L(B) + L(C) + L(D) = \frac{p-1}{2} \cdot \frac{q-1}{2} \ ,$$

so that

$$L(B) + L(C) = \frac{p-1}{2} \cdot \frac{q-1}{2} - 2L(A) \ .$$

Thus, the law of quadratic reciprocity can be written

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{L(B)+L(C)} \ ,$$

and this will follow if we show that

$$\left(\frac{p}{q}\right) = (-1)^{L(B)}, \quad \left(\frac{q}{p}\right) = (-1)^{L(C)}.$$

For reasons of symmetry, it will suffice to prove one of these equations, say the latter. In view of how the Legendre symbol is calculated, this may be stated in the following form, known as Gauss's Lemma:

$$q^{\frac{p-1}{2}} = (-1)^{L(C)} + \text{a multiple of } p.$$

To prove Gauss's Lemma, consider all pairs of integers (x,y) in the following region:

$$E: \ 1/2 < x < p/2, \ \ 1/2 < y < q/2, \ \ -p/2 < f(x,y) < p/2.$$

In E,

$$y - 1/2 < \frac{qx}{p} < y + 1/2,$$

so that y is the integer *closest to* $\frac{qx}{p}$; we shall write $y = g(x)$ to indicate that y is completely determined by x. Clearly, E contains C; in fact, $(x,y) \in C$ if and only if $(x,y) \in E$ and $f(x,y) > 0$.

As x ranges from 1 to $\frac{p-1}{2}$, and therefore y from 1 to $\frac{q-1}{2}$, the absolute value of $f(x,y) = f(x,g(x))$ takes exactly $\frac{p-1}{2}$ values. For, if $|f(x,y)| = |f(x',y')|$, where x' and y' range as x and y, then

$$q(x \pm x') = p(y \pm y'),$$

hence $x \pm x' = kp$ for some integer k, so that $x = x'$ and therefore $y = y'$.

Since, in E, $|f(x,y)| < p/2$, these $\frac{p-1}{2}$ values are $1, \ldots, \frac{p-1}{2}$ and hence

$$\prod_{x=1}^{\frac{p-1}{2}} |f(x,y)| = \left(\tfrac{p-1}{2}\right)!$$

Finally, we calculate

$$q^{\frac{p-1}{2}} \left(\tfrac{p-1}{2}\right)! = \prod_{x=1}^{\frac{p-1}{2}} qx = \prod_{x=1}^{\frac{p-1}{2}} (py - f(x,g(x))).$$

This differs by a multiple of p from

$$\prod_{x=1}^{\frac{p-1}{2}} (-f(x,g(x))) = (-1)^{L(C)} \prod_{x=1}^{\frac{p-1}{2}} |f(x,g(x))| = (-1)^{L(C)} \left(\tfrac{p-1}{2}\right)!\,,$$

since $-f(x,g(x))$ will be negative if and only if $(x,g(x)) \in C$. Since p does not divide $(\frac{p-1}{2})!$, it follows that $q^{\frac{p-1}{2}}$ differs from $(-1)^{L(C)}$ by a multiple of p. This completes the proof of Gauss's Lemma and therefore the proof of the Law of Quadratic Reciprocity.

Part II
Topics in the Foundations of Mathematics

1
The Number System

Much of 19th and 20th century mathematics is not accessible or meaningful at the undergraduate level. Still, we plan to examine some important and up to date material, including some exciting recent discoveries.

The following letters are used to represent sets of numbers:

N the natural numbers 0, 1, 2, . . .;

Z the integers (Z being for 'Zahlen');

Q the rationals (Q being for 'quotients');

R the reals;

C the complex numbers $a + bi$;

H the quaternions (H being for Hamilton, the person who introduced them).

This now widely accepted notation was first proposed by N. Bourbaki, actually a slowly changing group of French mathematicians who have been engaged for half a century in writing up the *Elements of Mathematics* in a systematic and trend setting fashion. One of the founding members was the late Jean Dieudonné, to whom many of their early decisions may be attributed.

The number systems are arranged here in what mathematicians conceive to be the correct logical order, which differs from the historical one (see the Introduction). Zero was not originally considered to be a natural number