proper subset which is simultaneously closed and open.) We say that X is *connected* iff it is non-empty and not disconnected.

We declare the empty set $\emptyset$ as being special - it is neither connected nor disconnected; one could think of the empty set as "unconnected".

Example 13.4.2. Consider the set $X := [1,2] \cup [3,4]$, with the usual metric. This set is disconnected because the sets $[1,2]$ and $[3,4]$ are open relative to X (why?).

Intuitively, a disconnected set is one which can be separated into two disjoint open sets; a connected set is one which cannot be separated in this manner. We defined what it means for a metric space to be connected; we can also define what it means for a set to be connected.

Definition 13.4.3 (Connected sets). Let (X, d) be a metric space, and let Y be a subset of X. We say that Y is *connected* iff the metric space $(Y, d|_{Y\times Y})$ is connected, and we say that Y is *disconnected* iff the metric space $(Y, d|_{Y\times Y})$ is disconnected.

Remark 13.4.4. This definition is intrinsic; whether a set Y is connected or not depends only on what the metric is doing on Y, but not on what ambient space X one placing Y in.

On the real line, connected sets are easy to describe.

Theorem 13.4.5. *Let X be a subset of the real line $\mathbf{R}$. Then the following statements are equivalent.*

(a) *X is connected.*

(b) *Whenever $x, y \in X$ and $x < y$, the interval $[x, y]$ is also contained in X.*

(c) *X is an interval (in the sense of Definition 9.1.1).*

Proof. First we show that (a) implies (b). Suppose that X is connected, and suppose for sake of contradiction that we could find points $x < y$ in X such that $[x, y]$ is *not* contained in X. Then there exists a real number $x < z < y$ such that $z \notin X$. Thus the sets $(-\infty, z) \cap X$ and $(z, \infty) \cap X$ will cover X. But these sets are non-empty (because they contain x and y respectively) and are open relative to X, and so X is disconnected, a contradiction.

Now we show that (b) implies (a). Let X be a set obeying the property (b). Suppose for sake of contradiction that X is disconnected. Then there exist disjoint non-empty sets V, W which are open relative to X, such that $V \cup W = X$. Since V and W are non-empty, we may choose an $x \in V$ and $y \in W$. Since V and W are disjoint, we have $x \neq y$; without loss of generality we may assume $x < y$. By property (b), we know that the entire interval $[x, y]$ is contained in X.

Now consider the set $[x, y] \cap V$. This set is both bounded and non-empty (because it contains x). Thus it has a supremum

$$z := \sup([x, y] \cap V).$$

Clearly $z \in [x, y]$, and hence $z \in X$. Thus either $z \in V$ or $z \in W$. Suppose first that $z \in V$. Then $z \neq y$ (since $y \in W$ and V is disjoint from W). But V is open relative to X, which contains $[x, y]$, so there is some ball $B_{([x,y],d)}(z, r)$ which is contained in V. But this contradicts the fact that z is the supremum of $[x, y] \cap V$. Now suppose that $z \in W$. Then $z \neq x$ (since $x \in V$ and V is disjoint from W). But W is open relative to X, which contains $[x, y]$, so there is some ball $B_{([x,y],d)}(z, r)$ which is contained in W. But this again contradicts the fact that z is the supremum of $[x, y] \cap V$. Thus in either case we obtain a contradiction, which means that X cannot be disconnected, and must therefore be connected.

It remains to show that (b) and (c) are equivalent; we leave this to Exercise 13.4.3. □

Continuous functions map connected sets to connected sets:

Theorem 13.4.6 (Continuity preserves connectedness). *Let $f : X \to Y$ be a continuous map from one metric space (X, d_X) to* *another* (Y, d_Y). *Let* E *be any connected subset of* X. *Then* $f(E)$ *is also connected.*

Proof. See Exercise 13.4.4. □

An important corollary of this result is the intermediate value theorem, generalizing Theorem 9.7.1.

Corollary 13.4.7 (Intermediate value theorem). *Let* $f : X \to \mathbf{R}$ *be a continuous map from one metric space* (X, d_X) *to the real line. Let* E *be any connected subset of* X, *and let* a, b *be any two elements of* E. *Let* y *be a real number between* $f(a)$ *and* $f(b)$, *i.e., either* $f(a) \leq y \leq f(b)$ *or* $f(a) \geq y \geq f(b)$. *Then there exists* $c \in E$ *such that* $f(c) = y$.

Proof. See Exercise 13.4.5. □

Exercise 13.4.1. Let (X, d_{disc}) be a metric space with the discrete metric. Let E be a subset of X which contains at least two elements. Show that E is disconnected.

Exercise 13.4.2. Let $f : X \to Y$ be a function from a connected metric space (X, d) to a metric space (Y, d_{disc}) with the discrete metric. Show that f is continuous if and only if it is constant. (Hint: use Exercise 13.4.1.)

Exercise 13.4.3. Prove the equivalence of statements (b) and (c) in Theorem 13.4.5.

Exercise 13.4.4. Prove Theorem 13.4.6. (Hint: the formulation of continuity in Theorem 13.1.5(c) is the most convenient to use.)

Exercise 13.4.5. Use Theorem 13.4.6 to prove Corollary 13.4.7.

Exercise 13.4.6. Let (X, d) be a metric space, and let $(E_\alpha)_{\alpha \in I}$ be a collection of connected sets in X. Suppose also that $\bigcap_{\alpha \in I} E_\alpha$ is non-empty. Show that $\bigcup_{\alpha \in I} E_\alpha$ is connected.

Exercise 13.4.7. Let (X, d) be a metric space, and let E be a subset of X. We say that E is *path-connected* iff, for every $x, y \in E$, there exists a continuous function $\gamma : [0, 1] \to E$ from the unit interval $[0, 1]$ to E such that $\gamma(0) = x$ and $\gamma(1) = y$. Show that every path-connected set is connected. (The converse is false, but is a bit tricky to show and will not be detailed here.)

Exercise 13.4.8. Let (X, d) be a metric space, and let E be a subset of X. Show that if E is connected, then the closure $\overline{E}$ of E is also connected. Is the converse true?

Exercise 13.4.9. Let (X, d) be a metric space. Let us define a relation $x \sim y$ on X by declaring $x \sim y$ iff there exists a connected subset of X which contains both x and y. Show that this is an equivalence relation (i.e., it obeys the reflexive, symmetric, and transitive axioms). Also, show that the equivalence classes of this relation (i.e., the sets of the form $\{y \in X : y \sim x\}$ for some $x \in X$) are all closed and connected. (Hint: use Exercise 13.4.6 and Exercise 13.4.8.) These sets are known as the *connected components* of X.

Exercise 13.4.10. Combine Proposition 13.3.2 and Corollary 13.4.7 to deduce a theorem for continuous functions on a compact connected domain which generalizes Corollary 9.7.4.

13.5 Topological spaces (Optional)

The concept of a metric space can be generalized to that of a *topological space*. The idea here is not to view the metric d as the fundamental object; indeed, in a general topological space there is no metric at all. Instead, it is the collection of *open sets* which is the fundamental concept. Thus, whereas in a metric space one introduces the metric d first, and then uses the metric to define first the concept of an open ball and then the concept of an open set, in a topological space one starts just with the notion of an open set. As it turns out, starting from the open sets, one cannot necessarily reconstruct a usable notion of a ball or metric (thus not all topological spaces will be metric spaces), but remarkably one can still define many of the concepts in the preceding sections.

We will not use topological spaces at all in this text, and so we shall be rather brief in our treatment of them here. A more complete study of these spaces can of course be found in any topology textbook, or a more advanced analysis text.

Definition 13.5.1 (Topological spaces). A *topological space* is a pair $(X, \mathcal{F})$, where X is a set, and $\mathcal{F} \subset 2^X$ is a collection of subsets of X, whose elements are referred to as *open sets*. Furthermore, the collection $\mathcal{F}$ must obey the following properties:

- The empty set $\emptyset$ and the whole set X are open; in other words, $\emptyset \in \mathcal{F}$ and $X \in \mathcal{F}$.
- Any finite intersection of open sets is open. In other words, if $V_1, \ldots, V_n$ are elements of $\mathcal{F}$, then $V_1 \cap \ldots \cap V_n$ is also in $\mathcal{F}$.
- Any arbitrary union of open sets is open (including infinite unions). In other words, if $(V_\alpha)_{\alpha \in I}$ is a family of sets in $\mathcal{F}$, then $\bigcup_{\alpha \in I} V_\alpha$ is also in $\mathcal{F}$.

In many cases, the collection $\mathcal{F}$ of open sets can be deduced from context, and we shall refer to the topological space $(X, \mathcal{F})$ simply as X.

From Proposition 12.2.15 we see that every metric space (X, d) is automatically also a topological space (if we set $\mathcal{F}$ equal to the collection of sets which are open in (X, d)). However, there do exist topological spaces which do not arise from metric spaces (see Exercise 13.5.1, 13.5.6).

We now develop the analogues of various notions in this chapter and the previous chapter for topological spaces. The notion of a ball must be replaced by the notion of a *neighbourhood.*

Definition 13.5.2 (Neighbourhoods). Let $(X, \mathcal{F})$ be a topological space, and let $x \in X$. A *neighbourhood of* x is defined to be any open set in $\mathcal{F}$ which contains x.

Example 13.5.3. If (X, d) is a metric space, $x \in X$, and $r > 0$, then $B(x, r)$ is a neighbourhood of x.

Definition 13.5.4 (Topological convergence). Let m be an integer, $(X, \mathcal{F})$ be a topological space and let $(x^{(n)})_{n=m}^\infty$ be a sequence of points in X. Let x be a point in X. We say that $(x^{(n)})_{n=m}^\infty$ *converges to* x if and only if, for every neighbourhood V of x, there exists an $N \geq m$ such that $x^{(n)} \in V$ for all $n \geq N$.

This notion is consistent with that of convergence in metric spaces (Exercise 13.5.2). One can then ask whether one has the basic property of uniqueness of limits (Proposition 12.1.20). The answer turns out to usually be yes - if the topological space has an additional property known as the *Hausdorff property* - but the answer can be no for other topologies; see Exercise 13.5.4.

Definition 13.5.5 (Interior, exterior, boundary). Let $(X, \mathcal{F})$ be a topological space, let E be a subset of X, and let x_0 be a point in X. We say that x_0 is an *interior point of* E if there exists a neighbourhood V of x_0 such that $V \subseteq E$. We say that x_0 is an *exterior point of* E if there exists a neighbourhood V of x_0 such that $V \cap E = \emptyset$. We say that x_0 is a *boundary point of* E if it is neither an interior point nor an exterior point of E.

This definition is consistent with the corresponding notion for metric spaces (Exercise 13.5.3).

Definition 13.5.6 (Closure). Let $(X, \mathcal{F})$ be a metric space, let E be a subset of X, and let x_0 be a point in X. We say that x_0 is an *adherent point* of E if every neighbourhood V of x_0 has a non-empty intersection with E. The set of all adherent points of E is called the *closure* of E and is denoted $\overline{E}$.

There is a partial analogue of Theorem 12.2.10, see Exercise 13.5.10.

We define a set K in a topological space $(X, \mathcal{F})$ to be *closed* iff its complement $X \backslash K$ is open; this is consistent with the metric space definition, thanks to Proposition 12.2.15(e). Some partial analogues of that Proposition are true (see Exercise 13.5.11).

To define the notion of a relative topology, we cannot use Definition 12.3.3 as this requires a metric function. However, we can instead use Proposition 12.3.4 as our starting point:

Definition 13.5.7 (Relative topology). Let $(X, \mathcal{F})$ be a topological space, and Y be a subset of X. Then we define $\mathcal{F}_Y := \{V \cap Y : V \in \mathcal{F}\}$, and refer this as the topology on Y *induced* by $(X, \mathcal{F})$. We call $(Y, \mathcal{F}_Y)$ a *topological subspace* of $(X, \mathcal{F})$. This is indeed a topological space, see Exercise 13.5.12.

From Proposition 12.3.4 we see that this notion is compatible with the one for metric spaces.

Next we define the notion of continuity.

Definition 13.5.8 (Continuous functions). Let $(X, \mathcal{F}_X)$ and $(Y, \mathcal{F}_Y)$ be topological spaces, and let $f : X \to Y$ be a function. If $x_0 \in X$, we say that f is *continuous at* x_0 iff for every neighbourhood V of $f(x_0)$, there exists a neighbourhood U of x_0 such that $f(U) \subseteq V$. We say that f is *continuous* iff it is continuous at every point $x \in X$.

This definition is consistent with that in Definition 13.1.1 (Exercise 13.5.15). Partial analogues of Theorems 13.1.4 and 13.1.5 are available (Exercise 13.5.16). In particular, a function is continuous iff the pre-images of every open set is open.

There is unfortunately no notion of a Cauchy sequence, a complete space, or a bounded space, for topological spaces. However, there is certainly a notion of a compact space, as we can see by taking Theorem 12.5.8 as our starting point:

Definition 13.5.9 (Compact topological spaces). Let $(X, \mathcal{F})$ be a topological space. We say that this space is *compact* if every open cover of X has a finite subcover. If Y is a subset of X, we say that Y is compact if the topological space on Y induced by $(X, \mathcal{F})$ is compact.

Many basic facts about compact metric spaces continue to hold true for compact topological spaces, notably Theorem 13.3.1 and Proposition 13.3.2 (Exercise 13.5.17). However, there is no notion of uniform continuity, and so there is no analogue of Theorem 13.3.5.

We can also define the notion of connectedness by repeating Definition 13.4.1 verbatim, and also repeating Definition 13.4.3 (but with Definition 13.5.7 instead of Definition 12.3.3). Many of the results and exercises in Section 13.4 continue to hold for topological spaces (with almost no changes to any of the proofs!).

Exercise 13.5.1. Let X be an arbitrary set, and let $\mathcal{F} := \{\emptyset, X\}$. Show that $(X, \mathcal{F})$ is a topology (called the *trivial topology* on X). If X contains more than one element, show that the trivial topology cannot be obtained from by placing a metric d on X. Show that this topological space is both compact and connected.

Exercise 13.5.2. Let (X, d) be a metric space (and hence a topological space). Show that the two notions of convergence of sequences in Definition 12.1.14 and Definition 13.5.4 coincide.

Exercise 13.5.3. Let (X, d) be a metric space (and hence a topological space). Show that the two notions of interior, exterior, and boundary in Definition 12.2.5 and Definition 13.5.5 coincide.

Exercise 13.5.4. A topological space $(X, \mathcal{F})$ is said to be *Hausdorff* if given any two distinct points $x, y \in X$, there exists a neighbourhood V of x and a neighbourhood W of y such that $V \cap W = \emptyset$. Show that any topological space coming from a metric space is Hausdorff, and show that the trivial topology is not Hausdorff. Show that the analogue of Proposition 12.1.20 holds for Hausdorff topological spaces, but give an example of a non-Hausdorff topological space in which Proposition 12.1.20 fails. (In practice, most topological spaces one works with are Hausdorff; non-Hausdorff topological spaces tend to be so pathological that it is not very profitable to work with them.)

Exercise 13.5.5. Given any totally ordered set X with order relation $\leq$, declare a set $V \subset X$ to be *open* if for every $x \in V$ there exist $a, b \in X$ such that the "interval" $\{y \in X : a < y < b\}$ contains x and is contained in V. Let $\mathcal{F}$ be the set of all open subsets of X. Show that $(X, \mathcal{F})$ is a topology (this is the *order topology* on the totally ordered set $(X, \leq)$) which is Hausdorff in the sense of Exercise 13.5.4. Show that on the real line $\mathbf{R}$ (with the standard ordering $\leq$), the order topology matches the standard topology (i.e., the topology arising from the standard metric). If instead one applies this to the extended real line $\mathbf{R}^*$, show that $\mathbf{R}$ is an open set with boundary $\{-\infty, +\infty\}$. If $(x_n)_{n=1}^\infty$ is a sequence of numbers in $\mathbf{R}$ (and hence in $\mathbf{R}^*$), show that x_n converges to $+\infty$ if and only if $\liminf_{n\to\infty} x_n = +\infty$, and x_n converges to $-\infty$ if and only if $\limsup_{n\to\infty} x_n = -\infty$.

Exercise 13.5.6. Let X be an uncountable set, and let $\mathcal{F}$ be the collection of all subsets E in X which are either empty or co-finite (which means that $X \backslash E$ is finite). Show that $(X, \mathcal{F})$ is a topology (this is called the *cofinite topology* on X) which is Hausdorff in the sense of Exercise 13.5.4, and is compact and connected. Also, show that if $x \in X$ $(V_n)_{n=1}^\infty$ is any countable collection of open sets containing x, then $\bigcap_{n=1}^{\infty} V_n \neq \{x\}$. Use this to show that the cofinite topology cannot be obtained by placing a metric d on X. (Hint: what is the set $\bigcap_{n=1}^{\infty} B(x, 1/n)$ equal to in a metric space?)

Exercise 13.5.7. Let X be an uncountable set, and let $\mathcal{F}$ be the collection of all subsets E in X which are either empty or co-countable (which means that $X \backslash E$ is at most countable). Show that $(X, \mathcal{F})$ is a topology (this is called the *cocountable topology* on X) which is Hausdorff in the sense of Exercise 13.5.4, and connected, but cannot arise from a metric space and is not compact.

Exercise 13.5.8. Let X be an uncountable set, and let ∞ be an element of X. Let $\mathcal{F}$ be the collection of all subsets E in X which are either empty, or are co-countable and contain ∞. Show that $(X, \mathcal{F})$ is a compact topological space; however show that not every sequence in X has a convergent subsequence.

Exercise 13.5.9. Let $(X, \mathcal{F})$ be a compact topological space. Show that every sequence in X has a convergent subsequence, by modifying Exercise 12.5.11. Explain why this does not contradict Exercise 13.5.8.

Exercise 13.5.10. Prove the following partial analogue of Proposition 12.2.10 for topological spaces: (c) implies both (a) and (b), which are equivalent to each other. Show that in the co-countable topology in Exercise 13.5.7, it is possible for (a) and (b) to hold without (c) holding.

Exercise 13.5.11. Let E be a subset of a topological space $(X, \mathcal{F})$. Show that E is open if and only if every element of E is an interior point, and show that E is closed if and only if E contains all of its adherent points. Prove analogues of Proposition 12.2.15(e)-(h) (some of these are automatic by definition). If we assume in addition that X is Hausdorff, prove an analogue of Proposition 12.2.15(d) also, but give an example to show that (d) can fail when X is not Hausdorff.

Exercise 13.5.12. Show that the pair $(Y, \mathcal{F}_Y)$ defined in Definition 13.5.7 is indeed a topological space.

Exercise 13.5.13. Generalize Corollary 12.5.9 to compact sets in a topological space.

Exercise 13.5.14. Generalize Theorem 12.5.10 to compact sets in a topological space.

Exercise 13.5.15. Let (X, d_X) and (Y, d_Y) be metric spaces (and hence a topological space). Show that the two notions continuity (both at a point, and on the whole domain) of a function $f : X \to Y$ in Definition 13.1.1 and Definition 13.5.8 coincide.

Exercise 13.5.16. Show that when Theorem 13.1.4 is extended to topological spaces, that (a) implies (b). (The converse is false, but constructing an example is difficult.) Show that when Theorem 13.1.5 is extended to topological spaces, that (a), (c), (d) are all equivalent to each other, and imply (b). (Again, the converse implications are false, but difficult to prove.)

Exercise 13.5.17. Generalize both Theorem 13.3.1 and Proposition 13.3.2 to compact sets in a topological space.

Chapter 14

Uniform convergence

In the previous two chapters we have seen what it means for a sequence $(x^{(n)})_{n=1}^{\infty}$ of points in a metric space (X, d_X) to converge to a limit x; it means that $\lim_{n\to\infty} d_X(x^{(n)}, x) = 0$, or equivalently that for every $\varepsilon > 0$ there exists an $N > 0$ such that $d_X(x^{(n)}, x) < \varepsilon$ for all $n > N$. (We have also generalized the notion of convergence to topological spaces $(X, \mathcal{F})$, but in this chapter we will focus on metric spaces.)

In this chapter, we consider what it means for a sequence of *functions* $(f^{(n)})_{n=1}^{\infty}$ from one metric space (X, d_X) to another (Y, d_Y) to converge. In other words, we have a sequence of functions $f^{(1)}, f^{(2)}, \ldots$, with each function $f^{(n)} : X \to Y$ being a function from X to Y, and we ask what it means for this sequence of functions to converge to some limiting function f.

It turns out that there are several different concepts of convergence of functions; here we describe the two most important ones, *pointwise convergence* and *uniform convergence.* (There are other types of convergence for functions, such as L^1 convergence, L^2 convergence, convergence in measure, almost everywhere convergence, and so forth, but these are beyond the scope of this text.) The two notions are related, but not identical; the relationship between the two is somewhat analogous to the relationship between continuity and uniform continuity.

Once we work out what convergence means for functions, and thus can make sense of such statements as $\lim_{n\to\infty} f^{(n)} = f$, we will then ask how these limits interact with other concepts. For instance, we already have a notion of limiting values of functions: $\lim_{x\to x_0;x\in X} f(x)$. Can we interchange limits, i.e.

$$\lim_{n\to\infty} \lim_{x\to x_0;x\in X} f^{(n)}(x) = \lim_{x\to x_0;x\in X} \lim_{n\to\infty} f^{(n)}(x)?$$

As we shall see, the answer depends on what type of convergence we have for $f^{(n)}$. We will also address similar questions involving interchanging limits and integrals, or limits and sums, or sums and integrals.

14.1 Limiting values of functions

Before we talk about limits of sequences of functions, we should first discuss a similar, but distinct, notion, that of limiting values of functions. We shall focus on the situation for metric spaces, but there are similar notions for topological spaces (Exercise 14.1.3).

Definition 14.1.1 (Limiting value of a function). Let (X, d_X) and (Y, d_Y) be metric spaces, let E be a subset of X, and let $f : X \to Y$ be a function. If $x_0 \in X$ is an adherent point of E, and $L \in Y$, we say that $f(x)$ *converges to L in Y as x converges to x_0 in E*, or write $\lim_{x\to x_0;x\in E} f(x) = L$, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $d_Y(f(x), L) < \varepsilon$ for all $x \in E$ such that $d_X(x, x_0) < \delta$.

Remark 14.1.2. Some authors exclude the case $x = x_0$ from the above definition, thus requiring $0 < d_X(x, x_0) < \delta$. In our current notation, this would correspond to removing x_0 from E, thus one would consider $\lim_{x\to x_0;x\in E\backslash\{x_0\}} f(x)$ instead of $\lim_{x\in x_0;x\in E} f(x)$. See Exercise 14.1.1 for a comparison of the two concepts.

Comparing this with Definition 13.1.1, we see that f is continuous at x_0 if and only if

$$\lim_{x\to x_0;x\in X} f(x) = f(x_0).$$

Thus f is continuous on X iff we have

$$\lim_{x \to x_0; x \in X} f(x) = f(x_0) \text{ for all } x_0 \in X.$$

Example 14.1.3. If $f : \mathbf{R} \to \mathbf{R}$ is the function $f(x) = x^2 - 4$, then

$$\lim_{x \to 1} f(x) = f(1) = 1 - 4 = -3$$

since f is continuous.

Remark 14.1.4. Often we shall omit the condition $x \in X$, and abbreviate $\lim_{x \to x_0; x \in X} f(x)$ as simply $\lim_{x \to x_0} f(x)$ when it is clear what space x will range in.

One can rephrase Definition 14.1.1 in terms of sequences:

Proposition 14.1.5. *Let (X, d_X) and (Y, d_Y) be metric spaces, let E be a subset of X, and let $f : X \to Y$ be a function. Let $x_0 \in X$ be an adherent point of E and $L \in Y$. Then the following four statements are logically equivalent:*

(a) $\lim_{x \to x_0; x \in E} f(x) = L$.

(b) *For every sequence $(x^{(n)})_{n=1}^\infty$ in E which converges to x_0 with respect to the metric d_X, the sequence $(f(x^{(n)}))_{n=1}^\infty$ converges to L with respect to the metric d_Y.*

(c) *For every open set $V \subset Y$ which contains L, there exists an open set $U \subset X$ containing x_0 such that $f(U \cap E) \subseteq V$.*

(d) *If one defines the function $g : E \cup \{x_0\} \to Y$ by defining $g(x_0) := L$, and $g(x) := f(x)$ for $x \in E \backslash \{x_0\}$, then g is continuous at x_0.*

Proof. See Exercise 14.1.2. $\square$

Remark 14.1.6. Observe from Proposition 14.1.5(b) and Proposition 12.1.20 that a function $f(x)$ can converge to at most one limit L as x converges to x_0. In other words, if the limit

$$\lim_{x \to x_0; x \in E} f(x)$$

exists at all, then it can only take at most one value.

Remark 14.1.7. The requirement that x_0 be an adherent point of E is necessary for the concept of limiting value to be useful, otherwise x_0 will lie in the exterior of E, the notion that $f(x)$ converges to L as x converges to x_0 in E is vacuous (for δ sufficiently small, there are no points $x \in E$ so that $d(x, x_0) < \delta$).

Remark 14.1.8. Strictly speaking, we should write

$$d_Y - \lim_{x \to x_0; x \in E} f(x) \text{ instead of } \lim_{x \to x_0; x \in E} f(x),$$

since the convergence depends on the metric d_Y. However in practice it will be obvious what the metric d_Y is and so we will omit the d_Y- prefix from the notation.

Exercise 14.1.1. Let (X, d_X) and (Y, d_Y) be metric spaces, let E be a subset of X, let $f : E \to Y$ be a function, and let x_0 be an element of E. Show that the limit $\lim_{x \to x_0; x \in E} f(x)$ exists if and only if the limit $\lim_{x \to x_0; x \in E \backslash \{x_0\}} f(x)$ exists and is equal to $f(x_0)$. Also, show that if the limit $\lim_{x \to x_0; x \in E} f(x)$ exists at all, then it must equal $f(x_0)$.

Exercise 14.1.2. Prove Proposition 14.1.5. (Hint: review your proof of Theorem 13.1.4.)

Exercise 14.1.3. Use Proposition 14.1.5(c) to define a notion of a limiting value of a function $f : X \to Y$ from one topological space $(X, \mathcal{F}_X)$ to another $(Y, \mathcal{F}_Y)$. Then prove the equivalence of Proposition 14.1.5(c) and 14.1.5(d). If in addition Y is a Hausdorff topological space (see Exercise 13.5.4), prove an analogue of Remark 14.1.6. Is the same statement true if Y is not Hausdorff?

Exercise 14.1.4. Recall from Exercise 13.5.5 that the extended real line $\mathbf{R}^*$ comes with a standard topology (the order topology). We view the natural numbers $\mathbf{N}$ as a subspace of this topological space, and $+\infty$ as an adherent point of $\mathbf{N}$ in $\mathbf{R}^*$. Let $(a_n)_{n=0}^{\infty}$ be a sequence taking values in a topological space $(Y, \mathcal{F}_Y)$, and let $L \in Y$. Show that $\lim_{n \to +\infty; n \in \mathbf{N}} a_n = L$ (in the sense of Exercise 14.1.3) if and only if $\lim_{n \to \infty} a_n = L$ (in the sense of Definition 13.5.4). This shows that the notions of limiting values of a sequence, and limiting values of a function, are compatible.

Exercise 14.1.5. Let (X, d_X), (Y, d_Y), (Z, d_Z) be metric spaces, and let $x_0 \in X$, $y_0 \in Y$, $z_0 \in Z$. Let $f : X \to Y$ and $g : Y \to Z$ be functions, and let E be a set. If we have $\lim_{x \to x_0; x \in E} f(x) = y_0$ and $\lim_{y \to y_0; y \in f(E)} g(x) = z_0$, conclude that $\lim_{x \in x_0; x \in E} g \circ f(x_0) = z_0$.