In this case we simply say β is the equivalence between the two extensions. As noted in Section 10.5, equivalence of extensions is reflexive, symmetric and transitive. We also observe that

equivalent extensions define the same G-module structure on A.

To see this assume (29) is an equivalence, let g be any element of G and let e_g be any element of E_1 mapping onto g by π_1. The action of g on A given by conjugation in E_1 maps each a to $\iota_1^{-1}(e_g\iota_1(a)e_g^{-1})$. Let $e'_g = \beta(e_g)$. Since the diagram commutes, $\pi_2(e'_g) = g$, so the action of g on A in the second extension is given by conjugation by e'_g. This conjugation maps a to $\iota_2^{-1}(e'_g\iota_2(a)e'^{-1}_g)$. Since ι_1, ι_2 and β are injective, the two actions of g on a are equal if and only if they result in the same image in E_2, i.e., $\beta \circ \iota_1(\iota_1^{-1}(e_g\iota_1(a)e_g^{-1})) = e'_g\iota_2(a)e'^{-1}_g$. This equality is now immediate from the definition of e'_g and the commutativity of the diagram.

We next see how an extension as in (28) defines a 2-cocycle in $Z^2(G, A)$. For simplicity we identify A as a subgroup of E via ι and we identify G as E/A via π.

Definition. A map $\mu : G \to E$ with $\pi \circ \mu(g) = g$ and $\mu(1) = 0$, i.e., so that for each $g \in G$, $\mu(g)$ is a representative of the coset Ag of E and the identity of E (which is the zero of A) represents the identity coset, is called a *normalized section* of π.

Fix a section μ of π in (28). Each element of E may be written uniquely in the form $a\mu(g)$, where $a \in A$ and $g \in G$. For $g, h \in G$ the product $\mu(g)\mu(h)$ in E lies in the coset Agh, so there is a unique element $f(g, h)$ in A such that

$$\mu(g)\mu(h) = f(g, h)\mu(gh) \qquad \text{for all } g, h \in G. \tag{17.30}$$

If in addition μ is normalized at the identity we also have

$$f(g, 1) = 0 = f(1, g) \qquad \text{for all } g \in G. \tag{17.31}$$

Definition. The function f defined by equation (30) is called the *factor set* for the extension E associated to the section μ. If f also satisfies (31) then f is called a *normalized* factor set.

We shall see in the examples following that it is possible for different sections μ to give the same factor set f.

We now verify that the factor set f is in fact a 2-cocycle. First note that the group operation in E may be written

$$\begin{aligned}(a_1\mu(g))(a_2\mu(h)) &= (a_1 + \mu(g)a_2\mu(g)^{-1})\mu(g)\mu(h)\\ &= (a_1 + g \cdot a_2)(\mu(g)\mu(h))\\ &= (a_1 + g \cdot a_2 + f(g, h))\mu(gh)\end{aligned} \tag{17.32}$$

where $g \cdot a_2$ denotes the G-module action of g on a_2 given by conjugation in E. Now use (32) and the associative law in E to compute the product $\mu(g)\mu(h)\mu(k)$ in two different ways:

$$\begin{aligned}(\mu(g)\mu(h))\mu(k) &= (f(g, h) + f(gh, k))\mu(ghk)\\ \mu(g)(\mu(h)\mu(k)) &= (gf(h, k) + f(g, hk))\mu(ghk).\end{aligned} \tag{17.33}$$

It follows that the factors in A of the two right hand sides in (33) are equal for every $g, h, k \in G$, and this is precisely the 2-cocycle condition (26) for f. This shows that the factor set associated to the extension E and any choice of section μ is an element in $Z^2(G, A)$.

We next see how the factor set f depends on the choice of section μ. Suppose μ' is another section for the same extension E in (28), and let f' be its associated factor set. Then for all $g \in G$ both $\mu(g)$ and $\mu'(g)$ lie in the same coset Ag, so there is a function $f_1 : G \to A$ such that $\mu'(g) = f_1(g)\mu(g)$ for all g. Then

$$\mu'(g)\mu'(h) = f'(g, h)\mu'(gh) = (f'(g, h) + f_1(gh))\mu(gh).$$

We also have

$$\begin{aligned}\mu'(g)\mu'(h) &= (f_1(g)\mu(g))(f_1(h)\mu(h)) = (f_1(g) + g \cdot f_1(h))(\mu(g)\mu(h)) \\ &= (f_1(g) + g \cdot f_1(h) + f(g, h))\mu(gh).\end{aligned}$$

Equating the factors in A in these two expressions for $\mu'(g)\mu'(h)$ shows that

$$f'(g, h) = f(g, h) + (gf_1(h) - f_1(gh) + f_1(g)) \qquad \text{for all } g, h \in G,$$

in other words f and f' differ by the 2-coboundary of f_1 as in (27).

We have shown that the factor sets associated to the extension E corresponding to different choices of sections give 2-cocycles in $Z^2(G, A)$ that differ by a coboundary in $B^2(G, A)$. Hence associated to the extension E is a well defined cohomology class in $H^2(G, A)$ determined by the factor set in (30) for any choice of section μ.

If the extension E of G by A is a *split* extension (which is to say that $E = A \rtimes G$ is the semidirect product of G by A with the given conjugation action of G on A), then there is a section μ of G that is a *homomorphism* from G to E. In this case the factor set f in (30) is identically 0: $f(g, h) = 0$ for all $g, h \in G$. Hence the cohomology class in $H^2(G, A)$ defined by a split extension is the trivial class.

Suppose now that β is an equivalence between the extension in (28) and an extension E':

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r, "\iota"] \arrow[d, "\mathrm{id}"] & E \arrow[r, "\pi"] \arrow[d, "\beta"] & G \arrow[r] \arrow[d, "\mathrm{id}"] & 1 \\ 1 \arrow[r] & A \arrow[r, "\iota'"] & E' \arrow[r, "\pi'"] & G \arrow[r] & 1. \end{tikzcd}$$

If μ is a section of π, then $\mu' = \beta \circ \mu$ is a section of π', so what we have just proved can be used to determine the cohomology class in $H^2(G, A)$ corresponding to E'. Applying the homomorphism β to equation (30) gives

$$\beta(\mu(g))\beta(\mu(h)) = \beta(f(g, h))\beta(\mu(gh)) \qquad \text{for all } g, h \in G.$$

Since β restricts to the identity map on A, this is

$$\mu'(g)\mu'(h) = f(g, h)\mu'(gh) \qquad \text{for all } g, h \in G,$$

which shows that the factor set for E' associated to μ' is the same as the factor set for E associated to μ. This proves that equivalent extensions define the same cohomology class in $H^2(G, A)$.

We next show how this procedure may be reversed: Given a class in $H^2(G, A)$ we construct an extension E_f whose corresponding factor set is in the given class in $H^2(G, A)$. The process generalizes the semidirect product construction of Section 5.5 (which is the special case when f is the zero cocycle representing the trivial class).

Note first that any 2-cocycle arising from the factor set of an extension as above where the section μ is normalized satisfies the condition in (31).

Definition. A 2-cocycle f such that $f(g, 1) = 0 = f(1, g)$ for all $g \in G$ is called a *normalized* 2-cocycle.

The construction of E_f is a little simpler when f is a normalized cocycle and for simplicity we indicate the construction in this case (the minor modifications necessary when f is not normalized are indicated in Exercise 4).

We first see that any 2-cocycle f lies in the same cohomology class as a normalized 2-cocycle. Let $d_1 f_1$ be the 2-coboundary of the constant function f_1 on G whose value is $f(1, 1)$. Then $f(1, 1) = d_1 f_1(1, 1)$, and one easily checks from the 2-cocycle condition that $f - d_1 f_1$ is normalized.

We may therefore assume that our cohomology class in $H^2(G, A)$ is represented by the normalized 2-cocycle f. Let E_f be the set $A \times G$, and define a binary operation on E_f by

$$(a_1, g)(a_2, h) = (a_1 + g \cdot a_2 + f(g, h), gh) \tag{17.34}$$

where, as usual, $g \cdot a_2$ denotes the module action of G on A. It is straightforward to check that the group axioms hold: Since f is normalized, the identity element is (0,1) and inverses are given by

$$(a, g)^{-1} = (-g^{-1} \cdot a - f(g^{-1}, g), g^{-1}). \tag{17.35}$$

The cocycle condition implies the associative law by calculations similar to (32) and (33) earlier — the details are left as exercises.

Since f is a normalized 2-cocycle, $A^* = \{(a, 1) \mid a \in A\}$ is a subgroup of E_f, and the map $\iota^* : a \mapsto (a, 1)$ is an isomorphism from A to A^*. Moreover, from (34) and (35) it follows that

$$(0, g)(a, 1)(0, g)^{-1} = (g \cdot a, 1) \qquad \text{for all } g \in G \text{ and all } a \in A. \tag{17.36}$$

Since E_f is generated by A^* together with the set of elements $(0, g)$ for $g \in G$, (36) implies that A^* is a normal subgroup of E_f. Furthermore, it is immediate from (34) that the map $\pi^* : (a, g) \mapsto g$ is a surjective homomorphism from E_f to G with kernel A^*, i.e., $E_f/A^* \cong G$. Thus

$$1 \longrightarrow A \xrightarrow{\iota^*} E_f \xrightarrow{\pi^*} G \longrightarrow 1 \tag{17.37}$$

is a specific extension of G by A, where (36) ensures also that the action of G on A by conjugation in this extension is the module action specified in determining the 2-cocycle f in $H^2(G, A)$. The extension sequence (37) shows that this extension has the normalized section $\mu(g) = (0, g)$ whose corresponding normalized factor set is f. Note that this proves not only that every cohomology class in $H^2(G, A)$ arises from some extension E, but that every normalized 2-cocycle arises as the normalized factor set of some extension.

Finally, suppose f' is another normalized 2-cocycle in the same cohomology class in $H^2(G, A)$ as f and let $E_{f'}$ be the corresponding extension. If f and f' differ by the coboundary of $f_1 : G \to A$ then $f(g, h) - f'(g, h) = gf_1(h) - f_1(gh) + f_1(g)$ for all $g, h \in G$. Setting $g = h = 1$ shows that $f_1(1) = 0$. Define

$$\beta : E_f \longrightarrow E_{f'} \qquad \text{by} \qquad \beta((a, g)) = (a + f_1(g), g).$$

It is immediate that β is a bijection, and

$$\begin{aligned}
\beta((a_1, g)(a_2, h)) &= \beta((a_1 + g \cdot a_2 + f(g, h), gh)) \\
&= (a_1 + g \cdot a_2 + f(g, h) + f_1(gh), gh)) \\
&= (a_1 + f_1(g) + g \cdot (a_2 + f_1(h)) + f'(g, h), gh) \\
&= (a_1 + f_1(g), g)(a_2 + f_1(h), h) = \beta((a_1, g))\beta((a_2, h))
\end{aligned}$$

shows that β is an isomorphism from E_f to $E_{f'}$.

The restriction of β to A is given by $\beta((a, 1)) = (a + f_1(1), 1) = (a, 1)$, so β is the identity map on A. Similarly β is the identity map on the second component of (a, g), so β induces the identity map on the quotient G. It follows that β defines an equivalence between the extensions E_f and $E_{f'}$. This shows that the equivalence class of the extension E_f depends only on the cohomology class of f in $H^2(G, A)$.

We summarize this discussion in the following theorem.

Theorem 36. Let A be a G-module. Then

(1) A function $f : G \times G \to A$ is a normalized factor set of some extension E of G by A (with conjugation given by the G-module action on A) if and only if f is a normalized 2-cocycle in $Z^2(G, A)$.

(2) There is a bijection between the equivalence classes of extensions E as in (1) and the cohomology classes in $H^2(G, A)$. The bijection takes an extension E into the class of a normalized factor set f for E associated to any normalized section μ of G into E, and takes a cohomology class c in $H^2(G, A)$ to the extension E_f defined by the extension (37) for any normalized cocycle f in the class c.

(3) Under the bijection in (2), split extensions correspond to the trivial cohomology class.

Corollary 37. Every extension of G by the abelian group A splits if and only if $H^2(G, A) = 0$.

Corollary 38. If A is a finite abelian group and $(|A|, |G|) = 1$ then every extension of G by A splits.

Proof: This follows immediately from Corollary 29 in Section 2.

We can use Corollary 38 to prove the same result without the restriction that A be an abelian group.