The regularity of μ shows that A contains a closed elementary set F such that $\mu(F) \geq \mu(A) - \varepsilon$; and since F is compact, we have

$$F \subset A_1 \cup \cdots \cup A_N$$

for some N. Hence

$$\mu(A) \leq \mu(F) + \varepsilon \leq \mu(A_1 \cup \cdots \cup A_N) + \varepsilon \leq \sum_1^N \mu(A_n) + \varepsilon \leq \mu^*(A) + 2\varepsilon.$$

In conjunction with (20), this proves (a).

Next, suppose $E = \bigcup E_n$, and assume that $\mu^*(E_n) < +\infty$ for all n. Given $\varepsilon > 0$, there are coverings $\{A_{nk}\}$, $k = 1, 2, 3, \ldots$, of E_n by open elementary sets such that

$$\sum_{k=1}^{\infty} \mu(A_{nk}) \leq \mu^*(E_n) + 2^{-n}\varepsilon. \tag{21}$$

Then

$$\mu^*(E) \leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \mu(A_{nk}) \leq \sum_{n=1}^{\infty} \mu^*(E_n) + \varepsilon,$$

and (19) follows. In the excluded case, i.e., if $\mu^*(E_n) = +\infty$ for some n, (19) is of course trivial.

11.9 Definition For any $A \subset R^p$, $B \subset R^p$, we define

$$S(A, B) = (A - B) \cup (B - A), \tag{22}$$

$$d(A, B) = \mu^*(S(A, B)). \tag{23}$$

We write $A_n \to A$ if

$$\lim_{n \to \infty} d(A, A_n) = 0.$$

If there is a sequence $\{A_n\}$ of elementary sets such that $A_n \to A$, we say that A is *finitely μ-measurable* and write $A \in \mathfrak{M}_F(\mu)$.

If A is the union of a countable collection of finitely μ-measurable sets, we say that A is *μ-measurable* and write $A \in \mathfrak{M}(\mu)$.

$S(A, B)$ is the so-called "symmetric difference" of A and B. We shall see that $d(A, B)$ is essentially a distance function.

The following theorem will enable us to obtain the desired extension of μ.

11.10 Theorem *$\mathfrak{M}(\mu)$ is a σ-ring, and μ^* is countably additive on $\mathfrak{M}(\mu)$.*

Before we turn to the proof of this theorem, we develop some of the properties of $S(A, B)$ and $d(A, B)$. We have

$$S(A, B) = S(B, A), \qquad S(A, A) = 0. \tag{24}$$

$$S(A, B) \subset S(A, C) \cup S(C, B). \tag{25}$$

$$\left.\begin{aligned} S(A_1 \cup A_2, B_1 \cup B_2) \\ S(A_1 \cap A_2, B_1 \cap B_2) \\ S(A_1 - A_2, B_1 - B_2) \end{aligned}\right\} \subset S(A_1, B_1) \cup S(A_2, B_2). \tag{26}$$

(24) is clear, and (25) follows from

$$(A - B) \subset (A - C) \cup (C - B), \qquad (B - A) \subset (C - A) \cup (B - C).$$

The first formula of (26) is obtained from

$$(A_1 \cup A_2) - (B_1 \cup B_2) \subset (A_1 - B_1) \cup (A_2 - B_2).$$

Next, writing E^c for the complement of E, we have

$$\begin{aligned} S(A_1 \cap A_2, B_1 \cap B_2) &= S(A_1^c \cup A_2^c, B_1^c \cup B_2^c) \\ &\subset S(A_1^c, B_1^c) \cup S(A_2^c, B_2^c) = S(A_1, B_1) \cup S(A_2, B_2); \end{aligned}$$

and the last formula of (26) is obtained if we note that

$$A_1 - A_2 = A_1 \cap A_2^c.$$

By (23), (19), and (18), these properties of $S(A, B)$ imply

$$d(A, B) = d(B, A), \qquad d(A, A) = 0, \tag{27}$$

$$d(A, B) \le d(A, C) + d(C, B), \tag{28}$$

$$\left.\begin{aligned} d(A_1 \cup A_2, B_1 \cup B_2) \\ d(A_1 \cap A_2, B_1 \cap B_2) \\ d(A_1 - A_2, B_1 - B_2) \end{aligned}\right\} \le d(A_1, B_1) + d(A_2, B_2). \tag{29}$$

The relations (27) and (28) show that $d(A, B)$ satisfies the requirements of Definition 2.15, except that $d(A, B) = 0$ does not imply $A = B$. For instance, if $\mu = m$, A is countable, and B is empty, we have

$$d(A, B) = m^*(A) = 0;$$

to see this, cover the nth point of A by an interval I_n such that

$$m(I_n) < 2^{-n}\varepsilon.$$

But if we define two sets A and B to be equivalent, provided

$$d(A, B) = 0,$$

we divide the subsets of R^p into equivalence classes, and $d(A, B)$ makes the set of these equivalence classes into a metric space. $\mathfrak{M}_F(\mu)$ is then obtained as the closure of $\mathscr{E}$. This interpretation is not essential for the proof, but it explains the underlying idea.

We need one more property of $d(A, B)$, namely,

$$|\mu^*(A) - \mu^*(B)| \le d(A, B), \tag{30}$$

if at least one of $\mu^*(A)$, $\mu^*(B)$ is finite. For suppose $0 \le \mu^*(B) \le \mu^*(A)$. Then (28) shows that

$$d(A,0) \le d(A, B) + d(B, 0),$$

that is,

$$\mu^*(A) \le d(A, B) + \mu^*(B).$$

Since $\mu^*(B)$ is finite, it follows that

$$\mu^*(A) - \mu^*(B) \le d(A, B).$$

Proof of Theorem 11.10 Suppose $A \in \mathfrak{M}_F(\mu)$, $B \in \mathfrak{M}_F(\mu)$. Choose $\{A_n\}$, $\{B_n\}$ such that $A_n \in \mathscr{E}$. $B_n \in \mathscr{E}$, $A_n \to A$, $B_n \to B$. Then (29) and (30) show that

$$A_n \cup B_n \to A \cup B, \tag{31}$$

$$A_n \cap B_n \to A \cap B, \tag{32}$$

$$A_n - B_n \to A - B, \tag{33}$$

$$\mu^*(A_n) \to \mu^*(A), \tag{34}$$

and $\mu^*(A) < +\infty$ since $d(A_n, A) \to 0$. By (31) and (33), $\mathfrak{M}_F(\mu)$ is a ring. By (7),

$$\mu(A_n) + \mu(B_n) = \mu(A_n \cup B_n) + \mu(A_n \cap B_n).$$

Letting $n \to \infty$, we obtain, by (34) and Theorem 11.8(a),

$$\mu^*(A) + \mu^*(B) = \mu^*(A \cup B) + \mu^*(A \cap B).$$

If $A \cap B = 0$, then $\mu^*(A \cap B) = 0$.

It follows that μ^* is additive on $\mathfrak{M}_F(\mu)$.

Now let $A \in \mathfrak{M}(\mu)$. Then A can be represented as the union of a countable collection of *disjoint* sets of $\mathfrak{M}_F(\mu)$. For if $A = \bigcup A_n'$ with $A_n' \in \mathfrak{M}_F(\mu)$, write $A_1 = A_1'$, and

$$A_n = (A_1' \cup \cdots \cup A_n') - (A_n' \cup \cdots \cup A_{n-1}') \qquad (n = 2, 3, 4, \ldots).$$

Then

$$A = \bigcup_{n=1}^{\infty} A_n \tag{35}$$

is the required representation. By (19)

$$\mu^*(A) \le \sum_{n=1}^{\infty} \mu^*(A_n). \tag{36}$$

On the other hand, $A \supset A_1 \cup \cdots \cup A_n$; and by the additivity of μ^* on $\mathfrak{M}_F(\mu)$ we obtain

$$\mu^*(A) \geq \mu^*(A_1 \cup \cdots \cup A_n) = \mu^*(A_1) + \cdots + \mu^*(A_n). \tag{37}$$

Equations (36) and (37) imply

$$\mu^*(A) = \sum_{n=1}^{\infty} \mu^*(A_n). \tag{38}$$

Suppose $\mu^*(A)$ is finite. Put $B_n = A_1 \cup \cdots \cup A_n$. Then (38) shows that

$$d(A, B_n) = \mu^*\Big(\bigcup_{i=n+1}^{\infty} A_i\Big) = \sum_{i=n+1}^{\infty} \mu^*(A_i) \to 0$$

as $n \to \infty$. Hence $B_n \to A$; and since $B_n \in \mathfrak{M}_F(\mu)$, it is easily seen that $A \in \mathfrak{M}_F(\mu)$.

We have thus shown that $A \in \mathfrak{M}_F(\mu)$ if $A \in \mathfrak{M}(\mu)$ and $\mu^*(A) < +\infty$.

It is now clear that μ^* is countably additive on $\mathfrak{M}(\mu)$. For if

$$A = \bigcup A_n,$$

where $\{A_n\}$ is a sequence of disjoint sets of $\mathfrak{M}(\mu)$, we have shown that (38) holds if $\mu^*(A_n) < +\infty$ for every n, and in the other case (38) is trivial.

Finally, we have to show that $\mathfrak{M}(\mu)$ is a σ-ring. If $A_n \in \mathfrak{M}(\mu)$, $n = 1, 2, 3, \ldots$, it is clear that $\bigcup A_n \in \mathfrak{M}(\mu)$ (Theorem 2.12). Suppose $A \in \mathfrak{M}(\mu)$, $B \in \mathfrak{M}(\mu)$, and

$$A = \bigcup_{n=1}^{\infty} A_n, \qquad B = \bigcup_{n=1}^{\infty} B_n,$$

where $A_n, B_n \in \mathfrak{M}_F(\mu)$. Then the identity

$$A_n \cap B = \bigcup_{i=1}^{\infty} (A_n \cap B_i)$$

shows that $A_n \cap B \in \mathfrak{M}(\mu)$; and since

$$\mu^*(A_n \cap B) \leq \mu^*(A_n) < +\infty,$$

$A_n \cap B \in \mathfrak{M}_F(\mu)$. Hence $A_n - B \in \mathfrak{M}_F(\mu)$, and $A - B \in \mathfrak{M}(\mu)$ since $A - B = \bigcup_{n=1}^{\infty} (A_n - B)$.

We now replace $\mu^*(A)$ by $\mu(A)$ if $A \in \mathfrak{M}(\mu)$. Thus μ, originally only defined on $\mathscr{E}$, is extended to a countably additive set function on the σ-ring $\mathfrak{M}(\mu)$. This extended set function is called a *measure*. The special case $\mu = m$ is called the *Lebesgue measure* on R^p.

11.11 Remarks

(*a*) If A is open, then $A \in \mathfrak{M}(\mu)$. For every open set in R^p is the union of a countable collection of open intervals. To see this, it is sufficient to construct a countable base whose members are open intervals.

By taking complements, it follows that every closed set is in $\mathfrak{M}(\mu)$.

(*b*) If $A \in \mathfrak{M}(\mu)$ and $\varepsilon > 0$, there exist sets F and G such that

$$F \subset A \subset G,$$

F is closed, G is open, and

$$\mu(G - A) < \varepsilon, \qquad \mu(A - F) < \varepsilon. \tag{39}$$

The first inequality holds since μ^* was defined by means of coverings by *open* elementary sets. The second inequality then follows by taking complements.

(*c*) We say that E is a *Borel set* if E can be obtained by a countable number of operations, starting from open sets, each operation consisting in taking unions, intersections, or complements. The collection $\mathscr{B}$ of all Borel sets in R^p is a σ-ring; in fact, it is the smallest σ-ring which contains all open sets. By Remark (*a*), $E \in \mathfrak{M}(\mu)$ if $E \in \mathscr{B}$.

(*d*) If $A \in \mathfrak{M}(\mu)$, there exist Borel sets F and G such that $F \subset A \subset G$, and

$$\mu(G - A) = \mu(A - F) = 0. \tag{40}$$

This follows from (*b*) if we take $\varepsilon = 1/n$ and let $n \to \infty$.

Since $A = F \cup (A - F)$, we see that every $A \in \mathfrak{M}(\mu)$ is the union of a Borel set and a set of measure zero.

The Borel sets are μ-measurable for every μ. But the sets of measure zero [that is, the sets E for which $\mu^*(E) = 0$] may be different for different μ's.

(*e*) For every μ, the sets of measure zero form a σ-ring.

(*f*) In case of the Lebesgue measure, every countable set has measure zero. But there are uncountable (in fact, perfect) sets of measure zero. The Cantor set may be taken as an example: Using the notation of Sec. 2.44, it is easily seen that

$$m(E_n) = (\tfrac{2}{3})^n \qquad (n = 1, 2, 3, \ldots);$$

and since $P = \bigcap E_n$, $P \subset E_n$ for every n, so that $m(P) = 0$.

MEASURE SPACES

11.12 Definition Suppose X is a set, not necessarily a subset of a euclidean space, or indeed of any metric space. X is said to be a *measure space* if there exists a σ-ring $\mathfrak{M}$ of subsets of X (which are called measurable sets) and a non-negative countably additive set function μ (which is called a measure), defined on $\mathfrak{M}$.

If, in addition, $X \in \mathfrak{M}$, then X is said to be a *measurable space*.

For instance, we can take $X = R^p$, $\mathfrak{M}$ the collection of all Lebesgue-measurable subsets of R^p, and μ Lebesgue measure.

Or, let X be the set of all positive integers, $\mathfrak{M}$ the collection of all subsets of X, and $\mu(E)$ the number of elements of E.

Another example is provided by probability theory, where events may be considered as sets, and the probability of the occurrence of events is an additive (or countably additive) set function.

In the following sections we shall always deal with measurable spaces. It should be emphasized that the integration theory which we shall soon discuss would not become simpler in any respect if we sacrificed the generality we have now attained and restricted ourselves to Lebesgue measure, say, on an interval of the real line. In fact, the essential features of the theory are brought out with much greater clarity in the more general situation, where it is seen that everything depends only on the countable additivity of μ on a σ-ring.

It will be convenient to introduce the notation

$$\{x \mid P\} \tag{41}$$

for the set of all elements x which have the property P.

MEASURABLE FUNCTIONS

11.13 Definition Let f be a function defined on the measurable space X, with values in the extended real number system. The function f is said to be *measurable* if the set

$$\{x \mid f(x) > a\} \tag{42}$$

is measurable for every real a.

11.14 Example If $X = R^p$ and $\mathfrak{M} = \mathfrak{M}(\mu)$ as defined in Definition 11.9, every continuous f is measurable, since then (42) is an open set.

11.15 Theorem *Each of the following four conditions implies the other three:*

$$\{x \mid f(x) > a\} \text{ is measurable for every real } a. \tag{43}$$

$$\{x \mid f(x) \geq a\} \text{ is measurable for every real } a. \tag{44}$$

$$\{x \mid f(x) < a\} \text{ is measurable for every real } a. \tag{45}$$

$$\{x \mid f(x) \leq a\} \text{ is measurable for every real } a. \tag{46}$$

Proof The relations

$$\{x \mid f(x) \geq a\} = \bigcap_{n=1}^{\infty} \left\{x \mid f(x) > a - \frac{1}{n}\right\},$$

$$\{x \mid f(x) < a\} = X - \{x \mid f(x) \geq a\},$$

$$\{x \mid f(x) \leq a\} = \bigcap_{n=1}^{\infty} \left\{x \mid f(x) < a + \frac{1}{n}\right\},$$

$$\{x \mid f(x) > a\} = X - \{x \mid f(x) \leq a\}$$

show successively that (43) implies (44), (44) implies (45), (45) implies (46), and (46) implies (43).

Hence any of these conditions may be used instead of (42) to define measurability.

11.16 Theorem *If f is measurable, then $|f|$ is measurable.*

Proof

$$\{x \mid |f(x)| < a\} = \{x \mid f(x) < a\} \cap \{x \mid f(x) > -a\}.$$

11.17 Theorem *Let $\{f_n\}$ be a sequence of measurable functions. For $x \in X$, put*

$$g(x) = \sup f_n(x) \qquad (n = 1, 2, 3, \ldots),$$

$$h(x) = \limsup_{n \to \infty} f_n(x).$$

Then g and h are measurable.

The same is of course true of the inf and lim inf.

Proof

$$\{x \mid g(x) > a\} = \bigcup_{n=1}^{\infty} \{x \mid f_n(x) > a\},$$

$$h(x) = \inf g_m(x),$$

where $g_m(x) = \sup f_n(x)$ $(n \geq m)$.

Corollaries

(*a*) *If f and g are measurable, then* $\max(f, g)$ *and* $\min(f, g)$ *are measurable. If*

$$f^+ = \max(f, 0), \qquad f^- = -\min(f, 0), \tag{47}$$

it follows, in particular, that f^+ and f^- are measurable.

(*b*) *The limit of a convergent sequence of measurable functions is measurable.*

11.18 Theorem *Let f and g be measurable real-valued functions defined on X, let F be real and continuous on R^2, and put*

$$h(x) = F(f(x), g(x)) \qquad (x \in X).$$

Then h is measurable.

In particular, $f + g$ and fg are measurable.

Proof Let

$$G_a = \{(u, v) \mid F(u, v) > a\}.$$

Then G_a is an open subset of R^2, and we can write

$$G_a = \bigcup_{n=1}^{\infty} I_n,$$

where $\{I_n\}$ is a sequence of open intervals:

$$I_n = \{(u, v) \mid a_n < u < b_n, c_n < v < d_n\}.$$

Since

$$\{x \mid a_n < f(x) < b_n\} = \{x \mid f(x) > a_n\} \cap \{x \mid f(x) < b_n\}$$

is measurable, it follows that the set

$$\{x \mid (f(x), g(x)) \in I_n\} = \{x \mid a_n < f(x) < b_n\} \cap \{x \mid c_n < g(x) < d_n\}$$

is measurable. Hence the same is true of

$$\begin{aligned}\{x \mid h(x) > a\} &= \{x \mid (f(x), g(x)) \in G_a\} \\ &= \bigcup_{n=1}^{\infty} \{x \mid (f(x), g(x)) \in I_n\}.\end{aligned}$$

Summing up, we may say that all ordinary operations of analysis, including limit operations, when applied to measurable functions, lead to measurable functions; in other words, all functions that are ordinarily met with are measurable.

That this is, however, only a rough statement is shown by the following example (based on Lebesgue measure, on the real line): If $h(x) = f(g(x))$, where