2

Greek Geometry

2.1 The Deductive Method

> He was 40 years old before he looked on Geometry; which happened accidentally. Being in a Gentleman's Library, Euclid's Elements lay open, and 'twas the 47 El. libri I. He read the Proposition. *By* G——sayd he (he would now and then sweare an emphaticall Oath by way of emphasis) *this is impossible*! So he reads the Demonstration of it, which referred him back to such a Proposition; which proposition he read. That referred him back to another, which he also read ... that at last he was demonstratively convinced of that trueth. This made him in love with Geometry.

This quotation about the philosopher Thomas Hobbes (1588–1679), from Aubrey's *Brief Lives*, beautifully captures the force of Greece's most important contribution to mathematics, the deductive method. (The proposition mentioned, incidentally, is Pythagoras' theorem.)

We have already seen that significant results were *known* before the period of classical Greece, but the Greeks were the first to construct mathematics by deduction from previously established results, resting ultimately on the most evident possible statements, called *axioms*. Thales (624–547 BCE) is thought to be the originator of this method [see Heath (1921), p. 128], and by 300 BCE it had become so sophisticated that Euclid's *Elements* set the standard for mathematical rigor until the nineteenth century. The *Elements* were in fact too subtle for most mathematicians, let alone

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their students, so that in time they were boiled down to the simplest and driest propositions about straight lines, triangles, and circles. This part of the *Elements* is based on the following axioms [in the translation of Heath (1925), p. 154], which Euclid called *postulates* and *common notions*.

Postulates

Let the following be postulated:

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any center and distance.
4. That all right angles are equal to one another.
5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Common Notions

1. Things which are equal to the same thing are also equal to one another.
2. If equals be added to equals, the wholes are equal.
3. If equals be subtracted from equals, the remainders are equal.
4. Things which coincide with one another are equal to one another.
5. The whole is greater than the part.

It appears that Euclid's intention was to deduce geometric propositions from visually evident statements (the postulates) using evident principles of logic (the common notions). Actually, he often made unconscious use of visually plausible assumptions that are not among his postulates. His very first proposition used the unstated assumption that two circles meet if the center of each is on the circumference of the other [Heath (1925), p. 242]. Nevertheless, such flaws were not noticed until the nineteenth century, and they were rectified by Hilbert (1899). By themselves, they probably would not have been enough to end the *Elements*' run of 22 centuries as a leading textbook. The *Elements* was overthrown by more serious mathematical upheavals in the nineteenth century. The so-called noneuclidean geometries, using alternatives to Euclid's fifth postulate (the *parallel axiom*), developed to the point where the old axioms could no longer be considered self-evident (see Chapter 18). At the same time, the concept of number matured to the point where irrational numbers became acceptable, and indeed preferable to intuitive geometric concepts, in view of the doubts about what the self-evident truths of geometry really were.

The outcome was a more adaptable language for geometry in which "points," "lines," and so on, could be defined, usually in terms of numbers, so as to suit the type of geometry under investigation. Such a development was long overdue, because even in Euclid's time the Greeks were investigating curves more complicated than circles, which did not fit conveniently in Euclid's system. Descartes (1637) introduced the coordinate method, which gives a single framework for handling both Euclid's geometry and higher curves (see Chapter 7), but it was not at first realized that coordinates allowed geometry to be entirely rebuilt on numerical foundations.

The comparatively trivial step (for us) of passing to axioms about numbers from axioms about points had to wait until the nineteenth century, when geometric axioms about points lost authority and number-theoretic axioms gained it. We shall say more about these developments later (and of problems with the authority of axioms in general, which arose in the twentieth century). For the remainder of this chapter we shall look at some important nonelementary topics in Greek geometry, using the coordinate framework where convenient.

Exercises

Euclid's Common Notions 1 and 4 define what we now call an *equivalence relation*, which is not necessarily the equality relation. In fact, the kind of relation Euclid had in mind was equality in *some* geometric quantity such as length or angle (but not necessarily equality in all respects—the latter is what he meant by "coinciding"). An equivalence relation $\cong$ is normally defined by three properties. For any a, b and c:

$$\begin{aligned} &a \cong a, && \text{(reflexive)}\\ a \cong b \quad &\Longrightarrow \quad b \cong a, && \text{(symmetric)}\\ a \cong b \text{ and } b \cong c \quad &\Longrightarrow \quad a \cong c. && \text{(transitive)} \end{aligned}$$

2.1.1 Explain how Common Notions 1 and 4 may be interpreted as the transitive and reflexive properties. Note that the natural way to write Common Notion 1 symbolically is slightly different from the statement of transitivity above.

2.1.2 Show that the symmetric property follows from Euclid's Common Notions 1 and 4.

Hilbert (1899) took advantage of Euclid's Common Notions 1 and 4 in his rectification of Euclid's axiom system. He *defined* equality of length by postulating a transitive and reflexive relation on line segments, and stated transitivity in the style of Euclid, so that the symmetric property was a consequence.

2.2 The Regular Polyhedra

Greek geometry is virtually complete as far as the elementary properties of plane figures are concerned. It is fair to say that only a handful of interesting elementary propositions about triangles and circles have been discovered since Euclid's time. Solid geometry is much more challenging, even today, so it is understandable that it was left in a less complete state by the Greeks. Nevertheless, they made some very impressive discoveries and managed to complete one of the most beautiful chapters in solid geometry, the enumeration of the regular polyhedra. The five possible regular polyhedra are shown in Figure 2.1.

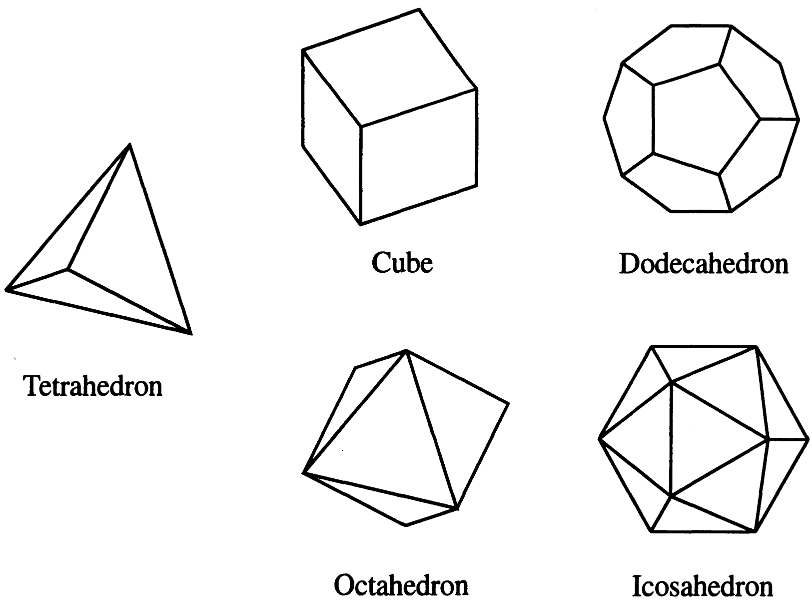

Figure 2.1: The regular polyhedra