However, we shall stick with the Peano axiomatic approach for now.

How are we to define what the natural numbers are? Informally, we could say

Definition 2.1.1. (Informal) A *natural number* is any element of the set

$$\mathbf{N} := \{0, 1, 2, 3, 4, \ldots\},$$

which is the set of all the numbers created by starting with 0 and then counting forward indefinitely. We call $\mathbf{N}$ the *set of natural numbers*.

Remark 2.1.2. In some texts the natural numbers start at 1 instead of 0, but this is a matter of notational convention more than anything else. In this text we shall refer to the set $\{1, 2, 3, \ldots\}$ as the *positive integers* $\mathbf{Z}^+$ rather than the natural numbers. Natural numbers are sometimes also known as *whole numbers*.

In a sense, this definition solves the problem of what the natural numbers are: a natural number is any element of the set[1] $\mathbf{N}$. However, it is not really that satisfactory, because it begs the question of what $\mathbf{N}$ is. This definition of "start at 0 and count indefinitely" seems like an intuitive enough definition of $\mathbf{N}$, but it is not entirely acceptable, because it leaves many questions unanswered. For instance: how do we know we can keep counting indefinitely, without cycling back to 0? Also, how do you perform operations such as addition, multiplication, or exponentiation?

We can answer the latter question first: we can define complicated operations in terms of simpler operations. Exponentiation is nothing more than repeated multiplication: 5^3 is nothing more than three fives multiplied together. Multiplication is nothing more than repeated addition; 5×3 is nothing more than three fives added together. (Subtraction and division will not be covered here, because they are not operations which are well-suited

[1]Strictly speaking, there is another problem with this informal definition: we have not yet defined what a "set" is, or what "element of" is. Thus for the rest of this chapter we shall avoid mention of sets and their elements as much as possible, except in informal discussion.

to the natural numbers; they will have to wait for the integers and rationals, respectively.) And addition? It is nothing more than the repeated operation of *counting forward*, or *incrementing*. If you add three to five, what you are doing is incrementing five three times. On the other hand, incrementing seems to be a fundamental operation, not reducible to any simpler operation; indeed, it is the first operation one learns on numbers, even before learning to add.

Thus, to define the natural numbers, we will use two fundamental concepts: the zero number 0, and the increment operation. In deference to modern computer languages, we will use $n{+}{+}$ to denote the increment or *successor* of n, thus for instance $3{+}{+} = 4$, $(3{+}{+}){+}{+} = 5$, etc. This is a slightly different usage from that in computer languages such as C, where $n{+}{+}$ actually *redefines* the value of n to be its successor; however in mathematics we try not to define a variable more than once in any given setting, as it can often lead to confusion; many of the statements which were true for the old value of the variable can now become false, and vice versa.

So, it seems like we want to say that $\mathbf{N}$ consists of 0 and everything which can be obtained from 0 by incrementing: $\mathbf{N}$ should consist of the objects

$$0, 0{+}{+}, (0{+}{+}){+}{+}, ((0{+}{+}){+}{+}){+}{+}, \text{etc.}$$

If we start writing down what this means about the natural numbers, we thus see that we should have the following axioms concerning 0 and the increment operation $++$:

Axiom 2.1. *0 is a natural number.*

Axiom 2.2. *If n is a natural number, then $n{+}{+}$ is also a natural number.*

Thus for instance, from Axiom 2.1 and two applications of Axiom 2.2, we see that $(0{+}{+}){+}{+}$ is a natural number. Of course, this notation will begin to get unwieldy, so we adopt a convention to write these numbers in more familiar notation:

Definition 2.1.3. We define 1 to be the number 0++, 2 to be the number (0++)++, 3 to be the number ((0++)++)++, etc. (In other words, $1 := 0{+}{+}$, $2 := 1{+}{+}$, $3 := 2{+}{+}$, etc. In this text I use "$x := y$" to denote the statement that x is *defined* to equal y.)

Thus for instance, we have

Proposition 2.1.4. *3 is a natural number.*

Proof. By Axiom 2.1, 0 is a natural number. By Axiom 2.2, $0{+}{+} = 1$ is a natural number. By Axiom 2.2 again, $1{+}{+} = 2$ is a natural number. By Axiom 2.2 again, $2{+}{+} = 3$ is a natural number. □

It may seem that this is enough to describe the natural numbers. However, we have not pinned down completely the behavior of **N**:

Example 2.1.5. Consider a number system which consists of the numbers $0, 1, 2, 3$, in which the increment operation wraps back from 3 to 0. More precisely 0++ is equal to 1, 1++ is equal to 2, 2++ is equal to 3, but 3++ is equal to 0 (and also equal to 4, by definition of 4). This type of thing actually happens in real life, when one uses a computer to try to store a natural number: if one starts at 0 and performs the increment operation repeatedly, eventually the computer will overflow its memory and the number will wrap around back to 0 (though this may take quite a large number of incrementation operations, for instance a two-byte representation of an integer will wrap around only after $65,536$ increments). Note that this type of number system obeys Axiom 2.1 and Axiom 2.2, even though it clearly does not correspond to what we intuitively believe the natural numbers to be like.

To prevent this sort of "wrap-around issue" we will impose another axiom:

Axiom 2.3. *0 is not the successor of any natural number; i.e., we have $n{+}{+} \neq 0$ for every natural number n.*

Now we can show that certain types of wrap-around do not occur: for instance we can now rule out the type of behavior in Example 2.1.5 using

Proposition 2.1.6. 4 *is not equal to* 0.

Don't laugh! Because of the way we have defined 4 - it is the increment of the increment of the increment of the increment of 0 - it is not necessarily true *a priori* that this number is not the same as zero, even if it is "obvious". ("a priori" is Latin for "beforehand" - it refers to what one already knows or assumes to be true before one begins a proof or argument. The opposite is "a posteriori" - what one knows to be true after the proof or argument is concluded.) Note for instance that in Example 2.1.5, 4 was indeed equal to 0, and that in a standard two-byte computer representation of a natural number, for instance, 65536 is equal to 0 (using our definition of 65536 as equal to 0 incremented sixty-five thousand, five hundred and thirty-six times).

Proof. By definition, $4 = 3{+}{+}$. By Axioms 2.1 and 2.2, 3 is a natural number. Thus by Axiom 2.3, $3{+}{+} \neq 0$, i.e., $4 \neq 0$. □

However, even with our new axiom, it is still possible that our number system behaves in other pathological ways:

Example 2.1.7. Consider a number system consisting of five numbers 0,1,2,3,4, in which the increment operation hits a "ceiling" at 4. More precisely, suppose that $0{+}{+} = 1$, $1{+}{+} = 2$, $2{+}{+} = 3$, $3{+}{+} = 4$, but $4{+}{+} = 4$ (or in other words that $5 = 4$, and hence $6 = 4$, $7 = 4$, etc.). This does not contradict Axioms 2.1,2.2,2.3. Another number system with a similar problem is one in which incrementation wraps around, but not to zero, e.g. suppose that $4{+}{+} = 1$ (so that $5 = 1$, then $6 = 2$, etc.).

There are many ways to prohibit the above types of behavior from happening, but one of the simplest is to assume the following axiom:

Axiom 2.4. *Different natural numbers must have different successors; i.e., if* n, m *are natural numbers and* $n \neq m$, *then* $n{+}{+} \neq m{+}{+}$. *Equivalently*[2], *if* $n{+}{+} = m{+}{+}$, *then we must have* $n = m$.

Thus, for instance, we have

Proposition 2.1.8. 6 *is not equal to* 2.

Proof. Suppose for sake of contradiction that $6 = 2$. Then $5{+}{+} = 1{+}{+}$, so by Axiom 2.4 we have $5 = 1$, so that $4{+}{+} = 0{+}{+}$. By Axiom 2.4 again we then have $4 = 0$, which contradicts our previous proposition. □

As one can see from this proposition, it now looks like we can keep all of the natural numbers distinct from each other. There is however still one more problem: while the axioms (particularly Axioms 2.1 and 2.2) allow us to confirm that $0, 1, 2, 3, \ldots$ are distinct elements of $\mathbf{N}$, there is the problem that there may be other "rogue" elements in our number system which are not of this form:

Example 2.1.9. (Informal) Suppose that our number system $\mathbf{N}$ consisted of the following collection of integers and half-integers:

$$\mathbf{N} := \{0, 0.5, 1, 1.5, 2, 2.5, 3, 3.5, \ldots\}.$$

(This example is marked "informal" since we are using real numbers, which we're not supposed to use yet.) One can check that Axioms 2.1-2.4 are still satisfied for this set.

What we want is some axiom which says that the only numbers in $\mathbf{N}$ are those which can be obtained from 0 and the increment operation - in order to exclude elements such as 0.5. But it is difficult to quantify what we mean by "can be obtained from" without already using the natural numbers, which we are trying to define. Fortunately, there is an ingenious solution to try to capture this fact:

[2]This is an example of reformulating an implication using its *contrapositive*; see Section A.2 for more details.

Axiom 2.5 (Principle of mathematical induction). *Let $P(n)$ be any property pertaining to a natural number n. Suppose that $P(0)$ is true, and suppose that whenever $P(n)$ is true, $P(n{+}{+})$ is also true. Then $P(n)$ is true for every natural number n.*

Remark 2.1.10. We are a little vague on what "property" means at this point, but some possible examples of $P(n)$ might be "n is even"; "n is equal to 3"; "n solves the equation $(n+1)^2 = n^2+2n+1$"; and so forth. Of course we haven't defined many of these concepts yet, but when we do, Axiom 2.5 will apply to these properties. (A logical remark: Because this axiom refers not just to *variables*, but also *properties*, it is of a different nature than the other four axioms; indeed, Axiom 2.5 should technically be called an *axiom schema* rather than an *axiom* - it is a template for producing an (infinite) number of axioms, rather than being a single axiom in its own right. To discuss this distinction further is far beyond the scope of this text, though, and falls in the realm of logic.)

The informal intuition behind this axiom is the following. Suppose $P(n)$ is such that $P(0)$ is true, and such that whenever $P(n)$ is true, then $P(n{+}{+})$ is true. Then since $P(0)$ is true, $P(0{+}{+}) = P(1)$ is true. Since $P(1)$ is true, $P(1{+}{+}) = P(2)$ is true. Repeating this indefinitely, we see that $P(0)$, $P(1)$, $P(2)$, $P(3)$, etc. are all true - however this line of reasoning will never let us conclude that $P(0.5)$, for instance, is true. Thus Axiom 2.5 should not hold for number systems which contain "unnecessary" elements such as 0.5. (Indeed, one can give a "proof" of this fact. Apply Axiom 2.5 to the property $P(n) = n$ "is not a half-integer", i.e., an integer plus 0.5. Then $P(0)$ is true, and if $P(n)$ is true, then $P(n{+}{+})$ is true. Thus Axiom 2.5 asserts that $P(n)$ is true for all natural numbers n, i.e., no natural number can be a half-integer. In particular, 0.5 cannot be a natural number. This "proof" is not quite genuine, because we have not defined such notions as "integer", "half-integer", and "0.5" yet, but it should give you some idea as to how the principle of induction is supposed to prohibit any numbers other than the "true" natural numbers from appearing in $\mathbf{N}$.)

The principle of induction gives us a way to prove that a property $P(n)$ is true for every natural number n. Thus in the rest of this text we will see many proofs which have a form like this:

Proposition 2.1.11. *A certain property $P(n)$ is true for every natural number n.*

Proof. We use induction. We first verify the base case $n = 0$, i.e., we prove $P(0)$. (Insert proof of $P(0)$ here). Now suppose inductively that n is a natural number, and $P(n)$ has already been proven. We now prove $P(n{+\!+})$. (Insert proof of $P(n{+\!+})$, assuming that $P(n)$ is true, here). This closes the induction, and thus $P(n)$ is true for all numbers n. $\square$

Of course we will not necessarily use the exact template, wording, or order in the above type of proof, but the proofs using induction will generally be something like the above form. There are also some other variants of induction which we shall encounter later, such as backwards induction (Exercise 2.2.6), strong induction (Proposition 2.2.14), and transfinite induction (Lemma 8.5.15).

Axioms 2.1-2.5 are known as the *Peano axioms* for the natural numbers. They are all very plausible, and so we shall make

Assumption 2.6. (*Informal*) *There exists a number system $\mathbf{N}$, whose elements we will call natural numbers, for which Axioms 2.1-2.5 are true.*

We will make this assumption a bit more precise once we have laid down our notation for sets and functions in the next chapter.

Remark 2.1.12. We will refer to this number system $\mathbf{N}$ as *the* natural number system. One could of course consider the possibility that there is more than one natural number system, e.g., we could have the Hindu-Arabic number system $\{0, 1, 2, 3, \dots\}$ and the Roman number system $\{O, I, II, III, IV, V, VI, \dots\}$, and if we really wanted to be annoying we could view these number systems as different. But these number systems are clearly equivalent (the technical term is *isomorphic*), because one can create a one-to-one correspondence $0 \leftrightarrow O$, $1 \leftrightarrow I$, $2 \leftrightarrow II$, etc. which maps the zero of the Hindu-Arabic system with the zero of the Roman system, and which is preserved by the increment operation (e.g., if 2 corresponds to II, then $2{+}{+}$ will correspond to $II{+}{+}$). For a more precise statement of this type of equivalence, see Exercise 3.5.13. Since all versions of the natural number system are equivalent, there is no point in having distinct natural number systems, and we will just use a single natural number system to do mathematics.

We will not prove Assumption 2.6 (though we will eventually include it in our axioms for set theory, see Axiom 3.7), and it will be the only assumption we will ever make about our numbers. A remarkable accomplishment of modern analysis is that just by starting from these five very primitive axioms, and some additional axioms from set theory, we can build all the other number systems, create functions, and do all the algebra and calculus that we are used to.

Remark 2.1.13. (Informal) One interesting feature about the natural numbers is that while each individual natural number is finite, the *set* of natural numbers is infinite; i.e., $\mathbf{N}$ is infinite but consists of individually finite elements. (The whole is greater than any of its parts.) There are no infinite natural numbers; one can even prove this using Axiom 2.5, provided one is comfortable with the notions of finite and infinite. (Clearly 0 is finite. Also, if n is finite, then clearly $n{+}{+}$ is also finite. Hence by Axiom 2.5, all natural numbers are finite.) So the natural numbers can *approach* infinity, but never actually reach it; infinity is not one of the natural numbers. (There are other number systems which admit "infinite" numbers, such as the cardinals, ordinals, and p-adics, but they do not obey the principle of induction, and in any event are beyond the scope of this text.)

Remark 2.1.14. Note that our definition of the natural numbers is *axiomatic* rather than *constructive*. We have not told you what the natural numbers *are* (so we do not address such questions as what the numbers are made of, are they physical objects, what do they measure, etc.) - we have only listed some things you can do with them (in fact, the only operation we have defined on them right now is the increment one) and some of the properties that they have. This is how mathematics works - it treats its objects *abstractly*, caring only about what properties the objects have, not what the objects are or what they mean. If one wants to do mathematics, it does not matter whether a natural number means a certain arrangement of beads on an abacus, or a certain organization of bits in a computer's memory, or some more abstract concept with no physical substance; as long as you can increment them, see if two of them are equal, and later on do other arithmetic operations such as add and multiply, they qualify as numbers for mathematical purposes (provided they obey the requisite axioms, of course). It is possible to construct the natural numbers from other mathematical objects - from sets, for instance - but there are multiple ways to construct a working model of the natural numbers, and it is pointless, at least from a mathematician's standpoint, as to argue about which model is the "true" one - as long as it obeys all the axioms and does all the right things, that's good enough to do maths.

Remark 2.1.15. Historically, the realization that numbers could be treated axiomatically is very recent, not much more than a hundred years old. Before then, numbers were generally understood to be inextricably connected to some external concept, such as counting the cardinality of a set, measuring the length of a line segment, or the mass of a physical object, etc. This worked reasonably well, until one was forced to move from one number system to another; for instance, understanding numbers in terms of counting beads, for instance, is great for conceptualizing the numbers 3 and 5, but doesn't work so well for -3 or $1/3$ or $\sqrt{2}$ or $3+4i$; thus each great advance in the theory of numbers - negative numbers, irrational numbers, complex numbers, even the number zero - led to a lot of unnecessary philosophical anguish. The great discovery of the late nineteenth century was that numbers can be understood abstractly via axioms, without necessarily needing a concrete model; of course a mathematician can use any of these models when it is convenient, to aid his or her intuition and understanding, but they can also be just as easily discarded when they begin to get in the way.

One consequence of the axioms is that we can now define sequences *recursively*. Suppose we want to build a sequence a_0, a_1, $a_2, \ldots$ of numbers by first defining a_0 to be some base value, e.g., $a_0 := c$ for some number c, and then by letting a_1 be some function of a_0, $a_1 := f_0(a_0)$, a_2 be some function of a_1, $a_2 := f_1(a_1)$, and so forth. In general, we set $a_{n++} := f_n(a_n)$ for some function f_n from $\mathbf{N}$ to $\mathbf{N}$. By using all the axioms together we will now conclude that this procedure will give a single value to the sequence element a_n for each natural number n. More precisely[3]:

Proposition 2.1.16 (Recursive definitions). *Suppose for each natural number n, we have some function $f_n : \mathbf{N} \to \mathbf{N}$ from the natural numbers to the natural numbers. Let c be a natural number. Then we can assign a unique natural number a_n to each natural number n, such that $a_0 = c$ and $a_{n++} = f_n(a_n)$ for each natural number n.*

Proof. (Informal) We use induction. We first observe that this procedure gives a single value to a_0, namely c. (None of the other definitions $a_{n++} := f_n(a_n)$ will redefine the value of a_0, because of Axiom 2.3.) Now suppose inductively that the procedure gives a single value to a_n. Then it gives a single value to a_{n++}, namely $a_{n++} := f_n(a_n)$. (None of the other definitions $a_{m++} := f_m(a_m)$ will redefine the value of a_{n++}, because of Axiom 2.4.) This completes the induction, and so a_n is defined for each natural number n, with a single value assigned to each a_n. □

[3]Strictly speaking, this proposition requires one to define the notion of a *function*, which we shall do in the next chapter. However, this will not be circular, as the concept of a function does not require the Peano axioms. Proposition 2.1.16 can be formalized more rigourously in the language of set theory; see Exercise 3.5.12.

Note how all of the axioms had to be used here. In a system which had some sort of wrap-around, recursive definitions would not work because some elements of the sequence would constantly be redefined. For instance, in Example 2.1.5, in which $3{+}{+} = 0$, then there would be (at least) two conflicting definitions for a_0, either c or $f_3(a_3)$). In a system which had superfluous elements such as 0.5, the element $a_{0.5}$ would never be defined.

Recursive definitions are very powerful; for instance, we can use them to define addition and multiplication, to which we now turn.

2.2 Addition

The natural number system is very bare right now: we have only one operation - increment - and a handful of axioms. But now we can build up more complex operations, such as addition.

The way it works is the following. To add three to five should be the same as incrementing five three times - this is one increment more than adding two to five, which is one increment more than adding one to five, which is one increment more than adding zero to five, which should just give five. So we give a recursive definition for addition as follows.

Definition 2.2.1 (Addition of natural numbers). Let m be a natural number. To add zero to m, we define $0 + m := m$. Now suppose inductively that we have defined how to add n to m. Then we can add $n{+}{+}$ to m by defining $(n{+}{+}) + m := (n + m){+}{+}$.

Thus $0 + m$ is m, $1 + m = (0{+}{+}) + m$ is $m{+}{+}$; $2 + m = (1{+}{+}) + m = (m{+}{+}){+}{+}$; and so forth; for instance we have $2+3 = (3{+}{+}){+}{+} = 4{+}{+} = 5$. From our discussion of recursion in the previous section we see that we have defined $n + m$ for every integer n. Here we are specializing the previous general discussion to the setting where $a_n = n + m$ and $f_n(a_n) = a_n{+}{+}$. Note that this definition is asymmetric: $3 + 5$ is incrementing 5 three times, while $5 + 3$ is incrementing 3 five times. Of course, they both yield the same value of 8. More generally, it is a fact (which we shall prove shortly) that $a+b=b+a$ for all natural numbers a, b, although this is not immediately clear from the definition.

Notice that we can prove easily, using Axioms 2.1, 2.2, and induction (Axiom 2.5), that the sum of two natural numbers is again a natural number (why?).

Right now we only have two facts about addition: that $0+m = m$, and that $(n{+}{+})+m = (n+m){+}{+}$. Remarkably, this turns out to be enough to deduce everything else we know about addition. We begin with some basic lemmas[4].

Lemma 2.2.2. *For any natural number n, $n + 0 = n$.*

Note that we cannot deduce this immediately from $0+m = m$ because we do not know yet that $a + b = b + a$.

Proof. We use induction. The base case $0 + 0 = 0$ follows since we know that $0 + m = m$ for every natural number m, and 0 is a natural number. Now suppose inductively that $n + 0 = n$. We wish to show that $(n{+}{+}) + 0 = n{+}{+}$. But by definition of addition, $(n{+}{+}) + 0$ is equal to $(n + 0){+}{+}$, which is equal to $n{+}{+}$ since $n + 0 = n$. This closes the induction. □

Lemma 2.2.3. *For any natural numbers n and m, $n + (m{+}{+}) = (n + m){+}{+}$.*

Again, we cannot deduce this yet from $(n{+}{+})+m = (n+m){+}{+}$ because we do not know yet that $a + b = b + a$.

Proof. We induct on n (keeping m fixed). We first consider the base case $n = 0$. In this case we have to prove $0 + (m{+}{+}) = (0 +$

[4] From a logical point of view, there is no difference between a lemma, proposition, theorem, or corollary - they are all claims waiting to be proved. However, we use these terms to suggest different levels of importance and difficulty. A lemma is an easily proved claim which is helpful for proving other propositions and theorems, but is usually not particularly interesting in its own right. A proposition is a statement which is interesting in its own right, while a theorem is a more important statement than a proposition which says something definitive on the subject, and often takes more effort to prove than a proposition or lemma. A corollary is a quick consequence of a proposition or theorem that was proven recently.

$m)$++. But by definition of addition, $0 + (m{++}) = m{++}$ and $0 + m = m$, so both sides are equal to $m{++}$ and are thus equal to each other. Now we assume inductively that $n + (m{++}) = (n + m){++}$; we now have to show that $(n{++}) + (m{++}) = ((n{++}) + m){++}$. The left-hand side is $(n + (m{++})){++}$ by definition of addition, which is equal to $((n + m){++}){++}$ by the inductive hypothesis. Similarly, we have $(n{++}) + m = (n + m){++}$ by the definition of addition, and so the right-hand side is also equal to $((n + m){++}){++}$. Thus both sides are equal to each other, and we have closed the induction. □

As a particular corollary of Lemma 2.2.2 and Lemma 2.2.3 we see that $n{++} = n + 1$ (why?).

As promised earlier, we can now prove that $a + b = b + a$.

Proposition 2.2.4 (Addition is commutative). *For any natural numbers n and m, $n + m = m + n$.*

Proof. We shall use induction on n (keeping m fixed). First we do the base case $n = 0$, i.e., we show $0 + m = m + 0$. By the definition of addition, $0 + m = m$, while by Lemma 2.2.2, $m + 0 = m$. Thus the base case is done. Now suppose inductively that $n + m = m + n$, now we have to prove that $(n{++}) + m = m + (n{++})$ to close the induction. By the definition of addition, $(n{++}) + m = (n + m){++}$. By Lemma 2.2.3, $m + (n{++}) = (m + n){++}$, but this is equal to $(n + m){++}$ by the inductive hypothesis $n + m = m + n$. Thus $(n{++}) + m = m + (n{++})$ and we have closed the induction. □

Proposition 2.2.5 (Addition is associative). *For any natural numbers a, b, c, we have $(a + b) + c = a + (b + c)$.*

Proof. See Exercise 2.2.1. □

Because of this associativity we can write sums such as $a + b + c$ without having to worry about which order the numbers are being added together.

Now we develop a cancellation law.

Proposition 2.2.6 (Cancellation law). *Let a, b, c be natural numbers such that $a + b = a + c$. Then we have $b = c$.*

Note that we cannot use subtraction or negative numbers yet to prove this proposition, because we have not developed these concepts yet. In fact, this cancellation law is crucial in letting us define subtraction (and the integers) later on in these notes, because it allows for a sort of "virtual subtraction" even before subtraction is officially defined.

Proof. We prove this by induction on a. First consider the base case $a = 0$. Then we have $0 + b = 0 + c$, which by definition of addition implies that $b = c$ as desired. Now suppose inductively that we have the cancellation law for a (so that $a+b = a+c$ implies $b = c$); we now have to prove the cancellation law for $a{+}{+}$. In other words, we assume that $(a{+}{+}) + b = (a{+}{+}) + c$ and need to show that $b = c$. By the definition of addition, $(a{+}{+}) + b = (a + b){+}{+}$ and $(a{+}{+}) + c = (a + c){+}{+}$ and so we have $(a + b){+}{+} = (a + c){+}{+}$. By Axiom 2.4, we have $a + b = a + c$. Since we already have the cancellation law for a, we thus have $b = c$ as desired. This closes the induction. □

We now discuss how addition interacts with positivity.

Definition 2.2.7 (Positive natural numbers). A natural number n is said to be *positive* iff it is not equal to 0. ("iff" is shorthand for "if and only if" - see Section A.1).

Proposition 2.2.8. *If a is positive and b is a natural number, then $a + b$ is positive (and hence $b + a$ is also, by Proposition 2.2.4).*

Proof. We use induction on b. If $b = 0$, then $a + b = a + 0 = a$, which is positive, so this proves the base case. Now suppose inductively that $a + b$ is positive. Then $a + (b{+}{+}) = (a + b){+}{+}$, which cannot be zero by Axiom 2.3, and is hence positive. This closes the induction. □

Corollary 2.2.9. *If a and b are natural numbers such that $a+b = 0$, then $a = 0$ and $b = 0$.*

Proof. Suppose for sake of contradiction that $a \neq 0$ or $b \neq 0$. If $a \neq 0$ then a is positive, and hence $a + b = 0$ is positive by Proposition 2.2.8, a contradiction. Similarly if $b \neq 0$ then b is positive, and again $a + b = 0$ is positive by Proposition 2.2.8, a contradiction. Thus a and b must both be zero. □

Lemma 2.2.10. *Let a be a positive number. Then there exists exactly one natural number b such that $b{+}{+} = a$.*

Proof. See Exercise 2.2.2. □

Once we have a notion of addition, we can begin defining a notion of *order*.

Definition 2.2.11 (Ordering of the natural numbers). Let n and m be natural numbers. We say that n is *greater than or equal to* m, and write $n \geq m$ or $m \leq n$, iff we have $n = m + a$ for some natural number a. We say that n is *strictly greater than* m, and write $n > m$ or $m < n$, iff $n \geq m$ and $n \neq m$.

Thus for instance $8 > 5$, because $8 = 5 + 3$ and $8 \neq 5$. Also note that $n{+}{+} > n$ for any n; thus there is no largest natural number n, because the next number $n{+}{+}$ is always larger still.

Proposition 2.2.12 (Basic properties of order for natural numbers). *Let a, b, c be natural numbers. Then*

(a) *(Order is reflexive) $a \geq a$.*

(b) *(Order is transitive) If $a \geq b$ and $b \geq c$, then $a \geq c$.*

(c) *(Order is anti-symmetric) If $a \geq b$ and $b \geq a$, then $a = b$.*

(d) *(Addition preserves order) $a \geq b$ if and only if $a + c \geq b + c$.*

(e) *$a < b$ if and only if $a{+}{+} \leq b$.*

(f) *$a < b$ if and only if $b = a + d$ for some positive number d.*

Proof. See Exercise 2.2.3. □