Theorem 8. Let $0 \to L \to M \to N \to 0$ be a short exact sequence of R-modules. Then there is a long exact sequence of abelian groups

$$\begin{aligned} 0 \to \operatorname{Hom}_R(N, D) \to \operatorname{Hom}_R(M, D) \to \operatorname{Hom}_R(L, D) &\xrightarrow{\delta_0} \operatorname{Ext}^1_R(N, D) \\ \to \operatorname{Ext}^1_R(M, D) \to \operatorname{Ext}^1_R(L, D) &\xrightarrow{\delta_1} \operatorname{Ext}^2_R(N, D) \to \cdots \end{aligned} \tag{17.12}$$

where the maps between groups at the same level n are as in Proposition 5 and the connecting homomorphisms δ_n are given by Theorem 2.

Proof: Take a simultaneous projective resolution of the short exact sequence as in Proposition 7 and take homomorphisms into D. To obtain the cohomology groups Ext^n_R from the resulting diagram, as noted in the discussion preceding Proposition 3 we replace the lowest nonzero row in the transformed diagram with a row of zeros to get the following commutative diagram:

$$\begin{tikzcd} & \vdots & \vdots & \vdots & \\ 0 \arrow[r] & \operatorname{Hom}_R(\overline{P}_1, D) \arrow[r] \arrow[u] & \operatorname{Hom}_R(P_1 \oplus \overline{P}_1, D) \arrow[r] \arrow[u] & \operatorname{Hom}_R(P_1, D) \arrow[r] \arrow[u] & 0 \\ 0 \arrow[r] & \operatorname{Hom}_R(\overline{P}_0, D) \arrow[r] \arrow[u] & \operatorname{Hom}_R(P_0 \oplus \overline{P}_0, D) \arrow[r] \arrow[u] & \operatorname{Hom}_R(P_0, D) \arrow[r] \arrow[u] & 0 \\ & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & \end{tikzcd} \tag{17.13}$$

The columns of (13) are cochain complexes, and the rows are split by Proposition 29(2) of Section 10.5 and the discussion following it. Thus (13) is a short exact sequence of cochain complexes. Theorem 2 then gives a long exact sequence of cohomology groups whose terms are, by definition, the groups $\operatorname{Ext}^n_R(\_\_, D)$, for $n \geq 0$. The 0^{th} order terms are identified by Proposition 3, completing the proof.

Theorem 8 shows how the exact sequence (2) can be extended in a natural way and shows that the group $\operatorname{Ext}^1_R(N, D)$ is the first measure of the failure of (2) to be exact on the right — in fact (2) can be extended to a short exact sequence on the right if and only if the connecting homomorphism δ_0 in (12) is the zero homomorphism. In particular, if $\operatorname{Ext}^1_R(N, D) = 0$ for all R-modules N, then (2) will be exact on the right for *every* exact sequence (1). We have already seen (Corollary 35 in Section 10.5) that this implies the R-module D is injective. Part of the next result shows that the converse is also true and characterizes injective modules in terms of Ext_R groups.

Proposition 9. For an R-module Q the following are equivalent:
(1) Q is injective,
(2) $\operatorname{Ext}^1_R(A, Q) = 0$ for all R-modules A, and
(3) $\operatorname{Ext}^n_R(A, Q) = 0$ for all R-modules A and all $n \geq 1$.

Proof: We showed (2) implies (1) above, and (3) implies (2) is trivial, so it remains to show that if Q is injective then $\text{Ext}^n_R(A, Q) = 0$ for all R-modules A and all $n \geq 1$. Take a projective resolution

$$\cdots \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

for A. Since Q is injective, the sequence

$$0 \to \text{Hom}_R(A, Q) \to \text{Hom}_R(P_0, Q) \to \cdots \to \text{Hom}_R(P_{n-1}, Q) \to \text{Hom}_R(P_n, Q) \to \cdots$$

is still exact (Corollary 35 in Section 10.5), so all of the cohomology groups for this cochain complex are 0. In particular, the groups $\text{Ext}^n_R(A, Q)$ for $n \geq 1$ are all trivial, which is (3).

For a fixed R-module D, the result in Theorem 8 can be viewed as explaining what happens to the short exact sequence $0 \to L \to M \to N \to 0$ on the right after applying the left exact functor $\text{Hom}_R(\_\_, D)$. This is why the (contravariant) functors $\text{Ext}^n_R(\_\_, D)$ are called the *right derived functors* for the functor $\text{Hom}_R(\_\_, D)$.

One can also consider the effect of applying the left exact functor $\text{Hom}_R(D, \_\_)$, i.e., by taking homomorphisms *from* D rather than *into* D. The next theorem shows that in fact the same Ext_R groups define the (covariant) right derived functors for $\text{Hom}_R(D, \_\_)$ as well.

Theorem 10. Let $0 \to L \to M \to N \to 0$ be a short exact sequence of R-modules. Then there is a long exact sequence of abelian groups

$$\begin{aligned} 0 \to \text{Hom}_R(D, L) \to \text{Hom}_R(D, M) \to \text{Hom}_R(D, N) &\xrightarrow{\gamma_0} \text{Ext}^1_R(D, L) \\ \to \text{Ext}^1_R(D, M) \to \text{Ext}^1_R(D, N) &\xrightarrow{\gamma_1} \text{Ext}^2_R(D, L) \to \cdots. \end{aligned} \tag{17.14}$$

Proof: Let $0 \to L \to M \to N \to 0$ be a short exact sequence of R-modules. By taking a projective resolution of D and then applying $\text{Hom}_R(\_\_, L)$, $\text{Hom}_R(\_\_, M)$ and $\text{Hom}_R(\_\_, N)$ to this resolution one obtains the columns in a commutative diagram similar to (13), but with L, M and N in the second positions rather than the first. Applying the Long Exact Sequence Theorem to this array gives (14).

Theorem 10 shows that the group $\text{Ext}^1_R(D, L)$ measures whether the exact sequence

$$0 \longrightarrow \text{Hom}_R(D, L) \longrightarrow \text{Hom}_R(D, M) \longrightarrow \text{Hom}_R(D, N)$$

can be extended to a short exact sequence — it can be extended if and only if γ_0 is the zero homomorphism. In particular, this will always be the case if the module D has the property that $\text{Ext}^1_R(D, B) = 0$ for all R-modules B; in this case it follows by Corollary 32 in Section 10.5 that D is a projective R-module. As in the situation of injective R-modules in Proposition 9, the vanishing of these cohomology groups in fact characterizes projective R-modules:

Proposition 11. For an R-module P the following are equivalent:

(1) P is projective,

(2) $\mathrm{Ext}^1_R(P, B) = 0$ for all R-modules B, and

(3) $\mathrm{Ext}^n_R(P, B) = 0$ for all R-modules B and all $n \geq 1$.

Proof: We proved (2) implies (1) above, and (3) implies (2) is trivial, so it remains to prove that (1) implies (3). If P is a projective R-module, then the simple exact sequence

$$0 \longrightarrow P \xrightarrow{1} P \longrightarrow 0$$

given by the identity map on P is a projective resolution of P. Taking homomorphisms into B gives the simple cochain complex

$$0 \to \mathrm{Hom}_R(P, B) \xrightarrow{1} \mathrm{Hom}_R(P, B) \to 0 \to \cdots \to 0 \to \cdots$$

from which it follows by definition that $\mathrm{Ext}^n_R(P, B) = 0$ for all $n \geq 1$, which gives (3).

Examples

(1) Since $\mathbb{Z}^m$ is a free, hence projective, $\mathbb{Z}$-module, it follows from Proposition 11 that

$$\mathrm{Ext}^n_{\mathbb{Z}}(\mathbb{Z}^m, B) = 0$$

for all abelian groups B, all $m \geq 1$, and all $n \geq 1$.

(2) It is not difficult to show that $\mathrm{Ext}^n_R(A_1 \oplus A_2, B) \cong \mathrm{Ext}^n_R(A_1, B) \oplus \mathrm{Ext}^n_R(A_2, B)$ for all $n \geq 0$ (cf. Exercise 10), so the previous example together with the example following Proposition 3 determines $\mathrm{Ext}^n_{\mathbb{Z}}(A, B)$ for all finitely generated abelian groups A. In particular, $\mathrm{Ext}^n_{\mathbb{Z}}(A, B) = 0$ for all finitely generated groups A, all abelian groups B, and all $n \geq 2$.

We have chosen to define the cohomology group $\mathrm{Ext}^n_R(A, B)$ using a projective resolution of A. There is a parallel development using an *injective resolution* of B:

$$0 \to B \to Q_0 \to Q_1 \to \cdots$$

where each Q_i is injective. In this situation one defines $\mathrm{Ext}^n_R(A, B)$ as the n^{th} cohomology group of the cochain sequence obtained by applying $\mathrm{Hom}_R(A, \_\_)$ to the resolution for B. The theory proceeds in a manner analogous to the development of this section. Ultimately one shows that there is a natural isomorphism between the groups $\mathrm{Ext}^n_R(A, B)$ constructed using both methods.

Examples

(1) Suppose $R = \mathbb{Z}$ and A and B are $\mathbb{Z}$-modules, i.e., are abelian groups. Recall that a $\mathbb{Z}$-module is injective if and only if it is divisible (Proposition 36 in Section 10.5). The group B can be embedded in an injective $\mathbb{Z}$-module Q_0 (Corollary 37 in Section 10.5) and the quotient, Q_1, of Q_0 by the image of B is again injective. Hence we have an injective resolution

$$0 \longrightarrow B \longrightarrow Q_0 \longrightarrow Q_1 \longrightarrow 0$$

of B. Applying $\mathrm{Hom}_{\mathbb{Z}}(A, \_\_)$ to this sequence gives the cochain complex

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(A, B) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(A, Q_0) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(A, Q_1) \longrightarrow 0 \longrightarrow \cdots$$

from which it follows immediately that

$$\mathrm{Ext}^n_{\mathbb{Z}}(A, B) = 0$$

for all abelian groups A and B and all $n \geq 2$, showing that the result of the previous example holds also when A is not finitely generated.

(2) Suppose A is a torsion abelian group. Then we have $\mathrm{Ext}^0(A, \mathbb{Z}) \cong \mathrm{Hom}(A, \mathbb{Z}) = 0$ since $\mathbb{Z}$ is torsion free. The sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ gives an injective resolution of $\mathbb{Z}$. Applying $\mathrm{Hom}(A, \_\_)$ gives the cochain complex

$$0 \longrightarrow \mathrm{Hom}(A, \mathbb{Z}) \longrightarrow \mathrm{Hom}(A, \mathbb{Q}) \longrightarrow \mathrm{Hom}(A. \mathbb{Q}/\mathbb{Z}) \longrightarrow 0 \longrightarrow \cdots$$

and since $\mathbb{Q}$ is also torsion free, this shows that

$$\mathrm{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z}).$$

The group $\mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$ is called the *Pontriagin dual group* to A. If A is a finite abelian group the Pontriagin dual of A is isomorphic to A (cf. Exercise 14, Section 5.2). In particular, $\mathrm{Ext}^1(A, \mathbb{Z}) \cong A$ is nonzero for all nonzero finite abelian groups A. We have $\mathrm{Ext}^n(A, \mathbb{Z}) = 0$ for all $n \geq 2$ by the previous example.

We record an important property of Ext^1_R, which helps to explain the name for these cohomology groups. Recall that equivalent extensions were defined at the beginning of Section 10.5.

Theorem 12. For any R-modules N and L there is a bijection between $\mathrm{Ext}^1_R(N, L)$ and the set of equivalence classes of extensions of N by L.

Although we shall not prove this result, in Section 4 we establish a similar bijection between equivalence classes of group extensions of G by A and elements of a certain cohomology group, where G is any finite group and A is any $\mathbb{Z}G$-module.

Example

Suppose $R = \mathbb{Z}$ and $A = B = \mathbb{Z}/p\mathbb{Z}$. We showed above that $\mathrm{Ext}^1_R(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$, so by Theorem 12 there are precisely p equivalence classes of extensions of $\mathbb{Z}/p\mathbb{Z}$ by $\mathbb{Z}/p\mathbb{Z}$. These are given by the direct sum $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ (which corresponds to the trivial class in $\mathrm{Ext}^1_R(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$) and the $p-1$ extensions

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{Z}/p^2\mathbb{Z} \xrightarrow{\;i\;} \mathbb{Z}/p\mathbb{Z} \longrightarrow 0$$

defined by the map $i(x) = ix \bmod p$ for $i = 1, 2, \ldots, p-1$. Note that while these are inequivalent as extensions, they all determine the same group $\mathbb{Z}/p^2\mathbb{Z}$.