is *complete* in the ν-adic topology). This is similar to the situation of the completion $\mathbb{R}$ of $\mathbb{Q}$ with respect to the usual Euclidean metric. This allows the application of ideas from analysis to the study of such rings, and is an important tool in the study of algebraic number fields and in algebraic geometry.

Fractional Ideals

We complete our discussion of Discrete Valuation Rings by giving another characterization of D.V.R.s in terms of "fractional ideals," which can be defined for any integral domain:

Definition. For any integral domain R with fraction field K, a *fractional ideal* of R is an R-submodule A of K such that $dA \subseteq R$ for some nonzero $d \in R$ (equivalently, a submodule of the form $d^{-1}I$ for some nonzero $d \in R$ and ideal I of R).

The equivalence of these two definitions follows from the observation that dA is an R-submodule (i.e., an ideal) of R.

The notion of a fractional ideal in K depends on the ring R. Loosely speaking, a fractional ideal is an ideal of R up to a fixed "denominator" d. The ideals of R are also fractional ideals of R (with denominator $d = 1$) and are the fractional ideals that are contained in R. For clarity these are occasionally called the *integral ideals* of R. When R is a Noetherian integral domain, a fractional ideal of R is the same as a finitely generated R-submodule of K (cf. Exercise 6).

For any $x \in K$ the (cyclic) R-module $Rx = \{rx \mid r \in R\}$ is called the *principal fractional ideal* generated by x.

If A and B are fractional ideals, their product, AB, is defined to be the set of all finite sums of elements of the form ab where $a \in A$ and $b \in B$. If $A = d^{-1}I$ and $B = (d')^{-1}J$ for ideals I, J in R and nonzero $d, d' \in R$, then $AB = (dd')^{-1}IJ$ where IJ is the usual product ideal. In particular, this shows that the product of two fractional ideals is a fractional ideal.

Definition. The fractional ideal A is said to be *invertible* if there exists a fractional ideal B with $AB = R$, in which case B is called the *inverse* of A and denoted A^{-1}.

If A is an invertible fractional ideal, the fractional ideal B with $AB = R$ is unique: $AB = AC = R$ implies $B = B(AC) = (BA)C = C$.

Proposition 9. Let R be an integral domain and let A be a fractional ideal of R.

(1) If A is a nonzero principal fractional ideal then A is invertible.

(2) If A is nonzero then the set $A' = \{x \in K \mid xA \subseteq R\}$ is a fractional ideal of R. In general we have $AA' \subseteq R$ and $AA' = R$ if and only if A is invertible, in which case $A^{-1} = A'$.

(3) If A is an invertible fractional ideal of R then A is finitely generated.

(4) The set of invertible fractional ideals is an abelian group under multiplication with identity R. The set of nonzero principal fractional ideals is a subgroup of the invertible fractional ideals.

Proof: If $A = xR$ is a nonzero principal fractional ideal, then taking $B = x^{-1}R$ shows that A is invertible, proving (1).

One easily sees that A' is an R-submodule of K. If A is a nonzero fractional ideal there is some nonzero element $d \in R$ such that $dA \subseteq R$, so A contains nonzero elements of R. Let a be any nonzero element of A contained in R. Then by definition of A' we have $aA' \subseteq R$, so A' is a fractional ideal. Also by definition, $AA' \subseteq R$. If $AA' = R$ then A is invertible with inverse $A^{-1} = A'$. Conversely, if $AB = R$, then $B \subseteq A'$ by definition of A'. Then $R = AB \subseteq AA' \subseteq R$, showing that $AA' = R$, proving (2).

If A is invertible, then $AA' = R$ by (2) and so $1 = a_1a_1' + \cdots + a_na_n'$ for some $a_1, \ldots, a_n \in A$ and $a_1', \ldots, a_n' \in A'$. If $a \in A$, then $a = (aa_1')a_1 + \cdots + (aa_n')a_n$, where each $aa_i' \in R$ by definition of A'. It follows that A is generated over R by $a_1, \ldots, a_n$ and so A is finitely generated, proving (3).

Finally, it is clear that the product of two invertible fractional ideals is again invertible. This product is commutative, associative, and $RA = A$ for any fractional ideal. The inverse of an invertible fractional ideal is an invertible fractional ideal by definition, proving the first statement in (4). The second statement in (4) is immediate since the product of xR and yR is $(xy)R$ and the inverse of xR is $x^{-1}R$.

Definition. If R is an integral domain, then the quotient of the group of invertible fractional ideals of R by the subgroup of nonzero principal fractional ideals of R is called the *class group* of R. The order of the class group of R is called the *class number* of R.

The class group of R is the trivial group and the class number of R is 1 if and only if R is a P.I.D. The class group of R measures how close the ideals of R are to being principal.

Whether a fractional ideal A of R is invertible is also related to whether A is *projective* as an R-module. Recall that an R-module M is projective over R if and only if M is a direct summand of a free module (Proposition 30, Section 10.5). Equivalently, M is projective if and only if there is a free R-module F and R-module homomorphisms $f : F \to M$ and $g : M \to F$ with $f \circ g = 1$ (Proposition 25, Section 10.5).

Proposition 10. Let R be an integral domain with fraction field K and let A be a nonzero fractional ideal of R. Then A is invertible if and only if A is a projective R-module.

Proof: Assume first that A is invertible, so $\sum_{i=1}^n a_ia_i' = 1$ for some $a_i \in A$ and $a_i' \in A'$ as in (2) of Proposition 9. Let F be the free R-module on $y_1, \ldots, y_n$. Define $f : F \to A$ by $f(\sum_{i=1}^n r_iy_i) = \sum_{i=1}^n r_ia_i$ and $g : A \to F$ by $f(c) = \sum_{i=1}^n (ca_i')y_i$. It is immediate that both f and g are R-module homomorphisms (note that $ca_i' \in R$ by definition of A'). Since

$$(f \circ g)(c) = f\left(\sum_{i=1}^n (ca_i')y_i\right) = \sum_{i=1}^n (ca_i')a_i = c\left(\sum_{i=1}^n a_ia_i'\right) = c,$$

so $f \circ g = 1$ and A is a direct summand of F, hence is projective.

Conversely, suppose that A is nonzero and projective, so there is a free R-module F and R-homomorphisms $f : F \to A$ and $g : A \to F$ with $f \circ g = 1$. Fix any $0 \neq a \in A$ and suppose $g(a) = \sum_{i=1}^{n} \tilde{a}_i y_i$ where $\tilde{a}_i \in R$ and $y_1, \ldots, y_n$ is part of a set of free generators for F. Define $a_i = f(y_i)$ and $a_i' = \tilde{a}_i / a \in K$ for $i = 1, \ldots, n$. For any $b \in A$ we have $bg(a) = ag(b) = g(ab)$ since g is an R-module homomorphism. Write $g(b) = \sum_{i=1}^{n} \tilde{b}_i y_i + \sum_{j \in \mathcal{J}} \tilde{b}_j y_j$ where $\{y_j\}$ for $j \in \mathcal{J}$ are the remaining elements in the set of free generators for F. Then

$$\sum_{i=1}^{n} (b\tilde{a}_i) y_i = \sum_{i=1}^{n} (a\tilde{b}_i) y_i + \sum_{j \in \mathcal{J}} (a\tilde{b}_j) y_j .$$

We may equate coefficients of the elements in the free R-module basis for F in this equation and it follows that $g(b) = \sum_{i=1}^{n} \tilde{b}_i y_i$ where $\tilde{b}_i \in R$ and that $b\tilde{a}_i = a\tilde{b}_i$ for $i = 1, \ldots, n$. In particular, it follows from the definition of a_i' that $ba_i' = b(\tilde{a}_i / a) = \tilde{b}_i$ is an element of R for every element b of A. This shows that $a_i' \in A'$ for $i = 1, \ldots, n$. Since $f \circ g = 1$, we have

$$a = f \circ g(a) = f\left(\sum_{i=1}^{n} \tilde{a}_i y_i\right) = \sum_{i=1}^{n} \tilde{a}_i a_i = \sum_{i=1}^{n} (a a_i') a_i = a\left(\sum_{i=1}^{n} a_i a_i'\right) ,$$

and so $\sum_{i=1}^{n} a_i a_i' = 1$. It follows that $AA' = R$ and so A is invertible by Proposition 9, completing the proof.

The next result shows that if the integral domain R is also a local ring, then whether fractional ideals are invertible determines whether R is a D.V.R.

Proposition 11. Suppose the integral domain R is a local ring that is not a field. Then R is a Discrete Valuation Ring if and only if every nonzero fractional ideal of R is invertible.

Proof: If R is a D.V.R. with uniformizing parameter t, then by Proposition 5 every nonzero ideal of R is of the form (t^n) for some $n \geq 0$ and every element d in R can be written in the form ut^m for some unit $u \in R$ and some $m \geq 0$. It follows that every nonzero fractional ideal of R is of the form $t^N R$ for some $N \in \mathbb{Z}$, so is a principal fractional ideal and hence invertible by the previous proposition.

Conversely, suppose that every nonzero fractional ideal of R is invertible. Then every nonzero ideal of R is finitely generated by (3) of Proposition 9, so R is Noetherian. Let M be the unique maximal ideal of R. If $M = M^2$ then $M = 0$ by Nakayama's Lemma, and then R would be a field, contrary to hypothesis. Hence there is an element t with $t \in M - M^2$. By assumption M is invertible, and since $t \in M$, the fractional ideal tM^{-1} is a nonzero ideal in R. If $tM^{-1} \subseteq M$, then $t \in M^2$, contrary to the choice of t. Hence $tM^{-1} = R$, so $(t) = M$, and M is a nonzero principal ideal. It follows by the equivalent condition 4 of Theorem 7 that R is a D.V.R., completing the proof.

We end this section with an application to algebraic geometry.