Exercises

Given three points A,B,C and the points $f(A),f(B),f(C)$ to which they are sent by an isometry f, it is possible to find three reflections that combine to form f by following the steps in the proof above. However, if one merely wants to know what *kind* of isometry f is—translation, rotation, or glide reflection—then the answer can be found more simply.

To fix ideas, we take the initial three points to be $A=(0,1)$, $B=(0,0)$, and $C=(1,0)$. You will probably find it helpful to sketch the triples of points $f(A)$, $f(B)$, $f(C)$ given in the following exercises.

3.7.1 Suppose that $f(A)=(1.4,2)$, $f(B)=(1.4,1)$, and $f(C)=(2.4,1)$. Is f a translation or a rotation? How can you tell that f is not a glide reflection?

3.7.2 Suppose that $f(A)=(0.4,1.8)$, $f(B)=(1,1)$, and $f(C)=(1.8,1.6)$. We can tell that f is not a translation or glide reflection (hence, it must be a rotation). How?

3.7.3 Suppose that $f(A)=(1.8,1.6)$, $f(B)=(1,1)$, and $f(C)=(0.4,1.8)$. How do I know that this is a glide reflection?

3.7.4 State a simple test for telling whether f is a translation, rotation, or glide reflection from the positions of $f(A)$, $f(B)$, and $f(C)$.

3.8 Discussion

The discovery of coordinates is rightly considered a turning point in the development of mathematics because it reveals a vast new panorama of geometry, open to exploration in at least three different directions.

- Description of curves by equations, and their analysis by algebra. This direction is called *algebraic geometry*, and the curves described by polynomial equations are called *algebraic curves*. Straight lines, described by the linear equations $ax+by+c=0$, are called curves of *degree 1*. Circles, described by the equations $(x-a)^2+(y-b)^2=r^2$, are curves of *degree 2*, and so on.

  One can see that there are curves of arbitrarily high degree, so most of algebraic geometry is beyond the scope of this book. Even the curves of degree 3 are worth a book of their own, so for them, and other algebraic curves, we refer readers elsewhere. Two excellent books, which show how algebraic geometry relates to other parts of mathematics, are *Elliptic Curves* by H. P. McKean and V. Moll and *Plane Algebraic Curves* by E. Brieskorn and H. Knörrer.

- Algebraic study of objects described by linear equations (such as lines and planes). Even this is a big subject, called *linear algebra*. Although it is technically part of algebraic geometry, it has a special flavor, very close to that of Euclidean geometry. We explore plane geometry from the viewpoint of linear algebra in Chapter 4, and later we make some brief excursions into three and four dimensions.

  The real strength of linear algebra is its ability to describe spaces of any number of dimensions in geometric language. Again, this investigation is beyond our scope, but we will recommend additional reading at the appropriate places.

- The study of transformations, which draws on the special branch of algebra known as *group theory*. Because many geometric transformations are described by linear equations, this study overlaps with linear algebra. The role of transformations was first emphasized by the German mathematician Felix Klein, in an address he delivered at the University of Erlangen in 1872. His address, known by its German name the *Erlanger Programm*, characterizes geometry as the study of *transformation groups* and their *invariants*.

So far, we have seen only one transformation group and a handful of invariants—the group of isometries of $\mathbb{R}^2$ and what it leaves invariant (length, angle, straightness)—so the importance of Klein's idea can hardly be clear yet. However, in Chapter 4 we introduce a very different group of transformations and a very different invariant—the *projective transformations* and the *cross-ratio*—so readers are asked to bear with us. In Chapters 7 and 8, we develop Klein's idea in some generality and give another significant example, the geometry of the "non-Euclidean" plane.

4

Vectors and Euclidean spaces

PREVIEW

In this chapter, we process coordinates by *linear algebra*. We view points as *vectors* that can be added and multiplied by numbers, and we introduce the *inner product* of vectors, which gives an efficient algebraic method to deal with both lengths and angles.

We revisit some theorems of Euclid to see where they fit in the world of vector geometry, and we become acquainted with some theorems that are particularly natural in this environment.

For plane geometry, the appropriate vectors are ordered pairs (x,y) of real numbers. We *add* pairs according to the rule

$$(u_1,u_2)+(v_1,v_2)=(u_1+v_1,u_2+v_2),$$

and *multiply a pair by a real number a* according to the rule

$$a(u_1,u_2)=(au_1,au_2).$$

These vector operations do not involve the concept of length or distance; yet they enable us to discuss certain ratios of lengths and to prove the theorems of Thales and Pappus.

The concept of distance is introduced through the concept of *inner product* $\mathbf{u}\cdot\mathbf{v}$ of vectors $\mathbf{u}$ and $\mathbf{v}$. If $\mathbf{u}=(u_1,u_2)$ and $\mathbf{v}=(v_1,v_2)$, then

$$\mathbf{u}\cdot\mathbf{v}=u_1v_1+u_2v_2.$$

The inner product gives us distance because $\mathbf{u}\cdot\mathbf{u}=|\mathbf{u}|^{\mathbf{2}}$, where $|\mathbf{u}|$ is the distance of $\mathbf{u}$ from the origin $\mathbf{0}$. It also gives us angle because

$$\mathbf{u}\cdot\mathbf{v}=|\mathbf{u}||\mathbf{v}|\cos\theta,$$

where θ is the angle between the directions of $\mathbf{u}$ and $\mathbf{v}$ from $\mathbf{0}$.

4.1 Vectors

Vectors are mathematical objects that can be added, and multiplied by numbers, subject to certain rules. The real numbers are the simplest example of vectors, and the rules for sums and multiples of any vectors are just the following properties of sums and multiples of numbers:

$$\begin{aligned} \mathbf{u}+\mathbf{v} &= \mathbf{v}+\mathbf{u} & 1\mathbf{u} &= \mathbf{u} \\ \mathbf{u}+(\mathbf{v}+\mathbf{w}) &= (\mathbf{u}+\mathbf{v})+\mathbf{w} & a(\mathbf{u}+\mathbf{v}) &= a\mathbf{u}+a\mathbf{v} \\ \mathbf{u}+\mathbf{0} &= \mathbf{u} & (a+b)\mathbf{u} &= a\mathbf{u}+b\mathbf{u} \\ \mathbf{u}+(-\mathbf{u}) &= \mathbf{0} & a(b\mathbf{u}) &= (ab)\mathbf{u}. \end{aligned}$$

These rules obviously hold when $a, b, 1, \mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{0}$ are all numbers, and $\mathbf{0}$ is the ordinary zero.

They also hold when $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are *points in the plane* $\mathbb{R}^2$, if we interpret $\mathbf{0}$ as $(0,0)$, $+$ as the *vector sum* defined for $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ by

$$(u_1, u_2) + (v_1 + v_2) = (u_1 + v_1, u_2 + v_2),$$

and $a\mathbf{u}$ as the *scalar multiple* defined by

$$a(u_1, u_2) = (au_1, au_2).$$

The vector sum is geometrically interesting, because $\mathbf{u}+\mathbf{v}$ is the fourth vertex of a parallelogram formed by the points $\mathbf{0}$, $\mathbf{u}$, and $\mathbf{v}$ (Figure 4.1).

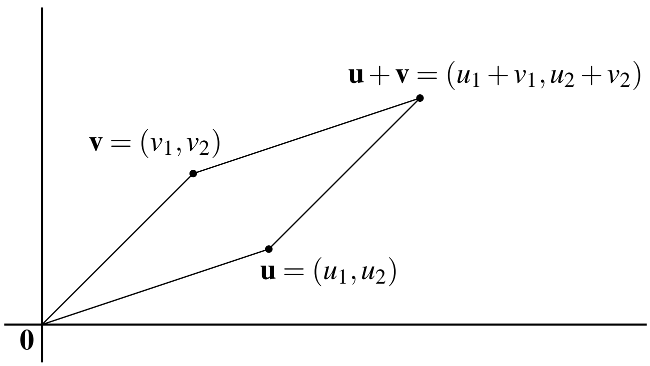

Figure 4.1: The parallelogram rule for vector sum

In fact, the rule for forming the sum of two vectors is often called the "parallelogram rule."

Scalar multiplication by a is also geometrically interesting, because it represents magnification by the factor a. It magnifies, or *dilates*, the whole plane by the factor a, transforming each figure into a similar copy of itself. Figure 4.2 shows an example of this with $a = 2.5$.

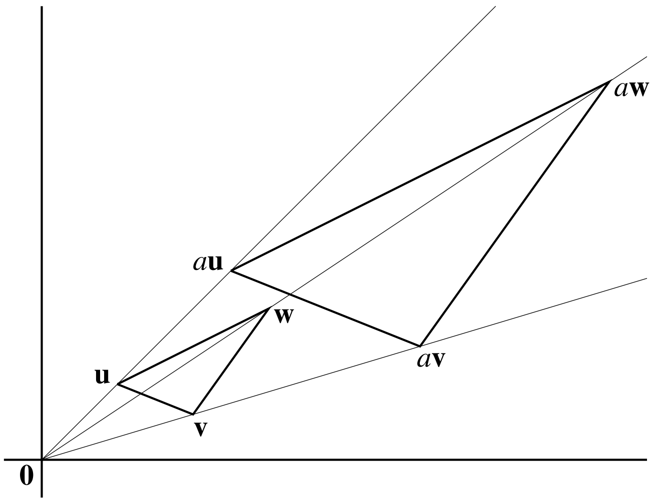

Figure 4.2: Scalar multiplication as a dilation of the plane

Real vector spaces

It seems that the operations of vector addition and scalar multiplication capture some geometrically interesting features of a space. With this in mind, we define a *real vector space* to be a set V of objects, called *vectors*, with operations of vector addition and scalar multiplication satisfying the following conditions:

- If $\mathbf{u}$ and $\mathbf{v}$ are in V, then so are $\mathbf{u}+\mathbf{v}$ and $a\mathbf{u}$ for any real number a.
- There is a *zero vector* $\mathbf{0}$ such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$ for each vector $\mathbf{u}$. Each $\mathbf{u}$ in V has a *additive inverse* $-\mathbf{u}$ such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
- Vector addition and scalar multiplication on V have the eight properties listed at the beginning of this section.

It turns out that real vector spaces are a natural setting for Euclidean geometry. We must introduce extra structure, which is called the *inner product*, before we can talk about length and angle. But once the inner product is there, we can prove all theorems of Euclidean geometry, often more efficiently than before. Also, we can uniformly extend geometry to *any number of dimensions* by considering the space $\mathbb{R}^n$ of *ordered n-tuples* of real numbers $(x_1, x_2, \ldots, x_n)$.

For example, we can study three-dimensional Euclidean geometry in the space of ordered triples

$$\mathbb{R}^3 = \{(x_1, x_2, x_3) : x_1, x_2, x_3 \in \mathbb{R}\},$$

where the sum of $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ is defined by

$$(u_1, u_2, u_3) + (v_1, v_2, v_3) = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$$

and the scalar multiple $a\mathbf{u}$ is defined by

$$a(u_1, u_2, u_3) = (au_1, au_2, au_3).$$

Exercises

It is obvious that $\mathbb{R}^2$ has the eight properties of a real vector space. However, it is worth noting that $\mathbb{R}^2$ "inherits" these eight properties from the corresponding properties of real numbers. For example, the property $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (called the *commutative law*) for vector addition is inherited from the corresponding commutative law for number addition, $u + v = v + u$, as follows:

$$\begin{aligned}
\mathbf{u} + \mathbf{v} &= (u_1, u_2) + (v_1 + v_2) \\
&= (u_1 + v_1, u_2 + v_2) \quad \text{by definition of vector addition} \\
&= (v_1 + u_1, v_2 + u_2) \quad \text{by commutative law for numbers} \\
&= (v_1, v_2) + (u_1, u_2) \quad \text{by definition of vector addition} \\
&= \mathbf{v} + \mathbf{u}.
\end{aligned}$$

4.1.1 Check that the other seven properties of a vector space for $\mathbb{R}^2$ are inherited from corresponding properties of $\mathbb{R}$.

4.1.2 Similarly check that $\mathbb{R}^n$ has the eight properties of a vector space.

The term "dilation" for multiplication of all vectors in $\mathbb{R}^2$ (or $\mathbb{R}^n$ for that matter) by a real number a goes a little beyond the everyday meaning of the word in the case when a is smaller than 1 or negative.

4.1.3 What is the geometric meaning of the transformation of $\mathbb{R}^2$ when every vector is multiplied by -1? Is it a rotation?

4.1.4 Is it a rotation of $\mathbb{R}^3$ when every vector is multiplied by -1?

4.2 Direction and linear independence

Vectors give a concept of *direction* in $\mathbb{R}^2$ by representing lines through $\mathbf{0}$. If $\mathbf{u}$ is a nonzero vector, then the real multiples $a\mathbf{u}$ of $\mathbf{u}$ make up the line through $\mathbf{0}$ and $\mathbf{u}$, so we call them the points "in direction $\mathbf{u}$ from $\mathbf{0}$." (You may prefer to say that $-\mathbf{u}$ is in the direction *opposite* to $\mathbf{u}$, but it is simpler to associate direction with a whole line, rather than a half line.)

Nonzero vectors $\mathbf{u}$ and $\mathbf{v}$, therefore, have *different directions from* $\mathbf{0}$ if neither is a multiple of the other. It follows that such $\mathbf{u}$ and $\mathbf{v}$ are *linearly independent*; that is, there are no real numbers a and b, not both zero, with

$$a\mathbf{u}+b\mathbf{v}=\mathbf{0}.$$

Because, if one of a, b is not zero in this equation, we can divide by it and hence express one of $\mathbf{u}$, $\mathbf{v}$ as a multiple of the other.

The concept of direction has an obvious generalization: $\mathbf{w}$ *has direction* $\mathbf{u}$ *from* $\mathbf{v}$ (or *relative to* $\mathbf{v}$) if $\mathbf{w}-\mathbf{v}$ is a multiple of $\mathbf{u}$. We also say that "$\mathbf{w}-\mathbf{v}$ has direction $\mathbf{u}$," and there is no harm in viewing $\mathbf{w}-\mathbf{v}$ as an abbreviation for the line segment from $\mathbf{v}$ to $\mathbf{w}$. As in coordinate geometry, we say that line segments from $\mathbf{v}$ to $\mathbf{w}$ and from $\mathbf{s}$ to $\mathbf{t}$ are *parallel* if they have the same direction; that is, if

$$\mathbf{w}-\mathbf{v}=a(\mathbf{t}-\mathbf{s}) \quad \text{for some real number } a\neq 0.$$

Figure 4.3 shows an example of parallel line segments, from $\mathbf{v}$ to $\mathbf{w}$ and from $\mathbf{s}$ to $\mathbf{t}$, both of which have direction $\mathbf{u}$.

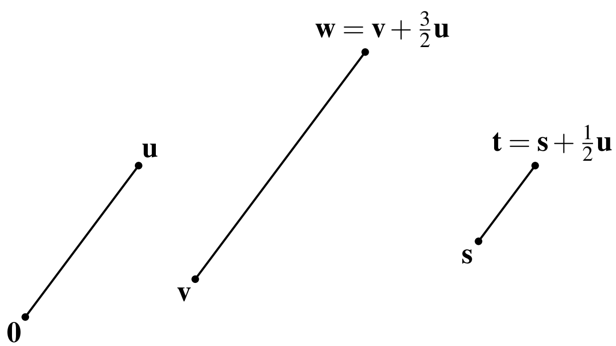

Figure 4.3: Parallel line segments with direction $\mathbf{u}$

Here we have

$$\mathbf{w}-\mathbf{v}=\frac{3}{2}\mathbf{u} \quad \text{and} \quad \mathbf{t}-\mathbf{s}=\frac{1}{2}\mathbf{u}, \quad \text{so} \quad \mathbf{w}-\mathbf{v}=3(\mathbf{t}-\mathbf{s}).$$

Now let us try out the vector concept of parallels on two important theorems from previous chapters. The first is a version of the Thales theorem that parallels cut a pair of lines in proportional segments.

Vector Thales theorem. *If* $\mathbf{s}$ *and* $\mathbf{v}$ *are on one line through* $\mathbf{0}$*,* $\mathbf{t}$ *and* $\mathbf{w}$ *are on another, and* $\mathbf{w}-\mathbf{v}$ *is parallel to* $\mathbf{t}-\mathbf{s}$*, then* $\mathbf{v}=a\mathbf{s}$ *and* $\mathbf{w}=a\mathbf{t}$ *for some number* a*.*

If $\mathbf{w}-\mathbf{v}$ is parallel to $\mathbf{t}-\mathbf{s}$, then

$$\mathbf{w}-\mathbf{v}=a(\mathbf{t}-\mathbf{s})=a\mathbf{t}-a\mathbf{s}\quad\text{for some real number } a.$$

Because $\mathbf{v}$ is on the same line through $\mathbf{0}$ as $\mathbf{s}$, we have $\mathbf{v}=b\mathbf{s}$ for some b, and similarly $\mathbf{w}=c\mathbf{t}$ for some c (this is a good moment to draw a picture). It follows that

$$\mathbf{w}-\mathbf{v}=c\mathbf{t}-b\mathbf{s}=a\mathbf{t}-a\mathbf{s},$$

and therefore,

$$(c-a)\mathbf{t}+(a-b)\mathbf{s}=\mathbf{0}.$$

But $\mathbf{s}$ and $\mathbf{t}$ are in different directions from $\mathbf{0}$, hence linearly independent, so

$$c-a=a-b=0.$$

Thus, $\mathbf{v}=a\mathbf{s}$ and $\mathbf{w}=a\mathbf{t}$, as required. □

As in axiomatic geometry (Exercise 1.4.3), the Pappus theorem follows from the Thales theorem. However, "proportionality" is easier to handle with vectors.

Vector Pappus theorem. *If* $\mathbf{r}$*,* $\mathbf{s}$*,* $\mathbf{t}$*,* $\mathbf{u}$*,* $\mathbf{v}$*,* $\mathbf{w}$ *lie alternately on two lines through* $\mathbf{0}$*, with* $\mathbf{u}-\mathbf{v}$ *parallel to* $\mathbf{s}-\mathbf{r}$ *and* $\mathbf{t}-\mathbf{s}$ *parallel to* $\mathbf{v}-\mathbf{w}$*, then* $\mathbf{u}-\mathbf{t}$ *is parallel to* $\mathbf{w}-\mathbf{r}$*.*

Figure 4.4 shows the situation described in the theorem.

Because $\mathbf{u}-\mathbf{v}$ is parallel to $\mathbf{s}-\mathbf{r}$, we have $\mathbf{u}=a\mathbf{s}$ and $\mathbf{v}=a\mathbf{r}$ for some number a. Because $\mathbf{t}-\mathbf{s}$ is parallel to $\mathbf{v}-\mathbf{w}$, we have $\mathbf{s}=b\mathbf{w}$ and $\mathbf{t}=b\mathbf{v}$ for some number b.

From these two facts, we conclude that

$$\mathbf{u}=a\mathbf{s}=ab\mathbf{w}\quad\text{and}\quad\mathbf{t}=b\mathbf{v}=ba\mathbf{r},$$

hence,

$$\mathbf{u}-\mathbf{t}=ab\mathbf{w}-ba\mathbf{r}=ab(\mathbf{w}-\mathbf{r}),$$

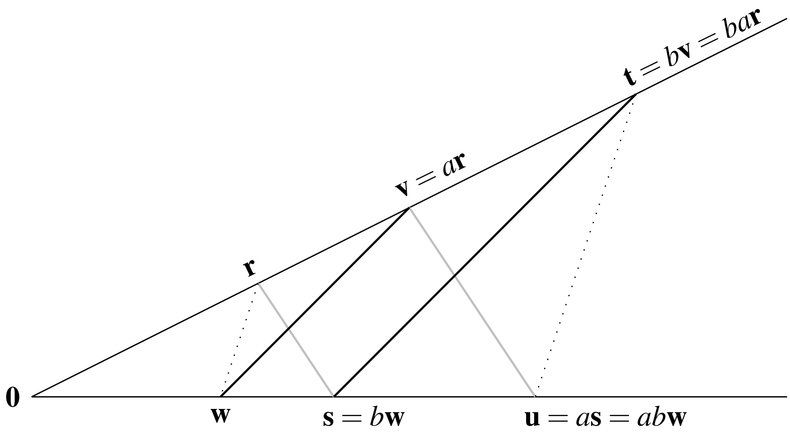

Figure 4.4: The parallel Pappus configuration, labeled by vectors

and therefore, $\mathbf{u} - \mathbf{t}$ is parallel to $\mathbf{w} - \mathbf{r}$. □

The last step in this proof, where we exchange ba for ab, is of course a trifle, because $ab = ba$ for any real numbers a and b. But it is a big step in Chapter 6, where we try to develop geometry without numbers. There we have to build an arithmetic of line segments, and the Pappus theorem is crucial in getting multiplication to behave properly.

Exercises

In Chapter 1, we mentioned that a second theorem about parallels, the Desargues theorem, often appears alongside the Pappus theorem in the foundations of geometry. This situation certainly holds in vector geometry, where the appropriate Desargues theorem likewise follows from the vector Thales theorem.

4.2.1 Following the setup explained in Exercise 1.4.4, and the formulation of the vector Pappus theorem above, formulate a "vector Desargues theorem."

4.2.2 Prove your vector Desargues theorem with the help of the vector Thales theorem.

4.3 Midpoints and centroids

The definition of a real vector space does not include a definition of distance, but we can speak of the midpoint of the line segment from $\mathbf{u}$ to $\mathbf{v}$ and, more generally, of the point that divides this segment in a given ratio.

To see why, first observe that $\mathbf{v}$ is obtained from $\mathbf{u}$ by adding $\mathbf{v}-\mathbf{u}$, the vector that represents the position of $\mathbf{v}$ *relative* to $\mathbf{u}$. More generally, adding any scalar multiple $a(\mathbf{v}-\mathbf{u})$ to $\mathbf{u}$ produces a point whose *direction* relative to $\mathbf{u}$ is the same as that of $\mathbf{v}$. Thus, the points $\mathbf{u}+a(\mathbf{v}-\mathbf{u})$ are precisely those on the line through $\mathbf{u}$ and $\mathbf{v}$. In particular, the midpoint of the segment between $\mathbf{u}$ and $\mathbf{v}$ is obtained by adding $\frac{1}{2}(\mathbf{v}-\mathbf{u})$ to $\mathbf{u}$, and hence,

$$\text{midpoint of line segment between } \mathbf{u} \text{ and } \mathbf{v} = \mathbf{u}+\frac{1}{2}(\mathbf{v}-\mathbf{u}) = \frac{1}{2}(\mathbf{u}+\mathbf{v}).$$

One might describe this result by saying that the midpoint of the line segment between $\mathbf{u}$ and $\mathbf{v}$ is the *vector average* of $\mathbf{u}$ and $\mathbf{v}$.

This description of the midpoint gives a very short proof of the theorem from Exercise 2.2.1, that the diagonals of a parallelogram bisect each other. By choosing one of the vertices of the parallelogram at $\mathbf{0}$, we can assume that the other vertices are at $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u}+\mathbf{v}$ (Figure 4.5).

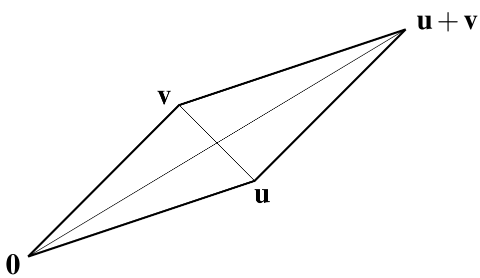

Figure 4.5: Diagonals of a parallelogram

Then the midpoint of the diagonal from $\mathbf{0}$ to $\mathbf{u}+\mathbf{v}$ is $\frac{1}{2}(\mathbf{u}+\mathbf{v})$. And, by the result just proved, this is also the midpoint of the other diagonal—the line segment between $\mathbf{u}$ and $\mathbf{v}$. □

The vector average of two or more points is physically significant because it is the *barycenter* or *center of mass* of the system obtained by placing equal masses at the given points. The geometric name for this vector average point is the *centroid*.

In the case of a triangle, the centroid has an alternative geometric description, given by the following classical theorem about *medians*: the lines from the vertices of a triangle to the midpoints of the respective opposite sides.

Concurrence of medians. *The medians of any triangle pass through the same point, the centroid of the triangle.*

To prove this theorem, suppose that the vertices of the triangle are $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. Then the median from $\mathbf{u}$ goes to the midpoint $\frac{1}{2}(\mathbf{v}+\mathbf{w})$, and so on, as shown in Figure 4.6.

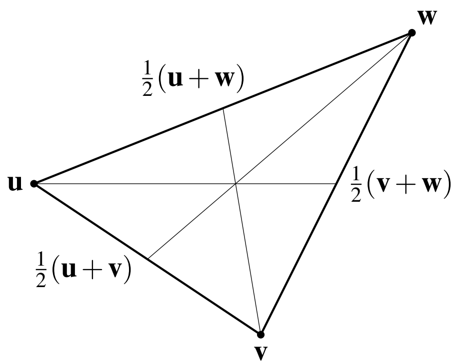

Figure 4.6: The medians of a triangle

Looking at this figure, it seems likely that the medians meet at the point 2/3 of the way from $\mathbf{u}$ to $\frac{1}{2}(\mathbf{v}+\mathbf{w})$, that is, at the point

$$\mathbf{u}+\frac{2}{3}\left(\frac{1}{2}(\mathbf{v}+\mathbf{w})-\mathbf{u}\right)=\mathbf{u}+\frac{1}{3}(\mathbf{v}+\mathbf{w})-\frac{2}{3}\mathbf{u}=\frac{1}{3}(\mathbf{u}+\mathbf{v}+\mathbf{w}).$$

Voilà! This is the centroid, and a similar argument shows that it lies 2/3 of the way between $\mathbf{v}$ and $\frac{1}{2}(\mathbf{u}+\mathbf{w})$ and 2/3 of the way between $\mathbf{w}$ and $\frac{1}{2}(\mathbf{u}+\mathbf{v})$. That is, the centroid is the common point of all three medians.□

You can of course check by calculation that $\frac{1}{3}(\mathbf{u}+\mathbf{v}+\mathbf{w})$ lies 2/3 of the way between $\mathbf{v}$ and $\frac{1}{2}(\mathbf{u}+\mathbf{w})$ and also 2/3 of the way between $\mathbf{w}$ and $\frac{1}{2}(\mathbf{u}+\mathbf{v})$. But the smart thing is not to *do* the calculation but to *predict the result*. We know that calculating the point 2/3 of the way between $\mathbf{u}$ and $\frac{1}{2}(\mathbf{v}+\mathbf{w})$ gives

$$\frac{1}{3}(\mathbf{u}+\mathbf{v}+\mathbf{w}),$$

a result that is unchanged when we permute the letters $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. The other two calculations are the same, except for the ordering of the letters $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. Hence, they lead to the same result.

Exercises

4.3.1 Show that a square with vertices $\mathbf{t}$, $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ has center $\frac{1}{4}(\mathbf{t}+\mathbf{u}+\mathbf{v}+\mathbf{w})$.

The theorem about concurrence of medians generalizes beautifully to three dimensions, where the figure corresponding to a triangle is a *tetrahedron*: a solid with four vertices joined by six lines that bound the tetrahedron's four triangular faces (Figure 4.7).

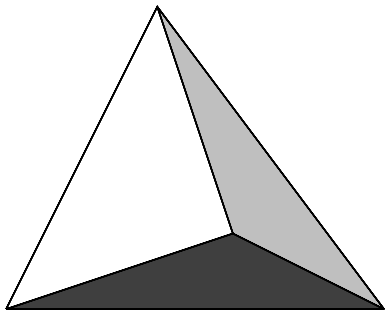

Figure 4.7: A tetrahedron

4.3.2 Suppose that the tetrahedron has vertices $\mathbf{t}$, $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. Show that the centroid of the face opposite to $\mathbf{t}$ is $\frac{1}{3}(\mathbf{u}+\mathbf{v}+\mathbf{w})$, and write down the centroids of the other three faces.

4.3.3 Now consider each line joining a vertex to the centroid of the opposite face. In particular, show that the point 3/4 of the way from $\mathbf{t}$ to the centroid of the opposite face is $\frac{1}{4}(\mathbf{t}+\mathbf{u}+\mathbf{v}+\mathbf{w})$—the centroid of the tetrahedron.

4.3.4 Explain why the point $\frac{1}{4}(\mathbf{t}+\mathbf{u}+\mathbf{v}+\mathbf{w})$ lies on the other three lines from a vertex to the centroid of the opposite face.

4.3.5 Deduce that the four lines from vertex to centroid of opposite face meet at the centroid of the tetrahedron.

4.4 The inner product

If $\mathbf{u}=(u_1,u_2)$ and $\mathbf{v}=(v_1,v_2)$ are vectors in $\mathbb{R}^2$, we define their *inner product* $\mathbf{u}\cdot\mathbf{v}$ to be $u_1v_1+u_2v_2$. Thus, the inner product of two vectors is not another vector, but a real number or "scalar." For this reason, $\mathbf{u}\cdot\mathbf{v}$ is also called the *scalar product* of $\mathbf{u}$ and $\mathbf{v}$.

It is easy to check, from the definition, that the inner product has the algebraic properties

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= \mathbf{v}\cdot\mathbf{u},\\ \mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) &= \mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w},\\ (a\mathbf{u})\cdot\mathbf{v} &= \mathbf{u}\cdot(a\mathbf{v}) = a(\mathbf{u}\cdot\mathbf{v}),\end{aligned}$$

which immediately give information about length and angle:

- The length $|\mathbf{u}|$ is the distance of $\mathbf{u}=(u_1,u_2)$ from $\mathbf{0}$, which is $\sqrt{u_1^2+u_2^2}$ by the definition of distance in $\mathbb{R}^2$ (Section 3.3). Hence,

  $$|\mathbf{u}|^2 = u_1^2+u_2^2 = \mathbf{u}\cdot\mathbf{u}.$$

  It follows that the square of the distance $|\mathbf{v}-\mathbf{u}|$ from $\mathbf{u}$ to $\mathbf{v}$ is

  $$|\mathbf{v}-\mathbf{u}|^2 = (\mathbf{v}-\mathbf{u})\cdot(\mathbf{v}-\mathbf{u}) = |\mathbf{u}|^2+|\mathbf{v}|^2-2\mathbf{u}\cdot\mathbf{v}.$$

- Vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular if and only if $\mathbf{u}\cdot\mathbf{v}=0$. Because $\mathbf{u}$ has slope u_2/u_1 and $\mathbf{v}$ has slope v_2/v_1, and we know from Section 3.5 that they are perpendicular if and only the product of their slopes is -1. That means

  $$\frac{u_2}{u_1} = -\frac{v_1}{v_2} \quad \text{and hence} \quad u_2v_2 = -u_1v_1,$$

  multiplying both sides by u_1v_2. This equation holds if and only if

  $$0 = u_1v_1+u_2v_2 = \mathbf{u}\cdot\mathbf{v}.$$

We will see in the next section how to extract more information about angle from the inner product. The formula above for $|\mathbf{v}-\mathbf{u}|^2$ turns out to be the "cosine rule" or "law of cosines" from high-school trigonometry. But even the criterion for perpendicularity gives a simple proof of a far-from-obvious theorem:

Concurrence of altitudes. *In any triangle, the perpendiculars from the vertices to opposite sides (the* altitudes*) have a common point.*

To prove this theorem, take $\mathbf{0}$ at the intersection of two altitudes, say those through the vertices $\mathbf{u}$ and $\mathbf{v}$ (Figure 4.8). Then it remains to show that the line from $\mathbf{0}$ to the third vertex $\mathbf{w}$ is perpendicular to the side $\mathbf{v}-\mathbf{u}$.

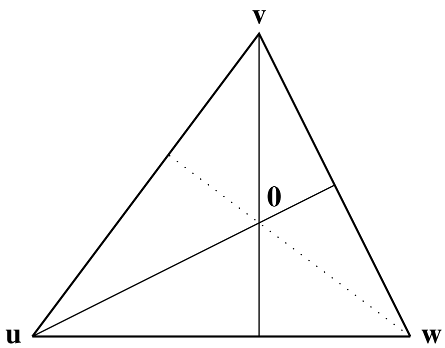

Figure 4.8: Altitudes of a triangle

Because $\mathbf{u}$ is perpendicular to the opposite side $\mathbf{w}-\mathbf{v}$, we have

$$\mathbf{u}\cdot(\mathbf{w}-\mathbf{v})=\mathbf{0}, \quad \text{that is,} \quad \mathbf{u}\cdot\mathbf{w}-\mathbf{u}\cdot\mathbf{v}=\mathbf{0}.$$

Because $\mathbf{v}$ is perpendicular to the opposite side $\mathbf{u}-\mathbf{w}$, we have

$$\mathbf{v}\cdot(\mathbf{u}-\mathbf{w})=\mathbf{0}, \quad \text{that is,} \quad \mathbf{v}\cdot\mathbf{u}-\mathbf{v}\cdot\mathbf{w}=\mathbf{0}.$$

Adding these two equations, and bearing in mind that $\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u}$, we get

$$\mathbf{u}\cdot\mathbf{w}-\mathbf{v}\cdot\mathbf{w}=\mathbf{0}, \quad \text{that is,} \quad \mathbf{w}\cdot(\mathbf{v}-\mathbf{u})=\mathbf{0}.$$

Thus, $\mathbf{w}$ is perpendicular to $\mathbf{v}-\mathbf{u}$, as required. □

Exercises

The inner product criterion for directions to be perpendicular, namely that their inner product is zero, gives a neat way to prove the theorem in Exercise 2.2.2 about the diagonals of a rhombus.

4.4.1 Suppose that a parallelogram has vertices at $\mathbf{0}$, $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u}+\mathbf{v}$. Show that its diagonals have directions $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$.

4.4.2 Deduce from Exercise 4.4.1 that the inner product of these directions is $|\mathbf{u}|^2-|\mathbf{v}|^2$, and explain why this is zero for a rhombus.

The inner product also gives a concise way to show that the equidistant line of two points is the perpendicular bisector of the line connecting them (thus proving more than we did in Section 3.3).