be an $n \times n$ matrix over the field F such that $\det A \neq 0$. If $y_1, \ldots, y_n$ are any scalars in F, the unique solution $X = A^{-1}Y$ of the system of equations $AX = Y$ is given by

$$x_j = \frac{\det B_j}{\det A}, \qquad j = 1, \ldots, n$$

where B_j is the $n \times n$ matrix obtained from A by replacing the jth column of A by Y.

In concluding this chapter, we should like to make some comments which serve to place determinants in what we believe to be the proper perspective. From time to time it is necessary to compute specific determinants, and this section has been partially devoted to techniques which will facilitate such work. However, the principal role of determinants in this book is theoretical. There is no disputing the beauty of facts such as Cramer's rule. But Cramer's rule is an inefficient tool for solving systems of linear equations, chiefly because it involves too many computations. So one should concentrate on what Cramer's rule says, rather than on how to compute with it. Indeed, while reflecting on this entire chapter, we hope that the reader will place more emphasis on understanding what the determinant function is and how it behaves than on how to compute determinants of specific matrices.

Exercises

1. Use the classical adjoint formula to compute the inverses of each of the following 3×3 real matrices.

$$\begin{bmatrix} -2 & 3 & 2 \\ 6 & 0 & 3 \\ 4 & 1 & -1 \end{bmatrix}, \qquad \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

2. Use Cramer's rule to solve each of the following systems of linear equations over the field of rational numbers.

(a)
$$\begin{aligned} x + y + z &= 11 \\ 2x - 6y - z &= 0 \\ 3x + 4y + 2z &= 0. \end{aligned}$$

(b)
$$\begin{aligned} 3x - 2y &= 7 \\ 3y - 2z &= 6 \\ 3z - 2x &= -1. \end{aligned}$$

3. An $n \times n$ matrix A over a field F is **skew-symmetric** if $A^t = -A$. If A is a skew-symmetric $n \times n$ matrix with complex entries and n is odd, prove that $\det A = 0$.

4. An $n \times n$ matrix A over a field F is called **orthogonal** if $AA^t = I$. If A is orthogonal, show that $\det A = \pm 1$. Give an example of an orthogonal matrix for which $\det A = -1$.

5. An $n \times n$ matrix A over the field of complex numbers is said to be **unitary** if $AA^* = I$ (A^* denotes the conjugate transpose of A). If A is unitary, show that $|\det A| = 1$.

6. Let T and U be linear operators on the finite dimensional vector space V. Prove
(a) $\det(TU) = (\det T)(\det U)$;
(b) T is invertible if and only if $\det T \neq 0$.

7. Let A be an $n \times n$ matrix over K, a commutative ring with identity. Suppose A has the block form

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & A_k \end{bmatrix}$$

where A_j is an $r_j \times r_j$ matrix. Prove

$$\det A = (\det A_1)(\det A_2) \cdots (\det A_k).$$

8. Let V be the vector space of $n \times n$ matrices over the field F. Let B be a fixed element of V and let T_B be the linear operator on V defined by $T_B(A) = AB - BA$. Show that $\det T_B = 0$.

9. Let A be an $n \times n$ matrix over a field, $A \neq 0$. If r is any positive integer between 1 and n, an $r \times r$ **submatrix** of A is any $r \times r$ matrix obtained by deleting $(n - r)$ rows and $(n - r)$ columns of A. The **determinant rank** of A is the largest positive integer r such that some $r \times r$ submatrix of A has a non-zero determinant. Prove that the determinant rank of A is equal to the row rank of A (= column rank A).

10. Let A be an $n \times n$ matrix over the field F. Prove that there are at most n distinct scalars c in F such that $\det(cI - A) = 0$.

11. Let A and B be $n \times n$ matrices over the field F. Show that if A is invertible there are at most n scalars c in F for which the matrix $cA + B$ is not invertible.

12. If V is the vector space of $n \times n$ matrices over F and B is a fixed $n \times n$ matrix over F, let L_B and R_B be the linear operators on V defined by $L_B(A) = BA$ and $R_B(A) = AB$. Show that
(a) $\det L_B = (\det B)^n$;
(b) $\det R_B = (\det B)^n$.

13. Let V be the vector space of all $n \times n$ matrices over the field of complex numbers, and let B be a fixed $n \times n$ matrix over C. Define a linear operator M_B on V by $M_B(A) = BAB^*$, where $B^* = \overline{B^t}$. Show that

$$\det M_B = |\det B|^{2n}.$$

Now let H be the set of all Hermitian matrices in V, A being Hermitian if $A = A^*$. Then H is a vector space over the field of *real* numbers. Show that the function T_B defined by $T_B(A) = BAB^*$ is a linear operator on the real vector space H, and then show that $\det T_B = |\det B|^{2n}$. (*Hint:* In computing $\det T_B$, show that V has a basis consisting of Hermitian matrices and then show that $\det T_B = \det M_B$.)

14. Let A, B, C, D be *commuting* $n \times n$ matrices over the field F. Show that the determinant of the $2n \times 2n$ matrix

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

is $\det(AD - BC)$.

5.5. Modules

If K is a commutative ring with identity, a module over K is an algebraic system which behaves like a vector space, with K playing the role of the scalar field. To be precise, we say that V is a **module over** K (or a K-**module**) if

1. there is an addition $(\alpha, \beta) \rightarrow \alpha + \beta$ on V, under which V is a commutative group;
2. there is a multiplication $(c, \alpha) \rightarrow c\alpha$ of elements α in V and c in K such that

$$\begin{aligned} (c_1 + c_2)\alpha &= c_1\alpha + c_2\alpha \\ c(\alpha_1 + \alpha_2) &= c\alpha_1 + c\alpha_2 \\ (c_1c_2)\alpha &= c_1(c_2\alpha) \\ 1\alpha &= \alpha. \end{aligned}$$

For us, the most important K-modules will be the n-tuple modules K^n. The matrix modules $K^{m \times n}$ will also be important. If V is any module, we speak of linear combinations, linear dependence and linear independence, just as we do in a vector space. We must be careful not to apply to V any vector space results which depend upon division by non-zero scalars, the one field operation which may be lacking in the ring K. For example, if $\alpha_1, \ldots, \alpha_k$ are linearly dependent, we cannot conclude that some α_i is a linear combination of the others. This makes it more difficult to find bases in modules.

A **basis** for the module V is a linearly independent subset which spans (or generates) the module. This is the same definition which we gave for vector spaces; and, the important property of a basis $\mathcal{B}$ is that each element of V can be expressed uniquely as a linear combination of (some finite number of) elements of $\mathcal{B}$. If one admits into mathematics the Axiom of Choice (see Appendix), it can be shown that every vector space has a basis. The reader is well aware that a basis exists in any vector space which is spanned by a finite number of vectors. But this is not the case for modules. Therefore we need special names for modules which have bases and for modules which are spanned by finite numbers of elements.

Definition. *The* K*-module* V *is called a* **free module** *if it has a basis. If* V *has a finite basis containing* n *elements, then* V *is called a* **free K-module with n generators.**

Definition. *The module* V *is* **finitely generated** *if it contains a finite subset which spans* V. *The* **rank** *of a finitely generated module is the smallest integer* k *such that some* k *elements span* V.

We repeat that a module may be finitely generated without having a finite basis. If V is a free K-module with n generators, then V is isomorphic to the module K^n. If $\{\beta_1, \ldots, \beta_n\}$ is a basis for V, there is an isomorphism which sends the vector $c_1\beta_1 + \cdots + c_n\beta_n$ onto the n-tuple $(c_1, \ldots, c_n)$ in K^n. It is not immediately apparent that the same module V could not also be a free module on k generators, with $k \neq n$. In other words, it is not obvious that any two bases for V must contain the same number of elements. The proof of that fact is an interesting application of determinants.

Theorem 5. *Let* K *be a commutative ring with identity. If* V *is a free* K*-module with* n *generators, then the rank of* V *is* n.

Proof. We are to prove that V cannot be spanned by less than n of its elements. Since V is isomorphic to K^n, we must show that, if $m < n$, the module K^n is not spanned by n-tuples $\alpha_1, \ldots, \alpha_m$. Let A be the matrix with rows $\alpha_1, \ldots, \alpha_m$. Suppose that each of the standard basis vectors $\epsilon_1, \ldots, \epsilon_n$ is a linear combination of $\alpha_1, \ldots, \alpha_m$. Then there exists a matrix P in $K^{n\times m}$ such that

$$PA = I$$

where I is the $n \times n$ identity matrix. Let $\tilde{A}$ be the $n \times n$ matrix obtained by adjoining $n - m$ rows of 0's to the bottom of A, and let $\tilde{P}$ be any $n \times n$ matrix which has the columns of P as its first n columns. Then

$$\tilde{P}\tilde{A} = I.$$

Therefore $\det \tilde{A} \neq 0$. But, since $m < n$, at least one row of $\tilde{A}$ has all 0 entries. This contradiction shows that $\alpha_1, \ldots, \alpha_m$ do not span K^n. ▌

It is interesting to note that Theorem 5 establishes the uniqueness of the dimension of a (finite-dimensional) vector space. The proof, based upon the existence of the determinant function, is quite different from the proof we gave in Chapter 2. From Theorem 5 we know that 'free module of rank n' is the same as 'free module with n generators.'

If V is a module over K, the **dual module** V^* consists of all linear functions f from V into K. If V is a free module of rank n, then V^* is also a free module of rank n. The proof is just the same as for vector spaces. If $\{\beta_1, \ldots, \beta_n\}$ is an ordered basis for V, there is an associated **dual basis** $\{f_1, \ldots, f_n\}$ for the module V^*. The function f_i assigns to each α in V its ith coordinate relative to $\{\beta_1, \ldots, \beta_n\}$:

$$\alpha = f_1(\alpha)\beta_1 + \cdots + f_n(\alpha)\beta_n.$$

If f is a linear function on V, then

$$f = f(\beta_1)f_1 + \cdots + f(\beta_n)f_n.$$

5.6. Multilinear Functions

The purpose of this section is to place our discussion of determinants in what we believe to be the proper perspective. We shall treat alternating multilinear forms on modules. These forms are the natural generalization of determinants as we presented them. The reader who has not read (or does not wish to read) the brief account of modules in Section 5.5 can still study this section profitably by consistently reading 'vector space over F of dimension n' for 'free module over K of rank n.'

Let K be a commutative ring with identity and let V be a module over K. If r is a positive integer, a function L from $V^r = V \times V \times \cdots \times V$ into K is called **multilinear** if $L(\alpha_1, \ldots, \alpha_r)$ is linear as a function of each α_i when the other α_j's are held fixed, that is, if for each i

$$L(\alpha_1, \ldots, c\alpha_i + \beta_i, \ldots, \alpha_r) = cL(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_r + L(\alpha_1, \ldots, \beta_i, \ldots, \alpha_r).$$

A multilinear function on V^r will also be called an **r-linear form** on V or a **multilinear form of degree** r on V. Such functions are sometimes called **r-tensors** on V. The collection of all multilinear functions on V^r will be denoted by $M^r(V)$. If L and M are in $M^r(V)$, then the sum $L + M$:

$$(L + M)(\alpha_1, \ldots, \alpha_r) = L(\alpha_1, \ldots, \alpha_r) + M(\alpha_1, \ldots, \alpha_r)$$

is also multilinear; and, if c is an element of K, the product cL:

$$(cL)(\alpha_1, \ldots, \alpha_r) = cL(\alpha_1, \ldots, \alpha_r)$$

is multilinear. Therefore $M^r(V)$ is a K-module—a submodule of the module of all functions from V^r into K.

If $r = 1$ we have $M^1(V) = V^*$, the dual module of linear functions on V. Linear functions can also be used to construct examples of multilinear forms of higher order. If $f_1, \ldots, f_r$ are linear functions on V, define

$$L(\alpha_1, \ldots, \alpha_r) = f_1(\alpha_1)f_2(\alpha_2) \cdots f_r(\alpha_r).$$

Clearly L is an r-linear form on V.

Example 9. If V is a module, a 2-linear form on V is usually called a **bilinear form** on V. Let A be an $n \times n$ matrix with entries in K. Then

$$L(X, Y) = Y^tAX$$

defines a bilinear form L on the module $K^{n\times 1}$. Similarly,

$$M(\alpha, \beta) = \alpha A\beta^t$$

defines a bilinear form M on K^n.

Example 10. The determinant function associates with each $n \times n$