If $\phi_1, \ldots, \phi_n$ are the components of Φ, then

$$\omega_\Phi = a(\Phi(\mathbf{u}))\, d\phi_{i_1} \wedge \cdots \wedge d\phi_{i_k}.$$

The theorem will follow if we can show that

$$d\phi_{i_1} \wedge \cdots \wedge d\phi_{i_k} = J(\mathbf{u})\, du_1 \wedge \cdots \wedge du_k, \tag{72}$$

where

$$J(\mathbf{u}) = \frac{\partial(x_{i_1}, \ldots, x_{i_k})}{\partial(u_1, \ldots, u_k)},$$

since (72) implies

$$\begin{aligned}\int_\Phi \omega &= \int_D a(\Phi(\mathbf{u}))J(\mathbf{u})\, d\mathbf{u} \\ &= \int_\Delta a(\Phi(\mathbf{u}))J(\mathbf{u})\, du_1 \wedge \cdots \wedge du_k = \int_\Delta \omega_\Phi.\end{aligned}$$

Let $[A]$ be the k by k matrix with entries

$$\alpha(p, q) = (D_q \phi_{i_p})(\mathbf{u}) \qquad (p, q = 1, \ldots, k).$$

Then

$$d\phi_{i_p} = \sum_q \alpha(p, q)\, du_q$$

so that

$$d\phi_{i_1} \wedge \cdots \wedge d\phi_{i_k} = \sum \alpha(1, q_1) \cdots \alpha(k, q_k)\, du_{q_1} \wedge \cdots \wedge du_{q_k}.$$

In this last sum, $q_1, \ldots, q_k$ range independently over $1, \ldots, k$. The anticommutative relation (42) implies that

$$du_{q_1} \wedge \cdots \wedge du_{q_k} = s(q_1, \ldots, q_k)\, du_1 \wedge \cdots \wedge du_k,$$

where s is as in Definition 9.33; applying this definition, we see that

$$d\phi_{i_1} \wedge \cdots \wedge d\phi_{i_k} = \det [A]\, du_1 \wedge \cdots \wedge du_k;$$

and since $J(\mathbf{u}) = \det [A]$, (72) is proved.

The final result of this section combines the two preceding theorems.

10.25 Theorem *Suppose T is a $\mathscr{C}'$-mapping of an open set $E \subset R^n$ into an open set $V \subset R^m$, Φ is a k-surface in E, and ω is a k-form in V.*
Then

$$\int_{T\Phi} \omega = \int_\Phi \omega_T.$$

Proof Let D be the parameter domain of Φ (hence also of $T\Phi$) and define Δ as in Theorem 10.24.

Then

$$\int_{T\Phi} \omega = \int_{\Delta} \omega_{T\Phi} = \int_{\Delta} (\omega_T)_\Phi = \int_{\Phi} \omega_T .$$

The first of these equalities is Theorem 10.24, applied to $T\Phi$ in place of Φ. The second follows from Theorem 10.23. The third is Theorem 10.24, with ω_T in place of ω.

SIMPLEXES AND CHAINS

10.26 Affine simplexes A mapping $\mathbf{f}$ that carries a vector space X into a vector space Y is said to be *affine* if $\mathbf{f} - \mathbf{f}(\mathbf{0})$ is linear. In other words, the requirement is that

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{0}) + A\mathbf{x} \tag{73}$$

for some $A \in L(X, Y)$.

An affine mapping of R^k into R^n is thus determined if we know $\mathbf{f}(\mathbf{0})$ and $\mathbf{f}(\mathbf{e}_i)$ for $1 \le i \le k$; as usual, $\{\mathbf{e}_1, \dots, \mathbf{e}_k\}$ is the standard basis of R^k.

We define the *standard simplex* Q^k to be the set of all $\mathbf{u} \in R^k$ of the form

$$\mathbf{u} = \sum_{i=1}^{k} \alpha_i \mathbf{e}_i \tag{74}$$

such that $\alpha_i \ge 0$ for $i = 1, \dots, k$ and $\Sigma \alpha_i \le 1$.

Assume now that $\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_k$ are points of R^n. The *oriented affine k-simplex*

$$\sigma = [\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_k] \tag{75}$$

is defined to be the k-surface in R^n with parameter domain Q^k which is given by the affine mapping

$$\sigma(\alpha_1 \mathbf{e}_1 + \cdots + \alpha_k \mathbf{e}_k) = \mathbf{p}_0 + \sum_{i=1}^{k} \alpha_i(\mathbf{p}_i - \mathbf{p}_0). \tag{76}$$

Note that σ is characterized by

$$\sigma(\mathbf{0}) = \mathbf{p}_0, \qquad \sigma(\mathbf{e}_i) = \mathbf{p}_i \qquad (\text{for } 1 \le i \le k), \tag{77}$$

and that

$$\sigma(\mathbf{u}) = \mathbf{p}_0 + A\mathbf{u} \qquad (\mathbf{u} \in Q^k) \tag{78}$$

where $A \in L(R^k, R^n)$ and $A\mathbf{e}_i = \mathbf{p}_i - \mathbf{p}_0$ for $1 \le i \le k$.

We call σ *oriented* to emphasize that the ordering of the vertices $\mathbf{p}_0, \ldots, \mathbf{p}_k$ is taken into account. If

$$(79) \qquad \bar{\sigma} = [p_{i_0}, p_{i_1}, \ldots, p_{i_k}],$$

where $\{i_0, i_1, \ldots, i_k\}$ is a permutation of the ordered set $\{0, 1, \ldots, k\}$, we adopt the notation

$$(80) \qquad \bar{\sigma} = s(i_0, i_1, \ldots, i_k)\sigma,$$

where s is the function defined in Definition 9.33. Thus $\bar{\sigma} = \pm\sigma$, depending on whether $s = 1$ or $s = -1$. Strictly speaking, having adopted (75) and (76) as the definition of σ, we should not write $\bar{\sigma} = \sigma$ unless $i_0 = 0, \ldots, i_k = k$, even if $s(i_0, \ldots, i_k) = 1$; what we have here is an equivalence relation, not an equality. However, for our purposes the notation is justified by Theorem 10.27.

If $\bar{\sigma} = \varepsilon\sigma$ (using the above convention) and if $\varepsilon = 1$, we say that $\bar{\sigma}$ and σ have the *same orientation*; if $\varepsilon = -1$, $\bar{\sigma}$ and σ are said to have *opposite orientations*. Note that we have not defined what we mean by the "orientation of a simplex." What we have defined is a relation between pairs of simplexes having the same set of vertices, the relation being that of "having the same orientation."

There is, however, one situation where the orientation of a simplex can be defined in a natural way. This happens when $n = k$ and when the vectors $\mathbf{p}_i - \mathbf{p}_0$ $(1 \le i \le k)$ are *independent*. In that case, the linear transformation A that appears in (78) is invertible, and its determinant (which is the same as the Jacobian of σ) is not 0. Then σ is said to be *positively* (or *negatively*) oriented if $\det A$ is positive (or negative). In particular, the simplex $[\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_k]$ in R^k, given by the identity mapping, has positive orientation.

So far we have assumed that $k \ge 1$. An *oriented* 0-*simplex* is defined to be a point with a sign attached. We write $\sigma = +\mathbf{p}_0$ or $\sigma = -\mathbf{p}_0$. If $\sigma = \varepsilon\mathbf{p}_0$ $(\varepsilon = \pm 1)$ and if f is a 0-form (i.e., a real function), we define

$$\int_\sigma f = \varepsilon f(p_0).$$

10.27 Theorem *If σ is an oriented rectilinear k-simplex in an open set $E \subset R^n$ and if $\bar{\sigma} = \varepsilon\sigma$ then*

$$(81) \qquad \int_{\bar{\sigma}} \omega = \varepsilon \int_\sigma \omega$$

for every k-form ω in E.

Proof For $k = 0$, (81) follows from the preceding definition. So we assume $k \ge 1$ and assume that σ is given by (75).

Suppose $1 \le j \le k$, and suppose $\bar{\sigma}$ is obtained from σ by interchanging $\mathbf{p}_0$ and $\mathbf{p}_j$. Then $\varepsilon = -1$, and

$$\bar{\sigma}(\mathbf{u}) = \mathbf{p}_j + B\mathbf{u} \qquad (\mathbf{u} \in Q^k),$$

where B is the linear mapping of R^k into R^n defined by $B\mathbf{e}_j = \mathbf{p}_0 - \mathbf{p}_j$, $B\mathbf{e}_i = \mathbf{p}_i - \mathbf{p}_j$ if $i \ne j$. If we write $A\mathbf{e}_i = \mathbf{x}_i$ $(1 \le i \le k)$, where A is given by (78), the column vectors of B (that is, the vectors $B\mathbf{e}_i$) are

$$\mathbf{x}_1 - \mathbf{x}_j, \ldots, \mathbf{x}_{j-1} - \mathbf{x}_j, -\mathbf{x}_j, \mathbf{x}_{j+1} - \mathbf{x}_j, \ldots, \mathbf{x}_k - \mathbf{x}_j.$$

If we subtract the jth column from each of the others, none of the determinants in (35) are affected, and we obtain columns $\mathbf{x}_1, \ldots, \mathbf{x}_{j-1}, -\mathbf{x}_j, \mathbf{x}_{j+1}, \ldots, \mathbf{x}_k$. These differ from those of A only in the sign of the jth column. Hence (81) holds for this case.

Suppose next that $0 < i < j \le k$ and that $\bar{\sigma}$ is obtained from σ by interchanging $\mathbf{p}_i$ and $\mathbf{p}_j$. Then $\bar{\sigma}(\mathbf{u}) = \mathbf{p}_0 + C\mathbf{u}$, where C has the same columns as A, except that the ith and jth columns have been interchanged. This again implies that (81) holds, since $\varepsilon = -1$.

The general case follows, since every permutation of $\{0, 1, \ldots, k\}$ is a composition of the special cases we have just dealt with.

10.28 Affine chains An *affine k-chain* Γ in an open set $E \subset R^n$ is a collection of finitely many oriented affine k-simplexes $\sigma_1, \ldots, \sigma_r$ in E. These need not be distinct; a simplex may thus occur in Γ with a certain multiplicity.

If Γ is as above, and if ω is a k-form in E, we define

$$\int_\Gamma \omega = \sum_{i=1}^{r} \int_{\sigma_i} \omega. \tag{82}$$

We may view a k-surface Φ in E as a function whose domain is the collection of all k-forms in E and which assigns the number $\int_\Phi \omega$ to ω. Since real-valued functions can be added (as in Definition 4.3), this suggests the use of the notation

$$\Gamma = \sigma_1 + \cdots + \sigma_r \tag{83}$$

or, more compactly,

$$\Gamma = \sum_{i=1}^{r} \sigma_i \tag{84}$$

to state the fact that (82) holds for every k-form ω in E.

To avoid misunderstanding, we point out explicitly that the notations introduced by (83) and (80) have to be handled with care. The point is that every oriented affine k-simplex σ in R^n is a function in two ways, with different domains and different ranges, and that therefore two entirely different operations of addition are possible. Originally, σ was defined as an R^n-valued function with domain Q^k; accordingly, $\sigma_1 + \sigma_2$ *could* be interpreted to be the function σ that assigns the vector $\sigma_1(\mathbf{u}) + \sigma_2(\mathbf{u})$ to every $\mathbf{u} \in Q^k$; note that σ is then again an oriented affine k-simplex in R^n! This is *not* what is meant by (83).

For example, if $\sigma_2 = -\sigma_1$ as in (80) (that is to say, if σ_1 and σ_2 have the same set of vertices but are oppositely oriented) and if $\Gamma = \sigma_1 + \sigma_2$, then $\int_\Gamma \omega = 0$ for all ω, and we may express this by writing $\Gamma = 0$ or $\sigma_1 + \sigma_2 = 0$. This does not mean that $\sigma_1(\mathbf{u}) + \sigma_2(\mathbf{u})$ is the null vector of R^n.

10.29 Boundaries For $k \geq 1$, the *boundary* of the oriented affine k-simplex

$$\sigma = [\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_k]$$

is defined to be the affine $(k-1)$-chain

$$\partial\sigma = \sum_{j=0}^{k} (-1)^j [\mathbf{p}_0, \ldots, \mathbf{p}_{j-1}, \mathbf{p}_{j+1}, \ldots, \mathbf{p}_k]. \tag{85}$$

For example, if $\sigma = [\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2]$, then

$$\partial\sigma = [\mathbf{p}_1, \mathbf{p}_2] - [\mathbf{p}_0, \mathbf{p}_2] + [\mathbf{p}_0, \mathbf{p}_1] = [\mathbf{p}_0, \mathbf{p}_1] + [\mathbf{p}_1, \mathbf{p}_2] + [\mathbf{p}_2, \mathbf{p}_0],$$

which coincides with the usual notion of the oriented boundary of a triangle.

For $1 \leq j \leq k$, observe that the simplex $\sigma_j = [\mathbf{p}_0, \ldots, \mathbf{p}_{j-1}, \mathbf{p}_{j+1}, \ldots, \mathbf{p}_k]$ which occurs in (85) has Q^{k-1} as its parameter domain and that it is defined by

$$\sigma_j(\mathbf{u}) = \mathbf{p}_0 + B\mathbf{u} \qquad (\mathbf{u} \in Q^{k-1}), \tag{86}$$

where B is the linear mapping from R^{k-1} to R^n determined by

$$B\mathbf{e}_i = \mathbf{p}_i - \mathbf{p}_0 \qquad (\text{if} \quad 1 \leq i \leq j-1),$$
$$B\mathbf{e}_i = \mathbf{p}_{i+1} - \mathbf{p}_0 \qquad (\text{if} \quad j \leq i \leq k-1).$$

The simplex

$$\sigma_0 = [\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_k],$$

which also occurs in (85), is given by the mapping

$$\sigma_0(\mathbf{u}) = \mathbf{p}_1 + B\mathbf{u},$$

where $B\mathbf{e}_i = \mathbf{p}_{i+1} - \mathbf{p}_1$ for $1 \leq i \leq k-1$.

10.30 Differentiable simplexes and chains Let T be a $\mathscr{C}''$-mapping of an open set $E \subset R^n$ into an open set $V \subset R^m$; T need not be one-to-one. If σ is an oriented affine k-simplex in E, then the composite mapping $\Phi = T \circ \sigma$ (which we shall sometimes write in the simpler form $T\sigma$) is a k-surface in V, with parameter domain Q^k. We call Φ an *oriented k-simplex of class $\mathscr{C}''$*.

A finite collection Ψ of oriented k-simplexes $\Phi_1, \ldots, \Phi_r$ of class $\mathscr{C}''$ in V is called a *k-chain of class $\mathscr{C}''$* in V. If ω is a k-form in V, we define

$$\int_{\Psi} \omega = \sum_{i=1}^{r} \int_{\Phi_i} \omega \tag{87}$$

and use the corresponding notation $\Psi = \Sigma \Phi_i$.

If $\Gamma = \Sigma \sigma_i$ is an affine chain and if $\Phi_i = T \circ \sigma_i$, we also write $\Psi = T \circ \Gamma$, or

$$T(\sum \sigma_i) = \sum T\sigma_i. \tag{88}$$

The boundary $\partial\Phi$ of the oriented k-simplex $\Phi = T \circ \sigma$ is defined to be the $(k-1)$ chain

$$\partial\Phi = T(\partial\sigma). \tag{89}$$

In justification of (89), observe that if T is affine, then $\Phi = T \circ \sigma$ is an oriented affine k-simplex, in which case (89) is not a matter of definition, but is seen to be a *consequence* of (85). Thus (89) generalizes this special case.

It is immediate that $\partial\Phi$ is of class $\mathscr{C}''$ if this is true of Φ.

Finally, we define the boundary $\partial\Psi$ of the k-chain $\Psi = \Sigma\Phi_i$ to be the $(k-1)$ chain

$$\partial\Psi = \sum \partial\Phi_i. \tag{90}$$

10.31 Positively oriented boundaries So far we have associated boundaries to chains, not to subsets of R^n. This notion of boundary is exactly the one that is most suitable for the statement and proof of Stokes' theorem. However, in applications, especially in R^2 or R^3, it is customary and convenient to talk about "oriented boundaries" of certain sets as well. We shall now describe this briefly.

Let Q^n be the standard simplex in R^n, let σ_0 be the identity mapping with domain Q^n. As we saw in Sec. 10.26, σ_0 may be regarded as a positively oriented n-simplex in R^n. Its boundary $\partial\sigma_0$ is an affine $(n-1)$-chain. This chain is called the *positively oriented boundary of the set Q^n*.

For example, the positively oriented boundary of Q^3 is

$$[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] - [\mathbf{0}, \mathbf{e}_2, \mathbf{e}_3] + [\mathbf{0}, \mathbf{e}_1, \mathbf{e}_3] - [\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2].$$

Now let T be a 1-1 mapping of Q^n into R^n, of class $\mathscr{C}''$, whose Jacobian is positive (at least in the interior of Q^n). Let $E = T(Q^n)$. By the inverse function theorem, E is the closure of an open subset of R^n. We define the positively oriented boundary of the set E to be the $(n-1)$-chain

$$\partial T = T(\partial\sigma_0),$$

and we may denote this $(n-1)$-chain by ∂E.

An obvious question occurs here: If $E = T_1(Q^n) = T_2(Q^n)$, and if both T_1 and T_2 have positive Jacobians, is it true that $\partial T_1 = \partial T_2$? That is to say, does the equality

$$\int_{\partial T_1} \omega = \int_{\partial T_2} \omega$$

hold for every $(n-1)$-form ω? The answer is yes, but we shall omit the proof. (To see an example, compare the end of this section with Exercise 17.)

One can go further. Let

$$\Omega = E_1 \cup \cdots \cup E_r,$$

where $E_i = T_i(Q^n)$, each T_i has the properties that T had above, and the interiors of the sets E_i are pairwise disjoint. Then the $(n-1)$-chain

$$\partial T_1 + \cdots + \partial T_r = \partial \Omega$$

is called the positively oriented boundary of Ω.

For example, the unit square I^2 in R^2 is the union of $\sigma_1(Q^2)$ and $\sigma_2(Q^2)$, where

$$\sigma_1(\mathbf{u}) = \mathbf{u}, \qquad \sigma_2(\mathbf{u}) = \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{u}.$$

Both σ_1 and σ_2 have Jacobian $1 > 0$. Since

$$\sigma_1 = [\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2], \qquad \sigma_2 = [\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2, \mathbf{e}_1]$$

we have

$$\begin{aligned} \partial\sigma_1 &= [\mathbf{e}_1, \mathbf{e}_2] - [\mathbf{0}, \mathbf{e}_2] + [\mathbf{0}, \mathbf{e}_1], \\ \partial\sigma_2 &= [\mathbf{e}_2, \mathbf{e}_1] - [\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1] + [\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2]; \end{aligned}$$

The sum of these two boundaries is

$$\partial I^2 = [\mathbf{0}, \mathbf{e}_1] + [\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2] + [\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2] + [\mathbf{e}_2, \mathbf{0}],$$

the positively oriented boundary of I^2. Note that $[\mathbf{e}_1, \mathbf{e}_2]$ canceled $[\mathbf{e}_2, \mathbf{e}_1]$.

If Φ is a 2-surface in R^m, with parameter domain I^2, then Φ (regarded as a function on 2-forms) is the same as the 2-chain

$$\Phi \circ \sigma_1 + \Phi \circ \sigma_2 .$$

Thus

$$\begin{aligned} \partial\Phi &= \partial(\Phi \circ \sigma_1) + \partial(\Phi \circ \sigma_2) \\ &= \Phi(\partial\sigma_1) + \Phi(\partial\sigma_2) = \Phi(\partial I^2). \end{aligned}$$

In other words, if the parameter domain of Φ is the square I^2, we need not refer back to the simplex Q^2, but can obtain $\partial\Phi$ directly from ∂I^2.

Other examples may be found in Exercises 17 to 19.