Similarly, the pitch is lowered by m octaves when length is multiplied by 2^m, and by n perfect fifths when length is multiplied by $(3/2)^n$. *But there are no natural numbers such that*

$$2^m = \left(\frac{3}{2}\right)^n,$$

as this implies

$$2^{m+n} = 3^n,$$

which is absurd because 2^{m+n} is even and 3^n is odd. Thus there are no natural numbers m and n such that m octaves equals n perfect fifths—in other words, the ratio of the octave to the perfect fifth is irrational.

Geometry, Measurement, and Numbers

I have claimed that the discovery of irrational lengths led to the separation of geometry from number theory in Greek mathematics. But perhaps this was only because rigor demanded a separate development of geometry, as long as there was no rigorous definition of irrational number, it was necessary to work with lengths. Rigor and precision are necessary for communication of mathematics to the public, but they are only *last stage* in the mathematician's own thought. New ideas generally emerge from confusion and obscurity, so they cannot be grasped precisely until they have first been grasped vaguely and even inconsistently. We know, for example, that Archimedes discovered results on the area and volume of curved figures by dubious methods, then revised his proofs to make them rigorous. Only the rigorous versions were known until 1906, when Heiberg discovered a lost manuscript revealing Archimedes' original methods (see Heath (1912)). Thus it is quite possible that the Greeks thought about irrational numbers but wrote about lengths for public consumption.

Even if this is so, shouldn't geometry be about lengths? Its name means "land measurement," after all. Well, this probably has more to do with the legendary origins of geometry than its actual use in ancient Greece. Plato believed that geometry was not really about land measurement but about different types of numbers. In the

Epinomis, a work due to Plato or one of his disciples, we find the remarkable statement:

> ... what is called by the very ridiculous name *mensuration* (*geometria*), but is really a manifest assimilation to one another of numbers which are naturally dissimilar, effected by reference to areas. Now to a man who can comprehend this, 'twill be plain that this is no mere feat of human skill, but a miracle of God's contrivance. [From the translation of the *Epinomis* by Taylor (1972), p. 249.]

Even in a profession where measurement is important, the ancients were more impressed with theory. In the 1st century B.C., Vitruvius wrote in his *Ten Books on Architecture*, Introduction to Book IX:

> Pythagoras showed that a right angle can be formed without the contrivances of the artisan. Thus the result which carpenters reach very laboriously, but scarcely to exactness, with their squares, can be demonstrated to perfection from the reasoning and methods of his teaching.

Presumably he was thinking of the converse Pythagorean theorem, according to which lengths a, b, and c for which $a^2 + b^2 = c^2$ make a triangle with a right angle between the sides a and b. (This follows easily from Pythagoras' theorem itself, and the side-side-side congruence axiom. Namely, construct a right angle with sides a and b, so the hypotenuse joining these sides has length c by Pythagoras' theorem. Then we have a right-angled triangle with sides a, b, and c, and it is the *only* triangle with these sides, by the congruence axiom.) He then described the construction of a right angle by combining rods of lengths 3, 4, and 5 in a triangle, and finally he suggested an application:

> This theorem affords a useful means of measuring many things, and it is particularly serviceable in the building of staircases in buildings, so that steps may be at their proper levels.

It has often been claimed that the $(3, 4, 5)$ triangle was used in ancient times to construct right angles, but this is the oldest reference to it that I know.

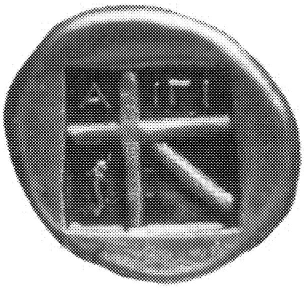

FIGURE 2.27 Geometric algebra on a Greek coin.

Just as we cannot be sure how the Greeks viewed irrationals, we cannot tell how they viewed the so-called *geometric algebra* in Euclid's *Elements*. One would not expect a full understanding of algebra, especially not with the Greeks' unsuitable notation. But they may have caught a glimpse of it. The geometric interpretation of $(a+b)^2 = a^2 + 2ab + b^2$ (Exercise 2.6.2) was so well known in ancient Greece that the figure actually appeared on coins. Figure 2.27 shows a photograph of one; this photograph was given to me by Benno Artmann.

The diagonal line in the figure is probably a construction line; the corresponding figure in Euclid (Book II, Proposition 4) uses this line (but drawn all the way across) to divide the vertical line into the same two parts, a and b, as the horizontal line.