their minds with the view that the future is determined by fate (Kneale p. 48).

Today we tend to dismiss the detailed elaboration of Aristotelian logic by the medieval Scholastics and recognize as a major advance only the ideas of Gottfried W. Leibniz (1646–1716). Leibniz conceived of a universal symbolic language which would be adequate not only for mathematics, but for all of science. Unfortunately, he did not get around to publishing the details of his proposal, perhaps because he was preoccupied by his controversy with Newton concerning the invention of 'the' calculus and by his successful diplomatic efforts to put George I on the throne of England.

It was only in 1847 that full-blown *symbolic logic* finally saw the light of day. This was the year in which both George Boole (1815–1864) and Augustus DeMorgan (1806–1871) published their first works in logic. The former saw propositional logic as a branch of algebra, distinguished from the usual algebra of 'quantities' by the 'idempotent' law: $p \times p = p$. De Morgan is remembered for his laws expressing the duality between disjunction ('or') and conjunction ('and'):

$$\neg(p \wedge q) = \neg p \vee \neg q, \qquad \neg(p \vee q) = \neg p \wedge \neg q.$$

The next major step was taken by Gottlob Frege (1848–1925),who was the first to have a modern view of universal and existential quantifiers, although without using the modern notation: $\forall$ and $\exists$. This was surprisingly recent, considering that every student nowadays is familiar with these notions. Frege also attempted to express all of mathematics in terms of logical symbols, in fact, reducing mathematics to logic, thus espousing a philosophical position called 'Logicism'.

Crucial to Frege's project is the assumption that, corresponding to any property expressed by a predicate of his language, there is a uniquely determined set whose elements are just the entities with that property. Frege expressed this assumption by the following *comprehension scheme*:

$$\exists_y \forall_x (x \in y \Longleftrightarrow P(x)),$$

where $P(x)$ is any formula, possibly containing the free variable x. (The double arrow means 'if and only if' or 'just in case'.) This scheme is accompanied by the *axiom of extensionality* to ensure the uniqueness of the set y whose existence has been asserted above:

$$\forall_y \forall_z (\forall_x (x \in y \Longleftrightarrow x \in z) \Longrightarrow y = z).$$

The axiom of extensionality implies, in particular, that there can be at most one entity y with no elements, the *empty set*. This axiom implicitly contains the assumption that all entities discussed by Frege's language are sets.

Frege had just written a book propounding these views when he received a letter from Bertrand Russell (1872–1970), pointing out that there was a serious problem when $P(x)$ was the formula $\neg(x \in x)$. Indeed, if y was such that $\forall(x \in y \Longleftrightarrow \neg(x \in x))$, one would obtain as a special case

$$y \in y \Longleftrightarrow \neg(y \in y),$$

which is a contradiction.

This argument is known as 'Russell's paradox'. One way to avoid the contradiction is to forbid expressions such as $x \in x$. Russell and Whitehead propose a *theory of types*, according to which each symbol denoting an entity should have attached to it a certain natural number, its *type*, and the formula $a \in b$ is permitted only if the type of b is one higher than the type of a. This theory was developed in excruciating detail in the three-volume *Principia Mathematica*, an unnecessarily complicated treatise that is more talked about than studied.

Although up-to-date versions of type theory are now available, most mathematicians prefer other methods for avoiding Russell's and similar paradoxes. On the whole, if mathematicians worry about such problems at all, they subscribe to the set theory of Gödel and Bernays, which distinguishes between *sets* and *classes*: only the former can be elements. Unfortunately, one has to add a number of axioms to specify which classes are sets. Logicians, on the other hand, prefer the set theory of Ernst Zermelo (1871–1953) and Abraham Fraenkel (1891–1965), who modify the comprehension as follows:

$$\forall_z \exists_y \forall_x (x \in y \Longleftrightarrow (x \in z \wedge P(x))).$$

They too had to introduce additional axioms, which spoiled the simplicity of Frege's project.

The fact that Frege's simple and natural comprehension scheme led to contradictions startled many mathematicians, and some became sceptical about all but the most basic procedures. Other mathematicians were already sceptical. We shall discuss L. Kronecker (1823–1891) and H. Poincaré (1854–1912) in this chapter, leaving L. E. J. Brouwer to the next chapter.

It was Kronecker who said 'God made the whole numbers, all the rest is the work of man'. He was suspicious of Cantor's infinite cardinals; but most of all he rejected *nonconstructive* arguments such as the following proof that there exist irrational numbers α and β such that α^β is rational.

Consider $\sqrt{2}^{\sqrt{2}}$. If it is rational, we are done, since we know that $\sqrt{2}$ is irrational. Suppose $\sqrt{2}^{\sqrt{2}}$ is irrational. Call it α and take $\beta = \sqrt{2}$. Then

$$\alpha^\beta = \sqrt{2}^{(\sqrt{2} \times \sqrt{2})} = \sqrt{2}^2 = 2,$$

which is surely rational.