called $\{x \in A : P(x) \text{ is true}\}$ *(or simply* $\{x \in A : P(x)\}$ *for short), whose elements are precisely the elements* x *in* A *for which* $P(x)$ *is true. In other words, for any object* y,

$$y \in \{x \in A : P(x) \text{ is true}\} \iff (y \in A \text{ and } P(y) \text{ is true}).$$

This axiom is also known as the *axiom of separation.* Note that $\{x \in A : P(x) \text{ is true}\}$ is always a subset of A (why?), though it could be as large as A or as small as the empty set. One can verify that the axiom of substitution works for specification, thus if $A = A'$ then $\{x \in A : P(x)\} = \{x \in A' : P(x)\}$ (why?).

Example 3.1.22. Let $S := \{1, 2, 3, 4, 5\}$. Then the set $\{n \in S : n < 4\}$ is the set of those elements n in S for which $n < 4$ is true, i.e., $\{n \in S : n < 4\} = \{1, 2, 3\}$. Similarly, the set $\{n \in S : n < 7\}$ is the same as S itself, while $\{n \in S : n < 1\}$ is the empty set.

We sometimes write $\{x \in A | P(x)\}$ instead of $\{x \in A : P(x)\}$; this is useful when we are using the colon ":" to denote something else, for instance to denote the range and domain of a function $f : X \to Y$).

We can use this axiom of specification to define some further operations on sets, namely intersections and difference sets.

Definition 3.1.23 (Intersections). The *intersection* $S_1 \cap S_2$ of two sets is defined to be the set

$$S_1 \cap S_2 := \{x \in S_1 : x \in S_2\}.$$

In other words, $S_1 \cap S_2$ consists of all the elements which belong to both S_1 and S_2. Thus, for all objects x,

$$x \in S_1 \cap S_2 \iff x \in S_1 \text{ and } x \in S_2.$$

Remark 3.1.24. Note that this definition is well-defined (i.e., it obeys the axiom of substitution, see Section A.7) because it is defined in terms of more primitive operations which were already known to obey the axiom of substitution. Similar remarks apply to future definitions in this chapter and will usually not be mentioned explicitly again.

Examples 3.1.25. We have $\{1,2,4\} \cap \{2,3,4\} = \{2,4\}$, $\{1,2\} \cap \{3,4\} = \emptyset$, $\{2,3\} \cup \emptyset = \{2,3\}$, and $\{2,3\} \cap \emptyset = \emptyset$.

Remark 3.1.26. By the way, one should be careful with the English word "and": rather confusingly, it can mean either union or intersection, depending on context. For instance, if one talks about a set of "boys and girls", one means the *union* of a set of boys with a set of girls, but if one talks about the set of people who are single and male, then one means the *intersection* of the set of single people with the set of male people. (Can you work out the rule of grammar that determines when "and" means union and when "and" means intersection?) Another problem is that "and" is also used in English to denote addition, thus for instance one could say that "2 and 3 is 5", while also saying that "the elements of $\{2\}$ and the elements of $\{3\}$ form the set $\{2,3\}$" and "the elements in $\{2\}$ and $\{3\}$ form the set $\emptyset$". This can certainly get confusing! One reason we resort to mathematical symbols instead of English words such as "and" is that mathematical symbols always have a precise and unambiguous meaning, whereas one must often look very carefully at the context in order to work out what an English word means.

Two sets A, B are said to be *disjoint* if $A \cap B = \emptyset$. Note that this is not the same concept as being *distinct*, $A \neq B$. For instance, the sets $\{1,2,3\}$ and $\{2,3,4\}$ are distinct (there are elements of one set which are not elements of the other) but not disjoint (because their intersection is non-empty). Meanwhile, the sets $\emptyset$ and $\emptyset$ are disjoint but not distinct (why?).

Definition 3.1.27 (Difference sets). Given two sets A and B, we define the set $A - B$ or $A \backslash B$ to be the set A with any elements of B removed:

$$A \backslash B := \{x \in A : x \notin B\};$$

for instance, $\{1,2,3,4\} \backslash \{2,4,6\} = \{1,3\}$. In many cases B will be a subset of A, but not necessarily.

We now give some basic properties of unions, intersections, and difference sets.

Proposition 3.1.28 (Sets form a boolean algebra). *Let A, B, C be sets, and let X be a set containing A, B, C as subsets.*

(a) *(Minimal element) We have $A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$.*

(b) *(Maximal element) We have $A \cup X = X$ and $A \cap X = A$.*

(c) *(Identity) We have $A \cap A = A$ and $A \cup A = A$.*

(d) *(Commutativity) We have $A \cup B = B \cup A$ and $A \cap B = B \cap A$.*

(e) *(Associativity) We have $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$.*

(f) *(Distributivity) We have $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.*

(g) *(Partition) We have $A \cup (X \backslash A) = X$ and $A \cap (X \backslash A) = \emptyset$.*

(h) *(De Morgan laws) We have $X \backslash (A \cup B) = (X \backslash A) \cap (X \backslash B)$ and $X \backslash (A \cap B) = (X \backslash A) \cup (X \backslash B)$.*

Remark 3.1.29. The de Morgan laws are named after the logician Augustus De Morgan (1806–1871), who identified them as one of the basic laws of set theory.

Proof. See Exercise 3.1.6. □

Remark 3.1.30. The reader may observe a certain symmetry in the above laws between $\cup$ and $\cap$, and between X and $\emptyset$. This is an example of *duality* - two distinct properties or objects being dual to each other. In this case, the duality is manifested by the complementation relation $A \mapsto X \backslash A$; the de Morgan laws assert that this relation converts unions into intersections and vice versa. (It also interchanges X and the empty set.) The above laws are collectively known as the *laws of Boolean algebra*, after the mathematician George Boole (1815–1864), and are also applicable to a number of other objects other than sets; it plays a particularly important rôle in logic.

We have now accumulated a number of axioms and results about sets, but there are still many things we are not able to do yet. One of the basic things we wish to do with a set is take each of the objects of that set, and somehow transform each such object into a new object; for instance we may wish to start with a set of numbers, say $\{3, 5, 9\}$, and increment each one, creating a new set $\{4, 6, 10\}$. This is not something we can do directly using only the axioms we already have, so we need a new axiom:

Axiom 3.6 (Replacement). *Let A be a set. For any object $x \in A$, and any object y, suppose we have a statement $P(x, y)$ pertaining to x and y, such that for each $x \in A$ there is at most one y for which $P(x, y)$ is true. Then there exists a set $\{y : P(x, y) \text{ is true for some } x \in A\}$, such that for any object z,*

$$\begin{aligned} z \in &\{y : P(x, y) \text{ is true for some } x \in A\} \\ &\iff P(x, z) \text{ is true for some } x \in A. \end{aligned}$$

Example 3.1.31. Let $A := \{3, 5, 9\}$, and let $P(x, y)$ be the statement $y = x{+}{+}$, i.e., y is the successor of x. Observe that for every $x \in A$, there is exactly one y for which $P(x, y)$ is true - specifically, the successor of x. Thus the above axiom asserts that the set $\{y : y = x{+}{+} \text{ for some } x \in \{3, 5, 9\}\}$ exists; in this case, it is clearly the same set as $\{4, 6, 10\}$ (why?).

Example 3.1.32. Let $A = \{3, 5, 9\}$, and let $P(x, y)$ be the statement $y = 1$. Then again for every $x \in A$, there is exactly one y for which $P(x, y)$ is true - specifically, the number 1. In this case $\{y : y = 1 \text{ for some } x \in \{3, 5, 9\}\}$ is just the singleton set $\{1\}$; we have replaced each element $3, 5, 9$ of the original set A by the same object, namely 1. Thus this rather silly example shows that the set obtained by the above axiom can be "smaller" than the original set.

We often abbreviate a set of the form

$$\{y : y = f(x) \text{ for some } x \in A\}$$

as $\{f(x) : x \in A\}$ or $\{f(x) | x \in A\}$. Thus for instance, if $A = \{3,5,9\}$, then $\{x{+}{+} : x \in A\}$ is the set $\{4,6,10\}$. We can of course combine the axiom of replacement with the axiom of specification, thus for instance we can create sets such as $\{f(x) : x \in A; P(x) \text{ is true}\}$ by starting with the set A, using the axiom of specification to create the set $\{x \in A : P(x) \text{ is true}\}$, and then applying the axiom of replacement to create $\{f(x) : x \in A; P(x) \text{ is true}\}$. Thus for instance $\{n{+}{+} : n \in \{3,5,9\}; n < 6\} = \{4,6\}$.

In many of our examples we have implicitly assumed that natural numbers are in fact objects. Let us formalize this as follows.

Axiom 3.7 (Infinity). *There exists a set* $\mathbf{N}$*, whose elements are called natural numbers, as well as an object 0 in* $\mathbf{N}$*, and an object* $n{+}{+}$ *assigned to every natural number* $n \in \mathbf{N}$*, such that the Peano axioms (Axioms 2.1 - 2.5) hold.*

This is the more formal version of Assumption 2.6. It is called the axiom of infinity because it introduces the most basic example of an infinite set, namely the set of natural numbers $\mathbf{N}$. (We will formalize what finite and infinite mean in Section 3.6.) From the axiom of infinity we see that numbers such as 3, 5, 7, etc. are indeed objects in set theory, and so (from the pair set axiom and pairwise union axiom) we can indeed legitimately construct sets such as $\{3,5,9\}$ as we have been doing in our examples.

One has to keep the concept of a set distinct from the elements of that set; for instance, the set $\{n+3 : n \in \mathbf{N}, 0 \le n \le 5\}$ is not the same thing as the expression or function $n+3$. We emphasize this with an example:

Example 3.1.33. (Informal) This example requires the notion of subtraction, which has not yet been formally introduced. The following two sets are equal,

$$\{n+3 : n \in \mathbf{N}, 0 \le n \le 5\} = \{8-n : n \in \mathbf{N}, 0 \le n \le 5\}, \quad (3.1)$$

(see below), even though the expressions $n+3$ and $8-n$ are never equal to each other for any natural number n. Thus, it is a good idea to remember to use those curly braces $\{\}$ when you talk about sets, lest you accidentally confuse a set with its elements. One reason for this counter-intuitive situation is that the letter n is being used in two different ways on the two sides of (3.1). To clarify the situation, let us rewrite the set $\{8-n : n \in \mathbf{N}, 0 \leq n \leq 5\}$ by replacing the letter n by the letter m, thus giving $\{8-m : m \in \mathbf{N}, 0 \leq m \leq 5\}$. This is exactly the same set as before (why?), so we can rewrite (3.1) as

$$\{n+3 : n \in \mathbf{N}, 0 \leq n \leq 5\} = \{8-m : m \in \mathbf{N}, 0 \leq m \leq 5\}.$$

Now it is easy to see (using (3.1.4)) why this identity is true: every number of the form $n+3$, where n is a natural number between 0 and 5, is also of the form $8-m$ where $m := 5-n$ (note that m is therefore also a natural number between 0 and 5); conversely, every number of the form $8-m$, where n is a natural number between 0 and 5, is also of the form $n+3$, where $n := 5-m$ (note that n is therefore a natural number between 0 and 5). Observe how much more confusing the above explanation of (3.1) would have been if we had not changed one of the n's to an m first!

Exercise 3.1.1. Show that the definition of equality in (3.1.4) is reflexive, symmetric, and transitive.

Exercise 3.1.2. Using only Definition 3.1.4, Axiom 3.2, and Axiom 3.3, prove that the sets $\emptyset$, $\{\emptyset\}$, $\{\{\emptyset\}\}$, and $\{\emptyset, \{\emptyset\}\}$ are all distinct (i.e., no two of them are equal to each other).

Exercise 3.1.3. Prove the remaining claims in Lemma 3.1.13.

Exercise 3.1.4. Prove the remaining claims in Proposition 3.1.18.

Exercise 3.1.5. Let A, B be sets. Show that the three statements $A \subseteq B$, $A \cup B = B$, $A \cap B = A$ are logically equivalent (any one of them implies the other two).

Exercise 3.1.6. Prove Proposition 3.1.28. (Hint: one can use some of these claims to prove others. Some of the claims have also appeared previously in Lemma 3.1.13.)

Exercise 3.1.7. Let A, B, C be sets. Show that $A \cap B \subseteq A$ and $A \cap B \subseteq B$. Furthermore, show that $C \subseteq A$ and $C \subseteq B$ if and only if $C \subseteq A \cap B$. In a similar spirit, show that $A \subseteq A \cup B$ and $B \subseteq A \cup B$, and furthermore that $A \subseteq C$ and $B \subseteq C$ if and only if $A \cup B \subseteq C$.

Exercise 3.1.8. Let A, B be sets. Prove the *absorption laws* $A \cap (A \cup B) = A$ and $A \cup (A \cap B) = A$.

Exercise 3.1.9. Let A, B, X be sets such that $A \cup B = X$ and $A \cap B = \emptyset$. Show that $A = X \backslash B$ and $B = X \backslash A$.

Exercise 3.1.10. Let A and B be sets. Show that the three sets $A \backslash B$, $A \cap B$, and $B \backslash A$ are disjoint, and that their union is $A \cup B$.

Exercise 3.1.11. Show that the axiom of replacement implies the axiom of specification.

3.2 Russell's paradox (Optional)

Many of the axioms introduced in the previous section have a similar flavor: they both allow us to form a set consisting of all the elements which have a certain property. They are both plausible, but one might think that they could be unified, for instance by introducing the following axiom:

Axiom 3.8 (Universal specification). *(Dangerous!) Suppose for every object x we have a property $P(x)$ pertaining to x (so that for every x, $P(x)$ is either a true statement or a false statement). Then there exists a set $\{x : P(x) \text{ is true}\}$ such that for every object y,*

$$y \in \{x : P(x) \textit{ is true}\} \iff P(y) \textit{ is true.}$$

This axiom is also known as the *axiom of comprehension*. It asserts that every property corresponds to a set; if we assumed that axiom, we could talk about the set of all blue objects, the set of all natural numbers, the set of all sets, and so forth. This axiom also implies most of the axioms in the previous section (Exercise 3.2.1). Unfortunately, this axiom cannot be introduced into set theory, because it creates a logical contradiction known as *Russell's paradox*, discovered by the philosopher and logician Bertrand Russell (1872–1970) in 1901. The paradox runs as follows. Let $P(x)$ be the statement

$$P(x) \iff \text{“}x \text{ is a set, and } x \notin x\text{”};$$

i.e., $P(x)$ is true only when x is a set which does not contain itself. For instance, $P(\{2,3,4\})$ is true, since the set $\{2,3,4\}$ is not one of the three elements 2, 3, 4 of $\{2,3,4\}$. On the other hand, if we let S be the set of all sets (which we would know to exist from the axiom of universal specification), then since S is itself a set, it is an element of S, and so $P(S)$ is false. Now use the axiom of universal specification to create the set

$$\Omega := \{x : P(x) \text{ is true}\} = \{x : x \text{ is a set and } x \notin x\},$$

i.e., the set of all sets which do not contain themselves. Now ask the question: does Ω contain itself, i.e. is $\Omega \in \Omega$? If Ω did contain itself, then by definition this means that $P(\Omega)$ is true, i.e., Ω is a set and $\Omega \notin \Omega$. On the other hand, if Ω did not contain itself, then $P(\Omega)$ would be true, and hence $\Omega \in \Omega$. Thus in either case we have both $\Omega \in \Omega$ and $\Omega \notin \Omega$, which is absurd.

The problem with the above axiom is that it creates sets which are far too "large" - for instance, we can use that axiom to talk about the set of *all* objects (a so-called "universal set"). Since sets are themselves objects (Axiom 3.1), this means that sets are allowed to contain themselves, which is a somewhat silly state of affairs. One way to informally resolve this issue is to think of objects as being arranged in a hierarchy. At the bottom of the hierarchy are the *primitive objects* - the objects that are not sets[1], such as the natural number 37. Then on the next rung of the hierarchy there are sets whose elements consist only of primitive objects, such as $\{3,4,7\}$ or the empty set $\emptyset$; let's call these "primitive sets" for now. Then there are sets whose elements consist only of primitive objects and primitive sets, such as $\{3,4,7,\{3,4,7\}\}$. Then we can form sets out of these objects, and so forth. The point is that at each stage of the hierarchy we only see sets whose elements consist of objects at lower stages of the hierarchy, and so at no stage do we ever construct a set which contains itself.

To actually formalize the above intuition of a hierarchy of objects is actually rather complicated, and we will not do so here.

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[1] In pure set theory, there will be no primitive objects, but there will be one primitive set $\emptyset$ on the next rung of the hierarchy.

Instead, we shall simply postulate an axiom which ensures that absurdities such as Russell's paradox do not occur.

Axiom 3.9 (Regularity). *If A is a non-empty set, then there is at least one element x of A which is either not a set, or is disjoint from A.*

The point of this axiom (which is also known as the *axiom of foundation*) is that it is asserting that at least one of the elements of A is so low on the hierarchy of objects that it does not contain any of the other elements of A. For instance, if $A = \{\{3,4\}, \{3,4,\{3,4\}\}\}$, then the element $\{3,4\} \in A$ does not contain any of the elements of A (neither 3 nor 4 lies in A), although the element $\{3,4,\{3,4\}\}$, being somewhat higher in the hierarchy, does contain an element of A, namely $\{3,4\}$. One particular consequence of this axiom is that sets are no longer allowed to contain themselves (Exercise 3.2.2).

One can legitimately ask whether we really need this axiom in our set theory, as it is certainly less intuitive than our other axioms. For the purposes of doing analysis, it turns out in fact that this axiom is never needed; all the sets we consider in analysis are typically very low on the hierarchy of objects, for instance being sets of primitive objects, or sets of sets of primitive objects, or at worst sets of sets of sets of primitive objects. However it is necessary to include this axiom in order to perform more advanced set theory, and so we have included this axiom in the text (but in an optional section) for sake of completeness.

Exercise 3.2.1. Show that the universal specification axiom, Axiom 3.8, if assumed to be true, would imply Axioms 3.2, 3.3, 3.4, 3.5, and 3.6. (If we assume that all natural numbers are objects, we also obtain Axiom 3.7.) Thus, this axiom, if permitted, would simplify the foundations of set theory tremendously (and can be viewed as one basis for an intuitive model of set theory known as "naive set theory"). Unfortunately, as we have seen, Axiom 3.8 is "too good to be true"!

Exercise 3.2.2. Use the axiom of regularity (and the singleton set axiom) to show that if A is a set, then $A \notin A$. Furthermore, show that if A and B are two sets, then either $A \notin B$ or $B \notin A$ (or both).

Exercise 3.2.3. Show (assuming the other axioms of set theory) that the universal specification axiom, Axiom 3.8, is equivalent to an axiom postulating the existence of a "universal set" Ω consisting of all objects (i.e., for all objects x, we have $x \in \Omega$). In other words, if Axiom 3.8 is true, then a universal set exists, and conversely, if a universal set exists, then Axiom 3.8 is true. (This may explain why Axiom 3.8 is called the axiom of *universal* specification). Note that if a universal set Ω existed, then we would have $\Omega \in \Omega$ by Axiom 3.1, contradicting Exercise 3.2.2. Thus the axiom of foundation specifically rules out the axiom of universal specification.

3.3 Functions

In order to do analysis, it is not particularly useful to just have the notion of a set; we also need the notion of a *function* from one set to another. Informally, a function $f : X \to Y$ from one set X to another set Y is an operation which assigns to each element (or "input") x in X, a single element (or "output") $f(x)$ in Y; we have already used this informal concept in the previous chapter when we discussed the natural numbers. The formal definition is as follows.

Definition 3.3.1 (Functions). Let X, Y be sets, and let $P(x, y)$ be a property pertaining to an object $x \in X$ and an object $y \in Y$, such that for every $x \in X$, there is exactly one $y \in Y$ for which $P(x, y)$ is true (this is sometimes known as the *vertical line test*). Then we define the *function* $f : X \to Y$ *defined by* P *on the domain* X *and range* Y to be the object which, given any input $x \in X$, assigns an output $f(x) \in Y$, defined to be the unique object $f(x)$ for which $P(x, f(x))$ is true. Thus, for any $x \in X$ and $y \in Y$,

$$y = f(x) \iff P(x, y) \text{ is true.}$$

Functions are also referred to as *maps* or *transformations*, depending on the context. They are also sometimes called *morphisms*, although to be more precise, a morphism refers to a more general class of object, which may or may not correspond to actual functions, depending on the context.

Example 3.3.2. Let $X = \mathbf{N}$, $Y = \mathbf{N}$, and let $P(x, y)$ be the property that $y = x{+}{+}$. Then for each $x \in \mathbf{N}$ there is exactly one y for which $P(x, y)$ is true, namely $y = x{+}{+}$. Thus we can define a function $f : \mathbf{N} \to \mathbf{N}$ associated to this property, so that $f(x) = x{+}{+}$ for all x; this is the *increment* function on $\mathbf{N}$, which takes a natural number as input and returns its increment as output. Thus for instance $f(4) = 5$, $f(2n + 3) = 2n + 4$ and so forth. One might also hope to define a *decrement* function $g : \mathbf{N} \to \mathbf{N}$ associated to the property $P(x, y)$ defined by $y{+}{+} = x$, i.e., $g(x)$ would be the number whose increment is x. Unfortunately this does not define a function, because when $x = 0$ there is no natural number y whose increment is equal to x (Axiom 2.3). On the other hand, we can legitimately define a decrement function $h : \mathbf{N}\backslash\{0\} \to \mathbf{N}$ associated to the property $P(x, y)$ defined by $y{+}{+} = x$, because when $x \in \mathbf{N}\backslash\{0\}$ there is indeed exactly one natural number y such that $y{+}{+} = x$, thanks to Lemma 2.2.10. Thus for instance $h(4) = 3$ and $h(2n+3) = h(2n+2)$, but $h(0)$ is undefined since 0 is not in the domain $N\backslash\{0\}$.

Example 3.3.3. (Informal) This example requires the real numbers $\mathbf{R}$, which we will define in Chapter 5. One could try to define a square root function $\sqrt{} : \mathbf{R} \to \mathbf{R}$ by associating it to the property $P(x, y)$ defined by $y^2 = x$, i.e., we would want $\sqrt{x}$ to be the number y such that $y^2 = x$. Unfortunately there are two problems which prohibit this definition from actually creating a function. The first is that there exist real numbers x for which $P(x, y)$ is never true, for instance if $x = -1$ then there is no real number y such that $y^2 = x$. This problem however can be solved by restricting the domain from $\mathbf{R}$ to the right half-line $[0, +\infty)$. The second problem is that even when $x \in [0, +\infty)$, it is possible for there to be more than one y in the range $\mathbf{R}$ for which $y^2 = x$, for instance if $x = 4$ then both $y = 2$ and $y = -2$ obey the property $P(x, y)$, i.e., both $+2$ and -2 are square roots of 4. This problem can however be solved by restricting the range of $\mathbf{R}$ to $[0, +\infty)$. Once one does this, then one can correctly define a square root function $\sqrt{} : [0, +\infty) \to [0, +\infty)$ using the relation $y^2 = x$, thus $\sqrt{x}$ is the unique number $y \in [0, +\infty)$ such that $y^2 = x$.

One common way to define a function is simply to specify its domain, its range, and how one generates the output $f(x)$ from each input; this is known as an *explicit* definition of a function. For instance, the function f in Example 3.3.2 could be defined explicitly by saying that f has domain and range equal to $\mathbf{N}$, and $f(x) := x{+}{+}$ for all $x \in \mathbf{N}$. In other cases we only define a function f by specifying what property $P(x, y)$ links the input x with the output $f(x)$; this is an *implicit* definition of a function. For instance, the square root function $\sqrt{x}$ in Example 3.3.3 was defined implicitly by the relation $(\sqrt{x})^2 = x$. Note that an implicit definition is only valid if we know that for every input there is exactly one output which obeys the implicit relation. In many cases we omit specifying the domain and range of a function for brevity, and thus for instance we could refer to the function f in Example 3.3.2 as "the function $f(x) := x{+}{+}$", "the function $x \mapsto x{+}{+}$", "the function $x{+}{+}$", or even the extremely abbreviated "$++$". However, too much of this abbreviation can be dangerous; sometimes it is important to know what the domain and range of the function is.

We observe that functions obey the axiom of substitution: if $x = x'$, then $f(x) = f(x')$ (why?). In other words, equal inputs imply equal outputs. On the other hand, unequal inputs do not necessarily ensure unequal outputs, as the following example shows:

Example 3.3.4. Let $X = \mathbf{N}$, $Y = \mathbf{N}$, and let $P(x, y)$ be the property that $y = 7$. Then certainly for every $x \in \mathbf{N}$ there is exactly one y for which $P(x, y)$ is true, namely the number 7. Thus we can create a function $f : \mathbf{N} \to \mathbf{N}$ associated to this property; it is simply the *constant function* which assigns the output of $f(x) = 7$ to each input $x \in \mathbf{N}$. Thus it is certainly possible for different inputs to generate the same output.

Remark 3.3.5. We are now using parentheses () to denote several different things in mathematics; on one hand, we are using them to clarify the order of operations (compare for instance $2 + (3 \times 4) = 14$ with $(2 + 3) \times 4 = 20$), but on the other hand we also use parentheses to enclose the argument $f(x)$ of a function or of a property such as $P(x)$. However, the two usages of parentheses usually are unambiguous from context. For instance, if a is a number, then $a(b+c)$ denotes the expression $a \times (b+c)$, whereas if f is a function, then $f(b+c)$ denotes the output of f when the input is $b+c$. Sometimes the argument of a function is denoted by subscripting instead of parentheses; for instance, a sequence of natural numbers $a_0, a_1, a_2, a_3, \ldots$ is, strictly speaking, a function from $\mathbf{N}$ to $\mathbf{N}$, but is denoted by $n \mapsto a_n$ rather than $n \mapsto a(n)$.

Remark 3.3.6. Strictly speaking, functions are not sets, and sets are not functions; it does not make sense to ask whether an object x is an element of a function f, and it does not make sense to apply a set A to an input x to create an output $A(x)$. On the other hand, it is possible to start with a function $f : X \to Y$ and construct its *graph* $\{(x, f(x)) : x \in X\}$, which describes the function completely: see Section 3.5.

We now define some basic concepts and notions for functions. The first notion is that of equality.

Definition 3.3.7 (Equality of functions). Two functions $f : X \to Y$, $g : X \to Y$ with the same domain and range are said to be *equal*, $f = g$, if and only if $f(x) = g(x)$ for *all* $x \in X$. (If $f(x)$ and $g(x)$ agree for some values of x, but not others, then we do not consider f and g to be equal[2].)

Example 3.3.8. The functions $x \mapsto x^2+2x+1$ and $x \mapsto (x+1)^2$ are equal on the domain $\mathbf{R}$. The functions $x \mapsto x$ and $x \mapsto |x|$ are equal on the positive real axis, but are not equal on $\mathbf{R}$; thus the concept of equality of functions can depend on the choice of domain.

Example 3.3.9. A rather boring example of a function is the *empty function* $f : \emptyset \to X$ from the empty set to an arbitrary set X. Since the empty set has no elements, we do not need

[2]In Chapter 19, we shall introduce a weaker notion of equality, that of two functions being *equal almost everywhere*.