A concrete category is a category whose objects are sets with structure and whose arrows are the structure preserving functions between them. In concrete categories, 1_A is the identity map on A, and $\circ$ is function composition.

EXAMPLE 32.1
Let A be any set. Let the class of objects be A. Let the class of arrows be A. Let $S(a) = T(a) = a$ for all $a \in A$. Let 1_a be a and let $a \circ a = a$. Then the conditions for a category are satisfied and we have what is called the *discrete* category corresponding to A. So a set may be viewed as a category.

EXAMPLE 32.2
Let $(A, 1, \cdot)$ be a monoid (with special object 1 and binary operation $\cdot$). For the class of objects take the singleton set $\{*\}$. For the class of arrows, take A. Let $S(a) = T(a) = *$ for all $a \in A$. Let 1_* be 1 (the monoid identity) and let $a \circ b$ be $a \cdot b$. The graph we have just constructed is a category, thanks to the structure of the monoid. In this way a monoid may be viewed as a category. If we consider the monoid of the natural numbers with addition, in this way, as a category, then what is the number 2? It is the unique arrow in the natural number monoid, viewed as a category, which can be written as a composition of nonidentity arrows in exactly one way.

EXAMPLE 32.3
Let $(A, \leq)$ be a pre-ordered set. Let the class of objects be A, and let the class of arrows be $\{(a, b) | a \leq b\}$. (Here a and b are assumed to be elements of A.) Let $S((a, b)) = a$ and $T((a, b)) = b$. By the reflexivity of $\leq$, (a, a) is an arrow for all $a \in A$. Let this be the identity arrow associated with a, so that $1_a = (a, a)$. Define the composition of arrows thus: $(b, c) \circ (a, b) = (a, c)$. By transitivity of $\leq$, we know that (a, c) is an arrow if (a, b) and (b, c) are arrows. Again, we have a category.

The examples from Chapter 31 show that *many interesting objects in mathematics congregate in categories.* The examples in this chapter illustrate that *interesting mathematical entities may often be viewed as categories.*

Exercise

1. Show that there is a category with exactly two objects and exactly one nonidentity arrow. (This category is sometimes said to *be* the number 2.)

33

Functors

If A and B are two categories, a *functor* F from A to B is a mapping sending objects of A to objects of B and, at the same time, a mapping sending arrows of A to arrows of B, so that

1. if g is any arrow in A with source a and target a', then $F(g)$ is an arrow in B with source $F(a)$ and target $F(a')$;
2. $F(1_a) = 1_{F(a)}$;
3. $F(g \circ h) = F(g) \circ F(h)$.

We saw in Chapter 32 that sets, monoids and pre-ordered sets may all be viewed as categories. How do the structure preserving mappings between such entities compare with the functors between them when they are viewed as categories? We answer the question as follows.

EXAMPLE 33.1

Suppose A and B are sets, each viewed as a category (Example 32.1). Let F be any map from A to B. If $a \in A$, then F maps a to $F(a)$. But a is just 1_a and $F(a)$ is just $1_{F(a)}$. Hence $F(1_a) = 1_{F(a)}$. If a and b are any elements of A, then they are also arrows of A. If $a \neq b$ then $S(a) = T(a) = a \neq b = S(b) = T(b)$, so they cannot be composed. If $a = b$ then $F(a \circ a) = F(a) = F(a) \circ F(a)$. Hence F is a functor.

EXAMPLE 33.2

Suppose A and B are monoids, each viewed as a category. Let F be a monoid homomorphism from A to B. Without loss of generality, we may suppose that $\{*\}$ is the class of objects for both categories A and B (see Example 32.2). Suppose $F(*) = *$. Since F is a homomorphism, $F(1_*) = 1_{F(*)}$. Moreover, $F(a \circ a') = F(a \cdot a') = F(a) \cdot F(a') = F(a) \circ F(a')$. Hence F is a functor.

EXAMPLE 33.3

A monotone mapping between pre-ordered sets may be viewed as a functor. The details are left as an exercise.

The next three examples illustrate the observation that *many entities of interest in mathematics may be viewed as functors.*

EXAMPLE 33.4

A set may be viewed as a category, as we saw in Example 32.1. It can also be viewed as a functor. Let A be the discrete one-element category and B the category of sets. For any set S there is a unique functor from A to B such that $F(1_A)$ is the identity function on S. This functor can be viewed as the set S.

EXAMPLE 33.5

Let A be a category with two objects, $\mathbf{a}$ and $\mathbf{o}$, and with four arrows: $1_{\mathbf{a}}$, $1_{\mathbf{o}}$, $s : \mathbf{a} \to \mathbf{o}$, and $t : \mathbf{a} \to \mathbf{o}$. This category may be pictured thus:

$$\mathbf{a} \ \underset{\rightarrow}{\overset{\rightarrow}{}} \ \mathbf{o}$$

Suppose F maps $\mathbf{a}$ to a set X and $\mathbf{o}$ to a set Y. Suppose $F(s)$ and $F(t)$ are functions with domain X and codomain Y. Then F is a functor from A to the category of sets. We can think of this functor as a graph with class of objects Y, class of arrows X, source mapping $F(s)$ and target mapping $F(t)$.

EXAMPLE 33.6

Let M be a monoid and X a set. Suppose $m : M \times X \to X$ is a function such that, for elements a and a' of M and b of X, $m(a \cdot a', b) = m(a, m(a', b))$. Suppose also that $m(1, b) = b$. Then (M, X, m) is an *M-set.* For example, M might be the monoid of positive integers with multiplication, and X might be the set of segments constructible in Euclidean geometry, and $m(a, CD)$ might be the function mapping a segment CD to