Exercises

1. Find the 25 primes less than 100 and express 100 as the sum of two primes.

2. Prove that there exist 1,000 consecutive positive integers none of which is prime. (Hint: start with $1001! + 2$.)

3. Prove that there are infinitely many prime numbers of the form

$$4m - 1.$$

(Hint: consider $n = 4q_1q_2q_3 \cdots q_k - 1$, where the q_i are primes of the form $4m - 1$, and show that not all prime divisors of n can be of the form $4m + 1$.)

4

Sumerian-Babylonian Mathematics

The Sumerians were a people of unknown linguistic affinity, who lived in the southern part of Mesopotamia (Iraq), and whose civilization was absorbed by the Semitic Babylonians around 2000 BC. Babylonian culture reached its peak in about 575 BC, under Nebuchadnezzar, but most of the mathematical achievements we shall discuss in this chapter and in Chapter 5 are much older, going back as far as 2000 BC — about the time when the biblical patriarch Abraham was said to have been born in the Sumerian city of Ur.

As we shall see, Mesopotamian mathematics is quite impressive. However, we should remember that, like the ancient Egyptians, the Mesopotamians never gave what we would call 'proofs' for their results; the first people to do so were the Greeks.

In representing numbers up to (and including) 59, the Sumerians and Babylonians used a decimal system. For example, they wrote 35 as follows, where we have approximated the original cuneiform figures by ours:

$$\begin{array}{cc} <<< & YYY \\ & YY \end{array}$$

On the other hand, 60 is again denoted by Y, and so is 60^2, as well as 60^{-1}, 60^{-2}, etc. It is usually clear from the context which is meant. Here are some further examples:

$$<<< \quad = \quad 30, \text{ or } 30/60 = 1/2;$$

$$< YY \quad = \quad 12, \text{ or } 1/5;$$

$$Y << \begin{matrix} Y & Y & Y \\ & Y & \end{matrix} \quad = \quad 84, \text{ or } 7/5.$$

The Babylonian use of scale 60 was taken over into Greek astronomy around 150 BC by Hipparchus of Nicaea and it is still used today in measuring time and angles. To remove ambiguities in the above three examples, we would write

$$30^\circ \text{ or } 30',$$

$$12^\circ \text{ or } 12',$$

$$1^\circ 24' \text{ or } 1'24''.$$

The scale 60, or *sexagesimal system*, was also employed for weights of silver: 60 shekels = 1 mina; 60 minas = 1 talent. The prophet Ezekiel, living in Babylon, wrote in 573 BC:

> The Lord Yahweh says this: ... Twenty shekels, twenty-five shekels and fifteen shekels are to make one mina (*Ezekiel* 45:9–12).

The later Babylonians even introduced a symbol for zero:

$$Y \begin{matrix} < \\ < \end{matrix} \begin{matrix} Y & Y & Y \\ & Y & \end{matrix} = 60^2 + 4 = 3604.$$

Ptolemy (150 AD) replaced this symbol by a small circle, probably from the Greek word 'ouden', meaning 'nothing'.

In order to divide, the Babylonians made use of the fact that $a/b = a{\cdot}b^{-1}$. To this end, they constructed tables of inverses, like the one given in Table 4.1 (taken from Neugebauer [1969]). Note that the scribe did not list the inverses of any integers having a prime factor other than 2, 3 or 5. It seems he was afraid of repeating sexagesimals!

The Babylonians also had tables of squares, cubes, square roots, cube roots, and even roots of the equations

$$x^2(x \pm 1) = a.$$

Their method for extracting square roots is sometimes called *Heron's method* after Heron of Alexandria (60 AD), who included it in his *Metrica*. Let a_1 be a rational number between $\sqrt{a}$ and $\sqrt{a}+1$, where a is a positive non-square integer; let $a_{n+1} = (a_n + a/a_n)/2$; then $a_n \to \sqrt{a}$ as $n \to \infty$. Indeed, if $e = a_1 - \sqrt{a}$, we have $0 < e < 1$ and

$$0 < a_{n+1} - \sqrt{a} < 2\sqrt{a}(e/2\sqrt{a})^{2^n}$$

(see Exercise 4). As $n \to \infty$, this tends to 0.

b	1/b	b	1/b
2	30′	16	3′45″
3	20′	18	3′20″
4	15′	20	3′
5	12′	24	2′30″
6	10′	25	2′24″
8	7′30′	27	2′13″20″
9	6′40′	30	2′
10	6′	32	1′52″30″
12	5′	36	1′40″
15	4′	40	1′30″

TABLE 4.1. Mesopotamian table of inverses (scale 60)

For example, if $a = 2$, $a_1 = 3/2$, then $a_2 = 17/12$ and $a_3 = 577/408$. In sexagesimal notation, $577/408 = 1°24'51''10'''35''''\ldots$. The fourth approximation $a_4 = 665857/470832$, which is $1°24'51''10'''7''''\ldots$ in sexagesimal notation. The difference between a_4 and $\sqrt{2}$ is less than

$$2\sqrt{2}\left(\frac{3/2-\sqrt{2}}{2\sqrt{2}}\right)^{2^4} < 10^{-23}.$$

The Babylonian tablet YBC7289, dating from about 1600 BC, gives $\sqrt{2}$ as $1°24'51''10'''$.

Exercises

1. Write 5000 in the Babylonian manner. (You may use our degrees, minutes and seconds.)

2. Let a/b be a proper, reduced fraction (with a and b positive integers). Let $e_1 = 60a/b$ and $e_{n+1} = 60(e_n - [e_n])$ – where $[e_n]$ is the greatest integer less than or equal to e_n. Prove that the Babylonian sexagesimal expansion for a/b is

$$(.[e_1][e_2][e_3]\ldots)_{60}\,.$$

3. Express $1/7$ as a repeating sexagesimal.

4. Prove by mathematical induction that

$$0 < a_{n+1} - \sqrt{a} < 2\sqrt{a}(e/2\sqrt{a})^{2^n}.$$

5. Use the Babylonian method to find $\sqrt{3}$ to within 60^{-10}.

6. Let a/b be a proper, reduced fraction (with a and b positive integers). Prove that a/b has an infinitely repeating sexagesimal expansion if and only if b has a prime factor which does not divide 60.