APPENDIX I

Cartesian Products and Zorn's Lemma

Section 1 of this appendix contains the definition of the Cartesian product of an arbitrary collection of sets. In the text we shall primarily be interested in products of finitely many (or occasionally countably many) sets. We indicate how the general definition agrees with the familiar "ordered n-tuple" notion of a Cartesian product in these cases. Section 2 contains a discussion of Zorn's Lemma and related topics.

1. CARTESIAN PRODUCTS

A set I is called an *indexing set* or *index set* if the elements of I are used to index some collection of sets. In particular, if A and I are sets, we can form the collection $\{A_i \mid i \in I\}$ by specifying that $A_i = A$ for all $i \in I$. Thus *any* set can be an indexing set; we use this term to emphasize that the elements are used as indices.

Definition.

(1) Let I be an indexing set and let $\{A_i \mid i \in I\}$ be a collection of sets. A *choice function* is any function

$$f : I \to \bigcup_{i \in I} A_i$$

such that $f(i) \in A_i$ for all $i \in I$.

(2) Let I be an indexing set and for all $i \in I$ let A_i be a set. The *Cartesian product* of $\{A_i \mid i \in I\}$ is the set of all choice functions from I to $\cup_{i \in I} A_i$ and is denoted by $\prod_{i \in I} A_i$ (where if either I or any of the sets A_i are empty the Cartesian product is the empty set). The elements of this Cartesian product are written as $\prod_{i \in I} a_i$, where this denotes the choice function f such that $f(i) = a_i$ for each $i \in I$.

(3) For each $j \in I$ the set A_j is called the j^{th} *component* of the Cartesian product $\prod_{i \in I} A_i$ and a_j is the j^{th} *coordinate* of the element $\prod_{i \in I} a_i$.

(4) For $j \in I$ the *projection map* of $\prod_{i \in I} A_i$ onto the j^{th} coordinate, A_j, is defined by $\prod_{i \in I} a_i \mapsto a_j$.

Each choice function f in the Cartesian product $\prod_{i \in I} A_i$ may be thought of as a way of "choosing" an element $f(i)$ from each set A_i.

If $I = \{1, 2, \dots, n\}$ for some $n \in \mathbb{Z}^+$ and if f is a choice function from I to $A_1 \cup \cdots \cup A_n$, where each A_i is nonempty, we can associate to f a unique (ordered) n-tuple:

$$f \to (f(1), f(2), \dots, f(n)).$$

Note that by definition of a choice function, $f(i) \in A_i$ for all i, so the n-tuple above has an element of A_i in the i^{th} position for each i .

Conversely, given an n-tuple $(a_1, a_1, \ldots, a_n)$, where $a_i \in A_i$ for all $i \in I$, there is a unique choice function, f, from I to $\cup_{i \in I} A_i$ associated to it, namely

$$f(i) = a_i, \qquad \text{for all } i \in I.$$

It is clear that this map from n-tuples to choice functions is the inverse to the map described in the preceding paragraph. Thus *there is a bijection between ordered n-tuples and elements of* $\prod_{i \in I} A_i$. Henceforth when $I = \{1, 2, \ldots, n\}$ we shall write

$$\prod_{i=1}^{n} A_i \quad \text{or} \quad A_1 \times A_2 \times \cdots \times A_n$$

for the Cartesian product and we shall describe the elements as ordered n-tuples.

If $I = \mathbb{Z}^+$, we shall similarly write: $\prod_{i=1}^{\infty} A_i$ or $A_1 \times A_2 \times \cdots$ for the Cartesian product of the A_i's. We shall write the elements as ordered tuples: $(a_1, a_2, \ldots)$, i.e., as infinite sequences whose i^{th} terms are in A_i.

Note that when $I = \{1, 2, \ldots, n\}$ or $I = \mathbb{Z}^+$ we have used the natural ordering on I to arrange the elements of our Cartesian products into n-tuples. Any other ordering of I (or any ordering on a finite or countable index set) gives a different representation of the elements of the same Cartesian product.

Examples

(1) $A \times B = \{(a, b) \mid a \in A, b \in B\}$.

(2) $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ (n factors) is the usual set of n-tuples with real number entries, Euclidean n-space.

(3) Suppose $I = \mathbb{Z}^+$ and A_i is the same set A, for all $i \in I$. The Cartesian product $\prod_{i \in \mathbb{Z}^+} A$ is the set of all (infinite) sequences $a_1, a_2, a_3 \ldots$ of elements of A. In particular, if $A = \mathbb{R}$, then the Cartesian product $\prod_{i \in \mathbb{Z}^+} \mathbb{R}$ is the set of all real sequences.

(4) Suppose I is any indexing set and A_i is the same set A, for all $i \in I$. The Cartesian product $\prod_{i \in I} A$ is just the set of all functions from I to A, where the function $f : I \to A$ corresponds to the element $\prod_{i \in I} f(i)$ in the Cartesian product. This Cartesian product is often (particularly in topology books) denoted by A^I. Note that for each fixed $j \in I$ the projection map onto the j^{th} coordinate sends the function f to $f(j)$, i.e., is evaluation at j.

(5) Let R be a ring and let x be an indeterminate over R. The definition of the ring $R[x]$ of polynomials in x with coefficients from R may be given in terms of Cartesian products rather than in the more intuitive and familiar terms of "formal sums" (in Chapters 7 and 9 we introduced them in the latter form since this is the way we envision and work with them). Let I be the indexing set $\mathbb{Z}^+ \cup \{0\}$ and let $R[x]$ be the subset of the Cartesian product $\prod_{i=0}^{\infty} R$ consisting of elements $(a_0, a_1, a_2, \ldots)$ such that only finitely many of the a_i's are nonzero. If $(a_0, a_1, a_2, \ldots, a_n, 0, 0, \ldots)$ is such a sequence we represent it by the more familiar "formal sum" $\sum_{i=0}^{n} a_i x^i$. Addition and multiplication of these sequences is defined so that the usual rules for addition and multiplication of polynomials hold.

Proposition 1. Let I be a nonempty countable set and for each $i \in I$ let A_i be a set. The cardinality of the Cartesian product is the product of the cardinalities of the sets A_i, i.e.,

$$|\prod_{i \in I} A_i| = \prod_{i \in I} |A_i|,$$

(where if some A_i is an infinite set or if I is infinite and an infinite number of A_i's have cardinality ≥ 2, both sides of this equality are infinity). In particular,

$$|A_1 \times A_2 \times \cdots \times A_n| = |A_1| \times |A_2| \times \cdots \times |A_n|.$$

Proof: In order to count the number of choice functions note that each $i \in I$ may be mapped to any of the $|A_i|$ elements of A_i and for $i \neq j$ the values of choice functions at i and j may be chosen completely independently. Thus the number of choice functions is the product of the cardinalities of the A_i's, as claimed.

For Cartesian products of finitely many sets, $A_1 \times A_2 \times \cdots \times A_n$, one can see this easily from the n-tuple representation: the elements of $A_1 \times A_2 \times \cdots \times A_n$ are n-tuples $(a_1, a_2, \ldots, a_n)$ and each a_i may be chosen as any of the $|A_i|$ elements of A_i. Since these choices are made independently for $i \neq j$, there are $|A_1| \cdot |A_2| \cdots |A_n|$ elements in the Cartesian product.

EXERCISE

1. Let I and J be any two indexing sets and let A be an arbitrary set. For any function $\varphi : J \to I$ define

$$\varphi^* : \prod_{i \in I} A \to \prod_{j \in J} A \quad \text{by} \quad \varphi^*(f) = f \circ \varphi \quad \text{for all choice functions } f \in \prod_{i \in I} A.$$

   (a) Let $I = \{1, 2\}$, let $J = \{1, 2, 3\}$ and let $\varphi : J \to I$ be defined by $\varphi(1) = 2$, $\varphi(2) = 2$ and $\varphi(3) = 1$. Describe explicitly how a 3-tuple in $A \times A \times A$ maps to an ordered pair in $A \times A$ under this φ^*.
   (b) Let $I = J = \{1, 2, \ldots, n\}$ and assume φ is a permutation of I. Describe in terms of n-tuples in $A \times A \times \cdots \times A$ the function φ^*.

2. PARTIALLY ORDERED SETS AND ZORN'S LEMMA

We shall have occasion to use Zorn's Lemma as a form of "infinite induction" in a few places in the text where it is desirable to know the existence of some set which is *maximal* with respect to certain specified properties. For example, Zorn's Lemma is used to show that every vector space has a basis. In this situation a basis of a vector space V is a subset of V which is maximal as a set consisting of linearly independent vectors (the maximality ensures that these vectors span V). For finite dimensional spaces this can be proved by induction; however, for spaces of arbitrary dimension Zorn's Lemma is needed to establish this. By having results which hold in full generality the theory often becomes a little neater in places, although the main results of the text do not require its use.

A specific instance in the text where a maximal object which helps to simplify matters is constructed by Zorn's Lemma is the algebraic closure of a field. An algebraic closure of a field F is an extension of F which is maximal among any collection of algebraic extensions. Such a field contains (up to isomorphism) all elements which are algebraic over F, hence all manipulations involving such algebraic elements can be effected in this one larger field. In any particular situation the use of an algebraic closure can be avoided by adjoining the algebraic elements involved to the base field F, however this becomes tedious (and often obscures matters) in complicated proofs. For the specific fields appearing as examples in this text the use of Zorn's Lemma to construct an algebraic closure can be avoided (for example, the construction of an algebraic closure of any subfield of the complex numbers or of any finite field does not require it).

The first example of the use of Zorn's Lemma appears in the proof of Proposition 11 in Section 7.4.

In order to state Zorn's Lemma we need some terminology.

Definition. A *partial order* on a nonempty set A is a relation $\leq$ on A satisfying

(1) $x \leq x$ for all $x \in A$ (reflexive),

(2) if $x \leq y$ and $y \leq x$ then $x = y$ for all $x, y \in A$ (antisymmetric),

(3) if $x \leq y$ and $y \leq z$ then $x \leq z$ for all $x, y, z \in A$ (transitive).

We shall usually say that A is a partially ordered set under the ordering $\leq$ or that A is partially ordered by $\leq$.

Definition. Let the nonempty set A be partially ordered by $\leq$.

(1) A subset B of A is called a *chain* if for all $x, y \in B$, either $x \leq y$ or $y \leq x$.

(2) An *upper bound* for a subset B of A is an element $u \in A$ such that $b \leq u$, for all $b \in B$.

(3) A *maximal element* of A is an element $m \in A$ such that if $m \leq x$ for any $x \in A$, then $m = x$.

In the literature a chain is also called a *tower* or called a *totally ordered* or *linearly ordered* or *simply ordered* subset.

Some examples below highlight the distinction between upper bounds and maximal elements. Also note that if m is a *maximal* element of A, it is not necessarily the case that $x \leq m$ for all $x \in A$ (i.e., m is not necessarily a *maximum* element).

Examples

(1) Let A be the power set (i.e., set of all subsets) of some set X and $\leq$ be set containment: $\subseteq$. Notice that this is only a *partial* ordering since some subsets of X may not be comparable, e.g. singletons: if $x \neq y$ then $\{x\} \not\subseteq \{y\}$ and $\{y\} \not\subseteq \{x\}$. In this situation an example of a chain is a collection of subsets of X such as

$$X_1 \subseteq X_2 \subseteq X_3 \subseteq \cdots.$$

Any subset B of A has an upper bound, b, namely,

$$b = \bigcup_{x \in B} x.$$