N. Since there are only countably many finite strings of symbols, there are only countably many expressions of this form. Call them $\alpha_1, \alpha_2, \ldots$.

We next enumerate all the proofs of L. There are only countably many — assuming that proofs are finite — since there are only countably many finite sequences of formulas. Call the proofs $P_1, P_2, \ldots$.

Let $R(m,n)$ be the relation which is satisfied just in case P_n is a proof of the formula $S^m0 \in \alpha_m$. $R(m,n)$ is recursive, since we can decide whether it is true for a given m and n by looking to see whether $S^m0 \in \alpha_m$ is the last line of the proof P_n.

By Gödel's Lemma, there is a formula $F(x,y)$ such that, if $R(m,n)$, then $\vdash F(S^m0, S^n0)$ and, if not $R(m,n)$, then $\vdash \neg F(S^m0, S^n0)$. Consider $\{x \in N | \neg\exists_{y\in N} F(x,y)\}$. This is one of the $\alpha_1, \alpha_2, \ldots$, say α_g. Then, from the definition of α_g, it is provable in L that

$$S^g0 \in \alpha_g \text{ if } \neg\exists_{y\in N} F(S^g0, y). \qquad (\clubsuit)$$

In both intuitionistic and ordinary logic, it follows that

$$\vdash \neg S^g0 \in \alpha_g \iff \vdash \exists_{y\in N} F(S^g0, y).$$

Consider the formula $S^g0 \in \alpha_g$. If it is provable in L, then there is some natural number n such that P_n is a proof of $S^g0 \in \alpha_g$. In that case, $R(g,n)$. But then $\vdash F(S^g0, S^n0)$, and hence $\vdash \exists_{y\in N} F(S^g0, y)$. We thus have $\vdash \neg S^g0 \in \alpha_g$.

Hence it is not the case that $\vdash S^g0 \in \alpha_g$ (assuming L is consistent). That is, for all n, not $R(g,n)$, and hence, for all n, $\vdash \neg F(S^g0, S^n0)$ (by Gödel's Lemma). Call this result $(*)$, for future reference.

From the soundness result, we may conclude that, for all n, $\neg F(S^g0, S^n0)$ is true in all models of L. Hence in any ω-complete model of L, the proposition $\forall_{y\in N} \neg F(S^g0, y)$ is true.

Even intuitionistic logic allows us to infer $\neg\exists_y G(y)$ from $\forall_y \neg G(y)$. Hence we may conclude that $\neg\exists_{y\in N} F(S^g0, y)$ is true in any ω-complete model, and thus $S^g0 \in \alpha_g$ is true in any such model by ($\clubsuit$). Thus, although $S^g0 \in \alpha_g$ is not provable in L (since L is consistent), $S^g0 \in \alpha_g$ is nonetheless true in all ω-complete models.

Exercises

1. Prove the two claims made about inferences in intuitionistic logic.
2. If $R(m,n)$ means 'm and n both equal 3', find the corresponding formula $F(x,y)$ whose existence is asserted by Gödel's Lemma.

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More about Gödel's Theorems

We say that a language L is *ω-consistent* provided, for any formula $A(x)$ of L, if $A(S^n 0)$ is provable for each natural number n, then it is not the case that $\vdash \neg\forall_{y\in N} A(y)$. It is not hard to show that ω-consistency implies consistency.

If L is ω-consistent, then it follows from $(*)$ in the previous chapter that it is not the case that

$$\vdash \neg\forall_{y\in N}\neg F(S^g 0, y).$$

From ($\clubsuit$) in the previous chapter, it now follows that it is not the case that $\vdash \neg S^g 0 \in a_g$. This gives us

Theorem 30.1.
(Gödel's Incompleteness Theorem (Syntactic Version))
If L is ω-consistent, there is a formula G such that neither G nor $\neg G$ is provable in L.

Rosser showed that ω-consistency here can be replaced by plain consistency. His proof is short, but tricky, and we shall skip it.

If neither a statement nor its negation is provable in a language, it is called *undecidable* relative to that language. There is no way to get rid of all undecidable statements in a language by adjoining a finite number of new axioms. If we added $S^g 0 \in a_g$ to the axioms of L, then there would still be a g' such that $S^{g'} 0 \in a_{g'}$ is an undecidable statement relative to this new language.

Hilbert's second problem was to prove the consistency of arithmetic using only formal arithmetic to do so. Gödel's Incompleteness Theorem implies that this is impossible. For let *Cons* be a statement in L which expresses the idea that there is no n such that P_n is a proof of $\perp$. Then, if L is consistent, and *if there is a proof in L of this*, we shall have $\vdash$ *Cons*. If L is not consistent, we can prove anything, so we can have $\vdash \neg Cons$ as well as $\vdash$ *Cons*.

One can formalize the proof of Gödel's Incompleteness Theorem to show that $\vdash Cons \Rightarrow S^g 0 \in a_g$. Hence, if there were a proof in L that L was consistent, namely, if $\vdash$ *Cons*, then we would have $\vdash S^g 0 \in a_g$. But this we have seen is not the case.

Gödel's Incompleteness Theorem, being a metamathematical result, has different implications depending on one's conception of mathematics. A classical *formalist* views arithmetic as nothing more than strings of symbols, manipulated according to certain rules. He does not want to rely on any interpretation of these symbols in order to ensure the consistency of the rules. He would hope that consistency could be established within the system itself. Gödel's Theorem shows, against the hopes of the classical formalist, that the consistency of arithmetic cannot be demonstrated within arithmetic. Moreover, it seems that the idea of truth cannot be captured by the notion of provability.

From an *intuitionist* point of view, Gödel's Incompleteness Theorem is not unwelcome. For intuitionists, 'true' should mean 'provable' (with a finite proof). They maintain that there are some statements which are neither true nor false; Gödel's result merely confirms this belief by showing that there are statements which are neither provable nor disprovable.

For the classical *Platonist*, the Incompleteness Theorem shows that there are statements true in the real world which are not provable. In other words, the realities of mathematics are too profound to be captured by any finite axiom system. There will always be truths in mathematics which cannot be cranked out by a computer, but which must await new philosophical insights for their discovery.