

$\psi : H \rtimes_{\varphi_1} K \rightarrow H \rtimes_{\varphi_2} K$ defined by $\psi((h, k)) = (\sigma(h), k^a)$ is a homomorphism. Show ψ is bijective by constructing a 2-sided inverse.]

7. This exercise describes thirteen isomorphism types of groups of order 56. (It is not too difficult to show that every group of order 56 is isomorphic to one of these.)

- (a) Prove that there are three abelian groups of order 56.
- (b) Prove that every group of order 56 has either a normal Sylow 2-subgroup or a normal Sylow 7-subgroup.
- (c) Construct the following non-abelian groups of order 56 which have a normal Sylow 7-subgroup and whose Sylow 2-subgroup S is as specified:
 - one group when $S \cong Z_2 \times Z_2 \times Z_2$
 - two nonisomorphic groups when $S \cong Z_4 \times Z_2$
 - one group when $S \cong Z_8$
 - two nonisomorphic groups when $S \cong Q_8$
 - three nonisomorphic groups when $S \cong D_8$.

[For a particular S , two groups are not isomorphic if the kernels of the maps from S into $\text{Aut}(Z_7)$ are not isomorphic.]

- (d) Let G be a group of order 56 with a nonnormal Sylow 7-subgroup. Prove that if S is the Sylow 2-subgroup of G then $S \cong Z_2 \times Z_2 \times Z_2$. [Let an element of order 7 act by conjugation on the seven nonidentity elements of S and deduce that they all have the same order.]
- (e) Prove that there is a unique group of order 56 with a nonnormal Sylow 7-subgroup. [For existence use the fact that $|GL_3(\mathbb{F}_2)| = 168$; for uniqueness use Exercise 6.]

8. Construct a non-abelian group of order 75. Classify all groups of order 75 (there are three of them). [Use Exercise 6 to show that the non-abelian group is unique.] (The classification of groups of order pq^2 , where p and q are primes with $p < q$ and p not dividing $q - 1$, is quite similar.)

9. Show that the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 4 \end{pmatrix}$ is an element of order 5 in $GL_2(\mathbb{F}_{19})$. Use this matrix to construct a non-abelian group of order 1805 and give a presentation of this group. Classify groups of order 1805 (there are three isomorphism types). [Use Exercise 6 to prove uniqueness of the non-abelian group.] (A general method for finding elements of prime order in $GL_n(\mathbb{F}_p)$ is described in the exercises in Section 12.2; this particular matrix of order 5 in $GL_2(\mathbb{F}_{19})$ appears in Exercise 16 of that section as an illustration of the method.)

10. This exercise classifies the groups of order 147 (there are six isomorphism types).

- (a) Prove that there are two abelian groups of order 147.
- (b) Prove that every group of order 147 has a normal Sylow 7-subgroup.
- (c) Prove that there is a unique non-abelian group whose Sylow 7-subgroup is cyclic.
- (d) Let $t_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and $t_2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ be elements of $GL_2(\mathbb{F}_7)$. Prove $P = \langle t_1, t_2 \rangle$ is a Sylow 3-subgroup of $GL_2(\mathbb{F}_7)$ and that $P \cong Z_3 \times Z_3$. Deduce that every subgroup of $GL_2(\mathbb{F}_7)$ of order 3 is conjugate in $GL_2(\mathbb{F}_7)$ to a subgroup of P .
- (e) By Example 3 in Section 1 the group P has four subgroups of order 3 and these are: $P_1 = \langle t_1 \rangle$, $P_2 = \langle t_2 \rangle$, $P_3 = \langle t_1 t_2 \rangle$, and $P_4 = \langle t_1 t_2^2 \rangle$. For $i = 1, 2, 3, 4$ let $G_i = (Z_7 \times Z_7) \rtimes_{\varphi_i} Z_3$, where φ_i is an isomorphism of Z_3 with the subgroup P_i of $\text{Aut}(Z_7 \times Z_7)$. For each i describe G_i in terms of generators and relations. Deduce that $G_1 \cong G_2$.
- (f) Prove that G_1 is not isomorphic to either G_3 or G_4 . [Show that the center of G_1 has

order 7 whereas the centers of G_3 and G_4 are trivial.]

- (g) Prove that G_3 is not isomorphic to G_4 . [Show that every subgroup of order 7 in G_3 is normal in G_3 but that G_4 has subgroups of order 7 that are not normal.]
 - (h) Classify the groups of order 147 by showing that the six nonisomorphic groups described above (two from part (a), one from part (c) and G_1 , G_3 , and G_4) are all the groups of order 147. [Use Exercise 6 and part (d).] (The classification of groups of order pq^2 , where p and q are primes with $p < q$ and $p \nmid q - 1$, is quite similar.)
11. Classify groups of order 28 (there are four isomorphism types).
 12. Classify the groups of order 20 (there are five isomorphism types).
 13. Classify groups of order $4p$, where p is a prime greater than 3. [There are four isomorphism types when $p \equiv 3 \pmod{4}$ and five isomorphism types when $p \equiv 1 \pmod{4}$.]
 14. This exercise classifies the groups of order 60 (there are thirteen isomorphism types). Let G be a group of order 60, let P be a Sylow 5-subgroup of G and let Q be a Sylow 3-subgroup of G .
 - (a) Prove that if P is not normal in G then $G \cong A_5$. [See Section 4.5.]
 - (b) Prove that if $P \leq G$ but Q is not normal in G then $G \cong A_4 \times Z_5$. [Show in this case that $P \leq Z(G)$, $G/P \cong A_4$, a Sylow 2-subgroup T of G is normal and $TQ \cong A_4$.]
 - (c) Prove that if both P and Q are normal in G then $G \cong Z_{15} \rtimes T$ where $T \cong Z_4$ or $Z_2 \times Z_2$. Show in this case that there are six isomorphism types when T is cyclic (one abelian) and there are five isomorphism types when T is the Klein 4-group (one abelian). [Use the same ideas as in the classifications of groups of orders 30 and 20.]
 15. Let p be an odd prime. Prove that every element of order 2 in $GL_2(\mathbb{F}_p)$ is conjugate to a diagonal matrix with ± 1 's on the diagonal. Classify the groups of order $2p^2$. [If A is a 2×2 matrix with $A^2 = I$ and v_1, v_2 is a basis for the underlying vector space, look at A acting on the vectors $w_1 = v_1 + v_2$ and $w_2 = v_1 - v_2$.]
 16. Show that there are exactly 4 distinct homomorphisms from Z_2 into $\text{Aut}(Z_8)$. Prove that the resulting semidirect products are the groups: $Z_8 \times Z_2$, D_{16} , the quasidihedral group QD_{16} and the modular group M (cf. the exercises in Section 2.5).
 17. Show that for any $n \geq 3$ there are exactly 4 distinct homomorphisms from Z_2 into $\text{Aut}(Z_{2^n})$. Prove that the resulting semidirect products give 4 nonisomorphic groups of order 2^{n+1} . [Recall Exercises 21 to 23 in Section 2.3.] (These four groups together with the cyclic group and the generalized quaternion group, $Q_{2^{n+1}}$, are all the groups of order 2^{n+1} which possess a cyclic subgroup of index 2.)
 18. Show that if H is any group then there is a group G that contains H as a normal subgroup with the property that for every automorphism σ of H there is an element $g \in G$ such that conjugation by g when restricted to H is the given automorphism σ , i.e., every automorphism of H is obtained as an inner automorphism of G restricted to H .
 19. Let H be a group of order n , let $K = \text{Aut}(H)$ and form $G = \text{Hol}(H) = H \rtimes K$ (where φ is the identity homomorphism). Let G act by left multiplication on the left cosets of K in G and let π be the associated permutation representation $\pi : G \rightarrow S_n$.
 - (a) Prove the elements of H are coset representatives for the left cosets of K in G and with this choice of coset representatives π restricted to H is the regular representation of H .
 - (b) Prove $\pi(G)$ is the normalizer in S_n of $\pi(H)$. Deduce that under the regular representation of any finite group H of order n , the normalizer in S_n of the image of H is isomorphic to $\text{Hol}(H)$. [Show $|G| = |N_{S_n}(\pi(H))|$ using Exercises 1 and 2 above.]
 - (c) Deduce that the normalizer of the group generated by an n -cycle in S_n is isomorphic to $\text{Hol}(Z_n)$ and has order $n\varphi(n)$.

20. Let p be an odd prime. Prove that if P is a non-cyclic p -group then P contains a normal subgroup U with $U \cong Z_p \times Z_p$. Deduce that for odd primes p a p -group that contains a unique subgroup of order p is cyclic. (For $p = 2$ it is a theorem that the generalized quaternion groups Q_{2^n} are the only non-cyclic 2-groups which contain a unique subgroup of order 2). [Proceed by induction on $|P|$. Let Z be a subgroup of order p in $Z(P)$ and let $\bar{P} = P/Z$. If \bar{P} is cyclic then P is abelian by Exercise 36 in Section 3.1 — show the result is true for abelian groups. When \bar{P} is not cyclic use induction to produce a normal subgroup \bar{H} of \bar{P} with $\bar{H} \cong Z_p \times Z_p$. Let H be the complete preimage of \bar{H} in P , so $|H| = p^3$. Let $H_0 = \{x \in H \mid x^p = 1\}$ so that H_0 is a characteristic subgroup of H of order p^2 or p^3 by Exercise 9 in Section 4. Show that a suitable subgroup of H_0 gives the desired normal subgroup U .]
21. Let p be an odd prime and let P be a p -group. Prove that if every subgroup of P is normal then P is abelian. (Note that Q_8 is a non-abelian 2-group with this property, so the result is false for $p = 2$.) [Use the preceding exercises and Exercise 15 of Section 4.]
22. Let F be a field let n be a positive integer and let G be the group of upper triangular matrices in $GL_n(F)$ (cf. Exercise 16, Section 2.1)
- Prove that G is the semidirect product $U \rtimes D$ where U is the set of upper triangular matrices with 1's down the diagonal (cf. Exercise 17, Section 2.1) and D is the set of diagonal matrices in $GL_n(F)$.
 - Let $n=2$. Recall that $U \cong F$ and $D \cong F^\times \times F^\times$ (cf. Exercise 11 in Section 3.1). Describe the homomorphism from D into $\text{Aut}(U)$ explicitly in terms of these isomorphisms (i.e., show how each element of $F^\times \times F^\times$ acts as an automorphism on F).
23. Let K and L be groups, let n be a positive integer, let $\rho : K \rightarrow S_n$ be a homomorphism and let H be the direct product of n copies of L . In Exercise 8 of Section 1 an injective homomorphism ψ from S_n into $\text{Aut}(H)$ was constructed by letting the elements of S_n permute the n factors of H . The composition $\psi \circ \rho$ is a homomorphism from G into $\text{Aut}(H)$. The *wreath product* of L by K is the semidirect product $H \rtimes K$ with respect to this homomorphism and is denoted by $L \wr K$ (this wreath product depends on the choice of permutation representation ρ of K — if none is given explicitly, ρ is assumed to be the left regular representation of K).
- Assume K and L are finite groups and ρ is the left regular representation of K . Find $|L \wr K|$ in terms of $|K|$ and $|L|$.
 - Let p be a prime, let $K = L = Z_p$ and let ρ be the left regular representation of K . Prove that $Z_p \wr Z_p$ is a non-abelian group of order p^{p+1} and is isomorphic to a Sylow p -subgroup of S_{p^2} . [The p copies of Z_p whose direct product makes up H may be represented by p disjoint p -cycles; these are cyclically permuted by K .]
24. Let n be an integer > 1 . Prove the following classification: every group of order n is abelian if and only if $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$, where p_1, \dots, p_r are distinct primes, $\alpha_i = 1$ or 2 for all $i \in \{1, \dots, r\}$ and p_i does not divide $p_j^{\alpha_j} - 1$ for all i and j . [See Exercise 56 in Section 4.5.]
25. Let $H(\mathbb{F}_p)$ be the Heisenberg group over the finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ (cf. Exercise 20 in Section 4). Prove that $H(\mathbb{F}_2) \cong D_8$, and that $H(\mathbb{F}_p)$ has exponent p and is isomorphic to the first non-abelian group in Example 7.