Some Remarks on Fourier Analysis and Group Characters

This brief discussion is intended to indicate some connections of the results above with other areas of mathematics.

The theory of group representations described to this point is a special branch of an area of mathematics called Harmonic Analysis. Readers may already be familiar with the basic theory of Fourier series which also falls into this realm. We make some observations which show how representation theory for finite groups corresponds to "Fourier series" for some infinite groups (in particular, to Fourier series on the circle). To be mathematically precise one needs the Lebesgue integral to ensure completeness of certain (Hilbert) spaces but readers may get the flavor of things by replacing "Lebesgue" by "Riemann."

Let G be the multiplicative group of points on the unit circle in $\mathbb{C}$:

$$G = \{z \in \mathbb{C} \mid |z| = 1\}.$$

We shall usually view G as the interval $[0, 2\pi]$ in $\mathbb{R}$ with the two end points identified, i.e., as the additive group $\mathbb{R}/2\pi\mathbb{Z}$ (the isomorphism is: the real number x corresponds to the complex number e^{ix}). Note that G has a translation invariant measure, namely the Lebesgue measure, and the measure of the circle is 2π. For finite groups, the counting measure is the translation invariant measure (so the measure of a subset H is the number of elements in that subset, $|H|$) and integrals on a finite group with respect to this counting measure are just finite sums.

The space

$$L^2(G) = \{f : G \to \mathbb{C} \mid f \text{ is measurable and } |f|^2 \text{ is integrable over } G\}$$

plays the role of the group algebra of the infinite group G. This space becomes a commutative ring with 1 under the convolution of functions: for $f, g \in L^2(G)$ the product $f * g : G \to \mathbb{C}$ is defined by

$$(f * g)(x) = \frac{1}{2\pi}\int_0^{2\pi} f(x-y)g(y)\,dy \qquad \text{for all } x \in G.$$

(Recall that for a finite group H, the group algebra is also formally the ring of $\mathbb{C}$-valued functions on H under a convolution multiplication and that these functions are written as formal sums – the element $\sum \alpha_g g \in \mathbb{C}G$ denotes the function which sends g to $\alpha_g \in \mathbb{C}$ for all $g \in G$.)

The complete set of continuous homomorphisms of G into $GL_1(\mathbb{C})$ is given by

$$e_n(x) = e^{inx}, \quad x \in [0, 2\pi], \quad n \in \mathbb{Z}.$$

(Recall that for a finite abelian group, all irreducible representations are 1-dimensional and for 1-dimensional representations, characters and representations may be identified.)

The ring $L^2(G)$ admits an Hermitian inner product: for $f, g \in L^2(G)$

$$(f, g) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{g(t)}\,dt.$$

Under this inner product, $\{e_n \mid n \in \mathbb{Z}\}$ is an orthonormal basis (where the term "basis" is used in the analytic sense that these are independent and 0 is the only function orthogonal to all of them). Moreover,

$$L^2(G) = \widehat{\bigoplus_{n\in\mathbb{Z}} E_n}$$

where E_n is the 1-dimensional subspace spanned by e_n, the hat over the direct sum denotes taking the closure of the direct sum in the L^2-topology, and equality indicates equality in the L^2 sense. (Recall that the group algebra of a finite abelian group is the direct sum of the irreducible 1-dimensional submodules, each occurring with multiplicity one.) These facts imply the well known result from Fourier analysis that every square integrable function $f(x)$ on $[0, 2\pi]$ has a Fourier series

$$\sum_{n=-\infty}^{\infty} c_n e^{inx}$$

where the Fourier coefficients, c_n, are given by

$$c_n = (f, e_n) = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}\, dt.$$

This brief description indicates how the representation theory of finite groups extends to certain infinite groups and the results we have proved may already be familiar in the latter context. In fact, there is a completely analogous theory for arbitrary (not necessarily abelian) compact Lie groups — here the irreducible (complex) representations need not be 1-dimensional but they are all finite dimensional and $L^2(G)$ decomposes as a direct sum of them, each appearing with multiplicity equal to its degree. The emphasis (at least at the introductory level) in this theory is often on the importance of being able to represent functions as (Fourier) series and then using these series to solve other problems (e.g., solve differential equations). The underlying group provides the "symmetry" on which to build this "harmonic analysis," rather than being itself the principal object of study.

EXERCISES

Let G be a finite group. Unless stated otherwise all representations and characters are over $\mathbb{C}$.

1. Prove that $\operatorname{tr} AB = \operatorname{tr} BA$ for $n \times n$ matrices A and B with entries from any commutative ring.
2. In each of (a) to (c) let ψ be the character afforded by the specified representation φ.
   (a) Let φ be the degree 2 representation of D_{10} described in Example 6 in the second set of examples in Section 1 (here $n = 5$) and show that $\| \psi \|^2 = 1$ (hence φ is irreducible).
   (b) Let φ be the degree 2 representation of Q_8 described in Example 7 in the second set of examples in Section 1 and show that $\| \psi \|^2 = 1$ (hence φ is irreducible).
   (c) Let φ be the degree 4 representation of Q_8 described in Example 8 in the second set of examples in Section 1 and show that $\| \psi \|^2 = 4$ (hence even though φ is irreducible over $\mathbb{R}$, φ decomposes over $\mathbb{C}$ as twice an irreducible representation of degree 2).
3. If χ is an irreducible character of G, prove that the χ-isotypic subspace of a $\mathbb{C}G$-module is unique.

4. Prove that if N is any irreducible $\mathbb{C}G$-module and $M = N \oplus N$, then M has infinitely many direct sum decompositions into two copies of N.
5. Prove that a class function is a character if and only if it is a positive integral linear combination of irreducible characters.
6. Let $\varphi : G \to GL(V)$ be a representation with character ψ. Let W be the subspace $\{v \in V \mid \varphi(g)(v) = v \text{ for all } g \in G\}$ of V fixed pointwise by all elements of G. Prove that $\dim W = (\psi, \chi_1)$, where χ_1 is the principal character of G.
7. Assume V is a $\mathbb{C}G$-module on which G acts by permuting the basis $\mathcal{B} = \{e_1, \ldots, e_n\}$. Write $\mathcal{B}$ as a disjoint union of the orbits $\mathcal{B}_1, \ldots, \mathcal{B}_t$ of G on $\mathcal{B}$.
   (a) Prove that V decomposes as a $\mathbb{C}G$-module as $V_1 \oplus \cdots \oplus V_t$, where V_i is the span of $\mathcal{B}_i$.
   (b) Prove that if v_i is the sum of the vectors in $\mathcal{B}_i$ then the 1-dimensional subspace of V_i spanned by v_i is the unique $\mathbb{C}G$-submodule of V_i affording the trivial representation (in other words, any vector in V_i that is fixed under the action of G is a multiple of v_i). [Use the fact that G is transitive on $\mathcal{B}_i$. See also Exercise 8 in Section 1.]
   (c) Let $W = \{v \in V \mid \varphi(g)(v) = v \text{ for all } g \in G\}$ be the subspace of V fixed pointwise by all elements of G. Deduce that $\dim W = t =$ the number of orbits of G on $\mathcal{B}$.
8. Prove the following result (sometimes called Burnside's Lemma although its origin is with Frobenius): let G be a subgroup of S_n and for each $\sigma \in G$ let $\text{Fix}(\sigma)$ denote the number of fixed points of σ on $\{1, \ldots, n\}$. Let t be the number of orbits of G on $\{1, \ldots, n\}$. Then
$$t|G| = \sum_{g \in G} \text{Fix}(g).$$
   [Use the preceding two exercises.]
9. Let G be a nontrivial, transitive group of permutations on the finite set Ω and let ψ be the character afforded by the linear representation over $\mathbb{C}$ obtained from Ω (cf. Example 4 in Section 1) so $\psi(\sigma)$ is the number of fixed points of σ on Ω. Now let G act on the set $\Omega \times \Omega$ by $g \cdot (\omega_1, \omega_2) = (g \cdot \omega_1, g \cdot \omega_2)$ and let π be the character afforded by the linear representation obtained from this action.
   (a) Prove that $\pi = \psi^2$.
   (b) Prove that the number of orbits of G on $\Omega \times \Omega$ is given by the inner product (ψ, ψ). [By the preceding exercises, the number of orbits on $\Omega \times \Omega$ is equal to (π, χ_1), where χ_1 is the principal character.]
   (c) Recall that G is said to be *doubly transitive* on Ω if it has precisely 2 orbits in its action on $\Omega \times \Omega$ (it always has at least 2 orbits since the diagonal, $\{(\omega, \omega) \mid \omega \in \Omega\}$, is one orbit). Prove that if G is doubly transitive on Ω then $\psi = \chi_1 + \chi_2$, where χ_1 is the principal character and χ_2 is a nonprincipal irreducible character of G.
   (d) Let $\Omega = \{1, 2, \ldots, n\}$ and let $G = S_n$ act on Ω in the natural fashion. Show that the character of the associated linear representation decomposes as the principal character plus an irreducible character of degree $n - 1$.
10. Let ψ be the character of any 2-dimensional representation of a group G and let x be an element of order 2 in G. Prove that $\psi(x) = 2$, 0 or -2. Generalize this to n-dimensional representations.
11. Let χ be an irreducible character of G. Prove that for every element z in the center of G we have $\chi(z) = \epsilon\chi(1)$, where ϵ is some root of 1 in $\mathbb{C}$. [Use Schur's Lemma.]
12. Let ψ be the character of some representation φ of G. Prove that for $g \in G$ the following hold:
   (a) if $\psi(g) = \psi(1)$ then $g \in \ker \varphi$, and