A *variable* is a symbol, such as n or x, which denotes a certain type of mathematical object - an integer, a vector, a matrix, that kind of thing. In almost all circumstances, the type of object that the variable represents should be declared, otherwise it will be difficult to make well-formed statements using it. (There are very few true statements that one can make about variables without knowing the type of variables involved. For instance, given a variable x of any type whatsoever, it is true that $x = x$, and if we also know that $x = y$, then we can conclude that $y = x$. But one cannot say, for instance, that $x + y = y + x$, until we know what type of objects x and y are and whether they support the operation of addition; for instance, the above statement is ill-formed if x is a matrix and y is a vector. Thus if one actually wants to do some useful mathematics, then every variable should have an explicit type.)

One can form expressions and statements involving variables, for instance, if x is a real variable (i.e., a variable which is a real number), $x + 3$ is an expression, and $x + 3 = 5$ is a statement. But now the truth of a statement may depend on the value of the variables involved; for instance the statement $x + 3 = 5$ is true if x is equal to 2, but is false if x is not equal to 2. Thus the truth of a statement involving a variable may depend on the *context* of the statement - in this case, it depends on what x is supposed to be. (This is a modification of the rule for propositional logic, in which all statements have a definite truth value.)

Sometimes we do not set a variable to be anything (other than specifying its type). Thus, we could consider the statement $x+3 = 5$ where x is an unspecified real number. In such a case we call this variable a *free variable*; thus we are considering $x+3 = 5$ with x a free variable. Statements with free variables might not have a definite truth value, as they depend on an unspecified variable. For instance, we have already remarked that $x + 3 = 5$ does not have a definite truth value if x is a free real variable, though of course for each given value of x the statement is either true or false. On the other hand, the statement $(x + 1)^2 = x^2 + 2x + 1$ is true for every real number x, and so we can regard this as a true statement even when x is a free variable.

At other times, we *set* a variable to equal a fixed value, by using a statement such as "Let $x = 2$" or "Set x equal to 2". In this case, the variable is known as a *bound variable*, and statements involving only bound variables and no free variables do have a definite truth value. For instance, if we set $x = 342$, then the statement "$x+135 = 477$" now has a definite truth value, whereas if x is a free real variable then "$x+135 = 477$" could be either true or false, depending on what x is. Thus, as we have said before, the truth of a statement such as "$x + 135 = 477$" depends on the context - whether x is free or bound, and if it is bound, what it is bound to.

One can also turn a free variable into a bound variable by using the quantifiers "for all" or "for some". For instance, the statement

$$(x + 1)^2 = x^2 + 2x + 1$$

is a statement with one free variable x, and need not have a definite truth value, but the statement

$$(x + 1)^2 = x^2 + 2x + 1 \text{ for all real numbers } x$$

is a statement with one bound variable x, and now has a definite truth value (in this case, the statement is true). Similarly, the statement

$$x + 3 = 5$$

has one free variable, and does not have a definite truth value, but the statement

$$x + 3 = 5 \text{ for some real number } x$$

is true, since it is true for $x = 2$. On the other hand, the statement

$$x + 3 = 5 \text{ for all real numbers } x$$

is false, because there are some (indeed, there are many) real numbers x for which $x + 3$ is not equal to 5.

Universal quantifiers. Let $P(x)$ be some statement depending on a free variable x. The statement "$P(x)$ is true for all x of type T" means that given any x of type T, the statement $P(x)$ is true regardless of what the exact value of x is. In other words, the statement is the same as saying "if x is of type T, then $P(x)$ is true". Thus the usual way to prove such a statement is to let x be a free variable of type T (by saying something like "Let x be any object of type T"), and then proving $P(x)$ for that object. The statement becomes false if one can produce even a single counterexample, i.e., an element x which lies in T but for which $P(x)$ is false. For instance, the statement "x^2 is greater than x for all positive x" can be shown to be false by producing a single example, such as $x = 1$ or $x = 1/2$, where x^2 is not greater than x.

On the other hand, producing a single example where $P(x)$ *is* true will not show that $P(x)$ is true for *all* x. For instance, just because the equation $x + 3 = 5$ has a solution when $x = 2$ does not imply that $x + 3 = 5$ for all real numbers x; it only shows that $x + 3 = 5$ is true for some real number x. (This is the source of the often-quoted, though somewhat inaccurate, slogan "One cannot prove a statement just by giving an example". The more precise statement is that one cannot prove a "for all" statement by examples, though one can certainly prove "for some" statements this way, and one can also *disprove* "for all" statements by a single counterexample.)

It occasionally happens that there are in fact no variables x of type T. In that case the statement "$P(x)$ is true for all x of type T" is *vacuously true* - it is true but has no content, similar to a vacuous implication. For instance, the statement

$$6 < 2x < 4 \text{ for all } 3 < x < 2$$

is true, and easily proven, but is vacuous. (Such a vacuously true statement can still be useful in an argument, though this doesn't happen very often.)

One can use phrases such as "For every" or "For each" instead of "For all", e.g., one can rephrase "$(x + 1)^2 = x^2 + 2x + 1$ for all real numbers x" as "For each real number x, $(x+1)^2$ is equal to x^2+2x+1". For the purposes of logic these rephrasings are equivalent. The symbol $\forall$ can be used instead of "For all", thus for instance "$\forall x \in X : P(x)$ is true" or "$P(x)$ is true $\forall x \in X$" is synonymous with "$P(x)$ is true for all $x \in X$".

Existential quantifiers The statement "$P(x)$ is true for some x of type T" means that there exists at least one x of type T for which $P(x)$ is true, although it may be that there is more than one such x. (One would use a quantifier such as "for exactly one x" instead of "for some x" if one wanted both existence and uniqueness of such an x.) To prove such a statement it suffices to provide a single example of such an x. For instance, to show that

$$x^2 + 2x - 8 = 0 \text{ for some real number } x$$

all one needs to do is find a single real number x for which $x^2 + 2x - 8 = 0$, for instance $x = 2$ will do. (One could also use $x = -4$, but one doesn't need to use both.) Note that one has the freedom to select x to be anything one wants when proving a for-some statement; this is in contrast to proving a for-all statement, where one has to let x be arbitrary. (One can compare the two statements by thinking of two games between you and an opponent. In the first game, the opponent gets to pick what x is, and then you have to prove $P(x)$; if you can always win this game, then you have proven that $P(x)$ is true for *all* x. In the second game, *you* get to choose what x is, and then you prove $P(x)$; if you can win this game, you have proven that $P(x)$ is true for *some* x.)

Usually, saying something is true for *all* x is much stronger than just saying it is true for *some* x. There is one exception though, if the condition on x is impossible to satisfy, then the for-all statement is vacuously true, but the for-some statement is false. For instance

$$6 < 2x < 4 \text{ for all } 3 < x < 2$$

is true, but

$$6 < 2x < 4 \text{ for some } 3 < x < 2$$

is false.

One can use phrases such as "For at least one" or "There exists ... such that" instead of "For some". For instance, one can rephrase "$x^2+2x-8=0$ for some real number x" as "There exists a real number x such that $x^2+2x-8=0$". The symbol $\exists$ can be used instead of "There exists ... such that", thus for instance "$\exists x \in X : P(x)$ is true" is synonymous with "$P(x)$ is true for some $x \in X$".

A.5 Nested quantifiers

One can nest two or more quantifiers together. For instance, consider the statement

For every positive number x, there exists a
positive number y such that $y^2 = x$.

What does this statement mean? It means that for each positive number x, the statement

There exists a positive number y such that $y^2 = x$

is true. In other words, one can find a positive square root of x for each positive number x. So the statement is saying that every positive number has a positive square root.

To continue the gaming metaphor, suppose you play a game where your opponent first picks a positive number x, and then you pick a positive number y. You win the game if $y^2 = x$. If you can always win the game regardless of what your opponent does, then you have proven that for every positive x, there exists a positive y such that $y^2 = x$.

Negating a universal statement produces an existential statement. The negation of "All swans are white" is not "All swans are not white", but rather "There is some swan which is not white". Similarly, the negation of "For every $0 < x < \pi/2$, we have $\cos(x) \geq 0$" is "For some $0 < x < \pi/2$, we have $\cos(x) < 0$, **not** "For every $0 < x < \pi/2$, we have $\cos(x) < 0$".

Negating an existential statement produces a universal statement. The negation of "There exists a black swan" is not "There exists a swan which is non-black", but rather "All swans are non-black". Similarly, the negation of "There exists a real number x such that $x^2 + x + 1 = 0$" is "For every real number x, $x^2 + x + 1 \neq 0$", **not** "There exists a real number x such that $x^2 + x + 1 \neq 0$". (The situation here is very similar to how "and" and "or" behave with respect to negations.)

If you know that a statement $P(x)$ is true for all x, then you can set x to be anything you want, and $P(x)$ will be true for that value of x; this is what "for all" means. Thus for instance if you know that

$$(x + 1)^2 = x^2 + 2x + 1 \text{ for all real numbers } x,$$

then you can conclude for instance that

$$(\pi + 1)^2 = \pi^2 + 2\pi + 1,$$

or for instance that

$$(\cos(y) + 1)^2 = \cos(y)^2 + 2\cos(y) + 1 \text{ for all real numbers } y$$

(because if y is real, then $\cos(y)$ is also real), and so forth. Thus universal statements are very versatile in their applicability - you can get $P(x)$ to hold for whatever x you wish. Existential statements, by contrast, are more limited; if you know that

$$x^2 + 2x - 8 = 0 \text{ for some real number } x$$

then you cannot simply substitute in any real number you wish, e.g., π, and conclude that $\pi^2 + 2\pi - 8 = 0$. However, you can of course still conclude that $x^2 + 2x - 8 = 0$ for some real number x, it's just that you don't get to pick which x it is. (To continue the gaming metaphor, you can make $P(x)$ hold, but your opponent gets to pick x for you, you don't get to choose for yourself.)

Remark A.5.1. In the history of logic, quantifiers were formally studied thousands of years before Boolean logic was. Indeed, *Aristotlean logic*, developed of course by Aristotle (384BC – 322BC) and his school, deals with objects, their properties, and quantifiers such as "for all" and "for some". A typical line of reasoning (or *syllogism*) in Aristotlean logic goes like this: "All men are mortal. Socrates is a man. Hence, Socrates is mortal". Aristotlean logic is a subset of mathematical logic, but is not as expressive because it lacks the concept of logical connectives such as and, or, or if-then (although "not" is allowed), and also lacks the concept of a binary relation such as $=$ or $<$.

Swapping the order of two quantifiers may or may not make a difference to the truth of a statement. Swapping two "for all" quantifiers is harmless: a statement such as

For all real numbers a, and for all real numbers b,
we have $(a+b)^2 = a^2 + 2ab + b^2$

is logically equivalent to the statement

For all real numbers b, and for all real numbers a,
we have $(a+b)^2 = a^2 + 2ab + b^2$

(why? The reason has nothing to do with whether the identity $(a+b)^2 = a^2+2ab+b^2$ is actually true or not). Similarly, swapping two "there exists" quantifiers has no effect:

There exists a real number a, and there exists a real number b,
such that $a^2 + b^2 = 0$

is logically equivalent to

There exists a real number b, and there exists a real number a,
such that $a^2 + b^2 = 0$.

However, swapping a "for all" with a "there exists" makes a lot of difference. Consider the following two statements:

(a) For every integer n, there exists an integer m which is larger than n.

(b) There exists an integer m such that m is larger than n for every integer n.

Statement (a) is obviously true: if your opponent hands you an integer n, you can always find an integer m which is larger than n. But Statement (b) is false: if you choose m first, then you cannot ensure that m is larger than every integer n; your opponent can easily pick a number n bigger than m to defeat that. The crucial difference between the two statements is that in Statement (a), the integer n was chosen *first*, and integer m could then be chosen in a manner depending on n; but in Statement (b), one was forced to choose m first, without knowing in advance what n is going to be. In short, the reason why the order of quantifiers is important is that the inner variables may possibly depend on the outer variables, but not vice versa.

Exercise A.5.1. What does each of the following statements mean, and which of them are true? Can you find gaming metaphors for each of these statements?

(a) For every positive number x, and every positive number y, we have $y^2 = x$.

(b) There exists a positive number x such that for every positive number y, we have $y^2 = x$.

(c) There exists a positive number x, and there exists a positive number y, such that $y^2 = x$.

(d) For every positive number y, there exists a positive number x such that $y^2 = x$.

(e) There exists a positive number y such that for every positive number x, we have $y^2 = x$.

A.6 Some examples of proofs and quantifiers

Here we give some simple examples of proofs involving the "for all" and "there exists" quantifiers. The results themselves are simple, but you should pay attention instead to how the quantifiers are arranged and how the proofs are structured.

Proposition A.6.1. *For every $\varepsilon > 0$ there exists a $\delta > 0$ such that $2\delta < \varepsilon$.*

Proof. Let $\varepsilon > 0$ be arbitrary. We have to show that there exists a $\delta > 0$ such that $2\delta < \varepsilon$. We only need to pick one such δ; choosing $\delta := \varepsilon/3$ will work, since one then has $2\delta = 2\varepsilon/3 < \varepsilon$. □

Notice how ε has to be arbitrary, because we are proving something for *every* ε; on the other hand, δ can be chosen as you wish, because you only need to show that there *exists* a δ which does what you want. Note also that δ can depend on ε, because the δ-quantifier is nested inside the ε-quantifier. If the quantifiers were reversed, i.e., if you were asked to prove "There exists a $\delta > 0$ such that for every $\varepsilon > 0$, $2\delta < \varepsilon$", then you would have to select δ *first* before being given ε. In this case it is impossible to prove the statement, because it is false (why?).

Normally, when one has to prove a "There exists..." statement, e.g., "Prove that there exists an $\varepsilon > 0$ such that X is true", one proceeds by selecting ε carefully, and then showing that X is true for that ε. However, this sometimes requires a lot of foresight, and it is legitimate to defer the selection of ε until later in the argument, when it becomes clearer what properties ε needs to satisfy. The only thing to watch out for is to make sure that ε does not depend on any of the bound variables nested inside X. For instance:

Proposition A.6.2. *There exists an $\varepsilon > 0$ such that* $\sin(x) > x/2$ *for all* $0 < x < \varepsilon$.

Proof. We pick $\varepsilon > 0$ to be chosen later, and let $0 < x < \varepsilon$. Since the derivative of $\sin(x)$ is $\cos(x)$, we see from the mean-value theorem we have

$$\frac{\sin(x)}{x} = \frac{\sin(x) - \sin(0)}{x - 0} = \cos(y)$$

for some $0 \leq y \leq x$. Thus in order to ensure that $\sin(x) > x/2$, it would suffice to ensure that $\cos(y) > 1/2$. To do this, it would suffice to ensure that $0 \leq y < \pi/3$ (since the cosine function takes the value of 1 at 0, takes the value of $1/2$ at $\pi/3$, and is decreasing in between). Since $0 \leq y \leq x$ and $0 < x < \varepsilon$, we see that $0 \leq y < \varepsilon$. Thus if we pick $\varepsilon := \pi/3$, then we have $0 \leq y < \pi/3$ as desired, and so we can ensure that $\sin(x) > x/2$ for all $0 < x < \varepsilon$. □

Note that the value of ε that we picked at the end did not depend on the nested variables x and y. This makes the above argument legitimate. Indeed, we can rearrange it so that we don't have to postpone anything:

Proof. We choose $\varepsilon := \pi/3$; clearly $\varepsilon > 0$. Now we have to show that for all $0 < x < \pi/3$, we have $\sin(x) > x/2$. So let $0 < x < \pi/3$ be arbitrary. By the mean-value theorem we have

$$\frac{\sin(x)}{x} = \frac{\sin(x) - \sin(0)}{x - 0} = \cos(y)$$

for some $0 \leq y \leq x$. Since $0 \leq y \leq x$ and $0 < x < \pi/3$, we have $0 \leq y < \pi/3$. Thus $\cos(y) > \cos(\pi/3) = 1/2$, since cos is decreasing on the interval $[0, \pi/3]$. Thus we have $\sin(x)/x > 1/2$ and hence $\sin(x) > x/2$ as desired. □

If we had chosen ε to depend on x and y then the argument would not be valid, because ε is the outer variable and x, y are nested inside it.

A.7 Equality

As mentioned before, one can create statements by starting with expressions (such as $2 \times 3 + 5$) and then asking whether an expression obeys a certain property, or whether two expressions are related by some sort of relation ($=$, $\leq$, $\in$, etc.). There are many relations, but the most important one is *equality*, and it is worth spending a little time reviewing this concept.

Equality is a relation linking two objects x, y of the same type T (e.g., two integers, or two matrices, or two vectors, etc.). Given two such objects x and y, the statement $x = y$ may or may not be true; it depends on the value of x and y and also on how equality is defined for the class of objects under consideration. For instance, as real numbers, the two numbers $0.9999\ldots$ and 1 are equal. In modular arithmetic with modulus 10 (in which numbers are considered equal to their remainders modulo 10), the numbers 12 and 2 are considered equal, $12 = 2$, even though this is not the case in ordinary arithmetic.

How equality is defined depends on the class T of objects under consideration, and to some extent is just a matter of definition. However, for the purposes of logic we require that equality obeys the following four *axioms of equality*:

- (Reflexive axiom). Given any object x, we have $x = x$.
- (Symmetry axiom). Given any two objects x and y of the same type, if $x = y$, then $y = x$.
- (Transitive axiom). Given any three objects x, y, z of the same type, if $x = y$ and $y = z$, then $x = z$.
- (Substitution axiom). Given any two objects x and y of the same type, if $x = y$, then $f(x) = f(y)$ for all functions or operations f. Similarly, for any property $P(x)$ depending on x, if $x = y$, then $P(x)$ and $P(y)$ are equivalent statements.

The first three axioms are clear, together, they assert that equality is an *equivalence relation*. To illustrate the substitution we give some examples.

Example A.7.1. Let x and y be real numbers. If $x = y$, then $2x = 2y$, and $\sin(x) = \sin(y)$. Furthermore, $x + z = y + z$ for any real number z.

Example A.7.2. Let n and m be integers. If n is odd and $n = m$, then m must also be odd. If we have a third integer k, and we know that $n > k$ and $n = m$, then we also know that $m > k$.

Example A.7.3. Let x, y, z be real numbers. If we know that $x = \sin(y)$ and $y = z^2$, then (by the substitution axiom) we have

$\sin(y) = \sin(z^2)$, and hence (by the transitive axiom) we have $x = \sin(z^2)$.

Thus, from the point of view of logic, we can define equality on a class of objects however we please, so long as it obeys the reflexive, symmetry, and transitive axioms, and is consistent with all other operations on the class of objects under discussion in the sense that the substitution axiom was true for all of those operations. For instance, if we decided one day to modify the integers so that 12 was now equal to 2, one could only do so if one also made sure that 2 was now equal to 12, and that $f(2) = f(12)$ for any operation f on these modified integers. For instance, we now need $2 + 5$ to be equal to $12 + 5$. (In this case, pursuing this line of reasoning will eventually lead to modular arithmetic with modulus 10.)

Exercise A.7.1. Suppose you have four real numbers a, b, c, d and you know that $a = b$ and $c = d$. Use the above four axioms to deduce that $a + d = b + c$.

Chapter B

Appendix: the decimal system

In Chapters 2, 4, 5 we painstakingly constructed the basic number systems of mathematics: the natural numbers, integers, rationals, and reals. Natural numbers were simply postulated to exist, and to obey five axioms; the integers then came via (formal) differences of the natural numbers; the rationals then came from (formal) quotients of the integers, and the reals then came from (formal) limits of the rationals.

This is all very well and good, but it does seem somewhat alien to one's prior experience with these numbers. In particular, very little use was made of the *decimal system*, in which the digits $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ are combined to represent these numbers. Indeed, except for a number of examples which were not essential to the main construction, the only decimals we really used were the numbers 0, 1, and 2, and the latter two can be rewritten as $0{+}{+}$ and $(0{+}{+}){+}{+}$.

The basic reason for this is that *the decimal system itself is not essential to mathematics.* It is very convenient for computations, and we have grown accustomed to it thanks to a thousand years of use, but in the history of mathematics it is actually a comparatively recent invention. Numbers have been around for about ten thousand years (starting from scratch marks on cave walls), but the modern Hindu-Arabic base 10 system for representing numbers only dates from the 11th century or so. Some early civilizations relied on other bases; for instance the Babylonians used a base 60 system (which still survives in our time system of hours, minutes, and seconds, and in our angular system of degrees, minutes, and seconds). And the ancient Greeks were able to do quite advanced mathematics, despite the fact that the most advanced number representation system available to them was the Roman numeral system $I, II, III, IV, \ldots$, which was horrendous for computations of even two-digit numbers. And of course modern computing relies on binary, hexadecimal, or byte-based (base 256) arithmetic instead of decimal, while analog computers such as the slide rule do not really rely on any number representation system at all. In fact, now that computers can do the menial work of number-crunching, there is very little use for decimals in modern mathematics. Indeed, we rarely use any numbers other than one-digit numbers or one-digit fractions (as well as e, π, i) explicitly in modern mathematical work; any more complicated numbers usually get called more generic names such as n.

Nevertheless, the subject of decimals does deserve an appendix, because it is so integral to the way we use mathematics in our everyday life, and also because we do want to use such notation as $3.14159\ldots$ to refer to real numbers, as opposed to the far clunkier "$\mathrm{LIM}_{n\to\infty} a_n$, where $a_1 = 3.1, a_2 := 3.14, a_3 := 3.141, \ldots$".

We begin by reviewing how the decimal system works for the positive integers, and then turn to the reals. Note that in this discussion we shall freely use all the results from earlier chapters.

B.1 The decimal representation of natural numbers

In this section we will avoid the usual convention of abbreviating $a \times b$ as ab, since this would mean that decimals such as 34 might be misconstrued as 3×4.

Definition B.1.1 (Digits). A *digit* is any one of the ten symbols $0, 1, 2, 3, \ldots, 9$. We equate these digits with natural numbers by the formulae $0 \equiv 0$, $1 \equiv 0{++}$, $2 \equiv 1{++}$, etc. all the way up to $9 \equiv 8{++}$. We also define the number ten by the formula ten $:= 9{++}$. (We cannot use the decimal notation 10 to denote ten yet, because that presumes knowledge of the decimal system and would be circular.)

Definition B.1.2 (Positive integer decimals). A *positive integer decimal* is any string $a_n a_{n-1} \ldots a_0$ of digits, where $n \geq 0$ is a natural number, and the first digit a_n is non-zero. Thus, for instance, 3049 is a positive integer decimal, but 0493 or 0 is not. We equate each positive integer decimal with a positive integer by the formula

$$a_n a_{n-1} \ldots a_0 \equiv \sum_{i=0}^{n} a_i \times \text{ten}^i.$$

Remark B.1.3. Note in particular that this definition implies that

$$10 = 0 \times \text{ten}^0 + 1 \times \text{ten}^1 = \text{ten}$$

and thus we can write ten as the more familiar 10. Also, a single digit integer decimal is exactly equal to that digit itself, e.g., the decimal 3 by the above definition is equal to

$$3 = 3 \times \text{ten}^0 = 3$$

so there is no possibility of confusion between a single digit, and a single digit decimal. (This is a subtle distinction, and not one which is worth losing much sleep over.)

Now we show that this decimal system indeed represents the positive integers. It is clear from the definition that every positive decimal representation gives a positive integer, since the sum consists entirely of natural numbers, and the last term $a_n \text{ten}^n$ is non-zero by definition.

Theorem B.1.4 (Uniqueness and existence of decimal representations). *Every positive integer m is equal to exactly one positive integer decimal.*

Proof. We shall use the principle of strong induction (Proposition 2.2.14, with $m_0 := 1$). For any positive integer m, let $P(m)$ denote the statement "m is equal to exactly one positive integer decimal".

Suppose we already know $P(m')$ is true for all positive integers $m' < m$; we now wish to prove $P(m)$.

First observe that either $m \geq$ ten or $m \in \{1,2,3,4,5,6,7,8,9\}$. (This is easily proved by ordinary induction.) Suppose first that $m \in \{1,2,3,4,5,6,7,8,9\}$. Then m clearly is equal to a positive integer decimal consisting of a single digit, and there is only one single-digit decimal which is equal to m. Furthermore, no decimal consisting of two or more digits can equal m, since if $a_n \ldots a_0$ is such a decimal (with $n > 0$) we have

$$a_n \ldots a_0 = \sum_{i=0}^{n} a_i \times \text{ten}^i \geq a_n \times \text{ten}^i \geq \text{ten} > m.$$

Now suppose that $m \geq$ ten. Then by the Euclidean algorithm (Proposition 2.3.9) we can write

$$m = s \times \text{ten} + r$$

where s is a positive integer, and $r \in \{0,1,2,3,4,5,6,7,8,9\}$. Since

$$s < s \times \text{ten} \leq s \times \text{ten} + r = m$$

we can use the strong induction hypothesis and conclude that $P(s)$ is true. In particular, s has a decimal representation

$$s = b_p \ldots b_0 = \sum_{i=0}^{p} b_i \times \text{ten}^i.$$

Multiplying by ten, we see that

$$s \times \text{ten} = \sum_{i=0}^{p} b_i \times \text{ten}^{i+1} = b_p \ldots b_0 0,$$

and then adding r we see that

$$m = s \times \text{ten} + r = \sum_{i=0}^{p} b_i \times \text{ten}^{i+1} + r = b_p \ldots b_0 r.$$

Thus m has at least one decimal representation. Now we need to show that m has at most one decimal representation. Suppose for sake of contradiction that we have at least two different representations

$$m = a_n \ldots a_0 = a'_{n'} \ldots a'_0.$$

First observe by the previous computation that

$$a_n \ldots a_0 = (a_n \ldots a_1) \times \text{ten} + a_0$$

and

$$a'_{n'} \ldots a'_0 = (a'_{n'} \ldots a'_1) \times \text{ten} + a'_0$$

and so after some algebra we obtain

$$a'_0 - a_0 = (a_n \ldots a_1 - a'_{n'} \ldots a'_1) \times \text{ten}.$$

The right-hand side is a multiple of ten, while the left-hand side lies strictly between $-$ten and $+$ten. Thus both sides must be equal to 0. This means that $a_0 = a'_0$ and $a_n \ldots a_1 = a'_{n'} \ldots a'_1$. But by previous arguments, we know that $a_n \ldots a_1$ is a smaller integer than $a_n \ldots a_0$. Thus by the strong induction hypothesis, the number $a_n \ldots a_0$ has only one decimal representation, which means that n' must equal n and a'_i must equal a_i for all $i = 1, \ldots, n$. Thus the decimals $a_n \ldots a_0$ and $a'_{n'} \ldots a'_0$ are in fact identical, contradicting the assumption that they were different. □

We refer to the decimal given by the above theorem as the *decimal representation* of m. Once one has this decimal representation, one can then derive the usual laws of long addition and long multiplication to connect the decimal representation of $x+y$ or $x \times y$ to that of x or y (Exercise B.1.1).

Once one has decimal representation of positive integers, one can of course represent negative integers decimally as well by using the $-$ sign. Finally, we let 0 be a decimal as well. This gives decimal representations of all integers. Every rational is then the ratio of two decimals, e.g., $335/113$ or $-1/2$ (with the denominator required to be non-zero, of course), though there may be more