

The isometries of \mathbb{S}^2

The simplest isometries of \mathbb{R}^3 that fix O are reflections in planes through O . The corresponding isometries of \mathbb{S}^2 are the *reflections in great circles*. Two planes \mathcal{P}_1 and \mathcal{P}_2 meet in a line \mathcal{L} through O , and the product of reflections in \mathcal{P}_1 and \mathcal{P}_2 is a rotation about \mathcal{L} (through twice the angle between \mathcal{P}_1 and \mathcal{P}_2). This situation is completely analogous to that in \mathbb{R}^2 , where the product of reflections through O is a rotation (through twice the angle between the lines).

Finally, there are products of reflections in three planes that are different from products of reflections in one or two planes. One such isometry is the *antipodal map* sending each point (x, y, z) to its *antipodal point* $(-x, -y, -z)$. This map is the product of

- reflection in the (y, z) -plane, which sends (x, y, z) to $(-x, y, z)$,
- reflection in the (z, x) -plane, which sends (x, y, z) to $(x, -y, z)$,
- reflection in the (x, y) -plane, which sends (x, y, z) to $(x, y, -z)$.

As in \mathbb{R}^2 , there is a “three reflections theorem” that any isometry of \mathbb{S}^2 is the product of one, two, or three reflections. The proof is similar to the proof for \mathbb{R}^2 in Sections 3.3 and 3.7 (see the exercises below). This three reflections theorem shows why all isometries of \mathbb{S}^2 are restrictions of isometries of \mathbb{R}^3 , namely, because this is true of reflections in great circles.

Exercises

The proof of the three reflections theorem begins, as it did for \mathbb{R}^2 , by considering the equidistant set of two points.

7.4.1 Show that the equidistant set of two points in \mathbb{R}^3 is a plane. Show also that the plane passes through O if the two points are both at distance 1 from O .

7.4.2 Deduce from Exercise 7.4.1 that the equidistant set of two points on \mathbb{S}^2 is a “line” (great circle) on \mathbb{S}^2 .

Next, we establish that there is a unique point on \mathbb{S}^2 at given distances from three points not in a “line.”

7.4.3 Suppose that two points $P, Q \in \mathbb{S}^2$ have the same distances from three points $A, B, C \in \mathbb{S}^2$ not in a “line.” Deduce from Exercise 7.4.2 that $P = Q$.

7.4.4 Deduce from Exercise 7.4.3 that an isometry of \mathbb{S}^2 is determined by the images of three points A, B, C not in a “line.”

Thus, it remains to show the following. Any three points $A, B, C \in \mathbb{S}^2$ not in a “line” can be mapped to any other three points $A', B', C' \in \mathbb{S}^2$, which are separated by the same respective distances, by one, two, or three reflections.

7.4.5 Complete this proof of the three reflections theorem by imitating the argument in Section 3.7.

7.5 The rotation group of the sphere

The group $\text{Isom}(\mathbb{S}^2)$ of all isometries of \mathbb{S}^2 has a subgroup $\text{Isom}^+(\mathbb{S}^2)$ consisting of the isometries that are products of an even number of reflections. Like $\text{Isom}^+(\mathbb{R}^2)$, this is the “orientation-preserving” subgroup. But, unlike $\text{Isom}^+(\mathbb{R}^2)$, $\text{Isom}^+(\mathbb{S}^2)$ includes no “translations”—only rotations. We already know that the product of two reflections of \mathbb{S}^2 is a rotation. Hence, to show that the product of any even number of reflections is a rotation, it remains to show that *the product of any two rotations of \mathbb{S}^2 is a rotation*.

Suppose that the two rotations of \mathbb{S}^2 are

- a rotation through angle θ about point P (that is, a rotation with axis through P and its antipodal point $-P$),
- a rotation through angle φ about point Q .

We have established that a rotation through θ about P is the product of reflections in “lines” (great circles) through P . Moreover, they can be *any* “lines” \mathcal{L} and \mathcal{M} through P as long as the angle between \mathcal{L} and \mathcal{M} is $\theta/2$. In particular, we can take the line \mathcal{M} to go through P and Q . Similarly, a rotation through φ about Q is the product of reflections in *any* lines through Q meeting at angle $\varphi/2$, so we can take the first “line” to be \mathcal{M} . The second “line” of reflection through Q is then the “line” \mathcal{N} at angle $\varphi/2$ from \mathcal{M} (Figure 7.5).

If $\bar{r}_{\mathcal{L}}, \bar{r}_{\mathcal{M}}, \bar{r}_{\mathcal{N}}$ denote the reflections in $\mathcal{L}, \mathcal{M}, \mathcal{N}$, respectively, then

rotation through θ about $P = \bar{r}_{\mathcal{M}}\bar{r}_{\mathcal{L}}$,

rotation through φ about $Q = \bar{r}_{\mathcal{N}}\bar{r}_{\mathcal{M}}$.

(Bear in mind that products of transformations are read from right to left, as this is the order in which functions are applied.) Hence, the product of these rotations is

$$\bar{r}_{\mathcal{N}}\bar{r}_{\mathcal{M}}\bar{r}_{\mathcal{M}}\bar{r}_{\mathcal{L}} = \bar{r}_{\mathcal{N}}\bar{r}_{\mathcal{L}}, \quad \text{because } \bar{r}_{\mathcal{M}}\bar{r}_{\mathcal{M}} \text{ is the identity.}$$

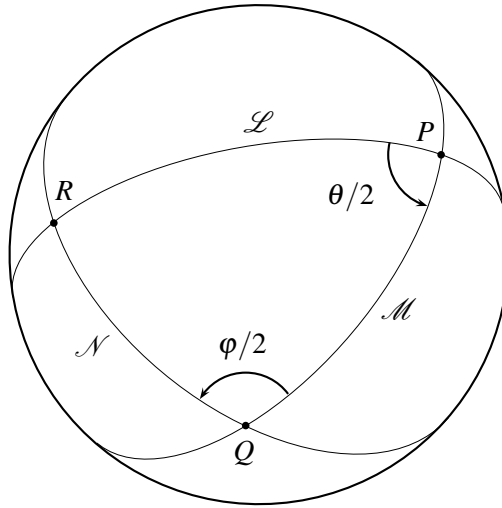


Figure 7.5: Reflection “lines” on the sphere

And it is clear from Figure 7.5 that $\bar{r}_N \bar{r}_L$ is a rotation (about the point R where \mathcal{L} meets \mathcal{N}). \square

Some special rotations

Before trying to obtain an overview of the rotation group of the sphere, it is helpful to look at the rotation group of the circle, which is analogous but considerably simpler.

The circle can be viewed as the unit *one-dimensional sphere* \mathbb{S}^1 in \mathbb{R}^2 , and its rotations are products of reflections in lines through O . This circumstance is what makes the rotation group of the circle similar to the rotation group of the sphere. What makes it a lot simpler is the fact that *each rotation of \mathbb{S}^1 corresponds to a point of \mathbb{S}^1* , because each rotation of \mathbb{S}^1 is determined by the point to which it sends the specific point $(1, 0)$. In other words, each rotation of the circle corresponds to an *angle*, namely the angle between the initial and final positions of any line through O . Also, rotations of \mathbb{S}^1 *commute*, because rotation through θ followed by rotation through φ results in rotation through $\theta + \varphi$, which is also the result of rotation through φ followed by rotation through θ .

In contrast to \mathbb{S}^1 , a rotation of \mathbb{S}^2 depends on three numbers: two angles that give the direction of its *axis*, and the amount of turn about this axis. Thus, the rotations of \mathbb{S}^2 cannot correspond to the points of \mathbb{S}^2 , although they do correspond to the points of an interesting three-dimensional space, as we shall see in Section 7.6.

Rotations of \mathbb{S}^2 generally do *not* commute, as can be seen by combining a quarter turn $z_{1/4}$ around the z -axis with a half-turn $x_{1/2}$ around the x -axis. Supposing that the quarter turn is in the direction that takes $(1, 0, 0)$ to $(0, 1, 0)$, we have

$$(1, 0, 0) \xrightarrow{z_{1/4}} (0, 1, 0) \xrightarrow{x_{1/2}} (0, -1, 0),$$

whereas

$$(1, 0, 0) \xrightarrow{x_{1/2}} (1, 0, 0) \xrightarrow{z_{1/4}} (0, 1, 0).$$

Exercises

In the Euclidean plane \mathbb{R}^2 , the product of a rotation about a point P and a rotation about a point Q is *not* necessarily a rotation.

7.5.1 Give an example of two rotations of \mathbb{R}^2 whose product is a translation.

7.5.2 By imitating the construction of rotations of \mathbb{S}^2 via reflections, explain how to decide whether the product of two rotations of \mathbb{R}^2 is a rotation and, if so, how to find its center and angle.

The group of all isometries of \mathbb{R}^2 , unlike the group of rotations of \mathbb{R}^2 about O , is not commutative.

7.5.3 Find a rotation and reflection of \mathbb{R}^2 that do not commute.

7.6 Representing space rotations by quaternions

The most elegant (and practical) way to describe rotations of \mathbb{R}^3 or \mathbb{S}^2 is with the help of the quaternions, which were introduced in Section 6.6. Because they appeared there only in exercises, we now review their basic properties for the sake of completeness.

A *quaternion* is a 2×2 matrix of the form

$$\mathbf{q} = \begin{pmatrix} a + ib & c + id \\ -c + id & a - ib \end{pmatrix}, \quad \text{where } a, b, c, d \in \mathbb{R} \text{ and } i^2 = -1.$$

We also write \mathbf{q} in the form $\mathbf{q} = a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, where

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

The various products of \mathbf{i} , \mathbf{j} , and \mathbf{k} are easily worked out by matrix multiplication, and one finds for example that $\mathbf{ij} = \mathbf{k} = -\mathbf{ji}$ and $\mathbf{i}^2 = -\mathbf{1}$.

Because \mathbf{q} corresponds to the quadruple (a, b, c, d) of real numbers, we can view \mathbf{q} as a point in \mathbb{R}^4 . If \mathbf{p} is an arbitrary point in \mathbb{R}^4 then the map sending $\mathbf{p} \mapsto \mathbf{pq}$ multiplies all distances in \mathbb{R}^4 by $|\mathbf{q}|$, the distance of \mathbf{q} from the origin. To see why, notice that

$$\det \mathbf{q} = a^2 + b^2 + c^2 + d^2 = |\mathbf{q}|^2.$$

Then it follows from the multiplicative property of determinants that

$$|\mathbf{pq}|^2 = \det(\mathbf{pq}) = (\det \mathbf{p})(\det \mathbf{q}) = |\mathbf{p}|^2 |\mathbf{q}|^2 \quad \text{and hence} \quad |\mathbf{pq}| = |\mathbf{p}| |\mathbf{q}|.$$

It follows that, for any points $\mathbf{p}_1, \mathbf{p}_2 \in \mathbb{R}^4$,

$$|\mathbf{p}_1 \mathbf{q} - \mathbf{p}_2 \mathbf{q}| = |(\mathbf{p}_1 - \mathbf{p}_2) \mathbf{q}| = |\mathbf{p}_1 - \mathbf{p}_2| |\mathbf{q}|.$$

Hence, the distance $|\mathbf{p}_1 - \mathbf{p}_2|$ between any two points is multiplied by the constant $|\mathbf{q}|$. In particular, *if $|\mathbf{q}| = 1$, then the map $\mathbf{p} \mapsto \mathbf{pq}$ is an isometry of \mathbb{R}^4 .*

The map $\mathbf{p} \mapsto \mathbf{qp}$ (which is not necessarily the same as the map $\mathbf{p} \mapsto \mathbf{pq}$, because quaternion multiplication is not commutative) is likewise an isometry when $|\mathbf{q}| = 1$. These maps are useful for studying rotations of \mathbb{R}^4 but, more surprisingly, also for studying rotations of \mathbb{R}^3 .

Rotations of $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ -space

If \mathbf{p} is any quaternion in $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ -space,

$$\mathbf{p} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}, \quad \text{where } x, y, z \in \mathbb{R},$$

and if \mathbf{q} is any nonzero quaternion, then it turns out that \mathbf{qpq}^{-1} also lies in $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ -space. Thus, *if $|\mathbf{q}| = 1$, then the map $\mathbf{p} \mapsto \mathbf{qpq}^{-1}$ defines an isometry of \mathbb{R}^3* , because $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ -space is just the space of real triples (x, y, z) and hence a copy of \mathbb{R}^3 .

Moreover, any quaternion with $|\mathbf{q}| = 1$ can be written in the form

$$\mathbf{q} = \cos \frac{\theta}{2} + (l\mathbf{i} + m\mathbf{j} + n\mathbf{k}) \sin \frac{\theta}{2}, \quad \text{where } l^2 + m^2 + n^2 = 1,$$

and the isometry $\mathbf{p} \mapsto \mathbf{qpq}^{-1}$ is a rotation of $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ -space through angle θ about the axis through $\mathbf{0}$ and $l\mathbf{i} + m\mathbf{j} + n\mathbf{k}$.

These facts can be confirmed by calculation, but we verify them only for the special case in which the axis of rotation is in the \mathbf{i} direction, and for special points \mathbf{p} that easily determine the nature of the isometry. Notice how the angles $\theta/2$ in \mathbf{q} and \mathbf{q}^{-1} combine to produce angle of rotation θ .

Example. The map $\mathbf{p} \mapsto \mathbf{qpq}^{-1}$, where $\mathbf{q} = \cos \frac{\theta}{2} + \mathbf{i} \sin \frac{\theta}{2}$.

First we check that any point $x\mathbf{i}$ on the \mathbf{i} -axis is fixed by this map.

$$\begin{aligned} \mathbf{qxiq}^{-1} &= \left(\cos \frac{\theta}{2} + \mathbf{i} \sin \frac{\theta}{2} \right) x\mathbf{i} \left(\cos \frac{\theta}{2} - \mathbf{i} \sin \frac{\theta}{2} \right) \\ &= \left(\cos \frac{\theta}{2} + \mathbf{i} \sin \frac{\theta}{2} \right) \left(x\mathbf{i} \cos \frac{\theta}{2} + x\mathbf{1} \sin \frac{\theta}{2} \right) \quad \text{because } \mathbf{i}^2 = -1 \\ &= x\mathbf{i} \left(\cos^2 \frac{\theta}{2} + \sin^2 \frac{\theta}{2} \right) + \mathbf{1} \left(\sin \frac{\theta}{2} \cos \frac{\theta}{2} - \sin \frac{\theta}{2} \cos \frac{\theta}{2} \right) \\ &= x\mathbf{i}. \end{aligned}$$

Next we check that the point \mathbf{j} is rotated through angle θ in the (\mathbf{j}, \mathbf{k}) -plane, to the point $\mathbf{j} \cos \theta + \mathbf{k} \sin \theta$.

$$\begin{aligned} \mathbf{qjq}^{-1} &= \left(\cos \frac{\theta}{2} + \mathbf{i} \sin \frac{\theta}{2} \right) \mathbf{j} \left(\cos \frac{\theta}{2} - \mathbf{i} \sin \frac{\theta}{2} \right) \\ &= \left(\cos \frac{\theta}{2} + \mathbf{i} \sin \frac{\theta}{2} \right) \left(\mathbf{j} \cos \frac{\theta}{2} + \mathbf{k} \sin \frac{\theta}{2} \right) \quad \text{because } \mathbf{ji} = -\mathbf{k} \\ &= \mathbf{j} \left(\cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2} \right) + \mathbf{k} \left(2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} \right) \quad \text{because } \mathbf{ik} = \mathbf{j}, \mathbf{ij} = \mathbf{k} \\ &= \mathbf{j} \cos \theta + \mathbf{k} \sin \theta. \end{aligned}$$

It can be similarly checked that $\mathbf{qkq}^{-1} = -\mathbf{k} \sin \theta + \mathbf{j} \cos \theta$. Hence the isometry $\mathbf{p} \mapsto \mathbf{qpq}^{-1}$ is a rotation of the (\mathbf{j}, \mathbf{k}) -plane through θ , because this is certainly what such a rotation does to the points $\mathbf{0}$, \mathbf{j} , and \mathbf{k} , and we know from Section 3.7 that any isometry of a plane is determined by what it does to three points not in a line.

Thus, the isometry $\mathbf{p} \mapsto \mathbf{qpq}^{-1}$ of $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ -space leaves the \mathbf{i} -axis fixed and rotates the (\mathbf{j}, \mathbf{k}) -plane through angle θ , so it is a rotation through θ about the \mathbf{i} -axis. \square

It should be emphasized that if the quaternion \mathbf{q} represents a certain rotation of \mathbb{R}^3 , then so does the opposite quaternion $-\mathbf{q}$, because $\mathbf{qpq}^{-1} = (-\mathbf{q})\mathbf{p}(-\mathbf{q})^{-1}$. Thus, rotations of \mathbb{R}^3 actually correspond to *pairs* of quaternions $\pm\mathbf{q}$ with $|\mathbf{q}| = 1$. This has interesting consequences when we try to interpret the group of rotations of \mathbb{R}^3 as a geometric object in its own right (Section 7.8).

Exercises

The representation of space rotations by quaternions is analogous to the representation of plane rotations by complex numbers, which was described in Section 4.7. As a warmup for the study of a *finite group of space rotations* in Section 7.7, we look here at some finite groups of plane rotations and the geometric objects they preserve. We take the plane to be \mathbb{C} , the complex numbers.

7.6.1 Consider the square with vertices $1, i, -1$, and $-i$. There is a group of four rotations of \mathbb{C} that map the square onto itself. These rotations correspond to multiplying \mathbb{C} by which four numbers?

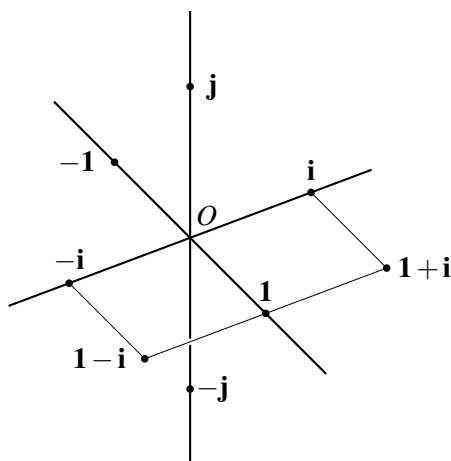
7.6.2 The *cyclic group* C_n is the group of n rotations that maps a regular n -gon onto itself. These rotations correspond to multiplying \mathbb{C} by which n complex numbers?

The noncommutative multiplication of quaternions is a blessing when we want to use them to represent space rotations, because we know that products of space rotations do not generally commute. Nevertheless, one wonders whether there is a reasonable commutative “product” operation on any \mathbb{R}^n , for any $n \geq 3$. “Reasonable” here includes the property $|uv| = |u||v|$ that holds for products on \mathbb{R} and \mathbb{R}^2 (the real and complex numbers), and the field axioms from Section 6.5.

In particular, there should be a *multiplicative identity*: a point $\mathbf{1}$ such that $|\mathbf{1}| = 1$ and $\mathbf{u}\mathbf{1} = \mathbf{u}$ for any point \mathbf{u} . Moreover, because $n \geq 3$, we can find points \mathbf{i} and \mathbf{j} , also of absolute value 1, such that $\mathbf{1}, \mathbf{i}$, and \mathbf{j} are in mutually perpendicular directions from O . Figure 7.6 shows these points, together with their negatives.

7.6.3 Show that $|\mathbf{1} + \mathbf{i}| = \sqrt{2} = |\mathbf{1} - \mathbf{i}|$, and deduce from the assumptions about the product operation that $2 = |\mathbf{1} - \mathbf{i}^2|$, which means that the point $\mathbf{1} - \mathbf{i}^2$ is at distance 2 from O .

7.6.4 Show also that $|\mathbf{i}^2| = 1$, so the point $\mathbf{1} - \mathbf{i}^2$ is at distance 1 from $\mathbf{1}$. Conclude from this and Exercise 7.6.3 that $\mathbf{i}^2 = -\mathbf{1}$.

Figure 7.6: Points in perpendicular directions from O

7.6.5 Show similarly that $\mathbf{u}^2 = -\mathbf{1}$ for any point \mathbf{u} whose direction from O is perpendicular to the direction of $\mathbf{1}$, and whose absolute value is 1.

7.6.6 Given that \mathbf{i} and \mathbf{j} are in perpendicular directions, show (multiplying the whole space by \mathbf{i}) that so are $\mathbf{i}^2 = -\mathbf{1}$ and \mathbf{ij} , and hence so too are $\mathbf{1}$ and \mathbf{ij} .

7.6.7 Thus, \mathbf{ij} is one point \mathbf{u} for which $\mathbf{u}^2 = -\mathbf{1}$, by Exercise 7.6.5. Deduce that

$$-\mathbf{1} = (\mathbf{ij})^2 = (\mathbf{ij})(\mathbf{ij}) = \mathbf{jijj} \quad \text{by the commutative and associative laws}$$

and show that this leads to the contradiction $-\mathbf{1} = \mathbf{1}$.

Therefore, when $n \geq 3$, there is no product on \mathbb{R}^n that satisfies all the field axioms.

7.7 A finite group of space rotations

\mathbb{R}^3 is home to the *regular polyhedra*, the remarkable symmetric objects discussed in Section 1.6 and shown in Figure 1.19. The best known of them is the cube, and the simplest of them is the *tetrahedron*, which fits inside the cube as shown in Figure 7.7.

Also shown in this figure are some rotations of the tetrahedron that are called *symmetries* because they preserve its appearance. If we choose a fixed position of the tetrahedron—a “tetrahedral hole” in space as it were—then these rotations bring the tetrahedron to positions where it once again

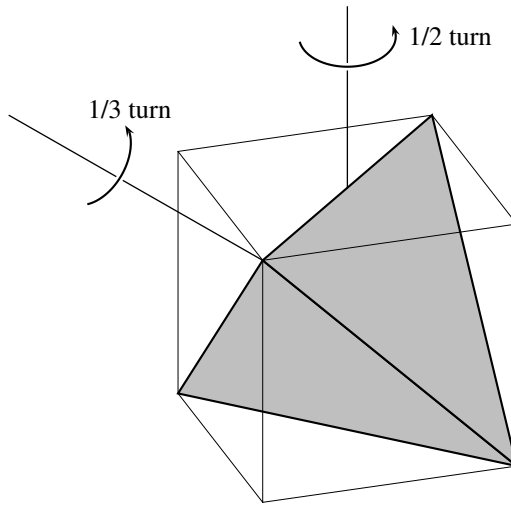


Figure 7.7: The tetrahedron and axes of rotation

fits in the hole. Altogether there are 12 such rotations. We can choose any one of the four faces to match a fixed face of the hole, say, the *front* face. Each of the four faces that can go in front has three edges that can match a given edge, say, the *bottom* edge, in the front face of the hole. Thus, we have $4 \times 3 = 12$ ways in which the tetrahedron can occupy the same position, each corresponding to a different symmetry. But once we have chosen a particular face to go in front, and a particular edge of that face to go on the bottom, we know where everything goes, so the symmetry is completely determined. Hence, there are exactly 12 rotational symmetries.

Each symmetry can be obtained, from a given initial position, by rotations like those shown in Figure 7.7. First there is the *trivial rotation*, which gives the *identity symmetry*, obtained by rotation through angle zero (about any axis). Then there are 11 nontrivial rotations, divided into two different types:

- The first type is a $1/2$ turn about an axis through centers of opposite edges of the tetrahedron (which also goes through opposite face centers of the cube). There are three such axes. Hence, there are three rotations of this type.
- The second type is a $1/3$ turn about an axis through a vertex and the center of the face opposite to it (which also goes through opposite

vertices of the cube). There are four such axes, and hence eight rotations of this type—because the $1/3$ turn clockwise is different from the $1/3$ turn anticlockwise.

Notice also that each $1/2$ turn moves all four vertices, whereas each $1/3$ turn leaves one vertex fixed and moves the remaining three. Thus, the 11 nontrivial rotations are all different. Therefore, together with the trivial rotation, they account for all 12 symmetries of the tetrahedron.

The quaternions representing rotations of the tetrahedron

As explained in Section 7.6, a rotation of $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ -space through angle θ about axis $l\mathbf{i} + m\mathbf{j} + n\mathbf{k}$ corresponds to a quaternion pair $\pm \mathbf{q}$, where

$$\mathbf{q} = \cos \frac{\theta}{2} + (l\mathbf{i} + m\mathbf{j} + n\mathbf{k}) \sin \frac{\theta}{2}.$$

If we choose coordinate axes so that the sides of the cube in Figure 7.7 are parallel to the \mathbf{i} , \mathbf{j} , and \mathbf{k} axes, then the axes of rotation are virtually immediate, and the corresponding quaternions are easy to work out.

- We can take the lines through opposite face centers of the cube to be the \mathbf{i} , \mathbf{j} , and \mathbf{k} axes. For a $1/2$ turn, the angle $\theta = \pi$, and hence $\theta/2 = \pi/2$. Therefore, because $\cos \frac{\pi}{2} = 0$ and $\sin \frac{\pi}{2} = 1$, the $1/2$ turns about the \mathbf{i} , \mathbf{j} , and \mathbf{k} axes are given by the quaternions \mathbf{i} , \mathbf{j} , and \mathbf{k} themselves.

Thus, the three $1/2$ turns are represented by the three quaternion pairs

$$\pm \mathbf{i}, \quad \pm \mathbf{j}, \quad \pm \mathbf{k}.$$

- Given the choice of \mathbf{i} , \mathbf{j} , and \mathbf{k} axes, the four rotation axes through opposite vertices of the cube correspond to four quaternion pairs, which together make up the eight combinations

$$\frac{1}{\sqrt{3}}(\pm \mathbf{i} \pm \mathbf{j} \pm \mathbf{k}) \quad (\text{independent choices of } + \text{ or } - \text{ sign}).$$

The factor $\frac{1}{\sqrt{3}}$ is to give each of these quaternions the absolute value 1, as specified for the representation of rotations.