10.14 Basic k-forms If $i_1, \ldots, i_k$ are integers such that $1 \le i_1 < i_2 < \cdots < i_k \le n$, and if I is the ordered k-tuple $\{i_1, \ldots, i_k\}$, then we call I an *increasing k-index*, and we use the brief notation

$$dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}. \tag{44}$$

These forms dx_I are the so-called *basic k-forms in R^n*.

It is not hard to verify that there are precisely $n!/k!(n-k)!$ basic k-forms in R^n; we shall make no use of this, however.

Much more important is the fact that every k-form can be represented in terms of basic k-forms. To see this, note that every k-tuple $\{j_1, \ldots, j_k\}$ of distinct integers can be converted to an increasing k-index J by a finite number of interchanges of pairs; each of these amounts to a multiplication by -1, as we saw in Sec. 10.13; hence

$$dx_{j_1} \wedge \cdots \wedge dx_{j_k} = \varepsilon(j_1, \ldots, j_k)\, dx_J \tag{45}$$

where $\varepsilon(j_1, \ldots, j_k)$ is 1 or -1, depending on the number of interchanges that are needed. In fact, it is easy to see that

$$\varepsilon(j_1, \ldots, j_k) = s(j_1, \ldots, j_k) \tag{46}$$

where s is as in Definition 9.33.

For example,

$$dx_1 \wedge dx_5 \wedge dx_3 \wedge dx_2 = -dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_5$$

and

$$dx_4 \wedge dx_2 \wedge dx_3 = dx_2 \wedge dx_3 \wedge dx_4.$$

If every k-tuple in (34) is converted to an increasing k-index, then we obtain the so-called *standard presentation* of ω:

$$\omega = \sum_I b_I(\mathbf{x})\, dx_I. \tag{47}$$

The summation in (47) extends over all increasing k-indices I. [Of course, every increasing k-index arises from many (from $k!$, to be precise) k-tuples. Each b_I in (47) may thus be a sum of several of the coefficients that occur in (34).]

For example,

$$x_1\, dx_2 \wedge dx_1 - x_2\, dx_3 \wedge dx_2 + x_3\, dx_2 \wedge dx_3 + dx_1 \wedge dx_2$$

is a 2-form in R^3 whose standard presentation is

$$(1 - x_1)\, dx_1 \wedge dx_2 + (x_2 + x_3)\, dx_2 \wedge dx_3.$$

The following uniqueness theorem is one of the main reasons for the introduction of the standard presentation of a k-form.

10.15 Theorem *Suppose*

$$\omega = \sum_I b_I(\mathbf{x})\, dx_I \tag{48}$$

is the standard presentation of a k-form ω in an open set $E \subset R^n$. If $\omega = 0$ in E, then $b_I(\mathbf{x}) = 0$ for every increasing k-index I and for every $\mathbf{x} \in E$.

Note that the analogous statement would be false for sums such as (34), since, for example,

$$dx_1 \wedge dx_2 + dx_2 \wedge dx_1 = 0.$$

Proof Assume, to reach a contradiction, that $b_J(\mathbf{v}) > 0$ for some $\mathbf{v} \in E$ and for some increasing k-index $J = \{j_1, \ldots, j_k\}$. Since b_J is continuous, there exists $h > 0$ such that $b_J(\mathbf{x}) > 0$ for all $\mathbf{x} \in R^n$ whose coordinates satisfy $|x_i - v_i| \le h$. Let D be the k-cell in R^k such that $\mathbf{u} \in D$ if and only if $|u_r| \le h$ for $r = 1, \ldots, k$. Define

$$\Phi(\mathbf{u}) = \mathbf{v} + \sum_{r=1}^{k} u_r \mathbf{e}_{j_r} \qquad (\mathbf{u} \in D). \tag{49}$$

Then Φ is a k-surface in E, with parameter domain D, and $b_J(\Phi(\mathbf{u})) > 0$ for every $\mathbf{u} \in D$.

We claim that

$$\int_\Phi \omega = \int_D b_J(\Phi(\mathbf{u}))\, d\mathbf{u}. \tag{50}$$

Since the right side of (50) is positive, it follows that $\omega(\Phi) \ne 0$. Hence (50) gives our contradiction.

To prove (50), apply (35) to the presentation (48). More specifically, compute the Jacobians that occur in (35). By (49),

$$\frac{\partial(x_{j_1}, \ldots, x_{j_k})}{\partial(u_1, \ldots, u_k)} = 1.$$

For any other increasing k-index $I \ne J$, the Jacobian is 0, since it is the determinant of a matrix with at least one row of zeros.

10.16 Products of basic k-forms Suppose

$$I = \{i_1, \ldots, i_p\}, \qquad J = \{j_1, \ldots, j_q\} \tag{51}$$

where $1 \le i_1 < \cdots < i_p \le n$ and $1 \le j_1 < \cdots < j_q \le n$. The *product* of the corresponding basic forms dx_I and dx_J in R^n is a $(p+q)$-form in R^n, denoted by the symbol $dx_I \wedge dx_J$, and defined by

$$dx_I \wedge dx_J = dx_{i_1} \wedge \cdots \wedge dx_{i_p} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_q}. \tag{52}$$

If I and J have an element in common, then the discussion in Sec. 10.13 shows that $dx_I \wedge dx_J = 0$.

If I and J have no element in common, let us write $[I, J]$ for the increasing $(p+q)$-index which is obtained by arranging the members of $I \cup J$ in increasing order. Then $dx_{[I, J]}$ is a basic $(p+q)$-form. We claim that

$$dx_I \wedge dx_J = (-1)^\alpha \, dx_{[I, J]} \tag{53}$$

where α is the number of differences $j_t - i_s$ that are *negative*. (The number of positive differences is thus $pq - \alpha$.)

To prove (53), perform the following operations on the numbers

$$i_1, \ldots, i_p ; j_1, \ldots, j_q . \tag{54}$$

Move i_p to the right, step by step, until its right neighbor is larger than i_p. The number of steps is the number of subscripts t such that $i_p < j_t$. (Note that 0 steps are a distinct possibility.) Then do the same for $i_{p-1}, \ldots, i_1$. The total number of steps taken is α. The final arrangement reached is $[I, J]$. Each step, when applied to the right side of (52), multiplies $dx_I \wedge dx_J$ by -1. Hence (53) holds.

Note that the right side of (53) is the standard presentation of $dx_I \wedge dx_J$.

Next, let $K = (k_1, \ldots, k_r)$ be an increasing r-index in $\{1, \ldots, n\}$. We shall use (53) to prove that

$$(dx_I \wedge dx_J) \wedge dx_K = dx_I \wedge (dx_J \wedge dx_K). \tag{55}$$

If any two of the sets I, J, K have an element in common, then each side of (55) is 0, hence they are equal.

So let us assume that I, J, K are pairwise disjoint. Let $[I, J, K]$ denote the increasing $(p+q+r)$-index obtained from their union. Associate β with the ordered pair (J, K) and γ with the ordered pair (I, K) in the way that α was associated with (I, J) in (53). The left side of (55) is then

$$(-1)^\alpha \, dx_{[I, J]} \wedge dx_K = (-1)^\alpha (-1)^{\beta+\gamma} \, dx_{[I, J, K]}$$

by two applications of (53), and the right side of (55) is

$$(-1)^\beta \, dx_I \wedge dx_{[J, K]} = (-1)^\beta (-1)^{\alpha+\gamma} \, dx_{[I, J, K]} .$$

Hence (55) is correct.

10.17 Multiplication Suppose ω and λ are p- and q-forms, respectively, in some open set $E \subset R^n$, with standard presentations

$$\omega = \sum_I b_I(\mathbf{x}) \, dx_I , \qquad \lambda = \sum_J c_J(\mathbf{x}) \, dx_J \tag{56}$$

where I and J range over all increasing p-indices and over all increasing q-indices taken from the set $\{1, \ldots, n\}$.

Their product, denoted by the symbol $\omega \wedge \lambda$, is defined to be

$$\omega \wedge \lambda = \sum_{I,J} b_I(\mathbf{x})c_J(\mathbf{x})\, dx_I \wedge dx_J. \tag{57}$$

In this sum, I and J range independently over their possible values, and $dx_I \wedge dx_J$ is as in Sec. 10.16. Thus $\omega \wedge \lambda$ is a $(p+q)$-form in E.

It is quite easy to see (we leave the details as an exercise) that the distributive laws

$$(\omega_1 + \omega_2) \wedge \lambda = (\omega_1 \wedge \lambda) + (\omega_2 \wedge \lambda)$$

and

$$\omega \wedge (\lambda_1 + \lambda_2) = (\omega \wedge \lambda_1) + (\omega \wedge \lambda_2)$$

hold, with respect to the addition defined in Sec. 10.13. If these distributive laws are combined with (55), we obtain the associative law

$$(\omega \wedge \lambda) \wedge \sigma = \omega \wedge (\lambda \wedge \sigma) \tag{58}$$

for arbitrary forms ω, λ, σ in E.

In this discussion it was tacitly assumed that $p \geq 1$ and $q \geq 1$. The product of a 0-form f with the p-form ω given by (56) is simply defined to be the p-form

$$f\omega = \omega f = \sum_I f(\mathbf{x})b_I(\mathbf{x})\, dx_I.$$

It is customary to write $f\omega$, rather than $f \wedge \omega$, when f is a 0-form.

10.18 Differentiation We shall now define a differentiation operator d which associates a $(k+1)$-form $d\omega$ to each k-form ω of class $\mathscr{C}'$ in some open set $E \subset R^n$.

A 0-form of class $\mathscr{C}'$ in E is just a real function $f \in \mathscr{C}'(E)$, and we define

$$df = \sum_{i=1}^{n} (D_i f)(\mathbf{x})\, dx_i. \tag{59}$$

If $\omega = \Sigma b_I(\mathbf{x})\, dx_I$ is the standard presentation of a k-form ω, and $b_I \in \mathscr{C}'(E)$ for each increasing k-index I, then we define

$$d\omega = \sum_I (db_I) \wedge dx_I. \tag{60}$$

10.19 Example Suppose E is open in R^n, $f \in \mathscr{C}'(E)$, and γ is a continuously differentiable curve in E, with domain $[0, 1]$. By (59) and (35),

$$\int_\gamma df = \int_0^1 \sum_{i=1}^{n} (D_i f)(\gamma(t))\gamma_i'(t)\, dt. \tag{61}$$

By the chain rule, the last integrand is $(f \circ \gamma)'(t)$. Hence

$$\int_\gamma df = f(\gamma(1)) - f(\gamma(0)), \tag{62}$$

and we see that $\int_\gamma df$ is the same for all γ with the same initial point and the same end point, as in (*a*) of Example 10.12.

Comparison with Example 10.12(*b*) shows therefore that the 1-form $x\,dy$ is not the derivative of any 0-form f. This could also be deduced from part (*b*) of the following theorem, since

$$d(x\,dy) = dx \wedge dy \neq 0.$$

10.20 Theorem

(*a*) *If ω and λ are k- and m-forms, respectively, of class $\mathscr{C}'$ in E, then*

$$d(\omega \wedge \lambda) = (d\omega) \wedge \lambda + (-1)^k \omega \wedge d\lambda. \tag{63}$$

(*b*) *If ω is of class $\mathscr{C}''$ in E, then $d^2\omega = 0$.*

Here $d^2\omega$ means, of course, $d(d\omega)$.

Proof Because of (57) and (60), (*a*) follows if (63) is proved for the special case

$$\omega = f\,dx_I, \qquad \lambda = g\,dx_J \tag{64}$$

where $f, g \in \mathscr{C}'(E)$, dx_I is a basic k-form, and dx_J is a basic m-form. [If k or m or both are 0, simply omit dx_I or dx_J in (64); the proof that follows is unaffected by this.] Then

$$\omega \wedge \lambda = fg\,dx_I \wedge dx_J.$$

Let us assume that I and J have no element in common. [In the other case each of the three terms in (63) is 0.] Then, using (53),

$$d(\omega \wedge \lambda) = d(fg\,dx_I \wedge dx_J) = (-1)^\alpha d(fg\,dx_{[I,J]}).$$

By (59), $d(fg) = f\,dg + g\,df$. Hence (60) gives

$$\begin{aligned} d(\omega \wedge \lambda) &= (-1)^\alpha (f\,dg + g\,df) \wedge dx_{[I,J]} \\ &= (g\,df + f\,dg) \wedge dx_I \wedge dx_J. \end{aligned}$$

Since dg is a 1-form and dx_I is a k-form, we have

$$dg \wedge dx_I = (-1)^k dx_I \wedge dg,$$

by (42). Hence

$$d(\omega \wedge \lambda) = (df \wedge dx_I) \wedge (g\, dx_J) + (-1)^k (f\, dx_I) \wedge (dg \wedge dx_J)$$
$$= (d\omega) \wedge \lambda + (-1)^k \omega \wedge d\lambda,$$

which proves (*a*).

Note that the associative law (58) was used freely.

Let us prove (*b*) first for a 0-form $f \in \mathscr{C}''$:

$$\begin{aligned} d^2 f &= d\left(\sum_{j=1}^{n} (D_j f)(\mathbf{x})\, dx_j\right) \\ &= \sum_{j=1}^{n} d(D_j f) \wedge dx_j \\ &= \sum_{i,j=1}^{n} (D_{ij} f)(\mathbf{x})\, dx_i \wedge dx_j . \end{aligned}$$

Since $D_{ij} f = D_{ji} f$ (Theorem 9.41) and $dx_i \wedge dx_j = -dx_j \wedge dx_i$, we see that $d^2 f = 0$.

If $\omega = f\, dx_I$, as in (64), then $d\omega = (df) \wedge dx_I$. By (60), $d(dx_I) = 0$. Hence (63) shows that

$$d^2 \omega = (d^2 f) \wedge dx_I = 0.$$

10.21 Change of variables Suppose E is an open set in R^n, T is a $\mathscr{C}'$-mapping of E into an open set $V \subset R^m$, and ω is a k-form in V, whose standard presentation is

$$\omega = \sum_I b_I(\mathbf{y})\, dy_I . \tag{65}$$

(We use $\mathbf{y}$ for points of V, $\mathbf{x}$ for points of E.)

Let $t_1, \ldots, t_m$ be the components of T: If

$$\mathbf{y} = (y_1, \ldots, y_m) = T(\mathbf{x})$$

then $y_i = t_i(\mathbf{x})$. As in (59),

$$dt_i = \sum_{j=1}^{n} (D_j t_i)(\mathbf{x})\, dx_j \qquad (1 \le i \le m). \tag{66}$$

Thus each dt_i is a 1-form in E.

The mapping T transforms ω into a k-form ω_T in E, whose definition is

$$\omega_T = \sum_I b_I(T(\mathbf{x}))\, dt_{i_1} \wedge \cdots \wedge dt_{i_k} . \tag{67}$$

In each summand of (67), $I = \{i_1, \ldots, i_k\}$ is an increasing k-index.

Our next theorem shows that addition, multiplication, and differentiation of forms are defined in such a way that they commute with changes of variables.

10.22 Theorem *With E and T as in Sec. 10.21, let ω and λ be k- and m-forms in V, respectively. Then*

(*a*) $(\omega + \lambda)_T = \omega_T + \lambda_T$ *if* $k = m$;
(*b*) $(\omega \wedge \lambda)_T = \omega_T \wedge \lambda_T$;
(*c*) $d(\omega_T) = (d\omega)_T$ *if ω is of class $\mathscr{C}'$ and T is of class $\mathscr{C}''$.*

Proof Part (*a*) follows immediately from the definitions. Part (*b*) is almost as obvious, once we realize that

$$(dy_{i_1} \wedge \cdots \wedge dy_{i_r})_T = dt_{i_1} \wedge \cdots \wedge dt_{i_r} \tag{68}$$

regardless of whether $\{i_1, \ldots, i_r\}$ is increasing or not; (68) holds because the same number of minus signs are needed on each side of (68) to produce increasing rearrangements.

We turn to the proof of (*c*). If f is a 0-form of class $\mathscr{C}'$ in V, then

$$f_T(\mathbf{x}) = f(T(\mathbf{x})), \qquad df = \sum_i (D_i f)(\mathbf{y})\, dy_i.$$

By the chain rule, it follows that

$$\begin{aligned} d(f_T) &= \sum_j (D_j f_T)(\mathbf{x})\, dx_j \\ &= \sum_j \sum_i (D_i f)(T(\mathbf{x}))(D_j t_i)(\mathbf{x})\, dx_j \\ &= \sum_i (D_i f)(T(\mathbf{x}))\, dt_i \\ &= (df)_T. \end{aligned} \tag{69}$$

If $dy_I = dy_{i_1} \wedge \cdots \wedge dy_{i_k}$, then $(dy_I)_T = dt_{i_1} \wedge \cdots \wedge dt_{i_k}$, and Theorem 10.20 shows that

$$d((dy_I)_T) = 0. \tag{70}$$

(This is where the assumption $T \in \mathscr{C}''$ is used.)

Assume now that $\omega = f\, dy_I$. Then

$$\omega_T = f_T(\mathbf{x})\, (dy_I)_T$$

and the preceding calculations lead to

$$\begin{aligned} d(\omega_T) &= d(f_T) \wedge (dy_I)_T = (df)_T \wedge (dy_I)_T \\ &= ((df) \wedge dy_I)_T = (d\omega)_T. \end{aligned}$$

The first equality holds by (63) and (70), the second by (69), the third by part (*b*), and the last by the definition of $d\omega$.

The general case of (*c*) follows from the special case just proved, if we apply (*a*). This completes the proof.

Our next objective is Theorem 10.25. This will follow directly from two other important transformation properties of differential forms, which we state first.

10.23 Theorem *Suppose T is a $\mathscr{C}'$-mapping of an open set $E \subset R^n$ into an open set $V \subset R^m$, S is a $\mathscr{C}'$-mapping of V into an open set $W \subset R^p$, and ω is a k-form in W, so that ω_S is a k-form in V and both $(\omega_S)_T$ and ω_{ST} are k-forms in E, where ST is defined by $(ST)(\mathbf{x}) = S(T(\mathbf{x}))$. Then*

$$(\omega_S)_T = \omega_{ST}. \tag{71}$$

Proof If ω and λ are forms in W, Theorem 10.22 shows that

$$((\omega \wedge \lambda)_S)_T = (\omega_S \wedge \lambda_S)_T = (\omega_S)_T \wedge (\lambda_S)_T$$

and

$$(\omega \wedge \lambda)_{ST} = \omega_{ST} \wedge \lambda_{ST}.$$

Thus if (71) holds for ω and for λ, it follows that (71) also holds for $\omega \wedge \lambda$. Since every form can be built up from 0-forms and 1-forms by addition and multiplication, and since (71) is trivial for 0-forms, it is enough to prove (71) in the case $\omega = dz_q$, $q = 1, \ldots, p$. (We denote the points of E, V, W by $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, respectively.)

Let $t_1, \ldots, t_m$ be the components of T, let $s_1, \ldots, s_p$ be the components of S, and let $r_1, \ldots, r_p$ be the components of ST. If $\omega = dz_q$, then

$$\omega_S = ds_q = \sum_j (D_j s_q)(\mathbf{y})\, dy_j,$$

so that the chain rule implies

$$\begin{aligned}(\omega_S)_T &= \sum_j (D_j s_q)(T(\mathbf{x}))\, dt_j \\ &= \sum_j (D_j s_q)(T(\mathbf{x})) \sum_i (D_i t_j)(\mathbf{x})\, dx_i \\ &= \sum_i (D_i r_q)(\mathbf{x})\, dx_i = dr_q = \omega_{ST}.\end{aligned}$$

10.24 Theorem *Suppose ω is a k-form in an open set $E \subset R^n$, Φ is a k-surface in E, with parameter domain $D \subset R^k$, and Δ is the k-surface in R^k, with parameter domain D, defined by $\Delta(\mathbf{u}) = \mathbf{u}\,(\mathbf{u} \in D)$. Then*

$$\int_\Phi \omega = \int_\Delta \omega_\Phi.$$

Proof We need only consider the case

$$\omega = a(\mathbf{x})\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$