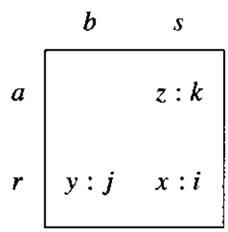
Assign labels $1, 2, \ldots, n$ so that vertex (r, s) has label $r + s - 1 \bmod n$. Define an orientation D of $K_n \square K_n$ by directing edges from vertex (r, s) with label i to the vertices in column s with lower labels and the vertices in row r with higher labels. Since i is higher than $i - 1$ other labels, (r, s) has $i - 1$ successors in its column and $n - i$ successors in its row. Hence $d^+(r, s) = d^-(r, s) = n - 1$.

We prove that D is kernel-perfect. Given $U \subseteq V(D)$, we obtain a kernel for the subdigraph $D[U]$ by solving a stable matching problem. When $(r, b) \in U$ and $(r, s) \to (r, b)$ in D, we want r to prefer b to s. Thus for row r, the preferences among columns begin with $\{s\colon (r, s) \in U\}$ in decreasing order of vertex labels, followed by any order among $\{s\colon (r, s) \notin U\}$. Similarly, for column s, the preferences among rows begin with $\{r\colon (r, s) \in U\}$ in increasing order of vertex labels, followed by any order among $\{r\colon (r, s) \notin U\}$.

The Gale–Shapley Proposal Algorithm (Algorithm 3.2.17) yields a stable matching M for these preferences. Viewing the matched pairs in M as positions in the grid, let $S = M \cap U$. Because M is a matching, S has no two positions in the same row or column; hence S is an independent set in D. We show that each $x \in U - S$ has a successor in S.

Let i be the label of position $x = (r, s) \in U - S$. Since $S = M \cap U$, we have $x \notin M$. Thus M has a position $y = (r, b)$ with some label j and a position $z = (a, s)$ with some label k. Because M is stable, we cannot have both r preferring s to b and s preferring r to a. From this statement we deduce by the steps below that x has y or z as a successor in S. ■



$$\text{not}\quad \big[(r \text{ prefers } s \text{ to } b) \;\text{ and }\; (s \text{ prefers } r \text{ to } a)\big]$$
$$\text{not}\quad \big[(y \notin U \;\text{ or }\; i > j) \;\text{ and }\; (z \notin U \;\text{ or }\; i < k)\big]$$
$$(y \in U \;\text{ and }\; i < j) \;\text{ or }\; (z \in U \;\text{ and }\; i > k)$$
$$(x \to y \in S) \;\text{ or }\; (x \to z \in S)$$

8.4.31. Remark. The List Coloring Conjecture relates to another conjecture. A **total coloring** of G assigns a color to each vertex and to each edge so that colored objects have different colors when they are adjacent or incident. The Total Coloring Conjecture (Behzad [1965]) states that every simple graph G has a total coloring with at most $\Delta(G) + 2$ colors. Rosenfeld [1971] and Behzad [1971] provide results on special classes. The List Coloring Conjecture would yield an upper bound of $\Delta(G) + 3$, since every graph G has a total coloring with at most $\chi_l'(G) + 2$ colors (Exercise 25). ■

The List Coloring Conjecture has been studied for planar graphs. Ellingham and Goddyn [1996] proved that every k-regular k-edge-colorable planar graph is k-edge-choosable (using the Four Color Theorem).

The discussion of planar graphs brings us back to list coloring of vertices. Although planar graphs have chromatic number at most 4, Vizing [1976] and

Erdős–Rubin–Taylor [1979] conjectured that the maximum choice number on this class is 5. Voigt [1993] constructed a non-4-choosable planar graph with 238 vertices; Mirzakhani [1996] (Exercise 26) reduced this to 63 vertices (both examples generalize to infinite families). In fact, there are 3-colorable planar graphs that are not 4-choosable (Gutner [1996], Voigt–Wirth [1997]).

Thomassen [1994b] proved the upper bound (and also [1995] that planar graphs of girth 5 are 3-choosable). Often in inductive proofs for planar graphs, the vertices on the unbounded face ("external vertices") play a special role.

8.4.32. Theorem. (Thomassen [1994b]) Planar graphs are 5-choosable.

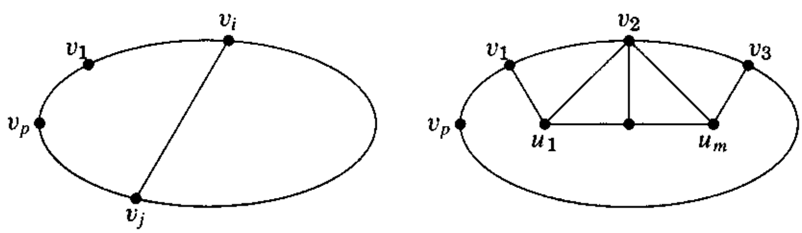
Proof: Adding edges cannot reduce the list chromatic number, so we may restrict our attention to plane graphs where the outer face is a cycle and every bounded face is a triangle. By induction on $n(G)$, we prove the stronger result that a coloring can be chosen even when two adjacent external vertices have distinct lists of size 1 and the other external vertices have lists of size 3. For the basis step ($n = 3$), a color remains available for the third vertex.

Now consider $n > 3$. Let v_p, v_1 be the vertices with fixed colors on the external cycle C. Let $v_1, \ldots, v_p$ be $V(C)$ in clockwise order.

Case 1: C has a chord v_iv_j with $1 \le i \le j-2 \le p-2$. We apply the induction hypothesis to the graph consisting of the cycle $v_1, \ldots, v_i, v_j, \ldots, v_p$ and its interior. This selects a proper coloring in which v_i, v_j receive some fixed colors. Next we apply the induction hypothesis to the graph consisting of the cycle $v_i, v_{i+1}, \ldots, v_j$ and its interior to complete the list coloring of G.

Case 2: C has no chord. Let $v_1, u_1, \ldots, u_m, v_3$ be the neighbors of v_2 in order ($3 = p$ is possible). Because bounded faces are triangles, G contains the path P with vertices $v_1, u_1, \ldots, u_m, v_3$. Since C is chordless, $u_1, \ldots, u_m$ are internal vertices, and the outer face of $G' = G - v_2$ is bounded by a cycle C' in which P replaces v_1, v_2, v_3.

Let c be the color assigned to v_1. Since $|L(v_2)| \ge 3$, we may choose distinct colors $x, y \in L(v_2) - \{c\}$. We reserve x, y for possible use on v_2 by forbidding x, y from $u_1, \ldots, u_m$. Since $|L(u_i)| \ge 5$, we have $|L(u_i) - \{x, y\}| \ge 3$. Hence we can apply the induction hypothesis to G', with $u_1, \ldots, u_m$ having lists of size at least 3 and other vertices having the same lists as in G. In the resulting coloring, v_1 and $u_1, \ldots, u_m$ have colors outside $\{x, y\}$. We extend this coloring to G by choosing for v_2 a color in $\{x, y\}$ that does not appear on v_3 in the coloring of G'. ∎



Case 1 Case 2

PARTITIONS USING PATHS AND CYCLES

We have considered the **F-decomposition** problem: partitioning $E(G)$ into the minimum number of subgraphs in a family **F**. This has been studied for many families **F**, such as cliques (Theorem 8.4.3), bipartite graphs (Exercise 3), complete bipartite graphs (Theorem 8.6.20), stars (vertex cover number—Section 3.1), and forests (arboricity—Corollary 8.2.57). Before considering extremal problems for decomposition of graphs into paths and cycles, we discuss an easier problem: covering the vertices of a digraph using the fewest paths.

Comparability graphs are those having transitive orientations; a digraph is **transitive** if $x \to y$ and $y \to z$ imply $x \to z$. The vertices of a path in a transitive digraph induce a tournament. Comparability graphs are perfect (Proposition 5.3.25), meaning that a transitive digraph D in which the largest tournament has ω vertices can be properly ω-colored. By the Perfect Graph Theorem (Theorem 8.1.6), we also know that $V(D)$ can be covered using $\alpha(D)$ tournaments in D, where $\alpha(D)$ is the maximum size of an independent set.

Letting paths be "chains" and independent sets be "antichains", this becomes Dilworth's Theorem for transitive loopless digraphs: The maximum size of an antichain equals the minimum number of chains needed to partition $V(D)$. In addition to following from the Perfect Graph Theorem, Dilworth's Theorem is equivalent to the König–Egerváry Theorem (Exercise 27), and a generalization of it follows from the Matroid Intersection Theorem (Exercise 8.2.50). Here we present a further generalization that has a short and self-contained proof.

8.4.33. Theorem. (Gallai–Milgram [1960]) The vertices of a digraph D can be covered using at most $\alpha(D)$ pairwise disjoint paths.

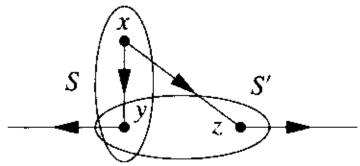
Proof: Since $V(D)$ can be covered using n disjoint paths of length 0, it suffices to prove a stronger claim: If **C** is a set of pairwise disjoint paths covering $V(D)$, and S is the set of sources (initial vertices) of these paths, then $V(D)$ can be covered using at most $\alpha(D)$ pairwise disjoint paths with sources in S. The proof is by induction on $n(D)$, with a trivial basis step for $n(D) = 1$. The added statement about the sources helps the induction step work.

Suppose that $n > 1$ and that **C** is a covering of $V(D)$ by k paths with source set S. The claim holds unless $|\mathbf{C}| = k > \alpha(D)$, in which case we construct a cover using fewer paths, all with sources in S. Since $k > \alpha$, there exists an edge xy with $x, y \in S$. Let A and B be the paths in **C** starting with x and y, respectively. We may assume that A has an edge xz, else we could add x to the beginning of B and save one path.

By deleting x from the start of A, we obtain a cover $\mathbf{C}'$ of $V(D - x)$ by k paths having sources in $S' = S - x + z$. Since $\alpha(D - x) \le \alpha(D)$, the induction hypothesis yields a cover $\mathbf{C}''$ of $V(D - x)$ using fewer than k paths, all with sources in S'. All elements of S' belong to S except z.

If z is the source of a path in $\mathbf{C}''$, then we add x at the beginning of that path. If z is not a source but y is, then we add x at the beginning of the path starting with y. If neither y nor z is a source, then at most $|S'| - 2 = k - 2$ paths have been used. and we can add x as a path by itself to obtain the desired cover of $V(D)$ using $k-1$ paths. In all cases, the resulting paths are pairwise disjoint and have sources in S.

By repeating this argument as long as $k > \alpha$, we can reduce the number of paths to α. ■



We return to the decomposition problem. Gallai conjectured that every n-vertex graph can be decomposed using $\lceil n/2 \rceil$ paths. Equality holds for cliques (Exercise 28). Other graphs have fewer edges, but the lack of connections could require more paths. Hajós conjectured analogously that an n-vertex even graph can be decomposed into $\lfloor n/2 \rfloor$ cycles. Both conjectures remain open, but Lovász proved the optimal bound when both paths and cycles are allowed. The **size** of a decomposition is the number of subgraphs used.

8.4.34. Theorem. (Lovász [1968b]) Every n-vertex graph can be decomposed into $\lfloor n/2 \rfloor$ paths and cycles.

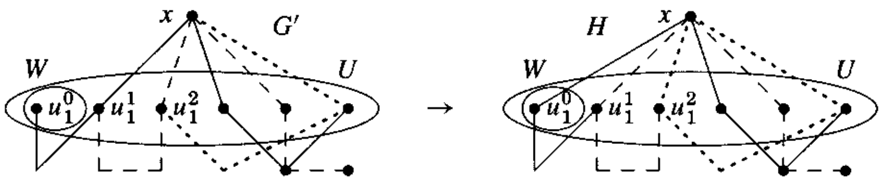
Proof: Let $\mathbf{F}$ be the family of all paths and cycles, and let $n'(G)$ be the number of non-isolated vertices in a graph G. By induction on $\lambda(G) = 2e(G) - n'(G)$, we prove that G has an $\mathbf{F}$-decomposition of size at most $\lfloor n'(G)/2 \rfloor$. Each component of G with more than one edge contributes positively to $\lambda(G)$. Hence $\lambda(G) \geq 0$, with equality only when each nontrivial component is an edge. The claim holds with equality when $\lambda(G) = 0$.

In the induction step, $\lambda(G) > 0$. We consider two cases. **Case 1**: If G has a vertex y of positive even degree, choose $x \in N(y)$, and let $W = \{z \in N(x)\colon d(z)$ is even$\}$. In this case, let $G' = G - \{xz : z \in W\}$. In obtaining G', we lose at least one edge (xy) and we isolate at most one vertex (x), so $\lambda(G') < \lambda(G)$. **Case 2**: If G has no vertex of positive even degree, then $\lambda(G) > 0$ forces $\Delta(G) > 1$. Let x be a vertex of degree at least 3, and form G^+ by introducing a new vertex y to subdivide an edge xx'. Let $W = \{y\}$, and let $G' = G^+ - xy$. Now $e(G') = e(G)$, but $n'(G') > n'(G)$, so $\lambda(G') < \lambda(G)$.

In each case, the induction hypothesis yields an $\mathbf{F}$-decomposition $\mathbf{D}$ of G' with $|\mathbf{D}| \leq \lfloor n'(G')/2 \rfloor$. We convert $\mathbf{D}$ into an $\mathbf{F}$-decomposition of size $|\mathbf{D}|$ for the graph H obtained from G' by adding edges from x to W. In Case 1, $H = G$ and $n'(G') \leq n'(G)$, so this is the desired decomposition. In Case 2, $H = G^+$ and $n'(G') = n'(G^+)$. Since $n'(G)$ is even, $\lfloor n'(G)/2 \rfloor = \lfloor n'(G^+)/2 \rfloor$. In an $\mathbf{F}$-decomposition of G^+, the $n'(G)$ vertices of odd degree must all be endpoints of paths; thus the added vertex y of degree 2 cannot be the end of a path. This means that xy and yx' belong to the same subgraph and can be replaced by xx' to obtain the desired decomposition of G.

The two cases now combine; we need only obtain the decomposition of H from $\mathbf{D}$. Let $U = N_H(x)$. Every vertex of U has odd degree in G', so for each

$u \in U$ there is a path $P(u)$ in $\mathbf{D}$ with endpoint u. For $u \in W$, we would like to extend $P(u)$ to absorb ux. This cannot be done if $P(u)$ reaches but does not end at x, since then the subgraph $P(u) \cup ux$ is not in $\mathbf{F}$. The idea is to cut the edge $u'x$ on which $P(u)$ reaches x, use the path $P(u) \cup ux - u'x$, and use $u'x$ to extend $P(u')$ instead. This generates a sequence of changes from each $u \in W$. We must show that the sequences terminate and do not conflict with each other.

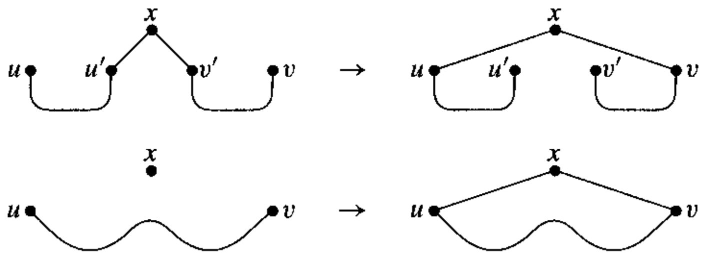


Let $W = w_1, \dots, w_t$. For $w_i \in W$, we form a list $u_i^0, u_i^2, \cdots$ with $u_i^0 = w_i$ and each $u_i^j \in U$. If in the ith list we have chosen a vertex u_i^j, we check whether x is an internal vertex of $P(u_i^j)$. If not, then we stop and do not define u_i^{j+1}. If so, then we set u_i^{j+1} to be the vertex on $P(u_i^j)$ just before x; this is the "u'" suggested above. The path $P(u_i^j)$ for $j \geq 1$ cannot start along the edge $u_i^j x$, because that edge is internal to $P(u_i^{j-1})$. (Our picture of G' shows three successive paths: $P(u_1^0)$ solid, $P(u_1^1)$ dashed, $P(u_1^2)$ dotted.)

We prove next that no vertex of U appears twice in the lists. Since $xu_i^j \in E(G')$ if $j \geq 1$, the vertices of W appear only as initial vertices. Let u_i^j, u_k^l be a repeated vertex with $\min\{j, l\}$ minimal; we have shown that $j, l > 0$. By minimality, $u_i^{j-1} \neq u_k^{l-1}$, and hence the paths $P(u_i^{j-1})$ and $P(u_k^{l-1})$ start at distinct vertices. If $u_i^j = u_k^l$, then the two paths share the edge $u_i^j x$ and must be the same path. This happens from distinct vertices only if u_i^{j-1} and u_k^{l-1} are opposite ends of the path, but then they cannot both visit u_i^j before x. Hence no repetition occurs.

Let $W' = \{u_i^j\}$. If $u = u_i^j$ and u is not the end of its list, let $u' = u_i^{j+1}$. We define an $\mathbf{F}$-decomposition of G consisting of one path or cycle Q' corresponding to each $Q \in \mathbf{D}$. If $Q \neq P(u)$ for some $u \in W'$, let $Q' = Q$. If $Q = P(u)$, let $Q' = Q + ux$ or $Q' = Q + ux - u'x$ depending on whether u is or is not the last vertex in its list. Always Q' is a path, except that Q' is a cycle when Q ends at x (and then u' is not defined). The union of the new paths corresponding to $\{P(u_i^j)\}$ is the same as $\cup P(u_i^j)$, except that the edges $\{xw_i\}$ are absorbed. Since $u \in W'$ appears only once in the lists, the edge ux winds up in only one of the new paths, and $\{Q' \colon Q \in \mathbf{D}\}$ is a decomposition of H. ∎

Note that in this proof Q may be the selected path from each of its endpoints $u, v \in W'$. This is not a problem, because the adjustments to Q made from the two ends do not conflict. The path may visit x (thus defining u' and v') or not, as sketched below.



CIRCUMFERENCE

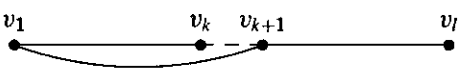
When a sufficient condition for Hamiltonian cycles fails slightly, we might expect that the graph still must have a fairly long cycle. The length of the longest cycle in G is the **circumference** $c(G)$. We first consider the number of edges needed to force a cycle of length at least c in an n-vertex graph. In this section, $P(v, w)$ denotes the v, w-portion of a path P containing v and w. Also P, Q denotes the concatenation of paths P and Q when the last vertex of P is the first vertex of Q.

8.4.35. Theorem. (Erdős–Gallai [1959]) For $m \geq 2$, every simple n-vertex graph with more than $m(n-1)/2$ edges has a cycle of length more than m.

Proof: (Woodall [1972]) We use induction on n for fixed m. When $n = m + 1$, fewer than $(n-1)/2$ edges are missing, so $\delta(G) \geq n/2$ and G is Hamiltonian. Suppose that $n > m+1$ and $c(G) \leq m$. If $d(x) \leq m/2$, then $e(G-x) \geq m(n-2)/2$. Applying the induction hypothesis to $G - x$ yields $c(G-x) > m$. Hence we may assume that $\delta(G) > m/2$. Similarly, we may assume that G is connected.

Among all longest paths in G, choose $P = v_1, \dots, v_l$ to maximize the degree d of v_1; since G is connected, we have $v_1 \not\leftrightarrow v_l$ (otherwise an edge from $V(P)$ to $V(G) - V(P)$ would yield a longer cycle). Let $W = \{v_i \colon v_1 \leftrightarrow v_{i+1}\}$. All neighbors of v_1 lie on P, so $|W| = d$. For $v_k \in W$, the path $P(v_k, v_1), v_1 v_{k+1}, P(v_{k+1}, v_l)$ also has length l; hence $N(v_k) \subseteq V(P)$, and the choice of P yields $d(v_k) \leq d$. Furthermore, no $v_k \in W$ has a neighbor v_j such that $j > m$, because then we could complete the long cycle by adding $v_j v_k$ to $P(v_k, v_1), v_1 v_{k+1}, P(v_{k+1}, v_j)$.

By limiting the edges incident to W, we force many edges into $G - W$. Let $Z = \{v_1, \dots, v_r\}$, where $r = \min\{l, m\}$. For each $v_k \in W$, we have shown that $N(v_k) \subseteq Z$. Hence there are $|[W, Z - W]| + e(G[W])$ edges incident to W. For fixed degree-sum in W, this is maximized when $[W, Z - W]$ is a complete bipartite graph. We further maximize by letting each vertex of W have degree d. The resulting count is $\frac{1}{2}|W|(d + |Z - W|) = dr/2 \leq dm/2$. Therefore, $G - W$ has $n - d$ vertices and more than $m(n-d-1)/2$ edges. By the induction hypothesis $c(G - W) > m$. (If the number of edges forced into $G - W$ is too large to exist, then this case cannot occur, and an earlier case applies.) ■



Most sufficient conditions for Hamiltonian cycles have "long cycle" versions. The long cycle version of Dirac's Theorem says that a 2-connected graph G has a cycle of length at least $\min\{n(G), 2\delta(G)\}$ (Dirac [1952b]). Requiring 2-connectedness eliminates the example $K_1 \vee 2K_\delta$ with circumference $\delta + 1$.

The long cycle version of Ore's Theorem [1960] came much later. It is implicit in Bondy [1971b] and was made explicit in Bermond [1976] and in Linial [1976]. The fundamental argument used in many long cycle results appears in Bondy [1971b]. It strengthens the Ore/Dirac switching argument (Theorem 7.2.8) by considering "gaps".

8.4.36. Lemma. (Bondy [1971b]) If $P = v_1, \ldots, v_l$ is a longest path in a 2-connected graph G, then $c(G) \ge \min\{n(G), d(v_1) + d(v_l)\}$.

Proof: (See also Linial [1976]). Let $m = d(v_1) + d(v_l)$, and suppose that $c(G) < \min\{n(G), m\}$. Since G is connected, an l-cycle would yield a longer path; thus $v_1 \not\leftrightarrow v_l$. If $v_1 \leftrightarrow v_j$ and $v_i \leftrightarrow v_l$ for some $i < j$, then i, j is a *crossover* with *gap* $j - i$. If we add $v_1 v_j$ and $v_l v_i$ to $P(j, l)$ and $P(i, 1)$, we obtain a cycle with length $l - (j - i - 1)$. Hence $l - (j - i - 1) < m$ when i, j is a crossover.

Let $x = v_1$ and $y = v_l$. If P has a crossover, let i, j be one with smallest gap. Thus x and y have no neighbors between v_i and v_j on P. Also $N(y)$ contains no predecessor on P of a neighbor of x, since an l-cycle yields a longer path. Hence $N(y)$ lies in $V(P) - \{y\}$ but avoids $\{v_{i+1}, \ldots, v_{j-2}\}$ and $\{v_{r-1} : v_r \leftrightarrow x\}$. Thus $d(y) \le (l - 1) - (j - 2 - i) - d(x)$. Since $l - (j - i - 1) < m$, we have $d(x) + d(y) < m$, which contradicts the hypothesis. Hence there is no crossover.

With $t_0 = \max\{i : x \leftrightarrow v_i\}$ and $u = \min\{i : y \leftrightarrow v_i\}$, we have proved that $t_0 \le u$. We will construct a cycle containing x and y and all their neighbors. Since the absence of crossovers implies that $|N(x) \cap N(y)| \le 1$, such a cycle has length at least $d(x) + d(y) + 1 > m$.

We iteratively define paths $P_1, P_2, \cdots$. Given t_{i-1}, we choose integers $s_i < t_{i-1} < t_i$ to maximize t_i such that G has a v_{s_i}, v_{t_i}-path P_i internally disjoint from P. Such a path exists because $G - v_{t-1}$ is connected. These paths are disjoint; if P_i shares a vertex with a later path P_j, then we can choose P_i as an s_i, t_j-path, which contradicts the maximality of t_i. Similarly, $s_{i+1} \ge t_{i-1}$, since otherwise P_{i+1} would be chosen instead of P_i.

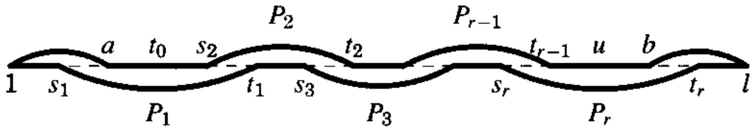
Let r be the smallest index such that $t_r > u$. Set

$$a = \min\{j : x \leftrightarrow v_j \text{ and } j > s_1\}, \qquad b = \max\{j : y \leftrightarrow v_j \text{ and } j < t_r\}.$$

Since $s_1 < t_0$ and $t_r > u$, the indices a, b are well-defined. We use the even-indexed paths P_i to build one x, y-path and the odd-indexed paths to build another x, y-path. When r is odd, the two paths are formed by the following concatenations.

$$x v_a,\ P(a, s_2),\ P_2,\ P(t_2, s_4),\ P_4,\ \ldots,\ P(t_{r-1}, b),\ v_b y$$

$$P(1, s_1),\ P_1,\ P(t_1, s_3),\ P_3,\ P(t_3, s_5),\ \ldots,\ P_r,\ P(t_r, l)$$



When r is even, the path starting with xv_a reaches t_r and ends with $P(t_r, l)$, while the other path reaches v_b and ends with $v_b y$.

We have observed that $s_{i+1} \geq t_{i-1}$. Hence

$$s_1 < a \leq t_0 \leq s_2 < t_1 \leq s_3 < t_2 \cdots < t_{r-1} \leq u \leq b < t_r$$

This implies that the two concatenations described are paths and that their union is a cycle. By the definition of a, we have $N(x) \subseteq P(1, s_1) \cup P(a, t_0)$, and similarly $N(y) \subseteq P(u, b) \cup P(t_r, l)$. With x and y themselves, the cycle thus has length at least $2 + d(x) + d(y) - 1 > m$. ■

Ore proved that G is Hamiltonian if $d(u) + d(v) \geq n(G)$ when $u \not\leftrightarrow v$. Bondy's Lemma implies the long cycle version of this, which strengthens the long cycle version of Dirac's Theorem.

8.4.37. Theorem. (Bondy [1971b], Bermond [1976], Linial [1976]) If G is 2-connected and $d(u) + d(v) \geq s$ for every nonadjacent pair $u, v \in V(G)$, then $c(G) \geq \min\{n(G), s\}$.

Proof: Ore's Theorem guarantees a Hamiltonian cycle if $s \geq n$, so we may assume that $s < n$. Suppose that P is a longest path in G, with endpoints x and y. Since G is connected, the maximality of P implies that $x \not\leftrightarrow y$. Now the condition $d(x) + d(y) \geq s$ allows us to invoke Lemma 8.4.36. ■

Bermond extended this to a "long cycle" combination of Chvátal's condition and Las Vergnas' condition. The technique of edge-switches involving an endpoint of a longest path was used in Theorem 8.4.35. Our statement is slightly weaker than that of Bermond but has a simpler proof.

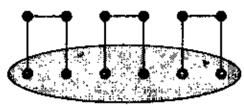
8.4.38. Theorem. (Bermond [1976]) Let G be a 2-connected graph with degree sequence $d_1 \leq \cdots \leq d_n$. If G has no nonadjacent pair x, y with degrees i, j such that $d_i \leq i < c/2$, $d_{j+1} \leq j$, and $i + j < c$, then $c(G) \geq c$.

Proof: Among the longest paths in G, let $P = v_1, \ldots, v_l$ with endpoints $x = v_1$ and $y = v_l$ be chosen to maximize $d(v_1) + d(v_l)$. If $d(x) + d(y) \geq c$, then we apply Bondy's Lemma. If $d(x) + d(y) < c$, then we claim that x, y contradicts the hypotheses. As usual, an l-cycle would yield a longer path (since G is connected), so $x \not\leftrightarrow y$. We may assume that $d(x) \leq d(y)$ and set $i = d(x)$ and $j = d(y)$.

All neighbors of x and y lie in P. If $x \leftrightarrow v_k$, then $P(v_{k-1}, x)$, xv_k, $P(v_k, y)$ is another longest path ending at y; thus $d(v_{k-1}) \leq d(x) = i$, by the choice of P. Since this holds for each of the i neighbors of x, we have $d_i \leq i$. Similarly, the j neighbors of y each have degree at most j. Also $d(x) \leq j$, so $d_{j+1} \leq j$. By hypothesis, $i + j = d(x) + d(y) < c$, which completes the contradiction. ■

G.-H. Fan [1984] strengthened Theorem 8.4.37 by weakening the degree condition and by requiring it only for nonadjacent pairs with common neighbors. T. Feng [1988] used Bondy's Lemma to shorten the proof. The result includes a sufficient condition for Hamiltonian cycles that does not require the closure to be complete.

8.4.39. Example. *A Hamiltonian graph.* For even n, let $G_1 = K_{n/2}$ and $G_2 = (n/4)K_2$, and form G by adding a matching between disjoint copies of G_1 and G_2. The Hamiltonian closure of G is G itself, so our previous sufficient conditions do not apply. Even though G has $n/2$ vertices of degree 2, Fan's Theorem implies that G is Hamiltonian. ∎

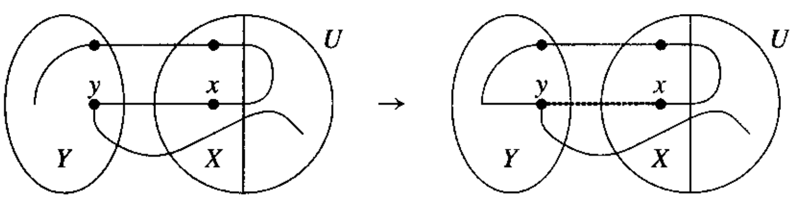


8.4.40. Theorem. (Fan [1984]) If G is 2-connected, and $d_G(u,v) = 2$ implies $\max\{d(u), d(v)\} \geq c/2$, then $c(G) \geq \min\{n(G), c\}$.

Proof: (Feng [1988]) Let $U = \{v \in V(G): d(v) \geq c/2\}$. By Bondy's Lemma, it suffices to find a longest path having both endpoints in U. Among the paths of maximum length, let $P = v_1, \ldots, v_m$ be one that has the maximum number of endpoints in U. If P fails to have both endpoints in U, then we will find a longer path or a path of the same length with more of its endpoints in U. We may assume that $v_1 \notin U$.

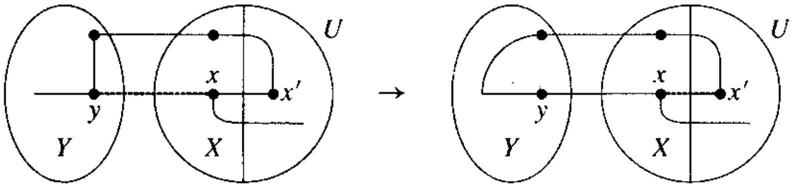
Since $d(v) < c/2$ for all $v \notin U$, the hypothesis on pairs with distance 2 implies that $G - U$ is a disjoint union of complete graphs. Let Y be the one containing v_1. Let X be the set of vertices in U having neighbors in Y. By the hypotheses, vertices of X have neighbors only in $Y \cup U$. Also $|X| \geq 2$, because G is 2-connected.

Let $r = |Y|$. We first show that P begins by visiting all of Y. If P omits some vertex of Y, then we can absorb it before the first exit from Y. If P leaves and returns to Y, then it returns via an edge xy. Because $G[Y]$ is complete, we can replace xy in P with v_1y, obtaining an x, v_m-path having the same length as P but more endpoints in U. Hence we may assume that $Y = \{v_1, \ldots, v_r\}$.



Consider $x \in X - v_{r+1}$. Suppose first that x has a neighbor $y \in Y$ other than the exit vertex v_r of P. If $x \notin V(P)$, then we can instead start with xy, absorb the rest of Y up to v_r, and thus complete an x, v_l-path longer than P. If

$x \in V(P)$, then we let x' be the vertex before x on P. Since $x \neq v_{r+1}$, we have $x' \in U$. We replace $x'x$ in P with yx, obtaining an x', v_l-path with the same length as P but more endpoints in U.



Hence we may assume for $x \in X - v_{r+1}$ that x has no neighbor in Y other than v_r. If $|Y| \geq 2$, this makes v_r a cut-vertex unless v_{r+1} has another neighbor $y \in Y - v_r$. Now we rearrange P to start with $v_r, \ldots, y, v_{r+1}$ instead of $v_1, \ldots, v_r, v_{r+1}$. This puts us in the case just discussed.

The remaining case is $|Y| = 1$ and $N(v_1) = X$. With $x \in X - v_{r+1}$ as before, we append x to the beginning of P or replace $x'x$ with xv_1. ∎

Finally, we present one result about digraphs that strengthens Ghouilà-Houri's sufficient condition (Theorem 7.2.22) for Hamiltonian cycles. We consider only loopless digraphs having at most one copy of each ordered pair as an edge; call these **strict** digraphs. For digraphs, we use "u, v nonadjacent" to mean $uv, vu \notin E(G)$. Also, we define $d(v) = d^+(v) + d^-(v)$.

Ghouilà-Houri [1960] actually proved that a digraph G is Hamiltonian if $d(v) \geq n(G)$ for each v; this is stronger than Theorem 7.2.22 as stated. Woodall [1972] proved that it suffices to have $d^+(u) + d^-(v) \geq n(G)$ whenever u, v are nonadjacent. This generalizes Ore's Theorem for undirected graphs (Exercise 33). Meyniel [1973] proved that a strict strong digraph G is Hamiltonian if $d(u)+d(v) \geq 2n(G)-1$ for all nonadjacent pairs u, v. Meyniel's Theorem implies Ghouilà-Houri's Theorem and Woodall's Theorem (Exercise 33).

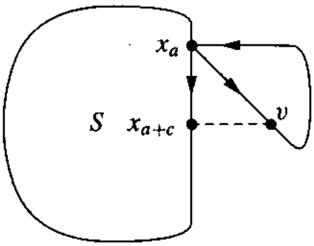
8.4.41. Example. *Meyniel's Theorem is best possible.* Let G consist of two doubly-directed cliques sharing a vertex. The digraph is strongly connected, and the only pairs of nonadjacent vertices consist of one vertex from each clique. If the cliques have order k and order $n + 1 - k$, then the total degrees for any nonadjacent pair are $2k - 2$ and $2n - 2k$, which sum to $2n - 2$. ∎

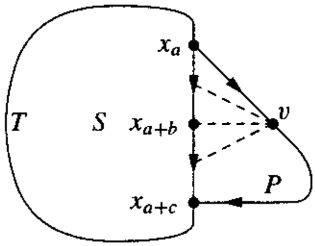
8.4.42. Theorem. (Meyniel [1973]) If G is a strict strongly connected digraph such that $d(u) + d(v) \geq 2n - 1$ whenever u, v are distinct nonadjacent vertices, then G is Hamiltonian.

Proof: (Bondy–Thomassen [1977]). We prove a technical lemma: if $T = v_1, \ldots, v_k$ is a path that cannot absorb the vertex v internally (between two of its vertices), then the number of edges from v to T plus the number of edges from T to v is at most $k+1$. This follows by counting. For $1 \leq i \leq k-1$, only one of the edges $v_i v$ and vv_{i+1} is permitted. Also vv_1 and $v_k v$ are permitted; there is no restriction on absorption at the end.

We use this to prove the following statement: If G is a strict strong non-Hamiltonian digraph, and S is a maximal vertex subset having a spanning cycle $(x_1, \ldots, x_m)$ in G, then there exist $v \in \overline{S}$ and integers a, b with $1 \le a \le m$ and $1 \le b < m$ such that (1) $x_a v \in E(G)$, (2) v is not adjacent to any x_{a+i} with $1 \le i \le b$, and (3) $d(v) + d(x_{a+b}) \le 2n - 1 - b$. Since $b \ge 1$, the conclusion of this statement is impossible under the hypothesis of the theorem, which will imply that the only maximal vertex set having a spanning cycle is $V(G)$.

Suppose first that no path leaves S and returns to it. Since G is strong and $S \ne V(G)$, some cycle C of length at least 2 shares exactly one vertex with S. Let this vertex be x_a, and let v be the successor of x_a on C. By the path condition, there is no path between v and $S - \{x_a\}$ in either direction. In particular, each vertex outside $S \cup \{v\}$ is incident to at most two edges also incident to v or v_{a+1}. Furthermore, v is incident to at most two edges also incident to S (the other endpoint must be v_a). Finally, each vertex of $S - v_{a+1}$ is incident to at most two edges also incident to v_{a+1}. Summing the allowed contributions yields $d(v) + d(x_{a+1}) \le 2n - 2$. Hence the desired condition holds with $b = 1$.





Now suppose that some path leaves S and returns to it. Choose such a path P so that the distance c along S from the start of P to the end of P is minimal. Let x_a be the start of P, and let v be its successor on P. The maximality of S implies that $c > 1$. Let T be the portion of S from x_{a+c} to x_a; this has $m - c + 1$ vertices. The maximality of S implies that v cannot be absorbed internally by T. Hence our technical lemma implies that v belongs to at most $m - c + 2$ edges incident to T. The minimality of c makes v nonadjacent to $x_{a+1}, \ldots, x_{c-1}$.

Let b be the largest integer in $[c]$ such that G has a path from x_{a+c} to x_a with vertex set $S - \{x_{a+b}, \ldots, x_{a+c-1}\}$. Let R be such a path (the path T with $b = 1$ implies that R exists.) Since $P \cup R$ is a cycle, the maximality of S yields $b < c$. By the maximality of b, x_{a+b} is not absorbed internally by R. Hence, by our technical lemma, x_{a+b} belongs to at most $m - c + b + 1$ edges incident to R.

Now we count $d(v) + d(x_{a+b})$. Each vertex outside $S \cup \{v\}$ is incident to at most two edges also incident to $\{v, x_{a+b}\}$, because the minimality of c prevents a path of length 2 between v and x_{a+b} (in either direction) using a vertex not in S. We have observed that v belongs to at most $m - c + 2$ edges incident to S. We have observed that v_{a+b} belongs to at most $m - c + b + 2$ edges incident to R. Finally, x_{a+b} belongs to at most $2(c - b - 1)$ edges incident to $S - R$. Hence $d(v) + d(x_{a+b}) \le 2(n - m - 1) + (m - c + 2) + (m - c + b + 1) + 2(c - b - 1) = 2n - 1 - b$. Again we have obtained the desired condition. ■

EXERCISES

8.4.1. Let $m = \lfloor n^2/4 \rfloor$. Prove that every n-vertex graph has an intersection representation using subsets of $[m]$ such that each element of $[m]$ appears in at most three sets. Equivalently, every n-vertex graph decomposes into at most m edges and triangles.

8.4.2. Prove that the following conditions on a graph G with no isolated vertices are equivalent. (Choudom–Parthasarathy–Ravindra [1975])

A) $\theta'(G) = \alpha(G)$.
B) $\theta'(G \vee G) = (\theta'(G))^2$.
C) $\theta'(G) = \theta(G)$.
D) Every clique in a minimum clique cover of $E(G)$ uses a simplicial vertex of G.

8.4.3. (+) Let $b(G)$ be the minimum number of bipartite graphs needed to partition $E(G)$ (called **biparticity**). Let $a(G)$ denote the minimum number of classes needed to partition $E(G)$ such that every cycle of G contains a non-zero even number of edges from some class. Prove that these parameters both equal $\lceil \lg \chi(G) \rceil$. (Hint: Prove $\lg \chi(G) \le b(G) \le a(G) \le \lceil \lg \chi(G) \rceil$.) (Harary–Hsu–Miller [1977], Alon–Egawa [1985])

8.4.4. Determine all the n-vertex graphs that have product dimension $n-1$. (Lovász–Nešetřil–Pultr [1980])

8.4.5. Prove that pdim $G \le 2$ if and only if G is the complement of the line graph of a bipartite graph (Lovász–Nešetřil–Pultr [1980])

8.4.6. Given r, compute pdim $(K_r + mK_1)$ for all $m \ge 1$. (Lovász–Nešetřil–Pultr [1980])

8.4.7. (−) Compute the product dimension of the three-dimensional cube.

8.4.8. Obtain upper and lower bounds on the product dimension of the Petersen graph that differ by 1 (the upper bound will most likely be the correct value, but showing that it cannot be improved is tedious).

8.4.9. Let $f(n)$ be the maximum value of pdim $G \cdot$ pdim $\overline{G}$ over all graphs on n vertices. Prove that $\lfloor n^2/4 \rfloor \le f(n) \le (n-1)^2$.

8.4.10. For $n \ge 4$, prove that pdim $P_n = \lceil \lg(n-1) \rceil$. For $n \ge 3$, prove that pdim $C_{2n} = 1 + \lceil \lg(n-1) \rceil$ and $1 + \lceil \lg n \rceil \le$ pdim $C_{2n+1} \le 2 + \lceil \lg n \rceil$. (Lovász–Nešetřil–Pultr [1980]) (Comment: Evans–Fricke–Maneri–McKee–Perkel [1994] showed that pdim $C_{2n+1} = 1 + \lceil \lg n \rceil$ except possibly when n is a power of 2.)

8.4.11. Prove that C_{2k+1} is not isometrically embeddable in any cartesian product of cliques if $k > 1$.

8.4.12. Determine the squashed-cube dimension of C_5.

8.4.13. (+) Determine the squashed-cube dimension of $K_{3,3}$. (Hint: Use symmetry to reduce case analysis.)

8.4.14. (!) Use Edmonds' Branching Theorem (Theorem 8.4.20) to prove the edge version of Menger's Theorem in digraphs: $\lambda'(x, y) = \kappa'(x, y)$. (Hint: Devise an appropriate graph transformation to obtain a short proof.)

8.4.15. (!) The gossip problem is also called the "telephone problem", and the corresponding problem for directed graphs is called the "telegraph problem". As a function of n, determine the minimum number of one-way transmissions among n people so that each person has a transmission path to every other. (Harary–Schwenk [1974])

8.4.16. Let D be a digraph solving the telegraph problem in which each vertex receives information from each other vertex exactly once. Prove that in D at least $n-1$ vertices hear their own information. For each n, construct such a D in which only $n-1$ vertices hear their own information, but for each $x \neq y$ there is exactly one increasing x, y-path. (Seress [1987])

8.4.17. *The NOHO property.*

a) Let G be a connected graph with $2n-4$ edges having a linear ordering that solves the gossip problem and satisfies NOHO (no increasing cycle). Suppose also that $n(G) > 8$ and that at most two vertices have degree 2. Prove that the graph obtained by deleting the first calls and last calls of vertices in G has 4 components, of which two are isolated vertices and two are caterpillars having the same size. (West [1982a])

b) For every even $n \geq 4$, construct a connected ordered graph with $2n-4$ edges that satisfies the NOHO property. (Hint: Make use of the structural properties proved in part (a) to guide the search.)

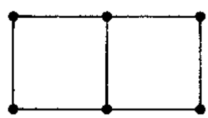
8.4.18. A **NODUP scheme** (NO DUPlicate transmission) is a connected ordered graph that has exactly one increasing path from each vertex to every other.

a) (−) Prove that every NODUP scheme has the NOHO property.

b) Prove that there is no NODUP scheme when $n \in \{6, 10, 14, 18\}$. (Comment: Seress [1986] proved that these are the only even values of n for which NODUP schemes do not exist, constructing them for all other values. For $n = 4k$, West [1982b] constructed NODUP schemes with $9n/4 - 6$ calls, and Seress [1986] proved that these are optimal.)

8.4.19. A vertex in a simple graph G wishes to broadcast information to all other vertices. In each time unit, each vertex that already knows the information can make one call to a neighbor that does not know the information. The time required to broadcast from v is the minimum number of time units in which all vertices can learn the information. Construct an n-vertex graph G with fewer than $2n$ edges such that every vertex of G can broadcast in time at most $1 + \lg n$. (Grigni–Peleg [1991])

8.4.20. (!) Prove that the graph below is not 2-choosable.



8.4.21. Prove that $K_{k,m}$ is k-choosable if and only if $m < k^k$ (Erdős–Rubin–Taylor [1979])

8.4.22. Prove that $\chi_l(G) \leq 1 + \max_{H \subseteq G} \delta(G)$ and that $\chi_l(G) + \chi_l(\overline{G}) \leq n+1$. Prove also that $\chi_l'(G) \leq 2\Delta(G) - 1$.

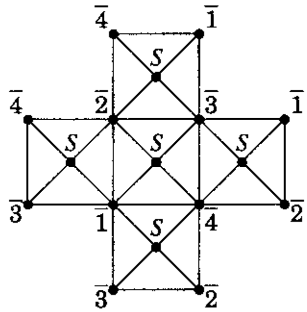
8.4.23. Prove that every chordal graph G is $\chi(G)$-choosable.

8.4.24. Prove that a connected graph G has a proper list coloring from lists such that $|L(v)| \geq d(v)$ for all v if there is strict inequality for at least one vertex.

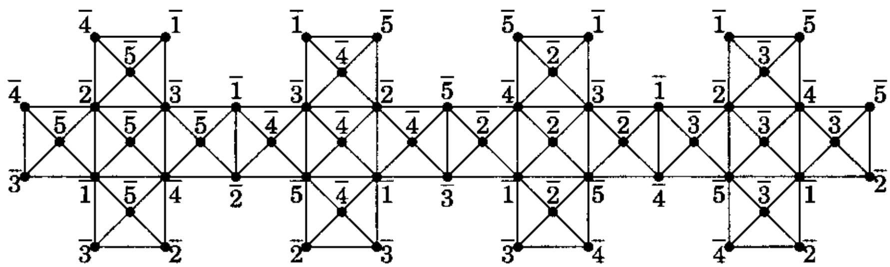
8.4.25. (!) Prove that G has a total coloring (Remark 8.4.31) with at most $\chi_l'(G) + 2$ colors.

8.4.26. (!) *Non-4-choosable planar graph of order 63.*

a) In the list assignments for the graph below, S denotes $[4]$ and $\bar{i}$ denotes $S - \{i\}$. Prove that this graph has no proper coloring chosen from these lists.



b) In the list assignments for the graph G below, $\bar{i}$ denotes $[5] - \{i\}$; each list has size 4. Let G' be the graph obtained from G by adding one vertex with list $\bar{1}$ adjacent to all vertices on the outside face of this drawing of G. Prove that G' has no proper coloring chosen from these lists. (Mirzakhani [1996])



8.4.27. (!) *Equivalence of Dilworth's Theorem and König–Egerváry Theorem.*

a) Given a bipartite graph G, apply Dilworth's Theorem to a transitive orientation of it to obtain the König–Egerváry Theorem.

b) Given a transitive digraph D, let G be the split of D as defined in Definition 1.4.20. Apply the König–Egerváry Theorem to G to obtain Dilworth's Theorem for D.

8.4.28. (!) Prove that K_n decomposes into $\lceil n/2 \rceil$ paths. Prove that K_n decomposes into $\lfloor n/2 \rfloor$ cycles when n is odd.

8.4.29. (!) *Decomposition of K_n into spanning connected subgraphs.*

a) Prove that if K_n decomposes into k spanning connected subgraphs, then $n \geq 2k$.

b) Prove that K_{2k} decomposes into k spanning trees of diameter 3. (Hint: Let the central edges of these trees form a perfect matching.) (Palumbíny [1973])

8.4.30. Prove that every 2-edge-connected 3-regular simple planar graph decomposes into paths of length 3. Prove the same statement for planar triangulations. (Jünger–Reinelt–Pulleyblank [1985])

8.4.31. Prove that Theorem 8.4.35 is best possible when $m-1$ divides $n-1$.

8.4.32. Let G be a graph such that $\overline{G}$ is triangle-free and not a forest. Prove that G has a cycle of length at least $n(G)/2$. (Hint: Use Theorem 8.4.37.) (N. Graham)

8.4.33. Use Woodall's Theorem to prove Ore's Theorem, and use Meyniel's Theorem to prove Woodall's Theorem.

8.4.34. Use Meyniel's Theorem to prove that a strict n-vertex digraph has a spanning path if $d(u) + d(v) \geq 2n - 3$ for every pair u, v of distinct nonadjacent vertices.