

elementary means was discovered by Harriot around 1590. It is the curve defined by the polar equation

$$r = e^{k\theta}$$

known as the *logarithmic* or *equiangular* spiral.

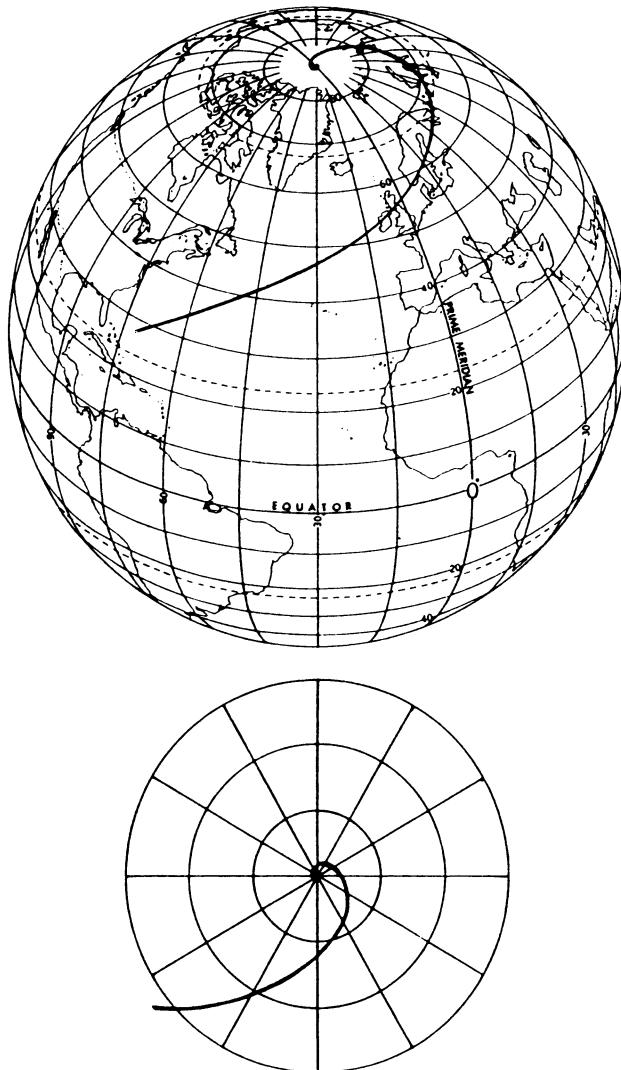


Figure 17.1: A rhumb line and its projection

Harriot did not have the exponential function and knew the curve only by its equiangular property, which is that the tangent makes a constant angle ϕ (depending on k) with the radius vector. [The spiral turned up in his researches on navigation and map projections (Section 16.2) as the projection of a *rhumb line* on the sphere (Figure 17.1). A rhumb line is a curve that meets the meridians at a constant angle; in practical terms, it represents the course of a ship sailing in a fixed compass direction.]

Not having the tools of calculus, Harriot had to rely on ingenious geometry and a simple limit argument. His construction is illustrated in Figure 17.2 [from Lohne (1979) p. 273]. The spiral of angle 55° is approximated by a polygon with sides s_1, s_2, s_3, \dots , which yield triangles T_1, T_2, T_3, \dots when connected to the origin p . T_1, T_2, T_3, \dots can be reassembled to form triangle ABT , whose area therefore equals that of the spiral (when the areas of overlapping turns are added together). Also

$$BT + TA = s_1 + s_2 + s_3 + \dots = \text{length of the spiral}$$

When the approximation is made with shorter sides s'_1, s'_2, s'_3, \dots , but otherwise in the same way, the *same triangle* ABT results: the isosceles triangle with base a and base angles 55°. Hence we have also found length and area of the smooth curve.

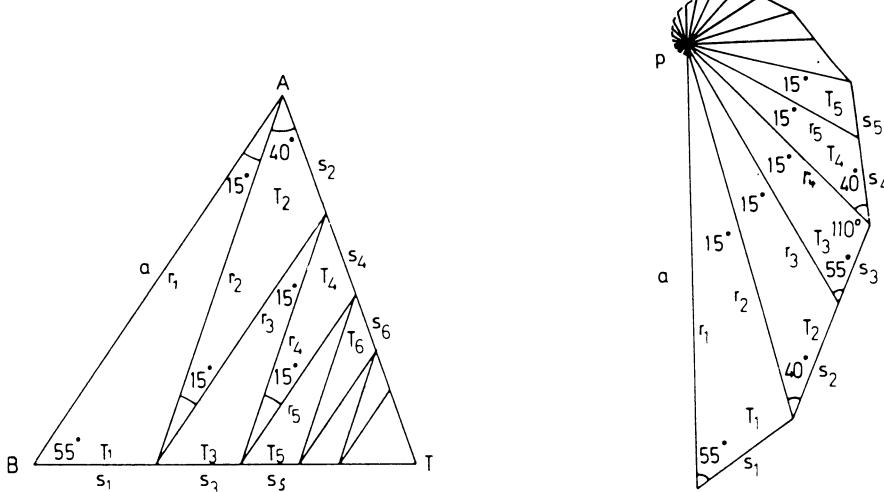


Figure 17.2: Constructing the area of a spiral

Harriot's work was not published, and the arc length of the equiangular spiral was rediscovered by Torricelli (1645). Gradually the problem of arc length became understood more systematically as a problem of integration, though usually a rather intractable one. The first solution for an algebraic curve was for the "semicubical parabola" $y^2 = x^3$, by Neil and Heuraet in 1657. Soon after this Wren solved the problem for the cycloid, and his solution was given by Wallis (1659). A remarkable feature of Wren's result is that the length of one arch of the cycloid is a rational multiple (namely, 4) of the diameter of the generating circle.

As mentioned in Section 13.3, other extraordinary properties of the cycloid are related to mechanics, and one of these will be reinterpreted geometrically in the next section. One transcendental curve that we did not discuss in connection with mechanics is the *tractrix* of Newton (1676b). Newton defined this curve by the property that the length of its tangent from point of contact to the x -axis is constant (Figure 17.3). It follows that the curve satisfies

$$\frac{dy}{ds} = \frac{y}{a},$$

where s denotes arc length. By using $ds = \sqrt{dx^2 + dy^2}$, this differential equation can be solved to give

$$x = a \log \frac{a + \sqrt{a^2 - y^2}}{y} - \sqrt{a^2 - y^2},$$

the equation for the curve given, in more geometric language, by Huygens (1693b). Huygens pointed out that the curve could be interpreted as the path of a stone pulled by a string of length a (hence the name "tractix"). Thus the tractrix, too, has some mechanical significance. In fact it can be constructed from the famous mechanical curve, the catenary, by a method we shall see in the next section. However, its most important role was in the generation of the *pseudosphere*, a surface discussed in Section 17.4.

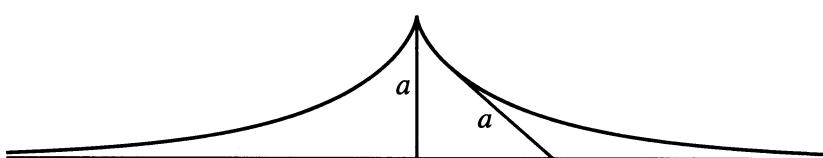


Figure 17.3: The tractrix

EXERCISES

The arc length of $y^2 = x^3$ is today a fairly routine exercise with the arc length integral $\int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$.

17.1.1 Show that arc length of $y = x^{3/2}$ between O and $x = a$ is

$$\frac{8}{27} \left(\left(1 + \frac{9a}{4} \right)^{3/2} - 1 \right).$$

Likewise, it is easy for us to derive properties of the logarithmic spiral from its polar equation and knowledge of the exponential function.

17.1.2 Show that the logarithmic spiral is *self-similar*. That is, magnifying $r = e^{k\theta}$ by a factor m to $r = me^{k\theta}$ gives a curve that is congruent to the original (in fact, it results from a rotation of the original).

Jakob Bernoulli was so impressed by this property of the logarithmic spiral that he arranged to have the spiral engraved on his tombstone, with a motto: *Eadem mutata resurgo* (“Though changed, I arise again the same”). [See Jakob Bernoulli (1692) p. 213.]

17.1.3 Deduce the equiangular property of the logarithmic spiral from its self-similarity.

The equation of the tractrix given above can be derived as follows.

17.1.4 Explain why the constant tangent property implies $\frac{dy}{ds} = \frac{y}{a}$, then multiply both sides of this equation by $\frac{ds}{dx} = \sqrt{1 + (\frac{dy}{dx})^2}$, and deduce that

$$\frac{dx}{dy} = \pm \frac{\sqrt{a^2 - y^2}}{y}.$$

17.1.5 Check by differentiation that $x = a \log \frac{a + \sqrt{a^2 - y^2}}{y} - \sqrt{a^2 - y^2}$ satisfies the differential equation found in Exercise 17.1.4, and also show that x has the appropriate value when $y = a$.

17.2 Curvature of Plane Curves

One of the most important ideas in differential geometry is that of *curvature*. The development of this idea from curves to surfaces then to higher-dimensional spaces has had many important consequences for mathematics

and physics, among them clarification of both the mathematical and physical meaning of “space,” “space-time,” and “gravitation.” In this section we shall look at the beginnings of the theory of curvature in the seventeenth-century theory of curves. As discussed here, this theory concerns *plane* curves only; space curves involve an additional consideration of *torsion* (twisting), which will not concern us.

Just as the direction of a curve C at point P is determined by its straight-line approximation, that is, tangent, at P , the curvature of C at P is determined by an approximating circle. Newton (1665c) was the first to single out the circle that defines the curvature: the circle through P whose center R is the limiting position of the intersection of the normal through P and the normal through a nearby point Q on the curve (Figure 17.4). R is called the *center of curvature*, $RP = \rho$ the *radius of a curvature*, and $1/\rho = \kappa$ the *curvature*. It follows that the circle of radius r has constant curvature $1/r$. The only other curve of constant curvature is the straight line, which has curvature 0. This is a consequence of the formula for curvature discovered by Newton (1671):

$$\rho = \frac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2}.$$

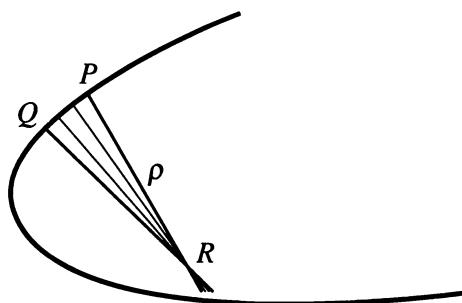


Figure 17.4: Normals through nearby points on a curve

There is an interesting relationship between a curve C and the locus C' of the center of curvature of C . C is the so-called *involute* of C' , which, intuitively speaking, is the path of the end of a piece of string as it is unwound from C' (Figure 17.5). It is intuitively clear that Q , the end of the string, is instantaneously moving in a circle with center at P , the point where the string is tangential to C' .

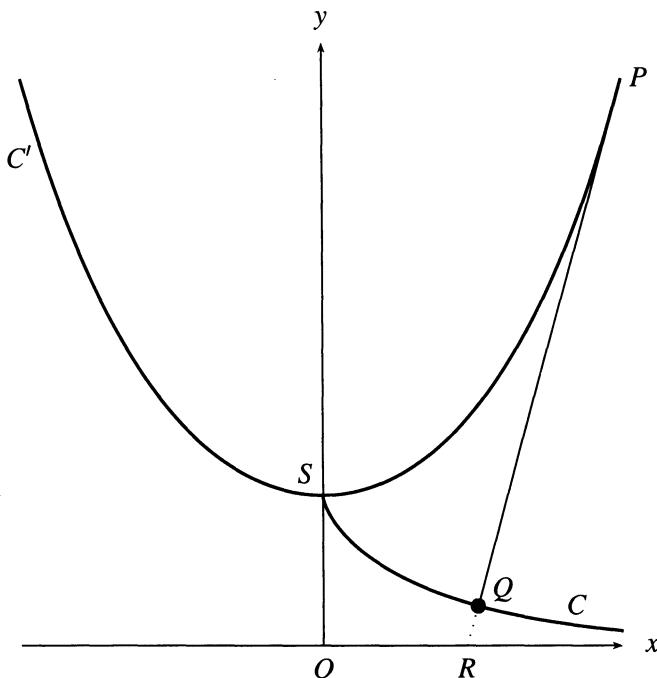


Figure 17.5: Construction of the involute

The geometric property of the cycloid that Huygens (1673) used to design the cycloidal pendulum (Section 13.3) can now be seen as simply this: the involute of a cycloid is another cycloid. Two other stunning results on involutes were obtained by the Bernoulli brothers. Jakob Bernoulli (1692) found that the involute of the logarithmic spiral is another logarithmic spiral, and Johann Bernoulli (1691) found that the tractrix is the involute of the catenary.

Another useful and intuitive definition of curvature, which turns out to be equivalent to the preceding one, was given by Kaestner (1761). He defined curvature as the rate at which the tangent turns, that is, $d\theta/ds = \lim_{\Delta s \rightarrow 0} (\Delta\theta/\Delta s)$, where $\Delta\theta$ is the angle between the tangents at points separated by an arc of the curve of length Δs . It follows from this definition that $\int_C \kappa ds = 2\pi$ for a simple closed curve C , since the tangent makes one complete turn on a circuit around C . We shall see in Section 17.6 that this result has a very interesting generalization for curves on nonplanar surfaces.

EXERCISES

Despite the complexity of the Newton curvature formula, it is easy enough to solve for y when the curvature is zero.

- 17.2.1** Use the formula to show that $\kappa = 0$ implies y is a linear function of x .
- 17.2.2** Show that $d\theta/ds = 1/r$ for the circle of radius r , and deduce that $d\theta/ds = \kappa$ for any curve.

The description of the tractrix as the involute of the catenary is convenient for studying the pseudosphere. We therefore work out some steps in this approach in the following exercises. The curve C' in Figure 17.5 is now assumed to be the catenary $y = \cosh x$, which meets the y -axis at the point S where $y = 1$.

- 17.2.3** Using the arc length integral on the catenary $y = \cosh x$ between $S = (0, 1)$ and $P = (\sigma, \cosh \sigma)$ show that

$$\text{arc length } PS = \sinh \sigma = PQ.$$

- 17.2.4** Also find the equation of tangent of P , and use it to show that $R = (\sigma - \coth \sigma, 0)$. Then use the value of PQ to show that

$$QR = \frac{1}{\sinh \sigma} = \frac{1}{PQ}.$$

- 17.2.5** Finally, use the length of PQ again to show that $Q = (\sigma - \tanh \sigma, \operatorname{sech} \sigma)$, and show that the parametric equations of the tractrix C ,

$$x = \sigma - \tanh \sigma, \quad y = \operatorname{sech} \sigma,$$

imply the cartesian equation of the tractrix (with $a = 1$),

$$x = \log \frac{1 + \sqrt{1 - y^2}}{y} - \sqrt{1 - y^2}.$$

17.3 Curvature of Surfaces

The first approach to defining curvature at a point P of a surface S in three-dimensional space was to express it in terms of the curvature of the plane curves, by considering sections of S by planes through the normal at P . Of course, different planes normal to the surface at P may cut the surface in quite different curves, with different curvatures, as the example of the cylinder shows (Figure 17.6).

However, among these curves there will be one of maximum curvature and one of minimum curvature (which may be negative, since we give a

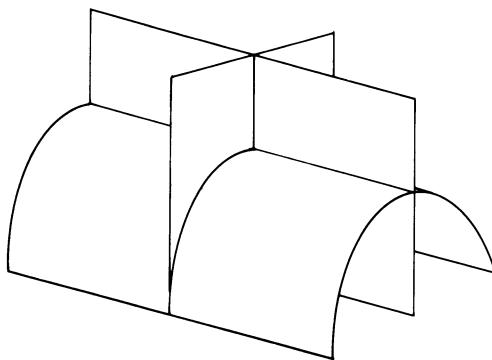


Figure 17.6: Sections of the cylinder

sign to curvature according to the side on which the center of curvature lies). Euler (1760) showed that these two curvatures κ_1 and κ_2 , called the *principal curvatures*, occur in perpendicular sections and that together they determine the curvature κ in a section at angle α to one of the principal sections by

$$\kappa = \kappa_1 \cos^2 \alpha + \kappa_2 \sin^2 \alpha.$$

This is as far as one can go as long as the curvature of surfaces is subordinated to the curvature of plane curves. A deeper idea occurred to Gauss in the course of his work in geodesy (surveying and mapmaking): curvature of a surface may be detectable *intrinsically*, that is, by measurements that take place entirely on the surface. The curvature of the earth, for example, was known on the basis of measurements made by explorers and surveyors, *not* (in the time of Gauss) by viewing it from space. Gauss (1827) made the extraordinary discovery that the quantity $\kappa_1 \kappa_2$ can be defined intrinsically and hence can serve as an intrinsic measure of curvature. He was so proud of this result that he called it the *theorema egregium* (excellent theorem). It follows in particular that $\kappa_1 \kappa_2$, which is called the *Gaussian curvature*, is unaffected by bending (without creasing or stretching).

The plane, for example, has $\kappa_1 = \kappa_2 = 0$ and thus zero Gaussian curvature. Hence so has any surface obtained by bending a plane, such as a cylinder. We can verify the *theorema egregium* in this case, because one of the principal curvatures of a cylinder is obviously zero.

Surfaces S_1, S_2 obtained from each other by bending are said to be *isometric*. More precisely, S_1 and S_2 are isometric if there is a one-to-one