basis of V. Then for each $g \in G$ the matrix, $\varphi(g)$, of g acting on V with respect to this basis is of the form

$$\varphi(g) = \begin{pmatrix} \varphi_1(g) & \psi(g) \\ 0 & \varphi_2(g) \end{pmatrix}$$

where $\varphi_1 = \varphi|_U$ (with respect to the chosen basis of U) and φ_2 is the representation of G on V/U (and ψ is not necessarily a homomorphism — $\psi(g)$ need not be a square matrix). So reducible representations are those with a corresponding matrix representation whose matrices are in block upper triangular form.

Assume further that the FG-module V is decomposable, $V = U \oplus U'$. Take for a basis of V the union of a basis of U and a basis of U'. With this choice of basis the matrix for each $g \in G$ is of the form

$$\varphi(g) = \begin{pmatrix} \varphi_1(g) & 0 \\ 0 & \varphi_2(g) \end{pmatrix}$$

(i.e., $\psi(g) = 0$ for all $g \in G$). Thus decomposable representations are those with a corresponding matrix representation whose matrices are in block diagonal form.

Examples

(1) As noted above, all degree 1 representations are irreducible, indecomposable and completely reducible. In particular, this applies to the trivial representation and to the representations described in Example 5 above.

(2) If $|G| > 1$, the regular representation of G is reducible (the augmentation ideal and the trace ideal are proper nonzero submodules). We shall later determine the conditions under which this representation is completely reducible and how it decomposes into a direct sum.

(3) For $n > 1$ the FS_n-module described in Example 10 above is reducible since N and I are proper, nonzero submodules. The module N is irreducible (being 1-dimensional) and if the characteristic of the field F does not divide n, then I is also irreducible.

(4) The degree 2 representation of the dihedral group $D_{2n} = G$ described in Example 6 above is irreducible for $n \geq 3$. There are no G-invariant 1-dimensional subspaces since a rotation by $2\pi/n$ radians sends no line in $\mathbb{R}^2$ to itself. Similarly, the degree 2 complex representation of Q_8 described in Example 7 is irreducible since the given matrix $\varphi(i)$ has exactly two 1-dimensional eigenspaces (corresponding to its distinct eigenvalues $\pm\sqrt{-1}$) and these are not invariant under the matrix $\varphi(j)$. The degree 4 representation $\varphi : Q_8 \to GL_4(\mathbb{R})$ described in Example 8 can also be shown to be irreducible (see the exercises). We shall see, however, that if we view φ as a complex representation $\varphi : Q_8 \to GL_4(\mathbb{C})$ (just by considering the real entries of the matrices to be complex entries) then there is a *complex* matrix P such that $P^{-1}\varphi(g)P$ is a direct sum of 2×2 block matrices for all $g \in Q_8$. Thus an irreducible representation over a field F may become reducible when the field is extended.

(5) Let $G = \langle g \rangle$ be cyclic of order n and assume F contains all the n^{th} roots of 1. As noted in Example 1 in the set of examples of group algebras, $F\langle g \rangle \cong F[x]/(x^n - 1)$. Thus the FG-modules are precisely the $F[x]$-modules annihilated by $x^n - 1$. The latter (finite dimensional) modules are described, up to equivalence, by the Jordan Canonical Form Theorem.

If the minimal polynomial of g acting on an $F\langle g \rangle$-module V has distinct roots in F, there is a basis of V such that g (hence all its powers) is represented by a diagonal matrix (cf. Corollary 25, Section 12.3). In this case, V is a completely reducible $F\langle g\rangle$-module (being a direct sum of 1-dimensional $\langle g\rangle$-invariant subspaces). In general, the minimal polynomial of g acting on V divides x^n-1 so if x^n-1 has distinct roots in F, then V is a completely reducible $F\langle g\rangle$-module. The polynomial x^n-1 has distinct roots in F if and only if the characteristic of F does not divide n. This gives a sufficient condition for every $F\langle g\rangle$-module to be completely reducible.

If the minimal polynomial of g acting on V does *not* have distinct roots (so the characteristic of F does divide n), the Jordan canonical form of g must have an elementary Jordan block of size >1. Since every linear transformation has a unique Jordan canonical form, g cannot be represented by a diagonal matrix, i.e., V is not completely reducible. It follows from results on cyclic modules in Section 12.3 that the (1-dimensional) eigenspace of g in any Jordan block of size >1 admits no $\langle g\rangle$-invariant complement, i.e., V is reducible but not completely reducible.

Specifically, let p be a prime, let $F=\mathbb{F}_p$ and let g be of order p. Let V be the 2-dimensional space over $\mathbb{F}_p$ with basis v, w and define an action of g on V by

$$g\cdot v=v \qquad \text{and} \qquad g\cdot w=v+w.$$

This endomorphism of V does have order p (in $GL(V)$) and the matrix of g with respect to this basis is the elementary Jordan block

$$\varphi(g)=\begin{pmatrix}1&1\\0&1\end{pmatrix}.$$

Now V is reducible (span$\{v\}$ is a $\langle g\rangle$-invariant subspace) but V is indecomposable (the above 2×2 elementary Jordan matrix is not similar to a diagonal matrix).

The first fundamental result in the representation theory of finite groups shows how Example 5 generalizes to noncyclic groups.

Theorem 1. *(Maschke's Theorem)* Let G be a finite group and let F be a field whose characteristic does not divide $|G|$. If V is any FG-module and U is any submodule of V, then V has a submodule W such that $V=U\oplus W$ (i.e., every submodule is a direct summand).

Remark: The hypothesis of Maschke's Theorem applies to any finite group when F has characteristic 0.

Proof: The idea of the proof of Maschke's Theorem is to produce an FG-module homomorphism

$$\pi: V\to U$$

which is a projection onto U, i.e., which satisfies the following two properties:

(i) $\pi(u)=u$ for all $u\in U$

(ii) $\pi(\pi(v))=\pi(v)$ for all $v\in V$ (i.e., $\pi^2=\pi$)

(in fact (ii) is implied by (i) and the fact that $\pi(V)\subseteq U$).

Suppose first that we can produce such an FG-module homomorphism and let $W=\ker\pi$. Since π is a module homomorphism, W is a submodule. We see that W is a direct sum complement to U as follows. If $v\in U\cap W$ then by (i), $v=\pi(v)$ whereas by definition of W, $\pi(v)=0$. This shows $U\cap W=0$. To show $V=U+W$ let v be an arbitrary element of V and write $v = \pi(v) + (v - \pi(v))$. By definition, $\pi(v) \in U$. By property (ii) of π,

$$\pi(v - \pi(v)) = \pi(v) - \pi(\pi(v)) = \pi(v) - \pi(v) = 0,$$

i.e., $v - \pi(v) \in W$. This shows $V = U + W$ and hence $V = U \oplus W$. To establish Maschke's Theorem it therefore suffices to find such an FG-module projection π.

Since U is a subspace it has a vector space direct sum complement W_0 in V (take a basis $\mathcal{B}_1$ of U, build it up to a basis $\mathcal{B}$ of V and let W_0 be the span of $\mathcal{B} - \mathcal{B}_1$). Thus $V = U \oplus W_0$ as vector spaces but W_0 need not be G-stable (i.e., need not be an FG-submodule). Let $\pi_0 : V \to U$ be the vector space projection of V onto U associated to this direct sum decomposition, i.e., π_0 is defined by

$$\pi_0(u + w) = u \qquad \text{for all } u \in U,\ w \in W_0.$$

The key idea of the proof is to "average" π_0 over G to form an FG-module projection π. For each $g \in G$ define

$$g\pi_0 g^{-1} : V \to U \qquad \text{by} \qquad g\pi_0 g^{-1}(v) = g \cdot \pi_0(g^{-1} \cdot v), \quad \text{for all } v \in V$$

(here $\cdot$ denotes the action of elements of the ring FG). Since π_0 maps V into U and U is stable under the action of g we have that $g\pi_0 g^{-1}$ maps V into U. Both g and g^{-1} act as F-linear transformations, so $g\pi_0 g^{-1}$ is a linear transformation. Furthermore, if u is in the G-stable space U then so is $g^{-1}u$, and by definition of π_0 we have $\pi_0(g^{-1}u) = g^{-1}u$. From this we obtain that for all $g \in G$,

$$g\pi_0 g^{-1}(u) = u \qquad \text{for all } u \in U$$

(i.e., $g\pi_0 g^{-1}$ is also a vector space projection of V onto U).

Let $n = |G|$ and view n as an element of F ($n = 1 + \cdots + 1$, n times). By hypothesis n is not zero in F and so has an inverse in F. Define

$$\pi = \frac{1}{n} \sum_{g \in G} g\pi_0 g^{-1}.$$

Since π is a scalar multiple of a sum of linear transformations from V to U, it is also a linear transformation from V to U. Furthermore, each term in the sum defining π restricts to the identity map on the subspace U and so $\pi|_U$ is $1/n$ times the sum of n copies of the identity. These observations prove the following:

$$\begin{gathered} \pi : V \to U \text{ is a linear transformation} \\ \pi(u) = u \qquad \text{for all } u \in U \\ \pi^2(v) = \pi(v) \qquad \text{for all } v \in V. \end{gathered}$$

It remains to show that π is an FG-module homomorphism (i.e., is FG-linear). It suffices to prove that for all $h \in G$, $\pi(hv) = h\pi(v)$, for $v \in V$. In this case

$$\begin{aligned}\pi(hv) &= \frac{1}{n}\sum_{g\in G} g\pi_0(g^{-1}hv)\\ &= \frac{1}{n}\sum_{g\in G} h(h^{-1}g)\pi_0((g^{-1}h)v)\\ &= \frac{1}{n}\sum_{\substack{k=h^{-1}g\\ g\in G}} h(k\pi_0(k^{-1}v) = h\pi(v)\end{aligned}$$

(as g runs over all elements of G, so does $k = h^{-1}g$ and the module element h may be brought outside the summation by the distributive law in modules). This establishes the existence of the FG-module projection π and so completes the proof.

The applications of Maschke's Theorem will be to finitely generated FG-modules. Unlike the situation of $F[x]$-modules, however, finitely generated FG-modules are automatically finite dimensional vector spaces (the difference being that FG itself is finite dimensional, whereas $F[x]$ is not). Let V be an FG-module. If V is a finite dimensional vector space over F, then a fortiori V is finitely generated as an FG-module (any F basis gives a set of generators over FG). Conversely, if V is finitely generated as an FG-module, say by $v_1, \ldots, v_k$, then one easily sees that V is spanned as a vector space by the finite set $\{g \cdot v_i \mid g \in G,\ 1 \le i \le k\}$. Thus

an FG-module is finitely generated if and only if it is finite dimensional.

Corollary 2. If G is a finite group and F is a field whose characteristic does not divide $|G|$, then every finitely generated FG-module is completely reducible (equivalently, every F-representation of G of finite degree is completely reducible).

Proof: Let V be a finitely generated FG-module. As noted above, V is finite dimensional over F, so we may proceed by induction on its dimension. If V is irreducible, it is completely reducible and the result holds. Suppose therefore that V has a proper, nonzero FG-submodule U. By Maschke's Theorem U has an FG-submodule complement W, i.e., $V = U \oplus W$. By induction, each of U and W are direct sums of irreducible submodules, hence so is V. This completes the induction.

Corollary 3. Let G be a finite group, let F be a field whose characteristic does not divide $|G|$ and let $\varphi : G \to GL(V)$ be a representation of G of finite degree. Then there is a basis of V such that for each $g \in G$ the matrix of $\varphi(g)$ with respect to this basis is block diagonal:

$$\begin{pmatrix}\varphi_1(g) & & & \\ & \varphi_2(g) & & \\ & & \ddots & \\ & & & \varphi_m(g)\end{pmatrix}$$

where φ_i is an irreducible matrix representation of G, $1 \le i \le m$.