Now $f$ is neither 1:1 nor onto, whereas $f_0$ is both 1:1 and onto. The latter statement simply says that each non-negative number is the square of exactly one non-negative number. The inverse function $f_0^{-1}$ is the function from $X_0$ into $X_0$ defined by $f_0^{-1}(x) = \sqrt{x}$.

(b) Let $X$ be the set of real numbers, and let $f$ be the function from $X$ into $X$ defined by $f(x) = x^3 + x^2 + 1$. The range of $f$ is all of $X$, and so $f$ is onto. The function $f$ is certainly not 1:1, e.g., $f(-1) = f(0)$. But $f$ is 1:1 on $X_0$, the set of non-negative real numbers, because the derivative of $f$ is positive for $x > 0$. As $x$ ranges over all non-negative numbers, $f(x)$ ranges over all real numbers $y$ such that $y \geq 1$. If we let $Y_0$ be the set of all $y \geq 1$, and let $f_0$ be the function from $X_0$ into $Y_0$ defined by $f_0(x) = f(x)$, then $f_0$ is a 1:1 function from $X_0$ onto $Y_0$. Accordingly, $f_0$ has an inverse function $f_0^{-1}$ from $Y_0$ onto $X_0$. Any formula for $f_0^{-1}(y)$ is rather complicated.

(c) Again let $X$ be the set of real numbers, and let $f$ be the sine function, that is, the function from $X$ into $X$ defined by $f(x) = \sin x$. The range of $f$ is the set of all $y$ such that $-1 \leq y \leq 1$; hence, $f$ is not onto. Since $f(x + 2\pi) = f(x)$, we see that $f$ is not 1:1. If we let $X_0$ be the interval $-\pi/2 \leq x \leq \pi/2$, then $f$ is 1:1 on $X_0$. Let $Y_0$ be the interval $-1 \leq y \leq 1$, and let $f_0$ be the function from $X_0$ into $Y_0$ defined by $f_0(x) = \sin x$. Then $f_0$ is a restriction of $f$ to the interval $X_0$, and $f_0$ is both 1:1 and onto. This is just another way of saying that, on the interval from $-\pi/2$ to $\pi/2$, the sine function takes each value between $-1$ and 1 exactly once. The function $f_0^{-1}$ is the inverse sine function:

$$f_0^{-1}(y) = \sin^{-1} y = \text{arc sin } y.$$

(d) This is a general example of a restriction of a function. It is much more typical of the type of restriction we shall use in this book than are the examples in (b) and (c) above. The example in (a) is a special case of this one. Let $X$ be a set and $f$ a function from $X$ into itself. Let $X_0$ be a subset of $X$. We say that $X_0$ is **invariant under** $f$ if for each $x$ in $X_0$ the element $f(x)$ is in $X_0$. If $X_0$ is invariant under $f$, then $f$ induces a function $f_0$ from $X_0$ into itself, by restricting the domain of its definition to $X_0$. The importance of invariance is that by restricting $f$ to $X_0$ we can obtain a function from $X_0$ into itself, rather than simply a function from $X_0$ into $X$.

## *A.3. Equivalence Relations*

An equivalence relation is a specific type of relation between pairs of elements in a set. To define an equivalence relation, we must first decide what a 'relation' is.

Certainly a formal definition of 'relation' ought to encompass such familiar relations as '$x = y$,' '$x < y$,' '$x$ is the mother of $y$,' and '$x$ is older than $y$.' If $X$ is a set, what does it take to determine a relation between pairs of elements of $X$? What it takes, evidently, is a rule for determining whether, for any two given elements $x$ and $y$ in $X$, $x$ stands in the given relationship to $y$ or not. Such a rule $R$, we shall call a (binary) **relation** on $X$. If we wish to be slightly more precise, we may proceed as follows. Let $X \times X$ denote the set of all ordered pairs $(x, y)$ of elements of $X$. A binary relation on $X$ is a function $R$ from $X \times X$ into the set $\{0, 1\}$. In other words, $R$ assigns to each ordered pair $(x, y)$ either a 1 or a 0. The idea is that if $R(x, y) = 1$, then $x$ stands in the given relationship to $y$, and if $R(x, y) = 0$, it does not.

If $R$ is a binary relation on the set $X$, it is convenient to write $xRy$ when $R(x, y) = 1$. A binary relation $R$ is called

(1) **reflexive,** if $xRx$ for each $x$ in $X$;
(2) **symmetric,** if $yRx$ whenever $xRy$;
(3) **transitive,** if $xRz$ whenever $xRy$ and $yRz$.

An **equivalence relation** on $X$ is a reflexive, symmetric, and transitive binary relation on $X$.

EXAMPLE 5. (a) On any set, equality is an equivalence relation. In other words, if $xRy$ means $x = y$, then $R$ is an equivalence relation. For, $x = x$, if $x = y$ then $y = x$, if $x = y$ and $y = z$ then $x = z$. The relation '$x \neq y$' is symmetric, but neither reflexive nor transitive.

(b) Let $X$ be the set of real numbers, and suppose $xRy$ means $x < y$. Then $R$ is not an equivalence relation. It is transitive, but it is neither reflexive nor symmetric. The relation '$x \leq y$' is reflexive and transitive, but not symmetric.

(c) Let $E$ be the Euclidean plane, and let $X$ be the set of all triangles in the plane $E$. Then congruence is an equivalence relation on $X$, that is, '$T_1 \cong T_2$' ($T_1$ is congruent to $T_2$) is an equivalence relation on the set of all triangles in a plane.

(d) Let $X$ be the set of all integers:

$$\ldots, -2, -1, 0, 1, 2, \ldots.$$

Let $n$ be a fixed positive integer. Define a relation $R_n$ on $X$ by: $xR_ny$ if and only if $(x - y)$ is divisible by $n$. The relation $R_n$ is called **congruence modulo** $n$. Instead of $xR_ny$, one usually writes

$$x \equiv y, \bmod n \qquad (x \text{ is congruent to } y \text{ modulo } n)$$

when $(x - y)$ is divisible by $n$. For each positive integer $n$, congruence modulo $n$ is an equivalence relation on the set of integers.

(e) Let $X$ and $Y$ be sets and $f$ a function from $X$ into $Y$. We define a relation $R$ on $X$ by: $x_1Rx_2$ if and only if $f(x_1) = f(x_2)$. It is easy to verify that $R$ is an equivalence relation on the set $X$. As we shall see, this one example actually encompasses all equivalence relations.

Suppose $R$ is an equivalence relation on the set $X$. If $x$ is an element of $X$, we let $E(x; R)$ denote the set of all elements $y$ in $X$ such that $xRy$. This set $E(x; R)$ is called the **equivalence class** of $x$ (for the equivalence relation $R$). Since $R$ is an equivalence relation, the equivalence classes have the following properties:

(1) Each $E(x; R)$ is non-empty; for, since $xRx$, the element $x$ belongs to $E(x; R)$.

(2) Let $x$ and $y$ be elements of $X$. Since $R$ is symmetric, $y$ belongs to $E(x; R)$ if and only if $x$ belongs to $E(y; R)$.

(3) If $x$ and $y$ are elements of $X$, the equivalence classes $E(x; R)$ and $E(y; R)$ are either identical or they have no members in common. First, suppose $xRy$. Let $z$ be any element of $E(x; R)$ i.e., an element of $X$ such that $xRz$. Since $R$ is symmetric, we also have $zRx$. By assumption $xRy$, and because $R$ is transitive, we obtain $zRy$ or $yRz$. This shows that any member of $E(x; R)$ is a member of $E(y; E)$. By the symmetry of $R$, we likewise see that any member of $E(y; R)$ is a member of $E(x; R)$; hence $E(x; R) = E(y; R)$. Now we argue that if the relation $xRy$ does not hold, then $E(x; R) \cap E(y; R)$ is empty. For, if $z$ is in both these equivalence classes, we have $xRz$ and $yRz$, thus $xRz$ and $zRy$, thus $xRy$.

If we let $\mathfrak{F}$ be the family of equivalence classes for the equivalence relation $R$, we see that (1) each set in the family $\mathfrak{F}$ is non-empty, (2) each element $x$ of $X$ belongs to one and only one of the sets in the family $\mathfrak{F}$, (3) $xRy$ if and only if $x$ and $y$ belong to the same set in the family $\mathfrak{F}$. Briefly, the equivalence relation $R$ subdivides $X$ into the union of a family of non-overlapping (non-empty) subsets. The argument also goes in the other direction. Suppose $\mathfrak{F}$ is any family of subsets of $X$ which satisfies conditions (1) and (2) immediately above. If we define a relation $R$ by (3), then $R$ is an equivalence relation on $X$ and $\mathfrak{F}$ is the family of equivalence classes for $R$.

EXAMPLE 6. Let us see what the equivalence classes are for the equivalence relations in Example 5.

(a) If $R$ is equality on the set $X$, then the equivalence class of the element $x$ is simply the set $\{x\}$, whose only member is $x$.

(b) If $X$ is the set of all triangles in a plane, and $R$ is the congruence relation, about all one can say at the outset is that the equivalence class of the triangle $T$ consists of all triangles which are congruent to $T$. One of the tasks of plane geometry is to give other descriptions of these equivalence classes.

(c) If $X$ is the set of integers and $R_n$ is the relation 'congruence modulo $n$,' then there are precisely $n$ equivalence classes. Each integer $x$ is uniquely expressible in the form $x = qn + r$, where $q$ and $r$ are integers and $0 \leq r \leq n - 1$. This shows that each $x$ is congruent modulo $n$ to exactly one of the $n$ integers $0, 1, 2, \ldots, n-1$. The equivalence classes are

$$\begin{aligned}
E_0 &= \{\ldots, -2n, -n, 0, n, 2n, \ldots\} \\
E_1 &= \{\ldots, 1-2n, 1-n, 1+n, 1+2n, \ldots\} \\
\vdots\ &= \qquad\qquad \vdots \\
E_{n-1} &= \{\ldots, n-1-2n, n-1-n, n-1, n-1+n, \\
&\qquad\qquad n-1+2n, \ldots\}.
\end{aligned}$$

(d) Suppose $X$ and $Y$ are sets, $f$ is a function from $X$ into $Y$, and $R$ is the equivalence relation defined by: $x_1 R x_2$ if and only if $f(x_1) = f(x_2)$. The equivalence classes for $R$ are just the largest subsets of $X$ on which $f$ is 'constant.' Another description of the equivalence classes is this. They are in 1:1 correspondence with the members of the range of $f$. If $y$ is in the range of $f$, the set of all $x$ in $X$ such that $f(x) = y$ is an equivalence class for $R$; and this defines a 1:1 correspondence between the members of the range of $f$ and the equivalence classes of $R$.

Let us make one more comment about equivalence relations. Given an equivalence relation $R$ on $X$, let $\mathfrak{F}$ be the family of equivalence classes for $R$. The association of the equivalence class $E(x; R)$ with the element $x$, defines a function $f$ from $X$ into $\mathfrak{F}$ (indeed, onto $\mathfrak{F}$):

$$f(x) = E(x; R).$$

This shows that $R$ is the equivalence relation associated with a function whose domain is $X$, as in Example 5(e). What this tells us is that every equivalence relation on the set $X$ is determined as follows. We have a rule (function) $f$ which associates with each element $x$ of $X$ an object $f(x)$, and $xRy$ if and only if $f(x) = f(y)$. Now one should think of $f(x)$ as some property of $x$, so that what the equivalence relation does (roughly) is to lump together all those elements of $X$ which have this property in common. If the object $f(x)$ is the equivalence class of $x$, then all one has said is that the common property of the members of an equivalence class is that they belong to the same equivalence class. Obviously this doesn't say much. Generally, there are many different functions $f$ which determine the given equivalence relation as above, and one objective in the study of equivalence relations is to find such an $f$ which gives a meaningful and elementary description of the equivalence relation. In Section A.5 we shall see how this is accomplished for a few special equivalence relations which arise in linear algebra.

## A.4. Quotient Spaces

Let $V$ be a vector space over the field $F$, and let $W$ be a subspace of $V$. There are, in general, many subspaces $W'$ which are complementary to $W$, i.e., subspaces with the property that $V = W \oplus W'$. If we have