7

The Fundamental Theorem of Algebra

The negative integers, fractions, irrationals and imaginary numbers were all introduced in order to supply solutions to polynomial equations. Do we need more numbers? Is there a polynomial equation with complex coefficients with a root that is not a complex number? The answer is no, and the fact that no new numbers are required is called the Fundamental Theorem of Algebra. It was first stated by Albert Girard (1595–1632) in 1629, and proofs were given by Jean d'Alembert (1717–1783) and Carl Friedrich Gauss (1777–1855). However, John Stillwell [1989] argues that there is a flaw in Gauss's proof, and the first rigorous proof was given only after Weierstrass established the basic properties of continuous functions.

Theorem 7.1. (The Fundamental Theorem of Algebra)
Every polynomial equation with complex coefficients has complex solutions.

Proof: We assume some simple facts about continuity. For example, we assume that if a continuous closed curve which loops around the origin n times gradually changes so that it loops around the origin only $n-1$ times, then, at some time, part of it passes through the origin. Let

$$w = p(z) = z^m + c_1 z^{m-1} + \cdots + c_{m-1}z + c_m \qquad (c_m \neq 0),$$

where the c_k's are complex numbers. Without loss of generality, we may take our polynomial equation to be $p(z) = 0$. Let

$$g(z) = \frac{p(z)}{z^m} = 1 + c_1/z + \cdots + c_m/z^m.$$

Then

$$\begin{aligned}
|g(z) - 1| &= |c_1/z + \cdots + c_m/z^m| \\
&\le |c_1/z| + \cdots + |c_m/z^m| \\
&= \frac{|c_1|}{|z|} + \cdots + \frac{|c_m|}{|z|^m} \\
&\le m\left(\frac{\max(|c_k|)}{|z|}\right)
\end{aligned}$$

if $|z| > 1$. Geometrically, this means that, when $|z|$ is large, $g(z)$ is represented by a point close to $(1,0)$. In particular, if z moves around a large circle with centre at the origin, then $g(z)$ will move in some continuous closed curve near $(1,0)$ and not loop around $(0,0)$.

We consider two planes, one to represent $z = x+yi$, and one to represent $w = u + vi$. As z moves around a circle of radius $|z| = r$ and centre $(0,0)$, w moves in some continuous closed curve in its own plane.

Suppose that z moves around a circle with radius large enough to keep $g(z)$ from winding around $(0,0)$. How many times does w loop around $(0,0)$? Equivalently, how does the angle of w change as z moves around its large circle? In the previous chapter we noted that the angle of a product of two complex numbers is the sum of the angles of those two numbers. Since $w = z^m g(z)$, the angle of w changes by an amount equal to the change in the angle of z^m plus the change in the angle of $g(z)$. Since $g(z)$ stays close to $(1,0)$ the net change in the angle of $g(z)$ as z goes around the large circle is 0. Thus the change in the angle of w is just the change in the angle of z^m. But the angle of a product of complex numbers is equal to the sum of their angles, so the change in the angle of w is just m times the change in the angle of z. As z moves around a circle, the change in the angle of z is 2π. Hence the change in the angle of w is $2\pi m$. In other words, as z moves around the large circle, w loops around the origin m times.

Now imagine the radius of the large circle gradually decreasing to 0. When it is close to 0, w moves in a continuous closed curve close to $p(0) \neq 0$. (We are assuming that $c_m \neq 0$.) Thus w no longer loops around the origin; at some point as the radius of the large circle decreased, w moved through the origin, that is, $p(z)$ was equal to 0 and the polynomial equation had a complex number solution.

Corollary 7.2. *A polynomial of degree m with complex coefficients has exactly m linear factors.*

Proof: We have seen that $p(z) = 0$ has a complex number solution, call it z_1. If $p(z) = q(z)(x - z_1) + R$ then $R = p(z_1) = 0$, so $x - z_1$ is a factor of $p(z)$. Now $q(z)$ has degree $m-1$, and it also has a complex root. Continuing the process, we can write $p(z) = (z - z_1)(z - z_2) \cdots (z - z_m)$.

It is a pity that the proof of the fundamental theorem of algebra makes use of the notion of continuity, which belongs to analysis rather than to algebra. It often happens in mathematics that, when we want to prove a basic property of a certain system, we have to go outside that system. Still, the above proof can be employed constructively for solving polynomial equations. Before the advent of modern computers, there was a mechanical device which traced out the w curve for any given radius $|z| = r$. As r was gradually reduced, a bell would ring as soon as the w curve passed through the origin and a solution had been found.

Exercises

1. Let $w = z^2 - z - 2$. Graph w as z moves around the origin on a circle of radius 2.1.

2. Repeat the above exercise with a circle of radius 2.

3. Assuming that the coefficients of $p(z)$ are all real numbers, show that if $x + yi$ is a root of $p(z)$ then so is $x - yi$.