(f) Hence

$$f(x) = c + \int_0^x \phi(t, f(t))\, dt.$$

This f is a solution of the given problem.

26. Prove an analogous existence theorem for the initial-value problem

$$\mathbf{y}' = \mathbf{\Phi}(x, \mathbf{y}), \qquad \mathbf{y}(0) = \mathbf{c},$$

where now $\mathbf{c} \in R^k$, $\mathbf{y} \in R^k$, and $\mathbf{\Phi}$ is a continuous bounded mapping of the part of R^{k+1} defined by $0 \le x \le 1$, $\mathbf{y} \in R^k$ into R^k. (Compare Exercise 28, Chap. 5.) *Hint:* Use the vector-valued version of Theorem 7.25.

8

SOME SPECIAL FUNCTIONS

POWER SERIES

In this section we shall derive some properties of functions which are represented by power series, i.e., functions of the form

(1) $$f(x) = \sum_{n=0}^{\infty} c_n x^n$$

or, more generally,

(2) $$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n.$$

These are called *analytic functions*.

We shall restrict ourselves to real values of x. Instead of circles of convergence (see Theorem 3.39) we shall therefore encounter intervals of convergence.

If (1) converges for all x in $(-R, R)$, for some $R > 0$ (R may be $+\infty$), we say that f is expanded in a power series about the point $x = 0$. Similarly, if (2) converges for $|x - a| < R$, f is said to be expanded in a power series about the point $x = a$. As a matter of convenience, we shall often take $a = 0$ without any loss of generality.

8.1 Theorem *Suppose the series*

$$\sum_{n=0}^{\infty} c_n x^n \tag{3}$$

converges for $|x| < R$, *and define*

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \qquad (|x| < R). \tag{4}$$

Then (3) *converges uniformly on* $[-R+\varepsilon, R-\varepsilon]$, *no matter which* $\varepsilon > 0$ *is chosen. The function* f *is continuous and differentiable in* $(-R, R)$, *and*

$$f'(x) = \sum_{n=1}^{\infty} nc_n x^{n-1} \qquad (|x| < R). \tag{5}$$

Proof Let $\varepsilon > 0$ be given. For $|x| \le R - \varepsilon$, we have

$$|c_n x^n| \le |c_n(R-\varepsilon)^n|;$$

and since

$$\Sigma c_n(R-\varepsilon)^n$$

converges absolutely (every power series converges absolutely in the interior of its interval of convergence, by the root test), Theorem 7.10 shows the uniform convergence of (3) on $[-R+\varepsilon, R-\varepsilon]$.

Since $\sqrt[n]{n} \to 1$ as $n \to \infty$, we have

$$\limsup_{n\to\infty} \sqrt[n]{n|c_n|} = \limsup_{n\to\infty} \sqrt[n]{|c_n|},$$

so that the series (4) and (5) have the same interval of convergence.

Since (5) is a power series, it converges uniformly in $[-R+\varepsilon, R-\varepsilon]$, for every $\varepsilon > 0$, and we can apply Theorem 7.17 (for series instead of sequences). It follows that (5) holds if $|x| \le R - \varepsilon$.

But, given any x such that $|x| < R$, we can find an $\varepsilon > 0$ such that $|x| < R - \varepsilon$. This shows that (5) holds for $|x| < R$.

Continuity of f follows from the existence of f' (Theorem 5.2).

Corollary *Under the hypotheses of Theorem* 8.1, f *has derivatives of all orders in* $(-R, R)$, *which are given by*

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)c_n x^{n-k}. \tag{6}$$

In particular,

$$f^{(k)}(0) = k!c_k \qquad (k = 0, 1, 2, \ldots). \tag{7}$$

(Here $f^{(0)}$ means f, and $f^{(k)}$ is the kth derivative of f, for $k = 1, 2, 3, \ldots$).

Proof Equation (6) follows if we apply Theorem 8.1 successively to f, $f', f'', \ldots$. Putting $x = 0$ in (6), we obtain (7).

Formula (7) is very interesting. It shows, on the one hand, that the coefficients of the power series development of f are determined by the values of f and of its derivatives at a single point. On the other hand, if the coefficients are given, the values of the derivatives of f at the center of the interval of convergence can be read off immediately from the power series.

Note, however, that although a function f may have derivatives of all orders, the series $\Sigma c_n x^n$, where c_n is computed by (7), need not converge to $f(x)$ for any $x \neq 0$. In this case, f cannot be expanded in a power series about $x = 0$. For if we had $f(x) = \Sigma a_n x^n$, we should have

$$n!a_n = f^{(n)}(0);$$

hence $a_n = c_n$. An example of this situation is given in Exercise 1.

If the series (3) converges at an endpoint, say at $x = R$, then f is continuous not only in $(-R, R)$, but also at $x = R$. This follows from Abel's theorem (for simplicity of notation, we take $R = 1$):

8.2 Theorem *Suppose Σc_n converges. Put*

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \qquad (-1 < x < 1).$$

Then

$$\lim_{x \to 1} f(x) = \sum_{n=0}^{\infty} c_n. \tag{8}$$

Proof Let $s_n = c_0 + \cdots + c_n$, $s_{-1} = 0$. Then

$$\sum_{n=0}^{m} c_n x^n = \sum_{n=0}^{m} (s_n - s_{n-1})x^n = (1 - x)\sum_{n=0}^{m-1} s_n x^n + s_m x^m.$$

For $|x| < 1$, we let $m \to \infty$ and obtain

$$f(x) = (1 - x)\sum_{n=0}^{\infty} s_n x^n. \tag{9}$$

Suppose $s = \lim_{n \to \infty} s_n$. Let $\varepsilon > 0$ be given. Choose N so that $n > N$ implies

$$|s - s_n| < \frac{\varepsilon}{2}.$$

Then, since

$$(1-x)\sum_{n=0}^{\infty} x^n = 1 \qquad (|x|<1),$$

we obtain from (9)

$$|f(x)-s| = \left|(1-x)\sum_{n=0}^{\infty}(s_n - s)x^n\right| \le (1-x)\sum_{n=0}^{N}|s_n - s|\,|x|^n + \frac{\varepsilon}{2} \le \varepsilon$$

if $x > 1-\delta$, for some suitably chosen $\delta > 0$. This implies (8).

As an application, let us prove Theorem 3.51, which asserts: *If* Σa_n, Σb_n, Σc_n, *converge to* A, B, C, *and if* $c_n = a_0 b_n + \cdots + a_n b_0$, *then* $C = AB$. We let

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad g(x) = \sum_{n=0}^{\infty} b_n x^n, \qquad h(x) = \sum_{n=0}^{\infty} c_n x^n,$$

for $0 \le x \le 1$. For $x < 1$, these series converge absolutely and hence may be multiplied according to Definition 3.48; when the multiplication is carried out, we see that

$$f(x)\cdot g(x) = h(x) \qquad (0 \le x < 1). \tag{10}$$

By Theorem 8.2,

$$f(x) \to A, \qquad g(x) \to B, \qquad h(x) \to C \tag{11}$$

as $x \to 1$. Equations (10) and (11) imply $AB = C$.

We now require a theorem concerning an inversion in the order of summation. (See Exercises 2 and 3.)

8.3 Theorem *Given a double sequence* $\{a_{ij}\}$, $i = 1, 2, 3, \ldots, j = 1, 2, 3, \ldots,$ *suppose that*

$$\sum_{j=1}^{\infty} |a_{ij}| = b_i \qquad (i = 1, 2, 3, \ldots) \tag{12}$$

and Σb_i *converges. Then*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}. \tag{13}$$

Proof We could establish (13) by a direct procedure similar to (although more involved than) the one used in Theorem 3.55. However, the following method seems more interesting.

Let E be a countable set, consisting of the points $x_0, x_1, x_2, \ldots$, and suppose $x_n \to x_0$ as $n \to \infty$. Define

$$f_i(x_0) = \sum_{j=1}^{\infty} a_{ij} \qquad (i = 1, 2, 3, \ldots), \tag{14}$$

$$f_i(x_n) = \sum_{j=1}^{n} a_{ij} \qquad (i, n = 1, 2, 3, \ldots), \tag{15}$$

$$g(x) = \sum_{i=1}^{\infty} f_i(x) \qquad (x \in E). \tag{16}$$

Now, (14) and (15), together with (12), show that each f_i is continuous at x_0. Since $|f_i(x)| \le b_i$ for $x \in E$, (16) converges uniformly, so that g is continuous at x_0 (Theorem 7.11). It follows that

$$\begin{aligned} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} &= \sum_{i=1}^{\infty} f_i(x_0) = g(x_0) = \lim_{n\to\infty} g(x_n) \\ &= \lim_{n\to\infty} \sum_{i=1}^{\infty} f_i(x_n) = \lim_{n\to\infty} \sum_{i=1}^{\infty} \sum_{j=1}^{n} a_{ij} \\ &= \lim_{n\to\infty} \sum_{j=1}^{n} \sum_{i=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}. \end{aligned}$$

8.4 Theorem *Suppose*

$$f(x) = \sum_{n=0}^{\infty} c_n x^n,$$

the series converging in $|x| < R$. *If* $-R < a < R$, *then* f *can be expanded in a power series about the point* $x = a$ *which converges in* $|x - a| < R - |a|$, *and*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \qquad (|x-a| < R - |a|). \tag{17}$$

This is an extension of Theorem 5.15 and is also known as *Taylor's theorem.*

Proof We have

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} c_n[(x-a)+a]^n \\ &= \sum_{n=0}^{\infty} c_n \sum_{m=0}^{n} \binom{n}{m} a^{n-m}(x-a)^m \\ &= \sum_{m=0}^{\infty} \left[\sum_{n=m}^{\infty} \binom{n}{m} c_n a^{n-m} \right] (x-a)^m. \end{aligned}$$

This is the desired expansion about the point $x = a$. To prove its validity, we have to justify the change which was made in the order of summation. Theorem 8.3 shows that this is permissible if

$$\sum_{n=0}^{\infty} \sum_{m=0}^{n} \left| c_n \binom{n}{m} a^{n-m}(x-a)^m \right| \tag{18}$$

converges. But (18) is the same as

$$\sum_{n=0}^{\infty} |c_n| \cdot (|x-a| + |a|)^n, \tag{19}$$

and (19) converges if $|x - a| + |a| < R$.

Finally, the form of the coefficients in (17) follows from (7).

It should be noted that (17) may actually converge in a larger interval than the one given by $|x - a| < R - |a|$.

If two power series converge to the same function in $(-R, R)$, (7) shows that the two series must be identical, i.e., they must have the same coefficients. It is interesting that the same conclusion can be deduced from much weaker hypotheses:

8.5 Theorem *Suppose the series $\Sigma a_n x^n$ and $\Sigma b_n x^n$ converge in the segment $S = (-R, R)$. Let E be the set of all $x \in S$ at which*

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n. \tag{20}$$

If E has a limit point in S, then $a_n = b_n$ for $n = 0, 1, 2, \ldots$. Hence (20) *holds for all $x \in S$.*

Proof Put $c_n = a_n - b_n$ and

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \qquad (x \in S). \tag{21}$$

Then $f(x) = 0$ on E.

Let A be the set of all limit points of E in S, and let B consist of all other points of S. It is clear from the definition of "limit point" that B is open. Suppose we can prove that A is open. Then A and B are disjoint open sets. Hence they are separated (Definition 2.45). Since $S = A \cup B$, and S is connected, one of A and B must be empty. By hypothesis, A is not empty. Hence B is empty, and $A = S$. Since f is continuous in S, $A \subset E$. Thus $E = S$, and (7) shows that $c_n = 0$ for $n = 0, 1, 2, \ldots$, which is the desired conclusion.

Thus we have to prove that A is open. If $x_0 \in A$, Theorem 8.4 shows that

$$f(x) = \sum_{n=0}^{\infty} d_n(x - x_0)^n \qquad (|x - x_0| < R - |x_0|). \tag{22}$$

We claim that $d_n = 0$ for all n. Otherwise, let k be the smallest non-negative integer such that $d_k \neq 0$. Then

$$f(x) = (x - x_0)^k g(x) \qquad (|x - x_0| < R - |x_0|), \tag{23}$$

where

$$g(x) = \sum_{m=0}^{\infty} d_{k+m}(x - x_0)^m. \tag{24}$$

Since g is continuous at x_0 and

$$g(x_0) = d_k \neq 0,$$

there exists a $\delta > 0$ such that $g(x) \neq 0$ if $|x - x_0| < \delta$. It follows from (23) that $f(x) \neq 0$ if $0 < |x - x_0| < \delta$. But this contradicts the fact that x_0 is a limit point of E.

Thus $d_n = 0$ for all n, so that $f(x) = 0$ for all x for which (22) holds, i.e., in a neighborhood of x_0. This shows that A is open, and completes the proof.

THE EXPONENTIAL AND LOGARITHMIC FUNCTIONS

We define

$$E(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \tag{25}$$

The ratio test shows that this series converges for every complex z. Applying Theorem 3.50 on multiplication of absolutely convergent series, we obtain

$$\begin{aligned} E(z)E(w) &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{m=0}^{\infty} \frac{w^m}{m!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{z^k w^{n-k}}{k!(n-k)!} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k} = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!}, \end{aligned}$$

which gives us the important addition formula

$$E(z + w) = E(z)E(w) \qquad (z, w \text{ complex}). \tag{26}$$

One consequence is that

$$E(z)E(-z) = E(z - z) = E(0) = 1 \qquad (z \text{ complex}). \tag{27}$$

This shows that $E(z) \neq 0$ for all z. By (25), $E(x) > 0$ if $x > 0$; hence (27) shows that $E(x) > 0$ for all real x. By (25), $E(x) \to +\infty$ as $x \to +\infty$; hence (27) shows that $E(x) \to 0$ as $x \to -\infty$ along the real axis. By (25), $0 < x < y$ implies that $E(x) < E(y)$; by (27), it follows that $E(-y) < E(-x)$; hence E is strictly increasing on the whole real axis.

The addition formula also shows that

$$\lim_{h=0} \frac{E(z+h) - E(z)}{h} = E(z) \lim_{h=0} \frac{E(h) - 1}{h} = E(z); \tag{28}$$

the last equality follows directly from (25).

Iteration of (26) gives

$$E(z_1 + \cdots + z_n) = E(z_1) \cdots E(z_n). \tag{29}$$

Let us take $z_1 = \cdots = z_n = 1$. Since $E(1) = e$, where e is the number defined in Definition 3.30, we obtain

$$E(n) = e^n \qquad (n = 1, 2, 3, \ldots). \tag{30}$$

If $p = n/m$, where n, m are positive integers, then

$$[E(p)]^m = E(mp) = E(n) = e^n, \tag{31}$$

so that

$$E(p) = e^p \qquad (p > 0, p \text{ rational}). \tag{32}$$

It follows from (27) that $E(-p) = e^{-p}$ if p is positive and rational. Thus (32) holds for all rational p.

In Exercise 6, Chap. 1, we suggested the definition

$$x^y = \sup x^p, \tag{33}$$

where the sup is taken over all rational p such that $p < y$, for any real y, and $x > 1$. If we thus define, for any real x,

$$e^x = \sup e^p \qquad (p < x, p \text{ rational}), \tag{34}$$

the continuity and monotonicity properties of E, together with (32), show that

$$E(x) = e^x \tag{35}$$

for all real x. Equation (35) explains why E is called the exponential function.

The notation $\exp(x)$ is often used in place of e^x, expecially when x is a complicated expression.

Actually one may very well use (35) instead of (34) as the definition of e^x; (35) is a much more convenient starting point for the investigation of the properties of e^x. We shall see presently that (33) may also be replaced by a more convenient definition [see (43)].

We now revert to the customary notation, e^x, in place of $E(x)$, and summarize what we have proved so far.

8.6 Theorem *Let e^x be defined on R^1 by* (35) *and* (25). *Then*

(*a*) *e^x is continuous and differentiable for all x;*

(*b*) $(e^x)' = e^x$;

(*c*) *e^x is a strictly increasing function of x, and $e^x > 0$;*

(*d*) $e^{x+y} = e^x e^y$;

(*e*) *$e^x \to +\infty$ as $x \to +\infty$, $e^x \to 0$ as $x \to -\infty$;*

(*f*) $\lim_{x\to+\infty} x^n e^{-x} = 0$, *for every n.*

Proof We have already proved (*a*) to (*e*); (25) shows that

$$e^x > \frac{x^{n+1}}{(n+1)!}$$

for $x > 0$, so that

$$x^n e^{-x} < \frac{(n+1)!}{x},$$

and (*f*) follows. Part (*f*) shows that e^x tends to $+\infty$ "faster" than any power of x, as $x \to +\infty$.

Since E is strictly increasing and differentiable on R^1, it has an inverse function L which is also strictly increasing and differentiable and whose domain is $E(R^1)$, that is, the set of all positive numbers. L is defined by

$$E(L(y)) = y \qquad (y > 0), \tag{36}$$

or, equivalently, by

$$L(E(x)) = x \qquad (x \text{ real}). \tag{37}$$

Differentiating (37), we get (compare Theorem 5.5)

$$L'(E(x)) \cdot E(x) = 1.$$

Writing $y = E(x)$, this gives us

$$L'(y) = \frac{1}{y} \qquad (y > 0). \tag{38}$$

Taking $x = 0$ in (37), we see that $L(1) = 0$. Hence (38) implies

$$L(y) = \int_1^y \frac{dx}{x}. \tag{39}$$