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Hypercomplex Numbers

20.1 Complex Numbers in Hindsight

We saw in Chapter 14 how a need for complex numbers was first recognized in the sixteenth century, with the solution of cubic equations. Mathematicians were forced to include $\sqrt{-1}$ among the numbers, in order to reconcile obvious real solutions of cubic equations with solutions given by the Cardano formula. As time went by, complex numbers were also found to be indispensable in geometry and analysis, as we saw in Chapters 15 and 16. With hindsight, one realizes there is nothing "impossible" or "imaginary" about complex numbers. They are just as real as the so-called "real" numbers, because two dimensions are just as real as one. And they have just as much right to be called "numbers," because they have the same arithmetic behavior as the real numbers.

But if complex numbers are so real—and not merely a lucky side effect of the Cardano formula—they should have been observed independently, and earlier, in the history of mathematics. There is a comparable situation in the history of astronomy that may help to make this point. The planet Neptune was discovered through the calculations of Adams and Leverrier in 1846, as we know from Section 13.2. But of course, Neptune had always been there, so it could have been observed earlier, before its special importance was understood. This actually happened! A check of Galileo's records later showed that he had observed Neptune in 1612, without realizing that it was a planet.

There was an analogous "observation" of complex numbers, without recognizing all their properties, by Diophantus. He gave no thought to

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$i = \sqrt{-1}$, which we tend to regard as the starting point of complex numbers today, but he did something else that is equally crucial—he operated on *pairs* of ordinary numbers. This happens in his work on sums of two squares, and it is significant because similar observations on sums of four and eight squares foreshadowed the discovery of the four-dimensional and eight-dimensional "numbers" that are the main subject of this chapter. Because these "numbers" have higher dimension than the complex numbers, they are called *hypercomplex.* We shall see how much they deserve to be called "numbers," but first it will be helpful to set the scene by recounting the discovery of Diophantus.

20.2 The Arithmetic of Pairs

In Book III, Problem 19 of his *Arithmetica*, Diophantus remarks

> 65 is naturally divided into two squares in two ways, namely into 7^2+4^2 and 8^2+1^2, which is due to the fact that 65 is the product of 13 and 5, each of which is the sum of two squares.

Apparently, he knows that the product of sums of two squares is itself the sum of two squares, because of the identity

$$(a_1^2+b_1^2)(a_2^2+b_2^2) = (a_1a_2 \mp b_1b_2)^2 + (b_1a_2 \pm a_1b_2)^2.$$

As usual, Diophantus merely illustrates the general result, in this case taking $a_1 = 3$, $b_1 = 2$, $a_2 = 2$, and $b_2 = 1$. But later mathematicians realized what he was driving at: the general identity was observed by al-Khazin around 950 CE, commenting on this very problem in Diophantus, and it was proved in Fibonacci's *Book of Squares* in 1225.

Although Diophantus talks in terms of products of sums of squares a^2+b^2, he is really operating on *pairs* (a,b), because he regards a^2+b^2 as the square on the hypotenuse of the right-angled triangle with the pair of sides a, b. Taking the upper signs in his identity, he is describing a rule for taking two triangles (a_1,b_1), (a_2,b_2) and producing a third triangle, $(a_1a_2 - b_1b_2, b_1a_2 + a_1b_2)$, whose hypotenuse is the product of the hypotenuses of the two triangles given initially.

Now if we interpret (a,b) as $a+ib$ instead of a triangle, Diophantus' rule is nothing but the rule for *multiplying complex numbers* because

$$(a_1+ib_1)(a_2+ib_2) = (a_1a_2 - b_1b_2) + i(b_1a_2 + a_1b_2).$$

His hypotenuse $\sqrt{a^2+b^2}$ is what we call the *absolute value* $|a+ib|$ of $a+ib$, and his identity (with the upper signs) is the *multiplicative property of the absolute value*:

$$|a_1+ib_1||a_2+ib_2| = |(a_1+ib_1)(a_2+ib_2)|.$$

Thus in some sense Diophantus "observed" the rule for multiplying complex numbers, and also the multiplicative property it implies for the absolute value. Admittedly, there is no *addition rule*, which takes the pairs (a_1,b_1), (a_2,b_2) and produces the pair (a_1+a_2,b_1+b_2), so Diophantus had no real arithmetic of pairs—but this could wait.

The concept of complex number had to emerge in algebra, and take charge of geometry and analysis, before mathematicians felt compelled to ask: what *is* a complex number? The definitive answer was given by Hamilton (1835): *a complex number is an ordered pair (a,b) of real numbers, and these pairs are added and multiplied according to the rules*

$$(a_1,b_1)+(a_2,b_2) = (a_1+a_2,b_1+b_2),$$
$$(a_1,b_1)\times(a_2,b_2) = (a_1a_2-b_1b_2,b_1a_2+a_1b_2).$$

The reason for replacing $a+ib$ by the pair of real numbers (a,b), of course, is to remove the controversial object $i=\sqrt{-1}$. Once this is done, it is easy to find the rules for adding and multiplying (a_1,b_1) and (a_2,b_2). Just rewrite the rules for adding and multiplying a_1+ib_1 and a_2+ib_2 in terms of pairs. This seems like a sly trick—using $i^2=-1$ to find the multiplication rule, then removing the i—until we remember that Diophantus found the multiplication rule without any help from $\sqrt{-1}$.

Hamilton realized that multiplying pairs of real numbers was an important question in its own right. In fact, he was interested in the bigger question of multiplying triples, quadruples, and so on. There is an obvious way to add triples for example, the *vector addition*

$$(a_1,b_1,c_1)+(a_2,b_2,c_2) = (a_1+a_2,b_1+b_2,c_1+c_2),$$

which generalizes to n-tuples for any n. But what would it mean to multiply triples? The multiplication rule for pairs does not generalize in any obvious way. Hamilton was tormented by this problem for years, and for a long time his arithmetic of pairs was the only progress he had to report. As we shall see, it played an important role in clarifying what arithmetic is in one and two dimensions, and what it should be in higher dimensions.