5.3.10. Let G be a connected graph with $\chi(G;k) = \sum_{i=0}^{n-1}(-1)^i a_i k^{n-i}$. For $1 \le i \le n$, prove that $a_i \ge \binom{n-1}{i}$. (Hint: Use the chromatic recurrence.)

5.3.11. (!) Prove that the sum of the coefficients of $\chi(G;k)$ is 0 unless G has no edges. (Hint: When a function is a polynomial, how can one obtain the sum of the coefficients?)

5.3.12. (+) *Coefficients of* $\chi(G;k)$.

a) Prove that the last nonzero term in the chromatic polynomial of G is the term whose exponent is the number of components of G.

b) Use part (a) to prove that if $p(k) = k^n - ak^{n-1} + \cdots \pm ck^r$ and $a > \binom{n-r+1}{2}$, then p is not a chromatic polynomial. (For example, this immediately implies that the polynomial in Exercise 5.3.3 is not a chromatic polynomial.)

5.3.13. Let G and H be graphs, possibly overlapping.

a) Prove that $\chi(G \cup H; k) = \frac{\chi(G;k)\chi(H;k)}{\chi(G\cap H;k)}$ when $G \cap H$ is a complete graph.

b) Consider two paths whose union is a cycle to show that the formula may fail when $G \cap H$ is not a complete graph.

c) Apply part (a) to conclude that the chromatic number of a graph is the maximum of the chromatic numbers of its blocks.

5.3.14. (!) Let P be the Petersen graph. By Brooks' Theorem, the Petersen graph is 3-colorable, and hence by the pigeonhole principle it has an independent set S of size 4.

a) Prove that $P - S = 3K_2$.

b) Using part (a) and symmetry, determine the number of vertex partitions of P into three independent sets.

c) In general, how can the number of partitions into the minimum number of independent sets be obtained from the chromatic polynomial of G?

5.3.15. Prove that a graph with chromatic number k has at most k^{n-k} vertex partitions into k independent sets, with equality achieved only by $K_k + (n-k)K_1$ (a k-clique plus $n-k$ isolated vertices). (Hint: Use induction on n and consider the deletion of a single vertex.) (Tomescu [1971])

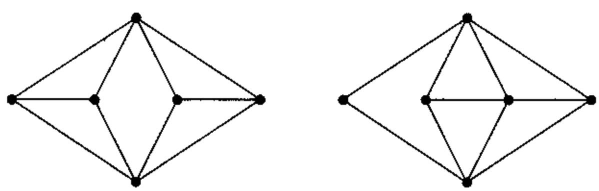
5.3.16. Let G be a simple graph with n vertices and m edges. Prove that G has at most $\frac{1}{3}\binom{m}{2}$ triangles. Conclude that the coefficient of k^{n-2} in $\chi(G;k)$ is positive, unless G has at most one edge. (Hint: Use Theorem 5.3.10.)

5.3.17. (∗) Use the inclusion-exclusion principle to prove Theorem 5.3.10 directly.

5.3.18. (!) Consider the chromatic polynomials of the graphs below.

a) Without computing them, give a short proof that they are equal.

b) Express this chromatic polynomial as the sum of the chromatic polynomials of two chordal graphs, and use this to give a one-line computation of it.



5.3.19. (−) Let G be the graph obtained from K_6 by subdividing one edge. Use the chromatic recurrence to Compute $\chi(G;k)$ as a product of linear factors (factors of the form $k - c_i$). Show that G is not a chordal graph. (Read [1975], Dmitriev [1980])

5.3.20. Let G be a chordal graph. Use a simplicial elimination ordering of G to prove the following statements.

a) G has at most n maximal cliques, with equality if and only if G has no edges. (Fulkerson–Gross [1965])

b) Every maximal clique of G containing no simplicial vertex of G is a separating set.

5.3.21. The **Szekeres–Wilf number** of a graph G is $1 + \max_{H \subseteq G} \delta(H)$. Prove that a graph G is chordal if and only if in every induced subgraph the Szekeres–Wilf number equals the clique number. (Voloshin [1982])

5.3.22. Let $k_r(G)$ be the number of r-cliques in a connected chordal graph G. Prove that $\sum_{r\ge1}(-1)^{r-1}k_r(G) = 1$. (Hint: Use induction on $n(G)$. Note that the binomial formula (Appendix A) implies that $\sum_{j\ge0}(-1)^j\binom{m}{j} = 0$ when $m \in \mathbb{N}$.)

5.3.23. Let S be the vertex set of a cycle in a chordal graph G. Prove that G has a cycle whose vertex set consists of all but one element of S. (Comment: When G has a spanning cycle and $S \subset V(G)$, Hendry conjectured that G also has a cycle whose vertex set consists of S plus one vertex.) (Hendry [1990])

5.3.24. Let e be a edge of a cycle C in a chordal graph. Prove that e forms a triangle with a third vertex of C.

5.3.25. Let Q be a maximal clique in a chordal graph G. Prove that if $G - Q$ is connected, then Q contains a simplicial vertex. (Voloshin–Gorgos [1982])

5.3.26. Exercise 5.3.13 establishes the formula $\chi(G \cup H; k) = \frac{\chi(G;k)\chi(H;k)}{\chi(G\cap H;k)}$ when $G \cap H$ is a complete graph.

a) Prove that the formula holds when $G \cup H$ is a chordal graph regardless of whether $G \cap H$ is a complete graph.

b) Prove that if x is a vertex in a chordal graph G, then

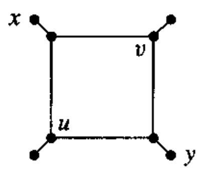
$$\chi(G; k) = \chi(G - x; k)k\frac{\chi(G[N(x)]; k-1)}{\chi(G[N(x)]; k)}.$$

(Comment: Part (b) allows the chromatic polynomial of a chordal graph to be computed via an arbitrary elimination ordering. For example, eliminating the central vertex of P_5 yields $\chi(P_5; k) = [k(k-1)]^2k\frac{(k-1)^2}{k^2} = k(k-1)^4$.) (Voloshin [1982])

5.3.27. (+) A **minimal vertex separator** in a graph G is a set $S \subseteq V(G)$ that for some pair x, y is a minimal set whose deletion separates x and y. Every minimal separating set is a minimal vertex separator, but u, v below show that the converse need not hold.

a) Prove that if every minimal vertex separator in G is a clique, then the same property holds in every induced subgraph of G.

b) Prove that a graph G is chordal if and only if every minimal vertex separator is a clique. (Dirac [1961])



5.3.28. (!) Let G be an interval graph. Prove that G is a chordal graph and that $\overline{G}$ is a comparability graph.

5.3.29. Determine the smallest imperfect graph G such that $\chi(G) = \omega(G)$.

5.3.30. An edge in an acyclic orientation of G is **dependent** if reversing it yields a cycle.

a) Prove that every acyclic orientation of a connected n-vertex graph has at least $n - 1$ independent edges.

b) Prove that if $\chi(G)$ is less than the girth of G, then G has an orientation with no dependent edges. (Hint: Use the technique in the proof of Theorem 5.1.21.)

5.3.31. (∗) The number $a(G)$ of acyclic orientations of G satisfies the recurrence $a(G) = a(G - e) + a(G \cdot e)$ (Theorem 5.3.27). The number of spanning trees of G appears to satisfy the same recurrence; does the number of acyclic orientations of G always equal the number of spanning trees? Why or why not?

5.3.32. (∗) Let D be an acyclic orientation of G, and let f be a coloring of $V(G)$ from the set $[k]$. We say that (D, f) is a **compatible pair** if $u \to v$ in D implies $f(u) \le f(v)$. Let $\eta(G; k)$ be the number of compatible pairs. Prove that $\eta(G; k) = (-1)^{n(G)}\chi(G; k)$. (Stanley [1973])

Chapter 6

Planar Graphs

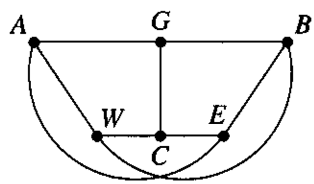
6.1. Embeddings and Euler's Formula

Topological graph theory, broadly conceived, is the study of graph layouts. Initial motivation involved the famous Four Color Problem: can the regions of every map on a globe be colored with four colors so that regions sharing a nontrivial boundary have different colors? Later motivation involves circuit layouts on silicon chips. Wire crossings cause problems in layouts, so we ask which circuits have layouts without crossings.

DRAWINGS IN THE PLANE

The following brain teaser appeared as early as Dudeney [1917].

6.1.1. Example. *Gas–water–electricity.* Three sworn enemies A, B, C live in houses in the woods. We must cut paths so that each has a path to each of three utilities, which by tradition are gas, water, and electricity. In order to avoid confrontations, we don't want any of the paths to cross. Can this be done? This asks whether $K_{3,3}$ can be drawn in the plane without edge crossings; we will give two proofs that it cannot. ■

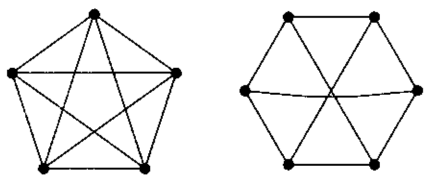


Arguments about drawings of graphs in the plane are based on the fact that every closed curve in the plane separates the plane into two regions (the inside and the outside). In elementary graph theory, we take this as an intuitive notion, but the full details in topology are quite difficult. Before discussing a way to make the arguments precise for graph theory, we show informally how this result is used to prove impossibility for planar drawings.

6.1.2. Proposition. K_5 and $K_{3,3}$ cannot be drawn without crossings.

Proof: Consider a drawing of K_5 or $K_{3,3}$ in the plane. Let C be a spanning cycle. If the drawing does not have crossing edges, then C is drawn as a closed curve. Chords of C must be drawn inside or outside this curve. Two chords conflict if their endpoints on C occur in alternating order. When two chords conflict, we can draw only one inside C and one outside C.

A 6-cycle in $K_{3,3}$ has three pairwise conflicting chords. We can put at most one inside and one outside, so it is not possible to complete the embedding. When C is a 5-cycle in K_5, at most two chords can go inside or outside. Since there are five chords, again it is not possible to complete the embeddings. Hence neither of these graphs is planar. ■



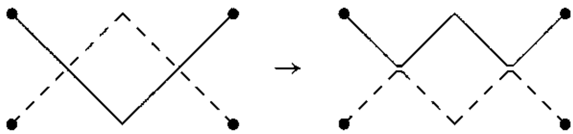
We need a precise notion of "drawing". We have used curves for edges. Using only curves formed from line segments avoids topological difficulties. These can approximate any curve well enough that the eye cannot tell the difference.

6.1.3. Definition. A **curve** is the image of a continuous map from $[0, 1]$ to $\mathbb{R}^2$. A **polygonal curve** is a curve composed of finitely many line segments. It is a **polygonal** u, v**-curve** when it starts at u and ends at v.

A **drawing** of a graph G is a function f defined on $V(G) \cup E(G)$ that assigns each vertex v a point $f(v)$ in the plane and assigns each edge with endpoints u, v a polygonal $f(u), f(v)$-curve. The images of vertices are distinct. A point in $f(e) \cap f(e')$ that is not a common endpoint is a **crossing**.

It is common to use the same name for a graph G and a particular drawing of G, referring to the points and curves in the drawing as the vertices and edges of G. Since the endpoint relation between the points and curves is the same as the incidence relation between the vertices and edges, the drawing can be viewed as a member of the isomorphism class containing G.

By moving edges slightly, we can ensure that no three edges have a common internal point, that an edge contains no vertex except its endpoints, and that no two edges are tangent. If two edges cross more than once, then modifying them as shown below reduces the number of crossings; thus we also require that edges cross at most once. We consider only drawings with these properties.



6.1.4. Definition. A graph is **planar** if it has a drawing without crossings. Such a drawing is a **planar embedding** of G. A **plane graph** is a particular planar embedding of a planar graph.

A curve is **closed** if its first and last points are the same. It is **simple** if it has no repeated points except possibly first=last.

A planar embedding of a graph cuts the plane into pieces. These pieces are fundamental objects of study.

6.1.5. Definition. An **open set** in the plane is a set $U \subseteq \mathbb{R}^2$ such that for every $p \in U$, all points within some small distance from p belong to U. A **region** is an open set U that contains a polygonal u, v-curve for every pair $u, v \in U$. The **faces** of a plane graph are the maximal regions of the plane that contain no point used in the embedding.

A finite plane graph G has one unbounded face (also called the **outer face**). The faces are pairwise disjoint. Points $p, q \in \mathbb{R}^2$ lying in no edge of G are in the same face if and only if there is a polygonal p, q-curve that crosses no edge.

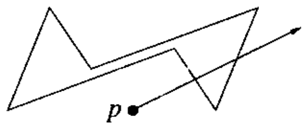
In a plane graph, every cycle is embedded as a simple closed curve. Some faces lie inside it, some outside. This again relies on the fact that a simple closed curve cuts the plane into two regions. As we have suggested, this is not too difficult for polygonal curves. We present some detail of this case in order to explain how to compute whether a point is in the inside or the outside. This proof appears in Tverberg [1980].

6.1.6.* Theorem. (Restricted Jordan Curve Theorem) A simple closed polygonal curve C consisting of finitely many segments partitions the plane into exactly two faces, each having C as boundary.

Proof: Because the list of segments is finite, nonintersecting segments cannot be arbitrarily close. Hence we can leave a face only by crossing C. As we follow C, the nearby points on our right are in a single face, and similarly for the points on the left. (There is a precise algebraic definition for "left" and "right" here.) If $x \notin C$ and $y \in C$, the segment xy first intersects C somewhere, approaching it from the right or the left. Hence every point not along C lies in the same face with at least one of the two sets we have described.

To prove that the points on the left and right lie in different faces, we consider rays in the plane. A ray emanating from a point p is "bad" if it contains an endpoint of a segment of C. Since C has finitely many segments, there are finitely many bad rays from p.

Since the list of segments is finite, each good ray from p crosses C finitely often. As the direction changes, the number of crossings changes only at a bad direction. Before and after such a direction, the parity of the number of crossings is the same. We say that p is an *even point* when every good ray from p crosses C an even number of times; otherwise p is an *odd point*.



Given points x and y in the same face of C, let P be a polygonal x, y-curve that avoids C. Since C has finitely many segments, the endpoints of segments on P can be adjusted slightly so that the rays along segments on P are good for their endpoints. A segment of P belongs to a ray from one end that contains the other; both points have good rays in the same direction. Since the segment does not intersect C, the two points have the same parity. Hence every two points in the same face have the same parity

Because the endpoints of a short segment intersecting C exactly once have opposite parity, there are two distinct faces. The even points and odd points form the outside face and the inside face, respectively. ■

DUAL GRAPHS

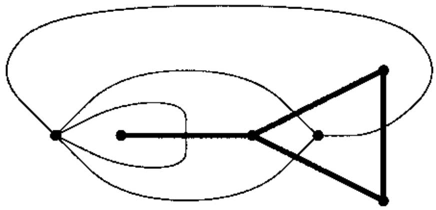
A map on the plane or the sphere can be viewed as a plane graph in which the faces are the territories, the vertices are places where boundaries meet, and the edges are the portions of the boundaries that join two vertices. We allow the full generality of loops and multiple edges. From any plane graph G, we can form a related plane graph called its "dual".

6.1.7. Definition. The **dual graph** G^* of a plane graph G is a plane graph whose vertices correspond to the faces of G. The edges of G^* correspond to the edges of G as follows: if e is an edge of G with face X on one side and face Y on the other side, then the endpoints of the dual edge $e^* \in E(G^*)$ are the vertices x, y of G^* that represent the faces X, Y of G. The order in the plane of the edges incident to $x \in V(G^*)$ is the order of the edges bounding the face X of G in a walk around its boundary.

6.1.8. Example. Every planar embedding of K_4 has four faces, and these pairwise share boundary edges. Hence the dual is another copy of K_4.

Every planar embedding of the cube Q_3 has eight vertices, 12 edges, and six faces. Opposite faces have no common boundary; the dual is a planar embedding of $K_{2,2,2}$, which has six vertices, 12 edges, and eight faces.

Taking the dual can introduce loops and multiple edges. For example, let G be the paw, drawn below in bold edges as a plane graph. Its dual graph G^* is drawn in solid edges. Since G has four vertices, four edges, and two faces, G^* has four faces, four edges, and two vertices. ∎



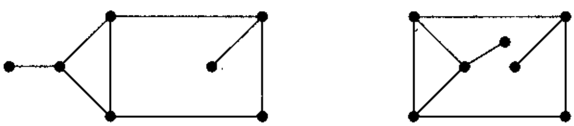
6.1.9. Remark. 1) Example 6.1.8 shows that a simple plane graph may have loops and multiple edges in its dual. A cut-edge of G becomes a loop in G^*, because the faces on both sides of it are the same. Multiple edges arise in the dual when distinct faces of G have more than one common boundary edge.

2) Some arguments require more careful geometric description of the dual. For each face X of G, we place the dual vertex x in the interior of X, so each face of G contains one vertex of G^*. For each edge e in the boundary of X, we draw a curve from x to a point on e; these do not cross. Each such curve meets another from the other side of e at the same point on e to form the edge of G^* that is dual to e. No other edges enter X. Hence G^* is a plane graph, and each edge of G^* in this layout crosses exactly one edge of G.

Such arguments lead to a proof that $(G^*)^*$ is isomorphic to G if and only if G is connected (Exercise 18). Mathematicians often use the word "dual" in a setting when performing an operation twice returns the original object. ∎

6.1.10. Example. Two embeddings of a planar graph may have nonisomorphic duals. Each embedding shown below has three faces, so in each case the dual has three vertices. In the embedding on the right, the dual vertex corresponding to the outside face has degree 4. In the embedding on the left, no dual vertex has degree 4, so the duals are not isomorphic.

This does not happen with 3-connected graphs. Every 3-connected planar graph has essentially one embedding (see Exercise 8.2.45). ∎



When a plane graph is connected, the boundary of each face is a closed walk. When the graph is not connected, there are faces whose boundary consists of more than one closed walk.

6.1.11. Definition. The **length** of a face in a plane graph G is the total length of the closed walk(s) in G bounding the face.

6.1.12. Example. A cut-edge belongs to the boundary of only one face, and it contributes twice to its length. Each graph in Example 6.1.10 has three faces. In the embedding on the left the lengths are 3, 6, 7; on the right they are 3, 4, 9. The sum of the lengths is 16 in each case, which is twice the number of edges. ■

6.1.13. Proposition. If $l(F_i)$ denotes the length of face F_i in a plane graph G, then $2e(G) = \sum l(F_i)$.

Proof: The face lengths are the degrees of the dual vertices. Since $e(G) = e(G^*)$, the statement $2e(G) = \sum l(F_i)$ is thus the same as the degree-sum formula $2e(G^*) = \sum d_{G^*}(x)$ for G^*. (Both sums count each edge twice.) ■

Proposition 6.1.13 illustrates that statements about a connected plane graph becomes statements about the dual graph when we interchange the roles of vertices and faces. Edges incident to a vertex become edges bounding a face, and vice versa, so the roles of face lengths and vertex degrees interchange.

We can also interpret coloring of G^* in terms of G. The edges of G^* represent shared boundaries between faces of G. Hence the chromatic number of G^* equals the number of colors needed to properly color the faces of G. Since the dual of the dual of a connected plane graph is the original graph, this means that four colors suffice to properly color the regions in every planar map if and only if every planar graph has chromatic number at most four.

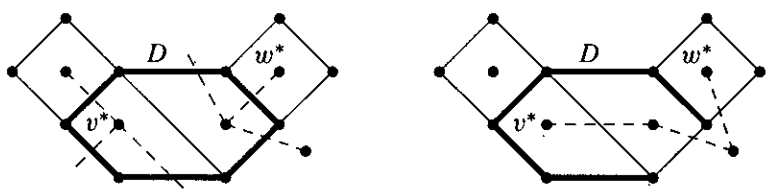
The Jordan Curve Theorem states that a simple closed curve cuts its interior from its exterior. In plane graphs, this duality between curve and cut becomes a duality between cycles and bonds.

6.1.14. Theorem. Edges in a plane graph G form a cycle in G if and only if the corresponding dual edges form a bond in G^*.

Proof: Consider $D \subseteq E(G)$. Suppose first that D is the edge set of a cycle in G. The corresponding edge set $D^* \subseteq E(G^*)$ contains all dual edges joining faces inside D to faces outside D (the Jordan Curve Theorem implies that there is at least one of each). Thus D^* contains an edge cut.

If D contains a cycle and more, then D^* contains an edge cut and more. If D contains no cycle in G, then it encloses no region (see Exercise 24a). It remains possible to reach the unbounded face of G from every other without crossing D. Hence $G^* - D^*$ is connected, and D^* contains no edge cut.

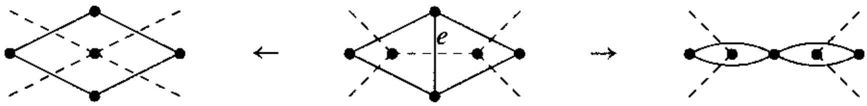
Thus D^* is a minimal edge cut if and only if D is a cycle. ■



The next remark yields an inductive proof of Theorem 6.1.14 (Exercise 19).

6.1.15. Remark. Deleting a non-cut edge of G has the effect of contracting an edge in G^*, as two faces of G merge into one. Contracting a non-loop edge of G has the effect of deleting an edge in G^*. Letting G be the central solid graph below, we have $G - e$ on the left and $G \cdot e$ on the right.

Note that to maintain this duality, we keep multiple edges and loops that arise from edge contraction in plane graphs. ■



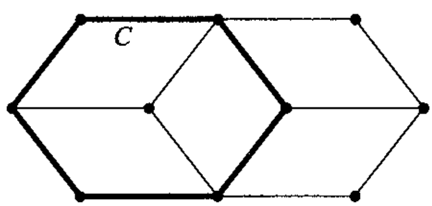
Face boundaries allow us to characterize bipartite planar graphs. The characterization can also be proved by induction (Exercise 20).

6.1.16. Theorem. The following are equivalent for a plane graph G.
A) G is bipartite.
B) Every face of G has even length.
C) The dual graph G^* is Eulerian.

Proof: A $\Rightarrow$ B. A face boundary consists of closed walks. Every odd closed walk contains an odd cycle. Therefore, in a bipartite plane graph the contributions to the length of faces are all even.

B $\Rightarrow$ A. Let C be a cycle in G. Since G has no crossings, C is laid out as a simple closed curve; let F be the region enclosed by C. Every region of G is wholly within F or wholly outside F. If we sum the face lengths for the regions inside F, we obtain an even number, since each face length is even. This sum counts each edge of C once. It also counts each edge inside F twice, since each such edge belongs twice to faces in F. Hence the parity of the length of C is the same as the parity of the full sum, which is even.

B $\Leftrightarrow$ C. The dual graph G^* is connected, and its vertex degrees are the face lengths of G. ■



Many questions we consider for general planar graphs can be answered rather easily for a special class of planar graphs.

6.1.17. Definition. A graph is **outerplanar** if it has an embedding with every vertex on the boundary of the unbounded face. An **outerplane graph** is such an embedding of an outerplanar graph.

The graph in Example 6.1.10 is outerplanar, but another embedding is needed to demonstrate this.

6.1.18. Proposition. The boundary of the outer face a 2-connected outerplane graph is a spanning cycle.

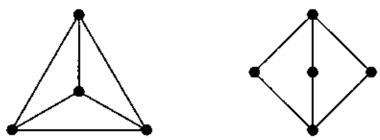
Proof: This boundary contains all the vertices. If it is not a cycle, then it passes through some vertex more than once. Such a vertex would be a cut-vertex. ■

6.1.19. Proposition. K_4 and $K_{2,3}$ are planar but not outerplanar.

Proof: The figure below shows that K_4 and $K_{2,3}$ are planar.

To show that they are not outerplanar, observe that they are 2-connected. Thus an outerplane embedding requires a spanning cycle. There is no spanning cycle in $K_{2,3}$, since it would be a cycle of length 5 in a bipartite graph.

There is a spanning cycle in K_4, but the endpoints of the remaining two edges alternate along it. Hence these chords conflict and cannot both be drawn inside. Drawing a chord outside separates a vertex from the outer face. ■



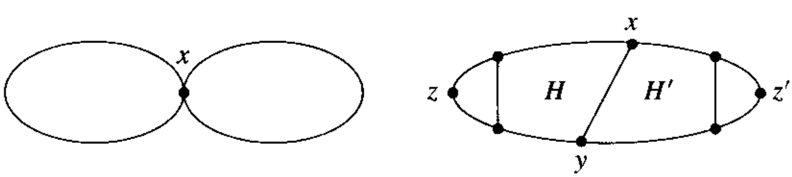
6.1.20. Proposition. Every simple outerplanar graph has a vertex of degree at most 2.

Proof: It suffices to prove the statement for connected graphs. We use induction on $n(G)$; when $n(G) \le 3$, every vertex has degree at most 2. For $n(G) \ge 4$, we prove the stronger statement that G has two nonadjacent vertices of degree at most 2.

Basis step ($n(G) = 4$): Since K_4 is not outerplanar, G has nonadjacent vertices, and two nonadjacent vertices have degree at most 2.

Induction step ($n(G) \ge 4$): If G has a cut-vertex x, then each $\{x\}$-lobe of G has a vertex of degree at most 2 other than x, and these are nonadjacent in G.

If G is 2-connected, then the outer face boundary is a cycle C. If C has no chords, then G is 2-regular. If xy is a chord of C, then the vertex sets of the two x,y-paths on C both induce outerplanar subgraphs. By the induction hypothesis, these subgraphs H, H' contain vertices z, z' of degree at most 2 that are not in $\{x, y\}$ (this includes the case where H or H' is K_3). Since no chord of C can be drawn outside C or cross xy, we have $z \not\leftrightarrow z'$. Thus z, z' is the desired pair of vertices. ■



EULER'S FORMULA

Euler's Formula ($n - e + f = 2$) is the basic counting tool relating vertices, edges, and faces in planar graphs.

6.1.21. Theorem. (Euler [1758]): If a connected plane graph G has exactly n vertices, e edges, and f faces, then $n - e + f = 2$.

Proof: We use induction on n. Basis step ($n = 1$): G is a "bouquet" of loops, each a closed curve in the embedding. If $e = 0$, then $f = 1$, and the formula holds. Each added loop passes through a face and cuts it into two faces (by the Jordan Curve Theorem). This augments the edge count and the face count each by 1. Thus the formula holds when $n = 1$ for any number of edges.

Induction step ($n > 1$): Since G is connected, we can find an edge that is not a loop. When we contract such an edge, we obtain a plane graph G' with n' vertices, e' edges, and f' faces. The contraction does not change the number of faces (we merely shortened boundaries), but it reduces the number of edges and vertices by 1, so $n' = n - 1$, $e' = e - 1$, and $f' = f$. Applying the induction hypothesis yields

$$n - e + f = n' + 1 - (e' + 1) + f' = n' - e' + f' = 2. \qquad \blacksquare$$

$n = 1$ $\qquad\qquad$ $n > 1$

6.1.22. Remark. 1) By Euler's Formula, all planar embeddings of a connected graph G have the same number of faces. Although the dual may depend on the embedding chosen for G, the number of vertices in the dual does not.

2) Euler's Formula as stated fails for disconnected graphs. If a plane graph G has k components, then adding $k-1$ edges to G yields a connected plane graph without changing the number of faces. Hence Euler's Formula generalizes for plane graphs with k components as $n - e + f = k + 1$ (for example, consider a graph with n vertices and no edges). ■

Euler's Formula has many applications, particularly for simple plane graphs, where all faces have length at least 3.

6.1.23. Theorem. If G is a simple planar graph with at least three vertices, then $e(G) \le 3n(G) - 6$. If also G is triangle-free, then $e(G) \le 2n(G) - 4$.

Proof: It suffices to consider connected graphs; otherwise we could add edges. Euler's Formula will relate $n(G)$ and $e(G)$ if we can dispose of f.

Proposition 6.1.13 provides an inequality between e and f. Every face boundary in a simple graph contains at least three edges (if $n(G) \ge 3$). Letting $\{f_i\}$ be the list of face lengths, this yields $2e = \sum f_i \ge 3f$. Substituting into $n - e + f = 2$ yields $e \le 3n - 6$.

When G is triangle-free, the faces have length at least 4. In this case $2e = \sum f_i \geq 4f$, and we obtain $e \leq 2n - 4$. ■

6.1.24. Example. Nonplanarity of K_5 and $K_{3,3}$ follows immediately from Theorem 6.1.23. For K_5, we have $e = 10 > 9 = 3n - 6$. Since $K_{3,3}$ is triangle-free, we have $e = 9 > 8 = 2n - 4$. These graphs have too many edges to be planar. ■

6.1.25. Definition. A **maximal planar graph** is a simple planar graph that is not a spanning subgraph of another planar graph. A **triangulation** is a simple plane graph where every face boundary is a 3-cycle.

6.1.26. Proposition. For a simple n-vertex plane graph G, the following are equivalent.
A) G has $3n - 6$ edges.
B) G is a triangulation.
C) G is a maximal plane graph.

Proof: A $\Leftrightarrow$ B. For a simple n-vertex plane graph, the proof of Theorem 6.1.23 shows that having $3n - 6$ edges is equivalent to $2e = 3f$, which occurs if and only if every face is a 3-cycle.

B $\Leftrightarrow$ C. There is a face that is longer than a 3-cycle if and only if there is a way to add an edge to the drawing and obtain a larger simple plane graph. ■

6.1.27. Remark. A graph embeds in the plane if and only if it embeds on a sphere. Given an embedding on a sphere, we can puncture the sphere inside a face and project the embedding onto a plane tangent to the opposite point. This yields a planar embedding in which the punctured face on the sphere becomes the unbounded face in the plane. The process is reversible. ■

6.1.28. Application. *Regular polyhedra.* Informally, we think of a regular polyhedron as a solid whose boundary consists of regular polygons of the same length, with the same number of faces meeting at each vertex. When we expand the polyhedron out to a sphere and then lay out the drawing in the plane as in Remark 6.1.27, we obtain a regular plane graph with faces of the same length. Hence the dual also is a regular graph.

Let G be a plane graph with n vertices, e edges, and f faces. Suppose that G is regular of degree k and that all faces have length l. The degree-sum formula for G and for G^* yields $kn = 2e = lf$. By substituting for n and f in Euler's Formula, we obtain $e(\frac{2}{k} - 1 + \frac{2}{l}) = 2$. Since e and 2 are positive, the other factor must also be positive, which yields $(2/k) + (2/l) > 1$, and hence $2l + 2k > kl$. This inequality is equivalent to $(k-2)(l-2) < 4$.

Because the dual of a 2-regular graph is not simple, we require that $k, l \geq 3$. Now $(k-2)(l-2) < 4$ also requires $k, l \leq 5$. The only integer pairs satisfying these requirements for (k, l) are $(3,3)$, $(3,4)$, $(3,5)$, $(4,3)$, and $(5,3)$.

Once we specify k and l, there is only one way to lay out the plane graph when we start with any face. Hence there are only the five Platonic solids listed below, one for each pair (k, l) that satisfying the requirements. ■

k	l	$(k-2)(l-2)$	e	n	f	name
3	3	1	6	4	4	tetrahedron
3	4	2	12	8	6	cube
4	3	2	12	6	8	octahedron
3	5	3	30	20	12	dodecahedron
5	3	3	30	12	20	icosahedron

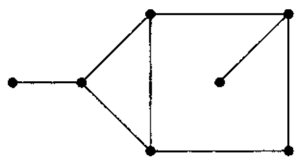
EXERCISES

6.1.1. (−) Prove or disprove:
a) Every subgraph of a planar graph is planar.
b) Every subgraph of a nonplanar graph is nonplanar.

6.1.2. (−) Show that the graphs formed by deleting one edge from K_5 and $K_{3,3}$ are planar.

6.1.3. (−) Determine all r, s such that $K_{r,s}$ is planar.

6.1.4. (−) Determine the number of isomorphism classes of planar graphs that can be obtained as planar duals of the graph below



6.1.5. (−) Prove that a plane graph has a cut-vertex if and only if its dual has a cut-vertex.

6.1.6. (−) Prove that a plane graph is 2-connected if and only if for every face, the bounding walk is a cycle.

6.1.7. (−) A **maximal outerplanar graph** is a simple outerplanar graph that is not a spanning subgraph of a larger simple outerplanar graph. Let G be a maximal outerplanar graph with at least three vertices. Prove that G is 2-connected.

6.1.8. (−) Prove that every simple planar graph has a vertex of degree at most 5.

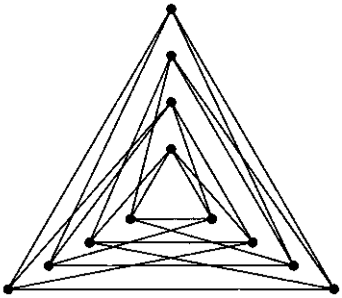
6.1.9. (−) Use Theorem 6.1.23 to prove that every simple planar graph with fewer than 12 vertices has a vertex of degree at most 4.

6.1.10. (−) Prove or disprove: There is no simple bipartite planar graph with minimum degree at least 4.

6.1.11. (−) Let G be a maximal planar graph. Prove that G^* is 2-edge-connected and 3-regular.

6.1.12. (−) Draw the five regular polyhedra as planar graphs. Show that the octahedron is the dual of the cube and the icosahedron is the dual of the dodecahedron.

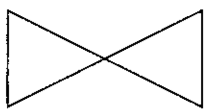
6.1.13. Find a planar embedding of the graph below.



6.1.14. Prove or disprove: For each $n \in \mathbb{N}$, there is a simple connected 4-regular planar graph with more than n vertices.

6.1.15. Construct a 3-regular planar graph of diameter 3 with 12 vertices. (Comment: T. Barcume proved that no such graph has more than 12 vertices.)

6.1.16. Let F be a figure drawn continuously in the plane without retracing any segment, ending at the start (this can be viewed as an Eulerian graph). Prove that F can be drawn without allowing the pencil point to cross what has already been drawn. For example, the figure below has two traversals; one crosses itself and the other does not.



6.1.17. Prove or disprove: If G is a 2-connected simple plane graph with minimum degree 3, then the dual graph G^* is simple.

6.1.18. Given a plane graph G, draw the dual graph G^* so that each dual edge intersects its corresponding edge in G and no other edge. Prove the following.
a) G^* is connected.
b) If G is connected, then each face of G^* contains exactly one vertex of G.
c) $(G^*)^* = G$ if and only if G is connected.

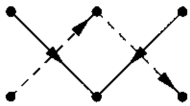
6.1.19. Let G be a plane graph. Use induction on $e(G)$ to prove Theorem 6.1.14: a set $D \subseteq E(G)$ is a cycle in G if and only if the corresponding set $D^* \subseteq E(G^*)$ is a bond in G^*. (Hint: Contract an edge of D and apply Remark 6.1.15.)

6.1.20. Prove by induction on the number of faces that a plane graph G is bipartite if and only if every face has even length.

6.1.21. (!) Prove that a set of edges in a connected plane graph G forms a spanning tree of G if and only if the duals of the remaining edges form a spanning tree of G^*.

6.1.22. The **weak dual** of a plane graph G is the graph obtained from the dual G^* by deleting the vertex for the unbounded face of G. Prove that the weak dual of an outerplane graph is a forest.

6.1.23. (!) *Directed plane graphs.* Let G be a plane graph, and let D be an orientation of G. The **dual** D^* is an orientation of G^* such that when an edge of D is traversed from tail to head, the dual edge in D^* crosses it from right to left. For example, if the solid edges below are in D, then the dashed edges are in D^*.



Prove that if D is strongly connected, then D^* has no cycle, and $\delta^-(D^*) = \delta^+(D^*) = 0$. Conclude that if D is strongly connected, then D has a face on which the edges form a clockwise cycle and another face on which the edges form a counterclockwise cycle.

6.1.24. (!) *Alternative proof of Euler's Formula.*

a) Use polygonal curves (not Euler's Formula) to prove by induction on $n(G)$ that every planar embedding of a tree G has one face.

b) Prove Euler's Formula by induction on the number of cycles.

6.1.25. (!) Prove that every n-vertex plane graph isomorphic to its dual has $2n-2$ edges. For all $n \ge 4$, construct a simple n-vertex plane graph isomorphic to its dual.

6.1.26. Determine the maximum number of edges in a simple outerplane graph with n vertices, giving three proofs.

a) By induction on n.

b) By using Euler's Formula.

c) By adding a vertex in the unbounded face and using Theorem 6.1.23.

6.1.27. Let G be a connected 3-regular plane graph in which every vertex lies on one face of length 4, one face of length 6, and one face of length 8.

a) In terms of $n(G)$, determine the number of faces of each length.

b) Use Euler's Formula and part (a) to determine the number of faces of G.

6.1.28. Let C be a closed curve bounding a convex region in the plane. Suppose that m chords of C are drawn so that no three share a point and no two share an endpoint. Let p be the number of pairs of chords that cross. In terms of m and p, compute the number of segments and the number of regions formed inside C. (Alexanderson–Wetzel [1977])

6.1.29. Prove that the complement of a simple planar graph with at least 11 vertices is nonplanar. Construct a self-complementary simple planar graph with 8 vertices.

6.1.30. (!) Let G be an n-vertex simple planar graph with girth k. Prove that G has at most $(n-2)\frac{k}{k-2}$ edges. Use this to prove that the Petersen graph is nonplanar.

6.1.31. Let G be the simple graph with vertex set $v_1, \dots, v_n$ whose edges are $\{v_i v_j \colon |i-j| \le 3\}$. Prove that G is a maximal planar graph.

6.1.32. Let G be a maximal planar graph. Prove that if S is a separating 3-set of G^*, then $G^* - S$ has two components. (Chappell)

6.1.33. (!) Let G be a triangulation, and let n_i be the number of vertices of degree i in G. Prove that $\sum(6-i)n_i = 12$.

6.1.34. Construct an infinite family of simple planar graphs with minimum degree 5 such that each has exactly 12 vertices of degree 5. (Hint: Modify the dodecahedron.)

6.1.35. (!) Prove that every simple planar graph with at least four vertices has at least four vertices with degree less than 6. For each even value of n with $n \ge 8$, construct an n-vertex simple planar graph G that has exactly four vertices with degree less than 6. (Grünbaum–Motzkin [1963])

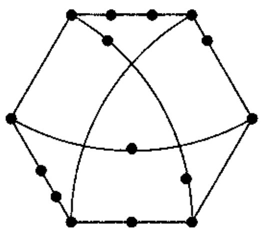
6.1.36. Let S be a set of n points in the plane such that for all $x, y \in S$, the distance in the plane between x and y is at least 1. Prove that there are at most $3n - 6$ pairs u, v in S such that the distance in the plane between u and v is exactly 1.

6.1.37. Given integers $k \geq 2$, $l \geq 1$, and kl even, construct a planar graph with exactly k faces in which every face has length l.

6.2. Characterization of Planar Graphs

Which graphs embed in the plane? We have proved that K_5 and $K_{3,3}$ do not. In fact, these are the crucial graphs and lead to a characterization of planar graphs known as Kuratowski's Theorem. Kasimir Kuratowski once asked Frank Harary about the origin of the notation for K_5 and $K_{3,3}$. Harary replied, "The K in K_5 stands for Kasimir, and the K in $K_{3,3}$ stands for Kuratowski!"

Recall that a subdivision of a graph is a graph obtained from it by replacing edges with pairwise internally-disjoint paths (Definition 5.2.19).



a subdivision of $K_{3,3}$

6.2.1. Proposition. If a graph G has a subgraph that is a subdivision of K_5 or $K_{3,3}$, then G is nonplanar.

Proof: Every subgraph of a planar graph is planar, so it suffices to show that subdivisions of K_5 and $K_{3,3}$ are nonplanar. Subdividing edges does not affect planarity; the curves in an embedding of a subdivision of G can be used to obtain an embedding of G, and vice versa. ■

By Proposition 6.2.1, avoiding subdivisions of K_5 and $K_{3,3}$ is a necessary condition for being a planar graph. Kuratowski proved TONCAS:

6.2.2. Theorem. (Kuratowski [1930]) A graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$. ■

Kuratowski's Theorem is our goal in the first half of this section, after which we will comment on other characterizations of planar graphs.

When G is planar, we can seek a planar embedding with additional properties. Wagner [1936], Fáry [1948], and Stein [1951] showed that every finite simple planar graph has an embedding in which all edges are straight line segments; this is known as **Fáry's Theorem** (Exercise 6). For 3-connected planar graphs, we will prove the stronger property that there exists an embedding in which every face is a convex polygon.

PREPARATION FOR KURATOWSKI'S THEOREM

We introduce short names for subgraphs that demonstrate nonplanarity.

6.2.3. Definition. A **Kuratowski subgraph** of G is a subgraph of G that is a subdivision of K_5 or $K_{3,3}$. A **minimal nonplanar graph** is a nonplanar graph such that every proper subgraph is planar.

We will prove that a minimal nonplanar graph with no Kuratowski subgraph must be 3-connected. Showing that every 3-connected graph with no Kuratowski subgraph is planar then completes the proof of Kuratowski's Theorem.

6.2.4. Lemma. If F is the edge set of a face in a planar embedding of G, then G has an embedding with F being the edge set of the unbounded face.

Proof: Project the embedding onto the sphere, where the edge sets of regions remain the same and all regions are bounded, and then return to the plane by projecting from inside the face bounded by F. ■

6.2.5. Lemma. Every minimal nonplanar graph is 2-connected.

Proof: Let G be a minimal nonplanar graph. If G is disconnected, then we embed one component of G inside one face of an embedding of the rest.

If G has a cut-vertex v, let $G_1, \dots, G_k$ be the $\{v\}$-lobes of G. By the minimality of G, each G_i is planar. By Lemma 6.2.4, we can embed each G_i with v on the outside face. We squeeze each embedding to fit in an angle smaller than $360/k$ degrees at v, after which we combine the embeddings at v to obtain an embedding of G. ■

6.2.6. Lemma. Let $S = \{x, y\}$ be a separating 2-set of G. If G is nonplanar, then adding the edge xy to some S-lobe of G yields a nonplanar graph.

Proof: Let $G_1, \dots, G_k$ be the S-lobes of G, and let $H_i = G_i \cup xy$. If H_i is planar, then by Lemma 6.2.4 it has an embedding with xy on the outside face. For each $i > 1$, this allows H_i to be attached to an embedding of $\bigcup_{j=1}^{i-1} H_j$ by embedding H_i in a face that has xy on its boundary. Afterwards, deleting the edge xy if it is not in G yields a planar embedding of G. ■

The next lemma allows us to restrict our attention to 3-connected graphs in order to prove Kuratowski's Theorem. The hypothesized graph doesn't exist, but if it did, it would be 3-connected.