$R_2+2R_1 \mapsto R_2$, $R_3-(x-1)R_1 \mapsto R_3$, $C_3+(x-1)C_1 \mapsto C_3$, $C_4+2C_1 \mapsto C_4$, $R_2 \leftrightarrow R_4$, $-R_2$, $R_3+2R_2 \mapsto R_3$, $R_4-(x+1)R_2 \mapsto R_4$, $C_3+2C_2 \mapsto C_3$, $C_4+(x-3)C_2 \mapsto C_4$).

I. (*Invariant Factor Decomposition*) If e_1, e_2, e_3, e_4 is a basis for V in this case, then using the row operations in this diagonalization as in the previous example we see that the generators of V corresponding to the factors above are $(x-1)e_1-2e_2-e_3=0$, $-2e_1+(x+1)e_2-e_4=0$, e_1, e_2. Hence a vector space basis for the two direct factors in the invariant decomposition of V in this case is given by e_1, Te_1 and e_2, Te_2 where T is the linear transformation defined by D, i.e., e_1, $e_1+2e_2+e_3$ and e_2, $2e_1-e_2+e_4$. The corresponding matrix P relating these bases is

$$P=\begin{pmatrix}1&1&0&2\\0&2&1&-1\\0&1&0&0\\0&0&0&1\end{pmatrix}$$

so that $P^{-1}DP$ is in rational canonical form:

$$P^{-1}DP=\begin{pmatrix}0&-1&0&0\\1&2&0&0\\0&0&0&-1\\0&0&1&2\end{pmatrix}$$

as can easily be checked.

II. (*Converting D Directly to Rational Canonical Form*) As in Example 2 we determine the matrix P' of the algorithm from the row operations used in the diagonalization of $xI-D$:

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\xrightarrow[C_1\leftrightarrow C_3]{}\begin{pmatrix}0&0&1&0\\0&1&0&0\\1&0&0&0\\0&0&0&1\end{pmatrix}\xrightarrow[-C_1]{}\begin{pmatrix}0&0&1&0\\0&1&0&0\\-1&0&0&0\\0&0&0&1\end{pmatrix}\to$$

$$\xrightarrow[\substack{C_1-2C_2\\ \mapsto C_1}]{}\begin{pmatrix}0&0&1&0\\-2&1&0&0\\-1&0&0&0\\0&0&0&1\end{pmatrix}\xrightarrow[\substack{C_1+(D-I)C_3\\ \mapsto C_1}]{}\begin{pmatrix}0&0&1&0\\0&1&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix}\xrightarrow[C_2\leftrightarrow C_4]{}\begin{pmatrix}0&0&1&0\\0&0&0&1\\0&0&0&0\\0&1&0&0\end{pmatrix}\to$$

$$\xrightarrow[-C_2]{}\begin{pmatrix}0&0&1&0\\0&0&0&1\\0&0&0&0\\0&-1&0&0\end{pmatrix}\xrightarrow[\substack{C_2-2C_3\\ \mapsto C_2}]{}\begin{pmatrix}0&-2&1&0\\0&0&0&1\\0&0&0&0\\0&-1&0&0\end{pmatrix}\xrightarrow[\substack{C_2+(D+I)C_4\\ \mapsto C_2}]{}\begin{pmatrix}0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix}=P'.$$

Here we have $d_1=2$ and $d_2=2$, corresponding to the third and fourth nonzero columns of P'. The columns of P are therefore given by

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\quad D\begin{pmatrix}1\\0\\0\\0\end{pmatrix}=\begin{pmatrix}1\\2\\1\\0\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\quad D\begin{pmatrix}0\\1\\0\\0\end{pmatrix}=\begin{pmatrix}2\\-1\\0\\1\end{pmatrix},$$

respectively, which again gives the matrix P above.

(4) In this example we determine all similarity classes of matrices A with entries from $\mathbb{Q}$ with characteristic polynomial $(x^4-1)(x^2-1)$. First note that any matrix with a degree

6 characteristic polynomial must be a 6×6 matrix. The polynomial $(x^4-1)(x^2-1)$ factors into irreducibles in $\mathbb{Q}[x]$ as $(x-1)^2(x+1)^2(x^2+1)$. Since the minimal polynomial $m_A(x)$ for A has the same roots as $c_A(x)$ it follows that $(x-1)(x+1)(x^2+1)$ divides $m_A(x)$. Suppose $a_1(x), \dots, a_m(x)$ are the invariant factors of some A, so $a_m(x) = m_A(x)$, $a_i(x) \mid a_{i+1}(x)$ (in particular, all the invariant factors divide $m_A(x)$) and $a_1(x)a_2(x)\cdots a_m(x) = (x^4-1)(x^2-1)$. One easily sees that the only permissible lists under these constraints are

(a) $(x-1)(x+1), \quad (x-1)(x+1)(x^2+1)$
(b) $x-1, \quad (x-1)(x+1)^2(x^2+1)$
(c) $x+1, \quad (x-1)^2(x+1)(x^2+1)$
(d) $(x-1)^2(x+1)^2(x^2+1)$.

One can now easily write out the corresponding direct sums of companion matrices to obtain representatives of the 4 similarity classes. We shall see in the next section that there are still only 4 similarity classes even in $M_6(\mathbb{C})$.

(5) In this example we find all similarity classes of 3×3 matrices A with entries from $\mathbb{Q}$ satisfying $A^6 = I$. For each such A, its minimal polynomial divides x^6-1 and in $\mathbb{Q}[x]$ the complete factorization of this polynomial is

$$x^6 - 1 = (x-1)(x+1)(x^2-x+1)(x^2+x+1).$$

Conversely, if B is any 3×3 matrix whose minimal polynomial divides x^6-1, then $B^6 = I$. The only restriction on the minimal polynomial for B is that its degree is at most 3 (by the Cayley–Hamilton Theorem). The only possibilities for the minimal polynomial of such a matrix A are therefore

(a) $x-1$
(b) $x+1$
(c) x^2-x+1
(d) x^2+x+1
(e) $(x-1)(x+1)$
(f) $(x-1)(x^2-x+1)$
(g) $(x-1)(x^2+x+1)$
(h) $(x+1)(x^2-x+1)$
(i) $(x+1)(x^2+x+1)$.

Under the constraints of the rational canonical form these give rise to the following permissible lists of invariant factors:

(i) $x-1, \quad x-1, \quad x-1$
(ii) $x+1, \quad x+1, \quad x+1$
(iii) $x-1, \quad (x-1)(x+1)$
(iv) $x+1, \quad (x-1)(x+1)$
(v) $(x-1)(x^2-x+1)$
(vi) $(x-1)(x^2+x+1)$
(vii) $(x+1)(x^2-x+1)$
(viii) $(x+1)(x^2+x+1)$.

Note that it is impossible to have a suitable set of invariant factors if the minimal polynomial is x^2+x+1 or x^2-x+1 . One can now write out the corresponding rational canonical forms; for example, (i) is I, (ii) is $-I$, and (iii) is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Note also that another way of phrasing this result is that any 3×3 matrix with entries from $\mathbb{Q}$ whose order (multiplicatively, of course) divides 6 is similar to one of these 8 matrices, so this example determines all elements of orders 1,2,3 and 6 in the group $GL_3(\mathbb{Q})$ (up to similarity).

EXERCISES

1. Prove that similar linear transformations of V (or $n \times n$ matrices) have the same characteristic and the same minimal polynomial.
2. Let M be as in Lemma 19. Prove that the minimal polynomial of M is the least common multiple of the minimal polynomials of $A_1, \dots, A_k$.
3. Prove that two 2×2 matrices over F which are not scalar matrices are similar if and only if they have the same characteristic polynomial.
4. Prove that two 3×3 matrices are similar if and only if they have the same characteristic and same minimal polynomials. Give an explicit counterexample to this assertion for 4×4 matrices.
5. Prove directly from the fact that the collection of *all* linear transformations of an n dimensional vector space V over F to itself form a vector space over F of dimension n^2 that the minimal polynomial of a linear transformation T has degree at most n^2.
6. Prove that the constant term in the characteristic polynomial of the $n \times n$ matrix A is $(-1)^n \det A$ and that the coefficient of x^{n-1} is the negative of the sum of the diagonal entries of A (the sum of the diagonal entries of A is called the *trace* of A). Prove that $\det A$ is the product of the eigenvalues of A and that the trace of A is the sum of the eigenvalues of A.
7. Determine the eigenvalues of the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

8. Verify that the characteristic polynomial of the companion matrix

$$\begin{pmatrix} 0 & 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & 0 & \dots & 0 & -a_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -a_{n-1} \end{pmatrix}$$

is

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

9. Find the rational canonical forms of

$$\begin{pmatrix} 0 & -1 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} c & 0 & -1 \\ 0 & c & 1 \\ -1 & 1 & c \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 422 & 465 & 15 & -30 \\ -420 & -463 & -15 & 30 \\ 840 & 930 & 32 & -60 \\ -140 & -155 & -5 & 12 \end{pmatrix}.$$

10. Find all similarity classes of 6×6 matrices over $\mathbb{Q}$ with minimal polynomial $(x+2)^2(x-1)$ (it suffices to give all lists of invariant factors and write out some of their corresponding matrices).

11. Find all similarity classes of 6×6 matrices over $\mathbb{C}$ with characteristic polynomial $(x^4 - 1)(x^2 - 1)$.

12. Find all similarity classes of 3×3 matrices A over $\mathbb{F}_2$ satisfying $A^6 = I$ (compare with the answer we computed over $\mathbb{Q}$). Do the same for 4×4 matrices B satisfying $B^{20} = I$.

13. Prove that the number of similarity classes of 3×3 matrices over $\mathbb{Q}$ with a given characteristic polynomial in $\mathbb{Q}[x]$ is the same as the number of similarity classes over any extension field of $\mathbb{Q}$. Give an example to show that this is not true in general for 4×4 matrices.

14. Determine all possible rational canonical forms for a linear transformation with characteristic polynomial $x^2(x^2+1)^2$.

15. Determine up to similarity all 2×2 rational matrices (i.e., $\in M_2(\mathbb{Q})$) of precise order 4 (multiplicatively, of course). Do the same if the matrix has entries from $\mathbb{C}$.

16. Show that $x^5 - 1 = (x-1)(x^2 - 4x + 1)(x^2 + 5x + 1)$ in $\mathbb{F}_{19}[x]$. Use this to determine up to similarity all 2×2 matrices with entries from $\mathbb{F}_{19}$ of (multiplicative) order 5.

17. Determine representatives for the conjugacy classes for $GL_3(\mathbb{F}_2)$. [Compare your answer with Theorem 15 and Proposition 14 of Chapter 6.]

18. Let V be a finite dimensional vector space over $\mathbb{Q}$ and suppose T is a nonsingular linear transformation of V such that $T^{-1} = T^2 + T$. Prove that the dimension of V is divisible by 3. If the dimension of V is precisely 3 prove that all such transformations T are similar.

19. Let V be the infinite dimensional real vector space

$$\mathbb{R}^\infty = \{(a_0, a_1, a_2, \ldots) \mid a_0, a_1, a_2, \cdots \in \mathbb{R}\}.$$

Define the map $T : V \to V$ by $T(a_0, a_1, a_2, \ldots) = (0, a_0, a_1, a_2, \ldots)$. Prove that T has no eigenvectors.

20. Let ℓ be a prime and let $\Phi_\ell(x) = \frac{x^\ell - 1}{x-1} = x^{\ell-1} + x^{\ell-2} + \ldots + x + 1 \in \mathbb{Z}[x]$ be the ℓ^{th} cyclotomic polynomial, which is irreducible over $\mathbb{Q}$ (Example 4 following Corollary 9.14). This exercise determines the smallest degree of a factor of $\Phi_\ell(x)$ modulo p for any prime p and so in particular determines when $\Phi_\ell(x)$ is irreducible modulo p. (This actually determines the complete factorization of $\Phi_\ell(x)$ modulo p — cf. Exercise 8 of Section 13.6.)

(a) Show that if $p = \ell$ then $\Phi_\ell(x)$ is divisible by $x - 1$ in $\mathbb{F}_\ell[x]$.

(b) Suppose $p \neq \ell$ and let f denote the order of p in $\mathbb{F}_\ell^\times$, i.e., f is the smallest power of p with $p^f \equiv 1 \bmod \ell$. Show that $m = f$ is the first value of m for which the group $GL_m(\mathbb{F}_p)$ contains an element A of order ℓ. [Use the formula for the order of this group at the end of Section 11.1.]

(c) Show that $\Phi_\ell(x)$ is not divisible by any polynomial of degree smaller than f in $\mathbb{F}_p[x]$ [consider the companion matrix for such a divisor and use (b)]. Let $m_A(x) \in \mathbb{F}_p[x]$ denote the minimal polynomial for the matrix A in (b) and conclude that $m_A(x)$ is irreducible of degree f and divides $\Phi_\ell(x)$ in $\mathbb{F}_p[x]$.