Although this gives us a power series $y = \sum a_k x^k$ which formally satisfies (6.44), the ratio test shows that this power series converges *only* for $x = 0$. Thus, there is no power-series solution of (6.44) valid in any open interval about $x_0 = 0$. This example does not violate Theorem 6.13 because when we put Equation (6.44) in the form (6.43) we find that the coefficients P_1 and P_2 are given by

$$P_1(x) = -\frac{1}{x^2} \qquad \text{and} \qquad P_2(x) = -\frac{1}{x^2}.$$

These functions do not have power-series expansions about the origin. The difficulty here is that the coefficient of y'' in (6.44) has the value 0 when $x = 0$; in other words, the differential equation has a singular point at $x = 0$.

A knowledge of the theory of functions of a complex variable is needed to appreciate the difficulties encountered in the investigation of differential equations near a singular point. However, some important special cases of equations with singular points can be treated by elementary methods. For example, suppose the differential equation in (6.43) is equivalent to an equation of the form

$$(x - x_0)^2 y'' + (x - x_0)P(x)y' + Q(x)y = 0, \tag{6.45}$$

where P and Q have power-series expansions in some open interval $(x_0 - r, x_0 + r)$. In this case we say that x_0 is a *regular* singular point of the equation. If we divide both sides of (6.45) by $(x - x_0)^2$ the equation becomes

$$y'' + \frac{P(x)}{x - x_0}\,y' + \frac{Q(x)}{(x - x_0)^2}\,y = 0$$

for $x \neq x_0$. If $P(x_0) \neq 0$ or $Q(x_0) \neq 0$, or if $Q(x_0) = 0$ and $Q'(x_0) \neq 0$, either the coefficient of y' or the coefficient of y will not have a power-series expansion about the point x_0, so Theorem 6.13 will not be applicable. In 1873 the German mathematician Georg Frobenius (1849-1917) developed a useful method for treating such equations. We shall describe the theorem of Frobenius but we shall not present its proof.† In the next section we give the details of the proof for an important special case, the Bessel equation.

Frobenius' theorem splits into two parts, depending on the nature of the roots of the quadratic equation

$$t(t - 1) + P(x_0)t + Q(x_0) = 0. \tag{6.46}$$

This quadratic equation is called the ***indicial equation*** of the given differential equation (6.45). The coefficients $P(x_0)$ and $Q(x_0)$ are the constant terms in the power-series expansions of P and Q. Let α_1 and α_2 denote the roots of the indicial equation. These roots may be real or complex, equal or distinct. The type of solution obtained by the Frobenius method depends on whether or not these roots differ by an integer.

† For a proof see E. Hille, Analysis, Vol. II, Blaisdell Publishing Co., 1966, or E. A. Coddington, ***An Introduction to Ordinary Differential Equations,*** Prentice-Hall, 1961.

THEOREM 6.14. FIRST CASE OF FROBENIUS' THEOREM. *Let α_1 and α_2 be the roots of the indicial equation and assume that $\alpha_1 - \alpha_2$ is not an integer. Then the differential equation (6.45) has two independent solutions u_1 and u_2 of the form*

$$u_1(x) = |x - x_0|^{\alpha_1} \sum_{n=0}^{\infty} a_n (x - x_0)^n, \qquad \text{with } a_0 = 1, \tag{6.47}$$

and

$$u_2(x) = |x - x_0|^{\alpha_2} \sum_{n=0}^{\infty} b_n (x - x_0)^n, \qquad \text{with } b_0 = 1. \tag{6.48}$$

Both series converge in the interval $|x - x_0| < r$, and the differential equation is satisfied for $0 < |x - x_0| < r$.

THEOREM 6.15. SECOND CASE OF FROBENIUS' THEOREM. *Let α_1, α_2 be the roots of the indicial equation and assume that $\alpha_1 - \alpha_2 = N$, a nonnegative integer. Then the differential equation (6.45) has a solution u_1 of the form (6.47) and another independent solution u_2 of the form*

$$u_2(x) = |x - x_0|^{\alpha_2} \sum_{n=0}^{\infty} b_n (x - x_0)^n + C\, u_1(x) \log |x - x_0|, \tag{6.49}$$

where $b_0 = 1$. The constant C is nonzero if $N = 0$. If $N > 0$, the constant C may or may not be zero. As in Case 1, both series converge in the interval $|x - x_0| < r$, and the solutions are valid for $0 < |x - x_0| < r$.

6.23 The Bessel equation

In this section we use the method suggested by Frobenius to solve the Bessel equation

$$x^2 y'' + xy' + (x^2 - \alpha^2) y = 0,$$

where α is a nonnegative constant. This equation is used in problems concerning vibrations of membranes, heat flow in cylinders, and propagation of electric currents in cylindrical conductors. Some of its solutions are known as *Bessel functions*. Bessel functions also arise in certain problems in Analytic Number Theory. The equation is named after the German astronomer F. W. Bessel (1784–1846), although it appeared earlier in the researches of Daniel Bernoulli (1732) and Euler (1764).

The Bessel equation has the form (6.45) with $x_0 = 0$, $P(x) = 1$, and $Q(x) = x^2 - \alpha^2$, so the point x_0 is a regular singular point. Since P and Q are analytic on the entire real line, we try to find solutions of the form

$$y = |x|^t \sum_{n=0}^{\infty} a_n x^n, \tag{6.50}$$

with $a_0 \neq 0$, valid for all real x with the possible exception of $x = 0$.

First we keep $x > 0$, so that $|x|^t = x^t$. Differentiation of (6.50) gives us

$$y' = t x^{t-1} \sum_{n=0}^{\infty} a_n x^n + x^t \sum_{n=0}^{\infty} n a_n x^{n-1} = x^{t-1} \sum_{n=0}^{\infty} (n + t) a_n x^n.$$

Similarly, we obtain

$$y'' = x^{t-2}\sum_{n=0}^{\infty}(n+t)(n+t-1)a_n x^n.$$

If $L(y) = x^2y'' + xy' + (x^2 - \alpha^2)y$, we find

$$L(y) = x^t\sum_{n=0}^{\infty}(n+t)(n+t-1)a_nx^n + x^t\sum_{n=0}^{\infty}(n+t)a_nx^n$$
$$+ x^t\sum_{n=0}^{\infty}a_nx^{n+2} - x^t\sum_{n=0}^{\infty}\alpha^2a_nx^n = x^t\sum_{n=0}^{\infty}[(n+t)^2-\alpha^2]a_nx^n + x^t\sum_{n=0}^{\infty}a_nx^{n+2}.$$

Now we put $L(y) = 0$, cancel x^t, and try to determine the a_n so that the coefficient of each power of x will vanish. For the constant term we need $(t^2 - \alpha^2)a_0 = 0$. Since we seek a solution with $a_0 \neq 0$, this requires that

$$t^2 - \alpha^2 = 0. \tag{6.51}$$

This is the *indicial equation.* Its roots α and $-\alpha$ are the only possible values of t that can give us a solution of the desired type.

Consider first the choice $t = \alpha$. For this t the remaining equations for determining the coefficients become

$$[(1+\alpha)^2 - \alpha^2]a_1 = 0 \quad \text{and} \quad [(n+\alpha)^2 - \alpha^2]a_n + a_{n-2} = 0 \tag{6.52}$$

for $n \geq 2$. Since $\alpha \geq 0$, the first of these implies that $a_1 = 0$. The second formula can be written as

$$a_n = -\frac{a_{n-2}}{(n+\alpha)^2 - \alpha^2} = -\frac{a_{n-2}}{n(n+2\alpha)}, \tag{6.53}$$

so $a_3 = a_5 = a_7 = \cdots = 0$. For the coefficients with even subscripts we have

$$a_2 = \frac{-a_0}{2(2+2\alpha)} = \frac{-a_0}{2^2(1+\alpha)}, \qquad a_4 = \frac{-a_2}{4(4+2\alpha)} = \frac{(-1)^2a_0}{2^4 2!\,(1+\alpha)(2+\alpha)},$$

$$a_6 = \frac{-a_4}{6(6+2\alpha)} = \frac{(-1)^3a_0}{2^6 3!\,(1+\alpha)(2+\alpha)(3+\alpha)},$$

and, in general,

$$a_{2n} = \frac{(-1)^na_0}{2^{2n}n!\,(1+\alpha)(2+\alpha)\cdots(n+\alpha)}.$$

Therefore the choice $t = \alpha$ gives us the solution

$$y = a_0x^{\alpha}\left(1 + \sum_{n=1}^{\infty}\frac{(-1)^nx^{2n}}{2^{2n}n!\,(1+\alpha)(2+\alpha)\cdots(n+\alpha)}\right).$$

The ratio test shows that the power series appearing in this formula converges for all real x.

In this discussion we assumed that x > 0. If x < 0 we can repeat the discussion with x^t replaced by $(-x)^t$. We again find that t must satisfy the equation $t^2 - \alpha^2 = 0$. Taking $t = \alpha$ we then obtain the same solution, except that the outside factor x^α is replaced by $(-x)^\alpha$. Therefore the function f_α given by the equation

(6.54)
$$f_\alpha(x) = a_0 \, |x|^\alpha \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} n!\,(1+\alpha)(2+\alpha)\cdots(n+\alpha)}\right)$$

is a solution of the Bessel equation valid for all real $x \neq 0$. For those values of α for which $f'_\alpha(0)$ and $f''_\alpha(0)$ exist the solution is also valid for x = 0.

Now consider the root $t = -\alpha$ of the indicial equation. We obtain, in place of (6.52), the equations

$$[(1-\alpha)^2 - \alpha^2]a_1 = 0 \qquad \text{and} \qquad [(n-\alpha)^2 - \alpha^2]a_n + a_{n-2} = 0,$$

which become

$$(1 - 2\alpha)a_1 = 0 \qquad \text{and} \qquad n(n-2\alpha)a_n + a_{n-2} = 0.$$

If 2α is not an integer these equations give us $a,$ = 0 and

$$a, = -\frac{a_{n-2}}{n(n-2\alpha)}$$

for $n \geq 2$. Since this recursion formula is the same as (6.53), with α replaced by $-\alpha$, we are led to the solution

(6.55)
$$f_{-\alpha}(x) = a_0 \, |x|^{-\alpha} \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} n!\,(1-\alpha)(2-\alpha)\cdots(n-\alpha)}\right)$$

valid for all real x $\neq$ 0.

The solution $f_{-\alpha}$ was obtained under the hypothesis that 2α is not a positive integer. However, the series for $f_{-\alpha}$ is meaningful even if 2α is a positive integer, so long as α is not a positive integer. It can be verified that $f_{-\alpha}$ satisfies the Bessel equation for such α. Therefore, for each $\alpha \geq 0$ we have the series solution f_α, given by Equation (6.54); and if α is not a nonnegative integer we have found another solution $f_{-\alpha}$ given by Equation (6.55). The two solutions f_α and $f_{-\alpha}$ are independent, since one of them $\to \infty$ as x $\to 0$, and the other does not. Next we shall simplify the form of the solutions. To do this we need some properties of Euler's gamma function, and we digress briefly to recall these properties.

For each real $s > 0$ we define $\Gamma(s)$ by the improper integral

$$\Gamma(s) = \int_{0+}^{\infty} t^{s-1} e^{-t}\, dt\,.$$

This integral converges if $s > 0$ and diverges if $s \leq 0$. Integration by parts leads to the functional equation

$$\Gamma(s + 1) = s\,\Gamma(s). \tag{6.56}$$

This implies that

$$\Gamma(s + 2) = (s + 1)\Gamma(s + 1) = (s + 1)s\,\Gamma(s),$$

$$\Gamma(s + 3) = (s + 2)\Gamma(s + 2) = (s + 2)(s + 1)s\,\Gamma(s),$$

and, in general,

$$\Gamma(s + n) = (s + n - 1)\cdots(s + 1)s\,\Gamma(s) \tag{6.57}$$

for every positive integer n. Since I'(1) = $\int_0^\infty e^{-t}\,dt = 1$, when we put $s = 1$ in (6.57) we find

$$\Gamma(n + 1) = n!.$$

Thus, the gamma function is an extension of the factorial function from integers to positive real numbers.

The functional equation (6.56) can be used to extend the definition of l?(s) to negative values of s that are not integers. We write (6.56) in the form

$$\Gamma(s) = \frac{\Gamma(s + 1)}{s}. \tag{6.58}$$

The right-hand member is meaningful ifs + 1 > 0 and $s \neq 0$. Therefore, we can use this equation to *define* I'(s) if $-1 < s < 0$. The right-hand member of (6.58) is now meaningful if $s + 2 > 0$, $s \neq -1$, $s \neq 0$, and we can use this equation to define l?(s) for -2 < $s < -1$. Continuing in this manner, we can extend the definition of l?(s) by induction to every open interval of the form $-n < s < -n + 1$, where n is a positive integer. The functional equation (6.56) and its extension in (6.57) are now valid for all real s for which both sides are meaningful.

We return now to the discussion of the Bessel equation. The series for f_α in Equation (6.54) contains the product $(1 + \alpha)(2 + \alpha)\cdots(n + \alpha)$. We can express this product in terms of the gamma function by taking $s = 1 + \alpha$ in (6.57). This gives us

$$(1 + \alpha)(2 + \alpha)\cdots(n + \alpha) = \frac{\Gamma(n + 1 + \alpha)}{\Gamma(1 + \alpha)}.$$

Therefore, if we choose *a,,* = $2^{-\alpha}/\Gamma(1 + \alpha)$ in Equation (6.54) and denote the resulting function $f_\alpha(x)$ by $J_\alpha(x)$ when $x > 0$, the solution for $x > 0$ can be written as

$$J_\alpha(x) = \left(\frac{x}{2}\right)^\alpha \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n + 1 + \alpha)} \frac{x}{2}\left(\frac{x}{2}\right)^{2n}. \tag{6.59}$$

The function J_α defined by this equation for x $>$ 0 and $\alpha \geq 0$ is called the *Bessel/unction of the first kind of order* α. When α is a nonnegative integer, say $\alpha = p$, the Bessel function J_p is given by the power series

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(n+p)!}\left(\frac{x}{2}\right)^{2n+p} \qquad (p = 0,\ 1, 2, \ldots).$$

This is also a solution of the Bessel equation for x < 0. Extensive tables of Bessel functions have been constructed. The graphs of the two functions J_0 and J_1 are shown in Figure 6.2.

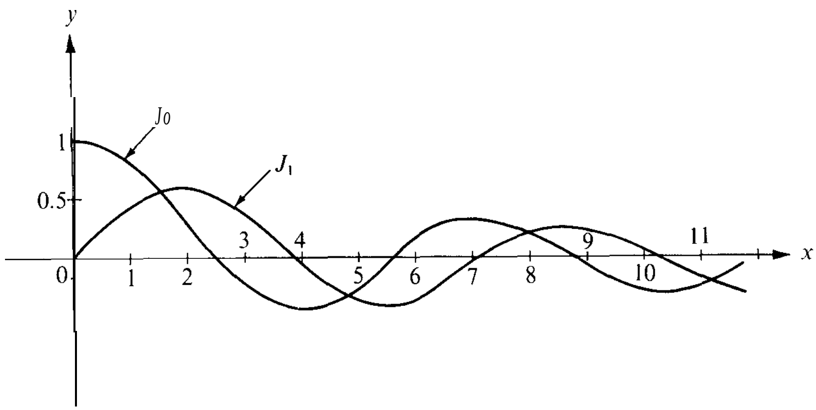

FIGURE 6.2 Graphs of the Bessel functions J_0 and J_1.

We can define a new function $J_{-\alpha}$ by replacing α by $-\alpha$ in Equation (6.59), if α is such that $\Gamma(n + 1 - \alpha)$ is meaningful; that is, if α is not a positive integer. Therefore, if x > 0 and $\alpha > 0$, $\alpha \neq 1, 2, 3, \ldots,$ wedeline

$$J_{-\alpha}(x) = \left(\frac{x}{2}\right)^{-\alpha} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+1-\alpha)}\left(\frac{x}{2}\right)^{2n}.$$

Taking $s = 1 - \alpha$ in (6.57) we obtain

$$\Gamma(n + 1 - \alpha) = (1 - \alpha)(2 - \alpha) \ldots (n - \alpha)\,\Gamma(1 - \alpha)$$

and we see that the series for $J_{-\alpha}(x)$ is the same as that for $f_{-\alpha}(x)$ in Equation (6.55) with $a, = 2^\alpha/\Gamma(1 - \alpha)$, x > 0. Therefore, if α is not a positive integer, $J_{-\alpha}$ is a solution of the Bessel equation for x > 0.

If α is not an integer, the two solutions $J,(x)$ and $J_{-\alpha}(x)$ are linearly independent on the positive real axis (since their ratio is not constant) and the general solution of the Bessel equation for x $>$ 0 is

$$y = c_1 J_\alpha(x) + c_2 J_{-\alpha}(x).$$

If α is a nonnegative integer, say $\alpha = p$, we have found only the solution J_p and its constant multiples valid for x > 0. Another solution, independent of this one, can be found by the method described in Exercise 4 of Section 6.16. This states that if u_1 is a solution of $y'' + P_1 y' + P_2 y = 0$ that never vanishes on an interval I, a second solution u_2 independent of u_1 is given by the integral

$$u_2(x) = u_1(x) \int_c^x \frac{Q(t)}{[u_1(t)]^2}\, dt,$$

where $Q(x) = e^{-\int P_1(x)\,dx}$. For the Bessel equation we have $P_1(x) = 1/x$, so $Q(x) = 1/x$ and a second solution u_2 is given by the formula

$$u_2(x) = J_p(x) \int_c^x \frac{1}{t[J_p(t)]^2}\, dt, \tag{6.60}$$

if c and x lie in an interval I in which J_p does not vanish.

This second solution can be put in other forms. For example, from Equation (6.59) we may write

$$\frac{1}{[J_p(t)]^2} = \frac{1}{t^{2p}}\, g_p(t),$$

where $g_p(0) \neq 0$. In the interval I the function g_p has a power-series expansion

$$g_p(t) = \sum_{n=0}^{\infty} A_n t^n$$

which could be determined by equating coefficients in the identity $g_p(t)\,[J_p(t)]^2 = t^{2p}$. If we assume the existence of such an expansion, the integrand in (6.60) takes the form

$$\frac{1}{t[J_p(t)]^2} = \frac{1}{t^{2p+1}} \sum_{n=0}^{\infty} A_n t^n .$$

Integrating this formula term by term from c to x we obtain a logarithmic term $A_{2p} \log x$ (from the power t^{-1}) plus a series of the form $x^{-2p} \sum B_n x^n$. Therefore Equation (6.60) takes the form

$$u_2(x) = A_{2p} J_p(x) \log x + J_p(x) x^{-2p} \sum_{n=0}^{\infty} B_n x^n .$$

It can be shown that the coefficient $A_{2p} \neq 0$. If we multiply $u_2(x)$ by $1/A_{2p}$ the resulting solution is denoted by $K_p(x)$ and has the form

$$K_p(x) = J_p(x) \log x + x^{-p} \sum_{n=0}^{\infty} C_n x^n .$$

This is the form of the solution promised by the second case of Frobenius' theorem.

Having arrived at this formula, we can verify that a solution of this form actually exists by substituting the right-hand member in the Bessel equation and determining the coefficients C_n so as to satisfy the equation. The details of this calculation are lengthy and will be omitted. The final result can be expressed as

$$K_p(x) = J_p(x) \log x - \frac{1}{2}\left(\frac{x}{2}\right)^{-p} \sum_{n=0}^{p-1} \frac{(p-n-1)!}{n!}\left(\frac{x}{2}\right)^{2n} - \frac{1}{2}\left(\frac{x}{2}\right)^{p} \sum_{n=0}^{\infty}(-1)^n \frac{h_n + h_{n+p}}{n!\,(n+p)!}\left(\frac{x}{2}\right)^{2n},$$

where $h_0 = 0$ and $h_n = 1 + \frac{1}{2} + \cdots + 1/n$ for $n \geq 1$. The series on the right converges for all real x. The function K_p defined for x > 0 by this formula is called the *Besselfunction of the second kind of orderp.* Since K_p is not a constant multiple of J_p , the general solution of the Bessel equation in this case for x > 0 is

$$y = c_1 J_p(x) + c_2 K_p(x).$$

Further properties of the Bessel functions are discussed in the next set of exercises.

6.24 Exercises

1. (a) Let f be any solution of the Bessel equation of order α and let $g(x) = x^{1/2} f(x)$ for x > 0. Show that g satisfies the differential equation

$$y'' + \left(1 + \frac{1 - 4\alpha^2}{4x^2}\right) y = 0.$$

(b) When $4\alpha^2 = 1$ the differential equation in (a) becomes $y'' + y = 0$; its general solution is $y = A \cos x + B \sin x$. Use this information and the equation† $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ to show that, for $x > 0$,

$$J_{1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \sin x \quad \text{and} \quad J_{-1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \cos x.$$

(c) Deduce the formulas in part (b) directly from the series for $J_{1/2}(x)$ and $J_{-1/2}(x)$.

2. Use the series representation for Bessel functions to show that

(a) $\dfrac{d}{dx}(x^\alpha J_\alpha(x)) = x^\alpha J_{\alpha-1}(x)$,

(b) $\dfrac{d}{dx}(x^{-\alpha} J_\alpha(x)) = -x^{-\alpha} J_{\alpha+1}(x)$

3. Let $F_\alpha(x) = x^\alpha J_\alpha(x)$ and $G_\alpha(x) = x^{-\alpha} J_\alpha(x)$ for $x > 0$. Note that each positive zero of J_α is a zero of F_α and is also a zero of G_α. Use Rolle's theorem and Exercise 2 to prove that the positive zeros of J_α and $J_{\alpha+1}$ interlace. That is, there is a zero of J_α between each pair of positive zeros of $J_{\alpha+1}$, and a zero of $J_{\alpha+1}$ between each pair of positive zeros of J_α. (See Figure 6.2.)

† The change of variable $t = u^2$ gives us

$$\Gamma(\tfrac{1}{2}) = \int_{0+}^{\infty} t^{-1/2} e^{-t}\, dt = 2\int_0^{\infty} e^{-u^2}\, du = \sqrt{\pi}.$$

(See Exercise 16 of Section 11.28 for a proof that $2\int_0^\infty e^{-u^2}\, du = \sqrt{\pi}$.)

4. (a) From the relations in Exercise 2 deduce the recurrence relations

$$\frac{\alpha}{x} J_\alpha(x) + J'_\alpha(x) = J_{\alpha-1}(x) \qquad \text{and} \qquad \frac{\alpha}{x} J_\alpha(x) - J'_\alpha(x) = J_{\alpha+1}(x).$$

(b) Use the relations in part (a) to deduce the formulas

$$J_{\alpha-1}(x) + J_{\alpha+1}(x) = \frac{2\alpha}{x} J_\alpha(x) \qquad \text{and} \qquad J_{\alpha-1}(x) - J_{\alpha+1}(x) = 2J'_\alpha(x).$$

5. Use Exercise 1 (b) and a suitable recurrence formula to show that

$$J_{3/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2}\left(\frac{\sin x}{x} - \cos x\right).$$

Find a similar formula for $J_{-3/2}(x)$. Note: $J_\alpha(x)$ is an elementary function for every α which is half an odd integer.

6. Prove that

$$\frac{1}{2}\frac{d}{dx}\left(J_\alpha^2(x) + J_{\alpha+1}^2(x)\right) = \frac{\alpha}{x} J_\alpha^2(x) - \frac{\alpha+1}{x} J_{\alpha+1}^2(x)$$

and

$$\frac{d}{dx}\left(xJ_\alpha(x)J_{\alpha+1}(x)\right) = x\left(J_\alpha^2(x) - J_{\alpha+1}^2(x)\right).$$

7. (a) Use the identities in Exercise 6 to show that

$$J_0^2(x) + 2\sum_{n=1}^{\infty} J_n^2(x) = 1 \qquad \text{and} \qquad \sum_{n=0}^{\infty}(2n+1)J_n(x)J_{n+1}(x) = \tfrac{1}{2}x.$$

(b) From part (a), deduce that $|J_0(x)| \le 1$ and $|J_n(x)| \le \frac{1}{2}\sqrt{2}$ for $n = 1, 2, 3, \ldots$, and all $x \ge 0$.

8. Let $g_\alpha(x) = x^{1/2} f_\alpha(ax^b)$ for $x > 0$, where a and b are nonzero constants. Show that g_α satisfies the differential equation

$$x^2y'' + (a^2b^2x^{2b} + \tfrac{1}{4} - \alpha^2b^2)y = 0$$

if, and only if, f_α is a solution of the Bessel equation of order α.

9. Use Exercise 8 to express the general solution of each of the following differential equations in terms of Bessel functions for $x > 0$.
(a) $y'' + xy = 0$.
(b) $y'' + x^2y = 0$.
(c) $y'' + x^my = 0$.
(d) $x^2y'' + (x^4 + \frac{1}{8})y = 0$.

10. Generalize Exercise 8 when f_α and g_α are related by the equation $g_\alpha(x) = x^c f_\alpha(ax^b)$ for $x > 0$. Then find the general solution of each of the following equations in terms of Bessel functions for $x > 0$.
(a) $xy'' + 6y' + y = 0$.
(b) $xy'' + 6y' + xy = 0$.
(c) $xy'' + 6y' + x^4y = 0$.
(d) $x^2y'' - xy' + (x + 1)y = 0$.

11. A Bessel function identity exists of the form

$$J_2(x) - J_0(x) = aJ''_c(x),$$

where a and c are constants. Determine a and c.

12. Find a power series solution of the differential equation $xy'' + y' + y = 0$ convergent for $-\infty < x < +\infty$. Show that for $x > 0$ it can be expressed in terms of a Bessel function.
13. Consider a linear second-order differential equation of the form

$$x^2A(x)y'' + xP(x)y' + Q(x)y = 0,$$

where A(x), P(x), and Q(x) have power series expansions,

$$A(x) = \sum_{k=0}^{\infty} a_k x^k, \qquad P(x) = \sum_{k=0}^{\infty} p_k x^k, \qquad Q(x) = \sum_{k=0}^{\infty} q_k x^k,$$

with $a_0 \neq 0$, each convergent in an open interval $(-r, r)$. If the differential equation has a series solution of the form

$$y = x^t \sum_{n=0}^{\infty} c_n x^n,$$

valid for $0 < x < r$, show that t satisfies a quadratic equation of the form $t^2 + bt + c = 0$, and determine b and c in terms of coefficients of the series for A(x), P(x), and Q(x).
14. Consider a special case of Exercise 13 in which $A(x) = 1 - x$, $P(x) = \frac{1}{2}$, and $Q(x) = -\frac{1}{4}x$. Find a series solution with t not an integer.
15. The differential equation $2x^2y'' + (x^2 - x)y' + y = 0$ has two independent solutions of the form

$$y = x^t \sum_{n=0}^{\infty} c_n x^n,$$

valid for $x > 0$. Determine these solutions.
16. The nonlinear differential equation $y'' + y + \alpha y^2 = 0$ is only "mildly" nonlinear if α is a small nonzero constant. Assume there is a solution which can be expressed as a power series in a of the form

$$y = \sum_{n=0}^{\infty} u_n(x)\alpha^n \qquad \text{(valid in some interval } 0 < \alpha < r)$$

and that this solution satisfies the initial conditions $y = 1$ and $y' = 0$ when $x = 0$. To conform with these initial conditions, we try to choose the coefficients $u_n(x)$ so that $u_0(0) = 1$, $u_0'(0) = 0$ and $u_n(0) = u_n'(0) = 0$ for $n \geq 1$. Substitute this series in the differential equation, equate suitable powers of α and thereby determine $u_0(x)$ and $u_1(x)$.

7

SYSTEMS OF DIFFERENTIAL EQUATIONS

7.1 Introduction

Although the study of differential equations began in the 17th century, it was not until the 19th century that mathematicians realized that relatively few differential equations could be solved by elementary means. The work of Cauchy, Liouville, and others showed the importance of establishing general theorems to guarantee the existence of solutions to certain specific classes of differential equations. Chapter 6 illustrated the use of an existence-uniqueness theorem in the study of linear differential equations. This chapter is concerned with a proof of this theorem and related topics.

Existence theory for differential equations of higher order can be reduced to the first-order case by the introduction of systems of equations. For example, the second-order equation

$$y'' + 2ty' - y = e^t \tag{7.1}$$

can be transformed to a system of two first-order equations by introducing two unknown functions y_1 and y_2, where

$$y_1 = y, \qquad y_2 = y_1'.$$

Then we have $y_2' = y_1'' = y''$, so (7.1) can be written as a system of two first-order equations :

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_1 - 2ty_2 + e^t. \end{aligned} \tag{7.2}$$

We cannot solve the equations separately by the methods of Chapter 6 because each of them involves two unknown functions.

In this chapter we consider systems consisting of n linear differential equations of first order involving n unknown functions $y_1, \ldots, y_n$. These systems have the form

$$\begin{aligned} y_1' &= p_{11}(t)y_1 + p_{12}(t)y_2 + \cdots + p_{1n}(t)y_n + q_1(t) \\ &\;\;\vdots \\ y_n' &= p_{n1}(t)y_1 + p_{n2}(t)y_2 + \cdots + p_{nn}(t)y_n + q_n(t). \end{aligned} \tag{7.3}$$

The functions p_{ik} and q_i which appear in (7.3) are considered as given functions defined on a given interval J. The functions $y_1, \ldots, y_n$ are unknown functions to be determined. Systems of this type are called *first-order linear systems.* In general, each equation in the system involves more than one unknown function so the equations cannot be solved separately.

A linear differential equation of order n can always be transformed to a linear system. Suppose the given nth order equation is

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = R(t), \tag{7.4}$$

where the coefficients a_i are given functions. To transform this to a system we write $y_1 = y$ and introduce a new unknown function for each of the successive derivatives of y. That is, we put

$$y_1 = y, \qquad y_2 = y_1', \qquad y_3 = y_2', \ldots, y_n = y_{n-1}',$$

and rewrite (7.4) as the system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ &\vdots \\ y_{n-1}' &= y_n \\ y_n' &= -a_n y_1 - a_{n-1} y_2 - \cdots - a_1 y_n + R(t). \end{aligned} \tag{7.5}$$

The discussion of systems may be simplified considerably by the use of vector and matrix notation. Consider the general system (7.3) and introduce vector-valued functions $Y = (y_1, \ldots, y_n)$, $Q = (q_1, \ldots, q_n)$, and a matrix-valued function $P = [p_{ij}]$, defined by the equations

$$Y(t) = (y_1(t), \ldots, y_n(t)), \qquad Q(t) = (q_1(t), \ldots, q_n(t)), \qquad P(t) = [p_{ji}(t)]$$

for each t in J. We regard the vectors as $n \times 1$ column matrices and write the system (7.3) in the simpler form

$$Y' = P(t)\, Y + Q(t). \tag{7.6}$$

For example, in system (7.2) we have

$$Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad P(t) = \begin{bmatrix} 0 & 1 \\ 1 & -2t \end{bmatrix}, \quad Q(t) = \begin{bmatrix} 0 \\ e^t \end{bmatrix}.$$

In system (7.5) we have

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \qquad P(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}, \qquad Q(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ R(t) \end{bmatrix}$$

An initial-value problem for system (7.6) is to find a vector-valued function Y which satisfies (7.6) and which also satisfies an initial condition of the form $Y(a) = B$, where $a \in J$ and $B = (b_1, \ldots, b_n)$ is a given n-dimensional vector.

In the case $n = 1$ (the scalar case) we know from Theorem 6.1 that, if P and Q are continuous on J, all solutions of (7.6) are given by the explicit formula

$$Y(x) = e^{A(x)}Y(a) + e^{A(x)} \int_a^x e^{-A(t)}Q(t)\, dt, \tag{7.7}$$

where $A(x) = \int_a^x P(t)\, dt$, and a is any point in J. We will show that this formula can be suitably generalized for systems, that is, when $P(t)$ is an $n \times n$ matrix function and $Q(t)$ is an n-dimensional vector function. To do this we must assign a meaning to integrals of matrices and to exponentials of matrices. Therefore, we digress briefly to discuss the calculus of matrix functions.

7.2 Calculus of matrix functions

The generalization of the concepts of integral and derivative for matrix functions is straightforward. If $P(t) = [p_{ij}(t)]$, we define the integral $\int_a^b P(t)\, dt$ by the equation

$$\int_a^b P(t)\, dt = \left[\int_a^b p_{ij}(t)\, dt\right].$$

That is, the integral of matrix $P(t)$ is the matrix obtained by integrating each entry of $P(t)$, assuming of course, that each entry is integrable on $[a, b]$. The reader can verify that the linearity property for integrals generalizes to matrix functions.

Continuity and differentiability of matrix functions are also defined in terms of the entries. We say that a matrix function $P = [p_{ij}]$ is continuous at t if each entry p_{ij} is continuous at t. The derivative P' is defined by differentiating each entry,

$$P'(t) = [p'_{ij}(t)],$$

whenever all derivatives $p'_{ij}(t)$ exist. It is easy to verify the basic differentiation rules for sums and products. For example, if P and Q are differentiable matrix functions, we have

$$(P + Q)' = P' + Q'$$

if P and Q are of the same size, and we also have

$$(PQ)' = PQ' + P'Q$$

if the product PQ is defined. The chain rule also holds. That is, if $F(t) = P[g(t)]$, where P is a differentiable matrix function and g is a differentiable scalar function, then $F'(t) = g'(t)P'[g(t)]$. The zero-derivative theorem, and the first and second fundamental theorems of calculus are also valid for matrix functions. Proofs of these properties are requested in the next set of exercises.

The definition of the exponential of a matrix is not so simple and requires further preparation. This is discussed in the next section.

7.3 Infinite series of matrices. Norms of matrices

Let $A = [a_{ij}]$ be an $n \times n$ matrix of real or complex entries. We wish to define the exponential e^A in such a way that it possesses some of the fundamental properties of the ordinary real or complex-valued exponential. In particular, we shall require the law of exponents in the form

(7.8) $$e^{tA}e^{sA} = e^{(t+s)A} \qquad \text{for all real } s \text{ and } t\,,$$

and the relation

(7.9) $$e^{O} = I\,,$$

where O and I are the $n \times n$ zero and identity matrices, respectively. It might seem natural to define e^A to be the matrix $[e^{a_{ij}}]$. However, this is unacceptable since it satisfies neither of properties (7.8) or (7.9). Instead, we shall define e^A by means of a power series expansion,

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}\,.$$

We know that this formula holds if A is a real or complex number, and we will prove that it implies properties (7.8) and (7.9) if A is a matrix. Before we can do this we need to explain what is meant by a convergent series of matrices.

DEFINITION OF CONVERGENT SERIES OF MATRICES. *Given an infinite sequence of $m \times n$ matrices $\{C_k\}$ whose entries are real or complex numbers, denote the ij-entry of C_k by $c_{ij}^{(k)}$. If all mn series*

(7.10) $$\sum_{k=1}^{\infty} c_{ij}^{(k)} \qquad (i = 1, \ldots, m;\ j = 1, \ldots, n)$$

are convergent, then we say the series of matrices $\sum_{k=1}^{\infty} C_k$ is convergent, and its sum is defined to be the $m \times n$ matrix whose ij-entry is the series in (7.10).