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John Stillwell

The Four Pillars of Geometry

With 138 Illustrations

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To Elaine

Preface

Many people think there is only one "right" way to teach geometry. For two millennia, the "right" way was Euclid's way, and it is still good in many respects. But in the 1950s the cry "Down with triangles!" was heard in France and new geometry books appeared, packed with linear algebra but with no diagrams. Was this the new "right" way, or was the "right" way something else again, perhaps transformation groups?

In this book, I wish to show that geometry can be developed in four fundamentally different ways, and that *all* should be used if the subject is to be shown in all its splendor. Euclid-style construction and axiomatics seem the best way to start, but linear algebra smooths the later stages by replacing some tortuous arguments by simple calculations. And how can one avoid projective geometry? It not only explains why objects look the way they do; it also explains why geometry is entangled with algebra. Finally, one needs to know that there is not one geometry, but many, and transformation groups are the best way to distinguish between them.

Two chapters are devoted to each approach: The first is concrete and introductory, whereas the second is more abstract. Thus, the first chapter on Euclid is about straightedge and compass constructions; the second is about axioms and theorems. The first chapter on linear algebra is about coordinates; the second is about vector spaces and the inner product. The first chapter on projective geometry is about perspective drawing; the second is about axioms for projective planes. The first chapter on transformation groups gives examples of transformations; the second constructs the hyperbolic plane from the transformations of the real projective line.

I believe that students are shortchanged if they miss any of these four approaches to the subject. Geometry, of all subjects, should be about *taking different viewpoints*, and geometry is unique among the mathematical disciplines in its ability to look different from different angles. Some prefer to approach it visually, others algebraically, but the miracle is that they are all looking at the same thing. (It is as if one discovered that number theory need not use addition and multiplication, but could be based on, say, the exponential function.)

The many faces of geometry are not only a source of amazement and delight. They are also a great help to the learner and teacher. We all know that some students prefer to visualize, whereas others prefer to reason or to calculate. Geometry has something for everybody, and all students will find themselves building on their strengths at some times, and working to overcome weaknesses at other times. We also know that Euclid has some beautiful proofs, whereas other theorems are more beautifully proved by algebra. In the multifaceted approach, every theorem can be given an elegant proof, and theorems with radically different proofs can be viewed from different sides.

This book is based on the course Foundations of Geometry that I taught at the University of San Francisco in the spring of 2004. It should be possible to cover it all in a one-semester course, but if time is short, some sections or chapters can be omitted according to the taste of the instructor. For example, one could omit Chapter 6 or Chapter 8. (But with regret, I am sure!)

Acknowledgements

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Thanks also go to my wife Elaine, who proofread the first version of the book, and to Robin Hartshorne, John Howe, Marc Ryser, Abe Shenitzer, and Michael Stillwell, who carefully read the revised version and saved me from many mathematical and stylistic errors.

Finally, I am grateful to the M. C. Escher Company – Baarn – Holland for permission to reproduce the Escher work *Circle Limit I* shown in Figure 8.19, and the explicit mathematical transformation of it shown in Figure 8.10. This work is copyright (2005) The M. C. Escher Company.

JOHN STILLWELL
San Francisco, November 2004
South Melbourne, April 2005

Contents

1

Straightedge and compass

PREVIEW

For over 2000 years, mathematics was almost synonymous with the geometry of Euclid's *Elements*, a book written around 300 BCE and used in school mathematics instruction until the 20th century. *Euclidean geometry*, as it is now called, was thought to be the foundation of all exact science.

Euclidean geometry plays a different role today, because it is no longer expected to support everything else. "Non-Euclidean geometries" were discovered in the early 19th century, and they were found to be more useful than Euclid's in certain situations. Nevertheless, non-Euclidean geometries arose as deviations from the Euclidean, so one first needs to know *what* they deviate from.

A naive way to describe Euclidean geometry is to say it concerns the geometric figures that can be drawn (or *constructed* as we say) by straightedge and compass. Euclid assumes that it is possible to draw a straight line between any two given points, and to draw a circle with given center and radius. All of the propositions he proves are about figures built from straight lines and circles.

Thus, to understand Euclidean geometry, one needs some idea of the scope of straightedge and compass constructions. This chapter reviews some basic constructions, to give a quick impression of the extent of Euclidean geometry, and to suggest why *right angles* and *parallel lines* play a special role in it.

Constructions also help to expose the role of length, area, and angle in geometry. The deeper meaning of these concepts, and the related role of *numbers* in geometry, is a thread we will pursue throughout the book.

1.1 Euclid's construction axioms

Euclid assumes that certain constructions can be done and he states these assumptions in a list called his *axioms* (traditionally called *postulates*). He assumes that it is possible to:

1. Draw a straight line segment between any two points.
2. Extend a straight line segment indefinitely.
3. Draw a circle with given center and radius.

Axioms 1 and 2 say we have a *straightedge*, an instrument for drawing arbitrarily long line segments. Euclid and his contemporaries tried to avoid infinity, so they worked with line segments rather than with whole lines. This is no real restriction, but it involves the annoyance of having to extend line segments (or "produce" them, as they say in old geometry books). Today we replace Axioms 1 and 2 by the single axiom that a *line* can be drawn through any two points.

The straightedge (unlike a ruler) has no scale marked on it and hence can be used *only* for drawing lines—not for measurement. Euclid separates the function of measurement from the function of drawing straight lines by giving measurement functionality only to the *compass*—the instrument assumed in Axiom 3. The compass is used to draw the circle through a given point B, with a given point A as center (Figure 1.1).

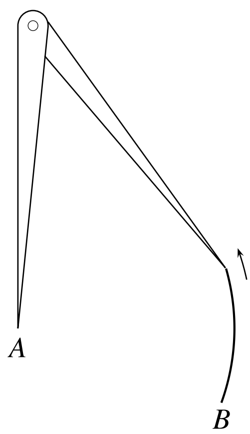

Figure 1.1: Drawing a circle

To do this job, the compass must rotate rigidly about A after being initially set on the two points A and B. Thus, it "stores" the length of the radius AB and allows this length to be transferred elsewhere. Figure 1.2 is a classic view of the compass as an instrument of measurement. It is William Blake's painting of Isaac Newton as the measurer of the universe.

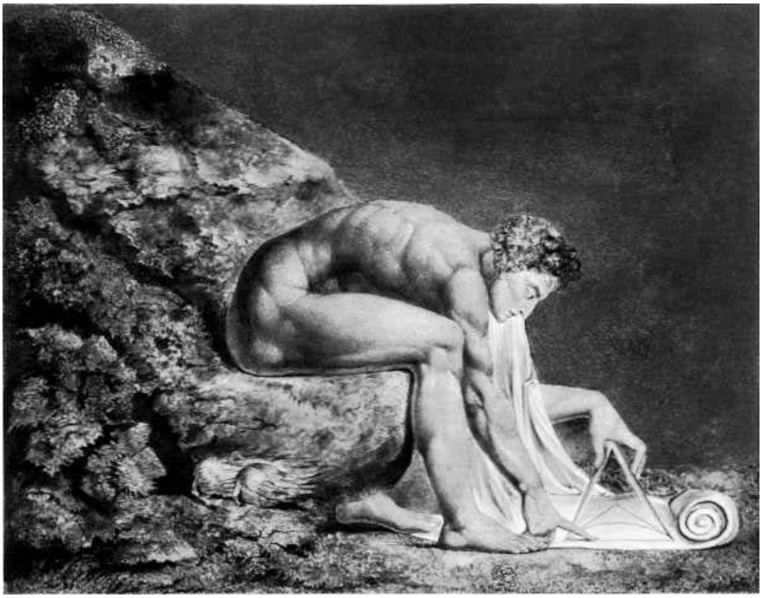

Figure 1.2: Blake's painting of Newton the measurer

The compass also enables us to *add* and *subtract* the length $|AB|$ of AB from the length $|CD|$ of another line segment CD by picking up the compass with radius set to $|AB|$ and describing a circle with center D (Figure 1.3, also *Elements*, Propositions 2 and 3 of Book I). By adding a fixed length repeatedly, one can construct a "scale" on a given line, effectively creating a ruler. This process illustrates how the power of measuring lengths resides in the compass. Exactly which lengths can be measured in this way is a deep question, which belongs to algebra and analysis. The full story is beyond the scope of this book, but we say more about it below.

Separating the concepts of "straightness" and "length," as the straightedge and the compass do, turns out to be important for understanding the foundations of geometry. The same separation of concepts reappears in different approaches to geometry developed in Chapters 3 and 5.

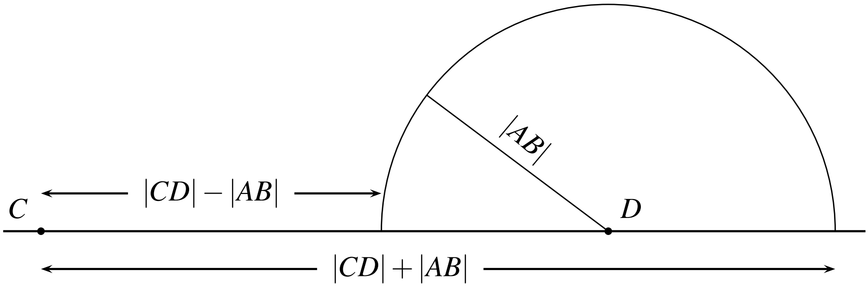

Figure 1.3: Adding and subtracting lengths

1.2 Euclid's construction of the equilateral triangle

Constructing an equilateral triangle on a given side AB is the first proposition of the *Elements*, and it takes three steps:

1. Draw the circle with center A and radius AB.
2. Draw the circle with center B and radius AB.
3. Draw the line segments from A and B to the intersection C of the two circles just constructed.

The result is the triangle ABC with sides AB, BC, and CA in Figure 1.4.

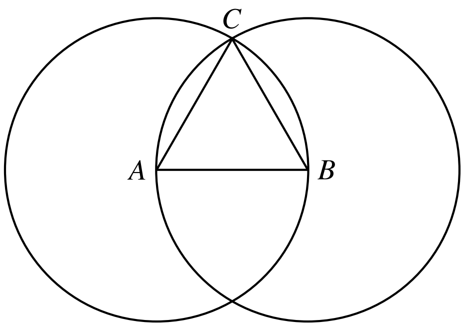

Figure 1.4: Constructing an equilateral triangle

Sides AB and CA have equal length because they are both radii of the first circle. Sides AB and BC have equal length because they are both radii of the second circle. Hence, all three sides of triangle ABC are equal. $\square$

This example nicely shows the interplay among

- *construction axioms*, which guarantee the existence of the construction lines and circles (initially the two circles on radius AB and later the line segments BC and CA),
- *geometric axioms*, which guarantee the existence of points required for later steps in the construction (the intersection C of the two circles),
- and *logic*, which guarantees that certain conclusions follow. In this case, we are using a principle of logic that says that things equal to the same thing (both $|BC|$ and $|CA|$ equal $|AB|$) are equal to each other (so $|BC| = |CA|$).

We have not yet discussed Euclid's geometric axioms or logic. We use the same logic for all branches of mathematics, so it can be assumed "known," but geometric axioms are less clear. Euclid drew attention to one and used others unconsciously (or, at any rate, without stating them). History has shown that Euclid correctly identified the most significant geometric axiom, namely the *parallel axiom*. We will see some reasons for its significance in the next section. The ultimate reason is that *there are important geometries in which the parallel axiom is false*.

The other axioms are not significant in this sense, but they should also be identified for completeness, and we will do so in Chapter 2. In particular, it should be mentioned that Euclid states no axiom about the intersection of circles, so he has not justified the existence of the point C used in his very first proposition!

A question arising from Euclid's construction

The equilateral triangle is an example of a *regular polygon*: a geometric figure bounded by equal line segments that meet at equal angles. Another example is the regular hexagon in Exercise 1.2.1. If the polygon has n sides, we call it an *n-gon*, so the regular 3-gon and the regular 6-gon are constructible. *For which n is the regular n-gon constructible?*

We will not completely answer this question, although we will show that the regular 4-gon and 5-gon are constructible. The question for general n turns out to belong to algebra and number theory, and a complete answer depends on a problem about prime numbers that has not yet been solved: For which m is $2^{2^m} + 1$ a prime number?