Examples

(1) A group G is abelian if and only if $G' = 1$.

(2) Sometimes it is possible to compute the commutator subgroup of a group without actually calculating commutators explicitly. For instance, if $G = D_8$, then since $Z(D_8) = \langle r^2 \rangle \trianglelefteq D_8$ and $D_8/Z(D_8)$ is abelian (the Klein 4-group), the commutator subgroup D_8' is a subgroup of $Z(D_8)$. Since D_8 is not itself abelian its commutator subgroup is nontrivial. The only possibility is that $D_8' = Z(D_8)$. By a similar argument, $Q_8' = Z(Q_8) = \langle -1 \rangle$. More generally, if G is any non-abelian group of order p^3, where p is a prime, $G' = Z(G)$ and $|G'| = p$ (Exercise 7).

(3) Let $D_{2n} = \langle r, s \mid r^n = s^2 = 1, s^{-1}rs = r^{-1} \rangle$. Since $[r, s] = r^{-2}$, we have $\langle r^{-2} \rangle = \langle r^2 \rangle \leq D_{2n}'$. Furthermore, $\langle r^2 \rangle \trianglelefteq D_{2n}$ and the images of r and s in $D_{2n}/\langle r^2 \rangle$ generate this quotient. They are commuting elements of order ≤ 2, so the quotient is abelian and $D_{2n}' \leq \langle r^2 \rangle$. Thus $D_{2n}' = \langle r^2 \rangle$. Finally, note that if n ($= |r|$) is odd, $\langle r^2 \rangle = \langle r \rangle$ whereas if n is even, $\langle r^2 \rangle$ is of index 2 in $\langle r \rangle$. Hence D_{2n}' is of index 2 or 4 in D_{2n} according to whether n is odd or even, respectively.

(4) Since conjugation by $g \in G$ is an automorphism of G, $[a^g, b^g] = [a, b]^g$ for all $a, b \in G$ by (3) of the proposition, i.e., conjugates of commutators are also commutators. For example, once we exhibit an element of one cycle type in S_n as a commutator, every element of the same cycle type is also a commutator (cf. Section 4.3). For example, every 5-cycle is a commutator in S_5 as follows: labelling the vertices of a pentagon as $1, \ldots, 5$ we see that $D_{10} \leq S_5$ (a subgroup of A_5 in fact). By the preceding example an element of order 5 is a commutator in D_{10}, hence also in S_5. Explicitly, $(1\,4\,2\,5\,3) = [(1\,2\,3\,4\,5), (2\,5)(4\,3)]$.

The next result actually follows from the proof of Proposition 3.13 but we isolate it explicitly for reference:

Proposition 8. Let H and K be subgroups of the group G. The number of distinct ways of writing each element of the set HK in the form hk, for some $h \in H$ and $k \in K$ is $|H \cap K|$. In particular, if $H \cap K = 1$, then each element of HK can be written uniquely as a product hk, for some $h \in H$ and $k \in K$.

Proof: Exercise.

The main result of this section is the following *recognition theorem*.

Theorem 9. Suppose G is a group with subgroups H and K such that

(1) H and K are normal in G, and

(2) $H \cap K = 1$.

Then $HK \cong H \times K$.

Proof: Observe that by hypothesis (1), HK is a subgroup of G (see Corollary 3.15). Let $h \in H$ and let $k \in K$. Since $H \trianglelefteq G$, $k^{-1}hk \in H$, so that $h^{-1}(k^{-1}hk) \in H$. Similarly, $(h^{-1}k^{-1}h)k \in K$. Since $H \cap K = 1$ it follows that $h^{-1}k^{-1}hk = 1$, i.e., $hk = kh$ so that every element of H commutes with every element of K.

By the preceding proposition each element of HK can be written uniquely as a product hk, with $h \in H$ and $k \in K$. Thus the map

$$\begin{aligned} \varphi : HK &\to H \times K \\ hk &\mapsto (h, k) \end{aligned}$$

is well defined. To see that φ is a homomorphism note that if $h_1, h_2 \in H$ and $k_1, k_2 \in K$, then we have seen that h_2 and k_1 commute. Thus

$$(h_1k_1)(h_2k_2) = (h_1h_2)(k_1k_2)$$

and the latter product is the unique way of writing $(h_1k_1)(h_2k_2)$ in the form hk with $h \in H$ and $k \in K$. This shows that

$$\begin{aligned} \varphi(h_1k_1h_2k_2) &= \varphi(h_1h_2k_1k_2) \\ &= (h_1h_2, k_1k_2) \\ &= (h_1, k_1)(h_2, k_2) = \varphi(h_1k_1)\varphi(h_2k_2) \end{aligned}$$

so that φ is a homomorphism. The homomorphism φ is a bijection since the representation of each element of HK as a product of the form hk is unique, which proves that φ is an isomorphism.

Definition. If G is a group and H and K are normal subgroups of G with $H \cap K = 1$, we call HK the *internal direct product* of H and K. We shall (when emphasis is called for) call $H \times K$ the *external direct product* of H and K.

The distinction between internal and external direct product is (by Theorem 9) purely notational: the elements of the internal direct product are written in the form hk, whereas those of the external direct product are written as ordered pairs (h, k). We have in previous instances passed between these. For example, when $Z_n = \langle a \rangle$ and $Z_m = \langle b \rangle$ we wrote $x = (a, 1)$ and $y = (1, b)$ so that every element of $Z_n \times Z_m$ was written in the form $x^r y^s$.

Examples

(1) If n is a positive odd integer, we show $D_{4n} \cong D_{2n} \times Z_2$. To see this let

$$D_{4n} = \langle r, s \mid r^{2n} = s^2 = 1,\ srs = r^{-1} \rangle$$

be the usual presentation of D_{4n}. Let $H = \langle s, r^2 \rangle$ and let $K = \langle r^n \rangle$. Geometrically, if D_{4n} is the group of symmetries of a regular $2n$-gon, H is the group of symmetries of the regular n-gon inscribed in the $2n$-gon by joining vertex $2i$ to vertex $2i + 2$, for all $i \bmod 2n$ (and if one lets $r_1 = r^2$, H has the usual presentation of the dihedral group of order $2n$ with generators r_1 and s). Note that $H \trianglelefteq D_{4n}$ (it has index 2). Since $|r| = 2n$, $|r^n| = 2$. Since $srs = r^{-1}$, we have $sr^ns = r^{-n} = r^n$, that is, s centralizes r^n. Since clearly r centralizes r^n, $K \leq Z(D_{4n})$. Thus $K \trianglelefteq D_{4n}$. Finally, $K \not\leq H$ since r^2 has odd order (or because r^n sends vertex i into vertex $i + n$, hence does not preserve the set of even vertices of the $2n$-gon). Thus $H \cap K = 1$ by Lagrange. Theorem 9 now completes the proof.

(2) Let I be a subset of $\{1, 2, \ldots, n\}$ and let G be the setwise stabilizer of I in S_n, i.e.,

$$G = \{\sigma \in S_n \mid \sigma(i) \in I \text{ for all } i \in I\}.$$

Let $J = \{1, 2, \dots, n\} - I$ be the complement of I and note that G is also the setwise stabilizer of J. Let H be the *pointwise* stabilizer of I and let K be the *pointwise* stabilizer of $\{1, 2, \dots, n\} - I$, i.e.,

$$H = \{\sigma \in G \mid \sigma(i) = i \text{ for all } i \in I\}$$
$$K = \{\tau \in G \mid \tau(j) = j \text{ for all } j \in J\}.$$

It is easy to see that H and K are normal subgroups of G (in fact they are kernels of the actions of G on I and J, respectively). Since any element of $H \cap K$ fixes all of $\{1, 2, \dots, n\}$, we have $H \cap K = 1$. Finally, since every element σ of G stabilizes the sets I and J, each cycle in the cycle decomposition of σ involves only elements of I or only elements of J. Thus σ may be written as a product $\sigma_I \sigma_J$, where $\sigma_I \in H$ and $\sigma_J \in K$. This proves $G = HK$. By Theorem 9, $G \cong H \times K$. Now any permutation of J can be extended to a permutation in S_n by letting it act as the identity on I. These are precisely the permutations in H (and similarly the permutations in K are the permutations of I which are the identity on J), so

$$H \cong S_J \quad K \cong S_I \quad \text{and} \quad G \cong S_m \times S_{n-m},$$

where $m = |J|$ (and, by convention, $S_\emptyset = 1$).

(3) Let $\sigma \in S_n$ and let I be the subset of $\{1, 2, \dots, n\}$ fixed pointwise by σ:

$$I = \{i \in \{1, 2, \dots, n\} \mid \sigma(i) = i\}.$$

If $C = C_{S_n}(\sigma)$, then by Exercise 18 of Section 4.3, C stabilizes the set I and its complement J. By the preceding example, C is isomorphic to a subgroup of $H \times K$, where H is the subgroup of all permutations in S_n fixing I pointwise and K is the set of all permutations fixing J pointwise. Note that $\sigma \in H$. Thus each element, α, of C can be written (uniquely) as $\alpha = \alpha_I \alpha_J$, for some $\alpha_I \in H$ and $\alpha_J \in K$. Note further that if τ is any permutation of $\{1, 2, \dots, n\}$ which fixes each $j \in J$ (i.e., any element of K), then σ and τ commute (since they move no common integers). Thus C contains all such τ, i.e., C contains the subgroup K. This proves that the group C consists of all elements $\alpha_I \alpha_J \in H \times K$ such that α_J is arbitrary in K and α_I commutes with σ in H:

$$\begin{aligned} C_{S_n}(\sigma) &= C_H(\sigma) \times K \\ &\cong C_{S_J}(\sigma) \times S_I. \end{aligned}$$

In particular, if σ is an m-cycle in S_n,

$$C_{S_n}(\sigma) = \langle \sigma \rangle \times S_{n-m}.$$

The latter group has order $m(n-m)!$, as computed in Section 4.3.

EXERCISES

Let G be a group.

1. Prove that if $x, y \in G$ then $[y, x] = [x, y]^{-1}$. Deduce that for any subsets A and B of G, $[A, B] = [B, A]$ (recall that $[A, B]$ is the *subgroup* of G generated by the commutators $[a, b]$).

2. Prove that a subgroup H of G is normal if and only if $[G, H] \le H$.

3. Let $a, b, c \in G$. Prove that
(a) $[a, bc] = [a, c](c^{-1}[a, b]c)$

(b) $[ab, c] = (b^{-1}[a, c]b)[b, c]$.

4. Find the commutator subgroups of S_4 and A_4.
5. Prove that A_n is the commutator subgroup of S_n for all $n \geq 5$.
6. Exhibit a representative of each cycle type of A_5 as a commutator in S_5.
7. Prove that if p is a prime and P is a non-abelian group of order p^3 then $P' = Z(P)$.
8. Assume $x, y \in G$ and both x and y commute with $[x, y]$. Prove that for all $n \in \mathbb{Z}^+$, $(xy)^n = x^n y^n [y, x]^{\frac{n(n-1)}{2}}$.
9. Prove that if p is an odd prime and P is a group of order p^3 then the p^{th} power map $x \mapsto x^p$ is a homomorphism of P into $Z(P)$. If P is not cyclic, show that the kernel of the p^{th} power map has order p^2 or p^3. Is the squaring map a homomorphism in non-abelian groups of order 8? Where is the oddness of p needed in the above proof? [Use Exercise 8.]
10. Prove that a finite abelian group is the direct product of its Sylow subgroups.
11. Prove that if $G = HK$ where H and K are characteristic subgroups of G with $H \cap K = 1$ then $\text{Aut}(G) \cong \text{Aut}(H) \times \text{Aut}(K)$. Deduce that if G is an abelian group of finite order then $\text{Aut}(G)$ is isomorphic to the direct product of the automorphism groups of its Sylow subgroups.
12. Use Theorem 4.17 to describe the automorphism group of a finite cyclic group.
13. Prove that D_{8n} is not isomorphic to $D_{4n} \times Z_2$.
14. Let $G = \{(a_{ij}) \in GL_n(F) \mid a_{ij} = 0 \text{ if } i > j, \text{ and } a_{11} = a_{22} = \cdots = a_{nn}\}$, where F is a field, be the group of upper triangular matrices all of whose diagonal entries are equal. Prove that $G \cong D \times U$, where D is the group of all nonzero multiples of the identity matrix and U is the group of upper triangular matrices with 1's down the diagonal.
15. If A and B are normal subgroups of G such that G/A and G/B are both abelian, prove that $G/(A \cap B)$ is abelian.
16. Prove that if K is a normal subgroup of G then $K' \trianglelefteq G$.
17. If K is a normal subgroup of G and K is cyclic, prove that $G' \leq C_G(K)$. [Recall that the automorphism group of a cyclic group is abelian.]
18. Let $K_1, K_2, \ldots, K_n$ be non-abelian simple groups and let $G = K_1 \times K_2 \times \cdots \times K_n$. Prove that every normal subgroup of G is of the form G_I for some subset I of $\{1, 2, \ldots, n\}$ (where G_I is defined in Exercise 2 of Section 1). [If $N \trianglelefteq G$ and $x = (a_1, \ldots, a_n) \in N$ with some $a_i \neq 1$, then show that there is some $g_i \in G_i$ not commuting with a_i. Show $[(1, \ldots, g_i, \ldots, 1), x] \in K_i \cap N$ and deduce $K_i \leq N$.]
19. A group H is called *perfect* if $H' = H$ (i.e., H equals its own commutator subgroup).
    (a) Prove that every non-abelian simple group is perfect.
    (b) Prove that if H and K are perfect subgroups of a group G then $\langle H, K \rangle$ is also perfect. Extend this to show that the subgroup of G generated by any collection of perfect subgroups is perfect.
    (c) Prove that any conjugate of a perfect subgroup is perfect.
    (d) Prove that any group G has a unique maximal perfect subgroup and that this subgroup is normal.
20. Let $H(F)$ be the Heisenberg group over the field F, cf. Exercise 11 of Section 1.4. Find an explicit formula for the commutator $[X, Y]$, where $X, Y \in H(F)$, and show that the commutator subgroup of $H(F)$ equals the center of $H(F)$ (cf. Section 2.2, Exercise 14).