Corollary *The set of all rational numbers is countable.*

Proof We apply Theorem 2.13, with $n = 2$, noting that every rational r is of the form b/a, where a and b are integers. The set of pairs (a, b), and therefore the set of fractions b/a, is countable.

In fact, even the set of all algebraic numbers is countable (see Exercise 2).

That not all infinite sets are, however, countable, is shown by the next theorem.

2.14 Theorem *Let A be the set of all sequences whose elements are the digits* 0 *and* 1. *This set A is uncountable.*

The elements of A are sequences like $1, 0, 0, 1, 0, 1, 1, 1, \ldots$.

Proof Let E be a countable subset of A, and let E consist of the sequences $s_1, s_2, s_3, \ldots$. We construct a sequence s as follows. If the nth digit in s_n is 1, we let the nth digit of s be 0, and vice versa. Then the sequence s differs from every member of E in at least one place; hence $s \notin E$. But clearly $s \in A$, so that E is a proper subset of A.

We have shown that every countable subset of A is a proper subset of A. It follows that A is uncountable (for otherwise A would be a proper subset of A, which is absurd).

The idea of the above proof was first used by Cantor, and is called Cantor's diagonal process; for, if the sequences $s_1, s_2, s_3, \ldots$ are placed in an array like (16), it is the elements on the diagonal which are involved in the construction of the new sequence.

Readers who are familiar with the binary representation of the real numbers (base 2 instead of 10) will notice that Theorem 2.14 implies that the set of all real numbers is uncountable. We shall give a second proof of this fact in Theorem 2.43.

METRIC SPACES

2.15 Definition A set X, whose elements we shall call *points*, is said to be a *metric space* if with any two points p and q of X there is associated a real number $d(p, q)$, called the *distance* from p to q, such that

(*a*) $d(p, q) > 0$ if $p \neq q$; $d(p, p) = 0$;
(*b*) $d(p, q) = d(q, p)$;
(*c*) $d(p, q) \leq d(p, r) + d(r, q)$, for any $r \in X$.

Any function with these three properties is called a *distance function*, or a *metric*.

2.16 Examples The most important examples of metric spaces, from our standpoint, are the euclidean spaces R^k, especially R^1 (the real line) and R^2 (the complex plane); the distance in R^k is defined by

$$d(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}| \qquad (\mathbf{x}, \mathbf{y} \in R^k). \tag{19}$$

By Theorem 1.37, the conditions of Definition 2.15 are satisfied by (19).

It is important to observe that every subset Y of a metric space X is a metric space in its own right, with the same distance function. For it is clear that if conditions (*a*) to (*c*) of Definition 2.15 hold for $p, q, r \in X$, they also hold if we restrict p, q, r to lie in Y.

Thus every subset of a euclidean space is a metric space. Other examples are the spaces $\mathscr{C}(K)$ and $\mathscr{L}^2(\mu)$, which are discussed in Chaps. 7 and 11, respectively.

2.17 Definition By the *segment* (a, b) we mean the set of all real numbers x such that $a < x < b$.

By the *interval* $[a, b]$ we mean the set of all real numbers x such that $a \leq x \leq b$.

Occasionally we shall also encounter "half-open intervals" $[a, b)$ and $(a, b]$; the first consists of all x such that $a \leq x < b$, the second of all x such that $a < x \leq b$.

If $a_i < b_i$ for $i = 1, \ldots, k$, the set of all points $\mathbf{x} = (x_1, \ldots, x_k)$ in R^k whose coordinates satisfy the inequalities $a_i \leq x_i \leq b_i$ $(1 \leq i \leq k)$ is called a *k-cell*. Thus a 1-cell is an interval, a 2-cell is a rectangle, etc.

If $\mathbf{x} \in R^k$ and $r > 0$, the *open* (or *closed*) *ball* B with center at $\mathbf{x}$ and radius r is defined to be the set of all $\mathbf{y} \in R^k$ such that $|\mathbf{y} - \mathbf{x}| < r$ (or $|\mathbf{y} - \mathbf{x}| \leq r$).

We call a set $E \subset R^k$ *convex* if

$$\lambda\mathbf{x} + (1 - \lambda)\mathbf{y} \in E$$

whenever $\mathbf{x} \in E$, $\mathbf{y} \in E$, and $0 < \lambda < 1$.

For example, *balls are convex*. For if $|\mathbf{y} - \mathbf{x}| < r$, $|\mathbf{z} - \mathbf{x}| < r$, and $0 < \lambda < 1$, we have

$$\begin{aligned} |\lambda\mathbf{y} + (1 - \lambda)\mathbf{z} - \mathbf{x}| &= |\lambda(\mathbf{y} - \mathbf{x}) + (1 - \lambda)(\mathbf{z} - \mathbf{x})| \\ &\leq \lambda|\mathbf{y} - \mathbf{x}| + (1 - \lambda)|\mathbf{z} - \mathbf{x}| < \lambda r + (1 - \lambda)r \\ &= r. \end{aligned}$$

The same proof applies to closed balls. It is also easy to see that k-cells are convex.

2.18 Definition Let X be a metric space. All points and sets mentioned below are understood to be elements and subsets of X.

(*a*) *A neighborhood* of p is a set $N_r(p)$ consisting of all q such that $d(p, q) < r$, for some $r > 0$. The number r is called the *radius* of $N_r(p)$.
(*b*) A point p is a *limit point* of the set E if *every* neighborhood of p contains a point $q \neq p$ such that $q \in E$.
(*c*) If $p \in E$ and p is not a limit point of E, then p is called an *isolated point* of E.
(*d*) E is *closed* if every limit point of E is a point of E.
(*e*) A point p is an *interior* point of E if there is a neighborhood N of p such that $N \subset E$.
(*f*) E is *open* if every point of E is an interior point of E.
(*g*) The *complement* of E (denoted by E^c) is the set of all points $p \in X$ such that $p \notin E$.
(*h*) E is *perfect* if E is closed and if every point of E is a limit point of E.
(*i*) E is *bounded* if there is a real number M and a point $q \in X$ such that $d(p, q) < M$ for all $p \in E$.
(*j*) E is *dense in* X if every point of X is a limit point of E, or a point of E (or both).

Let us note that in R^1 neighborhoods are segments, whereas in R^2 neighborhoods are interiors of circles.

2.19 Theorem *Every neighborhood is an open set.*

Proof Consider a neighborhood $E = N_r(p)$, and let q be any point of E. Then there is a positive real number h such that

$$d(p, q) = r - h.$$

For all points s such that $d(q, s) < h$, we have then

$$d(p, s) \leq d(p, q) + d(q, s) < r - h + h = r,$$

so that $s \in E$. Thus q is an interior point of E.

2.20 Theorem *If p is a limit point of a set E, then every neighborhood of p contains infinitely many points of E.*

Proof Suppose there is a neighborhood N of p which contains only a finite number of points of E. Let $q_1, \ldots, q_n$ be those points of $N \cap E$, which are distinct from p, and put

$$r = \min_{1 \leq m \leq n} d(p, q_m)$$

[we use this notation to denote the smallest of the numbers $d(p, q_1), \ldots, d(p, q_n)$]. The minimum of a finite set of positive numbers is clearly positive, so that $r > 0$.

The neighborhood $N_r(p)$ contains no point q of E such that $q \neq p$, so that p is not a limit point of E. This contradiction establishes the theorem.

Corollary *A finite point set has no limit points.*

2.21 Examples Let us consider the following subsets of R^2:

(*a*) The set of all complex z such that $|z| < 1$.
(*b*) The set of all complex z such that $|z| \leq 1$.
(*c*) A nonempty finite set.
(*d*) The set of all integers.
(*e*) The set consisting of the numbers $1/n$ ($n = 1, 2, 3, \ldots$). Let us note that this set E has a limit point (namely, $z = 0$) but that no point of E is a limit point of E; we wish to stress the difference between having a limit point and containing one.
(*f*) The set of all complex numbers (that is, R^2).
(*g*) The segment (a, b).

Let us note that (*d*), (*e*), (*g*) can be regarded also as subsets of R^1.
Some properties of these sets are tabulated below:

	Closed	*Open*	*Perfect*	*Bounded*
(*a*)	No	Yes	No	Yes
(*b*)	Yes	No	Yes	Yes
(*c*)	Yes	No	No	Yes
(*d*)	Yes	No	No	No
(*e*)	No	No	No	Yes
(*f*)	Yes	Yes	Yes	No
(*g*)	No		No	Yes

In (*g*), we left the second entry blank. The reason is that the segment (a, b) is not open if we regard it as a subset of R^2, but it is an open subset of R^1.

2.22 Theorem *Let $\{E_\alpha\}$ be a (finite or infinite) collection of sets E_α. Then*

$$\left(\bigcup_\alpha E_\alpha\right)^c = \bigcap_\alpha (E_\alpha^c). \tag{20}$$

Proof Let A and B be the left and right members of (20). If $x \in A$, then $x \notin \bigcup_\alpha E_\alpha$, hence $x \notin E_\alpha$ for any α, hence $x \in E_\alpha^c$ for every α, so that $x \in \bigcap E_\alpha^c$. Thus $A \subset B$.

Conversely, if $x \in B$, then $x \in E_\alpha^c$ for every α, hence $x \notin E_\alpha$ for any α, hence $x \notin \bigcup_\alpha E_\alpha$, so that $x \in (\bigcup_\alpha E_\alpha)^c$. Thus $B \subset A$.

It follows that $A = B$.

2.23 Theorem *A set E is open if and only if its complement is closed.*

Proof First, suppose E^c is closed. Choose $x \in E$. Then $x \notin E^c$, and x is not a limit point of E^c. Hence there exists a neighborhood N of x such that $E^c \cap N$ is empty, that is, $N \subset E$. Thus x is an interior point of E, and E is open.

Next, suppose E is open. Let x be a limit point of E^c. Then every neighborhood of x contains a point of E^c, so that x is not an interior point of E. Since E is open, this means that $x \in E^c$. It follows that E^c is closed.

Corollary *A set F is closed if and only if its complement is open.*

2.24 Theorem

(*a*) *For any collection $\{G_\alpha\}$ of open sets, $\bigcup_\alpha G_\alpha$ is open.*
(*b*) *For any collection $\{F_\alpha\}$ of closed sets, $\bigcap_\alpha F_\alpha$ is closed.*
(*c*) *For any finite collection $G_1, \ldots, G_n$ of open sets, $\bigcap_{i=1}^n G_i$ is open.*
(*d*) *For any finite collection $F_1, \ldots, F_n$ of closed sets, $\bigcup_{i=1}^n F_i$ is closed.*

Proof Put $G = \bigcup_\alpha G_\alpha$. If $x \in G$, then $x \in G_\alpha$ for some α. Since x is an interior point of G_α, x is also an interior point of G, and G is open. This proves (*a*).

By Theorem 2.22,

$$\left(\bigcap_\alpha F_\alpha\right)^c = \bigcup_\alpha (F_\alpha^c), \tag{21}$$

and F_α^c is open, by Theorem 2.23. Hence (*a*) implies that (21) is open so that $\bigcap_\alpha F_\alpha$ is closed.

Next, put $H = \bigcap_{i=1}^n G_i$. For any $x \in H$, there exist neighborhoods N_i of x, with radii r_i, such that $N_i \subset G_i$ $(i = 1, \ldots, n)$. Put

$$r = \min(r_1, \ldots, r_n),$$

and let N be the neighborhood of x of radius r. Then $N \subset G_i$ for $i = 1, \ldots, n$, so that $N \subset H$, and H is open.

By taking complements, (*d*) follows from (*c*):

$$\left(\bigcup_{i=1}^n F_i\right)^c = \bigcap_{i=1}^n (F_i^c).$$

2.25 Examples In parts (*c*) and (*d*) of the preceding theorem, the finiteness of the collections is essential. For let G_n be the segment $\left(-\frac{1}{n}, \frac{1}{n}\right)$ ($n = 1, 2, 3, \ldots$). Then G_n is an open subset of R^1. Put $G = \bigcap_{n=1}^{\infty} G_n$. Then G consists of a single point (namely, $x = 0$) and is therefore not an open subset of R^1.

Thus the intersection of an infinite collection of open sets need not be open. Similarly, the union of an infinite collection of closed sets need not be closed.

2.26 Definition If X is a metric space, if $E \subset X$, and if E' denotes the set of all limit points of E in X, then the *closure* of E is the set $\bar{E} = E \cup E'$.

2.27 Theorem *If X is a metric space and $E \subset X$, then*

(*a*) *$\bar{E}$ is closed,*
(*b*) *$E = \bar{E}$ if and only if E is closed,*
(*c*) *$\bar{E} \subset F$ for every closed set $F \subset X$ such that $E \subset F$.*

By (*a*) and (*c*), $\bar{E}$ is the *smallest* closed subset of X that contains E.

Proof
(*a*) If $p \in X$ and $p \notin \bar{E}$ then p is neither a point of E nor a limit point of E. Hence p has a neighborhood which does not intersect E. The complement of $\bar{E}$ is therefore open. Hence $\bar{E}$ is closed.
(*b*) If $E = \bar{E}$, (*a*) implies that E is closed. If E is closed, then $E' \subset E$ [by Definitions 2.18(*d*) and 2.26], hence $\bar{E} = E$.
(*c*) If F is closed and $F \supset E$, then $F \supset F'$, hence $F \supset E'$. Thus $F \supset \bar{E}$.

2.28 Theorem *Let E be a nonempty set of real numbers which is bounded above. Let $y = \sup E$. Then $y \in \bar{E}$. Hence $y \in E$ if E is closed.*

Compare this with the examples in Sec. 1.9.

Proof If $y \in E$ then $y \in \bar{E}$. Assume $y \notin E$. For every $h > 0$ there exists then a point $x \in E$ such that $y - h < x < y$, for otherwise $y - h$ would be an upper bound of E. Thus y is a limit point of E. Hence $y \in \bar{E}$.

2.29 Remark Suppose $E \subset Y \subset X$, where X is a metric space. To say that E is an open subset of X means that to each point $p \in E$ there is associated a positive number r such that the conditions $d(p, q) < r$, $q \in X$ imply that $q \in E$. But we have already observed (Sec. 2.16) that Y is also a metric space, so that our definitions may equally well be made within Y. To be quite explicit, let us say that E is *open relative to* Y if to each $p \in E$ there is associated an $r > 0$ such that $q \in E$ whenever $d(p, q) < r$ and $q \in Y$. Example 2.21(*g*) showed that a set