The Fibonacci numbers (1, 1, 2, 3, 5, 8, ...) proved to be so interesting that now there is a whole journal, called the *Fibonacci Quarterly*, devoted to them. Two examples of their relevance are the following.

The seeds of a sunflower head are arranged in such a way that they form two sets of arcs, emanating from the center. The number of clockwise arcs might be 21, and the number of counterclockwise arcs might be 13. In every case, these numbers are consecutive Fibonacci numbers.

One of the great feats of 20th century number theory was Matiyasevič's proof that there is no general procedure for solving Diophantine equations. We shall examine this proof in the Section on Hilbert's Tenth Problem in Part II, Chapter 21. A crucial element in this proof was Matiyasvič's use of the Fibonacci numbers.

To conclude this section, we derive the formula for the nth Fibonacci number u_n, where $u_0 = 0$ (which we include for convenience), $u_1 = 1$ and $u_{n+2} = u_n + u_{n+1}$ for all $n \geq 0$. Put

$$U(x) = u_0 + u_1 x + u_2 x^2 + \cdots = \Sigma_{n=0}^{\infty} u_n x^n.$$

This formal power series is usually called the *generating function* of u_n, although it is not, strictly speaking, a function. We easily calculate

$$U(x)(1 - x - x^2) = u_0 + (u_1 - u_0)x,$$

since $(u_{n+2} - u_{n+1} - u_n)x^{n+2} = 0$ for all $n \geq 0$.

Hence

$$U(x) = \frac{x}{1 - x - x^2} = \frac{A}{1 - \alpha x} + \frac{B}{1 - \beta x} = A\Sigma_{n=0}^{\infty}\alpha^n x^n + B\Sigma_{n=0}^{\infty}\beta^n x^n$$

in partial fractions, where

$$(1 - \alpha x)(1 - \beta x) \equiv 1 - x - x^2,$$

$$A(1 - \beta x) + B(1 - \alpha x) \equiv x.$$

We see from the first identity that $\alpha + \beta = 1$ and $\alpha\beta = -1$, hence

$$\alpha = \tfrac{1}{2}(1 + \sqrt{5}), \quad \beta = \tfrac{1}{2}(1 - \sqrt{5}).$$

From the second identity one easily determines

$$A = \frac{1}{\alpha - \beta} = \frac{1}{\sqrt{5}}, \quad B = \frac{1}{\beta - \alpha} = \frac{-1}{\sqrt{5}}.$$

Comparing the two expansions of $U(x)$ above we obtain

$$u_n = A\alpha^n + B\beta^n = \frac{1}{\sqrt{5}}\left(\frac{1 + \sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1 - \sqrt{5}}{2}\right)^n.$$

Furthermore, $\beta^n/\sqrt{5}$ is close to 0 and it is not hard to show that u_n is the integer nearest $\alpha^n/\sqrt{5}$.

The number α is called the 'golden ratio'. It played a role in Euclid's construction of the regular pentagon. The above formula for u_n was discovered and proved by A. de Moivre (1730). (See Section 9.6 in Stillwell's *Mathematics and its History.*)

The Fibonacci numbers arose in connection with the following problem from the *Liber abaci*: how many pairs of rabbits will be produced in a year, beginning with a single pair, if in every month each pair bears a new pair which becomes productive from the second month on?

Exercises

1. Solve Gerbert's problem.
2. Solve Jordanus's problem.
3. Solve problem (1) on Frederick II's math contest.
4. Solve problem (2) on Frederick II's math contest.
5. Starting with the formula $u_n = (\alpha^n - \beta^n)/\sqrt{5}$, show that u_n is the integer nearest $\alpha^n/\sqrt{5}$.
6. Show that u_{n+1}/u_n tends to the golden ratio as n tends to infinity.
7. Show that u_n and u_{n+1} are relatively prime.
8. Prove that the greatest common divisor of any two Fibonacci numbers is also a Fibonacci number.

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Mathematics in the Renaissance

Aside from the invention of the Indian numerals, and aside from the work of a few persons of talent, such as Pappus and Fibonacci, no significant advances in mathematics had taken place in the thousand years following Diophantus. In the 15th and 16th centuries there was a sudden spurt of activity, aided by the Chinese invention of printing, which reached Europe in 1450 and which carried mathematics, both pure and applied, beyond the achievements of the ancients. It is hard to overemphasize the importance of printing for the spread of mathematical knowledge. Copying mathematical texts by hand required much time and labour. In ancient times, many texts existed only in a single copy, which would be found in the library of Alexandria. This is why, for about 800 years, all mathematical activity was concentrated in one place. Now such texts were available all over the civilized world and people could learn mathematics even in such outlying places as Bohemia or Scotland. In this chapter, and in the next two chapters, we shall discuss advances in the following areas:

1. mathematical notation,
2. the theory of equations,
3. the invention of logarithms,
4. mechanics and astronomy.

(1) **Mathematical notation**

Johannes Regiomontanus (1436–1476) of Königsberg, then in Germany, gave the first systematic exposition of plane and spherical trigonometry,