Chapter 14

14.4 Exercises (page 513)

1. (b) $X \leq b$
2. (a) $\{\omega \mid X(\omega) \in (a, b], Y(\omega) \in (c, d]\}$
   (c) $X \leq a$, $Y \leq d$
   (d) $P(a < X \leq b, c < Y \leq d) = P(X \leq b, Y \leq d) - P(X \leq a, Y \leq d) - P(X \leq b, Y \leq c) + P(X \leq a, Y \leq c)$
3. (a) $\{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$, $\{(5, 6), (6, 5)\}$, $\{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1), (5, 6), (6, 5)\}$
   (b) $P(X = 7) = \frac{1}{6}$; $P(X = 11) = \frac{1}{18}$; $P(X = 7 \text{ or } X = 11) = \frac{2}{9}$
4. $Y = X_1 + X_2 + X_3 + X_4$; $P(Y = 0) = \frac{1}{16}$; $P(Y = 1) = \frac{1}{4}$; $P(Y \leq 1) = \frac{5}{16}$
5. $Y = 7X$ if $0 \leq X \leq 100$; $Y = 10X - 300$ if $X > 100$
6. (a) $Z = Y - 1$
   (b) $U = Y_1 + Y_2 - 1$

14.8 Exercises (page 523)

2.

t	2	3	4	5	6	7	8	9	10	11	12
$p_x(t)$	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$

3. (b) $p(-2) = \frac{1}{2}$, $p(0) = p(2) = \frac{1}{4}$
   (c) $0, \frac{3}{4}, \frac{3}{4}, \frac{1}{4}, 1, 0, \frac{1}{4}, a$
5. (a) $c = \frac{1}{3}$
   (b) $\frac{1}{9}, \frac{1}{3}, \frac{2}{3}, \frac{8}{9}$
   (c) $F(t) = 0$ for $t < 0$, $F(t) = \frac{1}{9}$ for $0 \leq t < 1$, $F(t) = \frac{1}{3}$ for $1 \leq t < 2$, $F(t) = 1$ for $t \geq 2$
   (d) No such t
   (e) $t = 2$
6. (a)

k	0	1	2	3	4
$p(k)$	$\frac{16}{81}$	$\frac{32}{81}$	$\frac{8}{27}$	$\frac{8}{81}$	$\frac{1}{81}$

   $p(t) = 0$ for $t \neq 0, 1, 2, 3, 4$
   (b) $F(t) = 0$ for $t < 0$, $F(t) = \frac{16}{81}$ on $[0, 1)$, $F(t) = \frac{16}{27}$ on $[1, 2)$, $F(t) = \frac{8}{9}$ on $[2, 3)$, $F(t) = \frac{80}{81}$ on $[3, 4)$, $F(t) = 1$ for $t \geq 4$
   (c) $\frac{8}{27}, \frac{56}{81}$
7. (b) $P(X = 0) = (1 - p)^2$; $P(X = 1) = 2p(1 - p)$
8. $p_X(k) = \dfrac{2k}{n(n + 1)}$; $F_X(t) = \dfrac{[t]([t] + 1)}{n(n + 1)}$ for $0 \leq t \leq n$, where $[t]$ denotes the greatest integer $\leq t$; $F_X(t) = 0$ for $t < 0$; $F_X(t) = 1$ for $t > n$
9. $p(k) = e^{-c}\dfrac{c^k}{k!}$, $k = 0, 1, 2, 3, \ldots; c \geq 0$

   $p(t) = 0$ for $t \neq 0, 1, 2, 3, \ldots$
10. (a) $p_X(t) = \frac{1}{2}$ at $t = -1$ and $t = +1$; $p_X(t) = 0$ elsewhere
    $F_X(t) = 0$ for $t < -1$; $F_X(t) = \frac{1}{2}$ for $-1 \leq t < 1$; $F_X(t) = 1$ for $t \geq 1$
11. $P(A) = \frac{2}{3}$; $P(B) = \frac{2}{3}$; $P(A \cap B) = \frac{1}{3}$; $P(B \mid A) = \frac{1}{2}$; $P(A \cup B) = 1$

14.12 Exercises (page 532)

1. (a) c = 1 ; f(t) = 1 if $0 \le t \le 1$; f(t) = 0 otherwise
   (b) $0, \frac{1}{3}, \frac{1}{3}$
2. $c = \frac{1}{2}$; $F(t) = 0$ if $t < -\pi/2$; $F(t) = \frac{1}{2}\cos t$ if $-\pi/2 \le t < 0$; $F(t) = 1 - \frac{1}{2}\cos t$ if $0 \le t < \pi/2$; $F(t) = 1$ if $t \ge \pi/2$
3. $c = \frac{3}{8}$; $F(t) = 0$ if $t < 0$; $F(t) = \frac{1}{4}(3t^2 - t^3)$ if $0 \le t \le 2$; F(t) = 1 if $t > 2$
4. (a) $\frac{1}{2}$
   (b) $\frac{1}{2}$
   (c) $\frac{1}{4}$
5. (a) $f(t) = 0$ if $t < \frac{1}{4}$; $f(t) = 8t - 1$ if $\frac{1}{4} \le t < \frac{1}{2}$; $f(t) = 7 - 8t$ if $\frac{1}{2} \le t < \frac{3}{4}$; $f(t) = 0$ if $t \ge \frac{3}{4}$
   (b) $F(t) = 0$ if $t < \frac{1}{4}$; $F(t) = 4t^2 - t$ if $\frac{1}{4} \le t < \frac{1}{2}$; $F(t) = -4t^2 + 7t - 2$ if $\frac{1}{2} \le t < \frac{3}{4}$; F(t) = 1 if $t \ge \frac{3}{4}$
   (c) $1, 1, \frac{1}{2}, 0, \frac{5}{16}$
6. (a) $0, \frac{5}{6}, \frac{2}{3}, \frac{1}{2}$
   (b) $t = 1$
7. (a)

k	5	10	15	20	25	30
$P(X \ge k)$	$\frac{5}{6}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$	0

   (b) Let each Styx train arrive 5 minutes before a Lethe train
8. $F_Y(t) = 0$ if $t < b$; $F_Y(t) = (t - b)/a$ if $b \le t \le b + a$; $F_Y(t) = 1$ if $t > b + a$
9. (a) $\frac{2}{37}$
   (b) $\frac{11}{37}$
   (c) $\frac{36}{37}$
10. $F(t) = \frac{1}{2} + \frac{1}{\pi}\arctan\frac{t - b}{a}$

14.16 Exercises (page 540)

1. (a) 105
   (b) 10.05
2. (a) $\frac{1}{2}(\sqrt{2} - 1)$
   (b) $\frac{1}{4}(2 - \sqrt{2})$
   (c) $\frac{1}{2}(2 - \sqrt{2})$
   (d) $\frac{1}{14}(4 - \sqrt{2})$
3. (a) $1 - e^{-1}$
   (b) e^{-2}
   (c) $(e - 1)/e^2$
   (d) e^{-3}
4. F(t) = 0 if $t <$ c ; F(t) = $1 - e^{-\lambda(t-c)}$ if $t \ge$ c
6. $\lambda' = \lambda/a$, $c' = b + ac$
9. (a) 0.5000
   (b) 0.1359
   (c) 0.9974
   (d) 0.0456
10. (a) 0.675
    (b) 0.025
11. (a) 0.6750
    (b) 0.025σ

12. (a) 0.8185
    (b) 0.8400
13. 75.98 inches
14. mean = b, variance = a^2
15. $F_Y(t)$ = Cl if $t < 0$; $f_Y(t) = 0$ if $t \leq 0$; $f_Y(t) = e^{-t/2}/\sqrt{2\pi t}$ if $t > 0$

14.18 Exercises (page 542)

1. (a) $F_Y(t) = 0$ if $t < 1$; $F_Y(t) = (t - 1)/3$ if $1 \leq t \leq 4$; $F_Y(t) = 1$ if $t > 4$; $f_Y(t) = \frac{1}{3}$ if $1 \leq t \leq 4$; $f_Y(t) = 0$ otherwise
   (b) $F_Y(t) = 0$ if $t < -2$; $F_Y(t) = (t + 2)/3$ if $-2 \leq t \leq 1$; $F_Y(t) = 1$ if $t > 1$; $f_Y(t) = \frac{1}{3}$ if $-2 \leq t \leq 1$; $f_Y(t) = 0$ otherwise
   (c) $F_Y(t) = 0$ if $t < 0$; $F_Y(t) = t^{1/2}$ if $0 \leq t \leq 1$; $F_Y(t) = 1$ if $t > 1$; $f_Y(t) = (2t)^{-1/2}$ if $0 \leq t \leq 1$; $f_Y(t) = 0$ otherwise
   (d) $F_Y(t) = e^t$ if $t \leq 0$; $F_Y(t) = 1$ if $t > 0$; $f_Y(t) = e^t$ if $t \leq 0$; $f_Y(t) = 0$ if $t > 0$
   (e) $F_Y(t) = e^{t/2}$ if $t \leq 0$; $F_Y(t) = 1$ if $t > 0$; $f_Y(t) = \frac{1}{2}e^{t/2}$ if $t \leq 0$; $f_Y(t) = 0$ if $t > 0$
   (f) $F_Y(t) = 0$ if $t < 1$; $F_Y(t) = \log t$ if $1 \leq t \leq e$; $F_Y(t) = 1$ if $t >$ e; $f_Y(t) =$ l/t if $1 \leq t \leq e$; $f_Y(t) = 0$ otherwise
2. Let ψ be the inverse of φ, defined on the open interval (a, b).
   Then $F_Y(t) = 0$ if $t \leq a$; $F_Y(t) = F_X[\psi(t)]$ if a $< t <$ b ; $F_Y(t) = 1$ if $t \geq$ b ; $f_Y(t) = f_X[\psi(t)]\psi'(t)$ if a $< t < b$; $f_Y(t) = 0$ otherwise
3. (a) $f_Y(t) = 0$ if $t \leq 0$; $f_Y(t) = (2\pi t)^{-1/2}e^{-t/2}$ if $t > 0$
   (b) $f_Y(t) = 0$ if $t < 0$; $f_Y(t) = 4t(2\pi)^{-1/2}e^{-t^4/2}$ if $t \geq 0$
   (C) $f_Y(t) = 0$ if $t \leq 0$; $f_Y(t) = (2\pi t^2)^{-1/2}e^{-(\log t)^2/2}$ if $t > 0$
   (d) $f_Y(t) = (2\pi)^{-1/2} \sec^2 t\, e^{-(\tan^2 t)/2}$ if $|t| < \pi/2$; $f_Y(t) = 0$ if $|t| \geq \pi/2$

14.22 Exercises (page 548)

2. (a) $P(X = x_1) = P(X = x_2) = P(Y = y_1) = P(Y = y_2) = \frac{1}{2}(p + q)$
   (b) $p = q = \frac{1}{2}$
3. (a) F(x,y) = $\left(\frac{x-a}{b-a}\right)\left(\frac{y-c}{d-c}\right)$ if a Ix $\leq b$ and c $\leq y \leq d$,

   $F(x, y) = \frac{x-a}{b-a}$ if a Ix $\leq b$ and y $> d$, $F(x, y) = \frac{y-c}{d-c}$ if x>b and $c \leq y \leq d$,

   $F(x, y) = 1$ if x $> b$ and y $> d$, $F(x, y) = 0$ otherwise
   (b) $F_X(x) = (x - a)/(b - a)$ if $a \leq x \leq b$; $F_X(x) = 0$ if x < a ; $F_X(x) = 1$ if $x > b$; $F_Y(y) = (y - c)/(d - c)$ if $c \leq y \leq d$; $F_Y(y) = 0$ if $y < c$; $F_Y(y) = 1$ if $y > d$
   (c) X and Y are independent
5. $P(Y = 0) = \frac{1}{2}$; $P(Y = 1) = P(Y = 2) = \frac{1}{4}$
7. $\frac{1}{5}, \frac{4}{5}$
8. $\frac{1}{3}$
9. (b) $f(x, y) = \frac{1}{2}$ if $(x, y) \in Q$; $f(x, y) = 0$ if $(x, y) \notin Q$;
   $f_X(x) = 1 - |x|$ if $|x| \leq 1$; $f_X(x) = 0$ if $|x| > 1$;
   $f_Y(y) = 1 - |y|$ if $|y| \leq 1$; $f_Y(y) = 0$ if $|y| > 1$.
   X and Y are not independent
10. $g(u, v) = f(u +$ a, $v + b)$

14.24 Exercises (page 553)

1. (b) $f_V(v) = 1 + v$ if $-1 \leq v < 0$; $f_V(v) = 1 -$ v if $0 \leq v \leq 1$; $f_V(v) = 0$ if $|v| > 1$
   (c) U and V are not independent

2. (b) $f_V(t) = 2 - 2t$ if $0 \leq t \leq 1$; $f_V(t) = 0$ otherwise
   (c) U and V are independent
3. (b) $g(u, v) = ue^{-u}$ if $u > 0$, $0 < v < 1$; $g(u,$ v$) = 0$ otherwise
   (c) $f_U(u) = ue^{-u}$ if $u > 0$; $f_U(u) = 0$ if $u \leq 0$
   (d) $f_V(v) = 1$ if $0 < v < 1$; $f_V(v) = 0$ otherwise
4. (b) $g(u, v) = (v/2\pi)e^{-(1+u^2)v^2/2}$ if $v \geq 0$
   (c) $f_U(u) = [\pi(1 + u^2)]^{-1}$
5. $f_Z(t) = \pi^{-\frac{1}{2}}e^{-t^2}$
6. (b) $f_U(u) = 0$ if $u < 0$; $f_U(u) = (u/\sigma^2) \exp\left(-\frac{u^2}{2\sigma^2}\right)$ if $u > 0$; $F_U(t) = 0$ if $t <$ O;
   $F_U(t) = 1 - \exp\left(-\frac{t^2}{2\sigma^2}\right)$ if $t \geq 0$

14.27 **Exercises (page 560)**

1. $E(X) = \frac{7}{2}$, Var $(X) = \frac{35}{12}$
7. (a) $E(X)$ = Var (X) = λ
   (b) None
   (c) $E(X) = 1/\lambda$, Var (X) = $1/\lambda^2$
   (d) $E(X) = m$, Var (X) = σ^2
8. (a) C(r) = $(r - 1)/2$
   (b) $F_X(t) = \frac{1}{2}|t|^{1-r}$ if $t < -1$; $F_X(t) = \frac{1}{2}$ if $-1 \leq t \leq 1$; $F_X(t) = 1 - \frac{1}{2}t^{1-r}$ if $t > 1$
   (c) $P(X < 5) = 1 - 5^{r-1}/2$; $P(5 < X < 10) = (5^{1-r} - 10^{1-r})/2$
   (d) X has a finite expectation when $r > 2$; $E(X) = Cl$
   (e) Variance is finite for $r > 3$; Var (X) = $(r - 1)/(r - 3)$
9. $E(X) = E(Y) = -\frac{1}{37}$; $E(Z) = -1767/50653$; $E(X + Y + Z) = -4505/50653$
10. $E(X) + \infty$ as $n \to \infty$
12. (a) $(2/\pi)^{\frac{1}{2}}$
    (b) $e^{\frac{1}{2}}$
    (c) $e^2 - e$
    (d) $(\pi/2)^{\frac{1}{2}}$

14.31 Exercises (page 568)

4. 251
5. 0
6. Chebyshev's inequality gives $\frac{1}{9}$; tables give 0.0027
8. (b) 0.6826
9. (b) 0.0796
10. (a) 0.0090
    (b) 0.0179

Chapter 15

15.5 Exercises (page 577)

2. (a) No
   (b) Yes
   (c) Yes
   (d) No
   (e) Yes
   (f) No
   (g) Yes
3. (a) Neither
   (b) Seminorm
   (c) Seminorm
   (d) Neither
   (e) Seminorm
   (f) Norm
   (g) Neither
   (h) Neither

6. (a), (b), (c)
8. (b) The polynomial in (a) plus $\frac{7}{8}(5x^3 - 3x)\int_{-1}^{1}(5t^3 - 3t)f(t)\,dt + \frac{9}{128}(35x^4 - 30x^2 + 3) \times \int_{-1}^{1}(35t^4 - 30t^2 + 3)f(t)\,dt$
9. $a = -3e/4 + 33/(4e)$, $b = 3/e$, $c = \frac{15}{4}(e - 7/e)$
10. $\frac{15}{128}(1 + 14x2 - 7x^4)$
11. (b) $\|f - P\|^2 = \frac{1}{96}$
12. (a) $\|P - f\|^2 = \dfrac{n - 1}{n} - \dfrac{\log 2\ n}{n - 1}$

    (b) $P(x) = \left(\dfrac{12}{(n-1)^2} - \dfrac{6(n+1)}{(n-1)^3}\log n\right)x + \dfrac{4(n^3 - 1)\log n}{(n-1)^4} - \dfrac{6(n+1)}{(n-1)^2}$;

    $\|P - f\|^2 = 36\ \log 2 - 28\ \log^2 2 - \frac{23}{2} = 0.0007$ when $n = 2$
13. (a) $\|P - f\|^2 = \dfrac{1}{2n}[(n - 2)e^{2n} + 4e^n - n - 2]$

    (b) P(x) = (18 − 6e)x + 4e − 10; $\|P - f\|^2 = 20e - \frac{7}{2}e^2 - \frac{57}{2} = 0.0038$
14. (b) $\varphi_{k+1}(x) = \dfrac{\sqrt{(2k+1)(2k+3)}}{k+1}x\varphi_k(x) - \dfrac{k}{k+1}\sqrt{\dfrac{2k+3}{2k-1}}\,\varphi_{k-1}(x)$, where $\varphi_k = P_k/\|P_k\|$
15. (b) $\displaystyle\sum_{k=0}^{m} P_k^2(x) = \frac{p_m}{p_{m+1}}[P_m(x)P'_{m+1}(x) - P_{m+1}(x)P'_m(x)]$

15.9 Exercises (page 585)

1. (a) $P(x) = \frac{1}{6}(x^2 + 13x + 12)$
   (b) $P(x) = \frac{1}{2}(x^2 - 5x + 6)$
   (c) $P(x) = -\frac{1}{6}(x^3 - 6x^2 + 5x - 6)$
   (d) $P(x) = 2x^3 + x^2 - x - 2$
   (e) $P(x) = -5x^3 - x^2 + 10x - 5$
2. $P(x) = \frac{1}{640}(9x^4 - 196x^2 + 640)$
4. (a) $Q(x) = 2x^3 + 3x^2 - x - 3$
   (b) $Q(x) = 4x^3 + 7x^2 - 3x - 7$
5. (a) $P(32) = \frac{31}{21}$; $f(32) - P(32) = \frac{43}{42}$
   (b) $P(32) = \frac{992}{255}$; $f(32) - P(32) = -\frac{709}{510}$
   (c) $P(32) = \frac{43}{15}$; $f(32) - P(32) = -\frac{11}{30}$
   (d) $P(32) = -\frac{403}{1530}$; $f(32) - P(32) = \frac{2114}{765}$
7. (a) $L_0(x) = \frac{1}{72}(u - 1)(u - 3)(u - 4)(u - 6)$; $L_1(x) = -\frac{1}{30}u(u - 3)(u - 4)(u - 6)$; $L_2(x) = \frac{1}{18}u(u - 1)(u - 4)(u - 6)$; $L_3(x) = -\frac{1}{24}u(u - 1)(u - 3)(u - 6)$; $L_4(x) = \frac{1}{180}u(u - 1)(u - 3)(u - 4)$
   (b) $P(2.6) = 20$
8. (b) $x \geq 1.581$
   (c) $h \leq 0.0006$
12. (a) $a = 0$, $b = 1$
    (b) $c = 1$, $d = -2L'_k(x_k)$
13. (d) Let $B_0(x) = 1$ and let $B_n(x) = (x - x_0)(x - x_0 - nh)^{n-1}/n!$ for $n \geq 1$; the one and only polynomial P of degree $\leq n$ satisfying the conditions $P(x_0) = c_0$, $P'(x_1) = c_1$, $P''(x_2) = c_2, \ldots, P^{(n)}(x_n) = c_n$ is given by $P(x) = c_0B_0(x) + \ldots + c_nB_n(x)$
16. x

15.13 Exercises (page 593)

4. (b)

8	-5040	13068	-13132	6769	-1960	322	-28	1	
9	40320	-109584	118124	-67284	22449	-4536	546	-36	1
10	-3628800	1026576	-1172700	723680	-269325	63273	-9450	870	-45	1

(c) $1 + 2x + 2x2 - 3x3 + x^4$

5. (c)

8	1	127	966	1701	1050	266	28	1	
9	1	255	3025	7770	6951	2646	462	36	1
10	1	511	9330	34105	42525	22827	5880	750	45	1

(d) $-1 + 6x^{(1)} + 16x^{(2)} + 9x^{(3)} + x^{(4)}$

7. (a) $\frac{4}{3}n^3 + \frac{11}{2}n^2 + \frac{61}{6}n$
(b) $\frac{1}{4}n^4 + \frac{5}{6}n^3 + \frac{3}{4}n^2 + \frac{1}{6}n$
(c) $\frac{1}{4}n^4 + \frac{3}{2}n^3 + \frac{11}{4}n^2 + \frac{3}{2}n$
(d) $\frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$

15.18 Exercises **(page 600)**

2. (b) $T_n(1) = n^2$, $T_n'(-1) = (-1)^{n-1}n^2$

5. $\sin\theta \sin n\theta = \dfrac{1-x^2}{n} T_n'(x)$; degree $= n + 1$

7. $Q(x) = x^{n+1} - 2^{-n}T_{n+1}(x)$

8. $Q(x) = -\frac{5}{2}x^4 + \frac{35}{16}x^3 - \frac{25}{32}x^2 + \frac{25}{256}x - \frac{1}{512}$

14. (b) $U_0(x) = 1$, $U_1(x) = 2x$, $U_2(x) = 4x^2 - 1$, $U_3(x) = 8x^3 - 4x$, $U_4(x) = 16x^4 - 12x^2 + 1$, $U_5(x) = 32x^5 - 32x^3 + 6x$

15.21 Exercises (page 610)

1. (a) $0.693773 - \epsilon$, where $0.000208 \leq \epsilon \leq 0.001667$. This gives the inequalities $0.6921 < \log 2 < 0.6936$
(b) $n = 578$

2. (a) $c = \sqrt{3}/3$

(b) $c_1 = \dfrac{a+b}{2} - \dfrac{b-a\sqrt{3}}{2\,3}$, $c_2 = \dfrac{a+b}{2} - \dfrac{b-a\sqrt{3}}{2}\cdot\dfrac{2}{3}$

3. (a) $c = \sqrt{2}/2$

(b) $c_1 = \dfrac{a+b}{2} - \dfrac{b-a\sqrt{2}}{2\,2}$, $c_2 = \dfrac{a+b}{2} - \dfrac{b-a\sqrt{2}}{2}\cdot\dfrac{2}{2}$

4. $a = 2 + \sqrt{2}$, $b = 2 - \sqrt{2}$

5. $c = \sqrt{\frac{3}{2}}$

7. (d) $$\int_a^b f(x)\,dx = \frac{b-a}{18}\left[5f\left(\frac{b+a}{2} - \frac{b-a}{2}\sqrt{\frac{3}{5}}\right) + 8f\left(\frac{b+a}{2}\right) + 5f\left(\frac{b+a}{2} + \frac{b-a}{2}\sqrt{\frac{3}{5}}\right)\right]$$

10. (a) $\log 2 = 0.693254 - \epsilon$, where $0.000016 \leq \epsilon \leq 0.000521$; this leads to the inequalities ***0.69273 < log 2 < 0.69324***
    (b) $\log 2 = 0.69315023 - \epsilon$, where $0.00000041 \leq \epsilon \leq 0.00001334$; this leads to the inequalities $0.693136 < \log 2 < 0.693149$
11. (d) $\log 2 = 0.693750 - \epsilon$, where $0.000115 \leq \epsilon \leq 0.003704$; this leads to the inequalities ***0.69004 < log 2 < 0.69364***

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