introduced numbers as coordinates and used algebra to simplify the description and manipulation of figures. However, coordinates were just a means to describe figures; they were not considered geometric in themselves. It was not until Dedekind and others clarified the concept of number in the late 19th century that numbers could be seen as the raw material for creating points, lines, and planes. This was virtually a reversal of Euclid's viewpoint, and it included a reversal of the role of the Pythagorean theorem—from a theorem about triangles to the definition of distance between two points in the plane.

Other basic theorems of Euclid's geometry can be proved, in this new setup, with the help of the concept of isometry, which is a rigorous counterpart of Euclid's idea of "movement."

In 1872, Klein made yet another dramatic shift in viewpoint when he realized that *the isometries make the geometry*. In particular, Euclidean plane geometry is *everything that is preserved by Euclidean isometries*. Indeed, the fundamental quantity preserved by Euclidean isometries is the distance $d(P_1, P_1')$ between points. We saw in Section 3.6 that any reflection preserves distance and that any isometry is a composite of reflections, hence $d(P_1, P_1')$ is preserved by all isometries. In principle, we could start with the set of reflections and "discover" the idea of Euclidean distance $d(P_1, P_1')$, by calculating the quantity

$$\sqrt{(x_2' - x_2)^2 + (y_2' - y_2)^2},$$

for the mirror images $P_2 = (x_2, y_2)$ and $P_2' = (x_2', y_2')$ of points $P_1 = (x_1, y_1)$ and $P_1' = (x_1', y_1')$ under any reflection, and finding it equal to the corresponding quantity for $P_1 = (x_1, y_1)$ and $P_1' = (x_1', y_1')$,

$$\sqrt{(x_1' - x_1)^2 + (y_1' - y_1)^2}.$$

As we know, the concepts of *line* (the equidistant set of two points) and *circle* (the equidistant set of one point) can be defined in terms of distance, so as soon as the distance function is derived from the isometries, we have the whole Euclidean plane geometry.

Thus Klein's idea gives yet another way to put Euclid's geometry on a rigorous foundation. As usual, the test of a new viewpoint is whether it enables us to see anything more clearly than before.

Klein's viewpoint shows us that Euclidean geometry is just one of several structurally similar geometries. One of them, of course, is the geometry of Euclidean *space*, for which the isometries are composites of reflections in planes.

Among the geometries of surfaces, the most familiar relative of Euclidean geometry is the geometry of the sphere. Its isometries are composites of reflections in planes through the sphere's center. The "equidistant sets" of these reflections are the intersections of the planes with the sphere, the so-called *great circles*. These are the "lines" of spherical geometry, and their basic properties are found by arguments similar to those we used for the Euclidean plane.

1. Any two "lines" have a point in common.
2. Hence the composite of two reflections is always a rotation.
3. Any isometry is the composite of one, two or three reflections.

An important part of Klein's concept of geometry is that the isometries form a *group of transformations*, a set of one-to-one functions closed under composites and inverses. Such a set is obtained by taking composites of reflections because each reflection is its own inverse; that is, the composite of a reflection with itself is the *identity function*, which sends each point to itself. It follows that each composite $f_1 f_2 \cdots f_k$ of reflections also has an inverse, namely, the composite (in reverse order) of their inverses, $f_k^{-1} \cdots f_2^{-1} f_1^{-1}$. Because if we compose these two composites we get

$$\begin{aligned} f_1 f_2 \cdots f_{k-1} f_k \cdot f_k^{-1} f_{k-1}^{-1} \cdots f_2^{-1} f_1^{-1} &= f_1 f_2 \cdots f_{k-1} \cdot f_{k-1}^{-1} \cdots f_2^{-1} f_1^{-1} \\ &\;\vdots \\ &= f_1 f_2 \cdot f_2^{-1} f_1^{-1} \\ &= f_1 \cdot f_1^{-1} \\ &= \text{identity function} \end{aligned}$$

by successive cancellation of inverses. Thus isometries of both the Euclidean plane and the sphere have inverses, and hence the corresponding sets of isometries are groups.

Viewing the set of isometries as a group draws our attention to *subgroups*—subsets of isometries that also form groups—and these also throw new light on geometry.

For example, consider the isometries that are composites of an *even* number of reflections. The inverse of such an isometry is also the composite of an even number of reflections, and so is the composite of two such isometries. Thus the composites of even numbers of reflections form a subgroup. Intuitively speaking, it is the subgroup that preserves "orientation," or "handedness," or "clockwiseness" or whatever you want to call those aspects of geometry that are *not* preserved by reflections. One such aspect is the cyclic order of the numbers 1, 3, 6, 12 on a clock face, or at least it seems to be. We can escape the tricky problem of defining these aspects by *letting the subgroup define them.* That is, we say that a property *depends on orientation* if it is not preserved by the whole group but preserved by the subgroup of composites of even numbers of reflections. We call the latter group the *orientation-preserving subgroup.*

This idea depends on the fact that the orientation-preserving subgroup is not the whole isometry group; it would fail if a composite of an even number of isometries was also a composite of an odd number of reflections. The example of the clock face makes this unlikely, but we can prove it by considering composites of one or two reflections and what they do to lines. A single reflection maps one line onto itself. A composite of two reflections is either a rotation, which maps no line onto itself, or else a translation, which maps infinitely many lines onto themselves. Thus a reflection cannot be a composite of two reflections, and it follows that it cannot be a composite of any even number of reflections, because all such products are rotations or translations (see exercises). Thus the orientation-preserving subgroup is not the whole isometry group, and hence there is such a thing as orientation!

## Exercises

One reason that composites of even numbers of reflections are rotations or translations is that a rotation is the composite of reflections in two intersecting lines. It appears intuitively clear (see Figure 3.9 again) that the rotation depends only on the point of intersection and the angle between the lines, hence it should be possible to represent any rotation

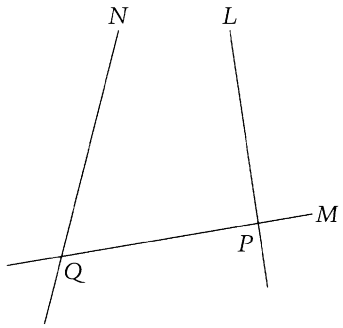


**FIGURE 3.11** Reflection lines for the composite of two rotations.

about $P$ as the composite of reflections in two lines, *one of which is given.* The first exercise confirms this intuition.

3.8.1. Let $L$, $M$, and $L'$ be any lines through a point $P$. Show, using the three reflections theorem or otherwise, that there is a line $M'$ through $P$ such that

$$\mathrm{ref}_M \mathrm{ref}_L = \mathrm{ref}_{M'} \mathrm{ref}_{L'}.$$

Also prove that if $M'$ is given, we can find $L'$ to satisfy this equation.

3.8.2. Deduce from Exercise 3.8.1 that rotations about two points $P$ and $Q$ can be expressed as $\mathrm{ref}_M \mathrm{ref}_L$ and $\mathrm{ref}_N \mathrm{ref}_M$, where $L$, $M$, and $N$ meet the points $P$ and $Q$ as indicated in Figure 3.11.

So far, these arguments apply to both the Euclidean plane and the sphere. The possibility of a translation arises in the next exercise, but only in the Euclidean plane.

3.8.3. Conclude from Exercise 3.8.2 that the composite of rotations is either a rotation or a translation. If it is a rotation, about which point?

It follows that, on the sphere, the composite of two rotations is a rotation, and hence the orientation-preserving subgroup of the sphere consists entirely of rotations. In the Euclidean plane, we still have to find how translations interact with each other and with rotations.

3.8.5. Show that the composite of translations is a translation.

3.8.6. By imitating the arguments in Exercises 3.8.1 and 3.8.3, or otherwise, show that the composite of a translation and a rotation is either a translation or a rotation.

## 3.9* The Non-Euclidean Plane

A beautiful example of the way isometries create geometry is the non-Euclidean plane of Henri Poincaré (1882). Poincaré found a geometry in which there is more than one parallel to a given "line" through a given point. His "plane" is the upper half ($y > 0$) of $\mathbb{R} \times \mathbb{R}$, and his "reflection" is a generalization of ordinary reflection called *reflection in a circle*. The non-Euclidean *isometries* are composites of reflections, and non-Euclidean *distances* are equal if there is a non-Euclidean isometry carrying one to the other.

The reflection of a point $P$ in a circle $C$ with center $Z$ and radius $r$ is defined to be the point $P'$ on the Euclidean line $ZP$ such that

$$ZP \cdot ZP' = r^2.$$

See Figure 3.12. Ordinary reflection can be regarded as the limiting case of reflection in a circle as the center $Z$ tends to infinity.

In fact, the reflections generating the isometries of the non-Euclidean plane include ordinary reflections in the vertical lines $x$ = constant. The other reflections used are reflections in circles with their centers on the $x$-axis. Thus the "lines" of the non-Euclidean plane are obtained immediately as the fixed point sets of the "reflections," namely, the vertical Euclidean half-lines $x$ = constant and the Euclidean semicircles with centers on the $x$-axis (Figure 3.13).

We can see from Figure 3.13 that the parallel axiom fails: $\mathcal{M}$ and $\mathcal{N}$ are two "lines" through the point $P$ that do not meet the "line" $\mathcal{L}$.

Apart from the parallel axiom, all other axioms of Euclid's geometry hold in the non-Euclidean plane. For example, there is exactly one "line" through any two points $P$ and $Q$, and in fact it can be found by a ruler and compass construction, as Figure 3.14 shows. One draws the Euclidean line $PQ$ and, if $PQ$ is not vertical, con-

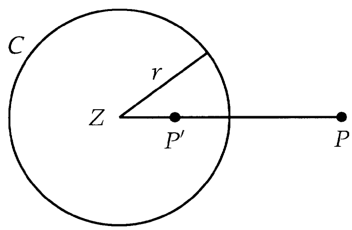


**FIGURE 3.12** Reflection of a point in a circle.

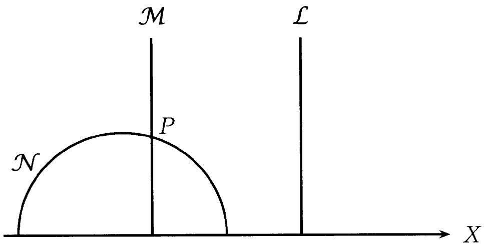


**FIGURE 3.13** Some "lines" of the non-Euclidean plane.

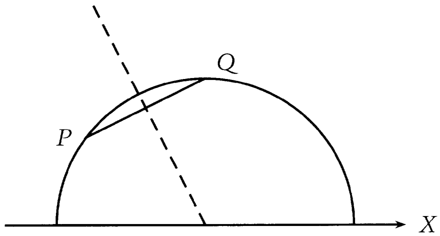


**FIGURE 3.14** Construction of a non-Euclidean "line."

structs its perpendicular bisector. The latter meets the $x$-axis at the center of the semicircle that is the non-Euclidean "line" $PQ$.

Another pleasant property of Poincaré's non-Euclidean plane is that its "angles" are ordinary angles. The only difference is that they are not angles between Euclidean lines but between "lines," that is, between Euclidean circles, or between a circle and a vertical Euclidean line. The angle between circles is the angle between their tangents at the point of intersection. For example, perpendicular "lines" are either perpendicular semicircles with their centers on the $x$-axis, or a vertical Euclidean half-line and a semicircle with its center at the lower end of the half-line (see Figures 3.17 and 3.16).

It turns out that a non-Euclidean "circle"—the set of points at constant non-Euclidean distance from a single point—is a Euclidean circle. Its Euclidean center is *not* the same as its non-Euclidean center (because Euclidean distance is not the same as non-Euclidean distance, as we shall see). However, the circle can also be constructed from its non-Euclidean center and a point on its circumference by ruler and compass. This means that the natural non-Euclidean "constructions" can all be done within Euclidean geometry. In particular, non-Euclidean "constructible points" are constructible points of the

Euclidean half-plane. Some of these constructions are pursued in the exercises.

The isometries of the non-Euclidean plane are particularly interesting. Those that preserve orientation are composites of two reflections, and there are three types, as the "lines" of reflection meet at a point of the half-plane, meet on the $x$-axis, or do not meet at all. The first are non-Euclidean *rotations*, the second are called *limit rotations* (because the center of rotation is infinitely far away, in terms of non-Euclidean distance), and the third are non-Euclidean *translations*.

An example of a non-Euclidean "translation" is the composite of reflections in the semicircles with center $O$ and radii 1 and 2. It is easy to check that this "translation" sends each $(x, y)$ in the upper half-plane to $(4x, 4y)$. Hence each of the points $x = 0$, $y = 1, 1/4, 1/4^2, 1/4^3, \ldots$ is mapped onto its predecessor by this translation, and so they are *equally spaced* in the sense of non-Euclidean distance. This explains why the $x$-axis is infinitely far away from all points of the non-Euclidean plane and shows that non-Euclidean lines are infinitely long.

This may seem to be a strange geometry, with semicircles called "lines" of infinite "length," but because all but one of Euclid's axioms hold it is feasible to use ordinary geometric reasoning. Poincaré, in fact, introduced this non-Euclidean plane because he wanted to study transformations generated by reflections in circles, and he found that geometric language made them easier to understand.

# Exercises

Reflection in a circle occurs frequently in mathematics, and it can be described in many ways. Its connection with ruler and compass constructions is established by Figure 3.15.

3.9.1. By comparison of similar right-angled triangles, show that $ZP \cdot ZP' = r^2$, and hence describe a ruler and compass construction of $P'$ from $P$, and $P$ from $P'$.

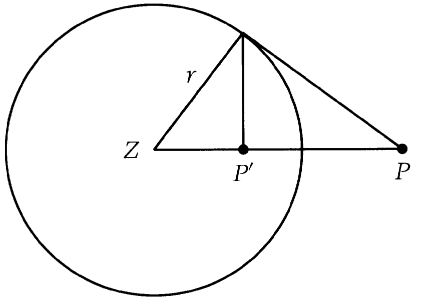


**FIGURE 3.15** Construction of reflection in a circle.

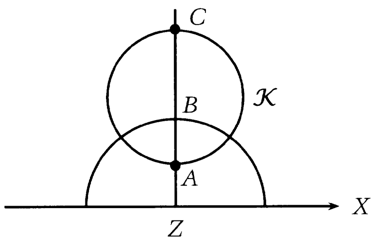


**FIGURE 3.16** Perpendicular diameters of a non-Euclidean circle.

The most important properties of reflection in a circle are that it preserves circles and angles, hence the angles and circles in the non-Euclidean plane look the same as Euclidean angles and circles. (Except that non-Euclidean angles usually occur between circles rather than between Euclidean straight lines.) We shall assume these facts in the exercises that follow. Their aim is to construct any non-Euclidean circle, given its non-Euclidean center $B$ and a point $A$ on its circumference. The simplest case is shown in Figure 3.16, where $A$ and $B$ are on a vertical Euclidean line.

3.9.2. Show that the reflection $C$ of $A$ in the semicircle through $B$ with center at $Z$ is the point opposite $A$ on the non-Euclidean circle $\mathcal{K}$ with radius $BA$. Hence give a ruler and compass construction of $\mathcal{K}$.

The general case is where $A$ and $B$ are not on the same vertical line (Figure 3.17). In this case we first construct the "line" $AB$ (semicircle through $A$ and $B$).

3.9.3. Describe a ruler and compass construction of the "line" $\mathcal{L}$ through $B$ perpendicular to the "line" $AB$.