For example, if Δ is a triangle in the plane $\mathbb{R}^2$, and G consists of all transformations of $\mathbb{R}^2$ that preserve length, then $\{g(\Delta) : g \in G\}$ consists of all triangles congruent to Δ. This example shows that members of the same G-orbit are "equivalent" in a sense that depends on G. In fact, we always get an equivalence relation from a group in this way. Here is another example.

19.5.1 If $G = \{\text{similarities of } \mathbb{R}^2\}$, what is $\{g(\Delta) : g \in G\}$ for a triangle Δ?

For any group G of transformations, we define a relation $X \cong_G Y$ ("X is G-equivalent to Y") between subsets X, Y of S by

$$X \cong_G Y \quad \Longleftrightarrow \quad X \text{ is in the } G\text{-orbit of } Y.$$

Then the group properties of G imply the following properties of the relation $\cong_G$.

19.5.2 Show that the relation $\cong_G$ has the properties

$$\begin{aligned} & X \cong_G X, && \text{(reflexive)} \\ X \cong_G Y \quad &\Longrightarrow \quad Y \cong_G X, && \text{(symmetric)} \\ X \cong_G Y \text{ and } Y \cong_G Z \quad &\Longrightarrow \quad X \cong_G Z. && \text{(transitive)} \end{aligned}$$

19.5.3 At which points does your solution of Exercise 19.5.2 involve the existence of an identity, existence of inverses, and existence of products in G?

The properties in Exercise 19.5.2 show that $\cong_G$ is an *equivalence relation*, according to the definition in the exercises for Section 2.1. It was also noted there that the reflexive and transitive properties actually imply symmetry, provided that transitivity is stated in the manner of Euclid's Common Notion 1: "Things equivalent to the same thing are equivalent to each other."

19.5.4 Prove Common Notion 1 for $\cong_G$:

$$X \cong_G Y \text{ and } Z \cong_G Y \quad \Longrightarrow \quad X \cong_G Z.$$

You will notice that this proof involves inverses, which previously were needed only to prove symmetry. This confirms that Euclid's Common Notion 1 is in some sense a combination of both transitivity and symmetry.

19.6 Combinatorial Group Theory

As mentioned in Section 19.4, the groups of the regular polyhedra were the first to be defined in terms of generators and relations. With finite groups such as these, however, one is concerned mainly with the simplicity and elegance of a presentation; the question of *existence* does not arise.

For any finite group G one can trivially obtain a finite set of generators (namely, *all* the elements $g_1, \ldots, g_n$ of G) and defining relations (namely, all equations $g_i g_j = g_k$ holding among the generators). Of course the same argument gives an infinite set of generators and defining relations for an infinite group, but this is also not interesting. The real problem is to find finite sets of generators and defining relations for infinite groups where possible.

This problem was first solved for the symmetry groups of certain regular tessellations, and such examples were the basis of the first systematic study of generators and relations, by Klein's student Dyck. Dyck's papers (1882, 1883) laid the foundations of this approach to group theory, now called *combinatorial*. For more technical information, as well as detailed history of the development of combinatorial group theory, see Chandler and Magnus (1982).

Figure 19.5 illustrates how generators and relations arise naturally from tessellations. This tessellation is based on the regular tessellation of the Euclidean plane by unit squares, but each square has been subdivided into black and white triangles to eliminate symmetries by rotation and reflection. The symmetries that remain are generated by

1. horizontal translation of length 1
2. vertical translation of length 1

These generators are subject to the obvious relation

$$ab = ba,$$

which implies that any element of the group can be written in the form $a^m b^n$. If $g = a^{m_1} b^{n_1}$ and $h = a^{m_2} b^{n_2}$, then $g = h$ only if $m_1 = m_2$ and $n_1 = n_2$, that is, only if $g = h$ is a *consequence* of the relation $ab = ba$. Thus all relations $g = h$ in the group follow from $ab = ba$, which means that the latter relation is a defining relation of the group.

The obviousness of the defining relation in this case blinds us to a fact that becomes more evident with tessellations of the hyperbolic plane: *the generators and relations can be read off the tessellation.* Group elements correspond to cells in the tessellation, squares in the present example. If we fix the square corresponding to the identity element 1, then the square to which square 1 is sent by the group element g may be called square g. The generators $a^{\pm 1}$, $b^{\pm 1}$ are the elements that send square 1 to adjacent

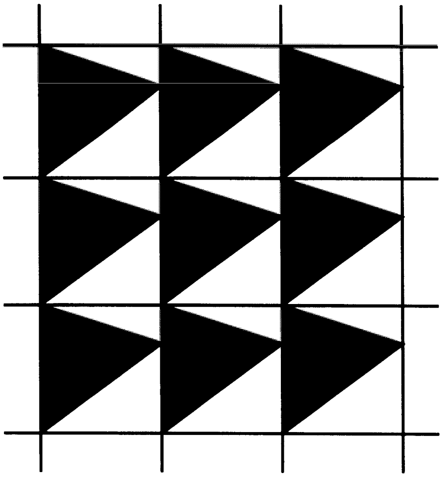

Figure 19.5: A tessellation of the plane

squares. They generate the group because square 1 can be sent to any other square by a series of moves from square to adjacent square. Relations correspond to equal sequences of moves or, what amounts to the same thing, to sequences of moves that return square 1 to its starting position. These sequences can all be derived from a circuit around a vertex (Figure 19.6), that is, the sequence $aba^{-1}b^{-1}$. Thus all relations are derived from $aba^{-1}b^{-1} = 1$, or, equivalently, $ab = ba$.

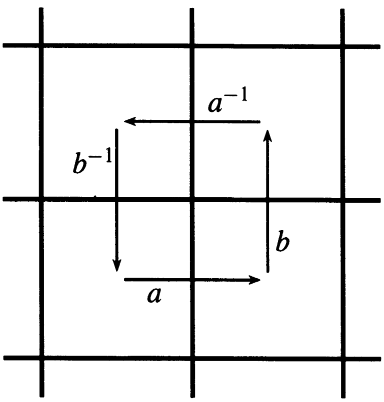

Figure 19.6: Circuit around a vertex

Generalizing these ideas, Poincaré (1882) showed that the symmetry groups of all regular tessellations, whether of the sphere, Euclidean plane, or hyperbolic plane, can be represented by finitely many generators and relations. Generators correspond to moves of the basic cell to adjacent cells, and hence to the sides of the basic cell; defining relations correspond to its vertices. These results are also important for topology, as we shall see in Chapter 22.

The notion of group abstracted from such examples was expressed in a somewhat technical way, involving normal subgroups, by Dyck (1882). The following, simpler, approach was worked out by Dehn and used by Dehn's student Magnus (1930). A group G is defined by a set $\{a_1, a_2, \ldots,\}$ of *generators* and a set $\{W_1 = W_1', W_2 = W_2', \ldots,\}$ of *defining relations*. Each generator a_i is called a *letter*; a_i has an *inverse* a_i^{-1}, and arbitrary finite sequences ("products") of letters and inverse letters are called *words*.

Words W, W' are called *equivalent* if $W = W'$ is a consequence of the defining relations, that is, if W may be converted to W' by a sequence of replacements of subwords W_i by W_i' (or vice versa) and cancellation (or insertion) of subwords $a_i a_i^{-1}$, $a_i^{-1} a_i$. The elements of G are the equivalence classes

$$[W] = \{W' : W' \text{ is equivalent to } W\},$$

and the product of elements $[U], [V]$ is defined by

$$[U][V] = [UV],$$

where UV denotes the result of concatenating the words U, V. It has to be checked that this product is well defined, but once this is done, the group properties (i), (ii), and (iii) of Section 19.1 follow easily.

EXERCISES

Here is how one verifies that the classes $[W]$ have the group properties.

19.6.1 If U is equivalent to U', show that UV is equivalent to $U'V$. Conclude, using this and a similar result for V', that the product $[U][V]$ is independent of the choice of representatives for $[U], [V]$.

19.6.2 $[U]([V][W]) = ([U][V])[W]$ is trivial. Why?

19.6.3 Show that $1 =$ equivalence class of the empty word.

19.6.4 Show that $[W]^{-1} = [W^{-1}]$, where W^{-1} is the result of writing W backward and changing the sign of each exponent.

19.7 Biographical Notes: Galois

Evariste Galois (Figure 19.7) was born in the town of Bourg-la-Reine, near Paris, in 1811, and died, from wounds received in a duel, in Paris in 1832. The tragedy and mystery of his short life make him the most romantic figure in mathematics, and several biographers have been tempted to cast Galois in the role of misunderstood genius and victim of the Establishment. However, it has been amply documented by Rothman (1982) that Galois does not easily fit this role. Though the known facts of his life should satisfy anyone's appetite for drama, his tragedy is of the more classical kind, whose seeds lie in the character of the victim himself.

Figure 19.7: Evariste Galois at the age of 15

Galois was the second of three children of Nicholas-Gabriel Galois, the director of a boarding school and later mayor of Bourg-la-Reine, and Adelaïde-Marie Demante, who came from a family of jurists. Both parents were well educated, and Galois seems to have had a happy, if unconventional, childhood. Up to the age of 12, he was educated entirely by his mother, a severe classicist who instilled in him a knowledge of Latin and Greek and a respect for Stoic morality. His father was far less of a stoic, but unconventional in a different way, being a republican at a time when France was returning to the monarchy. In October 1823 Evariste entered the Lycée of Louis-le-Grand, a well-known school whose pupils had included Robespierre and Victor Hugo and would later include the mathematician Charles Hermite, who found the transcendental solution to the quintic equation. There does not seem to have been any mathematics in Galois' family background, and he did not begin studying it at school until February 1827. To make the progress he did, he must have devoured mathematics at a greater rate than anyone in history, except perhaps Newton in 1665–6, so it is no wonder that for the first time his school reports noted unsatisfactory progress in other subjects. Comments about his character being "singular" and "closed" but "original" also began to be made. At this stage Galois was studying Legendre's *Geometry* and Lagrange's works on the theory of equations and analytic functions. He believed he was ready to enter the École Polytechnique but, due to his lack of preparation in the standard syllabus, failed the entrance examination.

In 1828 he had the good fortune to study mathematics under a teacher who recognized his genius, Louis-Paul-Emile Richard. This led to Galois' first publication, a paper on continued fractions that appeared in the *Annales* of Gergonne in March 1829. Thanks to Richard, many pieces of Galois' early work still exist and have been published in Bourgne and Azra (1962). They include class papers saved by Richard and later preserved for posterity by Hermite. One fragment from 1828 shows that Galois, like Abel, initially believed he could solve the quintic equation.

One might think that publication in a respected journal was reasonable encouragement for a 17-year-old mathematician, but it was not enough for Galois. He nursed a grudge against the examiners of the École Polytechnique for failing him, and he was supported by Richard, who declared that he should be admitted without examination. Needless to say, this did not happen, but worse disappointments were to follow.

Galois had already begun working on his theory of equations and submitted his first paper on the subject to the Paris Academy in May 1829. Cauchy was referee and even seemed to be favorably impressed [see Rothman (1982), p. 89], but months went by and the paper failed to appear. Then, in July 1829, Galois' father committed suicide. The cause was trivial, even childish—a spiteful attack on him by the priest of Bourg-la-Reine—but it unleashed political passions with which Galois senior could not cope. Nor could Evariste cope with the loss of his father. His distrust of the political and educational establishment deepened into paranoia, and the sacrifice of his own life must suddenly have seemed a real possibility. It was almost the last straw when, a few days after his father's death, he failed the entrance examination for the École Polytechnique a second time.

Despite these crushing blows, Galois persevered with examinations and succeeded in entering the less prestigious École Normale in November 1829. In early 1830 he got his theory of equations into print (though not through the Academy) with the publication of three papers. The more decisive event of 1830, however, was the July revolution against the Bourbon monarchy. It gave Galois the ideal focus for his rage over the death of his father and his own humiliations, and he emerged as a republican firebrand. He made friends with the republican leaders Blanqui and Raspail and began political agitation at the École Normale—until he was expelled in December 1830 for an article he wrote against its director. In the same month, the Bourbons fled France and, as mentioned in Section 16.7, Cauchy fled with them.

Immediately after leaving the École Normale, Galois joined the Artillery of the National Guard, a republican stronghold, to concentrate on revolutionary activity. At a republican banquet on May 9, 1831, he proposed a toast with a dagger in his hand, implying a threat against the life of the new king, Louis-Philippe. Galois was arrested the following day and held until June 15 in Sainte-Pélagie prison. He was then tried for threatening the life of the king, but he was acquitted almost immediately, evidently on the grounds that he was young and foolish. The acquittal was an act of considerable leniency, as Galois gave full vent to his opinions during the trial. He admitted that he still intended to kill the king "if he betrays" and added his view that the king "will soon turn traitor if he has not done so already."

Galois was arrested a second time on Bastille Day 1831, for illegal possession of weapons and for wearing the uniform of the Artillery Guard (which had been disbanded at the end of 1830). He was held in Sainte-Pélagie prison until October and then sentenced to a further six months. Galois became very despondent and once, thinking of his father, attempted suicide. Thus he was not in a receptive mood when he finally heard from the Academy—that they were returning his manuscript—even though he was invited to submit a more complete account of his theory. Galois did in fact begin to revise his work, but he poured most of his energy into the preface, a scorching condemnation of the scientific establishment and Academicians in particular "who already have the death of Abel on their consciences." The last six weeks of his imprisonment were spent in a nursing home. Some prisoners were transferred there as a measure against cholera, which was then epidemic in Paris. In these relatively pleasant surroundings, Galois resumed his research and managed to write a few philosophical essays.

He was released on April 29, 1832. Frustratingly little is known about the next, final, month of his life. He wrote to his friend Chevalier on May 25, expressing his complete disenchantment with life and hinting that a broken love affair was the reason. It appears that the woman was Stéphanie Dumotel, daughter of the resident physician at the nursing home. Two letters from her to Galois exist, though they are defaced (presumably by Galois himself) so as to be only partly readable. One, dated May 14, says "Please let us break up this affair." The other mentions sorrows someone else had caused her, in such a way that Galois might have felt obliged to come to her defense. Whether this was the cause of the fatal duel we do not know. It is also possible that Galois felt the duel had been hanging over his head for a long time. When he first went to prison in 1831, one of his comrades was Raspail, who, in a letter from prison on July 25 that year, quoted Galois as follows: "And I tell you, I will die in a duel over some low class coquette. Why? Because she will invite me to avenge her honour which another has compromised" [Raspail (1839), p. 89]. In letters he wrote to friends on the night before the duel, Galois again spoke of an "infamous coquette."

He also wrote: "Forgive those who kill me for they are of good faith." His opponent was in fact a fellow republican, Pescheux d'Herbinville. Authors who like conspiracy theories have since conjectured that d'Herbinville was really a police agent, but no evidence exists for this. His revolutionary credentials were as good as those of Galois. The police agent theory seems rather to reflect twentieth-century bafflement over dueling, something we no longer understand or sympathize with (though we still applaud successful duelists, such as Bolyai and Weierstrass). There may be *no* rational explanation for the duel, but no doubt the suicide of his father and Galois' own self-destructive tendencies were among the conditions that made it possible. Galois was convinced he was going to die over something small and contemptible, and the tragedy is that he let it happen.

The tragedy for mathematics was the incompleteness of Galois' work at the time of his death. The night before the duel, he wrote a long letter to Chevalier outlining his discoveries and hoping "some men will find it profitable to sort out this mess." Chevalier and Alfred Galois (Evariste's younger brother) later copied the mathematical papers and sent them to Gauss and Jacobi, but there was no response. The first to study them conscientiously was Liouville, who became convinced of their importance in 1843 and arranged to have them published. They finally appeared in 1846, and by the 1850s the algebraic part of the theory began to creep into textbooks. But, as mentioned in Section 19.2, there was more. Galois also talked of connections between algebraic equations and transcendental functions and made a cryptic reference to a "theory of ambiguity." The latter probably concerned the many-valuedness of algebraic functions, and we may be fairly sure that whatever Galois did was later superseded by Riemann. As for the transcendental functions, we also know that Hermite (1858) successfully completed one of Galois' investigations into solving the quintic equation by means of elliptic modular functions, and that Jordan (1870) exposed the group theory governing the behavior of such functions. However, these results only scratch the surface, and it is still possible that a bigger "Galois theory" remains to be discovered.