Corollary 25. If A is an $n \times n$ matrix with entries from F and F contains all the eigenvalues of A, then A is similar to a diagonal matrix over F if and only if the minimal polynomial of A has no repeated roots.

Proof: Suppose A is similar to a diagonal matrix. The minimal polynomial of a diagonal matrix has no repeated roots (its roots are precisely the distinct elements along the diagonal). Since similar matrices have the same minimal polynomial it follows that the minimal polynomial for A has no repeated roots.

Conversely, suppose the minimal polynomial for A has no repeated roots and let B be the Jordan canonical form of A. The matrix B is a block diagonal matrix with elementary Jordan matrices down the diagonal. By the exercises at the end of the preceding section the minimal polynomial for B is the least common multiple of the minimal polynomials of the Jordan blocks. It is easy to see directly that a Jordan block of size k with eigenvalue λ has minimal polynomial $(x-\lambda)^k$ (note that this is immediate from the fact that each elementary Jordan matrix gives the action on a *cyclic* $F[x]$-submodule whose annihilator is $(x-\lambda)^k$). Since A and B have the same minimal polynomial, the least common multiple of the $(x-\lambda)^k$ cannot have any repeated roots. It follows that k must be 1, i.e., that each Jordan block must be of size one and B is a diagonal matrix.

Changing From One Canonical Form to Another

We continue to assume that the field F contains all the eigenvalues of T (or A) so both the rational and Jordan canonical forms exist over F. The process of passing from one form to the other is exactly the same algorithm described in Section 5.2 for finite abelian groups (where the elementary divisors were determined from the list of invariant factors and vice versa).

In brief summary, recall that the elementary divisors are the prime power divisors of the invariant factors. They are obtained from the invariant factors by writing each invariant factor as a product of distinct linear factors to powers; the resulting set of powers of linear polynomials is the set of elementary divisors. For example, if the invariant factors of T are

$$(x-1)(x-3)^3, \quad (x-1)(x-2)(x-3)^3, \quad (x-1)(x-2)^2(x-3)^3$$

then the elementary divisors are

$$(x-1), \quad (x-3)^3, \quad (x-1), \quad (x-2), \quad (x-3)^3, \quad (x-1), \quad (x-2)^2, \quad (x-3)^3.$$

The largest invariant factor is the product of the largest of the distinct prime powers among the elementary divisors, the next largest invariant factor is the product of the largest of the distinct prime powers among the remaining elementary divisors, and so on. Given a list of elementary divisors we can find the list of invariant factors by first arranging the elementary divisors into n separate lists, one for each eigenvalue. In each of these n lists arrange the polynomials in increasing (i.e., nondecreasing) degree. Next arrange for all n lists to have the same length by appending an appropriate number of the constant polynomial 1. Now form the i^{th} invariant factor by taking the product of the i^{th} polynomial in each of these lists. For example, if the elementary divisors of T are

$$(x-1)^3,\ (x+4),\ (x+4)^2,\ (x-5)^2,\ (x-1)^5,\ (x-1)^3,\ (x-5)^3,\ (x-1)^4,\ (x+4)^3$$

then the intermediate lists are

$$\begin{array}{lllll} (1) & (x-1)^3, & (x-1)^3, & (x-1)^4, & (x-1)^5 \\ (2) & 1, & x+4, & (x+4)^2, & (x+4)^3 \\ (3) & 1, & 1, & (x-5)^2, & (x-5)^3 \end{array}$$

so the list of invariant factors is

$$(x-1)^3, \quad (x-1)^3(x+4), \quad (x-1)^4(x+4)^2(x-5)^2, \quad (x-1)^5(x+4)^3(x-5)^3.$$

Elementary Divisor Decomposition Algorithm: Converting to Jordan Canonical Forms

Theorem 21 indicates a computational procedure to determine the invariant factors of any given matrix A. Factorization of these invariant factors produces the elementary divisors of A, hence determines the Jordan canonical form for A as above.

The Invariant Factor Decomposition Algorithm following Theorem 21 starts with a basis $e_1, \dots, e_n$ for V and produces a set $f_1, \dots, f_m$ of elements of V which are $F[x]$-module generators for the cyclic factors in the invariant factor decomposition of V (with annihilators $(a_1(x)), \dots, (a_m(x))$, respectively). Since the elementary divisor decomposition is obtained from the invariant factor decomposition by applying the Chinese Remainder Theorem to the cyclic modules $F[x]/(a_i(x))$, this gives a set of $F[x]$-module generators for the cyclic factors in the elementary divisor decomposition of V. These elements then give rise to an explicit vector space basis for V with respect to which the linear transformation corresponding to A is in Jordan canonical form (equivalently, an explicit matrix P such that $P^{-1}AP$ is in Jordan canonical form). As for the Invariant Factor Decomposition Algorithm we state the result first in the general context of decomposing a vector space and then describe the algorithm to convert a given $n \times n$ matrix A to Jordan canonical form.

Explicit numerical examples of this algorithm are given later in Examples 2 and 3.

Elementary Divisor Decomposition Algorithm

(1) to (3): The first three steps in the algorithm are those from the Invariant Factor Decomposition Algorithm following Theorem 21.

(4) For each invariant factor $a(x)$ computed for A write

$$a(x) = (x-\lambda_1)^{\alpha_1}(x-\lambda_2)^{\alpha_2}\dots(x-\lambda_s)^{\alpha_s}$$

where $\lambda_1, \dots, \lambda_s \in F$ are distinct. Let $f \in V$ be the $F[x]$-module generator for the cyclic factor corresponding to the invariant factor $a(x)$ computed in (3). Then the elements

$$\frac{a(x)}{(x-\lambda_1)^{\alpha_1}}f, \quad \frac{a(x)}{(x-\lambda_2)^{\alpha_2}}f, \quad \dots, \quad \frac{a(x)}{(x-\lambda_s)^{\alpha_s}}f$$

(note that the $\dfrac{a(x)}{(x-\lambda_i)^{\alpha_i}} \in F[x]$ are polynomials) are $F[x]$-module generators for the cyclic factors of V corresponding to the elementary divisors

$$(x-\lambda_1)^{\alpha_1}, \quad (x-\lambda_2)^{\alpha_2}, \quad \ldots, \quad (x-\lambda_s)^{\alpha_s},$$

respectively.

(5) If $g_i = \dfrac{a(x)}{(x-\lambda_i)^{\alpha_i}} f$ is the $F[x]$-module generator for the cyclic factor of V corresponding to the elementary divisor $(x-\lambda_i)^{\alpha_i}$ then the corresponding *vector space* basis for this cyclic factor of V is given by the elements

$$(T-\lambda_i)^{\alpha_i-1}g_i, \quad (T-\lambda_i)^{\alpha_i-2}g_i, \quad \ldots, \quad (T-\lambda_i)g_i, \quad g_i.$$

(6) Write the k^{th} element of the vector space basis computed in (5) in terms of the original vector space basis $[e_1, e_2, \ldots, e_n]$ for V and use the coordinates for the k^{th} column of an $n \times n$ matrix P. Then $P^{-1}AP$ is in Jordan canonical form (with Jordan blocks appearing in the order used in (5) for the cyclic factors of V).

Converting an $n \times n$ Matrix to Jordan Canonical Form

(1) to **(2)**: The first two steps are those from the algorithm for Converting an $n \times n$ matrix to Rational Canonical Form following Theorem 21.

(3) When $xI - A$ has been diagonalized to the form in Theorem 21 the first $n-m$ columns of the matrix P' are 0 (providing a useful numerical check on the computations) and the remaining m columns of P' are nonzero. For each successive $i = 1, 2, \ldots, m$:

(a) Factor the i^{th} nonconstant diagonal element (which is of degree d_i):

$$a(x) = (x-\lambda_1)^{\alpha_1}(x-\lambda_2)^{\alpha_2}\ldots(x-\lambda_s)^{\alpha_s}$$

where $\lambda_1, \ldots, \lambda_s \in F$ are distinct (here $a(x) = a_i(x)$ is the i^{th} nonconstant diagonal element and s depends on i).

(b) Multiply the i^{th} nonzero column of P' successively by the d_i matrices:

$$\begin{array}{lll}
(A-\lambda_1 I)^{\alpha_1-1}(A-\lambda_2 I)^{\alpha_2} & & \ldots(A-\lambda_s I)^{\alpha_s} \\
(A-\lambda_1 I)^{\alpha_1-2}(A-\lambda_2 I)^{\alpha_2} & & \ldots(A-\lambda_s I)^{\alpha_s} \\
\qquad\vdots & & \\
(A-\lambda_1 I)^{0} & (A-\lambda_2 I)^{\alpha_2} & \ldots(A-\lambda_s I)^{\alpha_s} \\
 & & \\
(A-\lambda_1 I)^{\alpha_1} & (A-\lambda_2 I)^{\alpha_2-1} & \ldots(A-\lambda_s I)^{\alpha_s} \\
(A-\lambda_1 I)^{\alpha_1} & (A-\lambda_2 I)^{\alpha_2-2} & \ldots(A-\lambda_s I)^{\alpha_s} \\
 & \qquad\vdots & \\
(A-\lambda_1 I)^{\alpha_1} & (A-\lambda_2 I)^{0} & \ldots(A-\lambda_s I)^{\alpha_s} \\
 & \qquad\vdots &
\end{array}$$

$$(A-\lambda_1 I)^{\alpha_1}(A-\lambda_2 I)^{\alpha_2}\dots(A-\lambda_s I)^{\alpha_s-1}$$
$$(A-\lambda_1 I)^{\alpha_1}(A-\lambda_2 I)^{\alpha_2}\dots(A-\lambda_s I)^{\alpha_s-2}$$
$$\vdots$$
$$(A-\lambda_1 I)^{\alpha_1}(A-\lambda_2 I)^{\alpha_2}\dots(A-\lambda_s I)^{0}.$$

(c) Use the column vectors resulting from (b) (in that order) as the next d_i columns of an $n \times n$ matrix P.

Then $P^{-1}AP$ is in Jordan canonical form (whose Jordan blocks correspond to the ordering of the factors in (a)).

Examples

We can use Jordan canonical forms to carry out the same analysis of matrices that we did as examples of the use of rational canonical forms. In some instances, when the field is enlarged, the number of similarity classes increases (the number of similarity classes can never decrease when we extend the field by Corollary 18(2)).

(1) Let A, B and C be the matrices in Example 1 of the previous section and let $F = \mathbb{Q}$. Note that $\mathbb{Q}$ contains all the eigenvalues for these matrices. Since we have already determined the invariant factors of these matrices we can immediately obtain their elementary divisors. The elementary divisors of A are $x-2$, $x-2$ and $x-3$ and the elementary divisors of B and C are $(x-2)^2$ and $x-3$ so the respective Jordan canonical forms are:

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \qquad \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \qquad \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Notice that A is similar to a diagonal matrix but, by Corollary 25, B and C are not.

(2) For the matrix A, we determined in Example 2 of the previous section that $f_1 = -7e_1 + 7e_2 + e_3$ and $f_2 = -e_1 + e_2$ were $\mathbb{Q}[x]$-module generators for the two cyclic factors of V in its invariant factor decomposition, corresponding to the invariant factors $x-2$ and $(x-2)(x-3)$, respectively. Using the first algorithm described above, the elements f_1, $(x-3)f_2$ and $(x-2)f_2$ are therefore $\mathbb{Q}[x]$-module generators for the three cyclic factors of V in its elementary divisor decomposition, corresponding to the elementary divisors $x-2$, $x-2$, and $x-3$. An easy computation shows that these are the elements $-7e_1 + 7e_2 + e_3$, $-e_1$ and $-2e_1 + e_2$, respectively. Then the matrix

$$P = \begin{pmatrix} -7 & -1 & -2 \\ 7 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

conjugates A into its Jordan canonical form:

$$P^{-1}AP = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix},$$

as one easily checks.

The columns of this matrix can also be obtained following the second algorithm above, using the nonzero columns of the matrix P' computed in Example 2 of the