until they surface in the last nonzero remainder. The last nonzero remainder is thus the gcd of the original integers. The above calculations can be rewritten as follows:

$$\frac{1604}{502} = 3 + \frac{1}{\frac{502}{98}} = 3 + \frac{1}{5 + \frac{12}{98}} = 3 + \frac{1}{5 + \frac{1}{\frac{98}{12}}} = 3 + \frac{1}{5 + \frac{1}{8 + \frac{2}{12}}} = 3 + \frac{1}{5 + \frac{1}{8 + \frac{1}{6}}}$$

The final expression is a *(simple) continued fraction.*

Standard abbreviations for continued fractions allow us to write the above fraction as $3 + \frac{1}{5+} \frac{1}{8+} \frac{1}{6}$ or as (3,5,8,6). Thus (a_0, a_1, a_2) is the fraction

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2}} \, .$$

When one divides by a positive integer, one always obtains a remainder which is less than the divisor. Thus the sequence of divisors in Euclid's Algorithm steadily decreases. Using Euclid's Algorithm, we can write any positive rational number as a finite continued fraction $(a_0, a_1, a_2, \ldots, a_n)$, where the a_i are natural numbers and, for $i > 0$, $a_i > 0$.

However, even irrational real numbers can be written as continued fractions, and this too was known to the ancient Greeks, according to Fowler. We then obtain infinite continued fractions $(a_0, a_1, a_2, \ldots)$; but these may be approximated by finite continued fractions $c_0 = a_0$, $c_1 = (a_0, a_1)$, $c_2 = (a_0, a_1, a_2)$, etc., called *convergents*. Ultimately we are interested in the case when the a_n are all positive integers, with the possible exception of a_0. For the following argument only, we shall allow the a_n to be positive rational numbers for $n > 0$.

It is easily seen that $c_0 = \frac{a_0}{1}$ and $c_1 = \frac{a_0 a_1 + 1}{a_1}$. To calculate c_n when $n > 1$ we define two sequences of rationals (ultimately integers):

$$p_0 = a_0, \ p_1 = a_0 a_1 + 1, \ p_n = a_n p_{n-1} + p_{n-2} \text{ if } n > 1,$$

$$q_0 = 1, \ q_1 = a_1, \ q_n = a_n q_{n-1} + q_{n-2} \text{ if } n > 1.$$

We claim that $c_n = \frac{p_n}{q_n}$ for all n. This is evidently so when $n = 0$ or 1, so assume the result for n for any admissible $(a_0, a_1, \ldots, a_n)$; we shall show that it also holds for $n + 1$. Now c_{n+1} can be obtained from

$$c_n = \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}}$$

by replacing a_n by $a_n + \frac{1}{a_{n+1}}$, with the help of the induction assumption applied to $(a_0, \ldots, a_n + \frac{1}{a_{n+1}})$. Multiplying top and bottom of the resulting

ratio by a_{n+1}, the top becomes

$$
\begin{aligned}
a_{n+1}\left(\left(a_n + \frac{1}{a_{n+1}}\right)p_{n-1} + p_{n-2}\right) &= a_{n+1}(a_n p_{n-1} + p_{n-2}) + p_{n-1} \\
&= a_{n+1}p_n + p_{n-1} = p_{n+1}.
\end{aligned}
$$

Similarly, the bottom becomes q_{n+1}, so $c_{n+1} = p_{n+1}/q_{n+1}$, and therefore the result holds by mathematical induction.

The proof involved the rational number $a_n + \frac{1}{a_{n+1}}$ in the induction assumption, but from now on we shall stick to integer a_n. We have proved:

Theorem 14.1. *If $a_0 \in \mathbf{N}$ and $0 < a_n \in \mathbf{N}$ for $n > 0$, the nth convergent of $(a_0, a_1, a_2, \ldots)$ is p_n/q_n, where the p_n and q_n are defined inductively as above.*

Note that, if the a_n are all natural numbers, as we are now supposing, the inductively defined p_n and q_n will be positive integers, in fact, strictly increasing sequences of positive integers.

Theorem 14.2. *Let p_n and q_n be defined inductively as above. For all $n > 0$, $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$.*

Proof: This is evidently so when $n = 1$ or 2. Assume the result for n. Then

$$
\begin{aligned}
p_{n+1}q_n - p_n q_{n+1} &= (a_{n+1}p_n + p_{n-1})q_n - p_n(a_{n+1}q_n + q_{n-1}) \\
&= p_{n-1}q_n - p_n q_{n-1} \\
&= -(-1)^{n-1} \\
&= (-1)^n.
\end{aligned}
$$

An immediate consequence of this theorem is

Corollary 14.3. *For all $n \in \mathbf{N}$, $\gcd(p_n, q_n) = 1$; in other words, p_n/q_n is in lowest terms.*

Another immediate consequence is

Corollary 14.4. *For all $n > 0$, $c_n - c_{n-1} = \frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{(-1)^{n-1}}{q_n q_{n-1}}$.*

It follows that, for odd n,

$$
c_n - c_{n-2} = \frac{1}{q_n q_{n-1}} - \frac{1}{q_{n-2}q_{n-1}} < 0,
$$

since the q_n are strictly increasing, and similarly, for even n, $c_n - c_{n-2} > 0$. Thus we have a strictly decreasing sequence

$$
c_1 > c_3 > c_5 > \cdots
$$

and a strictly increasing sequence

$$c_2 < c_4 < c_6 < \cdots.$$

Moreover, by Corollary 14.4, the difference of the two sequences tends to 0, hence they must have a common limit a and we write $a = (a_0, a_1, a_2, \ldots)$.

For example, the infinite continued fraction $(1, 1, 1, \ldots)$ has a limit x, hence $x = 1 + \frac{1}{x}$, so $x^2 - x - 1 = 0$, hence $x = \frac{1}{2}(1 + \sqrt{5})$.

Exercises

1. Find the gcd of 10403 and 2987.

2. Show that $\gcd(a, b) = \gcd(a, a - b)$.

3. Calculate $(1,2,1,2,1,2, \ldots)$.

4. Show that every positive rational number b/c can be written as a simple continued fraction with an even number of a_i.

5. Let b/c be a positive proper fraction in lowest terms. Using Exercise 4, show that b/c can be written in the form $1/d + e/f$ where $b > e$ and $d > c > f$.

6. Using Exercise 5, show that every proper reduced fraction can be expressed as a sum of distinct *unit fractions* (that is, fractions with numerator 1 and a positive integer denominator).

7. Use the method of Exercises 5 and 6 to express $67/120$ as a sum of distinct unit fractions.

8. Associate with each letter the number of its place in the (English) alphabet. Then each word is associated with a sequence $a_0, a_1, \ldots, a_n$. Encode the word into $(a_0, a_1, \ldots, a_n)$, simplified into an ordinary fraction. How would you decipher such a fraction?

9. Show that $(2a, a, 2a, a, 2a, a, \ldots) = a + \sqrt{a^2 + 2}$.