Proposition 46. For any commutative ring R with 1, let R_P be the localization of R at the prime ideal P and let ${}^eP$ be the extension of P to R_P.

(1) The ring R_P is a local ring with unique maximal ideal ${}^eP$. The contraction of ${}^eP$ to R is P, i.e., ${}^c({}^eP) = P$, and the map from R to R_P induces an injection of the integral domain R/P into $R_P/{}^eP$. The quotient $R_P/{}^eP$ is a field and is isomorphic to the fraction field of the integral domain R/P.

(2) If R is an integral domain, then R_P is an integral domain. The ring R injects into the local ring R_P, and, identifying R with its image in R_P, the unique maximal ideal of R_P is PR_P.

(3) The prime ideals in R_P are in bijective correspondence with the prime ideals of R contained in P.

(4) If P is a minimal nonzero prime ideal of R then R_P has a unique nonzero prime ideal.

(5) If $P = M$ is a maximal ideal and I is any M-primary ideal of R then $R_M/{}^eI \cong R/I$. In particular, $R_M/{}^eM \cong R/M$ and $({}^eM)/({}^eM)^n \cong M/M^n$ for all $n \geq 1$.

Proof: If P' is a prime ideal of R, then $P' \cap (R - P) = \emptyset$ if and only if $P' \subseteq P$, so (3) is immediate from (3) in Proposition 38, and (4) follows. Since ${}^eP \neq R_P$ by (2) of Proposition 38, it follows from (3) that R_P is a local ring with unique maximal ideal ${}^eP$, which proves the first statement in (1).

By Proposition 38(2) the contraction ${}^c({}^eP)$ is the set $\{r \in R \mid dr \in P \text{ for some } d \in R - P\}$, and since P is prime, $dr \in P$ with $d \notin P$ implies $r \in P$. This shows that ${}^c({}^eP) = P$, which is the second statement in (1).

The kernel of the map from R to $R_P/{}^eP$ is ${}^c({}^eP) = P$, so the induced map from R/P into $R_P/{}^eP$ is injective. The quotient $R_P/{}^eP$ is a field by the first part of (1), so there is an induced homomorphism from the fraction field of the integral domain R/P into $R_P/{}^eP$. The universal property of the localization R_P shows there is an inverse homomorphism from $R_P/{}^eP$ to the fraction field of R/P (since every element of R not in P maps to a unit in R/P). It follows that $R_P/{}^eP$ is isomorphic to the fraction field of R/P.

If R is an integral domain, then $R - P$ has no zero divisors, so R injects into R_P by Corollary 37; if R is identified with its image in R_P then ${}^eP = PR_P$, so (2) follows.

To prove (5), by Proposition 42(1) we may pass to the quotient R/I and so reduce to the case $I = 0$. In this case the maximal ideal $P = M$ in R is the nilradical of R, hence is the unique maximal ideal of R. By Proposition 45 every element of $R - M$ is a unit, so $R_P = R$, and each of the statements in (5) follows immediately, completing the proof of the proposition.

Example

The results of (5) of the proposition are not true in general if P is a prime ideal that is not maximal. For example, $P = (0)$ in $R = \mathbb{Z}$ has $R/P = \mathbb{Z}$ and $R_P/PR_P = \mathbb{Q}$; in this case $(PR_P)/(PR_P)^n \cong P/P^n = 0$ for all $n \geq 1$ (cf. the exercises).

Definition. Let M be an R-module, let P be a prime ideal of R and set $D = R - P$. The R_P-module $D^{-1}M$ is called the *localization of M at $P*, and is denoted by M_P.

By Proposition 41, M_P can also be identified with the tensor product $R_P \otimes_R M$. When R is an integral domain and $P = (0)$, then $M_{(0)}$ is a module over the field of fractions F of R, i.e., is a vector space over F.

The element $m/1$ is zero in M_P if and only if $rm = 0$ for some $r \in R - P$, so localizing at P annihilates the P'-torsion elements of M for primes P' not contained in P. In particular, *localizing at (0) over an integral domain annihilates the torsion subgroup of M.*

Definition. If R is an integral domain, then the *rank* of the R-module M is the dimension of the localization $M_{(0)}$ as a vector space over the field of fractions of R.

It is easy to see that this definition of rank agrees with the notion of rank introduced in Chapter 12.

Example

Let $R = \mathbb{Z}$ and let $\mathbb{Z}_{(p)}$ be the localization of $\mathbb{Z}$ at the nonzero prime ideal (p). Any abelian group M is a $\mathbb{Z}$-module so we may localize M at (p) by forming $M_{(p)}$. This abelian group is the same as the quotient of M with respect to the subgroup of elements whose order is finite and not divisible by p. If M is a finite (or, more generally, torsion) abelian group, then $M_{(p)}$ is a p-group, and is the Sylow p-subgroup or p-primary component of M. The localization $M_{(0)}$ of M at (0) is the trivial group. For a specific example, let $M = \mathbb{Z}/6\mathbb{Z}$ be the cyclic group of order 6, considered as a $\mathbb{Z}$-module. Then the localization of M at $p = 2$ is $\mathbb{Z}/2\mathbb{Z}$, at $p = 3$ is $\mathbb{Z}/3\mathbb{Z}$, and reduces to 0 at all other prime ideals of $\mathbb{Z}$.

Localization of a module M at a prime P in general produces a simpler module M_P whose properties are easier to determine. It is then of interest to translate these "local" properties of M_P back into "global" information about the module M itself. For example, the most basic question of whether a module M is 0 can be answered locally:

Proposition 47. Let M be an R-module. Then the following are equivalent:

(1) $M = 0$,
(2) $M_P = 0$ for all prime ideals P of R, and
(3) $M_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m}$ of R.

Proof: The implications (1) implies (2) implies (3) are obvious, so it remains to prove that (3) implies (1). Suppose m is a nonzero element in M, and consider the annihilator I of m in R, i.e., the ideal of elements $r \in R$ with $rm = 0$. Since m is nonzero I is a proper ideal in R. Let $\mathfrak{m}$ be a maximal ideal of R containing I and consider the element $m/1$ in the corresponding localization $M_{\mathfrak{m}}$ of M. If this element were 0, then $rm = 0$ for some $r \in R - \mathfrak{m}$. But then r would be an element in I not contained in $\mathfrak{m}$, a contradiction. Hence $M_{\mathfrak{m}} \neq 0$, which proves that (3) implies (1).

It is not in general true that a property shared by all of the localizations of a module M is also shared by M. For example, all of the localizations of a ring R can be integral domains without R itself being an integral domain (for example, $\mathbb{Z}/6\mathbb{Z}$ above). Nevertheless, a great deal of information *can* be ascertained from studying the various possible localizations, and this is what makes this technique so useful. If R is an integral domain, for example, then each of the localizations R_P can be considered as a subring of the fraction field F of R that contains R; the next proposition shows that the elements of R are the only elements of F contained in every localization.

Proposition 48. Let R be an integral domain. Then R is the intersection of the localizations of R: $R = \cap_P R_P$. In fact, $R = \cap_{\mathfrak{m}} R_{\mathfrak{m}}$ is the intersection of the localizations of R at the maximal ideals $\mathfrak{m}$ of R.

Proof: As mentioned, $R \subseteq \cap_{\mathfrak{m}} R_{\mathfrak{m}}$. Suppose now that a is an element of the fraction field F of R that is contained in $R_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ of R, and consider

$$I_a = \{d \in R \mid da \in R\}.$$

It is easy to check that I is an ideal of R, and that $a \in R$ if and only if $1 \in I_a$, i.e., $I_a = R$. Suppose that $I_a \neq R$. Then there is a maximal ideal $\mathfrak{m}$ containing I_a, and since $a \in R_{\mathfrak{m}}$ we have $a = r/d$ for some $r \in R$ and $d \in R - \mathfrak{m}$. But then $d \in I_a$ and $d \notin \mathfrak{m}$, a contradiction. Hence $a \in R$, so $\cap_{\mathfrak{m}} R_{\mathfrak{m}} \subseteq R$, and we have proved the second assertion in the proposition. The first is then immediate.

Another important property of a ring R that can be detected locally is normality:

Proposition 49. Let R be an integral domain. Then the following are equivalent:

(1) R is normal, i.e., R is integrally closed (in its field of fractions)
(2) R_P is normal for all prime ideals P of R
(3) $R_{\mathfrak{m}}$ is normal for all maximal ideals $\mathfrak{m}$ of R.

Proof: Let F be the field of fractions of R, so all of the various localizations of R may be considered as subrings of F.

Assume first that R is integrally closed and suppose $y \in F$ is integral over R_P. Then y is a root of a monic polynomial of degree n with coefficients of the form a_i/d_i for some $d_i \notin P$. The element $y' = y(d_0 d_1 \cdots d_{n-1})^n$ is then a root of a monic polynomial of degree n with coefficients from R, i.e., y' is integral over R. Since R is assumed normal, this implies $y' \in R$, and so $y = y'/(d_0 \cdots d_{n-1}) \in R_P$, which proves that (1) implies (2). The implication (2) implies (3) is trivial. Suppose now that $R_{\mathfrak{m}}$ is normal for all maximal ideals $\mathfrak{m}$ of R and let y be an element of F that is integral over R. Since $R \subseteq R_{\mathfrak{m}}$, y is in particular also integral over $R_{\mathfrak{m}}$ and so $y \in R_{\mathfrak{m}}$ for every maximal ideal by assumption. Then $y \in R$ by the previous proposition, which proves that (3) implies (1).

We now may easily prove the first part of the Going–up Theorem (cf. Section 3) that was used in the proof of Corollary 27.

Corollary 50. Let R be a subring of the commutative ring S with $1 \in R$, and assume that S is integral over R. If P is a prime ideal in R, then there is a prime ideal Q of S with $P = Q \cap R$.