CHAPTER 1 Arithmetic

1.1 The Natural Numbers

The beauty and fascination of numbers can be summed up by one simple fact: anyone can count $1, 2, 3, 4, \ldots$, but no one knows all the implications of this simple process. Let me elaborate. We all realize that the sequence $1, 2, 3, 4, \ldots$ continues $5, 6, 7, 8, \ldots$, and that we can continue *indefinitely* adding 1. The objects produced by the counting process are what mathematicians call the *natural numbers*. Thus if we want to say what it is that $1, 2, 3, 17, 643, 100097801$, and 4514517888888856 have in common, in short, what a natural number *is*, we can only say that each is produced by the counting process. This is slightly troubling when you think about it: the simplest, and most finite, mathematical objects are defined by an infinite process. However, the concept of *natural number* is inseparable from the concept of infinity, so we must learn to live with it and, if possible, use it to our advantage.

In fact, one of the most powerful methods in mathematics draws its strength from the infinite counting process. This is *mathematical induction*, which we usually just call *induction* for short. It may be formulated in several ways, each basically a restatement of the fact that any natural number can be reached by counting.

J. Stillwell, *Numbers and Geometry*

The first form of induction we consider (and apparently the first actually used) expresses the fact that from each natural number we can "count down" to 1, by finitely often subtracting 1. It follows that an infinite descending sequence of natural numbers is *impossible*. And *nonexistence* of natural numbers with certain properties often follows by hypothetical construction of an infinite descending sequence. This form of induction is called *infinite descent*,[1] or simply *descent*. Possibly the oldest example is the following, which goes back to around 500 B.C. To abbreviate the proof and show its simple logical structure, we use the symbol $\Rightarrow$ for "hence" or "implies."

The proof shows that no natural number square is twice another, but the result is better known as the "irrationality of $\sqrt{2}$."

Irrationality of $\sqrt{2}$ *There are no natural numbers m and n such that $m^2 = 2n^2$.*

Proof The hypothetical equation $m^2 = 2n^2$ leads to a similar equation, but with smaller numbers, as follows:

$$\begin{aligned} m^2 = 2n^2 &\Rightarrow m^2 \text{ even} \\ &\Rightarrow m \text{ even, say, } m = 2m_1 \\ &\Rightarrow 4m_1^2 = m^2 = 2n^2 \\ &\Rightarrow n^2 \text{ even} \\ &\Rightarrow n \text{ even, say, } n = 2n_1 \\ &\Rightarrow m_1^2 = 2n_1^2 \text{ and } m > m_1 > 0 \\ &\Rightarrow m_2^2 = 2n_2^2 \text{ and } m > m_1 > m_2 > 0, \text{ similarly,} \end{aligned}$$

and so on. Thus we get an infinite descending sequence, $m > m_1 > m_2 > \cdots$, which is impossible. Hence there are no natural numbers m and n such that $m^2 = 2n^2$. □

As most readers will know, $\sqrt{2}$ is defined as the number x such that $2 = x^2$. The proof shows that $\sqrt{2}$ is *not a ratio m/n* of natural numbers m, n, as this would imply $2 = m^2/n^2$ and hence $m^2 = 2n^2$. This is why $\sqrt{2}$ is called *irrational*; it simply means "not a ratio." As is typical when we wish to prove a negative statement, we argue by

[1]The unsettling experience of infinite descent has been used as the basis of a horror story by Marghanita Laski, called *The Tower*.

contradiction: the existence of a ratio $m/n = \sqrt{2}$ is shown to imply an impossibility.

But if $\sqrt{2}$ is not a ratio, what is it? Does it even exist? These questions have had an enormous influence on the development of mathematics, and their answers fill a large part of this book. For the moment, it is enough to say that, whatever the whole story of $\sqrt{2}$ may be, its irrationality is a fact about natural numbers.

Exercises

A problem at least as old as the meaning of $\sqrt{2}$, though less subtle, also leads to an interesting descent argument. Again, this is a case where a question about fractions reduces to a question about natural numbers.

About 4000 years ago, the Egyptians invented a curious arithmetic of fractions that depended on expressing each fraction between 0 and 1 as a sum of distinct *unit fractions*, that is, fractions of the form $\frac{1}{n}$. For example, $\frac{2}{3}$ is the sum of the unit fractions $\frac{1}{2}$ and $\frac{1}{6}$. Such sums are called *Egyptian fractions*. As another example, an Egyptian fraction for $\frac{3}{5}$ is $\frac{1}{2} + \frac{1}{10}$.

1.1.1. Express $\frac{4}{5}$, $\frac{9}{10}$, and $\frac{11}{12}$ as Egyptian fractions.

1.1.2. Find two different Egyptian fractions for $\frac{7}{12}$.

We do not know the Egyptian methods for finding such sums. They seem to involve many special tricks for avoiding unnecessarily large denominators, and it is difficult to capture them all in a rule that works in all cases. A more systematic approach was developed in the book *Liber Abacci*, written in 1202 by Leonardo of Pisa, better known as Fibonacci.

The method of the *Liber Abacci* also includes several tricks, but one of them can be used on its own to express any fraction between 0 and 1 as an Egyptian fraction. The trick is to repeatedly *remove the largest unit fraction*. Thus if $\frac{a}{b}$ is a (nonunit) fraction between 0 and 1, in lowest terms, let $\frac{1}{n}$ be the largest unit fraction less than $\frac{a}{b}$, and form the new fraction $\frac{a'}{b'} = \frac{a}{b} - \frac{1}{n}$.

1.1.3. Assuming $\frac{a'}{b'}$ is in lowest terms, show that $0 < a' < a$.

1.1.4. Hence conclude, by descent, that finitely many such removals split $\frac{a}{b}$ into distinct unit fractions.