The matrix A is the direct sum

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & A_k \end{bmatrix} \tag{7-27}$$

of matrices $A_1, \ldots, A_k$. Each A_i is of the form

$$A_i = \begin{bmatrix} J_1^{(i)} & 0 & \cdots & 0 \\ 0 & J_2^{(i)} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & J_{n_i}^{(i)} \end{bmatrix}$$

where each $J_j^{(i)}$ is an elementary Jordan matrix with characteristic value c_i. Also, within each A_i, the sizes of the matrices $J_j^{(i)}$ decrease as j increases. An $n \times n$ matrix A which satisfies all the conditions described so far in this paragraph (for some distinct scalars $c_1, \ldots, c_k$) will be said to be in **Jordan form.**

We have just pointed out that if T is a linear operator for which the characteristic polynomial factors completely over the scalar field, then there is an ordered basis for V in which T is represented by a matrix which is in Jordan form. We should like to show now that this matrix is something uniquely associated with T, up to the order in which the characteristic values of T are written down. In other words, if two matrices are in Jordan form and they are similar, then they can differ only in that the order of the scalars c_i is different.

The uniqueness we see as follows. Suppose there is some ordered basis for V in which T is represented by the Jordan matrix A described in the previous paragraph. If A_i is a $d_i \times d_i$ matrix, then d_i is clearly the multiplicity of c_i as a root of the characteristic polynomial for A, or for T. In other words, the characteristic polynomial for T is

$$f = (x - c_1)^{d_1} \cdots (x - c_k)^{d_k}.$$

This shows that $c_1, \ldots, c_k$ and $d_1, \ldots, d_k$ are unique, up to the order in which we write them. The fact that A is the direct sum of the matrices A_i gives us a direct sum decomposition $V = W_1 \oplus \cdots \oplus W_k$ invariant under T. Now note that W_i must be the null space of $(T - c_iI)^n$, where $n = \dim V$; for, $A_i - c_iI$ is clearly nilpotent and $A_j - c_iI$ is non-singular for $j \neq i$. So we see that the subspaces W_i are unique. If T_i is the operator induced on W_i by T, then the matrix A_i is uniquely determined as the rational form for $(T_i - c_iI)$.

Now we wish to make some further observations about the operator T and the Jordan matrix A which represents T in some ordered basis. We shall list a string of observations:

(1) Every entry of A not on or immediately below the main diagonal

is 0. On the diagonal of A occur the k distinct characteristic values $c_1, \ldots, c_k$ of T. Also, c_i is repeated d_i times, where d_i is the multiplicity of c_i as a root of the characteristic polynomial, i.e., $d_i = \dim W_i$.

(2) For each i, the matrix A_i is the direct sum of n_i elementary Jordan matrices $J_j^{(i)}$ with characteristic value c_i. The number n_i is precisely the dimension of the space of characteristic vectors associated with the characteristic value c_i. For, n_i is the number of elementary nilpotent blocks in the rational form for $(T_i - c_iI)$, and is thus equal to the dimension of the null space of $(T - c_iI)$. In particular notice that T is diagonalizable if and only if $n_i = d_i$ for each i.

(3) For each i, the first block $J_1^{(i)}$ in the matrix A_i is an $r_i \times r_i$ matrix, where r_i is the multiplicity of c_i as a root of the *minimal* polynomial for T. This follows from the fact that the minimal polynomial for the nilpotent operator $(T_i - c_iI)$ is x^{r_i}.

Of course we have as usual the straight matrix result. If B is an $n \times n$ matrix over the field F and if the characteristic polynomial for B factors completely over F, then B is similar over F to an $n \times n$ matrix A in Jordan form, and A is unique up to a rearrangement of the order of its characteristic values. We call A the **Jordan form** of B.

Also, note that if F is an algebraically closed field, then the above remarks apply to every linear operator on a finite-dimensional space over F, or to every $n \times n$ matrix over F. Thus, for example, every $n \times n$ matrix over the field of complex numbers is similar to an essentially unique matrix in Jordan form.

Example 5. Suppose T is a linear operator on C^2. The characteristic polynomial for T is either $(x - c_1)(x - c_2)$ where c_1 and c_2 are distinct complex numbers, or is $(x - c)^2$. In the former case, T is diagonalizable and is represented in some ordered basis by

$$\begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}.$$

In the latter case, the minimal polynomial for T may be $(x - c)$, in which case $T = cI$, or may be $(x - c)^2$, in which case T is represented in some ordered basis by the matrix

$$\begin{bmatrix} c & 0 \\ 1 & c \end{bmatrix}.$$

Thus every 2×2 matrix over the field of complex numbers is similar to a matrix of one of the two types displayed above, possibly with $c_1 = c_2$.

Example 6. Let A be the complex 3×3 matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ a & 2 & 0 \\ b & c & -1 \end{bmatrix}.$$

The characteristic polynomial for A is obviously $(x - 2)^2(x + 1)$. Either this is the minimal polynomial, in which case A is similar to

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

or the minimal polynomial is $(x - 2)(x + 1)$, in which case A is similar to

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Now

$$(A - 2I)(A + I) = \begin{bmatrix} 0 & 0 & 0 \\ 3a & 0 & 0 \\ ac & 0 & 0 \end{bmatrix}$$

and thus A is similar to a diagonal matrix if and only if $a = 0$.

Example 7. Let

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & a & 2 \end{bmatrix}.$$

The characteristic polynomial for A is $(x - 2)^4$. Since A is the direct sum of two 2×2 matrices, it is clear that the minimal polynomial for A is $(x - 2)^2$. Now if $a = 0$ or if $a = 1$, then the matrix A is in Jordan form. Notice that the two matrices we obtain for $a = 0$ and $a = 1$ have the same characteristic polynomial and the same minimal polynomial, but are not similar. They are not similar because for the first matrix the solution space of $(A - 2I)$ has dimension 3, while for the second matrix it has dimension 2.

Example 8. Linear differential equations with constant coefficients (Example 14, Chapter 6) provide a nice illustration of the Jordan form. Let $a_0, \ldots, a_{n-1}$ be complex numbers and let V be the space of all n times differentiable functions f on an interval of the real line which satisfy the differential equation

$$\frac{d^n f}{dx^n} + a_{n-1}\frac{d^{n-1}f}{dx^{n-1}} + \cdots + a_1\frac{df}{dx} + a_0 f = 0.$$

Let D be the differentiation operator. Then V is invariant under D, because V is the null space of $p(D)$, where

$$p = x^n + \cdots + a_1 x + a_0.$$

What is the Jordan form for the differentiation operator on V?

Let $c_1, \ldots, c_k$ be the distinct complex roots of p:

$$p = (x - c_1)^{r_1} \cdots (x - c_k)^{r_k}.$$

Let V_i be the null space of $(D - c_iI)^{r_i}$, that is, the set of solutions to the differential equation

$$(D - c_iI)^{r_i}f = 0.$$

Then as we noted in Example 15, Chapter 6 the primary decomposition theorem tells us that

$$V = V_1 \oplus \cdots \oplus V_k.$$

Let N_i be the restriction of $D - c_iI$ to V_i. The Jordan form for the operator D (on V) is then determined by the rational forms for the nilpotent operators $N_1, \ldots, N_k$ on the spaces $V_1, \ldots, V_k$.

So, what we must know (for various values of c) is the rational form for the operator $N = (D - cI)$ on the space V_c, which consists of the solutions of the equation

$$(D - cI)^r f = 0.$$

How many elementary nilpotent blocks will there be in the rational form for N? The number will be the nullity of N, i.e., the dimension of the characteristic space associated with the characteristic value c. That dimension is 1, because any function which satisfies the differential equation

$$Df = cf$$

is a scalar multiple of the exponential function $h(x) = e^{cx}$. Therefore, the operator N (on the space V_c) has a cyclic vector. A good choice for a cyclic vector is $g = x^{r-1}h$:

$$g(x) = x^{r-1}e^{cx}.$$

This gives

$$\begin{aligned} Ng &= (r-1)x^{r-2}h \\ &\vdots \\ N^{r-1}g &= (r-1)!h \end{aligned}$$

The preceding paragraph shows us that the Jordan form for D (on the space V) is the direct sum of k elementary Jordan matrices, one for each root c_i.

Exercises

1. Let N_1 and N_2 be 3×3 nilpotent matrices over the field F. Prove that N_1 and N_2 are similar if and only if they have the same minimal polynomial.

2. Use the result of Exercise 1 and the Jordan form to prove the following: Let

A and B be $n \times n$ matrices over the field F which have the *same* characteristic polynomial

$$f = (x - c_1)^{d_1} \cdots (x - c_k)^{d_k}$$

and the same minimal polynomial. If no d_i is greater than 3, then A and B are similar.

3. If A is a complex 5×5 matrix with characteristic polynomial

$$f = (x - 2)^3(x + 7)^2$$

and minimal polynomial $p = (x - 2)^2(x + 7)$, what is the Jordan form for A?

4. How many possible Jordan forms are there for a 6×6 complex matrix with characteristic polynomial $(x + 2)^4(x - 1)^2$?

5. The differentiation operator on the space of polynomials of degree less than or equal to 3 is represented in the 'natural' ordered basis by the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

What is the Jordan form of this matrix? (F a subfield of the complex numbers.)

6. Let A be the complex matrix

$$\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 \\ 1 & 1 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}.$$

Find the Jordan form for A.

7. If A is an $n \times n$ matrix over the field F with characteristic polynomial

$$f = (x - c_1)^{d_1} \cdots (x - c_k)^{d_k}$$

what is the trace of A?

8. Classify up to similarity all 3×3 complex matrices A such that $A^3 = I$.

9. Classify up to similarity all $n \times n$ complex matrices A such that $A^n = I$.

10. Let n be a positive integer, $n \geq 2$, and let N be an $n \times n$ matrix over the field F such that $N^n = 0$ but $N^{n-1} \neq 0$. Prove that N has no square root, i.e., that there is no $n \times n$ matrix A such that $A^2 = N$.

11. Let N_1 and N_2 be 6×6 nilpotent matrices over the field F. Suppose that N_1 and N_2 have the same minimal polynomial and the same nullity. Prove that N_1 and N_2 are similar. Show that this is not true for 7×7 nilpotent matrices.

12. Use the result of Exercise 11 and the Jordan form to prove the following: Let A and B be $n \times n$ matrices over the field F which have the same characteristic polynomial

$$f = (x - c_1)^{d_1} \cdots (x - c_k)^{d_k}$$