and these vectors form an independent set. For suppose

$$\Sigma\, x_i\alpha_i + \Sigma\, y_j\beta_j + \Sigma\, z_r\gamma_r = 0.$$

Then

$$-\Sigma\, z_r\gamma_r = \Sigma\, x_i\alpha_i + \Sigma\, y_j\beta_j$$

which shows that $\Sigma\, z_r\gamma_r$ belongs to W_1. As $\Sigma\, z_r\gamma_r$ also belongs to W_2 it follows that

$$\Sigma\, z_r\gamma_r = \Sigma\, c_i\alpha_i$$

for certain scalars $c_1, \ldots, c_k$. Because the set

$$\{\alpha_1, \ldots, \alpha_k, \quad \gamma_1, \ldots, \gamma_n\}$$

is independent, each of the scalars $z_r = 0$. Thus

$$\Sigma\, x_i\alpha_i + \Sigma\, y_j\beta_j = 0$$

and since

$$\{\alpha_1, \ldots, \alpha_k, \quad \beta_1, \ldots, \beta_m\}$$

is also an independent set, each $x_i = 0$ and each $y_j = 0$. Thus,

$$\{\alpha_1, \ldots, \alpha_k, \quad \beta_1, \ldots, \beta_m, \quad \gamma_1, \ldots, \gamma_n\}$$

is a basis for $W_1 + W_2$. Finally

$$\begin{aligned} \dim W_1 + \dim W_2 &= (k + m) + (k + n) \\ &= k + (m + k + n) \\ &= \dim (W_1 \cap W_2) + \dim (W_1 + W_2). \quad \blacksquare \end{aligned}$$

Let us close this section with a remark about linear independence and dependence. We defined these concepts for sets of vectors. It is useful to have them defined for finite sequences (ordered n-tuples) of vectors: $\alpha_1, \ldots, \alpha_n$. We say that the vectors $\alpha_1, \ldots, \alpha_n$ are **linearly dependent** if there exist scalars $c_1, \ldots, c_n$, not all 0, such that $c_1\alpha_1 + \cdots + c_n\alpha_n = 0$. This is all so natural that the reader may find that he has been using this terminology already. What is the difference between a finite sequence $\alpha_1, \ldots, \alpha_n$ and a set $\{\alpha_1, \ldots, \alpha_n\}$? There are two differences, identity and order.

If we discuss the set $\{\alpha_1, \ldots, \alpha_n\}$, usually it is presumed that no two of the vectors $\alpha_1, \ldots, \alpha_n$ are identical. In a sequence $\alpha_1, \ldots, \alpha_n$ all the α_i's may be the same vector. If $\alpha_i = \alpha_j$ for some $i \neq j$, then the sequence $\alpha_1, \ldots, \alpha_n$ is linearly dependent:

$$\alpha_i + (-1)\alpha_j = 0.$$

Thus, if $\alpha_1, \ldots, \alpha_n$ are linearly independent, they are distinct and we may talk about the set $\{\alpha_1, \ldots, \alpha_n\}$ and know that it has n vectors in it. So, clearly, no confusion will arise in discussing bases and dimension. The dimension of a finite-dimensional space V is the largest n such that some n-tuple of vectors in V is linearly independent—and so on. The reader who feels that this paragraph is much ado about nothing might ask himself whether the vectors

$$\alpha_1 = (e^{\pi/2}, 1)$$
$$\alpha_2 = (\sqrt[3]{110}, 1)$$

are linearly independent in R^2.

The elements of a sequence are enumerated in a specific order. A set is a collection of objects, with no specified arrangement or order. Of course, to describe the set we may list its members, and that requires choosing an order. But, the order is not part of the set. The sets $\{1, 2, 3, 4\}$ and $\{4, 3, 2, 1\}$ are identical, whereas 1, 2, 3, 4 is quite a different sequence from 4, 3, 2, 1. The order aspect of sequences has no bearing on questions of independence, dependence, etc., because dependence (as defined) is not affected by the order. The sequence $\alpha_n, \ldots, \alpha_1$ is dependent if and only if the sequence $\alpha_1, \ldots, \alpha_n$ is dependent. In the next section, order will be important.

Exercises

1. Prove that if two vectors are linearly dependent, one of them is a scalar multiple of the other.

2. Are the vectors

$$\alpha_1 = (1, 1, 2, 4), \qquad \alpha_2 = (2, -1, -5, 2)$$
$$\alpha_3 = (1, -1, -4, 0), \qquad \alpha_4 = (2, 1, 1, 6)$$

linearly independent in R^4?

3. Find a basis for the subspace of R^4 spanned by the four vectors of Exercise 2.

4. Show that the vectors

$$\alpha_1 = (1, 0, -1), \qquad \alpha_2 = (1, 2, 1), \qquad \alpha_3 = (0, -3, 2)$$

form a basis for R^3. Express each of the standard basis vectors as linear combinations of α_1, α_2, and α_3.

5. Find three vectors in R^3 which are linearly dependent, and are such that any two of them are linearly independent.

6. Let V be the vector space of all 2×2 matrices over the field F. Prove that V has dimension 4 by exhibiting a basis for V which has four elements.

7. Let V be the vector space of Exercise 6. Let W_1 be the set of matrices of the form

$$\begin{bmatrix} x & -x \\ y & z \end{bmatrix}$$

and let W_2 be the set of matrices of the form

$$\begin{bmatrix} a & b \\ -a & c \end{bmatrix}.$$

(a) Prove that W_1 and W_2 are subspaces of V.

(b) Find the dimensions of W_1, W_2, $W_1 + W_2$, and $W_1 \cap W_2$.

8. Again let V be the space of 2×2 matrices over F. Find a basis $\{A_1, A_2, A_3, A_4\}$ for V such that $A_j^2 = A_j$ for each j.

9. Let V be a vector space over a subfield F of the complex numbers. Suppose α, β, and γ are linearly independent vectors in V. Prove that $(\alpha + \beta)$, $(\beta + \gamma)$, and $(\gamma + \alpha)$ are linearly independent.

10. Let V be a vector space over the field F. Suppose there are a finite number of vectors $\alpha_1, \ldots, \alpha_r$ in V which span V. Prove that V is finite-dimensional.

11. Let V be the set of all 2×2 matrices A with *complex* entries which satisfy $A_{11} + A_{22} = 0$.

(a) Show that V is a vector space over the field of *real* numbers, with the usual operations of matrix addition and multiplication of a matrix by a scalar.

(b) Find a basis for this vector space.

(c) Let W be the set of all matrices A in V such that $A_{21} = -\bar{A}_{12}$ (the bar denotes complex conjugation). Prove that W is a subspace of V and find a basis for W.

12. Prove that the space of all $m \times n$ matrices over the field F has dimension mn, by exhibiting a basis for this space.

13. Discuss Exercise 9, when V is a vector space over the field with two elements described in Exercise 5, Section 1.1.

14. Let V be the set of real numbers. Regard V as a vector space over the field of *rational* numbers, with the usual operations. Prove that this vector space is *not* finite-dimensional.

2.4. Coordinates

One of the useful features of a basis $\mathcal{B}$ in an n-dimensional space V is that it essentially enables one to introduce coordinates in V analogous to the 'natural coordinates' x_i of a vector $\alpha = (x_1, \ldots, x_n)$ in the space F^n. In this scheme, the coordinates of a vector α in V relative to the basis $\mathcal{B}$ will be the scalars which serve to express α as a linear combination of the vectors in the basis. Thus, we should like to regard the natural coordinates of a vector α in F^n as being defined by α and the standard basis for F^n; however, in adopting this point of view we must exercise a certain amount of care. If

$$\alpha = (x_1, \ldots, x_n) = \Sigma\, x_i \epsilon_i$$

and $\mathcal{B}$ is the standard basis for F^n, just how are the coordinates of α determined by $\mathcal{B}$ and α? One way to phrase the answer is this. A given vector α has a unique expression as a linear combination of the standard basis vectors, and the ith coordinate x_i of α is the coefficient of ϵ_i in this expression. From this point of view we are able to say which is the ith coordinate because we have a 'natural' ordering of the vectors in the standard basis, that is, we have a rule for determining which is the 'first' vector in the basis, which is the 'second,' and so on. If $\mathcal{B}$ is an arbitrary basis of the n-dimensional space V, we shall probably have no natural ordering of the vectors in $\mathcal{B}$, and it will therefore be necessary for us to impose some order on these vectors before we can define 'the ith coordinate of α relative to $\mathcal{B}$.' To put it another way, coordinates will be defined relative to sequences of vectors rather than sets of vectors.

Definition. *If V is a finite-dimensional vector space, an* **ordered basis** *for V is a finite sequence of vectors which is linearly independent and spans V.*

If the sequence $\alpha_1, \ldots, \alpha_n$ is an ordered basis for V, then the set $\{\alpha_1, \ldots, \alpha_n\}$ is a basis for V. The ordered basis is the set, together with the specified ordering. We shall engage in a slight abuse of notation and describe all that by saying that

$$\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}$$

is an ordered basis for V.

Now suppose V is a finite-dimensional vector space over the field F and that

$$\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}$$

is an ordered basis for V. Given α in V, there is a unique n-tuple $(x_1, \ldots, x_n)$ of scalars such that

$$\alpha = \sum_{i=1}^{n} x_i\alpha_i.$$

The n-tuple is unique, because if we also have

$$\alpha = \sum_{i=1}^{n} z_i\alpha_i$$

then

$$\sum_{i=1}^{n} (x_i - z_i)\alpha_i = 0$$

and the linear independence of the α_i tells us that $x_i - z_i = 0$ for each i. We shall call x_i the ith **coordinate of α relative to the ordered basis**

$$\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}.$$

If

$$\beta = \sum_{i=1}^{n} y_i\alpha_i$$

then

$$\alpha + \beta = \sum_{i=1}^{n} (x_i + y_i)\alpha_i$$

so that the ith coordinate of $(\alpha + \beta)$ in this ordered basis is $(x_i + y_i)$.

Similarly, the ith coordinate of $(c\alpha)$ is cx_i. One should also note that every n-tuple $(x_1, \ldots, x_n)$ in F^n is the n-tuple of coordinates of some vector in V, namely the vector

$$\sum_{i=1}^{n} x_i\alpha_i.$$

To summarize, each ordered basis for V determines a one-one correspondence

$$\alpha \rightarrow (x_1, \ldots, x_n)$$

between the set of all vectors in V and the set of all n-tuples in F^n. This correspondence has the property that the correspondent of $(\alpha + \beta)$ is the sum in F^n of the correspondents of α and β, and that the correspondent of $(c\alpha)$ is the product in F^n of the scalar c and the correspondent of α.

One might wonder at this point why we do not simply select some ordered basis for V and describe each vector in V by its corresponding n-tuple of coordinates, since we would then have the convenience of operating only with n-tuples. This would defeat our purpose, for two reasons. First, as our axiomatic definition of vector space indicates, we are attempting to learn to reason with vector spaces as abstract algebraic systems. Second, even in those situations in which we use coordinates, the significant results follow from our ability to change the coordinate system, i.e., to change the ordered basis.

Frequently, it will be more convenient for us to use the **coordinate matrix of α relative to the ordered basis $\mathcal{B}$**:

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

rather than the n-tuple $(x_1, \ldots, x_n)$ of coordinates. To indicate the dependence of this coordinate matrix on the basis, we shall use the symbol

$$[\alpha]_{\mathcal{B}}$$

for the coordinate matrix of the vector α relative to the ordered basis $\mathcal{B}$. This notation will be particularly useful as we now proceed to describe what happens to the coordinates of a vector α as we change from one ordered basis to another.

Suppose then that V is n-dimensional and that

$$\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\} \quad \text{and} \quad \mathcal{B}' = \{\alpha_1', \ldots, \alpha_n'\}$$

are two ordered bases for V. There are unique scalars P_{ij} such that

$$\text{(2-13)} \qquad \alpha_j' = \sum_{i=1}^{n} P_{ij}\alpha_i, \qquad 1 \leq j \leq n.$$

Let $x_1', \ldots, x_n'$ be the coordinates of a given vector α in the ordered basis $\mathcal{B}'$. Then

$$\begin{aligned}\alpha &= x_1'\alpha_1' + \cdots + x_n'\alpha_n' \\ &= \sum_{j=1}^{n} x_j'\alpha_j' \\ &= \sum_{j=1}^{n} x_j' \sum_{i=1}^{n} P_{ij}\alpha_i \\ &= \sum_{j=1}^{n} \sum_{i=1}^{n} (P_{ij}x_j')\alpha_i \\ &= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} P_{ij}x_j' \right) \alpha_i.\end{aligned}$$

Thus we obtain the relation

$$\alpha = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} P_{ij}x_j' \right) \alpha_i. \tag{2-14}$$

Since the coordinates $x_1, x_2, \ldots, x_n$ of α in the ordered basis $\mathcal{B}$ are uniquely determined, it follows from (2-14) that

$$x_i = \sum_{j=1}^{n} P_{ij}x_j', \qquad 1 \leq i \leq n. \tag{2-15}$$

Let P be the $n \times n$ matrix whose i, j entry is the scalar P_{ij}, and let X and X' be the coordinate matrices of the vector α in the ordered bases $\mathcal{B}$ and $\mathcal{B}'$. Then we may reformulate (2-15) as

$$X = PX'. \tag{2-16}$$

Since $\mathcal{B}$ and $\mathcal{B}'$ are linearly independent sets, $X = 0$ if and only if $X' = 0$. Thus from (2-16) and Theorem 7 of Chapter 1, it follows that P is invertible. Hence

$$X' = P^{-1}X. \tag{2-17}$$

If we use the notation introduced above for the coordinate matrix of a vector relative to an ordered basis, then (2-16) and (2-17) say

$$\begin{aligned}[\alpha]_{\mathcal{B}} &= P[\alpha]_{\mathcal{B}'} \\ [\alpha]_{\mathcal{B}'} &= P^{-1}[\alpha]_{\mathcal{B}}.\end{aligned}$$

Thus the preceding discussion may be summarized as follows.

Theorem 7. *Let* V *be an* n*-dimensional vector space over the field* F, *and let* $\mathcal{B}$ *and* $\mathcal{B}'$ *be two ordered bases of* V. *Then there is a unique, necessarily invertible,* n $\times$ n *matrix* P *with entries in* F *such that*

$$\begin{aligned}&\text{(i)} & [\alpha]_{\mathcal{B}} &= \mathrm{P}[\alpha]_{\mathcal{B}'} \\ &\text{(ii)} & [\alpha]_{\mathcal{B}'} &= \mathrm{P}^{-1}[\alpha]_{\mathcal{B}}\end{aligned}$$

for every vector α *in* V. *The columns of* P *are given by*

$$\mathrm{P}_j = [\alpha_j']_{\mathcal{B}}, \qquad j = 1, \ldots, n.$$