Thus $(L_M)^* = L_{M^*}$. In the computation above, we twice used the characteristic property of the trace function: $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

EXAMPLE 20. Let V be the space of polynomials over the field of complex numbers, with the inner product

$$(f|g) = \int_0^1 f(t)\overline{g(t)}\,dt.$$

If f is a polynomial, $f = \sum a_k x^k$, we let $\bar{f} = \sum \bar{a}_k x^k$. That is, $\bar{f}$ is the polynomial whose associated polynomial function is the complex conjugate of that for f:

$$\bar{f}(t) = \overline{f(t)}, \qquad t \text{ real}$$

Consider the operator 'multiplication by f,' that is, the linear operator M_f defined by $M_f(g) = fg$. Then this operator has an adjoint, namely, multiplication by $\bar{f}$. For

$$\begin{aligned}(M_f(g)|h) &= (fg|h)\\ &= \int_0^1 f(t)g(t)\overline{h(t)}\,dt\\ &= \int_0^1 g(t)[\overline{\overline{f(t)}h(t)}]\,dt\\ &= (g|\bar{f}h)\\ &= (g|M_{\bar{f}}(h))\end{aligned}$$

and so $(M_{\bar{f}})^* = M_f$.

EXAMPLE 21. In Example 20, we saw that some linear operators on an infinite-dimensional inner product space do have an adjoint. As we commented earlier, some do not. Let V be the inner product space of Example 21, and let D be the differentiation operator on $C[x]$. Integration by parts shows that

$$(Df|g) = f(1)g(1) - f(0)g(0) - (f|Dg).$$

Let us fix g and inquire when there is a polynomial D^*g such that $(Df|g) = (f|D^*g)$ for all f. If such a D^*g exists, we shall have

$$(f|D^*g) = f(1)g(1) - f(0)g(0) - (f|Dg)$$

or

$$(f|D^*g + Dg) = f(1)g(1) - f(0)g(0).$$

With g fixed, $L(f) = f(1)g(1) - f(0)g(0)$ is a linear functional of the type considered in Example 16 and cannot be of the form $L(f) = (f|h)$ unless $L = 0$. If D^*g exists, then with $h = D^*g + Dg$ we do have $L(f) = (f|h)$, and so $g(0) = g(1) = 0$. The existence of a suitable polynomial D^*g implies $g(0) = g(1) = 0$. Conversely, if $g(0) = g(1) = 0$, the polynomial $D^*g = -Dg$ satisfies $(Df|g) = (f|D^*g)$ for all f. If we choose any g for which $g(0) \neq 0$ or $g(1) \neq 0$, we cannot suitably define D^*g, and so we conclude that D has no adjoint.

We hope that these examples enhance the reader's understanding of the adjoint of a linear operator. We see that the adjoint operation, passing from T to T^*, behaves somewhat like conjugation on complex numbers. The following theorem strengthens the analogy.

Theorem 9. *Let* V *be a finite-dimensional inner product space. If* T *and* U *are linear operators on* V *and* c *is a scalar,*

(i) $(T + U)^* = T^* + U^*$;
(ii) $(cT)^* = \bar{c}T^*$;
(iii) $(TU)^* = U^*T^*$;
(iv) $(T^*)^* = T$.

Proof. To prove (i), let α and β be any vectors in V. Then

$$\begin{aligned} ((T + U)\alpha|\beta) &= (T\alpha + U\alpha|\beta) \\ &= (T\alpha|\beta) + (U\alpha|\beta) \\ &= (\alpha|T^*\beta) + (\alpha|U^*\beta) \\ &= (\alpha|T^*\beta + U^*\beta) \\ &= (\alpha|(T^* + U^*)\beta). \end{aligned}$$

From the uniqueness of the adjoint we have $(T + U)^* = T^* + U^*$. We leave the proof of (ii) to the reader. We obtain (iii) and (iv) from the relations

$$(TU\alpha|\beta) = (U\alpha|T^*\beta) = (\alpha|U^*T^*\beta)$$
$$(T^*\alpha|\beta) = \overline{(\beta|T^*\alpha)} = \overline{(T\beta|\alpha)} = (\alpha|T\beta). \quad \blacksquare$$

Theorem 9 is often phrased as follows: The mapping $T \rightarrow T^*$ is a conjugate-linear anti-isomorphism of period 2. The analogy with complex conjugation which we mentioned above is, of course, based upon the observation that complex conjugation has the properties $\overline{(z_1 + z_2)} = \bar{z}_1 + \bar{z}_2$, $\overline{(z_1 z_2)} = \bar{z}_1\bar{z}_2$, $\bar{\bar{z}} = z$. One must be careful to observe the reversal of order in a product, which the adjoint operation imposes: $(UT)^* = T^*U^*$. We shall mention extensions of this analogy as we continue our study of linear operators on an inner product space. We might mention something along these lines now. A complex number z is real if and only if $z = \bar{z}$. One might expect that the linear operators T such that $T = T^*$ behave in some way like the real numbers. This is in fact the case. For example, if T is a linear operator on a finite-dimensional *complex* inner product space, then

$$T = U_1 + iU_2 \tag{8-15}$$

where $U_1 = U_1^*$ and $U_2 = U_2^*$. Thus, in some sense, T has a 'real part' and an 'imaginary part.' The operators U_1 and U_2 satisfying $U_1 = U_1^*$, and $U_2 = U_2^*$, and (8-15) are unique, and are given by

$$U_1 = \frac{1}{2}(T + T^*)$$

$$U_2 = \frac{1}{2i}(T - T^*).$$

A linear operator T such that $T = T^*$ is called **self-adjoint** (or **Hermitian**). If $\mathfrak{B}$ is an orthonormal basis for V, then

$$[T^*]_{\mathfrak{B}} = [T]^*_{\mathfrak{B}}$$

and so T is self-adjoint if and only if its matrix in every orthonormal basis is a self-adjoint matrix. Self-adjoint operators are important, not simply because they provide us with some sort of real and imaginary part for the general linear operator, but for the following reasons: (1) Self-adjoint operators have many special properties. For example, for such an operator there is an orthonormal basis of characteristic vectors. (2) Many operators which arise in practice are self-adjoint. We shall consider the special properties of self-adjoint operators later.

Exercises

1. Let V be the space C^2, with the standard inner product. Let T be the linear operator defined by $T\epsilon_1 = (1, -2)$, $T\epsilon_2 = (i, -1)$. If $\alpha = (x_1, x_2)$, find $T^*\alpha$.

2. Let T be the linear operator on C^2 defined by $T\epsilon_1 = (1 + i, 2)$, $T\epsilon_2 = (i, i)$. Using the standard inner product, find the matrix of T^* in the standard ordered basis. Does T commute with T^*?

3. Let V be C^3 with the standard inner product. Let T be the linear operator on V whose matrix in the standard ordered basis is defined by

$$A_{jk} = i^{j+k}, \qquad (i^2 = -1).$$

Find a basis for the null space of T^*.

4. Let V be a finite-dimensional inner product space and T a linear operator on V. Show that the range of T^* is the orthogonal complement of the null space of T.

5. Let V be a finite-dimensional inner product space and T a linear operator on V. If T is invertible, show that T^* is invertible and $(T^*)^{-1} = (T^{-1})^*$.

6. Let V be an inner product space and β, γ fixed vectors in V. Show that $T\alpha = (\alpha|\beta)\gamma$ defines a linear operator on V. Show that T has an adjoint, and describe T^* explicitly.

Now suppose V is C^n with the standard inner product, $\beta = (y_1, \ldots, y_n)$, and $\gamma = (x_1, \ldots, x_n)$. What is the j, k entry of the matrix of T in the standard ordered basis? What is the rank of this matrix?

7. Show that the product of two self-adjoint operators is self-adjoint if and only if the two operators commute.

8. Let V be the vector space of the polynomials over R of degree less than or equal to 3, with the inner product

$$(f|g) = \int_0^1 f(t)g(t)\,dt.$$

If t is a real number, find the polynomial g_t in V such that $(f|g_t) = f(t)$ for all f in V.

9. Let V be the inner product space of Exercise 8, and let D be the differentiation operator on V. Find D^*.

10. Let V be the space of $n \times n$ matrices over the complex numbers, with the inner product $(A, B) = \operatorname{tr}(AB^*)$. Let P be a fixed invertible matrix in V, and let T_P be the linear operator on V defined by $T_P(A) = P^{-1}AP$. Find the adjoint of T_P.

11. Let V be a finite-dimensional inner product space, and let E be an idempotent linear operator on V, i.e., $E^2 = E$. Prove that E is self-adjoint if and only if $EE^* = E^*E$.

12. Let V be a finite-dimensional *complex* inner product space, and let T be a linear operator on V. Prove that T is self-adjoint if and only if $(T\alpha|\alpha)$ is real for every α in V.

8.4. Unitary Operators

In this section, we consider the concept of an isomorphism between two inner product spaces. If V and W are vector spaces, an isomorphism of V onto W is a one-one linear transformation from V onto W, i.e., a one-one correspondence between the elements of V and those of W, which 'preserves' the vector space operations. Now an inner product space consists of a vector space and a specified inner product on that space. Thus, when V and W are inner product spaces, we shall require an isomorphism from V onto W not only to preserve the linear operations, but also to preserve inner products. An isomorphism of an inner product space onto itself is called a 'unitary operator' on that space. We shall consider various examples of unitary operators and establish their basic properties.

Definition. *Let* V *and* W *be inner product spaces over the same field, and let* T *be a linear transformation from* V *into* W. *We say that* T **preserves inner products** *if* $(T\alpha|T\beta) = (\alpha|\beta)$ *for all* α, β *in* V. *An* **isomorphism** *of* V *onto* W *is a vector space isomorphism* T *of* V *onto* W *which also preserves inner products.*

If T preserves inner products, then $||T\alpha|| = ||\alpha||$ and so T is necessarily non-singular. Thus an isomorphism from V onto W can also be defined as a linear transformation from V onto W which preserves inner products. If T is an isomorphism of V onto W, then T^{-1} is an isomorphism of W onto V; hence, when such a T exists, we shall simply say V and W are **isomorphic.** Of course, isomorphism of inner product spaces is an equivalence relation.

Theorem 10. *Let* V *and* W *be finite-dimensional inner product spaces over the same field, having the same dimension. If* T *is a linear transformation from* V *into* W, *the following are equivalent.*

(i) T *preserves inner products.*

(ii) T *is an (inner product space) isomorphism.*

(iii) T *carries every orthonormal basis for* V *onto an orthonormal basis for* W.

(iv) T *carries some orthonormal basis for* V *onto an orthonormal basis for* W.

Proof. (i) $\rightarrow$ (ii) If T preserves inner products, then $||T\alpha|| = ||\alpha||$ for all α in V. Thus T is non-singular, and since $\dim V = \dim W$, we know that T is a vector space isomorphism.

(ii) $\rightarrow$ (iii) Suppose T is an isomorphism. Let $\{\alpha_1, \ldots, \alpha_n\}$ be an orthonormal basis for V. Since T is a vector space isomorphism and $\dim W = \dim V$, it follows that $\{T\alpha_1, \ldots, T\alpha_n\}$ is a basis for W. Since T also preserves inner products, $(T\alpha_j|T\alpha_k) = (\alpha_j|\alpha_k) = \delta_{jk}$.

(iii) $\rightarrow$ (iv) This requires no comment.

(iv) $\rightarrow$ (i) Let $\{\alpha_1, \ldots, \alpha_n\}$ be an orthonormal basis for V such that $\{T\alpha_1, \ldots, T\alpha_n\}$ is an orthonormal basis for W. Then

$$(T\alpha_j|T\alpha_k) = (\alpha_j|\alpha_k) = \delta_{jk}.$$

For any $\alpha = x_1\alpha_1 + \cdots + x_n\alpha_n$ and $\beta = y_1\alpha_1 + \cdots + y_n\alpha_n$ in V, we have

$$\begin{aligned}(\alpha|\beta) &= \sum_{j=1}^{n} x_j\bar{y}_j \\ (T\alpha|T\beta) &= \Big(\sum_j x_jT\alpha_j \Big| \sum_k y_kT\alpha_k\Big) \\ &= \sum_j \sum_k x_j\bar{y}_k(T\alpha_j|T\alpha_k) \\ &= \sum_{j=1}^{n} x_j\bar{y}_j\end{aligned}$$

and so T preserves inner products. ∎

Corollary. *Let* V *and* W *be finite-dimensional inner product spaces over the same field. Then* V *and* W *are isomorphic if and only if they have the same dimension.*

Proof. If $\{\alpha_1, \ldots, \alpha_n\}$ is an orthonormal basis for V and $\{\beta_1, \ldots, \beta_n\}$ is an orthonormal basis for W, let T be the linear transformation from V into W defined by $T\alpha_j = \beta_j$. Then T is an isomorphism of V onto W. ∎