the following calculation shows:

$$\begin{aligned} ax^2 + bxy + cy^2 = {} & a(x'^2 \cos^2\theta + 2x'y' \cos\theta \sin\theta + y'^2 \sin^2\theta) \\ & +b(-x'^2 \cos\theta \sin\theta + x'y'(\cos^2\theta - \sin^2\theta) \\ & \qquad +y'^2 \cos\theta \sin\theta) \\ & +c(x'^2 \sin^2\theta - 2x'y' \cos\theta \sin\theta + y'^2 \cos^2\theta). \end{aligned}$$

The coefficient of $x'y'$ is

$$a \sin 2\theta + b \cos 2\theta - c \sin 2\theta = (a - c) \sin 2\theta + b \cos 2\theta;$$

hence to make it zero we need

$$(c - a) \sin 2\theta = b \cos 2\theta.$$

If $c - a = 0$, we can satisfy this equation by choosing θ so that $\cos 2\theta = 0$. Otherwise we choose θ so that $\tan 2\theta = b/(c - a)$, which is always possible because $\tan 2\theta$ takes all real values.

We are assuming $b \neq 0$ (otherwise there is no need to rotate the axes), hence in both cases $\sin 2\theta \neq 0$. In the first case, this is because $\cos 2\theta = 0$, and in the second because $\tan 2\theta \neq 0$. We can use this fact to show that the coefficients of x'^2 and y'^2,

$$\begin{aligned} a' &= a \cos^2\theta - b \cos\theta \sin\theta + c \sin^2\theta \\ \text{and} \quad c' &= a \sin^2\theta + b \cos\theta \sin\theta + c \cos^2\theta, \end{aligned}$$

are not both zero. If they are, then adding the equations $a' = 0$ and $c' = 0$ gives $a + c = 0$ or $c = -a$. But in this case, subtracting the equation $a' = 0$ from $c' = 0$ gives $0 = b \sin 2\theta$, contrary to the facts that $b \neq 0$ and $\sin 2\theta \neq 0$.

2. Relative to the new axes, the curve has an equation of the form

$$a'x'^2 + c'y'^2 + d'x' + e'y' + f' = 0,$$

with a' and c' not both zero. Now we shift the origin, substituting $x'' + A$ for x' and $y'' + C$ for y'. This gives the equation

$$a'(x''^2 + 2Ax'' + A^2) + c'(y''^2 + 2Cy'' + C^2) + d'(x'' + A) + e'(y'' + C) + f' = 0.$$

If $a' \neq 0$ we make the coefficient $2a'A + d'$ of x'' zero by choosing $A = -d'/2a'$. If $b' \neq 0$ we make the coefficient $2c'C + e'$ of y'' zero by choosing $C = -e'/2c'$. Because a' and c' are not both zero, this gives an equation of one of the forms:

$$\begin{aligned} a''x''^2 + c''y''^2 &= f'', \\ \text{or} \quad a''x''^2 &= e''y'' + f'', \\ \text{or} \quad c''y''^2 &= d''x'' + f''. \end{aligned}$$

A further shift of origin and renaming of variables and constants converts the latter two of these equations to the standard equation of the parabola,

$$y = ax^2.$$

The former equation becomes the standard equation of the hyperbola or the ellipse, according as a'' and c'' have opposite or equal signs—unless the constant $f'' = 0$, in which case it represents a degenerate quadratic curve.

Exercises

The examples of degenerate quadratic curves given above—the pair of lines $x = 3y$ and $x = 4y$ and the single point $(0, 0)$—could be regarded as conic sections. After all, a plane can cut a cone in a pair of intersecting lines or a single point.

8.3.1. Show, however, that there is a degenerate quadratic curve that is *not* a section of a cone. Is it a section of a "degenerate cone"?

8.3.2. The lines $\frac{y}{b} = \pm\frac{x}{a}$ are called the *asymptotes* of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Describe how the hyperbola is related to its asymptotes, both in terms of degeneration and in terms of the cone.

Rotation of axes allows us to identify the quadratic curve $xy = 1$ as a hyperbola. Can you guess in advance how far the axes should be rotated?

8.3.3. Show that the curve $xy = 1$ is the same (after suitable rotation of axes and renaming of variables) as the hyperbola $x^2 - y^2 = 2$.

This raises the question: how might we tell in advance whether a quadratic curve is a hyperbola, ellipse, or parabola? The answer is certainly known after step 1. The curve is a parabola if one of a' or c' is zero, it is a hyperbola if they have opposite signs, and it is an ellipse if they have the same sign. Thus the problem is to detect these properties of a' and c' from properties of a, b, and c.

These properties are most easily brought to light using matrices, so for the rest of this exercise set we shall assume a basic knowledge of matrices and determinants. (The only facts we actually need are that the product of matrices corresponds to the composition of substitutions, and that the determinant of a product is the product of the determinants. A really tenacious reader may be able to prove these facts from first principles.)

The substitution

$$\begin{aligned} x &= x'\cos\theta + y'\sin\theta \\ y &= -x'\sin\theta + y'\cos\theta, \end{aligned}$$

is written in matrix notation as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}.$$

The usefulness of matrices here is due to the fact that we can also write the quadratic form $ax^2 + bxy + cy^2$ as a matrix product, namely,

$$ax^2 + bxy + cy^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

8.3.4. Deduce that if $a'x'^2 + b'x'y' + c'y'^2 = \begin{bmatrix} x' & y' \end{bmatrix} \begin{bmatrix} a' & b'/2 \\ b'/2 & c' \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$ is the quadratic form resulting from $ax^2 + bxy + cy^2$ by the substitution, then

$$\begin{bmatrix} a' & b'/2 \\ b'/2 & c' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

8.3.5. Deduce from Exercise 8.3.4, by taking determinants of both sides, that

$$b'^2 - 4a'c' = b^2 - 4ac.$$

8.3.6. Conclude from Exercise 8.3.5 that a nondegenerate quadratic curve

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

is

- a hyperbola if $b^2 - 4ac > 0$,
- an ellipse if $b^2 - 4ac < 0$,
- a parabola if $b^2 - 4ac = 0$.

8.4* Intersections

As far as we know, the conic sections were first studied by Menaechmus, a Greek mathematician who lived around 350 B.C. Menaechmus was searching for a construction of $\sqrt[3]{2}$, which, as we now know (from Exercises 3.2.5* to 3.2.8*, for example), is not constructible by ruler and compass. This was not known in ancient Greece, but apparently it was suspected, because constructions of $\sqrt[3]{2}$ were sought using curves other than straight lines and circles. Menaechmus' solution was the simplest, because it used only quadratic curves, the hyperbola, and the parabola.

In terms of coordinates, his construction is almost a triviality. One takes the parabola $y = x^2/2$ and intersects it with the hyperbola $xy = 1$. At the intersection, $y = x^2/2 = 1/x$: therefore $x^3 = 2$ and so $x = \sqrt[3]{2}$. It is also not difficult to regard the parabola and hyperbola as "constructible" in a reasonable sense. We have seen the "thread construction" of the ellipse in Exercise 8.2.3, and there are many other mechanical constructions of conic sections. Assuming such constructions are available and that they allow the curves to be constructed from their coefficients, we can study numbers *constructible from conic sections*. As with ruler and compass constructions, the idea is to form intersections of conic sections and straight lines, then to use the resulting points to construct further lines and conics, and so on. Because the circle is a particular conic section, these constructions will include the ruler and compass constructions, and square roots in particular.

It is not hard to generalize Menaechmus' construction to show that if a number is constructible from conic sections, then so is its cube root. Thus *the numbers constructible from conic sections include* *all numbers obtainable from 1 by rational operations, square roots, and cube roots.*

The converse statement is also true. It depends on finding the equations that arise from intersections of conic sections, and solving these equations by rational operations, square roots, and cube roots. Again, I shall not do all the details, but I hope to explain why square roots and cube roots are crucial. (The details of this part can be completed by doing the exercises, however.)

As we know from the last two sections, conic sections are quadratic curves, and any conic may be brought into one of the standard forms

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{(hyperbola)}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{(ellipse)}$$

$$y = ax^2 \qquad \text{(parabola)}$$

by rotation of axes and shift of origin. The amount of shift is determined by a rational computation, and the rotation depends on finding $\cos\theta$ and $\sin\theta$ when $\tan 2\theta$ is known. Because

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{\sin 2\theta}{\sqrt{1 - \sin^2 2\theta}},$$

we can find $\sin 2\theta$ from $\tan 2\theta$ by solving a quadratic equation. Then because

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\sin\theta\sqrt{1-\sin^2\theta},$$

we can find $\sin\theta$ from $\sin 2\theta$ by solving another quadratic equation, and we can find $\cos\theta$ similarly. This means we can find the coefficients of the standard form conic by rational operations and square roots. It also means that the coefficients of *any* given conic, relative to the new axes, are computable by rational operations and square roots.

The good thing about the standard form equations is that they can all be solved for y as a function of x using rational operations and (at most one) square root. Namely:

$$y = b\sqrt{\frac{x^2}{a^2} - 1} \qquad \text{(hyperbola),}$$

$$y = b\sqrt{1 - \frac{x^2}{a^2}} \qquad \text{(ellipse)},$$

$$y = ax^2 \qquad \text{(parabola)}.$$

Now suppose we wish to find the intersection of the standard form conic C_1 with any other conic C_2, whose equation relative to the new axes is, say,

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0. \qquad (C_2)$$

If C_1 is $y = ax^2$, we substitute ax^2 for y in C_2, obtaining a *quartic* (fourth-degree) equation for y. As will be seen in Exercises 8.4.1 and 8.4.2, this equation can be solved by rational operations, square roots, and cube roots.

If C_1 is $y = b\sqrt{\pm\left(1 - \frac{x^2}{a^2}\right)}$, we substitute this for y in C_2, obtaining

$$Ax^2 + Bbx\sqrt{\pm\left(1 - \frac{x^2}{a^2}\right)} \pm Cb^2\left(1 - \frac{x^2}{a^2}\right) + Dx + Eb\sqrt{\pm\left(1 - \frac{x^2}{a^2}\right)} + F = 0$$

or

$$Ax^2 \pm Cb^2\left(1 - \frac{x^2}{a^2}\right) + Dx + F = b(-Bx - E)\sqrt{\pm\left(1 - \frac{x^2}{a^2}\right)}.$$

Squaring both sides of this gives a quartic equation for x, hence it again follows that it can be solved by rational operations, square roots, and cube roots.

To sum up, what we have shown in outline is the following: *the numbers constructible by conic sections are precisely those obtainable from 1 by rational operations, square roots, and cube roots.*

Exercises

The solution of the quartic equation was discovered by Cardano's student Lodovico Ferrari in 1545. The first step is a small simplification of the equation by change of variable.

8.4.1. Show the general quartic equation $x^4 + ax^3 + bx^2 + cx + d = 0$ takes the form $x^4 + px^2 + qx + r = 0$ when x is replaced by $x + s$ for a suitable constant s.

The latter quartic equation can be rewritten

$$(x^2 + p)^2 = px^2 - qx + p^2 - r.$$

(Why?) This suggests that both sides might be made into squares simultaneously by adding an appropriate quantity. If so, we can reduce the quartic to a quadratic by taking the square root of both sides.

8.4.2. Show that the previous equation implies

$$(x^2 + p + y)^2 = (p + 2y)x^2 - qx + (p^2 - r + 2py + y^2)$$

for any y.

The left-hand side is a square by construction. The right-hand side is a quadratic in x, and we aim to make it a square by finding a suitable y.

8.4.3. Show that $Ax^2 + Bx + C = A(x + B/2A)^2$ when $B^2 - 4AC = 0$.

8.4.4. When $Ax^2 + Bx + C = (p + 2y)x^2 - qx + (p^2 - r + 2py + y^2)$, check that $B^2 - 4AC$ is a cubic in y. Hence conclude from the previous exercises, and Exercises 7.2.1 and 7.2.2, that the general quartic equation may be solved by rational operations, square roots, and cube roots.

8.5 Integer Points on Conics

Because conic sections are quadratic curves, we understand in principle all the rational points on a conic section C. Assuming the equation of C has rational coefficients, Section 4.4 gives the following description of all its rational points: they consist of any single rational point P on C, together with the other points where C is met by lines through P with rational slope. The rational points include the integer points, of course, but describing the integer points alone is another story entirely. The hyperbola, ellipse, and parabola all require different methods, with the hyperbola being the most difficult and interesting. To keep the story as simple as possible, I shall stick to equations in standard form.

Finding the integer points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is easiest, at least in principle. All its points lie within distance $\max(|a|, |b|)$ of