5.1.30. (+) Let $S = \binom{[n]}{2}$ denote the collection of 2-sets of the n-element set $[n]$. Define the graph G_n by $V(G_n) = S$ and $E(G_n) = \{(ij, jk): 1 \le i < j < k \le n\}$ (disjoint pairs, for example, are nonadjacent). Prove that $\chi(G_n) = \lceil \lg n \rceil$. (Hint: Prove that G_n is r-colorable if and only if $[r]$ has at least n distinct subsets. Comment: G_n is called the **shift graph** of K_n.) (attributed to A. Hajnal)

5.1.31. (!) Prove that a graph G is m-colorable if and only if $\alpha(G \,\square\, K_m) \ge n(G)$. (Berge [1973, p379–80])

5.1.32. (!) Prove that a graph G is 2^k-colorable if and only if G is the union of k bipartite graphs. (Hint: This generalizes Theorem 1.2.23.)

5.1.33. (!) Prove that every graph G has a vertex ordering relative to which greedy coloring uses $\chi(G)$ colors.

5.1.34. (!) For all $k \in \mathbb{N}$, construct a tree T_k with maximum degree k and an ordering σ of $V(T_k)$ such that greedy coloring relative to the ordering σ uses $k+1$ colors. (Hint: Use induction and construct the tree and ordering simultaneously. Comment: This result shows that the performance ratio of greedy coloring to optimal coloring can be as bad as $(\Delta(G)+1)/2$.) (Bean [1976])

5.1.35. Let G be a graph having no induced subgraph isomorphic to P_4. Prove that for every vertex ordering, greedy coloring produces an optimal coloring of G. (Hint: Suppose that the algorithm uses k colors for the ordering $v_1, \dots, v_n$, and let i be the smallest integer such that G has a clique consisting of vertices assigned colors i through k in this coloring. Prove that $i = 1$. Comment: P_4-free graphs are also called **cographs**.)

5.1.36. Given a vertex ordering $\sigma = v_1, \dots, v_n$ of a graph G, let $G_i = G[\{v_1, \dots, v_i\}]$ and $f(\sigma) = 1 + \max_i d_{G_i}(v_i)$. Greedy coloring relative to σ yields $\chi(G) \le f(\sigma)$. Define σ^* by letting v_n be a minimum degree vertex of G and letting v_i for $i < n$ be a minimum degree vertex of $G - \{v_{i+1}, \dots, v_n\}$. Show that $f(\sigma^*) = 1 + \max_{H \subseteq G} \delta(H)$, and thus that σ^* minimizes $f(\sigma)$. (Halin [1967], Matula [1968], Finck–Sachs [1969], Lick–White [1970])

5.1.37. Prove that $V(G)$ can be partitioned into $1 + \max_{H \subseteq G} \delta(H)/r$ classes such that every subgraph whose vertices lie in a single class has a vertex of degree less than r. (Hint: Consider ordering σ^* of Exercise 5.1.36. Comment: This generalizes Theorem 5.1.19. See also Chartrand–Kronk [1969] when $r = 2$.)

5.1.38. (!) Prove that $\chi(G) = \omega(G)$ when $\overline{G}$ is bipartite. (Hint: Phrase the claim in terms of $\overline{G}$ and apply results on bipartite graphs.)

5.1.39. (!) Prove that every k-chromatic graph has at least $\binom{k}{2}$ edges. Use this to prove that if G is the union of m complete graphs of order m, then $\chi(G) \le 1 + m\sqrt{m-1}$. (Comment: This bound is near tight, but the Erdős–Faber–Lovász Conjecture (see Erdős [1981]) asserts that $\chi(G) = m$ when the complete graphs are pairwise edge-disjoint.)

5.1.40. Prove that $\chi(G) \cdot \chi(\overline{G}) \ge n(G)$, use this to prove that $\chi(G) + \chi(\overline{G}) \ge 2\sqrt{n(G)}$, and provide a construction achieving these bounds whenever $\sqrt{n(G)}$ is an integer. (Nordhaus–Gaddum [1956], Finck [1968])

5.1.41. (!) Prove that $\chi(G) + \chi(\overline{G}) \le n(G) + 1$. (Hint: Use induction on $n(G)$.) (Nordhaus–Gaddum [1956])

5.1.42. (!) *Looseness of* $\chi(G) \ge n(G)/\alpha(G)$. Let G be an n-vertex graph, and let $c = (n+1)/\alpha(G)$. Use Exercise 5.1.41 to prove that $\chi(G) \cdot \chi(\overline{G}) \le (n+1)^2/4$, and use this to prove that $\chi(G) \le c(n+1)/4$. For each odd n, construct a graph such that $\chi(G) = c(n+1)/4$. (Nordhaus–Gaddum [1956], Finck [1968])

5.1.43. (!) *Paths and chromatic number in digraphs.*

a) Let $G = F \cup H$. Prove that $\chi(G) \le \chi(F)\chi(H)$.

b) Consider an orientation D of G and a function $f\colon V(G) \to \mathbb{R}$. Use part (a) and Theorem 5.1.21 to prove that if $\chi(G) > rs$, then D has a path $u_0 \to \cdots \to u_r$ with $f(u_0) \le \cdots \le f(u_r)$ or a path $v_0 \to \cdots \to v_s$ with $f(v_0) > \cdots > f(v_s)$.

c) Use part (b) to prove that every sequence of $rs + 1$ distinct real numbers has an increasing subsequence of size $r + 1$ or a decreasing subsequence of size $s + 1$. (Erdős–Szekeres [1935])

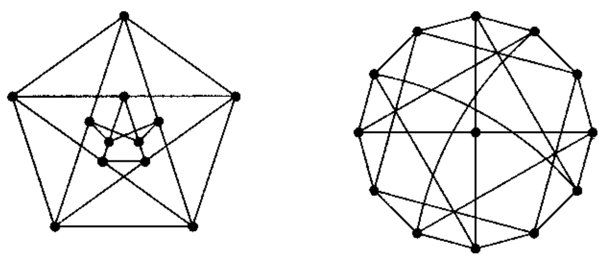
5.1.44. (!) *Minty's Theorem* (Minty [1962]). An **acyclic orientation** of a loopless graph is an orientation having no cycle. For each acyclic orientation D of G, let $r(D) = \max_C \lceil a/b \rceil$, where C is a cycle in G and a, b count the edges of C that are forward in D or backward in D, respectively. Fix a vertex $x \in V(G)$, and let W be a walk in G beginning at x. Let $g(W) = a - b \cdot r(D)$, where a is the number of steps along W that are forward edges in D and b is the number that are backward in D. For each $y \in V(G)$, let $g(y)$ be the maximum of $g(W)$ such that W is an x, y-walk (assume that G is connected).

a) Prove that $g(y)$ is finite and thus well-defined, and use $g(y)$ to obtain a proper $1 + r(D)$-coloring of G. Thus G is $1 + r(D)$-colorable.

b) Prove that $\chi(G) = \min_{D \in \mathbf{D}}$, where $\mathbf{D}$ is the set of acyclic orientations of G.

5.1.45. (+) Use Minty's Theorem (Exercise 5.1.44) to prove Theorem 5.1.21. (Hint: Prove that $l(D)$ is maximized by some acyclic orientation of G.)

5.1.46. (+) Prove that the 4-regular triangle-free graphs below are 4-chromatic. (Hint: Consider the maximum independent sets. Comment: Chvátal [1970] showed that the graph on the left is the smallest triangle-free 4-regular 4-chromatic graph.)



5.1.47. (!) Prove that Brooks' Theorem is equivalent to the following statement: every $k - 1$-regular k-critical graph is a complete graph or an odd cycle.

5.1.48. Let G be a simple graph with n vertices and m edges and maximum degree at most 3. Suppose that no component of G is a complete graph on 4 vertices. Prove that G contains a bipartite subgraph with at least $m - n/3$ edges. (Hint: Apply Brooks' Theorem, and then show how to delete a few edges to change a proper 3-coloring of G into a proper 2-coloring of a large subgraph of G.)

5.1.49. (−) Prove that the Petersen graph can be 2-colored so that the subgraph induced by each color class consists of isolated edges and vertices.

5.1.50. (!) *Improvement of Brooks' Theorem.*

a) Given a graph G, let $k_1, \ldots, k_t$ be nonnegative integers with $\sum k_i \ge \Delta(G) - t + 1$. Prove that $V(G)$ can be partitioned into sets $V_1, \ldots, V_t$ so that for each i, the subgraph G_i induced by V_i has maximum degree at most k_i. (Hint: Prove that the partition minimizing $\sum e(G_i)/k_i$ has the desired property.) (Lovász [1966])

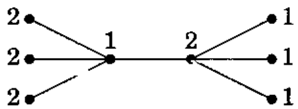
b) For $4 \le r \le \Delta(G) + 1$, use part (a) to prove that $\chi(G) \le \lceil \frac{r-1}{r}(\Delta(G) + 1) \rceil$ when G has no r-clique. (Borodin–Kostochka [1977], Catlin [1978], Lawrence [1978])

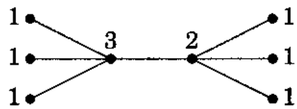
5.1.51. (!) Let G be an k-colorable graph, and let P be a set of vertices in G such that $d(x, y) \ge 4$ whenever $x, y \in P$. Prove that every coloring of P with colors from $[k + 1]$ extends to a proper $k + 1$ coloring of G. (Albertson–Moore [1999])

5.1.52. Prove that every graph G can be $\lceil(\Delta(G) + 1)/j\rceil$-colored so that each color class induces a subgraph having no j-edge-connected subgraph. For $j > 1$, prove that no smaller number of classes suffices when G is a j-regular j-edge-connected graph or is a complete graph with order congruent to 1 modulo j. (Comment: For $j = 1$, the restriction reduces to ordinary proper coloring.) (Matula [1973])

5.1.53. (+) Let $G_{n,k}$ be the $2k$-regular graph of Exercise 5.1.23. For $k \le 4$, determine the values of n such that $G_{n,k}$ can be 2-colored so that each color class induces a subgraph with maximum degree at most k. (Weaver–West [1994])

5.1.54. Let f be a proper coloring of a graph G in which the colors are natural numbers. The **color sum** is $\sum_{v \in V(G)} f(v)$. Minimizing the color sum may require using more than $\chi(G)$ colors. In the tree below, for example, the best proper 2-coloring has color sum 12, while there is a proper 3-coloring with color sum 11. Construct a sequence of trees in which the kth tree T_k use k colors in a proper coloring that minimizes the color sum. (Kubicka–Schwenk [1989])





5.1.55. (+) *Chromatic number is bounded by one plus longest odd cycle length.*

a) Let G be a 2-connected nonbipartite graph containing an even cycle C. Prove that there exist vertices x, y on C and an x, y-path P internally disjoint from C such that $d_C(x, y) \ne d_P(x, y) \bmod 2$.

b) Let G be a simple graph with no odd cycle having length at least $2k + 1$. Prove that if $\delta(G) \ge 2k$, then G has a cycle of length at least $4k$. (Hint: Consider the neighbors of an endpoint of a maximal path.)

c) Let G be a 2-connected nonbipartite graph with no odd cycle longer than $2k - 1$. Prove that $\chi(G) \le 2k$. (Erdős–Hajnal [1966])

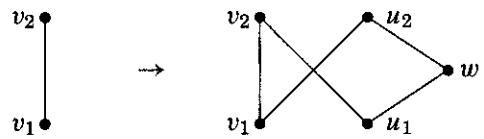
5.2. Structure of k-chromatic Graphs

We have observed that $\chi(H) \ge \omega(H)$ for all H. When equality holds in this bound for G and all its induced subgraphs (as for interval graphs), we say that G is **perfect**; we discuss such graphs in Sections 5.3 and 8.1. Our concern with the bound $\chi(G) \ge \omega(G)$ in this section is how *bad* it can be. Almost always $\chi(G)$ is much larger than $\omega(G)$, in a sense discussed precisely in Section 8.5. (The average values of $\omega(G)$, $\alpha(G)$, and $\chi(G)$ over all graphs with vertex set $[n]$ are very close to $2\lg n$, $2\lg n$, and $n/(2\lg n)$, respectively. Hence $\omega(G)$ is generally a bad lower bound on $\chi(G)$, and $n/\alpha(G)$ is generally a good lower bound.)

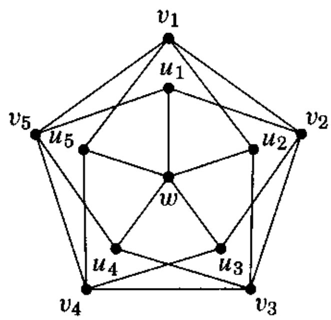
GRAPHS WITH LARGE CHROMATIC NUMBER

The bound $\chi(G) \geq \omega(G)$ can be tight, but it can also be very loose. There have been many constructions of graphs without triangles that have arbitrarily large chromatic number. We present one such construction here; others appear in Exercises 12–13.

5.2.1. Definition. From a simple graph G, **Mycielski's construction** produces a simple graph G' containing G. Beginning with G having vertex set $\{v_1, \dots, v_n\}$, add vertices $U = \{u_1, \dots, u_n\}$ and one more vertex w. Add edges to make u_i adjacent to all of $N_G(v_i)$, and finally let $N(w) = U$.



5.2.2. Example. From the 2-chromatic graph K_2, one iteration of Mycielski's construction yields the 3-chromatic graph C_5, as shown above. Below we apply the construction to C_5, producing the 4-chromatic **Grötzsch graph**. ■



5.2.3. Theorem. (Mycielski [1955]) From a k-chromatic triangle-free graph G, Mycielski's construction produces a $k+1$-chromatic triangle-free graph G'.

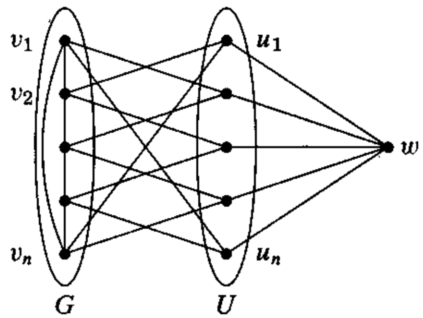
Proof: Let $V(G) = \{v_1, \dots, v_n\}$, and let G' be the graph produced from it by Mycielski's construction. Let $u_1, \dots, u_n$ be the copies of $v_1, \dots, v_n$, with w the additional vertex. Let $U = \{u_1, \dots, u_n\}$.

By construction, U is an independent set in G'. Hence the other vertices of any triangle containing u_i belong to $V(G)$ and are neighbors of v_i. This would complete a triangle in G, which can't exist. We conclude that G' is triangle-free.

A proper k-coloring f of G extends to a proper $k+1$-coloring of G' by setting $f(u_i) = f(v_i)$ and $f(w) = k+1$; hence $\chi(G') \leq \chi(G) + 1$. We prove equality by showing that $\chi(G) < \chi(G')$. To prove this we consider any proper coloring of G' and obtain from it a proper coloring of G using fewer colors.

Let g be a proper k-coloring of G' By changing the names of colors, we may assume that $g(w) = k$. This restricts g to $\{1, \dots, k-1\}$ on U. On $V(G)$, it may use all k colors. Let A be the set of vertices in G on which g uses color k; we change the colors used on A to obtain a proper $k-1$-coloring of G.

For each $v_i \in A$, we change the color of v_i to $g(u_i)$. Because all vertices of A have color k under g, no two edges of A are adjacent. Thus we need only check edges of the form $v_i v'$ with $v_i \in A$ and $v' \in V(G) - A$. If $v' \leftrightarrow v_i$, then by construction also $v' \leftrightarrow u_i$, which yields $g(v') \neq g(u_i)$. Since we change the color on v_i to $g(u_i)$, our change does not violate the edge $v_i v'$. We have shown that the modified coloring of $V(G)$ is a proper $k-1$-coloring of G. ■



If G is color-critical, then the graph G' resulting from Mycielski's construction is also color-critical (Exercise 9).

5.2.4.* Remark. Starting with $G_2 = K_2$, iterating Mycielski's construction produces a sequence $G_2, G_3, G_4, \ldots$ of graphs. The first three are K_2, C_5, and the Grötzsch graph. These are the smallest triangle-free 2-chromatic, 3-chromatic, and 4-chromatic graphs. The graphs then grow rapidly: $n(G_k) = 2n(G_{k-1}) + 1$. With $n(G_2) = 2$, this yields $n(G_k) = 3 \cdot 2^{k-2} - 1$ (exponential growth).

Let $f(k)$ be the minimum number of vertices in a triangle-free k-chromatic graph. Using probabilistic (non-constructive) methods, Erdős [1959] proved that $f(k) \leq ck^{2+\epsilon}$, where ϵ is any positive constant and c depends on ϵ but not on k. Using Ramsey numbers (Section 8.3), it is now known (non-constructively) that there are constants c_1, c_2 such that $c_1 k^2 \log k \leq f(k) \leq c_2 k^2 \log k$. Exercise 15 develops a quadratic lower bound.

Blanche Descartes[†] [1947, 1954] constructed color-critical graphs with girth 6 (Exercise 13). Using probabilistic methods, Erdős [1959] proved that graphs exist with chromatic number at least k and girth at least g (Theorem 8.5.11). Later, explicit constructions were found (Lovász [1968a], Nešetřil–Rödl [1979], Lubotzsky–Phillips–Sarnak [1988], Kriz [1989]).

By all these constructions, forbidding K_r from G does not place a bound on $\chi(G)$. Gyárfás [1975] and Sumner [1981]) conjectured that forbidding a fixed clique and a fixed forest as an *induced* subgraph does bound the chromatic number. Exercise 11 proves this when the forest is $2K_2$. (See also Kierstead–Penrice [1990, 1994], Kierstead [1992, 1997], Kierstead–Rödl [1996]) ■

[†]This pseudonym was used by W.T. Tutte and also by three others.

EXTREMAL PROBLEMS AND TURÁN'S THEOREM

Perhaps extremal questions can shed some light on the structure of k-chromatic graphs. For example, which are the smallest and largest k-chromatic graphs with n vertices?

5.2.5. Proposition. Every k-chromatic graph with n vertices has at least $\binom{k}{2}$ edges. Equality holds for a complete graph plus isolated vertices.

Proof: An optimal coloring of a graph has an edge with endpoints of colors i and j for each pair i, j of colors. Otherwise, colors i and j could be combined into a single color class and use fewer colors. Since there are $\binom{k}{2}$ distinct pairs of colors, there must be $\binom{k}{2}$ distinct edges. ∎

Exercise 6 asks for the minimum size among connected k-chromatic graphs with n vertices.

The maximization problem is more interesting (of course, it makes sense only when restricted to simple graphs). Given a proper k-coloring, we can continue to add edges without increasing the chromatic number as long as two vertices in different color classes are nonadjacent. Thus we may restrict our attention to graphs without such pairs.

5.2.6. Definition. A **complete multipartite graph** is a simple graph G whose vertices can be partitioned into sets so that $u \leftrightarrow v$ if and only if u and v belong to different sets of the partition. Equivalently, every component of $\overline{G}$ is a complete graph. When $k \geq 2$, we write $K_{n_1,\dots,n_k}$ for the complete k-partite graph with partite sets of sizes $n_1, \dots, n_k$ and complement $K_{n_1} + \cdots + K_{n_k}$.

We use this notation only for $k > 1$, since K_n denotes a complete graph. A complete k-partite graph is k-chromatic; the partite sets are the color classes in the only proper k-coloring. Also, since a vertex in a partite set of size t has degree $n(G) - t$, the edges can be counted using the degree-sum formula (Exercise 18). Which distribution of vertices to partite sets maximizes $e(G)$?

5.2.7. Example. *The Turán graph.* The **Turán graph** $T_{n,r}$ is the complete r-partite graph with n vertices whose partite sets differ in size by at most 1. By the pigeonhole principle (see Appendix A), some partite set has size at least $\lceil n/r \rceil$ and some has size at most $\lfloor n/r \rfloor$. Therefore, differing by at most 1 means that they all have size $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$.

Let $a = \lfloor n/r \rfloor$. After putting a vertices in each partite set, $b = n - ra$ remain, so $T_{n,r}$ has b partite sets of size $a + 1$ and $r - b$ partite sets of size a. Thus the defining condition on $T_{n,r}$ specifies a single isomorphism class. ∎

5.2.8. Lemma. Among simple r-partite (that is, r-colorable) graphs with n vertices, the Turán graph is the unique graph with the most edges.

Proof: As noted before Definition 5.2.6, we need only consider complete r-partite graphs. Given a complete r-partite graph with partite sets differing by more than 1 in size, we move a vertex v from the largest class (size i) to the smallest class (size j). The edges not involving v are the same as before, but v gains $i-1$ neighbors in its old class and loses j neighbors in its new class. Since $i-1>j$, the number of edges increases. Hence we maximize the number of edges only by equalizing the sizes as in $T_{n,r}$. ■

We used the idea of this local alteration previously in Theorem 1.3.19 and in Theorem 1.3.23; we are finding the largest r-partite subgraph of K_n.

What happens if we have more edges and thus force chromatic number at least $r+1$? We have seen that there are graphs with chromatic number $r+1$ that have no triangles. Nevertheless, if we go beyond the maximum number of edges in an r-colorable graph with n vertices, then we are forced not only to use $r+1$ colors but also to have K_{r+1} as a subgraph.

This famous result of Turán generalizes Theorem 1.3.23 and is viewed as the origin of extremal graph theory.

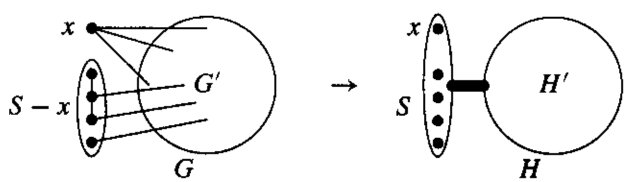
5.2.9. Theorem. (Turán [1941]) Among the n-vertex simple graphs with no $r+1$-clique, $T_{n,r}$ has the maximum number of edges.

Proof: The Turán graph $T_{n,r}$, like every r-colorable graph, has no $r+1$-clique, since each partite set contributes at most one vertex to each clique. If we can prove that the maximum is achieved by an r-partite graph, then Lemma 5.2.8 implies that the maximum is achieved by $T_{n,r}$. Thus it suffices to prove that if G has no $r+1$-clique, then there is an r-partite graph H with the same vertex set as G and at least as many edges.

We prove this by induction on r. When $r=1$, G and H have no edges. For the induction step, consider $r>1$. Let G be an n-vertex graph with no $r+1$-clique, and let $x \in V(G)$ be a vertex of degree $k=\Delta(G)$. Let G' be the subgraph of G induced by the neighbors of x. Since x is adjacent to every vertex in G' and G has no $r+1$-clique, the graph G' has no r-clique. We can thus apply the induction hypothesis to G'; this yields an $r-1$-partite graph H' with vertex set $N(x)$ such that $e(H') \geq e(G')$.

Let H be the graph formed from H' by joining all of $N(x)$ to all of $S = V(G) - N(x)$. Since S is an independent set, H is r-partite. We claim that $e(H) \geq e(G)$. By construction, $e(H) = e(H') + k(n-k)$. We also have $e(G) \leq e(G') + \sum_{v \in S} d_G(v)$, since the sum counts each edge of G once for each endpoint it has outside $V(G')$. Since $\Delta(G)=k$, we have $d_G(v) \leq k$ for each $v \in S$, and $|S| = n-k$, so $\sum_{v \in S} d_G(v) \leq k(n-k)$. As desired, we have

$$e(G) \leq e(G') + (n-k)k \leq e(H') + k(n-k) = e(H)$$ ■

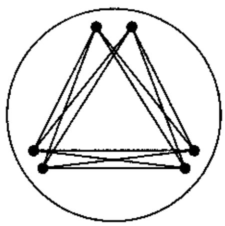


In fact, the Turán graph is the unique extremal graph (Exercise 21). Exercises 16–24 pertain to Turán's Theorem, including alternative proofs, the value of $e(T_{n,r})$, and applications. The argument used in Theorem 1.3.23 was simply one instance of the induction step in Theorem 5.2.9.

Turán's theorem applies to extremal problems when some condition forbids cliques of a given order; we describe a geometric application from Bondy–Murty [1976, p113–115].

5.2.10.* Example. *Distant pairs of points.* In a circular city of diameter 1, we might want to locate n police cars to maximize the number of pairs that are far apart, say separated by distance more than $d = 1/\sqrt{2}$. If six cars occupy equally spaced points on the circle, then the only pairs not more than d apart are the consecutive pairs around the outside: there are nine good pairs.

Instead, putting two cars each near the vertices of an equilateral triangle with side-length $\sqrt{3}/2$ yields three bad pairs and twelve good pairs. (This may not be the socially best criterion!) In general, with $\lceil n/3 \rceil$ or $\lfloor n/3 \rfloor$ cars near each vertex of this triangle, the good pairs correspond to edges of the tripartite Turán graph. We show next that this construction is best. ■

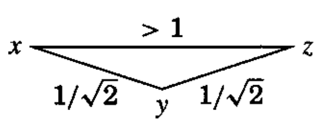


5.2.11.* Application. In a set of n points in the plane with no pair more than distance 1 apart, the maximum number of pairs separated by distance more than $1/\sqrt{2}$ is $\lfloor n^2/3 \rfloor$.

Proof: Draw a graph G on these points by making points adjacent when the distance between them exceeds $1/\sqrt{2}$. By Turán's Theorem and the construction in Example 5.2.10, it suffices to show that G has no K_4.

Among any four points, some three form an angle of at least 90°: if the four form a convex quadrilateral, then the interior angles sum to 360°, and if one point is inside the triangle formed by the others, then with them it forms three angles summing to 360°.

Suppose that G has a 4-vertex clique with points w, x, y, z, where $\angle xyz \geq 90°$. Since the lengths of xy and yz exceed $1/\sqrt{2}$, xz is longer than the hypotenuse of a right triangle with legs of length $1/\sqrt{2}$. Hence the distance between x and z exceeds 1, which contradicts the hypothesis. ■



Even without the full structural statement of Turán's Theorem, one can prove directly a rough bound on the number of edges in an n-vertex graph with no K_{r+1} (Exercise 16). Turning this around yields a sharp lower bound on the chromatic number of a graph in terms of the number of vertices and number of edges (Exercise 17).

COLOR-CRITICAL GRAPHS

The Turán graph solves a problem that is somehow opposite to understanding what forces chromatic number k. It considers the maximal graphs that *avoid* needing k colors instead of the minimal graphs that *do* need k colors.

Every k-chromatic graph has a k-critical subgraph, since we can continue discarding edges and isolated vertices without reducing the chromatic number until we reach a point where every such deletion reduces the chromatic number. Thus knowing the k-critical graphs could help us test for $k-1$-colorability. We begin with elementary properties of k-critical graphs.

5.2.12. Remark. A graph G with no isolated vertices is color-critical if and only if $\chi(G-e) < \chi(G)$ for every $e \in E(G)$. Hence when we prove that a connected graph is color-critical, we need only compare it with subgraphs obtained by deleting a single edge. ■

5.2.13. Proposition. Let G be a k-critical graph.

a) For $v \in V(G)$, there is a proper k-coloring of G in which the color on v appears nowhere else, and the other $k-1$ colors appear on $N(v)$.

b) For $e \in E(G)$, every proper $k-1$-coloring of $G-e$ gives the same color to the two endpoints of e.

Proof: (a) Given a proper $k-1$-coloring f of $G-v$, adding color k on v alone completes a proper k-coloring of G. The other colors must all appear on $N(v)$, since otherwise assigning a missing color to v would complete a proper $k-1$-coloring of G.

(b) If some proper $k-1$-coloring of $G-e$ gave distinct colors to the endpoints of e, then adding e would yield a proper $k-1$-coloring of G. ■

For any graph G, Proposition 5.2.13a holds for every $v \in V(G)$ such that $\chi(G-v) < \chi(G) = k$, and Proposition 5.2.13b holds for every $e \in E(G)$ such that $\chi(G-e) < \chi(G) = k$.

5.2.14. Example. The graph $C_5 \vee K_s$ of Example 5.1.8 is color-critical. In general, the join of two color-critical graphs is always color-critical. This is easy to prove using Remark 5.2.12 and Proposition 5.2.13 by considering cases for the deletion of an edge; the deleted edge e may belong to G or H or have an endpoint in each (Exercise 3). ■

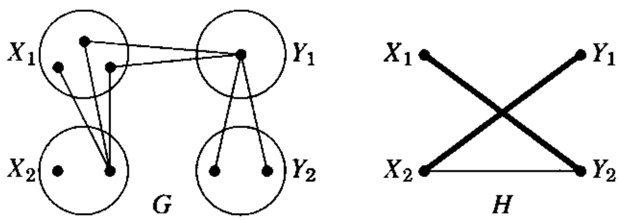
We proved in Lemma 5.1.18 that $\delta(G) \geq k-1$ when G is a k-critical graph. We can strengthen this to $\kappa'(G) \geq k-1$ by using the König–Egerváry Theorem.

5.2.15. Lemma. (Kainen) Let G be a graph with $\chi(G) > k$, and let X, Y be a partition of $V(G)$. If $G[X]$ and $G[Y]$ are k-colorable, then the edge cut $[X, Y]$ has at least k edges.

Proof: Let $X_1, \ldots, X_k$ and $Y_1, \ldots, Y_k$ be the partitions of X and Y formed by the color classes in proper k-colorings of $G[X]$ and $G[Y]$. If there is no edge between X_i and Y_j, then $X_i \cup Y_j$ is an independent set in G. We show that if $|[X, Y]| < k$, then we can combine color classes from $G[X]$ and $G[Y]$ in pairs to form a proper k-coloring of G.

Form a bipartite graph H with vertices $X_1, \ldots, X_k$ and $Y_1, \ldots, Y_k$, putting $X_iY_j \in E(H)$ if in G there is *no edge* between the set X_i and the set Y_j. If $|[X, Y]| < k$, then H has more than $k(k-1)$ edges. Since m vertices can cover at most km edges in a subgraph of $K_{k,k}$, $E(H)$ cannot be covered by $k-1$ vertices. By the König–Egerváry Theorem, H therefore has a perfect matching M.

In G, we give color i to all of X_i and all of the set Y_j to which it is matched by M. Since there are no edges joining X_i and Y_j, doing this for all i produces a proper k-coloring of G, which contradicts the hypothesis that $\chi(G) > k$. Hence we conclude that $|[X, Y]| \geq k$. ∎

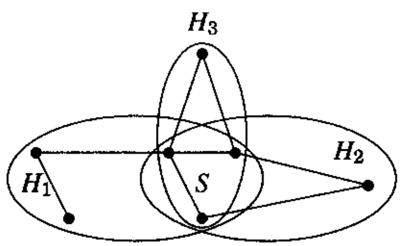


5.2.16. Theorem. (Dirac [1953]) Every k-critical graph is $k-1$-edge-connected.

Proof: Let G be a k-critical graph, and let $[X, Y]$ be a minimum edge cut. Since G is k-critical, $G[X]$ and $G[Y]$ are $k-1$-colorable. Applied with $k-1$ as the parameter, Lemma 5.2.15 then states that $|[X, Y]| \geq k-1$. ∎

Although a k-critical graph must be $k-1$-edge-connected, it need not be $k-1$-connected; Exercise 32 shows how to construct k-critical graphs that have connectivity 2. Nevertheless, we can restrict the behavior of small vertex cutsets in k-critical graphs.

5.2.17. Definition. Let S be a set of vertices in a graph G. An S-**lobe** of G is an induced subgraph of G whose vertex set consists of S and the vertices of a component of $G - S$.



For every $S \subseteq V(G)$, the graph G is the union of its S-lobes. We use this to prove a statement about vertex cutsets in k-critical graphs that will be useful in the next theorem. Exercise 33 strengthens the result when $|S| = 2$.

5.2.18. Proposition. If G is k-critical, then G has no cutset consisting of pairwise adjacent vertices. In particular, if G has a cutset $S = \{x, y\}$, then $x \not\leftrightarrow y$ and G has an S-lobe H such that $\chi(H + xy) = k$.

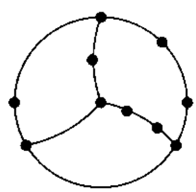
Proof: Let S be a cutset in a k-critical graph G. Let $H_1, \ldots, H_t$ be the S-lobes of G. Since each H_i is a proper subgraph of a k-critical graph, each H_i is $k-1$-colorable. If each H_i has a proper $k-1$-coloring giving distinct colors to the vertices of S, then the names of the colors in these $k-1$-colorings can be permuted to agree on S. The colorings then combine to form a $k-1$-coloring of G, which is impossible.

Hence some S-lobe H has no proper $k-1$-coloring with distinct colors on S. This implies that S is not a clique. If $S = \{x, y\}$, then every $k-1$-coloring of H assigns the same color to x and y, and hence $H + xy$ is not $k-1$-colorable. ■

FORCED SUBDIVISIONS

We need not have a k-clique to have chromatic number k, but perhaps we must have some weakened form of a k-clique.

5.2.19. Definition. An **H-subdivision** (or **subdivision of** H) is a graph obtained from a graph H by successive edge subdivisions (Definition 5.2.19). Equivalently, it is a graph obtained from H by replacing edges with pairwise internally disjoint paths.



a subdivision of K_4

5.2.20. Theorem. (Dirac [1952a]) Every graph with chromatic number at least 4 contains a K_4-subdivision.

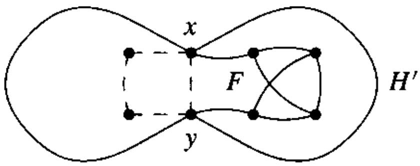
Proof: We use induction on $n(G)$.

Basis step: $n(G) = 4$. The graph G can only be K_4 itself.

Induction step: $n(G) > 4$. Since $\chi(G) \geq 4$, we may let H be a a 4-critical subgraph of G. By Proposition 5.2.18, H has no cut-vertex. If $\kappa(H) = 2$ and $S = \{x, y\}$ is a cutset of size 2, then by Proposition 5.2.18 $x \not\leftrightarrow y$ and H has an S-lobe H' such that $\chi(H' + xy) \geq 4$. Since $n(H') < n(G)$, we can apply the induction hypothesis to obtain a K_4-subdivision in H'.

This K_4-subdivision F appears also in G unless it contains xy (see figure below). In this case, we modify F to obtain a K_4-subdivision in G by replacing the edge xy with an x, y-path through another S-lobe of H. Such a path exists because the minimality of the cutset S implies that each vertex of S has a neighbor in each component of $H - S$.

Hence we may assume that H is 3-connected. Select a vertex $x \in V(G)$. Since $H - x$ is 2-connected, it has a cycle C of length at least 3. (Let x be the central vertex and C the outside cycle in the figure above.) Since H is 3-connected, the Fan Lemma (Theorem 4.2.23) yields an $x, V(C)$-fan of size 3 in H. These three paths, together with C, form a K_4-subdivision in H. ■



5.2.21.* Remark. Hajós [1961] conjectured that every k-chromatic graph contains a subdivision of K_k. For $k = 2$, the statement says that every 2-chromatic graph has a nontrivial path. For $k = 3$, it says that every 3-chromatic graph has a cycle. Theorem 5.2.20 proves it for $k = 4$, and it is open for $k \in \{5, 6\}$.

Hajós' Conjecture is false for $k \geq 7$ (Catlin [1979]—see Exercise 40). Hadwiger [1943] proposed a weaker conjecture: every k-chromatic graph has a subgraph that becomes K_k via edge contractions. This is weaker because a K_k-subdivision is a special subgraph of this type. For $k = 4$, Hadwiger's Conjecture is equivalent to Theorem 5.2.20. For $k = 5$, it is equivalent to the Four Color Theorem (Chapter 6). For $k = 6$, it was proved using the Four Color Theorem by Robertson, Seymour, and Thomas [1993]. For $k \geq 7$, it remains open. ■

Some results about k-critical graphs extend to the larger class of graphs with $\delta(G) \geq k - 1$. For example, every graph with minimum degree at least 3 has a K_4-subdivision (Exercise 38); this strengthens Theorem 5.2.20.

Dirac [1965] and Jung [1965] proved that sufficiently large chromatic number forces a K_k-subdivision in G. Mader improved this by weakening the hypothesis and generalizing the conclusion: for a simple graph F, every simple graph G with $\delta(G) \geq 2^{e(F)}$ contains a subdivision of F. The threshold $2^{e(F)}$ is larger than necessary but permits a short proof.

5.2.22.* Lemma. (Mader [1967], see Thomassen [1988]) If G is a simple graph with minimum degree at least $2k$, then G contains disjoint subgraphs G' and H such that 1) H is connected, 2) $\delta(G') \geq k$, and 3) each vertex of G' has a neighbor in H.

Proof: We may assume that G is connected. Let $G \cdot H'$ denote the graph obtained from G by contracting the edges of a connected subgraph H' and delete extra copies of multiple edges. In $G \cdot H'$, the set $V(H')$ becomes a single vertex. Consider all connected subgraphs H' of G such that $G \cdot H'$ has at least

$k(n(G) - n(H') + 1)$ edges. Since $\delta(G) \geq 2k$, every 1-vertex subgraph of G is such a subgraph. Since such subgraphs exist, we may choose H to be a maximal subgraph with this property.

Let S be the set of vertices outside H with neighbors in H, and let $G' = G[S]$. We need only show that $\delta(G') \geq k$. Each $x \in V(G')$ has a neighbor $y \in V(H)$. In $G \cdot (H \cup xy)$, the edges incident to x in G' collapse onto edges from $V(G')$ to H that appear in $G \cdot H$, and the edge xy contracts. Hence $e(G \cdot H) - e(G \cdot (H \cup xy)) = d_{G'}(x) + 1$. By the choice of H, this difference is more than k, and hence $\delta(G') \geq k$. ■

5.2.23.* Theorem. (Mader [1967], see Thomassen [1988]) If F and G are simple graphs with $e(F) = m$ and $\delta(F) \geq 1$, then $\delta(G) \geq 2^m$ implies that G contains a subdivision of F.

Proof: We use induction on m. The claim is trivial for $m \leq 1$. Consider $m \geq 2$. By Lemma 5.2.22, we may choose disjoint subgraphs H and G' in G such that H is connected, $\delta(G') \geq 2^{m-1}$, and every vertex of G' has a neighbor in H.

If F has an edge $e = xy$ such that $\delta(F - e) \geq 1$, then the induction hypothesis yields a subdivision J of $F - e$ in G'. A path through H can be added between the vertices of J representing x and y to complete a subdivision of F.

If $\delta(F - e) = 0$ for all $e \in E(F)$, then every edge of F is incident to a leaf. Now F is a forest of stars, and $\delta(G) \geq 2^m \geq 2m$ allows us to find F itself in G; we leave this claim to Exercise 42. ■

5.2.24.* Remark. The case when F is a complete graph remains of particular interest. Let $f(k)$ be the minimum d such that every graph with minimum degree at least d contains a K_k-subdivision. Theorem 5.2.23 yields $f(k) \leq 2^{\binom{k}{2}}$. Komlós–Szemerédi [1996] and Bollobás–Thomason [1998] proved that $f(k) < ck^2$ for some constant c (the latter shows $c \leq 256$). Since $K_{m,m-1}$ has no K_{2k}-subdivision when $m = k(k+1)/2$ (Exercise 41), we have $f(k) > k^2/8$.

Exercise 38 yields $f(4) = 3$. Furthermore, $f(5) = 6$. The icosahedron (Exercise 7.3.8) yields $f(5) \geq 6$, since this graph is 5-regular and has no K_5-subdivision. On the other hand, Mader [1998] proved Dirac's conjecture [1964] that every n-vertex graph with at least $3n - 5$ edges contains a K_5-subdivision. By the degree-sum formula, $\delta(G) \geq 6$ yields at least $3n$ edges; hence $f(5) \leq 6$.

Finally, we note that Scott [1997] proved a subdivision version of the Gyárfas–Sumner Conjecture (Remark 5.2.4) for each tree T and integer k: If G has with no k-clique but $\chi(G)$ is sufficiently large, then G contains a subdivision of T as an *induced* subgraph. ■

EXERCISES

5.2.1. (−) Let G be a graph such that $\chi(G - x - y) = \chi(G) - 2$ for all pairs x, y of distinct vertices. Prove that G is a complete graph. (Comment: Lovász conjectured that the conclusion also holds when the condition is imposed only on pairs of adjacent vertices.)

5.2.2. $(-)$ Prove that a simple graph is a complete multipartite graph if and only if it has no 3-vertex induced subgraph with one edge.

5.2.3. $(-)$ The results below imply that there is no k-critical graph with $k+1$ vertices.

a) Let x and y be vertices in a k-critical graph G. Prove that $N(x) \subseteq N(y)$ is impossible. Conclude that no k-critical graph has $k+1$ vertices.

b) Prove that $\chi(G \vee H) = \chi(G) + \chi(H)$, and that $G \vee H$ is color-critical if and only if both G and H are color-critical. Conclude that $C_5 \vee K_{k-3}$, with $k+2$ vertices, is k-critical.

5.2.4. For $n \in \mathbb{N}$, let G be the graph with vertex set $\{v_0, \ldots, v_{3n}\}$ defined by $v_i \leftrightarrow v_j$ if and only if $|i-j| \le 2$ and $i+j$ is not divisible by 6.

a) Determine the blocks of G.

b) Prove that adding the edge $v_0 v_{3n}$ to G creates a 4-critical graph.

5.2.5. $(-)$ Find a subdivision of K_4 in the Grötzsch graph (Example 5.2.2).

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5.2.6. Determine the minimum number of edges in a connected n-vertex graph with chromatic number k. (Hint: Consider a k-critical subgraph.) (Eršov–Kožuhin [1962]—see Bhasker–Samad–West [1994] for higher connectivity.)

5.2.7. (!) Given an optimal coloring of a k-chromatic graph, prove that for each color i there is a vertex with color i that is adjacent to vertices of the other $k-1$ colors.

5.2.8. Use properties of color-critical graphs to prove Proposition 5.1.14 again: $\chi(G) \le 1 + \max_i \min\{d_i, i-1\}$, where $d_1 \ge \cdots \ge d_n$ are the vertex degrees in G.

5.2.9. (!) Prove that if G is a color-critical graph, then the graph G' generated from it by applying Mycielski's construction is also color-critical.

5.2.10. Given a graph G with vertex set $v_1, \ldots, v_n$, let G' be the graph generated from G by Mycielski's construction. Let H be a subgraph of G. Let G'' be the graph obtained from G' by adding the edges $\{u_i u_j \colon v_i v_j \in E(H)\}$. Prove that $\chi(G'') = \chi(G) + 1$ and that $\omega(G'') = \max\{\omega(G), \omega(H) + 1\}$. (Pritikin)

5.2.11. (!) Prove that if G has no induced $2K_2$, then $\chi(G) \le \binom{\omega(G)+1}{2}$. (Hint: Use a maximum clique to define a collection of $\binom{\omega(G)}{2} + \omega(G)$ independent sets that cover the vertices. Comment: This is a special case of the Gyárfás–Sumner Conjecture—Remark 5.2.4) (Wagon [1980])

5.2.12. (!) Let $G_1 = K_1$. For $k > 1$, construct G_k as follows. To the disjoint union $G_1 + \cdots + G_{k-1}$, and add an independent set T of size $\prod_{i=1}^{k-1} n(G_i)$. For each choice of $(v_1, \ldots, v_{k-1})$ in $V(G_1) \times \cdots \times V(G_{k-1})$, let one vertex of T have neighborhood $\{v_1, \ldots, v_{k-1}\}$. (In the sketch of G_4 below, neighbors are shown for only two elements of T.)

a) Prove that $\omega(G_k) = 2$ and $\chi(G_k) = k$. (Zykov [1949])

b) Prove that G_k is k-critical. (Schäuble [1969])