31

Concrete Categories

In the 20th century, we find a great deal of concrete, practical mathematics. Statistics is flourishing, the computer has proved the Four Colour Theorem, and numbers with upwards of two hundred digits can be factored.

Another trend in the 20th century is a degree of abstraction never seen before in mathematics. For example, the study of the Euclidean plane has been replaced by the study of vector spaces and topological spaces that abstract some of its properties. A prominant and influential proponent of this trend in algebra was Emmy Noether (1882–1935). The supreme abstraction is the notion of a *category*, to which we shall turn our attention in this chapter.

Nowadays, when studying vector spaces, we are forced to look also at linear transformations. Similarly, when studying topological spaces, we are led to continuous mappings. When studying groups (themselves an abstraction of permutation groups), we have to look at homomorphisms. To abstract the properties which these examples have in common, we introduce the notion of a 'concrete category'.

A *concrete category* is a class of sets, each endowed with a certain structure, together with the class of all functions which map one set to another while preserving this structure.

EXAMPLE 31.1
The class of sets together with the class of all functions between them is a concrete category. Here there is no structure to preserve, so the condition on the functions is trivially satisfied.

EXAMPLE 31.2
A *monoid* is a set containing a special *identity* element 1 together with a binary operation $\cdot$ between members of that set, such that $(a \cdot b) \cdot c = a \cdot (b \cdot c)$, and $1 \cdot a = a \cdot 1 = a$. A *monoid homomorphism* is a function f from a monoid A to another monoid A' which preserves structure: $f(a \cdot b) = f(a) \cdot f(b)$ and $f(1) = 1$. (When we write '$f(1) = 1$' it is understood that the first 1 is the special element of A and the second 1 is the special element of A'.) For example, taking 0 as the special element and $+$ as the binary operation, the natural numbers form a monoid. As another example, the singleton set $\{1\}$ is a monoid with $1 \cdot 1 = 1$. The mapping from the natural numbers to $\{1\}$ is a monoid homomorphism. Note that group homomorphisms are a special case of monoid homomorphisms. The class of monoids, together with the monoid homomorphisms forms a concrete category.

EXAMPLE 31.3
A *pre-ordered set* is a set, together with a binary relation $\leq$ on that set which is reflexive and transitive: $a \leq a$, and if $a \leq b$ and $b \leq c$ then $a \leq c$. A *monotone mapping* is a function f from a pre-ordered set A to a pre-ordered set A' that preserves the order: if $a \leq b$ then $f(a) \leq f(b)$.

For example, both the natural numbers and the even numbers together with the relation $\leq$ are pre-ordered sets. The doubling function maps the natural numbers into the set of even numbers in an order preserving way, and is thus a monotone mapping. The collection of all pre-ordered sets together with all monotone mappings between them forms a concrete category.

Note that in all cases the identity function on a set will preserve its structure. Also, the composition of two structure preserving functions will preserve structure.

More abstract than a concrete category is a *category*, which we shall define in Chapter 32.

Exercise

1. Verify that groups and group homomorphisms form a concrete category.

32

Graphs and Categories

A *graph* (more precisely: an *oriented multigraph*) consists of a class of *arrows* (or *directed edges*) together with a class of *objects* (or *nodes*), and also two mappings from the class of arrows to the class of objects. The mappings are called S (*source* or *domain*) and T (*target* or *codomain*).

$$\{\text{arrows}\} \underset{\text{target}}{\overset{\text{source}}{\rightrightarrows}} \{\text{objects}\}$$

If f is an arrow, $S(f) = A$ and $T(f) = B$, we write

$$f : A \to B \text{ or } A \xrightarrow{f} B.$$

A *category* is a graph subject to the following conditions:

1. associated with any two arrows $f : A \to B$ and $g : B \to C$ (so that $T(f) = S(g)$) is an arrow $g \circ f : A \to C$;

2. if $f : A \to B$, $g : B \to C$ and $h : C \to D$, then $(h \circ g) \circ f = h \circ (g \circ f)$;

3. associated with each object A, there is an identity arrow 1_A, whose source and target are A;

4. if $f : A \to B$ then $f \circ 1_A = f$; if $g : B \to A$ then $1_A \circ g = g$.