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Renaissance Mathematics Continued

(3) **The invention of logarithms**

John Napier (1550–1617), a Scottish aristocrat, spent 20 years of his life on the construction of logarithms. Like Stifel, he was interested in proving that the Pope was the Antichrist (the opponent of Jesus, who is supposed to appear just prior to the latter's prophesied second coming). Napier was convinced that the world would end before 1700.

Napier describes his technique for calculating logs in his *Mirifici Logarithmorum Canonis Constructio*, which was published in 1619, two years after the author's death.

This *Construction of the Wonderful Canon of Logarithms* is tricky to read, because what Napier calls the 'logarithm' of x is actually what we call $10^7 \log_{1/e}(x/10^7)$. The book is also a bit tedious because Napier pays minute attention to error bounds. However, if we simplify and modernize Napier's presentation somewhat — as we shall do below — we obtain a very lucid piece of mathematics.

Napier's basic ideas are these. Suppose there is a particle on the negative half of the real number line, moving towards the origin at a speed proportional to its distance from the origin. At time 0, it is at -1. For any distance d, there is a time when the particle is that distance from the origin. Call this time the 'logarithm' of d.

Assume that the constant of proportionality is 1. Then $dx/dt = -x$ and $x(0) = -1$. Hence $x(t) = -e^{-t}$, and so $t = -\log_e d = \log_{1/e} d$. In this section '$\log x$' shall mean $\log_{1/e} x$. Note that $\log_{1/e} d$ decreases as d increases from $1/2$ to 1.

Napier used the following three ideas to calculate his tables.

1. When z is very small, $\log(1-z) \approx z$.

2. The natural number powers of $(1-10^{-m})$ — where m is a natural number — are easy to calculate. For, if $s = 1-10^{-m}$, then $s^2 = s(1-10^{-m}) = s - s/10^m$. Similarly, $s^3 = s^2 - s^2/10^m$. It is always just a matter of shifting the decimal point m places and subtracting. Thus Napier has

$$\begin{aligned}
1-10^{-5} &= 0.999\,990\,000\,000\,000, \\
(1-10^{-5})^2 &= 0.999\,990\,000\,000\,000 \\
&\quad -.000\,009\,999\,900\,000 \\
&= 0.999\,980\,000\,100\,000, \\
(1-10^{-5})^3 &= 0.999\,980\,000\,100\,000 \\
&\quad -.000\,009\,999\,800\,001 \\
&= 0.999\,970\,000\,299\,999.
\end{aligned}$$

3. Near 1, the log is smooth, and linear interpolations give excellent approximations to it.

Near -1, Napier's particle is moving at about 1 unit/second, and we can assume that $\log d = 1 - d$ for $d = 1 - 10^{-5} = 0.99999$. As in (2) above, Napier calculates $d, d^2, \ldots, d^{50}$. He finds that $d^{50} = 0.9995001225$. Hence $\log 0.9995001225 = 50 \times \log d = 50 \times 0.99999$. With linear interpolation, Napier can thus obtain a very accurate value for $\log u$, where $u = 0.9995$.

Next, using ideas similar to those in (2), Napier quickly and accurately calculates $u^2, u^3, \ldots, u^{20}$. He obtains $u^{20} = 0.990047358$. Using interpolation, he then gets a very accurate value for $\log w$, where $w = 0.99$. (Your pocket calculator will not be more accurate.)

For $a = 1, 2, \ldots, 20$ and $b = 0, 1, \ldots, 68$, Napier calculates $u^a w^b$. This gives him 1380 points between $u = 0.9995$ and $u^{20}w^{68} = 0.499860940$ for calculating logarithms in that range.

For example, $u^{19}w^{68} = 0.500110996 > 1/2 > 0.499860940 = u^{20}w^{68}$. If

$$k = \frac{\log(u^{19}w^{68}) - \log(u^{20}w^{68})}{u^{19}w^{68} - u^{20}w^{68}},$$

then k is the slope of the line joining two given points. Thus, by linear interpolation we have

$$\begin{aligned}
\log 1/2 &\approx \log(u^{20}w^{68}) + (\tfrac{1}{2} - u^{20}w^{68})k \\
&\approx 20 \log u + 68 \log w - 0.000278 \\
&\approx 0.693147.
\end{aligned}$$

This value of $\log 1/2$ is accurate to six decimal places.