$$f(x_1, x_2, x_3, x_4) = c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4$$

are in the annihilator of W?

8. Let W be the subspace of R^5 which is spanned by the vectors

$$\alpha_1 = \epsilon_1 + 2\epsilon_2 + \epsilon_3, \qquad \alpha_2 = \epsilon_2 + 3\epsilon_3 + 3\epsilon_4 + \epsilon_5$$
$$\alpha_3 = \epsilon_1 + 4\epsilon_2 + 6\epsilon_3 + 4\epsilon_4 + \epsilon_5.$$

Find a basis for W^0.

9. Let V be the vector space of all 2×2 matrices over the field of real numbers, and let

$$B = \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix}.$$

Let W be the subspace of V consisting of all A such that $AB = 0$. Let f be a linear functional on V which is in the annihilator of W. Suppose that $f(I) = 0$ and $f(C) = 3$, where I is the 2×2 identity matrix and

$$C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Find $f(B)$.

10. Let F be a subfield of the complex numbers. We define n linear functionals on F^n $(n \geq 2)$ by

$$f_k(x_1, \ldots, x_n) = \sum_{j=1}^{n} (k - j)x_j, \qquad 1 \leq k \leq n.$$

What is the dimension of the subspace annihilated by $f_1, \ldots, f_n$?

11. Let W_1 and W_2 be subspaces of a finite-dimensional vector space V.

(a) Prove that $(W_1 + W_2)^0 = W_1^0 \cap W_2^0$.
(b) Prove that $(W_1 \cap W_2)^0 = W_1^0 + W_2^0$.

12. Let V be a finite-dimensional vector space over the field F and let W be a subspace of V. If f is a linear functional on W, prove that there is a linear functional g on V such that $g(\alpha) = f(\alpha)$ for each α in the subspace W.

13. Let F be a subfield of the field of complex numbers and let V be any vector space over F. Suppose that f and g are linear functionals on V such that the function h defined by $h(\alpha) = f(\alpha)g(\alpha)$ is also a linear functional on V. Prove that either $f = 0$ or $g = 0$.

14. Let F be a field of characteristic zero and let V be a finite-dimensional vector space over F. If $\alpha_1, \ldots, \alpha_m$ are finitely many vectors in V, each different from the zero vector, prove that there is a linear functional f on V such that

$$f(\alpha_i) \neq 0, \qquad i = 1, \ldots, m.$$

15. According to Exercise 3, similar matrices have the same trace. Thus we can define the trace of a linear operator on a finite-dimensional space to be the trace of any matrix which represents the operator in an ordered basis. This is well-defined since all such representing matrices for one operator are similar.

Now let V be the space of all 2×2 matrices over the field F and let P be a fixed 2×2 matrix. Let T be the linear operator on V defined by $T(A) = PA$. Prove that trace $(T) = 2$ trace (P).

16. Show that the trace functional on $n \times n$ matrices is unique in the following sense. If W is the space of $n \times n$ matrices over the field F and if f is a linear functional on W such that $f(AB) = f(BA)$ for each A and B in W, then f is a scalar multiple of the trace function. If, in addition, $f(I) = n$, then f is the trace function.

17. Let W be the space of $n \times n$ matrices over the field F, and let W_0 be the subspace spanned by the matrices C of the form $C = AB - BA$. Prove that W_0 is exactly the subspace of matrices which have trace zero. (*Hint:* What is the dimension of the space of matrices of trace zero? Use the matrix 'units,' i.e., matrices with exactly one non-zero entry, to construct enough linearly independent matrices of the form $AB - BA$.)

3.6. The Double Dual

One question about dual bases which we did not answer in the last section was whether every basis for V^* is the dual of some basis for V. One way to answer that question is to consider V^{**}, the dual space of V^*.

If α is a vector in V, then α induces a linear functional L_α on V^* defined by

$$L_\alpha(f) = f(\alpha), \qquad f \text{ in } V^*.$$

The fact that L_α is linear is just a reformulation of the definition of linear operations in V^*:

$$\begin{aligned} L_\alpha(cf + g) &= (cf + g)(\alpha) \\ &= (cf)(\alpha) + g(\alpha) \\ &= cf(\alpha) + g(\alpha) \\ &= cL_\alpha(f) + L_\alpha(g). \end{aligned}$$

If V is finite-dimensional and $\alpha \neq 0$, then $L_\alpha \neq 0$; in other words, there exists a linear functional f such that $f(\alpha) \neq 0$. The proof is very simple and was given in Section 3.5: Choose an ordered basis $\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}$ for V such that $\alpha_1 = \alpha$ and let f be the linear functional which assigns to each vector in V its first coordinate in the ordered basis $\mathcal{B}$.

Theorem 17. *Let* V *be a finite-dimensional vector space over the field* F. *For each vector α in* V *define*

$$\mathrm{L}_\alpha(\mathrm{f}) = \mathrm{f}(\alpha), \qquad \mathrm{f} \textit{ in } \mathrm{V}^*.$$

The mapping $\alpha \to \mathrm{L}_\alpha$ is then an isomorphism of V *onto* V**.

Proof. We showed that for each α the function L_α is linear. Suppose α and β are in V and c is in F, and let $\gamma = c\alpha + \beta$. Then for each f in V^*

$$\begin{aligned} L_\gamma(f) &= f(\gamma) \\ &= f(c\alpha + \beta) \\ &= cf(\alpha) + f(\beta) \\ &= cL_\alpha(f) + L_\beta(f) \end{aligned}$$

and so

$$L_\gamma = cL_\alpha + L_\beta.$$

This shows that the mapping $\alpha \rightarrow L_\alpha$ is a linear transformation from V into V^{**}. This transformation is non-singular; for, according to the remarks above $L_\alpha = 0$ if and only if $\alpha = 0$. Now $\alpha \rightarrow L_\alpha$ is a non-singular linear transformation from V into V^{**}, and since

$$\dim V^{**} = \dim V^* = \dim V$$

Theorem 9 tells us that this transformation is invertible, and is therefore an isomorphism of V onto V^{**}. ▌

Corollary. *Let* V *be a finite-dimensional vector space over the field* F. *If* L *is a linear functional on the dual space* V* *of* V, *then there is a unique vector* α *in* V *such that*

$$\mathrm{L}(\mathrm{f}) = \mathrm{f}(\alpha)$$

for every f *in* V*.

Corollary. *Let* V *be a finite-dimensional vector space over the field* F. *Each basis for* V* *is the dual of some basis for* V.

Proof. Let $\mathcal{B}^* = \{f_1, \ldots, f_n\}$ be a basis for V^*. By Theorem 15, there is a basis $\{L_1, \ldots, L_n\}$ for V^{**} such that

$$L_i(f_j) = \delta_{ij}.$$

Using the corollary above, for each i there is a vector α_i in V such that

$$L_i(f) = f(\alpha_i)$$

for every f in V^*, i.e., such that $L_i = L_{\alpha_i}$. It follows immediately that $\{\alpha_1, \ldots, \alpha_n\}$ is a basis for V and that $\mathcal{B}^*$ is the dual of this basis. ▌

In view of Theorem 17, we usually identify α with L_α and say that V 'is' the dual space of V^* or that the spaces V, V^* are naturally in duality with one another. Each is the dual space of the other. In the last corollary we have an illustration of how that can be useful. Here is a further illustration.

If E is a subset of V^*, then the annihilator E^0 is (technically) a subset of V^{**}. If we choose to identify V and V^{**} as in Theorem 17, then E^0 is a subspace of V, namely, the set of all α in V such that $f(\alpha) = 0$ for all f in E. In a corollary of Theorem 16 we noted that each subspace W is determined by its annihilator W^0. How is it determined? The answer is that W is the subspace annihilated by all f in W^0, that is, the intersection of the null spaces of all f's in W^0. In our present notation for annihilators, the answer may be phrased very simply: $W = (W^0)^0$.

Theorem 18. *If* S *is any subset of a finite-dimensional vector space* V, *then* $(\mathrm{S}^0)^0$ *is the subspace spanned by* S.

Proof. Let W be the subspace spanned by S. Clearly $W^0 = S^0$. Therefore, what we are to prove is that $W = W^{00}$. We have given one proof. Here is another. By Theorem 16

$$\dim W + \dim W^0 = \dim V$$
$$\dim W^0 + \dim W^{00} = \dim V^*$$

and since $\dim V = \dim V^*$ we have

$$\dim W = \dim W^{00}.$$

Since W is a subspace of W^{00}, we see that $W = W^{00}$. ∎

The results of this section hold for arbitrary vector spaces; however, the proofs require the use of the so-called Axiom of Choice. We want to avoid becoming embroiled in a lengthy discussion of that axiom, so we shall not tackle annihilators for general vector spaces. But, there are two results about linear functionals on arbitrary vector spaces which are so fundamental that we should include them.

Let V be a vector space. We want to define hyperspaces in V. Unless V is finite-dimensional, we cannot do that with the dimension of the hyperspace. But, we can express the idea that a space N falls just one dimension short of filling out V, in the following way:

1. N is a proper subspace of V;
2. if W is a subspace of V which contains N, then either $W = N$ or $W = V$.

Conditions (1) and (2) together say that N is a proper subspace and there is no larger proper subspace, in short, N is a maximal proper subspace.

Definition. *If* V *is a vector space, a* **hyperspace** *in* V *is a maximal proper subspace of* V.

Theorem 19. *If* f *is a non-zero linear functional on the vector space* V, *then the null space of* f *is a hyperspace in* V. *Conversely, every hyperspace in* V *is the null space of a (not unique) non-zero linear functional on* V.

Proof. Let f be a non-zero linear functional on V and N_f its null space. Let α be a vector in V which is not in N_f, i.e., a vector such that $f(\alpha) \neq 0$. We shall show that every vector in V is in the subspace spanned by N_f and α. That subspace consists of all vectors

$$\gamma + c\alpha, \qquad \gamma \text{ in } N_f,\ c \text{ in } F.$$

Let β be in V. Define

$$c = \frac{f(\beta)}{f(\alpha)}$$

which makes sense because $f(\alpha) \neq 0$. Then the vector $\gamma = \beta - c\alpha$ is in N_f since

$$\begin{aligned} f(\gamma) &= f(\beta - c\alpha) \\ &= f(\beta) - cf(\alpha) \\ &= 0. \end{aligned}$$

So β is in the subspace spanned by N_f and α.

Now let N be a hyperspace in V. Fix some vector α which is not in N. Since N is a maximal proper subspace, the subspace spanned by N and α is the entire space V. Therefore each vector β in V has the form

$$\beta = \gamma + c\alpha, \qquad \gamma \text{ in } N,\ c \text{ in } F.$$

The vector γ and the scalar c are uniquely determined by β. If we have also

$$\beta = \gamma' + c'\alpha, \qquad \gamma' \text{ in } N,\ c' \text{ in } F.$$

then

$$(c' - c)\alpha = \gamma - \gamma'.$$

If $c' - c \neq 0$, then α would be in N; hence, $c' = c$ and $\gamma' = \gamma$. Another way to phrase our conclusion is this: If β is in V, there is a unique scalar c such that $\beta - c\alpha$ is in N. Call that scalar $g(\beta)$. It is easy to see that g is a linear functional on V and that N is the null space of g. ▮

Lemma. *If* f *and* g *are linear functionals on a vector space* V, *then* g *is a scalar multiple of* f *if and only if the null space of* g *contains the null space of* f, *that is, if and only if* $f(\alpha) = 0$ *implies* $g(\alpha) = 0$.

Proof. If $f = 0$ then $g = 0$ as well and g is trivially a scalar multiple of f. Suppose $f \neq 0$ so that the null space N_f is a hyperspace in V. Choose some vector α in V with $f(\alpha) \neq 0$ and let

$$c = \frac{g(\alpha)}{f(\alpha)}.$$

The linear functional $h = g - cf$ is 0 on N_f, since both f and g are 0 there, and $h(\alpha) = g(\alpha) - cf(\alpha) = 0$. Thus h is 0 on the subspace spanned by N_f and α—and that subspace is V. We conclude that $h = 0$, i.e., that $g = cf$. ▮

Theorem 20. *Let* $g, f_1, \ldots, f_r$ *be linear functionals on a vector space* V *with respective null spaces* $N, N_1, \ldots, N_r$. *Then* g *is a linear combination of* $f_1, \ldots, f_r$ *if and only if* N *contains the intersection* $N_1 \cap \cdots \cap N_r$.

Proof. If $g = c_1f_1 + \cdots + c_rf_r$ and $f_i(\alpha) = 0$ for each i, then clearly $g(\alpha) = 0$. Therefore, N contains $N_1 \cap \cdots \cap N_r$.

We shall prove the converse (the 'if' half of the theorem) by induction on the number r. The preceding lemma handles the case $r = 1$. Suppose we know the result for $r = k - 1$, and let $f_1, \ldots, f_k$ be linear functionals with null spaces $N_1, \ldots, N_k$ such that $N_1 \cap \cdots \cap N_k$ is contained in N, the