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John Stillwell

Numbers and Geometry

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Springer

John Stillwell
Monash University
Department of Mathematics
Clayton, Victoria
3168 Australia

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To my mother and my late father
who taught me how to count

Preface

What should every aspiring mathematician know? The answer for most of the 20th century has been: calculus. For 2000 years before that, the answer was Euclid. It now seems a good time to raise the question again, because the old answers are no longer convincing. Mathematics today is much more than Euclid, but it is also much more than calculus; and the calculus now taught is, sadly, much less than it used to be. Little by little, calculus has been deprived of the algebra, geometry, and logic it needs to sustain it, until many institutions have had to put it on high-tech life-support systems. A subject struggling to survive is hardly a good introduction to the vigor of real mathematics.

But if it were only a matter of putting the guts back into calculus it would not be necessary to write a new book. It would be enough to recommend, for example, Spivak's *Calculus*, or Hardy's *Pure Mathematics*. In the current situation, we need to revive not only calculus, but also algebra, geometry, and the whole idea that mathematics is a rigorous, cumulative discipline in which each mathematician stands on the shoulders of giants.

The best way to teach real mathematics, I believe, is to start deeper down, with the elementary ideas of number and space. Everyone concedes that these are fundamental, but they have been scandalously neglected, perhaps in the naive belief that anyone learning calculus has outgrown them. In fact, arithmetic, algebra, and geometry can never be outgrown, and the most rewarding path to higher mathematics sustains their development alongside the "advanced" branches such as calculus. Also, by maintaining ties between these disciplines, it is possible to present a more unified view of mathematics, yet at the same time to include more spice and variety.

The aim of this book, then, is to give a broad view of arithmetic, geometry, and algebra at the level of calculus, without being a calculus book (or a "precalculus" book). Its roots are in arithmetic and geometry, the two opposite poles of mathematics, and the source of historic conceptual conflict. The resolution of this conflict, and its role in the development of mathematics, is one of the main stories in the book. The key is algebra, which brings arithmetic and geometry together and allows them to flourish and branch out in new directions. To keep the story as simple as possible, I link everything to the algebraic themes of linear and quadratic equations.

The restriction to low-degree equations is not as dreadful as high school algebra might suggest. Even linear equations are interesting when only integer solutions are sought, and they neatly motivate a whole introductory course in number theory, from the Euclidean algorithm to unique prime factorization. Quadratic equations are even more interesting from the integer point of view, with Pythagorean triples and Pell's equation leading deep into algebra, geometry, and analysis. From the point of view of geometry, quadratic equations represent the conic sections—a fascinating topic in itself—and the areas bounded by these curves define the circular, logarithmic, and hyperbolic functions. In this way we are led to the subject matter of a first calculus course, but with less machinery and more time to probe fundamental questions such as the nature of numbers, curves, and area. It is worth mentioning that we also cover the main ideas of Euclid—geometry, arithmetic, and the theory of real numbers—but with 2000 years of extra insights added.

In fact, this book could be described as a deeper look at ordinary things. Most of mathematics is about numbers, curves, and functions, and the links between these concepts can be suggested by a thorough study of simple examples, such as the circle and the square. I hope to show that mathematics, like the world, fits William Blake's description:

> To see a World in a Grain of Sand,
> And a Heaven in a Wild Flower,
> Hold Infinity in the palm of your hand
> And Eternity in an hour.

Because it is virtually impossible to learn mathematics by mere reading, this book includes many exercises and every encouragement to do them. There is a set of exercises at the end of each section, so new ideas can be instantly tested, clarified, and reinforced. The exercises are often variations or generalizations of the results in the main text; in some cases I think they are even more interesting! In particular, they include simplified arrangements of many classic proofs by great mathematicians, from Euclid to Hilbert. Each set of exercises is accompanied by a commentary to make its purpose and significance clear. I hope this will be useful, particularly when several exercises have to be linked together to produce a big result.

Who is this book for? Because it presupposes only high school algebra, it can in principle be read by any well-prepared student entering university. It complements the usual courses, hence it can be offered as a "hard option" to students who are not sufficiently extended by the standard material at that level. On the other hand, it has so little in common with the calculus and linear algebra that dominates the standard curriculum that it may well be a revelation even to senior undergraduates. Many students now graduate in mathematics without having done a course in number theory, geometry, or foundations—for example, without having seen the fundamental theorem of arithmetic, non-Euclidean geometry, or the definition of real numbers. For such students, this book could serve as a capstone course in the senior year, presenting a unified approach to mathematics and proving many of the classic results that are normally only mentioned.

It could also be used in conventional courses. Chapters 1, 4, 6, 7, and parts of 8 and 9 contain most of the standard material for a first number theory course. Chapters 2, 3, 5, and 8 could serve as a first course in geometry. But naturally it would be best if the two courses were coordinated—perhaps run in parallel—to take advantage of the links between them. The whole is greater than the sum of its parts.

A glance at the table of contents and the index will reveal that this book contains a lot of material, some of it quite hard. This is because I want to provide many interesting paths to follow, for students of all levels. However, it is not necessary to follow each path to its end. The harder sections and exercises are marked with stars, and they can be omitted without losing access to most of the material that follows. There are also informal discussions at the ends of chapters, intended to help readers see the big picture even while some of the details remain confused. Each discussion deals with a few main themes, placing them in historical and mathematical perspective, linking them to other parts of the book, and sometimes extending their development and suggesting further reading.

The book grew out of a talk I gave at Oberwolfach in November 1995, following a suggestion of Urs Kirchgraber. Several parts of it have been used in courses at Monash, from first year to fourth year, and the book in its entirety has benefited from the comments of Mark Aarons, Benno Artmann, Tristan Needham, and Aldo Taranto. To them, and as usual to my wife Elaine, I offer my sincere thanks.

Clayton, Victoria, Australia

John Stillwell
February 1997

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