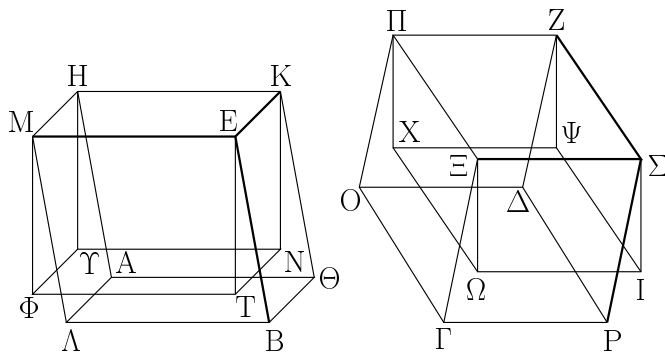


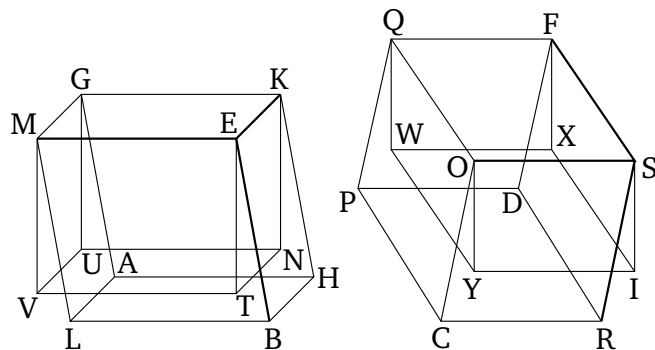
τῷ  $\Gamma\Delta$  ἔστιν ἴσον. ἄλλο δὲ τὸ  $\Delta\Gamma$  ἔστιν ἄρα ὡς ἡ  $\Gamma\Delta$  βάσις πρὸς τὴν  $\Delta\Gamma$ , οὕτως ἡ  $\Omega\Gamma$  πρὸς τὴν  $\Delta\Gamma$ . καὶ ἐπεὶ στερεὸν παραλληλεπίπεδον τὸ  $\Pi$  ἐπιπέδῳ τῷ  $PZ$  τέμνεται παραλλήλῳ ὄντι τοῖς ἀπεναντίον ἐπιπέδοις, ἔστιν ὡς ἡ  $\Gamma\Delta$  βάσις πρὸς τὴν  $\Delta\Gamma$  βάσιν, οὕτως τὸ  $\Gamma Z$  στερεὸν πρὸς τὸ  $\Pi$  στερεόν. διὰ τὰ αὐτὰ δὴ, ἐπεὶ στερεὸν παραλληλεπίπεδον τὸ  $\Omega$  ἐπιπέδῳ τῷ  $P\Psi$  τέμνεται παραλλήλῳ ὄντι τοῖς ἀπεναντίον ἐπιπέδοις, ἔστιν ὡς ἡ  $\Omega\Gamma$  βάσις πρὸς τὴν  $\Gamma\Delta$  βάσιν, οὕτως τὸ  $\Omega\Psi$  στερεὸν πρὸς τὸ  $\Pi$ . ἀλλ' ὡς ἡ  $\Gamma\Delta$  βάσις πρὸς τὴν  $\Delta\Gamma$ , οὕτως ἡ  $\Omega\Gamma$  πρὸς τὴν  $\Delta\Gamma$ . καὶ ὡς ἄρα τὸ  $\Gamma Z$  στερεὸν πρὸς τὸ  $\Pi$  στερεόν, οὕτως τὸ  $\Omega\Psi$  στερεὸν πρὸς τὸ  $\Pi$ . ἐκάτερον ἄρα τῶν  $\Gamma Z$ ,  $\Omega\Psi$  στερεῶν πρὸς τὸ  $\Pi$  τὸν αὐτὸν ἔχει λόγον· ἴσον ἄρα ἐστὶ τὸ  $\Gamma Z$  στερεὸν τῷ  $\Omega\Psi$  στερεῷ. ἀλλὰ τὸ  $\Omega\Psi$  τῷ  $AE$  ἐδείχθη ἴσον· καὶ τὸ  $AE$  ἄρα τῷ  $\Gamma Z$  ἔστιν ἴσον.



Μὴ ἔστωσαν δὴ αἱ ἐφεστηκυῖαι αἱ  $AH$ ,  $\Theta K$ ,  $BE$ ,  $\Lambda M$ ,  $\Gamma\Xi$ ,  $O\Pi$ ,  $\Delta Z$ ,  $P\Sigma$  πρὸς ὀρθὰς ταῖς  $AB$ ,  $\Gamma\Delta$  βάσεσιν· λέγω πάλιν, ὅτι ἴσον τὸ  $AE$  στερεὸν τῷ  $\Gamma Z$  στερεῷ. ἤχθωσαν γὰρ ἀπὸ τῶν  $K$ ,  $E$ ,  $H$ ,  $M$ ,  $\Pi$ ,  $Z$ ,  $\Xi$ ,  $\Sigma$  σημείων ἐπὶ τὸ ὑποκείμενον ἐπίπεδον κάθετοι αἱ  $KN$ ,  $ET$ ,  $HY$ ,  $M\Phi$ ,  $\Pi X$ ,  $Z\Psi$ ,  $\Xi\Omega$ ,  $\Sigma I$ , καὶ συμβαλλέτωσαν τῷ ἐπιπέδῳ κατὰ τὰ  $N$ ,  $T$ ,  $Y$ ,  $\Phi$ ,  $X$ ,  $\Psi$ ,  $\Omega$ ,  $I$  σημεία, καὶ ἐπεζεύχθωσαν αἱ  $NT$ ,  $NU$ ,  $Y\Phi$ ,  $T\Phi$ ,  $X\Psi$ ,  $X\Omega$ ,  $\Omega I$ ,  $I\Psi$ . ἴσον δὴ ἐστὶ τὸ  $K\Phi$  στερεὸν τῷ  $\Pi I$  στερεῷ· ἐπὶ τε γὰρ ἴσων βάσεων εἰσι τῶν  $KM$ ,  $\Pi\Sigma$  καὶ ὑπὸ τὸ αὐτὸ ὕψος, ὧν αἱ ἐφεστῶσαι πρὸς ὀρθὰς εἰσι ταῖς βάσεσιν. ἀλλὰ τὸ μὲν  $K\Phi$  στερεὸν τῷ  $AE$  στερεῷ ἔστιν ἴσον, τὸ δὲ  $\Pi I$  τῷ  $\Gamma Z$ · ἐπὶ τε γὰρ τῆς αὐτῆς βάσεως εἰσι καὶ ὑπὸ τὸ αὐτὸ ὕψος, ὧν αἱ ἐφεστῶσαι οὐκ εἰσιν ἐπὶ τῶν αὐτῶν εὐθειῶν. καὶ τὸ  $AE$  ἄρα στερεὸν τῷ  $\Gamma Z$  στερεῷ ἔστιν ἴσον.

Τὰ ἄρα ἐπὶ ἴσων βάσεων ὄντα στερεὰ παραλληλεπίπεδα καὶ ὑπὸ τὸ αὐτὸ ὕψος ἴσα ἀλλήλοις ἐστίν· ὅπερ ἔδει δεῖξαι.

solid  $XY$  is also equal to solid  $AE$ . And since parallelogram  $RUWT$  is equal to parallelogram  $YT$ . For they are on the same base  $RT$ , and between the same parallels  $RT$  and  $YW$  [Prop. 1.35]. But,  $RUWT$  is equal to  $CD$ , since (it is) also (equal) to  $AB$ . Parallelogram  $YT$  is thus also equal to  $CD$ . And  $DT$  is another (parallelogram). Thus, as base  $CD$  is to  $DT$ , so  $YT$  (is) to  $DT$  [Prop. 5.7]. And since the parallelepiped solid  $CI$  has been cut by the plane  $RF$ , which is parallel to the opposite planes (of  $CI$ ), as base  $CD$  is to base  $DT$ , so solid  $CF$  (is) to solid  $RI$  [Prop. 11.25]. So, for the same (reasons), since the parallelepiped solid  $YI$  has been cut by the plane  $RX$ , which is parallel to the opposite planes (of  $YI$ ), as base  $YT$  is to base  $TD$ , so solid  $YX$  (is) to solid  $RI$  [Prop. 11.25]. But, as base  $CD$  (is) to  $DT$ , so  $YT$  (is) to  $DT$ . And, thus, as solid  $CF$  (is) to solid  $RI$ , so solid  $YX$  (is) to solid  $RI$ . Thus, solids  $CF$  and  $YX$  each have the same ratio to  $RI$  [Prop. 5.11]. Thus, solid  $CF$  is equal to solid  $YX$  [Prop. 5.9]. But,  $YX$  was shown (to be) equal to  $AE$ . Thus,  $AE$  is also equal to  $CF$ .



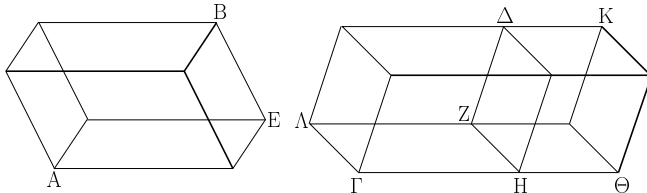
And so let the (straight-lines) standing up,  $AG$ ,  $HK$ ,  $BE$ ,  $LM$ ,  $CO$ ,  $PQ$ ,  $DF$ , and  $RS$ , not be at right-angles to the bases  $AB$  and  $CD$ . Again, I say that solid  $AE$  (is) equal to solid  $CF$ . For let  $KN$ ,  $ET$ ,  $GU$ ,  $MV$ ,  $QW$ ,  $FX$ ,  $OY$ , and  $SI$  have been drawn from points  $K$ ,  $E$ ,  $G$ ,  $M$ ,  $Q$ ,  $F$ ,  $O$ , and  $S$  (respectively) perpendicular to the reference plane (i.e., the plane of the bases  $AB$  and  $CD$ ), and let them have met the plane at points  $N$ ,  $T$ ,  $U$ ,  $V$ ,  $W$ ,  $X$ ,  $Y$ , and  $I$  (respectively). And let  $NT$ ,  $NU$ ,  $UV$ ,  $TV$ ,  $WX$ ,  $WY$ ,  $YI$ , and  $IX$  have been joined. So solid  $KV$  is equal to solid  $QI$ . For they are on the equal bases  $KM$  and  $QS$ , and (have) the same height, and the (straight-lines) standing up in them are at right-angles to their bases (see first part of proposition). But, solid  $KV$  is equal to solid  $AE$ , and  $QI$  to  $CF$ . For they are on the same base, and (have) the same height, and the (straight-lines) standing up in them are not on the same straight-lines [Prop. 11.30]. Thus, solid  $AE$  is also equal to solid  $CF$ .

Thus, parallelepiped solids which are on equal bases,

and (have) the same height, are equal to one another. (Which is) the very thing it was required to show.

λβ'.

Τὰ ὑπὸ τὸ αὐτὸ ὕψος ὄντα στερεὰ παραλληλεπίπεδα πρὸς ἄλληλά ἐστιν ὡς αἱ βάσεις.



Ἐστω ὑπὸ τὸ αὐτὸ ὕψος στερεὰ παραλληλεπίπεδα τὰ  $AB$ ,  $\Gamma\Delta$ . λέγω, ὅτι τὰ  $AB$ ,  $\Gamma\Delta$  στερεὰ παραλληλεπίπεδα πρὸς ἄλληλά ἐστιν ὡς αἱ βάσεις, τουτέστιν ὅτι ἐστὶν ὡς ἡ  $AE$  βάσις πρὸς τὴν  $\Gamma Z$  βάσιν, οὕτως τὸ  $AB$  στερεὸν πρὸς τὸ  $\Gamma\Delta$  στερεόν.

Παραβεβλήσθω γὰρ παρὰ τὴν  $ZH$  τῷ  $AE$  ἴσον τὸ  $Z\Theta$ , καὶ ἀπὸ βάσεως μὲν τῆς  $Z\Theta$ , ὕψους δὲ τοῦ αὐτοῦ τῷ  $\Gamma\Delta$  στερεὸν παραλληλεπίπεδον συμπληρώσθω τὸ  $HK$ . ἴσον δὴ ἐστὶ τὸ  $AB$  στερεὸν τῷ  $HK$  στερεῷ· ἐπὶ τε γὰρ ἴσων βάσεων εἰσι τῶν  $AE$ ,  $Z\Theta$  καὶ ὑπὸ τὸ αὐτὸ ὕψος. καὶ ἐπεὶ στερεὸν παραλληλεπίπεδον τὸ  $\Gamma K$  ἐπιπέδῳ τῷ  $\Delta H$  τέτμηται παραλλήλῳ ὄντι τοῖς ἀπεναντίον ἐπιπέδοις, ἐστὶν ἄρα ὡς ἡ  $\Gamma Z$  βάσις πρὸς τὴν  $Z\Theta$  βάσιν, οὕτως τὸ  $\Gamma\Delta$  στερεὸν πρὸς τὸ  $\Delta\Theta$  στερεόν. ἴση δὲ ἡ μὲν  $Z\Theta$  βάσις τῇ  $AE$  βάσει, τὸ δὲ  $HK$  στερεὸν τῷ  $AB$  στερεῷ· ἐστὶν ἄρα καὶ ὡς ἡ  $AE$  βάσις πρὸς τὴν  $\Gamma Z$  βάσιν, οὕτως τὸ  $AB$  στερεὸν πρὸς τὸ  $\Gamma\Delta$  στερεόν.

Τὰ ἄρα ὑπὸ τὸ αὐτὸ ὕψος ὄντα στερεὰ παραλληλεπίπεδα πρὸς ἄλληλά ἐστιν ὡς αἱ βάσεις· ὅπερ ἔδει δεῖξαι.

λγ'.

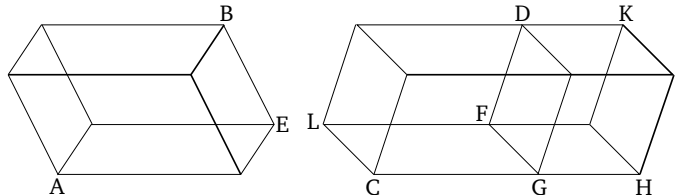
Τὰ ὅμοια στερεὰ παραλληλεπίπεδα πρὸς ἄλληλα ἐν τριπλασίονι λόγῳ εἰσὶ τῶν ὁμολόγων πλευρῶν.

Ἐστω ὅμοια στερεὰ παραλληλεπίπεδα τὰ  $AB$ ,  $\Gamma\Delta$ , ὁμόλογος δὲ ἔστω ἡ  $AE$  τῇ  $\Gamma Z$ . λέγω, ὅτι τὸ  $AB$  στερεὸν πρὸς τὸ  $\Gamma\Delta$  στερεὸν τριπλασίονα λόγον ἔχει, ἥπερ ἡ  $AE$  πρὸς τὴν  $\Gamma Z$ .

Ἐκβεβλήσθωσαν γὰρ ἐπ' εὐθείας ταῖς  $AE$ ,  $HE$ ,  $\Theta E$  αἱ  $EK$ ,  $EL$ ,  $EM$ , καὶ κείσθω τῇ μὲν  $\Gamma Z$  ἴση ἡ  $EK$ , τῇ δὲ  $ZN$  ἴση ἡ  $EL$ , καὶ ἔτι τῇ  $ZP$  ἴση ἡ  $EM$ , καὶ συμπληρώσθω τὸ  $KL$  παραλληλόγραμμον καὶ τὸ  $KO$  στερεόν.

### Proposition 32

Parallelepiped solids which (have) the same height are to one another as their bases.



Let  $AB$  and  $CD$  be parallelepiped solids (having) the same height. I say that the parallelepiped solids  $AB$  and  $CD$  are to one another as their bases. That is to say, as base  $AE$  is to base  $CF$ , so solid  $AB$  (is) to solid  $CD$ .

For let  $FH$ , equal to  $AE$ , have been applied to  $FG$  (in the angle  $FGH$  equal to angle  $LCG$ ) [Prop. 1.45]. And let the parallelepiped solid  $GK$ , (having) the same height as  $CD$ , have been completed on the base  $FH$ . So solid  $AB$  is equal to solid  $GK$ . For they are on the equal bases  $AE$  and  $FH$ , and (have) the same height [Prop. 11.31]. And since the parallelepiped solid  $CK$  has been cut by the plane  $DG$ , which is parallel to the opposite planes (of  $CK$ ), thus as the base  $CF$  is to the base  $FH$ , so the solid  $CD$  (is) to the solid  $DH$  [Prop. 11.25]. And base  $FH$  (is) equal to base  $AE$ , and solid  $GK$  to solid  $AB$ . And thus as base  $AE$  is to base  $CF$ , so solid  $AB$  (is) to solid  $CD$ .

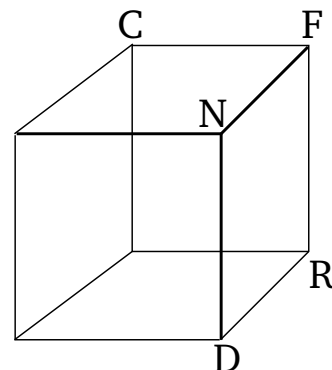
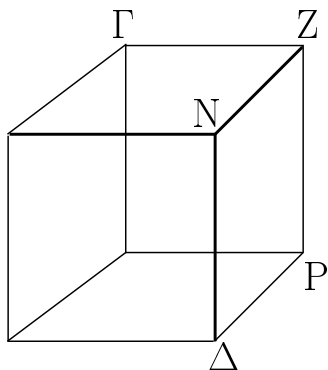
Thus, parallelepiped solids which (have) the same height are to one another as their bases. (Which is) the very thing it was required to show.

### Proposition 33

Similar parallelepiped solids are to one another as the cubed ratio of their corresponding sides.

Let  $AB$  and  $CD$  be similar parallelepiped solids, and let  $AE$  correspond to  $CF$ . I say that solid  $AB$  has to solid  $CD$  the cubed ratio that  $AE$  (has) to  $CF$ .

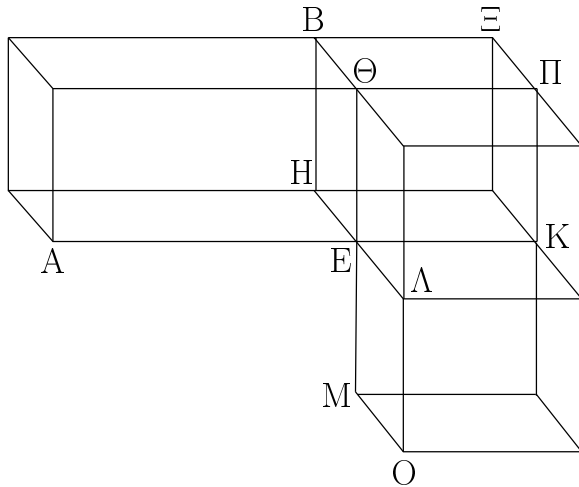
For let  $EK$ ,  $EL$ , and  $EM$  have been produced in a straight-line with  $AE$ ,  $GE$ , and  $HE$  (respectively). And let  $EK$  be made equal to  $CF$ , and  $EL$  equal to  $FN$ , and, further,  $EM$  equal to  $FR$ . And let the parallelogram  $KL$  have been completed, and the solid  $KP$ .



Καὶ ἐπεὶ δύο αἱ  $KE$ ,  $EA$  δυσὶ ταῖς  $\Gamma Z$ ,  $ZN$  ἴσαι εἰσίν, ἀλλὰ καὶ γωνία ἡ ὑπὸ  $KEA$  γωνία τῇ ὑπὸ  $\Gamma ZN$  ἐστὶν ἴση, ἐπειδὴ περ καὶ ἡ ὑπὸ  $AEH$  τῇ ὑπὸ  $\Gamma ZN$  ἐστὶν ἴση διὰ τὴν ὁμοιότητα τῶν  $AB$ ,  $\Gamma\Delta$  στερεῶν, ἴσον ἄρα ἐστὶ [καὶ ὅμοιον] τὸ  $KA$  παραλληλόγραμμον τῷ  $\Gamma N$  παραλληλογράμμῳ. διὰ τὰ αὐτὰ δὴ καὶ τὸ μὲν  $KM$  παραλληλόγραμμον ἴσον ἐστὶ καὶ ὅμοιον τῷ  $\Gamma P$  [παραλληλογράμμῳ] καὶ ἔτι τὸ  $EO$  τῷ  $\Delta Z$ · τρία ἄρα παραλληλόγραμμα τοῦ  $KO$  στερεοῦ τρισὶ παραλληλογράμμοις τοῦ  $\Gamma\Delta$  στερεοῦ ἴσα ἐστὶ καὶ ὅμοια. ἀλλὰ τὰ μὲν τρία τρισὶ τοῖς ἀπεναντίον ἴσα ἐστὶ καὶ ὅμοια, τὰ δὲ τρία τρισὶ τοῖς ἀπεναντίον ἴσα ἐστὶ καὶ ὅμοια· ὅλον ἄρα τὸ  $KO$  στερεὸν ὅλῳ τῷ  $\Gamma\Delta$  στερεῷ ἴσον ἐστὶ καὶ ὅμοιον. συμπεπληρώσθω τὸ  $HK$  παραλληλόγραμμον, καὶ ἀπὸ βάσεων μὲν τῶν  $HK$ ,  $KA$  παραλληλόγραμμων, ὕψους δὲ τοῦ αὐτοῦ τῷ  $AB$  στερεᾷ συμπεπληρώσθω τὰ  $E\Xi$ ,  $\Lambda\Pi$ . καὶ ἐπεὶ διὰ τὴν ὁμοιότητα τῶν  $AB$ ,  $\Gamma\Delta$  στερεῶν ἐστὶν ὡς ἡ  $AE$  πρὸς τὴν  $\Gamma Z$ , οὕτως ἡ  $EH$  πρὸς τὴν  $ZN$ , καὶ ἡ  $E\Theta$  πρὸς τὴν  $ZP$ , ἴση δὲ ἡ μὲν  $\Gamma Z$  τῇ  $EK$ , ἡ δὲ  $ZN$  τῇ  $EA$ , ἡ δὲ  $ZP$  τῇ  $EM$ , ἔστιν ἄρα ὡς ἡ  $AE$  πρὸς τὴν  $EK$ , οὕτως ἡ  $HE$  πρὸς τὴν  $EA$  καὶ ἡ  $\Theta E$  πρὸς τὴν  $EM$ . ἀλλ' ὡς μὲν ἡ  $AE$  πρὸς τὴν  $EK$ , οὕτως τὸ  $AH$  [παραλληλόγραμμον] πρὸς τὸ  $HK$  παραλληλόγραμμον, ὡς δὲ ἡ  $HE$  πρὸς τὴν  $EA$ , οὕτως τὸ  $HK$  πρὸς τὸ  $KA$ , ὡς δὲ ἡ  $\Theta E$  πρὸς  $EM$ , οὕτως τὸ  $\Pi E$  πρὸς τὸ  $KM$ · καὶ ὡς ἄρα τὸ  $AH$  παραλληλόγραμμον πρὸς τὸ  $HK$ , οὕτως τὸ  $HK$  πρὸς τὸ  $KA$  καὶ τὸ  $\Pi E$  πρὸς τὸ  $KM$ . ἀλλ' ὡς μὲν τὸ  $AH$  πρὸς τὸ  $HK$ , οὕτως τὸ  $AB$  στερεὸν πρὸς τὸ  $E\Xi$  στερεόν, ὡς δὲ τὸ  $HK$  πρὸς τὸ  $KA$ , οὕτως τὸ  $\Xi E$  στερεὸν πρὸς τὸ  $\Pi\Lambda$  στερεόν, ὡς δὲ τὸ  $\Pi E$  πρὸς τὸ  $KM$ , οὕτως τὸ  $\Pi\Lambda$  στερεὸν πρὸς τὸ  $KO$  στερεόν· καὶ ὡς ἄρα τὸ  $AB$  στερεὸν πρὸς τὸ  $E\Xi$ , οὕτως τὸ  $E\Xi$  πρὸς τὸ  $\Pi\Lambda$  καὶ τὸ  $\Pi\Lambda$  πρὸς τὸ  $KO$ . ἐὰν δὲ τέσσαρα μεγέθη κατὰ τὸ συνεχὲς ἀνάλογον ᾗ, τὸ πρῶτον πρὸς τὸ τέταρτον τριπλασίονα λόγον ἔχει ἥπερ πρὸς τὸ δεύτερον· τὸ  $AB$  ἄρα στερεὸν πρὸς τὸ  $KO$  τριπλασίονα λόγον ἔχει ἥπερ τὸ  $AB$  πρὸς τὸ  $E\Xi$ . ἀλλ' ὡς τὸ  $AB$  πρὸς τὸ  $E\Xi$ , οὕτως τὸ  $AH$  παραλληλόγραμμον πρὸς τὸ  $HK$  καὶ ἡ  $AE$  εὐθεῖα πρὸς τὴν  $EK$ · ὥστε καὶ τὸ  $AB$  στερεὸν πρὸς τὸ  $KO$  τριπλασίονα λόγον ἔχει ἥπερ ἡ  $AE$  πρὸς τὴν  $EK$ . ἴσον δὲ τὸ [μὲν]  $KO$  στερεὸν τῷ  $\Gamma\Delta$  στερεῷ, ἡ δὲ  $EK$  εὐθεῖα τῇ  $\Gamma Z$ · καὶ τὸ  $AB$  ἄρα στερεὸν πρὸς τὸ  $\Gamma\Delta$  στερεόν τρι-

And since the two (straight-lines)  $KE$  and  $EL$  are equal to the two (straight-lines)  $CF$  and  $FN$ , but angle  $KEL$  is also equal to angle  $CFN$ , inasmuch as  $AEG$  is also equal to  $CFN$ , on account of the similarity of the solids  $AB$  and  $CD$ , parallelogram  $KL$  is thus equal [and similar] to parallelogram  $CN$ . So, for the same (reasons), parallelogram  $KM$  is also equal and similar to [parallelogram]  $CR$ , and, further,  $EP$  to  $DF$ . Thus, three parallelograms of solid  $KP$  are equal and similar to three parallelograms of solid  $CD$ . But the three (former parallelograms) are equal and similar to the three opposite (parallelograms), and the three (latter parallelograms) are equal and similar to the three opposite (parallelograms) [Prop. 11.24]. Thus, the whole of solid  $KP$  is equal and similar to the whole of solid  $CD$  [Def. 11.10]. Let parallelogram  $GK$  have been completed. And let the solids  $EO$  and  $LQ$ , with bases the parallelograms  $GK$  and  $KL$  (respectively), and with the same height as  $AB$ , have been completed. And since, on account of the similarity of solids  $AB$  and  $CD$ , as  $AE$  is to  $CF$ , so  $EG$  (is) to  $FN$ , and  $EH$  to  $FR$  [Defs. 6.1, 11.9], and  $CF$  (is) equal to  $EK$ , and  $FN$  to  $EL$ , and  $FR$  to  $EM$ , thus as  $AE$  is to  $EK$ , so  $GE$  (is) to  $EL$ , and  $HE$  to  $EM$ . But, as  $AE$  (is) to  $EK$ , so [parallelogram]  $AG$  (is) to parallelogram  $GK$ , and as  $GE$  (is) to  $EL$ , so  $GK$  (is) to  $KL$ , and as  $HE$  (is) to  $EM$ , so  $QE$  (is) to  $KM$  [Prop. 6.1]. And thus as parallelogram  $AG$  (is) to  $GK$ , so  $GK$  (is) to  $KL$ , and  $QE$  (is) to  $KM$ . But, as  $AG$  (is) to  $GK$ , so solid  $AB$  (is) to solid  $EO$ , and as  $GK$  (is) to  $KL$ , so solid  $OE$  (is) to solid  $QL$ , and as  $QE$  (is) to  $KM$ , so solid  $QL$  (is) to solid  $KP$  [Prop. 11.32]. And, thus, as solid  $AB$  is to  $EO$ , so  $EO$  (is) to  $QL$ , and  $QL$  to  $KP$ . And if four magnitudes are continuously proportional then the first has to the fourth the cubed ratio that (it has) to the second [Def. 5.10]. Thus, solid  $AB$  has to  $KP$  the cubed ratio which  $AB$  (has) to  $EO$ . But, as  $AB$  (is) to  $EO$ , so parallelogram  $AG$  (is) to  $GK$ , and the straight-line  $AE$  to  $EK$  [Prop. 6.1]. Hence, solid  $AB$  also has to  $KP$  the cubed ratio that  $AE$  (has) to  $EK$ . And solid  $KP$  (is)

πλασίονα λόγον ἔχει ἥπερ ἡ ὁμόλογος αὐτοῦ πλευρὰ ἡ  $AE$  πρὸς τὴν ὁμόλογον πλευρὰν τὴν  $\Gamma Z$ .



Τὰ ἄρα ὅμοια στερεὰ παραλληλεπίπεδα ἐν τριπλασίονι λόγῳ ἐστὶ τῶν ὁμολόγων πλευρῶν· ὅπερ ἔδει δεῖξαι.

### Πόρισμα.

Ἐκ δὴ τούτου φανερόν, ὅτι ἐὰν τέσσαρες εὐθεῖαι ἀνάλογον ὦσιν, ἔσται ὡς ἡ πρώτη πρὸς τὴν τετάρτην, οὕτω τὸ ἀπὸ τῆς πρώτης στερεὸν παραλληλεπίπεδον πρὸς τὸ ἀπὸ τῆς δευτέρας τὸ ὅμοιον καὶ ὁμοίως ἀναγραφόμενον, ἐπεὶ περ καὶ ἡ πρώτη πρὸς τὴν τετάρτην τριπλασίονα λόγον ἔχει ἥπερ πρὸς τὴν δευτέραν.

λδ'.

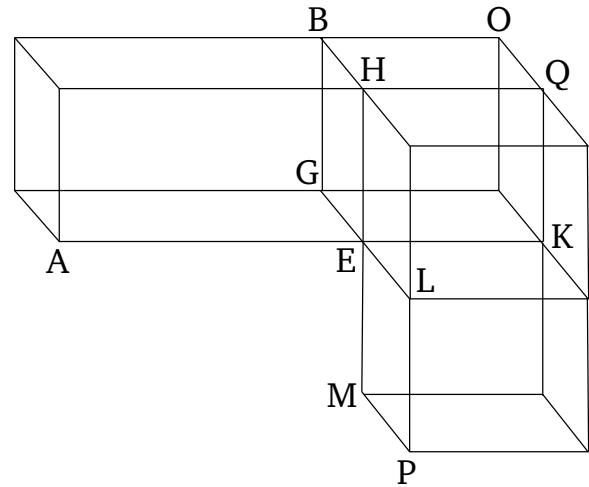
Τῶν ἴσων στερεῶν παραλληλεπιπέδων ἀντιπεπύονθαι αἱ βάσεις τοῖς ὕψεσιν· καὶ ὧν στερεῶν παραλληλεπιπέδων ἀντιπεπύονθαι αἱ βάσεις τοῖς ὕψεσιν, ἴσα ἐστὶν ἐκείνα.

Ἐστω ἴσα στερεὰ παραλληλεπίπεδα τὰ  $AB$ ,  $\Gamma\Delta$ · λέγω, ὅτι τῶν  $AB$ ,  $\Gamma\Delta$  στερεῶν παραλληλεπιπέδων ἀντιπεπύονθαι αἱ βάσεις τοῖς ὕψεσιν, καὶ ἐστὶν ὡς ἡ  $EO$  βάσις πρὸς τὴν  $NI$  βάσιν, οὕτως τὸ τοῦ  $\Gamma\Delta$  στερεοῦ ὕψος πρὸς τὸ τοῦ  $AB$  στερεοῦ ὕψος.

Ἐστώσαν γὰρ πρότερον αἱ ἐφεστηκυῖαι αἱ  $AH$ ,  $EZ$ ,  $AB$ ,  $\Theta K$ ,  $GM$ ,  $N\Xi$ ,  $OD$ ,  $IP$  πρὸς ὀρθὰς ταῖς βάσεσιν αὐτῶν· λέγω, ὅτι ἐστὶν ὡς ἡ  $EO$  βάσις πρὸς τὴν  $NI$  βάσιν, οὕτως ἡ  $GM$  πρὸς τὴν  $AH$ .

Εἰ μὲν οὖν ἴση ἐστὶν ἡ  $EO$  βάσις τῇ  $NI$  βάσει, ἔστι δὲ καὶ τὸ  $AB$  στερεὸν τῷ  $\Gamma\Delta$  στερεῷ ἴσον, ἔσται καὶ ἡ  $GM$  τῇ  $AH$  ἴση. τὰ γὰρ ὑπὸ τὸ αὐτὸ ὕψος στερεὰ παραλληλεπίπεδα πρὸς ἄλληλά ἐστιν ὡς αἱ βάσεις. καὶ ἔσται ὡς ἡ  $EO$  βάσις πρὸς τὴν  $NI$ , οὕτως ἡ  $GM$  πρὸς τὴν  $AH$ , καὶ φανερόν, ὅτι

equal to solid  $CD$ , and straight-line  $EK$  to  $CF$ . Thus, solid  $AB$  also has to solid  $CD$  the cubed ratio which its corresponding side  $AE$  (has) to the corresponding side  $CF$ .



Thus, similar parallelepiped solids are to one another as the cubed ratio of their corresponding sides. (Which is) the very thing it was required to show.

### Corollary

So, (it is) clear, from this, that if four straight-lines are (continuously) proportional then as the first is to the fourth, so the parallelepiped solid on the first will be to the similar, and similarly described, parallelepiped solid on the second, since the first also has to the fourth the cubed ratio that (it has) to the second.

### Proposition 34<sup>†</sup>

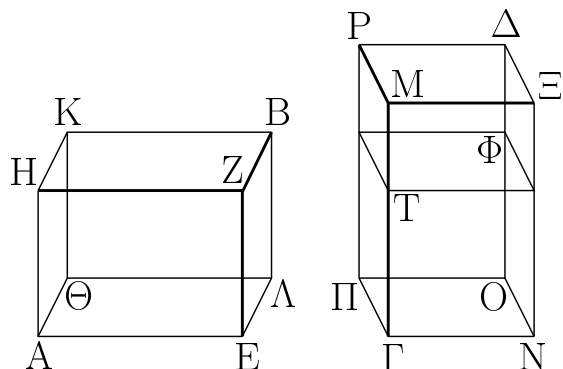
The bases of equal parallelepiped solids are reciprocally proportional to their heights. And those parallelepiped solids whose bases are reciprocally proportional to their heights are equal.

Let  $AB$  and  $CD$  be equal parallelepiped solids. I say that the bases of the parallelepiped solids  $AB$  and  $CD$  are reciprocally proportional to their heights, and (so) as base  $EH$  is to base  $NQ$ , so the height of solid  $CD$  (is) to the height of solid  $AB$ .

For, first of all, let the (straight-lines) standing up,  $AG$ ,  $EF$ ,  $LB$ ,  $HK$ ,  $CM$ ,  $NO$ ,  $PD$ , and  $QR$ , be at right-angles to their bases. I say that as base  $EH$  is to base  $NQ$ , so  $CM$  (is) to  $AG$ .

Therefore, if base  $EH$  is equal to base  $NQ$ , and solid  $AB$  is also equal to solid  $CD$ ,  $CM$  will also be equal to  $AG$ . For parallelepiped solids of the same height are to one another as their bases [Prop. 11.32]. And as base

τῶν  $AB$ ,  $\Gamma\Delta$  στερεῶν παραλληλεπιπέδων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν.



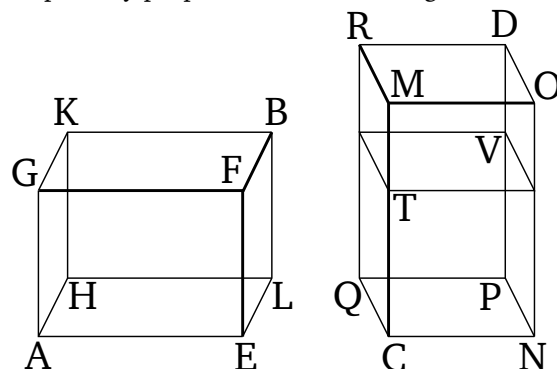
Μὴ ἔστω δὴ ἴση ἡ  $E\Theta$  βάσις τῇ  $\Pi\Pi$  βάσει, ἀλλ' ἔστω μείζων ἡ  $E\Theta$ . ἔστι δὲ καὶ τὸ  $AB$  στερεὸν τῷ  $\Gamma\Delta$  στερεῷ ἴσον· μείζων ἄρα ἐστὶ καὶ ἡ  $\Gamma M$  τῆς  $AH$ . κείσθω οὖν τῇ  $AH$  ἴση ἡ  $\Gamma T$ , καὶ συμπληρώσθω ἀπὸ βάσεως μὲν τῆς  $\Pi\Pi$ , ὕψους δὲ τοῦ  $\Gamma T$ , στερεὸν παραλληλεπίπεδον τὸ  $\Phi\Gamma$ . καὶ ἐπεὶ ἴσον ἐστὶ τὸ  $AB$  στερεὸν τῷ  $\Gamma\Delta$  στερεῷ, ἔξωθεν δὲ τὸ  $\Gamma\Phi$ , τὰ δὲ ἴσα πρὸς τὸ αὐτὸ τὸν αὐτὸν ἔχει λόγον, ἔστιν ἄρα ὡς τὸ  $AB$  στερεὸν πρὸς τὸ  $\Gamma\Phi$  στερεόν, οὕτως τὸ  $\Gamma\Delta$  στερεὸν πρὸς τὸ  $\Gamma\Phi$  στερεόν. ἀλλ' ὡς μὲν τὸ  $AB$  στερεὸν πρὸς τὸ  $\Gamma\Phi$  στερεόν, οὕτως ἡ  $E\Theta$  βάσις πρὸς τὴν  $\Pi\Pi$  βάσιν· ἰσοῦψή γὰρ τὰ  $AB$ ,  $\Gamma\Phi$  στερεά· ὡς δὲ τὸ  $\Gamma\Delta$  στερεὸν πρὸς τὸ  $\Gamma\Phi$  στερεόν, οὕτως ἡ  $M\Pi$  βάσις πρὸς τὴν  $T\Pi$  βάσιν καὶ ἡ  $\Gamma M$  πρὸς τὴν  $\Gamma T$ · καὶ ὡς ἄρα ἡ  $E\Theta$  βάσις πρὸς τὴν  $\Pi\Pi$  βάσιν, οὕτως ἡ  $M\Pi$  πρὸς τὴν  $\Gamma T$ . ἴση δὲ ἡ  $\Gamma T$  τῇ  $AH$ · καὶ ὡς ἄρα ἡ  $E\Theta$  βάσις πρὸς τὴν  $\Pi\Pi$  βάσιν, οὕτως ἡ  $M\Pi$  πρὸς τὴν  $AH$ . τῶν  $AB$ ,  $\Gamma\Delta$  ἄρα στερεῶν παραλληλεπιπέδων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν.

Πάλιν δὴ τῶν  $AB$ ,  $\Gamma\Delta$  στερεῶν παραλληλεπιπέδων ἀντιπεπονθέντων αἱ βάσεις τοῖς ὕψεσιν, καὶ ἔστω ὡς ἡ  $E\Theta$  βάσις πρὸς τὴν  $\Pi\Pi$  βάσιν, οὕτως τὸ τοῦ  $\Gamma\Delta$  στερεοῦ ὕψος πρὸς τὸ τοῦ  $AB$  στερεοῦ ὕψος· λέγω, ὅτι ἴσον ἐστὶ τὸ  $AB$  στερεὸν τῷ  $\Gamma\Delta$  στερεῷ.

Ἐστώσαν [γὰρ] πάλιν αἱ ἐφεστηκυῖαι πρὸς ὀρθὰς ταῖς βάσεσιν. καὶ εἰ μὲν ἴση ἐστὶν ἡ  $E\Theta$  βάσις τῇ  $\Pi\Pi$  βάσει, καὶ ἐστὶν ὡς ἡ  $E\Theta$  βάσις πρὸς τὴν  $\Pi\Pi$  βάσιν, οὕτως τὸ τοῦ  $\Gamma\Delta$  στερεοῦ ὕψος πρὸς τὸ τοῦ  $AB$  στερεοῦ ὕψος, ἴσον ἄρα ἐστὶ καὶ τὸ τοῦ  $\Gamma\Delta$  στερεοῦ ὕψος τῷ τοῦ  $AB$  στερεοῦ ὕψει. τὰ δὲ ἐπὶ ἴσων βάσεων στερεὰ παραλληλεπίπεδα καὶ ὑπὸ τὸ αὐτὸ ὕψος ἴσα ἀλλήλοις ἐστίν· ἴσον ἄρα ἐστὶ τὸ  $AB$  στερεὸν τῷ  $\Gamma\Delta$  στερεῷ.

Μὴ ἔστω δὴ ἡ  $E\Theta$  βάσις τῇ  $\Pi\Pi$  [βάσει] ἴση, ἀλλ' ἔστω μείζων ἡ  $E\Theta$ · μείζον ἄρα ἐστὶ καὶ τὸ τοῦ  $\Gamma\Delta$  στερεοῦ ὕψος τοῦ τοῦ  $AB$  στερεοῦ ὕψους, τουτέστιν ἡ  $\Gamma M$  τῆς  $AH$ . κείσθω τῇ  $AH$  ἴση πάλιν ἡ  $\Gamma T$ , καὶ συμπληρώσθω ὁμοίως τὸ  $\Gamma\Phi$  στερεόν. ἐπεὶ ἐστὶν ὡς ἡ  $E\Theta$  βάσις πρὸς τὴν  $\Pi\Pi$  βάσιν, οὕτως ἡ  $M\Pi$  πρὸς τὴν  $AH$ , ἴση δὲ ἡ  $AH$  τῇ  $\Gamma T$ ,

$EH$  (is) to  $NQ$ , so  $CM$  will be to  $AG$ . And (so it is) clear that the bases of the parallelepiped solids  $AB$  and  $CD$  are reciprocally proportional to their heights.

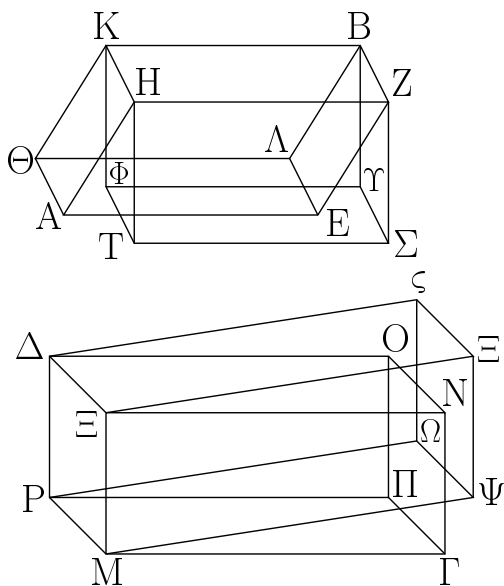


So let base  $EH$  not be equal to base  $NQ$ , but let  $EH$  be greater. And solid  $AB$  is also equal to solid  $CD$ . Thus,  $CM$  is also greater than  $AG$ . Therefore, let  $CT$  be made equal to  $AG$ . And let the parallelepiped solid  $VC$  have been completed on the base  $NQ$ , with height  $CT$ . And since solid  $AB$  is equal to solid  $CD$ , and  $CV$  (is) extrinsic (to them), and equal (magnitudes) have the same ratio to the same (magnitude) [Prop. 5.7], thus as solid  $AB$  is to solid  $CV$ , so solid  $CD$  (is) to solid  $CV$ . But, as solid  $AB$  (is) to solid  $CV$ , so base  $EH$  (is) to base  $NQ$ . For the solids  $AB$  and  $CV$  (are) of equal height [Prop. 11.32]. And as solid  $CD$  (is) to solid  $CV$ , so base  $MQ$  (is) to base  $TQ$  [Prop. 11.25], and  $CM$  to  $CT$  [Prop. 6.1]. And, thus, as base  $EH$  is to base  $NQ$ , so  $MC$  (is) to  $AG$ . And  $CT$  (is) equal to  $AG$ . And thus as base  $EH$  (is) to base  $NQ$ , so  $MC$  (is) to  $AG$ . Thus, the bases of the parallelepiped solids  $AB$  and  $CD$  are reciprocally proportional to their heights.

So, again, let the bases of the parallelepiped solids  $AB$  and  $CD$  be reciprocally proportional to their heights, and let base  $EH$  be to base  $NQ$ , as the height of solid  $CD$  (is) to the height of solid  $AB$ . I say that solid  $AB$  is equal to solid  $CD$ . [For] let the (straight-lines) standing up again be at right-angles to the bases. And if base  $EH$  is equal to base  $NQ$ , and as base  $EH$  is to base  $NQ$ , so the height of solid  $CD$  (is) to the height of solid  $AB$ , the height of solid  $CD$  is thus also equal to the height of solid  $AB$ . And parallelepiped solids on equal bases, and also with the same height, are equal to one another [Prop. 11.31]. Thus, solid  $AB$  is equal to solid  $CD$ .

So, let base  $EH$  not be equal to [base]  $NQ$ , but let  $EH$  be greater. Thus, the height of solid  $CD$  is also greater than the height of solid  $AB$ , that is to say  $CM$  (greater) than  $AG$ . Let  $CT$  again be made equal to  $AG$ , and let the solid  $CV$  have been similarly completed. Since as base  $EH$  is to base  $NQ$ , so  $MC$  (is) to  $AG$ ,

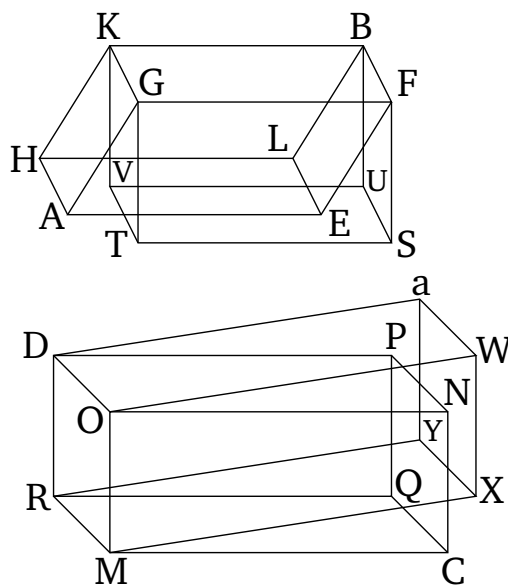
ἔστιν ἄρα ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως ἡ ΓΜ πρὸς τὴν ΓΤ. ἀλλ' ὡς μὲν ἡ ΕΘ [βάσις] πρὸς τὴν ΝΠ βάσιν, οὕτως τὸ ΑΒ στερεὸν πρὸς τὸ ΓΦ στερεόν· ἰσοῦψή γάρ ἐστι τὰ ΑΒ, ΓΦ στερεά· ὡς δὲ ἡ ΓΜ πρὸς τὴν ΓΤ, οὕτως ἡ τε ΜΠ βάσις πρὸς τὴν ΠΤ βάσιν καὶ τὸ ΓΔ στερεὸν πρὸς τὸ ΓΦ στερεόν. καὶ ὡς ἄρα τὸ ΑΒ στερεὸν πρὸς τὸ ΓΦ στερεόν, οὕτως τὸ ΓΔ στερεὸν πρὸς τὸ ΓΦ στερεόν· ἐκάτερον ἄρα τῶν ΑΒ, ΓΔ πρὸς τὸ ΓΦ τὸν αὐτὸν ἔχει λόγον. ἴσον ἄρα ἐστὶ τὸ ΑΒ στερεὸν τῷ ΓΔ στερεῷ.



Μὴ ἔστωσαν δὴ αἱ ἐφεστηκυῖαι αἱ ΖΕ, ΒΛ, ΗΑ, ΚΘ, ΕΝ, ΔΟ, ΜΓ, ΡΠ πρὸς ὀρθὰς ταῖς βάσεσιν αὐτῶν, καὶ ἤχθωσαν ἀπὸ τῶν Ζ, Η, Β, Κ, Ξ, Μ, Ρ, Δ σημείων ἐπὶ τὰ διὰ τῶν ΕΘ, ΝΠ ἐπίπεδα κάθετοι καὶ συμβαλλέτωσαν τοῖς ἐπιπέδοις κατὰ τὰ Σ, Τ, Υ, Φ, Χ, Ψ, Ω, ς, καὶ συμπληρώσθω τὰ ΖΦ, ΞΩ στερεά· λέγω, ὅτι καὶ οὕτως ἴσων ὄντων τῶν ΑΒ, ΓΔ στερεῶν ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν, καὶ ἐστὶν ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως τὸ τοῦ ΓΔ στερεοῦ ὕψος πρὸς τὸ τοῦ ΑΒ στερεοῦ ὕψος.

Ἐπεὶ ἴσον ἐστὶ τὸ ΑΒ στερεὸν τῷ ΓΔ στερεῷ, ἀλλὰ τὸ μὲν ΑΒ τῷ ΒΤ ἐστὶν ἴσον· ἐπὶ τε γὰρ τῆς αὐτῆς βάσεως εἰσι τῆς ΖΚ καὶ ὑπὸ τὸ αὐτὸ ὕψος· τὸ δὲ ΓΔ στερεὸν τῷ ΔΨ ἐστὶν ἴσον· ἐπὶ τε γὰρ πάλιν τῆς αὐτῆς βάσεως εἰσι τῆς ΡΞ καὶ ὑπὸ τὸ αὐτὸ ὕψος· καὶ τὸ ΒΤ ἄρα στερεὸν τῷ ΔΨ στερεῷ ἴσον ἐστίν. ἔστιν ἄρα ὡς ἡ ΖΚ βάσις πρὸς τὴν ΞΡ βάσιν, οὕτως τὸ τοῦ ΔΨ στερεοῦ ὕψος πρὸς τὸ τοῦ ΒΤ στερεοῦ ὕψος. ἴση δὲ ἡ μὲν ΖΚ βάσις τῇ ΕΘ βάσει, ἡ δὲ ΞΡ βάσις τῇ ΝΠ βάσει· ἔστιν ἄρα ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως τὸ τοῦ ΔΨ στερεοῦ ὕψος πρὸς τὸ τοῦ ΒΤ στερεοῦ ὕψος. τὰ δ' αὐτὰ ὕψη ἐστὶ τῶν ΔΨ, ΒΤ στερεῶν καὶ τῶν ΔΓ, ΒΑ· ἔστιν ἄρα ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ

and  $AG$  (is) equal to  $CT$ , thus as base  $EH$  (is) to base  $NQ$ , so  $CM$  (is) to  $CT$ . But, as [base]  $EH$  (is) to base  $NQ$ , so solid  $AB$  (is) to solid  $CV$ . For solids  $AB$  and  $CV$  are of equal heights [Prop. 11.32]. And as  $CM$  (is) to  $CT$ , so (is) base  $MQ$  to base  $QT$  [Prop. 6.1], and solid  $CD$  to solid  $CV$  [Prop. 11.25]. And thus as solid  $AB$  (is) to solid  $CV$ , so solid  $CD$  (is) to solid  $CV$ . Thus,  $AB$  and  $CD$  each have the same ratio to  $CV$ . Thus, solid  $AB$  is equal to solid  $CD$  [Prop. 5.9].



So, let the (straight-lines) standing up,  $FE$ ,  $BL$ ,  $GA$ ,  $KH$ ,  $ON$ ,  $DP$ ,  $MC$ , and  $RQ$ , not be at right-angles to their bases. And let perpendiculars have been drawn to the planes through  $EH$  and  $NQ$  from points  $F$ ,  $G$ ,  $B$ ,  $K$ ,  $O$ ,  $M$ ,  $R$ , and  $D$ , and let them have joined the planes at (points)  $S$ ,  $T$ ,  $U$ ,  $V$ ,  $W$ ,  $X$ ,  $Y$ , and  $a$  (respectively). And let the solids  $FV$  and  $OY$  have been completed. In this case, also, I say that the solids  $AB$  and  $CD$  being equal, their bases are reciprocally proportional to their heights, and (so) as base  $EH$  is to base  $NQ$ , so the height of solid  $CD$  (is) to the height of solid  $AB$ .

Since solid  $AB$  is equal to solid  $CD$ , but  $AB$  is equal to  $BT$ . For they are on the same base  $FK$ , and (have) the same height [Props. 11.29, 11.30]. And solid  $CD$  is equal to  $DX$ . For, again, they are on the same base  $RO$ , and (have) the same height [Props. 11.29, 11.30]. Solid  $BT$  is thus also equal to solid  $DX$ . Thus, as base  $FK$  (is) to base  $OR$ , so the height of solid  $DX$  (is) to the height of solid  $BT$  (see first part of proposition). And base  $FK$  (is) equal to base  $EH$ , and base  $OR$  to  $NQ$ . Thus, as base  $EH$  is to base  $NQ$ , so the height of solid  $DX$  (is) to

βάσιν, οὕτως τὸ τοῦ ΔΓ στερεοῦ ὕψος πρὸς τὸ τοῦ ΑΒ στερεοῦ ὕψος. τῶν ΑΒ, ΓΔ ἄρα στερεῶν παραλληλεπιπέδων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν.

Πάλιν δὴ τῶν ΑΒ, ΓΔ στερεῶν παραλληλεπιπέδων ἀντιπεπονθέντων αἱ βάσεις τοῖς ὕψεσιν, καὶ ἔστω ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως τὸ τοῦ ΓΔ στερεοῦ ὕψος πρὸς τὸ τοῦ ΑΒ στερεοῦ ὕψος· λέγω, ὅτι ἴσον ἐστὶ τὸ ΑΒ στερεὸν τῷ ΓΔ στερεῷ.

Τῶν γὰρ αὐτῶν κατασκευασθέντων, ἐπεὶ ἐστὶν ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως τὸ τοῦ ΓΔ στερεοῦ ὕψος πρὸς τὸ τοῦ ΑΒ στερεοῦ ὕψος, ἴση δὲ ἡ μὲν ΕΘ βάσις τῇ ΖΚ βάσει, ἡ δὲ ΝΠ τῇ ΞΡ, ἔστιν ἄρα ὡς ἡ ΖΚ βάσις πρὸς τὴν ΞΡ βάσιν, οὕτως τὸ τοῦ ΓΔ στερεοῦ ὕψος πρὸς τὸ τοῦ ΑΒ στερεοῦ ὕψος. τὰ δ' αὐτὰ ὕψη ἐστὶ τῶν ΑΒ, ΓΔ στερεῶν καὶ τῶν ΒΤ, ΔΨ· ἔστιν ἄρα ὡς ἡ ΖΚ βάσις πρὸς τὴν ΞΡ βάσιν, οὕτως τὸ τοῦ ΔΨ στερεοῦ ὕψος πρὸς τὸ τοῦ ΒΤ στερεοῦ ὕψος. τῶν ΒΤ, ΔΨ ἄρα στερεῶν παραλληλεπιπέδων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν· ἴσον ἄρα ἐστὶ τὸ ΒΤ στερεὸν τῷ ΔΨ στερεῷ. ἀλλὰ τὸ μὲν ΒΤ τῷ ΒΑ ἴσον ἐστίν· ἐπὶ τε γὰρ τῆς αὐτῆς βάσεως [εἰσι] τῆς ΖΚ καὶ ὑπὸ τὸ αὐτὸ ὕψος. τὸ δὲ ΔΨ στερεὸν τῷ ΔΓ στερεῷ ἴσον ἐστίν. καὶ τὸ ΑΒ ἄρα στερεὸν τῷ ΓΔ στερεῷ ἐστὶν ἴσον· ὅπερ ἔδει δεῖξαι.

the height of solid  $BT$ . And solids  $DX$ ,  $BT$  are the same height as (solids)  $DC$ ,  $BA$  (respectively). Thus, as base  $EH$  is to base  $NQ$ , so the height of solid  $DC$  (is) to the height of solid  $AB$ . Thus, the bases of the parallelepiped solids  $AB$  and  $CD$  are reciprocally proportional to their heights.

So, again, let the bases of the parallelepiped solids  $AB$  and  $CD$  be reciprocally proportional to their heights, and (so) let base  $EH$  be to base  $NQ$ , as the height of solid  $CD$  (is) to the height of solid  $AB$ . I say that solid  $AB$  is equal to solid  $CD$ .

For, with the same construction (as before), since as base  $EH$  is to base  $NQ$ , so the height of solid  $CD$  (is) to the height of solid  $AB$ , and base  $EH$  (is) equal to base  $FK$ , and  $NQ$  to  $OR$ , thus as base  $FK$  is to base  $OR$ , so the height of solid  $CD$  (is) to the height of solid  $AB$ . And solids  $AB$ ,  $CD$  are the same height as (solids)  $BT$ ,  $DX$  (respectively). Thus, as base  $FK$  is to base  $OR$ , so the height of solid  $DX$  (is) to the height of solid  $BT$ . Thus, the bases of the parallelepiped solids  $BT$  and  $DX$  are reciprocally proportional to their heights. Thus, solid  $BT$  is equal to solid  $DX$  (see first part of proposition). But,  $BT$  is equal to  $BA$ . For [they are] on the same base  $FK$ , and (have) the same height [Props. 11.29, 11.30]. And solid  $DX$  is equal to solid  $DC$  [Props. 11.29, 11.30]. Thus, solid  $AB$  is also equal to solid  $CD$ . (Which is) the very thing it was required to show.

† This proposition assumes that (a) if two parallelepipeds are equal, and have equal bases, then their heights are equal, and (b) if the bases of two equal parallelepipeds are unequal, then that solid which has the lesser base has the greater height.

λε'.

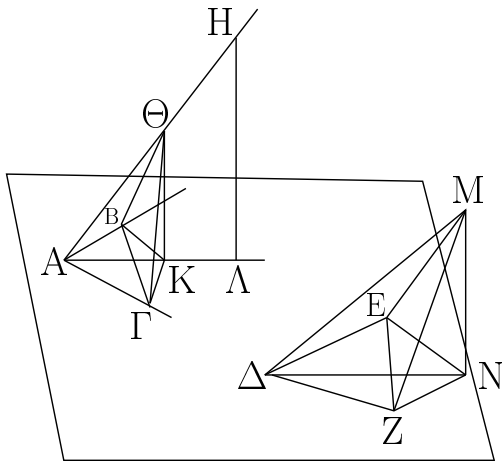
### Proposition 35

Ἐάν ὦσι δύο γωνίαι ἐπίπεδοι ἴσαι, ἐπὶ δὲ τῶν κορυφῶν αὐτῶν μετέωροι εὐθεῖαι ἐπισταθῶσιν ἴσας γωνίας περιέχουσαι μετὰ τῶν ἐξ ἀρχῆς εὐθειῶν ἑκατέραν ἑκατέρᾳ, ἐπὶ δὲ τῶν μετέωρων ληφθῇ τυχόντα σημεία, καὶ ἀπ' αὐτῶν ἐπὶ τὰ ἐπίπεδα, ἐν οἷς εἰσιν αἱ ἐξ ἀρχῆς γωνίαι, κάθετοι ἀχθῶσιν, ἀπὸ δὲ τῶν γενομένων σημείων ἐν τοῖς ἐπιπέδοις ἐπὶ τὰς ἐξ ἀρχῆς γωνίας ἐπιξευχθῶσιν εὐθεῖαι, ἴσας γωνίας περιέξουσιν μετὰ τῶν μετέωρων.

Ἐστωσαν δύο γωνίαι εὐθύγραμμοι ἴσαι αἱ ὑπὸ ΒΑΓ, ΕΔΖ, ἀπὸ δὲ τῶν Α, Δ σημείων μετέωροι εὐθεῖαι ἐφεστώσων αἱ ΑΗ, ΔΜ ἴσας γωνίας περιέχουσιν μετὰ τῶν ἐξ ἀρχῆς εὐθειῶν ἑκατέραν ἑκατέρᾳ, τὴν μὲν ὑπὸ ΜΔΕ τῇ ὑπὸ ΗΑΒ, τὴν δὲ ὑπὸ ΜΔΖ τῇ ὑπὸ ΗΑΓ, καὶ εἰλήφθω ἐπὶ τῶν ΑΗ, ΔΜ τυχόντα σημεία τὰ Η, Μ, καὶ ἤχθωσαν ἀπὸ τῶν Η, Μ σημείων ἐπὶ τὰ διὰ τῶν ΒΑΓ, ΕΔΖ ἐπίπεδα κάθετοι αἱ ΗΛ, ΜΝ, καὶ συμβαλλέτωσαν τοῖς ἐπιπέδοις κατὰ τὰ Α, Ν, καὶ ἐπεξεύχθωσαν αἱ ΛΑ, ΝΔ· λέγω, ὅτι ἴση ἐστὶν ἡ ὑπὸ ΗΑΛ γωνία τῇ ὑπὸ ΜΔΝ γωνία.

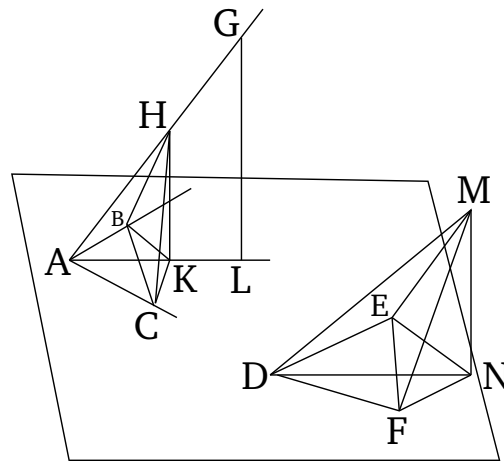
If there are two equal plane angles, and raised straight-lines are stood on the apexes of them, containing equal angles respectively with the original straight-lines (forming the angles), and random points are taken on the raised (straight-lines), and perpendiculars are drawn from them to the planes in which the original angles are, and straight-lines are joined from the points created in the planes to the (vertices of the) original angles, then they will enclose equal angles with the raised (straight-lines).

Let  $BAC$  and  $EDF$  be two equal rectilinear angles. And let the raised straight-lines  $AG$  and  $DM$  have been stood on points  $A$  and  $D$ , containing equal angles respectively with the original straight-lines. (That is)  $MDE$  (equal) to  $GAB$ , and  $MDF$  (to)  $GAC$ . And let the random points  $G$  and  $M$  have been taken on  $AG$  and  $DM$  (respectively). And let the  $GL$  and  $MN$  have been drawn from points  $G$  and  $M$  perpendicular to the planes through



Κείσθω τῇ ΔΜ ἴση ἡ ΑΘ, καὶ ἤχθω διὰ τοῦ Θ σημείου τῇ ΗΛ παράλληλος ἡ ΘΚ. ἡ δὲ ΗΛ κάθετός ἐστιν ἐπὶ τὸ διὰ τῶν ΒΑΓ ἐπίπεδον· καὶ ἡ ΘΚ ἄρα κάθετός ἐστιν ἐπὶ τὸ διὰ τῶν ΒΑΓ ἐπίπεδον. ἤχθωσαν ἀπὸ τῶν Κ, Ν σημείων ἐπὶ τὰς ΑΓ, ΔΖ, ΑΒ, ΔΕ εὐθείας κάθετοι αἱ ΚΓ, ΝΖ, ΚΒ, ΝΕ, καὶ ἐπεζεύχθωσαν αἱ ΘΓ, ΓΒ, ΜΖ, ΖΕ. ἐπεὶ τὸ ἀπὸ τῆς ΘΑ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΘΚ, ΚΑ, τῷ δὲ ἀπὸ τῆς ΚΑ ἴσα ἐστὶ τὰ ἀπὸ τῶν ΚΓ, ΓΑ, καὶ τὸ ἀπὸ τῆς ΘΑ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΘΚ, ΚΓ, ΓΑ. τοῖς δὲ ἀπὸ τῶν ΘΚ, ΚΓ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΘΓ· τὸ ἄρα ἀπὸ τῆς ΘΑ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΘΓ, ΓΑ. ὀρθὴ ἄρα ἐστὶν ἡ ὑπὸ ΘΓΑ γωνία. διὰ τὰ αὐτὰ δὴ καὶ ἡ ὑπὸ ΔΖΜ γωνία ὀρθὴ ἐστίν. ἴση ἄρα ἐστὶν ἡ ὑπὸ ΑΓΘ γωνία τῇ ὑπὸ ΔΖΜ. ἔστι δὲ καὶ ἡ ὑπὸ ΘΑΓ τῇ ὑπὸ ΜΔΖ ἴση. δύο δὴ τρίγωνά ἐστι τὰ ΜΔΖ, ΘΑΓ δύο γωνίας δυσὶ γωνίαις ἴσας ἔχοντα ἑκατέραν ἑκατέρᾳ καὶ μίαν πλευρὰν μὲν πλεωρᾶ ἴσην τὴν ὑποτείνουσιν ὑπὸ μίαν τῶν ἴσων γωνιῶν τὴν ΘΑ τῇ ΜΔ· καὶ τὰς λοιπὰς ἄρα πλευρὰς ταῖς λοιπαῖς πλευραῖς ἴσας ἔξει ἑκατέραν ἑκατέρᾳ. ἴση ἄρα ἐστὶν ἡ ΑΓ τῇ ΔΖ. ὁμοίως δὴ δεῖξομεν, ὅτι καὶ ἡ ΑΒ τῇ ΔΕ ἐστὶν ἴση. ἐπεὶ οὖν ἴση ἐστὶν ἡ μὲν ΑΓ τῇ ΔΖ, ἡ δὲ ΑΒ τῇ ΔΕ, δύο δὴ αἱ ΓΑ, ΑΒ δυσὶ ταῖς ΖΔ, ΔΕ ἴσαι εἰσίν. ἀλλὰ καὶ γωνία ἡ ὑπὸ ΓΑΒ γωνία τῇ ὑπὸ ΖΔΕ ἐστὶν ἴση· βάσις ἄρα ἡ ΒΓ βάσει τῇ ΕΖ ἴση ἐστὶ καὶ τὸ τρίγωνον τῷ τριγώνῳ καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις· ἴση ἄρα ἡ ὑπὸ ΑΓΒ γωνία τῇ ὑπὸ ΔΖΕ. ἔστι δὲ καὶ ὀρθὴ ἡ ὑπὸ ΑΓΚ ὀρθὴ τῇ ὑπὸ ΔΖΝ ἴση· καὶ λοιπὴ ἄρα ἡ ὑπὸ ΒΓΚ λοιπὴ τῇ ὑπὸ ΕΖΝ ἐστὶν ἴση. διὰ τὰ αὐτὰ δὴ καὶ ἡ ὑπὸ ΓΒΚ τῇ ὑπὸ ΖΕΝ ἐστὶν ἴση. δύο δὴ τρίγωνά ἐστι τὰ ΒΓΚ, ΕΖΝ [τὰς] δύο γωνίας δυσὶ γωνίαις ἴσας ἔχοντα ἑκατέραν ἑκατέρᾳ καὶ μίαν πλευρὰν μὲν πλεωρᾶ ἴσην τὴν πρὸς ταῖς ἴσαις γωνίαις τὴν ΒΓ τῇ ΕΖ· καὶ τὰς λοιπὰς ἄρα πλευρὰς ταῖς λοιπαῖς πλευραῖς ἴσας ἔξουσιν. ἴση ἄρα ἐστὶν ἡ ΓΚ τῇ ΖΝ. ἔστι δὲ

$BAC$  and  $EDF$  (respectively). And let them have joined the planes at points  $L$  and  $N$  (respectively). And let  $LA$  and  $ND$  have been joined. I say that angle  $GAL$  is equal to angle  $MDN$ .



Let  $AH$  be made equal to  $DM$ . And let  $HK$  have been drawn through point  $H$  parallel to  $GL$ . And  $GL$  is perpendicular to the plane through  $BAC$ . Thus,  $HK$  is also perpendicular to the plane through  $BAC$  [Prop. 11.8]. And let  $KC$ ,  $NF$ ,  $KB$ , and  $NE$  have been drawn from points  $K$  and  $N$  perpendicular to the straight-lines  $AC$ ,  $DF$ ,  $AB$ , and  $DE$ . And let  $HC$ ,  $CB$ ,  $MF$ , and  $FE$  have been joined. Since the (square) on  $HA$  is equal to the (sum of the squares) on  $HK$  and  $KA$  [Prop. 1.47], and the (sum of the squares) on  $KC$  and  $CA$  is equal to the (square) on  $KA$  [Prop. 1.47], thus the (square) on  $HA$  is equal to the (sum of the squares) on  $HK$ ,  $KC$ , and  $CA$ . And the (square) on  $HC$  is equal to the (sum of the squares) on  $HK$  and  $KC$  [Prop. 1.47]. Thus, the (square) on  $HA$  is equal to the (sum of the squares) on  $HC$  and  $CA$ . Thus, angle  $HCA$  is a right-angle [Prop. 1.48]. So, for the same (reasons), angle  $DFM$  is also a right-angle. Thus, angle  $ACH$  is equal to (angle)  $DFM$ . And  $HAC$  is also equal to  $MDF$ . So,  $MDF$  and  $HAC$  are two triangles having two angles equal to two angles, respectively, and one side equal to one side—(namely), that subtending one of the equal angles—(that is),  $HA$  (equal) to  $MD$ . Thus, they will also have the remaining sides equal to the remaining sides, respectively [Prop. 1.26]. Thus,  $AC$  is equal to  $DF$ . So, similarly, we can show that  $AB$  is also equal to  $DE$ . Therefore, since  $AC$  is equal to  $DF$ , and  $AB$  to  $DE$ , so the two (straight-lines)  $CA$  and  $AB$  are equal to the two (straight-lines)  $FD$  and  $DE$  (respectively). But, angle  $CAB$  is also equal to angle  $FDE$ . Thus, base  $BC$  is equal to base  $EF$ , and triangle  $(ACB)$  to triangle  $(DFE)$ , and the remaining angles to the remaining angles (respectively) [Prop. 1.4].

καὶ ἡ  $ΑΓ$  τῇ  $ΔΖ$  ἴση· δύο δὲ αἱ  $ΑΓ$ ,  $ΓΚ$  δυοὶ ταῖς  $ΔΖ$ ,  $ΖΝ$  ἴσαι εἰσὶν· καὶ ὀρθὰς γωνίας περιέχουσιν. βάσις ἄρα ἡ  $ΑΚ$  βάσει τῇ  $ΔΝ$  ἴση ἐστίν. καὶ ἐπεὶ ἴση ἐστὶν ἡ  $ΑΘ$  τῇ  $ΔΜ$ , ἴσον ἐστὶ καὶ τὸ ἀπὸ τῆς  $ΑΘ$  τῷ ἀπὸ τῆς  $ΔΜ$ . ἀλλὰ τῷ μὲν ἀπὸ τῆς  $ΑΘ$  ἴσα ἐστὶ τὰ ἀπὸ τῶν  $ΑΚ$ ,  $ΚΘ$ · ὀρθὴ γὰρ ἡ ὑπὸ  $ΑΚΘ$ · τῷ δὲ ἀπὸ τῆς  $ΔΜ$  ἴσα τὰ ἀπὸ τῶν  $ΔΝ$ ,  $ΝΜ$ · ὀρθὴ γὰρ ἡ ὑπὸ  $ΔΝΜ$ · τὰ ἄρα ἀπὸ τῶν  $ΑΚ$ ,  $ΚΘ$  ἴσα ἐστὶ τοῖς ἀπὸ τῶν  $ΔΝ$ ,  $ΝΜ$ , ὣν τὸ ἀπὸ τῆς  $ΑΚ$  ἴσον ἐστὶ τῷ ἀπὸ τῆς  $ΔΝ$ · λοιπὸν ἄρα τὸ ἀπὸ τῆς  $ΚΘ$  ἴσον ἐστὶ τῷ ἀπὸ τῆς  $ΝΜ$ · ἴση ἄρα ἡ  $ΘΚ$  τῇ  $ΜΝ$ . καὶ ἐπεὶ δύο αἱ  $ΘΑ$ ,  $ΑΚ$  δυοὶ ταῖς  $ΜΔ$ ,  $ΔΝ$  ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ, καὶ βάσις ἡ  $ΘΚ$  βάσει τῇ  $ΜΝ$  ἐδείχθη ἴση, γωνία ἄρα ἡ ὑπὸ  $ΘΑΚ$  γωνία τῇ ὑπὸ  $ΜΔΝ$  ἐστὶν ἴση.

Ἐὰν ἄρα ὦσι δύο γωνίαι ἐπίπεδοι ἴσαι καὶ τὰ ἐξῆς τῆς προτάσεως [ὅπερ ἔδει δεῖξαι].

Thus, angle  $ACB$  (is) equal to  $DFE$ . And the right-angle  $ACK$  is also equal to the right-angle  $DFN$ . Thus, the remainder  $BCK$  is equal to the remainder  $EFN$ . So, for the same (reasons),  $CBK$  is also equal to  $FEN$ . So,  $BCK$  and  $EFN$  are two triangles having two angles equal to two angles, respectively, and one side equal to one side—(namely), that by the equal angles—(that is),  $BC$  (equal) to  $EF$ . Thus, they will also have the remaining sides equal to the remaining sides (respectively) [Prop. 1.26]. Thus,  $CK$  is equal to  $FN$ . And  $AC$  (is) also equal to  $DF$ . So, the two (straight-lines)  $AC$  and  $CK$  are equal to the two (straight-lines)  $DF$  and  $FN$  (respectively). And they enclose right-angles. Thus, base  $AK$  is equal to base  $DN$  [Prop. 1.4]. And since  $AH$  is equal to  $DM$ , the (square) on  $AH$  is also equal to the (square) on  $DM$ . But, the the (sum of the squares) on  $AK$  and  $KH$  is equal to the (square) on  $AH$ . For angle  $AKH$  (is) a right-angle [Prop. 1.47]. And the (sum of the squares) on  $DN$  and  $NM$  (is) equal to the square on  $DM$ . For angle  $DNM$  (is) a right-angle [Prop. 1.47]. Thus, the (sum of the squares) on  $AK$  and  $KH$  is equal to the (sum of the squares) on  $DN$  and  $NM$ , of which the (square) on  $AK$  is equal to the (square) on  $DN$ . Thus, the remaining (square) on  $KH$  is equal to the (square) on  $NM$ . Thus,  $HK$  (is) equal to  $MN$ . And since the two (straight-lines)  $HA$  and  $AK$  are equal to the two (straight-lines)  $MD$  and  $DN$ , respectively, and base  $HK$  was shown (to be) equal to base  $MN$ , angle  $HAK$  is thus equal to angle  $MDN$  [Prop. 1.8].

Thus, if there are two equal plane angles, and so on of the proposition. [(Which is) the very thing it was required to show].

### Πόρισμα.

Ἐκ δὲ τούτου φανερόν, ὅτι, ἐὰν ὦσι δύο γωνίαι ἐπίπεδοι ἴσαι, ἐπισταθῶσι δὲ ἐπ' αὐτῶν μετέωροι εὐθεῖαι ἴσαι ἴσας γωνίας περιέχουσαι μετὰ τῶν ἐξ ἀρχῆς εὐθειῶν ἑκατέραν ἑκατέρᾳ, αἱ ἀπ' αὐτῶν κάθετοι ἀγόμεναι ἐπὶ τὰ ἐπίπεδα, ἐν οἷς εἰσιν αἱ ἐξ ἀρχῆς γωνίαι, ἴσαι ἀλλήλαις εἰσὶν. ὅπερ ἔδει δεῖξαι.

### Corollary

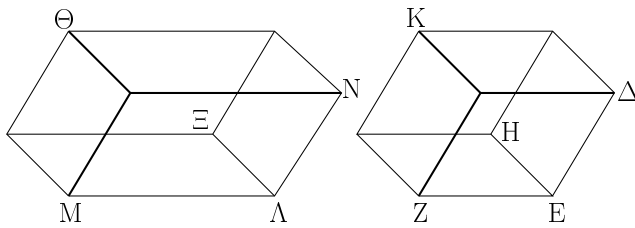
So, it is clear, from this, that if there are two equal plane angles, and equal raised straight-lines are stood on them (at their apexes), containing equal angles respectively with the original straight-lines (forming the angles), then the perpendiculars drawn from (the raised ends of) them to the planes in which the original angles lie are equal to one another. (Which is) the very thing it was required to show.

### λζ'.

Ἐὰν τρεῖς εὐθεῖαι ἀνάλογον ὦσιν, τὸ ἐκ τῶν τριῶν στερεὸν παραλληλεπίπεδον ἴσον ἐστὶ τῷ ἀπὸ τῆς μέσης στερεῷ παραλληλεπίπεδῳ ἰσοπλεύρῳ μὲν, ἰσογωνίῳ δὲ τῷ προειρημένῳ.

### Proposition 36

If three straight-lines are (continuously) proportional then the parallelepiped solid (formed) from the three (straight-lines) is equal to the equilateral parallelepiped solid on the middle (straight-line which is) equiangular to the aforementioned (parallelepiped solid).



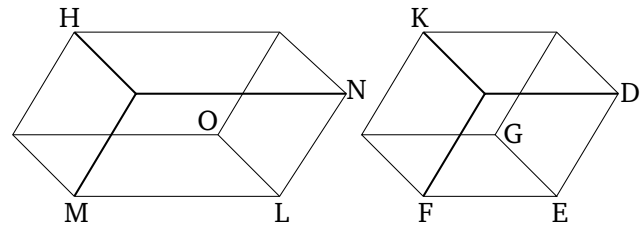
A \_\_\_\_\_  
 B \_\_\_\_\_  
 Γ \_\_\_\_\_

Ἐστωσαν τρεῖς εὐθεῖαι ἀνάλογον αἱ  $A, B, \Gamma$ , ὥς ἡ  $A$  πρὸς τὴν  $B$ , οὕτως ἡ  $B$  πρὸς τὴν  $\Gamma$ . λέγω, ὅτι τὸ ἐκ τῶν  $A, B, \Gamma$  στερεὸν ἴσον ἐστὶ τῷ ἀπὸ τῆς  $B$  στερεῷ ἰσοπλευρῷ μὲν, ἰσογωνίῳ δὲ τῷ προειρημένῳ.

Ἐκκείσθω στερεὰ γωνία ἡ πρὸς τῷ  $E$  περιεχομένη ὑπὸ τῶν ὑπὸ  $\Delta E H, H E Z, Z E \Delta$ , καὶ κείσθω τῇ μὲν  $B$  ἴση ἐκάστη τῶν  $\Delta E, H E, E Z$ , καὶ συμπληρώσθω τὸ  $E K$  στερεὸν παραλληλεπίπεδον, τῇ δὲ  $A$  ἴση ἡ  $\Lambda M$ , καὶ συνεστάτω πρὸς τῇ  $\Lambda M$  εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ  $\Lambda$  τῇ πρὸς τῷ  $E$  στερεᾷ γωνίᾳ ἴση στερεὰ γωνία ἡ περιεχομένη ὑπὸ τῶν  $\Lambda \Lambda \Xi, \Xi \Lambda M, M \Lambda N$ , καὶ κείσθω τῇ μὲν  $B$  ἴση ἡ  $\Lambda \Xi$ , τῇ δὲ  $\Gamma$  ἴση ἡ  $\Lambda N$ . καὶ ἐπεὶ ἐστὶν ὥς ἡ  $A$  πρὸς τὴν  $B$ , οὕτως ἡ  $B$  πρὸς τὴν  $\Gamma$ , ἴση δὲ ἡ μὲν  $A$  τῇ  $\Lambda M$ , ἡ δὲ  $B$  ἐκατέρᾳ τῶν  $\Lambda \Xi, E \Delta$ , ἡ δὲ  $\Gamma$  τῇ  $\Lambda N$ , ἐστὶν ἄρα ὥς ἡ  $\Lambda M$  πρὸς τὴν  $E Z$ , οὕτως ἡ  $\Delta E$  πρὸς τὴν  $\Lambda N$ . καὶ περὶ ἴσας γωνίας τὰς ὑπὸ  $\Lambda M, \Delta E Z$  αἱ πλευραὶ ἀντιπεπόνθασιν· ἴσον ἄρα ἐστὶ τὸ  $M N$  παραλληλόγραμμον τῷ  $\Delta Z$  παραλληλογράμμῳ. καὶ ἐπεὶ δύο γωνίαι ἐπίπεδοι εὐθύγραμμοι ἴσαι εἰσὶν αἱ ὑπὸ  $\Delta E Z, \Lambda M$ , καὶ ἐπ' αὐτῶν μετέωροι εὐθεῖαι ἐφεστᾶσιν αἱ  $\Lambda \Xi, E H$  ἴσαι τε ἀλλήλαις καὶ ἴσας γωνίας περιέχουσιν μετὰ τῶν ἐξ ἀρχῆς εὐθειῶν ἐκατέραν ἐκατέρᾳ, αἱ ἄρα ἀπὸ τῶν  $H, \Xi$  σημείων κάθετοι ἀγόμεναι ἐπὶ τὰ διὰ τῶν  $\Lambda M, \Delta E Z$  ἐπίπεδα ἴσαι ἀλλήλαις εἰσὶν· ὥστε τὰ  $\Lambda \Theta, E K$  στερεὰ ὑπὸ τὸ αὐτὸ ὕψος ἐστίν. τὰ δὲ ἐπὶ ἴσων βάσεων στερεὰ παραλληλεπίπεδα καὶ ὑπὸ τὸ αὐτὸ ὕψος ἴσα ἀλλήλοις ἐστίν· ἴσον ἄρα ἐστὶ τὸ  $\Theta \Lambda$  στερεὸν τῷ  $E K$  στερεῷ. καὶ ἐστὶ τὸ μὲν  $\Lambda \Theta$  τὸ ἐκ τῶν  $A, B, \Gamma$  στερεόν, τὸ δὲ  $E K$  τὸ ἀπὸ τῆς  $B$  στερεόν· τὸ ἄρα ἐκ τῶν  $A, B, \Gamma$  στερεὸν παραλληλεπίπεδον ἴσον ἐστὶ τῷ ἀπὸ τῆς  $B$  στερεῷ ἰσοπλευρῷ μὲν, ἰσογωνίῳ δὲ τῷ προειρημένῳ· ὃπερ εἶδει δεῖξαι.

λζ'.

Ἐὰν τέσσαρες εὐθεῖαι ἀνάλογον ᾤσιν, καὶ τὰ ἀπ' αὐτῶν



A \_\_\_\_\_  
 B \_\_\_\_\_  
 C \_\_\_\_\_

Let  $A, B$ , and  $C$  be three (continuously) proportional straight-lines, (such that) as  $A$  (is) to  $B$ , so  $B$  (is) to  $C$ . I say that the (parallelepiped) solid (formed) from  $A, B$ , and  $C$  is equal to the equilateral solid on  $B$  (which is) equiangular with the aforementioned (solid).

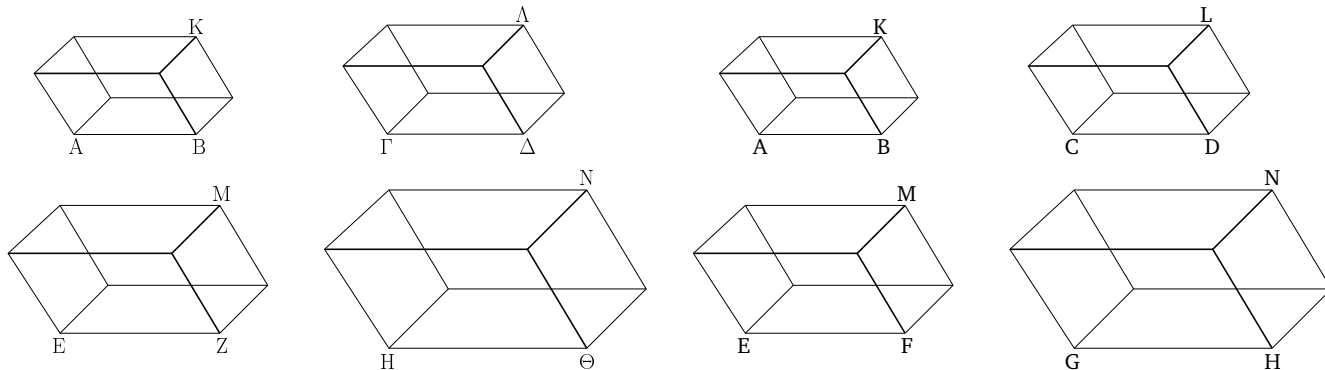
Let the solid angle at  $E$ , contained by  $DEG, GEF$ , and  $FED$ , be set out. And let  $DE, GE$ , and  $EF$  each be made equal to  $B$ . And let the parallelepiped solid  $E K$  have been completed. And (let)  $LM$  (be made) equal to  $A$ . And let the solid angle contained by  $NLO, OLM$ , and  $MLN$  have been constructed on the straight-line  $LM$ , and at the point  $L$  on it, (so as to be) equal to the solid angle  $E$  [Prop. 11.23]. And let  $LO$  be made equal to  $B$ , and  $LN$  equal to  $C$ . And since as  $A$  (is) to  $B$ , so  $B$  (is) to  $C$ , and  $A$  (is) equal to  $LM$ , and  $B$  to each of  $LO$  and  $ED$ , and  $C$  to  $LN$ , thus as  $LM$  (is) to  $EF$ , so  $DE$  (is) to  $LN$ . And (so) the sides around the equal angles  $NLM$  and  $DEF$  are reciprocally proportional. Thus, parallelogram  $MN$  is equal to parallelogram  $DF$  [Prop. 6.14]. And since the two plane rectilinear angles  $DEF$  and  $NLM$  are equal, and the raised straight-lines stood on them (at their apexes),  $LO$  and  $EG$ , are equal to one another, and contain equal angles respectively with the original straight-lines (forming the angles), the perpendiculars drawn from points  $G$  and  $O$  to the planes through  $NLM$  and  $DEF$  (respectively) are thus equal to one another [Prop. 11.35 corr.]. Thus, the solids  $LH$  and  $E K$  (have) the same height. And parallelepiped solids on equal bases, and with the same height, are equal to one another [Prop. 11.31]. Thus, solid  $HL$  is equal to solid  $E K$ . And  $LH$  is the solid (formed) from  $A, B$ , and  $C$ , and  $E K$  the solid on  $B$ . Thus, the parallelepiped solid (formed) from  $A, B$ , and  $C$  is equal to the equilateral solid on  $B$  (which is) equiangular with the aforementioned (solid). (Which is) the very thing it was required to show.

Proposition 37<sup>†</sup>

If four straight-lines are proportional then the similar,

στερεὰ παραλληλεπίπεδα ὁμοία τε καὶ ὁμοίως ἀναγραφόμενα ἀνάλογον ἔσται· καὶ ἐὰν τὰ ἀπ' αὐτῶν στερεὰ παραλληλεπίπεδα ὁμοία τε καὶ ὁμοίως ἀναγραφόμενα ἀνάλογον ᾗ, καὶ αὐταὶ αἱ εὐθεῖαι ἀνάλογον ἔσονται.

and similarly described, parallelepiped solids on them will also be proportional. And if the similar, and similarly described, parallelepiped solids on them are proportional then the straight-lines themselves will be proportional.



Ἐστωσαν τέσσαρες εὐθεῖαι ἀνάλογον αἱ  $AB$ ,  $\Gamma\Delta$ ,  $EZ$ ,  $H\Theta$ , ὡς ἡ  $AB$  πρὸς τὴν  $\Gamma\Delta$ , οὕτως ἡ  $EZ$  πρὸς τὴν  $H\Theta$ , καὶ ἀναγεγράφθωσαν ἀπὸ τῶν  $AB$ ,  $\Gamma\Delta$ ,  $EZ$ ,  $H\Theta$  ὁμοία τε καὶ ὁμοίως κείμενα στερεὰ παραλληλεπίπεδα τὰ  $KA$ ,  $\Lambda\Gamma$ ,  $ME$ ,  $NH$ · λέγω, ὅτι ἔστιν ὡς τὸ  $KA$  πρὸς τὸ  $\Lambda\Gamma$ , οὕτως τὸ  $ME$  πρὸς τὸ  $NH$ .

Ἐπεὶ γὰρ ὁμοίον ἐστὶ τὸ  $KA$  στερεὸν παραλληλεπίπεδον τῷ  $\Lambda\Gamma$ , τὸ  $KA$  ἄρα πρὸς τὸ  $\Lambda\Gamma$  τριπλασίονα λόγον ἔχει ἢ περ ἡ  $AB$  πρὸς τὴν  $\Gamma\Delta$ . διὰ τὰ αὐτὰ δὴ καὶ τὸ  $ME$  πρὸς τὸ  $NH$  τριπλασίονα λόγον ἔχει ἢ περ ἡ  $EZ$  πρὸς τὴν  $H\Theta$ . καὶ ἐστὶν ὡς ἡ  $AB$  πρὸς τὴν  $\Gamma\Delta$ , οὕτως ἡ  $EZ$  πρὸς τὴν  $H\Theta$ . καὶ ὡς ἄρα τὸ  $AK$  πρὸς τὸ  $\Lambda\Gamma$ , οὕτως τὸ  $ME$  πρὸς τὸ  $NH$ .

Ἀλλὰ δὴ ἔστω ὡς τὸ  $AK$  στερεὸν πρὸς τὸ  $\Lambda\Gamma$  στερεόν, οὕτως τὸ  $ME$  στερεὸν πρὸς τὸ  $NH$ · λέγω, ὅτι ἔστιν ὡς ἡ  $AB$  εὐθεῖα πρὸς τὴν  $\Gamma\Delta$ , οὕτως ἡ  $EZ$  πρὸς τὴν  $H\Theta$ .

Ἐπεὶ γὰρ πάλιν τὸ  $KA$  πρὸς τὸ  $\Lambda\Gamma$  τριπλασίονα λόγον ἔχει ἢ περ ἡ  $AB$  πρὸς τὴν  $\Gamma\Delta$ , ἔχει δὲ καὶ τὸ  $ME$  πρὸς τὸ  $NH$  τριπλασίονα λόγον ἢ περ ἡ  $EZ$  πρὸς τὴν  $H\Theta$ , καὶ ἐστὶν ὡς τὸ  $KA$  πρὸς τὸ  $\Lambda\Gamma$ , οὕτως τὸ  $ME$  πρὸς τὸ  $NH$ , καὶ ὡς ἄρα ἡ  $AB$  πρὸς τὴν  $\Gamma\Delta$ , οὕτως ἡ  $EZ$  πρὸς τὴν  $H\Theta$ .

Ἐὰν ἄρα τέσσαρες εὐθεῖαι ἀνάλογον ᾧσι καὶ τὰ ἐξῆς τῆς προτάσεως· ὅπερ ἔδει δεῖξαι.

Let  $AB$ ,  $CD$ ,  $EF$ , and  $GH$ , be four proportional straight-lines, (such that) as  $AB$  (is) to  $CD$ , so  $EF$  (is) to  $GH$ . And let the similar, and similarly laid out, parallelepiped solids  $KA$ ,  $LC$ ,  $ME$  and  $NG$  have been described on  $AB$ ,  $CD$ ,  $EF$ , and  $GH$  (respectively). I say that as  $KA$  is to  $LC$ , so  $ME$  (is) to  $NG$ .

For since the parallelepiped solid  $KA$  is similar to  $LC$ ,  $KA$  thus has to  $LC$  the cubed ratio that  $AB$  (has) to  $CD$  [Prop. 11.33]. So, for the same (reasons),  $ME$  also has to  $NG$  the cubed ratio that  $EF$  (has) to  $GH$  [Prop. 11.33]. And since as  $AB$  is to  $CD$ , so  $EF$  (is) to  $GH$ , thus, also, as  $AK$  (is) to  $LC$ , so  $ME$  (is) to  $NG$ .

And so let solid  $AK$  be to solid  $LC$ , as solid  $ME$  (is) to  $NG$ . I say that as straight-line  $AB$  is to  $CD$ , so  $EF$  (is) to  $GH$ .

For, again, since  $KA$  has to  $LC$  the cubed ratio that  $AB$  (has) to  $CD$  [Prop. 11.33], and  $ME$  also has to  $NG$  the cubed ratio that  $EF$  (has) to  $GH$  [Prop. 11.33], and as  $KA$  is to  $LC$ , so  $ME$  (is) to  $NG$ , thus, also, as  $AB$  (is) to  $CD$ , so  $EF$  (is) to  $GH$ .

Thus, if four straight-lines are proportional, and so on of the proposition. (Which is) the very thing it was required to show.

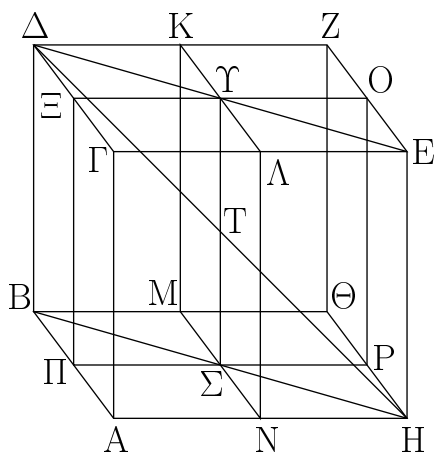
† This proposition assumes that if two ratios are equal then the cube of the former is also equal to the cube of the latter, and *vice versa*.

λη'.

### Proposition 38

Ἐὰν κύβου τῶν ἀπεναντίον ἐπιπέδων αἱ πλευραὶ δίχα τμηθῶσιν, διὰ δὲ τῶν τομῶν ἐπίπεδα ἐκβληθῇ, ἡ κοινὴ τομὴ τῶν ἐπιπέδων καὶ ἡ τοῦ κύβου διάμετρος δίχα τέμνουσιν ἀλλήλας.

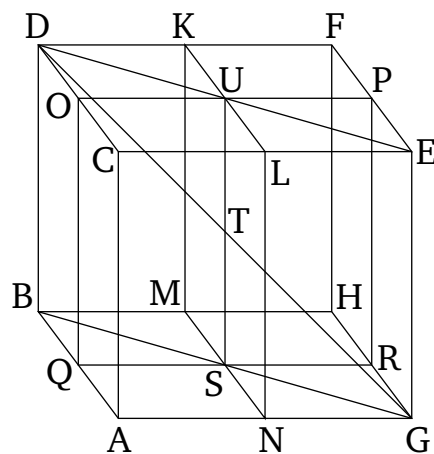
If the sides of the opposite planes of a cube are cut in half, and planes are produced through the pieces, then the common section of the (latter) planes and the diameter of the cube cut one another in half.



Κύβου γάρ τοῦ ΑΖ τῶν ἀπεναντίον ἐπιπέδων τῶν ΓΖ, ΑΘ αἱ πλευραὶ δίχα τετμήσθωσαν κατὰ τὰ Κ, Λ, Μ, Ν, Ξ, Π, Ο, Ρ σημεῖα, διὰ δὲ τῶν τομῶν ἐπίπεδα ἐκβεβλήσθω τὰ ΚΝ, ΞΡ, κοινὴ δὲ τομὴ τῶν ἐπιπέδων ἔστω ἡ ΥΣ, τοῦ δὲ ΑΖ κύβου διαγώνιος ἡ ΔΗ. λέγω, ὅτι ἴση ἔστιν ἡ μὲν ΥΤ τῇ ΤΣ, ἡ δὲ ΔΤ τῇ ΤΗ.

Ἐπεζεύχθωσαν γάρ αἱ ΔΥ, ΥΕ, ΒΣ, ΣΗ. καὶ ἐπεὶ παράλληλός ἐστιν ἡ ΔΞ τῇ ΟΕ, αἱ ἐναλλάξ γωνίαι αἱ ὑπὸ ΔΞΥ, ΥΟΕ ἴσαι ἀλλήλαις εἰσίν. καὶ ἐπεὶ ἴση ἔστιν ἡ μὲν ΔΞ τῇ ΟΕ, ἡ δὲ ΞΥ τῇ ΥΟ, καὶ γωνίας ἴσας περιέχουσιν, βάσεις ἄρα ἡ ΔΥ τῇ ΥΕ ἔστιν ἴση, καὶ τὸ ΔΞΥ τρίγωνον τῷ ΟΥΕ τριγώνῳ ἔστιν ἴσον καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι. ἴση ἄρα ἡ ὑπὸ ΞΥΔ γωνία τῇ ὑπὸ ΟΥΕ γωνίᾳ. διὰ δὲ τοῦτο εὐθεία ἐστιν ἡ ΔΥΕ. διὰ τὰ αὐτὰ δὲ καὶ ΒΣΗ εὐθεία ἐστιν, καὶ ἴση ἡ ΒΣ τῇ ΣΗ. καὶ ἐπεὶ ἡ ΓΑ τῇ ΔΒ ἴση ἐστὶ καὶ παράλληλος, ἀλλὰ ἡ ΓΑ καὶ τῇ ΕΗ ἴση τέ ἐστὶ καὶ παράλληλος, καὶ ἡ ΔΒ ἄρα τῇ ΕΗ ἴση τέ ἐστὶ καὶ παράλληλος. καὶ ἐπιζευγνύουσιν αὐτὰς εὐθεῖαι αἱ ΔΕ, ΒΗ· παράλληλος ἄρα ἐστὶν ἡ ΔΕ τῇ ΒΗ. ἴση ἄρα ἡ μὲν ὑπὸ ΕΔΤ γωνία τῇ ὑπὸ ΒΗΤ· ἐναλλάξ γάρ· ἡ δὲ ὑπὸ ΔΤΥ τῇ ὑπὸ ΗΤΣ. δύο δὲ τρίγωνα ἐστὶ τὰ ΔΤΥ, ΗΤΣ τὰς δύο γωνίας ταῖς δυσὶ γωνίαις ἴσας ἔχοντα καὶ μίαν πλευρὰν μιᾷ πλευρᾷ ἴσην τὴν ὑποτείνουσιν ὑπὸ μίαν τῶν ἴσων γωνιῶν τὴν ΔΥ τῇ ΗΣ· ἡμίσειαι γάρ εἰσι τῶν ΔΕ, ΒΗ· καὶ τὰς λοιπὰς πλευρὰς ταῖς λοιπαῖς πλευραῖς ἴσας ἔξει. ἴση ἄρα ἡ μὲν ΔΤ τῇ ΤΗ, ἡ δὲ ΥΤ τῇ ΤΣ.

Ἐὰν ἄρα κύβου τῶν ἀπεναντίον ἐπιπέδων αἱ πλευραὶ δίχα τμηθῶσιν, διὰ δὲ τῶν τομῶν ἐπίπεδα ἐκβληθῇ, ἡ κοινὴ τομὴ τῶν ἐπιπέδων καὶ ἡ τοῦ κύβου διάμετρος δίχα τέμνουσιν ἀλλήλας· ὅπερ εἶδει δεῖξαι.



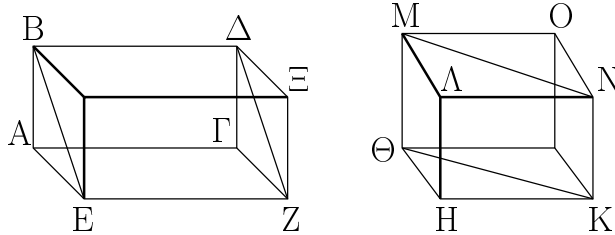
For let the opposite planes  $CF$  and  $AH$  of the cube  $AF$  have been cut in half at the points  $K, L, M, N, O, Q, P$ , and  $R$ . And let the planes  $KN$  and  $OR$  have been produced through the pieces. And let  $US$  be the common section of the planes, and  $DG$  the diameter of cube  $AF$ . I say that  $UT$  is equal to  $TS$ , and  $DT$  to  $TG$ .

For let  $DU, UE, BS$ , and  $SG$  have been joined. And since  $DO$  is parallel to  $PE$ , the alternate angles  $DOU$  and  $UPE$  are equal to one another [Prop. 1.29]. And since  $DO$  is equal to  $PE$ , and  $OU$  to  $UP$ , and they contain equal angles, base  $DU$  is thus equal to base  $UE$ , and triangle  $DOU$  is equal to triangle  $PUE$ , and the remaining angles (are) equal to the remaining angles [Prop. 1.4]. Thus, angle  $ODU$  (is) equal to angle  $PUE$ . So, for this (reason),  $DUE$  is a straight-line [Prop. 1.14]. So, for the same (reason),  $BSG$  is also a straight-line, and  $BS$  equal to  $SG$ . And since  $CA$  is equal and parallel to  $DB$ , but  $CA$  is also equal and parallel to  $EG$ ,  $DB$  is thus also equal and parallel to  $EG$  [Prop. 11.9]. And the straight-lines  $DE$  and  $BG$  join them.  $DE$  is thus parallel to  $BG$  [Prop. 1.33]. Thus, angle  $EDT$  (is) equal to  $BGT$ . For (they are) alternate [Prop. 1.29]. And (angle)  $DTU$  (is equal) to  $GTS$  [Prop. 1.15]. So,  $DTU$  and  $GTS$  are two triangles having two angles equal to two angles, and one side equal to one side—(namely), that subtended by one of the equal angles—(that is),  $DU$  (equal) to  $GS$ . For they are halves of  $DE$  and  $BG$  (respectively). (Thus), they will also have the remaining sides equal to the remaining sides [Prop. 1.26]. Thus,  $DT$  (is) equal to  $TG$ , and  $UT$  to  $TS$ .

Thus, if the sides of the opposite planes of a cube are cut in half, and planes are produced through the pieces, then the common section of the (latter) planes and the diameter of the cube cut one another in half. (Which is) the very thing it was required to show.

λθ'.

Ἐάν ᾗ δύο πρίσματα ἰσοῦψῃ, καὶ τὸ μὲν ἔχῃ βάσιν παραλληλόγραμμον, τὸ δὲ τρίγωνον, διπλάσιον δὲ ᾗ τὸ παραλληλόγραμμον τοῦ τριγώνου, ἴσα ἔσται τὰ πρίσματα.



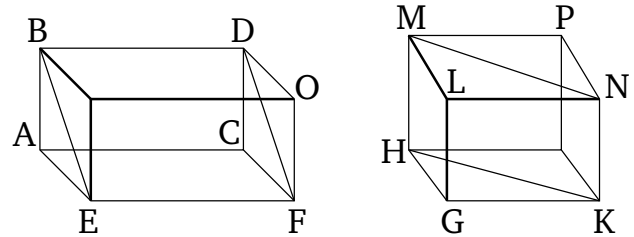
Ἐστω δύο πρίσματα ἰσοῦψῃ τὰ ΑΒΓΔΕΖ, ΗΘΚΛΜΝ, καὶ τὸ μὲν ἔχέτω βάσιν τὸ ΑΖ παραλληλόγραμμον, τὸ δὲ τὸ ΗΘΚ τρίγωνον, διπλάσιον δὲ ἔστω τὸ ΑΖ παραλληλόγραμμον τοῦ ΗΘΚ τριγώνου· λέγω, ὅτι ἴσον ἐστὶ τὸ ΑΒΓΔΕΖ πρίσμα τῷ ΗΘΚΛΜΝ πρίσματι.

Συμπεπληρώσθω γὰρ τὰ ΑΞ, ΗΟ στερεά. ἐπεὶ διπλάσιόν ἐστι τὸ ΑΖ παραλληλόγραμμον τοῦ ΗΘΚ τριγώνου, ἔστι δὲ καὶ τὸ ΘΚ παραλληλόγραμμον διπλάσιον τοῦ ΗΘΚ τριγώνου, ἴσον ἄρα ἐστὶ τὸ ΑΖ παραλληλόγραμμον τῷ ΘΚ παραλληλογράμμῳ. τὰ δὲ ἐπὶ ἴσων βάσεων ὄντα στερεὰ παραλληλεπίπεδα καὶ ὑπὸ τὸ αὐτὸ ὕψος ἴσα ἀλλήλοις ἐστίν· ἴσον ἄρα ἐστὶ τὸ ΑΞ στερεὸν τῷ ΗΟ στερεῷ. καὶ ἐστὶ τοῦ μὲν ΑΞ στερεοῦ ἡμισυ τὸ ΑΒΓΔΕΖ πρίσμα, τοῦ δὲ ΗΟ στερεοῦ ἡμισυ τὸ ΗΘΚΛΜΝ πρίσμα· ἴσον ἄρα ἐστὶ τὸ ΑΒΓΔΕΖ πρίσμα τῷ ΗΘΚΛΜΝ πρίσματι.

Ἐάν ἄρα ᾗ δύο πρίσματα ἰσοῦψῃ, καὶ τὸ μὲν ἔχῃ βάσιν παραλληλόγραμμον, τὸ δὲ τρίγωνον, διπλάσιον δὲ ᾗ τὸ παραλληλόγραμμον τοῦ τριγώνου, ἴσα ἔσται τὰ πρίσματα· ὅπερ ἔδει δεῖξαι.

## Proposition 39

If there are two equal height prisms, and one has a parallelogram, and the other a triangle, (as a) base, and the parallelogram is double the triangle, then the prisms will be equal.



Let  $ABCDEF$  and  $GHKLMN$  be two equal height prisms, and let the former have the parallelogram  $AF$ , and the latter the triangle  $GHK$ , as a base. And let parallelogram  $AF$  be twice triangle  $GHK$ . I say that prism  $ABCDEF$  is equal to prism  $GHKLMN$ .

For let the solids  $AO$  and  $GP$  have been completed. Since parallelogram  $AF$  is double triangle  $GHK$ , and parallelogram  $HK$  is also double triangle  $GHK$  [Prop. 1.34], parallelogram  $AF$  is thus equal to parallelogram  $HK$ . And parallelepiped solids which are on equal bases, and (have) the same height, are equal to one another [Prop. 11.31]. Thus, solid  $AO$  is equal to solid  $GP$ . And prism  $ABCDEF$  is half of solid  $AO$ , and prism  $GHKLMN$  half of solid  $GP$  [Prop. 11.28]. Prism  $ABCDEF$  is thus equal to prism  $GHKLMN$ .

Thus, if there are two equal height prisms, and one has a parallelogram, and the other a triangle, (as a) base, and the parallelogram is double the triangle, then the prisms are equal. (Which is) the very thing it was required to show.

# ELEMENTS BOOK 12

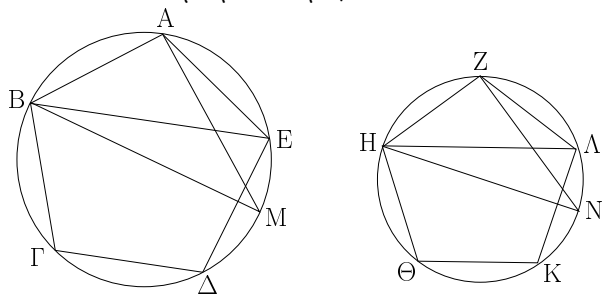
## *Proportional Stereometry*<sup>†</sup>

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<sup>†</sup>The novel feature of this book is the use of the so-called *method of exhaustion* (see Prop. 10.1), a precursor to integration which is generally attributed to Eudoxus of Cnidus.

α'.

Τὰ ἐν τοῖς κύκλοις ὅμοια πολύγωνα πρὸς ἀλλήλα ἐστὶν ὡς τὰ ἀπὸ τῶν διαμέτρων τετράγωνα.



Ἐστωσαν κύκλοι οἱ ΑΒΓ, ΖΗΘ, καὶ ἐν αὐτοῖς ὅμοια πολύγωνα ἔστω τὰ ΑΒΓΔΕ, ΖΗΘΚΛ, διαμέτροι δὲ τῶν κύκλων ἔστωσαν ΒΜ, ΗΝ· λέγω, ὅτι ἐστὶν ὡς τὸ ἀπὸ τῆς ΒΜ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΗΝ τετράγωνον, οὕτως τὸ ΑΒΓΔΕ πολύγωνον πρὸς τὸ ΖΗΘΚΛ πολύγωνον.

Ἐπεζεύχθωσαν γὰρ αἱ ΒΕ, ΑΜ, ΗΛ, ΖΝ. καὶ ἐπεὶ ὅμοιον τὸ ΑΒΓΔΕ πολύγωνον τῷ ΖΗΘΚΛ πολυγώνῳ, ἴση ἐστὶ καὶ ἡ ὑπὸ ΒΑΕ γωνία τῇ ὑπὸ ΗΖΛ, καὶ ἐστὶν ὡς ἡ ΒΑ πρὸς τὴν ΑΕ, οὕτως ἡ ΗΖ πρὸς τὴν ΖΛ. δύο δὲ τρίγωνά ἐστι τὰ ΒΑΕ, ΗΖΛ μίαν γωνίαν μιᾷ γωνίᾳ ἴσην ἔχοντα τὴν ὑπὸ ΒΑΕ τῇ ὑπὸ ΗΖΛ, περὶ δὲ τὰς ἴσας γωνίας τὰς πλευρὰς ἀνάλογον· ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΕ τρίγωνον τῷ ΖΗΛ τριγώνῳ. ἴση ἄρα ἐστὶν ἡ ὑπὸ ΑΕΒ γωνία τῇ ὑπὸ ΖΑΗ. ἀλλ' ἡ μὲν ὑπὸ ΑΕΒ τῇ ὑπὸ ΑΜΒ ἐστὶν ἴση· ἐπὶ γὰρ τῆς αὐτῆς περιφερείας βεβήκασιν· ἡ δὲ ὑπὸ ΖΑΗ τῇ ὑπὸ ΖΝΗ· καὶ ἡ ὑπὸ ΑΜΒ ἄρα τῇ ὑπὸ ΖΝΗ ἐστὶν ἴση. ἐστὶ δὲ καὶ ὀρθὴ ἡ ὑπὸ ΒΑΜ ὀρθὴ τῇ ὑπὸ ΗΖΝ ἴση· καὶ ἡ λοιπὴ ἄρα τῇ λοιπῇ ἐστὶν ἴση. ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΜ τρίγωνον τῷ ΖΗΝ τρίγωνῳ. ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΒΜ πρὸς τὴν ΗΝ, οὕτως ἡ ΒΑ πρὸς τὴν ΗΖ. ἀλλὰ τοῦ μὲν τῆς ΒΜ πρὸς τὴν ΗΝ λόγον διπλασίων ἐστὶν ὁ τοῦ ἀπὸ τῆς ΒΜ τετραγώνου πρὸς τὸ ἀπὸ τῆς ΗΝ τετράγωνον, τοῦ δὲ τῆς ΒΑ πρὸς τὴν ΗΖ διπλασίων ἐστὶν ὁ τοῦ ΑΒΓΔΕ πολυγώνου πρὸς τὸ ΖΗΘΚΛ πολύγωνον· καὶ ὡς ἄρα τὸ ἀπὸ τῆς ΒΜ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΗΝ τετράγωνον, οὕτως τὸ ΑΒΓΔΕ πολύγωνον πρὸς τὸ ΖΗΘΚΛ πολύγωνον.

Τὰ ἄρα ἐν τοῖς κύκλοις ὅμοια πολύγωνα πρὸς ἀλλήλα ἐστὶν ὡς τὰ ἀπὸ τῶν διαμέτρων τετράγωνα· ὅπερ εἶδει δεῖξαι.

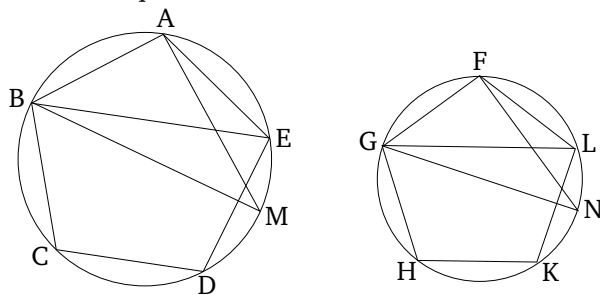
β'.

Οἱ κύκλοι πρὸς ἀλλήλους εἰσὶν ὡς τὰ ἀπὸ τῶν διαμέτρων τετράγωνα.

Ἐστωσαν κύκλοι οἱ ΑΒΓΔ, ΕΖΗΘ, διαμέτροι δὲ αὐτῶν

## Proposition 1

Similar polygons (inscribed) in circles are to one another as the squares on the diameters (of the circles).



Let  $ABC$  and  $FGH$  be circles, and let  $ABCDE$  and  $FGHKL$  be similar polygons (inscribed) in them (respectively), and let  $BM$  and  $GN$  be the diameters of the circles (respectively). I say that as the square on  $BM$  is to the square on  $GN$ , so polygon  $ABCDE$  (is) to polygon  $FGHKL$ .

For let  $BE$ ,  $AM$ ,  $GL$ , and  $FN$  have been joined. And since polygon  $ABCDE$  (is) similar to polygon  $FGHKL$ , angle  $BAE$  is also equal to (angle)  $GFL$ , and as  $BA$  is to  $AE$ , so  $GF$  (is) to  $FL$  [Def. 6.1]. So,  $BAE$  and  $GFL$  are two triangles having one angle equal to one angle, (namely),  $BAE$  (equal) to  $GFL$ , and the sides around the equal angles proportional. Triangle  $ABE$  is thus equiangular with triangle  $FGL$  [Prop. 6.6]. Thus, angle  $AEB$  is equal to (angle)  $FLG$ . But,  $AEB$  is equal to  $AMB$ , and  $FLG$  to  $FNG$ , for they stand on the same circumference [Prop. 3.27]. Thus,  $AMB$  is also equal to  $FNG$ . And the right-angle  $BAM$  is also equal to the right-angle  $GFN$  [Prop. 3.31]. Thus, the remaining (angle) is also equal to the remaining (angle) [Prop. 1.32]. Thus, triangle  $ABM$  is equiangular with triangle  $FNG$ . Thus, proportionally, as  $BM$  is to  $GN$ , so  $BA$  (is) to  $GF$  [Prop. 6.4]. But, the (ratio) of the square on  $BM$  to the square on  $GN$  is the square of the ratio of  $BM$  to  $GN$ , and the (ratio) of polygon  $ABCDE$  to polygon  $FGHKL$  is the square of the (ratio) of  $BA$  to  $GF$  [Prop. 6.20]. And, thus, as the square on  $BM$  (is) to the square on  $GN$ , so polygon  $ABCDE$  (is) to polygon  $FGHKL$ .

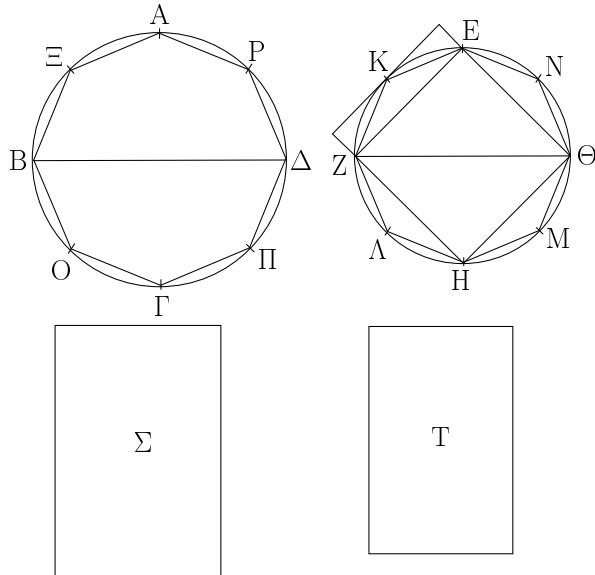
Thus, similar polygons (inscribed) in circles are to one another as the squares on the diameters (of the circles). (Which is) the very thing it was required to show.

## Proposition 2

Circles are to one another as the squares on (their) diameters.

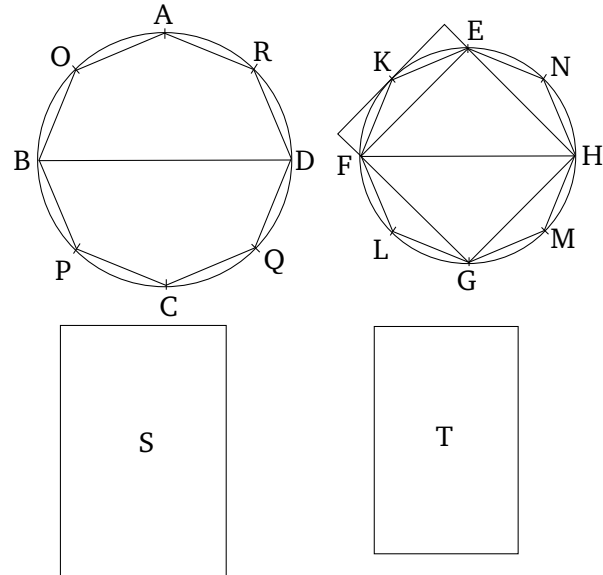
Let  $ABCD$  and  $EFGH$  be circles, and [let]  $BD$  and

[ἔστωσαν] αἱ  $B\Delta$ ,  $Z\Theta$ · λέγω, ὅτι ἐστὶν ὡς ὁ  $AB\Gamma\Delta$  κύκλος πρὸς τὸν  $EZH\Theta$  κύκλον, οὕτως τὸ ἀπὸ τῆς  $B\Delta$  τετράγωνον πρὸς τὸ ἀπὸ τῆς  $Z\Theta$  τετράγωνον.



Εἰ γὰρ μὴ ἐστὶν ὡς ὁ  $AB\Gamma\Delta$  κύκλος πρὸς τὸν  $EZH\Theta$ , οὕτως τὸ ἀπὸ τῆς  $B\Delta$  τετράγωνον πρὸς τὸ ἀπὸ τῆς  $Z\Theta$ , ἔσται ὡς τὸ ἀπὸ τῆς  $B\Delta$  πρὸς τὸ ἀπὸ τῆς  $Z\Theta$ , οὕτως ὁ  $AB\Gamma\Delta$  κύκλος ἤτοι πρὸς ἑλάσσον τι τοῦ  $EZH\Theta$  κύκλου χωρίον ἢ πρὸς μείζον. ἔστω πρότερον πρὸς ἑλάσσον τὸ  $\Sigma$ . καὶ ἐγγεγράφθω εἰς τὸν  $EZH\Theta$  κύκλον τετράγωνον τὸ  $EZH\Theta$ . τὸ δὴ ἐγγεγραμμένον τετράγωνον μείζον ἐστὶν ἢ τὸ ἥμισυ τοῦ  $EZH\Theta$  κύκλου, ἐπειδὴ περ ἐὰν διὰ τῶν  $E, Z, H, \Theta$  σημείων ἐφαπτομένης [εὐθείας] τοῦ κύκλου ἀγάγωμεν, τοῦ περιγραφομένου περὶ τὸν κύκλον τετραγώνου ἥμισυ ἐστὶ τὸ  $EZH\Theta$  τετράγωνον, τοῦ δὲ περιγραφέντος τετραγώνου ἐλάττων ἐστὶν ὁ κύκλος· ὥστε τὸ  $EZH\Theta$  ἐγγεγραμμένον τετράγωνον μείζον ἐστὶ τοῦ ἡμίσεως τοῦ  $EZH\Theta$  κύκλου. τετμήσθωσαν δίχα αἱ  $EZ, ZH, H\Theta, \Theta E$  περιφέρειαι κατὰ τὰ  $K, \Lambda, M, N$  σημεία, καὶ ἐπεζεύχθωσαν αἱ  $EK, KZ, Z\Lambda, \Lambda H, H M, M\Theta, \Theta N, N E$ · καὶ ἕκαστον ἄρα τῶν  $EKZ, Z\Lambda H, H M\Theta, \Theta N E$  τριγώνων μείζον ἐστὶν ἢ τὸ ἥμισυ τοῦ καθ' ἑαυτὸ τμήματος τοῦ κύκλου, ἐπειδὴ περ ἐὰν διὰ τῶν  $K, \Lambda, M, N$  σημείων ἐφαπτομένης τοῦ κύκλου ἀγάγωμεν καὶ ἀναπληρώσωμεν τὰ ἐπὶ τῶν  $EZ, ZH, H\Theta, \Theta E$  εὐθειῶν παραλληλόγραμμα, ἕκαστον τῶν  $EKZ, Z\Lambda H, H M\Theta, \Theta N E$  τριγώνων ἥμισυ ἔσται τοῦ καθ' ἑαυτὸ παραλληλογράμμου, ἀλλὰ τὸ καθ' ἑαυτὸ τμήμα ἐλαττόν ἐστὶ τοῦ παραλληλογράμμου· ὥστε ἕκαστον τῶν  $EKZ, Z\Lambda H, H M\Theta, \Theta N E$  τριγώνων μείζον ἐστὶ τοῦ ἡμίσεως τοῦ καθ' ἑαυτὸ τμήματος τοῦ κύκλου. τέμνοντες δὴ τὰς ὑπολειπομένας περιφερείας δίχα καὶ ἐπιζευγύνοντες εὐθείας καὶ τοῦτο αἰ ποιοῦντες καταλείβομεν τινα ἀποτμήματα τοῦ κύκλου, ἃ ἔσται ἐλάσσονα τῆς ὑπεροχῆς, ἢ ὑπερέχει ὁ  $EZH\Theta$  κύκλος τοῦ  $\Sigma$  χωρίου.

$FH$  [be] their diameters. I say that as circle  $ABCD$  is to circle  $EFGH$ , so the square on  $BD$  (is) to the square on  $FH$ .



For if the circle  $ABCD$  is not to the (circle)  $EFGH$ , as the square on  $BD$  (is) to the (square) on  $FH$ , then as the (square) on  $BD$  (is) to the (square) on  $FH$ , so circle  $ABCD$  will be to some area either less than, or greater than, circle  $EFGH$ . Let it, first of all, be (in that ratio) to (some) lesser (area),  $S$ . And let the square  $EFGH$  have been inscribed in circle  $EFGH$  [Prop. 4.6]. So the inscribed square is greater than half of circle  $EFGH$ , inasmuch as if we draw tangents to the circle through the points  $E, F, G$ , and  $H$ , then square  $EFGH$  is half of the square circumscribed about the circle [Prop. 1.47], and the circle is less than the circumscribed square. Hence, the inscribed square  $EFGH$  is greater than half of circle  $EFGH$ . Let the circumferences  $EF, FG, GH$ , and  $HE$  have been cut in half at points  $K, L, M$ , and  $N$  (respectively), and let  $EK, KF, FL, LG, GM, MH, HN$ , and  $NE$  have been joined. And, thus, each of the triangles  $EKF, FLG, GMH$ , and  $HNE$  is greater than half of the segment of the circle about it, inasmuch as if we draw tangents to the circle through points  $K, L, M$ , and  $N$ , and complete the parallelograms on the straight-lines  $EF, FG, GH$ , and  $HE$ , then each of the triangles  $EKF, FLG, GMH$ , and  $HNE$  will be half of the parallelogram about it, but the segment about it is less than the parallelogram. Hence, each of the triangles  $EKF, FLG, GMH$ , and  $HNE$  is greater than half of the segment of the circle about it. So, by cutting the circumferences remaining behind in half, and joining straight-lines, and doing this continually, we will (even-

ἐδείχθη γὰρ ἐν τῷ πρώτῳ θεωρήματι τοῦ δεκάτου βιβλίου, ὅτι δύο μεγεθῶν ἀνίσων ἐκκειμένων, ἐὰν ἀπὸ τοῦ μείζονος ἀφαιρεθῇ μείζον ἢ τὸ ἥμισυ καὶ τοῦ καταλειπομένου μείζον ἢ τὸ ἥμισυ, καὶ τοῦτο αἰεὶ γίγνεται, λειφθήσεται τι μέγεθος, ὃ ἔσται ἔλασσον τοῦ ἐκκειμένου ἐλάσσονος μεγέθους. λελείφθω οὖν, καὶ ἔστω τὰ ἐπὶ τῶν  $EK$ ,  $KZ$ ,  $ZA$ ,  $AH$ ,  $HM$ ,  $MO$ ,  $ON$ ,  $NE$  τμήματα τοῦ  $EZH\Theta$  κύκλου ἐλάττονα τῆς ὑπεροχῆς, ἥ ὑπερέχει ὁ  $EZH\Theta$  κύκλος τοῦ  $\Sigma$  χωρίου. λοιπὸν ἄρα τὸ  $EKZAHM\Theta N$  πολύγωνον μείζον ἔστι τοῦ  $\Sigma$  χωρίου. ἐγγεγράφθω καὶ εἰς τὸν  $AB\Gamma\Delta$  κύκλον τῷ  $EKZAHM\Theta N$  πολυγώνῳ ὁμοιον πολύγωνον τὸ  $A\Xi B O\Gamma\Pi\Delta P$ . ἔστιν ἄρα ὡς τὸ ἀπὸ τῆς  $B\Delta$  τετράγωνον πρὸς τὸ ἀπὸ τῆς  $Z\Theta$  τετράγωνον, οὕτως τὸ  $A\Xi B O\Gamma\Pi\Delta P$  πολύγωνον πρὸς τὸ  $EKZAHM\Theta N$  πολύγωνον. ἀλλὰ καὶ ὡς τὸ ἀπὸ τῆς  $B\Delta$  τετράγωνον πρὸς τὸ ἀπὸ τῆς  $Z\Theta$ , οὕτως ὁ  $AB\Gamma\Delta$  κύκλος πρὸς τὸ  $\Sigma$  χωρίον· καὶ ὡς ἄρα ὁ  $AB\Gamma\Delta$  κύκλος πρὸς τὸ  $\Sigma$  χωρίον, οὕτως τὸ  $A\Xi B O\Gamma\Pi\Delta P$  πολύγωνον πρὸς τὸ  $EKZAHM\Theta N$  πολύγωνον· ἐναλλάξ ἄρα ὡς ὁ  $AB\Gamma\Delta$  κύκλος πρὸς τὸ ἐν αὐτῷ πολύγωνον, οὕτως τὸ  $\Sigma$  χωρίον πρὸς τὸ  $EKZAHM\Theta N$  πολύγωνον. μείζων δὲ ὁ  $AB\Gamma\Delta$  κύκλος τοῦ ἐν αὐτῷ πολυγώνου· μείζον ἄρα καὶ τὸ  $\Sigma$  χωρίον τοῦ  $EKZAHM\Theta N$  πολυγώνου. ἀλλὰ καὶ ἔλαττον· ὅπερ ἔστιν ἀδύνατον. οὐκ ἄρα ἔστιν ὡς τὸ ἀπὸ τῆς  $B\Delta$  τετράγωνον πρὸς τὸ ἀπὸ τῆς  $Z\Theta$ , οὕτως ὁ  $AB\Gamma\Delta$  κύκλος πρὸς ἔλασσόν τι τοῦ  $EZH\Theta$  κύκλου χωρίου. ὁμοίως δὲ δεῖξομεν, ὅτι οὐδὲ ὡς τὸ ἀπὸ  $Z\Theta$  πρὸς τὸ ἀπὸ  $B\Delta$ , οὕτως ὁ  $EZH\Theta$  κύκλος πρὸς ἔλασσόν τι τοῦ  $AB\Gamma\Delta$  κύκλου χωρίου.

Λέγω δὴ, ὅτι οὐδὲ ὡς τὸ ἀπὸ τῆς  $B\Delta$  πρὸς τὸ ἀπὸ τῆς  $Z\Theta$ , οὕτως ὁ  $AB\Gamma\Delta$  κύκλος πρὸς μείζον τι τοῦ  $EZH\Theta$  κύκλου χωρίου.

Εἰ γὰρ δυνατόν, ἔστω πρὸς μείζον τὸ  $\Sigma$ . ἀνάπαλιν ἄρα [ἔστιν] ὡς τὸ ἀπὸ τῆς  $Z\Theta$  τετράγωνον πρὸς τὸ ἀπὸ τῆς  $B\Delta$ , οὕτως τὸ  $\Sigma$  χωρίον πρὸς τὸν  $AB\Gamma\Delta$  κύκλον. ἀλλ' ὡς τὸ  $\Sigma$  χωρίον πρὸς τὸν  $AB\Gamma\Delta$  κύκλον, οὕτως ὁ  $EZH\Theta$  κύκλος πρὸς ἔλαττον τι τοῦ  $AB\Gamma\Delta$  κύκλου χωρίου· καὶ ὡς ἄρα τὸ ἀπὸ τῆς  $Z\Theta$  πρὸς τὸ ἀπὸ τῆς  $B\Delta$ , οὕτως ὁ  $EZH\Theta$  κύκλος πρὸς ἔλασσόν τι τοῦ  $AB\Gamma\Delta$  κύκλου χωρίου· ὅπερ ἀδύνατον ἐδείχθη. οὐκ ἄρα ἔστιν ὡς τὸ ἀπὸ τῆς  $B\Delta$  τετράγωνον πρὸς τὸ ἀπὸ τῆς  $Z\Theta$ , οὕτως ὁ  $AB\Gamma\Delta$  κύκλος πρὸς μείζον τι τοῦ  $EZH\Theta$  κύκλου χωρίου. ἐδείχθη δέ, ὅτι οὐδὲ πρὸς ἔλασσον· ἔστιν ἄρα ὡς τὸ ἀπὸ τῆς  $B\Delta$  τετράγωνον πρὸς τὸ ἀπὸ τῆς  $Z\Theta$ , οὕτως ὁ  $AB\Gamma\Delta$  κύκλος πρὸς τὸν  $EZH\Theta$  κύκλον.

Οἱ ἄρα κύκλοι πρὸς ἀλλήλους εἰσὶν ὡς τὰ ἀπὸ τῶν διαμέτρων τετράγωνα· ὅπερ ἔδει δεῖξαι.

tually) leave behind some segments of the circle whose (sum) will be less than the excess by which circle  $EFGH$  exceeds the area  $S$ . For we showed in the first theorem of the tenth book that if two unequal magnitudes are laid out, and if (a part) greater than a half is subtracted from the greater, and (if from) the remainder (a part) greater than a half (is subtracted), and this happens continually, then some magnitude will (eventually) be left which will be less than the lesser laid out magnitude [Prop. 10.1]. Therefore, let the (segments) have been left, and let the (sum of the) segments of the circle  $EFGH$  on  $EK$ ,  $KF$ ,  $FL$ ,  $LG$ ,  $GM$ ,  $MH$ ,  $HN$ , and  $NE$  be less than the excess by which circle  $EFGH$  exceeds area  $S$ . Thus, the remaining polygon  $EKFLGMHN$  is greater than area  $S$ . And let the polygon  $AOBPCQDR$ , similar to the polygon  $EKFLGMHN$ , have been inscribed in circle  $ABCD$ . Thus, as the square on  $BD$  is to the square on  $FH$ , so polygon  $AOBPCQDR$  (is) to polygon  $EKFLGMHN$  [Prop. 12.1]. But, also, as the square on  $BD$  (is) to the square on  $FH$ , so circle  $ABCD$  (is) to area  $S$ . And, thus, as circle  $ABCD$  (is) to area  $S$ , so polygon  $AOBPCQDR$  (is) to polygon  $EKFLGMHN$  [Prop. 5.11]. Thus, alternately, as circle  $ABCD$  (is) to the polygon (inscribed) within it, so area  $S$  (is) to polygon  $EKFLGMHN$  [Prop. 5.16]. And circle  $ABCD$  (is) greater than the polygon (inscribed) within it. Thus, area  $S$  is also greater than polygon  $EKFLGMHN$ . But, (it is) also less. The very thing is impossible. Thus, the square on  $BD$  is not to the (square) on  $FH$ , as circle  $ABCD$  (is) to some area less than circle  $EFGH$ . So, similarly, we can show that the (square) on  $FH$  (is) not to the (square) on  $BD$  as circle  $EFGH$  (is) to some area less than circle  $ABCD$  either.

So, I say that neither (is) the (square) on  $BD$  to the (square) on  $FH$ , as circle  $ABCD$  (is) to some area greater than circle  $EFGH$ .

For, if possible, let it be (in that ratio) to (some) greater (area),  $S$ . Thus, inversely, as the square on  $FH$  [is] to the (square) on  $DB$ , so area  $S$  (is) to circle  $ABCD$  [Prop. 5.7 corr.]. But, as area  $S$  (is) to circle  $ABCD$ , so circle  $EFGH$  (is) to some area less than circle  $ABCD$  (see lemma). And, thus, as the (square) on  $FH$  (is) to the (square) on  $BD$ , so circle  $EFGH$  (is) to some area less than circle  $ABCD$  [Prop. 5.11]. The very thing was shown (to be) impossible. Thus, as the square on  $BD$  is to the (square) on  $FH$ , so circle  $ABCD$  (is) not to some area greater than circle  $EFGH$ . And it was shown that neither (is it in that ratio) to (some) lesser (area). Thus, as the square on  $BD$  is to the (square) on  $FH$ , so circle  $ABCD$  (is) to circle  $EFGH$ .

Thus, circles are to one another as the squares on