where there are $i-2$ trailing zeros, $2 \le i \le n$. Show that $\ge$ defines the grlex order with respect to $\{x_1, \dots, x_n\}$.

10. Suppose I is a monomial ideal generated by monomials $m_1, \dots, m_k$. Prove that the polynomial $f \in F[x_1, \dots, x_n]$ is in I if and only if every monomial term f_i of f is a multiple of one of the m_j. [For polynomials $a_1, \dots, a_k \in F[x_1, \dots, x_n]$ expand the polynomial $a_1m_1 + \cdots + a_km_k$ and note that every monomial term is a multiple of at least one of the m_j.] Show that $x^2yz+3xy^2$ is an element of the ideal $I = (xyz, y^2) \subset F[x, y, z]$ but is not an element of the ideal $I' = (xz^2, y^2)$.

11. Fix a monomial ordering on $R = F[x_1, \dots, x_n]$ and suppose $\{g_1, \dots, g_m\}$ is a Gröbner basis for the ideal I in R. Prove that $h \in LT(I)$ if and only if h is a sum of monomial terms each divisible by some $LT(g_i)$, $1 \le i \le m$. [Use the previous exercise.]

12. Suppose I is a monomial ideal with monomial generators $g_1, \dots, g_m$. Use the previous exercise to prove directly that $\{g_1, \dots, g_m\}$ is a Gröbner basis for I.

13. Suppose I is a monomial ideal with monomial generators $g_1, \dots, g_m$. Use Buchberger's Criterion to prove that $\{g_1, \dots, g_m\}$ is a Gröbner basis for I.

14. Suppose I is a monomial ideal in $R = F[x_1, \dots, x_n]$ and suppose $\{m_1, \dots, m_k\}$ is a minimal set of monomials generating I, i.e., each m_i is a monomial and no proper subset of $\{m_1, \dots, m_k\}$ generates I. Prove that the m_i, $1 \le i \le k$ are unique. [Use Exercise 10.]

15. Fix a monomial ordering on $R = F[x_1, \dots, x_n]$.

(a) Prove that $\{g_1, \dots, g_m\}$ is a minimal Gröbner basis for the ideal I in R if and only if $\{LT(g_1), \dots, LT(g_m)\}$ is a minimal generating set for $LT(I)$.

(b) Prove that the leading terms of a minimal Gröbner basis for I are uniquely determined and the number of elements in any two minimal Gröbner bases for I is the same. [Use (a) and the previous exercise.]

16. Fix a monomial ordering on $F[x_1, \dots, x_n]$ and suppose $G = \{g_1, \dots, g_m\}$ is a set of generators for the nonzero ideal I. Show that if $S(g_i, g_j) \not\equiv 0 \bmod G$ then the ideal $(LT(g_1), ..., LT(g_m), LT(S(g_i, g_j))$ is strictly larger than the ideal $(LT(g_1), ..., LT(g_m))$. Conclude that the algorithm for computing a Gröbner basis described following Proposition 26 terminates after a finite number of steps. [Use Exercise 1.]

17. Fix the lexicographic ordering $x > y$ on $F[x, y]$. Use Buchberger's Criterion to show that $\{x^2y - y^2, x^3 - xy\}$ is a Gröbner basis for the ideal $I = (x^2y - y^2, x^3 - xy)$.

18. Show $\{x - y^3, y^5 - y^6\}$ is the reduced Gröbner basis for the ideal $I = (x - y^3, -x^2 + xy^2)$ with respect to the lexicographic ordering defined by $x > y$ in $F[x, y]$.

19. Fix the lexicographic ordering $x > y$ on $F[x, y]$.

(a) Show that $\{x^3 - y, x^2y - y^2, xy^2 - y^2, y^3 - y^2\}$ is the reduced Gröbner basis for the ideal $I = (-x^3 + y, x^2y - y^2)$.

(b) Determine whether the polynomial $f = x^6 - x^5y$ is an element of the ideal I.

20. Fix the lexicographic ordering $x > y > z$ on $F[x, y, z]$. Show that $\{x^2 + xy + z, xyz + z^2, xz^2, z^3\}$ is the reduced Gröbner basis for the ideal $I = (x^2 + xy + z, xyz + z^2)$ and in particular conclude that the leading term ideal $LT(I)$ requires four generators.

21. Fix the lexicographic ordering $x > y$ on $F[x, y]$. Use Buchberger's Criterion to show that $\{x^2y - y^2, x^3 - xy\}$ is a Gröbner basis for the ideal $I = (x^2y - y^2, x^3 - xy)$.

22. Let $I = (x^2 - y, x^2y - z)$ in $F[x, y, z]$.

(a) Show that $\{x^2 - y, y^2 - z\}$ is the reduced Gröbner basis for I with respect to the lexicographic ordering defined by $x > y > z$.

(b) Show that $\{x^2 - y, z - y^2\}$ is the reduced Gröbner basis for I with respect to the lexicographic ordering defined by $z > x > y$ (note these are essentially the same polynomials as in (a)).

(c) Show that $\{y - x^2, z - x^4\}$ is the reduced Gröbner basis for I with respect to the lexicographic ordering defined by $z > y > x$.

23. Show that the ideals $I = (x^2y + xy^2 - 2y, x^2 + xy - x + y^2 - 2y, xy^2 - x - y + y^3)$ and $J = (x - y^2, xy - y, x^2 - y)$ in $F[x, y]$ are equal.

24. Use reduced Gröbner bases to show that the ideal $I = (x^3 - yz, yz + y)$ and the ideal $J = (x^3z + x^3, x^3 + y)$ in $F[x, y, z]$ are equal.

25. Show that the reduced Gröbner basis using the lexicographic ordering $x > y$ for the ideal $I = (x^2 + xy^2, x^2 - y^3, y^3 - y^2)$ is $\{x^2 - y^2, y^3 - y^2, xy^2 + y^2\}$.

26. Show that the reduced Gröbner basis for the ideal $I = (xy + y^2, x^2y + xy^2 + x^2)$ is $\{x^2, xy + y^2, y^3\}$ with respect to the lexicographic ordering $x > y$ and is $\{y^2 + yx, x^2\}$ with respect to the lexicographic ordering $y > x$.

There are generally substantial differences in computational complexity when using different monomial orders. The grevlex monomial ordering often provides the most efficient computation and produces simpler polynomials.

27. Show that $\{x^3 - y^3, x^2 + xy^2 + y^4, x^2y + xy^3 + y^2\}$ is a reduced Gröbner basis for the ideal I in the example following Corollary 28 with respect to the grlex monomial ordering. (Note that while this gives three generators for I rather than two for the lexicographic ordering as in the example, the degrees are smaller.)

28. Let $I = (x^4 - y^4 + z^3 - 1, x^3 + y^2 + z^2 - 1)$. Show that there are five elements in a reduced Gröbner basis for I with respect to the lexicographic ordering with $x > y > z$ (the maximum degree among the five generators is 12 and the maximum number of monomial terms among the five generators is 35), that there are two elements for the lexicographic ordering $y > z > x$ (maximum degree is 6 and maximum number of terms is 8), and that $\{x^3 + y^2 + z^2 - 1, xy^2 + xz^2 - x + y^4 - z^3 + 1\}$ is the reduced Gröbner basis for the grevlex monomial ordering.

29. Solve the system of equations $x^2 - yz = 3$, $y^2 - xz = 4$, $z^2 - xy = 5$ over $\mathbb{C}$.

30. Find a Gröbner basis for the ideal $I = (x^2 + xy + y^2 - 1, x^2 + 4y^2 - 4)$ for the lexicographic ordering $x > y$ and use it to find the four points of intersection of the ellipse $x^2 + xy + y^2 = 1$ with the ellipse $x^2 + 4y^2 = 4$ in $\mathbb{R}^2$.

31. Use Gröbner bases to find all six solutions to the system of equations $2x^3 + 2x^2y^2 + 3y^3 = 0$ and $3x^5 + 2x^3y^3 + 2y^5 = 0$ over $\mathbb{C}$.

32. Use Gröbner bases to show that $(x, z) \cap (y^2, x - yz) = (xy, x - yz)$ in $F[x, y, z]$.

33. Use Gröbner bases to compute the intersection of the ideals $(x^3y - xy^2 + 1, x^2y^2 - y^3 - 1)$ and $(x^2 - y^2, x^3 + y^3)$ in $F[x, y]$.

The following four exercises deal with the *ideal quotient* of two ideals I and J in a ring R.

Definition. The *ideal quotient* $(I : J)$ of two ideals I, J in a ring R is the ideal

$$(I : J) = \{r \in R \mid rJ \in I\}.$$

34. (a) Suppose R is an integral domain, $0 \neq f \in R$ and I is an ideal in R. Show that if $\{g_1, \dots, g_s\}$ are generators for the ideal $I \cap (f)$, then $\{g_1/f, \dots, g_s/f\}$ are generators for the ideal quotient $(I : (f))$.

(b) If I is an ideal in the commutative ring R and $f_1, \dots, f_s \in R$, show that the ideal quotient $(I : (f_1, \dots f_s))$ is the ideal $\cap_{i=1}^{s}(I : (f_i))$.

35. If $I = (x^2y + z^3, x + y^3 - z, 2y^4z - yz^2 - z^3)$ and $J = (x^2y^5, x^3z^4, y^3z^7)$ in $\mathbb{Q}[x, y, z]$ show $(I : J)$ is the ideal $(z^2, y+z, x-z)$. [Use the previous exercise and Proposition 30.]

36. Suppose that K is an ideal in R, that I is an ideal containing K, and J is any ideal. If $\overline{I}$ and $\overline{J}$ denote the images of I and J in the quotient ring R/K, show that $\overline{(I : J)} = (\overline{I} : \overline{J})$ where $\overline{(I : J)}$ is the image in R/K of the ideal quotient $(I : J)$.

37. Let K be the ideal $(y^5 - z^4)$ in $R = \mathbb{Q}[y, z]$. For each of the following pairs of ideals I and J, use the previous two exercises together with Proposition 30 to verify the ideal quotients $(\overline{I} : \overline{J})$ in the ring R/K:

i. $I = (y^3, y^5 - z^4)$, $J = (z)$, $(\overline{I} : \overline{J}) = (\bar{y}^3, \bar{z}^3)$.
ii. $I = (y^3, z, y^5 - z^4)$, $J = (y)$, $(\overline{I} : \overline{J}) = (\bar{y}^2, \bar{z})$.
iii. $I = (y, y^3, z, y^5 - z^4)$, $J = (1)$, $(\overline{I} : \overline{J}) = (\bar{y}, \bar{z})$.

Exercises 38 to 44 develop some additional elementary properties of monomial ideals in $F[x_1, \dots, x_n]$. It follows from Hilbert's Basis Theorem that ideals are finitely generated, however one need not assume this in these exercises—the arguments are the same for finitely or infinitely generated ideals. These exercises may be used to give an independent proof of Hilbert's Basis Theorem (Exercise 44). In these exercises, M and N are monomial ideals with monomial generators $\{m_i \mid i \in I\}$ and $\{n_j \mid j \in J\}$ for some index sets I and J respectively.

38. Prove that the sum and product of two monomial ideals is a monomial ideal by showing that $M + N = (m_i,\ n_j \mid i \in I,\ j \in J)$, and $MN = (m_i n_j \mid i \in I,\ j \in J)$.

39. Show that if $\{M_s \mid s \in S\}$ is any nonempty collection of monomial ideals that is totally ordered under inclusion then $\cup_{s \in S} M_s$ is a monomial ideal. (In particular, the union of any increasing sequence of monomial ideals is a monomial ideal, cf. Exercise 19, Section 7.3.)

40. Prove that the intersection of two monomial ideals is a monomial ideal by showing that $M \cap N = (e_{i,j} \mid i \in I,\ j \in J)$, where $e_{i,j}$ is the least common multiple of m_i and n_j. [Use Exercise 10.]

41. Prove that for any monomial n, the ideal quotient $(M : (n))$ is $(m_i/d_i \mid i \in I)$, where d_i is the greatest common divisor of m_i and n (cf. Exercise 34). Show that if N is finitely generated, then the ideal quotient $(M : N)$ of two monomial ideals is a monomial ideal.

42. (a) Show that M is a monomial prime ideal if and only if $M = (S)$ for some subset of S of $\{x_1, x_2, \dots, x_n\}$. (In particular, there are only finitely many monomial prime ideals, and each is finitely generated.)
(b) Show that $(x_1, \dots, x_n)$ is the only monomial maximal ideal.

43. *(Dickson's Lemma—*a special case of Hilbert's Basis Theorem) Prove that every monomial ideal in $F[x_1, \dots, x_n]$ is finitely generated as follows.
Let $\mathcal{S} = \{N \mid N$ is a monomial ideal that is not finitely generated$\}$, and assume by way of contradiction $\mathcal{S} \neq \emptyset$.
(a) Show that $\mathcal{S}$ contains a maximal element M. [Use Exercise 30 and Zorn's Lemma.]
(b) Show that there are monomials x, y not in M with $xy \in M$. [Use Exercise 33(a).]
(c) For x as in (b), show that M contains a finitely generated monomial ideal M_0 such that $M_0 + (x) = M + (x)$ and $M = M_0 + (x)(M : (x))$, where $(M : (x))$ is the (monomial) ideal defined in Exercise 32, and $(x)(M : (x))$ is the product of these two ideals. Deduce that M is finitely generated, a contradiction which proves $\mathcal{S} = \emptyset$. [Use the maximality of M and previous exercises.]

44. If I is a nonzero ideal in $F[x_1, \dots, x_n]$, use Dickson's Lemma to prove that $LT(I)$ is finitely generated. Conclude that I has a Gröbner basis and deduce Hilbert's Basis Theorem. [cf. Proposition 24.]