symmetric bilinear forms. When f is skew-symmetric, the matrix of f in any ordered basis will have all its diagonal entries 0. This just corresponds to the observation that $f(\alpha, \alpha) = 0$ for every α in V, since $f(\alpha, \alpha) = -f(\alpha, \alpha)$.

Let us suppose f is a non-zero skew-symmetric bilinear form on V. Since $f \neq 0$, there are vectors α, β in V such that $f(\alpha, \beta) \neq 0$. Multiplying α by a suitable scalar, we may assume that $f(\alpha, \beta) = 1$. Let γ be any vector in the subspace spanned by α and β, say $\gamma = c\alpha + d\beta$. Then

$$\begin{aligned} f(\gamma, \alpha) &= f(c\alpha + d\beta, \alpha) = df(\beta, \alpha) = -d \\ f(\gamma, \beta) &= f(c\alpha + d\beta, \beta) = cf(\alpha, \beta) = c \end{aligned}$$

and so

$$\gamma = f(\gamma, \beta)\alpha - f(\gamma, \alpha)\beta. \tag{10-7}$$

In particular, note that α and β are necessarily linearly independent; for, if $\gamma = 0$, then $f(\gamma, \alpha) = f(\gamma, \beta) = 0$.

Let W be the two-dimensional subspace spanned by α and β. Let $W^\perp$ be the set of all vectors δ in V such that $f(\delta, \alpha) = f(\delta, \beta) = 0$, that is, the set of all δ such that $f(\delta, \gamma) = 0$ for every γ in the subspace W. We claim that $V = W \oplus W^\perp$. For, let ϵ be any vector in V, and

$$\begin{aligned} \gamma &= f(\epsilon, \beta)\alpha - f(\epsilon, \alpha)\beta \\ \delta &= \epsilon - \gamma. \end{aligned}$$

Then γ is in W, and δ is in $W^\perp$, for

$$\begin{aligned} f(\delta, \alpha) &= f(\epsilon - f(\epsilon, \beta)\alpha + f(\epsilon, \alpha)\beta, \alpha) \\ &= f(\epsilon, \alpha) + f(\epsilon, \alpha)f(\beta, \alpha) \\ &= 0 \end{aligned}$$

and similarly $f(\delta, \beta) = 0$. Thus every ϵ in V is of the form $\epsilon = \gamma + \delta$, with γ in W and δ in $W^\perp$. From (9-7) it is clear that $W \cap W^\perp = \{0\}$, and so $V = W \oplus W^\perp$.

Now the restriction of f to $W^\perp$ is a skew-symmetric bilinear form on $W^\perp$. This restriction may be the zero form. If it is not, there are vectors α' and β' in $W^\perp$ such that $f(\alpha', \beta') = 1$. If we let W' be the two-dimensional subspace spanned by α' and β', then we shall have

$$V = W \oplus W' \oplus W_0$$

where W_0 is the set of all vectors δ in $W^\perp$ such that $f(\alpha', \delta) = f(\beta', \delta) = 0$. If the restriction of f to W_0 is not the zero form, we may select vectors α'', β'' in W_0 such that $f(\alpha'', \beta'') = 1$, and continue.

In the finite-dimensional case it should be clear that we obtain a finite sequence of pairs of vectors,

$$(\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_k, \beta_k)$$

with the following properties:

(a) $f(\alpha_j, \beta_j) = 1, j = 1, \ldots, k$.
(b) $f(\alpha_i, \alpha_j) = f(\beta_i, \beta_j) = f(\alpha_i, \beta_j) = 0, i \neq j$.
(c) If W_j is the two-dimensional subspace spanned by α_j and β_j, then

$$V = W_1 \oplus \cdots \oplus W_k \oplus W_0$$

where every vector in W_0 is 'orthogonal' to all α_j and β_j, and the restriction of f to W_0 is the zero form.

Theorem 6. *Let* V *be an* n*-dimensional vector space over a subfield of the complex numbers, and let* f *be a skew-symmetric bilinear form on* V. *Then the rank* r *of* f *is even, and if* r = 2k *there is an ordered basis for* V *in which the matrix of* f *is the direct sum of the* (n − r) × (n − r) *zero matrix and* k *copies of the* 2 × 2 *matrix*

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Proof. Let $\alpha_1, \beta_1, \ldots, \alpha_k, \beta_k$ be vectors satisfying conditions (a), (b), and (c) above. Let $\{\gamma_1, \ldots, \gamma_s\}$ be any ordered basis for the subspace W_0. Then

$$\mathcal{B} = \{\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_k, \beta_k, \gamma_1, \ldots, \gamma_s\}$$

is an ordered basis for V. From (a), (b), and (c) it is clear that the matrix of f in the ordered basis $\mathcal{B}$ is the direct sum of the $(n - 2k) \times (n - 2k)$ zero matrix and k copies of the 2×2 matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{10-8}$$

Furthermore, it is clear that the rank of this matrix, and hence the rank of f, is $2k$. ∎

One consequence of the above is that if f is a non-degenerate, skew-symmetric bilinear form on V, then the dimension of V must be even. If $\dim V = 2k$, there will be an ordered basis $\{\alpha_1, \beta_1, \ldots, \alpha_k, \beta_k\}$ for V such that

$$f(\alpha_i, \beta_j) = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

$$f(\alpha_i, \alpha_j) = f(\beta_i, \beta_j) = 0.$$

The matrix of f in this ordered basis is the direct sum of k copies of the 2×2 skew-symmetric matrix (10-8). We obtain another standard form for the matrix of a non-degenerate skew-symmetric form if, instead of the ordered basis above, we consider the ordered basis

$$\{\alpha_1, \ldots, \alpha_k, \beta_k, \ldots, \beta_1\}.$$

The reader should find it easy to verify that the matrix of f in the latter ordered basis has the block form

$$\begin{bmatrix} 0 & J \\ -J & 0 \end{bmatrix}$$

where J is the $k \times k$ matrix

$$\begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \iddots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{bmatrix}.$$

Exercises

1. Let V be a vector space over a field F. Show that the set of all skew-symmetric bilinear forms on V is a subspace of $L(V, V, F)$.

2. Find all skew-symmetric bilinear forms on R^3.

3. Find a basis for the space of all skew-symmetric bilinear forms on R^n.

4. Let f be a symmetric bilinear form on C^n and g a skew-symmetric bilinear form on C^n. Suppose $f + g = 0$. Show that $f = g = 0$.

5. Let V be an n-dimensional vector space over a subfield F of C. Prove the following.

(a) The equation $(Pf)(\alpha, \beta) = \frac{1}{2}f(\alpha, \beta) - \frac{1}{2}f(\beta, \alpha)$ defines a linear operator P on $L(V, V, F)$.

(b) $P^2 = P$, i.e., P is a projection.

(c) rank $P = \dfrac{n(n-1)}{2}$; nullity $P = \dfrac{n(n+1)}{2}$.

(d) If U is a linear operato on V, the equation $(U^\dagger f)(\alpha, \beta) = f(U\alpha, U\beta)$ defines a linear operator $U^\dagger$ on $L(V, V, F)$.

(e) For every linear operator U, the projection P commutes with $U^\dagger$.

6. Prove an analogue of Exercise 11 in Section 10.2 for non-degenerate, skew-symmetric bilinear forms.

7. Let f be a bilinear form on a vector space V. Let L_f and R_f be the mappings of V into V^* associated with f in Section 10.1. Prove that f is skew-symmetric if and only if $L_f = -R_f$.

8. Prove an analogue of Exercise 17 in Section 10.2 for skew-symmetric forms.

9. Let V be a finite-dimensional vector space and L_1, L_2 linear functionals on V. Show that the equation

$$f(\alpha, \beta) = L_1(\alpha)L_2(\beta) - L_1(\beta)L_2(\alpha)$$

defines a skew-symmetric bilinear form on V. Show that $f = 0$ if and only if L_1, L_2 are linearly dependent.

10. Let V be a finite-dimensional vector space over a subfield of the complex numbers and f a skew-symmetric bilinear form on V. Show hat f has rank 2 if and only if there exist linearly independent linear functionals L_1, L_2 on V such that

$$f(\alpha, \beta) = L_1(\alpha)L_2(\beta) - L_1(\beta)L_2(\alpha).$$

11. Let f be any skew-symmetric bilinear form on R^3. Prove that there are linear functionals L_1, L_2 such that

$$f(\alpha, \beta) = L_1(\alpha)L_2(\beta) - L_1(\beta)L_2(\alpha).$$

12. Let V be a finite-dimensional vector space over a subfield of the complex numbers, and let f, g be skew-symmetric bilinear forms on V. Show that there is an *invertible* linear operator T on V such that $f(T\alpha, T\beta) = g(\alpha, \beta)$ for all α, β if and only if f and g have the same rank.

13. Show that the result of Exercise 12 is valid for symmetric bilinear forms on a complex vector space, but is not valid for symmetric bilinear forms on a real vector space.

10.4. Groups Preserving Bilinear Forms

Let f be a bilinear form on the vector space V, and let T be a linear operator on V. We say that T **preserves** f if $f(T\alpha, T\beta) = f(\alpha, \beta)$ for all α, β in V. For any T and f the function g, defined by $g(\alpha, \beta) = f(T\alpha, T\beta)$, is easily seen to be a bilinear form on V. To say that T preserves f is simply to say $g = f$. The identity operator preserves every bilinear form. If S and T are linear operators which preserve f, the product ST also preserves f; for $f(ST\alpha, ST\beta) = f(T\alpha, T\beta) = f(\alpha, \beta)$. In other words, the collection of linear operators which preserve a given bilinear form is closed under the formation of (operator) products. In general, one cannot say much more about this collection of operators; however, if f is non-degenerate, we have the following.

Theorem 7. *Let* f *be a non-degenerate bilinear form on a finite-dimensional vector space* V. *The set of all linear operators on* V *which preserve* f *is a group under the operation of composition.*

Proof. Let G be the set of linear operators preserving f. We observed that the identity operator is in G and that whenever S and T are in G the composition ST is also in G. From the fact that f is non-degenerate, we shall prove that any operator T in G is invertible, and T^{-1} is also in G. Suppose T preserves f. Let α be a vector in the null space of T. Then for any β in V we have

$$f(\alpha, \beta) = f(T\alpha, T\beta) = f(0, T\beta) = 0.$$

Since f is non-degenerate, $\alpha = 0$. Thus T is invertible. Clearly T^{-1} also preserves f; for

$$f(T^{-1}\alpha, T^{-1}\beta) = f(TT^{-1}\alpha, TT^{-1}\beta) = f(\alpha, \beta). \quad \blacksquare$$

If f is a non-degenerate bilinear form on the finite-dimensional space V, then each ordered basis $\mathcal{B}$ for V determines a group of matrices 'preserving' f. The set of all matrices $[T]_{\mathcal{B}}$, where T is a linear operator preserving f, will be a group under matrix multiplication. There is an alternative description of this group of matrices, as follows. Let $A = [f]_{\mathcal{B}}$, so that if α and β are vectors in V with respective coordinate matrices X and Y relative to $\mathcal{B}$, we shall have

$$f(\alpha, \beta) = X^t A Y.$$

Let T be any linear operator on V and $M = [T]_{\mathcal{B}}$. Then

$$\begin{aligned} f(T\alpha, T\beta) &= (MX)^t A (MY) \\ &= X^t (M^t A M) Y. \end{aligned}$$

Accordingly, T preserves f if and only if $M^t A M = A$. In matrix language then, Theorem 7 says the following: If A is an invertible $n \times n$ matrix, the set of all $n \times n$ matrices M such that $M^t A M = A$ is a group under matrix multiplication. If $A = [f]_{\mathcal{B}}$, then M is in this group of matrices if and only if $M = [T]_{\mathcal{B}}$, where T is a linear operator which preserves f.

Before turning to some examples, let us make one further remark. Suppose f is a bilinear form which is symmetric. A linear operator T preserves f if and only if T preserves the quadratic form

$$q(\alpha) = f(\alpha, \alpha)$$

associated with f. If T preserves f, we certainly have

$$q(T\alpha) = f(T\alpha, T\alpha) = f(\alpha, \alpha) = q(\alpha)$$

for every α in V. Conversely, since f is symmetric, the polarization identity

$$f(\alpha, \beta) = \tfrac{1}{4} q(\alpha + \beta) - \tfrac{1}{4} q(\alpha - \beta)$$

shows us that T preserves f provided that $q(T\gamma) = q(\gamma)$ for each γ in V. (We are assuming here that the scalar field is a subfield of the complex numbers.)

Example 6. Let V be either the space R^n or the space C^n. Let f be the bilinear form

$$f(\alpha, \beta) = \sum_{j=1}^{n} x_j y_j$$

where $\alpha = (x_1, \ldots, x_n)$ and $\beta = (y_1, \ldots, y_n)$. The group preserving f is called the n-dimensional (real or complex) **orthogonal group.** The name 'orthogonal group' is more commonly applied to the associated group of matrices in the standard ordered basis. Since the matrix of f in the standard basis is I, this group consists of the matrices M which satisfy $M^t M = I$. Such a matrix M is called an $n \times n$ (real or complex) **orthogonal matrix.** The two $n \times n$ orthogonal groups are usually de-