If f is a quadratic polynomial, P is identical to f and the error R is zero. It is a remarkable fact that we also have $R = 0$ when f is a ***cubic*** polynomial. To prove this property we use the error estimate for Lagrange interpolation given by Theorem 15.3, and we write

$$f(x) - P(x) = (x - a)(x - m)(x - b)\frac{f'''(c)}{3!}, \tag{15.41}$$

where $c \in (a, b)$. When f is a cubic polynomial the third derivative f''' is constant, say $f'''(x) = C$, and the foregoing formula becomes

$$f(x) - P(x) = \frac{C}{6}(x - a)(x - m)(x - b) = \frac{C}{6}(t + h)t(t - h),$$

where $t = x - m$ and $h = (b - a)/2$. Therefore

$$R = \int_a^b [f(x) - P(x)]\, dx = \frac{C}{6}\int_{-h}^{h}(t^3 - h^2 t)\, dt = 0,$$

since the last integrand is an odd function. This property is illustrated in Figure 15.5. The dotted curve is the graph of a cubic polynomial f that agrees with P at a, m, b. In this case $R = \int_a^b [f(x) - P(x)]\, dx = A_1 - A_2$, where A_1 and A_2 are the areas of the two shaded regions. Since $R = 0$ the two regions have equal areas.

We have just seen that Equation (15.40) is valid if P is a polynomial of degree ≤ 3 that agrees with f at a, m, and b. By choosing this polynomial carefully we can considerably improve the error estimate in (15.41). We have already imposed three conditions on P, namely, $P(a) = f(a)$, $P(m) = f(m)$, $P(b) = f(b)$. Now we impose a fourth condition, $P'(m) = f'(m)$. This will give P and f the same slope at $(m, f(m))$, and we can hope that this will improve the approximation of f by P throughout $[a, b]$.

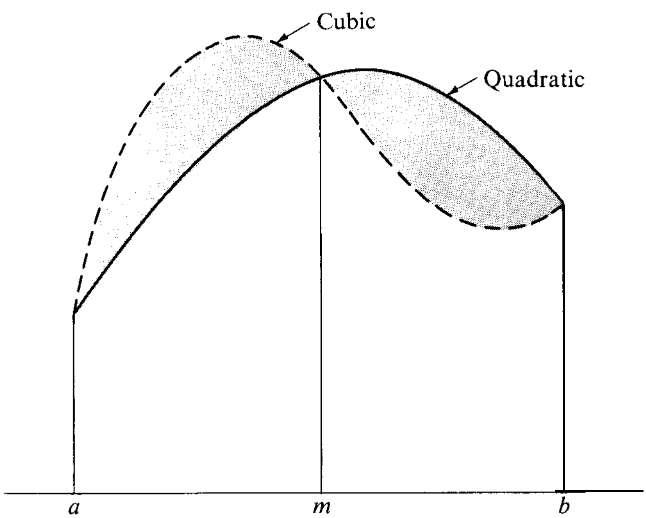

FIGURE 15.5 The two shaded regions have equal areas, for every cubic interpolating polynomial.

To show that such a P can always be chosen, we let Q be the quadratic polynomial that agrees with f at $a, m, b,$ and we let

$$P(x) = Q(x) + A(x - a)(x - m)(x - b),$$

where A is a constant to be determined. For any choice of A, this cubic polynomial P agrees with Q and hence with f at a, m, b. Now we choose A to make $P'(m) = f'(m)$. Differentiating the formula for $P(x)$ and putting $x = m$ we obtain

$$P'(m) = Q'(m) + A(m - a)(m - b).$$

Therefore if we take $A = [f'(m) - Q'(m)]/[(m - a)(m - b)]$ we also satisfy the condition $P'(m) = f'(m)$.

Next we show that for this choice of P we have

$$f(x) - P(x) = (x - a)(x - m)^2(x - b)\frac{f^{(4)}(z)}{4!} \tag{15.42}$$

for some z in (a, b), provided that the fourth derivative $f^{(4)}$ exists in $[a, b]$. To prove (15.42) we argue as in the proof of Theorem 15.3. First we note that (15.42) is trivially satisfied for any choice of z if $x = a, m,$ or b. Therefore, assume $x \neq a$, $x \neq m$, $x \neq b$, keep x fixed, and introduce a new function F defined on $[a, b]$ by the equation

$$F(t) = A(x)[f(t) - P(t)] - A(t)[f(x) - P(x)],$$

where

$$A(t) = (t - a)(t - m)^2(t - b).$$

Note that $F(t) = 0$ for $t = a, m, b,$ and x. By Rolle's theorem, $F'(t)$ vanishes in each of the three open intervals determined by these four points. In addition, $F'(m) = 0$ because $A'(m) = 0$ and $f'(m) = P'(m)$. Therefore $F'(t) = 0$ for at least four distinct points in (a, b). By Rolle's theorem $F''(t) = 0$ for at least three points, $F'''(t) = 0$ for at least two points, and $F^{(4)}(t) = 0$ for at least one point, say for $t = z$. From the definition of F we find

$$\begin{aligned} F^{(4)}(t) &= A(x)[f^{(4)}(t) - P^{(4)}(t)] - A^{(4)}(t)[f(x) - P(x)] \\ &= A(x)f^{(4)}(t) - 4!\,[f(x) - P(x)]. \end{aligned}$$

When we substitute $t = z$ in this equation we obtain (15.42).

Now it is a simple matter to prove Simpson's rule in the following form.

THEOREM 15.14. SIMPSON'S RULE. *Assume f has a continuous fourth derivative on $[a, b]$, and let $m = (a + b)/2$. Then we have*

$$\int_a^b f(x)\,dx = \frac{b - a}{6}[f(a) + 4f(m) + f(b)] - \frac{(b - a)^5}{2880} f^{(4)}(c) \tag{15.43}$$

for some c in $[a, b]$.

Proof. Let M_4 and m_4 denote, respectively, the maximum and minimum values of $f^{(4)}$ on [a, b], and let $B(x) = -(x - a)(x - m)^2(x - b)/4!$. Since $B(x) \geq 0$ for each x in [a, b], Equation (15.42) leads to the inequalities

$$m_4\, B(x) \leq P(x) - f(x) \leq M_4\, B(x).$$

Integrating, we find

$$m_4 \int_a^b B(x)\, dx \leq \int_a^b [P(x) - f(x)]\, dx \leq M_4 \int_a^b B(x)\, dx\,. \tag{15.44}$$

To evaluate the integral $\int_a^b B(x)\, dx$ we let $h = (b - a)/2$ and we have

$$\int_a^b B(x)\, dx = -\frac{1}{4!}\int_a^b (x - a)(x - m)^2(x - b)\, dx = -\frac{1}{4!}\int_{-h}^{h} (t + h)t^2(t - h)\, dt$$
$$= -\frac{2}{4!}\int_0^h t^2(t^2 - h^2)\,dt = \frac{1}{4!}\frac{4h^5}{15} = \frac{(b - a)^5}{2880}.$$

Therefore the inequalities in (15.44) give us

$$m_4 \leq \frac{2880}{(b - a)^5}\int_a^b [P(x) - f(x)]\, dx \leq M_4.$$

But since $f^{(4)}$ is continuous on [a, b], it assumes every value between its minimum m_4 and its maximum M_4 somewhere in [a, b]. Therefore

$$f^{(4)}(c) = \frac{2880}{(b - a)^5}\int_a^b [P(x) - f(x)]\, dx$$

for some c in [a, b]. Since $\int_a^b P(x)\, dx = \frac{1}{6}(b - a)[f(a) + 4f(m) + f(b)]$, this equation gives us (15.43).

Simpson's rule is of special interest because its accuracy is greater than might be expected from a knowledge of the function f at only three points. If the values of f are known at an odd number of equally spaced points, say at a, $a + h, \ldots, a + 2nh$, it is usually simpler to apply Simpson's rule successively to each of the intervals $[a, a + 2h]$, $[a + 2h, a + 4h]$, $\ldots$, rather than to use an interpolating polynomial of degree $\leq 2n$ over the full interval $[a, a + 2nh]$. Applying Simpson's rule in this manner, we obtain the following extension of Theorem 15.14.

THEOREM 15.15. EXTENDED SIMPSON'S RULE. *Assume f has a continuous fourth derivative in [a, b]. Let $h = (b - a)/(2n)$ and let $f_k = f(a + kh)$ for $k = 1, 2, \ldots, 2n - 1$. Then we have*

$$\int_a^b f(x)\, dx = \frac{b - a}{6n}\left(f(a) + 4\sum_{k=1}^{n} f_{2k-1} + 2\sum_{k=1}^{n-1} f_{2k} + f(b)\right) - \frac{(b - a)^5}{2880n^4} f^{(4)}(\bar{c})$$

for some $\bar{c}$ in $[a, b]$.

The proof of this theorem is requested in Exercise 9 of the next section.

15.21 Exercises

1. (a) Apply the trapezoidal rule with $n = 10$ to estimate the value of the integral
$$\log 2 = \int_1^2 \frac{dx}{x}.$$
Obtain upper and lower bounds for the error. (See Exercise 10(b) to compare the accuracy with that obtained from Simpson's rule.)
(b) What is the smallest value of n that would ensure six-place accuracy in the calculation of log 2 by this method?
2. (a) Show that there is a positive number c in the interval [0, 1] such that the formula
$$\int_{-1}^{1} f(x)\,dx = f(c) + f(-c)$$
is exact for all polynomials of degree ≤ 3.
(b) Generalize the result of part (a) for an arbitrary interval. That is, show that constants c_1 and c_2 exist in [a, b] such that the formula
$$\int_a^b f(x)\,dx = \frac{b-a}{2}[f(c_1) + f(c_2)]$$
is exact for all polynomials of degree ≤ 3. Express c_1 and c_2 in terms of a and b.
3. (a) Show that a positive constant c exists such that the formula
$$\int_{-1/2}^{1/2} f(x)\,dx = \tfrac{1}{3}[f(-c) + f(0) + f(c)]$$
is exact for all polynomials of degree ≤ 3.
(b) Generalize the result of part (a) for an arbitrary interval. That is, show that constants c_1 and c_2 exist in [a, b] such that the formula
$$\int_a^b f(x)\,dx = \frac{b-a}{3}\left[f(c_1) + f\left(\frac{a+b}{2}\right) + f(c_2)\right]$$
is exact for all polynomials of degree ≤ 3. Express c_1 and c_2 in terms of a and b.
4. Show that positive constants a and b exist such that the formula
$$\int_0^\infty e^{-x} f(x)\,dx = \tfrac{1}{4}[af(b) + bf(a)]$$
is exact for all polynomials of degree ≤ 3.
5. Show that a positive constant c exists such that the formula
$$\int_{-\infty}^{\infty} e^{-x^2} f(x)\,dx = \frac{\sqrt{\pi}}{6}[f(-c) + 4f(0) + f(c)]$$
is exact for all polynomials of degree ≤ 5.
6. Let P_n be the interpolation polynomial of degree $\leq n$ that agrees with fat $n + 1$ distinct points $x_0, x_1, \ldots, x_n$.
(a) Show that constants $A_0(n)$, A,(n), . . . , A,(n) exist, depending only on the numbers $x_0, x_1, \ldots, x_n$, a, and b, and not on f, such that
$$\int_a^b P_n(x)\,dx = \sum_{k=0}^{n} A_k(n) f(x_k).$$
The numbers $A_k(n)$ are called weights. (They are sometimes called Christoffel numbers.)

(b) For a given set of distinct interpolation points and a given interval [a, b], let $W_0(n)$, $W_1(n), \ldots, W_n(n)$ be $n + 1$ constants such that the formula

$$\int_a^b f(x)\, dx = \sum_{k=0}^{n} W_k(n) f(x_k)$$

is exact for all polynomials of degree $\leq n$. Prove that

$$\sum_{k=0}^{n} x_k^r W_k(n) = \frac{b^{r+1} - a^{r+1}}{r + 1} \qquad \text{for } r = 0, 1, \ldots, n.$$

This is a system of $n + 1$ linear equations that can be used to determine the weights. It can be shown that this system always has a unique solution. It can also be shown that for a suitable choice of interpolation points it is possible to make all the weights equal. When the weights are all equal the integration formula is called a Chebyshev integration formula. Exercises 2 and 3 give examples of Chebyshev integration formulas. The next exercise shows that for a proper choice of interpolation points the resulting integration formula is exact for all polynomials of degree $\leq 2n + 1$.

7. In this exercise you may use properties of the Legendre polynomials stated in Sections 6.19 and 6.20. Let $x_0, x_1, \ldots, x_n$ be the zeros of the Legendre polynomial $P_{n+1}(x)$. These zeros are distinct and they all lie in the interval $[-1, 1]$. Let $f(x)$ be any polynomial in x of degree $\leq 2n + 1$. Divide $f(x)$ by $P_{n+1}(x)$ and write

$$f(x) = P_{n+1}(x) Q(x) + R(x),$$

where the polynomials Q and R have degree $\leq n$.
(a) Show that the polynomial R agrees with f at the zeros of P_{n+1} and that

$$\int_{-1}^{1} f(x)\, dx = \int_{-1}^{1} R(x)\, dx.$$

(b) Show that $n + 1$ weights $W_0(n), \ldots, W_n(n)$ exist (independent of f) such that

$$\int_{-1}^{1} f(x)\, dx = \sum_{k=0}^{n} W_k(n) f(x_k).$$

This gives an integration formula with $n + 1$ interpolation points that is exact for all polynomials of degree $\leq 2n + 1$.
(c) Take $n = 2$ and show that the formula in part (b) becomes

$$\int_{-1}^{1} f(x)\, dx = \tfrac{5}{9} f\left(-\sqrt{\tfrac{3}{5}}\right) + \tfrac{8}{9} f(0) + \tfrac{5}{9} f\left(\sqrt{\tfrac{3}{5}}\right).$$

This is exact for all polynomials of degree ≤ 5.
(d) Introduce a suitable linear transformation and rewrite the formula in part (c) for an arbitrary interval [a, b].

8. This exercise describes a method of Peano for deriving the error formula in Simpson's rule.
(a) Use integration by parts repeatedly to deduce the relation

$$\int u(t) v'''(t)\, dt = u(t) v''(t) - u'(t) v'(t) + u''(t) v(t) - \int g(t)\, dt,$$

where $g(t) = u'''(t) v(t)$.

(b) Assume φ has a continuous fourth derivative in the interval $[-1, 1]$. Take

$$v(t) = t(1-t)^2/6, \qquad u(t) = \varphi(t) + \varphi(-t),$$

and use part (a) to show that

$$\int_{-1}^{1} \varphi(t)\, dt = \tfrac{1}{3}[\varphi(-1) + 4\varphi(0) + \varphi(1)] - \int_0^1 g(t)\, dt.$$

Then show that $\int_0^1 g(t)\, dt = \varphi^{(4)}(c)/90$ for some c in $[-1, 1]$.
(c) Introduce a suitable linear transformation to deduce Theorem 15.14 from the result of part (b).

9. (a) Let $a_1, a_2, \ldots, a_n$ be nonnegative numbers whose sum is 1. Assume φ is continuous on an interval $[a, b]$. If $c_1, c_2, \ldots, c_n$ are any n points in $[a, b]$ (not necessarily distinct), prove that there is at least one point c in $[a, b]$ such that

$$\sum_{k=1}^{n} a_k \varphi(c_k) = \varphi(c).$$

[Hint: Let M and m denote the maximum and minimum of φ on $[a, b]$ and use the inequality $m \le \varphi(c_k) \le M$.]

(b) Use part (a) and Theorem 15.14 to derive the extended form of Simpson's rule given in Theorem 15.15.

10. Compute log 2 from the formula $\log 2 = \int_1^2 x^{-1}\, dx$ by using the extension of Simpson's rule with (a) $n = 2$; (b) $n = 5$. Give upper and lower bounds for the error in each case.

11. (a) Let $\varphi(t)$ be a polynomial in t of degree ≤ 3. Express $\varphi(t)$ by Newton's interpolation formula and integrate to deduce the formula

$$\int_0^3 \varphi(t)\, dt = \tfrac{3}{8}[\varphi(0) + 3\varphi(1) + 3\varphi(2) + \varphi(3)].$$

(b) Let P be the interpolation polynomial of degree ≤ 3 that agrees with f at the points a, $a + h$, $a + 2h$, $a + 3h$, where $h > 0$. Use part (a) to prove that

$$\int_a^{a+3h} P(x)\, dx = \frac{3h}{8}[f(a) + 3f(a+h) + 3f(a+2h) + f(a+3h)].$$

(c) Assume f has a continuous fourth derivative in $[a, b]$, and let $h = (b - a)/3$. Prove that

$$\int_a^b f(x)\, dx = \frac{b-a}{8}[f(a) + 3f(a+h) + 3f(a+2h) + f(b)] - \frac{(b-a)^5}{6480} f^{(4)}(c)$$

for some c in $[a, b]$. This approximate integration formula is called ***Cotes' rule.***
(d) Use Cotes' rule to compute $\log 2 = \int_1^2 x^{-1}\, dx$ and give upper and lower bounds for the error.

12. (a) Use the vector equation $r(t) = a \sin t\, i + b \cos t\, j$, where $0 < b < a$, to show that the circumference L of an ellipse is given by the integral

$$L = 4a \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 t}\, dt,$$

where $k = \sqrt{a^2 - b^2}/a$.

(b) Show that Simpson's rule gives the formula

$$L = \frac{\pi}{3}\left[a + b + \sqrt{8(a^2 + b^2)}\right] - \frac{a\pi^5}{23040} f^{(4)}(c)$$

for some c in $[0, \pi/2]$, where $f(t) = \sqrt{1 - k^2 \sin^2 t}$.

15.22 The Euler summation formula

Let n be a positive integer. When the trapezoidal formula (Theorem 15.13) is applied to the interval $[0, n]$ it becomes

$$\int_0^n f(x)\,dx = \sum_{k=0}^{n-1} f(k) + \tfrac{1}{2}(f(n) - f(0)) - \frac{f''(c)n}{12}$$

for some c in $[0, n]$. If f is a quadratic polynomial then f'' is constant and hence $f''(c) = f''(0)$. In this case the formula can be rewritten in the form

$$\sum_{k=0}^{n} f(k) = \int_0^n f(x)\,dx + \frac{f(0) + f(n)}{2} + \frac{f''(0)n}{12}. \tag{15.45}$$

It is exact when f is any polynomial of degree ≤ 2.

Euler discovered a remarkable extension of this formula that is exact for any function with a continuous first derivative. It can be used to approximate integrals by sums, or, as is more often the case, to evaluate or estimate sums in terms of integrals. For this reason it is usually referred to as a "summation" formula rather than an integration formula. It can be stated as follows.

THEOREM 15.16. EULER'S SUMMATION FORMULA. *Assume f has a continuous derivative on $[0, n]$. Then we have*

$$\sum_{k=0}^{n} f(k) = \int_0^n f(x)\,dx + \frac{f(0) + f(n)}{2} + \int_0^n \left(x - [x] - \tfrac{1}{2}\right) f'(x)\,dx, \tag{15.46}$$

where $[x]$ denotes the greatest integer $\le x$.

Proof. Integration by parts gives us

$$\int_0^n \left(x - \tfrac{1}{2}\right) f'(x)\,dx = \left(n - \tfrac{1}{2}\right) f(n) + \tfrac{1}{2} f(0) - \int_0^n f(x)\,dx. \tag{15.47}$$

Now we consider the integral $\int_0^n [x] f'(x)\,dx$ and write it as a sum of integrals in each of which $[x]$ has a fixed value. Thus, we have

$$\begin{aligned}
\int_0^n [x] f'(x)\,dx &= \sum_{r=0}^{n-1} \int_r^{r+1} [x] f'(x)\,dx = \sum_{r=0}^{n-1} r \int_r^{r+1} f'(x)\,dx \\
&= \sum_{r=0}^{n-1} r(f(r+1) - f(r)) = \sum_{r=0}^{n-1} r f(r+1) - \sum_{r=0}^{n-1} r f(r) \\
&= -\sum_{r=0}^{n-1} f(r+1) + \sum_{r=0}^{n-1} (r+1) f(r+1) - \sum_{r=0}^{n-1} r f(r) \\
&= -\sum_{k=1}^{n} f(k) + n f(n) = -\sum_{k=0}^{n} f(k) + f(0) + n f(n).
\end{aligned}$$

Subtracting this from Equation (15.47) we obtain

$$\int_0^n (x - [x] - \tfrac{1}{2}) f'(x)\, dx = \sum_{k=0}^{n} f(k) - \frac{f(0) + f(n)}{2} - \int_0^n f(x)\, dx\,,$$

which is equivalent to (15.46).

The last integral on the right of (15.46) can be written as

$$\int_0^n (x - [x] - \tfrac{1}{2}) f'(x)\, dx = \int_0^n \varphi_1(x) f'(x)\, dx\,,$$

where φ_1 is the function defined by

$$\varphi_1(x) = \begin{cases} x - [x] - \tfrac{1}{2} & \text{if } x \text{ is not an integer,} \\ 0 & \text{if } x \text{ is an integer.} \end{cases}$$

The graph of φ_1 is shown in Figure 15.6(a). We note that $\varphi_1(x + 1) = \varphi_1(x)$, which means that φ_1 is periodic with period 1. Also, if $0 < x < 1$ we have $\varphi_1(x) = x - \frac{1}{2}$, so $\int_0^1 \varphi_1(t)\, dt = 0$.

Figure 15.6(b) shows the graph of φ_2, the indefinite integral of φ_1, given by

$$\varphi_2(x) = \int_0^x \varphi_1(t)\, dt\,.$$

It is easily verified that φ_2 is also periodic with period 1. Moreover, we have

$$\varphi_2(x) = \frac{x(x-1)}{2} \quad \text{if } 0 \le x \le 1\,.$$

This shows that $-\frac{1}{8} \le \varphi_2(x) \le 0$ for all x. The strict inequalities $-\frac{1}{8} < \varphi_2(x) < 0$ hold except when x is an integer or half an integer.

The next theorem describes another version of Euler's summation formula in terms of the function φ_2.

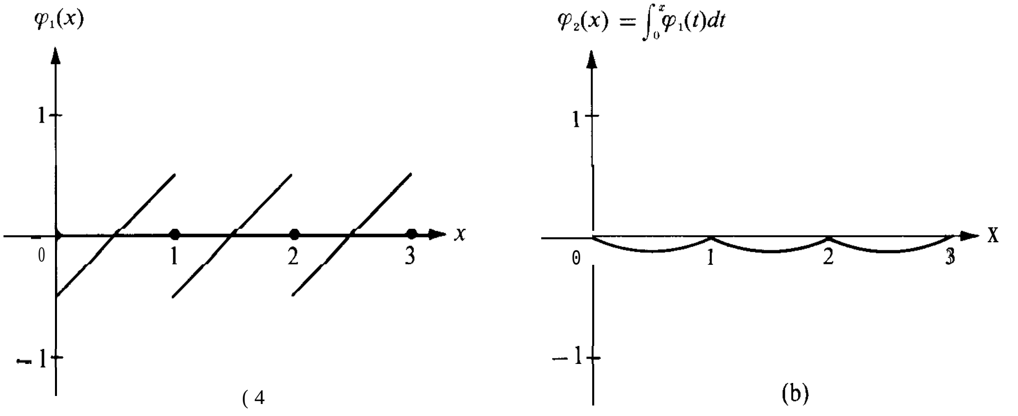

FIGURE 15.6 Graphs of the periodic functions φ_1 and φ_2.

THEOREM 15.17. *If f'' is continuous on [0, n] we have*

$$\sum_{k=0}^{n} f(k) = \int_0^n f(x)\,dx + \frac{f(0)+f(n)}{2} - \int_0^n \varphi_2(x) f''(x)\,dx\,. \tag{15.48}$$

Proof. Since $\varphi_2'(x) = \varphi_1(x)$ at the points of continuity of φ_1, we have

$$\int_0^n \varphi_1(x) f'(x)\,dx = \int_0^n \varphi_2'(x) f'(x)\,dx\,.$$

Integration by parts gives us

$$\int_0^n \varphi_2'(x) f'(x)\,dx = \varphi_2(x) f'(x)\Big|_0^n - \int_0^n \varphi_2(x) f''(x)\,dx = -\int_0^n \varphi_2(x) f''(x)\,dx\,,$$

since $\varphi_2(n) = \varphi_2(0) = 0$. Using this in (15.46) we obtain (15.48).

Note: Although Theorems 15.16 and 15.17 refer to the interval [0, n], both formulas are valid when 0 is replaced throughout by 1 or by any positive integer $< n$.

To illustrate the use of Euler's summation formula we derive the following formula for $\log n!$.

THEOREM 15.18. *For any positive integer n we have*

$$\log n! = (n + \tfrac{1}{2}) \log n - n + C + E(n), \tag{15.49}$$

where $0 < E(n) < 1/(8n)$ *and* $C = 1 + \int_1^\infty t^{-2}\varphi_2(t)\,dt$.

Proof. We take $f(x) = \log x$ and apply Theorem 15.17 to the interval $[1, n]$. This gives us

$$\sum_{k=1}^{n} \log k = \int_1^n \log x\,dx + \tfrac{1}{2}\log n + \int_1^n \frac{\varphi_2(x)}{x^2}\,dx\,.$$

Using the relation $\int \log t\,dt = t \log t - t$ we rewrite this equation as follows,

$$\log n! = (n + \tfrac{1}{2}) \log n - n + 1 + \int_1^n \frac{\varphi_2(t)}{t^2}\,dt\,. \tag{15.50}$$

Since $|\varphi_2(t)| \le \frac{1}{8}$ the improper integral $\int_1^\infty t^{-2}\varphi_2(t)\,dt$ converges absolutely and we can write

$$\int_1^n \frac{\varphi_2(t)}{t^2}\,dt = \int_1^\infty \frac{\varphi_2(t)}{t^2}\,dt - \int_n^\infty \frac{\varphi_2(t)}{t^2}\,dt\,.$$

Therefore Equation (15.50) becomes

$$\log n! = (n + \tfrac{1}{2}) \log n - n + C - \int_n^\infty \frac{\varphi_2(t)}{t^2}\,dt\,,$$

where $C = 1 + \int_1^\infty t^{-2}\varphi_2(t)\,dt$. Since we have $-\frac{1}{8} < \varphi_2(t) < 0$ except when t is an integer or half an integer, we obtain

$$0 < -\int_n^\infty \frac{\varphi_2(t)}{t^2}\,dt < \frac{1}{8n},$$

This proves (15.49), with $E(n) = -\int_n^\infty t^{-2}\varphi_2(t)\,dt$.

From Theorem 15.18 we can derive Stirling's formula for estimating $n!$.

THEOREM 15.19. STIRLING'S FORMULA. *If n is a positive integer we have*

$$\sqrt{2\pi}\, n^{n+1/2}e^{-n} < n! < \sqrt{2\pi}\, n^{n+1/2}e^{-n}\left(1 + \frac{1}{4n}\right).$$

Proof. Using Equation (15.49) and the inequalities for $E(n)$ we obtain

$$\exp\left((n + \tfrac{1}{2})\log n - n + C\right) < n! < \exp\left((n + \tfrac{1}{2})\log n - n + C + \frac{1}{8n}\right),$$

where $\exp(t) = e^t$. Using the relation $e^x < 1 + 2x$, with $x = 1/(8n)$, we can rewrite these inequalities as

$$An^{n+1/2}e^{-n} < n! < An^{n+1/2}e^{-n}\left(1 + \frac{1}{4n}\right), \tag{15.51}$$

where $A = e^C$. To complete the proof we need to show that $A = \sqrt{2\pi}$.

We shall deduce this from the inequality

$$\pi n \le \left(\frac{2^{2n}(n!)^2}{(2n)!}\right)^2 \le \frac{\pi(2n+1)}{2}, \tag{15.52}$$

discovered by John Wallis (1616-1703). First we show how Wallis' inequality implies $A = \sqrt{2\pi}$; then we discuss the proof of (15.52).

If we let

$$A_n = \frac{n!}{n^{n+1/2}e^{-n}}$$

the inequality in (15.51) implies

$$A < A_n < A\left(1 + \frac{1}{4n}\right).$$

This shows that $A_n \to A$ as $n \to \infty$. In (15.52) we write $n! = n^{n+1/2}e^{-n}A_n$ to obtain

$$\pi n \le \left(\frac{2^{2n}n^{2n+1}e^{-2n}A_n^2}{(2n)^{2n+1/2}e^{-2n}A_{2n}}\right)^2 \le \frac{\pi(2n+1)}{2},$$

which is equivalent to

$$\pi \le \frac{A_n^4}{2A_{2n}^2} \le \pi\,\frac{2n+1}{2n}.$$

We let $n \to$ co in this last inequality. Since A = e^C > 0 we obtain

$$\pi \le \frac{A^4}{2A^2} \le \pi .$$

This shows that $A^2 = 2\pi$, so $A = \sqrt{2\pi}$, as asserted.

It remains to prove Wallis' inequality (15.52). For this purpose we introduce the numbers

$$\text{I,} = \int_0^{\pi/2} \sin^n t\, dt ,$$

where n is any nonnegative integer. We note that $I_0 = \pi/2$ and $I_1 = 1$. For $0 \le t \le \pi/2$ we have $0 \le \sin t \le 1$; hence $0 \le \sin^{n+1} t \le \sin^n t$. This shows that the sequence {I,} is monotonic decreasing. Therefore we can write

(15.53) $$\frac{1}{I_{2n}I_{2n-1}} \le \frac{1}{I_{2n}^2} \le \frac{1}{I_{2n}I_{2n+1}} .$$

Now we shall evaluate each member of this inequality; this will lead at once to Wallis' inequality.

Integration of the identity

$$\frac{d}{dt}(\cos t \sin^{n+1} t) = (n + 1) \sin\text{”} t - (n + 2) \sin^{n+2} t$$

over the interval $[0, \pi/2]$ gives us

$$0 = (n + 1)I_n - (n + 2)I_{n+2},$$

or

(15.54) $$I_{n+2} = \frac{n+1}{n+2} I_n .$$

Using this recursion formula with n replaced by $2k - 2$ we find

$$\frac{I_{2k}}{I_{2k-2}} = \frac{2k-1}{2k} = \frac{2k(2k-1)}{(2k)^2} .$$

Multiplying these equations for $k = 1, 2, \ldots, n$, we find

$$\prod_{k=1}^{n} \frac{I_{2k}}{I_{2k-2}} = \prod_{k=1}^{n} \frac{2k(2k-1)}{(2k)^2} = \frac{(2n)!}{2^{2n}(n!)^2} .$$

The product on the left telescopes to I_{2n}/I_0. Since $I_0 = \pi/2$ we obtain

(15.55) $$I_{2n} = \frac{(2n)!}{2^{2n}(n!)^2} \cdot \frac{\pi}{2} .$$

In a similar way we apply the recursion formula (15.54) with n replaced by $2k - 1$ and multiply the resulting equations for $k = 1, 2, \ldots, n$ to obtain

$$\prod_{k=1}^{n} \frac{I_{2k+1}}{I_{2k-1}} = \prod_{k=1}^{n} \frac{(2k)^2}{2k(2k+1)} = \frac{2^{2n}(n!)^2}{(2n+1)!} = \frac{1}{2n+1} \cdot \frac{\pi}{2} \cdot \frac{1}{I_{2n}^2} .$$

The product on the left telescopes to $I_{2n+1}/I_1 = I_{2n+1}$, so we get

$$I_{2n}I_{2n+1} = \frac{\pi}{2(2n+1)} . \tag{15.56}$$

Since $I_{2n+1} = 2nI_{2n-1}/(2n + 1)$, Equation (15.56) implies

$$I_{2n}I_{2n-1} = \frac{\pi}{4n} .$$

We use this in (15.53), together with the two relations (15.55) and (15.56). Then we multiply by $\pi^2/4$ to obtain Wallis' inequality (15.52).

15.23 Exercises

1. If f is a polynomial of degree ≤ 2, show that Euler's summation formula (15.48) reduces to the trapezoidal formula, as expressed in Equation (15.45).
2. Euler's constant C is defined by the limit formula

$$C = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n \right).$$

(See Section 10.17 in Volume I.) Use Euler's summation formula to prove that

$$\sum_{k=1}^{n} \frac{1}{k} = \log n + C + \frac{1}{2n} - \frac{E(n)}{n^2} ,$$

where $0 \le E(n) \le \frac{1}{8}$. Also, show that

$$C = 1 - \int_1^{\infty} \frac{t - [t]}{t^2}\, dt .$$

3. (a) If $s > 0$, $s \ne 1$, use Euler's summation formula to prove that

$$\sum_{k=1}^{n} \frac{1}{k^s} = \frac{n^{1-s}}{1 - s} + C(s) + s\int_n^{\infty} \frac{t - [t]}{t^{s+1}}\, dt ,$$

where

$$C(s) = 1 + \frac{1}{s - 1} - s\int_1^{\infty} \frac{t - [t]}{t^{s+1}}\, dt .$$