This sort of question can be answered as soon as the so-called "real number system" is constructed.

1.1 Example We now show that the equation

$$p^2 = 2 \tag{1}$$

is not satisfied by any rational p. If there were such a p, we could write $p = m/n$ where m and n are integers that are not both even. Let us assume this is done. Then (1) implies

$$m^2 = 2n^2, \tag{2}$$

This shows that m^2 is even. Hence m is even (if m were odd, m^2 would be odd), and so m^2 is divisible by 4. It follows that the right side of (2) is divisible by 4, so that n^2 is even, which implies that n is even.

The assumption that (1) holds thus leads to the conclusion that both m and n are even, contrary to our choice of m and n. Hence (1) is impossible for rational p.

We now examine this situation a little more closely. Let A be the set of all positive rationals p such that $p^2 < 2$ and let B consist of all positive rationals p such that $p^2 > 2$. We shall show that *A contains no largest number and B contains no smallest.*

More explicitly, for every p in A we can find a rational q in A such that $p < q$, and for every p in B we can find a rational q in B such that $q < p$.

To do this, we associate with each rational $p > 0$ the number

$$q = p - \frac{p^2 - 2}{p + 2} = \frac{2p + 2}{p + 2}. \tag{3}$$

Then

$$q^2 - 2 = \frac{2(p^2 - 2)}{(p + 2)^2}. \tag{4}$$

If p is in A then $p^2 - 2 < 0$, (3) shows that $q > p$, and (4) shows that $q^2 < 2$. Thus q is in A.

If p is in B then $p^2 - 2 > 0$, (3) shows that $0 < q < p$, and (4) shows that $q^2 > 2$. Thus q is in B.

1.2 Remark The purpose of the above discussion has been to show that the rational number system has certain gaps, in spite of the fact that between any two rationals there is another: If $r < s$ then $r < (r + s)/2 < s$. The real number system fills these gaps. This is the principal reason for the fundamental role which it plays in analysis.

In order to elucidate its structure, as well as that of the complex numbers, we start with a brief discussion of the general concepts of *ordered set* and *field.*

Here is some of the standard set-theoretic terminology that will be used throughout this book.

1.3 Definitions If A is any set (whose elements may be numbers or any other objects), we write $x \in A$ to indicate that x is a member (or an element) of A.

If x is not a member of A, we write: $x \notin A$.

The set which contains no element will be called the *empty set*. If a set has at least one element, it is called *nonempty*.

If A and B are sets, and if every element of A is an element of B, we say that A is a subset of B, and write $A \subset B$, or $B \supset A$. If, in addition, there is an element of B which is not in A, then A is said to be a *proper* subset of B. Note that $A \subset A$ for every set A.

If $A \subset B$ and $B \subset A$, we write $A = B$. Otherwise $A \neq B$.

1.4 Definition Throughout Chap. 1, the set of all rational numbers will be denoted by Q.

ORDERED SETS

1.5 Definition Let S be a set. An *order* on S is a relation, denoted by $<$, with the following two properties:

(i) If $x \in S$ and $y \in S$ then one and only one of the statements

$$x < y, \qquad x = y, \qquad y < x$$

is true.

(ii) If $x, y, z \in S$, if $x < y$ and $y < z$, then $x < z$.

The statement "$x < y$" may be read as "x is less than y" or "x is smaller than y" or "x precedes y".

It is often convenient to write $y > x$ in place of $x < y$.

The notation $x \leq y$ indicates that $x < y$ or $x = y$, without specifying which of these two is to hold. In other words, $x \leq y$ is the negation of $x > y$.

1.6 Definition An *ordered set* is a set S in which an order is defined.

For example, Q is an ordered set if $r < s$ is defined to mean that $s - r$ is a positive rational number.

1.7 Definition Suppose S is an ordered set, and $E \subset S$. If there exists a $\beta \in S$ such that $x \leq \beta$ for every $x \in E$, we say that E is *bounded above*, and call β an *upper bound* of E.

Lower bounds are defined in the same way (with $\geq$ in place of $\leq$).

1.8 Definition Suppose S is an ordered set, $E \subset S$, and E is bounded above. Suppose there exists an $\alpha \in S$ with the following properties:

(i) α is an upper bound of E.
(ii) If $\gamma < \alpha$ then γ is not an upper bound of E.

Then α is called the *least upper bound of* E [that there is at most one such α is clear from (ii)] or the *supremum of* E, and we write

$$\alpha = \sup E.$$

The *greatest lower bound*, or *infimum*, of a set E which is bounded below is defined in the same manner: The statement

$$\alpha = \inf E$$

means that α is a lower bound of E and that no β with $\beta > \alpha$ is a lower bound of E.

1.9 Examples

(*a*) Consider the sets A and B of Example 1.1 as subsets of the ordered set Q. The set A is bounded above. In fact, the upper bounds of A are exactly the members of B. Since B contains no smallest member, *A has no least upper bound in Q.*

Similarly, B is bounded below: The set of all lower bounds of B consists of A and of all $r \in Q$ with $r \leq 0$. Since A has no lasgest member, *B has no greatest lower bound in Q.*

(*b*) If $\alpha = \sup E$ exists, then α may or may not be a member of E. For instance, let E_1 be the set of all $r \in Q$ with $r < 0$. Let E_2 be the set of all $r \in Q$ with $r \leq 0$. Then

$$\sup E_1 = \sup E_2 = 0,$$

and $0 \notin E_1$, $0 \in E_2$.

(*c*) Let E consist of all numbers $1/n$, where $n = 1, 2, 3, \ldots$. Then $\sup E = 1$, which is in E, and $\inf E = 0$, which is not in E.

1.10 Definition An ordered set S is said to have the *least-upper-bound property* if the following is true:

If $E \subset S$, E is not empty, and E is bounded above, then $\sup E$ exists in S.

Example 1.9(*a*) shows that Q does not have the least-upper-bound property.

We shall now show that there is a close relation between greatest lower bounds and least upper bounds, and that every ordered set with the least-upper-bound property also has the greatest-lower-bound property.

1.11 Theorem *Suppose S is an ordered set with the least-upper-bound property, $B \subset S$, B is not empty, and B is bounded below. Let L be the set of all lower bounds of B. Then*

$$\alpha = \sup L$$

exists in S, and $\alpha = \inf B$.

In particular, $\inf B$ *exists in S.*

Proof Since B is bounded below, L is not empty. Since L consists of exactly those $y \in S$ which satisfy the inequality $y \leq x$ for every $x \in B$, we see that *every $x \in B$ is an upper bound of L*. Thus L is bounded above. Our hypothesis about S implies therefore that L has a supremum in S; call it α.

If $\gamma < \alpha$ then (see Definition 1.8) γ is not an upper bound of L, hence $\gamma \notin B$. It follows that $\alpha \leq x$ for every $x \in B$. Thus $\alpha \in L$.

If $\alpha < \beta$ then $\beta \notin L$, since α is an upper bound of L.

We have shown that $\alpha \in L$ but $\beta \notin L$ if $\beta > \alpha$. In other words, α is a lower bound of B, but β is not if $\beta > \alpha$. This means that $\alpha = \inf B$.

FIELDS

1.12 Definition A *field* is a set F with two operations, called *addition* and *multiplication*, which satisfy the following so-called "field axioms" (A), (M), and (D):

(A) Axioms for addition

(A1) If $x \in F$ and $y \in F$, then their sum $x + y$ is in F.
(A2) Addition is commutative: $x + y = y + x$ for all $x, y \in F$.
(A3) Addition is associative: $(x + y) + z = x + (y + z)$ for all $x, y, z \in F$.
(A4) F contains an element 0 such that $0 + x = x$ for every $x \in F$.
(A5) To every $x \in F$ corresponds an element $-x \in F$ such that

$$x + (-x) = 0.$$

(M) Axioms for multiplication

(M1) If $x \in F$ and $y \in F$, then their product xy is in F.
(M2) Multiplication is commutative: $xy = yx$ for all $x, y \in F$.
(M3) Multiplication is associative: $(xy)z = x(yz)$ for all $x, y, z \in F$.
(M4) F contains an element $1 \neq 0$ such that $1x = x$ for every $x \in F$.
(M5) If $x \in F$ and $x \neq 0$ then there exists an element $1/x \in F$ such that

$$x \cdot (1/x) = 1.$$

(D) The distributive law

$$x(y + z) = xy + xz$$

holds for all $x, y, z \in F$.

1.13 Remarks

(*a*) One usually writes (in any field)

$$x - y, \frac{x}{y}, x + y + z, xyz, x^2, x^3, 2x, 3x, \ldots$$

in place of

$$x + (-y), x \cdot \left(\frac{1}{y}\right), (x + y) + z, (xy)z, xx, xxx, x + x, x + x + x, \ldots.$$

(*b*) The field axioms clearly hold in Q, the set of all rational numbers, if addition and multiplication have their customary meaning. Thus Q is a field.

(*c*) Although it is not our purpose to study fields (or any other algebraic structures) in detail, it is worthwhile to prove that some familiar properties of Q are consequences of the field axioms; once we do this, we will not need to do it again for the real numbers and for the complex numbers.

1.14 Proposition *The axioms for addition imply the following statements.*

(*a*) *If* $x + y = x + z$ *then* $y = z$.
(*b*) *If* $x + y = x$ *then* $y = 0$.
(*c*) *If* $x + y = 0$ *then* $y = -x$.
(*d*) $-(-x) = x$.

Statement (*a*) is a cancellation law. Note that (*b*) asserts the uniqueness of the element whose existence is assumed in (A4), and that (*c*) does the same for (A5).

Proof If $x + y = x + z$, the axioms (A) give

$$\begin{aligned} y = 0 + y = (-x + x) + y &= -x + (x + y) \\ &= -x + (x + z) = (-x + x) + z = 0 + z = z. \end{aligned}$$

This proves (*a*). Take $z = 0$ in (*a*) to obtain (*b*). Take $z = -x$ in (*a*) to obtain (*c*).

Since $-x + x = 0$, (*c*) (with $-x$ in place of x) gives (*d*).

1.15 Proposition *The axioms for multiplication imply the following statements.*

(*a*) *If* $x \neq 0$ *and* $xy = xz$ *then* $y = z$.
(*b*) *If* $x \neq 0$ *and* $xy = x$ *then* $y = 1$.
(*c*) *If* $x \neq 0$ *and* $xy = 1$ *then* $y = 1/x$.
(*d*) *If* $x \neq 0$ *then* $1/(1/x) = x$.

The proof is so similar to that of Proposition 1.14 that we omit it.

1.16 Proposition *The field axioms imply the following statements, for any* $x, y, z \in F$.

(*a*) $0x = 0$.
(*b*) *If* $x \neq 0$ *and* $y \neq 0$ *then* $xy \neq 0$.
(*c*) $(-x)y = -(xy) = x(-y)$.
(*d*) $(-x)(-y) = xy$.

Proof $0x + 0x = (0 + 0)x = 0x$. Hence 1.14(*b*) implies that $0x = 0$, and (*a*) holds.

Next, assume $x \neq 0$, $y \neq 0$, but $xy = 0$. Then (*a*) gives

$$1 = \left(\frac{1}{y}\right)\left(\frac{1}{x}\right)xy = \left(\frac{1}{y}\right)\left(\frac{1}{x}\right)0 = 0,$$

a contradiction. Thus (*b*) holds.

The first equality in (*c*) comes from

$$(-x)y + xy = (-x + x)y = 0y = 0,$$

combined with 1.14(*c*); the other half of (*c*) is proved in the same way. Finally,

$$(-x)(-y) = -[x(-y)] = -[-(xy)] = xy$$

by (*c*) and 1.14(*d*).

1.17 Definition An *ordered field* is a *field F* which is also an *ordered set*, such that

(i) $x + y < x + z$ if $x, y, z \in F$ and $y < z$,
(ii) $xy > 0$ if $x \in F$, $y \in F$, $x > 0$, and $y > 0$.

If $x > 0$, we call x *positive*; if $x < 0$, x is *negative*.

For example, Q is an ordered field.

All the familiar rules for working with inequalities apply in every ordered field: Multiplication by positive [negative] quantities preserves [reverses] inequalities, no square is negative, etc. The following proposition lists some of these.

1.18 Proposition *The following statements are true in every ordered field.*

(*a*) *If* $x > 0$ *then* $-x < 0$, *and vice versa.*
(*b*) *If* $x > 0$ *and* $y < z$ *then* $xy < xz$.
(*c*) *If* $x < 0$ *and* $y < z$ *then* $xy > xz$.
(*d*) *If* $x \neq 0$ *then* $x^2 > 0$. *In particular,* $1 > 0$.
(*e*) *If* $0 < x < y$ *then* $0 < 1/y < 1/x$.

Proof
(*a*) If $x > 0$ then $0 = -x + x > -x + 0$, so that $-x < 0$. If $x < 0$ then $0 = -x + x < -x + 0$, so that $-x > 0$. This proves (*a*).
(*b*) Since $z > y$, we have $z - y > y - y = 0$, hence $x(z - y) > 0$, and therefore

$$xz = x(z - y) + xy > 0 + xy = xy.$$

(*c*) By (*a*), (*b*), and Proposition 1.16(*c*),

$$-[x(z - y)] = (-x)(z - y) > 0,$$

so that $x(z - y) < 0$, hence $xz < xy$.
(*d*) If $x > 0$, part (ii) of Definition 1.17 gives $x^2 > 0$. If $x < 0$, then $-x > 0$, hence $(-x)^2 > 0$. But $x^2 = (-x)^2$, by Proposition 1.16(*d*). Since $1 = 1^2$, $1 > 0$.
(*e*) If $y > 0$ and $v \leq 0$, then $yv \leq 0$. But $y \cdot (1/y) = 1 > 0$. Hence $1/y > 0$. Likewise, $1/x > 0$. If we multiply both sides of the inequality $x < y$ by the positive quantity $(1/x)(1/y)$, we obtain $1/y < 1/x$.

THE REAL FIELD

We now state the *existence theorem* which is the core of this chapter.

1.19 Theorem *There exists an ordered field R which has the least-upper-bound property.*

Moreover, R contains Q as a subfield.

The second statement means that $Q \subset R$ and that the operations of addition and multiplication in R, when applied to members of Q, coincide with the usual operations on rational numbers; also, the positive rational numbers are positive elements of R.

The members of R are called *real numbers.*

The proof of Theorem 1.19 is rather long and a bit tedious and is therefore presented in an Appendix to Chap. 1. The proof actually constructs R from Q.