> interested mainly in offering a concrete counterpart of abstract geometry; however, we do not wish to omit a declaration that the validity of the new order of concepts does not depend on the possibility of such a counterpart.

In the second paper, Beltrami vindicates this ringing endorsement of Riemann's ideas with whole families of models of non-Euclidean geometry in any number of dimensions. Among them is the half-plane model used in this chapter, and its generalization to three dimensions, the "half-space model." The half-space model has

- "points" that are the points $(x,y,z) \in \mathbb{R}^3$ with $z > 0$,
- "lines" that are the vertical Euclidean half lines in $\mathbb{R}^3$ and the vertical semicircles with centers on the plane $z = 0$,
- "planes" that are the vertical Euclidean half planes in $\mathbb{R}^3$ and hemispheres with centers on $z = 0$.

It turns out that non-Euclidean distance on a plane $z = a$ is a constant multiple of Euclidean distance. This surprising result gives probably the simplest proof of a result first discovered by Friedrich Wachter, a member of Gauss's circle, in 1816: *Three-dimensional non-Euclidean geometry contains a model of the Euclidean plane*.

Another model of the hyperbolic plane, discovered by Beltrami, is the *conformal disk model*. It is like the half plane in being angle-preserving, but unlike it in being finite. Its "points" are the interior points of the unit disk (the points z with $|z| < 1$, if we work in the plane of complex numbers), and its "lines" are circular arcs perpendicular to the unit circle. Figure 8.19, which is the original M. C. Escher picture *Circle Limit I*, can be viewed as a picture of the conformal disk model. The fish are arranged along "lines," and they are all of the same hyperbolic length. As mentioned in connection with Figure 8.10, the transformed *Circle Limit I*, the function

$$z \mapsto \frac{1 - zi}{z - i}$$

maps the conformal disk model onto the half-plane model.

It should be stressed that *all models of non-Euclidean geometry, in a given dimension, are isomorphic to the half-space model.* For example, models of the non-Euclidean plane satisfy Hilbert's axioms (Section 2.9)

Figure 8.19: The conformal disk model

with the parallel axiom replaced by the non-Euclidean parallel hypothesis. And Hilbert in his *Grundlagen* showed that the "lines" satisfying these axioms have "ends" that behave like the points of $\mathbb{RP}^1$. Thus, any non-Euclidean plane is essentially the same as the half plane discussed in this chapter, so we can call it *the* non-Euclidean plane or *the* hyperbolic plane.

Non-Euclidean reality

In Beltrami's original model, the open disk in which "lines" are line segments ending on the unit circle, isometries map Euclidean lines to Euclidean lines, and so they are projective maps. For this reason, the model is often called the *projective disk*. It can also be constructed by methods of projective geometry, and indeed this is essentially what Cayley did in 1859. The first to connect all the dots between projective and non-Euclidean geometry was Klein in 1871. An English translation of his paper may be found in *Sources of Hyperbolic Geometry*. Although Klein had only to fill a few technical gaps, it was he who first made the important conceptual point that *a model of non-Euclidean geometry ensures that the non-Euclidean parallel hypothesis is not contradictory. Hence, Euclid's parallel axiom does not follow from his other axioms.*

In 1872, Klein also made the great advance of linking geometries to groups of transformations. This link gives a deeper reason for the presence of non-Euclidean geometry in projective geometry: The real projective line and the non-Euclidean plane have isomorphic groups of transformations.

The group of the non-Euclidean plane was first described explicitly by the French mathematician Henri Poincaré in 1882, along with its interpretation as the group of Möbius transformations of the half plane.The relevant parts of his work may also be found in *Sources of Hyperbolic Geometry*. Poincaré became interested in non-Euclidean geometry when he noticed that some functions of a complex variable have *non-Euclidean periodicity*.

An ordinary periodic function, such as $\cos x$, has *Euclidean periodicity* in the sense that its values repeat when x undergoes the Euclidean translation $x \mapsto x + 2\pi$. A complex function can have non-Euclidean periodicity, and one example is the *modular function* $j(z)$. Its definition is too long to explain here, but its periodicity is simple: The values of $j(z)$ repeat under the Möbius transformations $z \mapsto z + 1$ and $z \mapsto -1/z$. As we know, these are isometries of the half plane. If one applies them over and over, to the lines $x = 0$, $x = 1$, and the unit semicircle, they produce the non-Euclidean regular tessellation shown in Figure 8.20.

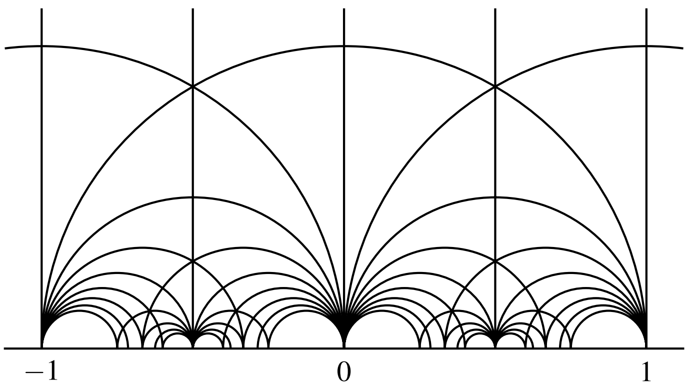

Figure 8.20: The modular tessellation

The modular function and its periodicity were already part of mathematical reality, having been known to Gauss and others since early in the 19th century. But Poincaré was the first to see its non-Euclidean symmetry. He used non-Euclidean geometry to study large classes of functions whose behavior had until then seemed intractable. Poincaré was also the first to view the half plane as an extension of the real projective line, as we have done in this chapter. In fact, he went much further, noticing that the half-space model of non-Euclidean space is a natural extension of the *complex* projective line $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$.

Just as the real projective line $\mathbb{R} \cup \{\infty\}$ comes with the linear fractional transformations

$$x \mapsto \frac{ax+b}{cx+d}, \quad \text{where } a,b,c,d \in \mathbb{R} \text{ and } ad-bc \neq 0,$$

the complex projective line $\mathbb{C} \cup \{\infty\}$ comes with the linear fractional transformations

$$z \mapsto \frac{az+b}{cz+d}, \quad \text{where } a,b,c,d \in \mathbb{C} \text{ and } ad-bc \neq 0.$$

And just as the linear fractional transformations of $\mathbb{R} \cup \{\infty\}$ extend to Möbius transformations of the half plane, the linear fractional transformations of $\mathbb{C} \cup \{\infty\}$ extend to Möbius transformations of the *half space*, for which there is likewise an invariant non-Euclidean distance, and the non-Euclidean "lines" and "planes" mentioned above.

It is a great advantage to have a concept of distance, even if the distance is non-Euclidean and one needs an extra dimension to acquire it. By passing to the third dimension, Poincaré could understand transformations of $\mathbb{C}$ whose behavior is almost incomprehensible when viewed in the plane. Understanding comes by viewing these transformations as compressed versions of isometries of non-Euclidean space, which behave quite simply (like isometries of the half plane). Thus, expanding from a projective line to a non-Euclidean space is not just an interesting theoretical possibility—it is sometimes the best way to understand the mysteries of projection.

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Index

1-sphere, 158
120-cell, 173
2-sphere, 155
24-cell, 166, 173
3-sphere, 167
600-cell, 173

addition
  algebraic properties, 42
  commutative law, 68
  of area, 27
  of lengths, 3, 42
  of ordered pairs, 65
  of ordered triples, 68
  of vectors, 66
  projective, 129
  vector, 66, 172
    parallelogram rule, 67
affine
  geometry, 150
    theorems of, 150
  map, 170
  transformation, 150
  view of cube, 172
Alberti, Leon Battista, 90
alternativity, 140
angle, 43
  alternate interior, 22
  and inner product, 68
  and slope, 55
  bisection, 6, 7
  congruence, 44
  division
    faulty, 9
  in a circle, 36
  in a semicircle, 36, 79
  in half plane, 176, 186
  infinitesimal view, 187
  of regular *n*-gon, 23
  right, 1
  sum
    and area, 190
    of convex *n*-gon, 23
    of quadrilateral, 23
    of triangle, 22
  vertically opposite, 27
antipodal
  map, 156
  point, 156, 168
Archimedean axiom, 45
area, 26
  and angle sum, 190
  as rectangle, 28
  equality
    Greek concept, 28
  Euclid's concept, 27
  Euclid's theory of, 42
  non-Euclidean, 189
  of geodesic triangle, 204
  of parallelogram, 29
  of spherical triangle, 190
  of triangle, 29
arithmetic
  axioms for, 43
  of segments, 42
  projective, 128
arithmetization, 46, 117
Artmann, Benno, 18
ASA, 20, 24, 56

and existence of parallels, 22
associative law
for projective addition
and scissors theorem, 136
for projective multiplication
and Desargues theorem, 136
of multiplication, 117
axiom
Archimedean, 45
Dedekind, 45
Desargues, 117
of circle intersection, 45
of unique line, 22
in $\mathbb{R}^2$, 50
Pappus, 117
parallel, 5, 20, 21
and foundations of geometry, 174
in $\mathbb{R}^2$, 50
modern, 22
Playfair's, 22
SAS, 24
axioms
congruence, 24, 44
construction, 2, 5
Euclid's, 47
model of, 50
field, 87, 113, 117, 120
for arithmetic, 43
for non-Euclidean plane, 209
for projective geometry, 43
geometric, 5
Hilbert, 42, 47, 86, 87
list of, 43
incidence, 42, 43, 95
order, 42, 44
projective plane, 94
models of, 95
vector space, 86

barycenter, 72
Beltrami, Eugenio, 206
models of non-Euclidean plane, 208
theorem on constant curvature, 206
betweenness, 43
bisection
of angle, 6, 7
of line segment, 6
Blake, William, 3
Bolyai, Janos, 204
Brieskorn, Egbert, 63

$\mathbb{C}$, 85
as a field, 114
as a plane, 85
Cauchy–Schwarz inequality, 80
algebraic proof, 82
Cayley, Arthur, 137
and quaternions, 173
discovery of octonions, 141
invariant length, 206
projective disk, 210
center of mass, 72
centroid, 71
as intersection of medians, 72
as vector average, 72
of tetrahedron, 74
circle
angles in, 36
equation of, 52
intersection axiom, 45
non-Euclidean, 196
Cohn-Vossen, Stefan, 19
coincidence, 121
Desargues, 121
in projective arithmetic, 128, 133
in tiled floor, 122
Pappus, 121
common notions, 26
commutative law, 68
for projective addition
and Pappus, 135
for projective multiplication
and Pappus, 134
for vector addition, 68
of multiplication, 117
fails for quaternions, 137
compass, 1
and projective addition, 129

axiom, 2
completeness of line, 45
complex conjugation, 178
concurrence
of altitudes, 75
of medians, 73
of tetrahedron, 74
of perpendicular bisectors, 77
configuration
Desargues, 13
little Desargues, 119
Pappus, 12
congruence, 43
of triangles, 24
axioms, 24, 44
of angles, 44
of line segments, 44
constructibility, 46
algebraic criterion, 55
constructible
figures, 1
numbers, 45
operations, 41
construction
axioms, 2, 5
bisection, 6
by straightedge alone, 88
of tiling, 92
by straightedge and compass, 1
equilateral triangle, 42
of irrational length, 11
of parallel, 8
of perpendicular, 7
of product of lengths, 10
of quotient of lengths, 10
of regular n-gon, 5
of regular 17-gon, 19
of regular pentagon, 18, 41
of regular polyhedra, 18
of right-angled triangle, 37
of square, 9
of given area, 33, 38
of square root, 38, 40
of square tiling, 9
coordinates, 46
and Pappus theorem, 115
homogeneous, 98
in a field, 116
in Pappian planes, 120
in projective geometry, 117
of a point in the plane, 47
cosine, 56
and inner product, 77
formula for inner product, 78, 86
function, 77
rule, 75, 78
and Pythagorean theorem, 79
costruzione legittima, 90
Coxeter, Harold Scott Macdonald, 19
$\mathbb{CP}^1$ and half-space model, 212
$\mathbb{CP}^2$, 98
$\mathbb{CP}^3$, 99
cross-ratio, 64, 108, 143
as defining invariant, 110
as fundamental invariant, 112
determination of fourth point, 110
group, 113
in half plane, 176, 191
invariance
discovered by Desargues, 115
is it visible?, 109
on non-Euclidean line, 192
preserved by linear fractional functions, 108
preserved by projection, 108
transformations of, 113
cube, 18, 163
affine view, 172
of a sum, 29
curvature of space, 206
curvature of surface, 204
curve
algebraic, 63
equidistant, 196
of degree 1, 63
of degree 2, 63

Dürer, Albrecht, 89
Dedekind axiom, 45
Densmore, Dana, 18
Desargues, Girard
  configuration, 13
  and involutions, 199
  and the cross-ratio, 115
  configuration
    spatial, 141
  theorem, 12, 42, 115
    and associative multiplication, 136, 140
    as an axiom, 117
    converse, 125
    fails in Moulton plane, 121
    fails in $\mathbb{OP}^2$, 141
    holds in $\mathbb{HP}^2$, 138
    in space, 141
    little, 119
    projective, 119
    vector version, 71
Descartes, René, 46
  Géométrie, 16, 41, 46
    constructible operations, 41, 55
determinant, 107, 148
  multiplicative property, 160
dilation, 67, 68
  and scalar multiplication, 67
  matrix representation, 150
direction, 69
  and parallel lines, 69
  relative, 69
distance
  great-circle, 155
  in $\mathbb{R}^2$, 51
  in $\mathbb{R}^3$, 81
  in $\mathbb{R}^n$, 81
  non-Euclidean, 191
    on y-axis, 193
distributive law, 138
  and Pappus theorem, 139
dividing by zero, 106
dodecahedron, 18
Einstein, Albert, 87
equation
  homogeneous, 98
  linear, 50
  of circle, 52
  of line, 49
  of non-Euclidean line, 180
equidistant
  curves, 196
  line, 52, 76
    is perpendicular bisector, 76
  point, 53
  set
    in $\mathbb{R}^3$, 156
    in $\mathbb{S}^2$, 156
Erlanger Programm, 64
Escher, Maurits, 194, 209
Euclid, 1
  and non-Euclidean geometry, 203
  common notions, 26
  concept of equal area, 27
  construction
    axioms, 2, 95
    of equilateral triangle, 4, 6
    of regular pentagon, 41
  Elements, 1, 3, 4, 17
    Heath translation, 17
  geometric axioms, 5
  proofs of Pythagorean theorem, 32, 38
  statement of parallel axiom, 21
  theory of area, 42
Euclidean
  geometry, 1, 43
    3-dimensional, 68
    and inner product, 68
    and linear algebra, 64
    and numbers, 43
    and parallel axiom, 20
    group, 145
    in vector spaces, 68
    model of, 47
    oriented, 145

via straightedge and compass, 1
inner product, 82
plane, 45, 46
space, 80

Fano plane, 115
Fermat, Pierre de, 46
field, 113
axioms, 87, 113, 117, 120
as coincidences, 133
in $\mathbb{R}^n$, 162
$\mathbb{C}$, 114
$\mathbb{F}_2$, 114
finite, 114
of rational numbers, 114
$\mathbb{R}$, 114
with absolute value, 172
foundations of geometry, 3, 12, 174, 177

gaps, 45, 47, 114
Gauss, Carl Friedrich, 204
and modular function, 211
construction of 17-gon, 19
Disquisitiones generales, 204
generating transformations, 103
are products of reflections, 183
of half plane, 179
of $\mathbb{RP}^1$, 178
preserve cross-ratio, 108
geodesic, 204
geometry
affine, 150
algebraic, 63
arithmetization of, 46
Cartesian, 46
coordinate, 46
differential, 174
Euclidean, 1, 20, 43, 143
3-dimensional, 68
and inner product, 68
and linear algebra, 64
in vector spaces, 68
plane of, 45
foundations of, 3, 12, 174, 177
hyperbolic, 143, 203
Klein's concept, 143
n-dimensional, 68
non-Euclidean, 1, 143
and parallel axiom, 174
origin of word, 170
projective, 43, 87, 143
as geometry of vision, 88
spherical, 154, 174
is non-Euclidean, 174
vector, 65, 143
glide reflection, 60
Graves, John, 141
great circle, 155
reflection in, 156
group
abstract, 168
continuous, 170
cyclic, 162
isomorphism, 169
of isometries, 64
of transformations, 64, 113, 143, 144, 168
theory, 64

$\mathbb{H}$, 137
half plane, 175
generating transformations of, 179
isomorphic to $\mathbb{RP}^1$, 194
tilings, 188
half space, 209
and $\mathbb{CP}^1$, 212
Hamilton, William Rowan, 137
and rotations of $\mathbb{R}^3$, 173
search for number systems, 172
Harriot, Thomas, 190
Hartshorne, Robin, 18, 43, 56
on the cross-ratio, 108
Harunobu, Suzuki, 170
Hessenberg, Gerhard, 140
hexagon, 5
regular, 5
construction, 6
tiling, 6

perspective view, 94
Hilbert, David, 19, 140
axioms, 42, 86, 87
for non-Euclidean plane, 209
list of, 43
constructed non-Pappian plane, 140
Grundlagen, 42, 140, 210
segment arithmetic, 42
theorem on negative curvature, 206
homogeneous coordinates, 98
horizon, 89, 100
horocycle *see* limit circle 198
$\mathbb{HP}^2$, 138
satisfies Desargues, 138
violates Pappus, 138
hyperbolic
geometry, 143, 203
plane, 210
uniqueness, 210

icosahedron, 18
identity function, 144
incidence, 42, 95
axioms, 43, 95
in projective space, 99
theorem, 118
infinity, 2
line at, 92, 100
point at, 92
projection from, 101
inner product, 65
algebraic properties, 75
and cosine, 77
and Euclidean geometry, 68
and length, 65, 75
and perpendicularity, 75
cosine formula, 78, 86
in $\mathbb{R}^2$, 74
$\mathbb{R}^n$, 81
positive definite, 82
intersection, 53
of circles, 4, 5, 54
axiom, 45
of lines, 54
invariant, 64, 110
fundamental, 112
length in half plane, 176
of group of transformations, 143
of isometry group, 143, 145
of projective transformations, 143
inversion in a circle, 179
involution, 199
irrational
length, 11, 28
number $\sqrt{2}$, 16, 17, 47
isometry, 57
and motion, 57
composite, 144
group, 64
of half plane, 196
of non-Euclidean space, 212
of $\mathbb{R}^2$, 61, 144
of $\mathbb{R}^3$, 155
of $\mathbb{S}^2$, 155
reason for name, 58
isomorphism, 169
between geometries, 194
between models, 209
isosceles triangle theorem, 24

Joyce, David, 18

Kaplansky, Irving, 87
Klein, Felix, 64, 143
and projective disk, 210
and the parallel axiom, 210
and transformation groups, 143, 211
concept of geometry, 143
Knörrer, Horst, 63

law of cosines, 75
length
addition, 3, 42
and inner product, 65, 68, 75
division, 10
in $\mathbb{R}^2$, 51
irrational, 11, 28
multiplication, 10, 42

non-Euclidean, 176
product
and Thales' theorem, 10
rational, 11
subtraction, 3
limit circle, 198
limit rotation, 198
line, 2
as fixed point set, 183
at infinity, 92, 100
broken, 120
completeness of, 45
defined by linear equation, 50
equation of, 49
equidistant, 52, 76
is perpendicular bisector, 76
non-Euclidean, 176
is infinite, 195
number, 47
of Moulton plane, 120
perpendicular, 7
projective, 97
algebraic definition, 107
modelled by circle, 97, 101
projection of, 101
real, 97
segment, 2, 43
bisection, 6
congruence of, 44
n-section, 8
slope of, 48
linear
algebra, 64, 65
equation, 50
independence, 69
transformation, 143, 146
inverse, 148
matrix representation, 148
preserves parallels, 147
preserves straightness, 147
preserves vector operations, 146
linear fractional functions, 104
are realized by projection, 104
behave like matrices, 107
characterization, 111
defining invariant of, 110
generators of, 108
on real projective line, 107
orientation-preserving, 182
preserve cross-ratio, 108
linear fractional transformations *see* linear fractional functions 104
lines
asymptotic, 203
parallel, 1, 21
and direction, 69
little Desargues theorem, 119
and alternative multiplication, 140
and projective addition, 129
and tiled-floor coincidence, 123
fails in Moulton plane, 121
implies little Pappus, 135
little Pappus theorem, 135
Lobachevsky, Nikolai Ivanovich, 204
hyperbolic formulas, 205
logic, 5

magnification *see* dilation 13
matrix, 83
determinant of, 148
of linear fractional function, 107
of linear transformation, 147
product, 148
rotation, 83
McKean, Henry, 63
medians, 72
concurrence of, 73
midpoint, 71
Minding, Ferdinand, 205
Minkowski space, 87
Möbius transformations, 182
and cross-ratio, 192
generating, 179
preserve angle, 186
preserve non-Euclidean lines, 184
restriction to $\mathbb{RP}^1$, 182
model

of Euclid's axioms, 50
of Euclidean plane geometry, 47
of non-Euclidean geometry, 204, 206
of non-Euclidean plane, 209
of non-Euclidean space, 209
of projective line, 97
of projective plane, 95
of the line, 47
modular function, 211
Moll, Victor, 63
motion, 24, 46, 56
is an isometry, 57
Moufang, Ruth, 140
and $\mathbb{HP}^2$, 141
and little Desargues theorem, 140
and $\mathbb{OP}^2$, 141
Moulton plane, 120, 141
lines of, 120
violates converse Desargues, 128
violates little Desargues, 121
violates tiled-floor coincidence, 124
Moulton, Forest Ray, 128
multiplication
algebraic properties, 42
associative law, 117
commutative law, 117
noncommutative, 137, 162
of lengths, 42
projective, 130

Newton, Isaac, 3
non-Euclidean
area, 189
circle, 196
distance, 191
is additive, 193
on y-axis, 193
geometry, 1, 143
and parallel axiom, 174
models of, 204, 206
length, 176
lines, 46, 176
are infinite, 195
equations of, 180
invariance, 184
uniqueness, 177
violate parallel axiom, 176
parallel hypothesis, 203
periodicity, 211
plane, 64
axioms, 209
from projective line, 174
uniqueness, 210
rotation, 196
space, 209, 212
isometries of, 212
symmetry, 211
translation, 196
triangle, 188
non-Pappian plane, 138, 141
of Hilbert, 140
numbers, 1
as coordinates, 46
complex, 84
and non-Euclidean plane, 174
and rotation, 84
constructible, 45
irrational, 47
n-dimensional, 172
nonconstructible, 45
found by Wantzel, 55
prime, 5, 19
rational, 47
real, 45, 47
as vectors, 66

$\mathbb{O}$, 141
octahedron, 18, 173
octonion projective plane, 141
discovered by Moufang, 141
satisfies little Desargues, 141
violates Desargues, 141
octonion projective space
does not exist, 142
octonions, 141
$\mathbb{OP}^2$, 141
satisfies little Desargues, 141
violates Desargues, 141

order, 42
  axioms, 44
ordered n-tuple, 68
ordered pair, 48
  addition, 65
  as vector, 65
  scalar multiple, 65
ordered triple, 68
origin, 47

Pappus, 24
  configuration, 12
    labeled by vectors, 71
  theorem, 12, 42, 65
    and commutative multiplication, 134, 140
    and coordinates, 115
    and distributive law, 139
    and projective addition, 132
    as an axiom, 117
    fails in $\mathbb{HP}^2$, 138
    implies Desargues, 140
    projective version, 118
    vector version, 70
parallel
  construction, 8
  lines, 1, 21
    and ASA, 22
    have same slope, 50
    in projective plane, 95
parallel axiom, 20, 21, 42
  and foundations of geometry, 174
  and non-Euclidean geometry, 174
  does not follow
    from other axioms, 210
  equivalents, 203
  Euclid's statement, 21
  fails for non-Euclidean lines, 176
  independent of the others, 177
  Playfair's statement, 22
parallelogram, 25
  area of, 29
  diagonals bisect, 26
    vector proof, 72
  opposite sides are equal, 25
  rule, 67
pentagon
  regular, 18
    construction of, 41
perpendicular, 7
  bisector, 76
  construction of, 7
perspective, 88
  drawing, 89
  view of equally-spaced points, 91
  view of $\mathbb{RP}^2$, 100
  view of tiling, 88
    by straightedge alone, 92
π, 45
plane
  hyperbolic, 210
  Moulton, 120
  non-Euclidean, 64, 174
  non-Pappian, 138, 141
  number, 47
  of Euclidean geometry, 45
  Pappian, 119
  projective, 94
  real number, 45
  real projective, 95
Playfair, John, 22
Poincaré, Henri, 211
point at infinity, 92
polygon
  regular, 5
  squaring, 38
polyhedra
  regular, 18
polytope, 166
positive definite, 82
postulates *see* axioms 2
product
  as group operation, 168
  of functions, 168
  of lengths
    as rectangle, 16, 28
    by straightedge and compass, 10

of matrices, 148
of rotations, 157, 159
projection, 100
axonometric, 171
from finite point, 102
from infinity, 101
in Japanese art, 170
is linear fractional, 104
of projective line, 101
preserves cross-ratio, 108
projective
addition, 129
arithmetic, 128
Desargues configuration, 119
disk, 210
distortion, 102
geometry, 43, 87
and non-Euclidean plane, 174
as geometry of vision, 88
axioms, 43
coordinates in, 117
reason for name, 100
line
algebraic definition, 107
real, 97
little Desargues configuration, 119
multiplication, 130
Pappus configuration, 118
plane, 94
axioms, 94
complex, 98
extends Euclidean plane, 96
Fano, 115
$\mathbb{FP}^2$, 113
model of axioms, 95
octonion, 141
quaternion, 138
real, 95
plane axioms
models of, 99
planes, 117
reflection, 182
space, 99
3-dimensional, 99
complex, 99
incidence properties, 99
real, 99, 141
three-dimensional, 167
transformations, 151
as linear transformations, 151
of $\mathbb{RP}^1$, 106, 184
pseudosphere *see* tractroid 205
Pythagorean theorem, 20
and distance in $\mathbb{R}^2$, 51
and distance in $\mathbb{R}^3$, 81
and cosine rule, 79
Euclid's first proof, 32
Euclid's second proof, 38
in $\mathbb{R}^2$, 51
Pythagoreans, 11

quaternion projective plane, 138
satisfies Desargues, 138
quaternions, 137
and the fourth dimension, 172
as complex matrices, 137, 159
noncommutative multiplication, 137, 162
opposite, 162
represent rotations, 159
quotient of lengths, 10

$\mathbb{R}$, 47
as a field, 114
as a line, 47
$\mathbb{R}^2$, 45
as a field, 172
as a plane, 48
as a vector space, 66
distance in, 51
isometry of, 61, 144
$\mathbb{R}^3$, 95
regular polyhedra in, 173
rotation of, 156
$\mathbb{R}^4$, 172
regular polytopes in, 173
rational

length, 11
numbers, 47
ray, 43
rectangle, 21
as product of lengths, 16
reflection
and Möbius transformations, 182
as linear transformation, 149
by generating transformations, 183
fixed point set is a line, 183
in a circle, 182
in a sphere, 182
in great circle, 156
in unit circle, 179
of half plane, 178
of $\mathbb{R}^2$
in x-axis, 60
in any line, 60
ordinary, 183
projective, 182
regular
17-gon, 19
hexagon, 5
n-gon, 162
pentagon, 18
polygon, 5
polyhedra, 18, 163
polytopes, 166
discovered by Schlafli, 173
relativity, 87
rhombus, 26
has perpendicular diagonals, 26
vector proof, 76
Riemann, Bernhard, 206
right angle, 1
$\mathbb{R}^n$, 68
as a Euclidean space, 81
as a vector space, 68
rotation
and complex numbers, 84
and multiplication by -1, 68
as linear transformation, 149
as product of reflections, 60
group of sphere, 157
limit, 198
matrix, 83
non-Euclidean, 196
of $\mathbb{R}^2$, 57, 59
of $\mathbb{R}^3$, 156, 159
of $\mathbb{S}^1$, 158
of $\mathbb{S}^2$, 157, 159
of tetrahedron, 163
$\mathbb{RP}^1$, 107, 151
as boundary of half plane, 175
isomorphic to half plane, 194
$\mathbb{RP}^2$, 95, 151
$\mathbb{RP}^3$, 99, 167
and geometry of the sphere, 99
as a group, 168

$\mathbb{S}^1$, 158
as a group, 169
rotation of, 158
$\mathbb{S}^2$, 155
isometry of, 155
noncommuting rotations, 159
rotation of, 157
$\mathbb{S}^3$, 167
as a group, 169
Saccheri, Girolamo, 203
SAS, 20, 56
statement, 24
scalar multiple, 66
scalar product *see* inner product 74
Schläfli, Ludwig, 173
scissors theorem, 126
and Desargues theorem, 126
and projective multiplication, 130, 132
little
and little Desargues, 128
fails in Moulton plane, 128
semicircle
angle in, 36, 79
as non-Euclidean line, 176
sine, 56
slope, 48

and angle, 55
and perpendicularity, 56
infinite, 49
reciprocal, 153
relative, 56
Snapper, Ernst, 87
space
3-dimensional, 68
curved, 206
Desargues theorem in, 141
Euclidean, 81
half, 209
Möbius transformations of, 212
n-dimensional, 68
non-Euclidean, 209, 212
isometries of, 212
projective, 99
complex, 99
n-dimensional, 141
real, 99
rotations of, 159
vector, 86
real, 67
sphere
0-dimensional, 184
1-dimensional, 158, 167
2-dimensional, 155, 167
3-dimensional, 167
in $\mathbb{R}^3$, 154
spherical
geometry, 154
lines of, 155
triangle, 188, 190
square, 18
construction, 9
diagonal of, 15
of a sum, 27
tiling
construction, 9
square root, 38
construction, 40
squaring
a polygon, 38
the circle, 38, 45
SSS, 24
straightedge, 1
axioms, 2
surface
curvature of, 204
incomplete, 205
of constant curvature, 204

tetrahedron, 18, 74, 163
centroid of, 74
rotations of, 163
as quaternions, 165
Thales, 8
theorem on right angles, 36
Thales theorem, 8, 20, 65
and product of lengths, 10
and quotient of lengths, 11
converse, 11
proof, 34
vector version, 70
theorem
Desargues, 12, 42, 115
converse, 125
projective, 119
vector version, 71
intermediate value, 202
isosceles triangle, 24
little Desargues, 119
fails in Moulton plane, 121
on concurrence of altitudes, 75
on concurrence of medians, 73
Pappus, 12, 42, 65
and coordinates, 115
little, 135
projective version, 118
vector version, 70
Pythagorean, 20
and cosine rule, 79
Euclid's first proof, 32
Euclid's second proof, 38
scissors, 126
Thales, 8, 20, 65
proof, 34