is not in general exact since $1 \otimes \psi$ need not be injective. If $0 \to L \xrightarrow{\psi} M \xrightarrow{\varphi} N \to 0$ is a *split* short exact sequence, however, then since tensor products commute with direct sums by Theorem 17, it follows that

$$0 \longrightarrow D \otimes_R L \xrightarrow{1\otimes\psi} D \otimes_R M \xrightarrow{1\otimes\varphi} D \otimes_R N \longrightarrow 0$$

is also a split short exact sequence.

The following result relating to modules D having the property that (13) can always be extended to a short exact sequence is immediate from Theorem 39:

Proposition 40. Let A be a right R-module. Then the following are equivalent:

(1) For any left R-modules L, M, and N, if

$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow 0$$

is a short exact sequence, then

$$0 \longrightarrow A \otimes_R L \xrightarrow{1\otimes\psi} A \otimes_R M \xrightarrow{1\otimes\varphi} A \otimes_R N \longrightarrow 0$$

is also a short exact sequence.

(2) For any left R-modules L and M, if $0 \to L \xrightarrow{\psi} M$ is an exact sequence of left R-modules (i.e., $\psi : L \to M$ is injective) then $0 \to A \otimes_R L \xrightarrow{1\otimes\psi} A \otimes_R M$ is an exact sequence of abelian groups (i.e., $1 \otimes \psi : A \otimes_R L \to A \otimes_R M$ is injective).

Definition. A right R-module A is called *flat* if it satisfies either of the two equivalent conditions of Proposition 40.

For a fixed right R-module D, the first part of Theorem 39 is referred to by saying that the functor $D \otimes_R$ __ is *right exact*.

Corollary 41. If D is a right R-module, then the functor $D \otimes_R$ __ from the category of left R-modules to the category of abelian groups is right exact. If D is an (S, R)-bimodule (for example when $S = R$ is commutative and D is given the standard R-module structure), then $D \otimes_R$ __ is a right exact functor from the category of left R-modules to the category of left S-modules. The functor is exact if and only if D is a flat R-module.

We have already seen some flat modules:

Corollary 42. Free modules are flat; more generally, projective modules are flat.

Proof: To show that the free R-module F is flat it suffices to show that for any injective map $\psi : L \to M$ of R-modules L and M the induced map $1 \otimes \psi : F \otimes_R L \to F \otimes_R M$ is also injective. Suppose first that $F \cong R^n$ is a finitely generated free R-module. In this case $F \otimes_R L = R^n \otimes_R L \cong L^n$ since $R \otimes_R L \cong L$ and tensor products commute with direct sums. Similarly $F \otimes_R M \cong M^n$ and under these isomorphisms the map $1 \otimes \psi : F \otimes_R L \to F \otimes_R M$ is just the natural map of L^n to M^n induced by the inclusion ψ in each component. In particular, $1 \otimes \psi$ is injective and it follows that any finitely generated free module is flat. Suppose now that F is an arbitrary free module and that the element $\sum f_i \otimes l_i \in F \otimes_R L$ is mapped to 0 by $1 \otimes \psi$. This means that the element $\sum (f_i, \psi(l_i))$ can be written as a sum of generators as in equation (6) in the previous section in the free group on $F \times M$. Since this sum of elements is finite, all of the first coordinates of the resulting equation lie in some finitely generated free submodule F' of F. Then this equation implies that $\sum f_i \otimes l_i \in F' \otimes_R L$ is mapped to 0 in $F' \otimes_R M$. Since F' is a finitely generated free module, the injectivity we proved above shows that $\sum f_i \otimes l_i$ is 0 in $F' \otimes_R L$ and so also in $F \otimes_R L$. It follows that $1 \otimes \psi$ is injective and hence that F is flat.

Suppose now that P is a projective module. Then P is a direct summand of a free module F (Proposition 30), say $F = P \oplus P'$. If $\psi : L \to M$ is injective then $1 \otimes \psi : F \otimes_R L \to F \otimes_R M$ is also injective by what we have already shown. Since $F = P \oplus P'$ and tensor products commute with direct sums, this shows that

$$1 \otimes \psi : (P \otimes_R L) \oplus (P' \otimes_R L) \to (P \otimes_R M) \oplus (P' \otimes_R M)$$

is injective. Hence $1 \otimes \psi : P \otimes_R L \to P \otimes_R M$ is injective, proving that P is flat.

Examples

(1) Since $\mathbb{Z}$ is a projective $\mathbb{Z}$-module it is flat. The example before Theorem 39 shows that $\mathbb{Z}/2\mathbb{Z}$ not a flat $\mathbb{Z}$-module.

(2) The $\mathbb{Z}$-module $\mathbb{Q}$ is a flat $\mathbb{Z}$-module, as follows. Suppose $\psi : L \to M$ is an injective map of $\mathbb{Z}$-modules. Every element of $\mathbb{Q} \otimes_{\mathbb{Z}} L$ can be written in the form $(1/d) \otimes l$ for some nonzero integer d and some $l \in L$ (Exercise 7 in Section 4). If $(1/d) \otimes l$ is in the kernel of $1 \otimes \psi$ then $(1/d) \otimes \psi(l)$ is 0 in $\mathbb{Q} \otimes_{\mathbb{Z}} M$. By Exercise 8 in Section 4 this means $c\psi(l) = 0$ in M for some nonzero integer c. Then $\psi(c \cdot l) = 0$, and the injectivity of ψ implies $c \cdot l = 0$ in L. But this implies that $(1/d) \otimes l = (1/cd) \otimes (c \cdot l) = 0$ in L, which shows that $1 \otimes \psi$ is injective.

(3) The $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$ is injective (by Proposition 36), but is not flat: the injective map $\psi(z) = 2z$ from $\mathbb{Z}$ to $\mathbb{Z}$ does not remain injective after tensoring with $\mathbb{Q}/\mathbb{Z}$ ($1 \otimes \psi : \mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \otimes \mathbb{Z}$ has the nonzero element $(\frac{1}{2} + \mathbb{Z}) \otimes 1$ in its kernel — identifying $\mathbb{Q}/\mathbb{Z} = \mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}$ this is the statement that multiplication by 2 has the element 1/2 in its kernel).

(4) The direct sum of flat modules is flat (Exercise 5). In particular, $\mathbb{Q} \oplus \mathbb{Z}$ is flat. This module is neither projective nor injective (since $\mathbb{Q}$ is not projective by Exercise 8 and $\mathbb{Z}$ is not injective by Proposition 36 (cf. Exercises 3 and 4).

We close this section with an important relation between Hom and tensor products:

Theorem 43. *(Adjoint Associativity)* Let R and S be rings, let A be a right R-module, let B be an (R, S)-bimodule and let C be a right S-module. Then there is an isomorphism of abelian groups:

$$\mathrm{Hom}_S(A \otimes_R B, C) \cong \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C))$$

(the homomorphism groups are right module homomorphisms—note that $\mathrm{Hom}_S(B, C)$ has the structure of a right R-module, cf. the exercises). If $R = S$ is commutative this is an isomorphism of R-modules with the standard R-module structures.

Proof: Suppose $\varphi : A \otimes_R B \to C$ is a homomorphism. For any fixed $a \in A$ define the map $\Phi(a)$ from B to C by $\Phi(a)(b) = \varphi(a \otimes b)$. It is easy to check that $\Phi(a)$ is a homomorphism of right S-modules and that the map Φ from A to $\mathrm{Hom}_S(B, C)$ given by mapping a to $\Phi(a)$ is a homomorphism of right R-modules. Then $f(\varphi) = \Phi$ defines a group homomorphism from $\mathrm{Hom}_S(A \otimes_R B, C)$ to $\mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C))$. Conversely, suppose $\Phi : A \to \mathrm{Hom}_S(B, C)$ is a homomorphism. The map from $A \times B$ to C defined by mapping (a, b) to $\Phi(a)(c)$ is an R-balanced map, so induces a homomorphism φ from $A \otimes_R B$ to C. Then $g(\Phi) = \varphi$ defines a group homomorphism inverse to f and gives the isomorphism in the theorem.

As a first application of Theorem 43 we give an alternate proof of the first result in Theorem 39 that the tensor product is right exact in the case where $S = R$ is a commutative ring. If $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ is exact, then by Theorem 33 the sequence

$$0 \longrightarrow \mathrm{Hom}_R(N, E) \longrightarrow \mathrm{Hom}_R(M, E) \longrightarrow \mathrm{Hom}_R(L, E)$$

is exact for every R-module E. Then by Theorem 28, the sequence

$$0 \to \mathrm{Hom}_R(D, \mathrm{Hom}_R(N, E)) \to \mathrm{Hom}_R(D, \mathrm{Hom}_R(M, E)) \to \mathrm{Hom}_R(D, \mathrm{Hom}_R(L, E))$$

is exact for all D and all E. By adjoint associativity, this means the sequence

$$0 \longrightarrow \mathrm{Hom}_R(D \otimes_R N, E) \longrightarrow \mathrm{Hom}_R(D \otimes_R M, E) \longrightarrow \mathrm{Hom}_R(D \otimes_R L, E)$$

is exact for any D and all E. Then, by the second part of Theorem 33, it follows that the sequence

$$D \otimes_R L \longrightarrow D \otimes_R M \longrightarrow D \otimes_R N \longrightarrow 0$$

is exact for all D, which is the right exactness of the tensor product.

As a second application of Theorem 43 we prove that the tensor product of two projective modules over a commutative ring R is again projective (see also Exercise 9 for a more direct proof).

Corollary 44. If R is commutative then the tensor product of two projective R-modules is projective.

Proof: Let P_1 and P_2 be projective modules. Then by Corollary 32, $\mathrm{Hom}_R(P_2, \_\_)$ is an exact functor from the category of R-modules to the category of R-modules. Then the composition $\mathrm{Hom}_R(P_1, \mathrm{Hom}_R(P_2, \_\_))$ is an exact functor by the same corollary. By Theorem 43 this means that $\mathrm{Hom}_R(P_1 \otimes_R P_2, \_\_)$ is an exact functor on R-modules. It follows again from Corollary 32 that $P_1 \otimes_R P_2$ is projective.

Summary

Each of the functors $\mathrm{Hom}_R(A, \_\_)$, $\mathrm{Hom}_R(\_\_, A)$, and $A \otimes_R \_\_$, map left R-modules to abelian groups; the functor $\_\_ \otimes_R A$ maps right R-modules to abelian groups. When R is commutative all four functors map R-modules to R-modules.

(1) Let A be a left R-module. The functor $\mathrm{Hom}_R(A, \_\_)$ is covariant and left exact; the module A is projective if and only if $\mathrm{Hom}_R(A, \_\_)$ is exact (i.e., is also right exact).

(2) Let A be a left R-module. The functor $\mathrm{Hom}_R(\_\_, A)$ is contravariant and left exact; the module A is injective if and only if $\mathrm{Hom}_R(\_\_, A)$ is exact.

(3) Let A be a right R-module. The functor $A \otimes_R \_\_$ is covariant and right exact; the module A is flat if and only if $A \otimes_R \_\_$ is exact (i.e., is also left exact).

(4) Let A be a left R-module. The functor $\_\_ \otimes_R A$ is covariant and right exact; the module A is flat if and only if $\_\_ \otimes_R A$ is exact.

(5) Projective modules are flat. The $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$ is injective but not flat. The $\mathbb{Z}$-module $\mathbb{Z} \oplus \mathbb{Q}$ is flat but neither projective nor injective.

EXERCISES

Let R be a ring with 1.

1. Suppose that

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[d, "\alpha"'] & B \arrow[r, "\varphi"] \arrow[d, "\beta"'] & C \arrow[d, "\gamma"'] \\ A' \arrow[r, "\psi'"] & B' \arrow[r, "\varphi'"] & C' \end{tikzcd}$$

is a commutative diagram of groups and that the rows are exact. Prove that

(a) if φ and α are surjective, and β is injective then γ is injective. [If $c \in \ker \gamma$, show there is a $b \in B$ with $\varphi(b) = c$. Show that $\varphi'(\beta(b)) = 0$ and deduce that $\beta(b) = \psi'(a')$ for some $a' \in A'$. Show there is an $a \in A$ with $\alpha(a) = a'$ and that $\beta(\psi(a)) = \beta(b)$. Conclude that $b = \psi(a)$ and hence $c = \varphi(b) = 0$.]

(b) if ψ', α, and γ are injective, then β is injective,

(c) if φ, α, and γ are surjective, then β is surjective,

(d) if β is injective, α and γ are surjective, then γ is injective,

(e) if β is surjective, γ and ψ' are injective, then α is surjective.

2. Suppose that

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & D \arrow[d, "\delta"'] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \end{tikzcd}$$

is a commutative diagram of groups, and that the rows are exact. Prove that

(a) if α is surjective, and β, δ are injective, then γ is injective.

(b) if δ is injective, and α, γ are surjective, then β is surjective.

3. Let P_1 and P_2 be R-modules. Prove that $P_1 \oplus P_2$ is a projective R-module if and only if both P_1 and P_2 are projective.

4. Let Q_1 and Q_2 be R-modules. Prove that $Q_1 \oplus Q_2$ is an injective R-module if and only if both Q_1 and Q_2 are injective.

5. Let A_1 and A_2 be R-modules. Prove that $A_1 \oplus A_2$ is a flat R-module if and only if both A_1 and A_2 are flat. More generally, prove that an arbitrary direct sum $\sum A_i$ of R-modules is flat if and only if each A_i is flat. [Use the fact that tensor product commutes with arbitrary direct sums.]

6. Prove that the following are equivalent for a ring R:

(i) Every R-module is projective.

(ii) Every R-module is injective.