matrix A an element $\det A$ in K. If $\det A$ is considered as a function of the rows of A:

$$\det A = D(\alpha_1, \ldots, \alpha_n)$$

then D is an n-linear form on K^n.

EXAMPLE 11. It is easy to obtain an algebraic expression for the general r-linear form on the module K^n. If $\alpha_1, \ldots, \alpha_r$ are vectors in V and A is the $r \times n$ matrix with rows $\alpha_1, \ldots, \alpha_r$, then for any function L in $M^r(K^n)$,

$$\begin{aligned} L(\alpha_1, \ldots, \alpha_r) &= L\left(\sum_{j=1}^{n} A_{1j}\epsilon_j, \alpha_2, \ldots, \alpha_r\right) \\ &= \sum_{j=1}^{n} A_{1j}L(\epsilon_j, \alpha_2, \ldots, \alpha_r) \\ &= \sum_{j=1}^{n} A_{1j}L\left(\epsilon_j, \sum_{j=1}^{n} A_{2k}\epsilon_k, \ldots, \alpha_r\right) \\ &= \sum_{j=1}^{n}\sum_{k=1}^{n} A_{1j}A_{2k}L(\epsilon_j, \epsilon_k, \alpha_3, \ldots, \alpha_r) \\ &= \sum_{j,k=1}^{n} A_{1j}A_{2k}L(\epsilon_j, \epsilon_k, \alpha_3, \ldots, \alpha_r). \end{aligned}$$

If we replace $\alpha_3, \ldots, \alpha_r$ in turn by their expressions as linear combinations of the standard basis vectors, and if we write $A(i, j)$ for A_{ij}, we obtain the following:

$$L(\alpha_1, \ldots, \alpha_r) = \sum_{j_1, \ldots, j_r=1}^{n} A(1, j_1) \cdots A(r, j_r)L(\epsilon_{j_1}, \ldots \epsilon_{j_r}). \tag{5-26}$$

In (5-26), there is one term for each r-tuple $J = (j_1, \ldots, j_r)$ of positive integers between 1 and n. There are n^r such r-tuples. Thus L is completely determined by (5-26) and the particular values:

$$c_J = L(\epsilon_{j_1}, \ldots, \epsilon_{j_r})$$

assigned to the n^r elements $(\epsilon_{j_1}, \ldots, \epsilon_{j_r})$. It is also easy to see that if for each r-tuple J we choose an element c_J of K then

$$L(\alpha_1, \ldots, \alpha_r) = \sum_J A(1, j_1) \cdots A(r, j_r)c_J \tag{5-27}$$

defines an r-linear form on K^n.

Suppose that L is a multilinear function on V^r and M is a multilinear function on V^s. We define a function $L \otimes M$ on V^{r+s} by

$$(L \otimes M)(\alpha_1, \ldots, \alpha_{r+s}) = L(\alpha_1, \ldots, \alpha_r)M(\alpha_{r+1}, \ldots, \alpha_{r+s}). \tag{5-28}$$

If we think of V^{r+s} as $V^r \times V^s$, then for α in V^r and β in V^s

$$(L \otimes M)(\alpha, \beta) = L(\alpha)M(\beta).$$

It is clear that $L \otimes M$ is multilinear on V^{r+s}. The function $L \otimes M$ is called the **tensor product** of L and M. The tensor product is not commutative. In fact, $M \otimes L \neq L \otimes M$ unless $L = 0$ or $M = 0$; however, the tensor product does relate nicely to the module operations in M^r and M^s.

Lemma. *Let* L, L_1 *be* r*-linear forms on* V, *let* M, M_1 *be* s*-linear forms on* V *and let* c *be an element of* K.

(a) $(\mathrm{cL} + \mathrm{L}_1) \otimes \mathrm{M} = \mathrm{c}(\mathrm{L} \otimes \mathrm{M}) + \mathrm{L}_1 \otimes \mathrm{M}$;
(b) $\mathrm{L} \otimes (\mathrm{cM} + \mathrm{M}_1) = \mathrm{c}(\mathrm{L} \otimes \mathrm{M}) + \mathrm{L} \otimes \mathrm{M}_1$.

Proof. Exercise.

Tensoring is associative, i.e., if L, M and N are (respectively) r-, s- and t-linear forms on V, then

$$(L \otimes M) \otimes N = L \otimes (M \otimes N).$$

This is immediate from the fact that the multiplication in K is associative. Therefore, if $L_1, L_2, \ldots, L_k$ are multilinear functions on $V^{r_1}, \ldots, V^{r_k}$, then the tensor product

$$L = L_1 \otimes \cdots \otimes L_k$$

is unambiguously defined as a multilinear function on V^r, where $r = r_1 + \cdots + r_k$. We mentioned a particular case of this earlier. If $f_1, \ldots, f_r$ are linear functions on V, then the tensor product

$$L = f_1 \otimes \cdots \otimes f_r$$

is given by

$$L(\alpha_1, \ldots, \alpha_r) = f_1(\alpha_1) \cdots f_r(\alpha_r).$$

Theorem 6. *Let* K *be a commutative ring with identity. If* V *is a free* K*-module of rank* n *then* $\mathrm{M}^{\mathrm{r}}(\mathrm{V})$ *is a free* K*-module of rank* n^{r}; *in fact, if* $\{\mathrm{f}_1, \ldots, \mathrm{f}_{\mathrm{n}}\}$ *is a basis for the dual module* V^*, *the* n^{r} *tensor products*

$$\mathrm{f}_{\mathrm{j}_1} \otimes \cdots \otimes \mathrm{f}_{\mathrm{j}_{\mathrm{r}}}, \qquad 1 \leq \mathrm{j}_1 \leq \mathrm{n}, \ldots, 1 \leq \mathrm{j}_{\mathrm{r}} \leq \mathrm{n}$$

form a basis for $\mathrm{M}^{\mathrm{r}}(\mathrm{V})$.

Proof. Let $\{f_1, \ldots, f_n\}$ be an ordered basis for V^* which is dual to the basis $\{\beta_1, \ldots, \beta_n\}$ for V. For each vector α in V we have

$$\alpha = f_1(\alpha)\beta_1 + \cdots + f_n(\alpha)\beta_n.$$

We now make the calculation carried out in Example 11. If L is an r-linear form on V and $\alpha_1, \ldots, \alpha_r$ are elements of V, then by (5-26)

$$L(\alpha_1, \ldots, \alpha_r) = \sum_{j_1, \ldots, j_r} f_{j_1}(\alpha_1) \cdots f_{j_r}(\alpha_r) L(\beta_{j_1}, \ldots, \beta_{j_r}).$$

In other words,

$$L = \sum_{j_1, \ldots, j_r} L(\beta_{j_1}, \ldots, \beta_{j_r}) f_{j_1} \otimes \cdots \otimes f_{j_r}. \tag{5-29}$$

This shows that the n^r tensor products

(5-30) $$E_J = f_{j_1} \otimes \cdots \otimes f_{j_r}$$

given by the r-tuples $J = (j_1, \ldots, j_r)$ span the module $M^r(V)$. We see that the various r-forms E_J are independent, as follows. Suppose that for each J we have an element c_J in K and we form the multilinear function

(5-31) $$L = \sum_J c_J E_J.$$

Notice that if $I = (i_1, \ldots, i_r)$, then

$$E_J(\beta_{i_1}, \ldots, \beta_{i_r}) = \begin{cases} 0, & I \neq J \\ 1, & I = J. \end{cases}$$

Therefore we see from (5-31) that

(5-32) $$c_I = L(\beta_{i_1}, \ldots, \beta_{i_r}).$$

In particular, if $L = 0$ then $c_I = 0$ for each r-tuple I. ▌

Definition. *Let* L *be an* r*-linear form on a* K*-module* V. *We say that* L *is* **alternating** *if* $L(\alpha_1, \ldots, \alpha_r) = 0$ *whenever* $\alpha_i = \alpha_j$ *with* $i \neq j$.

If L is an alternating multilinear function on V^r, then

$$L(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_j, \ldots, \alpha_r) = -L(\alpha_1, \ldots, \alpha_j, \ldots, \alpha_i, \ldots, \alpha_r).$$

In other words, if we transpose two of the vectors (with different indices) in the r-tuple $(\alpha_1, \ldots, \alpha_r)$ the associated value of L changes sign. Since every permutation σ is a product of transpositions, we see that $L(\alpha_{\sigma 1}, \ldots, \alpha_{\sigma r}) = (\operatorname{sgn} \sigma)\, L(\alpha_1, \ldots, \alpha_r)$.

We denote by $\Lambda^r(V)$ the collection of all alternating r-linear forms on V. It should be clear that $\Lambda^r(V)$ is a submodule of $M^r(V)$.

EXAMPLE 12. Earlier in this chapter, we showed that on the module K^n there is precisely one alternating n-linear form D with the property that $D(\epsilon_1, \ldots, \epsilon_n) = 1$. We also showed in Theorem 2 that if L is any form in $\Lambda^n(K^n)$ then

$$L = L(\epsilon_1, \ldots, \epsilon_n) D.$$

In other words, $\Lambda^n(K^n)$ is a free K-module of rank 1. We also developed an explicit formula (5-15) for D. In terms of the notation we are now using, that formula may be written

(5-33) $$D = \sum_\sigma (\operatorname{sgn} \sigma) f_{\sigma 1} \otimes \cdots \otimes f_{\sigma n}.$$

where $f_1, \ldots, f_n$ are the standard coordinate functions on K^n and the sum is extended over the $n!$ different permutations σ of the set $\{1, \ldots, n\}$. If we write the determinant of a matrix A as

$$\det A = \sum_\sigma (\operatorname{sgn} \sigma)\, A(\sigma 1, 1) \cdots A(\sigma n, n)$$

then we obtain a different expression for D:

(5-34)
$$D(\alpha_1, \ldots, \alpha_n) = \sum_\sigma (\operatorname{sgn} \sigma) f_1(\alpha_{\sigma 1}) \cdots f_n(\alpha_{\sigma n})$$
$$= \sum_\sigma (\operatorname{sgn} \sigma) L(\alpha_{\sigma 1}, \ldots, \alpha_{\sigma n})$$

where $L = f_1 \otimes \cdots \otimes f_n$.

There is a general method for associating an alternating form with a multilinear form. If L is an r-linear form on a module V and if σ is a permutation of $\{1, \ldots, r\}$, we obtain another r-linear function L_σ by defining

$$L_\sigma(\alpha_1, \ldots, \alpha_r) = L(\alpha_{\sigma 1}, \ldots, \alpha_{\sigma r}).$$

If L happens to be alternating, then $L_\sigma = (\operatorname{sgn} \sigma)L$. Now, for each L in $M^r(V)$ we define a function $\pi_r L$ in $M^r(V)$ by

(5-35)
$$\pi_r L = \sum_\sigma (\operatorname{sgn} \sigma) L_\sigma$$

that is,

(5-36)
$$(\pi_r L)(\alpha_1, \ldots, \alpha_r) = \sum_\sigma (\operatorname{sgn} \sigma) L(\alpha_{\sigma 1}, \ldots, \alpha_{\sigma r}).$$

Lemma. *π_r is a linear transformation from $M^r(V)$ into $\Lambda^r(V)$. If L is in $\Lambda^r(V)$ then $\pi_r L = r!L$.*

Proof. Let τ be any permutation of $\{1, \ldots, r\}$. Then

$$(\pi_r L)(\alpha_{\tau 1}, \ldots, \alpha_{\tau r}) = \sum_\sigma (\operatorname{sgn} \sigma) L(\alpha_{\tau\sigma 1}, \ldots, \alpha_{\tau\sigma r})$$
$$= (\operatorname{sgn} \tau) \sum_\sigma (\operatorname{sgn} \tau\sigma) L(\alpha_{\tau\sigma 1}, \ldots, \alpha_{\tau\sigma r}).$$

As σ runs (once) over all permutations of $\{1, \ldots, r\}$, so does $\tau\sigma$. Therefore,

$$(\pi_r L)(\alpha_{\tau 1}, \ldots, \alpha_{\tau r}) = (\operatorname{sgn} \tau)(\pi_r L)(\alpha_1, \ldots, \alpha_r).$$

Thus $\pi_r L$ is an alternating form.

If L is in $\Lambda^r(V)$, then $L(\alpha_{\sigma 1}, \ldots, \alpha_{\sigma r}) = (\operatorname{sgn} \sigma) L(\alpha_1, \ldots, \alpha_r)$ for each σ; hence $\pi_r L = r!L$. ∎

In (5-33) we showed that the determinant function D in $\Lambda^n(K^n)$ is

$$D = \pi_n(f_1 \otimes \cdots \otimes f_n)$$

where $f_1, \ldots, f_n$ are the standard coordinate functions on K^n. There is an important remark we should make in connection with the last lemma. If K is a field of characteristic zero, such that $r!$ is invertible in K, then π maps $M^r(V)$ onto $\Lambda^r(V)$. In fact, in that case it is more natural from one point of view to use the map $\pi_1 = (1/r!)\pi$ rather than π, because π_1 is a projection of $M^r(V)$ onto $\Lambda^r(V)$, i.e., a linear map of $M^r(V)$ onto $\Lambda^r(V)$ such that $\pi_1(L) = L$ if and only if L is in $\Lambda^r(V)$.

Theorem 7. *Let K be a commutative ring with identity and let V be a free K-module of rank n. If r > n, then* $\Lambda^r(V) = \{0\}$. *If* $1 \leq r \leq n$, *then* $\Lambda^r(V)$ *is a free K-module of rank* $\binom{n}{r}$.

Proof. Let $\{\beta_1, \ldots, \beta_n\}$ be an ordered basis for V with dual basis $\{f_1, \ldots, f_n\}$. If L is in $M^r(V)$, we have

$$L = \sum_J L(\beta_{j_1}, \ldots, \beta_{j_r}) f_{j_1} \otimes \cdots \otimes f_{j_r} \tag{5-37}$$

where the sum extends over all r-tuples $J = (j_1, \ldots, j_r)$ of integers between 1 and n. If L is alternating, then

$$L(\beta_{j_1}, \ldots, \beta_{j_r}) = 0$$

whenever two of the subscripts j_i are the same. If $r > n$, then in each r-tuple J some integer must be repeated. Thus $\Lambda^r(V) = \{0\}$ if $r > n$.

Now suppose $1 \leq r \leq n$. If L is in $\Lambda^r(V)$, the sum in (5-37) need be extended only over the r-tuples J for which $j_1, \ldots, j_r$ are distinct, because all other terms are 0. Each r-tuple of distinct integers between 1 and n is a permutation of an r-tuple $J = (j_1, \ldots, j_r)$ such that $j_1 < \cdots < j_r$. This special type of r-tuple is called an r-**shuffle** of $\{1, \ldots, n\}$. There are

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

such shuffles.

Suppose we fix an r-shuffle J. Let L_J be the sum of all the terms in (5-37) corresponding to permutations of the shuffle J. If σ is a permutation of $\{1, \ldots, r\}$, then

$$L(\beta_{j_{\sigma 1}}, \ldots, \beta_{j_{\sigma r}}) = (\operatorname{sgn} \sigma)\, L(\beta_{j_1}, \ldots, \beta_{j_r}).$$

Thus

$$L_J = L(\beta_{j_1}, \ldots, \beta_{j_r}) D_J \tag{5-38}$$

where

$$\begin{aligned} D_J &= \sum_\sigma (\operatorname{sgn} \sigma)\, f_{j_{\sigma 1}} \otimes \cdots \otimes f_{j_{\sigma r}} \\ &= \pi_r(f_{j_1} \otimes \cdots \otimes f_{j_r}). \end{aligned} \tag{5-39}$$

We see from (5-39) that each D_J is alternating and that

$$L = \sum_{\text{shuffles } J} L(\beta_{j_1}, \ldots, \beta_{j_r}) D_J \tag{5-40}$$

for every L in $\Lambda^r(V)$. The assertion is that the $\binom{n}{r}$ forms D_J constitute a basis for $\Lambda^r(V)$. We have seen that they span $\Lambda^r(V)$. It is easy to see that they are independent, as follows. If $I = (i_1, \ldots, i_r)$ and $J = (j_1, \ldots, j_r)$ are shuffles, then

$$D_J(\beta_{i_1}, \ldots, \beta_{i_r}) = \begin{cases} 1, & I = J \\ 0, & I \neq J \end{cases}. \tag{5-41}$$

Suppose we have a scalar c_J for each shuffle and we define

$$L = \sum_J c_J D_J.$$

From (5-40) and (5-41) we obtain

$$c_I = L(\beta_{i_1}, \ldots, \beta_{i_r}).$$

In particular, if $L = 0$ then $c_I = 0$ for each shuffle I. ∎

Corollary. *If* V *is a free* K*-module of rank* n, *then* $\Lambda^n(V)$ *is a free* K*-module of rank* 1. *If* T *is a linear operator on* V, *there is a unique element* c *in* K *such that*

$$L(T\alpha_1, \ldots, T\alpha_n) = cL(\alpha_1, \ldots, \alpha_n)$$

for every alternating n*-linear form* L *on* V.

Proof. If L is in $\Lambda^n(V)$, then clearly

$$L_T(\alpha_1, \ldots, \alpha_n) = L(T\alpha_1, \ldots, T\alpha_n)$$

defines an alternating n-linear form L_T. Let M be a generator for the rank 1 module $\Lambda^n(V)$. Each L in $\Lambda^n(V)$ is uniquely expressible as $L = aM$ for some a in K. In particular, $M_T = cM$ for a certain c. For $L = aM$ we have

$$\begin{aligned} L_T &= (aM)_T \\ &= aM_T \\ &= a(cM) \\ &= c(aM) \\ &= cL. \end{aligned}$$ ∎

Of course, the element c in the last corollary is called the **determinant** of T. From (5-39) for the case $r = n$ (when there is only one shuffle $J = (1, \ldots, n)$) we see that the determinant of T is the determinant of the matrix which represents T in any ordered basis $\{\beta_1, \ldots, \beta_n\}$. Let us see why. The representing matrix has i, j entry

$$A_{ij} = f_j(T\beta_i)$$

so that

$$\begin{aligned} D_J(T\beta_1, \ldots, T\beta_n) &= \sum_\sigma (\operatorname{sgn} \sigma) A(1, \sigma 1) \cdots A(n, \sigma n) \\ &= \det A. \end{aligned}$$

On the other hand,

$$\begin{aligned} D_J(T\beta_1, \ldots, T\beta_n) &= (\det T)\, D_J(\beta_1, \ldots, \beta_n) \\ &= \det T. \end{aligned}$$

The point of these remarks is that via Theorem 7 and its corollary we obtain a definition of the determinant of a linear operator which does not presume knowledge of determinants of matrices. Determinants of matrices can be defined in terms of determinants of operators instead of the other way around.