

The following are easy consequences of the definition.

1. Any set which contains a linearly dependent set is linearly dependent.
2. Any subset of a linearly independent set is linearly independent.
3. Any set which contains the 0 vector is linearly dependent; for $1 \cdot 0 = 0$.
4. A set S of vectors is linearly independent if and only if each finite subset of S is linearly independent, i.e., if and only if for any distinct vectors $\alpha_1, \dots, \alpha_n$ of S , $c_1\alpha_1 + \dots + c_n\alpha_n = 0$ implies each $c_i = 0$.

Definition. Let V be a vector space. A **basis** for V is a linearly independent set of vectors in V which spans the space V . The space V is **finite-dimensional** if it has a finite basis.

EXAMPLE 12. Let F be a subfield of the complex numbers. In F^3 the vectors

$$\begin{aligned}\alpha_1 &= (-3, 0, -3) \\ \alpha_2 &= (-1, 1, 2) \\ \alpha_3 &= (4, 2, -2) \\ \alpha_4 &= (2, 1, 1)\end{aligned}$$

are linearly dependent, since

$$2\alpha_1 + 2\alpha_2 - \alpha_3 + 0 \cdot \alpha_4 = 0.$$

The vectors

$$\begin{aligned}\epsilon_1 &= (1, 0, 0) \\ \epsilon_2 &= (0, 1, 0) \\ \epsilon_3 &= (0, 0, 1)\end{aligned}$$

are linearly independent

EXAMPLE 13. Let F be a field and in F^n let S be the subset consisting of the vectors $\epsilon_1, \epsilon_2, \dots, \epsilon_n$ defined by

$$\begin{aligned}\epsilon_1 &= (1, 0, 0, \dots, 0) \\ \epsilon_2 &= (0, 1, 0, \dots, 0) \\ &\quad \ddots \\ \epsilon_n &= (0, 0, 0, \dots, 1).\end{aligned}$$

Let x_1, x_2, \dots, x_n be scalars in F and put $\alpha = x_1\epsilon_1 + x_2\epsilon_2 + \dots + x_n\epsilon_n$. Then

$$(2-12) \qquad \alpha = (x_1, x_2, \dots, x_n).$$

This shows that $\epsilon_1, \dots, \epsilon_n$ span F^n . Since $\alpha = 0$ if and only if $x_1 = x_2 = \dots = x_n = 0$, the vectors $\epsilon_1, \dots, \epsilon_n$ are linearly independent. The set $S = \{\epsilon_1, \dots, \epsilon_n\}$ is accordingly a basis for F^n . We shall call this particular basis the **standard basis** of F^n .

EXAMPLE 14. Let P be an invertible $n \times n$ matrix with entries in the field F . Then P_1, \dots, P_n , the columns of P , form a basis for the space of column matrices, $F^{n \times 1}$. We see that as follows. If X is a column matrix, then

$$PX = x_1P_1 + \cdots + x_nP_n.$$

Since $PX = 0$ has only the trivial solution $X = 0$, it follows that $\{P_1, \dots, P_n\}$ is a linearly independent set. Why does it span $F^{n \times 1}$? Let Y be any column matrix. If $X = P^{-1}Y$, then $Y = PX$, that is,

$$Y = x_1P_1 + \cdots + x_nP_n.$$

So $\{P_1, \dots, P_n\}$ is a basis for $F^{n \times 1}$.

EXAMPLE 15. Let A be an $m \times n$ matrix and let S be the solution space for the homogeneous system $AX = 0$ (Example 7). Let R be a row-reduced echelon matrix which is row-equivalent to A . Then S is also the solution space for the system $RX = 0$. If R has r non-zero rows, then the system of equations $RX = 0$ simply expresses r of the unknowns x_1, \dots, x_n in terms of the remaining $(n - r)$ unknowns x_j . Suppose that the leading non-zero entries of the non-zero rows occur in columns k_1, \dots, k_r . Let J be the set consisting of the $n - r$ indices different from k_1, \dots, k_r :

$$J = \{1, \dots, n\} - \{k_1, \dots, k_r\}.$$

The system $RX = 0$ has the form

$$\begin{array}{rcl} x_{k_1} + \sum_j c_{1j}x_j & = & 0 \\ \vdots & \vdots & \vdots \\ x_{k_r} + \sum_j c_{rj}x_j & = & 0 \end{array}$$

where the c_{ij} are certain scalars. All solutions are obtained by assigning (arbitrary) values to those x_j 's with j in J and computing the corresponding values of x_{k_1}, \dots, x_{k_r} . For each j in J , let E_j be the solution obtained by setting $x_j = 1$ and $x_i = 0$ for all other i in J . We assert that the $(n - r)$ vectors E_j , j in J , form a basis for the solution space.

Since the column matrix E_j has a 1 in row j and zeros in the rows indexed by other elements of J , the reasoning of Example 13 shows us that the set of these vectors is linearly independent. That set spans the solution space, for this reason. If the column matrix T , with entries t_1, \dots, t_n , is in the solution space, the matrix

$$N = \sum_j t_j E_j$$

is also in the solution space and is a solution such that $x_j = t_j$ for each j in J . The solution with that property is unique; hence, $N = T$ and T is in the span of the vectors E_j .

EXAMPLE 16. We shall now give an example of an infinite basis. Let F be a subfield of the complex numbers and let V be the space of polynomial functions over F . Recall that these functions are the functions from F into F which have a rule of the form

$$f(x) = c_0 + c_1x + \cdots + c_nx^n.$$

Let $f_k(x) = x^k$, $k = 0, 1, 2, \dots$. The (infinite) set $\{f_0, f_1, f_2, \dots\}$ is a basis for V . Clearly the set spans V , because the function f (above) is

$$f = c_0f_0 + c_1f_1 + \cdots + c_nf_n.$$

The reader should see that this is virtually a repetition of the definition of polynomial function, that is, a function f from F into F is a polynomial function if and only if there exists an integer n and scalars c_0, \dots, c_n such that $f = c_0f_0 + \cdots + c_nf_n$. Why are the functions independent? To show that the set $\{f_0, f_1, f_2, \dots\}$ is independent means to show that each finite subset of it is independent. It will suffice to show that, for each n , the set $\{f_0, \dots, f_n\}$ is independent. Suppose that

$$c_0f_0 + \cdots + c_nf_n = 0.$$

This says that

$$c_0 + c_1x + \cdots + c_nx^n = 0$$

for every x in F ; in other words, every x in F is a root of the polynomial $f(x) = c_0 + c_1x + \cdots + c_nx^n$. We assume that the reader knows that a polynomial of degree n with complex coefficients cannot have more than n distinct roots. It follows that $c_0 = c_1 = \cdots = c_n = 0$.

We have exhibited an infinite basis for V . Does that mean that V is not finite-dimensional? As a matter of fact it does; however, that is not immediate from the definition, because for all we know V might also have a finite basis. That possibility is easily eliminated. (We shall eliminate it in general in the next theorem.) Suppose that we have a finite number of polynomial functions g_1, \dots, g_r . There will be a largest power of x which appears (with non-zero coefficient) in $g_1(x), \dots, g_r(x)$. If that power is k , clearly $f_{k+1}(x) = x^{k+1}$ is not in the linear span of g_1, \dots, g_r . So V is not finite-dimensional.

A final remark about this example is in order. Infinite bases have nothing to do with ‘infinite linear combinations.’ The reader who feels an irresistible urge to inject power series

$$\sum_{k=0}^{\infty} c_kx^k$$

into this example should study the example carefully again. If that does not effect a cure, he should consider restricting his attention to finite-dimensional spaces from now on.

Theorem 4. Let V be a vector space which is spanned by a finite set of vectors $\beta_1, \beta_2, \dots, \beta_m$. Then any independent set of vectors in V is finite and contains no more than m elements.

Proof. To prove the theorem it suffices to show that every subset S of V which contains more than m vectors is linearly dependent. Let S be such a set. In S there are distinct vectors $\alpha_1, \alpha_2, \dots, \alpha_n$ where $n > m$. Since β_1, \dots, β_m span V , there exist scalars A_{ij} in F such that

$$\alpha_j = \sum_{i=1}^m A_{ij}\beta_i.$$

For any n scalars x_1, x_2, \dots, x_n we have

$$\begin{aligned} x_1\alpha_1 + \cdots + x_n\alpha_n &= \sum_{j=1}^n x_j\alpha_j \\ &= \sum_{j=1}^n x_j \sum_{i=1}^m A_{ij}\beta_i \\ &= \sum_{j=1}^n \sum_{i=1}^m (A_{ij}x_j)\beta_i \\ &= \sum_{i=1}^m \left(\sum_{j=1}^n A_{ij}x_j \right) \beta_i. \end{aligned}$$

Since $n > m$, Theorem 6 of Chapter 1 implies that there exist scalars x_1, x_2, \dots, x_n not all 0 such that

$$\sum_{j=1}^n A_{ij}x_j = 0, \quad 1 \leq i \leq m.$$

Hence $x_1\alpha_1 + x_2\alpha_2 + \cdots + x_n\alpha_n = 0$. This shows that S is a linearly dependent set. ■

Corollary 1. If V is a finite-dimensional vector space, then any two bases of V have the same (finite) number of elements.

Proof. Since V is finite-dimensional, it has a finite basis

$$\{\beta_1, \beta_2, \dots, \beta_m\}.$$

By Theorem 4 every basis of V is finite and contains no more than m elements. Thus if $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is a basis, $n \leq m$. By the same argument, $m \leq n$. Hence $m = n$. ■

This corollary allows us to define the **dimension** of a finite-dimensional vector space as the number of elements in a basis for V . We shall denote the dimension of a finite-dimensional space V by $\dim V$. This allows us to reformulate Theorem 4 as follows.

Corollary 2. Let V be a finite-dimensional vector space and let $n = \dim V$. Then

- (a) any subset of V which contains more than n vectors is linearly dependent;
 (b) no subset of V which contains fewer than n vectors can span V .

EXAMPLE 17. If F is a field, the dimension of F^n is n , because the standard basis for F^n contains n vectors. The matrix space $F^{m \times n}$ has dimension mn . That should be clear by analogy with the case of F^n , because the mn matrices which have a 1 in the i, j place with zeros elsewhere form a basis for $F^{m \times n}$. If A is an $m \times n$ matrix, then the solution space for A has dimension $n - r$, where r is the number of non-zero rows in a row-reduced echelon matrix which is row-equivalent to A . See Example 15.

If V is any vector space over F , the zero subspace of V is spanned by the vector 0, but $\{0\}$ is a linearly dependent set and not a basis. For this reason, we shall agree that the zero subspace has dimension 0. Alternatively, we could reach the same conclusion by arguing that the empty set is a basis for the zero subspace. The empty set spans $\{0\}$, because the intersection of all subspaces containing the empty set is $\{0\}$, and the empty set is linearly independent because it contains no vectors.

Lemma. Let S be a linearly independent subset of a vector space V . Suppose β is a vector in V which is not in the subspace spanned by S . Then the set obtained by adjoining β to S is linearly independent.

Proof. Suppose $\alpha_1, \dots, \alpha_m$ are distinct vectors in S and that

$$c_1\alpha_1 + \cdots + c_m\alpha_m + b\beta = 0.$$

Then $b = 0$; for otherwise,

$$\beta = \left(-\frac{c_1}{b}\right)\alpha_1 + \cdots + \left(-\frac{c_m}{b}\right)\alpha_m$$

and β is in the subspace spanned by S . Thus $c_1\alpha_1 + \cdots + c_m\alpha_m = 0$, and since S is a linearly independent set each $c_i = 0$. ■

Theorem 5. If W is a subspace of a finite-dimensional vector space V , every linearly independent subset of W is finite and is part of a (finite) basis for W .

Proof. Suppose S_0 is a linearly independent subset of W . If S is a linearly independent subset of W containing S_0 , then S is also a linearly independent subset of V ; since V is finite-dimensional, S contains no more than $\dim V$ elements.

We extend S_0 to a basis for W , as follows. If S_0 spans W , then S_0 is a basis for W and we are done. If S_0 does not span W , we use the preceding lemma to find a vector β_1 in W such that the set $S_1 = S_0 \cup \{\beta_1\}$ is independent. If S_1 spans W , fine. If not, apply the lemma to obtain a vector β_2

in W such that $S_2 = S_1 \cup \{\beta_2\}$ is independent. If we continue in this way, then (in not more than $\dim V$ steps) we reach a set

$$S_m = S_0 \cup \{\beta_1, \dots, \beta_m\}$$

which is a basis for W . ■

Corollary 1. *If W is a proper subspace of a finite-dimensional vector space V , then W is finite-dimensional and $\dim W < \dim V$.*

Proof. We may suppose W contains a vector $\alpha \neq 0$. By Theorem 5 and its proof, there is a basis of W containing α which contains no more than $\dim V$ elements. Hence W is finite-dimensional, and $\dim W \leq \dim V$. Since W is a proper subspace, there is a vector β in V which is not in W . Adjoining β to any basis of W , we obtain a linearly independent subset of V . Thus $\dim W < \dim V$. ■

Corollary 2. *In a finite-dimensional vector space V every non-empty linearly independent set of vectors is part of a basis.*

Corollary 3. *Let A be an $n \times n$ matrix over a field F , and suppose the row vectors of A form a linearly independent set of vectors in F^n . Then A is invertible.*

Proof. Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be the row vectors of A , and suppose W is the subspace of F^n spanned by $\alpha_1, \alpha_2, \dots, \alpha_n$. Since $\alpha_1, \alpha_2, \dots, \alpha_n$ are linearly independent, the dimension of W is n . Corollary 1 now shows that $W = F^n$. Hence there exist scalars B_{ij} in F such that

$$\epsilon_i = \sum_{j=1}^n B_{ij} \alpha_j, \quad 1 \leq i \leq n$$

where $\{\epsilon_1, \epsilon_2, \dots, \epsilon_n\}$ is the standard basis of F^n . Thus for the matrix B with entries B_{ij} we have

$$BA = I. \quad \blacksquare$$

Theorem 6. *If W_1 and W_2 are finite-dimensional subspaces of a vector space V , then $W_1 + W_2$ is finite-dimensional and*

$$\dim W_1 + \dim W_2 = \dim (W_1 \cap W_2) + \dim (W_1 + W_2).$$

Proof. By Theorem 5 and its corollaries, $W_1 \cap W_2$ has a finite basis $\{\alpha_1, \dots, \alpha_k\}$ which is part of a basis

$$\{\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_m\} \quad \text{for } W_1$$

and part of a basis

$$\{\alpha_1, \dots, \alpha_k, \gamma_1, \dots, \gamma_n\} \quad \text{for } W_2.$$

The subspace $W_1 + W_2$ is spanned by the vectors

$$\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_m, \gamma_1, \dots, \gamma_n$$