(3) If $n = 5$, the partitions of 5 and corresponding representatives of the conjugacy classes (with 1-cycles not written) are as given in the following table:

Partition of 5	Representative of Conjugacy Class
1, 1, 1, 1, 1	1
1, 1, 1, 2	(1 2)
1, 1, 3	(1 2 3)
1, 4	(1 2 3 4)
5	(1 2 3 4 5)
1, 2, 2	(1 2)(3 4)
2, 3	(1 2)(3 4 5)

Proposition 11 and Proposition 6 can be used to exhibit the centralizers of some elements in S_n. For example, if σ is an m-cycle in S_n, then the number of conjugates of σ (i.e., the number of m-cycles) is

$$\frac{n \cdot (n-1) \cdots (n-m+1)}{m}.$$

By Proposition 6 this is the index of the centralizer of σ: $\dfrac{|S_n|}{|C_{S_n}(\sigma)|}$. Since $|S_n| = n!$ we obtain

$$|C_{S_n}(\sigma)| = m \cdot (n-m)!.$$

The element σ certainly commutes with $1, \sigma, \sigma^2, \ldots, \sigma^{m-1}$. It also commutes with any permutation in S_n whose cycles are disjoint from σ and there are $(n-m)!$ permutations of this type (the full symmetric group on the numbers not appearing in σ). The product of elements of these two types already accounts for $m \cdot (n-m)!$ elements commuting with σ. By the order computation above, this is the full centralizer of σ in S_n. Explicitly,

$$\text{if } \sigma \text{ is an } m\text{-cycle in } S_n, \text{ then } C_{S_n}(\sigma) = \{\sigma^i \tau \mid 0 \leq i \leq m-1,\ \tau \in S_{n-m}\}$$

where S_{n-m} denotes the subgroup of S_n which fixes all integers appearing in the m-cycle σ (and is the identity subgroup if $m = n$ or $m = n-1$).

For example, the centralizer of $\sigma = (1\ 3\ 5)$ in S_7 is the subgroup

$$\{(1\ 3\ 5)^i \tau \mid i = 0, 1 \text{ or } 2, \text{ and } \tau \text{ fixes } 1, 3 \text{ and } 5\}.$$

Note that $\tau \in S_A$ where $A = \{2, 4, 6, 7\}$, so there are $4!$ choices for τ and the centralizer has order $3 \cdot 4! = 72$.

We shall discuss centralizers of other elements of S_n in the next exercises and in Chapter 5.

We can use this discussion of the conjugacy classes in S_n to give a combinatorial proof of the simplicity of A_5. We first observe that normal subgroups of a group G are the union of conjugacy classes of G, i.e.,

$$\text{if } H \trianglelefteq G, \text{ then for every conjugacy class } \mathcal{K} \text{ of } G \text{ either } \mathcal{K} \subseteq H \text{ or } \mathcal{K} \cap H = \emptyset.$$

This is because if $x \in \mathcal{K} \cap H$, then $gxg^{-1} \in gHg^{-1}$ for all $g \in G$. Since H is normal, $gHg^{-1} = H$, so that H contains all the conjugates of x, i.e., $\mathcal{K} \subseteq H$.

Theorem 12. A_5 is a simple group.

Proof: We first work out the conjugacy classes of A_5 and their orders. Proposition 11 does not apply directly since two elements of the same cycle type (which are conjugate in S_5) need *not* be conjugate in A_5. Exercises 19 to 22 analyze the relation of classes in S_n to classes in A_n in detail.

We have already seen that representatives of the cycle types of even permutations can be taken to be

$$1, \quad (1\,2\,3), \quad (1\,2\,3\,4\,5) \quad \text{and} \quad (1\,2)(3\,4).$$

The centralizers of 3-cycles and 5-cycles in S_5 were determined above, and checking which of these elements are contained in A_5 we see that

$$C_{A_5}((1\,2\,3)) = \langle\, (1\,2\,3)\,\rangle \quad \text{and} \quad C_{A_5}((1\,2\,3\,4\,5)) = \langle\, (1\,2\,3\,4\,5)\,\rangle.$$

These groups have orders 3 and 5 (index 20 and 12), respectively, so there are 20 distinct conjugates of $(1\,2\,3)$ and 12 distinct conjugates of $(1\,2\,3\,4\,5)$ in A_5. Since there are a total of twenty 3-cycles in S_5 (Exercise 16, Section 1.3) and all of these lie in A_5, we see that

all twenty 3-cycles are conjugate in A_5.

There are a total of twenty-four 5-cycles in A_5 but only 12 distinct conjugates of the 5-cycle $(1\,2\,3\,4\,5)$. Thus some 5-cycle, σ, is *not* conjugate to $(1\,2\,3\,4\,5)$ in A_5 (in fact, $(1\,3\,5\,2\,4)$ is not conjugate in A_5 to $(1\,2\,3\,4\,5)$ since the method of proof in Proposition 11 shows that any element of S_5 conjugating $(1\,2\,3\,4\,5)$ into $(1\,3\,5\,2\,4)$ must be an odd permutation). As above we see that σ also has 12 distinct conjugates in A_5, hence

the 5-cycles lie in two conjugacy classes in A_5, each of which has 12 elements.

Since the 3-cycles and 5-cycles account for all the nonidentity elements of odd order, the 15 remaining nonidentity elements of A_5 must have order 2 and therefore have cycle type (2,2). It is easy to see that $(1\,2)(3\,4)$ commutes with $(1\,3)(2\,4)$ but does not commute with any element of odd order in A_5. It follows that $|C_{A_5}((12)(34))| = 4$. Thus $(1\,2)(3\,4)$ has 15 distinct conjugates in A_5, hence

all 15 elements of order 2 in A_5 are conjugate to $(1\,2)(3\,4)$.

In summary, the conjugacy classes of A_5 have orders $1, 15, 20, 12$ and 12.

Now, suppose H were a normal subgroup of A_5. Then as we observed above, H would be the union of conjugacy classes of A_5. Then the order of H would be both a divisor of 60 (the order of A_5) and be the sum of some collection of the integers $\{1, 12, 12, 15, 20\}$ (the sizes of the conjugacy classes in A_5). A quick check shows the only possibilities are $|H| = 1$ or $|H| = 60$, so that A_5 has no proper, nontrivial normal subgroups.

Right Group Actions

As noted in Section 1.7, in the definition of an action the group elements appear to the left of the set elements and so our notion of an action might more precisely be termed a *left group action*. One can analogously define the notion of a *right group action* of the group G on the nonempty set A as a map from $A \times G$ to A, denoted by $a \cdot g$ for $a \in A$ and $g \in G$, that satisfies the axioms:

(1) $(a \cdot g_1) \cdot g_2 = a \cdot (g_1 g_2)$ for all $a \in A$, and $g_1, g_2 \in G$, and
(2) $a \cdot 1 = a$ for all $a \in A$.

In much of the literature on group theory, conjugation is written as a right group action using the following notation:

$$a^g = g^{-1}ag \qquad \text{for all } g, a \in G.$$

Similarly, for subsets S of G one defines $S^g = g^{-1}Sg$. In this notation the two axioms for a right action are verified as follows:

$$(a^{g_1})^{g_2} = g_2^{-1}(g_1^{-1}ag_1)g_2 = (g_1g_2)^{-1}a(g_1g_2) = a^{(g_1g_2)}$$

and

$$a^1 = 1^{-1}a1 = a$$

for all $g_1, g_2, a \in G$. Thus the two axioms for this right action of a group on itself take the form of the familiar "laws of exponentiation." (Note that the integer power a^n of a group element a is easily distinguished from the conjugate a^g of a by the nature of the exponent: $n \in \mathbb{Z}$ but $g \in G$.) Because conjugation is so ubiquitous in the theory of groups, this notation is a useful and efficient shorthand (as opposed to always writing gag^{-1} or $g \cdot a$ for action on the left by conjugation).

For arbitrary group actions it is an easy exercise to check that if we are given a left group action of G on A then the map $A \times G \to A$ defined by $a \cdot g = g^{-1} \cdot a$ is a right group action. Conversely, given a right group action of G on A we can form a left group action by $g \cdot a = a \cdot g^{-1}$. Call these pairs *corresponding group actions*. Put another way, for corresponding group actions, g acts on the left in the same way that g^{-1} acts on the right. This is particularly transparent for the action of conjugation because the "left conjugate of a by g," namely gag^{-1}, is the same group element as the "right conjugate of a by g^{-1}," namely $a^{g^{-1}}$. Thus two elements or subsets of a group are "left conjugate" if and only if they are "right conjugate," and so the relation "conjugacy" is the same for the left and right corresponding actions. More generally, it is also an exercise (Exercise 1) to see that for any corresponding left and right actions the orbits are the same.

We have consistently used left actions since they are compatible with the notation of applying functions on the left (i.e., with the notation $\varphi(g)$); in this way left multiplication on the left cosets of a subgroup is a left action. Similarly, right multiplication on the right cosets of a subgroup is a right action and the associated permutation representation φ is a homomorphism provided the function $\varphi : G \to S_A$ is written on the right as $(g_1g_2)\varphi$ (and also provided permutations in S_A are written on the right as functions from A to itself). There are instances where a set admits two actions by a group G: one naturally on the left and the other on the right, so that it is useful to be comfortable with both types of actions.

EXERCISES

Let G be a group.

1. Suppose G has a left action on a set A, denoted by $g \cdot a$ for all $g \in G$ and $a \in A$. Denote the corresponding right action on A by $a \cdot g$. Prove that the (equivalence) relations $\sim$ and $\sim'$ defined by
$$a \sim b \qquad \text{if and only if} \qquad a = g \cdot b \quad \text{for some } g \in G$$
and
$$a \sim' b \qquad \text{if and only if} \qquad a = b \cdot g \quad \text{for some } g \in G$$
are the same relation (i.e., $a \sim b$ if and only if $a \sim' b$).
2. Find all conjugacy classes and their sizes in the following groups:
(a) D_8 **(b)** Q_8 **(c)** A_4.
3. Find all the conjugacy classes and their sizes in the following groups:
(a) $Z_2 \times S_3$ **(b)** $S_3 \times S_3$ **(c)** $Z_3 \times A_4$.
4. Prove that if $S \subseteq G$ and $g \in G$ then $gN_G(S)g^{-1} = N_G(gSg^{-1})$ and $gC_G(S)g^{-1} = C_G(gSg^{-1})$.
5. If the center of G is of index n, prove that every conjugacy class has at most n elements.
6. Assume G is a non-abelian group of order 15. Prove that $Z(G) = 1$. Use the fact that $\langle g \rangle \leq C_G(g)$ for all $g \in G$ to show that there is at most one possible class equation for G. [Use Exercise 36, Section 3.1.]
7. For $n = 3$, 4, 6 and 7 make lists of the partitions of n and give representatives for the corresponding conjugacy classes of S_n.
8. Prove that $Z(S_n) = 1$ for all $n \geq 3$.
9. Show that $|C_{S_n}((1\,2)(3\,4))| = 8 \cdot (n-4)!$ for all $n \geq 4$. Determine the elements in this centralizer explicitly.
10. Let σ be the 5-cycle $(1\ 2\ 3\ 4\ 5)$ in S_5. In each of (a) to (c) find an explicit element $\tau \in S_5$ which accomplishes the specified conjugation:
(a) $\tau\sigma\tau^{-1} = \sigma^2$
(b) $\tau\sigma\tau^{-1} = \sigma^{-1}$
(c) $\tau\sigma\tau^{-1} = \sigma^{-2}$.
11. In each of (a) – (d) determine whether σ_1 and σ_2 are conjugate. If they are, give an explicit permutation τ such that $\tau\sigma_1\tau^{-1} = \sigma_2$.
(a) $\sigma_1 = (1\ 2)(3\ 4\ 5)$ and $\sigma_2 = (1\ 2\ 3)(4\ 5)$
(b) $\sigma_1 = (1\ 5)(3\ 7\ 2)(10\ 6\ 8\ 11)$ and $\sigma_2 = (3\ 7\ 5\ 10)(4\ 9)(13\ 11\ 2)$
(c) $\sigma_1 = (1\ 5)(3\ 7\ 2)(10\ 6\ 8\ 11)$ and $\sigma_2 = \sigma_1^3$
(d) $\sigma_1 = (1\ 3)(2\ 4\ 6)$ and $\sigma_2 = (3\ 5)(2\ 4)(5\ 6)$.
12. Find a representative for each conjugacy class of elements of order 4 in S_8 and in S_{12}.
13. Find all finite groups which have exactly two conjugacy classes.
14. In Exercise 1 of Section 2 two labellings of the elements $\{1, a, b, c\}$ of the Klein 4-group V were chosen to give two versions of the left regular representation of V into S_4. Let π_1 be the version of regular representation obtained in part (a) of that exercise and let π_2 be the version obtained via the labelling in part (b). Let $\tau = (2\ 4)$. Show that $\tau \circ \pi_1(g) \circ \tau^{-1} = \pi_2(g)$ for each $g \in V$ (i.e., conjugation by τ sends the image of π_1 to the image of π_2 elementwise).