

**LEMMA 3.3.** If  $A$  is an  $m \times n$  matrix and  $B$  an  $n \times p$  matrix, then we have

$$(AB)_i = A_i B.$$

That is, the  $i$ th row of the product  $AB$  is equal to the product of row matrix  $A_i$  with  $B$ .

*Proof.* Let  $B^j$  denote the  $j$ th column of  $B$  and let  $C = AB$ . Then the sum in (3.12) can be regarded as the dot product of the  $i$ th row of  $A$  with the  $j$ th column of  $B$ ,

$$c_{ij} = A_i \cdot B^j.$$

Therefore the  $i$ th row  $C_i$  is the row matrix

$$C_i = [A_i \cdot B^1, A_i \cdot B^2, \dots, A_i \cdot B^p].$$

But this is also the result of matrix multiplication of row matrix  $A_i$  with  $B$ , since

$$A_i B = [a_{i1}, \dots, a_{in}] \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix} = [A_i \cdot B^1, \dots, A_i \cdot B^p].$$

Therefore  $C_i = A_i B$ , which proves the lemma.

**THEOREM 3.4. PRODUCT FORMULA FOR DETERMINANTS.** For any  $n \times n$  matrices  $A$  and  $B$  we have

$$\det(AB) = (\det A)(\det B).$$

*Proof.* Since  $(AB)_i = A_i B$ , we are to prove that

$$d(A_1 B, \dots, A_n B) = d(A_1, \dots, A_n) d(B_1, \dots, B_n).$$

Using the lemma again we also have  $B_i = (ZB) = I_i B$ , where  $Z$  is the  $n \times n$  identity matrix. Therefore  $d(B_1, \dots, B_n) = d(I_1 B, \dots, I_n B)$ , and we are to prove that

$$d(A_1 B, \dots, A_n B) = d(A_1, \dots, A_n) d(I_1 B, \dots, I_n B).$$

We keep  $B$  fixed and introduce a function  $\mathbf{f}$  defined by the formula

$$f(A_1, \dots, A_n) = d(A_1 B, \dots, A_n B).$$

The equation we wish to prove states that

$$(3.13) \quad f(A_1, \dots, A_n) = d(A_1, \dots, A_n) f(I_1, \dots, I_n).$$

But now it is a simple matter to verify that  $f$  satisfies Axioms 1, 2, and 3 for a determinant function so, by the uniqueness theorem, Equation (3.13) holds for every matrix  $A$ . This completes the proof.

Applications of the product formula are given in the next two sections.

### 3.8 The determinant of the inverse of a nonsingular matrix

We recall that a square matrix  $A$  is called nonsingular if it has a left inverse  $B$  such that  $BA = I$ . If a left inverse exists it is unique and is also a right inverse,  $AB = I$ . We denote the inverse by  $A^{-1}$ . The relation between  $\det A$  and  $\det A^{-1}$  is as natural as one could expect.

**THEOREM 3.5.** **If** matrix  $A$  is nonsingular, then  $\det A \neq 0$  and we have

$$(3.14) \quad \det A^{-1} = \frac{1}{\det A}.$$

*Proof.* From the product formula we have

$$(\det A)(\det A^{-1}) = \det (AA^{-1}) = \det I = 1.$$

Hence  $\det A \neq 0$  and (3.14) holds.

Theorem 3.5 shows that nonvanishing of  $\det A$  is a necessary condition for  $A$  to be nonsingular. Later we will prove that this condition is also sufficient. That is, if  $\det A \neq 0$  then  $A^{-1}$  exists.

### 3.9 Determinants and independence of vectors

A simple criterion for testing independence of vectors can be deduced from Theorem 3.5.

**THEOREM 3.6.** A set of  $n$  vectors  $A_1, \dots, A_n$  in  $n$ -space is independent if and only if  $d(A_1, \dots, A_n) \neq 0$ .

*Proof.* We already proved in Theorem 3.2(e) that dependence implies  $d(A_1, \dots, A_n) = 0$ . To prove the converse, we assume that  $A_1, \dots, A_n$  are independent and prove that  $d(A_1, \dots, A_n) \neq 0$ .

Let  $V_n$  denote the linear space of  $n$ -tuples of scalars. Since  $A_1, \dots, A_n$  are  $n$  independent elements in an  $n$ -dimensional space they form a basis for  $V_n$ . By Theorem 2.12 there is a linear transformation  $T: V_n \rightarrow V_n$  which maps these  $n$  vectors onto the unit coordinate vectors,

$$T(A_k) = I_k \quad \text{for } k = 1, 2, \dots, n.$$

Therefore there is an  $n \times n$  matrix  $B$  such that

$$A_k B = I, \quad \text{for } k = 1, 2, \dots, n.$$

But by Lemma 3.3 we have  $A_k B = (AB)$ , where  $A$  is the matrix with rows  $A_1, \dots, A_k$ . Hence  $AB = I$ , so  $A$  is nonsingular and  $\det A \neq 0$ .

### 3.10 The determinant of a block-diagonal matrix

A square matrix  $C$  of the form

$$C = \begin{bmatrix} A & \\ O & B \end{bmatrix},$$

where  $A$  and  $B$  are square matrices and each  $O$  denotes a matrix of zeros, is called a *block-diagonal matrix* with two diagonal blocks  $A$  and  $B$ . An example is the  $5 \times 5$  matrix

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 4 & 5 & 6 \\ 0 & 0 & 7 & 8 & 9 \end{bmatrix}.$$

The diagonal blocks in this case are

$$A = \begin{bmatrix} 1 & & \\ & 1 & \\ 0 & & \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

The next theorem shows that the determinant of a block-diagonal matrix is equal to the product of the determinants of its diagonal blocks.

**THEOREM 3.7.** *For any two square matrices  $A$  and  $B$  we have*

$$(3.15) \quad \det \begin{bmatrix} A & O \\ O & B \end{bmatrix} = (\det A)(\det B).$$

*Proof.* Assume  $A$  is  $n \times n$  and  $B$  is  $m \times m$ . We note that the given block-diagonal matrix can be expressed as a product of the form

$$\begin{bmatrix} A & O \\ O & B \end{bmatrix} = \begin{bmatrix} A & O \\ O & I_m \end{bmatrix} \begin{bmatrix} I_n & O \\ O & B \end{bmatrix}$$

where  $I_n$  and  $I_m$  are identity matrices of orders  $n$  and  $m$ , respectively. Therefore, by the product formula for determinants we have

$$(3.16) \quad \det \begin{bmatrix} A & O \\ O & B \end{bmatrix} = \det \begin{bmatrix} A & O \\ O & I_m \end{bmatrix} \det \begin{bmatrix} I_n & O \\ O & B \end{bmatrix}.$$

Now we regard the determinant  $\det \begin{bmatrix} A & O \\ O & I_m \end{bmatrix}$  as a function of the  $n$  rows of  $A$ . This is possible because of the block of zeros in the upper right-hand corner. It is easily verified that this function satisfies all four axioms for a determinant function of order  $n$ . Therefore, by the uniqueness theorem, we must have

$$\det \begin{bmatrix} A & O \\ O & I_m \end{bmatrix} = \det A.$$

A similar argument shows that  $\det \begin{bmatrix} I_n & O \\ O & B \end{bmatrix} = \det B$ . Hence (3.16) implies (3.15).

### 3.11 Exercises

- For each of the following statements about square matrices, give a proof or exhibit a counter example.
  - $\det(A + B) = \det A + \det B$ .
  - $\det \{(A + B)^2\} = \{\det(A + B)\}^2$
  - $\det \{(A + B)^2\} = \det(A^2 + 2AB + B^2)$
  - $\det \{(A + B)^2\} = \det(A^2 + B^2)$ .
- (a) Extend 'Theorem 3.7' to block-diagonal matrices with three diagonal blocks:

$$\det \begin{bmatrix} A & O & O \\ O & B & O \\ O & O & C \end{bmatrix} = (\det A)(\det B)(\det C).$$

- State and prove a generalization for block-diagonal matrices with any number of diagonal blocks.

- Let  $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a & b & c & d \\ e & f & g & h \end{bmatrix}$ ,  $B = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ . Prove that  $\det A = \det \begin{bmatrix} c & d \\ g & h \end{bmatrix}$  and that  $\det B = \det \begin{bmatrix} a & 1 \\ e & f \end{bmatrix}$ .

- State and prove a generalization of Exercise 3 for  $n \times n$  matrices.

- Let  $A = \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ e & f & g & h \end{bmatrix}$ . Prove that  $\det A = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \det \begin{bmatrix} g & h \\ z & w \end{bmatrix}$ .

$$\begin{bmatrix} x & y & z & w \end{bmatrix}$$

- State and prove a generalization of Exercise 5 for  $n \times n$  matrices of the form

$$A = \begin{bmatrix} B & O \\ C & D \end{bmatrix}$$

where  $B, C, D$  denote square matrices and  $O$  denotes a matrix of zeros.

7. Use Theorem 3.6 to determine whether the following sets of vectors are linearly dependent or independent.

- (a)  $A_1 = (1, -1, 0)$ ,  $A_2 = (0, 1, -1)$ ,  $A_3 = (2, 3, -1)$ .
- (b)  $A_1 = (1, -1, 2, 1)$ ,  $A_2 = (-1, 2, -1, 0)$ ,  $A_3 = (3, -1, 1, 0)$ ,  $A_4 = (1, 0, 0, 1)$ .
- (c)  $A_1 = (1, 0, 0, 0, 1)$ ,  $A_2 = (1, 1, 0, 0, 0)$ ,  $A_3 = (1, 0, 1, 0, 1)$ ,  $A_4 = (1, 1, 0, 1, 1)$ ,  
 $A_5 = (0, 1, 0, 1, 0)$ .

### 3.12 Expansion formulas for determinants. Minors and cofactors

We still have not shown that a determinant function actually exists, except in the  $2 \times 2$  case. In this section we exploit the linearity property and the uniqueness theorem to show that if determinants exist they can be computed by a formula which expresses every determinant of order  $n$  as a linear combination of determinants of order  $n - 1$ . Equation (3.2) in Section 3.1 is an example of this formula in the  $3 \times 3$  case. The general formula will suggest a method for proving existence of determinant functions by induction.

Every row of an  $n \times n$  matrix  $A$  can be expressed as a linear combination of the  $n$  unit coordinate vectors  $I_1, \dots, I_n$ . For example, the first row  $A_1$  can be written as follows:

$$A_1 = \sum_{j=1}^n a_{1j} I_j.$$

Since determinants are linear in the first row we have

$$(3.17) \quad d(A_1, A_2, \dots, A_n) = d\left(\sum_{j=1}^n a_{1j} I_j, A_2, \dots, A_n\right) = \sum_{j=1}^n a_{1j} d(I_j, A_2, \dots, A_n).$$

Therefore to compute  $\det A$  it suffices to compute  $d(I_j, A_2, \dots, A_n)$  for each unit coordinate vector  $I_j$ .

Let us use the notation  $A'_{1j}$  to denote the matrix obtained from  $A$  by replacing the first row  $A_1$  by the unit vector  $I_j$ . For example, if  $n = 3$  there are three such matrices:

$$A'_{11} = \begin{bmatrix} 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad A'_{12} = \begin{bmatrix} 0 & 1 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad A'_{13} = \begin{bmatrix} 0 & 0 & 1 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Note that  $\det A'_{1j} = d(I_j, A_2, \dots, A_n)$ . Equation (3.17) can now be written in the form

$$(3.18) \quad \det A = \sum_{j=1}^n a_{1j} \det A'_{1j}.$$

This is called an *expansion formula*; it expresses the determinant of  $A$  as a linear combination of the elements in its first row.

The argument used to derive (3.18) can be applied to the  $k$ th row instead of the first row. The result is an expansion formula in terms of elements of the  $k$ th row.

**THEOREM 3.8. EXPANSION BY COFACTORS.** Let  $A'_{kj}$  denote the matrix obtained from  $A$  by replacing the  $k$ th row  $A$ , by the unit coordinate vector  $I_j$ . Then we have the expansion formula

$$(3.19) \quad \det A = \sum_{j=1}^n a_{kj} \det A'_{kj}$$

which expresses the determinant of  $A$  as a linear combination of the elements of the  $k$ th row. The number  $\det A'_{kj}$  is called the cofactor of entry  $a_{kj}$ .

In the next theorem we shall prove that each cofactor is, except for a plus or minus sign, equal to a determinant of a matrix of order  $n - 1$ . These smaller matrices are called minors.

**DEFINITION.** Given a square matrix  $A$  of order  $n \geq 2$ , the square matrix of order  $n - 1$  obtained by deleting the  $k$ th row and the  $j$ th column of  $A$  is called the  $k, j$  minor of  $A$  and is denoted by  $A_{k,j}$ .

**EXAMPLE.** A matrix  $A = (a_{kj})$  of order 3 has nine minors. Three of them are

$$A_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}, \quad A_{12} = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}, \quad A_{13} = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}.$$

Equation (3.2) expresses the determinant of a  $3 \times 3$  matrix as a linear combination of determinants of these three minors. The formula can be written as follows:

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13}.$$

The next theorem extends this formula to the  $n \times n$  case for any  $n \geq 2$ .

**THEOREM 3.9. EXPANSION BY  $k$ TH-ROW MINORS.** For any  $n \times n$  matrix  $A$ ,  $n \geq 2$ , the cofactor of  $a_{kj}$  is related to the minor  $A_{kj}$  by the formula

$$(3.20) \quad \det A'_{kj} = (-1)^{k+j} \det A_{kj}.$$

Therefore the expansion of  $\det A$  in terms of elements of the  $k$ th row is given by

$$(3.21) \quad \det A = \sum_{j=1}^n (-1)^{k+j} a_{kj} \det A_{kj}.$$

*Proof.* We illustrate the idea of the proof by considering first the special case  $k = j = 1$ . The matrix  $A'_{11}$  has the form

$$A_{11} = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}$$

By applying elementary row operations of type (3) we can make every entry below the 1 in the first column equal to zero, leaving all the remaining entries intact. For example, if we multiply the first row of  $A'_{11}$  by  $-a_{21}$  and add the result to the second row, the new second row becomes  $(0, a_{22}, a_{23}, \dots, a_{2n})$ . By a succession of such elementary row operations we obtain a new matrix which we shall denote by  $A^0_{11}$  and which has the form

$$A^0_{11} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & a_{2n} \\ 0 & a_{32} & \cdots & a_{3n} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

Since row operations of type (3) leave the determinant unchanged we have

$$(3.22) \quad \det A^0_{11} = \det A'_{11}.$$

But  $A^0_{11}$  is a block-diagonal matrix so, by Theorem 3.7, we have

$$\det A^0_{11} = \det A_{11},$$

where  $A_{11}$  is the 1, 1 minor of  $A$ ,

$$A_{11} = \begin{bmatrix} a_{22} & \cdots & a_{2n} \\ a_{32} & \cdots & a_{3n} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

Therefore  $\det A'_{11} = \det A_{11}$ , which proves (3.20) for  $k = j = 1$ .

We consider next the special case  $k = 1, j$  arbitrary, and prove that

$$(3.23) \quad \det A'_{1j} = (-1)^{j-1} \det A_{1j}.$$

Once we prove (3.23) the more general formula (3.20) follows at once because matrix  $A'_{kj}$  can be transformed to a matrix of the form  $B'_{1j}$  by  $k - 1$  successive interchanges of adjacent rows. The determinant changes sign at each interchange so we have

$$(3.24) \quad \det A'_{kj} = (-1)^{k-1} \det B'_{1j},$$

where  $B$  is an  $n \times n$  matrix whose first row is  $I_j$  and whose 1,  $j$  minor  $B_{1j}$  is  $A_{kj}$ . By (3.23), we have

$$\det B'_{1j} = (-1)^{j-1} \det B_{1j} = (-1)^{j-1} \det A_{kj},$$

so (3.24) gives us

$$\det A'_{kj} = (-1)^{k-1} (-1)^{j-1} \det A_{kj} = (-1)^{k+j} \det A_{kj}.$$

Therefore if we prove (3.23) we also prove (3.20).

We turn now to the proof of (3.23). The matrix  $A'_{1j}$  has the form

$$A'_{1j} = \begin{bmatrix} 0 & \cdots & 1 & \cdots & 0 \\ a_{21} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & & \vdots & & \vdots \\ \cdot & & \cdot & & \cdot \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix}.$$

By elementary row operations of type (3) we introduce a column of zeros below the 1 and transform this to

$$A^0_{1j} = \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ a_{21} & \cdots & a_{2,j-1} & 0 & a_{2,j+1} & \cdots & a_{2n} \\ \vdots & & & & & & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & 0 & a_{n,j+1} & \cdots & a_{nn} \end{bmatrix}.$$

As before, the determinant is unchanged so  $\det A^0_{1j} = \det A'_{1j}$ . The 1, j minor  $A_{1j}$  has the form

$$A_{1j} = \begin{bmatrix} a_{21} & \cdots & a_{2,j-1} & a_{2,j+1} & \cdots & a_{2n} \\ \vdots & & \vdots & \vdots & & \vdots \\ \cdot & & \cdot & \cdot & & \cdot \\ a_{n1} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{nn} \end{bmatrix}.$$

Now we regard  $\det A^0_{1j}$  as a function of the  $n - 1$  rows of  $A_{1j}$ , say  $\det A^0_{1j} = f(A_{1j})$ . The function  $f$  satisfies the first three axioms for a determinant function of order  $n - 1$ . Therefore, by the uniqueness theorem we can write

$$(3.25) \quad f(A_{1j}) = f(J) \det A_{1j},$$

where  $J$  is the identity matrix of order  $n - 1$ . Therefore, to prove (3.23) we must show that  $f(J) = (-1)^{j-1}$ . Now  $f(J)$  is by definition the determinant of the matrix

$$C = \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 1 & & 0 & 0 & 0 & & 0 \\ \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & & \cdot \\ 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & & \cdot \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \quad \begin{array}{l} \leftarrow \text{jth row} \\ \uparrow \\ \text{jth column} \end{array}$$

The entries along the sloping lines are all 1. The remaining entries not shown are all 0. By interchanging the first row of  $C$  successively with rows  $2, 3, \dots, j$  we arrive at the  $n \times n$  identity matrix  $I$  after  $j - 1$  interchanges. The determinant changes sign at each interchange, so  $\det C = (-1)^{j-1}$ . Hence  $f(J) = (-1)^{j-1}$ , which proves (3.23) and hence (3.20).

### 3.13 Existence of the determinant function

In this section we use induction on  $n$ , the size of a matrix, to prove that determinant functions of every order exist. For  $n = 2$  we have already shown that a determinant function exists. We can also dispense with the case  $n = 1$  by defining  $\det [a_{11}] = a_{11}$ .

Assuming a determinant function exists of order  $n - 1$ , a logical candidate for a determinant function of order  $n$  would be one of the expansion formulas of Theorem 3.9, for example, the expansion in terms of the first-row minors. However, it is easier to verify the axioms if we use a different but analogous formula expressed in terms of the first-column minors.

**THEOREM 3.10.** *Assume determinants of order  $n - 1$  exist. For any  $n \times n$  matrix  $A = (a_{jk})$ , let  $f$  be the function defined by the formula*

$$(3.26) \quad f(A_1, \dots, A_n) = \sum_{j=1}^n (-1)^{j+1} a_{j1} \det A_{j1}.$$

*Then  $f$  satisfies all four axioms for a determinant function of order  $n$ . Therefore, by induction, determinants of order  $n$  exist for every  $n$ .*

**Proof.** We regard each term of the sum in (3.26) as a function of the rows of  $A$  and we write

$$f_j(A_1, \dots, A_n) = (-1)^{j+1} a_{j1} \det A_{j1}.$$

If we verify that each  $f_j$  satisfies Axioms 1 and 2 the same will be true for  $f$ .

Consider the effect of multiplying the first row of  $A$  by a scalar  $t$ . The minor  $A_{11}$  is not affected since it does not involve the first row. The coefficient  $a_{11}$  is multiplied by  $t$ , so we have

$$f_1(tA_1, A_2, \dots, A_n) = ta_{11} \det A_{11} = tf_1(A_1, A_2, \dots, A_n).$$

If  $j > 1$  the first row of each minor  $A_{j1}$  gets multiplied by  $t$  and the coefficient  $a_{j1}$  is not affected, so again we have

$$f_j(tA_1, A_2, \dots, A_n) = tf_j(A_1, A_2, \dots, A_n).$$

Therefore each  $f_j$  is homogeneous in the first row.

If the  $k$ th row of  $A$  is multiplied by  $t$ , where  $k > 1$ , the minor  $A_{11}$  is not affected but  $a_{k1}$  is multiplied by  $t$ , so  $f_k$  is homogeneous in the  $k$ th row. If  $j \neq k$ , the coefficient  $a_{j1}$  is not affected but some row of  $A_{j1}$  gets multiplied by  $t$ . Hence every  $f_j$  is homogeneous in the  $k$ th row.

A similar argument shows that each  $f_j$  is additive in every row, so  $\mathbf{f}$  satisfies Axioms 1 and 2. We prove next that  $\mathbf{f}$  satisfies Axiom 3', the weak version of Axiom 3. From Theorem 3.1, it then follows that  $\mathbf{f}$  satisfies Axiom 3.

To verify that  $\mathbf{f}$  satisfies Axiom 3', assume two adjacent rows of  $A$  are equal, say  $A_{\cdot k+1} = A_{\cdot k+1}$ . Then, except for minors  $A_{k1}$  and  $A_{k+1,1}$ , each minor  $A_{j1}$  has two equal rows so  $\det A_{j1} = 0$ . Therefore the sum in (3.26) consists only of the two terms corresponding to  $j = k$  and  $j = k + 1$ ,

$$(3.27) \quad f(A_1, \dots, A_n) = (-1)^{k+1} a_{k1} \det A_{k1} + (-1)^{k+2} a_{k+1,1} \det A_{k+1,1},$$

But  $A_{k1} = A_{k+1,1}$  and  $a_{k1} = a_{k+1,1}$  since  $A_{\cdot k} = A_{\cdot k+1}$ . Therefore the two terms in (3.27) differ only in sign, so  $\mathbf{f}(A_1, \dots, A_n) = 0$ . Thus,  $\mathbf{f}$  satisfies Axiom 3'.

Finally, we verify that  $\mathbf{f}$  satisfies Axiom 4. When  $A = I$  we have  $a_{11} = 1$  and  $a_{ij} = 0$  for  $j > 1$ . Also,  $A_{\cdot j}$  is the identity matrix of order  $n - 1$ , so each term in (3.26) is zero except the first, which is equal to 1. Hence  $\mathbf{f}(I_1, \dots, I_n) = 1$  so  $f$  satisfies Axiom 4.

In the foregoing proof we could just as well have used a function defined in terms of the  $k$ th-column minors  $A_{jk}$  instead of the first-column minors  $A_{j1}$ . In fact, if we let

$$(3.28) \quad f(A_1, \dots, A_n) = \sum_{j=1}^n (-1)^{j+k} a_{j1} \det A_{jk},$$

exactly the same type of proof shows that this  $\mathbf{f}$  satisfies all four axioms for a determinant function. Since determinant functions are unique, the expansion formulas in (3.28) and those in (3.21) are all equal to  $\det A$ .

The expansion formulas (3.28) not only establish the existence of determinant functions but also reveal a new aspect of the theory of determinants—a connection between row-properties and column-properties. This connection is discussed further in the next section.

### 3.14 The determinant of a transpose

Associated with each matrix  $A$  is another matrix called the *transpose* of  $A$  and denoted by  $A^t$ . The rows of  $A^t$  are the columns of  $A$ . For example, if

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad \text{then } A^t = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

A formal definition may be given as follows.

**DEFINITION OF TRANPOSE.** *The transpose of an  $m \times n$  matrix  $A = (a_{ij})_{i,j=1}^{m,n}$  is the  $n \times m$  matrix  $A^t$  whose  $i, j$  entry is  $a_{ji}$ .*

Although transposition can be applied to any rectangular matrix we shall be concerned primarily with square matrices. We prove next that transposition of a square matrix does not alter its determinant,

**THEOREM 3.11.** *For any  $n \times n$  matrix  $A$  we have*

$$\det A = \det A^t.$$

*Proof.* The proof is by induction on  $n$ . For  $n = 1$  and  $n = 2$  the result is easily verified. Assume, then, that the theorem is true for matrices of order  $n - 1$ . Let  $A = (a_{ij})$  and let  $B = A^t = (b_{ij})$ . Expanding  $\det A$  by its first-column minors and  $\det B$  by its first-row minors we have

$$\det A = \sum_{j=1}^n (-1)^{j+1} a_{j1} \det A_{j1}, \quad \det B = \sum_{j=1}^n (-1)^{j+1} b_{1j} \det B_{1j}.$$

But from the definition of transpose we have  $b_{1j} = a_{i,j}$ , and  $B_{1j} = (A_{j1})^t$ . Since we are assuming the theorem is true for matrices of order  $n - 1$  we have  $\det B_{1j} = \det A_{j1}$ . Hence the foregoing sums are equal term by term, so  $\det A = \det B$ .

### 3.15 The cofactor matrix

Theorem 3.5 showed that if  $A$  is nonsingular then  $\det A \neq 0$ . The next theorem proves the converse. That is, if  $\det A \neq 0$ , then  $A^{-1}$  exists. Moreover, it gives an explicit formula for expressing  $A^{-1}$  in terms of a matrix formed from the cofactors of the entries of  $A$ .

In Theorem 3.9 we proved that the  $i, j$  cofactor of  $a_{ij}$  is equal to  $(-1)^{i+j} \det A_{ij}$ , where  $A_{ij}$  is the  $i, j$  minor of  $A$ . Let us denote this cofactor by  $\text{cof } a_{ij}$ . Thus, by definition,

$$\text{cof } a_{ij} = (-1)^{i+j} \det A_{ij}.$$

**DEFINITION OF THE COFACTOR MATRIX.** *The matrix whose  $i, j$  entry is  $\text{cof } a_{ij}$  is called the cofactor matrix? of  $A$  and is denoted by  $\text{cof } A$ . Thus, we have*

$$\text{cof } A = (\text{cof } a_{ij})_{i,j=1}^n = ((-1)^{i+j} \det A_{ij})_{i,j=1}^n.$$

The next theorem shows that the product of  $A$  with the transpose of its cofactor matrix is, apart from a scalar factor, the identity matrix  $I$ .

**THEOREM 3.12.** *For any  $n \times n$  matrix  $A$  with  $n \geq 2$  we have*

$$(3.29) \quad A(\text{cof } A)^t = (\det A)I.$$

*In particular, if  $\det A \neq 0$  the inverse of  $A$  exists and is given by*

$$A^{-1} = \frac{1}{\det A} (\text{cof } A)^t.$$

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† In much of the matrix literature the transpose of the cofactor matrix is called the *adjugate* of  $A$ . Some of the older literature calls it the *adjoint* of  $A$ . However, current nomenclature reserves the term *adjoint* for an entirely different object, discussed in Section 5.8.

**Proof.** Using Theorem 3.9 we express  $\det A$  in terms of its  $k$ th-row cofactors by the formula

$$(3.30) \quad \det A = \sum_{j=1}^n a_{kj} \operatorname{cof} a_{kj}.$$

Keep  $\mathbf{k}$  fixed and apply this relation to a new matrix  $\mathbf{B}$  whose  $i$ th row is equal to the  $k$ th row of  $A$  for some  $\mathbf{i} \neq \mathbf{k}$ , and whose remaining rows are the same as those of  $A$ . Then  $\det \mathbf{B} = 0$  because the  $i$ th and  $k$ th rows of  $\mathbf{B}$  are equal. Expressing  $\det \mathbf{B}$  in terms of its  $i$ th-row cofactors we have

$$(3.31) \quad \det \mathbf{B} = \sum_{j=1}^n b_{ij} \operatorname{cof} b_{ij} = 0.$$

But since the  $i$ th row of  $\mathbf{B}$  is equal to the  $k$ th row of  $A$  we have

$$b_{ij} = a_{kj} \quad \text{and} \quad \operatorname{cof} b_{ij} = \operatorname{cof} a_{ij} \quad \text{for every } j.$$

Hence (3.31) states that

$$(3.32) \quad \sum_{j=1}^n a_{kj} \operatorname{cof} a_{ij} = 0 \quad \text{if } k \neq i.$$

Equations (3.30) and (3.32) together can be written as follows:

$$(3.33) \quad \sum_{j=1}^n a_{kj} \operatorname{cof} a_{ij} = \begin{cases} \det A & \text{if } \mathbf{i} = \mathbf{k} \\ \mathbf{0} & \text{if } i \neq k. \end{cases}$$

But the sum appearing on the left of (3.33) is the  $\mathbf{k}, \mathbf{i}$  entry of the product  $A(\operatorname{cof} A)^t$ . Therefore (3.33) implies (3.29).

As a direct corollary of Theorems 3.5 and 3.12 we have the following necessary and sufficient condition for a square matrix to be nonsingular.

**THEOREM 3.13.** *A square matrix  $A$  is nonsingular if and only if  $\det A \neq 0$ .*

### 3.16 Cramer's rule

Theorem 3.12 can also be used to give explicit formulas for the solutions of a system of linear equations with a nonsingular coefficient matrix. The formulas are called **Cramer's rule**, in honor of the Swiss mathematician Gabriel Cramer (1704-1752).

**THEOREM 3.14. CRAMER'S RULE.** *If a system of  $n$  linear equations in  $n$  unknowns  $x_1, \dots, x_n$ ,*

$$\sum_{j=1}^n a_{ij} x_j = b_i \quad (i = 1, 2, \dots, n)$$

has a nonsingular coefficient-matrix  $A = (a_{ij})$ , then there is a unique solution for the system given by the formulas

$$(3.34) \quad x_j = \frac{1}{\det A} \sum_{k=1}^n b_k \operatorname{cof} a_{kj}, \quad \text{for } j = 1, 2, \dots, n.$$

*Proof.* The system can be written as a matrix equation,

$$\cdot \quad AX = B,$$

where  $X$  and  $B$  are column matrices,  $X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ ,  $B = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ . Since  $A$  is nonsingular there is a unique solution  $X$  given by

$$(3.35) \quad X = A^{-1}B = \frac{1}{\det A} (\operatorname{cof} A)^t B.$$

The formulas in (3.34) follow by equating components in (3.35).

It should be noted that the formula for  $x_j$  in (3.34) can be expressed as the quotient of two determinants,

$$x_j = \frac{\det C_j}{\det A},$$

where  $C_j$  is the matrix obtained from  $A$  by replacing the  $j$ th column of  $A$  by the column matrix  $B$ .

### 3.17 Exercises

1. Determine the cofactor matrix of each of the following matrices :

$$(a) \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad (b) \begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 1 \\ -1 & -2 & 0 \end{bmatrix}, \quad (c) \begin{bmatrix} 3 & 1 & 2 & 4 \\ 2 & 0 & 5 & 1 \\ 1 & -1 & -2 & 6 \\ -2 & 3 & 2 & 3 \end{bmatrix}.$$

2. Determine the inverse of each of the nonsingular matrices in Exercise 1.

3. Find all values of the scalar  $\lambda$  for which the matrix  $\lambda I - A$  is singular, if  $A$  is equal to

$$(a) \begin{bmatrix} 0 & 3 \\ 2 & -1 \end{bmatrix}, \quad (b) \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix}, \quad (c) \begin{bmatrix} 11 & -2 & 8 \\ 19 & -3 & 14 \\ -8 & 2 & -5 \end{bmatrix}.$$

4. If  $A$  is an  $n \times n$  matrix with  $n \geq 2$ , prove each of the following properties of its cofactor matrix:
- $\text{cof}(A^t) = (\text{cof } A)^t$ .
  - $(\text{cof } A)^t A = (\det A)I$ .
  - $A(\text{cof } A)^t = (\text{cof } A)^t A$  ( $A$  commutes with the transpose of its cofactor matrix).
5. Use Cramer's rule to solve each of the following systems:
- $x + 2y + 3z = 8, \quad 2x - y + 4z = 7, \quad -y + z = 1$ .
  - $x + y + 2z = 0, \quad 3x - y - z = 3, \quad 2x + 5y + 3z = 4$ .
6. (a) Explain why each of the following is a Cartesian equation for a straight line in the  $xy$ -plane passing through two distinct points  $(x_1, y_1)$  and  $(x_2, y_2)$ .

$$\det \begin{bmatrix} x - x_1 & y - y_1 \\ x_2 - x_1 & y_2 - y_1 \end{bmatrix} = 0; \quad \det \begin{bmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{bmatrix} = 0.$$

- (b) State and prove corresponding relations for a plane in 3-space passing through three distinct points.
- (c) State and prove corresponding relations for a circle in the  $xy$ -plane passing through three noncollinear points.
7. Given  $n^2$  functions  $f_{ij}$ , each differentiable on an interval  $(a, b)$ , define  $F(x) = \det [f_{ij}(x)]$  for each  $x$  in  $(a, b)$ . Prove that the derivative  $F'(x)$  is a sum of  $n$  determinants,

$$F'(x) = \sum_{i=1}^n \det A_i(x),$$

- where  $A_i(x)$  is the matrix obtained by differentiating the functions in the  $i$ th row of  $[f_{ij}(x)]$ .
8. An  $n \times n$  matrix of functions of the form  $W(x) = [u_j^{(i-1)}(x)]$ , in which each row after the first is the derivative of the previous row, is called a *Wronskian* matrix in honor of the Polish mathematician J. M. H. Wronski (1778-1853). Prove that the derivative of the determinant of  $W(x)$  is the determinant of the matrix obtained by differentiating each entry in the last row of  $W(x)$ .

[Hint: Use Exercise 7.]

# 4

## EIGENVALUES AND EIGENVECTORS

### 4.1 Linear transformations with diagonal matrix representations

Let  $T: V \rightarrow V$  be a linear transformation on a finite-dimensional linear space  $V$ . Those properties of  $T$  which are independent of any coordinate system (basis) for  $V$  are called *intrinsic properties* of  $T$ . They are shared by all the matrix representations of  $T$ . If a basis can be chosen so that the resulting matrix has a particularly simple form it may be possible to detect some of the intrinsic properties directly from the matrix representation.

Among the simplest types of matrices are the diagonal matrices. Therefore we might ask whether every linear transformation has a diagonal matrix representation. In Chapter 2 we treated the problem of finding a diagonal matrix representation of a linear transformation  $T: V \rightarrow W$ , where  $\dim V = n$  and  $\dim W = m$ . In Theorem 2.14 we proved that there always exists a basis  $(e_1, \dots, e_n)$  for  $V$  and a basis  $(w_1, \dots, w_m)$  for  $W$  such that the matrix of  $T$  relative to this pair of bases is a diagonal matrix. In particular, if  $W = V$  the matrix will be a square diagonal matrix. The new feature now is that we want to use the *same basis* for both  $V$  and  $W$ . With this restriction it is not always possible to find a diagonal matrix representation for  $T$ . We turn, then, to the problem of determining which transformations do have a diagonal matrix representation.

*Notation:* If  $A = (a_{ij})$  is a diagonal matrix, we write  $A = \text{diag } (a_{11}, a_{22}, \dots, a_{nn})$ .

It is easy to give a necessary and sufficient condition for a linear transformation to have a diagonal matrix representation.

**THEOREM 4.1.** *Given a linear transformation  $T: V \rightarrow V$ , where  $\dim V = n$ . If  $T$  has a diagonal matrix representation, then there exists an independent set of elements  $u_1, \dots, u_n$  in  $V$  and a corresponding set of scalars  $\lambda_1, \dots, \lambda_n$  such that*

$$(4.1) \quad T(u_k) = \lambda_k u_k \quad \text{for } k = 1, 2, \dots, n.$$

*Conversely, if there is an independent set  $u_1, \dots, u_n$  in  $V$  and a corresponding set of scalars  $\lambda_1, \dots, \lambda_n$  satisfying (4.1), then the matrix*

$$A = \text{diag } (\lambda_1, \dots, \lambda_n)$$

*is a representation of  $T$  relative to the basis  $(u_1, \dots, u_n)$ .*

**Proof.** Assume first that  $\mathbf{T}$  has a diagonal matrix representation  $A = (a_{ik})$  relative to some basis  $(e_1, \dots, e_n)$ . The action of  $\mathbf{T}$  on the basis elements is given by the formula

$$T(e_k) = \sum_{i=1}^n a_{ik} e_i = a_{kk} e_k$$

since  $a_{ik} = 0$  for  $i \neq k$ . This proves (4.1) with  $u_k = e_k$  and  $\lambda_k = a_{kk}$ .

Now suppose independent elements  $u_1, \dots, u_n$  and scalars  $\lambda_1, \dots, \lambda_n$  exist satisfying (4.1). Since  $u_1, \dots, u_n$  are independent they form a basis for  $\mathbf{V}$ . If we define  $a_{ii} = \lambda_i$  and  $a_{ik} = 0$  for  $i \neq k$ , then the matrix  $A = (a_{ik})$  is a diagonal matrix which represents  $\mathbf{T}$  relative to the basis  $(u_1, \dots, u_n)$ .

Thus the problem of finding a diagonal matrix representation of a linear transformation has been transformed to another problem, that of finding independent elements  $u_1, \dots, u_n$  and scalars  $\lambda_1, \dots, \lambda_n$ , to satisfy (4.1). Elements  $u_k$  and scalars  $\lambda_k$  satisfying (4.1) are called **eigenvectors** and **eigenvalues** of  $T$ , respectively. In the next section we study eigenvectors and eigenvalues in a more general setting.

## 4.2 Eigenvectors and eigenvalues of a linear transformation

In this discussion  $\mathbf{V}$  denotes a linear space and  $S$  denotes a subspace of  $\mathbf{V}$ . The spaces  $S$  and  $\mathbf{V}$  are not required to be finite dimensional.

**DEFINITION.** Let  $T: S \rightarrow \mathbf{V}$  be a linear transformation of  $S$  into  $\mathbf{V}$ . A scalar  $\lambda$  is called an eigenvalue of  $T$  if there is a nonzero element  $x$  in  $S$  such that

$$(4.2) \quad T(x) = \lambda x.$$

The element  $x$  is called an eigenvector of  $T$  belonging to  $\lambda$ . The scalar  $\lambda$  is called an eigenvalue corresponding to  $x$ .

There is exactly one eigenvalue corresponding to a given eigenvector  $x$ . In fact, if we have  $T(x) = \lambda x$  and  $T(x) = \mu x$  for some  $x \neq O$ , then  $\lambda x = \mu x$  so  $\lambda = \mu$ .

**Note:** Although Equation (4.2) always holds for  $x = 0$  and any scalar  $\lambda$ , the definition excludes 0 as an eigenvector. One reason for this prejudice against 0 is to have exactly one eigenvalue  $\lambda$  associated with a given eigenvector  $x$ .

The following examples illustrate the meaning of these concepts.

**EXAMPLE 1. Multiplication by a fixed scalar.** Let  $T: S \rightarrow \mathbf{V}$  be the linear transformation defined by the equation  $T(x) = cx$  for each  $x$  in  $S$ , where  $c$  is a fixed scalar. In this example every nonzero element of  $S$  is an eigenvector belonging to the scalar  $c$ .

† The words *eigenvector* and *eigenvalue* are partial translations of the German words *Eigenvektor* and *Eigenwert*, respectively. Some authors use the terms *characteristic vector*, or *proper vector* as synonyms for eigenvector. Eigenvalues are also called *characteristic values*, *proper values*, or *latent roots*.

**EXAMPLE 2.** The eigenspace  $E(\lambda)$  consisting of all  $x$  such that  $T(x) = \lambda x$ . Let  $T: S \rightarrow V$  be a linear transformation having an eigenvalue  $\lambda$ . Let  $E(\lambda)$  be the set of all elements  $x$  in  $S$  such that  $T(x) = \lambda x$ . This set contains the zero element 0 and all eigenvectors belonging to  $\lambda$ . It is easy to prove that  $E(\lambda)$  is a subspace of  $S$ , because if  $x$  and  $y$  are in  $E(\lambda)$  we have

$$T(ax + by) = aT(x) + bT(y) = a\lambda x + b\lambda y = \lambda(ax + by)$$

for all scalars  $a$  and  $b$ . Hence  $(ax + by) \in E(\lambda)$  so  $E(\lambda)$  is a subspace. The space  $E(\lambda)$  is called the *eigenspace* corresponding to  $\lambda$ . It may be finite- or infinite-dimensional. If  $E(\lambda)$  is finite-dimensional then  $\dim E(\lambda) \geq 1$ , since  $E(\lambda)$  contains at least one nonzero element  $x$  corresponding to  $\lambda$ .

**EXAMPLE 3. Existence of zero eigenvalues.** If an eigenvector exists it cannot be zero, by definition. However, the zero scalar can be an eigenvalue. In fact, if 0 is an eigenvalue for  $x$  then  $T(x) = Ox = 0$ , so  $x$  is in the null space of  $T$ . Conversely, if the null space of  $T$  contains any nonzero elements then each of these is an eigenvector with eigenvalue 0. In general,  $E(0)$  is the null space of  $T - 0I$ .

**EXAMPLE 4. Rejection in the  $xy$ -plane.** Let  $S = V = V_1(R)$  and let  $T$  be a reflection in the  $xy$ -plane. That is, let  $T$  act on the basis vectors  $\mathbf{i}, \mathbf{j}, k$  as follows:  $T(i) = i$ ,  $T(j) = j$ ,  $T(k) = -k$ . Every nonzero vector in the  $xy$ -plane is an eigenvector with eigenvalue 1. The remaining eigenvectors are those of the form  $ck$ , where  $c \neq 0$ ; each of them has eigenvalue  $-1$ .

**EXAMPLE 5. Rotation of the plane through a fixed angle  $\alpha$ .** This example is of special interest because it shows that the existence of eigenvectors may depend on the underlying field of scalars. The plane can be regarded as a linear space in two different ways: (1) As a 2-dimensional real linear space,  $V = V_1(R)$ , with two basis elements  $(1, 0)$  and  $(0, 1)$ , and with real numbers as scalars; or (2) as a 1-dimensional complex linear space,  $V = V_1(C)$ , with one basis element  $1$ , and complex numbers as scalars.

Consider the second interpretation first. Each element  $z \neq 0$  of  $V_1(C)$  can be expressed in polar form,  $z = re^{i\theta}$ . If  $T$  rotates  $z$  through an angle  $\alpha$  then  $T(z) = re^{i(\theta+\alpha)} = e^{i\alpha}z$ . Thus, each  $z \neq 0$  is an eigenvector with eigenvalue  $\lambda = e^{i\alpha}$ . Note that the eigenvalue  $e^{i\alpha}$  is not real unless  $\alpha$  is an integer multiple of  $\pi$ .

Now consider the plane as a real linear space,  $V_1(R)$ . Since the scalars of  $V_1(R)$  are real numbers the rotation  $T$  has real eigenvalues only if  $\alpha$  is an integer multiple of  $\pi$ . In other words, if  $\alpha$  is not an integer multiple of  $\pi$  then  $T$  has no real eigenvalues and hence no eigenvectors. Thus the existence of eigenvectors and eigenvalues may depend on the choice of scalars for  $V$ .

**EXAMPLE 6. The differentiation operator.** Let  $V$  be the linear space of all real functions  $f$  having derivatives of every order on a given open interval. Let  $D$  be the linear transformation which maps each  $f$  onto its derivative,  $D(f) = f'$ . The eigenvectors of  $D$  are those nonzero functions  $f$  satisfying an equation of the form