16.1.1 Assuming that the series for e^y is also valid for $y = ix$, show that

$$e^{ix} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right).$$

16.1.2 Assuming it is valid to differentiate the sine series term by term, show that

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots,$$

and hence that $e^{ix} = \cos x + i \sin x$.

Another consequence of $e^{ix} = \cos x + i \sin x$ is that $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{i\pi/2}$, which allows us to evaluate the outlandish number i^i.

16.1.3 Show that i^i has a real value [Euler (1746)]. What is it?

16.1.4 Using the fact that $e^{2in\pi} = 1$ for any integer n, give a formula for all values of i^i [Euler (1746)].

16.2 Conformal Mapping

Another important general situation clarified by complex functions is the problem of conformal mapping. Mapping a sphere (the earth's surface) onto a plane is a practical problem that has attracted the attention of mathematicians since ancient times. Before the eighteenth century the most notable mathematical contributions to mapping were stereographic projection (Section 15.2), due to Ptolemy around 150 CE, and the Mercator projection used by G. Mercator in 1569 [this Mercator was Gerard, not the Nicholas who discovered the series for $\log(1+x)$]. Both these projections were conformal, that is, angle-preserving, or what eighteenth-century mathematicians preferred to call "similar in the small." This means that the image $f(R)$ of any region R tends toward an exact scale map of R as the size of R tends to 0. Since "similarity in the large" is clearly impossible—for example, a great circle cannot be mapped to a closed curve that divides the plane into two equal parts—conformality is the best one can do to preserve the appearance of regions on the sphere. Preservation of angles was intentional in the Mercator projection, whose purpose was to assist navigation, and in the case of stereographic projection conformality was first noticed by Harriot around 1590 [see Lohne (1979)].

Advances in the theory of conformal mapping were made by Lambert (1772), Euler (1777) (sphere onto plane), and Lagrange (1779) (general surface of revolution onto plane). All these authors used complex numbers, but Lagrange's presentation is the clearest and most general. Using the method of d'Alembert (1752), he combined a pair of differential equations in two real variables into a single equation in one complex variable and arrived at the result that any two conformal maps of a surface of revolution onto the (x,y)-plane were related via a complex function $f(x+iy)$ mapping the plane onto itself. These results were crowned by the result of Gauss (1822) generalizing Lagrange's theorem to conformal maps of an arbitrary surface onto the plane.

Conversely, a complex function $f(z)$ defines a map of the z plane onto itself, and it is easy to see that this map is conformal. In fact, this is a consequence of the differentiability of f. To say that the limit

$$\lim_{\delta z \to 0} \frac{f(z_0+\delta z)-f(z_0)}{\delta z}$$

exists is to say that the mapping of the disk $\{z : |z-z_0| < |\delta z|\}$ around z_0 to the region around $f(z_0)$ tends to a scale mapping as the radius $|\delta z|$ tends to 0. If the derivative is expressed in polar form as

$$f'(z_0) = re^{i\alpha},$$

then r is the scale factor of this limit mapping and α is the angle of rotation. Riemann (1851) seems to have been the first to take the conformal mapping property as a basis for the theory of complex functions. His deepest result in this direction was the *Riemann mapping theorem*, which states that any region of the plane bounded by a simple closed curve can be mapped onto the unit disk conformally, and hence by a complex function. The proof of this theorem in Riemann (1851) depends on properties of potential functions, which Riemann justified partly by appeal to physical intuition—the so-called *Dirichlet's principle*. Such reasoning went against the growing tendency toward rigor in nineteenth-century analysis, and stricter proofs were given by Schwarz (1870) and Neumann (1870). However, Riemann's faith in the physical roots of complex function theory was eventually justified when Hilbert (1900b) put Dirichlet's principle on a sound basis.

EXERCISES

The claim that differentiability of $f(z)$ implies that f is a conformal mapping should be qualified by the condition $f'(z) \neq 0$, because if the scale factor tends to 0 then f cannot be said to be a scale mapping. At points where $f'(z) = 0$ one may find that angles are altered. Here is an example.

16.2.1 Show that $f(z) = z^2$ defines a conformal mapping except at $z = 0$, where it doubles angles.

This is no surprise if we view $z \mapsto z^2$ as a two-sheeted covering of the plane $\mathbb{C}$ (compare with Section 15.4).

16.2.2 Show that the map $z \mapsto z^2$ is two-to-one except at $z = 0$, and relate the angle doubling at $z = 0$ to the branch point of the covering.

16.2.3 Similarly describe the behavior of the map $z \mapsto z^3$ at $z = 0$.

16.3 Cauchy's Theorem

We have seen that interesting complex functions arise from integration. For example, the elliptic functions come from inversion of elliptic integrals (Section 12.3). However, it is not at first clear what the integral $\int_{z_0}^{z} f(t)\,dt$ means when z_0, z are complex numbers. It is natural, and not technically difficult, to define $\int_{z_0}^{z} f(t)\,dt$ as $\int_{\mathscr{C}} f(t)\,dt$, the integral of f along a curve $\mathscr{C}$ from z_0 to z; the problem is that $\int_{\mathscr{C}} f(t)\,dt$ appears to depend on $\mathscr{C}$ and hence may not be anything like a function of z, as one would wish.

The first to recognize and resolve this problem seems to have been Gauss. In a letter to Bessel, Gauss (1811) raised the problem and claimed its resolution as follows:

> Now how is one to think of $\int \Phi(z)\,dz$ for $z = a + ib$? Evidently, if one wishes to start from clear concepts, one must assume that z changes by infinitely small increments (each of the form $\alpha + i\beta$) from that value for which the integral is to be 0 to $c = a + ib$, and then *sum* all the $\Phi(z)\,dz$. ...But now ...continuous transition from one value of z to another $a + ib$ takes place along a curve and hence is possible in infinitely many ways. I now conjecture that the integral $\int_0^c \Phi(z)\,dz$ will always have the same value after two different transitions if z never becomes infinite within the region enclosed by the two curves representing the transitions.
>
> [Translation of Gauss (1811) in Birkhoff (1973)]

In the same letter, Gauss also observed that if $\Phi(z)$ *does* become infinite in the region, then in general $\int_0^c \Phi(z)\,dz$ *will* take different values when integrated along different curves. He saw in particular that the infinitely many values of $\log c$ corresponded to the different ways a path from 1 to c could wind around $z = 0$, the point where $\Phi(z) = 1/z$ becomes infinite.

The theorem that $\int_{z_0}^{z} f(t)\,dt$ is independent of the path throughout a region where f is finite (and differentiable, which went without saying for Gauss) is now known as *Cauchy's theorem*, since Cauchy was the first to offer a proof and to develop the consequences of the theorem. An equivalent and more convenient statement is that $\int_{\mathscr{C}} f(t)\,dt = 0$ for any closed curve $\mathscr{C}$ in a region where f is differentiable. Cauchy presented a proof to the Paris Academy in 1814 but first published it later [Cauchy (1825)]. In Cauchy (1846) he presented a more transparent proof, based on the Cauchy–Riemann equations and the theorem of Green (1828) and Ostrogradsky (1828), which relates a line integral to a surface integral. The latter theorem, usually known as *Green's theorem*, is a generalization of the fundamental theorem of calculus to real functions $f(x,y)$ of two variables and can be stated as follows: if $\mathscr{C}$ is a simple closed curve bounding a region $\mathscr{R}$ and f is suitably smooth, then

$$\int_{\mathscr{C}} f\,dx = \iint_{\mathscr{R}} \frac{\partial f}{\partial y}\,dx\,dy,$$
$$\int_{\mathscr{C}} f\,dy = -\iint_{\mathscr{R}} \frac{\partial f}{\partial x}\,dx\,dy,$$

where $\iint_{\mathscr{R}}$ denotes the surface integral over $\mathscr{R}$ and $\int_{\mathscr{C}}$ denotes the line integral around $\mathscr{C}$ in the counterclockwise sense. (The difference in sign in the two formulas reflects the different sense of $\mathscr{C}$ when x and y are interchanged.)

Cauchy's theorem follows from Green's theorem by an easy calculation. If

$$f(t) = u(t) + iv(t)$$

is the decomposition of f into real and imaginary parts, and if we write

$$dt = dx + i\,dy,$$

then

$$\begin{aligned}\int_{\mathscr{C}} f(t)\,dt &= \int_{\mathscr{C}} (u+iv)(dx+idy)\\ &= \int_{\mathscr{C}} (udx - vdy) + i\int_{\mathscr{C}} (vdx+udy)\\ &= \iint_{\mathscr{R}} \left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) dxdy + i\iint_{\mathscr{R}} \left(\frac{\partial v}{\partial y}-\frac{\partial u}{\partial x}\right) dxdy\\ &= 0\end{aligned}$$

since

$$\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=0 \quad \text{and} \quad \frac{\partial v}{\partial y}-\frac{\partial u}{\partial x}=0$$

by the Cauchy–Riemann equations. This proof requires f to have a continuous first derivative in order to be able to apply Green's theorem. The restriction of continuity of $f'(t)$ in the proof was removed by Goursat (1900). As it happens, if f' exists, it will have not only continuity but also derivatives of all orders. This follows from one of the remarkable consequences Cauchy (1837) drew from the assumption $\int_{\mathscr{C}} f(t)\,dt = 0$, namely, that f has a power-series expansion. By Goursat (1900), then, differentiability of a complex function is enough to guarantee a power-series expansion. A generalization of this result to f that become infinite at isolated points was made by Laurent (1843) (f then has an expansion including negative powers; that is the *Laurent expansion*) and to "many-valued" f with branch points by Puiseux (1850) (f then has an expansion in fractional powers, the *Newton–Puiseux expansion*).

EXERCISES

The Cauchy–Riemann equations follow easily from the existence of $f'(z)$, that is, from the condition that

$$\lim_{\delta z\to 0}\frac{f(z+\delta z)-f(z)}{\delta z}$$

has the same value, regardless of the path along which $\delta z \to 0$.

16.3.1 Suppose $f(z) = u(x,y)+iv(x,y)$ and $\delta z = \delta x + i\delta y$. By letting $\delta z \to 0$ along the x-axis ($\delta y = 0$) and along the y-axis ($\delta x = 0$), and equating the resulting values of $f'(z)$, show that

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}.$$

These equations give a convenient test for a function $u(x,y) + iv(x,y)$ to be a differentiable function of $z = x + iy$.

16.3.2 Check that $u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$ satisfy the Cauchy–Riemann equations.

16.3.3 Express $x^2 - y^2 + 2ixy$ as a function of $z = x + iy$.

16.4 Double Periodicity of Elliptic Functions

The view of complex integration provided by Cauchy's theorem is one step toward understanding elliptic integrals such as $\int_0^z dt/\sqrt{t(t-\alpha)(t-\beta)}$. The other important step is the idea of a Riemann surface (Section 15.4), which enables us to visualize the possible paths of integration from 0 to z. The "function" $1/\sqrt{t(t-\alpha)(t-\beta)}$ is of course two-valued and, by an argument like that in Section 15.4, is represented by a two-sheeted covering of the t sphere, with branch points at 0, α, β, ∞. Thus the paths of integration, correctly viewed, are curves on this surface, which is topologically a torus (again, as in Section 15.4).

Now a torus contains certain closed curves that do not bound a piece of the surface, such as the curves $\mathscr{C}_1$ and $\mathscr{C}_2$ shown in Figure 16.1. There is no region $\mathscr{R}$ bounded by $\mathscr{C}_1$ or $\mathscr{C}_2$; hence Green's theorem does not apply, and we in fact obtain nonzero values

$$\omega_1 = \int_{\mathscr{C}_1} \frac{dt}{\sqrt{t(t-\alpha)(t-\beta)}},$$
$$\omega_2 = \int_{\mathscr{C}_2} \frac{dt}{\sqrt{t(t-\alpha)(t-\beta)}}.$$

Consequently the integral

$$\Phi^{-1}(z) = \int_0^z \frac{dt}{\sqrt{t(t-\alpha)(t-\beta)}}$$

will be ambiguous: for each value $\Phi^{-1}(z) = w$ obtained for a certain path $\mathscr{C}$ from 0 to z we also obtain the values $w + m\omega_1 + n\omega_2$ by adding to $\mathscr{C}$ a detour that winds m times around $\mathscr{C}_1$ and n times around $\mathscr{C}_2$. (For topological reasons, this is essentially the most general path of integration.)

It follows that the inverse relation $\Phi(w) = z$, the elliptic function corresponding to the integral, satisfies

$$\Phi(w) = \Phi(w + m\omega_1 + n\omega_2)$$

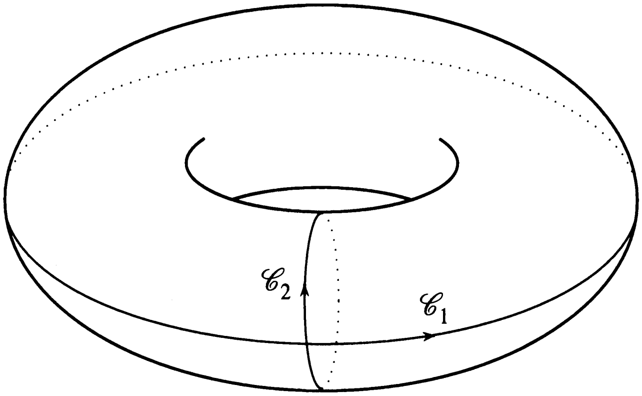

Figure 16.1: Nonbounding curves on the torus

for any integers m, n. That is, Φ is doubly periodic, with periods ω_1, ω_2. This intuitive explanation of double periodicity is due to Riemann (1851), who later [Riemann (1858a)] developed the theory of elliptic functions from this standpoint.

Remarkable series expansions of elliptic functions, which exhibit the double periodicity analytically, were discovered by Eisenstein (1847). The precedents for Eisenstein's series, as Eisenstein himself pointed out, were partial fraction expansions of circular functions discovered by Euler, for example

$$\pi \cot \pi x = \sum_{n=-\infty}^{\infty} \frac{1}{x+n}$$

[Euler (1748a), p. 191]. It is obvious (at least formally, though one has to be a little careful about the meaning of this summation to ensure convergence) that the sum is unchanged when x is replaced by $x+1$; hence the period 1 of $\pi \cot \pi x$ is exhibited directly by its series expansion. Eisenstein showed that doubly periodic functions could be obtained by analogous expressions, such as

$$\sum_{m,n=-\infty}^{\infty} \frac{1}{(z+m\omega_1+n\omega_2)^2},$$

which again (with suitable interpretation to ensure convergence) are obviously unchanged when z is replaced by $z+\omega_1$ or $z+\omega_2$. Hence we obtain a function with periods ω_1, ω_2. The function above is in fact identical (up to a constant) with the Weierstrass $\wp$-function, mentioned in Section 12.5