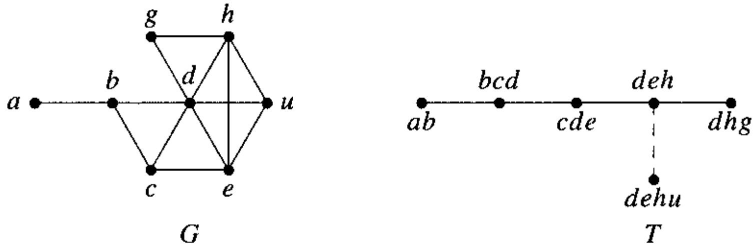
Let x be a leaf of T, and let u be a vertex of G such that $T(u)$ contains x but not its neighbor. The subtrees for neighbors of u in G must contain x and hence are pairwise intersecting. Thus u is simplicial in G. Deleting $T(u)$ yields a subtree representation of $G - u$. We complete a simplicial elimination order of G using such an ordering of $G - u$ given by the induction hypothesis. ■



Because the class of chordal graphs is hereditary, a simplicial elimination ordering can start with any simplicial vertex. Thus a brute-force approach to finding such an ordering would be to examine neighborhoods until we find a simplicial vertex, delete it, and iterate.

Rose–Tarjan–Lueker [1976] found a faster way, which was simplified further by Tarjan [1976]. The idea here, because there is always a simplicial vertex among the vertices farthest from a given vertex (proof of Theorem 5.3.17), is that a simplicial elimination ordering can *end* at any vertex. Thus we start with an arbitrary vertex and list the vertices clumped around it. The result is a simplicial construction ordering (the reverse of a simplicial elimination ordering) if and only if the graph is chordal. The algorithm was published with several applications in Tarjan–Yannakakis [1984]; we follow Golumbic [1984].

8.1.12. Algorithm. *Maximum Cardinality Search (MCS)*
Input: A graph G.
Output: A vertex numbering - a bijection $f: V(G) \to \{1, \ldots, n(G)\}$.
Idea: For each unnumbered vertex v, maintain a label $l(v)$ that is its degree among the vertices already numbered. The vertices at the end of a simplicial elimination ordering are those clumped around the last vertex, so in a simplicial construction ordering the vertices with high labels should be added first.
Initialization: Assign label 0 to every vertex. Set $i = 1$.
Iteration: Select any unnumbered vertex with maximum label. Number it i and add 1 to the label of its neighbors. Augment i and iterate. ■

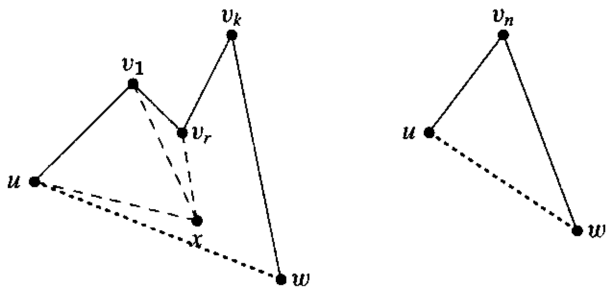
8.1.13. Example. The first vertex chosen in the MCS order is arbitrary. An application of MCS to the graph G above could start by setting $f(c) = 1$ and hence $l(b) = l(d) = l(e) = 1$. Next we could select $f(e) = 2$ and update $l(d) = 2$, $l(h) = l(u) = 1$. Now d is the only vertex with label as large as 2, and hence $f(d) = 3$. We update $l(b) = l(h) = l(u) = 2$, $l(g) = 1$, $l(a) = 0$. Continuing the procedure can produce the order c, e, d, b, h, g, a, u in increasing order of f. This is a simplicial construction ordering, and u, a, g, h, b, d, e, c is a simplicial elimination ordering. ■

8.1.14. Theorem. (Tarjan [1976]). A simple graph G is chordal if and only if the numbering $v_1, \dots, v_n$ produced by the Maximum Cardinality Search algorithm is a simplicial construction ordering of G.

Proof: If MCS produces a simplicial construction ordering, then G is chordal. Conversely, suppose that G is chordal, and let $f\colon V(G) \to [n]$ be the numbering produced by MCS. A *bridge* of f is a chordless path of length at least 2 whose lowest numbers occur at the endpoints. We prove first that f has no bridge. Otherwise, let $P = u, v_1, \dots, v_k, w$ be a bridge that minimizes $\max\{f(u), f(w)\}$. By symmetry, we may assume that $f(u) > f(w)$ (f is used as the vertical coordinate to position vertices in the illustration).

Since u is numbered in preference to v_k at time $f(u)$, and w is already numbered at that time, there exists a vertex $x \in N(u) - N(v_k)$ with $f(x) < f(u)$. Letting $v_0 = u$, set $r = \max\{j\colon x \leftrightarrow v_j\}$. The path $P' = x, v_r, \dots, v_k, w$ is chordless, since $x \leftrightarrow w$ would complete a chordless cycle. Since both of $f(x), f(w)$ are less than $f(u)$, P' is a bridge that contradicts the choice of P. Hence f has no bridge.

With this claim, the proof follows by induction on $n(G)$. It suffices to show that v_n is simplicial, since the application of MCS to $G - v_n$ produces the same numbering $v_1, \dots, v_{n-1}$ that leaves v_n at the end. If v_n is not simplicial, then v_n has nonadjacent neighbors u, w, in which case u, v_n, w is a bridge of f. ■



The MCS algorithm runs in time $O(n(G)+e(G))$ with careful implementation. For each j, we maintain a doubly linked list of the vertices with label j. For each vertex we store its label and pointers to its neighbors and to its position in the lists. When v is numbered, in time $O(1+d(v))$ we remove v from its list, augment its neighbors labels, and move its neighbors into the next higher lists. To complete the chordality test, we must also check whether the MCS order is a simplicial construction ordering (Exercise 10). Simplicial elimination or construction orderings quickly yield optimal colorings, cliques, stable sets, and clique coverings (Exercise 9).

The alternative algorithm found by Rose, Tarjan, and Leuker is known as Lexicographic Breadth First Search (LBFS). Closely related to the proof of Theorem 5.3.17, LBFS has been used for many applications in testing graph properties and computing graph parameters. Corneil–Olariu–Stewart [2000] provides a good introduction to this topic.

Given a simplicial elimination ordering, Theorem 8.1.14 computes a subtree representation. When the list of maximal cliques is known, Kruskal's algorithm (Theorem 2.3.3) can be used to compute a subtree representation without knowing a simplicial elimination ordering.

8.1.15. Definition. A tree T is a **clique tree** of G if there is a bijection between $V(T)$ and the maximal cliques of G such that for each $v \in V(G)$ the cliques containing v induce a subtree of T.

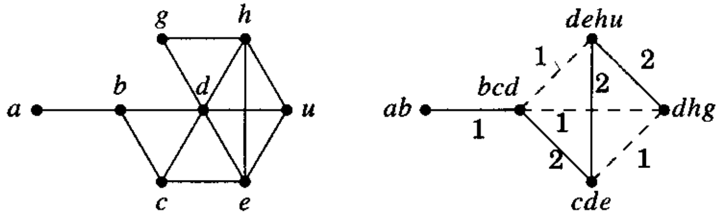
8.1.16. Lemma. Every tree of minimum order in which G has a subtree representation is a clique tree of G.

Proof: Let T be a host tree of minimum order for a subtree representation of G. By Lemma 8.1.10, the vertices of a maximal clique Q in G occur at a common vertex q of T. If the vertices of G assigned to some $q' \in V(T)$ form a proper subclique Q' of Q, then the subtrees for these vertices contain the entire q', q-path in T. The first edge of T on that path can be contracted without changing the intersection graph, which yields a smaller host tree. ∎

The **weighted intersection graph** of a collection **A** of finite sets is a weighted clique in which the elements of **A** are the vertices and the weight of each edge AA' is $|A \cap A'|$.

8.1.17. Theorem. (Acharya–Las Vergnas [1982]) Let $M(G)$ be the weighted intersection graph of the set of maximal cliques $\{Q_i\}$ of a simple graph G. If T is a spanning tree of $M(G)$, then $w(T) \le \sum n(Q_i) - n(G)$, with equality if and only T is a clique tree.

Proof: (McKee [1993]) Let T be a spanning tree of $M(G)$. Let T_v be the subgraph of T induced by $\{Q_i\colon v \in Q_i\}$. Each vertex $v \in V(G)$ contributes once to the weight of T for each edge of T_v; hence $w(T) = \sum_{v \in V(G)} e(T_v)$. Each T_v is a forest, so $e(T_v) \le n(T_v) - 1$, with equality if and only if T_v is a tree. The term $n(T_v)$ contributes 1 to the size of each clique containing v. Summing the inequality for each vertex yields $w(T) \le \sum n(Q_i) - n(G)$. Equality holds if and only if each T_v is a tree, which is true if and only if T is a clique tree. ∎



G $\qquad$ $M(G)$ $\quad$ (T solid)

$$w(T) = 7 = 15 - 8 = \sum n(Q_i) - n(G)$$

As a consequence of Theorem 8.1.17, we can test whether G is a chordal graph by finding the maximum weight of a spanning tree in $M(G)$. Furthermore, when G is chordal the clique trees are precisely the maximum-weight spanning trees of $M(G)$ (Bernstein–Goodman [1981], Shibata [1988]; see McKee [1993] for related material).

OTHER CLASSES OF PERFECT GRAPHS

Interval graphs are the intersection graphs of collections of intervals on a line. We proved directly in Proposition 5.1.16 that they are perfect; this also follows from being a subclass of the chordal graphs (Exercise 26). Interval graphs arise in linear scheduling problems having constraints on concurrent events (recall Example 5.1.15).

8.1.18. Example. *Classical applications of interval graphs.*

Analysis of DNA chains. Interval graphs were invented for the study of DNA. Benzer [1959] studied the linearity of the chain for higher organisms. Each gene is encoded as an interval, except that the relevant interval may contain a dozen or more irrelevant junk pieces called "introns" among the relevant pieces called "exons". Under the hypothesis that mutations arise from alterations of connected segments, changes in traits of microorganisms can be studied to determine whether their determining amino-acid sets could intersect. This establishes a graph with traits as vertices and "common alterations" as edges. Under the hypotheses of linearity and contiguity, the graph is an interval graph, and this aids in locating genes along the DNA sequence.

Timing of traffic lights. Given traffic streams at an intersection, a traffic engineer (or a person with common sense) can determine which pairs of streams may flow simultaneously. Given an "all-stop" moment in the cycle, the intersection graph of the green-light intervals must be an interval graph whose edges are a subset of the allowable pairs. These can be studied to optimize some criterion such as average waiting time (see Roberts [1978]).

Archeological seriation. Given pottery samples at an archeological dig, we seek a time-line of what styles were used when. Assume that each style was used during one time interval and that two styles appearing in the same grave were used concurrently. Let two styles be an edge if they appear together in a grave. If this graph is an interval graph, then its interval representations are the possible time-lines. Otherwise, the information is incomplete, and the desired interval graph requires additional edges. ■

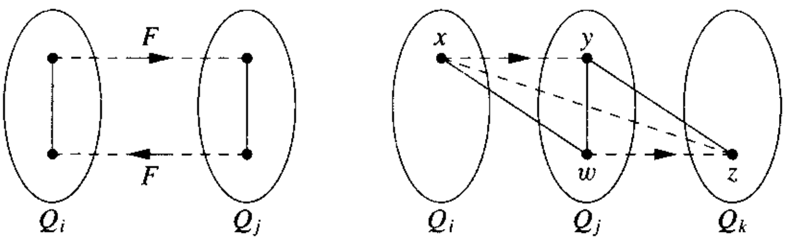
We present two characterizations of interval graphs. Property B in Theorem 8.1.20 is due to Gilmore and Hoffman [1964], and property C is due to Fulkerson and Gross [1965].

8.1.19. Definition. A 0,1-matrix has the **consecutive 1s property** (for columns) if its rows can be permuted so that the 1s in each column appear consecutively. The **clique-vertex incidence matrix** of G is the incidence matrix with rows indexed by the maximal cliques and columns indexed by $V(G)$.

8.1.20. Theorem. The following equivalent conditions on a graph G characterize the interval graphs.
A) G has an interval representation.
B) G is a chordal graph, and $\overline{G}$ is a comparability graph.
C) The clique-vertex incidence matrix has the consecutive 1s property.

Proof: We leave A $\Rightarrow$ B and A $\Leftrightarrow$ C to Exercises 26–27, proving B $\Rightarrow$ C here. Let G be a chordal graph such that $\overline{G}$ has a transitive orientation F. We use F and the absence of chordless cycles in G to establish an ordering on the maximal cliques of G that exhibits the consecutive 1s property for the clique-vertex incidence matrix M.

Let Q_i and Q_j be maximal cliques in G. By maximality, each vertex of one clique has a nonneighbor in the other. Suppose that under F, some edge of $\overline{G}$ points from Q_i to Q_j and some edge of $\overline{G}$ points from Q_j to Q_i. If these edges have a common vertex, then the transitivity of F forces an edge of a clique in G to belong to $\overline{G}$. Hence the situation is as on the left below, with the (dashed) edges of F having four distinct vertices. If the two remaining pairs among these four vertices form edges in G, then G has an induced C_4. Hence at least one diagonal is in $\overline{G}$, but each possible orientation of it in F contradicts transitivity. We conclude that all the edges of $\overline{G}$ between vertex sets Q_i and Q_j point in the same direction in F.



We can now define a tournament T with vertices corresponding to the maximal cliques of G. We put $Q_i \to Q_j$ in T when all edges of F between Q_i and Q_j point from Q_i to Q_j. By the preceding paragraph, T is an orientation of a complete graph. We claim that T is transitive. To prove this we need to show that $Q_i \to Q_j$ and $Q_j \to Q_k$ imply $Q_i \to Q_k$. Suppose that $x \to y$ and $w \to z$ in F with $x \in Q_i$, $y, w \in Q_j$, and $z \in Q_k$. If $y = w$, transitivity of F immediately implies $x \to z$. Otherwise, consider a pair xz as shown on the right above. Joining x and z in G would form an induced C_4 in G, so $x \not\leftrightarrow z$. Hence this pair appears in F, and it must be directed from $x \to z$ to avoid violating transitivity. We conclude that $Q_i \to Q_k$ in T.

A transitive tournament specifies a unique linear ordering of the vertices consistent with the edges; use the transitive tournament T to order the rows of M as $Q_1 \to \cdots \to Q_m$. Suppose that under this ordering there is some column x where the 1s do not appear consecutively. Then we have Q_i, Q_j, Q_k such that

$i < j < k$, $x \in Q_i, Q_k$, $x \notin Q_j$. Since $x \notin Q_j$, the clique Q_j must have some vertex y not adjacent to x, else Q_j could absorb x and would not be maximal. Now $x \in Q_i$ implies $x \to y$ in F, and $x \in Q_k$ implies $y \to x$ in F, which cannot both happen. ■

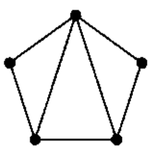
The interval graphs form a relatively small family of perfect graphs. We next discuss larger classes that maintain some of the nice properties of chordal graphs and comparability graphs.

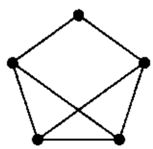
8.1.21. Definition. *Classes of perfect graphs* (conditions on odd cycles apply only for length at least 5).
o-triangulated: every odd cycle has a noncrossing pair of chords.
parity: every odd cycle has a crossing pair of chords.
Meyniel: every odd cycle has at least two chords.
weakly chordal: no induced cycle of length at least 5 in G or $\overline{G}$.
strongly perfect: every induced subgraph has a stable set meeting all its maximal cliques.

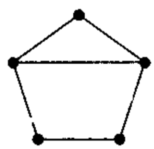
Gallai [1962] proved that o-triangulated graphs are perfect. Every chordal graph is o-triangulated (Exercise 34) and weakly chordal (Exercise 40). All o-triangulated and parity graphs are Meyniel graphs. Meyniel graphs are perfect (Meyniel [1976], Lovász [1983]) and also strongly perfect (Ravindra [1982]).

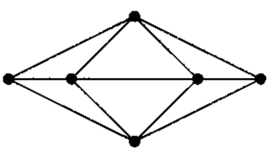
Parity graphs, shown to be perfect in Olaru [1969] and Sachs [1970], carry that name due to a later characterization by Burlet and Uhry [1984]: G is a parity graph if and only if, for every pair $x, y \in V(G)$, the chordless x, y-paths are all even or all odd (Exercise 36).

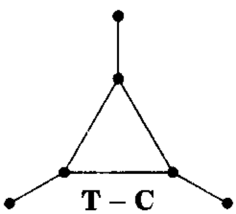
8.1.22. Example. The graphs below exhibit differences among these classes. Here **T**, **C**, **O**, **P**, **M**, **W** respectively denote the classes of chordal (Triangulated), comparability, o-triangulated, parity, Meyniel, and weakly chordal graphs. ■


T − P


P − O


(W ∩ C) − M


M − O − P


T − C

Strongly perfect graphs were introduced by Berge and Duchet [1984]. Changing maximal to maximum in the definition yields a weaker requirement equivalent to γ-perfection; a stable set meeting all maxim*um* cliques can be used as the first color class in an $\omega(G)$-coloring constructed inductively. Thus strongly perfect graphs are perfect.

The class of strongly perfect graphs does not contain all Meyniel graphs or all weakly chordal graphs (Exercises 39–40), but it does contain all chordal graphs and all comparability graphs. (As observed in Proposition 5.3.25, when G has a transitive orientation, each induced subgraph inherits a transitive orientation, and the vertices with indegree 0 in this orientation form a stable set that meets all the maximal cliques.)

Our next class is a subclass of the strongly perfect graphs (Exercises 37–38) that still contains all chordal graphs and comparability graphs. Introduced by Chvátal [1984], it has played an important role in the theory of perfect graphs.

8.1.23. Definition. A **perfect order** on a graph is a vertex ordering such that greedy coloring with respect to the ordering inherited by each induced subgraph produces an optimal coloring of that subgraph. A **perfectly orderable graph** is a graph having a perfect order.

In an orientation of G, an **obstruction** is an induced 4-vertex path a, b, c, d whose first and last edges are oriented toward the leaves. The orientation of G *associated with* a vertex ordering L orients each edge toward the vertex earlier in L: $u \leftarrow v$ if $u < v$. A vertex ordering is **obstruction-free** if its associated orientation has no obstruction.

The orientation associated with a perfect order is obstruction-free, because on an obstruction the greedy coloring would use three colors instead of two. Chvátal proved that a graph is perfectly orderable if and only if it has an obstruction-free ordering. The characterization implies that perfectly orderable graphs are perfect and that chordal graphs and comparability graphs are perfectly orderable.

8.1.24. Example. *Chordal graphs and comparability graphs are perfectly orderable.* The orientation of a chordal graph associated with a simplicial construction ordering has no induced $u \leftarrow v \rightarrow w$. A transitive orientation of a comparability graph has no induced $u \rightarrow v \rightarrow w$.

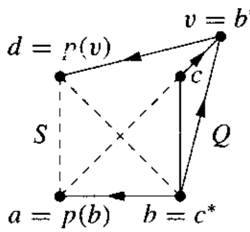
Every orientation with an obstruction has both an induced $u \rightarrow v \rightarrow w$ and an induced $u \leftarrow v \rightarrow w$. Hence if G is a comparability graph or a chordal graph, then G has an obstruction-free ordering. By Chvátal's characterization, such graphs are perfectly orderable. ■

8.1.25. Lemma. (Chvátal [1984]) Let G have a clique Q and a stable set S that are disjoint, and suppose that each $w \in Q$ is adjacent to some $p(w) \in S$. If

L is an obstruction-free ordering of G such that $p(w) < w$ for all $w \in Q$, then some $p(w) \in S$ is adjacent to all of Q.

Proof: We use induction on $n(G)$. For the basis step $n(G) = 1$, there is nothing to prove. Consider $n(G) > 1$. For each $w \in Q$, the graph $G - w$ satisfies the hypotheses using the clique $Q - w$ and the stable set $\{p(u) : u \in Q - w\}$. By the induction hypothesis, there is a vertex $w^* \in Q - w$ such that $p(w^*) \leftrightarrow Q - w$. We obtain $w \in Q$ such that $p(w^*) \leftrightarrow Q$ unless $p(w^*) \not\leftrightarrow w$ for all $w \in Q$. This assigns a unique w^* to every w, since $p(w^*)$ is nonadjacent only to w among Q. Mapping w to w^* thus defines a permutation on Q. Since $p(w) \leftrightarrow w$, the permutation has no fixed point.

We seek an obstruction in the orientation associated with L. Let v be the least vertex of Q in L. Let $b, c \in Q$ be the vertices such that $b^* = v$ and $c^* = b$ (possibly $c = v$). Let $a = p(b)$ and $d = p(v)$. Because $p(w^*) \not\leftrightarrow w$, we have $a \not\leftrightarrow c$ and $d \not\leftrightarrow b$, which implies $a \neq d$ in the stable set S and yields the picture below for the orientation associated with L.



Because $d = p(b^*)$, the only vertex of Q nonadjacent to d is b; thus $c \leftrightarrow d$. Since $d = p(v) < v \le c$ in L, we have $d \leftarrow c$. Now a, b, c, d induce an obstruction, which contradicts the hypothesis for L. Hence $p(w^*) \leftrightarrow w$ for some w, and $p(w^*)$ is the desired vertex of S. ∎

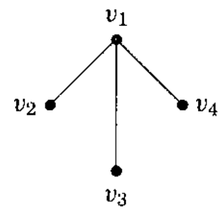
8.1.26. Theorem. (Chvátal [1984]) A vertex ordering of a simple graph G is a perfect order if and only if it is obstruction-free, and every graph with such an ordering is perfect.

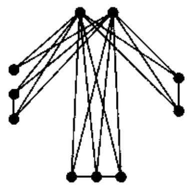
Proof: We have observed that the condition is necessary. Since the class of graphs with obstruction-free orderings is hereditary (the inherited ordering for an induced subgraph is also obstruction-free), it suffices to show that the greedy coloring of G relative to an obstruction-free ordering L is optimal. Let k be the number of colors used by the greedy coloring relative to L. To prove optimality, we show that G has a k-clique; this also inductively proves perfection.

Let $f\colon V(G) \to [k]$ be the resulting coloring. Let i be the least integer such that G has a clique $w_{i+1}, \dots, w_k$ such that $f(w_j) = j$. Since f uses color k on some vertex, i is well-defined. If $i = 0$, then G has a k-clique.

If $i > 0$, then for each w_j there is a vertex $p(w_j)$ such that $p(w_j) < w_j$ in L and $f(p(w_j)) = i$; otherwise the greedy coloring would use a lower color on w_j. Let $S = \{p(w_{i+1}), \dots, p(w_k)\}$. Since all of S has the same color, S is a stable set. Hence the conditions of Lemma 8.1.25 are satisfied, and some vertex of S can be added to the clique to become w_i. This contradicts the minimality of i. ∎

Next we consider a different way of generating perfect graphs. An operation that preserves perfection can enlarge a class of perfect graphs. Vertex multiplication, which expands each vertex into an independent set, is such a property. We generalize this. If $V(G) = \{v_1, \dots, v_n\}$, and $H_1, \dots, H_n$ are pairwise disjoint graphs, then the **composition** $G[H_1, \dots, H_n]$ is the graph $H_1 + \cdots + H_n$ together with $\{xy\colon x \in V(H_i), y \in V(H_j), v_iv_j \in E(G)\}$. The special case $G[\overline{K}_{h1}, \dots, \overline{K}_{hn}]$ is $G \circ h$. The example below uses $H_1 = 2K_1$, $H_2 = K_2 + K_1$, $H_3 = P_3$, $H_4 = K_2$, and $G = K_{1,3}$ with central vertex v_1.





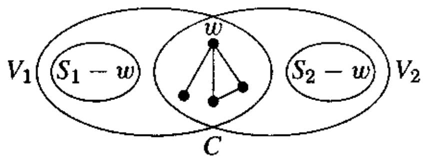
Lovász proved that composition preserves perfection. This is one corollary of Chvátal's Star-Cutset Lemma.

8.1.27. Definition. A **star-cutset** of G is a vertex cut S containing a vertex x adjacent to all of $S - \{x\}$. A **minimal imperfect graph** is an imperfect graph whose proper induced subgraphs are all perfect.

8.1.28. Lemma. (The Star-Cutset Lemma Lemma) If G has no stable set intersecting every maximum clique, and every proper induced subgraph of G is $\omega(G)$-colorable, then G has no star-cutset.

Proof: Suppose that G has a star-cutset C, with w adjacent to all of $C - \{w\}$. Since $G - C$ is disconnected, we can partition $V(G - C)$ into sets V_1, V_2 with no edge between them. Let $G_i = G[V_i \cup C]$, and let f_i be a proper $\omega(G)$-coloring of G_i. Let S_i be the set of vertices in G_i with the same color in f_i as w; this includes w but no other vertex of C. Since there are no edges between V_1 and V_2, the union $S = S_1 \cup S_2$ is a stable set.

If Q is a clique in $G - S$, then Q is contained in $G_1 - S_1$ or in $G_2 - S_2$. Since f_i provides an $\omega(G) - 1$-coloring of $G_i - S_i$, we have $|Q| \le \omega(G) - 1$. Since this applies to every clique Q in $G - S$, the stable set S meets every $\omega(G)$-clique of G, which contradicts the hypotheses. ∎



8.1.29. Theorem. (The Star-Cutset Lemma, Chvátal [1985b]) No minimal imperfect graph has a star-cutset.

Proof: If G is a minimal imperfect graph, then $\chi(G) > \omega(G)$ and deletion of any stable set S leaves a perfect graph. Hence we have

$$1 + \omega(G) \le \chi(G) \le 1 + \chi(G - S) = 1 + \omega(G - S) \le 1 + \omega(G).$$

This yields $\omega(G - S) = \omega(G)$, which means that no stable set meets every maximum clique. Furthermore, since G is minimally imperfect, every proper induced subgraph G' satisfies $\chi(G') = \omega(G') \le \omega(G)$, making it $\omega(G)$-colorable. Lemma 8.1.28 now implies that G has no star-cutset. ∎

The Replacement Lemma generalizes Lemma 8.1.4.

8.1.30. Corollary. (Replacement Lemma—Lovász [1972b]) Every composition of perfect graphs is perfect.

Proof: A composition can be constructed by a sequence of substitutions in which a single vertex v of G_1 is replaced by a graph G_2 and all edges added between $V(G_2)$ and $U = N_{G_1}(v)$ to form a graph G. Hence it suffices to prove that this operation preserves perfection. If the resulting graph G is not perfect, then it contains a minimal imperfect induced subgraph F. Such a subgraph cannot be contained in G_1 or G_2, which forces it to have at least two vertices of G_2 and at least one vertex of G_1.

If F has no vertex of G_1 outside U, then $F = F[U] \vee (F \cap G_2)$. The join operation preserves perfection, since $\chi(H \vee H') = \chi(H) + \chi(H')$ and $\omega(H \vee H') = \omega(H) + \omega(H')$ for all H, H'. Hence we may assume that F has a vertex of G_1 outside U. In this case, $V(F) \cap U$ together with one vertex of G_2 in F is a star-cutset of F. Hence the replacement of v with G_2 introduces no minimal imperfect subgraph F. ∎

The Star-Cutset Lemma also yields perfection of weakly chordal graphs. Hayward [1985] proved that G or $\overline{G}$ has a star-cutset when G is a weakly chordal graph that is not a clique or stable set. With the Star-Cutset Lemma and the Perfect Graph Theorem, this implies that no weakly chordal graph is a minimal imperfect graph. Since the class is hereditary, it follows that every weakly chordal graph is perfect.

IMPERFECT GRAPHS

The **p-critical** graphs are the minimal imperfect graphs. The Strong Perfect Graph Conjecture (SPGC) states that the only p-critical graphs are the odd cycles (of length at least 5) and their complements. With enough properties of p-critical graphs, perhaps we could prove that only odd cycles and their complements have all these properties; this would prove the SPGC. We begin with simple observations about p-critical graphs, some used earlier in discussing star-cutsets. (This presentation was originally modeled after Shmoys [1981].)

8.1.31. Lemma. If G is p-critical, then G is connected, $\overline{G}$ is p-critical, $\omega(G) \geq 2$, and $\alpha(G) \geq 2$. Furthermore, for every $x \in V(G)$, $\chi(G - x) = \omega(G)$ and $\theta(G - x) = \alpha(G)$.

Proof: G is perfect if and only if every component of G is perfect, and G is perfect if and only if $\overline{G}$ is perfect. Cliques and their complements are perfect. Finally, we observed in proving Theorem 8.1.29 that deleting a stable set from a p-critical graph cannot decrease the clique number. Since $G - x$ is perfect, we thus have $\chi(G - x) = \omega(G - x) = \omega(G)$. The condition $\theta(G - x) = \alpha(G)$ is this statement for $\overline{G}$. ■

More subtle properties of p-critical graphs follow from Lovász's extension of the PGT.

8.1.32. Theorem. (Lovász [1972b]) A graph G is perfect if and only if $\omega(G[A])\alpha(G[A]) \geq |A|$ for all $A \subseteq V(G)$. ■

The property "$\omega(G[A])\alpha(G[A]) \geq |A|$ for all $A \subseteq V(G)$" was suggested by Fulkerson; we call it **β-perfection**. It is implied by α-perfection or γ-perfection; if we can color G with $\omega(G)$ stable sets, then some stable set has at least $n(G)/\omega(G)$ vertices. The converse involves counting arguments like those we gave for the PGT, but more delicate. Since β-perfection is unchanged under complementation, Theorem 8.1.32 immediately implies the PGT.

8.1.33. Theorem. If G is p-critical, then $n(G) = \alpha(G)\omega(G) + 1$. Furthermore, for every $x \in V(G)$, $G - x$ has a partition into $\omega(G)$ stable sets of size $\alpha(G)$ and a partition into $\alpha(G)$ cliques of size $\omega(G)$.

Proof: When G is p-critical, the condition for β-perfection fails only for the full vertex set $A = V(G)$. Hence for each $x \in V(G)$ we have

$$n(G) - 1 \leq \alpha(G - x)\omega(G - x) = \alpha(G)\omega(G) \leq n(G) - 1.$$

Therefore, $n(G) = \alpha(G)\omega(G) + 1$. Since $\chi(G - x) = \omega(G - x) = \omega(G)$, we can cover $G - x$ by $\omega(G)$ stable sets. Having size at most $\alpha(G)$, these sets partition the $\alpha(G)\omega(G)$ vertices of $G - x$ into $\omega(G)$ stable sets of size $\alpha(G)$. Similarly, $\theta(G - x) = \alpha(G - x) = \alpha(G)$ yields a partition of $V(G - x)$ into $\alpha(G)$ cliques of size $\omega(G)$. ■

Study of p-critical graphs has benefitted by enlarging the class to include other graphs satisfying the properties in Theorem 8.1.33. Structural properties of the larger class are useful when proving the SPGC for special classes of graphs. Padberg [1974] began the study of these graphs. Several definitions were suggested to extend the class of p-critical graphs but turned out to be alternative characterizations of the same class. The definition we use originates in Bland–Huang–Trotter [1979].

8.1.34. Definition. For integers $a, w \geq 2$, a graph G is **a, w-partitionable** if it has $aw + 1$ vertices and for each $x \in V(G)$ the subgraph $G - x$ has a partition into a cliques of size w and a partition into w stable sets of size a.

8.1.35. Theorem. (Buckingham–Golumbic [1983]) A graph G of order $aw + 1$ is a, w-partitionable if and only if $\chi(G - x) = w$ and $\theta(G - x) = a$ for every $x \in V(G)$. Furthermore, $\omega(G) = w$ and $\alpha(G) = a$ for such graphs, and the inequalities $\chi(G - x) \le w$ and $\theta(G - x) \le a$ suffice.

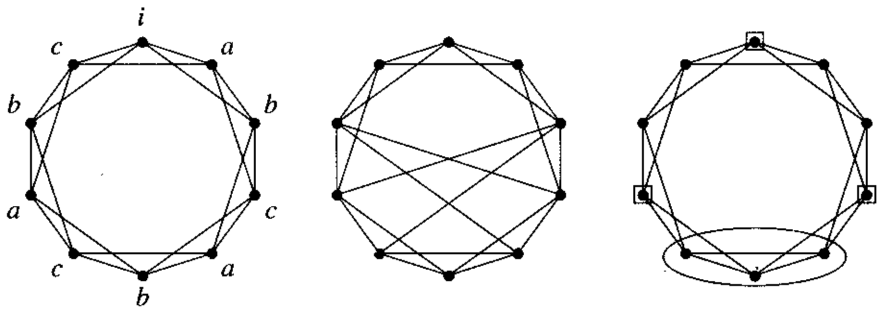
Proof: Let G be partitionable. Since $G - x$ is w-colorable and has a w-clique, $\chi(G - x) = w = \omega(G - x)$. Since $a \ge 2$, G is not a complete graph. Deleting a vertex x outside a maximum clique Q in G yields $\omega(G) = \omega(G - x) = w$. The same arguments for $\overline{G}$ yield the results for a.

Conversely, suppose that $\chi(G - x) \le w$ and $\theta(G - x) \le a$ for every x in $V(G)$. The latter inequality yields $\alpha(H) \le a$. Hence an optimal coloring of $G - x$ uses at most w stable sets of size at most a. Since $n(G - x) = aw$, such a coloring partitions $V(G - x)$ into w stable sets of size a. Similarly, a covering of $G - x$ by a cliques yields the desired clique partition. ∎

By Theorem 8.1.33 and Theorem 8.1.35, every p-critical graph is partitionable and every partitionable graph is imperfect. Furthermore, G is a, w-partitionable if and only if $\overline{G}$ is w, a-partitionable.

8.1.36. Example. *Cycle-powers.* The graph C_n^d is constructed by placing n vertices on a circle and making each vertex adjacent to the d nearest vertices in each direction on the circle. When $d = 1$, $C_n^d = C_n$. We view the vertices as the integers modulo n, in order. The graph C_{10}^2, shown on the left below, is neither perfect nor p-critical (the vertices $0, 2, 4, 6, 8$ induce C_5), but C_{10}^2 is 3,3-partitionable. When i is removed, the unique partition of the remaining nine vertices into three triangles is $\{(i + 1, i + 2, i + 3), (i + 4, i + 5, i + 6), (i + 7, i + 8, i + 9)\}$, and the unique partition into three stable sets is $\{(i + 1, i + 4, i + 7), (i + 2, i + 5, i + 8), (i + 3, i + 6, i + 9)\}$.

Always C_{aw+1}^{w-1} is a, w-partitionable. Every w consecutive vertices in $G - x$ form a clique, and every a vertices spaced by jumps of length w form a stable set. Showing that C_{aw+1}^{w-1} is p-critical if and only if $w = 2$ or $a = 2$ reduces the SPGC to the statement that G is p-critical if and only if $G = C_{\alpha(G)\omega(G)+1}^{\omega(G)-1}$. ∎



8.1.37. Example. *Other partitionable graphs.* Other partitionable graphs arise by adding unimportant edges to C_{aw+1}^{w-1}. In C_{10}^2, we can add any diagonal without changing the set of maximum cliques or the set of maximum stable sets, so the resulting graph is still partitionable. We will see that the SPGC would follow if all partitionable graphs came from cycle-powers by adding unimportant edges of this type.

Nevertheless, there are other partitionable graphs, such as the graph in the middle above (Chvátal–Graham–Perold–Whitesides [1979], Huang [1976]). Every edge in this graph belongs to a maximum clique, but it has two more edges than C_{10}^2. The partitions demonstrating that it is partitionable differ from those used for C_{10}^2 (Exercise 42). ■

8.1.38. Example. *Further properties of* C_{aw+1}^{w-1}. The graph C_{aw+1}^{w-1} has exactly n maximum cliques, each using w consecutive vertices on the cycle. Each vertex lies in w consecutive w-cliques. There are also exactly n maximum stable sets, each having $a-1$ gaps of length w and one gap of length $w+1$ between successive vertices. A maximum stable set containing x has a places for the larger gap, so each vertex x lies in a maximum stable sets.

Finally, a w-clique can avoid a maximum stable set only by fitting inside the gap of length $w+1$ (shown above Example 8.1.37 on the right). Thus there is a pairing $\{(Q_i, S_i)\}$ between the maximum stable sets and maximum cliques such that $Q_i \cap S_j = \emptyset$ if and only if $i = j$. ■

These "further properties" comprise the next characterization. The arguments are due to Padberg [1974], who used them in a polyhedral characterization of perfect graphs. Here combinatorial conclusions follow from properties of matrices in linear algebra. Other characterizations of partitionable graphs appeared in Bland–Huang–Trotter [1979], Golumbic [1980, p58-62], Tucker [1977], Chvátal–Graham–Perold–Whitesides [1979], and Buckingham [1980].

8.1.39. Theorem. A graph G of order $n = aw + 1$ is a, w-partitionable if and only if both conditions below hold:

1) $\alpha(G) = a$ and $\omega(G) = w$, and each vertex of G belongs to exactly w cliques of size w and a stable sets of size a.

2) G has exactly n maximum cliques $\{Q_i\}$ and exactly n maximum stable sets $\{S_j\}$, with $Q_i \cap S_j = \emptyset$ if and only if $i = j$ (Q_i and S_i are **mates**).

Proof: *Necessity.* We have proved $\chi(G - x) = w = \omega(G)$ and $\theta(G - x) = a = \alpha(G)$ for each $x \in V(G)$. Choose a clique Q of size w. For each $x \in Q$, $G - x$ has a partition into a cliques of size w. Together, Q and these w partitions form a list of $n = aw + 1$ maximum cliques $Q_1, \dots, Q_n$. Each vertex outside Q appears in one clique in each partition. Each vertex in Q appears in Q and once in $w-1$ partitions. Hence every vertex appears in exactly w cliques in the list.

For each Q_i, we obtain a maximum stable set S_i disjoint from Q_i. Choose $x \in Q_i$. The w maximum stable sets that partition $V(G - x)$ can meet Q_i only at the $w-1$ vertices other than x. Therefore, one of these stable sets misses Q_i; call it S_i. We will show that these two lists contain all the cliques and stable sets and have the desired intersection properties.

Let A be the incidence matrix with $a_{i,j} = 1$ if $x_j \in Q_i$ and $a_{i,j} = 0$ otherwise. Let B be the matrix with $b_{i,j} = 1$ if $x_j \in S_i$ and $b_{i,j} = 0$ otherwise. The ijth entry of AB^T is the dot product of row i of A with row j of B, which equals $|Q_i \cap S_j|$. By proving that $AB^T = J - I$, where J is the matrix of all 1s, we obtain $Q_i \cap S_j \neq \emptyset$ if and only if $i \neq j$. Since $J - I$ is nonsingular, this will also imply that A and B are nonsingular. Nonsingular matrices have distinct rows, and hence $Q_1, \ldots, Q_n$ and $S_1, \ldots, S_n$ will be distinct.

By construction, $|Q_i \cap S_i| = 0$. Since cliques and stable sets intersect at most once, to prove that $AB^T = J - I$ we need only show that each column of AB^T sums to $n-1$. Multiplying by the row vector $\mathbf{1}_n^T$ on the left computes these sums. We constructed A so that each column has w 1s (because each vertex appears in w cliques in the list) and B so that each row has a 1s (because each stable set has size a). Therefore,

$$\mathbf{1}_n^T(AB^T) = (\mathbf{1}_n^T A)B^T = w\mathbf{1}_n^T B^T = wa\mathbf{1}_n = (n-1)\mathbf{1}_n^T.$$

To prove that G has no other maximum cliques, we let q be the incidence vector of a maximum clique Q and show that q must be a row of A. Since A is nonsingular, its rows span $\mathbb{R}^n$, and we can write q as a linear combination: $q = tA$. To solve for t, we need A^{-1}. Since every row of A sums to ω, we have $A(\omega^{-1}J - B^T) = \omega^{-1}\omega J - (J - I) = I$, and hence $A^{-1} = \omega^{-1}J - B^T$. Thus,

$$t = qA^{-1} = q(\omega^{-1}J - B^T) = \omega^{-1}qJ - qB^T = \omega^{-1}\omega\mathbf{1}_n^T - qB^T.$$

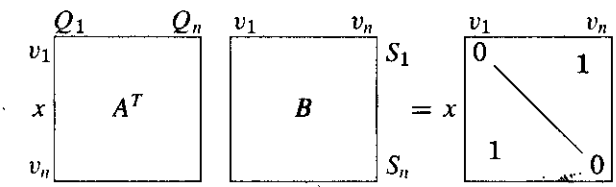
The ith column of B^T is the incidence vector of S_i; hence coordinate i of qB^T equals $|Q \cap S_i|$, which is 0 or 1. Hence t is a 0,1-vector and q is a sum of rows of A. Since q sums to ω, only one row can be used. Thus q is a row of A and $Q_1, \ldots, Q_n$ are the only maximum cliques.

The same argument applied to $\overline{G}$ shows that G has exactly n maximum stable sets, with each vertex appearing in a of them.

Sufficiency. By Theorem 8.1.35, we need only prove that $\chi(G - x) \leq w$ and $\theta(G - x) \leq a$ for all $x \in V(G)$. Given the cliques and stable sets as guaranteed by condition (2), define the incidence matrices A, B as above. By condition (1), each column of B has a 1s, and hence $JB = aJ = BJ$. The intersection requirements in condition (2) yield $AB^T = J - I$. This is nonsingular, so B is nonsingular and

$$A^T B = B^{-1}BA^T B = B^{-1}(J - I)B = B^{-1}BJ - I = J - I.$$

In the product $A^T B = J - I$, the row corresponding to $x \in V(G)$ states that $V(G - x)$ is covered by the mates of the w maximum cliques containing x (illustrated below), and hence $\chi(G - x) \leq w$. Similarly, the column corresponding to x states that $V(G - x)$ is covered by the mates of the a maximum stable sets containing x, and hence $\theta(G - x) \leq a$. ■



8.1.40. Corollary. If G is α, ω-partitionable and $\omega = 2$, then $G = C_{2\alpha+1}$; if $\alpha = 2$, then $G = \overline{C}_{2\omega+1}$. Hence the SPGC reduces to showing that p-critical graphs have $\omega = 2$ or $\alpha = 2$.

Proof: If $\omega = 2$, then every vertex belongs to exactly two cliques of size 2, so G is 2-regular. Furthermore, G is connected and has odd order $(2\alpha + 1)$, so G is an odd cycle. For $\alpha = 2$, consider $\overline{G}$. ■

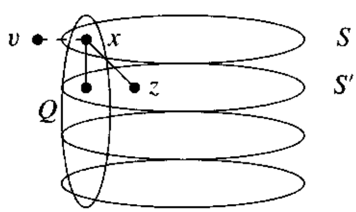
Henceforth we use $w, \omega(G), \omega$ interchangeably and $a, \alpha(G), \alpha$ interchangeably for partitionable graphs.

8.1.41. Theorem. (Tucker [1977]) Let x be a vertex in a partitionable graph G. The subgraph $G - x$ has a unique minimum coloring; denoted $X(G - x)$, it consists of the mates of the maximum cliques containing x. Similarly, $G - x$ has a unique minimum clique covering $X(G - x)$ consisting of the mates of the maximum stable sets containing x.

Proof: Since G is a, w-partitionable, $G - x$ is w-colorable using w stable sets of size a. Every w-clique containing x misses some color class, since the clique has only $w - 1$ vertices in $G - x$. Thus all w-cliques containing x have mates as color classes in the coloring. There are exactly w of these, so the coloring is unique. The other statement follows by complementation. ■

8.1.42. Theorem. (Buckingham–Golumbic [1983]) If x is a vertex of an α, ω-partitionable graph G, then $2\omega - 2 \le d(x) \le n - 2\alpha + 1$.

Proof: Select a vertex $v \not\leftrightarrow x$ (see illustration above). Let S be the stable set in $X(G - v)$ that contains x, and let S' be another stable set in $X(G - v)$. Choose $z \in N(x) \cap S_2$. In $\Theta(G - z)$, some clique Q contains x. Since $v \not\leftrightarrow x$, Q has one vertex in each stable set of $X(G - v)$, including S'. Since $Q \in \Theta(G - z)$ implies $z \notin Q$, this yields a second neighbor of x in S'. Thus x has at least two neighbors in each of the $\omega - 1$ stable sets in $X(G - v)$, yielding $d(x) \ge 2\omega - 2$. The same argument in $\overline{G}$ yields $n - 1 - d(x) = |N_{\overline{G}}(x)| \ge 2\alpha - 2$. ■



These bounds on vertex degrees in α, ω-partitionable graphs are sharp, as they hold with equality for powers of cycles.

8.1.43. Definition. An edge of a graph is **critical** if deleting it increases the independence number. A pair of nonadjacent vertices is **co-critical** if adding it increases the clique number.