Proof of Abel's theorem. It will suffice to prove the first claim, i.e., that

$$\lim_{x\to a+R: x\in(a-R,a+R)} \sum_{n=0}^{\infty} c_n(x-a)^n = \sum_{n=0}^{\infty} c_n R^n$$

whenever the sum $\sum_{n=0}^{\infty} c_n R^n$ converges; the second claim will then follow (why?) by replacing c_n by $(-1)^n c_n$ in the above claim. If we make the substitutions $d_n := c_n R^n$ and $y := \frac{x-a}{R}$, then the above claim can be rewritten as

$$\lim_{y\to 1: y\in(-1,1)} \sum_{n=0}^{\infty} d_n y^n = \sum_{n=0}^{\infty} d_n$$

whenever the sum $\sum_{n=0}^{\infty} d_n$ converges. (Why is this equivalent to the previous claim?)

Write $D := \sum_{n=0}^{\infty} d_n$, and for every $N \geq 0$ write

$$s_N := (\sum_{n=0}^{N-1} d_n) - D$$

so in particular $s_0 = -D$. Then observe that $\lim_{N\to\infty} S_N = 0$, and that $d_n = S_{n+1} - S_n$. Thus for any $y \in (-1, 1)$ we have

$$\sum_{n=0}^{\infty} d_n y^n = \sum_{n=0}^{\infty} (S_{n+1} - S_n) y^n.$$

Applying the summation by parts formula (Lemma 15.3.2), and noting that $\lim_{n\to\infty} y^n = 0$, we obtain

$$\sum_{n=0}^{\infty} d_n y^n = -S_0 y^0 - \sum_{n=0}^{\infty} S_{n+1}(y^{n+1} - y^n).$$

Observe that $-S_0 y^0 = +D$. Thus to finish the proof of Abel's theorem, it will suffice to show that

$$\lim_{y\to 1: y\in(-1,1)} \sum_{n=0}^{\infty} S_{n+1}(y^{n+1} - y^n) = 0.$$

Since y converges to 1, we may as well restrict y to $[0,1)$ instead of $(-1,1)$; in particular we may take y to be positive.

From the triangle inequality for series (Proposition 7.2.9), we have

$$\begin{aligned}|\sum_{n=0}^{\infty} S_{n+1}(y^{n+1}-y^n)| &\leq \sum_{n=0}^{\infty} |S_{n+1}(y^{n+1}-y^n)| \\ &= \sum_{n=0}^{\infty} |S_{n+1}|(y^n-y^{n+1}),\end{aligned}$$

so by the squeeze test (Corollary 6.4.14) it suffices to show that

$$\lim_{y\to 1:y\in(-1,1)} \sum_{n=0}^{\infty} |S_{n+1}|(y^n-y^{n+1}) = 0.$$

The expression $\sum_{n=0}^{\infty} |S_{n+1}|(y^n-y^{n+1})$ is clearly non-negative, so it will suffice to show that

$$\lim \sup_{y\to 1:y\in(-1,1)} \sum_{n=0}^{\infty} |S_{n+1}|(y^n-y^{n+1}) = 0.$$

Let $\varepsilon > 0$. Since S_n converges to 0, there exists an N such that $|S_n| \leq \varepsilon$ for all $n > N$. Thus we have

$$\sum_{n=0}^{\infty} |S_{n+1}|(y^n-y^{n+1}) \leq \sum_{n=0}^{N} |S_{n+1}|(y^n-y^{n+1}) + \sum_{n=N+1}^{\infty} \varepsilon(y^n-y^{n+1}).$$

The last summation is a telescoping series, which sums to εy^{N+1} (See Lemma 7.2.15, recalling from Lemma 6.5.2 that $y^n \to 0$ as $n \to \infty$), and thus

$$\sum_{n=0}^{\infty} |S_{n+1}|(y^n-y^{n+1}) \leq \sum_{n=0}^{N} |S_{n+1}|(y^n-y^{n+1}) + \varepsilon y^{N+1}.$$

Now take limits as $y \to 1$. Observe that $y^n - y^{n+1} \to 0$ as $y \to 1$ for every $n \in 0, 1, \ldots, N$. Since we can interchange limits and *finite* sums (Exercise 7.1.5), we thus have

$$\limsup_{n\to\infty} \sum_{n=0}^{\infty} |S_{n+1}|(y^n - y^{n+1}) \leq \varepsilon.$$

But $\varepsilon > 0$ was arbitrary, and thus we must have

$$\limsup_{n\to\infty} \sum_{n=0}^{\infty} |S_{n+1}|(y^n - y^{n+1}) = 0$$

since the left-hand side must be non-negative. The claim follows. □

Exercise 15.3.1. Prove Lemma 15.3.2. (Hint: first work out the relationship between the partial sums $\sum_{n=0}^{N}(a_{n+1}-a_n)b_n$ and $\sum_{n=0}^{N} a_{n+1}(b_{n+1}-b_n)$.)

15.4 Multiplication of power series

We now show that the product of two real analytic functions is again real analytic.

Theorem 15.4.1. *Let $f : (a-r, a+r) \to \mathbf{R}$ and $g : (a-r, a+r) \to \mathbf{R}$ be functions analytic on $(a-r, a+r)$, with power series expansions*

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

and

$$g(x) = \sum_{n=0}^{\infty} d_n (x-a)^n$$

respectively. Then $fg : (a-r, a+r) \to \mathbf{R}$ is also analytic on $(a-r, a+r)$, with power series expansion

$$f(x)g(x) = \sum_{n=0}^{\infty} e_n (x-a)^n$$

where $e_n := \sum_{m=0}^{n} c_m d_{n-m}$.

Remark 15.4.2. The sequence $(e_n)_{n=0}^{\infty}$ is sometimes referred to as the *convolution* of the sequences $(c_n)_{n=0}^{\infty}$ and $(d_n)_{n=0}^{\infty}$; it is closely related (though not identical) to the notion of convolution introduced in Definition 14.8.9.

Proof. We have to show that the series $\sum_{n=0}^{\infty} e_n(x-a)^n$ converges to $f(x)g(x)$ for all $x \in (a-r, a+r)$. Now fix x to be any point in $(a-r, a+r)$. By Theorem 15.1.6, we see that both f and g have radii of convergence at least r. In particular, the series $\sum_{n=0}^{\infty} c_n(x-a)^n$ and $\sum_{n=0}^{\infty} d_n(x-a)^n$ are absolutely convergent. Thus if we define

$$C := \sum_{n=0}^{\infty} |c_n(x-a)^n|$$

and

$$D := \sum_{n=0}^{\infty} |d_n(x-a)^n|$$

then C and D are both finite.

For any $N \geq 0$, consider the partial sum

$$\sum_{n=0}^{N} \sum_{m=0}^{\infty} |c_m(x-a)^m d_n(x-a)^n|.$$

We can rewrite this as

$$\sum_{n=0}^{N} |d_n(x-a)^n| \sum_{m=0}^{\infty} |c_m(x-a)^m|,$$

which by definition of C is equal to

$$\sum_{n=0}^{N} |d_n(x-a)^n| C,$$

which by definition of D is less than or equal to DC. Thus the above partial sums are bounded by DC for every N. In particular, the series

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} |c_m(x-a)^m d_n(x-a)^n|$$

is convergent, which means that the sum

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_m (x-a)^m d_n (x-a)^n$$

is absolutely convergent.

Let us now compute this sum in two ways. First of all, we can pull the $d_n(x-a)^n$ factor out of the $\sum_{m=0}^{\infty}$ summation, to obtain

$$\sum_{n=0}^{\infty} d_n (x-a)^n \sum_{m=0}^{\infty} c_m (x-a)^m.$$

By our formula for $f(x)$, this is equal to

$$\sum_{n=0}^{\infty} d_n (x-a)^n f(x);$$

by our formula for $g(x)$, this is equal to $f(x)g(x)$. Thus

$$f(x)g(x) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_m (x-a)^m d_n (x-a)^n.$$

Now we compute this sum in a different way. We rewrite it as

$$f(x)g(x) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_m d_n (x-a)^{n+m}.$$

By Fubini's theorem for series (Theorem 8.2.2), because the series was absolutely convergent, we may rewrite it as

$$f(x)g(x) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_m d_n (x-a)^{n+m}.$$

Now make the substitution $n' := n + m$, to rewrite this as

$$f(x)g(x) = \sum_{m=0}^{\infty} \sum_{n'=m}^{\infty} c_m d_{n'-m} (x-a)^{n'}.$$

If we adopt the convention that $d_j = 0$ for all negative j, then this is equal to

$$f(x)g(x) = \sum_{m=0}^{\infty} \sum_{n'=0}^{\infty} c_m d_{n'-m}(x-a)^{n'}.$$

Applying Fubini's theorem again, we obtain

$$f(x)g(x) = \sum_{n'=0}^{\infty} \sum_{m=0}^{\infty} c_m d_{n'-m}(x-a)^{n'},$$

which we can rewrite as

$$f(x)g(x) = \sum_{n'=0}^{\infty} (x-a)^{n'} \sum_{m=0}^{\infty} c_m d_{n'-m}.$$

Since d_j was 0 when j is negative, we can rewrite this as

$$f(x)g(x) = \sum_{n'=0}^{\infty} (x-a)^{n'} \sum_{m=0}^{n'} c_m d_{n'-m},$$

which by definition of e is

$$f(x)g(x) = \sum_{n'=0}^{\infty} e_{n'}(x-a)^{n'},$$

as desired. □

15.5 The exponential and logarithm functions

We can now use the machinery developed in the last few sections to develop a rigourous foundation for many standard functions used in mathematics. We begin with the exponential function.

Definition 15.5.1 (Exponential function). For every real number x, we define the *exponential function* $\exp(x)$ to be the real number

$$\exp(x) := \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Theorem 15.5.2 (Basic properties of exponential).

(a) *For every real number x, the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is absolutely convergent. In particular,* $\exp(x)$ *exists and is real for every $x \in \mathbf{R}$, the power series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ has an infinite radius of convergence, and* exp *is a real analytic function on $(-\infty, \infty)$.*

(b) exp *is differentiable on* $\mathbf{R}$, *and for every $x \in \mathbf{R}$,* $\exp'(x) = \exp(x)$.

(c) exp *is continuous on* $\mathbf{R}$, *and for every interval $[a, b]$, we have* $\int_{[a,b]} \exp(x)\, dx = \exp(b) - \exp(a)$.

(d) *For every $x, y \in \mathbf{R}$, we have* $\exp(x + y) = \exp(x) \exp(y)$.

(e) *We have* $\exp(0) = 1$. *Also, for every $x \in \mathbf{R}$,* $\exp(x)$ *is positive, and* $\exp(-x) = 1/\exp(x)$.

(f) exp *is strictly monotone increasing: in other words, if x, y are real numbers, then we have* $\exp(y) > \exp(x)$ *if and only if $y > x$.*

Proof. See Exercise 15.5.1. □

One can write the exponential function in a more compact form, introducing famous *Euler's number* $e = 2.71828183\ldots$, also known as the *base of the natural logarithm*:

Definition 15.5.3 (Euler's number). The number e is defined to be

$$e := \exp(1) = \sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots.$$

Proposition 15.5.4. *For every real number x, we have* $\exp(x) = e^x$.

Proof. See Exercise 15.5.3. □

In light of this proposition we can and will use e^x and $\exp(x)$ interchangeably.

Since $e > 1$ (why?), we see that $e^x \to +\infty$ as $x \to +\infty$, and $e^x \to 0$ as $x \to -\infty$. From this and the intermediate value theorem (Theorem 9.7.1) we see that the range of the function exp is $(0, \infty)$. Since exp is increasing, it is injective, and hence exp is a bijection from $\mathbf{R}$ to $(0, \infty)$, and thus has an inverse from $(0, \infty) \to \mathbf{R}$. This inverse has a name:

Definition 15.5.5 (Logarithm). We define the *natural logarithm function* $\log : (0, \infty) \to \mathbf{R}$ (also called ln) to be the inverse of the exponential function. Thus $\exp(\log(x)) = x$ and $\log(\exp(x)) = x$.

Since exp is continuous and strictly monotone increasing, we see that log is also continuous and strictly monotone increasing (see Proposition 9.8.3). Since exp is also differentiable, and the derivative is never zero, we see from the inverse function theorem (Theorem 10.4.2) that log is also differentiable. We list some other properties of the natural logarithm below.

Theorem 15.5.6 (Logarithm properties).

(a) *For every $x \in (0, \infty)$, we have $\ln'(x) = \frac{1}{x}$. In particular, by the fundamental theorem of calculus, we have $\int_{[a,b]} \frac{1}{x}\, dx = \ln(b) - \ln(a)$ for any interval $[a, b]$ in $(0, \infty)$.*

(b) *We have $\ln(xy) = \ln(x) + \ln(y)$ for all $x, y \in (0, \infty)$.*

(c) *We have $\ln(1) = 0$ and $\ln(1/x) = -\ln(x)$ for all $x \in (0, \infty)$.*

(d) *For any $x \in (0, \infty)$ and $y \in \mathbf{R}$, we have $\ln(x^y) = y \ln(x)$.*

(e) *For any $x \in (-1, 1)$, we have*

$$\ln(1 - x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}.$$

In particular, ln *is analytic at* 1, *with the power series expansion*

$$\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (x - 1)^n$$

for $x \in (0,2)$, with radius of convergence 1.

Proof. See Exercise 15.5.5. □

Example 15.5.7. We now give a modest application of Abel's theorem (Theorem 15.3.1): from the alternating series test we see that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is convergent. By Abel's theorem we thus see that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \lim_{x \to 2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (x-1)^n$$
$$= \lim_{x \to 2} \ln(x) = \ln(2),$$

thus we have the formula

$$\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots.$$

Exercise 15.5.1. Prove Theorem 15.5.2. (Hints: for part (a), use the ratio test. For parts (bc), use Theorem 15.1.6. For part (d), use Theorem 15.4.1. For part (e), use part (d). For part (f), use part (d), and prove that $\exp(x) > 1$ when x is positive. You may find the binomial formula from Exercise 7.1.4 to be useful.

Exercise 15.5.2. Show that for every integer $n \geq 3$, we have

$$0 < \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \ldots < \frac{1}{n!}.$$

(Hint: first show that $(n+k)! > 2^k n!$ for all $k = 1, 2, 3, \ldots$.) Conclude that $n!e$ is not an integer for every $n \geq 3$. Deduce from this that e is irrational. (Hint: prove by contradiction.)

Exercise 15.5.3. Prove Proposition 15.5.4. (Hint: first prove the claim when x is a natural number. Then prove it when x is an integer. Then prove it when x is a rational number. Then use the fact that real numbers are the limits of rational numbers to prove it for all real numbers. You may find the exponent laws (Proposition 6.7.3) to be useful.)

Exercise 15.5.4. Let $f : \mathbf{R} \to \mathbf{R}$ be the function defined by setting $f(x) := \exp(-1/x)$ when $x > 0$, and $f(x) := 0$ when $x < 0$. Prove that f is infinitely differentiable, and $f^{(k)}(0) = 0$ for every integer $k \geq 0$, but that f is not real analytic at 0.

Exercise 15.5.5. Prove Theorem 15.5.6. (Hints: for part (a), use the inverse function theorem (Theorem 10.4.2) or the chain rule (Theorem 10.1.15). For parts (bcd), use Theorem 15.5.2 and the exponent laws (Proposition 6.7.3). For part (e), start with the geometric series formula (Lemma 7.3.3) and integrate using Theorem 15.1.6).

Exercise 15.5.6. Prove that the natural logarithm function is real analytic on $(0, +\infty)$.

Exercise 15.5.7. Let $f : \mathbf{R} \to (0, \infty)$ be a positive, real analytic function such that $f'(x) = f(x)$ for all $x \in \mathbf{R}$. Show that $f(x) = Ce^x$ for some positive constant C; justify your reasoning. (Hint: there are basically three different proofs available. One proof uses the logarithm function, another proof uses the function e^{-x}, and a third proof uses power series. Of course, you only need to supply one proof.)

Exercise 15.5.8. Let $m > 0$ be an integer. Show that

$$\lim_{x \to +\infty} \frac{e^x}{x^m} = +\infty.$$

(Hint: what happens to the ratio between $e^{x+1}/(x+1)^m$ and e^x/x^m as $x \to +\infty$?)

Exercise 15.5.9. Let $P(x)$ be a polynomial, and let $c > 0$. Show that there exists a real number $N > 0$ such that $e^{cx} > |P(x)|$ for all $x > N$; thus an exponentially growing function, no matter how small the growth rate c, will eventually overtake any given polynomial $P(x)$, no matter how large. (Hint: use Exercise 15.5.8.)

Exercise 15.5.10. Let $f : (0, +\infty) \times \mathbf{R} \to \mathbf{R}$ be the exponential function $f(x, y) := x^y$. Show that f is continuous. (Hint: note that Propositions 9.4.10, 9.4.11 only show that f is continuous in each variable, which is insufficient, as Exercise 13.2.11 shows. The easiest way to proceed is to write $f(x, y) = \exp(y \ln x)$ and use the continuity of exp() and ln(). For an extra challenge, try proving this exercise without using the logarithm function.)

15.6 A digression on complex numbers

To proceed further we need the complex number system $\mathbf{C}$, which is an extension of the real number system $\mathbf{R}$. A full discussion of this important number system (and in particular the branch of mathematics known as *complex analysis*) is beyond the scope of this text; here, we need the system primarily because of a very useful mathematical operation, the *complex exponential function* $z \mapsto \exp(z)$, which generalizes the real exponential function $x \mapsto \exp(x)$ introduced in the previous section.

Informally, we could define the complex numbers as

Definition 15.6.1 (Informal definition of complex numbers). The complex numbers $\mathbf{C}$ are the set of all numbers of the form $a + bi$, where a, b are real numbers and i is a square root of -1, $i^2 = -1$.

However, this definition is a little unsatisfactory as it does not explain how to add, multiply, or compare two complex numbers. To construct the complex numbers rigourously we will first introduce a *formal* version of the complex number $a + bi$, which we shall temporarily denote as (a, b); this is similar to how in Chapter 4, when constructing the integers $\mathbf{Z}$, we needed a formal notion of subtraction a—b before the actual notion of subtraction $a - b$ could be introduced, or how when constructing the rational numbers, a formal notion of division $a//b$ was needed before it was superceded by the actual notion a/b of division. It is also similar to how, in the construction of the real numbers, we defined a formal limit $\mathrm{LIM}_{n\to\infty} a_n$ before we defined a genuine limit $\lim_{n\to\infty} a_n$.

Definition 15.6.2 (Formal definition of complex numbers). A *complex number* is any pair of the form (a, b), where a, b are real numbers, thus for instance $(2, 4)$ is a complex number. Two complex numbers $(a, b), (c, d)$ are said to be equal iff $a = c$ and $b = d$, thus for instance $(2 + 1, 3 + 4) = (3, 7)$, but $(2, 1) \neq (1, 2)$ and $(2, 4) \neq (2, -4)$. The set of all complex numbers is denoted $\mathbf{C}$.

At this stage the complex numbers $\mathbf{C}$ are indistinguishable from the Cartesian product $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$ (also known as the *Cartesian plane*). However, we will introduce a number of operations on the complex numbers, notably that of *complex multiplication*, which are not normally placed on the Cartesian plane $\mathbf{R}^2$. Thus one can think of the complex number system $\mathbf{C}$ as the Cartesian plane $\mathbf{R}^2$ equipped with a number of additional structures. We begin with the notion of addition and negation. Using the informal definition of the complex numbers, we expect

$$(a, b)+(c, d) = (a+bi)+(c+di) = (a+c)+(b+d)i = (a+c, b+d)$$

and similarly

$$-(a, b) = -(a + bi) = (-a) + (-b)i = (-a, -b).$$

As these derivations used the informal definition of the complex numbers, these identities have not yet been rigourously proven. However we shall simply *encode* these identities into our complex number system by defining the notion of addition and negation by the above rules:

Definition 15.6.3 (Complex addition, negation, and zero). If $z = (a, b)$ and $w = (c, d)$ are two complex numbers, we define their *sum* $z + w$ to be the complex number $z + w := (a + c, b + d)$. Thus for instance $(2, 4) + (3, -1) = (5, 3)$. We also define the *negation* $-z$ of z to be the complex number $-z := (-a, -b)$, thus for instance $-(3, -1) = (-3, 1)$. We also define the *complex zero* $0_{\mathbf{C}}$ to be the complex number $0_{\mathbf{C}} = (0, 0)$.

It is easy to see that notion of addition is well-defined in the sense that if $z = z'$ and $w = w'$ then $z + w = z' + w'$. Similarly for negation. The complex addition, negation, and zero operations obey the usual laws of arithmetic:

Lemma 15.6.4 (The complex numbers are an additive group). *If z_1, z_2, z_3 are complex numbers, then we have the commutative property $z_1 + z_2 = z_2 + z_1$, the associative property $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$, the identity property $z_1 + 0_{\mathbf{C}} = 0_{\mathbf{C}} + z_1 = z_1$, and the inverse property $z_1 + (-z_1) = (-z_1) + z_1 = 0_{\mathbf{C}}$.*

Proof. See Exercise 15.6.1. □

Next, we define the notion of complex multiplication and reciprocal. The informal justification of the complex multiplication rule is

$$\begin{aligned}(a,b)\cdot(c,d) &= (a+bi)(c+di)\\ &= ac+adi+bic+bidi\\ &= (ac-bd)+(ad+bc)i\\ &= (ac-bd, ad+bc)\end{aligned}$$

since i^2 is supposed to equal -1. Thus we define

Definition 15.6.5 (Complex multiplication). If $z = (a,b)$ and $w = (c,d)$ are complex numbers, then we define their *product* zw to be the complex number $zw := (ac-bd, ad+bc)$. We also introduce the *complex identity* $1_{\mathbf{C}} := (1,0)$.

This operation is easily seen to be well-defined, and also obeys the usual laws of arithmetic:

Lemma 15.6.6. *If z_1, z_2, z_3 are complex numbers, then we have the commutative property $z_1z_2 = z_2z_1$, the associative property $(z_1z_2)z_3 = z_1(z_2z_3)$, the identity property $z_11_{\mathbf{C}} = 1_{\mathbf{C}}z_1 = z_1$, and the distributivity properties $z_1(z_2+z_3) = z_1z_2 + z_1z_3$ and $(z_2+z_3)z_1 = z_2z_1 + z_3z_1$.*

Proof. See Exercise 15.6.2. □

The above lemma can also be stated more succinctly, as the assertion that $\mathbf{C}$ is a commutative ring. As is usual, we now write $z-w$ as shorthand for $z+(-w)$.

We now identify the real numbers $\mathbf{R}$ with a subset of the complex numbers $\mathbf{C}$ by identifying any real number x with the complex number $(x,0)$, thus $x \equiv (x,0)$. Note that this identification is consistent with equality (thus $x = y$ iff $(x,0) = (y,0)$), with addition ($x_1+x_2 = x_3$ iff $(x_1,0)+(x_2,0) = (x_3,0)$), with negation ($x = -y$ iff $(x,0) = -(y,0)$), and multiplication ($x_1x_2 = x_3$ iff $(x_1,0)(x_2,0) = (x_3,0)$), so we will no longer need to distinguish between "real addition" and "complex addition", and similarly for equality, negation, and multiplication. For instance, we can compute $3(2,4)$ by identifying the real number 3 with the complex number $(3,0)$ and then computing $(3,0)(2,4) = (3\times 2 - 0\times 4, 3\times 4 + 0\times 2) = (6,12)$. Note also that $0 \equiv 0_{\mathbf{C}}$ and $1 \equiv 1_{\mathbf{C}}$, so we can now drop the $\mathbf{C}$ subscripts from the zero 0 and the identity 1.

We now define i to be the complex number $i := (0,1)$. We can now reconstruct the informal definition of the complex numbers as a lemma:

Lemma 15.6.7. *Every complex number $z \in \mathbf{C}$ can be written as $z = a + bi$ for exactly one pair a, b of real numbers. Also, we have $i^2 = -1$, and $-z = (-1)z$.*

Proof. See Exercise 15.6.3. $\square$

Because of this lemma, we will now refer to complex numbers in the more usual notation $a + bi$, and discard the formal notation (a, b) henceforth.

Definition 15.6.8 (Real and imaginary parts). If z is a complex number with the representation $z = a + bi$ for some real numbers a, b, we shall call a the *real part* of z and denote $\Re(z) := a$, and call b the *imaginary part* of z and denote $\Im(z) := b$, thus for instance $\Re(3+4i) = 3$ and $\Im(3+4i) = 4$, and in general $z = \Re(z) + i\Im(z)$. Note that z is real iff $\Im(z) = 0$. We say that z is *imaginary* iff $\Re(z) = 0$, thus for instance $4i$ is imaginary, while $3 + 4i$ is neither real nor imaginary, and 0 is both real and imaginary. We define the *complex conjugate* $\overline{z}$ of z to be the complex number $\overline{z} := \Re(z) - i\Im(z)$, thus for instance $\overline{3+4i} = 3 - 4i$, $\overline{i} = -i$, and $\overline{3} = 3$.

The operation of complex conjugation has several nice properties:

Lemma 15.6.9 (Complex conjugation is an involution). *Let z, w be complex numbers, then $\overline{z+w} = \overline{z} + \overline{w}$, $\overline{-z} = -\overline{z}$, and $\overline{zw} = \overline{z}\,\overline{w}$. Also $\overline{\overline{z}} = z$. Finally, we have $\overline{z} = \overline{w}$ if and only if $z = w$, and $\overline{z} = z$ if and only if z is real.*

Proof. See Exercise 15.6.4. $\square$

The notion of absolute value $|x|$ was defined for rational numbers x in Definition 4.3.1, and this definition extends to real numbers in the obvious manner. However, we cannot extend this definition directly to the complex numbers, as most complex numbers are neither positive nor negative. (For instance, we do not classify i as either a positive or negative number; see Exercise 15.6.15 for some reasons why). However, we can still define absolute value by generalizing the formula $|x| = \sqrt{x^2}$ from Exercise 5.6.3:

Definition 15.6.10 (Complex absolute value). If $z = a + bi$ is a complex number, we define the *absolute value* $|z|$ of z to be the real number $|z| := \sqrt{a^2 + b^2} = (a^2 + b^2)^{1/2}$.

From Exercise 5.6.3 we see that this notion of absolute value generalizes the notion of real absolute value. The absolute value has a number of other good properties:

Lemma 15.6.11 (Properties of complex absolute value). *Let z, w be complex numbers. Then $|z|$ is a non-negative real number, and $|z| = 0$ if and only if $z = 0$. Also we have the identity $z\overline{z} = |z|^2$, and so $|z| = \sqrt{z\overline{z}}$. As a consequence we have $|zw| = |z||w|$ and $|\overline{z}| = |z|$. Finally, we have the inequalities*

$$-|z| \leq \Re(z) \leq |z|; \quad -|z| \leq \Im(z) \leq |z|; \quad |z| \leq |\Re(z)| + |\Im(z)|$$

as well as the triangle inequality $|z + w| \leq |z| + |w|$.

Proof. See Exercise 15.6.6. □

Using the notion of absolute value, we can define a notion of reciprocal:

Definition 15.6.12 (Complex reciprocal). If z is a non-zero complex number, we define the *reciprocal* z^{-1} of z to be the complex number $z^{-1} := |z|^{-2}\overline{z}$ (note that $|z|^{-2}$ is well-defined as a positive real number because $|z|$ is positive real, thanks to Lemma 15.6.11). Thus for instance $(1 + 2i)^{-1} = |1 + 2i|^{-2}(1 - 2i) = (1^2 + 2^2)^{-1}(1 - 2i) = \frac{1}{5} - \frac{2}{5}i$. If z is zero, $z = 0$, we leave the reciprocal 0^{-1} undefined.