1.2.2 Show that most of the numbers b are divisible by 60, and that the rest are divisible by 30 or 12.

Such numbers were in fact exceptionally "round" for the Babylonians, because 60 was the base for their system of numerals. It looks like they computed Pythagorean triples starting with the "round" numbers b and that the column of b values later broke off the tablet.

Euclid's formula for Pythagorean triples comes out of his theory of divisibility, which we shall take up in Section 3.3. Divisibility is also involved in some basic properties of Pythagorean triples, such as their evenness or oddness.

1.2.3 Show that any integer square leaves remainder 0 or 1 on division by 4.

1.2.4 Deduce from Exercise 1.2.3 that if (a,b,c) is a Pythagorean triple then a and b cannot both be odd.

1.3 Rational Points on the Circle

We know from Section 1.1 that a Pythagorean triple (a,b,c) can be realized by a triangle with sides a, b and hypotenuse c. This in turn yields a triangle with fractional (or *rational*) number sides $x = a/c$, $y = b/c$ and hypotenuse 1. All such triangles can be fitted inside the circle of radius 1 as shown in Figure 1.4. The sides x and y become what we now call the *coordinates* of

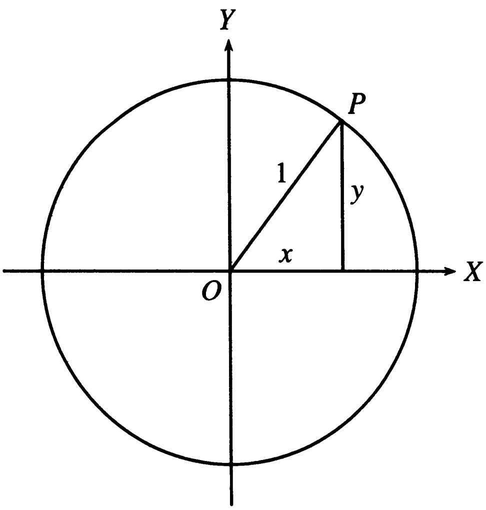

Figure 1.4: The unit circle

the point P on the circle. The Greeks did not use this language; however, they could derive the relationship between x and y we call the *equation of the circle*. Since

$$a^2+b^2=c^2 \tag{1}$$

we have

$$\left(\frac{a}{c}\right)^2+\left(\frac{b}{c}\right)^2=1,$$

so the relationship between $x=a/c$ and $y=b/c$ is

$$x^2+y^2=1. \tag{2}$$

Consequently, finding integer solutions of (1) is equivalent to finding rational solutions of (2), or finding *rational points* on the curve (2).

Such problems are now called *Diophantine*, after Diophantus, who was the first to deal with them seriously and successfully. *Diophantine equations* have acquired the more special connotation of equations for which integer solutions are sought, although Diophantus himself sought only rational solutions. [There is an interesting open problem that turns on this distinction. Matiyasevich (1970) proved that there is no algorithm for deciding which polynomial equations have integer solutions. It is not known whether there is an algorithm for deciding which polynomial equations have *rational* solutions.]

Most of the problems solved by Diophantus involve quadratic or cubic equations, usually with one obvious trivial solution. Diophantus used the obvious solution as a stepping stone to the nonobvious, but no account of his method survived. It was ultimately reconstructed by Fermat and Newton in the seventeenth century, and this so-called *chord-tangent construction* will be considered later. At present, we need it only for the equation $x^2+y^2=1$, which is an ideal showcase for the method in its simplest form.

A trivial solution of this equation is $x=-1$, $y=0$, which is the point Q on the unit circle (Figure 1.5). After a moment's thought, one realizes that a line through Q, with rational slope t,

$$y=t(x+1) \tag{3}$$

will meet the circle at a second rational point R. This is because substitution of $y=t(x+1)$ in $x^2+y^2=1$ gives a quadratic equation with rational coefficients and one rational solution ($x=-1$); hence the second solution must also be a rational value of x. But then the y value of this point will

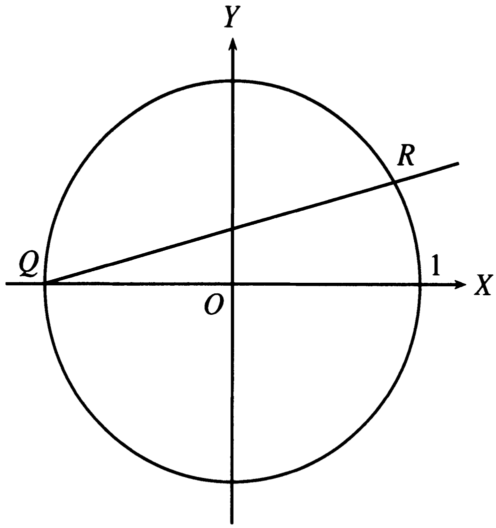

Figure 1.5: Construction of rational points

also be rational, since t and x will be rational in (3). Conversely, the chord joining Q to any other rational point R on the circle will have a rational slope. Thus by letting t run through all rational values, we find all rational points $R \neq Q$ on the unit circle.

What are these points? We find them by solving the equations just discussed. Substituting $y = t(x+1)$ in $x^2 + y^2 = 1$ gives

$$x^2 + t^2(x+1)^2 = 1$$

or

$$x^2(1+t^2) + 2t^2x + (t^2 - 1) = 0.$$

This quadratic equation in x has solutions -1 and $(1-t^2)/(1+t^2)$. The nontrivial solution $x = (1-t^2)/(1+t^2)$, when substituted in (3), gives $y = 2t/(1+t^2)$.

EXERCISES

The parameter t in the pair $\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$ runs through all rational numbers if $t = q/p$ and p, q run through all pairs of integers.

1.3.1 Deduce that if (a,b,c) is any Pythagorean triple then

$$\frac{a}{c} = \frac{p^2 - q^2}{p^2 + q^2}, \quad \frac{b}{c} = \frac{2pq}{p^2 + q^2}$$

for some integers p and q.

1.3.2 Use Exercise 1.3.1 to prove Euclid's formula for Pythagorean triples.

The triples (a,b,c) in Plimpton 322 seem to have been computed to provide right-angled triangles covering a range of shapes—their angles actually follow an increasing sequence in roughly equal steps. This raises the question: can the shape of any right-angled triangle be approximated by a Pythagorean triple?

1.3.3 Show that any right-angled triangle with hypotenuse 1 may be approximated arbitrarily closely by one with rational sides.

Some important information may be gleaned from Diophantus' method if we compare the angle at O in Figure 1.4 with the angle at Q in Figure 1.5. The two angles are shown in Figure 1.6, and hopefully you know from high school geometry the relation between them.

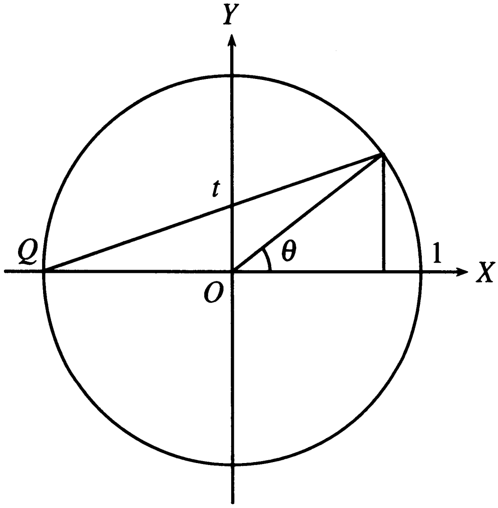

Figure 1.6: Angles in a circle

1.3.4 Use Figure 1.6 to show that $t = \tan\frac{\theta}{2}$ and

$$\cos\theta = \frac{1-t^2}{1+t^2}, \quad \sin\theta = \frac{2t}{1+t^2}.$$

1.4 Right-angled Triangles

It is high time we looked at Pythagoras' theorem from the traditional point of view, as a theorem about right-angled triangles; however, we shall be rather brief about its proof. It is not known how the theorem was first proved, but probably it was by simple manipulations of area, perhaps suggested by rearrangement of floor tiles. Just how easy it can be to prove Pythagoras' theorem is shown by Figure 1.7 [given by Heath (1925) in his edition of Euclid's *Elements*, Vol. 1, p. 354]. Each large square contains four copies of the given right-angled triangle. Subtracting these four triangles from the large square leaves, on the one hand (Figure 1.7, *left*), the sum of the squares on the two sides of the triangle. On the other hand (*right*), it also leaves the square on the hypotenuse. This proof, like the hundreds of others that have been given for Pythagoras' theorem, rests on certain geometric assumptions. It is in fact possible to transcend geometric assumptions by using numbers as the foundation for geometry, and Pythagoras' theorem then becomes true almost by definition, as an immediate consequence of the definition of distance (see Section 1.6).

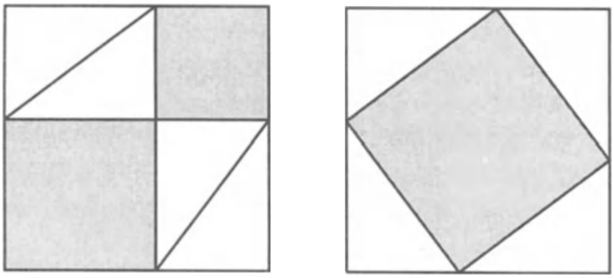

Figure 1.7: Proof of Pythagoras' theorem

To the Greeks, however, it did not seem possible to build geometry on the basis of numbers, due to a conflict between their notions of number and length. In the next section we shall see how this conflict arose.

EXERCISES

A way to see Pythagoras' theorem in a tiled floor was suggested by Magnus (1974), p. 159, and it is shown in Figure 1.8. (The dotted squares are not tiles; they are a hint.)

1.4.1 What has this figure to do with Pythagoras' theorem?

Euclid's first proof of Pythagoras' theorem, in Book I of the *Elements*, is also based on area. It depends only on the fact that triangles with the same base and height have equal area, though it involves a rather complicated figure. In Book VI, Proposition 31, he gives another proof, based on similar triangles (Figure 1.9).

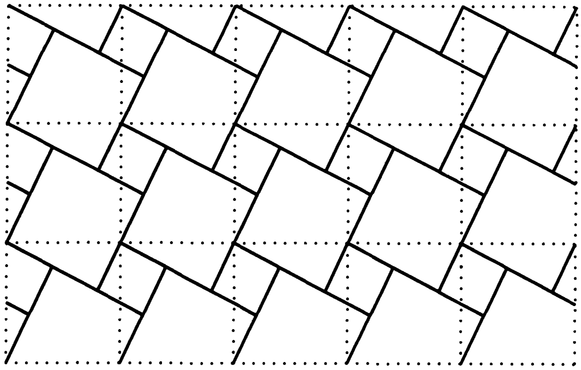

Figure 1.8: Pythagoras' theorem in a tiled floor

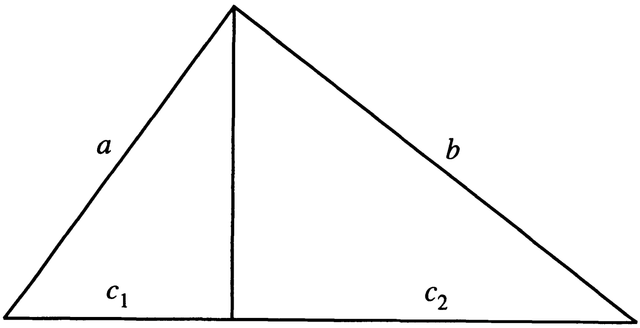

Figure 1.9: Another proof of Pythagoras' theorem

1.4.2 Show that the three triangles in Figure 1.9 are similar, and hence give another proof of Pythagoras' theorem by equating ratios of corresponding sides.

1.5 Irrational Numbers

We have mentioned that the Babylonians, although probably aware of the geometric meaning of Pythagoras' theorem, devoted most of their attention to the whole-number triples it had brought to light, the Pythagorean triples. Pythagoras and his followers were even more devoted to whole numbers. It was they who discovered the role of numbers in musical harmony: dividing a vibrating string in two raises its pitch by an octave, dividing in three raises the pitch another fifth, and so on. This great discovery, the first clue that the physical world might have an underlying mathematical structure, inspired them to seek numerical patterns, which to them meant *whole-number* patterns, everywhere. Imagine their consternation when they found that Pythagoras' theorem led to quantities that were not numerically computable. They found lengths that were *incommensurable*, that is, not measurable as integer multiples of the same unit. The ratio between such lengths is therefore not a ratio of whole numbers, hence in the Greek view not a ratio at all, or *irrational.*

The incommensurable lengths discovered by the Pythagoreans were the side and diagonal of the unit square. It follows immediately from Pythagoras' theorem that

$$(\text{diagonal})^2 = 1 + 1 = 2.$$

Hence if the diagonal and side are in the ratio m/n (where m and n can be assumed to have no common divisor), we have

$$\frac{m^2}{n^2} = 2,$$

whence

$$m^2 = 2n^2.$$

The Pythagoreans were interested in odd and even numbers, so they probably observed that the latter equation, which says m^2 is even, also implies that m is even, say $m = 2p$. But if

$$m = 2p,$$

then

$$2n^2 = m^2 = 4p^2;$$

hence

$$n^2 = 2p^2,$$

which similarly implies n is even, contrary to the hypothesis that m and n have no common divisor. (This proof appears in Aristotle's *Prior Analytics.* An alternative, more geometric proof is mentioned in Section 3.4.)

This discovery had profound consequences. Legend has it that the first Pythagorean to make the result public was drowned at sea [see Heath (1921), Vol. 1, pp. 65, 154]. It led to a split between the theories of number and space that was not healed until the nineteenth century (if then, some mathematicians would add). The Pythagoreans could not accept $\sqrt{2}$ as a number, but no one could deny that it was the diagonal of the unit square. Consequently, geometrical quantities had to be treated separately from numbers or, rather, without mentioning any numbers except rationals. Greek geometers thus developed ingenious techniques for precise handling of arbitrary lengths in terms of rationals, known as the *theory of proportions* and the *method of exhaustion*.

When Dedekind reconsidered these techniques in the nineteenth century, he realized that they provided an arithmetical interpretation of irrational quantities after all (Chapter 4). It was then possible, as Hilbert (1899) showed, to reconcile the apparent conflict between arithmetic and geometry. The key role of Pythagoras' theorem in this reconciliation is described in the next section.

EXERCISES

The crucial step in the proof that $\sqrt{2}$ is irrational is showing that m^2 even implies m is even or, equivalently, that m odd implies m^2 odd. It is worth taking a closer look at why this is true.

1.5.1 Writing an arbitrary odd number m in the form $2q+1$, for some integer q, show that m^2 also has the form $2r+1$, which shows that m^2 is also odd.

You probably did some algebra like this in Exercise 1.2.3, but if not, here is your chance:

1.5.2 Show that the square of $2q+1$ is in fact of the form $4s+1$, and hence explain why every integer square leaves remainder 0 or 1 on division by 4.

1.6 The Definition of Distance

The numerical interpretation of irrationals gave each length a numerical measure and hence made it possible to give coordinates x, y to each point P on the plane. The simplest way is to take a pair of perpendicular lines (*axes*) OX, OY and let x, y be the lengths of the perpendiculars from P to OX and OY respectively (Figure 1.10). Geometric properties of P are then reflected by arithmetical relations between x and y. This opens up the possibility of *analytic geometry*, whose development is discussed in Chapter 7. Here we want only to see how coordinates give a precise meaning to the basic geometric notion of *distance*.