

null space of  $g$ . Let  $g', f'_1, \dots, f'_{k-1}$  be the restrictions of  $g, f_1, \dots, f_{k-1}$  to the subspace  $N_k$ . Then  $g', f'_1, \dots, f'_{k-1}$  are linear functionals on the vector space  $N_k$ . Furthermore, if  $\alpha$  is a vector in  $N_k$  and  $f'_i(\alpha) = 0$ ,  $i = 1, \dots, k-1$ , then  $\alpha$  is in  $N_1 \cap \dots \cap N_k$  and so  $g'(\alpha) = 0$ . By the induction hypothesis (the case  $r = k-1$ ), there are scalars  $c_i$  such that

$$g' = c_1 f'_1 + \dots + c_{k-1} f'_{k-1}.$$

Now let

$$(3-16) \quad h = g - \sum_{i=1}^{k-1} c_i f_i.$$

Then  $h$  is a linear functional on  $V$  and (3-16) tells us that  $h(\alpha) = 0$  for every  $\alpha$  in  $N_k$ . By the preceding lemma,  $h$  is a scalar multiple of  $f_k$ . If  $h = c_k f_k$ , then

$$g = \sum_{i=1}^k c_i f_i. \quad \blacksquare$$

## Exercises

1. Let  $n$  be a positive integer and  $F$  a field. Let  $W$  be the set of all vectors  $(x_1, \dots, x_n)$  in  $F^n$  such that  $x_1 + \dots + x_n = 0$ .

- (a) Prove that  $W^0$  consists of all linear functionals  $f$  of the form

$$f(x_1, \dots, x_n) = c \sum_{j=1}^n x_j.$$

- (b) Show that the dual space  $W^*$  of  $W$  can be ‘naturally’ identified with the linear functionals

$$f(x_1, \dots, x_n) = c_1 x_1 + \dots + c_n x_n$$

on  $F^n$  which satisfy  $c_1 + \dots + c_n = 0$ .

2. Use Theorem 20 to prove the following. If  $W$  is a subspace of a finite-dimensional vector space  $V$  and if  $\{g_1, \dots, g_r\}$  is any basis for  $W^0$ , then

$$W = \bigcap_{i=1}^r N_{g_i}.$$

3. Let  $S$  be a set,  $F$  a field, and  $V(S; F)$  the space of all functions from  $S$  into  $F$ :

$$\begin{aligned} (f+g)(x) &= f(x) + g(x) \\ (cf)(x) &= cf(x). \end{aligned}$$

Let  $W$  be any  $n$ -dimensional subspace of  $V(S; F)$ . Show that there exist points  $x_1, \dots, x_n$  in  $S$  and functions  $f_1, \dots, f_n$  in  $W$  such that  $f_i(x_j) = \delta_{ij}$ .

## 3.7. The Transpose of a Linear Transformation

Suppose that we have two vector spaces over the field  $F$ ,  $V$ , and  $W$ , and a linear transformation  $T$  from  $V$  into  $W$ . Then  $T$  induces a linear

transformation from  $W^*$  into  $V^*$ , as follows. Suppose  $g$  is a linear functional on  $W$ , and let

$$(3-17) \quad f(\alpha) = g(T\alpha)$$

for each  $\alpha$  in  $V$ . Then (3-17) defines a function  $f$  from  $V$  into  $F$ , namely, the composition of  $T$ , a function from  $V$  into  $W$ , with  $g$ , a function from  $W$  into  $F$ . Since both  $T$  and  $g$  are linear, Theorem 6 tells us that  $f$  is also linear, i.e.,  $f$  is a linear functional on  $V$ . Thus  $T$  provides us with a rule  $T^t$  which associates with each linear functional  $g$  on  $W$  a linear functional  $f = T^t g$  on  $V$ , defined by (3-17). Note also that  $T^t$  is actually a linear transformation from  $W^*$  into  $V^*$ ; for, if  $g_1$  and  $g_2$  are in  $W^*$  and  $c$  is a scalar

$$\begin{aligned} [T^t(cg_1 + g_2)](\alpha) &= (cg_1 + g_2)(T\alpha) \\ &= cg_1(T\alpha) + g_2(T\alpha) \\ &= c(T^t g_1)(\alpha) + (T^t g_2)(\alpha) \end{aligned}$$

so that  $T^t(cg_1 + g_2) = cT^t g_1 + T^t g_2$ . Let us summarize.

**Theorem 21.** *Let  $V$  and  $W$  be vector spaces over the field  $F$ . For each linear transformation  $T$  from  $V$  into  $W$ , there is a unique linear transformation  $T^t$  from  $W^*$  into  $V^*$  such that*

$$(T^t g)(\alpha) = g(T\alpha)$$

for every  $g$  in  $W^*$  and  $\alpha$  in  $V$ .

We shall call  $T^t$  the **transpose** of  $T$ . This transformation  $T^t$  is often called the adjoint of  $T$ ; however, we shall not use this terminology.

**Theorem 22.** *Let  $V$  and  $W$  be vector spaces over the field  $F$ , and let  $T$  be a linear transformation from  $V$  into  $W$ . The null space of  $T^t$  is the annihilator of the range of  $T$ . If  $V$  and  $W$  are finite-dimensional, then*

- (i)  $\text{rank } (T^t) = \text{rank } (T)$
- (ii) *the range of  $T^t$  is the annihilator of the null space of  $T$ .*

*Proof.* If  $g$  is in  $W^*$ , then by definition

$$(T^t g)(\alpha) = g(T\alpha)$$

for each  $\alpha$  in  $V$ . The statement that  $g$  is in the null space of  $T^t$  means that  $g(T\alpha) = 0$  for every  $\alpha$  in  $V$ . Thus the null space of  $T^t$  is precisely the annihilator of the range of  $T$ .

Suppose that  $V$  and  $W$  are finite-dimensional, say  $\dim V = n$  and  $\dim W = m$ . For (i): Let  $r$  be the rank of  $T$ , i.e., the dimension of the range of  $T$ . By Theorem 16, the annihilator of the range of  $T$  then has dimension  $(m - r)$ . By the first statement of this theorem, the nullity of  $T^t$  must be  $(m - r)$ . But then since  $T^t$  is a linear transformation on an  $m$ -dimensional space, the rank of  $T^t$  is  $m - (m - r) = r$ , and so  $T$  and  $T^t$  have the same rank. For (ii): Let  $N$  be the null space of  $T$ . Every functional in the range

of  $T'$  is in the annihilator of  $N$ ; for, suppose  $f = T'g$  for some  $g$  in  $W^*$ ; then, if  $\alpha$  is in  $N$

$$f(\alpha) = (T'g)(\alpha) = g(T\alpha) = g(0) = 0.$$

Now the range of  $T'$  is a subspace of the space  $N^0$ , and

$$\dim N^0 = n - \dim N = \text{rank } (T) = \text{rank } (T')$$

so that the range of  $T'$  must be exactly  $N^0$ . ■

**Theorem 23.** *Let  $V$  and  $W$  be finite-dimensional vector spaces over the field  $F$ . Let  $\mathfrak{B}$  be an ordered basis for  $V$  with dual basis  $\mathfrak{B}^*$ , and let  $\mathfrak{B}'$  be an ordered basis for  $W$  with dual basis  $\mathfrak{B}'^*$ . Let  $T$  be a linear transformation from  $V$  into  $W$ ; let  $A$  be the matrix of  $T$  relative to  $(\mathfrak{B}, \mathfrak{B}')$  and let  $B$  be the matrix of  $T'$  relative to  $(\mathfrak{B}'^*, \mathfrak{B}^*)$ . Then  $B_{ij} = A_{ji}$ .*

*Proof.* Let

$$\begin{aligned} \mathfrak{B} &= \{\alpha_1, \dots, \alpha_n\}, & \mathfrak{B}' &= \{\beta_1, \dots, \beta_m\}, \\ \mathfrak{B}^* &= \{f_1, \dots, f_n\}, & \mathfrak{B}'^* &= \{g_1, \dots, g_m\}. \end{aligned}$$

By definition,

$$T\alpha_j = \sum_{i=1}^m A_{ij}\beta_i, \quad j = 1, \dots, n$$

$$T'g_j = \sum_{i=1}^n B_{ij}f_i, \quad j = 1, \dots, m.$$

On the other hand,

$$\begin{aligned} (T'g_j)(\alpha_i) &= g_j(T\alpha_i) \\ &= g_j \left( \sum_{k=1}^m A_{ki}\beta_k \right) \\ &= \sum_{k=1}^m A_{ki}g_j(\beta_k) \\ &= \sum_{k=1}^m A_{ki}\delta_{jk} \\ &= A_{ji}. \end{aligned}$$

For any linear functional  $f$  on  $V$

$$f = \sum_{i=1}^m f(\alpha_i)f_i.$$

If we apply this formula to the functional  $f = T'g_j$  and use the fact that  $(T'g_j)(\alpha_i) = A_{ji}$ , we have

$$T'g_j = \sum_{i=1}^n A_{ji}f_i$$

from which it immediately follows that  $B_{ij} = A_{ji}$ . ■

**Definition.** If  $A$  is an  $m \times n$  matrix over the field  $F$ , the transpose of  $A$  is the  $n \times m$  matrix  $A^t$  defined by  $A_{ij}^t = A_{ji}$ .

Theorem 23 thus states that if  $T$  is a linear transformation from  $V$  into  $W$ , the matrix of which in some pair of bases is  $A$ , then the transpose transformation  $T^t$  is represented in the dual pair of bases by the transpose matrix  $A^t$ .

**Theorem 24.** Let  $A$  be any  $m \times n$  matrix over the field  $F$ . Then the row rank of  $A$  is equal to the column rank of  $A$ .

*Proof.* Let  $\mathcal{G}$  be the standard ordered basis for  $F^n$  and  $\mathcal{G}'$  the standard ordered basis for  $F^m$ . Let  $T$  be the linear transformation from  $F^n$  into  $F^m$  such that the matrix of  $T$  relative to the pair  $\mathcal{G}, \mathcal{G}'$  is  $A$ , i.e.,

$$T(x_1, \dots, x_n) = (y_1, \dots, y_m)$$

where

$$y_i = \sum_{j=1}^n A_{ij}x_j.$$

The column rank of  $A$  is the rank of the transformation  $T$ , because the range of  $T$  consists of all  $m$ -tuples which are linear combinations of the column vectors of  $A$ .

Relative to the dual bases  $\mathcal{G}'^*$  and  $\mathcal{G}^*$ , the transpose mapping  $T^t$  is represented by the matrix  $A^t$ . Since the columns of  $A^t$  are the rows of  $A$ , we see by the same reasoning that the row rank of  $A$  (the column rank of  $A^t$ ) is equal to the rank of  $T^t$ . By Theorem 22,  $T$  and  $T^t$  have the same rank, and hence the row rank of  $A$  is equal to the column rank of  $A$ . ■

Now we see that if  $A$  is an  $m \times n$  matrix over  $F$  and  $T$  is the linear transformation from  $F^n$  into  $F^m$  defined above, then

$$\text{rank } (T) = \text{row rank } (A) = \text{column rank } (A)$$

and we shall call this number simply the **rank** of  $A$ .

**EXAMPLE 25.** This example will be of a general nature—more discussion than example. Let  $V$  be an  $n$ -dimensional vector space over the field  $F$ , and let  $T$  be a linear operator on  $V$ . Suppose  $\mathcal{G} = \{\alpha_1, \dots, \alpha_n\}$  is an ordered basis for  $V$ . The matrix of  $T$  in the ordered basis  $\mathcal{G}$  is defined to be the  $n \times n$  matrix  $A$  such that

$$T\alpha_j = \sum_{i=1}^n A_{ij}\alpha_i$$

in other words,  $A_{ij}$  is the  $i$ th coordinate of the vector  $T\alpha_j$  in the ordered basis  $\mathcal{G}$ . If  $\{f_1, \dots, f_n\}$  is the dual basis of  $\mathcal{G}$ , this can be stated simply

$$A_{ij} = f_i(T\alpha_j).$$

Let us see what happens when we change basis. Suppose

$$\mathfrak{B}' = \{\alpha'_1, \dots, \alpha'_n\}$$

is another ordered basis for  $V$ , with dual basis  $\{f'_1, \dots, f'_n\}$ . If  $B$  is the matrix of  $T$  in the ordered basis  $\mathfrak{B}'$ , then

$$B_{ij} = f'_i(T\alpha'_j).$$

Let  $U$  be the invertible linear operator such that  $U\alpha_j = \alpha'_j$ . Then the transpose of  $U$  is given by  $U^t f'_i = f_i$ . It is easy to verify that since  $U$  is invertible, so is  $U^t$  and  $(U^t)^{-1} = (U^{-1})^t$ . Thus  $f'_i = (U^{-1})^t f_i$ ,  $i = 1, \dots, n$ . Therefore,

$$\begin{aligned} B_{ij} &= [(U^{-1})^t f_i](T\alpha'_j) \\ &= f_i(U^{-1}T\alpha'_j) \\ &= f_i(U^{-1}TU\alpha_j). \end{aligned}$$

Now what does this say? Well,  $f_i(U^{-1}TU\alpha_j)$  is the  $i, j$  entry of the matrix of  $U^{-1}TU$  in the ordered basis  $\mathfrak{B}$ . Our computation above shows that this scalar is also the  $i, j$  entry of the matrix of  $T$  in the ordered basis  $\mathfrak{B}'$ . In other words

$$\begin{aligned} [T]_{\mathfrak{B}'} &= [U^{-1}TU]_{\mathfrak{B}} \\ &= [U^{-1}]_{\mathfrak{B}} [T]_{\mathfrak{B}} [U]_{\mathfrak{B}} \\ &= [U]_{\mathfrak{B}}^{-1} [T]_{\mathfrak{B}} [U]_{\mathfrak{B}} \end{aligned}$$

and this is precisely the change-of-basis formula which we derived earlier.

## Exercises

1. Let  $F$  be a field and let  $f$  be the linear functional on  $F^2$  defined by  $f(x_1, x_2) = ax_1 + bx_2$ . For each of the following linear operators  $T$ , let  $g = T'f$ , and find  $g(x_1, x_2)$ .

- (a)  $T(x_1, x_2) = (x_1, 0)$ ;
- (b)  $T(x_1, x_2) = (-x_2, x_1)$ ;
- (c)  $T(x_1, x_2) = (x_1 - x_2, x_1 + x_2)$ .

2. Let  $V$  be the vector space of all polynomial functions over the field of real numbers. Let  $a$  and  $b$  be fixed real numbers and let  $f$  be the linear functional on  $V$  defined by

$$f(p) = \int_a^b p(x) dx.$$

If  $D$  is the differentiation operator on  $V$ , what is  $D'f$ ?

3. Let  $V$  be the space of all  $n \times n$  matrices over a field  $F$  and let  $B$  be a fixed  $n \times n$  matrix. If  $T$  is the linear operator on  $V$  defined by  $T(A) = AB - BA$ , and if  $f$  is the trace function, what is  $T'f$ ?

4. Let  $V$  be a finite-dimensional vector space over the field  $F$  and let  $T$  be a linear operator on  $V$ . Let  $c$  be a scalar and suppose there is a non-zero vector  $\alpha$  in  $V$  such that  $T\alpha = c\alpha$ . Prove that there is a non-zero linear functional  $f$  on  $V$  such that  $T'f = cf$ .

5. Let  $A$  be an  $m \times n$  matrix with *real* entries. Prove that  $A = 0$  if and only if  $\text{trace}(A^t A) = 0$ .

6. Let  $n$  be a positive integer and let  $V$  be the space of all polynomial functions over the field of real numbers which have degree at most  $n$ , i.e., functions of the form

$$f(x) = c_0 + c_1 x + \cdots + c_n x^n.$$

Let  $D$  be the differentiation operator on  $V$ . Find a basis for the null space of the transpose operator  $D^t$ .

7. Let  $V$  be a finite-dimensional vector space over the field  $F$ . Show that  $T \rightarrow T^t$  is an isomorphism of  $L(V, V)$  onto  $L(V^*, V^*)$ .

8. Let  $V$  be the vector space of  $n \times n$  matrices over the field  $F$ .

(a) If  $B$  is a fixed  $n \times n$  matrix, define a function  $f_B$  on  $V$  by  $f_B(A) = \text{trace}(B^t A)$ . Show that  $f_B$  is a linear functional on  $V$ .

(b) Show that every linear functional on  $V$  is of the above form, i.e., is  $f_B$  for some  $B$ .

(c) Show that  $B \rightarrow f_B$  is an isomorphism of  $V$  onto  $V^*$ .

# 4. *Polynomials*

## 4.1. *Algebras*

The purpose of this chapter is to establish a few of the basic properties of the algebra of polynomials over a field. The discussion will be facilitated if we first introduce the concept of a linear algebra over a field.

**Definition.** Let  $F$  be a field. A **linear algebra over the field  $F$**  is a vector space  $\mathfrak{A}$  over  $F$  with an additional operation called **multiplication of vectors** which associates with each pair of vectors  $\alpha, \beta$  in  $\mathfrak{A}$  a vector  $\alpha\beta$  in  $\mathfrak{A}$  called the **product** of  $\alpha$  and  $\beta$  in such a way that

(a) *multiplication is associative,*

$$\alpha(\beta\gamma) = (\alpha\beta)\gamma$$

(b) *multiplication is distributive with respect to addition,*

$$\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma \quad \text{and} \quad (\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma$$

(c) *for each scalar  $c$  in  $F$ ,*

$$c(\alpha\beta) = (c\alpha)\beta = \alpha(c\beta).$$

If there is an element  $1$  in  $\mathfrak{A}$  such that  $1\alpha = \alpha 1 = \alpha$  for each  $\alpha$  in  $\mathfrak{A}$ , we call  $\mathfrak{A}$  a **linear algebra with identity over  $F$** , and call  $1$  the **identity** of  $\mathfrak{A}$ . The algebra  $\mathfrak{A}$  is called **commutative** if  $\alpha\beta = \beta\alpha$  for all  $\alpha$  and  $\beta$  in  $\mathfrak{A}$ .

**EXAMPLE 1.** The set of  $n \times n$  matrices over a field, with the usual operations, is a linear algebra with identity; in particular the field itself is an algebra with identity. This algebra is not commutative if  $n \geq 2$ . The field itself is (of course) commutative.