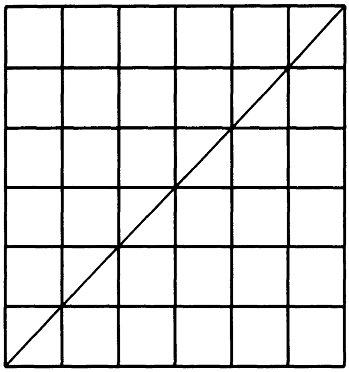

Figure 8.4: The actual floor

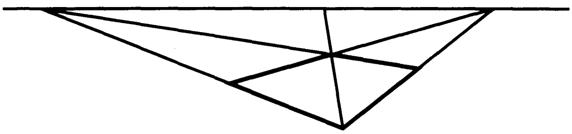

Figure 8.5: Tiled floor with arbitrary orientation

8.1.2 By using diagonals as in Exercise 8.1.1, show how to generate the lines in the tiling when the baseline is parallel to the horizon, without making any measurements.

8.2 Anamorphosis

It is clear from the Alberti veil construction that a perspective view will not look absolutely correct except when seen from the viewpoint used by the artist. Experience shows, however, that distortion is not noticeable except from extreme viewing positions. Following the mastery of perspective by the Italian artists, an interesting variation developed, in which the picture looks right from only one, extreme, viewpoint. The first known example of this style, known as *anamorphosis*, is an undated drawing by Leonardo da Vinci from the *Codex Atlanticus* (compiled between 1483 and 1518). Figure 8.6 shows part of this drawing, a child's face which looks correct when viewed with the eye near the right-hand edge of the page.

Figure 8.6: Leonardo's drawing of a face

The idea was taken up by German artists around 1530. The most famous example occurs in Holbein's painting *The Two Ambassadors* (1533). A mysterious streak across the bottom of the picture becomes a skull when viewed from near the picture's edge. For an excellent view of this picture and a history of anamorphosis, see Baltrušaitis (1977) and Wright (1983), pp. 146–156. The art of anamorphosis reached its technically most advanced form in France in the early seventeenth century. It seems no coincidence that this was also the time and place of the birth of projective geometry. In fact, key figures in the two fields, Niceron and Desargues, were well aware of each other's work.

Niceron (1613–1646) was a student of Mersenne and, like him, a monk in the order of Minims. He executed some extraordinary anamorphic wall paintings, up to 55 meters long, and also explained the theory in *La perspective curieuse* [Nicéron (1638)]. Figure 8.7 is his illustration of anamorphosis of a chair [from Baltrušaitis (1977), p. 44]. The anamorphosis, viewed normally, shows a chair like none ever seen, yet from a suitably extreme point one sees an ordinary chair in perspective.

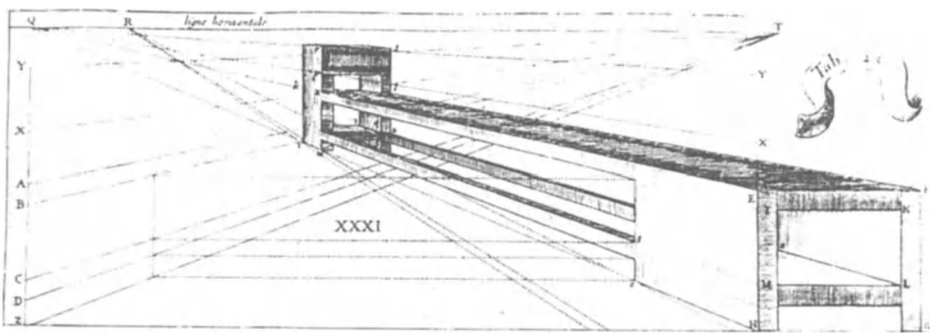

Figure 8.7: Niceron's chair

This example exposes an important mathematical fact: *a perspective view of a perspective view is not in general a perspective view*. Iteration of perspective views gives what we now call a *projective* view, and Niceron's chair shows that projectivity is a broader concept than perspectivity. As a consequence, *projective geometry*, which studies the properties that are invariant under projection, is broader than the theory of perspective. Perspective itself did not develop into a mathematical theory, *descriptive geometry*, until the end of the eighteenth century.

8.3 Desargues' Projective Geometry

The mathematical setting in which one can understand Alberti's veil is the family of lines ("light rays") through a point (the "eye"), together with a plane V (the "veil") (Figure 8.8). In this setting, the problems of perspective and anamorphosis were not very difficult, but the *concepts* were interesting and a challenge to traditional geometric thought. Contrary to Euclid, one had the following:

(i) Points at infinity ("vanishing points") where parallels met.

(ii) Transformations that changed lengths and angles (projections).

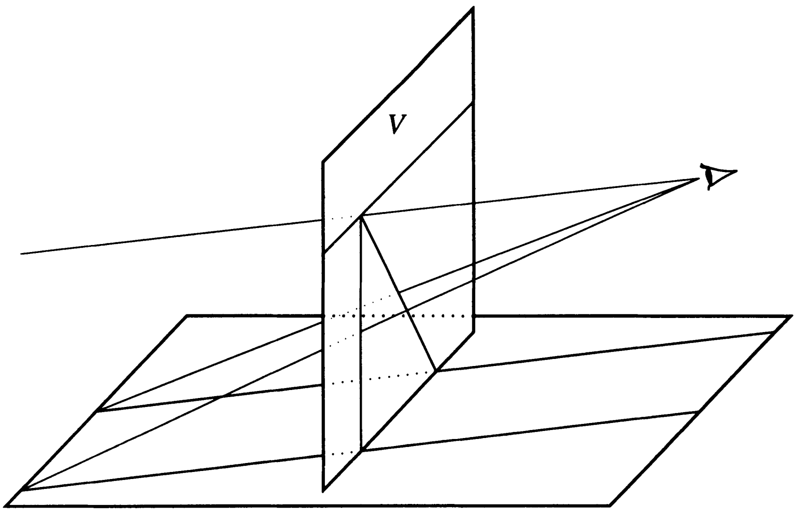

Figure 8.8: Seeing through Alberti's veil

The first to construct a mathematical theory incorporating these ideas was Desargues (1591–1661), although the idea of points at infinity had already been used by Kepler (1604), p. 93. The book of Desargues (1639), *Brouillon project d'une atteinte aux événemens des rencontres du cône avec un plan* (Schematic Sketch of What Happens When a Cone Meets a Plane), suffered an extreme case of delayed recognition, being completely lost for 200 years. Fortunately, his two most important theorems, the so-called Desargues' theorem and the invariance of the cross-ratio, were published in a book on perspective [Bosse (1648)]. The text of Desargues (1639) and a portion of Bosse (1648) containing Desargues' theorem may be found in Taton (1951). An English translation, with an extensive historical and mathematical analysis, is in Field and Gray (1987).

Kepler and Desargues both postulated one point at infinity on each line, closing the line to a "circle of infinite radius." All lines in a family of parallels share the same point at infinity. Nonparallel lines, having a finite point in common, do not have the same point at infinity. Thus any two distinct lines have exactly one point in common—a simpler axiom than Euclid's. Strangely enough, the line at infinity was only introduced into the theory by Poncelet (1822), even though it is the most obvious line in perspective drawing, the horizon. Desargues made extensive use of projections in the *Brouillon projet*; he was the first to use them to prove theorems about conic sections.

Desargues' theorem is a property of triangles in perspective illustrated by Figure 8.9. The theorem states that the points X, Y, Z at the intersections of corresponding sides lie in a line. This is obvious if the triangles are in space, since the line is the intersection of the planes containing them. The theorem in the plane is subtly but fundamentally different and requires a separate proof, as Desargues realized. In fact, Desargues' theorem was shown to play a key role in the foundations of projective geometry by Hilbert (1899) (see Section 20.7).

The invariance of the cross-ratio answers a natural question first raised by Alberti: since length and angle are not preserved by projection, what is? No property of three points on a line can be invariant because it is possible to project any three points on a line to any three others (Exercise 8.3.1). At least four points are therefore needed, and the cross-ratio is in fact a projective invariant of four points. The cross-ratio $(ABCD)$ of points A,B,C,D on a line (in that order) is $\frac{CA}{CB}/\frac{DA}{DB}$. Its invariance is most simply seen by reexpressing it in terms of angles using Figure 8.10. Let O be any

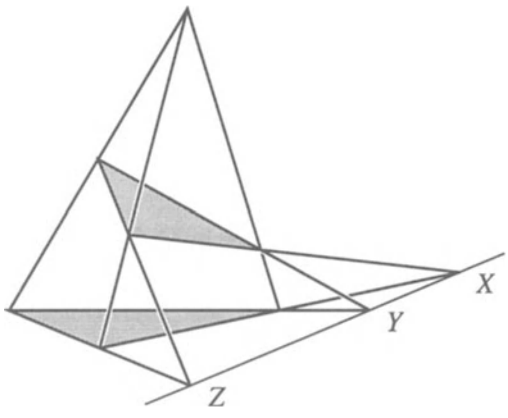

Figure 8.9: Desargues' theorem

point outside the line and consider the areas of the triangles OCA, OCB, ODA, and ODB. First compute them from bases on AB and height h, then recompute using OA and OB as bases and heights expressed in terms of the sines of angles at O:

$$\begin{aligned}
\frac{1}{2}h \cdot CA &= \text{area } OCA = \frac{1}{2}OA \cdot OC \sin\angle COA, \\
\frac{1}{2}h \cdot CB &= \text{area } OCB = \frac{1}{2}OB \cdot OC \sin\angle COB, \\
\frac{1}{2}h \cdot DA &= \text{area } ODA = \frac{1}{2}OA \cdot OD \sin\angle DOA, \\
\frac{1}{2}h \cdot DB &= \text{area } ADB = \frac{1}{2}OB \cdot OD \sin\angle DOB.
\end{aligned}$$

Substituting the values of CA, CB, DA, and DB from these equations we find [following Möbius (1827)] the cross-ratio in terms of angles at O:

$$\frac{CA}{CB}\bigg/\frac{DA}{DB} = \frac{\sin\angle COA}{\sin\angle COB}\bigg/\frac{\sin\angle DOA}{\sin\angle DOB}.$$

Any four points A', B', C', D' in perspective with A, B, C, D from a point O have the same angles (Figure 8.10), hence they will have the same cross-ratio. But then so will any four points A'', B'', C'', D'' projectively related to A, B, C, D, since a projectivity is by definition the product of a sequence of perspectivities.

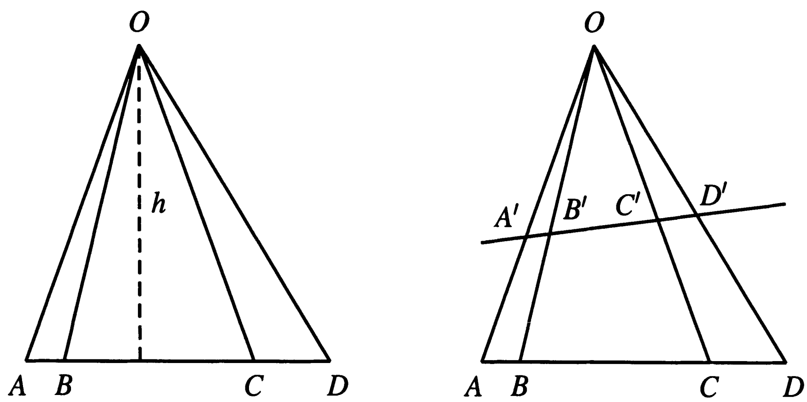

Figure 8.10: Evaluating the cross ratio

Exercises

As mentioned above, we cannot hope for an invariant that is simpler than the cross-ratio, because any three points in a line are projectively related to any other.

8.3.1 Show that any three points on a line can be sent to any other three points on a line by projection.

The case of Desargues' theorem where the two triangles lie in the same plane can be proved by viewing the plane in space. The setup for the proof is shown in Figure 8.11. The triangles $A_1B_1C_1$ and $A_2B_2C_2$ are in perspective from O in a plane Π, P is a point in space outside Π, and the line OD_1D_2 meets Π only at O.

8.3.2 Show that the triangles $A_1C_1D_1$ and $A_2C_2D_2$ are in different planes, and in perspective from O.

Thus it follows from the nonplanar version of Desargues' theorem that the intersections of the side pairs (A_1D_1, A_2D_2), (A_1C_1, A_2C_2), and (C_1D_1, C_2D_2) lie in a line.

8.3.3 Show that these intersections are projected from P to the intersections of the side pairs (A_1B_1, A_2B_2), (A_1C_1, A_2C_2), and (C_1B_1, C_2B_2), and hence deduce the planar Desargues' theorem.

8.3.4 Does this proof capture your intuitive idea of looking at the planar Desargues configuration (Figure 8.9) and interpreting it three-dimensionally? If so, what does the point P represent?

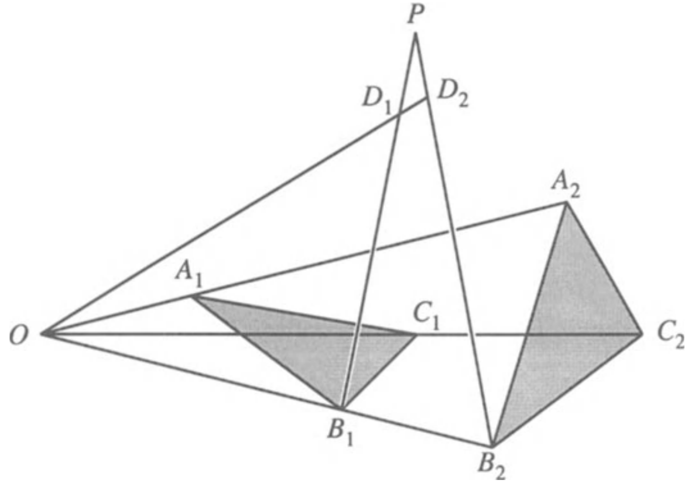

Figure 8.11: The planar Desargues' theorem

8.4 The Projective View of Curves

The problems of perspective drawing mainly involved the geometry of straight lines. There were, it is true, problems such as drawing ellipses to look like perspective views of circles, but artists were generally content to solve such problems by interpolating smooth-looking curves in a suitable straight-line framework. An example is the drawing of a chalice by Uccello (1397–1475) in Figure 8.12.

A mathematical theory of perspective for curves became possible with the advent of analytic geometry. When a curve is specified by an equation $f(x,y) = 0$, the equation of any perspective view is obtainable by suitably transforming x and y. However, this transformational viewpoint, even though quite simple algebraically, emerged only with Möbius (1827). The first works in projective geometry, by Desargues (1639) and Pascal (1640), used the language of classical geometry, even though the language of equations was available from Descartes (1637). This was understandable, not only because the analytic method was so obscure in Descartes, but also because the advantages of the projective method could be more clearly seen when it was used in a classical setting. Desargues and Pascal confined themselves to straight lines and conic sections, showing how projective geometry could easily reach and surpass the results obtained by the Greeks.