30. Let λ be an eigenvalue of the linear transformation T on the finite dimensional vector space V over the field F. Let $r_k = \dim_F(T-\lambda)^k V$ be the rank of the linear transformation $(T-\lambda)^k$ on V. For any $k \geq 1$, prove that $r_{k-1}-2r_k+r_{k+1}$ is the number of Jordan blocks of T corresponding to λ of size k [use Exercise 12 in Section 1]. (This gives an efficient method for determining the Jordan canonical form for T by computing the ranks of the matrices $(A-\lambda I)^k$ for a matrix A representing T, cf. Exercise 31(a) in Section 11.2.)

31. Let N be an $n \times n$ matrix with coefficients in the field F. The matrix N is said to be *nilpotent* if some power of N is the zero matrix, i.e., $N^k = 0$ for some k. Prove that any nilpotent matrix is similar to a block diagonal matrix whose blocks are matrices with 1's along the first superdiagonal and 0's elsewhere.

32. Prove that if N is an $n \times n$ nilpotent matrix then in fact $N^n = 0$.

33. Let A be a strictly upper triangular $n \times n$ matrix (all entries on and below the main diagonal are zero). Prove that A is nilpotent.

34. Prove that the trace of a nilpotent $n \times n$ matrix is 0 (recall the trace of a matrix is the sum of the diagonal elements).

35. For $0 \leq i \leq n$, let d_i be the g.c.d. of the determinants of all the $i \times i$ minors of $xI - A$, for A as in Theorem 21 (take the 0×0 minor to be 1). Prove that the i^{th} element along the diagonal of the Smith Normal Form for A is d_i/d_{i-1}. This gives the invariant factors for A. [Show these g.c.d.s do not change under elementary row and column operations.]

36. Let $V = \mathbb{C}^n$ be the usual n-dimensional vector space of n-tuples $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ of complex numbers. Let T be the linear transformation defined by setting $T(\alpha_1, \alpha_2, \ldots, \alpha_n)$ equal to $(0, \alpha_1, \alpha_2, \ldots, \alpha_{n-1})$. Determine the Jordan canonical form for T.

37. Let J be a Jordan block of size n with eigenvalue λ over $\mathbb{C}$.
    (a) Prove that the Jordan canonical form for the matrix J^2 is the Jordan block of size n with eigenvalue λ^2 if $\lambda \neq 0$.
    (b) If $\lambda = 0$ prove that the Jordan canonical form for J^2 has two blocks (with eigenvalues 0) of size $\frac{n}{2}, \frac{n}{2}$ if n is even and of size $\frac{n-1}{2}, \frac{n+1}{2}$ if n is odd.

38. Determine necessary and sufficient conditions for a matrix $A \in M_n(\mathbb{C})$ to have a square root, i.e., for there to exist another matrix $B \in M_n(\mathbb{C})$ such that $A = B^2$. [Suppose B is in Jordan canonical form and consider the Jordan canonical form for B^2 using the previous exercise.]

39. Let J be a Jordan block of size n with eigenvalue λ over a field F of characteristic 2. Determine the Jordan canonical form for the matrix J^2. Determine necessary and sufficient conditions for a matrix $A \in M_n(F)$ to have a square root, i.e., for there to exist another matrix $B \in M_n(F)$ such that $A = B^2$.

The remaining exercises explore functions (power series) of a matrix and introduce some applications of the Jordan canonical form to the theory of differential equations.

Throughout these exercises the matrices are assumed to be $n \times n$ matrices with entries from the field K, where K is either the real or complex numbers. Let

$$G(x) = \sum_{k=0}^{\infty} \alpha_k x^k$$

be a power series with coefficients from K. Let $G_N(x) = \sum_{k=0}^{N} \alpha_k x^k$ be the N^{th} partial sum of $G(x)$ and for each $A \in M_n(K)$ let $G_N(A)$ be the element of $M_n(K)$ obtained (as usual) by substituting A in this polynomial. For each fixed i, j we obtain a sequence of real or complex numbers c_{ij}^N, $N = 0, 1, 2, \dots$ by taking c_{ij}^N to be the i, j entry of the matrix $G_N(A)$. The series

$$G(A) = \sum_{k=0}^{\infty} \alpha_k A^k$$

is said to *converge* to the matrix C in $M_n(K)$ if for each $i, j \in \{1, 2, \dots, n\}$ the sequence c_{ij}^N, $N = 0, 1, 2, \dots$ converges to the i, j entry of C (in which case we write $G(A) = C$). Say $G(A)$ *converges* if there is some $C \in M_n(K)$ such that $G(A) = C$. If A is a 1×1 matrix, this is the usual notion of convergence of a series in K.

For $A = (a_{ij}) \in M_n(K)$ define

$$\| A \| = \sum_{i,j=1}^{n} |a_{ij}|$$

i.e., $\| A \|$ is the sum of the absolute values of all the entries of A.

40. Prove that for all $A, B \in M_n(K)$ and all $\alpha \in K$
 (a) $\| A + B \| \le \| A \| + \| B \|$
 (b) $\| AB \| \le \| A \| \cdot \| B \|$
 (c) $\| \alpha A \| = |\alpha| \cdot \| A \|$.

41. Let R be the radius of convergence of the real or complex power series $G(x)$ (where $R = \infty$ if $G(x)$ converges for all $x \in K$).
 (a) Prove that if $\| A \| < R$ then $G(A)$ converges.
 (b) Deduce that for *all* matrices A the following power series converge:

$$\sin(A) = A - \frac{A^3}{3!} + \frac{A^5}{5!} + \cdots + (-1)^k \frac{A^{2k+1}}{(2k+1)!} + \cdots$$

$$\cos(A) = I - \frac{A^2}{2!} + \frac{A^4}{4!} + \cdots + (-1)^k \frac{A^{2k}}{(2k)!} + \cdots$$

$$\exp(A) = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots + \frac{A^k}{k!} + \cdots$$

 where I is the $n \times n$ identity matrix.

In view of applications to the theory of differential equations we introduce a variable t at this point, so that for $A \in M_n(K)$ the matrix At is obtained from A by multiplying each entry by t (which is the same as multiplying A by the "scalar" matrix tI). We obtain a function from a subset of K into $M_n(K)$ defined by $t \mapsto G(At)$ at all points t where the series $G(At)$ converges. In particular, $\sin(At)$, $\cos(At)$ and $\exp(At)$ converge for all $t \in K$.

42. Let P be a nonsingular $n \times n$ matrix.
 (a) Prove that $PG(At)P^{-1} = G(PAtP^{-1}) = G(PAP^{-1}t)$. (This implies that, up to a change of basis, it suffices to compute $G(At)$ for matrices A in canonical form). [Take limits of partial sums to get the first equality. The second equality is immediate because the matrix tI commutes with every matrix.]
 (b) Prove that if A is the direct sum of matrices $A_1, A_2, \dots, A_m$, then $G(At)$ is the direct sum of the matrices $G(A_1t), G(A_2t), \dots, G(A_mt)$.
 (c) Show that if Z is the diagonal matrix with entries $z_1, z_2, \dots, z_n$ then $G(Zt)$ is the diagonal matrix with entries $G(z_1t), G(z_2t), \dots, G(z_nt)$.

The matrix $\exp(A)$ defined in Exercise 41(b) is called the *exponential* of A and is often denoted by e^A. The next three exercises lead to a formula for the matrix $\exp(Jt)$, where J is an elementary Jordan matrix.

43. Prove that if A and B are *commuting* matrices then $\exp(A+B) = \exp(A)\exp(B)$. [Treat A and B as commuting indeterminates and deduce this by comparing the power series on the left hand side with the product of the two power series on the right hand side.]

44. Use the preceding exercise to show that if M is any matrix and λ is any element of K then

$$\exp(\lambda It + M) = e^{\lambda t}\exp(M).$$

45. Let N be the $r \times r$ matrix with 1's on the first superdiagonal and zeros elsewhere. Compute the exponential of the following nilpotent $r \times r$ matrix:

$$\text{if } Nt = \begin{pmatrix} 0 & t & & & \\ & 0 & t & & \\ & & & \ddots & \\ & & & & t \\ & & & & 0 \end{pmatrix} \quad \text{then } \exp(Nt) = \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \cdots & \frac{t^{r-1}}{(r-1)!} \\ & 1 & t & \frac{t^2}{2!} & & \vdots \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & \ddots & t & \frac{t^2}{2!} \\ & & & & 1 & t \\ & & & & & 1 \end{pmatrix}.$$

Deduce that if J is the $r \times r$ elementary Jordan matrix with eigenvalue λ then

$$\exp(Jt) = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2!}e^{\lambda t} & \cdots & \cdots & \frac{t^{r-1}}{(r-1)!}e^{\lambda t} \\ & e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2!}e^{\lambda t} & & \vdots \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & \ddots & te^{\lambda t} & \frac{t^2}{2!}e^{\lambda t} \\ & & & & e^{\lambda t} & te^{\lambda t} \\ & & & & & e^{\lambda t} \end{pmatrix}.$$

[To do the first part use the observation that since Nt is a nilpotent matrix, $\exp(Nt)$ is a *polynomial* in Nt, i.e., all but a finite number of the terms in the power series are zero. To compute the exponential of Jt write Jt as $\lambda It + Nt$ and use Exercise 44 with $M = Nt$.]

Let $A \in M_n(K)$ and let P be a change of basis matrix such that $P^{-1}AP$ is in Jordan canonical form. Suppose $P^{-1}AP$ is the sum of elementary Jordan matrices $J_1, \ldots, J_m$. The preceding exercises (with $t = 1$) show that $\exp(A)$ can easily be found by writing $E = \exp(P^{-1}AP)$ as the direct sum of the matrices $\exp(J_1), \ldots, \exp(J_m)$ and then changing the basis back again to obtain $\exp(A) = PEP^{-1}$.

46. For the 4×4 matrices D and P given in Example 3 of this section:

$$D = \begin{pmatrix} 1 & 2 & -4 & 4 \\ 2 & -1 & 4 & -8 \\ 1 & 0 & 1 & -2 \\ 0 & 1 & -2 & 3 \end{pmatrix} \qquad P = \begin{pmatrix} 0 & 1 & 2 & 0 \\ 2 & 0 & -2 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

show that

$$E = \begin{pmatrix} e & e & 0 & 0 \\ 0 & e & 0 & 0 \\ 0 & 0 & e & e \\ 0 & 0 & 0 & e \end{pmatrix} \quad \text{and} \quad \exp(D) = \begin{pmatrix} e & 2e & -4e & 4e \\ 2e & -e & 4e & -8e \\ e & 0 & e & -2e \\ 0 & e & -2e & 3e \end{pmatrix}.$$

47. Compute the exponential of each of the following matrices:

(a) the matrix A in Example 2 of this section

(b) the matrix in Exercise 4 (where you computed the Jordan canonical form and a change of basis matrix)

(c) the matrix in Exercise 16.

48. Show that $\exp(0) = I$ (here 0 is the zero matrix and I is the identity matrix). Deduce that $\exp(A)$ is nonsingular with inverse $\exp(-A)$ for all matrices $A \in M_n(K)$.

49. Prove that $\det(\exp(A)) = e^{\mathrm{tr}(A)}$, where $\mathrm{tr}(A)$ is the trace of A (the sum of the diagonal entries of A).

50. Fix any $A \in M_n(K)$. Prove that the map

$$K \to GL_n(K) \qquad \text{defined by} \qquad t \mapsto \exp(At)$$

is a group homomorphism (here K is the additive group of the field). (Note how this generalizes the familiar exponential map from K to $K^\times$, which is the $n = 1$ case. The subgroup $\{\exp(At) \mid t \in K\}$ is called a *1-parameter subgroup* of $GL_n(K)$. These subgroups and the exponential map play an important role in the theory of *Lie groups* — $GL_n(K)$ being a particular example of a Lie group.).

Let $G(x)$ be a power series having an infinite radius of convergence and fix a matrix $A \in M_n(K)$. The entries of the matrix $G(At)$ are K-valued functions of the variable t that are defined for all t. Let $c_{ij}(t)$ be the function of t in the i, j entry of $G(At)$. The *derivative* of $G(At)$ with respect to t, denoted by $\frac{d}{dt}G(At)$, is the matrix whose i, j entry is $\frac{d}{dt}c_{ij}(t)$ obtained by differentiating each of the entries of $G(At)$. In other words, if we identify $M_n(K)$ with K^{n^2} by considering each $n \times n$ matrix as an n^2-tuple, then $t \mapsto G(At)$ is a map from K to K^{n^2} (i.e., is a vector valued function of t) whose derivative is just the usual (componentwise) derivative of this vector valued function.

51. Establish the following properties of derivatives:

(a) If $G(x) = \sum_{k=0}^{\infty} \alpha_k x^k$ then $\frac{d}{dt}G(At) = A \sum_{k=1}^{\infty} k\alpha_k (At)^{k-1}$.

(b) If v is an $n \times 1$ matrix with (constant) entries from K then

$$\frac{d}{dt}(G(At)v) = \left(\frac{d}{dt}G(At)\right) v.$$

52. Deduce from part (a) of the preceding exercise that

$$\frac{d}{dt}\exp(At) = A\exp(At).$$

Now let $y_1(t), \ldots, y_n(t)$ be differentiable functions of the real variable t that are related by the following linear system of first order differential equations with constant coefficients $a_{ij} \in K$:

$$\begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + \ldots + a_{1n}y_n \\ y_2' &= a_{21}y_1 + a_{22}y_2 + \ldots + a_{2n}y_n \\ &\ \vdots \\ y_n' &= a_{n1}y_1 + a_{n2}y_2 + \ldots + a_{nn}y_n \end{aligned} \tag{$*$}$$