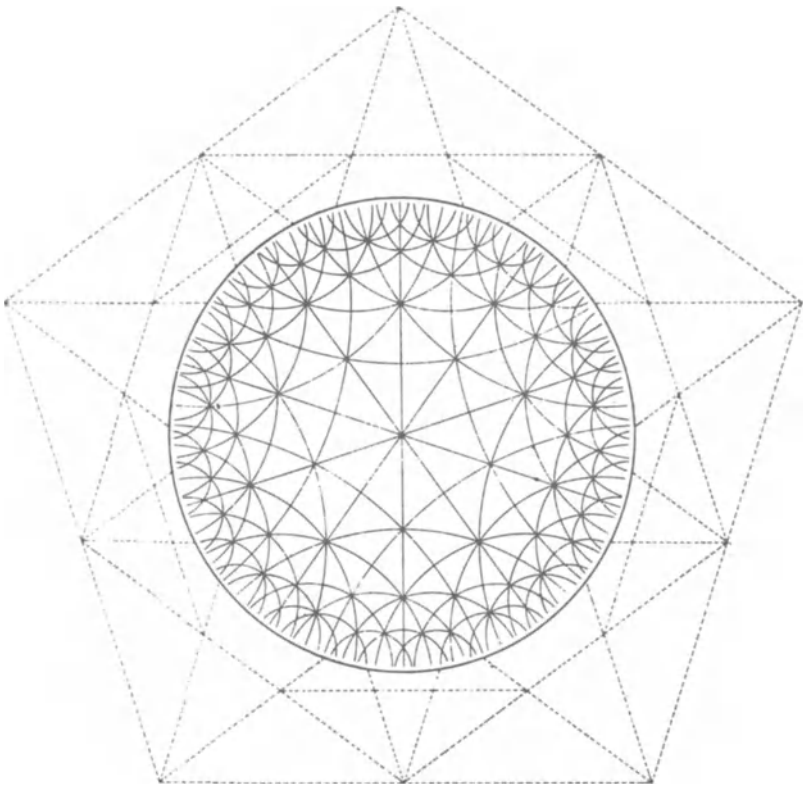

Figure 18.16: The Schwarz tessellation

of *linear fractional transformations*,

$$z \mapsto \frac{az+b}{cz+d},$$

special cases of which, as we have seen, express the rigid motions of the two-dimensional Euclidean, spherical, and hyperbolic geometries. He showed that each linear fractional transformation of the plane $\mathbb{C}$ is induced by hyperbolic motion of the three-dimensional half-space with boundary plane $\mathbb{C}$; thus Poincaré's theorem embraces those of Wachter and Beltrami on the representation of two-dimensional Euclidean, spherical, and hyperbolic geometry within three-dimensional hyperbolic geometry.

EXERCISES

18.6.1 Show that a triangle in the hyperbolic plane can have any angle sum $< \pi$.

18.6.2 Deduce that there are equilateral triangles with angle $2\pi/n$ for each $n \geq 7$.

18.6.3 Also deduce that triangles with angle zero exist, in a certain sense, and that their area is finite.

18.6.4 Find corresponding results for regular n-gons.

18.7 Biographical Notes: Bolyai and Lobachevsky

János Bolyai was born in 1802 in Kolozvár, then in the Transylvanian part of Hungary (and now Cluj, Romania) and died in Marosvásárhely in Hungary (now Târgu-Mureş, Romania) in 1860. His father, Farkas (also known by his German name, Wolfgang), was professor of mathematics, physics, and chemistry, and his mother, Susanna von Árkos, was the daughter of a surgeon. János received his early education from his father and also studied at the Evangelic-Reformed College, where his father taught, from 1815 to 1818. Farkas had been a fellow student of Gauss at Göttingen, and hoped that János would follow him there, but instead the younger Bolyai opted for a military career. He studied at the Vienna engineering academy from 1818 to 1822 and then entered the army.

In the army, János became known as an invincible duelist, but he suffered from bouts of fever and was eventually pensioned off in 1833. He returned to Marosvásárhely to live with his father, but the two did not get along, and in 1834 he moved to a small family estate. He set up house with his mistress, Rosalie von Orbán; they had three children. This could have been the start of a mathematical career, in the style of Descartes, as a leisured country gentleman. But, sad to say, Bolyai's mathematical career was already over in 1833, and it was not until after his death that the world knew he had accomplished anything.

János had inherited a passion for the foundations of geometry from his father, so much so that in 1820 Farkas tried almost desperately to steer him away from the problem of parallels: "You should not tempt the parallels in this way, I know this way until its end—I also have measured this bottomless night, I have lost in it every light, every joy of my life" [Stäckel (1913), pp. 76–77]. Of course János ignored this warning, but eventually he found the way out that Farkas had missed. After unsuccessful attempts to prove the Euclidean parallel axiom, he discarded it and proceeded to derive consequences of Axiom P_2. By 1823 his results seemed so complete and elegant they somehow had to be real, and he wrote triumphantly to his father: "From nothing I have created another entirely new world."

Farkas was unwilling to accept the new geometry, but in June 1831 he agreed to send his son's results to Gauss, who did not answer for over six months (admittedly, this was the time of his wife's death). When Gauss did answer it was in the most self-serving way imaginable:

> Now something about the work of your son. You will probably be shocked for a moment when I begin by saying *that I cannot praise it*, but I cannot do anything else, since to praise it would be to praise myself. The whole content of the paper, the path that your son has taken, and the results to which he has been led, agree almost everywhere with my own meditations, which have occupied me in part already for 30–35 years.
>
> [Gauss (1832b)]

Later in the letter, Gauss offered Bolyai the same backhanded thanks that he had offered to Abel (see Section 12.6) for "saving him the trouble" of writing up the results himself, and he raised the question of the volume of the tetrahedron as a problem for further research.

As we now know, Gauss *did* have many of the results of noneuclidean geometry by this time, including the answer to the volume problem he had raised to test his young rival [see Gauss (1832a)]. Nevertheless, Gauss was almost certainly wrong to imply that his understanding of noneuclidean geometry went back 35 years. As late as 1804, when Farkas Bolyai wrote to him about the problem of parallels, Gauss could offer no help except the hope that the problem would be settled one day [see Kaufmann-Bühler (1981), p. 100].

János Bolyai was disillusioned and embittered by Gauss' reply but did not give up immediately. He published his work as an appendix to his father's book the *Tentamen* [F. Bolyai (1832a)]. However, when there was no response from other mathematicians he became discouraged and never published again. He was also troubled by the possibility that there might, after all, be contradictions in his geometry. As we know, this possibility was not ruled out until 1868, and by then Gauss, Bolyai, and Lobachevsky were all dead.

Nikolai Ivanovich Lobachevsky (Figure 18.17) was born in Novgorod in 1792 and died in Kazan in 1856. He was the son of Ivan Maksimovich Lobachevsky and Praskovia Aleksandrovna. His father died when Nikolai was five years old, and his mother moved with her three sons to Kazan. By persistent efforts, she was able to secure scholarships for their education, and in 1807 Nikolai entered Kazan University, which had been founded just two years earlier. He was supervised by Martin Bartels, the former teacher of Gauss, but a link to Gauss' geometric ideas seems less likely than in the case of Bolyai, since Bartels had little contact with Gauss after his school days.

Figure 18.17: Nikolai Ivanovich Lobachevsky

Lobachevsky stayed at Kazan for the rest of his life, becoming professor in 1814 and making many contributions to the growth of the university.

He married the wealthy Lady Varvara Alekseeva Moisieva in 1832 and was raised to the nobility in 1837, in recognition of his services to education. The couple had seven children.

Lobachevsky's investigation of parallels began in 1816, when he lectured on geometry, and he at first thought he could prove the Euclidean axiom. Gradually he became aware of the way in which parallels regulate other geometric properties, such as areas, and in 1832 he wrote *Geometriya*, which separated theorems not requiring an assumption about parallels from those that did. He still believed in the Euclidean axiom, however, so Bolyai was ahead of him at this stage. Lobachevsky's publications in noneuclidean geometry began in 1829, but at first they attracted no attention, since they were in Russian and Kazan University was little known. He did gain a wider audience with an article in French in Crelle's journal in 1837, but Gauss seems to have been the only one to recognize its importance. Gauss was in fact so impressed that he collected Lobachevsky's obscure Kazan publications and taught himself Russian in order to read them, but once again he was unwilling to admit to others how impressed he was. It seems that he never contacted Lobachevsky at all, and it is only through a letter (1846b) published after his death that his opinion became public. As usual Gauss' first thought was to guard his own priority, and his memory of when he discovered noneuclidean geometry seems to have improved with age:

> Lobachevsky calls it imaginary geometry. You know that I have had the same conviction for 54 years (since 1792), with a certain later extension which I do not want to go into here. There was nothing materially new for me in Lobachevsky's paper, but he explains his theory in a way which is different from mine, and does this in a masterful way, in a truly geometric spirit.
>
> [Kaufmann-Bühler (1981), p. 150)]

Perhaps Lobachevsky was lucky *not* to hear the kind of praise Gauss bestowed on his competitors, although he was certainly less easily discouraged than Bolyai. Despite the silence of foreign mathematicians, opposition from mathematicians in Russia, and the handicap of blindness in his later years, he continued to refine and expand his theory. The final version of his work, *Pangéométrie*, was published in 1855–56, the last year of his life.