Exercises

By extending Euclid's construction of the equilateral triangle, construct:

1.2.1 A regular hexagon.

1.2.2 A tiling of the plane by equilateral triangles (solid lines in Figure 1.5).

1.2.3 A tiling of the plane by regular hexagons (dashed lines in Figure 1.5).

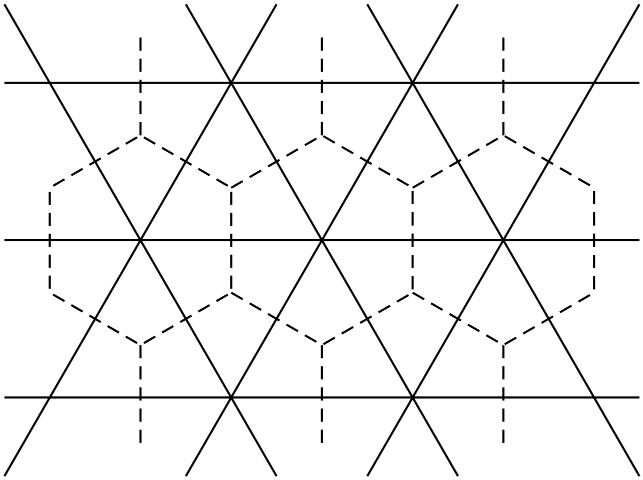

Figure 1.5: Triangle and hexagon tilings of the plane

1.3 Some basic constructions

The equilateral triangle construction comes first in the *Elements* because several other constructions follow from it. Among them are constructions for bisecting a line segment and bisecting an angle. ("Bisect" is from the Latin for "cut in two.")

Bisecting a line segment

To bisect a given line segment AB, draw the two circles with radius AB as above, but now consider both of their intersection points, C and D. The line CD connecting these points bisects the line segment AB (Figure 1.6).

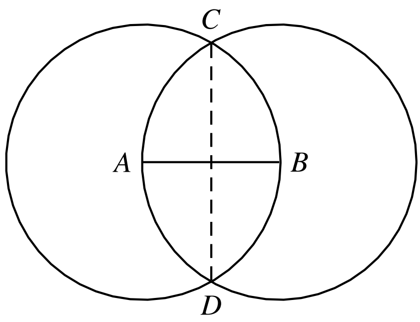

Figure 1.6: Bisecting a line segment AB

Notice also that BC is *perpendicular* to AB, so this construction can be adapted to construct perpendiculars.

- To construct the perpendicular to a line $\mathscr{L}$ at a point E on the line, first draw a circle with center E, cutting $\mathscr{L}$ at A and B. Then the line CD constructed in Figure 1.6 is the perpendicular through E.
- To construct the perpendicular to a line $\mathscr{L}$ through a point E not on $\mathscr{L}$, do the same; only make sure that the circle with center E is large enough to cut the line $\mathscr{L}$ at two different points.

Bisecting an angle

To bisect an angle POQ (Figure 1.7), first draw a circle with center O cutting OP at A and OQ at B. Then the perpendicular CD that bisects the line segment AB also bisects the angle POQ.

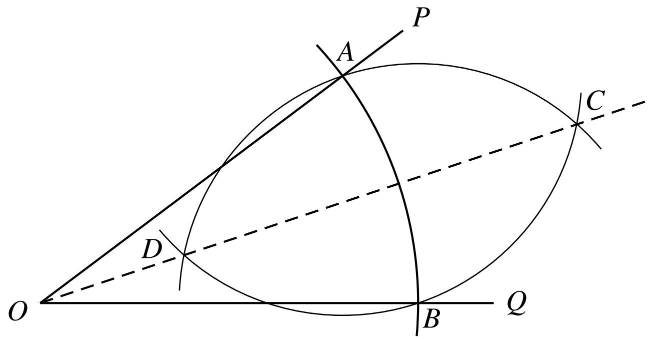

Figure 1.7: Bisecting an angle POQ

It seems from these two constructions that bisecting a line segment and bisecting an angle are virtually the same problem. Euclid bisects the angle before the line segment, but he uses two similar constructions (*Elements*, Propositions 9 and 10 of Book I). However, a distinction between line segments and angles emerges when we attempt division into three or more parts. There is a simple tool for dividing a line segment in any number of equal parts—*parallel lines*—but no corresponding tool for dividing angles.

Constructing the parallel to a line through a given point

We use the two constructions of perpendiculars noted above—for a point off the line and a point on the line. Given a line $\mathscr{L}$ and a point P outside $\mathscr{L}$, first construct the perpendicular line $\mathscr{M}$ to $\mathscr{L}$ through P. Then construct the perpendicular to $\mathscr{M}$ through P, which is the parallel to $\mathscr{L}$ through P.

Dividing a line segment into n equal parts

Given a line segment AB, draw any other line $\mathscr{L}$ through A and mark n successive, equally spaced points $A_1, A_2, A_3, \ldots, A_n$ along $\mathscr{L}$ using the compass set to any fixed radius. Figure 1.8 shows the case $n = 5$. Then connect A_n to B, and draw the parallels to BA_n through $A_1, A_2, \ldots, A_{n-1}$. These parallels divide AB into n equal parts.

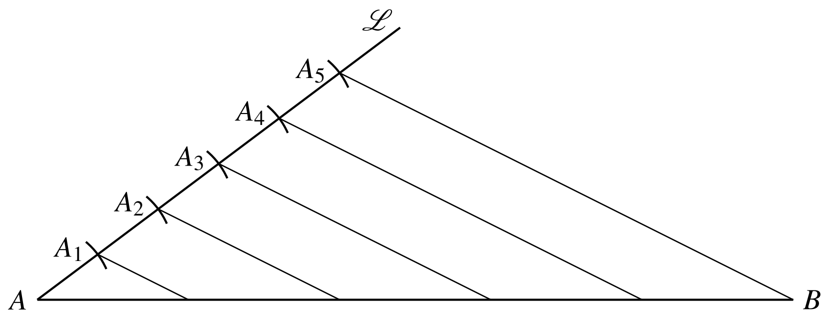

Figure 1.8: Dividing a line segment into equal parts

This construction depends on a property of parallel lines sometimes attributed to Thales (Greek mathematician from around 600 BCE): *parallels cut any lines they cross in proportional segments.* The most commonly used instance of this theorem is shown in Figure 1.9, where a parallel to one side of a triangle cuts the other two sides proportionally.

The line $\mathscr{L}$ parallel to the side BC cuts side AB into the segments AP and PB, side AC into AQ and QC, and $|AP|/|PB| = |AQ|/|QC|$.

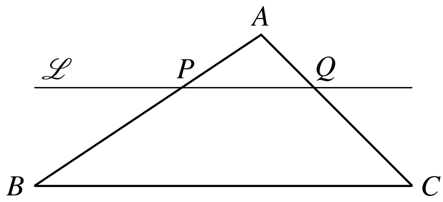

Figure 1.9: The Thales theorem in a triangle

This theorem of Thales is the key to using algebra in geometry. In the next section we see how it may be used to multiply and divide line segments, and in Chapter 2 we investigate how it may be derived from fundamental geometric principles.

Exercises

1.3.1 Check for yourself the constructions of perpendiculars and parallels described in words above.

1.3.2 Can you find a more direct construction of parallels?

Perpendiculars give another important polygon—the square.

1.3.3 Give a construction of the square on a given line segment.

1.3.4 Give a construction of the square tiling of the plane.

One might try to use division of a line segment into n equal parts to divide an angle into n equal parts as shown in Figure 1.10. We mark A on OP and B at equal distance on OQ as before, and then try to divide angle POQ by dividing line segment AB. However, this method is faulty even for division into three parts.

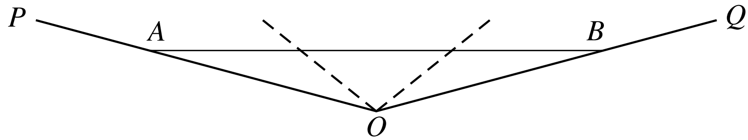

Figure 1.10: Faulty trisection of an angle

1.3.5 Explain why division of AB into three equal parts (trisection) does *not* always divide angle POQ into three equal parts. (Hint: Consider the case in which POQ is nearly a straight line.)

The version of the Thales theorem given above (referring to Figure 1.9) has an equivalent form that is often useful.

1.3.6 If A,B,C,P,Q are as in Figure 1.9, so that $|AP|/|PB| = |AQ|/|QC|$, show that this equation is equivalent to $|AP|/|AB| = |AQ|/|AC|$.

1.4 Multiplication and division

Not only can one add and subtract line segments (Section 1.1); one can also multiply and divide them. The *product* ab and *quotient* a/b of line segments a and b are obtained by the straightedge and compass constructions below. The key ingredients are parallels, and the key geometric property involved is the Thales theorem on the proportionality of line segments cut off by parallel lines.

To get started, it is necessary to choose a line segment as the *unit of length*, 1, which has the property that $1a = a$ for any length a.

Product of line segments

To multiply line segment b by line segment a, we first construct any triangle UOA with $|OU| = 1$ and $|OA| = a$. We then extend OU by length b to B_1 and construct the parallel to UA through B_1. Suppose this parallel meets the extension of OA at C (Figure 1.11).

By the Thales theorem, $|AC| = ab$.

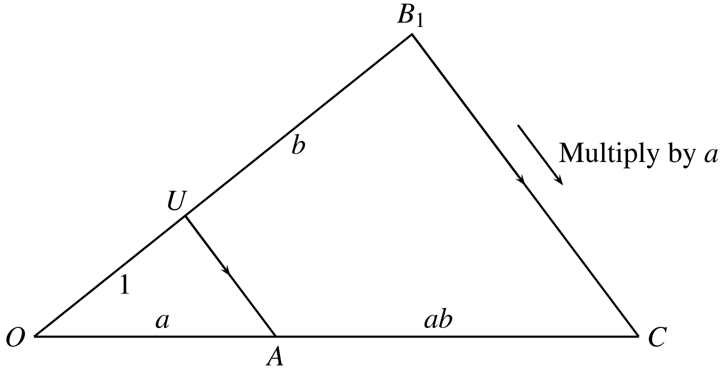

Figure 1.11: The product of line segments

Quotient of line segments

To divide line segment b by line segment a, we begin with the same triangle UOA with $|OU| = 1$ and $|OA| = a$. Then we extend OA by distance b to B_2 and construct the parallel to UA through B_2. Suppose that this parallel meets the extension of OU at D (Figure 1.12).

By the Thales theorem, $|UD| = b/a$.

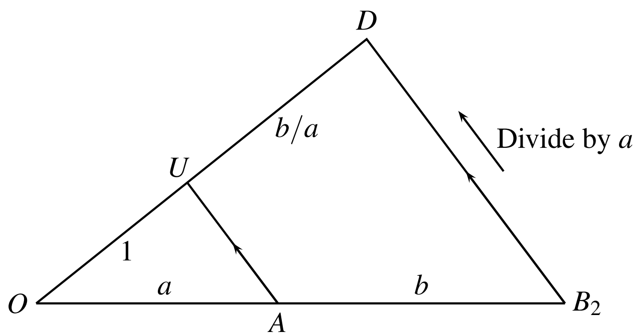

Figure 1.12: The quotient of line segments

The sum operation from Section 1.1 allows us to construct a segment n units in length, for any natural number n, simply by adding the segment 1 to itself n times. The quotient operation then allows us to construct a segment of length m/n, for any natural numbers m and $n \neq 0$. These are what we call the *rational* lengths. A great discovery of the Pythagoreans was that *some lengths are not rational*, and that some of these "irrational" lengths can be constructed by straightedge and compass. It is not known how the Pythagoreans made this discovery, but it has a connection with the Thales theorem, as we will see in the next section.

Exercises

Exercise 1.3.6 showed that if PQ is parallel to BC in Figure 1.9, then $|AP|/|AB| = |AQ|/|AC|$. That is, a parallel implies proportional (left and right) sides. The following exercise shows the converse: proportional sides imply a parallel, or (equivalently), a nonparallel implies nonproportional sides.

1.4.1 Using Figure 1.13, or otherwise, show that if PR is *not* parallel to BC, then $|AP|/|AB| \neq |AR|/|AC|$.

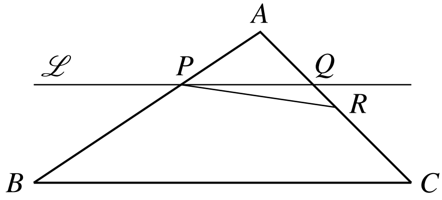

Figure 1.13: Converse of the Thales theorem

1.4.2 Conclude from Exercise 1.4.1 that if P is any point on AB and Q is any point on AC, then PQ is parallel to BC *if and only if* $|AP|/|AB| = |AQ|/|AC|$.

The "only if" direction of Exercise 1.4.2 leads to two famous theorems—the *Pappus* and *Desargues theorems*—that play an important role in the foundations of geometry. We will meet them in more general form later. In their simplest form, they are the following theorems about parallels.

1.4.3 (Pappus of Alexandria, around 300 CE) Suppose that A, B, C, D, E, F lie alternately on lines $\mathscr{L}$ and $\mathscr{M}$ as shown in Figure 1.14.

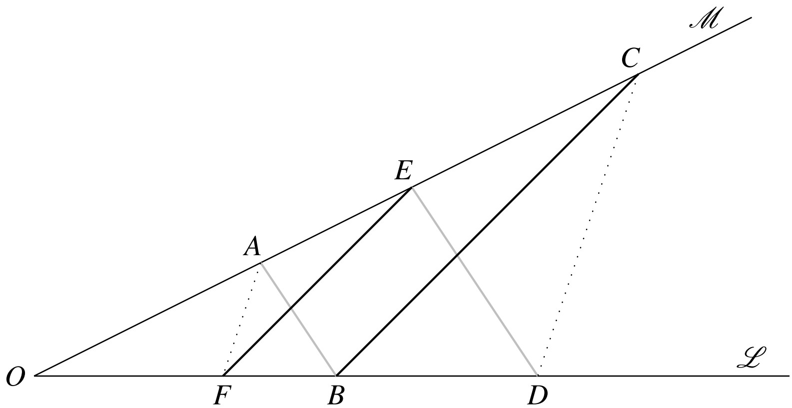

Figure 1.14: The parallel Pappus configuration

Use the Thales theorem to show that if AB is parallel to ED and FE is parallel to BC then

$$\frac{|OA|}{|OF|} = \frac{|OC|}{|OD|}.$$

Deduce from Exercise 1.4.2 that AF is parallel to CD.

1.4.4 (Girard Desargues, 1648) Suppose that points A, B, C, A', B', C' lie on concurrent lines $\mathscr{L}, \mathscr{M}, \mathscr{N}$ as shown in Figure 1.15. (The triangles ABC and $A'B'C'$ are said to be "in perspective from O.")

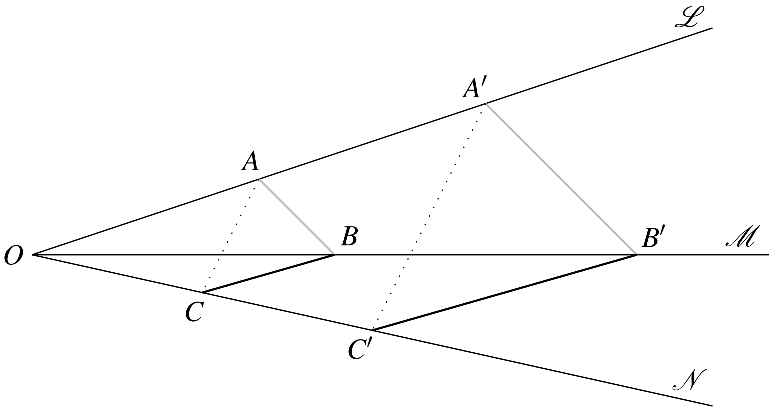

Figure 1.15: The parallel Desargues configuration

Use the Thales theorem to show that if AB is parallel to $A'B'$ and BC is parallel to $B'C'$, then

$$\frac{|OA|}{|OC|} = \frac{|OA'|}{|OC'|}.$$

Deduce from Exercise 1.4.2 that AC is parallel to $A'C'$.

1.5 Similar triangles

Triangles ABC and $A'B'C'$ are called *similar* if their corresponding angles are equal, that is, if

$$\begin{aligned}
\text{angle at } A &= \text{angle at } A' \quad (= \alpha \text{ say}),\\
\text{angle at } B &= \text{angle at } B' \quad (= \beta \text{ say}),\\
\text{angle at } C &= \text{angle at } C' \quad (= \gamma \text{ say}).
\end{aligned}$$

It turns out that equal angles imply that *all sides are proportional*, so we may say that one triangle is a magnification of the other, or that they have the same "shape." This important result extends the Thales theorem, and actually follows from it.

Why similar triangles have proportional sides

Imagine moving triangle ABC so that vertex A coincides with A' and sides AB and AC lie on sides $A'B'$ and $A'C'$, respectively. Then we obtain the situation shown in Figure 1.16. In this figure, b and c denote the side lengths of triangle ABC opposite vertices B and C, respectively, and b' and c' denote the side lengths of triangle $A'B'C'(=AB'C')$ opposite vertices B' and C', respectively.

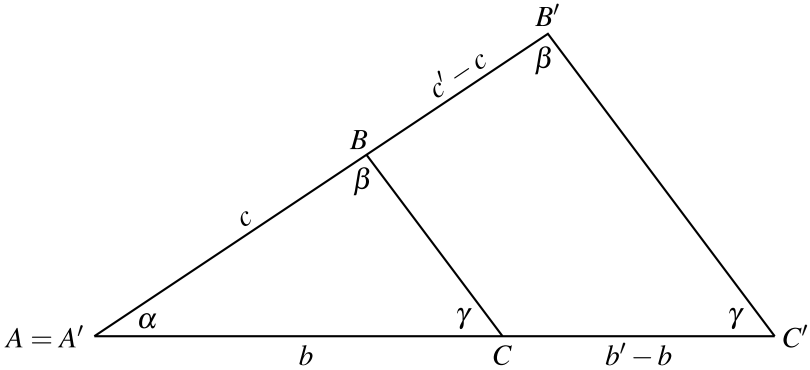

Figure 1.16: Similar triangles

Because BC and $B'C'$ both meet AB' at angle β, they are parallel, and so it follows from the Thales theorem (Section 1.3) that

$$\frac{b}{c}=\frac{b'-b}{c'-c}.$$

Multiplying both sides by $c(c'-c)$ gives $b(c'-c)=c(b'-b)$, that is,

$$bc'-bc=cb'-cb,$$

and hence

$$bc'=cb'.$$

Finally, dividing both sides by cc', we get

$$\frac{b}{c}=\frac{b'}{c'}.$$

That is, *corresponding sides of triangles ABC and $A'B'C'$ opposite to the angles β and γ are proportional.*

We got this result by making the angles α in the two triangles coincide. If we make the angles β coincide instead, we similarly find that the sides opposite to α and γ are proportional. Thus, in fact, *all corresponding sides of similar triangles are proportional.* $\square$

This consequence of the Thales theorem has many implications. In everyday life, it underlies the existence of scale maps, house plans, engineering drawings, and so on. In pure geometry, its implications are even more varied. Here is just one, which shows why square roots and irrational numbers turn up in geometry.

The diagonal of the unit square is $\sqrt{2}$

The diagonals of the unit square cut it into four quarters, each of which is a triangle similar to the half square cut off by a diagonal (Figure 1.17).

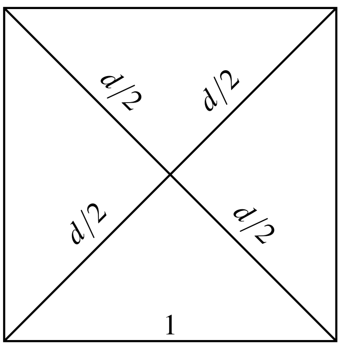

Figure 1.17: Quarters and halves of the square

Each of the triangles in question has one right angle and two half right angles, so it follows from the theorem above that corresponding sides of any two of these triangles are proportional. In particular, if we take the half square, with short side 1 and long side d, and compare it with the quarter square, with short side $d/2$ and long side 1, we get

$$\frac{\text{short}}{\text{long}} = \frac{1}{d} = \frac{d/2}{1}.$$

Multiplying both sides of the equation by $2d$ gives $2 = d^2$, so $d = \sqrt{2}$. $\square$

The great, but disturbing, discovery of the Pythagoreans is that $\sqrt{2}$ *is irrational*. That is, there are no natural numbers m and n such $\sqrt{2} = m/n$.

If there are such m and n we can assume that they have no common divisor, and then the assumption $\sqrt{2} = m/n$ implies

	$2 = m^2/n^2$	squaring both sides
hence	$m^2 = 2n^2$	multiplying both sides by n^2
hence	m^2 is even	
hence	m is even	since the square of an odd number is odd
hence	$m = 2l$	for some natural number l
hence	$m^2 = 4l^2 = 2n^2$	
hence	$n^2 = 2l^2$	
hence	n^2 is even	
hence	n is even	since the square of an odd number is odd.

Thus, m and n have the common divisor 2, contrary to assumption. Our original assumption is therefore false, so there are no natural numbers m and n such that $\sqrt{2} = m/n$. □

Lengths, products, and area

Geometry obviously has to include the diagonal of the unit square, hence *geometry includes the study of irrational lengths.* This discovery troubled the ancient Greeks, because they did not believe that irrational lengths could be treated like numbers. In particular, the idea of interpreting the product of line segments as another line segment is *not* in Euclid. It first appears in Descartes' *Géométrie* of 1637, where algebra is used systematically in geometry for the first time.

The Greeks viewed the product of line segments a and b as the *rectangle* with perpendicular sides a and b. If lengths are not necessarily numbers, then the product of two lengths is best interpreted as an area, and the product of three lengths as a volume—but then the product of four lengths seems to have no meaning at all. This difficulty perhaps explains why algebra appeared comparatively late in the development of geometry. On the other hand, interpreting the product of lengths as an area gives some remarkable insights, as we will see in Chapter 2. So it is also possible that algebra had to wait until the Greek concept of product had exhausted its usefulness.

Exercises

In general, two geometric figures are called similar if one is a magnification of the other. Thus, two rectangles are similar if the ratio $\frac{\text{long side}}{\text{short side}}$ is the same for both.

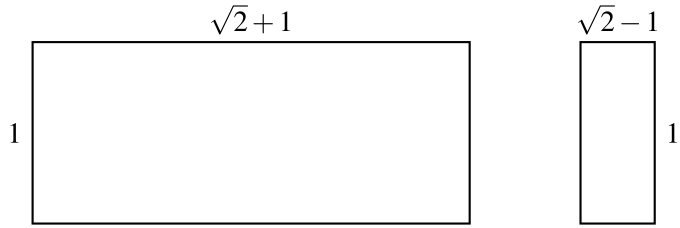

Figure 1.18: A pair of similar rectangles

1.5.1 Show that $\frac{\sqrt{2}+1}{1} = \frac{1}{\sqrt{2}-1}$ and hence that the two rectangles in Figure 1.18 are similar.

1.5.2 Deduce that if a rectangle with long side a and short side b has the same shape as the two above, then so has the rectangle with long side b and short side $a-2b$.

This simple observation gives another proof that $\sqrt{2}$ is irrational:

1.5.3 Suppose that $\sqrt{2}+1 = m/n$, where m and n are natural numbers with m as small as possible. Deduce from Exercise 1.5.2 that we also have $\sqrt{2}+1 = n/(m-2n)$. This is a contradiction. Why?

1.5.4 It follows from Exercise 1.5.3 that $\sqrt{2}+1$ is irrational. Why does this imply that $\sqrt{2}$ is irrational?

1.6 Discussion

Euclid's *Elements* is the most influential book in the history of mathematics, and anyone interested in geometry should own a copy. It is not easy reading, but you will find yourself returning to it year after year and noticing something new. The standard edition in English is Heath's translation, which is now available as a Dover reprint of the 1925 Cambridge University Press edition. This reprint is carried by many bookstores; I have even seen it for sale at Los Angeles airport! Its main drawback is its size—three bulky volumes—due to the fact that more than half the content consists of

Heath's commentary. You can find the Heath translation *without* the commentary in the Britannica *Great Books of the Western World*, Volume 11. These books can often be found in used bookstores. Another, more recent, one-volume edition of the Heath translation is *Euclid's Elements*, edited by Dana Densmore and published by Green Lion Press in 2003.

A second (slight) drawback of the Heath edition is that it is about 80 years old and beginning to sound a little antiquated. Heath's English is sometimes quaint, and his commentary does not draw on modern research in geometry. He does not even mention some important advances that were known to experts in 1925. For this reason, a modern version of the *Elements* is desirable. A perfect version for the 21st century does not yet exist, but there is a nice concise web version by David Joyce at

`http://aleph0.clarkeu.edu/~djoyce/java/elements/elements.html`

This *Elements* has a small amount of commentary, but I mainly recommend it for proofs in simple modern English and nice diagrams. The diagrams are "variable" by dragging points on the screen, so each diagram represents all possible situations covered by a theorem.

For modern commentary on Euclid, I recommend two books: *Euclid: the Creation of Mathematics* by Benno Artmann and *Geometry: Euclid and Beyond* by Robin Hartshorne, published by Springer-Verlag in 1999 and 2000, respectively. Both books take Euclid as their starting point. Artmann mainly fills in the Greek background, although he also takes care to make it understandable to modern readers. Hartshorne is more concerned with what came after Euclid, and he gives a very thorough analysis of the gaps in Euclid and the ways they were filled by modern mathematicians. You will find Hartshorne useful supplementary reading for Chapters 2 and 3, where we examine the logical structure of the *Elements* and some of its gaps.

The climax of the *Elements* is the theory of regular polyhedra in Book XIII. Only five regular polyhedra exist, and they are shown in Figure 1.19. Notice that three of them are built from equilateral triangles, one from squares, and one from regular pentagons. This remarkable phenomenon underlines the importance of equilateral triangles and squares, and draws attention to the regular pentagon. In Chapter 2, we show how to construct it. Some geometers believe that the material in the *Elements* was chosen very much with the theory of regular polyhedra in mind. For example, Euclid wants to construct the equilateral triangle, square, and pentagon in order to construct the regular polyhedra.

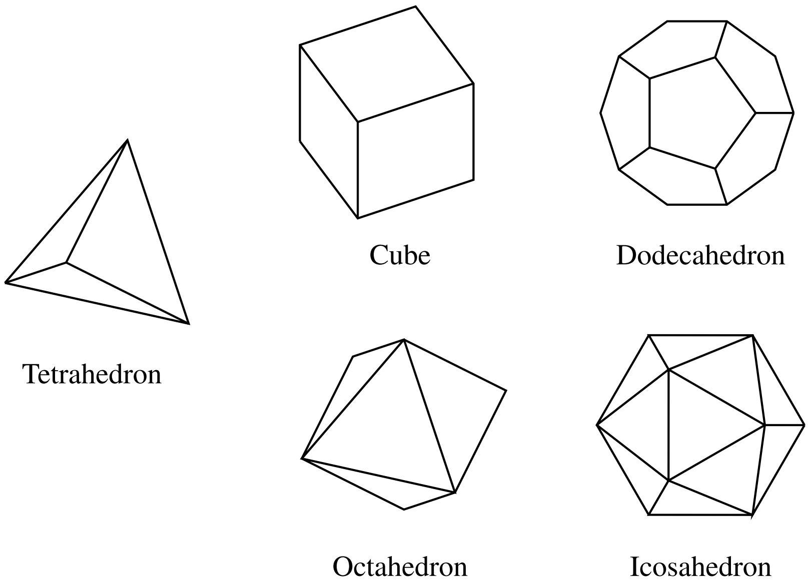

Figure 1.19: The regular polyhedra

It is fortunate that Euclid did not need regular polygons more complex than the pentagon, because none were constructed until modern times. The regular 17-gon was constructed by the 19-year-old Carl Friedrich Gauss in 1796, and his discovery was the key to the "question arising" from the construction of the equilateral triangle in Section 1.2: for which n is the regular n-gon constructible? Gauss showed (with some steps filled in by Pierre Wantzel in 1837) that a regular polygon with a prime number p of sides is constructible just in case p is of the form $2^{2^m}+1$. This result gives three constructible p-gons not known to the Greeks, because

$$2^4+1=17, \quad 2^8+1=257, \quad 2^{16}+1=65537$$

are all prime numbers. But no larger prime numbers of the form $2^{2^m}+1$ are known! Thus we do not know whether a larger constructible p-gon exists.

These results show that the *Elements* is not all of geometry, even if one accepts the same subject matter as Euclid. To see where Euclid fits in the general panorama of geometry, I recommend the books *Geometry and the Imagination* by D. Hilbert and S. Cohn-Vossen, and *Introduction to Geometry* by H. S. M. Coxeter (Wiley, 1969).

2

Euclid's approach to geometry

PREVIEW

Length is the fundamental concept of Euclid's geometry, but several important theorems seem to be "really" about angle or area—for example, the theorem on the sum of angles in a triangle and the Pythagorean theorem on the sum of squares. Also, Euclid often uses area to prove theorems about length, such as the Thales theorem.

In this chapter, we retrace some of Euclid's steps in the theory of angle and area to show how they lead to the Pythagorean theorem and the Thales theorem. We begin with his theory of angle, which shows most clearly the influence of his *parallel axiom*, the defining axiom of what is now called *Euclidean geometry*.

Angle is linked with length from the beginning by the so-called SAS ("side angle side") criterion for equal triangles (or "congruent triangles," as we now call them). We observe the implications of SAS for isosceles triangles and the properties of angles in a circle, and we note the related criterion, ASA ("angle side angle").

The theory of area depends on ASA, and it leads directly to a proof of the Pythagorean theorem. It leads more subtly to the Thales theorem and its consequences that we saw in Chapter 1. The theory of angle then combines nicely with the Thales theorem to give a second proof of the Pythagorean theorem.

In following these deductive threads, we learn more about the scope of straightedge and compass constructions, partly in the exercises. Interesting spinoffs from these investigations include a process for cutting any polygon into pieces that form a square, a construction for the square root of any length, and a construction of the regular pentagon.

2.1 The parallel axiom

In Chapter 1, we saw how useful it is to have *rectangles*: four-sided polygons whose angles are all right angles. Rectangles owe their existence to *parallel lines*—lines that do not meet—and fundamentally to the *parallel axiom* that Euclid stated as follows.

Euclid's parallel axiom. *If a straight line crossing two straight lines makes the interior angles on one side together less than two right angles, then the two straight lines will meet on that side.*

Figure 2.1 illustrates the situation described by Euclid's parallel axiom, which is what happens when the two lines are *not* parallel. If $\alpha+\beta$ is less than two right angles, then $\mathscr{L}$ and $\mathscr{M}$ meet somewhere on the right.

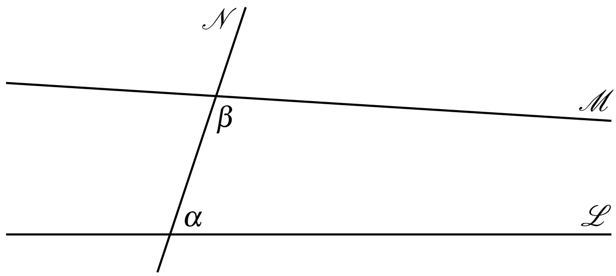

Figure 2.1: When lines are not parallel

It follows that if $\mathscr{L}$ and $\mathscr{M}$ do not meet on either side, then $\alpha+\beta=\pi$. In other words, if $\mathscr{L}$ and $\mathscr{M}$ *are* parallel, then α and β together make a straight angle and the angles made by $\mathscr{L}$, $\mathscr{M}$, and $\mathscr{N}$ are as shown in Figure 2.2.

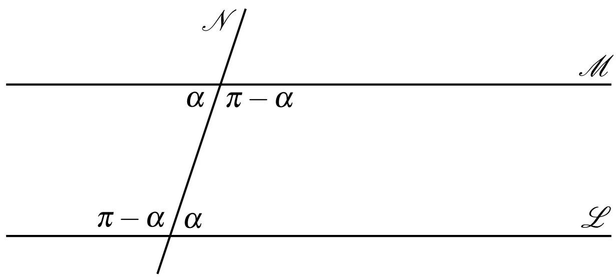

Figure 2.2: When lines are parallel