

in the literature when the theory is developed over fields such as  $\mathbb{R}$  and  $\mathbb{C}$ ). In this case the multiplication of two alternating tensors  $z$  and  $w$  is defined by first taking the product  $zw = z \otimes w$  in  $\mathcal{T}(M)$  and then projecting the resulting tensor into the submodule of alternating tensors. Note that the simple product of two alternating tensors need not be alternating (for example, the square of an alternating tensor is a symmetric tensor).

### Example

Let  $V$  be a vector space over a field  $F$  in which  $k! \neq 0$ . There are many *vector space complements* to  $\mathcal{A}^k(V)$  in  $\mathcal{T}^k(V)$  (just extend a basis for the subspace  $\mathcal{A}^k(V)$  to a basis for  $\mathcal{T}^k(V)$ , for example). These complements depend on choices of bases for  $\mathcal{T}^k(V)$  and so are indistinguishable from each other from vector space considerations alone. The additional structure on  $\mathcal{T}^k(V)$  given by the action of  $S_k$  singles out a unique complement to  $\mathcal{A}^k(V)$ , namely the subspace of alternating tensors in Proposition 40.

Suppose that  $k! \neq 0$  in  $F$  for all  $k \geq 2$  (i.e., the field  $F$  has “characteristic 0,” cf. Exercise 26 in Section 7.3), for example,  $F = \mathbb{Q}$ . Then the full exterior algebra  $\bigwedge(V) = \bigoplus_{k \geq 0} \bigwedge^k(V)$  can be identified with the collection of tensors whose homogeneous components are alternating (with respect to the appropriate symmetric groups  $S_k$ ).

Multiplication in  $\bigwedge(V)$  in terms of alternating tensors is rather cumbersome, however. For example let  $v_1, v_2, v_3$  be distinct basis vectors in  $V$ . The product of the two alternating tensors  $z = v_1$  and  $w = v_2 \otimes v_3 - v_3 \otimes v_2$  is obtained by first computing

$$z \otimes w = v_1 \otimes v_2 \otimes v_3 - v_1 \otimes v_3 \otimes v_2$$

in the full tensor algebra. This 3-tensor is not alternating — for example,

$$(1\ 2)(z \otimes w) = v_2 \otimes v_1 \otimes v_3 - v_3 \otimes v_1 \otimes v_2 \neq -z \otimes w$$

and also  $(1\ 2\ 3)(z \otimes w) = v_3 \otimes v_1 \otimes v_2 - v_2 \otimes v_1 \otimes v_3 \neq z \otimes w$ . The multiplication requires that we project this tensor into the subspace of alternating tensors. This projection is given by  $(1/3!)Alt(z \otimes w)$  and an easy computation shows that

$$\begin{aligned} \frac{1}{6}Alt(z \otimes w) &= \frac{1}{3}[v_1 \otimes v_2 \otimes v_3 + v_2 \otimes v_3 \otimes v_1 + v_3 \otimes v_1 \otimes v_2 \\ &\quad - v_1 \otimes v_3 \otimes v_2 - v_2 \otimes v_1 \otimes v_3 - v_3 \otimes v_2 \otimes v_1], \end{aligned}$$

so the right hand side is the product of  $z$  and  $w$  in terms of alternating tensors. The same product in terms of the quotient algebra  $\bigwedge(V)$  is simply

$$v_1 \wedge (2v_2 \wedge v_3) = 2v_1 \wedge v_2 \wedge v_3.$$

## EXERCISES

In these exercises  $R$  is a commutative ring with 1 and  $M$  is an  $R$ -module;  $F$  is a field and  $V$  is a finite dimensional vector space over  $F$ .

1. Prove that if  $M$  is a cyclic  $R$ -module then  $\mathcal{T}(M) = \mathcal{S}(M)$ , i.e., the tensor algebra  $\mathcal{T}(M)$  is commutative.
2. Fill in the details for the proof of Proposition 33 that  $S/I = \bigoplus_{k=0}^{\infty} S_k/I_k$ . [Show first that  $S_i I_j \subseteq I_{i+j}$ . Use this to show that the multiplication  $(S_i/I_i)(S_j/I_j) \subseteq S_{i+j}/I_{i+j}$  is well defined, and then check the ring axioms and verify the statements made in the proof of Proposition 33.]

3. Show that the image of the map  $\text{Sym}_2$  for the  $\mathbb{Z}$ -module  $\mathbb{Z}$  consists of the 2-tensors  $a(1 \otimes 1)$  where  $a$  is an even integer. Conclude in particular that the symmetric tensor  $1 \otimes 1$  in  $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}$  is not contained in the image of the map  $\text{Sym}$ .
4. Prove that  $m \wedge n_1 \wedge n_2 \wedge \cdots \wedge n_k = (-1)^k(n_1 \wedge n_2 \wedge \cdots \wedge n_k \wedge m)$ . In particular,  $x \wedge (y \wedge z) = (y \wedge z) \wedge x$  for all  $x, y, z \in M$ .
5. Prove that if  $M$  is a free  $R$ -module of rank  $n$  then  $\bigwedge^i(M)$  is a free  $R$ -module of rank  $\binom{n}{i}$  for  $i = 0, 1, 2, \dots$
6. If  $A$  is any  $R$ -algebra in which  $a^2 = 0$  for all  $a \in A$  and  $\varphi : M \rightarrow A$  is an  $R$ -module homomorphism, prove there is a unique  $R$ -algebra homomorphism  $\Phi : \bigwedge(M) \rightarrow A$  such that  $\Phi|_M = \varphi$ .
7. Let  $R = \mathbb{Z}[x, y]$  and  $I = (x, y)$ .
  - (a) Prove that if  $ax + by = a'x + b'y$  in  $R$  then  $a' = a + yf$  and  $b' = b - xf$  for some polynomial  $f(x, y) \in R$ .
  - (b) Prove that the map  $\varphi(ax+by, cx+dy) = ad - bc \text{ mod } (x, y)$  in the example following Corollary 37 is a well defined alternating  $R$ -bilinear map from  $I \times I$  to  $\mathbb{Z} = R/I$ .
8. Let  $R$  be an integral domain and let  $F$  be its field of fractions.
  - (a) Considering  $F$  as an  $R$ -module, prove that  $\bigwedge^2 F = 0$ .
  - (b) Let  $I$  be any  $R$ -submodule of  $F$  (for example, any ideal in  $R$ ). Prove that  $\bigwedge^i I$  is a torsion  $R$ -module for  $i \geq 2$  (i.e., for every  $x \in \bigwedge^i I$  there is some nonzero  $r \in R$  with  $rx = 0$ ).
  - (c) Give an example of an integral domain  $R$  and an  $R$ -module  $I$  in  $F$  with  $\bigwedge^i I \neq 0$  for every  $i \geq 0$  (cf. the example following Corollary 37).
9. Let  $R = \mathbb{Z}[G]$  be the group ring of the group  $G = \{1, \sigma\}$  of order 2. Let  $M = \mathbb{Z}e_1 + \mathbb{Z}e_2$  be a free  $\mathbb{Z}$ -module of rank 2 with basis  $e_1$  and  $e_2$ . Define  $\sigma(e_1) = e_1 + 2e_2$  and  $\sigma(e_2) = -e_2$ . Prove that this makes  $M$  into an  $R$ -module and that the  $R$ -module  $\bigwedge^2 M$  is a group of order 2 with  $e_1 \wedge e_2$  as generator.
10. Prove that  $z - (1/k!)Alt(z) = (1/k!) \sum_{\sigma \in S_k} (z - \epsilon(\sigma)\sigma z)$  for any  $k$ -tensor  $z$  and use this to prove that the kernel of the  $R$ -module homomorphism  $(1/k!)Alt$  in Proposition 40 is  $\mathcal{A}^k(M)$ .
11. Prove that the image of  $Alt_k$  is the unique largest subspace of  $\mathcal{T}^k(V)$  on which each permutation  $\sigma$  in the symmetric group  $S_k$  acts as multiplication by the scalar  $\epsilon(\sigma)$ .
12. (a) Prove that if  $f(x, y)$  is an alternating bilinear map on  $V$  (i.e.,  $f(x, x) = 0$  for all  $x \in V$ ) then  $f(x, y) = -f(y, x)$  for all  $x, y \in V$ .
- (b) Suppose that  $-1 \neq 1$  in  $F$ . Prove that  $f(x, y)$  is an alternating bilinear map on  $V$  (i.e.,  $f(x, x) = 0$  for all  $x \in V$ ) if and only if  $f(x, y) = -f(y, x)$  for all  $x, y \in V$ .
- (c) Suppose that  $-1 = 1$  in  $F$ . Prove that every alternating bilinear form  $f(x, y)$  on  $V$  is symmetric (i.e.,  $f(x, y) = f(y, x)$  for all  $x, y \in V$ ). Prove that there is a symmetric bilinear map on  $V$  that is not alternating. [One approach: show that  $C^2(V) \subset \mathcal{A}^2(V)$  and  $C^2(V) \neq \mathcal{A}^2(V)$  by counting dimensions. Alternatively, construct an explicit symmetric map that is not alternating.]
13. Let  $F$  be any field in which  $-1 \neq 1$  and let  $V$  be a vector space over  $F$ . Prove that  $V \otimes_F V = \mathcal{S}^2(V) \oplus \bigwedge^2(V)$  i.e., that every 2-tensor may be written uniquely as a sum of a symmetric and an alternating tensor.
14. Prove that if  $M$  is an  $R$ -module *direct factor* of the  $R$ -module  $N$  then  $\mathcal{T}(M)$  (respectively,  $\mathcal{S}(M)$  and  $\bigwedge(M)$ ) is an  $R$ -subalgebra of  $\mathcal{T}(N)$  (respectively,  $\mathcal{S}(N)$  and  $\bigwedge(N)$ ).

# Modules over Principal Ideal Domains

The main purpose of this chapter is to prove a structure theorem for finitely generated modules over particularly nice rings, namely Principal Ideal Domains. This theorem is an example of the ideal structure of the ring (which is particularly simple for P.I.D.s) being reflected in the structure of its modules. If we apply this result in the case where the P.I.D. is the ring of integers  $\mathbb{Z}$  then we obtain a proof of the Fundamental Theorem of Finitely Generated Abelian Groups (which we examined in Chapter 5 without proof). If instead we apply this structure theorem in the case where the P.I.D. is the ring  $F[x]$  of polynomials in  $x$  with coefficients in a field  $F$  we shall obtain the basic results on the so-called rational and Jordan canonical forms for a matrix. Before proceeding to the proof we briefly discuss these two important applications.

We have already discussed in Chapter 5 the result that any finitely generated abelian group is isomorphic to the direct sum of cyclic abelian groups, either  $\mathbb{Z}$  or  $\mathbb{Z}/n\mathbb{Z}$  for some positive integer  $n \neq 0$ . Recall also that an abelian group is the same thing as a  $\mathbb{Z}$ -module. Since the ideals of  $\mathbb{Z}$  are precisely the trivial ideal  $(0)$  and the principal ideals  $(n) = n\mathbb{Z}$  generated by positive integers  $n$ , we see that the Fundamental Theorem of Finitely Generated Abelian Groups in the language of modules says that any finitely generated  $\mathbb{Z}$ -module is the direct sum of modules of the form  $\mathbb{Z}/I$  where  $I$  is an ideal of  $\mathbb{Z}$  (these are the cyclic  $\mathbb{Z}$ -modules), together with a uniqueness statement when the direct sum is written in a particular form. Note the correspondence between the ideal structure of  $\mathbb{Z}$  and the structure of its (finitely generated) modules, the finitely generated abelian groups.

The Fundamental Theorem of Finitely Generated Modules over a P.I.D. states that the same result holds when the Principal Ideal Domain  $\mathbb{Z}$  is replaced by *any* P.I.D. In particular, we have seen in Chapter 10 that a module over the ring  $F[x]$  of polynomials in  $x$  with coefficients in the field  $F$  is the same thing as a vector space  $V$  together with a fixed linear transformation  $T$  of  $V$  (where the element  $x$  acts on  $V$  by the linear transformation  $T$ ). The Fundamental Theorem in this case will say that such a vector space is the direct sum of modules of the form  $F[x]/I$  where  $I$  is an ideal of  $F[x]$ , hence is either the trivial ideal  $(0)$  or a principal ideal  $(f(x))$  generated by some nonzero polynomial  $f(x)$  (these are the cyclic  $F[x]$ -modules), again with a uniqueness statement when the direct sum is written in a particular form. If this is translated back into the language of vector spaces and linear transformations we can obtain information on the