This is for $y > x$. By interchanging x and y we thus see that

$$|F(y) - F(x)| \leq M(x - y)$$

when $x > y$. Also, we have $F(y) - F(x) = 0$ when $x = y$. Thus in all three cases we have

$$|F(y) - F(x)| \leq M|x - y|.$$

Now let $x \in [a, b]$, and let $(x_n)_{n=0}^{\infty}$ be any sequence in $[a, b]$ converging to x. Then we have

$$-M|x_n - x| \leq F(x_n) - F(x) \leq M|x_n - x|$$

for each n. But $-M|x_n - x|$ and $M|x_n - x|$ both converge to 0 as $n \to \infty$, so by the squeeze test $F(x_n) - F(x)$ converges to 0 as $n \to \infty$, and thus $\lim_{n\to\infty} F(x_n) = F(x)$. Since this is true for all sequences $x_n \in [a, b]$ converging to x, we thus see that F is continuous at x. Since x was an arbitrary element of $[a, b]$, we thus see that F is continuous.

Now suppose that $x_0 \in [a, b]$, and f is continuous at x_0. Choose any $\varepsilon > 0$. Then by continuity, we can find a $\delta > 0$ such that $|f(x) - f(x_0)| \leq \varepsilon$ for all x in the interval $I := [x_0 - \delta, x_0 + \delta] \cap [a, b]$, or in other words

$$f(x_0) - \varepsilon \leq f(x) \leq f(x_0) + \varepsilon \text{ for all } x \in I.$$

We now show that

$$|F(y) - F(x_0) - f(x_0)(y - x_0)| \leq \varepsilon|y - x_0|$$

for all $y \in I$, since Proposition 10.1.7 will then imply that F is differentiable at x_0 with derivative $F'(x_0) = f(x_0)$ as desired.

Now fix $y \in I$. There are three cases. If $y = x_0$, then $F(y) - F(x_0) - f(x_0)(y - x_0) = 0$ and so the claim is obvious. If $y > x_0$, then

$$F(y) - F(x_0) = \int_{[x_0,y]} f.$$

Since $x_0, y \in I$, and I is a connected set, then $[x_0, y]$ is a subset of I, and thus we have

$$f(x_0) - \varepsilon \le f(x) \le f(x_0) + \varepsilon \text{ for all } x \in [x_0, y],$$

and thus

$$(f(x_0) - \varepsilon)(y - x_0) \le \int_{[x_0,y]} f \le (f(x_0) + \varepsilon)(y - x_0)$$

and so in particular

$$|F(y) - F(x_0) - f(x_0)(y - x_0)| \le \varepsilon |y - x_0|$$

as desired. The case $y < x_0$ is similar and is left to the reader. □

Example 11.9.2. Recall in Exercise 9.8.5 that we constructed a monotone function $f : \mathbf{R} \to \mathbf{R}$ which was discontinuous at every rational and continuous everywhere else. By Proposition 11.6.1, this monotone function is Riemann integrable on $[0, 1]$. If we define $F : [0, 1] \to \mathbf{R}$ by $F(x) := \int_{[0,x]} f$, then F is a continuous function which is differentiable at every irrational number. On the other hand, F is non-differentiable at every rational number; see Exercise 11.9.1.

Informally, the first fundamental theorem of calculus asserts that

$$(\int_{[a,x]} f)'(x) = f(x)$$

given a certain number of assumptions on f. Roughly, this means that the derivative of an integral recovers the original function. Now we show the reverse, that the integral of a derivative recovers the original function.

Definition 11.9.3 (Antiderivatives). Let I be a bounded interval, and let $f : I \to \mathbf{R}$ be a function. We say that a function $F : I \to \mathbf{R}$ is an *antiderivative* of f if F is differentiable on I and $F'(x) = f(x)$ for all $x \in I$.

Theorem 11.9.4 (Second Fundamental Theorem of Calculus). *Let $a < b$ be real numbers, and let $f : [a,b] \to \mathbf{R}$ be a Riemann integrable function. If $F : [a,b] \to \mathbf{R}$ is an antiderivative of f, then*

$$\int_{[a,b]} f = F(b) - F(a).$$

Proof. We will use Riemann sums. The idea is to show that

$$U(f,\mathbf{P}) \geq F(b) - F(a) \geq L(f,\mathbf{P})$$

for every partition $\mathbf{P}$ of $[a,b]$. The left inequality asserts that $F(b) - F(a)$ is a lower bound for $\{U(f,\mathbf{P}) : \mathbf{P}$ is a partition of $[a,b]\}$, while the right inequality asserts that $F(b) - F(a)$ is an upper bound for $\{L(f,\mathbf{P}) : \mathbf{P}$ is a partition of $[a,b]\}$. But by Proposition 11.3.12, this means that

$$\overline{\int}_{[a,b]} f \geq F(b) - F(a) \geq \underline{\int}_{[a,b]} f,$$

but since f is assumed to be Riemann integrable, both the upper and lower Riemann integral equal $\int_{[a,b]} f$. The claim follows.

We have to show the bound $U(f,\mathbf{P}) \geq F(b) - F(a) \geq L(f,\mathbf{P})$. We shall just show the first inequality $U(f,\mathbf{P}) \geq F(b) - F(a)$; the other inequality is similar.

Let $\mathbf{P}$ be a partition of $[a,b]$. From Lemma 11.8.4 we have

$$F(b) - F(a) = \sum_{J \in \mathbf{P}} F[J] = \sum_{J \in \mathbf{P} : J \neq \emptyset} F[J],$$

while from definition we have

$$U(f,\mathbf{P}) = \sum_{J \in \mathbf{P} : J \neq \emptyset} \sup_{x \in J} f(x)|J|.$$

Thus it will suffice to show that

$$F[J] \leq \sup_{x \in J} f(x)|J|$$

for all $J \in \mathbf{P}$ (other than the empty set).

When J is a point then the claim is clear, since both sides are zero. Now suppose that $J = [c, d], (c, d], [c, d)$, or (c, d) for some $c < d$. Then the left-hand side is $F[J] = F(d) - F(c)$. By the mean-value theorem, this is equal to $(d-c)F'(e)$ for some $e \in J$. But since $F'(e) = f(e)$, we thus have

$$F[J] = (d-c)f(e) = f(e)|J| \leq \sup_{x \in J} f(x)|J|$$

as desired. □

Of course, as you are all aware, one can use the second fundamental theorem of calculus to compute integrals relatively easily provided that you can find an anti-derivative of the integrand f. Note that the first fundamental theorem of calculus ensures that every *continuous* Riemann integrable function has an anti-derivative. For discontinuous functions, the situation is more complicated, and is a graduate-level real analysis topic which will not be discussed here. Also, not every function with an anti-derivative is Riemann integrable; as an example, consider the function $F : [-1, 1] \to \mathbf{R}$ defined by $F(x) := x^2 \sin(1/x^3)$ when $x \neq 0$, and $F(0) := 0$. Then F is differentiable everywhere (why?), so F' has an antiderivative, but F' is unbounded (why?), and so is not Riemann integrable.

We now pause to mention the infamous "$+C$" ambiguity in anti-derivatives:

Lemma 11.9.5. *Let I be a bounded interval, and let $f : I \to \mathbf{R}$ be a function. Let $F : I \to \mathbf{R}$ and $G : I \to \mathbf{R}$ be two antiderivatives of f. Then there exists a real number C such that $F(x) = G(x)+C$ for all $x \in I$.*

Proof. See Exercise 11.9.2. □

Exercise 11.9.1. Let $f : [0, 1] \to \mathbf{R}$ be the function in Exercise 9.8.5. Show that for every rational number $q \in \mathbf{Q} \cap [0, 1]$, the function f is not differentiable at q. (Hint: use the mean-value theorem, Corollary 10.2.9.)

Exercise 11.9.2. Prove Lemma 11.9.5. (Hint: apply the mean-value theorem, Corollary 10.2.9, to the function $F - G$. One can also prove this lemma using the second Fundamental theorem of calculus (how?), but one has to be careful since we do not assume f to be Riemann integrable.)

Exercise 11.9.3. Let $a < b$ be real numbers, and let $f : [a, b] \to \mathbf{R}$ be a monotone increasing function. Let $F : [a, b] \to \mathbf{R}$ be the function $F(x) := \int_{[a,x]} f$. Let x_0 be an element of $[a, b]$. Show that F is differentiable at x_0 if and only if f is continuous at x_0. (Hint: one direction is taken care of by one of the fundamental theorems of calculus. For the other, consider left and right limits of f and argue by contradiction.)

11.10 Consequences of the fundamental theorems

We can now give a number of useful consequences of the fundamental theorems of calculus (beyond the obvious application, that one can now compute any integral for which an anti-derivative is known). The first application is the familiar integration by parts formula.

Proposition 11.10.1 (Integration by parts formula). *Let $I = [a, b]$, and let $F : [a, b] \to \mathbf{R}$ and $G : [a, b] \to \mathbf{R}$ be differentiable functions on $[a, b]$ such that F' and G' are Riemann integrable on I. Then we have*

$$\int_{[a,b]} FG' = F(b)G(b) - F(a)G(a) - \int_{[a,b]} F'G.$$

Proof. See Exercise 11.10.1. □

Next, we show that under certain circumstances, one can write a Riemann-Stieltjes integral as a Riemann integral. We begin with piecewise constant functions.

Theorem 11.10.2. *Let $\alpha : [a, b] \to \mathbf{R}$ be a monotone increasing function, and suppose that α is also differentiable on $[a, b]$, with α' being Riemann integrable. Let $f : [a, b] \to \mathbf{R}$ be a piecewise* *constant function on* $[a, b]$. *Then* $f\alpha'$ *is Riemann integrable on* $[a, b]$, *and*

$$\int_{[a,b]} f\, d\alpha = \int_{[a,b]} f\alpha'.$$

Proof. Since f is piecewise constant, it is Riemann integrable, and since α' is also Riemann integrable, then $f\alpha'$ is Riemann integrable by Theorem 11.4.5.

Suppose that f is piecewise constant with respect to some partition $\mathbf{P}$ of $[a, b]$; without loss of generality we may assume that $\mathbf{P}$ does not contain the empty set. Then we have

$$\int_{[a,b]} f\, d\alpha = p.c. \int_{[\mathbf{P}]} f\, d\alpha = \sum_{J \in \mathbf{P}} c_J \alpha[J]$$

where c_J is the constant value of f on J. On the other hand, from Theorem 11.2.16(h) (generalized to partitions of arbitrary length - why is this generalization true?) we have

$$\int_{[a,b]} f\alpha' = \sum_{J \in \mathbf{P}} \int_J f\alpha' = \sum_{J \in \mathbf{P}} \int_J c_J \alpha' = \sum_{J \in \mathbf{P}} c_J \int_J \alpha'.$$

But by the second fundamental theorem of calculus (Theorem 11.9.4), $\int_J \alpha' = \alpha[J]$, and the claim follows. □

Corollary 11.10.3. *Let* $\alpha : [a, b] \to \mathbf{R}$ *be a monotone increasing function, and suppose that* α *is also differentiable on* $[a, b]$, *with* α' *being Riemann integrable. Let* $f : [a, b] \to \mathbf{R}$ *be a function which is Riemann-Stieltjes integrable with respect to* α *on* $[a, b]$. *Then* $f\alpha'$ *is Riemann integrable on* $[a, b]$, *and*

$$\int_{[a,b]} f\, d\alpha = \int_{[a,b]} f\alpha'.$$

Proof. Note that since f and α' are bounded, then $f\alpha'$ must also be bounded. Also, since α is monotone increasing and differentable, α' is non-negative.

Let $\varepsilon > 0$. Then, we can find a piecewise constant function $\overline{f}$ majorizing f on $[a, b]$, and a piecewise constant function $\underline{f}$ minorizing f on $[a, b]$, such that

$$\int_{[a,b]} f \, d\alpha - \varepsilon \leq \int_{[a,b]} \underline{f} \, d\alpha \leq \int_{[a,b]} \overline{f} \, d\alpha \leq \int_{[a,b]} f \, d\alpha + \varepsilon.$$

Applying Theorem 11.10.2, we obtain

$$\int_{[a,b]} f \, d\alpha - \varepsilon \leq \int_{[a,b]} \underline{f}\alpha' \leq \int_{[a,b]} \overline{f} \, \alpha' \leq \int_{[a,b]} f \, d\alpha + \varepsilon.$$

Since α' is non-negative and $\underline{f}$ minorizes f, then $\underline{f}\alpha'$ minorizes $f\alpha'$. Thus $\int_{[a,b]} \underline{f}\alpha' \leq \underline{\int}_{[a,b]} f\alpha'$ (why?). Thus

$$\int_{[a,b]} f \, d\alpha - \varepsilon \leq \underline{\int}_{[a,b]} f\alpha'.$$

Similarly we have

$$\overline{\int}_{[a,b]} f\alpha' \leq \int_{[a,b]} f \, d\alpha + \varepsilon.$$

Since these statements are true for any $\varepsilon > 0$, we must have

$$\int_{[a,b]} f \, d\alpha \leq \underline{\int}_{[a,b]} f\alpha' \leq \overline{\int}_{[a,b]} f\alpha' \leq \int_{[a,b]} f \, d\alpha$$

and the claim follows. $\square$

Remark 11.10.4. Informally, Corollary 11.10.3 asserts that $f \, d\alpha$ is essentially equivalent to $f \frac{d\alpha}{dx} dx$, when α is differentiable. However, the advantage of the Riemann-Stieltjes integral is that it still makes sense even when α is not differentiable.

We now build up to the familiar change of variables formula. We first need a preliminary lemma.

Lemma 11.10.5 (Change of variables formula I). *Let $[a, b]$ be a closed interval, and let $\phi : [a, b] \to [\phi(a), \phi(b)]$ be a continuous monotone increasing function. Let $f : [\phi(a), \phi(b)] \to \mathbf{R}$ be a piecewise constant function on $[\phi(a), \phi(b)]$. Then $f \circ \phi : [a, b] \to \mathbf{R}$ is also piecewise constant on $[a, b]$, and*

$$\int_{[a,b]} f \circ \phi \, d\phi = \int_{[\phi(a),\phi(b)]} f.$$

Proof. We give a sketch of the proof, leaving the gaps to be filled in Exercise 11.10.2. Let $\mathbf{P}$ be a partition of $[\phi(a), \phi(b)]$ such that f is piecewise constant with respect to $\mathbf{P}$; we may assume that $\mathbf{P}$ does not contain the empty set. For each $J \in \mathbf{P}$, let c_J be the constant value of f on J, thus

$$\int_{[\phi(a),\phi(b)]} f = \sum_{J \in \mathbf{P}} c_J |J|.$$

For each interval J, let $\phi^{-1}(J)$ be the set $\phi^{-1}(J) := \{x \in [a, b] : \phi(x) \in J\}$. Then $\phi^{-1}(J)$ is connected (why?), and is thus an interval. Furthermore, c_J is the constant value of $f \circ \phi$ on $\phi^{-1}(J)$ (why?). Thus, if we define $\mathbf{Q} := \{\phi^{-1}(J) : J \in \mathbf{P}\}$ (ignoring the fact that $\mathbf{Q}$ has been used to represent the rational numbers), then $\mathbf{Q}$ partitions $[a, b]$ (why?), and $f \circ \phi$ is piecewise constant with respect to $\mathbf{Q}$ (why?). Thus

$$\int_{[a,b]} f \circ \phi \, d\phi = \int_{[\mathbf{Q}]} f \circ \phi \, d\phi = \sum_{J \in \mathbf{P}} c_J \phi[\phi^{-1}(J)].$$

But $\phi[\phi^{-1}(J)] = |J|$ (why?), and the claim follows. $\square$

Proposition 11.10.6 (Change of variables formula II). *Let $[a, b]$ be a closed interval, and let $\phi : [a, b] \to [\phi(a), \phi(b)]$ be a continuous monotone increasing function. Let $f : [\phi(a), \phi(b)] \to \mathbf{R}$ be a Riemann integrable function on $[\phi(a), \phi(b)]$. Then $f \circ \phi : [a, b] \to \mathbf{R}$ is Riemann-Stieltjes integrable with respect to ϕ on $[a, b]$, and*

$$\int_{[a,b]} f \circ \phi \, d\phi = \int_{[\phi(a),\phi(b)]} f.$$

Proof. This will be obtained from Lemma 11.10.5 in a similar manner to how Corollary 11.10.3 was obtained from Theorem 11.10.2. First observe that since f is Riemann integrable, it is bounded, and then $f \circ \phi$ must also be bounded (why?).

Let $\varepsilon > 0$. Then, we can find a piecewise constant function $\overline{f}$ majorizing f on $[\phi(a), \phi(b)]$, and a piecewise constant function $\underline{f}$ minorizing f on $[\phi(a), \phi(b)]$, such that

$$\int_{[\phi(a),\phi(b)]} f - \varepsilon \leq \int_{[\phi(a),\phi(b)]} \underline{f} \leq \int_{[\phi(a),\phi(b)]} \overline{f} \leq \int_{[\phi(a),\phi(b)]} f + \varepsilon.$$

Applying Lemma 11.10.5, we obtain

$$\int_{[\phi(a),\phi(b)]} f - \varepsilon \leq \int_{[a,b]} \underline{f} \circ \phi \; d\phi \leq \int_{[a,b]} \overline{f} \circ \phi \; d\phi \leq \int_{[\phi(a),\phi(b)]} f + \varepsilon.$$

Since $\underline{f} \circ \phi$ is piecewise constant and minorizes $f \circ \phi$, we have

$$\int_{[a,b]} \underline{f} \circ \phi \; d\phi \leq \underline{\int}_{[a,b]} f \circ \phi \; d\phi$$

while similarly we have

$$\int_{[a,b]} \overline{f} \circ \phi \; d\phi \geq \overline{\int}_{[a,b]} f \circ \phi \; d\phi.$$

Thus

$$\int_{[\phi(a),\phi(b)]} f - \varepsilon \leq \underline{\int}_{[a,b]} f \circ \phi \; d\phi \leq \overline{\int}_{[a,b]} f \circ \phi \; d\phi \leq \int_{[\phi(a),\phi(b)]} f + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this implies that

$$\int_{[\phi(a),\phi(b)]} f \leq \underline{\int}_{[a,b]} f \circ \phi \; d\phi \leq \overline{\int}_{[a,b]} f \circ \phi \; d\phi \leq \int_{[\phi(a),\phi(b)]} f$$

and the claim follows. $\square$

Combining this formula with Corollary 11.10.3, one immediately obtains the following familiar formula:

Proposition 11.10.7 (Change of variables formula III). *Let $[a,b]$ be a closed interval, and let $\phi : [a,b] \to [\phi(a), \phi(b)]$ be a differentiable monotone increasing function such that ϕ' is Riemann integrable. Let $f : [\phi(a), \phi(b)] \to \mathbf{R}$ be a Riemann integrable function on $[\phi(a), \phi(b)]$. Then $(f \circ \phi)\phi' : [a,b] \to \mathbf{R}$ is Riemann integrable on $[a,b]$, and*

$$\int_{[a,b]} (f \circ \phi)\phi' = \int_{[\phi(a),\phi(b)]} f.$$

Exercise 11.10.1. Prove Proposition 11.10.1. (Hint: first use Corollary 11.5.2 and Theorem 11.4.5 to show that FG' and $F'G$ are Riemann integrable. Then use the product rule (Theorem 10.1.13(d)).)

Exercise 11.10.2. Fill in the gaps marked (why?) in the proof of Lemma 11.10.5.

Exercise 11.10.3. Let $a < b$ be real numbers, and let $f : [a,b] \to \mathbf{R}$ be a Riemann integrable function. Let $g : [-b,-a] \to \mathbf{R}$ be defined by $g(x) := f(-x)$. Show that g is also Riemann integrable, and $\int_{[-b,-a]} g = \int_{[a,b]} f$.

Exercise 11.10.4. What is the analogue of Proposition 11.10.7 when ϕ is monotone decreasing instead of monotone increasing? (When ϕ is neither monotone increasing or monotone decreasing, the situation becomes significantly more complicated.)

Chapter A

Appendix: the basics of mathematical logic

The purpose of this appendix is to give a quick introduction to *mathematical logic*, which is the language one uses to conduct rigourous mathematical proofs. Knowing how mathematical logic works is also very helpful for understanding the mathematical way of thinking, which once mastered allows you to approach mathematical concepts and problems in a clear and confident way - including many of the proof-type questions in this text.

Writing logically is a very useful skill. It is somewhat related to, but not the same as, writing clearly, or efficiently, or convincingly, or informatively; ideally one would want to do all of these at once, but sometimes one has to make compromises, though with practice you'll be able to achieve more of your writing objectives concurrently. Thus a logical argument may sometimes look unwieldy, excessively complicated, or otherwise appear unconvincing. The big advantage of writing logically, however, is that one can be absolutely sure that your conclusion will be correct, as long as all your hypotheses were correct and your steps were logical; using other styles of writing one can be reasonably convinced that something is true, but there is a difference between being convinced and being *sure*.

Being logical is not the only desirable trait in writing, and in fact sometimes it gets in the way; mathematicians for instance often resort to short informal arguments which are not logically rigourous when they want to convince other mathematicians of a statement without going through all of the long details, and the same is true of course for non-mathematicians as well. So saying that a statement or argument is "not logical" is not necessarily a bad thing; there are often many situations when one has good reasons not to be emphatic about being logical. However, one should be aware of the distinction between logical reasoning and more informal means of argument, and not try to pass off an illogical argument as being logically rigourous. In particular, if an exercise is asking for a proof, then it is expecting you to be logical in your answer.

Logic is a skill that needs to be learnt like any other, but this skill is also innate to all of you - indeed, you probably use the laws of logic unconsciously in your everyday speech and in your own internal (non-mathematical) reasoning. However, it does take a bit of training and practice to recognize this innate skill and to apply it to abstract situations such as those encountered in mathematical proofs. Because logic is innate, the laws of logic that you learn should *make sense* - if you find yourself having to memorize one of the principles or laws of logic here, without feeling a mental "click" or comprehending why that law should work, then you will probably *not* be able to use that law of logic correctly and effectively in practice. So, *please* don't study this appendix the way you might cram before a final - that is going to be useless. Instead, **put away your highlighter pen**, and *read* and *understand* this appendix rather than merely *studying* it!

A.1 Mathematical statements

Any mathematical argument proceeds in a sequence of *mathematical statements*. These are precise statements concerning various mathematical objects (numbers, vectors, functions, etc.) and relations between them (addition, equality, differentiation, etc.). These objects can either be constants or variables; more on this later. Statements[1] are either true or false.

[1]More precisely, statements with no free variables are either true or false. We shall discuss free variables later on in this appendix.

Example A.1.1. $2+2=4$ is a true statement; $2+2=5$ is a false statement.

Not every combination of mathematical symbols is a statement. For instance,

$$= 2 + +4 = - = 2$$

is not a statement; we sometimes call it *ill-formed* or *ill-defined*. The statements in the previous example are *well-formed* or *well-defined*. Thus well-formed statements can be either true or false; ill-formed statements are considered to be neither true nor false (in fact, they are usually not considered statements at all). A more subtle example of an ill-formed statement is

$$0/0 = 1;$$

division by zero is undefined, and so the above statement is ill-formed. A logical argument should not contain any ill-formed statements, thus for instance if an argument uses a statement such as $x/y = z$, it needs to first ensure that y is not equal to zero. Many purported proofs of "0=1" or other false statements rely on overlooking this "statements must be well-formed" criterion.

Many of you have probably written ill-formed or otherwise inaccurate statements in your mathematical work, while intending to mean some other, well-formed and accurate statement. To a certain extent this is permissible - it is similar to misspelling some words in a sentence, or using a slightly inaccurate or ungrammatical word in place of a correct one ("She ran good" instead of "She ran well"). In many cases, the reader (or grader) can detect this mis-step and correct for it. However, it looks unprofessional and suggests that you may not know what you are talking about. And if indeed you actually do not know what you are talking about, and are applying mathematical or logical rules blindly, then writing an ill-formed statement can quickly confuse you into writing more and more nonsense - usually of the sort which receives no credit in grading. So it is important, especially when just learning a subject, to take care in keeping statements well-formed and precise. Once you have more skill and confidence, of course you can afford once again to speak loosely, because you will know what you are doing and won't be in as much danger of veering off into nonsense.

One of the basic axioms of mathematical logic is that every well-formed statement is either true or false, but not both. (Though if there are free variables, the truth of a statement may depend on the values of these variables. More on this later.) Furthermore, the truth or falsity of a statement is intrinsic to the statement, and does not depend on the opinion of the person viewing the statement (as long as all the definitions and notations are agreed upon, of course). So to prove that a statement is true, it suffices to show that it is not false, while to show that a statement is false, it suffices to show that it is not true; this is the principle underlying the powerful technique of *proof by contradiction*, which we discuss later. This axiom is viable as long as one is working with precise concepts, for which the truth or falsity can be determined (at least in principle) in an objective and consistent manner. However, if one is working in very non-mathematical situations, then this axiom becomes much more dubious, and so it can be a mistake to apply mathematical logic to non-mathematical situations. (For instance, a statement such as "this rock weighs 52 pounds" is reasonably precise and objective, and so it is fairly safe to use mathematical reasoning to manipulate it, whereas vague statements such as "this rock is heavy", "this piece of music is beautiful" or "God exists" are much more problematic. So while mathematical logic is a very useful and powerful tool, it still does have some limitations of applicability.) One can still attempt to apply logic (or principles similar to logic) in these cases (for instance, by creating a *mathematical model* of a real-life phenomenon), but this is now science or philosophy, not mathematics, and we will not discuss it further here.

Remark A.1.2. There are other models of logic which attempt to deal with statements that are not definitely true or definitely false, such as modal logic, intuitionist logic, or fuzzy logic, but these are well beyond the scope of this text.

Being true is different from being *useful* or *efficient*. For instance, the statement

$$2 = 2$$

is true but unlikely to be very useful. The statement

$$4 \leq 4$$

is also true, but not very efficient (the statement $4 = 4$ is more precise). It may also be that a statement may be false yet still be useful, for instance

$$\pi = 22/7$$

is false, but is still useful as a first approximation. In mathematical reasoning, we only concern ourselves with truth rather than usefulness or efficiency; the reason is that truth is objective (everybody can agree on it) and we can deduce true statements from precise rules, whereas usefulness and efficiency are to some extent matters of opinion, and do not follow precise rules. Also, even if some of the individual steps in an argument may not seem very useful or efficient, it is still possible (indeed, quite common) for the final conclusion to be quite non-trivial (i.e., not obviously true) and useful.

Statements are different from *expressions*. Statements are true or false; expressions are a sequence of mathematical symbols which produces some mathematical object (a number, matrix, function, set, etc.) as its value. For instance

$$2 + 3 * 5$$

is an expression, not a statement; it produces a number as its value. Meanwhile,

$$2 + 3 * 5 = 17$$

is a statement, not an expression. Thus it does not make any sense to ask whether $2 + 3 * 5$ is true or false. As with statements, expressions can be well-defined or ill-defined; $2 + 3/0$, for instance,