(e) *If $x > 1$, then $x^{1/k}$ is a decreasing function of k. If $x < 1$, then $x^{1/k}$ is an increasing function of k. If $x = 1$, then $x^{1/k} = 1$ for all k.*

(f) *We have $(xy)^{1/n} = x^{1/n}y^{1/n}$.*

(g) *We have $(x^{1/n})^{1/m} = x^{1/nm}$.*

Proof. See Exercise 5.6.1. □

The observant reader may note that this definition of $x^{1/n}$ might possibly be inconsistent with our previous notion of x^n when $n = 1$, but it is easy to check that $x^{1/1} = x = x^1$ (why?), so there is no inconsistency.

One consequence of Lemma 5.6.6(b) is the following cancellation law: if y and z are positive and $y^n = z^n$, then $y = z$. (Why does this follow from Lemma 5.6.6(b)?) Note that this only works when y and z are positive; for instance, $(-3)^2 = 3^2$, but we cannot conclude from this that $-3 = 3$.

Now we define how to raise a positive number x to a *rational* exponent q.

Definition 5.6.7. Let $x > 0$ be a positive real number, and let q be a rational number. To define x^q, we write $q = a/b$ for some integer a and positive integer b, and define

$$x^q := (x^{1/b})^a.$$

Note that every rational q, whether positive, negative, or zero, can be written in the form a/b where a is an integer and b is positive (why?). However, the rational number q can be expressed in the form a/b in more than one way, for instance $1/2$ can also be expressed as $2/4$ or $3/6$. So to ensure that this definition is well-defined, we need to check that different expressions a/b give the same formula for x^q:

Lemma 5.6.8. *Let a, a' be integers and b, b' be positive integers such that $a/b = a'/b'$, and let x be a positive real number. Then we have $(x^{1/b'})^{a'} = (x^{1/b})^a$.*

Proof. There are three cases: $a = 0$, $a > 0$, $a < 0$. If $a = 0$, then we must have $a' = 0$ (why?) and so both $(x^{1/b'})^{a'}$ and $(x^{1/b})^a$ are equal to 1, so we are done.

Now suppose that $a > 0$. Then $a' > 0$ (why?), and $ab' = ba'$. Write $y := x^{1/(ab')} = x^{1/(ba')}$. By Lemma 5.6.6(g) we have $y = (x^{1/b'})^{1/a}$ and $y = (x^{1/b})^{1/a'}$; by Lemma 5.6.6(a) we thus have $y^a = x^{1/b'}$ and $y^{a'} = x^{1/b}$. Thus we have

$$(x^{1/b'})^{a'} = (y^a)^{a'} = y^{aa'} = (y^{a'})^a = (x^{1/b})^a$$

as desired.

Finally, suppose that $a < 0$. Then we have $(-a)/b = (-a')/b$. But $-a$ is positive, so the previous case applies and we have $(x^{1/b'})^{-a'} = (x^{1/b})^{-a}$. Taking the reciprocal of both sides we obtain the result. □

Thus x^q is well-defined for every rational q. Note that this new definition is consistent with our old definition for $x^{1/n}$ (why?) and is also consistent with our old definition for x^n (why?).

Some basic facts about rational exponentiation:

Lemma 5.6.9. *Let $x, y > 0$ be positive reals, and let q, r be rationals.*

(*a*) *x^q is a positive real.*

(*b*) *$x^{q+r} = x^q x^r$ and $(x^q)^r = x^{qr}$.*

(*c*) *$x^{-q} = 1/x^q$.*

(*d*) *If $q > 0$, then $x > y$ if and only if $x^q > y^q$.*

(*e*) *If $x > 1$, then $x^q > x^r$ if and only if $q > r$. If $x < 1$, then $x^q > x^r$ if and only if $q < r$.*

Proof. See Exercise 5.6.2. □

We still have to do real exponentiation; in other words, we still have to define x^y where $x > 0$ and y is a real number - but we will defer that until Section 6.7, once we have formalized the concept of limit.

In the rest of the text we shall now just assume the real numbers to obey all the usual laws of algebra, order, and exponentiation.

Exercise 5.6.1. Prove Lemma 5.6.6. (Hints: review the proof of Proposition 5.5.12. Also, you will find proof by contradiction a useful tool, especially when combined with the trichotomy of order in Proposition 5.4.7 and Proposition 5.4.12. The earlier parts of the lemma can be used to prove later parts of the lemma. With part (e), first show that if $x > 1$ then $x^{1/n} > 1$, and if $x < 1$ then $x^{1/n} < 1$.)

Exercise 5.6.2. Prove Lemma 5.6.9. (Hint: you should rely mainly on Lemma 5.6.6 and on algebra.)

Exercise 5.6.3. If x is a real number, show that $|x| = (x^2)^{1/2}$.

Chapter 6

Limits of sequences

6.1 Convergence and limit laws

In the previous chapter, we defined the real numbers as formal limits of rational (Cauchy) sequences, and we then defined various operations on the real numbers. However, unlike our work in constructing the integers (where we eventually replaced formal differences with actual differences) and rationals (where we eventually replaced formal quotients with actual quotients), we never really finished the job of constructing the real numbers, because we never got around to replacing formal limits $\mathrm{LIM}_{n\to\infty}a_n$ with actual limits $\lim_{n\to\infty} a_n$. In fact, we haven't defined limits at all yet. This will now be rectified.

We begin by repeating much of the machinery of ε-close sequences, etc. again - but this time, we do it for sequences of *real* numbers, not rational numbers. Thus this discussion will supercede what we did in the previous chapter. First, we define distance for real numbers:

Definition 6.1.1 (Distance between two real numbers). Given two real numbers x and y, we define their distance $d(x, y)$ to be $d(x, y) := |x - y|$.

Clearly this definition is consistent with Definition 4.3.2. Further, Proposition 4.3.3 works just as well for real numbers as it does for rationals, because the real numbers obey all the rules of algebra that the rationals do.

Definition 6.1.2 (ε-close real numbers). Let $\varepsilon > 0$ be a real number. We say that two real numbers x, y are *ε-close* iff we have $d(y, x) \leq \varepsilon$.

Again, it is clear that this definition of ε-close is consistent with Definition 4.3.4.

Now let $(a_n)_{n=m}^{\infty}$ be a sequence of *real* numbers; i.e., we assign a real number a_n for every integer $n \geq m$. The starting index m is some integer; usually this will be 1, but in some cases we will start from some index other than 1. (The choice of label used to index this sequence is unimportant; we could use for instance $(a_k)_{k=m}^{\infty}$ and this would represent exactly the same sequence as $(a_n)_{n=m}^{\infty}$.) We can define the notion of a Cauchy sequence in the same manner as before:

Definition 6.1.3 (Cauchy sequences of reals). Let $\varepsilon > 0$ be a real number. A sequence $(a_n)_{n=N}^{\infty}$ of real numbers starting at some integer index N is said to be *ε-steady* iff a_j and a_k are ε-close for every $j, k \geq N$. A sequence $(a_n)_{n=m}^{\infty}$ starting at some integer index m is said to be *eventually ε-steady* iff there exists an $N \geq m$ such that $(a_n)_{n=N}^{\infty}$ is ε-steady. We say that $(a_n)_{n=m}^{\infty}$ is a *Cauchy sequence* iff it is eventually ε-steady for every $\varepsilon > 0$.

To put it another way, a sequence $(a_n)_{n=m}^{\infty}$ of real numbers is a Cauchy sequence if, for every real $\varepsilon > 0$, there exists an $N \geq m$ such that $|a_n - a_{n'}| \leq \varepsilon$ for all $n, n' \geq N$. These definitions are consistent with the corresponding definitions for rational numbers (Definitions 5.1.3, 5.1.6, 5.1.8), although verifying consistency for Cauchy sequences takes a little bit of care:

Proposition 6.1.4. *Let $(a_n)_{n=m}^{\infty}$ be a sequence of rational numbers starting at some integer index m. Then $(a_n)_{n=m}^{\infty}$ is a Cauchy sequence in the sense of Definition 5.1.8 if and only if it is a Cauchy sequence in the sense of Definition 6.1.3.*

Proof. Suppose first that $(a_n)_{n=m}^{\infty}$ is a Cauchy sequence in the sense of Definition 6.1.3; then it is eventually ε-steady for every *real* $\varepsilon > 0$. In particular, it is eventually ε-steady for every *rational*

$\varepsilon > 0$, which makes it a Cauchy sequence in the sense of Definition 5.1.8.

Now suppose that $(a_n)_{n=m}^\infty$ is a Cauchy sequence in the sense of Definition 5.1.8; then it is eventually ε-steady for every *rational* $\varepsilon > 0$. If $\varepsilon > 0$ is a real number, then there exists a *rational* $\varepsilon' > 0$ which is smaller than ε, by Proposition 5.4.14. Since ε' is rational, we know that $(a_n)_{n=m}^\infty$ is eventually ε'-steady; since $\varepsilon' < \varepsilon$, this implies that $(a_n)_{n=m}^\infty$ is eventually ε-steady. Since ε is an arbitrary positive real number, we thus see that $(a_n)_{n=m}^\infty$ is a Cauchy sequence in the sense of Definition 6.1.3. □

Because of this proposition, we will no longer care about the distinction between Definition 5.1.8 and Definition 6.1.3, and view the concept of a Cauchy sequence as a single unified concept.

Now we talk about what it means for a sequence of real numbers to converge to some limit L.

Definition 6.1.5 (Convergence of sequences). Let $\varepsilon > 0$ be a real number, and let L be a real number. A sequence $(a_n)_{n=N}^\infty$ of real numbers is said to be *ε-close to L* iff a_n is ε-close to L for every $n \geq N$, i.e., we have $|a_n - L| \leq \varepsilon$ for every $n \geq N$. We say that a sequence $(a_n)_{n=m}^\infty$ is *eventually ε-close to L* iff there exists an $N \geq m$ such that $(a_n)_{n=N}^\infty$ is ε-close to L. We say that a sequence $(a_n)_{n=m}^\infty$ *converges to L* iff it is eventually ε-close to L for every real $\varepsilon > 0$.

One can unwrap all the definitions here and write the concept of convergence more directly; see Exercise 6.1.2.

Examples 6.1.6. The sequence

$$0.9, 0.99, 0.999, 0.9999, \ldots$$

is 0.1-close to 1, but is not 0.01-close to 1, because of the first element of the sequence. However, it is eventually 0.01-close to 1. In fact, for every real $\varepsilon > 0$, this sequence is eventually ε-close to 1, hence is convergent to 1.

Proposition 6.1.7 (Uniqueness of limits). *Let $(a_n)_{n=m}^{\infty}$ be a real sequence starting at some integer index m, and let $L \neq L'$ be two distinct real numbers. Then it is not possible for $(a_n)_{n=m}^{\infty}$ to converge to L while also converging to L'.*

Proof. Suppose for sake of contradiction that $(a_n)_{n=m}^{\infty}$ was converging to both L and L'. Let $\varepsilon = |L - L'|/3$; note that ε is positive since $L \neq L'$. Since $(a_n)_{n=m}^{\infty}$ converges to L, we know that $(a_n)_{n=m}^{\infty}$ is eventually ε-close to L; thus there is an $N \geq m$ such that $d(a_n, L) \leq \varepsilon$ for all $n \geq N$. Similarly, there is an $M \geq m$ such that $d(a_n, L') \leq \varepsilon$ for all $n \geq M$. In particular, if we set $n := \max(N, M)$, then we have $d(a_n, L) \leq \varepsilon$ and $d(a_n, L') \leq \varepsilon$, hence by the triangle inequality $d(L, L') \leq 2\varepsilon = 2|L - L'|/3$. But then we have $|L - L'| \leq 2|L - L'|/3$, which contradicts the fact that $|L - L'| > 0$. Thus it is not possible to converge to both L and L'. □

Now that we know limits are unique, we can set up notation to specify them:

Definition 6.1.8 (Limits of sequences). If a sequence $(a_n)_{n=m}^{\infty}$ converges to some real number L, we say that $(a_n)_{n=m}^{\infty}$ is *convergent* and that its *limit* is L; we write

$$L = \lim_{n \to \infty} a_n$$

to denote this fact. If a sequence $(a_n)_{n=m}^{\infty}$ is not converging to any real number L, we say that the sequence $(a_n)_{n=m}^{\infty}$ is *divergent* and we leave $\lim_{n\to\infty} a_n$ undefined.

Note that Proposition 6.1.7 ensures that a sequence can have at most one limit. Thus, if the limit exists, it is a single real number, otherwise it is undefined.

Remark 6.1.9. The notation $\lim_{n\to\infty} a_n$ does not give any indication about the starting index m of the sequence, but the starting index is irrelevant (Exercise 6.1.3). Thus in the rest of this discussion we shall not be too careful as to where these sequences start, as we shall be mostly focused on their limits.

We sometimes use the phrase "$a_n \to x$ as $n \to \infty$" as an alternate way of writing the statement "$(a_n)_{n=m}^\infty$ converges to x". Bear in mind, though, that the individual statements $a_n \to x$ and $n \to \infty$ do not have any rigourous meaning; this phrase is just a convention, though of course a very suggestive one.

Remark 6.1.10. The exact choice of letter used to denote the index (in this case n) is irrelevant: the phrase $\lim_{n\to\infty} a_n$ has exactly the same meaning as $\lim_{k\to\infty} a_k$, for instance. Sometimes it will be convenient to change the label of the index to avoid conflicts of notation; for instance, we might want to change n to k because n is simultaneously being used for some other purpose, and we want to reduce confusion. See Exercise 6.1.4.

As an example of a limit, we present

Proposition 6.1.11. *We have* $\lim_{n\to\infty} 1/n = 0$.

Proof. We have to show that the sequence $(a_n)_{n=1}^\infty$ converges to 0, where $a_n := 1/n$. In other words, for every $\varepsilon > 0$, we need to show that the sequence $(a_n)_{n=1}^\infty$ is eventually ε-close to 0. So, let $\varepsilon > 0$ be an arbitrary real number. We have to find an N such that $|a_n - 0| \leq \varepsilon$ for every $n \geq N$. But if $n \geq N$, then

$$|a_n - 0| = |1/n - 0| = 1/n \leq 1/N.$$

Thus, if we pick $N > 1/\varepsilon$ (which we can do by the Archimedean principle), then $1/N < \varepsilon$, and so $(a_n)_{n=N}^\infty$ is ε-close to 0. Thus $(a_n)_{n=1}^\infty$ is eventually ε-close to 0. Since ε was arbitrary, $(a_n)_{n=1}^\infty$ converges to 0. □

Proposition 6.1.12 (Convergent sequences are Cauchy). *Suppose that* $(a_n)_{n=m}^\infty$ *is a convergent sequence of real numbers. Then* $(a_n)_{n=m}^\infty$ *is also a Cauchy sequence.*

Proof. See Exercise 6.1.5. □

Example 6.1.13. The sequence $1, -1, 1, -1, 1, -1, \ldots$ is not a Cauchy sequence (because it is not eventually 1-steady), and is hence not a convergent sequence, by Proposition 6.1.12.

Remark 6.1.14. For a converse to Proposition 6.1.12, see Theorem 6.4.18 below.

Now we show that formal limits can be superceded by actual limits, just as formal subtraction was superceded by actual subtraction when constructing the integers, and formal division superceded by actual division when constructing the rational numbers.

Proposition 6.1.15 (Formal limits are genuine limits). *Suppose that $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence of rational numbers. Then $(a_n)_{n=1}^{\infty}$ converges to $\mathrm{LIM}_{n\to\infty} a_n$, i.e.*

$$\mathrm{LIM}_{n\to\infty} a_n = \lim_{n\to\infty} a_n.$$

Proof. See Exercise 6.1.6. □

Definition 6.1.16 (Bounded sequences). A sequence $(a_n)_{n=m}^{\infty}$ of real numbers is *bounded by* a real number M iff we have $|a_n| \leq M$ for all $n \geq m$. We say that $(a_n)_{n=m}^{\infty}$ is *bounded* iff it is bounded by M for some real number $M > 0$.

This definition is consistent with Definition 5.1.12; see Exercise 6.1.7.

Recall from Lemma 5.1.15 that every Cauchy sequence of rational numbers is bounded. An inspection of the proof of that Lemma shows that the same argument works for real numbers; every Cauchy sequence of real numbers is bounded. In particular, from Proposition 6.1.12 we see have

Corollary 6.1.17. *Every convergent sequence of real numbers is bounded.*

Example 6.1.18. The sequence $1, 2, 3, 4, 5, \ldots$ is not bounded, and hence is not convergent.

We can now prove the usual limit laws.

Theorem 6.1.19 (Limit Laws). *Let $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ be convergent sequences of real numbers, and let x, y be the real numbers $x := \lim_{n\to\infty} a_n$ and $y := \lim_{n\to\infty} b_n$.*

(a) *The sequence* $(a_n+b_n)_{n=m}^{\infty}$ *converges to* $x+y$*; in other words,*

$$\lim_{n\to\infty}(a_n+b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n.$$

(b) *The sequence* $(a_nb_n)_{n=m}^{\infty}$ *converges to* xy*; in other words,*

$$\lim_{n\to\infty}(a_nb_n) = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n).$$

(c) *For any real number* c*, the sequence* $(ca_n)_{n=m}^{\infty}$ *converges to* cx*; in other words,*

$$\lim_{n\to\infty}(ca_n) = c\lim_{n\to\infty} a_n.$$

(d) *The sequence* $(a_n-b_n)_{n=m}^{\infty}$ *converges to* $x-y$*; in other words,*

$$\lim_{n\to\infty}(a_n-b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n.$$

(e) *Suppose that* $y \neq 0$*, and that* $b_n \neq 0$ *for all* $n \geq m$*. Then the sequence* $(b_n^{-1})_{n=m}^{\infty}$ *converges to* y^{-1}*; in other words,*

$$\lim_{n\to\infty} b_n^{-1} = (\lim_{n\to\infty} b_n)^{-1}.$$

(f) *Suppose that* $y \neq 0$*, and that* $b_n \neq 0$ *for all* $n \geq m$*. Then the sequence* $(a_n/b_n)_{n=m}^{\infty}$ *converges to* x/y*; in other words,*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}.$$

(g) *The sequence* $(\max(a_n,b_n))_{n=m}^{\infty}$ *converges to* $\max(x,y)$*; in other words,*

$$\lim_{n\to\infty} \max(a_n,b_n) = \max(\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n).$$

(h) *The sequence* $(\min(a_n,b_n))_{n=m}^{\infty}$ *converges to* $\min(x,y)$*; in other words,*

$$\lim_{n\to\infty} \min(a_n,b_n) = \min(\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n).$$

Proof. See Exercise 6.1.8. □

Exercise 6.1.1. Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers, such that $a_{n+1} > a_n$ for each natural number n. Prove that whenever n and m are natural numbers such that $m > n$, then we have $a_m > a_n$. (We refer to these sequences as *increasing* sequences.)

Exercise 6.1.2. Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers, and let L be a real number. Show that $(a_n)_{n=m}^{\infty}$ converges to L if and only if, given any real $\varepsilon > 0$, one can find an $N \geq m$ such that $|a_n - L| \leq \varepsilon$ for all $n \geq N$.

Exercise 6.1.3. Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers, let c be a real number, and let $m' \geq m$ be an integer. Show that $(a_n)_{n=m}^{\infty}$ converges to c if and only if $(a_n)_{n=m'}^{\infty}$ converges to c.

Exercise 6.1.4. Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers, let c be a real number, and let $k \geq 0$ be a non-negative integer. Show that $(a_n)_{n=m}^{\infty}$ converges to c if and only if $(a_{n+k})_{n=m}^{\infty}$ converges to c.

Exercise 6.1.5. Prove Proposition 6.1.12. (Hint: use the triangle inequality, or Proposition 4.3.7.)

Exercise 6.1.6. Prove Proposition 6.1.15, using the following outline. Let $(a_n)_{n=m}^{\infty}$ be a Cauchy sequence of rationals, and write $L := LIM_{n\to\infty} a_n$. We have to show that $(a_n)_{n=m}^{\infty}$ converges to L. Let $\varepsilon > 0$. Assume for sake of contradiction that sequence a_n is *not* eventually ε-close to L. Use this, and the fact that $(a_n)_{n=m}^{\infty}$ is Cauchy, to show that there is an $N \geq m$ such that either $a_n > L + \varepsilon/2$ for all $n \geq N$, or $a_n < L - \varepsilon/2$ for all $n \geq N$. Then use Exercise 5.4.8.

Exercise 6.1.7. Show that Definition 6.1.16 is consistent with Definition 5.1.12 (i.e., prove an analogue of Proposition 6.1.4 for bounded sequences instead of Cauchy sequences).

Exercise 6.1.8. Prove Theorem 6.1.19. (Hint: you can use some parts of the theorem to prove others, e.g., (b) can be used to prove (c); (a),(c) can be used to prove (d); and (b), (e) can be used to prove (f). The proofs are similar to those of Lemma 5.3.6, Proposition 5.3.10, and Lemma 5.3.15. For (e), you may need to first prove the auxiliary result that any sequence whose elements are non-zero, and which converges to a non-zero limit, is bounded away from zero.)

Exercise 6.1.9. Explain why Theorem 6.1.19(f) fails when the limit of the denominator is 0. (To repair that problem requires *L'Hôpital's rule*, see Section 10.5.)

Exercise 6.1.10. Show that the concept of equivalent Cauchy sequence, as defined in Definition 5.2.6, does not change if ε is required to be positive real instead of positive rational. More precisely, if $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ are sequences of reals, show that $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ are eventually ε-close for every rational $\varepsilon > 0$ if and only if they are eventually ε-close for every real $\varepsilon > 0$. (Hint: modify the proof of Proposition 6.1.4.)

6.2 The extended real number system

There are some sequences which do not converge to any real number, but instead seem to be wanting to converge to $+\infty$ or $-\infty$. For instance, it seems intuitive that the sequence

$$1, 2, 3, 4, 5, \ldots$$

should be converging to $+\infty$, while

$$-1, -2, -3, -4, -5, \ldots$$

should be converging to $-\infty$. Meanwhile, the sequence

$$1, -1, 1, -1, 1, -1, \ldots$$

does not seem to be converging to anything (although we shall see later that it does have $+1$ and -1 as "limit points" - see below). Similarly the sequence

$$1, -2, 3, -4, 5, -6, \ldots$$

does not converge to any real number, and also does not appear to be converging to $+\infty$ or converging to $-\infty$. To make this precise we need to talk about something called the *extended real number system.*

Definition 6.2.1 (Extended real number system). The *extended real number system* $\mathbf{R}^*$ is the real line $\mathbf{R}$ with two additional elements attached, called $+\infty$ and $-\infty$. These elements are distinct from each other and also distinct from every real number. An extended real number x is called *finite* iff it is a real number, and *infinite* iff it is equal to $+\infty$ or $-\infty$. (This definition is not directly related to the notion of finite and infinite sets in Section 3.6, though it is of course similar in spirit.)

These new symbols, $+\infty$ and $-\infty$, at present do not have much meaning, since we have no operations to manipulate them (other than equality $=$ and inequality $\neq$). Now we place a few operations on the extended real number system.

Definition 6.2.2 (Negation of extended reals). The operation of negation $x \mapsto -x$ on $\mathbf{R}$, we now extend to $\mathbf{R}^*$ by defining $-(+\infty) := -\infty$ and $-(-\infty) := +\infty$.

Thus every extended real number x has a negation, and $-(-x)$ is always equal to x.

Definition 6.2.3 (Ordering of extended reals). Let x and y be extended real numbers. We say that $x \leq y$, i.e., x is less than or equal to y, iff one of the following three statements is true:

(a) x and y are real numbers, and $x \leq y$ as real numbers.

(b) $y = +\infty$.

(c) $x = -\infty$.

We say that $x < y$ if we have $x \leq y$ and $x \neq y$. We sometimes write $x < y$ as $y > x$, and $x \leq y$ as $y \geq x$.

Examples 6.2.4. $3 \leq 5$, $3 < +\infty$, and $-\infty < +\infty$, but $3 \not\leq -\infty$.

Some basic properties of order and negation on the extended real number system:

Proposition 6.2.5. *Let x, y, z be extended real numbers. Then the following statements are true:*

(a) (Reflexivity) We have $x \leq x$.

(b) *(Trichotomy) Exactly one of the statements $x < y$, $x = y$, or $x > y$ is true.*

(c) *(Transitivity) If $x \leq y$ and $y \leq z$, then $x \leq z$.*

(d) *(Negation reverses order) If $x \leq y$, then $-y \leq -x$.*

Proof. See Exercise 6.2.1. □

One could also introduce other operations on the extended real number system, such as addition, multiplication, etc. However, this is somewhat dangerous as these operations will almost certainly fail to obey the familiar rules of algebra. For instance, to define addition it seems reasonable (given one's intuitive notion of infinity) to set $+\infty + 5 = +\infty$ and $+\infty + 3 = +\infty$, but then this implies that $+\infty + 5 = +\infty + 3$, while $5 \neq 3$. So things like the cancellation law begin to break down once we try to operate involving infinity. To avoid these issues we shall simply not define any arithmetic operations on the extended real number system other than negation and order.

Remember that we defined the notion of *supremum* or *least upper bound* of a set E of reals; this gave an extended real number $\sup(E)$, which was either finite or infinite. We now extend this notion slightly.

Definition 6.2.6 (Supremum of sets of extended reals). Let E be a subset of $\mathbf{R}^*$. Then we define the *supremum* $\sup(E)$ or *least upper bound* of E by the following rule.

(a) If E is contained in $\mathbf{R}$ (i.e., $+\infty$ and $-\infty$ are not elements of E), then we let $\sup(E)$ be as defined in Definition 5.5.10.

(b) If E contains $+\infty$, then we set $\sup(E) := +\infty$.

(c) If E does not contain $+\infty$ but does contain $-\infty$, then we set $\sup(E) := \sup(E - \{-\infty\})$ (which is a subset of $\mathbf{R}$ and thus falls under case (a)).

We also define the *infimum* $\inf(E)$ of E (also known as the *greatest lower bound* of E by the formula

$$\inf(E) := -\sup(-E)$$

where $-E$ is the set $-E := \{-x : x \in E\}$.

Example 6.2.7. Let E be the negative integers, together with $-\infty$:

$$E = \{-1, -2, -3, -4, \ldots\} \cup \{-\infty\}.$$

Then $\sup(E) = \sup(E - \{-\infty\}) = -1$, while

$$\inf(E) = -\sup(-E) = -(+\infty) = -\infty.$$

Example 6.2.8. The set $\{0.9, 0.99, 0.999, 0.9999, \ldots\}$ has infimum 0.9 and supremum 1. Note that in this case the supremum does not actually belong to the set, but it is in some sense "touching it" from the right.

Example 6.2.9. The set $\{1, 2, 3, 4, 5\ldots\}$ has infimum 1 and supremum $+\infty$.

Example 6.2.10. Let E be the empty set. Then $\sup(E) = -\infty$ and $\inf(E) = +\infty$ (why?). This is the only case in which the supremum can be less than the infimum (why?).

One can intuitively think of the supremum of E as follows. Imagine the real line with $+\infty$ somehow on the far right, and $-\infty$ on the far left. Imagine a piston at $+\infty$ moving leftward until it is stopped by the presence of a set E; the location where it stops is the supremum of E. Similarly if one imagines a piston at $-\infty$ moving rightward until it is stopped by the presence of E, the location where it stops is the infimum of E. In the case when E is the empty set, the pistons pass through each other, the supremum landing at $-\infty$ and the infimum landing at $+\infty$.

The following theorem justifies the terminology "least upper bound" and "greatest lower bound":

Theorem 6.2.11. *Let E be a subset of $\mathbf{R}^*$. Then the following statements are true.*