Exercises

The English mathematician Thomas Harriot discovered that the area of a spherical triangle is proportional to (angle sum $-\ \pi$) in 1603. His argument is based on the two views of a spherical triangle shown in Figure 8.8.

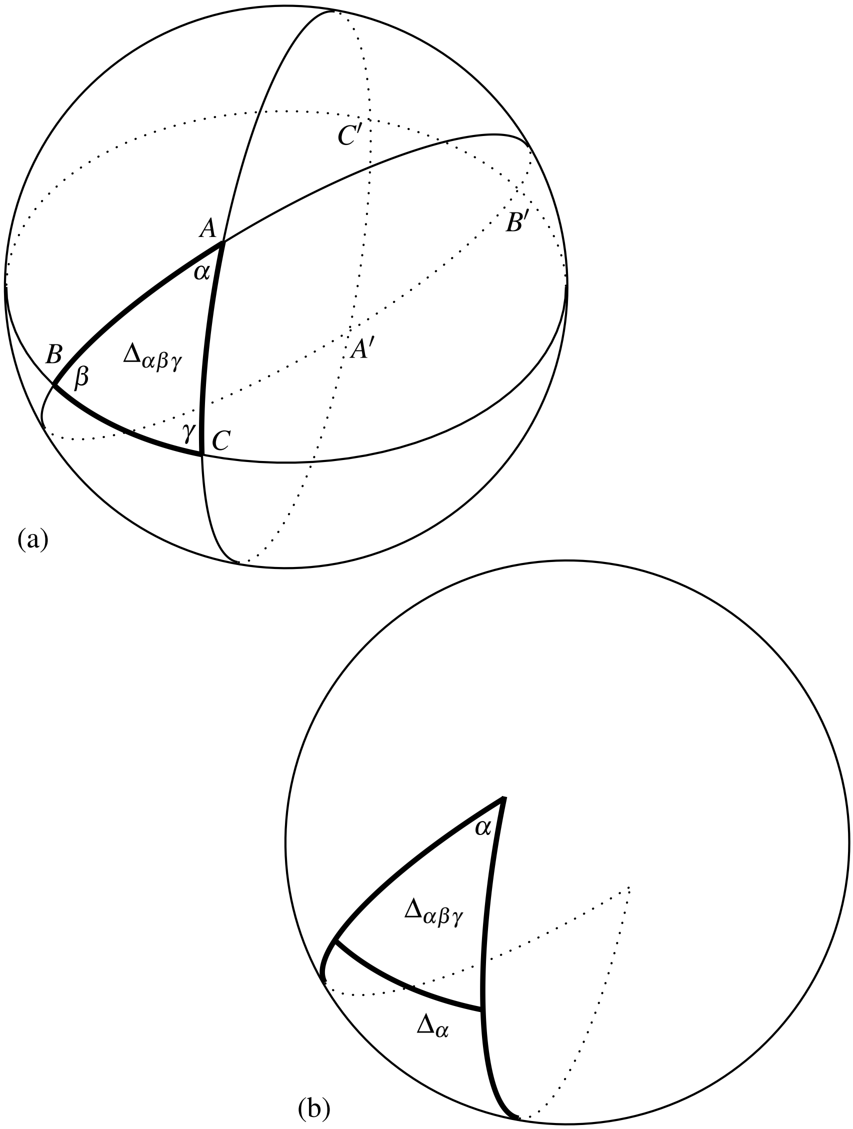

Figure 8.8: Area of a spherical triangle

View (a) shows all sides of the spherical triangle extended to great circles. These divide the sphere into eight spherical triangles, which are obviously congruent in antipodal pairs. View (b) shows the result of extending two sides, which is a "slice" of the sphere with area proportional to the angle at its two ends.

8.5.1 Letting the area of the triangle with angles α, β, γ be $\Delta_{\alpha\beta\gamma}$, and letting the areas of the other triangles be $\Delta_\alpha, \Delta_\beta, \Delta_\gamma$ as shown in view (a), prove that

$$2\left(\Delta_{\alpha\beta\gamma} + \Delta_\alpha + \Delta_\beta + \Delta_\gamma\right) = \text{area of sphere, call it } A. \tag{1}$$

8.5.2 Use view (b) to explain why

$$\Delta_{\alpha\beta\gamma} + \Delta_\alpha = \frac{\alpha}{2\pi}A, \quad \Delta_{\alpha\beta\gamma} + \Delta_\beta = \frac{\beta}{2\pi}A, \quad \Delta_{\alpha\beta\gamma} + \Delta_\gamma = \frac{\gamma}{2\pi}A.$$

8.5.3 Deduce from Exercise 8.5.2 that

$$3\Delta_{\alpha\beta\gamma} + \Delta_\alpha + \Delta_\beta + \Delta_\gamma = \frac{\alpha+\beta+\gamma}{2\pi}A \tag{2}$$

8.5.4 Deduce from equations (1) and (2) that $4\Delta_{\alpha\beta\gamma} = \frac{\alpha+\beta+\gamma-\pi}{\pi}A$, and hence that

$$\Delta_{\alpha\beta\gamma} = \text{constant} \times (\alpha + \beta + \gamma - \pi).$$

8.5.5 Using a formula for the area of the sphere, show that $\Delta_{\alpha\beta\gamma} = \alpha + \beta + \gamma - \pi$ on a sphere of radius 1.

8.6 Non-Euclidean distance

So far we have found invariants of Möbius transformations by geometrically inspired guesses that can be confirmed by calculations with linear fractional functions. But still up our sleeve is the cross-ratio card, which carries the fundamental invariant of linear fractional transformations, and to find out what non-Euclidean distance is we finally have to play it.

We know from Section 5.7 that the cross-ratio is invariant under the transformations $x \mapsto x + l$ and $x \mapsto kx$, and exactly the same calculations apply to $z \mapsto z + l$ and $z \mapsto kz$. It is invariant under $z \mapsto -1/z$, as can be shown by a calculation similar to, but shorter than, that given in Section 5.7 for $x \mapsto 1/x$. However, it is *not* generally invariant under the Möbius transformation $z \mapsto -\bar{z}$, because this replaces the cross-ratio by its complex conjugate. We can only say that *Möbius transformations either leave the cross-ratio invariant or change it to its complex conjugate.*

Luckily, this does not matter, because we are interested in the cross-ratio only when the four points lie on a non-Euclidean line. It turns out that *the cross-ratio of four points on a non-Euclidean line is real, and hence equal to its own complex conjugate.*

This is obvious when the points are pi, qi, ri, si on the upper y-axis, because, in this case, the cross-ratio equals the real number $\frac{(r-p)(s-q)}{(r-q)(s-p)}$ by cancellation of the i factors. It follows for any other non-Euclidean line $\mathscr{L}$ by mapping the upper y-axis onto $\mathscr{L}$ by a Möbius transformation.

- If $\mathscr{L}$ is another vertical line $x = l$, we map the upper y-axis to $\mathscr{L}$ by $z \mapsto z + l$.
- If $\mathscr{L}$ is a semicircle with center on the x-axis, we first map the upper y-axis to $x = 1$ by $z \mapsto z + 1$, and then to the semicircle with ends at 0 and 1 by $z \mapsto 1/\bar{z}$. Finally we map this semicircle to $\mathscr{L}$ by dilating it to the radius of $\mathscr{L}$ and then translating its center to the center of $\mathscr{L}$. $\square$

We know from the previous section that the transformations $z \mapsto z + l$, $z \mapsto kz$ for $k > 0$, $z \mapsto -\bar{z}$, and $z \mapsto -1/z$ generate all Möbius transformations, so we have now proved that *the cross-ratio of any four points on a non-Euclidean line is preserved by Möbius transformations.*

So far, so good, but distance is a function of two points, not four. If the cross-ratio is going to help us define distance, we need to specialize it to a function of two variables.

One of the beauties of a non-Euclidean line is that it lies between two endpoints. The non-Euclidean line represented by the upper y-axis, for example, consists of the points between 0 and ∞. The endpoints are not points of the line, but it is meaningful to include them in a cross-ratio, because Möbius transformations apply to all complex numbers, and ∞. If we take 0 and ∞ as the third and fourth members of the quadruple pi, qi, ri, si on the upper y-axis, then the cross-ratio of this quadruple simplifies as follows:

$$\begin{aligned}
\frac{(r-p)(s-q)}{(r-q)(s-p)} &= \frac{(r-p)(1-q/s)}{(r-q)(1-p/s)} \quad \text{dividing top and bottom by } s \\
&= \frac{r-p}{r-q} \quad \text{because } s = \infty \text{ and } 1/\infty = 0 \\
&= \frac{p}{q} \quad \text{because } r = 0.
\end{aligned}$$

Any Möbius transformation of the upper y-axis sends endpoints to endpoints, as one can see from the generating transformations, but it is possible for 0 and ∞ to be exchanged. If $r = \infty$ and $s = 0$, we find that the cross-ratio of pi, qi, ri, si is q/p, not p/q. Thus, q/p is not an invariant of Möbius transformations of the upper y-axis, *but* $|\log \frac{q}{p}|$ *is*, because $\log \frac{p}{q} = -\log \frac{q}{p}$ for any $p, q > 0$.

This prompts us to make the following definition.

Distance on the upper y-axis. The *non-Euclidean distance* $\mathrm{ndist}(pi, qi)$ between points pi and qi on the upper y-axis is $|\log \frac{q}{p}|$.

This definition of distance is appropriate for two reasons:

- As already shown, non-Euclidean distance on the upper y-axis is invariant under all Möbius transformations.

- Non-Euclidean distance is *additive*. That is, if pi, qi, ri lie on the upper y-axis in that order, then

$$\mathrm{ndist}(pi, ri) = \mathrm{ndist}(pi, qi) + \mathrm{ndist}(qi, ri).$$

  This is because

$$\left|\log \frac{r}{p}\right| = \left|\log \frac{q}{p}\frac{r}{q}\right| = \left|\log \frac{q}{p} + \log \frac{r}{q}\right| = \left|\log \frac{q}{p}\right| + \left|\log \frac{r}{q}\right|$$

  by the additive property of the logarithm function.

It follows from this definition that the infinity of points $2^n i$, for integers n, are *equally spaced* along the upper y-axis, in the sense of non-Euclidean distance. The faces shown in Figure 8.9 are of equal size in this sense. The upper y-axis is not only infinite in the upward direction, but also in the downward direction. There is infinite non-Euclidean distance between any of its points and the x-axis. Thus, the upper y-axis satisfies Euclid's second axiom for "lines": Any segment of it can be "extended indefinitely."

Having defined non-Euclidean distance on the upper y-axis, we can use the axis as a "ruler" to measure the distance between two points in the upper half plane. Given any two points u and v, we find the unique non-Euclidean line $\mathscr{L}$ through u and v as described in Section 8.1, and then map $\mathscr{L}$ onto the upper y-axis by a Möbius transformation f as described (in reverse) in the first part of this section. We take the non-Euclidean distance from u to v to be the non-Euclidean distance from $f(u)$ to $f(v)$, namely $\mathrm{ndist}(f(u), f(v))$.

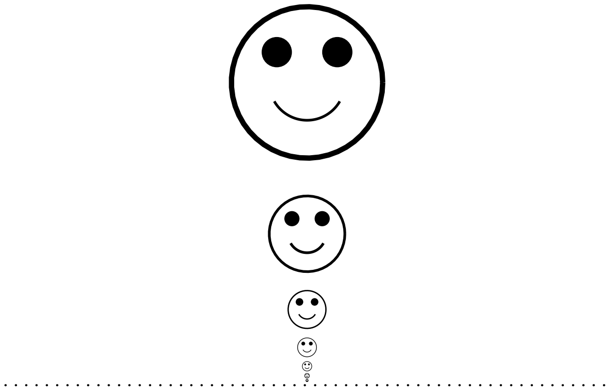

Figure 8.9: Faces of equal non-Euclidean size

The quantity $\mathrm{ndist}(f(u), f(v))$ does not depend on the Möbius transformation f used to map $\mathscr{L}$ onto the upper y-axis. If g is another Möbius transformation mapping $\mathscr{L}$ onto the upper y-axis, then fg^{-1} is a Möbius transformation that maps the upper y-axis onto itself and sends the points $g(u)$ and $g(v)$ to $f(u)$ and $f(v)$, respectively. Hence,

$$\mathrm{ndist}(g(u), g(v)) = \mathrm{ndist}(f(u), f(v))$$

by the invariance of non-Euclidean distance on the upper y-axis under Möbius transformations.

The hidden geometry of the projective line

As we mentioned in Section 7.1, Klein associated a "geometry" with each group of transformations. We have set up the group of transformations of the half plane to be isomorphic to the group of transformations of $\mathbb{RP}^1$. Hence, the half plane and $\mathbb{RP}^1$ have *isomorphic geometries* in the sense of Klein, even though they seem very different. Indeed, we transferred geometry from $\mathbb{RP}^1$ to the half plane mainly because of the difference: Geometry is much more *visible* in the half plane.

Figure 8.7 is one illustration of this, and Figure 8.10 is another—a regular tiling of the half plane by fish that are congruent in the sense of non-Euclidean length. Figure 8.10 is essentially the picture *Circle Limit I*, by M. C. Escher, but mapped to the half plane by the transformation

$$z \mapsto \frac{1 - zi}{z - i}.$$