τὸ ΔΒ καί ἐστιν ἴσον τῷ ΚΘ, ῥητὸν ἄρα ἐστὶ καὶ τὸ ΚΘ. καὶ παρὰ ῥητὴν τὴν ΕΖ παράκειται· ῥητὴ ἄρα ἐστὶν ἡ ΗΘ καὶ σύμμετρος τῇ ΕΖ μήκει. ἀλλά καὶ ἡ ΕΗ ῥητή ἐστι καὶ ἀσύμμετρος τῇ ΕΖ μήκει· ἀσύμμετρος ἄρα ἐστὶν ἡ ΕΗ τῇ ΗΘ μήκει. καί ἐστιν ὡς ἡ ΕΗ πρὸς τὴν ΗΘ, οὕτως τὸ ἀπὸ τῆς ΕΗ πρὸς τὸ ὑπὸ τῶν ΕΗ, ΗΘ· ἀσύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΕΗ τῷ ὑπὸ τῶν ΕΗ, ΗΘ. ἀλλὰ τῷ μὲν ἀπὸ τῆς ΕΗ σύμμετρά ἐστι τὰ ἀπὸ τῶν ΕΗ, ΗΘ τετράγωνα· ῥητὰ γὰρ ἀμφότερα· τῷ δὲ ὑπὸ τῶν ΕΗ, ΗΘ σύμμετρόν ἐστι τὸ δὶς ὑπὸ τῶν ΕΗ, ΗΘ· διπλάσιον γάρ ἐστιν αὐτοῦ· ἀσύμμετρα ἄρα ἐστὶ τὰ ἀπὸ τῶν ΕΗ, ΗΘ τῷ δὶς ὑπὸ τῶν ΕΗ, ΗΘ· καὶ συναμφότερα ἄρα τά τε ἀπὸ τῶν ΕΗ, ΗΘ καὶ τὸ δὶς ὑπὸ τῶν ΕΗ, ΗΘ, ὅπερ ἐστὶ τὸ ἀπὸ τῆς ΕΘ, ἀσύμμετρόν ἐστι τοῖς ἀπὸ τῶν ΕΗ, ΗΘ. ῥητὰ δὲ τὰ ἀπὸ τῶν ΕΗ, ΗΘ· ἄλογον ἄρα τὸ ἀπὸ τῆς ΕΘ. ἄλογος ἄρα ἐστὶν ἡ ΕΘ. ἀλλὰ καὶ ῥηρή· ὅπερ ἐστὶν ἀδύνατον.

Μέσον ἄρα μέσου οὐχ ὑπερέχει ῥητῷ· ὅπερ ἔδει δεῖξαι.

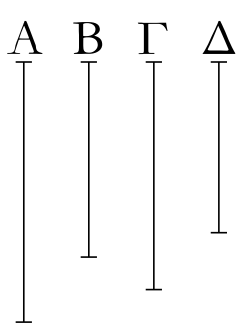
And they are applied to the rational (straight-line) EF. Thus, HE and EG are each rational, and incommensurable in length with EF [Prop. 10.22]. And since DB is rational, and is equal to KH, KH is thus also rational. And (KH) is applied to the rational (straight-line) EF. GH is thus rational, and commensurable in length with EF [Prop. 10.20]. But, EG is also rational, and incommensurable in length with EF. Thus, EG is incommensurable in length with GH [Prop. 10.13]. And as EG is to GH, so the (square) on EG (is) to the (rectangle contained) by EG and GH [Prop. 10.13 lem.]. Thus, the (square) on EG is incommensurable with the (rectangle contained) by EG and GH [Prop. 10.11]. But, the (sum of the) squares on EG and GH is commensurable with the (square) on EG. For (EG and GH are) both rational. And twice the (rectangle contained) by EG and GH is commensurable with the (rectangle contained) by EG and GH [Prop. 10.6]. For (the former) is double the latter. Thus, the (sum of the squares) on EG and GH is incommensurable with twice the (rectangle contained) by EG and GH [Prop. 10.13]. And thus the sum of the (squares) on EG and GH plus twice the (rectangle contained) by EG and GH, that is the (square) on EH [Prop. 2.4], is incommensurable with the (sum of the squares) on EG and GH [Prop. 10.16]. And the (sum of the squares) on EG and GH (is) rational. Thus, the (square) on EH is irrational [Def. 10.4]. Thus, EH is irrational [Def. 10.4]. But, (it is) also rational. The very thing is impossible.

Thus, a medial (area) does not exceed a medial (area) by a rational (area). (Which is) the very thing it was required to show.

† In other words, $\sqrt{k} - \sqrt{k'} \neq k''$.

κζ΄.

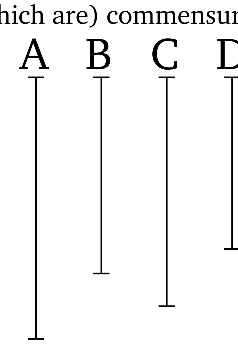
Μέσας εὑρεῖν δυνάμει μόνον συμμέτρους ῥητὸν περιεχούσας.



Ἐκκείσθωσαν δύο ῥηταὶ δυνάμει μόνον σύμμετροι αἱ Α, Β, καὶ εἰλήφθω τῶν Α, Β μέση ἀνάλογον ἡ Γ, καὶ γεγονέτω ὡς ἡ Α πρὸς τὴν Β, οὕτως ἡ Γ πρὸς τὴν Δ.

Proposition 27

To find (two) medial (straight-lines), containing a rational (area), (which are) commensurable in square only.



Let the two rational (straight-lines) A and B, (which are) commensurable in square only, be laid down. And let C—the mean proportional (straight-line) to A and B—

Καὶ ἐπεὶ αἱ Α, Β ῥηταί εἰσι δυνάμει μόνον σύμμετροι, τὸ ἄρα ὑπὸ τῶν Α, Β, τουτέστι τὸ ἀπὸ τῆς Γ, μέσον ἐστίν. μέση ἄρα ἡ Γ. καὶ ἐπεί ἐστιν ὡς ἡ Α πρὸς τὴν Β, [οὕτως] ἡ Γ πρὸς τὴν Δ, αἱ δὲ Α, Β δυνάμει μόνον [εἰσὶ] σύμμετροι, καὶ αἱ Γ, Δ ἄρα δυνάμει μόνον εἰσὶ σύμμετροι. καί ἐστι μέση ἡ Γ· μέση ἄρα καὶ ἡ Δ. αἱ Γ, Δ ἄρα μέσαι εἰσὶ δυνάμει μόνον σύμμετροι. λέγω, ὅτι καὶ ῥητὸν περιέχουσιν. ἐπεὶ γάρ ἐστιν ὡς ἡ Α πρὸς τὴν Β, οὕτως ἡ Γ πρὸς τὴν Δ, ἐναλλὰξ ἄρα ἐστὶν ὡς ἡ Α πρὸς τὴν Γ, ἡ Β πρὸς τὴν Δ. ἀλλ᾽ ὡς ἡ Α πρὸς τὴν Γ, ἡ Γ πρὸς τὴν Β· καὶ ὡς ἄρα ἡ Γ πρὸς τὴν Β, οὕτως ἡ Β πρὸς τὴν Δ· τὸ ἄρα ὑπὸ τῶν Γ, Δ ἴσον ἐστὶ τῷ ἀπὸ τῆς Β. ῥητὸν δὲ τὸ ἀπὸ τῆς Β· ῥητὸν ἄρα [ἐστὶ] καὶ τὸ ὑπὸ τῶν Γ, Δ.

Εὕρηνται ἄρα μέσαι δυνάμει μόνον σύμμετροι ῥητὸν περιέχουσαι· ὅπερ ἔδει δεῖξαι.

have been taken [Prop. 6.13]. And let it be contrived that as A (is) to B, so C (is) to D [Prop. 6.12].

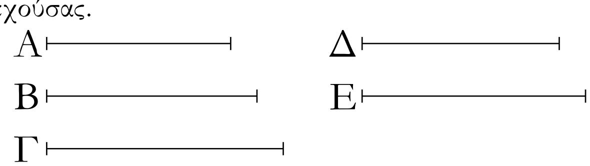
And since the rational (straight-lines) A and B are commensurable in square only, the (rectangle contained) by A and B—that is to say, the (square) on C [Prop. 6.17]—is thus medial [Prop 10.21]. Thus, C is medial [Prop. 10.21]. And since as A is to B, [so] C (is) to D, and A and B [are] commensurable in square only, C and D are thus also commensurable in square only [Prop. 10.11]. And C is medial. Thus, D is also medial [Prop. 10.23]. Thus, C and D are medial (straight-lines which are) commensurable in square only. I say that they also contain a rational (area). For since as A is to B, so C (is) to D, thus, alternately, as A is to C, so B (is) to D [Prop. 5.16]. But, as A (is) to C, (so) C (is) to B. And thus as C (is) to B, so B (is) to D [Prop. 5.11]. Thus, the (rectangle contained) by C and D is equal to the (square) on B [Prop. 6.17]. And the (square) on B (is) rational. Thus, the (rectangle contained) by C and D [is] also rational.

Thus, (two) medial (straight-lines, C and D), containing a rational (area), (which are) commensurable in square only, have been found.† (Which is) the very thing it was required to show.

† C and D have lengths $k^{1/4}$ and $k^{3/4}$ times that of A, respectively, where the length of B is $k^{1/2}$ times that of A.

κη΄.

Μέσας εὑρεῖν δυνάμει μόνον συμμέτρους μέσον περιεχούσας.



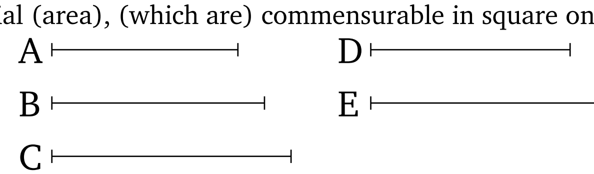
Ἐκκείσθωσαν [τρεῖς] ῥηταὶ δυνάμει μόνον σύμμετροι αἱ Α, Β, Γ, καὶ εἰλήφθω τῶν Α, Β μέση ἀνάλογον ἡ Δ, καὶ γεγονέτω ὡς ἡ Β πρὸς τὴν Γ, ἡ Δ πρὸς τὴν Ε.

Ἐπεὶ αἱ Α, Β ῥηταί εἰσι δυνάμει μόνον σύμμετροι, τὸ ἄρα ὑπὸ τῶν Α, Β, τουτέστι τὸ ἀπὸ τῆς Δ, μέσον ἐστὶν. μέση ἄρα ἡ Δ. καὶ ἐπεὶ αἱ Β, Γ δυνάμει μόνον εἰσὶ σύμμετροι, καί ἐστιν ὡς ἡ Β πρὸς τὴν Γ, ἡ Δ πρὸς τὴν Ε, καὶ αἱ Δ, Ε ἄρα δυνάμει μόνον εἰσὶ σύμμετροι. μέση δὲ ἡ Δ· μέση ἄρα καὶ ἡ Ε· αἱ Δ, Ε ἄρα μέσαι εἰσὶ δυνάμει μόνον σύμμετροι. λέγω δή, ὅτι καὶ μέσον περιέχουσιν. ἐπεὶ γάρ ἐστιν ὡς ἡ Β πρὸς τὴν Γ, ἡ Δ πρὸς τὴν Ε, ἐναλλὰξ ἄρα ὡς ἡ Β πρὸς τὴν Δ, ἡ Γ πρὸς τὴν Ε. ὡς δὲ ἡ Β πρὸς τὴν Δ, ἡ Δ πρὸς τὴν Α· καὶ ὡς ἄρα ἡ Δ πρὸς τὴν Α, ἡ Γ πρὸς τὴν Ε· τὸ ἄρα ὑπὸ τῶν Α, Γ ἴσον ἐστὶ τῷ ὑπὸ τῶν Δ, Ε. μέσον δὲ τὸ ὑπὸ τῶν Α, Γ· μέσον ἄρα καὶ τὸ ὑπὸ τῶν Δ, Ε.

Ἐὕρηνται ἄρα μέσαι δυνάμει μόνον σύμμετροι μέσον

Proposition 28

To find (two) medial (straight-lines), containing a medial (area), (which are) commensurable in square only.



Let the [three] rational (straight-lines) A, B, and C, (which are) commensurable in square only, be laid down. And let, D, the mean proportional (straight-line) to A and B, have been taken [Prop. 6.13]. And let it be contrived that as B (is) to C, (so) D (is) to E [Prop. 6.12].

Since the rational (straight-lines) A and B are commensurable in square only, the (rectangle contained) by A and B—that is to say, the (square) on D [Prop. 6.17]—is medial [Prop. 10.21]. Thus, D (is) medial [Prop. 10.21]. And since B and C are commensurable in square only, and as B is to C, (so) D (is) to E, D and E are thus commensurable in square only [Prop. 10.11]. And D (is) medial. E (is) thus also medial [Prop. 10.23]. Thus, D and E are medial (straight-lines which are) commensurable in square only. So, I say that they also enclose a medial (area). For since as B is to C, (so) D (is) to E, thus, περιέχουσαι· ὅπερ ἔδει δεῖξαι.

alternately, as B (is) to D, (so) C (is) to E [Prop. 5.16]. And as B (is) to D, (so) D (is) to A. And thus as D (is) to A, (so) C (is) to E. Thus, the (rectangle contained) by A and C is equal to the (rectangle contained) by D and E [Prop. 6.16]. And the (rectangle contained) by A and C is medial [Prop. 10.21]. Thus, the (rectangle contained) by D and E (is) also medial.

Thus, (two) medial (straight-lines, D and E), containing a medial (area), (which are) commensurable in square only, have been found. (Which is) the very thing it was required to show.

† D and E have lengths $k^{1/4}$ and $k'^{1/2}/k^{1/4}$ times that of A, respectively, where the lengths of B and C are $k^{1/2}$ and $k'^{1/2}$ times that of A, respectively.

Λῆμμα α΄.

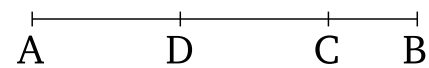
Εὑρειν δύο τετραγώνους ἀριθμούς, ὥστε καὶ τὸν συγκείμενον ἐξ αὐτῶν εἶναι τετράγωνον.

Ἐκκείσθωσαν δύο ἀριθμοὶ οἱ ΑΒ, ΒΓ, ἔστωσαν δὲ ἤτοι ἄρτιοι ἢ περιττοί. καὶ ἐπεί, ἐάν τε ἀπὸ ἀρτίου ἄρτιος ἀφαιρεθῇ, ἐάν τε ἀπὸ περισσοῦ περισσός, ὁ λοιπὸς ἄρτιός ἐστιν, ὁ λοιπὸς ἄρα ὁ ΑΓ ἄρτιός ἐστιν. τετμήσθω ὁ ΑΓ δίχα κατὰ τὸ Δ. ἔστωσαν δὲ καὶ οἱ ΑΒ, ΒΓ ἤτοι ὅμοιοι ἐπίπεδοι ἢ τετράγωνοι, οἳ καὶ αὐτοὶ ὅμοιοί εἰσιν ἐπίπεδοι· ὁ ἄρα ἐκ τῶν ΑΒ, ΒΓ μετὰ τοῦ ἀπὸ [τοῦ] ΓΔ τετραγώνου ἴσος ἐστὶ τῷ ἀπὸ τοῦ ΒΔ τετραγώνῳ. καί ἐστι τετράγωνος ὁ ἐκ τῶν ΑΒ, ΒΓ, ἐπειδήπερ ἐδείχθη, ὅτι, ἐὰν δύο ὅμοιοι ἐπίπεδοι πολλαπλασιάσαντες ἀλλήλους ποιῶσί τινα, ὁ γενόμενος τετράγωνός ἐστιν. εὕρηνται ἄρα δύο τετράγωνοι ἀριθμοὶ ὅ τε ἐκ τῶν ΑΒ, ΒΓ καὶ ὁ ἀπὸ τοῦ ΓΔ, οἳ συντεθέντες ποιοῦσι τὸν ἀπὸ τοῦ ΒΔ τετράγωνον.

Καὶ φανερόν, ὅτι εὕρηνται πάλιν δύο τετράγωνοι ὅ τε ἀπὸ τοῦ ΒΔ καὶ ὁ ἀπὸ τοῦ ΓΔ, ὥστε τὴν ὑπεροχὴν αὐτῶν τὸν ὑπὸ ΑΒ, ΒΓ εἶναι τετράγωνον, ὅταν οἱ ΑΒ, ΒΓ ὅμοιοι ὦσιν ἐπίπεδοι. ὅταν δὲ μὴ ὦσιν ὅμοιοι ἐπίπεδοι, εὕρηνται δύο τετράγωνοι ὅ τε ἀπὸ τοῦ ΒΔ καὶ ὁ ἀπὸ τοῦ ΔΓ, ὧν ἡ ὑπεροχὴ ὁ ὑπὸ τῶν ΑΒ, ΒΓ οὐκ ἔστι τετράγωνος· ὅπερ ἔδει δεῖξαι.

Lemma I

To find two square numbers such that the sum of them is also square.

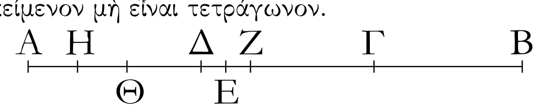


Let the two numbers AB and BC be laid down. And let them be either (both) even or (both) odd. And since, if an even (number) is subtracted from an even (number), or if an odd (number is subtracted) from an odd (number), then the remainder is even [Props. 9.24, 9.26], the remainder AC is thus even. Let AC have been cut in half at D. And let AB and BC also be either similar plane (numbers), or square (numbers)—which are themselves also similar plane (numbers). Thus, the (number created) from (multiplying) AB and BC, plus the square on CD, is equal to the square on BD [Prop. 2.6]. And the (number created) from (multiplying) AB and BC is square—inasmuch as it was shown that if two similar plane (numbers) make some (number) by multiplying one another then the (number so) created is square [Prop. 9.1]. Thus, two square numbers have been found—(namely,) the (number created) from (multiplying) AB and BC, and the (square) on CD—which, (when) added (together), make the square on BD.

And (it is) clear that two square (numbers) have again been found—(namely,) the (square) on BD, and the (square) on CD—such that their difference—(namely,) the (rectangle) contained by AB and BC—is square whenever AB and BC are similar plane (numbers). But, when they are not similar plane numbers, two square (numbers) have been found—(namely,) the (square) on BD, and the (square) on DC—between which the difference—(namely,) the (rectangle) contained by AB and BC—is not square. (Which is) the very thing it was required to show.

Λῆμμα β´.

Εὑρεῖν δύο τετραγώνους ἀριθμούς, ὥστε τὸν ἐξ αὐτῶν συγκείμενον μὴ εἶναι τετράγωνον.

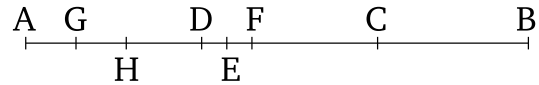


Ἔστω γὰρ ὁ ἐκ τῶν ΑΒ, ΒΓ, ὡς ἔφαμεν, τετράγωνος, καὶ ἄρτιος ὁ ΓΑ, καὶ τετμήσθω ὁ ΓΑ δίχα τῷ Δ. φανερὸν δή, ὅτι ὁ ἐκ τῶν ΑΒ, ΒΓ τετράγωνος μετὰ τοῦ ἀπὸ [τοῦ] ΓΔ τετραγώνου ἴσος ἐστὶ τῷ ἀπὸ [τοῦ] ΒΔ τετραγώνῳ. ἀφῃρήσθω μονὰς ἡ ΔΕ· ὁ ἄρα ἐκ τῶν ΑΒ, ΒΓ μετὰ τοῦ ἀπὸ [τοῦ] ΓΕ ἐλάσσων ἐστὶ τοῦ ἀπὸ [τοῦ] ΒΔ τετραγώνου. λέγω οὖν, ὅτι ὁ ἐκ τῶν ΑΒ, ΒΓ τετράγωνος μετὰ τοῦ ἀπὸ [τοῦ] ΓΕ οὐκ ἔσται τετράγωνος.

Εἰ γὰρ ἔσται τετράγωνος, ἤτοι ἴσος ἐστὶ τῷ ἀπὸ [τοῦ] ΒΕ ἢ ἐλάσσων τοῦ ἀπὸ [τοῦ] ΒΕ, οὐκέτι δὲ καὶ μείζων, ἵνα μὴ τμηθῇ ἡ μονάς. ἔστω, εἰ δυνατόν, πρότερον ὁ ἐκ τῶν ΑΒ, ΒΓ μετὰ τοῦ ἀπὸ ΓΕ ἴσος τῷ ἀπὸ ΒΕ, καὶ ἔστω τῆς ΔΕ μονάδος διπλασίων ὁ ΗΑ. ἐπεὶ οὖν ὅλος ὁ ΑΓ ὅλου τοῦ ΓΔ ἐστι διπλασίων, ὧν ὁ ΑΗ τοῦ ΔΕ ἐστι διπλασίων, καὶ λοιπὸς ἄρα ὁ ΗΓ λοιποῦ τοῦ ΕΓ ἐστι διπλασίων· δίχα ἄρα τέτμηται ὁ ΗΓ τῷ Ε. ὁ ἄρα ἐκ τῶν ΗΒ, ΒΓ μετὰ τοῦ ἀπὸ ΓΕ ἴσος ἐστὶ τῷ ἀπὸ ΒΕ τετραγώνῳ. ἀλλὰ καὶ ὁ ἐκ τῶν ΑΒ, ΒΓ μετὰ τοῦ ἀπὸ ΓΕ ἴσος ὑπόκειται τῷ ἀπὸ [τοῦ] ΒΕ τετραγώνῳ· ὁ ἄρα ἐκ τῶν ΗΒ, ΒΓ μετὰ τοῦ ἀπὸ ΓΕ ἴσος ἐστὶ τῷ ἐκ τῶν ΑΒ, ΒΓ μετὰ τοῦ ἀπὸ ΓΕ. καὶ κοινοῦ ἀφαιρεθέντος τοῦ ἀπὸ ΓΕ συνάγεται ὁ ΑΒ ἴσος τῷ ΗΒ· ὅπερ ἄτοπον. οὐκ ἄρα ὁ ἐκ τῶν ΑΒ, ΒΓ μετὰ τοῦ ἀπὸ [τοῦ] ΓΕ ἴσος ἐστὶ τῷ ἀπὸ ΒΕ. λέγω δή, ὅτι οὐδὲ ἐλάσσων τοῦ ἀπὸ ΒΕ. εἰ γὰρ δυνατόν, ἔστω τῷ ἀπὸ ΒΖ ἴσος, καὶ τοῦ ΔΖ διπλασίων ὁ ΘΑ. καὶ συναχθήσεται πάλιν διπλασίων ὁ ΘΓ τοῦ ΓΖ· ὥστε καὶ τὸν ΓΘ δίχα τετμῆσθαι κατὰ τὸ Ζ, καὶ διὰ τοῦτο τὸν ἐκ τῶν ΘΒ, ΒΓ μετὰ τοῦ ἀπὸ ΖΓ ἴσον γίνεσθαι τῷ ἀπὸ ΒΖ. ὑπόκειται δὲ καὶ ὁ ἐκ τῶν ΑΒ, ΒΓ μετὰ τοῦ ἀπὸ ΓΕ ἴσος τῷ ἀπὸ ΒΖ. ὥστε καὶ ὁ ἐκ τῶν ΘΒ, ΒΓ μετὰ τοῦ ἀπὸ ΓΖ ἴσος ἔσται τῷ ἐκ τῶν ΑΒ, ΒΓ μετὰ τοῦ ἀπὸ ΓΕ· ὅπερ ἄτοπον. οὐκ ἄρα ὁ ἐκ τῶν ΑΒ, ΒΓ μετὰ τοῦ ἀπὸ ΓΕ ἴσος ἐστὶ [τῷ] ἐλάσσονι τοῦ ἀπὸ ΒΕ. ἐδείχθη δέ, ὅτι οὐδὲ [αὐτῷ] τῷ ἀπὸ ΒΕ. οὐκ ἄρα ὁ ἐκ τῶν ΑΒ, ΒΓ μετὰ τοῦ ἀπὸ ΓΕ τετράγωνός ἐστιν. ὅπερ ἔδει δεῖξαι.

Lemma II

To find two square numbers such that the sum of them is not square.

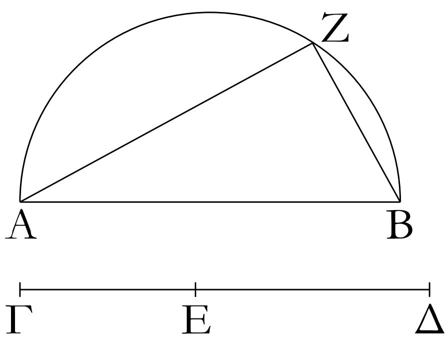


For let the (number created) from (multiplying) AB and BC, as we said, be square. And (let) CA (be) even. And let CA have been cut in half at D. So it is clear that the square (number created) from (multiplying) AB and BC, plus the square on CD, is equal to the square on BD [see previous lemma]. Let the unit DE have been subtracted (from BD). Thus, the (number created) from (multiplying) AB and BC, plus the (square) on CE, is less than the square on BD. I say, therefore, that the square (number created) from (multiplying) AB and BC, plus the (square) on CE, is not square.

For if it is square, it is either equal to the (square) on BE, or less than the (square) on BE, but cannot any more be greater (than the square on BE), lest the unit be divided. First of all, if possible, let the (number created) from (multiplying) AB and BC, plus the (square) on CE, be equal to the (square) on BE. And let GA be double the unit DE. Therefore, since the whole of AC is double the whole of CD, of which AG is double DE, the remainder GC is thus double the remainder EC. Thus, GC has been cut in half at E. Thus, the (number created) from (multiplying) GB and BC, plus the (square) on CE, is equal to the square on BE [Prop. 2.6]. But, the (number created) from (multiplying) AB and BC, plus the (square) on CE, was also assumed (to be) equal to the square on BE. Thus, the (number created) from (multiplying) GB and BC, plus the (square) on CE, is equal to the (number created) from (multiplying) AB and BC, plus the (square) on CE. And subtracting the (square) on CE from both, AB is inferred (to be) equal to GB. The very thing is absurd. Thus, the (number created) from (multiplying) AB and BC, plus the (square) on CE, is not equal to the (square) on BE. So I say that (it is) not less than the (square) on BE either. For, if possible, let it be equal to the (square) on BF. And (let) HA (be) double DF. And it can again be inferred that HC (is) double CF. Hence, CH has also been cut in half at F. And, on account of this, the (number created) from (multiplying) HB and BC, plus the (square) on FC, becomes equal to the (square) on BF [Prop. 2.6]. And the (number created) from (multiplying) AB and BC, plus the (square) on CE, was also assumed (to be) equal to the (square) on BF. Hence, the (number created) from (multiplying) HB and BC, plus the (square) on CF, will also be equal to the (number created) from (multiplying) AB and BC, plus the (square) on CE. The very thing is absurd. Thus, the (number created) from (multiplying) AB and BC, plus the (square) on CE, is not equal to less than the (square) on BE. And it was shown that (is it) not equal to the (square) on BE either. Thus, the (number created) from (multiplying) AB and BC, plus the square on CE, is not square. (Which is) the very thing it was required to show.

κθ΄.

Εὑρεῖν δύο ῥητὰς δυνάμει μόνον συμμέτρους, ὥστε τὴν μείζονα τῆς ἐλάσσονος μεῖζον δύνασθαι τῷ ἀπὸ συμμέτρου ἑαυτῇ μήκει.

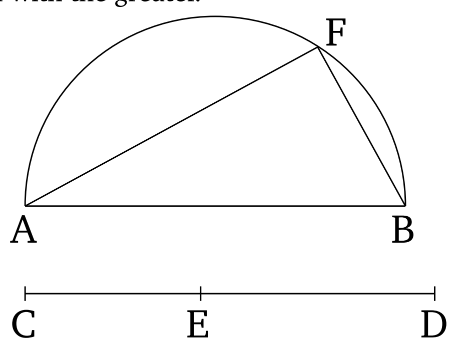


Ἐκκείσθω γάρ τις ῥητὴ ἡ ΑΒ καὶ δύο τετράγωνοι ἀριθμοὶ οἱ ΓΔ, ΔΕ, ὥστε τὴν ὑπεροχὴν αὐτῶν τὸν ΓΕ μὴ εἶναι τετράγωνον, καὶ γεγράφθω ἐπὶ τῆς ΑΒ ἡμικύκλιον τὸ ΑΖΒ, καὶ πεποιήσθω ὡς ὁ ΔΓ πρὸς τὸν ΓΕ, οὕτως τὸ ἀπὸ τῆς ΒΑ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΑΖ τετράγωνον, καὶ ἐπεζεύχθω ἡ ΖΒ.

Ἐπεὶ [οὖν] ἐστιν ὡς τὸ ἀπὸ τῆς ΒΑ πρὸς τὸ ἀπὸ τῆς ΑΖ, οὕτως ὁ ΔΓ πρὸς τὸν ΓΕ, τὸ ἀπὸ τῆς ΒΑ ἄρα πρὸς τὸ ἀπὸ τῆς ΑΖ λόγον ἔχει, ὃν ἀριθμὸς ὁ ΔΓ πρὸς ἀριθμὸν τὸν ΓΕ· σύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΒΑ τῷ ἀπὸ τῆς ΑΖ. ῥητὸν δὲ τὸ ἀπὸ τῆς ΑΒ· ῥητὸν ἄρα καὶ τὸ ἀπὸ τῆς ΑΖ· ῥητὴ ἄρα καὶ ἡ ΑΖ. καὶ ἐπεὶ ὁ ΔΓ πρὸς τὸν ΓΕ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδὲ τὸ ἀπὸ τῆς ΒΑ ἄρα πρὸς τὸ ἀπὸ τῆς ΑΖ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ ΑΒ τῇ ΑΖ μήκει· αἱ ΒΑ, ΑΖ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι. καὶ ἐπεί [ἐστιν] ὡς ὁ ΔΓ πρὸς τὸν ΓΕ, οὕτως τὸ ἀπὸ τῆς ΒΑ πρὸς τὸ ἀπὸ τῆς ΑΖ, ἀναστρέψαντι ἄρα ὡς ὁ ΓΔ πρὸς τὸν ΔΕ, οὕτως τὸ ἀπὸ τῆς ΑΒ πρὸς τὸ ἀπὸ τῆς ΒΖ. ὁ δὲ ΓΔ πρὸς τὸν ΔΕ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· καὶ τὸ ἀπὸ τῆς ΑΒ ἄρα πρὸς τὸ ἀπὸ τῆς ΒΖ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· σύμμετρος ἄρα ἐστὶν ἡ ΑΒ τῇ ΒΖ μήκει. καί ἐστι τὸ ἀπὸ τῆς ΑΒ ἴσον τοῖς ἀπὸ τῶν ΑΖ, ΖΒ· ἡ ΑΒ ἄρα τῆς ΑΖ μεῖζον δύναται τῇ ΒΖ συμμέτρῳ

Proposition 29

To find two rational (straight-lines which are) commensurable in square only, such that the square on the greater is larger than the (square on the) lesser by the (square) on (some straight-line which is) commensurable in length with the greater.



For let some rational (straight-line) AB be laid down, and two square numbers, CD and DE, such that the difference between them, CE, is not square [Prop. 10.28 lem. I]. And let the semi-circle AFB have been drawn on AB. And let it be contrived that as DC (is) to CE, so the square on BA (is) to the square on AF [Prop. 10.6 corr.]. And let FB have been joined.

[Therefore,] since as the (square) on BA is to the (square) on AF, so DC (is) to CE, the (square) on BA thus has to the (square) on AF the ratio which the number DC (has) to the number CE. Thus, the (square) on BA is commensurable with the (square) on AF [Prop. 10.6]. And the (square) on AB (is) rational [Def. 10.4]. Thus, the (square) on AF (is) also rational. Thus, AF (is) also rational. And since DC does not have to CE the ratio which (some) square number (has) to (some) square number, the (square) on BA thus does not have to the (square) on AF the ratio which (some) square number has to (some) square number either. Thus, AB is incommensurable in length with AF [Prop. 10.9]. Thus, the rational (straight-lines) BA and AF are commensurable in square only. And since as DC [is] to CE, so the (square) on BA (is) to the (square) on AF, thus, via conversion, as CD (is) to DE, so the (square) on AB (is) to the (square) on ἑαυτῇ.

Εὕρηνται ἄρα δύο ῥηταὶ δυνάμει μόνον σύμμετροι αἱ ΒΑ, ΑΖ, ὥστε τὴν μεῖζονα τὴν ΑΒ τῆς ἐλάσσονος τῆς ΑΖ μεῖζον δύνασθαι τῷ ἀπὸ τῆς ΒΖ συμμέτρου ἑαυτῇ μήκει· ὅπερ ἔδει δεῖξαι.

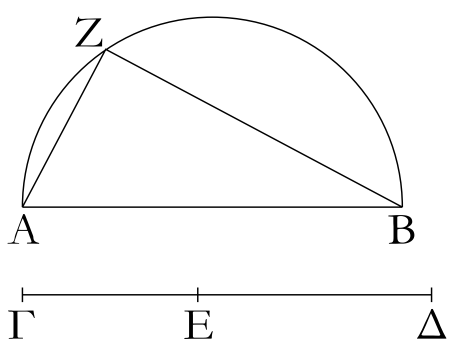
BF [Props. 5.19 corr., 3.31, 1.47]. And CD has to DE the ratio which (some) square number (has) to (some) square number. Thus, the (square) on AB also has to the (square) on BF the ratio which (some) square number has to (some) square number. AB is thus commensurable in length with BF [Prop. 10.9]. And the (square) on AB is equal to the (sum of the squares) on AF and FB [Prop. 1.47]. Thus, the square on AB is greater than (the square on) AF by (the square on) BF, (which is) commensurable (in length) with (AB).

Thus, two rational (straight-lines), BA and AF, commensurable in square only, have been found such that the square on the greater, AB, is larger than (the square on) the lesser, AF, by the (square) on BF, (which is) commensurable in length with (AB).† (Which is) the very thing it was required to show.

† BA and AF have lengths 1 and $\sqrt{1-k^2}$ times that of AB, respectively, where $k = \sqrt{DE/CD}$.

λ΄.

Εὑρεῖν δύο ῥητὰς δυνάμει μόνον συμμέτρους, ὥστε τὴν μείζονα τῆς ἐλάσσονος μεῖζον δύνασθαι τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ μήκει.

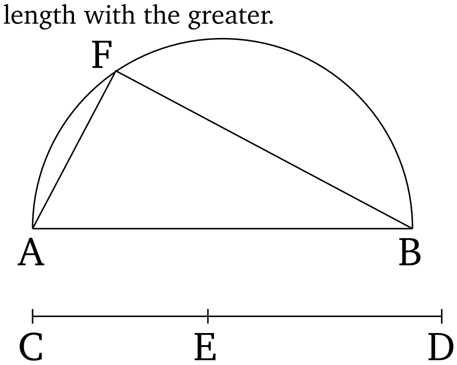


Ἐκκείσθω ῥητὴ ἡ ΑΒ καὶ δύο τετράγωνοι ἀριθμοὶ οἱ ΓΕ, ΕΔ, ὥστε τὸν συγκείμενον ἐξ αὐτῶν τὸν ΓΔ μὴ εἶναι τετράγωνον, καὶ γεγράφθω ἐπὶ τῆς ΑΒ ἡμικύκλιον τὸ ΑΖΒ, καὶ πεποιήσθω ὡς ὁ ΔΓ πρὸς τὸν ΓΕ, οὕτως τὸ ἀπὸ τῆς ΒΑ πρὸς τὸ ἀπὸ τῆς ΑΖ, καὶ ἐπεζεύχθω ἡ ΖΒ.

Ὁμοίως δὴ δείξομεν τῷ πρὸ τούτου, ὅτι αἱ ΒΑ, ΑΖ ῥηταί εἰσι δυνάμει μόνον σύμμετροι. καὶ ἐπεί ἐστιν ὡς ὁ ΔΓ πρὸς τὸν ΓΕ, οὕτως τὸ ἀπὸ τῆς ΒΑ πρὸς τὸ ἀπὸ τῆς ΑΖ, ἀναστρέψαντι ἄρα ὡς ὁ ΓΔ πρὸς τὸν ΔΕ, οὕτως τὸ ἀπὸ τῆς ΑΒ πρὸς τὸ ἀπὸ τῆς ΒΖ. ὁ δὲ ΓΔ πρὸς τὸν ΔΕ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· οὐδ᾽ ἄρα τὸ ἀπὸ τῆς ΑΒ πρὸς τὸ ἀπὸ τῆς ΒΖ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ ΑΒ τῇ ΒΖ μήκει. καὶ δύναται ἡ ΑΒ τῆς ΑΖ μεῖζον τῷ ἀπὸ τῆς ΖΒ ἀσυμμέτρου ἑαυτῇ.

Proposition 30

To find two rational (straight-lines which are) commensurable in square only, such that the square on the greater is larger than the (the square on) lesser by the (square) on (some straight-line which is) incommensurable in length with the greater.



Let the rational (straight-line) AB be laid out, and the two square numbers, CE and ED, such that the sum of them, CD, is not square [Prop. 10.28 lem. II]. And let the semi-circle AFB have been drawn on AB. And let it be contrived that as DC (is) to CE, so the (square) on BA (is) to the (square) on AF [Prop. 10.6 corr]. And let FB have been joined.

So, similarly to the (proposition) before this, we can show that BA and AF are rational (straight-lines which are) commensurable in square only. And since as DC is to CE, so the (square) on BA (is) to the (square) on AF, thus, via conversion, as CD (is) to DE, so the (square) on AB (is) to the (square) on BF [Props. 5.19 corr., 3.31, 1.47]. And CD does not have to DE the ratio which (some) square number (has) to (some) square number.

Αἱ ΑΒ, ΑΖ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι, καὶ ἡ ΑΒ τῆς ΑΖ μεῖζον δύναται τῷ ἀπὸ τῆς ΖΒ ἀσυμμέτρου ἑαυτῇ μήκει· ὅπερ ἔδει δεῖξαι.

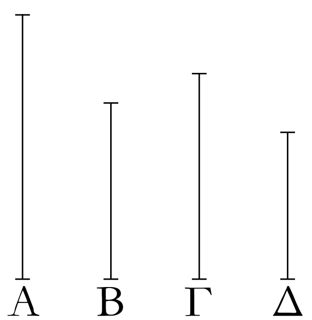
Thus, the (square) on AB does not have to the (square) on BF the ratio which (some) square number has to (some) square number either. Thus, AB is incommensurable in length with BF [Prop. 10.9]. And the square on AB is greater than the (square on) AF by the (square) on FB [Prop. 1.47], (which is) incommensurable (in length) with (AB).

Thus, AB and AF are rational (straight-lines which are) commensurable in square only, and the square on AB is greater than (the square on) AF by the (square) on FB, (which is) incommensurable (in length) with (AB).†
(Which is) the very thing it was required to show.

† AB and AF have lengths 1 and $1/\sqrt{1+k^2}$ times that of AB, respectively, where $k = \sqrt{DE/CE}$.

λα´.

Εὑρεῖν δύο μέσας δυνάμει μόνον συμμέτρους ῥητὸν περιεχούσας, ὥστε τὴν μείζονα τῆς ἐλάσσονος μεῖζον δύνασθαι τῷ ἀπὸ συμμέτρου ἑαυτῇ μήκει.

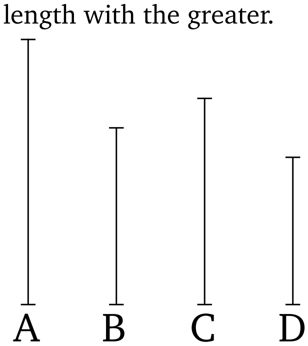


Ἐκκείσθωσαν δύο ῥηταὶ δυνάμει μόνον σύμμετροι αἱ Α, Β, ὥστε τὴν Α μείζονα οὖσαν τῆς ἐλάσσονος τῆς Β μεῖζον δύνασθαι τῷ ἀπὸ συμμέτρου ἑαυτῇ μήκει. καὶ τῷ ὑπὸ τῶν Α, Β ἴσον ἔστω τὸ ἀπὸ τῆς Γ. μέσον δὲ τὸ ὑπὸ τῶν Α, Β· μέσον ἄρα καὶ τὸ ἀπὸ τῆς Γ· μέση ἄρα καὶ ἡ Γ. τῷ δὲ ἀπὸ τῆς Β ἴσον ἔστω τὸ ὑπὸ τῶν Γ, Δ· ῥητὸν δὲ τὸ ἀπὸ τῆς Β· ῥητὸν ἄρα καὶ τὸ ὑπὸ τῶν Γ, Δ. καὶ ἐπεί ἐστιν ὡς ἡ Α πρὸς τὴν Β, οὕτως τὸ ὑπὸ τῶν Α, Β πρὸς τὸ ἀπὸ τῆς Β, ἀλλὰ τῷ μὲν ὑπὸ τῶν Α, Β ἴσον ἐστὶ τὸ ἀπὸ τῆς Γ, τῷ δὲ ἀπὸ τῆς Β ἴσον τὸ ὑπὸ τῶν Γ, Δ, ὡς ἄρα ἡ Α πρὸς τὴν Β, οὕτως τὸ ἀπὸ τῆς Γ πρὸς τὸ ὑπὸ τῶν Γ, Δ. ὡς δὲ τὸ ἀπὸ τῆς Γ πρὸς τὸ ὑπὸ τῶν Γ, Δ, οὕτως ἡ Γ πρὸς τὴν Δ· καὶ ὡς ἄρα ἡ Α πρὸς τὴν Β, οὕτως ἡ Γ πρὸς τὴν Δ. σύμμετρος δὲ ἡ Α τῇ Β δυνάμει μόνον· σύμμετρος ἄρα καὶ ἡ Γ τῇ Δ δυνάμει μόνον. καί ἐστι μέση ἡ Γ· μέση ἄρα καὶ ἡ Δ. καὶ ἐπεί ἐστιν ὡς ἡ Α πρὸς τὴν Β, ἡ Γ πρὸς τὴν Δ, ἡ δὲ Α τῆς Β μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ, καὶ ἡ Γ ἄρα τῆς Δ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ.

Εὕρηνται ἄρα δύο μέσαι δυνάμει μόνον σύμμετροι αἱ Γ,

Proposition 31

To find two medial (straight-lines), commensurable in square only, (and) containing a rational (area), such that the square on the greater is larger than the (square on the) lesser by the (square) on (some straight-line) commensurable in length with the greater.



Let two rational (straight-lines), A and B, commensurable in square only, be laid out, such that the square on the greater A is larger than the (square on the) lesser B by the (square) on (some straight-line) commensurable in length with (A) [Prop. 10.29]. And let the (square) on C be equal to the (rectangle contained) by A and B. And the (rectangle contained by) A and B (is) medial [Prop. 10.21]. Thus, the (square) on C (is) also medial. Thus, C (is) also medial [Prop. 10.21]. And let the (rectangle contained) by C and D be equal to the (square) on B. And the (square) on B (is) rational. Thus, the (rectangle contained) by C and D (is) also rational. And since as A is to B, so the (rectangle contained) by A and B (is) to the (square) on B [Prop. 10.21 lem.], but the (square) on C is equal to the (rectangle contained) by A and B, and the (rectangle contained) by C and D to the (square) on B, thus as A (is) to B, so the (square) on C (is) to the (rectangle contained) by C and D. And as the (square) on C (is) to the (rectangle contained) by

Δ ῥητὸν περιέχουσαι, καὶ ἡ Γ τῆς Δ μεῖζον δυνάται τῷ ἀπὸ συμμέτρου ἑαυτῇ μήκει.

Ὁμοίως δὴ δειχθήσεται καὶ τῷ ἀπὸ ἀσυμμέτρου, ὅταν ἡ Α τῆς Β μεῖζον δύνηται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ.

C and D, so C (is) to D [Prop. 10.21 lem.]. And thus as A (is) to B, so C (is) to D. And A is commensurable in square only with B. Thus, C (is) also commensurable in square only with D [Prop. 10.11]. And C is medial. Thus, D (is) also medial [Prop. 10.23]. And since as A is to B, (so) C (is) to D, and the square on A is greater than (the square on) B by the (square) on (some straight-line) commensurable (in length) with (A), the square on C is thus also greater than (the square on) D by the (square) on (some straight-line) commensurable (in length) with (C) [Prop. 10.14].

Thus, two medial (straight-lines), C and D, commensurable in square only, (and) containing a rational (area), have been found. And the square on C is greater than (the square on) D by the (square) on (some straight-line) commensurable in length with (C).†

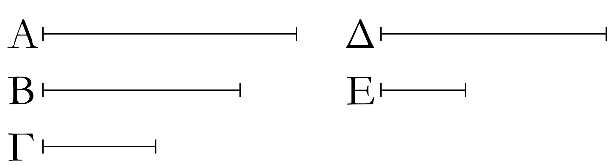
So, similarly, (the proposition) can also be demonstrated for (some straight-line) incommensurable (in length with C), provided that the square on A is greater than (the square on B) by the (square) on (some straight-line) incommensurable (in length) with (A) [Prop. 10.30].‡

† C and D have lengths $(1-k^2)^{1/4}$ and $(1-k^2)^{3/4}$ times that of A, respectively, where k is defined in the footnote to Prop. 10.29.

‡ C and D would have lengths $1/(1+k^2)^{1/4}$ and $1/(1+k^2)^{3/4}$ times that of A, respectively, where k is defined in the footnote to Prop. 10.30.

λβ′.

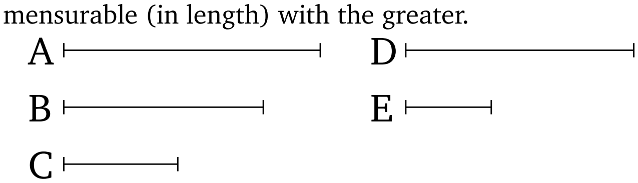
Εὑρεῖν δύο μέσας δυνάμει μόνον συμμέτρους μέσον περιεχούσας, ὥστε τὴν μείζονα τῆς ἐλάσσονος μεῖζον δύνασθαι τῷ ἀπὸ συμμέτρου ἑαυτῇ.



Ἐκκείσθωσαν τρεῖς ῥηταὶ δυνάμει μόνον σύμμετροι αἱ Α, Β, Γ, ὥστε τὴν Α τῆς Γ μεῖζον δύνασθαι τῷ ἀπὸ συμμέτρου ἑαυτῇ, καὶ τῷ μὲν ὑπὸ τῶν Α, Β ἴσον ἔστω τὸ ἀπὸ τῆς Δ. μέσον ἄρα τὸ ἀπὸ τῆς Δ· καὶ ἡ Δ ἄρα μέση ἐστίν. τῷ δὲ ὑπὸ τῶν Β, Γ ἴσον ἔστω τὸ ὑπὸ τῶν Δ, Ε. καὶ ἐπεί ἐστιν ὡς τὸ ὑπὸ τῶν Α, Β πρὸς τὸ ὑπὸ τῶν Β, Γ, οὕτως ἡ Α πρὸς τὴν Γ, ἀλλὰ τῷ μὲν ὑπὸ τῶν Α, Β ἴσον ἐστὶ τὸ ἀπὸ τῆς Δ, τῷ δὲ ὑπὸ τῶν Β, Γ ἴσον τὸ ὑπὸ τῶν Δ, Ε, ἔστιν ἄρα ὡς ἡ Α πρὸς τὴν Γ, οὕτως τὸ ἀπὸ τῆς Δ πρὸς τὸ ὑπὸ τῶν Δ, Ε. ὡς δὲ τὸ ἀπὸ τῆς Δ πρὸς τὸ ὑπὸ τῶν Δ, Ε, οὕτως ἡ Δ πρὸς τὴν Ε· καὶ ὡς ἄρα ἡ Α πρὸς τὴν Γ, οὕτως ἡ Δ πρὸς τὴν Ε. σύμμετρος δὲ ἡ Α τῇ Γ δυνάμει [μόνον]. σύμμετρος ἄρα καὶ ἡ Δ τῇ Ε δυνάμει μόνον. μέση

Proposition 32

To find two medial (straight-lines), commensurable in square only, (and) containing a medial (area), such that the square on the greater is larger than the (square on the) lesser by the (square) on (some straight-line) commensurable (in length) with the greater.



Let three rational (straight-lines), A, B and C, commensurable in square only, be laid out such that the square on A is greater than (the square on C) by the (square) on (some straight-line) commensurable (in length) with (A) [Prop. 10.29]. And let the (square) on D be equal to the (rectangle contained) by A and B. Thus, the (square) on D (is) medial. Thus, D is also medial [Prop. 10.21]. And let the (rectangle contained) by D and E be equal to the (rectangle contained) by B and C. And since as the (rectangle contained) by A and B is to the (rectangle contained) by B and C, so A (is) to C [Prop. 10.21 lem.], but the (square) on D is equal to the (rectangle contained) by A and B, and the (rectangle δὲ ἡ Δ· μέση ἄρα καὶ ἡ Ε. καὶ ἐπεί ἐστιν ὡς ἡ Α πρὸς τὴν Γ, ἡ Δ πρὸς τὴν Ε, ἡ δὲ Α τῆς Γ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ, καὶ ἡ Δ ἄρα τῆς Ε μεῖζον δυνήσεται τῷ ἀπὸ συμμέτρου ἑαυτῇ. λέγω δή, ὅτι καὶ μέσον ἐστὶ τὸ ὑπὸ τῶν Δ, Ε. ἐπεὶ γὰρ ἴσον ἐστὶ τὸ ὑπὸ τῶν Β, Γ τῷ ὑπὸ τῶν Δ, Ε, μέσον δὲ τὸ ὑπὸ τῶν Β, Γ [αἱ γὰρ Β, Γ ῥηταί εἰσι δυνάμει μόνον σύμμετροι], μέσον ἄρα καὶ τὸ ὑπὸ τῶν Δ, Ε.

Εὕρηνται ἄρα δύο μέσαι δυνάμει μόνον σύμμετροι αἱ Δ, Ε μέσον περιέχουσαι, ὥστε τὴν μείζονα τῆς ἐλάσσονος μεῖζον δύνασθαι τῷ ἀπὸ συμμέτρου ἑαυτῇ.

Ὁμοίως δὴ πάλιν διεχθήσεται καὶ τῷ ἀπὸ ἀσυμμέτρου, ὅταν ἡ Α τῆς Γ μεῖζον δύνηται τῷ ἀπὸ ἀσυμμέτρου ἑαυτη.

contained) by D and E to the (rectangle contained) by B and C, thus as A is to C, so the (square) on D (is) to the (rectangle contained) by D and E. And as the (square) on D (is) to the (rectangle contained) by D and E, so D (is) to E [Prop. 10.21 lem.]. And thus as A (is) to C, so D (is) to E. And A (is) commensurable in square [only] with C. Thus, D (is) also commensurable in square only with E [Prop. 10.11]. And D (is) medial. Thus, E (is) also medial [Prop. 10.23]. And since as A is to C, (so) D (is) to E, and the square on A is greater than (the square on) C by the (square) on (some straight-line) commensurable (in length) with (A), the square on D will thus also be greater than (the square on) E by the (square) on (some straight-line) commensurable (in length) with (D) [Prop. 10.14]. So, I also say that the (rectangle contained) by D and E is medial. For since the (rectangle contained) by B and C is equal to the (rectangle contained) by D and E, and the (rectangle contained) by B and C (is) medial [for B and C are rational (straight-lines which are) commensurable in square only] [Prop. 10.21], the (rectangle contained) by D and E (is) thus also medial.

Thus, two medial (straight-lines), D and E, commensurable in square only, (and) containing a medial (area), have been found such that the square on the greater is larger than the (square on the) lesser by the (square) on (some straight-line) commensurable (in length) with the greater.†.

So, similarly, (the proposition) can again also be demonstrated for (some straight-line) incommensurable (in length with the greater), provided that the square on A is greater than (the square on) C by the (square) on (some straight-line) incommensurable (in length) with (A) [Prop. 10.30].‡

† D and E have lengths $k'^{1/4}$ and $k'^{1/4}\sqrt{1-k^2}$ times that of A, respectively, where the length of B is $k'^{1/2}$ times that of A, and k is defined in the footnote to Prop. 10.29.

‡ D and E would have lengths $k'^{1/4}$ and $k'^{1/4}/\sqrt{1+k^2}$ times that of A, respectively, where the length of B is $k'^{1/2}$ times that of A, and k is defined in the footnote to Prop. 10.30.

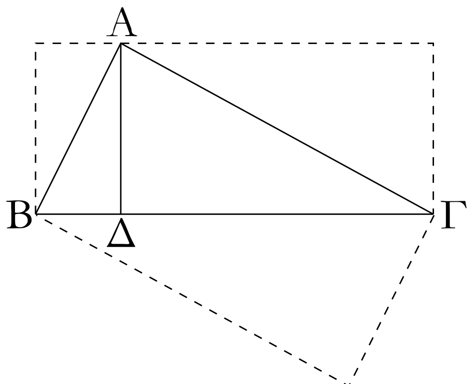
Λῆμμα.

Ἔστω τρίγωνον ὀρθογώνιον τὸ ΑΒΓ ὀρθὴν ἔχον τὴν Α, καὶ ἤχθω κάθετος ἡ ΑΔ· λέγω, ὅτι τὸ μὲν ὑπὸ τῶν ΓΒΔ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΒΑ, τὸ δὲ ὑπὸ τῶν ΒΓΑ ἴσον τῷ ἀπὸ τῆς ΓΑ, καὶ τὸ ὑπὸ τῶν ΒΔ, ΔΓ ἴσον τῷ ἀπὸ τῆς ΑΔ, καὶ ἔτι τὸ ὑπὸ τῶν ΒΓ, ΑΔ ἴσον [ἐστὶ] τῷ ὑπὸ τῶν ΒΑ, ΑΓ.

Καὶ πρῶτον, ὅτι τὸ ὑπὸ τῶν ΓΒΔ ἴσον [ἐστὶ] τῷ ἀπὸ τῆς ΒΑ.

Lemma

Let ABC be a right-angled triangle having the (angle) A a right-angle. And let the perpendicular AD have been drawn. I say that the (rectangle contained) by CBD is equal to the (square) on BA, and the (rectangle contained) by BCD (is) equal to the (square) on CA, and the (rectangle contained) by BD and DC (is) equal to the (square) on AD, and, further, the (rectangle contained) by BC and AD [is] equal to the (rectangle contained) by BA and AC.



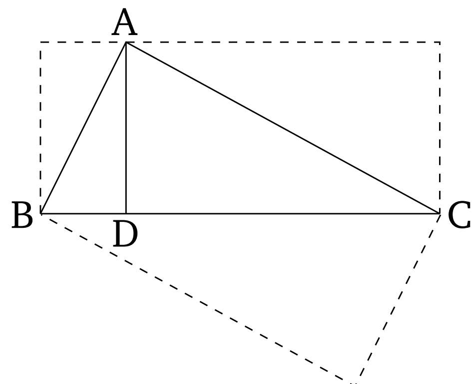
Ἐπεὶ γὰρ ἐν ὀρθογωνίῳ τριγώνῳ ἀπὸ τῆς ὀρθῆς γωνίας ἐπὶ τὴν βάσιν κάθετος ἦκται ἡ ΑΔ, τὰ ΑΒΔ, ΑΔΓ ἄρα τρίγωνα ὅμοιά ἐστι τῷ τε ὅλῳ τῷ ΑΒΓ καὶ ἀλλήλοις. καὶ ἐπεὶ ὅμοιόν ἐστι τὸ ΑΒΓ τρίγωνον τῷ ΑΒΔ τριγώνῳ, ἔστιν ἄρα ὡς ἡ ΓΒ πρὸς τὴν ΒΑ, οὕτως ἡ ΒΑ πρὸς τὴν ΒΔ· τὸ ἄρα ὑπὸ τῶν ΓΒΔ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΒ.

Διὰ τὰ αὐτὰ δὴ καὶ τὸ ὑπὸ τῶν ΒΓΔ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΓ.

Καὶ ἐπεί, ἐὰν ἐν ὀρθογωνίῳ τριγώνῳ ἀπὸ τῆς ὀρθῆς γωνίας ἐπὶ τὴν βάσιν κάθετος ἀχθῇ, ἡ ἀχθεῖσα τῶν τῆς βάσεως τμημάτων μέση ἀνάλογόν ἐστιν, ἔστιν ἄρα ὡς ἡ ΒΔ πρὸς τὴν ΔΑ, οὕτως ἡ ΑΔ πρὸς τὴν ΔΓ· τὸ ἄρα ὑπὸ τῶν ΒΔ, ΔΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΔΑ.

Λέγω, ὅτι καὶ τὸ ὑπὸ τῶν ΒΓ, ΑΔ ἴσον ἐστὶ τῷ ὑπὸ τῶν ΒΑ, ΑΓ. ἐπεὶ γάρ, ὡς ἔφαμεν, ὅμοιόν ἐστι τὸ ΑΒΓ τῷ ΑΒΔ, ἔστιν ἄρα ὡς ἡ ΒΓ πρὸς τὴν ΓΑ, οὕτως ἡ ΒΑ πρὸς τὴν ΑΔ. τὸ ἄρα ὑπὸ τῶν ΒΓ, ΑΔ ἴσον ἐστὶ τῷ ὑπὸ τῶν ΒΑ, ΑΓ· ὅπερ ἔδει δεῖξαι.

And, first of all, (let us prove) that the (rectangle contained) by CBD [is] equal to the (square) on BA.



For since AD has been drawn from the right-angle in a right-angled triangle, perpendicular to the base, ABD and ADC are thus triangles (which are) similar to the whole, ABC, and to one another [Prop. 6.8]. And since triangle ABC is similar to triangle ABD, thus as CB is to BA, so BA (is) to BD [Prop. 6.4]. Thus, the (rectangle contained) by CBD is equal to the (square) on AB [Prop. 6.17].

So, for the same (reasons), the (rectangle contained) by BCD is also equal to the (square) on AC.

And since if a (straight-line) is drawn from the right-angle in a right-angled triangle, perpendicular to the base, the (straight-line so) drawn is the mean proportional to the pieces of the base [Prop. 6.8 corr.], thus as BD is to DA, so AD (is) to DC. Thus, the (rectangle contained) by BD and DC is equal to the (square) on DA [Prop. 6.17].

I also say that the (rectangle contained) by BC and AD is equal to the (rectangle contained) by BA and AC. For since, as we said, ABC is similar to ABD, thus as BC is to CA, so BA (is) to AD [Prop. 6.4]. Thus, the (rectangle contained) by BC and AD is equal to the (rectangle contained) by BA and AC [Prop. 6.16]. (Which is) the very thing it was required to show.

λγ´.

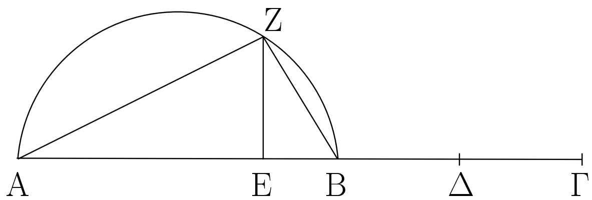
Εὑρεῖν δύο εὐθείας δυνάμει ἀσυμμέτρους ποιούσας τὸ μὲν συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων ῥητόν, τὸ δ᾽ ὑπ᾽ αὐτῶν μέσον.

Ἐκκείσθωσαν δύο ῥηταὶ δυνάμει μόνον σύμμετροι αἱ ΑΒ, ΒΓ, ὥστε τὴν μείζονα τὴν ΑΒ τῆς ἐλάσσονος τῆς ΒΓ μεῖζον δύνασθαι τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ, καὶ τετμήσθω ἡ ΒΓ δίχα κατὰ τὸ Δ, καὶ τῷ ἀφ᾽ ὁποτέρας τῶν ΒΔ, ΔΓ ἴσον παρὰ τὴν ΑΒ παραβεβλήσθω παραλληλόγραμμον ἐλλεῖπον εἴδει τετραγώνῳ, καὶ ἔστω τὸ ὑπὸ τῶν ΑΕΒ, καὶ γεγράφθω ἐπὶ τῆς ΑΒ ἡμικύκλιον τὸ ΑΖΒ, καὶ ἤχθω τῇ ΑΒ πρὸς

Proposition 33

To find two straight-lines (which are) incommensurable in square, making the sum of the squares on them rational, and the (rectangle contained) by them medial.

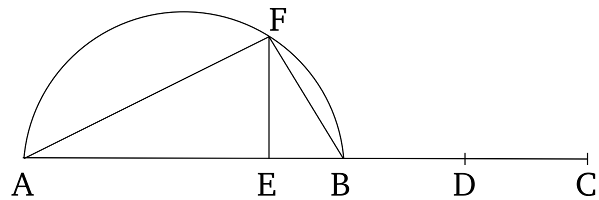
Let the two rational (straight-lines) AB and BC, (which are) commensurable in square only, be laid out such that the square on the greater, AB, is larger than (the square on) the lesser, BC, by the (square) on (some straight-line which is) incommensurable (in length) with (AB) [Prop. 10.30]. And let BC have been cut in half at D. And let a parallelogram equal to the (square) on eiὀρθὰς ἡ ΕΖ, καὶ ἐπεζεύχθωσαν αἱ ΑΖ, ΖΒ.



Καὶ ἐπεὶ [δύο] εὐθεῖαι ἄνισοί εἰσιν αἱ ΑΒ, ΒΓ, καὶ ἡ ΑΒ τῆς ΒΓ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ, τῷ δὲ τετάρτῳ τοῦ ἀπὸ τῆς ΒΓ, τουτέστι τῷ ἀπὸ τῆς ἡμισείας αὐτῆς, ἴσον παρὰ τὴν ΑΒ παραβέβληται παραλληλόγραμμον ἐλλεῖπον εἴδει τετραγώνῳ καὶ ποιεῖ τὸ ὑπὸ τῶν ΑΕΒ, ἀσύμμετρος ἄρα ἐστὶν ἡ ΑΕ τῇ ΕΒ. καί ἐστιν ὡς ἡ ΑΕ πρὸς ΕΒ, οὕτως τὸ ὑπὸ τῶν ΒΑ, ΑΕ πρὸς τὸ ὑπὸ τῶν ΑΒ, ΒΕ, ἴσον δὲ τὸ μὲν ὑπὸ τῶν ΒΑ, ΑΕ τῷ ἀπὸ τῆς ΑΖ, τὸ δὲ ὑπὸ τῶν ΑΒ, ΒΕ τῷ ἀπὸ τῆς ΒΖ· ἀσύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΑΖ τῷ ἀπὸ τῆς ΖΒ· αἱ ΑΖ, ΖΒ ἄρα δυνάμει εἰσὶν ἀσύμμετροι. καὶ ἐπεὶ ἡ ΑΒ ῥητή ἐστιν, ῥητὸν ἄρα ἐστὶ καὶ τὸ ἀπὸ τῆς ΑΒ· ὥστε καὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΖ, ΖΒ ῥητόν ἐστιν. καὶ ἐπεὶ πάλιν τὸ ὑπὸ τῶν ΑΕ, ΕΒ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΕΖ, ὑπόκειται δὲ τὸ ὑπὸ τῶν ΑΕ, ΕΒ καὶ τῷ ἀπὸ τῆς ΒΔ ἴσον, ἴση ἄρα ἐστὶν ἡ ΖΕ τῇ ΒΔ· διπλῆ ἄρα ἡ ΒΓ τῆς ΖΕ· ὥστε καὶ τὸ ὑπὸ τῶν ΑΒ, ΒΓ σύμμετρόν ἐστι τῷ ὑπὸ τῶν ΑΒ, ΕΖ. μέσον δὲ τὸ ὑπὸ τῶν ΑΒ, ΒΓ· μέσον ἄρα καὶ τὸ ὑπὸ τῶν ΑΒ, ΕΖ. ἴσον δὲ τὸ ὑπὸ τῶν ΑΒ, ΕΖ τῷ ὑπὸ τῶν ΑΖ, ΖΒ· μέσον ἄρα καὶ τὸ ὑπὸ τῶν ΑΖ, ΖΒ. ἐδείχθη δὲ καὶ ῥητὸν τὸ συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων.

Εὕρηνται ἄρα δύο εὐθεῖαι δυνάμει ἀσύμμετροι αἱ ΑΖ, ΖΒ ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων ῥητόν, τὸ δὲ ὑπ᾽ αὐτῶν μέσον· ὅπερ ἔδει δεῖξαι.

ther of BD or DC, (and) falling short by a square figure, have been applied to AB [Prop. 6.28], and let it be the (rectangle contained) by AEB. And let the semi-circle AFB have been drawn on AB. And let EF have been drawn at right-angles to AB. And let AF and FB have been joined.



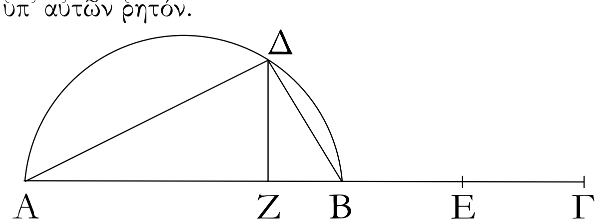
And since AB and BC are [two] unequal straight-lines, and the square on AB is greater than (the square on) BC by the (square) on (some straight-line which is) incommensurable (in length) with (AB). And a parallelogram, equal to one quarter of the (square) on BC—that is to say, (equal) to the (square) on half of it—(and) falling short by a square figure, has been applied to AB, and makes the (rectangle contained) by AEB. AE is thus incommensurable (in length) with EB [Prop. 10.18]. And as AE is to EB, so the (rectangle contained) by BA and AE (is) to the (rectangle contained) by AB and BE. And the (rectangle contained) by BA and AE (is) equal to the (square) on AF, and the (rectangle contained) by AB and BE to the (square) on BF [Prop. 10.32 lem.]. The (square) on AF is thus incommensurable with the (square) on FB [Prop. 10.11]. Thus, AF and FB are incommensurable in square. And since AB is rational, the (square) on AB is also rational. Hence, the sum of the (squares) on AF and FB is also rational [Prop. 1.47]. And, again, since the (rectangle contained) by AE and EB is equal to the (square) on EF, and the (rectangle contained) by AE and EB was assumed (to be) equal to the (square) on BD, FE is thus equal to BD. Thus, BC is double FE. And hence the (rectangle contained) by AB and BC is commensurable with the (rectangle contained) by AB and EF [Prop. 10.6]. And the (rectangle contained) by AB and BC (is) medial [Prop. 10.21]. Thus, the (rectangle contained) by AB and EF (is) also medial [Prop. 10.23 corr.]. And the (rectangle contained) by AB and EF (is) equal to the (rectangle contained) by AF and FB [Prop. 10.32 lem.]. Thus, the (rectangle contained) by AF and FB (is) also medial. And the sum of the squares on them was also shown (to be) rational.

Thus, the two straight-lines, AF and FB, (which are) incommensurable in square, have been found, making the sum of the squares on them rational, and the (rectangle contained) by them medial. (Which is) the very thing it was required to show.

† AF and FB have lengths $\sqrt{[1 + k/(1 + k^2)^{1/2}]/2}$ and $\sqrt{[1 - k/(1 + k^2)^{1/2}]/2}$ times that of AB, respectively, where k is defined in the footnote to Prop. 10.30.

λδ΄.

Εὑρεῖν δύο εὐθείας δυνάμει ἀσυμμέτρους ποιούσας τὸ μὲν συγκείμενον ἐκ τῶν ἀπ᾿ αὐτῶν τετραγώνων μέσον, τὸ δ᾿ ὑπ᾿ αὐτῶν ῥητόν.



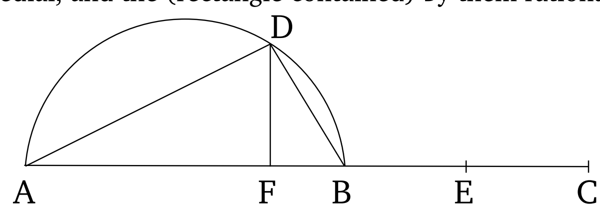
Ἐκκείσθωσαν δύο μέσαι δυνάμει μόνον σύμμετροι αἱ ΑΒ, ΒΓ ῥητὸν περιέχουσαι τὸ ὑπ᾿ αὐτῶν, ὥστε τὴν ΑΒ τῆς ΒΓ μεῖζον δύνασθαι τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ, καὶ γεγράφθω ἐπὶ τῆς ΑΒ τὸ ΑΔΒ ἡμικύκλιον, καὶ τετμήσθω ἡ ΒΓ δίχα κατὰ τὸ Ε, καὶ παραβεβλήσθω παρὰ τὴν ΑΒ τῷ ἀπὸ τῆς ΒΕ ἴσον παραλληλόγραμμον ἐλλεῖπον εἴδει τετραγώνῳ τὸ ὑπὸ τῶν ΑΖΒ· ἀσύμμετρος ἄρα [ἐστὶν] ἡ ΑΖ τῇ ΖΒ μήκει. καὶ ἤχθω ἀπὸ τοῦ Ζ τῇ ΑΒ πρὸς ὀρθὰς ἡ ΖΔ, καὶ ἐπεζεύχθωσαν αἱ ΑΔ, ΔΒ.

Ἐπεὶ ἀσύμμετρός ἐστιν ἡ ΑΖ τῇ ΖΒ, ἀσύμμετρον ἄρα ἐστὶ καὶ τὸ ὑπὸ τῶν ΒΑ, ΑΖ τῷ ὑπὸ τῶν ΑΒ, ΒΖ. ἴσον δὲ τὸ μὲν ὑπὸ τῶν ΒΑ, ΑΖ τῷ ἀπὸ τῆς ΑΔ, τὸ δὲ ὑπὸ τῶν ΑΒ, ΒΖ τῷ ἀπὸ τῆς ΔΒ· ἀσύμμετρον ἄρα ἐστὶ καὶ τὸ ἀπὸ τῆς ΑΔ τῷ ἀπὸ τῆς ΔΒ. καὶ ἐπεὶ μέσον ἐστὶ τὸ ἀπὸ τῆς ΑΒ, μέσον ἄρα καὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΔ, ΔΒ. καὶ ἐπεὶ διπλῆ ἐστιν ἡ ΒΓ τῆς ΔΖ, διπλάσιον ἄρα καὶ τὸ ὑπὸ τῶν ΑΒ, ΒΓ τοῦ ὑπὸ τῶν ΑΒ, ΖΔ. ῥητὸν δὲ τὸ ὑπὸ τῶν ΑΒ, ΒΓ· ῥητὸν ἄρα καὶ τὸ ὑπὸ τῶν ΑΒ, ΖΔ. τὸ δὲ ὑπὸ τῶν ΑΒ, ΖΔ ἴσον τῷ ὑπὸ τῶν ΑΔ, ΔΒ· ὥστε καὶ τὸ ὑπὸ τῶν ΑΔ, ΔΒ ῥητόν ἐστιν.

Εὕρηνται ἄρα δύο εὐθεῖαι δυνάμει ἀσύμμετροι αἱ ΑΔ, ΔΒ ποιοῦσαι τὸ [μὲν] συγκείμενον ἐκ τῶν ἀπ᾿ αὐτῶν τετραγώνων μέσον, τὸ δ᾿ ὑπ᾿ αὐτῶν ῥητόν· ὅπερ ἔδει δεῖξαι.

Proposition 34

To find two straight-lines (which are) incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them rational.



Let the two medial (straight-lines) AB and BC, (which are) commensurable in square only, be laid out having the (rectangle contained) by them rational, (and) such that the square on AB is greater than (the square on) BC by the (square) on (some straight-line) incommensurable (in length) with (AB) [Prop. 10.31]. And let the semi-circle ADB have been drawn on AB. And let BC have been cut in half at E. And let a (rectangular) parallelogram equal to the (square) on BE, (and) falling short by a square figure, have been applied to AB, (and let it be) the (rectangle contained by) AFB [Prop. 6.28]. Thus, AF [is] incommensurable in length with FB [Prop. 10.18]. And let FD have been drawn from F at right-angles to AB. And let AD and DB have been joined.

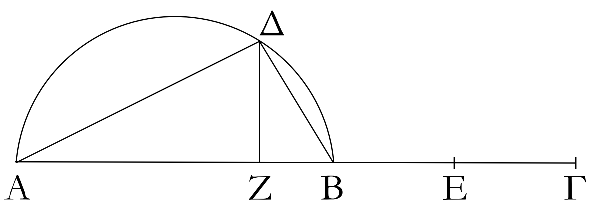
Since AF is incommensurable (in length) with FB, the (rectangle contained) by BA and AF is thus also incommensurable with the (rectangle contained) by AB and BF [Prop. 10.11]. And the (rectangle contained) by BA and AF (is) equal to the (square) on AD, and the (rectangle contained) by AB and BF to the (square) on DB [Prop. 10.32 lem.]. Thus, the (square) on AD is also incommensurable with the (square) on DB. And since the (square) on AB is medial, the sum of the (squares) on AD and DB (is) thus also medial [Props. 3.31, 1.47]. And since BC is double DF [see previous proposition], the (rectangle contained) by AB and BC (is) thus also double the (rectangle contained) by AB and FD. And the (rectangle contained) by AB and BC (is) rational. Thus, the (rectangle contained) by AB and FD (is) also rational [Prop. 10.6, Def. 10.4]. And the (rectangle contained) by AB and FD (is) equal to the (rectangle contained) by AD and DB [Prop. 10.32 lem.]. And hence the (rectangle contained) by AD and DB is rational.

Thus, two straight-lines, AD and DB, (which are) incommensurable in square, have been found, making the sum of the squares on them medial, and the (rectangle contained) by them rational.† (Which is) the very thing it was required to show.

† AD and DB have lengths $\sqrt{[(1+k^2)^{1/2}+k]/[2\,(1+k^2)]}$ and $\sqrt{[(1+k^2)^{1/2}-k]/[2\,(1+k^2)]}$ times that of AB, respectively, where k is defined in the footnote to Prop. 10.29.

λε′.

Εὑρεῖν δύο εὐθείας δυνάμει ἀσυμμέτρους ποιούσας τό τε συγκείμενον ἐκ τῶν ἀπ᾿ αὐτῶν τετραγώνων μέσον καὶ τὸ ὑπ᾿ αὐτῶν μέσον καὶ ἔτι ἀσύμμετρον τῷ συγκειμένῳ ἐκ τῶν ἀπ᾿ αὐτῶν τετραγώνῳ.



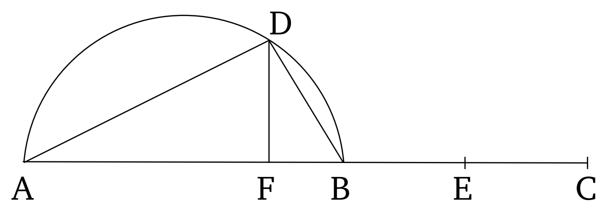
Ἐκκείσθωσαν δύο μέσαι δυνάμει μόνον σύμμετροι αἱ ΑΒ, ΒΓ μέσον περιέχουσαι, ὥστε τὴν ΑΒ τῆς ΒΓ μεῖζον δύνασθαι τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ, καὶ γεγράφθω ἐπὶ τῆς ΑΒ ἡμικύκλιον τὸ ΑΔΒ, καὶ τὰ λοιπὰ γεγονέτω τοῖς ἐπάνω ὁμοίως.

Καὶ ἐπεὶ ἀσύμμετρός ἐστιν ἡ ΑΖ τῇ ΖΒ μήκει, ἀσύμμετρός ἐστι καὶ ἡ ΑΔ τῇ ΔΒ δυνάμει. καὶ ἐπεὶ μέσον ἐστὶ τὸ ἀπὸ τῆς ΑΒ, μέσον ἄρα καὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΔ, ΔΒ. καὶ ἐπεὶ τὸ ὑπὸ τῶν ΑΖ, ΖΒ ἴσον ἐστὶ τῷ ἀφ᾿ ἑκατέρας τῶν ΒΕ, ΔΖ, ἴση ἄρα ἐστὶν ἡ ΒΕ τῇ ΔΖ· διπλῆ ἄρα ἡ ΒΓ τῆς ΖΔ· ὥστε καὶ τὸ ὑπὸ τῶν ΑΒ, ΒΓ διπλάσιόν ἐστι τοῦ ὑπὸ τῶν ΑΒ, ΖΔ. μέσον δὲ τὸ ὑπὸ τῶν ΑΒ, ΒΓ· μέσον ἄρα καὶ τὸ ὑπὸ τῶν ΑΒ, ΖΔ. καί ἐστιν ἴσον τῷ ὑπὸ τῶν ΑΔ, ΔΒ· μέσον ἄρα καὶ τὸ ὑπὸ τῶν ΑΔ, ΔΒ. καὶ ἐπεὶ ἀσύμμετρός ἐστιν ἡ ΑΒ τῇ ΒΓ μήκει, σύμμετρος δὲ ἡ ΓΒ τῇ ΒΕ, ἀσύμμετρος ἄρα καὶ ἡ ΑΒ τῇ ΒΕ μήκει· ὥστε καὶ τὸ ἀπὸ τῆς ΑΒ τῷ ὑπὸ τῶν ΑΒ, ΒΕ ἀσύμμετρόν ἐστιν. ἀλλὰ τῷ μὲν ἀπὸ τῆς ΑΒ ἴσα ἐστὶ τὰ ἀπὸ τῶν ΑΔ, ΔΒ, τῷ δὲ ὑπὸ τῶν ΑΒ, ΒΕ ἴσον ἐστὶ τὸ ὑπὸ τῶν ΑΒ, ΖΔ, τουτέστι τὸ ὑπὸ τῶν ΑΔ, ΔΒ· ἀσύμμετρον ἄρα ἐστὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΔ, ΔΒ τῷ ὑπὸ τῶν ΑΔ, ΔΒ.

Εὕρηνται ἄρα δύο εὐθεῖαι αἱ ΑΔ, ΔΒ δυνάμει ἀσύμμετροι ποιοῦσαι τό τε συγκείμενον ἐκ τῶν ἀπ᾿ αὐτῶν μέσον καὶ τὸ ὑπ᾿ αὐτῶν μέσον καὶ ἔτι ἀσύμμετρον τῷ συγκειμένῳ ἐκ τῶν ἀπ᾿ αὐτῶν τετραγώνων· ὅπερ ἔδει δεῖξαι.

Proposition 35

To find two straight-lines (which are) incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them medial, and, moreover, incommensurable with the sum of the squares on them.



Let the two medial (straight-lines) AB and BC, (which are) commensurable in square only, be laid out containing a medial (area), such that the square on AB is greater than (the square on) BC by the (square) on (some straight-line) incommensurable (in length) with (AB) [Prop. 10.32]. And let the semi-circle ADB have been drawn on AB. And let the remainder (of the figure) be generated similarly to the above (proposition).

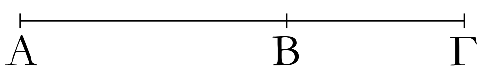
And since AF is incommensurable in length with FB [Prop. 10.18], AD is also incommensurable in square with DB [Prop. 10.11]. And since the (square) on AB is medial, the sum of the (squares) on AD and DB (is) thus also medial [Props. 3.31, 1.47]. And since the (rectangle contained) by AF and FB is equal to the (square) on each of BE and DF, BE is thus equal to DF. Thus, BC (is) double FD. And hence the (rectangle contained) by AB and BC is double the (rectangle) contained by AB and FD. And the (rectangle contained) by AB and BC (is) medial. Thus, the (rectangle contained) by AB and FD (is) also medial. And it is equal to the (rectangle contained) by AD and DB [Prop. 10.32 lem.]. Thus, the (rectangle contained) by AD and DB (is) also medial. And since AB is incommensurable in length with BC, and CB (is) commensurable (in length) with BE, AB (is) thus also incommensurable in length with BE [Prop. 10.13]. And hence the (square) on AB is also incommensurable with the (rectangle contained) by AB and BE [Prop. 10.11]. But the (sum of the squares) on AD and DB is equal to the (square) on AB [Prop. 1.47]. And the (rectangle contained) by AB and FD—that is to say, the (rectangle contained) by AD and DB—is equal to the (rectangle contained) by AB and BE. Thus, the sum of the (squares) on AD and DB is incommensurable with the (rectangle contained) by AD and DB.

Thus, two straight-lines, AD and DB, (which are) incommensurable in square, have been found, making the sum of the (squares) on them medial, and the (rectangle contained) by them medial, and, moreover, incommensurable with the sum of the squares on them.† (Which is) the very thing it was required to show.

† AD and DB have lengths $k'^{1/4}\sqrt{[1+k/(1+k^2)^{1/2}]/2}$ and $k'^{1/4}\sqrt{[1-k/(1+k^2)^{1/2}]/2}$ times that of AB, respectively, where k and k' are defined in the footnote to Prop. 10.32.

λϛʹ.

Ἐὰν δύο ῥηταὶ δυνάμει μόνον σύμμετροι συντεθῶσιν, ἡ ὅλη ἄλογός ἐστιν, καλείσθω δὲ ἐκ δύο ὀνομάτων.

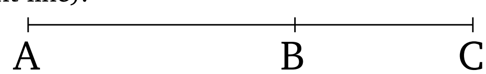


Συγκείσθωσαν γὰρ δύο ῥηταὶ δυνάμει μόνον σύμμετροι αἱ ΑΒ, ΒΓ· λέγω, ὅτι ὅλη ἡ ΑΓ ἄλογός ἐστιν.

Ἐπεὶ γὰρ ἀσύμμετρός ἐστιν ἡ ΑΒ τῇ ΒΓ μήκει· δυνάμει γὰρ μόνον εἰσὶ σύμμετροι· ὡς δὲ ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως τὸ ὑπὸ τῶν ΑΒΓ πρὸς τὸ ἀπὸ τῆς ΒΓ, ἀσύμμετρον ἄρα ἐστὶ τὸ ὑπὸ τῶν ΑΒ, ΒΓ τῷ ἀπὸ τῆς ΒΓ. ἀλλὰ τῷ μὲν ὑπὸ τῶν ΑΒ, ΒΓ σύμμετρόν ἐστι τὸ δὶς ὑπὸ τῶν ΑΒ, ΒΓ, τῷ δὲ ἀπὸ τῆς ΒΓ σύμμετρά ἐστι τὰ ἀπὸ τῶν ΑΒ, ΒΓ· αἱ γὰρ ΑΒ, ΒΓ ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἀσύμμετρον ἄρα ἐστὶ τὸ δὶς ὑπὸ τῶν ΑΒ, ΒΓ τοῖς ἀπὸ τῶν ΑΒ, ΒΓ. καὶ συνθέντι τὸ δὶς ὑπὸ τῶν ΑΒ, ΒΓ μετὰ τῶν ἀπὸ τῶν ΑΒ, ΒΓ, τουτέστι τὸ ἀπὸ τῆς ΑΓ, ἀσύμμετρόν ἐστι τῷ συγκειμένῳ ἐκ τῶν ἀπὸ τῶν ΑΒ, ΒΓ· ῥητὸν δὲ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΒ, ΒΓ· ἄλογον ἄρα [ἐστὶ] τὸ ἀπὸ τῆς ΑΓ· ὥστε καὶ ἡ ΑΓ ἄλογός ἐστιν, καλείσθω δὲ ἐκ δύο ὀνομάτων· ὅπερ ἔδει δεῖξαι.

Proposition 36

If two rational (straight-lines which are) commensurable in square only are added together then the whole (straight-line) is irrational—let it be called a binomial (straight-line).†



For let the two rational (straight-lines), AB and BC, (which are) commensurable in square only, be laid down together. I say that the whole (straight-line), AC, is irrational. For since AB is incommensurable in length with BC—for they are commensurable in square only—and as AB (is) to BC, so the (rectangle contained) by ABC (is) to the (square) on BC, the (rectangle contained) by AB and BC is thus incommensurable with the (square) on BC [Prop. 10.11]. But, twice the (rectangle contained) by AB and BC is commensurable with the (rectangle contained) by AB and BC [Prop. 10.6]. And (the sum of) the (squares) on AB and BC is commensurable with the (square) on BC—for the rational (straight-lines) AB and BC are commensurable in square only [Prop. 10.15]. Thus, twice the (rectangle contained) by AB and BC is incommensurable with (the sum of) the (squares) on AB and BC [Prop. 10.13]. And, via composition, twice the (rectangle contained) by AB and BC, plus (the sum of) the (squares) on AB and BC—that is to say, the (square) on AC [Prop. 2.4]—is incommensurable with the sum of the (squares) on AB and BC [Prop. 10.16]. And the sum of the (squares) on AB and BC (is) rational. Thus, the (square) on AC [is] irrational [Def. 10.4]. Hence, AC is also irrational [Def. 10.4]—let it be called a binomial (straight-line).‡ (Which is) the very thing it was required to show.

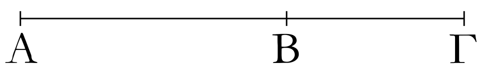
† Literally, "from two names".

‡ Thus, a binomial straight-line has a length expressible as $1 + k^{1/2}$ [or, more generally, $\rho\,(1 + k^{1/2})$, where ρ is rational—the same proviso applies to the definitions in the following propositions]. The binomial and the corresponding apotome, whose length is expressible as $1 - k^{1/2}$

(see Prop. 10.73), are the positive roots of the quartic $x^4 - 2\,(1+k)\,x^2 + (1-k)^2 = 0$.

λζ΄.

Ἐὰν δύο μέσαι δυνάμει μόνον σύμμετροι συντεθῶσι ῥητὸν περιέχουσαι, ἡ ὅλη ἄλογός ἐστιν, καλείσθω δὲ ἐκ δύο μέσων πρώτη.

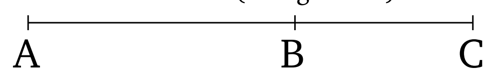


Συγκείσθωσαν γὰρ δύο μέσαι δυνάμει μόνον σύμμετροι αἱ ΑΒ, ΒΓ ῥητὸν περιέχουσαι· λέγω, ὅτι ὅλη ἡ ΑΓ ἄλογός ἐστιν.

Ἐπεὶ γὰρ ἀσύμμετρός ἐστιν ἡ ΑΒ τῇ ΒΓ μήκει, καὶ τὰ ἀπὸ τῶν ΑΒ, ΒΓ ἄρα ἀσύμμετρά ἐστι τῷ δὶς ὑπὸ τῶν ΑΒ, ΒΓ· καὶ συνθέντι τὰ ἀπὸ τῶν ΑΒ, ΒΓ μετὰ τοῦ δὶς ὑπὸ τῶν ΑΒ, ΒΓ, ὅπερ ἐστὶ τὸ ἀπὸ τῆς ΑΓ, ἀσύμμετρόν ἐστι τῷ ὑπὸ τῶν ΑΒ, ΒΓ. ῥητὸν δὲ τὸ ὑπὸ τῶν ΑΒ, ΒΓ· ὑπόκεινται γὰρ αἱ ΑΒ, ΒΓ ῥητὸν περιέχουσαι· ἄλογον ἄρα τὸ ἀπὸ τῆς ΑΓ· ἄλογος ἄρα ἡ ΑΓ, καλείσθω δὲ ἐκ δύο μέσων πρώτη· ὅπερ ἔδει δεῖξαι.

Proposition 37

If two medial (straight-lines), commensurable in square only, which contain a rational (area), are added together then the whole (straight-line) is irrational—let it be called a first bimedial (straight-line).†



For let the two medial (straight-lines), AB and BC, commensurable in square only, (and) containing a rational (area), be laid down together. I say that the whole (straight-line), AC, is irrational.

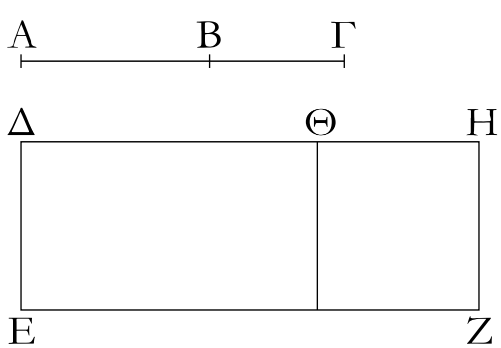
For since AB is incommensurable in length with BC, (the sum of) the (squares) on AB and BC is thus also incommensurable with twice the (rectangle contained) by AB and BC [see previous proposition]. And, via composition, (the sum of) the (squares) on AB and BC, plus twice the (rectangle contained) by AB and BC—that is, the (square) on AC [Prop. 2.4]—is incommensurable with the (rectangle contained) by AB and BC [Prop. 10.16]. And the (rectangle contained) by AB and BC (is) rational—for AB and BC were assumed to enclose a rational (area). Thus, the (square) on AC (is) irrational. Thus, AC (is) irrational [Def. 10.4]—let it be called a first bimedial (straight-line).‡ (Which is) the very thing it was required to show.

† Literally, "first from two medials".

‡ Thus, a first bimedial straight-line has a length expressible as $k^{1/4} + k^{3/4}$. The first bimedial and the corresponding first apotome of a medial, whose length is expressible as $k^{1/4} - k^{3/4}$ (see Prop. 10.74), are the positive roots of the quartic $x^4 - 2\sqrt{k}\,(1+k)\,x^2 + k\,(1-k)^2 = 0$.

λη΄.

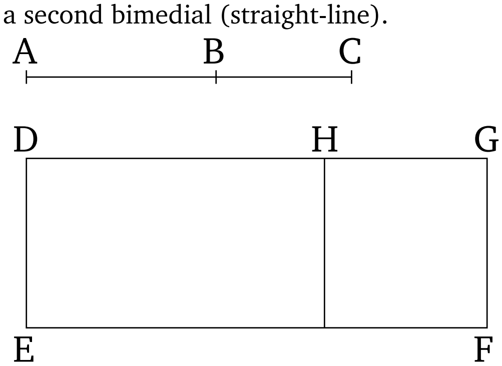
Ἐὰν δύο μέσαι δυνάμει μόνον σύμμετροι συντεθῶσι μέσον περιέχουσαι, ἡ ὅλη ἄλογός ἐστιν, καλείσθω δὲ ἐκ δύο μέσων δευτέρα.



Συγκείσθωσαν γὰρ δύο μέσαι δυνάμει μόνον σύμμετροι αἱ ΑΒ, ΒΓ μέσον περιέχουσαι· λέγω, ὅτι ἄλογός ἐστιν ἡ

Proposition 38

If two medial (straight-lines), commensurable in square only, which contain a medial (area), are added together then the whole (straight-line) is irrational—let it be called a second bimedial (straight-line).



For let the two medial (straight-lines), AB and BC, commensurable in square only, (and) containing a medial

ΑΓ.

Ἐκκείσθω γὰρ ῥητὴ ἡ ΔΕ, καὶ τῷ ἀπὸ τῆς ΑΓ ἴσον παρὰ τὴν ΔΕ παραβεβλήσθω τὸ ΔΖ πλάτος ποιοῦν τὴν ΔΗ. καὶ ἐπεὶ τὸ ἀπὸ τῆς ΑΓ ἴσον ἐστὶ τοῖς τε ἀπὸ τῶν ΑΒ, ΒΓ καὶ τῷ δὶς ὑπὸ τῶν ΑΒ, ΒΓ, παραβεβλήσθω δὴ τοῖς ἀπὸ τῶν ΑΒ, ΒΓ παρὰ τὴν ΔΕ ἴσον τὸ ΕΘ· λοιπὸν ἄρα τὸ ΘΖ ἴσον ἐστὶ τῷ δὶς ὑπὸ τῶν ΑΒ, ΒΓ. καὶ ἐπεὶ μέση ἐστὶν ἑκατέρα τῶν ΑΒ, ΒΓ, μέσα ἄρα ἐστὶ καὶ τὰ ἀπὸ τῶν ΑΒ, ΒΓ. μέσον δὲ ὑπόκειται καὶ τὸ δὶς ὑπὸ τῶν ΑΒ, ΒΓ. καί ἐστι τοῖς μὲν ἀπὸ τῶν ΑΒ, ΒΓ ἴσον τὸ ΕΘ, τῷ δὲ δὶς ὑπὸ τῶν ΑΒ, ΒΓ ἴσον τὸ ΖΘ· μέσον ἄρα ἑκάτερον τῶν ΕΘ, ΘΖ. καὶ παρὰ ῥητὴν τὴν ΔΕ παράκειται· ῥητὴ ἄρα ἐστὶν ἑκατέρα τῶν ΔΘ, ΘΗ καὶ ἀσύμμετρος τῇ ΔΕ μήκει. ἐπεὶ οὖν ἀσύμμετρός ἐστιν ἡ ΑΒ τῇ ΒΓ μήκει, καί ἐστιν ὡς ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως τὸ ἀπὸ τῆς ΑΒ πρὸς τὸ ὑπὸ τῶν ΑΒ, ΒΓ, ἀσύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΑΒ τῷ ὑπὸ τῶν ΑΒ, ΒΓ. ἀλλὰ τῷ μὲν ἀπὸ τῆς ΑΒ σύμμετρόν ἐστι τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΒ, ΒΓ τετραγώνων, τῷ δὲ ὑπὸ τῶν ΑΒ, ΒΓ σύμμετρόν ἐστι τὸ δὶς ὑπὸ τῶν ΑΒ, ΒΓ. ἀσύμμετρον ἄρα ἐστὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΒ, ΒΓ τῷ δὶς ὑπὸ τῶν ΑΒ, ΒΓ. ἀλλὰ τοῖς μὲν ἀπὸ τῶν ΑΒ, ΒΓ ἴσον ἐστὶ τὸ ΕΘ, τῷ δὲ δὶς ὑπὸ τῶν ΑΒ, ΒΓ ἴσον ἐστὶ τὸ ΘΖ. ἀσύμμετρον ἄρα ἐστὶ τὸ ΕΘ τῷ ΘΖ· ὥστε καὶ ἡ ΔΘ τῇ ΘΗ ἐστιν ἀσύμμετρος μήκει. αἱ ΔΘ, ΘΗ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι. ὥστε ἡ ΔΗ ἄλογός ἐστιν. ῥητὴ δὲ ἡ ΔΕ· τὸ δὲ ὑπὸ ἀλόγου καὶ ῥητῆς περιεχόμενον ὀρθογώνιον ἄλογόν ἐστιν· ἄλογον ἄρα ἐστὶ τὸ ΔΖ χωρίον, καὶ ἡ δυναμένη [αὐτὸ] ἄλογός ἐστιν. δύναται δὲ τὸ ΔΖ ἡ ΑΓ· ἄλογος ἄρα ἐστὶν ἡ ΑΓ, καλείσθω δὲ ἐκ δύο μέσων δευτέρα. ὅπερ ἔδει δεῖξαι.

(area), be laid down together [Prop. 10.28]. I say that AC is irrational.

For let the rational (straight-line) DE be laid down, and let (the rectangle) DF, equal to the (square) on AC, have been applied to DE, making DG as breadth [Prop. 1.44]. And since the (square) on AC is equal to (the sum of) the (squares) on AB and BC, plus twice the (rectangle contained) by AB and BC [Prop. 2.4], so let (the rectangle) EH, equal to (the sum of) the squares on AB and BC, have been applied to DE. The remainder HF is thus equal to twice the (rectangle contained) by AB and BC. And since AB and BC are each medial, (the sum of) the squares on AB and BC is thus also medial.‡ And twice the (rectangle contained) by AB and BC was also assumed (to be) medial. And EH is equal to (the sum of) the squares on AB and BC, and FH (is) equal to twice the (rectangle contained) by AB and BC. Thus, EH and HF (are) each medial. And they were applied to the rational (straight-line) DE. Thus, DH and HG are each rational, and incommensurable in length with DE [Prop. 10.22]. Therefore, since AB is incommensurable in length with BC, and as AB is to BC, so the (square) on AB (is) to the (rectangle contained) by AB and BC [Prop. 10.21 lem.], the (square) on AB is thus incommensurable with the (rectangle contained) by AB and BC [Prop. 10.11]. But, the sum of the squares on AB and BC is commensurable with the (square) on AB [Prop. 10.15], and twice the (rectangle contained) by AB and BC is commensurable with the (rectangle contained) by AB and BC [Prop. 10.6]. Thus, the sum of the (squares) on AB and BC is incommensurable with twice the (rectangle contained) by AB and BC [Prop. 10.13]. But, EH is equal to (the sum of) the squares on AB and BC, and HF is equal to twice the (rectangle) contained by AB and BC. Thus, EH is incommensurable with HF. Hence, DH is also incommensurable in length with HG [Props. 6.1, 10.11]. Thus, DH and HG are rational (straight-lines which are) commensurable in square only. Hence, DG is irrational [Prop. 10.36]. And DE (is) rational. And the rectangle contained by irrational and rational (straight-lines) is irrational [Prop. 10.20]. The area DF is thus irrational, and (so) the square-root [of it] is irrational [Def. 10.4]. And AC is the square-root of DF. AC is thus irrational—let it be called a second bimedial (straight-line).§ (Which is) the very thing it was required to show.

† Literally, "second from two medials".

‡ Since, by hypothesis, the squares on AB and BC are commensurable—see Props. 10.15, 10.23.

§ Thus, a second bimedial straight-line has a length expressible as $k^{1/4}+k'^{1/2}/k^{1/4}$. The second bimedial and the corresponding second apotome of a medial, whose length is expressible as $k^{1/4}-k'^{1/2}/k^{1/4}$ (see Prop. 10.75), are the positive roots of the quartic $x^4-2\,[(k+k')/\sqrt{k}\,]\,x^2+$