Theorem 1. *A non-empty subset W of V is a subspace of V if and only if for each pair of vectors α, β in W and each scalar c in F the vector $c\alpha + \beta$ is again in W.*

Proof. Suppose that W is a non-empty subset of V such that $c\alpha + \beta$ belongs to W for all vectors α, β in W and all scalars c in F. Since W is non-empty, there is a vector ρ in W, and hence $(-1)\rho + \rho = 0$ is in W. Then if α is any vector in W and c any scalar, the vector $c\alpha = c\alpha + 0$ is in W. In particular, $(-1)\alpha = -\alpha$ is in W. Finally, if α and β are in W, then $\alpha + \beta = 1\alpha + \beta$ is in W. Thus W is a subspace of V.

Conversely, if W is a subspace of V, α and β are in W, and c is a scalar, certainly $c\alpha + \beta$ is in W. ∎

Some people prefer to use the $c\alpha + \beta$ property in Theorem 1 as the definition of a subspace. It makes little difference. The important point is that, if W is a non-empty subset of V such that $c\alpha + \beta$ is in V for all α, β in W and all c in F, then (with the operations inherited from V) W is a vector space. This provides us with many new examples of vector spaces.

EXAMPLE 6.

(a) If V is any vector space, V is a subspace of V; the subset consisting of the zero vector alone is a subspace of V, called the **zero subspace** of V.

(b) In F^n, the set of n-tuples $(x_1, \ldots, x_n)$ with $x_1 = 0$ is a subspace; however, the set of n-tuples with $x_1 = 1 + x_2$ is not a subspace ($n \geq 2$).

(c) The space of polynomial functions over the field F is a subspace of the space of all functions from F into F.

(d) An $n \times n$ (square) matrix A over the field F is **symmetric** if $A_{ij} = A_{ji}$ for each i and j. The symmetric matrices form a subspace of the space of all $n \times n$ matrices over F.

(e) An $n \times n$ (square) matrix A over the field C of complex numbers is **Hermitian** (or **self-adjoint**) if

$$A_{jk} = \overline{A_{kj}}$$

for each j, k, the bar denoting complex conjugation. A 2×2 matrix is Hermitian if and only if it has the form

$$\begin{bmatrix} z & x + iy \\ x - iy & w \end{bmatrix}$$

where x, y, z, and w are real numbers. The set of all Hermitian matrices is *not* a subspace of the space of all $n \times n$ matrices over C. For if A is Hermitian, its diagonal entries A_{11}, A_{22}, . . . , are all real numbers, but the diagonal entries of iA are in general not real. On the other hand, it is easily verified that the set of $n \times n$ complex Hermitian matrices is a vector space over the field R of real numbers (with the usual operations).

EXAMPLE 7. **The solution space of a system of homogeneous linear equations.** Let A be an $m \times n$ matrix over F. Then the set of all $n \times 1$ (column) matrices X over F such that $AX = 0$ is a subspace of the space of all $n \times 1$ matrices over F. To prove this we must show that $A(cX + Y) = 0$ when $AX = 0$, $AY = 0$, and c is an arbitrary scalar in F. This follows immediately from the following general fact.

Lemma. *If* A *is an* m $\times$ n *matrix over* F *and* B, C *are* n $\times$ p *matrices over* F *then*

$$A(dB + C) = d(AB) + AC \tag{2-11}$$

for each scalar d *in* F.

Proof.

$$\begin{aligned}[A(dB + C)]_{ij} &= \sum_k A_{ik}(dB + C)_{kj} \\ &= \sum_k (dA_{ik}B_{kj} + A_{ik}C_{kj}) \\ &= d\sum_k A_{ik}B_{kj} + \sum_k A_{ik}C_{kj} \\ &= d(AB)_{ij} + (AC)_{ij} \\ &= [d(AB) + AC]_{ij}. \quad \blacksquare\end{aligned}$$

Similarly one can show that $(dB + C)A = d(BA) + CA$, if the matrix sums and products are defined.

Theorem 2. *Let* V *be a vector space over the field* F. *The intersection of any collection of subspaces of* V *is a subspace of* V.

Proof. Let $\{W_a\}$ be a collection of subspaces of V, and let $W = \bigcap_a W_a$ be their intersection. Recall that W is defined as the set of all elements belonging to every W_a (see Appendix). Since each W_a is a subspace, each contains the zero vector. Thus the zero vector is in the intersection W, and W is non-empty. Let α and β be vectors in W and let c be a scalar. By definition of W, both α and β belong to each W_a, and because each W_a is a subspace, the vector $(c\alpha + \beta)$ is in every W_a. Thus $(c\alpha + \beta)$ is again in W. By Theorem 1, W is a subspace of V. $\blacksquare$

From Theorem 2 it follows that if S is any collection of vectors in V, then there is a smallest subspace of V which contains S, that is, a subspace which contains S and which is contained in every other subspace containing S.

Definition. *Let* S *be a set of vectors in a vector space* V. *The* **subspace spanned** *by* S *is defined to be the intersection* W *of all subspaces of* V *which contain* S. *When* S *is a finite set of vectors,* S = $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, *we shall simply call* W *the* **subspace spanned by the vectors** $\alpha_1, \alpha_2, \ldots, \alpha_n$.

Theorem 3. *The subspace spanned by a non-empty subset S of a vector space V is the set of all linear combinations of vectors in S.*

Proof. Let W be the subspace spanned by S. Then each linear combination

$$\alpha = x_1\alpha_1 + x_2\alpha_2 + \cdots + x_m\alpha_m$$

of vectors $\alpha_1, \alpha_2, \ldots, \alpha_m$ in S is clearly in W. Thus W contains the set L of all linear combinations of vectors in S. The set L, on the other hand, contains S and is non-empty. If α, β belong to L then α is a linear combination,

$$\alpha = x_1\alpha_1 + x_2\alpha_2 + \cdots + x_m\alpha_m$$

of vectors α_i in S, and β is a linear combination,

$$\beta = y_1\beta_1 + y_2\beta_2 + \cdots + y_n\beta_n$$

of vectors β_j in S. For each scalar c,

$$c\alpha + \beta = \sum_{i=1}^{m} (cx_i)\alpha_i + \sum_{j=1}^{n} y_j\beta_j.$$

Hence $c\alpha + \beta$ belongs to L. Thus L is a subspace of V.

Now we have shown that L is a subspace of V which contains S, and also that any subspace which contains S contains L. It follows that L is the intersection of all subspaces containing S, i.e., that L is the subspace spanned by the set S. ▌

Definition. *If $S_1, S_2, \ldots, S_k$ are subsets of a vector space V, the set of all sums*

$$\alpha_1 + \alpha_2 + \cdots + \alpha_k$$

of vectors α_i in S_i is called the **sum** *of the subsets $S_1, S_2, \ldots, S_k$ and is denoted by*

$$S_1 + S_2 + \cdots + S_k$$

or by

$$\sum_{i=1}^{k} S_i.$$

If $W_1, W_2, \ldots, W_k$ are subspaces of V, then the sum

$$W = W_1 + W_2 + \cdots + W_k$$

is easily seen to be a subspace of V which contains each of the subspaces W_i. From this it follows, as in the proof of Theorem 3, that W is the subspace spanned by the union of $W_1, W_2, \ldots, W_k$.

EXAMPLE 8. Let F be a subfield of the field C of complex numbers. Suppose

$$\begin{aligned}\alpha_1 &= (1, 2, 0, 3, 0)\\ \alpha_2 &= (0, 0, 1, 4, 0)\\ \alpha_3 &= (0, 0, 0, 0, 1).\end{aligned}$$

By Theorem 3, a vector α is in the subspace W of F^5 spanned by α_1, α_2, α_3 if and only if there exist scalars c_1, c_2, c_3 in F such that

$$\alpha = c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3.$$

Thus W consists of all vectors of the form

$$\alpha = (c_1, 2c_1, c_2, 3c_1 + 4c_2, c_3)$$

where c_1, c_2, c_3 are arbitrary scalars in F. Alternatively, W can be described as the set of all 5-tuples

$$\alpha = (x_1, x_2, x_3, x_4, x_5)$$

with x_i in F such that

$$\begin{aligned}x_2 &= 2x_1\\ x_4 &= 3x_1 + 4x_3.\end{aligned}$$

Thus $(-3, -6, 1, -5, 2)$ is in W, whereas $(2, 4, 6, 7, 8)$ is not.

EXAMPLE 9. Let F be a subfield of the field C of complex numbers, and let V be the vector space of all 2×2 matrices over F. Let W_1 be the subset of V consisting of all matrices of the form

$$\begin{bmatrix} x & y \\ z & 0 \end{bmatrix}$$

where x, y, z are arbitrary scalars in F. Finally, let W_2 be the subset of V consisting of all matrices of the form

$$\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$$

where x and y are arbitrary scalars in F. Then W_1 and W_2 are subspaces of V. Also

$$V = W_1 + W_2$$

because

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix}.$$

The subspace $W_1 \cap W_2$ consists of all matrices of the form

$$\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}.$$

EXAMPLE 10. Let A be an $m \times n$ matrix over a field F. The **row vectors** of A are the vectors in F^n given by $\alpha_i = (A_{i1}, \ldots, A_{in})$, $i = 1, \ldots, m$. The subspace of F^n spanned by the row vectors of A is called the **row**

space of A. The subspace considered in Example 8 is the row space of the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

It is also the row space of the matrix

$$B = \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -4 & -8 & 1 & -8 & 0 \end{bmatrix}.$$

EXAMPLE 11. Let V be the space of all polynomial functions over F. Let S be the subset of V consisting of the polynomial functions $f_0, f_1, f_2, \ldots$ defined by

$$f_n(x) = x^n, \qquad n = 0, 1, 2, \ldots.$$

Then V is the subspace spanned by the set S.

Exercises

1. Which of the following sets of vectors $\alpha = (a_1, \ldots, a_n)$ in R^n are subspaces of R^n $(n \geq 3)$?

(a) all α such that $a_1 \geq 0$;
(b) all α such that $a_1 + 3a_2 = a_3$;
(c) all α such that $a_2 = a_1^2$;
(d) all α such that $a_1a_2 = 0$;
(e) all α such that a_2 is rational.

2. Let V be the (real) vector space of all functions f from R into R. Which of the following sets of functions are subspaces of V?

(a) all f such that $f(x^2) = f(x)^2$;
(b) all f such that $f(0) = f(1)$;
(c) all f such that $f(3) = 1 + f(-5)$;
(d) all f such that $f(-1) = 0$;
(e) all f which are continuous.

3. Is the vector $(3, -1, 0, -1)$ in the subspace of R^5 spanned by the vectors $(2, -1, 3, 2)$, $(-1, 1, 1, -3)$, and $(1, 1, 9, -5)$?

4. Let W be the set of all $(x_1, x_2, x_3, x_4, x_5)$ in R^5 which satisfy

$$\begin{aligned} 2x_1 - x_2 + \tfrac{4}{3}x_3 - x_4 \phantom{{}-3x_5} &= 0 \\ x_1 \phantom{{}-x_2} + \tfrac{2}{3}x_3 \phantom{{}-x_4} - x_5 &= 0 \\ 9x_1 - 3x_2 + 6x_3 - 3x_4 - 3x_5 &= 0. \end{aligned}$$

Find a finite set of vectors which spans W.

5. Let F be a field and let n be a positive integer ($n \geq 2$). Let V be the vector space of all $n \times n$ matrices over F. Which of the following sets of matrices A in V are subspaces of V?

(a) all invertible A;
(b) all non-invertible A;
(c) all A such that $AB = BA$, where B is some fixed matrix in V;
(d) all A such that $A^2 = A$.

6. (a) Prove that the only subspaces of R^1 are R^1 and the zero subspace.

(b) Prove that a subspace of R^2 is R^2, or the zero subspace, or consists of all scalar multiples of some fixed vector in R^2. (The last type of subspace is, intuitively, a straight line through the origin.)

(c) Can you describe the subspaces of R^3?

7. Let W_1 and W_2 be subspaces of a vector space V such that the set-theoretic union of W_1 and W_2 is also a subspace. Prove that one of the spaces W_i is contained in the other.

8. Let V be the vector space of all functions from R into R; let V_e be the subset of even functions, $f(-x) = f(x)$; let V_o be the subset of odd functions, $f(-x) = -f(x)$.

(a) Prove that V_e and V_o are subspaces of V.
(b) Prove that $V_e + V_o = V$.
(c) Prove that $V_e \cap V_o = \{0\}$.

9. Let W_1 and W_2 be subspaces of a vector space V such that $W_1 + W_2 = V$ and $W_1 \cap W_2 = \{0\}$. Prove that for each vector α in V there are *unique* vectors α_1 in W_1 and α_2 in W_2 such that $\alpha = \alpha_1 + \alpha_2$.

2.3. Bases and Dimension

We turn now to the task of assigning a dimension to certain vector spaces. Although we usually associate 'dimension' with something geometrical, we must find a suitable algebraic definition of the dimension of a vector space. This will be done through the concept of a basis for the space.

Definition. *Let* V *be a vector space over* F. *A subset* S *of* V *is said to be* **linearly dependent** (*or simply,* **dependent**) *if there exist distinct vectors* $\alpha_1, \alpha_2, \ldots, \alpha_n$ *in* S *and scalars* $c_1, c_2, \ldots, c_n$ *in* F, *not all of which are* 0, *such that*

$$c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n = 0.$$

A set which is not linearly dependent is called **linearly independent.** *If the set* S *contains only finitely many vectors* $\alpha_1, \alpha_2, \ldots, \alpha_n$, *we sometimes say that* $\alpha_1, \alpha_2, \ldots, \alpha_n$ *are dependent* (*or independent*) *instead of saying* S *is dependent* (*or independent*).