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Quaternions Applied to Physics

If Hamilton's intention had been to apply quaternions to physics, he was stymied by the fact that physical space has only three dimensions. Writing the quaternion x as $x = x_0 + \xi$, where $\xi = i_1x_1 + i_2x_2 + i_3x_3$, we call x_0 the *scalar part* and ξ the *vector part.* A 3-*vector* is then a quaternion whose scalar part is 0. Unfortunately, if we multiply ξ by another vector $\eta = y_1i_1 + y_2i_2 + y_3i_3$, we obtain not a vector, but the quaternion

$$-(x_1y_1 + x_2y_2 + x_3y_3) + (y_2x_3 - x_2y_3)i_1 + \cdots.$$

Oliver Heaviside pointed out the importance of the two separate parts of this product and wrote

$$\xi \circ \eta = x_1y_1 + x_2y_2 + x_3y_3,$$

$$\xi \times \eta = (y_2z_3 - x_2y_3)i_1 + \cdots$$

calling them the *scalar* and *vector product*, respectively. His *vector analysis* soon replaced the use of quaternions in physics. It enabled him to give a concise formulation of Maxwell's laws of electromagnetism. Writing $\nabla = i_1\frac{\partial}{\partial x_1} + i_2\frac{\partial}{\partial x_2} + i_3\frac{\partial}{\partial x_3}$ for the vector representing partial differentiation with respect to the space coordinates and letting $E = E_1i_1 + E_2i_2 + E_3i_3$ and $M = M_1i_1 + M_2i_2 + M_3i_3$ represent the electric and magnetic fields, respectively, Maxwell's equations may be written as follows:

$$\nabla \circ M = 0, \qquad \nabla \times E + \frac{\partial M}{c\partial t} = 0, \qquad \nabla \circ E = \rho, \qquad \nabla \times M - \frac{\partial E}{c\partial t} = \rho\frac{d\xi}{cdt},$$

where c is the velocity of light, ρ is the charge density, and $\frac{d\xi}{dt}$ the velocity of the matter bearing the electric charge.

Using the language of quaternions, albeit quaternions with complex components, these four equations may be combined into one, namely

$$\left(\frac{\partial}{c\partial t} - i\nabla\right)(M + iE) + \left(\rho + i\rho\frac{d\xi}{cdt}\right) = 0.$$

This was pointed out by Silberstein [1924] pp. 44-46; but one may wonder whether it wasn't already known to Maxwell himself, in view of his assertion that 'the invention of the calculus of quaternions is a step towards the knowledge of quantities related to space which can only be compared, for its importance, with the invention of triple coordinates by Descartes. The ideas of this calculus... are fitted to be of the greatest use in all parts of science.' (See Maxwell [1869].)

Einstein's theory of relativity made it clear that space and time should be combined into a single entity and that the expression

$$s^2 = c^2t^2 - x^2 - y^2 - z^2$$

should be invariant under coordinate transformations passing from a stationary to a moving platform. The minus sign in this expression suggests that we are talking about the norm of a quaternion

$$x = x_0 + i\xi, \qquad x_0 = ct, \qquad \xi = i_1x_1 + i_2x_2 + i_3x_3,$$

whose scalar part is real, but whose vector part is imaginary. Such a quaternion represents a point in so-called *Minkowski space.* Following Silberstein [1924], pp. 154-55, we observe that $s^2 = N(x)$ is invariant under a *Lorentz transformation,* which may itself be expressed with the help of 'biquaternions'.

A *biquaternion* $a = a_0 + i_1a_1 + i_2a_2 + i_3a_3$ has complex components $a_0, \ldots, a_3$. We must here distinguish the quaternion conjugate a^t of a from its complex conjugate a^c :

$$a^t = a_0 - i_1a_1 - i_2a_2 - i_3a_3,$$

$$a^c = \bar{a}_0 + i_1\bar{a}_1 + i_2\bar{a}_2 + i_3\bar{a}_3.$$

If a is represented as a 4×4 matrix $L(a)$ over $\mathbf{C}$ as in Chapter 8, $L(a^t) = L(a)^t$, the transpose of $L(a)$. We call a biquaternion x *Hermitian* if $x^t = x^c$; this is what characterizes the quaternion $x = ct + i\xi$ describing a point in Minkowski space, that is, an event in space-time. A *Lorentz transformation* sends x onto pxp^{ct}, where p is a biquaternion of norm 1. Indeed,

$$(pxp^{ct})^t = p^cx^tp^t = p^cx^cp^t = (pxp^{ct})^c,$$

so pxp^{ct} is also Hermitian. Moreover, $N(pxp^{ct}) = pxp^{ct}p^c x^t p^t = pxx^t p^t = N(x)N(p) = N(x)$, since $p^{ct}p^c = (p^t p)^c = N(p)^c = 1^c = 1$. Thus a Lorentz transformation preserves the norm of a Hermitian biquaternion.

Another Hermitian biquaternion which transforms like x is

$$m_0 \frac{dx}{ds} = m + im\frac{d\xi}{cdt},$$

where $m = m_0 \frac{cdt}{ds}$ is the mass of the moving particle and $m\frac{d\xi}{dt}$ is its momentum. Here m_0 is called the *rest mass*; it is assumed to be invariant under Lorentz transformations. As Einstein observed, the principle of conservation of momentum should carry with it also that of

$$m = m_0 \frac{cdt}{ds} = m_0(1 - v^2/c^2)^{-\frac{1}{2}},$$

since$(\frac{ds}{dt})^2 = c^2 - v^2$, where $v^2 = (\frac{dx_1}{dt})^2 + (\frac{dx_2}{dt})^2 + (\frac{dx_3}{dt})^2$ is the square of the velocity of the particle. If v is small compared to to c, we have

$$mc^2 = m_0c^2(1 - v^2/c^2)^{-\frac{1}{2}} \approx m_0c^2 + \tfrac{1}{2}m_0v^2,$$

which must then also be conserved. Einstein considered this to be the total energy $E = mc^2$ of the particle, consisting of the rest energy m_0c^2 and the kinetic energy $\frac{1}{2}m_0v^2$.

As we have seen, Maxwell's equations may be condensed into

$$\left(\frac{\partial}{c\partial t} - i\nabla\right)(M + iE) + \left(\rho + i\rho\frac{d\xi}{cdt}\right) = 0.$$

This may be written more concisely:

$$D^c F + J = 0,$$

where $D = \frac{\partial}{c\partial t} + i\nabla$ is sent by a Lorentz transformation onto $p^c D p^t$ and the so-called *six-vector* $F = M + iE$ is sent onto $p^c F p^{ct}$, so that $J = \rho + i\rho\frac{d\xi}{cdt}$ is sent onto pJp^{ct}. Thus J transforms like the mass-momentum biquaternion and may also be written as $\rho_0 \frac{dx}{ds}$, where

$$\rho = \rho_0 \frac{cdt}{ds} = \rho_0 \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}.$$

If the above rules for transforming D, F and J are adopted, it thus becomes clear that Maxwell's equations are preserved under Lorentz transformations. This fact, though without the aid of quaternions, appears to have first been noted by Poincaré.