the order of integration to obtain

$$\int_0^1 \left[\int_y^{\sqrt{y}} f(x, y)\, dx\right] dy.$$

EXAMPLE 3. A double integral of a positive function f, $\iint_S f(x, y)\, dx\, dy$, reduces to the repeated integral :

$$\int_0^3 \left[\int_{4y/3}^{\sqrt{25-y^2}} f(x, y)\, dx\right] dy.$$

Determine the region S and interchange the order of integration.

Solution. For each fixed y between 0 and 3, the integration with respect to x is over the interval from $4y/3$ to $\sqrt{25 - y^2}$. Therefore region S is of Type II and lies between the two curves $x = 4y/3$ and $x = \sqrt{25 - y^2}$. This region, shown in Figure 11.12, is a sector of a circle. When the order of integration is reversed the region must be split into two regions of Type I; the result is the sum of two integrals:

$$\int_0^4 \left[\int_0^{3x/4} f(x, y)\, dy\right] dx + \int_4^5 \left[\int_0^{\sqrt{25-x^2}} f(x, y)\, dy\right] dx.$$

11.15 Exercises

In Exercises 1 through 5, make a sketch of the region of integration and evaluate the double integral.

1. $\iint_S x \cos (x + y)\, dx\, dy$, where S is the triangular region whose vertices are (0, 0), $(\pi, 0)$, (π, π).
2. $\iint_S (1 + x) \sin y\, dx\, dy$, where S is the trapezoid with vertices (0, 0), (1, 0), (1, 2), (0, 1).
3. $\iint_S e^{x+y}\, dx\, dy$, where $S = \{(x, y) \mid |x| + |y| \le 1\}$.
4. $\iint_S x^2y^2\, dx\, dy$, where S is the bounded portion of the first quadrant lying between the two hyperbolas $xy = 1$ and $xy = 2$ and the two straight lines $y = x$ and $y = 4x$.
5. $\iint_S (x^2 - y^2)\, dx\, dy$, where S is bounded by the curve $y = \sin x$ and the interval $[0, \pi]$.
6. A pyramid is bounded by the three coordinate planes and the plane $x + 2y + 3z = 6$. Make a sketch of the solid and compute its volume by double integration.
7. A solid is bounded by the surface $z = x^2 - y^2$, the xy-plane, and the planes $x = 1$ and $x = 3$. Make a sketch of the solid and compute its volume by double integration.
8. Compute, by double integration, the volume of the ordinate set of f over S if:
   (a) $f(x, y) = x^2 + y^2$ and $S = \{(x, y) \mid |x| \le 1, |y| \le 1\}$.
   (b) $f(x, y) = 3x + y$ and $S = \{(x, y) \mid 4x^2 + 9y^2 \le 36, x > 0, y > 0\}$.
   (c) $f(x, y) = y + 2x + 20$ and $S = \{(x, y) \mid x^2 + y^2 \le 16\}$.

In Exercises 9 through 18 assume that the double integral of a positive function f extended over a region S reduces to the given repeated integral. In each case, make a sketch of the region S and interchange the order of integration.

9. $\int_0^1 \left[\int_0^y f(x, y)\, dx\right] dy.$

10. $\int_0^2 \left[\int_{y^2}^{2y} f(x, y)\, dx\right] dy.$

11. $\int_1^4 \left[\int_{\sqrt{x}}^2 f(x, y)\, dy\right] dx.$

12. $\int_1^2 \left[\int_{2-x}^{\sqrt{2x-x^2}} f(x, y)\, dy\right] dx.$

13. $\int_{-6}^{2} \left[\int_{(x^2-4)/4}^{2-x} f(x, y)\, dy\right] dx.$

14. $\int_1^e \left[\int_0^{\log x} f(x, y)\, dy\right] dx.$

15. $\int_{-1}^1 \left[\int_{-\sqrt{1-x^2}}^{1-x^2} f(x, y)\, dy\right] dx.$

16. $\int_0^1 \left[\int_{x^3}^{x^2} f(x, y)\, dy\right] dx.$

17. $\int_0^\pi \left[\int_{-\sin(x/2)}^{\sin x} f(x, y)\, dy\right] dx.$

18. $\int_0^4 \left[\int_{-\sqrt{4-y}}^{(y-4)/2} f(x, y)\, dx\right] dy.$

19. When a double integral was set up for the volume Yof the solid under the paraboloid $z = x^2 + y^2$ and above a region S of the xy-plane, the following sum of iterated integrals was obtained:

$$V = \int_0^1 \left[\int_0^y (x^2 + y^2)\, dx\right] dy + \int_1^2 \left[\int_0^{2-y} (x^2 + y^2)\, dx\right] dy .$$

Sketch the region S and express V as an iterated integral in which the order of integration is reversed. Also, carry out the integration and compute V.

20. When a double integral was set up for the volume V of the solid under the surface $z = f(x, y)$ and above a region S of the xy-plane, the following sum of iterated integrals was obtained:

$$V = \int_0^{a \sin c} \left[\int_{\sqrt{a^2-y^2}}^{\sqrt{b^2-y^2}} f(x, y)\, dx\right] dy + \int_{a \sin c}^{b \sin c} \left[\int_{y \cot c}^{\sqrt{b^2-y^2}} f(x, y)\, dx\right] dy.$$

Given that $0 < a < b$ and $0 < c < \pi/2$, sketch the region S, giving the equations of all curves which form its boundary.

21. When a double integral was set up for the volume V of the solid under the surface $z = f(x, y)$ and above a region S of the xy-plane, the following sum of iterated integrals was obtained:

$$V = \int_1^2 \left[\int_x^{x^3} f(x, y)\, dy\right] dx + \int_2^8 \left[\int_x^8 f(x, y)\, dy\right] dx.$$

(a) Sketch the region S and express V as an iterated integral in which the order of integration is reversed.

(b) Carry out the integration and compute V when $f(x, y) = e^x (x/y)^{1/2}$.

22. Let $A = \int_0^1 e^{-t^2}\, dt$ and $B = \int_0^{1/2} e^{-t^2}\, dt$. Evaluate the iterated integral

$$I = 2\int_{-1/2}^{1} \left[\int_0^x e^{-y^2}\, dy\right] dx$$

in terms of A and B. These are positive integers m and n such that

$$I = mA - nB + e^{-1} - e^{-1/4}.$$

Use this fact to check your answer.

23. A solid cone is obtained by connecting every point of a plane region S with a vertex not in the plane of S. Let A denote the area of S, and let h denote the altitude of the cone. Prove that:
(a) The cross-sectional area cut by a plane parallel to the base and at a distance t from the vertex is $(t/h)^2A$, if $0 \le t \le h$.
(b) The volume of the cone is $\frac{1}{3}Ah$.
24. Reverse the order of integration to derive the formula

$$\int_0^a\left[\int_0^y e^{m(a-x)} f(x)\,dx\right] dy = \int_0^a (a - x)e^{m(a-x)} f(x)\,dx,$$

where a and m are constants, $a > 0$.

11.16 Further applications of double integrals

We have already seen that double integrals can be used to compute volumes of solids and areas of plane regions. Many other concepts such as mass, center of mass, and moment of inertia can be defined and computed with the aid of double integrals. This section contains a brief discussion of these topics. They are of special importance in physics and engineering.

Let P denote the vector from the origin to an arbitrary point in 3-space. If n positive masses $m_1, m_2, \ldots, m_n$ are located at points $P_1, P_2, \ldots, P_n$, respectively, the *center of mass* of the system is defined to be the point C determined by the vector equation

$$C = \frac{\sum_{k=1}^n m_k P_k}{\sum_{k=1}^n m_k}.$$

The denominator, $\sum m_k$, is called the *total mass* of the system.

If each mass m_k is translated by a given vector A to a new point Q_k where $Q_k = P_k + A$, the center of mass is also translated by A, since we have

$$\frac{\sum m_k Q_k}{\sum m_k} = \frac{\sum m_k(P_k + A)}{\sum m_k} = \frac{\sum m_k P_k}{\sum m_k} + A = C + A.$$

This may also be described by saying that the location of the center of mass depends only on the points $P_1, P_2, \ldots, P_n$ and the masses, and not on the origin. The center of mass is a theoretically computed quantity which represents, so to speak, a fictitious "balance point" of the system.

If the masses lie in a plane at points with coordinates $(x_1, y_1), \ldots, (x_n, y_n)$, and if the center of mass has coordinates $(\bar{x}, \bar{y})$, the vector relation which defines C can be expressed as two scalar equations

$$\bar{x} = \frac{\sum m_k x_k}{\sum m_k} \quad \text{and} \quad \bar{y} = \frac{\sum m_k y_k}{\sum m_k}.$$

In the numerator of the quotient defining $\bar{x}$, the kth term of the sum, $m_k x_k$, is called the *moment* of the mass m_k about the y-axis. If a mass m equal to the total mass of the system were placed at the center of mass, its moment about the y-axis would be equal to the moment of the system,

$$m\bar{x} = \sum_{k=1}^n m_k x_k.$$

When we deal with a system whose total mass is distributed throughout some region in the plane rather than at a finite number of discrete points, the concepts of mass, center of mass, and moment are defined by means of integrals rather than sums. For example, consider a thin plate having the shape of a plane region S. Assume that matter is distributed over this plate with a known density (mass per unit area). By this we mean that there is a nonnegative function f defined on S and that $f(x, y)$ represents the mass per unit area at the point (x, y). If the plate is made of a homogeneous material the density is constant. In this case the total mass of the plate is defined to be the product of the density and the area of the plate.

When the density varies from point to point we use the double integral of the density as the definition of the total mass. In other words, if the density function f is integrable over S, we define the total mass m(S) of the plate by the equation

$$m(S) = \iint_S f(x, y)\, dx\, dy .$$

The quotient

$$\frac{\text{mass}}{\text{area}} = \frac{\iint_S f(x, y)\, dx\, dy}{\iint_S dx\, dy}$$

is called the ***average density*** of the plate. If S is thought of as a geometric configuration rather than as a thin plate, this quotient is called the ***average*** or ***mean value*** of the function f over the region S. In this case we do not require f to be nonnegative.

By analogy with the finite case, we define the ***center of mass*** of the plate to be the point $(\bar{x}, \bar{y})$ determined by the equations

$$(11.14) \qquad \bar{x}m(S) = \iint_S xf(x, y)\, dx\, dy \quad \text{and} \quad \bar{y}m(S) = \iint_S yf(x, y)\, dx\, dy .$$

The integrals on the right are called the moments of the plate about the y-axis and the x-axis, respectively. When the density is constant, say $f(x, y) = c$, a factor c cancels in each of Equations (11.14) and we obtain

$$\bar{x}a(S) = \iint_S x\, dx\, dy \quad \text{and} \quad \bar{y}a(S) = \iint_S y\, dx\, dy ,$$

where $a(S)$ is the area of S. In this case the point $(\bar{x}, \bar{y})$ is called the ***centroid*** of the plate (or of the region S).

If L is a line in the plane of the plate, let $\delta(x, y)$ denote the perpendicular distance from a point (x, y) in S to the line L. Then the number I_L defined by the equation

$$I_L = \iint_S \delta^2(x, y) f(x, y)\, dx\, dy$$

is called the ***moment of inertia*** of the plate about L. When $f(x, y) = 1$, I_L is called the moment of inertia or ***second moment*** of the region S about L. The moments of inertia about the x- and y-axes are denoted by I_x and I_y, respectively, and they are given by the integrals

$$I_x = \iint_S y^2 f(x, y)\, dx\, dy \qquad \text{and} \qquad I_y = \iint_S x^2 f(x, y)\, dx\, dy .$$

The sum of these two integrals is called the ***polar moment of inertia*** I_0, about the origin :

$$I_0 = I_x + I_y = \iint_S (x^2 + y^2) f(x, y)\, dx\, dy .$$

Note: The mass and center of mass of a plate are properties of the body and are independent of the location of the origin and of the directions chosen for the coordinates axes. The polar moment of inertia depends on the location of the origin but not on the directions chosen for the axes. The moments and the moments of inertia about the x- and y-axes depend on the location of the origin and on the directions chosen for the axes. If a plate of constant density has an axis of symmetry, its centroid will lie on this axis. If there are two axes of symmetry, the centroid will lie on their intersection. These facts, which can be proven from the foregoing definitions, often help to simplify calculations involving center of mass and moment of inertia.

EXAMPLE 1. A thin plate of constant density c is bounded by two concentric circles with radii a and b and center at the origin, where $0 < b < a$. Compute the polar moment of inertia.

Solution. The integral for I_0 is

$$I_0 = c \iint_S (x^2 + y^2)\, dx\, dy ,$$

where $S = \{(x, y) \mid b^2 \le x^2 + y^2 \le a^2\}$. To simplify the computations we write the integral as a difference of two integrals,

$$I_0 = c \iint_{S(a)} (x^2 + y^2)\, dx\, dy - c \iint_{S(b)} (x^2 + y^2)\, dx\, dy ,$$

where $S(a)$ and $S(b)$ are circular disks with radii a and b, respectively. We can use iterated integration to evaluate the integral over $S(a)$, and we find

$$\iint_{S(a)} (x^2 + y^2)\, dx\, dy = 4 \int_0^a \left[\int_0^{\sqrt{a^2 - x^2}} (x^2 + y^2)\, dy \right] dx = \frac{\pi a^4}{2} .$$

(We have omitted the details of the computation because this integral can be evaluated more easily with the use of polar coordinates, to be discussed in Section 11.27.) Therefore

$$I_0 = \frac{\pi c}{2} (a^4 - b^4) = \pi c(a^2 - b^2) \frac{(a^2 + b^2)}{2} = m \frac{a^2 + b^2}{2} ,$$

where $m = \pi c(a^2 - b^2)$, the mass of the plate.

EXAMPLE 2. Determine the centroid of the plane region bounded by one arch of a sine curve.

Solution. We take the region S bounded by the curve $y = \sin x$ and the interval $0 \le x \le \pi$. By symmetry, the x-coordinate of the centroid is $\bar{x} = \pi/2$. The y-coordinate, $\bar{y}$, is given by

$$\bar{y} = \frac{\iint_S y\,dx\,dy}{\iint_S dx\,dy} = \frac{\int_0^\pi \left[\int_0^{\sin x} y\,dy\right] dx}{\int_0^\pi \sin x\,dx} = \frac{\int_0^\pi \frac{1}{2}\sin^2 x\,dx}{2} = \frac{\pi}{8}.$$

11.17 Two theorems of Pappus

Pappus of Alexandria, who lived around 300 A.D., was one of the last geometers of the Alexandrian school of Greek mathematics. He wrote a compendium of eight books summarizing much of the mathematical knowledge of his time. (The last six and a part of the second are extant.) Pappus discovered a number of interesting properties of centroids, two of which are described in this section. The first relates the centroid of a plane region with the volume of the solid of revolution obtained by rotating the region about a line in its plane.

Consider a plane region Q lying between the graphs of two continuous functions f and g over an interval $[a, b]$, where $0 \le g \le f$. Let S be the solid of revolution generated by rotating Q about the x-axis. Let $a(Q)$ denote the area of Q, $v(S)$ the volume of S, and $\bar{y}$ the y-coordinate of the centroid of Q. As Q is rotated to generate S, the centroid travels along a circle of radius $\bar{y}$. Pappus' theorem states that ***the volume of S is equal to the circumference of this circle multiplied by the area of*** Q; that is,

$$v(S) = 2\pi\bar{y}a(Q) \tag{11.15}$$

To prove this formula we simply note that the volume is given by the integral

$$v(S) = \pi \int_a^b [f^2(x) - g^2(x)]\,dx$$

and that $\bar{y}$ is given by the formula

$$\bar{y}a(Q) = \iint_Q y\,dy\,dx = \int_a^b \left[\int_{g(x)}^{f(x)} y\,dy\right] dx = \int_a^b \tfrac{1}{2}[f^2(x) - g^2(x)]\,dx.$$

Comparing these two formulas we immediately obtain (11.15).

EXAMPLE 1. ***Volume of a torus.*** Let S be the torus generated by rotating a circular disk Q of radius R about an axis at a distance $b > R$ from the center of Q. The volume of S is easily calculated by Pappus' theorem. We have $\bar{y} = b$ and $a(Q) = \pi R^2$, so

$$v(S) = 2\pi\bar{y}a(Q) = 2\pi^2R^2b.$$

The next example shows that Pappus' theorem can also be used to calculate centroids.

EXAMPLE 2. *Centroid of a semicircular disk.* Let $\bar{y}$ denote the y-coordinate of the centroid of the semicircular disk

$$Q = \{(x, y) \mid x^2 + y^2 \le R^2, \quad y \ge 0\}.$$

The area of Q is $\frac{1}{2}\pi R^2$. When Q is rotated about the x-axis it generates a solid sphere of volume $\frac{4}{3}\pi R^3$. By Pappus' formula we have

$$\tfrac{4}{3}\pi R^3 = 2\pi\bar{y}(\tfrac{1}{2}\pi R^2),$$

so $\bar{y} = \dfrac{4R}{3\pi}$.

The next theorem of Pappus states that *the centroid of the union of two disjoint plane regions A and B lies on the line segment joining the centroid of A and the centroid of B.* More generally, let A and B be two thin plates that are either disjoint or intersect in a set of content zero. Let $m(A)$ and $m(B)$ denote their masses and let C_A and C_B denote vectors from an origin to their respective centers of mass. Then the union $A \cup B$ has mass $m(A) + m(B)$ and its center of mass is determined by the vector C, where

(11.16) $$C = \frac{m(A)C_A + m(B)C_B}{m(A) + m(B)}.$$

The quotient for C is a linear combination of the form $aC_A + bC_B$, where a and b are nonnegative scalars with sum 1. A linear combination of this form is called a *convex combination* of C_A and C_B. The endpoint of C lies on the line segment joining the endpoints of C_A and C_B.

Pappus' formula (11.16) follows at once from the definition of center of mass given in (11.14). The proof is left as an exercise for the reader. The theorem can be extended in an obvious way to the union of three or more regions. It is especially useful in practice when a plate of constant density is made up of several pieces, each of which has geometric symmetry. We determine the centroid of each piece and then form a suitable convex combination to find the centroid of the union. Examples are given in Exercise 21 of the next section.

11.18 Exercises

In Exercises 1 through 8 a region S is bounded by one or more curves described by the given equations. In each case sketch the region S and determine the coordinates $\bar{x}$ and $\bar{y}$ of the centroid.

1. $y = x^2, \quad x + y = 2.$
2. $y^2 = x + 3, \quad y^2 = 5 - x.$
3. $x - 2y + 8 = 0, \quad x + 3y + 5 = 0, \quad x = -2, \quad x = 4.$
4. $y = \sin^2 x, \quad y = 0, \quad 0 \le x \le \pi.$
5. $y = \sin x, \quad y = \cos x, \quad 0 \le x \le \frac{\pi}{4}.$
6. $y = \log x, \quad y \quad 0, \quad 1 \le x \le a.$
7. $\sqrt{x} + \sqrt{y} = 1, = \quad x = 0, \quad y = 0.$
8. $x^{2/3} + y^{2/3} = 1, \quad x = 0, \; y = 0,$ in first quadrant.

9. A thin plate is bounded by an arc of the parabola $y = 2x - x^2$ and the interval $0 \le x \le 2$. Determine its mass if the density at each point (x, y) is $(1 - y)/(1 + x)$.
10. Find the center of mass of a thin plate in the shape of a rectangle $ABCD$ if the density at any point is the product of the distances of the point from two adjacent sides AB and AD.

In Exercises 11 through 16, compute the moments of inertia I_x and I_y of a thin plate S in the xy-plane bounded by the one or more curves described by the given equations. In each case $f(x, y)$ denotes the density at an arbitrary point (x, y) of S.

11. $y = \sin^2 x, \quad y = -\sin^2 x, \quad -\pi \le x \le \pi; \quad f(x, y) = 1.$
12. $\frac{x}{a} + \frac{y}{b} = 1, \quad \frac{x}{c} + \frac{y}{b} = 1, \quad y = 0, \quad 0 < c < a, \quad b > 0; \quad f(x, y) = 1.$
13. $(x - r)^2 + (y - r)^2 = r^2, \quad x = 0, \quad y = 0, \quad 0 \le x \le r, \quad 0 \le y \le r; \quad f(x, y) = 1.$
14. $xy = 1, \quad xy = 2, \quad x = 2y, \quad y = 2x, \quad x > 0, \quad y > 0; \quad f(x, y) = 1.$
15. $y = e^x, \quad y = 0, \quad 0 \le x \le a; \quad f(x, y) = xy.$
16. $y = \sqrt{2x}, \quad y = 0, \quad 0 \le x \le 2; \quad f(x, y) = |x - y|.$
17. Let S be a thin plate of mass m, and let L_0 and L be two parallel lines in the plane of S, where L_0 passes through the center of mass of S. Prove *the parallel-axis theorem:*

$$I_L = I_{L_0} + mh^2,$$

where h is the perpendicular distance between the two lines L and L_0. *[Hint:* A careful choice of coordinates axes will simplify the work.]
18. The boundary of a thin plate is an ellipse with semiaxes a and b. Let L denote a line in the plane of the plate passing through the center of the ellipse and making an angle a with the axis of length $2a$. If the density is constant and if the mass of the plate is m, show that the moment of inertia I_L is equal to $\frac{1}{4}m(a^2 \sin^2 \alpha + b^2 \cos^2 \alpha)$.
19. Find the average distance from one comer of a square of side h to points inside the square.
20. Let δ denote the distance from an arbitrary point P inside a circle of radius r to a fixed point P_0 whose distance from the center of the circle is h. Find the average of the function δ^2 over the region enclosed by the circle.
21. Let A, B, C denote the following rectangles in the xy-plane:

$$A = [0, 4] \times [0, 1], \qquad B = [2, 3] \times [1, 3], \qquad C = [2, 4] \times [3, 4].$$

Use a theorem of Pappus to determine the centroid of each of the following figures:
(a) $A \cup B$. (c) $B \cup C$.
(b) $A \cup C$. (d) $A \cup B \cup C$.
22. An isosceles triangle T has base 1 and altitude h. The base of T coincides with one edge of a rectangle R of base 1 and altitude 2. Find a value of h so that the centroid of $R \cup T$ will lie on the edge common to R and T.
23. An isosceles triangle T has base $2r$ and altitude h. The base of T coincides with one edge of a semicircular disk D of radius r. Determine the relation that must hold between r and h so that the centroid of $T \cup D$ will lie inside the triangle.

11.19 Green's theorem in the plane

The second fundamental theorem of calculus for line integrals states that the line integral of a gradient ∇f along a path joining two points a and b may be expressed in terms of the function values f (a) and $f(b)$. There is a two-dimensional analog of the second fundamental theorem which expresses a double integral over a plane region R as a line integral taken along a closed curve forming the boundary of R. This theorem is usually referred to as ***Green's theorem.*** † It can be stated in several ways; the most common is in the form of the identity :

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy = \oint_C P\, dx + Q\, dy. \tag{11.17}$$

The curve C which appears on the right is the boundary of the region R, and the integration symbol $\oint$ indicates that the curve is to be traversed in the counterclockwise direction, as suggested by the example shown in Figure 11.13.

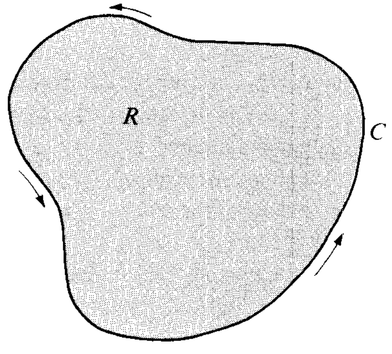

FIGURE 11.13 The curve C is the boundary of R, traversed in a counterclockwise direction.

Two types of assumptions are required for the validity of this identity. First, conditions are imposed on the functions P and Q to ensure the existence of the integrals. The usual assumptions are that P and Q are continuously differentiable on an open set S containing the region R. This implies continuity of P and Q on C as well as continuity of $\partial P/\partial y$ and $\partial Q/\partial x$ on R, although the theorem is also valid under less stringent hypotheses. Second, there are conditions of a geometric nature that are imposed on the region R and its boundary curve C. The curve C may be any ***rectifiable simple closed curve.*** The term "rectifiable" means, of course, that C has a finite arc length. To explain what is meant by a simple closed curve, we refer to the vector-valued function which describes the curve.

Suppose C is described by a continuous vector-valued function α defined on an interval $[a, b]$. If $\alpha(a) = \alpha(b)$, the curve is ***closed.*** A closed curve such that $\alpha(t_1) \neq \alpha(t_2)$ for every pair of values $t_1 \neq t_2$ in the half-open interval $(a, b]$ is called a ***simple*** closed curve. This means that, except for the end points of the interval $[a, b]$, distinct values of t lead to distinct points on the curve. A circle is the prototype of a simple closed curve.

Simple closed curves that lie in a plane are usually called ***Jordan curves*** in honor of Camille Jordan (1838–1922), a famous French mathematician who did much of the pioneering work on such concepts as simple closed curves and arc length. Every Jordan curve C

† In honor of George Green (1793–1841), an English mathematician who wrote on the applications of mathematics to electricity and magnetism, fluid flow, and the reflection and refraction of light and sound. The theorem which bears Green's name appeared earlier in the researches of Gauss and Lagrange.

decomposes the plane into two disjoint open connected sets having the curve C as their common boundary. One of these regions is *bounded* and is called the *interior* (or *inner region)* of C. (An example is the shaded region in Figure 11.13.) The other is unbounded and is called the *exterior* (or *outer region)* of C. For some familiar Jordan curves such as circles, ellipses, or elementary polygons, it is intuitively evident that the curve divides the plane into an inner and an outer region, but to prove that this is true for an *arbitrary* Jordan curve is not easy. Jordan was the first to point out that this statement requires proof; the result is now known as the *Jordan curve theorem.* Toward the end of the 19th century Jordan and others published incomplete proofs. In 1905 the American mathematician Oswald Veblen (1880-1960) gave the first complete proof of this theorem. Green's theorem is valid whenever C is a rectifiable Jordan curve, and the region R is the union of C and its interior.† Since we have not defined line integrals along arbitrary rectifiable curves, we restrict our discussion here to piecewise smooth curves.

There is another technical difficulty associated with the formulation of Green's theorem. We have already remarked that, for the validity of the identity in (11.17), the curve C must be traversed in the counterclockwise direction. Intuitively, this means that a man walking along the curve in this direction always has the region R to his left. Again, for some familiar Jordan curves, such as those mentioned earlier, the meaning of the expression "traversing a curve in the counterclockwise direction" is intuitively evident. However, in a strictly rigorous treatment of Green's theorem one would have to define this expression in completely analytic terms, that is, in terms of the vector-valued function $\boldsymbol{a}$ that describes the curve. One possible definition is outlined in Section 11.24.

Having pointed out some of the difficulties associated with the formulation of Green's theorem, we shall state the theorem in a rather general form and then indicate briefly why it is true for certain special regions. In this discussion the meaning of "counterclockwise" will be intuitive, so the treatment is not completely rigorous.

THEOREM 11.10. GREEN'S THEOREM FOR PLANE REGIONS BOUNDED BY PIECEWISE SMOOTH JORDAN CURVES. *Let P and Q be scalar fields that are continuously differentiable on an open set S in the xy-plane. Let C be a piecewise smooth Jordan curve, and let R denote the union of C and its interior. Assume R is a subset of S. Then we have the identity*

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy = \oint_C P\, dx + Q\, dy, \tag{11.18}$$

where the line integral is taken around C in the counterclockwise direction.

Note: The identity in (11.18) is equivalent to the *two* formulas

$$\iint_R \frac{\partial Q}{\partial x}\, dx\, dy = \oint_C Q\, dy \tag{11.19}$$

† A proof of Green's theorem for regions of this generality can be found in Chapter 10 of the author's *Mathematical Analysis.*

and

$$(11.20) \qquad -\iint_R \frac{\partial P}{\partial y}\,dx\,dy = \oint_C P\,dx\,.$$

In fact, if both of these are true, (11.18) follows by addition. Conversely, if (11.18) is true we may obtain (11.19) and (11.20) as special cases by taking $P = 0$ and $Q = 0$, respectively.

Proof for special regions. We shall prove (11.20) for a region R of Type I. Such a region has the form

$$R = \{(x, y) \mid a \le x \le b \ \text{ and } f(x) \le y \le g(x)\},$$

where f and g are continuous on $[a, b]$ with $f \le g$. The boundary C of R consists of four parts, a lower arc C_1 (the graph of f), an upper arc C_2 (the graph of g), and two vertical line segments, traversed in the directions indicated in Figure 11.14.

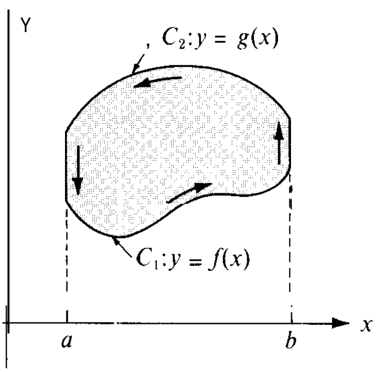

FIGURE 11.14 Proof of Green's theorem for a special region.

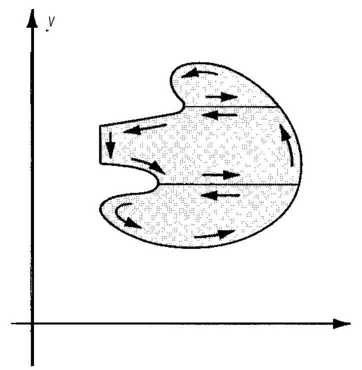

FIGURE 11.15 Proof of Green's theorem for a more general region.

First we evaluate the double integral $-\iint_R (\partial P/\partial y)\,dx\,dy$ by iterated integration. Integrating first with respect to y we have

$$(11.21) \qquad -\iint_R \frac{\partial P}{\partial y}\,dx\,dy = -\int_a^b \left[\int_{f(x)}^{g(x)} \frac{\partial P}{\partial y}\,dy\right] dx = \int_a^b \left[\int_{g(x)}^{f(x)} \frac{\partial P}{\partial y}\,dy\right] dx$$

$$= \int_a^b P[x, f(x)]\,dx - \int_a^b P[x, g(x)]\,dx\,.$$

On the other hand, the line integral $\int_C P\,dx$ can be written as follows:

$$\int_C P\,dx = \int_{C_1} P\,dx + \int_{C_2} P\,dx\,,$$

since the line integral along each vertical segment is zero. To evaluate the integral along

C_1 we use the vector representation $\boldsymbol{a}(t) = t\boldsymbol{i} + f(t)\boldsymbol{j}$ and obtain

$$\int_{C_1} P\,dx = \int_a^b P[t, f(t)]\,dt.$$

To evaluate the integral along C_2 we use the representation $\boldsymbol{a}(t) = t\boldsymbol{i} + g(t)\boldsymbol{j}$ and take into account the reversal in direction to obtain

$$\int_{C_2} P\,dx = -\int_a^b P[t, g(t)]\,dt.$$

Therefore we have

$$\int_C P\,dx = \int_a^b P[t, f(t)]\,dt - \int_a^b P[t, g(t)]\,dt.$$

Comparing this equation with the formula in (11.21) we obtain (11.20).

A similar argument can be used to prove (11.19) for regions of Type II. In this way a proof of Green's theorem is obtained for regions that are of both Type I and Type II. Once this is done, the theorem can be proved for those regions R that can be decomposed into a finite number of regions that are of both types. "Crosscuts" are introduced as shown in Figure 11.15, the theorem is applied to each subregion, and the results are added together. The line integrals along the crosscuts cancel in pairs, as suggested in the figure, and the sum of the line integrals along the boundaries of the subregions is equal to the line integral along the boundary of R.

11.20 Some applications of Green's theorem

The following examples illustrate some applications of Green's theorem.

EXAMPLE 1. Use Green's theorem to compute the work done by the force field $\boldsymbol{f}(x, y) = (y + 3x)\boldsymbol{i} + (2y - x)\boldsymbol{j}$ in moving a particle once around the ellipse $4x^2 + y^2 = 4$ in the counterclockwise direction.

Solution. The work is equal to $\int_C P\,dx + Q\,dy$, where $P = y + 3x$, $Q = 2y - x$, and C is the ellipse. Since $\partial Q/\partial x - \partial P/\partial y = -2$, Green's theorem gives us

$$\int_C P\,dx + Q\,dy = \iint_R (-2)\,dx\,dy = -2a(R),$$

where $a(R)$ is the area of the region enclosed by the ellipse. Since this ellipse has semiaxes $a = 1$ and $b = 2$, its area is $\pi ab = 2\pi$ and the value of the line integral is -4π.

EXAMPLE 2. Evaluate the line integral $\int_C (5 - xy - y^2)\,dx - (2xy - x^2)\,dy$, where C is the square with vertices (0, 0), (1, 0), (1, 1), (0, 1), traversed counterclockwise.

Solution. Here $P = 5 - xy - y^2$, $Q = x^2 - 2xy$, and $\partial Q/\partial x - \partial P/\partial y = 3x$. Hence, by Green's theorem, we have

$$\int_C P\,dx + Q\,dy = 3\iint_R x\,dx\,dy = 3\bar{x},$$