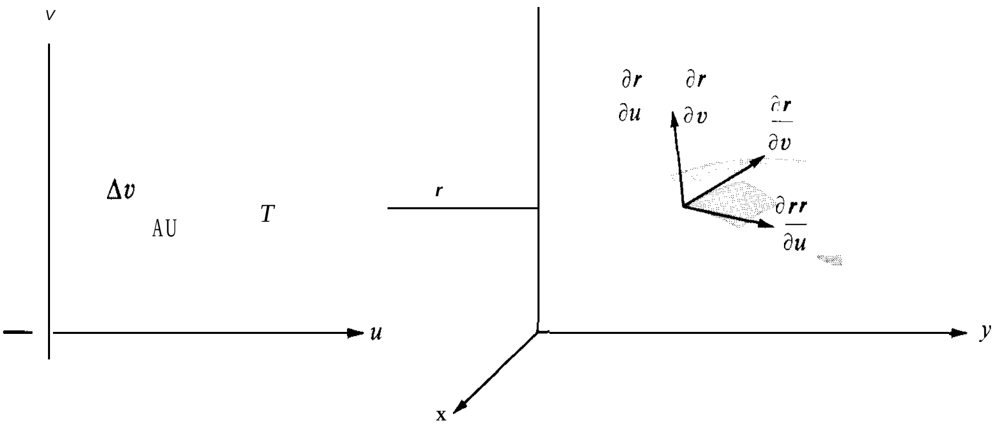

FIGURE 12.6 Geometric interpretation of the Vectors $\frac{\partial \mathbf{r}}{\partial u}$, $\frac{\partial \mathbf{r}}{\partial v}$, and $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$

If (u, v) is a point in T at which $\partial \mathbf{r}/\partial u$ and $\partial \mathbf{r}/\partial v$ are continuous and the fundamental vector product is nonzero, then the image point $\mathbf{r}(u, v)$ is called a *regular point* of $\mathbf{r}$. Points at which $\partial \mathbf{r}/\partial u$ or $\partial \mathbf{r}/\partial v$ fails to be continuous or $\partial \mathbf{r}/\partial u \times \partial \mathbf{r}/\partial v = \mathbf{0}$ are called *singular points* of r. A surface $\mathbf{r}(T)$ is called *smooth* if all its points are regular points. Every surface has more than one parametric representation. Some of the examples discussed below show that a point of a surface may be a regular point for one representation but a singular point for some other representation. The geometric significance of regular and singular points can be explained as follows:

Consider a horizontal line segment in T. Its image under $\mathbf{r}$ is a curve (called a u-curve) lying on the surface $\mathbf{r}(T)$. For fixed v, think of the parameter u as representing time. The vector $\partial \mathbf{r}/\partial u$ is the velocity vector of this curve. When u changes by an amount Au, a point originally at $\mathbf{r}(u, v)$ moves along a u-curve a distance approximately equal to $\|\partial \mathbf{r}/\partial u\|$ Au since $\|\partial \mathbf{r}/\partial u\|$ represents the speed along the u-curve. Similarly, for fixed u a point of a v-curve moves in time Δv a distance nearly equal to $\|\partial \mathbf{r}/\partial v\|\ \Delta v$. A rectangle in T having area Au Δv is traced onto a portion of $\mathbf{r}(T)$ which we shall approximate by the parallelogram determined by the vectors $(\partial \mathbf{r}/\partial u)$ Au and $(\partial \mathbf{r}/\partial v)\ \Delta v$. (See Figure 12.6.) The area of the parallelogram spanned by $(\partial \mathbf{r}/\partial u)$ Au and $(\partial \mathbf{r}/\partial v)\ \Delta v$ is the magnitude of their cross product,

$$\left\| \frac{\partial \mathbf{r}}{\partial u}\Delta u \times \frac{\partial \mathbf{r}}{\partial v}\Delta v \right\| = \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| \Delta u\, \Delta v .$$

Therefore the length of the fundamental vector product may be thought of as a local magnification factor for areas. At the points at which this vector product is zero the parallelogram collapses to a curve or point, and degeneracies occur. At each regular point the vectors $\partial \mathbf{r}/\partial u$ and $\partial \mathbf{r}/\partial v$ determine a plane having the vector $\partial \mathbf{r}/\partial u \times \partial \mathbf{r}/\partial v$ as a normal. In the next section we shall prove that $\partial \mathbf{r}/\partial u \times \partial \mathbf{r}/\partial v$ is normal to every smooth curve on the surface; for this reason the plane determined by $\partial \mathbf{r}/\partial u$ and $\partial \mathbf{r}/\partial v$ is called the *tangent plane* of the surface. Continuity of $\partial \mathbf{r}/\partial u$ and $\partial \mathbf{r}/\partial v$ implies continuity of $\partial \mathbf{r}/\partial u \times \partial \mathbf{r}/\partial v$; this, in turn, means that the tangent plane varies continuously on a smooth surface.

Thus we see that continuity of $\partial \boldsymbol{r}/\partial u$ and $\partial \boldsymbol{r}/\partial v$ prevents the occurrence of sharp edges or corners on the surface; the nonvanishing of $\partial \boldsymbol{r}/\partial u \times \partial \boldsymbol{r}/\partial v$ prevents degeneracies.

EXAMPLE 1. ***Surfaces with an explicit representation,*** $z = f(x, y)$. For a surface with an explicit representation of the form $z = f(x, y)$, we can use x and y as the parameters, which gives us the vector equation

$$\boldsymbol{r}(x, y) = x\boldsymbol{i} + y\boldsymbol{j} + f(x, y)\boldsymbol{k}.$$

This representation always gives a simple parametric surface. The region T is called the projection of the surface on the xy-plane. (An example is shown in Figure 12.7, p. 425.) To compute the fundamental vector product we note that

$$\frac{\partial \boldsymbol{r}}{\partial x} = \boldsymbol{i} + \frac{\partial f}{\partial x}\boldsymbol{k} \quad \text{and} \quad \frac{\partial \boldsymbol{r}}{\partial y} = \boldsymbol{j} + \frac{\partial f}{\partial y}\boldsymbol{k},$$

if f is differentiable. This gives us

$$\frac{\partial \boldsymbol{r}}{\partial x} \times \frac{\partial \boldsymbol{r}}{\partial y} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 0 & \dfrac{\partial f}{\partial x} \\ 0 & 1 & \dfrac{\partial f}{\partial y} \end{vmatrix} = -\frac{\partial f}{\partial x}\boldsymbol{i} - \frac{\partial f}{\partial y}\boldsymbol{j} + \boldsymbol{k}. \tag{12.5}$$

Since the z-component of $\partial \boldsymbol{r}/\partial x \times \partial \boldsymbol{r}/\partial y$ is 1, the fundamental vector product is never zero. Therefore the only singular points that can occur for this representation are points at which at least one of the partial derivatives $\partial f/\partial x$ or $\partial f/\partial y$ fails to be continuous.

A specific case is the equation $z = \sqrt{1 - x^2 - y^2}$, which represents a hemisphere of radius 1 and center at the origin, if $x^2 + y^2 \leq 1$. The vector equation

$$\boldsymbol{r}(x, y) = x\boldsymbol{i} + y\boldsymbol{j} + \sqrt{1 - x^2 - y^2}\,\boldsymbol{k}$$

maps the unit disk $T = \{(x, y) \mid x^2 + y^2 \leq 1\}$ onto the hemisphere in a one-to-one fashion. The partial derivatives $\partial \boldsymbol{r}/\partial x$ and $\partial \boldsymbol{r}/\partial y$ exist and are continuous everywhere in the interior of this disk, but they do not exist on the boundary. Therefore every point on the equator is a singular point of this representation.

EXAMPLE 2. We consider the same hemisphere as in Example 1, but this time as the image of the rectangle $T = [0, 2\pi] \times [0, \frac{1}{2}\pi]$ under the mapping

$$\boldsymbol{r}(u, v) = a\cos u\cos v\,\boldsymbol{i} + a\sin u\cos v\,\boldsymbol{j} + a\sin v\,\boldsymbol{k}.$$

The vectors $\partial \boldsymbol{r}/\partial u$ and $\partial \boldsymbol{r}/\partial v$ are given by the formulas

$$\frac{\partial \boldsymbol{r}}{\partial u} = -a \sin u \cos v\, \boldsymbol{i} + a \cos u \cos v\, \boldsymbol{j},$$

$$\frac{\partial \boldsymbol{r}}{\partial v} = -a \cos u \sin v\, \boldsymbol{i} - a \sin u \sin v\, \boldsymbol{j} + a \cos v\, \boldsymbol{k}.$$

An easy calculation shows that their cross product is equal to

$$\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} = a \cos v\, \boldsymbol{r}(u, v).$$

The image of T is not a simple parametric surface because this mapping is not one-to-one on T. In fact, every point on the line segment $v = \frac{1}{2}\pi$, $0 \le u \le 2\pi$, is mapped onto the point $(0, 0, a)$ (the North Pole). Also, because of the periodicity of the sine and cosine, $\boldsymbol{r}$ takes the same values at the points $(0, v)$ and $(2\pi, v)$, so the right and left edges of T are mapped onto the same curve, a circular arc joining the North Pole to the point $(a, 0, 0)$ on the equator. (See Figure 12.3.) The vectors $\partial \boldsymbol{r}/\partial u$ and $\partial \boldsymbol{r}/\partial v$ are continuous everywhere in T. Since $\|\partial \boldsymbol{r}/\partial u \times \partial \boldsymbol{r}/\partial v\| = a^2 \cos v$, the only singular points of this representation occur when $\cos v = 0$. The North Pole is the only such point.

12.3 The fundamental vector product as a normal to the surface

Consider a smooth parametric surface $\boldsymbol{r}(T)$, and let C^* be a smooth curve in T. Then the image $C = \boldsymbol{r}(C^*)$ is a smooth curve lying on the surface. We shall prove that at each point of C the vector $\partial \boldsymbol{r}/\partial u \times \partial \boldsymbol{r}/\partial v$ is normal to C, as illustrated in Figure 12.6.

Suppose that C^* is described by a function $\boldsymbol{\alpha}$ defined on an interval $[a, b]$, say

$$\boldsymbol{\alpha}(t) = U(t)\boldsymbol{i} + V(t)\boldsymbol{j}.$$

Then the image curve C is described by the composite function

$$\boldsymbol{\rho}(t) = \boldsymbol{r}[\boldsymbol{\alpha}(t)] = X[\boldsymbol{\alpha}(t)]\boldsymbol{i} + Y[\boldsymbol{\alpha}(t)]\boldsymbol{j} + Z[\boldsymbol{\alpha}(t)]\boldsymbol{k}.$$

We wish to prove that the derivative $\boldsymbol{\rho}'(t)$ is perpendicular to the vector $\partial \boldsymbol{r}/\partial u \times \partial \boldsymbol{r}/\partial v$ when the partial derivatives $\partial \boldsymbol{r}/\partial u$ and $\partial \boldsymbol{r}/\partial v$ are evaluated at $(U(t), V(t))$. To compute $\boldsymbol{\rho}'(t)$ we differentiate each component of $\boldsymbol{\rho}(t)$ by the chain rule (Theorem 8.8) to obtain

$$\boldsymbol{\rho}'(t) = \nabla X \cdot \boldsymbol{\alpha}'(t)\boldsymbol{i} + \nabla Y \cdot \boldsymbol{\alpha}'(t)\boldsymbol{j} + \nabla Z \cdot \boldsymbol{\alpha}'(t)\boldsymbol{k}, \tag{12.6}$$

where the gradient vectors ∇X, ∇Y, and ∇Z are evaluated at $(U(t), V(t))$. Equation (12.6) can be rewritten as

$$\boldsymbol{\rho}'(t) = \frac{\partial \boldsymbol{r}}{\partial u} U'(t) + \frac{\partial \boldsymbol{r}}{\partial v} V'(t),$$

where the derivatives $\partial \boldsymbol{r}/\partial u$ and $\partial \boldsymbol{r}/\partial v$ are evaluated at $(U(t), V(t))$. Since $\partial \boldsymbol{r}/\partial u$ and $\partial \boldsymbol{r}/\partial v$ are each perpendicular to the cross product $\partial \boldsymbol{r}/\partial u \times \partial \boldsymbol{r}/\partial v$, the same is true of $\boldsymbol{\rho}'(t)$. This proves that $\partial \boldsymbol{r}/\partial u \times \partial \boldsymbol{r}/\partial v$ is normal to C, as asserted. For this reason, the vector product $\partial \boldsymbol{r}/\partial u \times \partial \boldsymbol{r}/\partial v$ is said to be ***normal*** to the surface $\boldsymbol{r}(T)$. At each regular point P of $\boldsymbol{r}(T)$ the vector $\partial \boldsymbol{r}/\partial u \times \partial \boldsymbol{r}/\partial v$ is nonzero; the plane through P having this vector as a normal is called the ***tangent plane*** to the surface at P.

12.4 Exercises

In Exercises 1 through 6, eliminate the parameters u and v to obtain a Cartesian equation, thus showing that the given vector equation represents a portion of the surface named. Also, compute the fundamental vector product $\partial \boldsymbol{r}/\partial u \times \partial \boldsymbol{r}/\partial v$ in terms of u and v.

1. *Plane:*
$\boldsymbol{r}(u, v) = (x_0 + a_1u + b_1v)\boldsymbol{i} + (y_0 + a_2u + b_2v)\boldsymbol{j} + (z_0 + a_3u + b_3v)\boldsymbol{k}$.
2. *Elliptic paraboloid:*
$\boldsymbol{r}(u, v) = au\cos v\,\boldsymbol{i} + bu\sin v\,\boldsymbol{j} + u^2\boldsymbol{k}$.
3. *Ellipsoid:*
$\boldsymbol{r}(u, v) = a\sin u\cos v\,\boldsymbol{i} + b\sin u\sin v\,\boldsymbol{j} + c\cos u\,\boldsymbol{k}$.
4. *Surface of revolution:*
$\boldsymbol{r}(u, v) = u\cos v\,\boldsymbol{i} + u\sin v\,\boldsymbol{j} + f(u)\boldsymbol{k}$.
5. *Cylinder:*
$\boldsymbol{r}(u, v) = u\boldsymbol{i} + a\sin v\,\boldsymbol{j} + a\cos v\,\boldsymbol{k}$.
6. *Torus:*
$\boldsymbol{r}(u, v) = (a + b\cos u)\sin v\,\boldsymbol{i} + (a + b\cos u)\cos v\,\boldsymbol{j} + b\sin u\,\boldsymbol{k}$, where $0 < b < a$. What are the geometric meanings of a and b?

In Exercises 7 through 10 compute the magnitude of the vector product $\partial \boldsymbol{r}/\partial u \times \partial \boldsymbol{r}/\partial v$.

7. $\boldsymbol{r}(u, v) = a\sin u\cosh v\,\boldsymbol{i} + b\cos u\cosh v\,\boldsymbol{j} + c\sinh v\,\boldsymbol{k}$.
8. $\boldsymbol{r}(u, v) = (u + v)\boldsymbol{i} + (u - v)\boldsymbol{j} + 4v^2\boldsymbol{k}$.
9. $\boldsymbol{r}(u, v) = (u + v)\boldsymbol{i} + (u^2 + v^2)\boldsymbol{j} + (u^3 + v^3)\boldsymbol{k}$.
10. $\boldsymbol{r}(u, v) = u\cos v\,\boldsymbol{i} + u\sin v\,\boldsymbol{j} + \frac{1}{2}u^2\sin 2v\,\boldsymbol{k}$.

12.5 Area of a parametric surface

Let $S = \boldsymbol{r}(T)$ be a parametric surface described by a vector-valued function $\boldsymbol{r}$ defined on a region T in the uv-plane. In Section 12.2 we found that the length of the fundamental vector product $\partial \boldsymbol{r}/\partial u \times \partial \boldsymbol{r}/\partial v$ could be interpreted as a local magnification factor for areas. (See Figure 12.6.) A rectangle in T of area $\Delta u\,\Delta v$ is mapped by $\boldsymbol{r}$ onto a curvilinear parallelogram on S with area nearly equal to

$$\left\| \frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} \right\| \Delta u\,\Delta v .$$

This observation suggests the following definition.

DEFINITION OF AREA OF A PARAMETRIC SURFACE. ***The area of S, denoted by a(S), is defined by the double integral***

$$a(S) = \iint_T \left\| \frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} \right\| du\,dv . \tag{12.7}$$

In other words, to determine the area of S we first compute the fundamental vector product $\partial \boldsymbol{r}/\partial u \times \partial \boldsymbol{r}/\partial v$ and then integrate its length over the region T. When $\partial \boldsymbol{r}/\partial u \times \partial \boldsymbol{r}/\partial v$ is expressed in terms of its components, by means of Equation (12.4), we have

$$(12.8) \qquad a(S) = \iint_T \sqrt{\left(\frac{\partial(Y, Z)}{\partial(u, v)}\right)^2 + \left(\frac{\partial(Z, X)}{\partial(u, v)}\right)^2 + \left(\frac{\partial(X, Y)}{\partial(u, v)}\right)^2}\, du\, dv .$$

Written in this form, the integral for surface area resembles the integral for computing the arc length of a curve.†

If S is given explicitly by an equation of the form $z = f(x, y)$ we may use x and y as the parameters. The fundamental vector product is given by Equation (12.5), so we have

$$\left\| \frac{\partial \boldsymbol{r}}{\partial x} \times \frac{\partial \boldsymbol{r}}{\partial y} \right\| = \left\| -\frac{\partial f}{\partial x} \boldsymbol{i} - \frac{\partial f}{\partial y} \boldsymbol{j} + \boldsymbol{k} \right\| = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} .$$

In this case the integral for surface area becomes

$$(12.9) \qquad a(S) = \iint_T \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dx\, dy ,$$

where the region T is now the projection of S on the xy-plane, as illustrated in Figure 12.7.

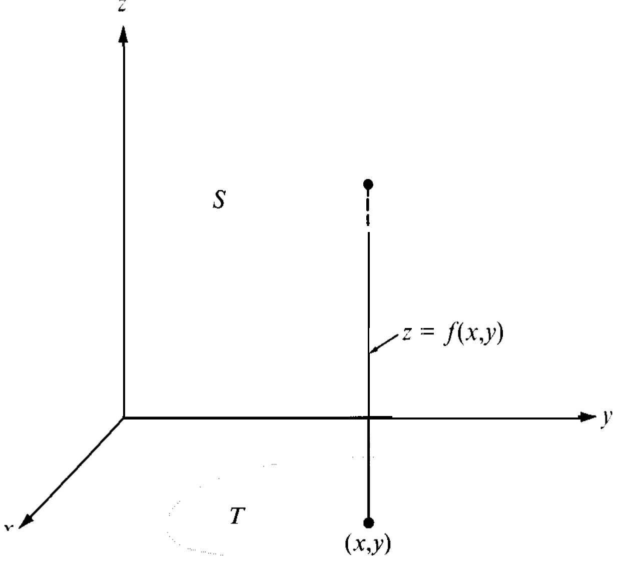

FIGURE 12.7 A surface S with an explicit representation, $z = f(x, y)$. The region T is the projection of S on the xy-plane.

† Since the integral in (12.7) involves $\boldsymbol{r}$, the area of a surface will depend on the function used to describe the surface. When we discuss surface integrals we shall prove (in Section 12.8) that under certain general conditions the area is independent of the parametric representation. The result is analogous to Theorem 10.1, in which we discussed the invariance of line integrals under a change of parameter.

When S lies in a plane parallel to the xy-plane, the function f is constant, so $\partial f/\partial x = \partial f/\partial y = 0$, and Equation (12.9) becomes

$$a(S) = \iint_T dx\,dy$$

This agrees with the usual formula for areas of plane regions.

Equation (12.9) can be written in another form that gives further insight into its geometric significance. At each point of S, let γ denote the angle between the normal vector $N = \partial \boldsymbol{r}/\partial x \times \partial \boldsymbol{r}/\partial y$ and the unit coordinate vector $\boldsymbol{k}$. (See Figure 12.8.) Since the z-component of N is 1, we have

$$\cos\gamma = \frac{\boldsymbol{N}\cdot\boldsymbol{k}}{\|\boldsymbol{N}\|\,\|\boldsymbol{k}\|} = \frac{1}{\|\boldsymbol{N}\|} = \frac{1}{\left\|\dfrac{\partial \boldsymbol{r}}{\partial x} \times \dfrac{\partial \boldsymbol{r}}{\partial y}\right\|},$$

and hence $\|\partial \boldsymbol{r}/\partial x \times \partial \boldsymbol{r}/\partial y\| = 1/\cos\gamma$. Therefore Equation (12.9) becomes

(12.10) $$a(S) = \iint_T \frac{1}{\cos\gamma}\,dx\,dy.$$

Suppose now that S lies in a plane not perpendicular to the xy-plane. Then γ is constant and Equation (12.10) states that the area of S = (area of T)/$\cos\gamma$, or that

(12.11) $$a(T) = a(S)\cos\gamma.$$

Equation (12.11) is sometimes referred to as the ***area cosine principle.*** It tells us that if a region S in one plane is projected onto a region T in another plane, making an angle γ with the first plane, the area of T is $\cos\gamma$ times that of S. This formula is obviously true

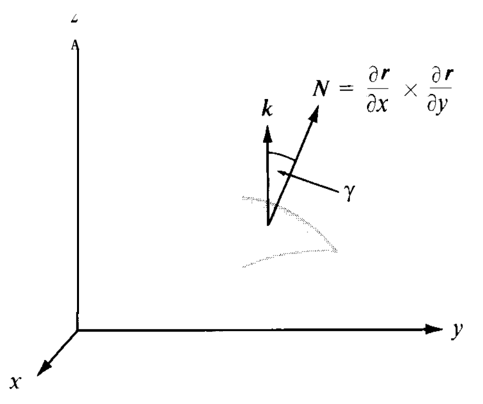

FIGURE 12.8 The length of $\dfrac{\partial \boldsymbol{r}}{\partial x} \times \dfrac{\partial \boldsymbol{r}}{\partial y}$ is $1/\cos\gamma$.

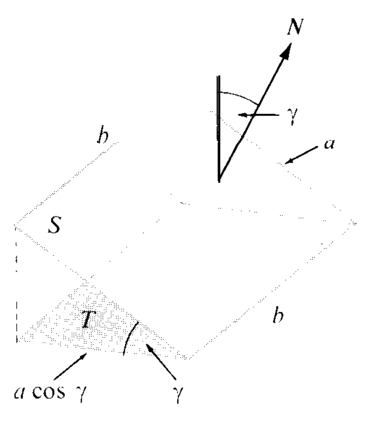

FIGURE 12.9 The area cosine principle for a rectangle.

when S is the rectangle shown in Figure 12.9, because distances in one direction are shortened by the factor $\cos \gamma$ while those in a perpendicular direction are unaltered by projection. Equation (12.11) extends this property to any plane region S having an area.

Suppose now that S is given by an implicit representation $F(x, y, z) = 0$. If S can be projected in a one-to-one fashion on the xy-plane, the equation $F(x, y, z) = 0$ defines z as a function of x and y, say $z = f(x, y)$, and the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ are related to those of F by the equations

$$\frac{\partial f}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z} \quad \text{and} \quad \frac{\partial f}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z}$$

for those points at which $\partial F/\partial z \neq 0$. Substituting these quotients in (12.9), we find

$$a(S) = \iint_T \frac{\sqrt{(\partial F/\partial x)^2 + (\partial F/\partial y)^2 + (\partial F/\partial z)^2}}{|\partial F/\partial z|}\, dx\, dy \tag{12.12}$$

EXAMPLE 1. ***Area of a hemisphere.*** Consider a hemisphere S of radius a and center at the origin. We have at our disposal the implicit representation $x^2 + y^2 + z^2 = a^2$, $z \geq 0$; the explicit representation $z = \sqrt{a^2 - x^2 - y^2}$; and the parametric representation

$$\mathbf{r}(u, v) = a \cos u \cos v\, \mathbf{i} + a \sin u \cos v\, \mathbf{j} + a \sin v\, \mathbf{k}. \tag{12.13}$$

To compute the area of S from the implicit representation we refer to Equation (12.12) with

$$F(x, y, z) = x^2 + y^2 + z^2 - a^2.$$

The partial derivatives of F are $\partial F/\partial x = 2x$, $\partial F/\partial y = 2y$, $\partial F/\partial z = 2z$. The hemisphere S projects in a one-to-one fashion onto the circular disk $D = \{(x, y) \mid x^2 + y^2 \leq a^2\}$ in the xy-plane. We cannot apply Equation (12.12) directly because the partial derivative $\partial F/\partial z$ is zero on the boundary of D. However, the derivative $\partial F/\partial z$ is nonzero everywhere in the interior of D, so we can consider the smaller concentric disk $D(R)$ of radius R, where $R < a$. If $S(R)$ denotes the corresponding portion of the upper hemisphere, Equation (12.12) is now applicable and we find

$$\begin{aligned} \text{area of } S(R) &= \iint_{D(R)} \frac{\sqrt{(2x)^2 + (2y)^2 + (2z)^2}}{|2z|}\, dx\, dy \\ &= \iint_{D(R)} \frac{a}{z}\, dx\, dy = a \iint_{D(R)} \frac{1}{\sqrt{a^2 - x^2 - y^2}}\, dx\, dy. \end{aligned}$$

The last integral can be easily evaluated by the use of polar coordinates, giving us

$$\text{area of } S(R) = a \int_0^{2\pi} \left[\int_0^R \frac{1}{\sqrt{a^2 - r^2}}\, r\, dr \right] d\theta = 2\pi a(a - \sqrt{a^2 - R^2}).$$

When $R \to a$ this approaches the limit $2\pi a^2$.

We can avoid the limiting process in the foregoing calculation by using the parametric representation in (12.13). The calculations of Example 2 in Section 12.2 show that

$$\left\| \frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} \right\| = \|a \cos v\, \boldsymbol{r}(u, v)\| = a^2 \, |\cos v| .$$

Therefore we can apply Equation (12.7), taking for the region T the rectangle $[0, 2\pi] \times [0, \frac{1}{2}\pi]$. We find

$$a(S) = a^2 \iint_T |\cos v| \, du \, dv = a^2 \int_0^{2\pi} \left[\int_0^{\pi/2} \cos v \, du \right] du = 2\pi a^2 .$$

EXAMPLE 2. ***Another theorem of Pappus.*** One of the theorems of Pappus states that a surface of revolution, obtained by rotating a plane curve of length L about an axis in the plane of the curve, has area $2\pi L h$, where h is the distance from the centroid of the curve to the axis of rotation. We shall use Equation (12.7) to prove this theorem.

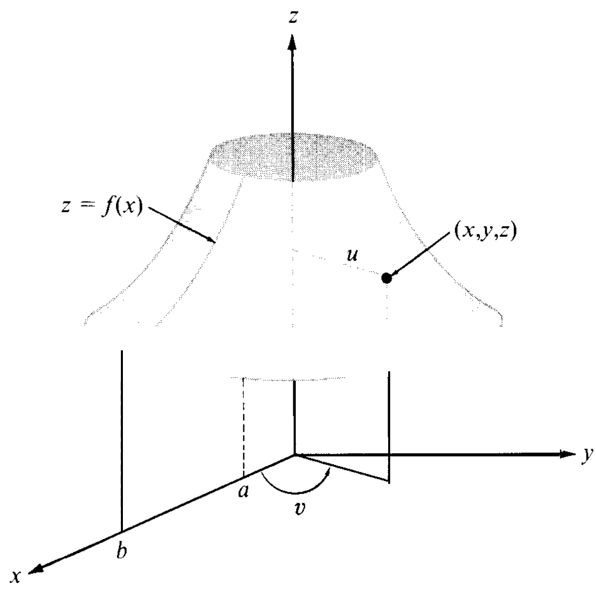

FIGURE 12.10 Area of a surface of revolution determined by Pappus' theorem.

Suppose a curve C, initially in the xz-plane, is rotated about the z-axis. Let its equation in the xz-plane be $z = f(x)$, where $a \le x \le b$, $a \ge 0$. The surface of revolution S so generated can be described by the vector equation

$$\boldsymbol{r}(u, v) = u \cos v\, \boldsymbol{i} + u \sin v\, \boldsymbol{j} + f(u)\boldsymbol{k},$$

where $(u, v) \in [a, b] \times [0, 2\pi]$. The parameters u and v can be interpreted as the radius and angle of polar coordinates, as illustrated in Figure 12.10. If $a \le u \le b$, all points (x, y, z) at a given distance u from the z-axis have the same z-coordinate, f(u), so they all lie on the surface. The fundamental vector product of this representation is

$$\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \cos v & \sin v & f'(u) \\ -u \sin v & u \cos v & 0 \end{vmatrix} = -uf'(u) \cos v \, \boldsymbol{i} - uf'(u) \sin v \, \boldsymbol{j} + u\boldsymbol{k},$$

and hence

$$\left\| \frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} \right\| = u\sqrt{1 + [f'(u)]^2}.$$

Therefore Equation (12.7) becomes

$$a(S) = \int_0^{2\pi} \left[\int_a^b u\sqrt{1 + [f'(u)]^2} \, du \right] dv = 2\pi \int_a^b u\sqrt{1 + [f'(u)]^2} \, du .$$

The last integral can be expressed as $\int_C x \, ds$, a line integral with respect to arc length along the curve C. As such, it is equal to $\bar{x}L$, where $\bar{x}$ is the x-coordinate of the centroid of C and L is the length of C. (See Section 10.8.) Therefore the area of S is $2\pi L\bar{x}$. This proves the theorem of Pappus.

12.6 Exercises

1. Let S be a parallelogram not parallel to any of the coordinate planes. Let S_1, S_2, and S_3 denote the areas of the projections of S on the three coordinate planes. Show that the area of S is $\sqrt{S_1^2 + S_2^2 + S_3^2}$.
2. Compute the area of the region cut from the plane $x + y + z = a$ by the cylinder $x^2 + y^2 = a^2$.
3. Compute the surface area of that portion of the sphere $x^2 + y^2 + z^2 = a^2$ lying within the cylinder $x^2 + y^2 = ay$, where $a > 0$.
4. Compute the area of that portion of the surface $z^2 = 2xy$ which lies above the first quadrant of the xy-plane and is cut off by the planes $x = 2$ and $y = 1$.
5. A parametric surface S is described by the vector equation

$$\boldsymbol{r}(u, v) = u \cos v \, \boldsymbol{i} + u \sin v \, \boldsymbol{j} + u^2 \boldsymbol{k},$$

where $0 \leq u \leq 4$ and $0 \leq v \leq 2\pi$.
(a) Show that S is a portion of a surface of revolution. Make a sketch and indicate the geometric meanings of the parameters u and v on the surface.
(b) Compute the fundamental vector product $\partial \boldsymbol{r}/\partial u \times \partial \boldsymbol{r}/\partial v$ in terms of u and v.
(c) The area of S is $\pi(65\sqrt{65} - 1)/n$, where n is an integer. Compute the value of n.
6. Compute the area of that portion of the conical surface $x^2 + y^2 = z^2$ which lies above the xy-plane and is cut off by the sphere $x^2 + y^2 + z^2 = 2ax$.
7. Compute the area of that portion of the conical surface $x^2 + y^2 = z^2$ which lies between the two planes $z = 0$ and $x + 2z = 3$.
8. Compute the area of that portion of the paraboloid $x^2 + z^2 = 2ay$ which is cut off by the plane $y = a$.
9. Compute the area of the torus described by the vector equation

$$\boldsymbol{r}(u, v) = (a + b \cos u) \sin v \, \boldsymbol{i} + (a + b \cos u) \cos v \, \boldsymbol{j} + b \sin u \, \boldsymbol{k},$$

where $0 < b < a$ and $0 \leq u \leq 2\pi$, $0 \leq v \leq 2\pi$. [*Hint:* Use the theorem of Pappus.]

10. A sphere is inscribed in a right circular cylinder. The sphere is sliced by two parallel planes perpendicular to the axis of the cylinder. Show that the portions of the sphere and cylinder lying between these planes have equal surface areas.
11. Let T be the unit disk in the uv-plane, $T = \{(u, v) \mid u^2 + v^2 \le 1\}$, and let

$$r(u, v) = \frac{2u}{u^2 + v^2 + 1}\, i + \frac{2v}{u^2 + v^2 + 1}\, j + \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1}\, k .$$

(a) Determine the image of each of the following sets under r: the unit circle $u^2 + v^2 = 1$; the interval $-1 \le u \le 1$; that part of the line $u = v$ lying in T.
(b) The surface $S = r(T)$ is a familiar surface. Name and sketch it.
(c) Determine the image of the uv-plane under r. Indicate by a sketch in the xyz-space the geometric meanings of the parameters u and v.

12.7 Surface integrals

Surface integrals are, in many respects, analogous to line integrals; the integration takes place along a surface rather than along a curve. We defined line integrals in terms of a parametric representation for the curve. Similarly, we shall define surface integrals in terms of a parametric representation for the surface. Then we shall prove that under certain general conditions the value of the integral is independent of the representation.

DEFINITION OF A SURFACE INTEGRAL. *Let $S = r(T)$ be a parametric surface described by a differentiable function r defined on a region T in the uv-plane, and let f be a scalar field defined and bounded on S. The surface integral of f over S is denoted by the symbol $\iint_{r(T)} f\, dS$ [or by $\iint_S f(x, y, z)\, dS$], and is defined by the equation*

$$\iint_{r(T)} f\, dS = \iint_T f[r(u, v)] \left\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right\| du\, dv \tag{12.14}$$

whenever the double integral on the right exists.

The following examples illustrate some applications of surface integrals.

EXAMPLE 1. ***Surface area.*** When $f = 1$, Equation (12.14) becomes

$$\iint_{r(T)} dS = \iint_T \left\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right\| du\, dv .$$

The double integral on the right is that used earlier in Section 12.5 to define surface area. Thus, the area of S is equal to the surface integral $\iint_{r(T)} dS$. For this reason, the symbol dS is sometimes referred to as an "element of surface area," and the surface integral $\iint_{r(T)} f\, dS$ is said to be an integral of f with respect to the element of surface area, extended over the surface $r(T)$.

EXAMPLE 2. ***Center of mass. Moment of inertia.*** If the scalar field f is interpreted as the density (mass per unit area) of a thin material in the shape of the surface S, the total mass m of the surface is defined by the equation

$$m = \iint_S f(x, y, z)\, dS .$$

Its center of mass is the point $(\bar{x}, \bar{y}, \bar{z})$ determined by the equations

$$\bar{x}m = \iint_S xf(x, y, z)\, dS, \qquad \bar{y}m = \iint_S yf(x, y, z)\, dS, \qquad \bar{z}m = \iint_S zf(x, y, z)\, dS .$$

The moment of inertia I_L of S about an axis L is defined by the equation

$$I_L = \iint_S \delta^2(x, y, z) f(x, y, z)\, dS ,$$

where $\delta(x, y, z)$ denotes the perpendicular distance from a general point (x, y, z) of S to the line L.

To illustrate, let us determine the center of mass of a uniform hemispherical surface of radius a. We use the parametric representation

$$\boldsymbol{r}(u, v) = a \cos u \cos v\, \boldsymbol{i} + a \sin u \cos v\, \boldsymbol{j} + a \sin v\, \boldsymbol{k} ,$$

where $(u, v) \in [0, 2\pi] \times [0, \frac{1}{2}\pi]$. This particular representation was discussed earlier in Example 2 of Section 12.2, where we found that the magnitude of the fundamental vector product is $a^2 |\cos v|$. In this example the density f is constant, say $f = c$, and the mass m is $2\pi a^2 c$, the area of S times c. Because of symmetry, the coordinates $\bar{x}$ and $\bar{y}$ of the center of mass are 0. The coordinate $\bar{z}$ is given by

$$\begin{aligned}\bar{z}m &= c \iint_S z\, dS = c \iint_T a \sin v \cdot a^2 |\cos v|\, du\, dv \\ &= 2\pi a^3 c \int_0^{\pi/2} \sin v \cos v\, dv = \pi a^3 c = \frac{a}{2} m ,\end{aligned}$$

so $\bar{z} = a/2$.

EXAMPLE 3. ***Fluid flow through a surface.*** We consider a fluid as a collection of points called ***particles.*** At each particle (x, y, z) we attach a vector $V(x, y, z)$ which represents the velocity of that particular particle. This is the velocity field of the flow. The velocity field may or may not change with time. We shall consider only ***steady-state*** flows, that is, flows for which the velocity $V(x, y, z)$ depends only on the position of the particle and not on time.

We denote by $\rho(x, y, z)$ the density (mass per unit volume) of the fluid at the point (x, y, z). If the fluid is incompressible the density ρ will be constant throughout the fluid. For a compressible fluid, such as a gas, the density may vary from point to point. In any case, the density is a scalar field associated with the flow. The product of the density and the velocity we denote by F; that is,

$$F(x, y, z) = \rho(x, y, z)V(x, y, z).$$

This is a vector field called the *flux density* of the flow. The vector $F(x, y, z)$ has the same direction as the velocity, and its length has the dimensions

$$\frac{\text{mass}}{\text{unit volume}} \cdot \frac{\text{distance}}{\text{unit time}} = \frac{\text{mass}}{(\text{unit area})(\text{unit time})}.$$

In other words, the flux density vector $F(x, y, z)$ tells us how much mass of fluid per unit area per unit time is flowing in the direction of $V(x, y, z)$ at the point (x, y, z).

Let $S = r(T)$ be a simple parametric surface. At each regular point of S let n denote the unit normal having the same direction as the fundamental vector product. That is, let

$$n = \frac{\dfrac{\partial r}{\partial u} \times \dfrac{\partial r}{\partial v}}{\left\| \dfrac{\partial r}{\partial u} \times \dfrac{\partial r}{\partial v} \right\|}. \tag{12.15}$$

The dot product $F \cdot n$ represents the component of the flux density vector in the direction of n. The mass of fluid flowing through S in unit time in the direction of n is defined to be the surface integral

$$\iint_{r(T)} F \cdot n \, dS = \iint_T F \cdot n \left\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right\| du\, dv.$$

12.8 Change of parametric representation

We turn now to a discussion of the independence of surface integrals under a change of parametric representation. Suppose a function r maps a region A in the uv-plane onto a parametric surface $r(A)$. Suppose also that A is the image of a region B in the st-plane under a one-to-one continuously differentiable mapping G given by

$$G(s, t) = U(s, t)i + V(s, t)j \qquad \text{if } (s, t) \in B. \tag{12.16}$$

Consider the function R defined on B by the equation

$$R(s, t) = r[G(s, t)]. \tag{12.17}$$

(See Figure 12.11.) Two functions r and R so related will be called *smoothly equivalent.* Smoothly equivalent functions describe the same surface. That is, $r(A)$ and $R(B)$ are identical as point sets. (This follows at once from the one-to-one nature of G.) The next theorem describes the relationship between their fundamental vector products.

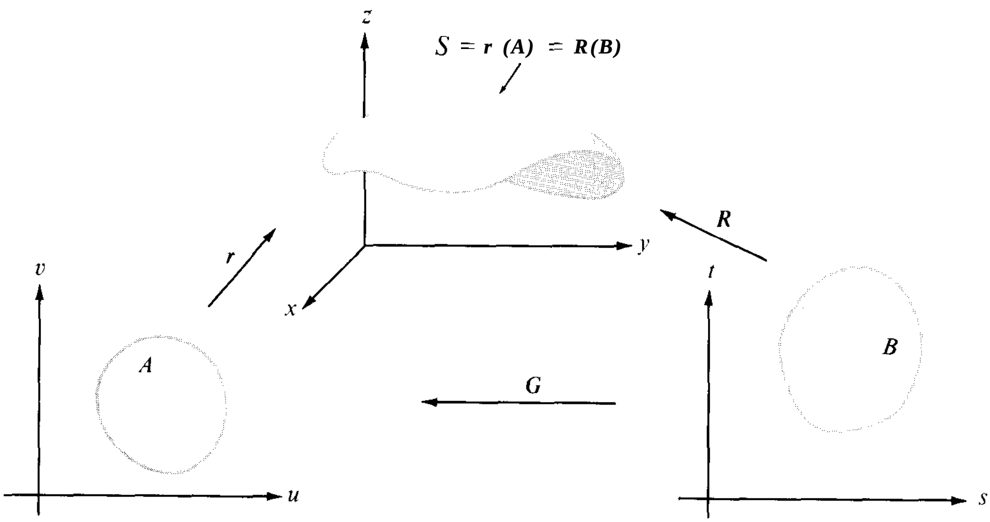

FIGURE 12.11 Two parametric representations of the same surface.

THEOREM 12.1. *Let* r *and R be smoothly equivalent functions related by Equation* (12.17), *where* $G = Ui + Vj$ *is a one-to-one continuously differentiable mapping of a region B in the st-plane onto a region A in the uv-plane given by Equation* (12.16). *Then we have*

$$\frac{\partial \boldsymbol{R}}{\partial s} \times \frac{\partial \boldsymbol{R}}{\partial t} = \left(\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}\right) \frac{\partial(U, V)}{\partial(s, t)}, \tag{12.18}$$

where the partial derivatives $\partial \boldsymbol{r}/\partial u$ *and* $\partial \boldsymbol{r}/\partial v$ *are to be evaluated at the point* $(U(s, t), V(s, t))$. *In other words, the fundamental vector product of R is equal to that of r, times the Jacobian determinant of the mapping G.*

Proof. The derivatives $\partial \boldsymbol{R}/\partial s$ and $\partial \boldsymbol{R}/\partial t$ can be computed by differentiation of Equation (12.17). If we apply the chain rule (Theorem 8.8) to each component of $\boldsymbol{R}$ and rearrange terms, we find that

$$\frac{\partial \boldsymbol{R}}{\partial s} = \frac{\partial \boldsymbol{r}}{\partial u}\frac{\partial U}{\partial s} + \frac{\partial \boldsymbol{r}}{\partial v}\frac{\partial V}{\partial s} \quad \text{and} \quad \frac{\partial \boldsymbol{R}}{\partial t} = \frac{\partial \boldsymbol{r}}{\partial u}\frac{\partial U}{\partial t} + \frac{\partial \boldsymbol{r}}{\partial v}\frac{\partial V}{\partial t},$$

where the derivatives $\partial \boldsymbol{r}/\partial u$ and $\partial \boldsymbol{r}/\partial v$ are evaluated at $(U(s, t), V(s, t))$. Now we cross multiply these two equations and, noting the order of the factors, we obtain

$$\frac{\partial \boldsymbol{R}}{\partial s} \times \frac{\partial \boldsymbol{R}}{\partial t} = \left(\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}\right)\left(\frac{\partial U}{\partial s}\frac{\partial V}{\partial t} - \frac{\partial U}{\partial t}\frac{\partial V}{\partial s}\right) = \left(\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}\right)\frac{\partial(U, V)}{\partial(s, t)}.$$

This completes the proof.

The invariance of surface integrals under smoothly equivalent parametric representations is now an easy consequence of Theorem 12.1.