is ill-defined. More subtle examples of ill-defined expressions arise when, for instance, attempting to add a vector to a matrix, or evaluating a function outside of its domain, e.g., $\sin^{-1}(2)$.

One can make statements out of expressions by using *relations* such as $=$, $<$, $\geq$, $\in$, $\subset$, etc. or by using *properties* (such as "is prime", "is continuous", "is invertible", etc.) For instance, "$30+5$ is prime" is a statement, as is "$30 + 5 \leq 42 - 7$". Note that mathematical statements are allowed to contain English words.

One can make a *compound statement* from more primitive statements by using *logical connectives* such as and, or, not, if-then, if-and-only-if, and so forth. We give some examples below, in decreasing order of intuitiveness.

Conjunction. If X is a statement and Y is a statement, the statement "X and Y" is true if X and Y are both true, and is false otherwise. For instance, "$2 + 2 = 4$ and $3 + 3 = 6$" is true, while "$2+2 = 4$ and $3+3 = 5$" is not. Another example: "$2+2 = 4$ and $2 + 2 = 4$" is true, even if it is a bit redundant; logic is concerned with truth, not efficiency.

Due to the expressiveness of the English language, one can reword the statement "X and Y" in many ways, e.g., "X and also Y", or "Both X and Y are true", etc. Interestingly, the statement "X, but Y" is logically the same statement as "X and Y", but they have different connotations (both statements affirm that X and Y are both true, but the first version suggests that X and Y are in contrast to each other, while the second version suggests that X and Y support each other). Again, logic is about truth, not about connotations or suggestions.

Disjunction. If X is a statement and Y is a statement, the statement "X or Y" is true if either X or Y is true, or both. For instance, "$2 + 2 = 4$ or $3 + 3 = 5$" is true, but "$2 + 2 = 5$ or $3 + 3 = 5$" is not. Also "$2 + 2 = 4$ or $3 + 3 = 6$" is true (even if it is a bit inefficient; it would be a stronger statement to say "$2 + 2 = 4$ and $3 + 3 = 6$"). Thus by default, the word "or" in mathematical logic defaults to *inclusive or*. The reason we do this is that with inclusive or, to verify "X or Y", it suffices to verify that just one of X or Y is true; we don't need to show that the other one is false. So we know, for instance, that "$2 + 2 = 4$ or $2353 + 5931 = 7284$" is true without having to look at the second equation. As in the previous discussion, the statement "$2+2=4$ or $2 + 2 = 4$" is true, even if it is highly inefficient.

If one really does want to use exclusive or, use a statement such as "Either X or Y is true, but not both" or "Exactly one of X or Y is true". Exclusive or does come up in mathematics, but nowhere near as often as inclusive or.

Negation. The statement "X is not true" or "X is false", or "It is not the case that X", is called the *negation* of X, and is true if and only if X is false, and is false if and only if X is true. For instance, the statement "It is not the case that $2+2=5$" is a true statement. Of course we could abbreviate this statement to "$2 + 2 \neq 5$".

Negations convert "and" into "or". For instance, the negation of "Jane Doe has black hair and Jane Doe has blue eyes" is "Jane Doe doesn't have black hair or doesn't have blue eyes", *not* "Jane Doe doesn't have black hair and doesn't have blue eyes" (can you see why?). Similarly, if x is an integer, the negation of "x is even and non-negative" is "x is odd or negative", not "x is odd and negative". (Note how it is important here that or is inclusive rather than exclusive.) Or the negation of "$x \geq 2$ and $x \leq 6$" (i.e., "$2 \leq x \leq 6$") is "$x < 2$ or $x > 6$", not "$x < 2$ and $x > 6$" or "$2 < x > 6.$".

Similarly, negations convert "or" into "and". The negation of "John Doe has brown hair or black hair" is "John Doe does not have brown hair and does not have black hair", or equivalently "John Doe has neither brown nor black hair". If x is a real number, the negation of "$x \geq 1$ or $x \leq -1$" is "$x < 1$ and $x > -1$" (i.e., $-1 < x < 1$).

It is quite possible that a negation of a statement will produce a statement which could not possibly be true. For instance, if x is an integer, the negation of "x is either even or odd" is "x is neither even nor odd", which cannot possibly be true. Remember, though, that even if a statement is false, it is still a statement, and it is definitely possible to arrive at a true statement using an argument which at times involves false statements. (Proofs by contradiction, for instance, fall into this category. Another example is proof by dividing into cases. If one divides into three mutually exclusive cases, Case 1, Case 2, and Case 3, then at any given time two of the cases will be false and only one will be true, however this does not necessarily mean that the proof as a whole is incorrect or that the conclusion is false.)

Negations are sometimes unintuitive to work with, especially if there are multiple negations; a statement such as "It is not the case that either x is not odd, or x is not larger than or equal to 3, but not both" is not particularly pleasant to use. Fortunately, one rarely has to work with more than one or two negations at a time, since often negations cancel each other. For instance, the negation of "X is not true" is just "X is true", or more succinctly just "X". Of course one should be careful when negating more complicated expressions because of the switching of "and" and "or", and similar issues.

If and only if (iff). If X is a statement, and Y is a statement, we say that "X is true if and only if Y is true", whenever X is true, Y has to be also, and whenever Y is true, X has to be also (i.e., X and Y are "equally true"). Other ways of saying the same thing are "X and Y are logically equivalent statements", or "X is true iff Y is true", or "$X \leftrightarrow Y$". Thus for instance, if x is a real number, then the statement "$x = 3$ if and only if $2x = 6$" is true: this means that whenever $x = 3$ is true, then $2x = 6$ is true, and whenever $2x = 6$ is true, then $x = 3$ is true. On the other hand, the statement "$x = 3$ if and only if $x^2 = 9$" is false; while it is true that whenever $x = 3$ is true, $x^2 = 9$ is also true, it is not the case that whenever $x^2 = 9$ is true, that $x = 3$ is also automatically true (think of what happens when $x = -3$).

Statements that are equally true, are also equally false: if X and Y are logically equivalent, and X is false, then Y has to be false also (because if Y were true, then X would also have to be true). Conversely, any two statements which are equally false will also be logically equivalent. Thus for instance $2 + 2 = 5$ if and only if $4 + 4 = 10$.

Sometimes it is of interest to show that more than two statements are logically equivalent; for instance, one might want to assert that three statements X, Y, and Z are all logically equivalent. This means whenever one of the statements is true, then all of the statements are true; and it also means that if one of the statements is false, then all of the statements are false. This may seem like a lot of logical implications to prove, but in practice, once one demonstrates enough logical implications between X, Y, and Z, one can often conclude all the others and conclude that they are all logicallly equivalent. See for instance Exercises A.1.5, A.1.6.

Exercise A.1.1. What is the negation of the statement "either X is true, or Y is true, but not both"?

Exercise A.1.2. What is the negation of the statement "X is true if and only if Y is true"? (There may be multiple ways to phrase this negation).

Exercise A.1.3. Suppose that you have shown that whenever X is true, then Y is true, and whenever X is false, then Y is false. Have you now demonstrated that X and Y are logically equivalent? Explain.

Exercise A.1.4. Suppose that you have shown that whenever X is true, then Y is true, and whenever Y is false, then X is false. Have you now demonstrated that X is true if and only if Y is true? Explain.

Exercise A.1.5. Suppose you know that X is true if and only if Y is true, and you know that Y is true if and only if Z is true. Is this enough to show that X, Y, Z are all logically equivalent? Explain.

Exercise A.1.6. Suppose you know that whenever X is true, then Y is true; that whenever Y is true, then Z is true; and whenever Z is true, then X is true. Is this enough to show that X, Y, Z are all logically equivalent? Explain.

A.2 Implication

Now we come to the least intuitive of the commonly used logical connectives - implication. If X is a statement, and Y is a statement, then "if X, then Y" is the implication from X to Y; it is also written "when X is true, Y is true", or "X implies Y" or

"Y is true when X is" or "X is true only if Y is true" (this last one takes a bit of mental effort to see). What this statement "if X, then Y" means depends on whether X is true or false. If X is true, then "if X, then Y" is true when Y is true, and false when Y is false. If however X is false, then "if X, then Y" is *always* true, regardless of whether Y is true or false! To put it another way, when X is true, the statement "if X, then Y" implies that Y is true. But when X is false, the statement "if X, then Y" offers no information about whether Y is true or not; the statement is true, but *vacuous* (i.e., does not convey any new information beyond the fact that the hypothesis is false).

Examples A.2.1. If x is an integer, then the statement "If $x = 2$, then $x^2 = 4$" is true, regardless of whether x is actually equal to 2 or not (though this statement is only likely to be useful when x *is* equal to 2). This statement does not assert that x is equal to 2, and does not assert that x^2 is equal to 4, but it does assert that when and if x is equal to 2, then x^2 is equal to 4. If x is not equal to 2, the statement is still true but offers no conclusion on x or x^2.

Some special cases of the above implication: the implication "If $2 = 2$, then $2^2 = 4$" is true (true implies true). The implication "If $3 = 2$, then $3^2 = 4$" is true (false implies false). The implication "If $-2 = 2$, then $(-2)^2 = 4$" is true (false implies true). The latter two implications are considered vacuous - they do not offer any new information since their hypothesis is false. (Nevertheless, it is still possible to employ vacuous implications to good effect in a proof - a vacously true statement is still true. We shall see one such example shortly.)

As we see, the falsity of the hypothesis does not destroy the truth of an implication, in fact it is just the opposite! (When a hypothesis is false, the implication is automatically true.) The only way to disprove an implication is to show that the hypothesis is true while the conclusion is false. Thus "If $2 + 2 = 4$, then $4 + 4 = 2$" is a false implication. (True does not imply false.)

One can also think of the statement "if X, then Y" as "Y is *at least as true as X*" - if X is true, then Y also has to be true, but if X is false, Y could be as false as X, but it could also be true. This should be compared with "X if and only if Y", which asserts that X and Y are *equally true.*

Vacuously true implications are often used in ordinary speech, sometimes without knowing that the implication is vacuous. A somewhat frivolous example is "If wishes were wings, then pigs would fly". (The statement "hell freezes over" is also a popular choice for a false hypothesis.) A more serious one is "If John had left work at 5pm, then he would be here by now." This kind of statement is often used in a situation in which the conclusion and hypothesis are both false; but the implication is still true regardless. This statement, by the way, can be used to illustrate the technique of proof by contradiction: if you believe that "If John had left work at 5pm, then he would be here by now", and you also know that "John is not here by now", then you can conclude that "John did not leave work at 5pm", because John leaving work at 5pm would lead to a contradiction. Note how a vacuous implication can be used to derive a useful truth.

To summarize, implications are sometimes vacuous, but this is not actually a problem in logic, since these implications are still true, and vacuous implications can still be useful in logical arguments. In particular, one can safely use statements like "If X, then Y" without necessarily having to worry about whether the hypothesis X is actually true or not (i.e., whether the implication is vacuous or not).

Implications can also be true even when there is no causal link between the hypothesis and conclusion. The statement "If $1 + 1 = 2$, then Washington D.C. is the capital of the United States" is true (true implies true), although rather odd; the statement "If $2 + 2 = 3$, then New York is the capital of the United States" is similarly true (false implies false). Of course, such a statement may be unstable (the capital of the United States may one day change, while $1 + 1$ will always remain equal to 2) but it is true, at least for the moment. While it is possible to use acausal implications in a logical argument, it is not recommended as it can cause unneeded confusion. (Thus, for instance, while it is true that a false statement can be used to imply any other statement, true or false, doing so arbitrarily would probably not be helpful to the reader.)

To prove an implication "If X, then Y", the usual way to do this is to first assume that X is true, and use this (together with whatever other facts and hypotheses you have) to deduce Y. This is still a valid procedure even if X later turns out to be false; the implication does not guarantee anything about the truth of X, and only guarantees the truth of Y conditionally on X first being true. For instance, the following is a valid proof of a true proposition, even though both hypothesis and conclusion of the proposition are false:

Proposition A.2.2. *If $2 + 2 = 5$, then $4 = 10 - 4$.*

Proof. Assume $2 + 2 = 5$. Multiplying both sides by 2, we obtain $4 + 4 = 10$. Subtracting 4 from both sides, we obtain $4 = 10 - 4$ as desired. □

On the other hand, a common error is to prove an implication by first assuming the *conclusion* and then arriving at the hypothesis. For instance, the following Proposition is correct, but the proof is not:

Proposition A.2.3. *Suppose that $2x + 3 = 7$. Show that $x = 2$.*

Proof. (Incorrect) $x = 2$; so $2x = 4$; so $2x + 3 = 7$. □

When doing proofs, it is important that you are able to distinguish the hypothesis from the conclusion; there is a danger of getting hopelessly confused if this distinction is not clear.

Here is a short proof which uses implications which are possibly vacuous.

Theorem A.2.4. *Suppose that n is an integer. Then $n(n + 1)$ is an even integer.*

Proof. Since n is an integer, n is even or odd. If n is even, then $n(n+1)$ is also even, since any multiple of an even number is even. If n is odd, then $n+1$ is even, which again implies that $n(n+1)$ is even. Thus in either case $n(n+1)$ is even, and we are done. □

Note that this proof relied on two implications: "if n is even, then $n(n+1)$ is even", and "if n is odd, then $n(n+1)$ is even". Since n cannot be both odd and even, at least one of these implications has a false hypothesis and is therefore vacuous. Nevertheless, both these implications are true, and one needs *both* of them in order to prove the theorem, because we don't know in advance whether n is even or odd. And even if we did, it might not be worth the trouble to check it. For instance, as a special case of this theorem we immediately know

Corollary A.2.5. *Let* $n = (253 + 142) * 123 - (423 + 198)^{342} + 538 - 213$. *Then* $n(n+1)$ *is an even integer.*

In this particular case, one can work out exactly which parity n is - even or odd - and then use only one of the two implications in the above Theorem, discarding the vacuous one. This may seem like it is more efficient, but it is a false economy, because one then has to determine what parity n is, and this requires a bit of effort - more effort than it would take if we had just left both implications, including the vacuous one, in the argument. So, somewhat paradoxically, the inclusion of vacuous, false, or otherwise "useless" statements in an argument can actually *save* you effort in the long run! (I'm not suggesting, of course, that you ought to pack your proofs with lots of time-wasting and irrelevant statements; all I'm saying here is that you need not be unduly concerned that some hypotheses in your argument might not be correct, as long as your argument is still structured to give the correct conclusion regardless of whether those hypotheses were true or false.)

The statement "If X, then Y" is not the same as "If Y, then X"; for instance, while "If $x = 2$, then $x^2 = 4$" is true, "If $x^2 = 4$, then $x = 2$" can be false if x is equal to -2. These two statements are called *converses* of each other; thus the converse of a true implication is not necessarily another true implication. We use the statement "X if and only if Y" to denote the statement that "If X, then Y; and if Y, then X". Thus for instance, we can say that $x = 2$ if and only if $2x = 4$, because if $x = 2$ then $2x = 4$, while if $2x = 4$ then $x = 2$. One way of thinking about an if-and-only-if statement is to view "X if and only if Y" as saying that X is just as true as Y; if one is true then so is the other, and if one is false, then so is the other. For instance, the statement "If $3 = 2$, then $6 = 4$" is true, since both hypothesis and conclusion are false. (Under this view, "If X, then Y" can be viewed as a statement that Y is at least as true as X.) Thus one could say "X and Y are equally true" instead of "X if and only if Y".

Similarly, the statement "If X is true, then Y is true" is **not** the same as "If X is false, then Y is false". Saying that "if $x = 2$, then $x^2 = 4$" does not imply that "if $x \neq 2$, then $x^2 \neq 4$", and indeed we have $x = -2$ as a counterexample in this case. If-then statements are not the same as if-and-only-if statements. (If we knew that "X is true if and only if Y is true", then we would also know that "X is false if and only if Y is false".) The statement "If X is false, then Y is false" is sometimes called the *inverse* of "If X is true, then Y is true"; thus the inverse of a true implication is not necessarily a true implication.

If you know that "If X is true, then Y is true", then it is also true that "If Y is false, then X is false" (because if Y is false, then X can't be true, since that would imply Y is true, a contradiction). For instance, if we knew that "If $x = 2$, then $x^2 = 4$", then we also know that "If $x^2 \neq 4$, then $x \neq 2$". Or if we knew "If John had left work at 5pm, he would be here by now", then we also know "If John isn't here now, then he could not have left work at 5pm". The statement "If Y is false, then X is false" is known as the *contrapositive* of "If X, then Y" and both statements are equally true.

In particular, if you know that X implies something which is known to be false, then X itself must be false. This is the idea behind *proof by contradiction* or *reductio ad absurdum*: to show something must be false, assume first that it is true, and show that this implies something which you know to be false (e.g., that a statement is simultaneously true and not true). For instance:

Proposition A.2.6. *Suppose that x be a positive number such that* $\sin(x) = 1$. *Then* $x \geq \pi/2$.

Proof. Suppose for sake of contradiction that $x < \pi/2$. Since x is positive, we thus have $0 < x < \pi/2$. Since $\sin(x)$ is increasing for $0 < x < \pi/2$, and $\sin(0) = 0$ and $\sin(\pi/2) = 1$, we thus have $0 < \sin(x) < 1$. But this contradicts the hypothesis that $\sin(x) = 1$. Hence $x \geq \pi/2$. □

Note that one feature of proof by contradiction is that at some point in the proof you assume a hypothesis (in this case, that $x < \pi/2$) which later turns out to be false. Note however that this does not alter the fact that the argument remains valid, and that the conclusion is true; this is because the ultimate conclusion does not rely on that hypothesis being true (indeed, it relies instead on it being false!).

Proof by contradiction is particularly useful for showing "negative" statements - that X is false, that a is not equal to b, that kind of thing. But the line between positive and negative statements is sort of blurry. (Is the statement $x \geq 2$ a positive or negative statement? What about its negation, that $x < 2$?) So this is not a hard and fast rule.

Logicians often use special symbols to denote logical connectives; for instance "X implies Y" can be written "$X \implies Y$", "X is not true" can be written "$\sim X$", "$!X$", or "$\neg X$", "X and Y" can be written "$X \wedge Y$" or "$X \& Y$", and so forth. But for general-purpose mathematics, these symbols are not often used; English words are often more readable, and don't take up much more space. Also, using these symbols tends to blur the line between expressions and statements; it's not as easy to understand "$((x = 3) \wedge (y = 5)) \implies (x + y = 8)$" as "If $x = 3$ and $y = 5$, then $x + y = 8$". So in general I would not recommend using these symbols (except possibly for $\implies$, which is a very intuitive symbol).

A.3 The structure of proofs

To prove a statement, one often starts by assuming the hypothesis and working one's way toward a conclusion; this is the *direct* approach to proving a statement. Such a proof might look something like the following:

Proposition A.3.1. *A implies B.*

Proof. Assume A is true. Since A is true, C is true. Since C is true, D is true. Since D is true, B is true, as desired. □

An example of such a direct approach is

Proposition A.3.2. *If $x = \pi$, then* $\sin(x/2) + 1 = 2$.

Proof. Let $x = \pi$. Since $x = \pi$, we have $x/2 = \pi/2$. Since $x/2 = \pi/2$, we have $\sin(x/2) = 1$. Since $\sin(x/2) = 1$, we have $\sin(x/2) + 1 = 2$. □

In the above proof, we started at the hypothesis and moved steadily from there toward a conclusion. It is also possible to work backwards from the conclusion and seeing what it would take to imply it. For instance, a typical proof of Proposition A.3.1 of this sort might look like the following:

Proof. To show B, it would suffice to show D. Since C implies D, we just need to show C. But C follows from A. □

As an example of this, we give another proof of Proposition A.3.2:

Proof. To show $\sin(x/2) + 1 = 2$, it would suffice to show that $\sin(x/2) = 1$. Since $x/2 = \pi/2$ would imply $\sin(x/2) = 1$, we just need to show that $x/2 = \pi/2$. But this follows since $x = \pi$. □

Logically speaking, the above two proofs of Proposition A.3.2 are the same, just arranged differently. Note how this proof style is different from the (incorrect) approach of starting with the conclusion and seeing what it would imply (as in Proposition A.2.3);

instead, we start with the conclusion and see what would imply it.

Another example of a proof written in this backwards style is the following:

Proposition A.3.3. *Let $0 < r < 1$ be a real number. Then the series $\sum_{n=1}^{\infty} nr^n$ is convergent.*

Proof. To show this series is convergent, it suffices by the ratio test to show that the ratio

$$\left|\frac{r^{n+1}(n+1)}{r^n n}\right| = r\frac{n+1}{n}$$

converges to something less than 1 as $n \to \infty$. Since r is already less than 1, it will be enough to show that $\frac{n+1}{n}$ converges to 1. But since $\frac{n+1}{n} = 1 + \frac{1}{n}$, it suffices to show that $\frac{1}{n} \to 0$. But this is clear since $n \to \infty$. □

One could also do any combination of moving forwards from the hypothesis and backwards from the conclusion. For instance, the following would be a valid proof of Proposition A.3.1:

Proof. To show B, it would suffice to show D. So now let us show D. Since we have A by hypothesis, we have C. Since C implies D, we thus have D as desired. □

Again, from a logical point of view this is exactly the same proof as before. Thus there are many ways to write the same proof down; how you do so is up to you, but certain ways of writing proofs are more readable and natural than others, and different arrangements tend to emphasize different parts of the argument. (Of course, when you are just starting out doing mathematical proofs, you're generally happy to get *some* proof of a result, and don't care so much about getting the "best" arrangement of that proof; but the point here is that a proof can take many different forms.)

The above proofs were pretty simple because there was just one hypothesis and one conclusion. When there are multiple hypotheses and conclusions, and the proof splits into cases, then proofs can get more complicated. For instance a proof might look as tortuous as this:

Proposition A.3.4. *Suppose that A and B are true. Then C and D are true.*

Proof. Since A is true, E is true. From E and B we know that F is true. Also, in light of A, to show D it suffices to show G. There are now two cases: H and I. If H is true, then from F and H we obtain C, and from A and H we obtain G. If instead I is true, then from I we have G, and from I and G we obtain C. Thus in both cases we obtain both C and G, and hence C and D. □

Incidentally, the above proof could be rearranged into a much tidier manner, but you at least get the idea of how complicated a proof could become. To show an implication there are several ways to proceed: you can work forward from the hypothesis; you can work backward from the conclusion; or you can divide into cases in the hope to split the problem into several easier sub-problems. Another is to argue by contradiction, for instance you can have an argument of the form

Proposition A.3.5. *Suppose that A is true. Then B is false.*

Proof. Suppose for sake of contradiction that B is true. This would imply that C is true. But since A is true, this implies that D is true; which contradicts C. Thus B must be false. □

As you can see, there are several things to try when attempting a proof. With experience, it will become clearer which approaches are likely to work easily, which ones will probably work but require much effort, and which ones are probably going to fail. In many cases there is really only one obvious way to proceed. Of course, there may definitely be multiple ways to approach a problem, so if you see more than one way to begin a problem, you can just try whichever one looks the easiest, but be prepared to switch to another approach if it begins to look hopeless.

Also, it helps when doing a proof to keep track of which statements are *known* (either as hypotheses, or deduced from the hypotheses, or coming from other theorems and results), and which statements are *desired* (either the conclusion, or something which would imply the conclusion, or some intermediate claim or lemma which will be useful in eventually obtaining the conclusion). Mixing the two up is almost always a bad idea, and can lead to one getting hopelessly lost in a proof.

A.4 Variables and quantifiers

One can get quite far in logic just by starting with primitive statements (such as "$2+2 = 4$" or "John has black hair"), then forming compound statements using logical connectives, and then using various laws of logic to pass from one's hypotheses to one's conclusions; this is known as *propositional logic* or *Boolean logic.* (It is possible to list a dozen or so such laws of propositional logic, which are sufficient to do everything one wants to do, but I have deliberately chosen not to do so here, because you might then be tempted to memorize that list, and that is **not** how one should learn how to do logic, unless one happens to be a computer or some other non-thinking device. However, if you really are curious as to what the formal laws of logic are, look up "laws of propositional logic" or something similar in the library or on the internet.)

However, to do mathematics, this level of logic is insufficient, because it does not incorporate the fundamental concept of *variables* - those familiar symbols such as x or n which denote various quantities which are unknown, or set to some value, or assumed to obey some property. Indeed we have already sneaked in some of these variables in order to illustrate some of the concepts in propositional logic (mainly because it gets boring after a while to talk endlessly about variable-free statements such as $2 + 2 = 4$ or "Jane has black hair"). *Mathematical logic* is thus the same as propositional logic but with the additional ingredient of variables added.