satisfies the linear differential equation

$$x^2y'' + (3x - x^2)y' + (1 - x - e^{2x})y = 0.$$

Use this information to determine the general solution of the equation on the interval $(0, +\infty)$.

6.17 Linear equations of second order with analytic coefficients

A function f is said to be analytic on an interval $(x_0 - r, x_0 + r)$ if f has a power-series expansion in this interval,

$$f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n,$$

convergent for $|x - x_0| < r$. If the coefficients of a homogeneous linear differential equation

$$y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_n(x)y = 0$$

are analytic in an interval $(x_0 - r, x_0 + r)$, then it can be shown that there exist n independent solutions $u_1, \ldots, u_n$, each of which is analytic on the same interval. We shall prove this theorem for equations of second order and then discuss an important example that occurs in many applications.

THEOREM 6.13. *Let P_1 and P_2 be analytic on an open interval $(x_0 - r, x_0 + r)$, say*

$$P_1(x) = \sum_{n=0}^{\infty} b_n(x - x_0)^n, \qquad P_2(x) = \sum_{n=0}^{\infty} c_n(x - x_0)^n.$$

Then the differential equation

$$y'' + P_1(x)y' + P_2(x)y = 0 \tag{6.31}$$

has two independent solutions u_1 and u_2 which are analytic on the same interval.

Proof. We try to find a power-series solution of the form

$$y = \sum_{n=0}^{\infty} a_n(x - x_0)^n, \tag{6.32}$$

convergent in the given interval. To do this, we substitute the given series for P_1 and P_2 in the differential equation and then determine relations which the coefficients a_n must satisfy so that the function y given by (6.32) will satisfy the equation.

The derivatives y' and y'' can be obtained by differentiating the power series for y term by term (see Theorem 11.9 in Volume I). This gives us

$$y' = \sum_{n=1}^{\infty} na_n(x - x_0)^{n-1} = \sum_{n=0}^{\infty}(n + 1)a_{n+1}(x - x_0)^n,$$

$$y'' = \sum_{n=2}^{\infty} n(n - 1)a_n(x - x_0)^{n-2} = \sum_{n=0}^{\infty}(n + 2)(n + 1)a_{n+2}(x - x_0)^n.$$

The products $P_1(x)y'$ and $P_2(x)y$ are given by the power series†

$$P_1(x)y' = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(k+1)a_{k+1}b_{n-k}\right)(x-x_0)^n$$

and

$$P_2(x)y = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}a_k c_{n-k}\right)(x-x_0)^n .$$

When these series are substituted in the differential equation (6.31) we find

$$\sum_{n=0}^{\infty}\left\{(n+2)(n+1)a_{n+2} + \sum_{k=0}^{n}[(k+1)a_{k+1}b_{n-k} + a_k c_{n-k}]\right\}(x-x_0)^n = 0 .$$

Therefore the differential equation will be satisfied if we choose the coefficients ***a,*** so that they satisfy the recursion formula

$$(6.33) \qquad (n+2)(n+1)a_{n+2} = -\sum_{k=0}^{n}[(k+1)a_{k+1}b_{n-k} + a_k c_{n-k}]$$

for $n = 0, 1, 2, \ldots$. This formula expresses a_{n+2} in terms of the earlier coefficients $a_0, a_1, \ldots, a_{n+1}$ and the coefficients of the given functions P_1 and P_2. We choose arbitrary values of the first two coefficients ***a,*** and ***a,*** and use the recursion formula to define the remaining coefficients $a_2, a_3, \ldots,$ in terms of ***a,*** and a,. This guarantees that the power series in (6.32) will satisfy the differential equation (6.31). The next step in the proof is to show that the series so defined actually converges for every x in the interval $(x_0 - r, x_0 + r)$. This is done by dominating the series in (6.32) by another power series known to converge. Finally, we show that we can choose ***a,*** and ***a,*** to obtain two independent solutions.

We prove now that the series (6.32) whose coefficients are defined by (6.33) converges in the required interval.

Choose a fixed point $x_1 \neq x_0$ in the interval $(x_0 - r, x_0 + r)$ and let $t = |x_1 - x_0|$. Since the series for P_1 and P_2 converge absolutely for $x = x_1$ the terms of these series are bounded, say

$$|b_k|\, t^k \le M_1 \qquad \text{and} \qquad |c_k|\, t^k \le M_2 ,$$

for some $M_1 > 0$, $M_2 > 0$. Let A4 be the larger of M_1 and tM_2. Then we have

$$|b_k| \le \frac{M}{t^k} \qquad \text{and} \qquad |c_k| \le \frac{M}{t^{k+1}} .$$

The recursion formula implies the inequality

$$\begin{aligned}(n+2)(n+1)\,|a_{n+2}| &\le \sum_{k=0}^{n}\left\{(k+1)\,|a_{k+1}|\frac{M}{t^{n-k}} + |a_k|\frac{M}{t^{n-k+1}}\right\}\\ &= \frac{M}{t^{n+1}}\left\{\sum_{k=0}^{n}(k+1)\,|a_{k+1}|\,t^{k+1} + \sum_{k=0}^{n}|a_{k+1}|\,t^{k+1} + |a_0| - |a_{n+1}|\,t^{n+1}\right\}\\ &\le \frac{M}{t^{n+1}}\left\{\sum_{k=0}^{n}(k+2)\,|a_{k+1}|\,t^{k+1} + |a_0|\right\} = \frac{M}{t^{n+1}}\sum_{k=0}^{n+1}(k+1)\,|a_k|\,t^k .\end{aligned}$$

† Those readers not familiar with multiplication of power series may consult Exercise 7 of Section 6.21.

Now let $A_0 = |a_0|$, $A_1 = |a_1|$, and define $A_2, A_3, \ldots$ successively by the recursion formula

(6.34) $$(n + 2)(n + 1)A_{n+2} = \frac{M}{t^{n+1}} \sum_{k=0}^{n+1} (k + 1)A_k t^k$$

for $n \geq 0$. Then $|a_n| \leq A_n$ for all $n \geq 0$, so the series $\sum a_n(x - x_0)^n$ is dominated by the series $\sum A_n |x - x_0|^n$. Now we use the ratio test to show that $\sum A_n |x - x_0|^n$ converges if $|x - x_0| < t$.

Replacing n by $n - 1$ in (6.34) and subtracting t^{-1} times the resulting equation from (6.34) we find that $(n + 2)(n + 1)A_{n+2} - t^{-1}(n + 1)nA_{n+1} = M(n + 2)A_{n+1}$. Therefore

$$A_{n+2} = A_{n+1} \frac{(n + 1)n + (n + 2)Mt}{(n + 2)(n + 1)t},$$

and we find

$$\frac{A_{n+2}\,|x - x_0|^{n+2}}{A_{n+1}\,|x - x_0|^{n+1}} = \frac{(n + 1)n + (n + 2)Mt}{(n + 2)(n + 1)t}|x - x_0| \to \frac{|x - x_0|}{t}$$

as $n \to \infty$. This limit is less than 1 if $|x - x_0| < t$. Hence $\sum a_n(x - x_0)^n$ converges if $|x - x_0| < t$. But since $t = |x_1 - x_0|$ and since x_1 was an arbitrary point in the interval $(x_0 - r, x_0 + r)$, the series $\sum a_n(x - x_0)^n$ converges for all x in $(x_0 - r, x_0 + r)$.

The first two coefficients a_0 and a_1 represent the initial values of y and its derivative at the point x_0. If we let u_1 be the power-series solution with $a_0 = 1$ and $a_1 = 0$, so that

$$u_1(x_0) = 1 \quad \text{and} \quad u_1'(x_0) = 0,$$

and let u_2 be the solution with $a_0 = 0$ and $a_1 = 1$, so that

$$u_2(x_0) = 0 \quad \text{and} \quad u_2'(x_0) = 1,$$

then the solutions u_1 and u_2 will be independent. This completes the proof.

6.18 The Legendre equation

In this section we find power-series solutions for the Legendre equation,

(6.35) $$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0,$$

where α is any real constant. This equation occurs in problems of attraction and in heat-flow problems with spherical symmetry. When α is a positive integer we shall find that the equation has polynomial solutions called ***Legendre polynomials.*** These are the same polynomials we encountered earlier in connection with the Gram-Schmidt process (Chapter 1, page 26).

The Legendre equation can be written as

$$[(x^2 - 1)y']' = \alpha(\alpha + 1)y,$$

which has the form

$$T(y) = \lambda y,$$

where T is a Sturm-Liouville operator, $T(f) = (pf')'$, with $p(x) = x^2 - 1$ and $\lambda = \alpha(\alpha + 1)$. Therefore the nonzero solutions of the Legendre equation are eigenfunctions of T belonging to the eigenvalue $\alpha(\alpha + 1)$. Since $p(x)$ satisfies the boundary conditions

$$p(1) = p(-1) = 0,$$

the operator T is symmetric with respect to the inner product

$$(f, g) = \int_{-1}^{1} f(x)g(x)\, dx .$$

The general theory of symmetric operators tells us that eigenfunctions belonging to distinct eigenvalues are orthogonal (Theorem 5.3).

In the differential equation treated in Theorem 6.13 the coefficient of y'' is 1. The Legendre equation can be put in this form if we divide through by $1 - x^2$. From (6.35) we obtain

$$y'' + P_1(x)y' + P_2(x)y = 0,$$

where

$$P_1(x) = -\frac{2x}{1 - x^2} \quad \text{and} \quad P_2(x) = \frac{\alpha(\alpha + 1)}{1 - x^2},$$

if $x^2 \neq 1$. Since $1/(1 - x^2) = \sum_{n=0}^{\infty} x^{2n}$ for $|x| < 1$, both P_1 and P_2 have power-series expansions in the open interval $(-1, 1)$ so Theorem 6.13 is applicable. To find the recursion formula for the coefficients it is simpler to leave the equation in the form (6.35) and try to find a power-series solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

valid in the open interval $(-1, 1)$. Differentiating this series term by term we obtain

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}.$$

Therefore we have

$$2xy' = \sum_{n=1}^{\infty} 2na_n x^n = \sum_{n=0}^{\infty} 2na_n x^n,$$

and

$$\begin{aligned}(1 - x^2)y'' &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n \\ &= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} n(n-1)a_n x^n \\ &= \sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} - n(n-1)a_n]x^n .\end{aligned}$$

If we substitute these series in the differential equation (6.35), we see that the equation will be satisfied if, and only if, the coefficients satisfy the relation

$$(n + 2)(n + 1)a_{n+2} - n(n - 1)a_n - 2na_n + \alpha(\alpha + 1)a_n = 0$$

for all $n \geq 0$. This equation is the same as

$$(n+2)(n+1)a_{n+2} - (n-\alpha)(n+1+\alpha)a_n = 0,$$

or

$$a_{n+2} = -\frac{(\alpha - n)(\alpha + n + 1)}{(n+1)(n+2)}\, a_n . \tag{6.36}$$

This relation enables us to determine $a_2, a_4, a_6, \ldots,$ successively in terms of a_0. Similarly, we can compute $a_3, a_5, a_7, \ldots,$ in terms of a_1. For the coefficients with even subscripts we have

$$a_2 = -\frac{\alpha(\alpha+1)}{1 \cdot 2}\, a_0,$$

$$a_4 = -\frac{(\alpha-2)(\alpha+3)}{3 \cdot 4}\, a_2 = (-1)^2 \frac{\alpha(\alpha-2)(\alpha+1)(\alpha+3)}{4!}\, a_0,$$

and, in general,

$$a_{2n} = (-1)^n \frac{\alpha(\alpha-2)\cdots(\alpha-2n+2)\cdot(\alpha+1)(\alpha+3)\cdots(\alpha+2n-1)}{(2n)!}\, a_0$$

This can be proved by induction. For the coefficients with odd subscripts we find

$$a_{2n+1} = (-1)^n \frac{(\alpha-1)(\alpha-3)\cdots(\alpha-2n+1)\cdot(\alpha+2)(\alpha+4)\cdots(\alpha+2n)}{(2n+1)!}\, a_1$$

Therefore the series for y can be written as

$$y = a_0 u_1(x) + a_1 u_2(x), \tag{6.37}$$

where

$$u_1(x) = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{\alpha(\alpha-2)\cdots(\alpha-2n+2)\cdot(\alpha+1)(\alpha+3)\cdots(\alpha+2n-1)}{(2n)!}\, x^{2n} \tag{6.38}$$

and

$$u_2(x) = x + \sum_{n=1}^{\infty} (-1)^n \frac{(\alpha-1)(\alpha-3)\cdots(\alpha-2n+1)\cdot(\alpha+2)(\alpha+4)\cdots(\alpha+2n)}{(2n+1)!}\, x^{2n+1}. \tag{6.39}$$

The ratio test shows that each of these series converges for $|x| < 1$. Also, since the relation (6.36) is satisfied separately by the even and odd coefficients, each of u_1 and u_2 is a solution of the differential equation (6.35). These solutions satisfy the initial conditions

$$u_1(0) = 1, \qquad u_1'(0) = 0, \qquad u_2(0) = 0, \qquad u_2'(0) = 1.$$

Since u_1 and u_2 are independent, the general solution of the Legendre equation (6.35) over the open interval $(-1, 1)$ is given by the linear combination (6.37) with arbitrary constants a, and a,.

When α is 0 or a positive even integer, say $\alpha = 2m$, the series for $u_1(x)$ becomes a polynomial of degree $2m$ containing only even powers of x. Since we have

$$\alpha(\alpha - 2) \ldots (\alpha - 2n + 2) = 2m(2m - 2) \cdots (2m - 2n + 2) = \frac{2^n m!}{(m - n)!}$$

and

$$(a + 1)(\alpha + 3) \cdots (\alpha + 2n - 1) = (2m + 1)(2m + 3) \cdots (2m + 2n - 1)$$
$$= \frac{(2m + 2n)!\, m!}{2^n (2m)!\, (m + n)!}$$

the formula for $u_1(x)$ in this case becomes

$$u_1(x) = 1 + \frac{(m!)^2}{(2m!)} \sum_{k=1}^{m} (-1)^k \frac{(2m + 2k)!}{(m - k)!\, (m + k)!\, (2k)!} x^{2k}. \tag{6.40}$$

For example, when $\alpha = 0, 2, 4, 6$ $(m = 0, 1, 2, 3)$ the corresponding polynomials are

$$u_1(x) = 1, \qquad 1 - 3x3, \qquad 1 - 10x^2 + \tfrac{35}{3}x^4, \qquad 1 - 21x^2 + 63x^4 - \tfrac{231}{5}x^6.$$

The series for $u_2(x)$ is not a polynomial when α is even because the coefficient of x^{2n+1} is never zero.

When α is an odd positive integer, the roles of u_1 and u_2 are reversed; the series for $u_2(x)$ becomes a polynomial and the series for $u_1(x)$ is not a polynomial. Specifically if, $\alpha = 2m + 1$ we have

$$u_2(x) = x + \frac{(m!)^2}{(2m + 1)!} \sum_{k=1}^{m} (-1)^k \frac{(2m + 2k + 1)!}{(m - k)!\, (m + k)!\, (2k + 1)!} x^{2k+1}. \tag{6.41}$$

For example, when $\alpha = 1, 3, 5$ $(m = 0, 1, 2)$, the corresponding polynomials are

$$u_2(x) = x, \qquad x - \tfrac{5}{3}x^3, \qquad x - \tfrac{14}{3}x^3 + \tfrac{21}{5}x^5.$$

6.19 The Legendre polynomials

Some of the properties of the polynomial solutions of the Legendre equation can be deduced directly from the differential equation or from the formulas in (6.40) and (6.41). Others are more easily deduced from an alternative formula for these polynomials which we shall now derive.

First we shall obtain a single formula which contains (aside from constant factors) both the polynomials in (6.40) and (6.41). Let

$$P_n(x) = \frac{1}{2^n} \sum_{r=0}^{[n/2]} \frac{(-1)^r (2n - 2r)!}{r!\, (n - r)!\, (n - 2r)!} x^{n-2r}, \tag{6.42}$$

where *[n/2]* denotes the greatest integer $\leq n/2$. We will show presently that this is the ***Legendre polynomial*** of degree n introduced in Chapter 1. When n is even, it is a constant multiple of the polynomial $u_1(x)$ in Equation (6.40); when n is odd, it is a constant multiple of the polynomial $u_2(x)$ in (6.41).† The first seven Legendre polynomials are given by the formulas

$$P_0(x) = 1, \qquad P_1(x) = x, \qquad P_2(x) = \tfrac{1}{2}(3x^2 - 1), \qquad P_3(x) = \tfrac{1}{2}(5x^3 - 3x),$$
$$P_4(x) = \tfrac{1}{8}(35x^4 - 30x^2 + 3), \quad P_5(x) = \tfrac{1}{8}(63x^5 - 70x^3 + 15x),$$
$$P_6(x) = \tfrac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5).$$

Figure 6.1 shows the graphs of the first five of these functions over the interval $[-1, 1]$.

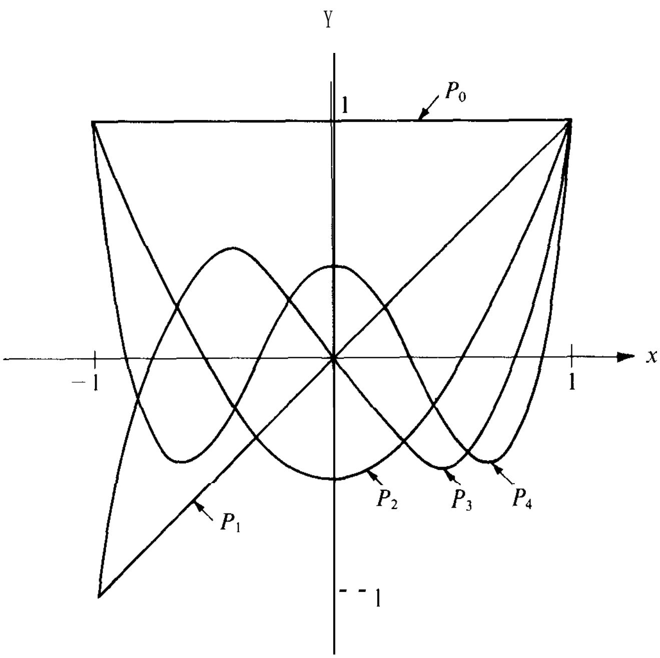

FIGURE 6.1 Graphs of Legendre polynomials over the interval $[-1, 1]$.

Now we can show that, except for scalar factors, the Legendre polynomials are those obtained by applying the Gram-Schmidt orthogonalization process to the sequence of polynomials $1, x, x^2, \ldots$, with the inner product

$$(f, g) = \int_{-1}^{1} f(x)g(x)\,dx.$$

† When n is even, say $n = 2m$, we may replace the index of summation k in Equation (6.40) by a new index r, where $r = m - k$; we find that the sum in (6.40) is a constant multiple of $P_n(x)$. Similarly, when n is odd, a change of index transforms the sum in (6.41) to a constant multiple of $P_n(x)$.

First we note that if $m \neq n$ the polynomials P_n and P_m are orthogonal because they are eigenfunctions of a symmetric operator belonging to distinct eigenvalues. Also, since P_n has degree n and $P_0 = 1$, the polynomials $P_0(x), P_1(x), \ldots, P_n(x)$ span the same subspace as $1, x, \ldots, x^n$. In Section 1.14, Example 2, we constructed another orthogonal set of polynomials $y_0, y_1, y_2, \ldots$, such that $y_0(x), y_1(x), \ldots, y_n(x)$ spans the same subspace as $1, x, \ldots, x^n$ for each n. The orthogonalization theorem (Theorem 1.13) tells us that, except for scalar factors, there is only one set of orthogonal functions with this property. Hence we must have

$$P_n(x) = c_n y_n(x)$$

for some scalars c_n. The coefficient of x^n in $y_n(x)$ is 1, so c_n is the coefficient of x^n in $P_n(x)$. From (6.42) we see that

$$c_n = \frac{(2n)!}{2^n(n!)^2}.$$

6.20 Rodrigues' formula for the Legendre polynomials

In the sum (6.42) defining $P_n(x)$ we note that

$$\frac{(2n-2r)!}{(n-2r)!}x^{n-2r} = \frac{d^n}{dx^n}x^{2n-2r} \quad \text{and} \quad \frac{1}{r!\,(n-r)!} = \frac{1}{n!}\binom{n}{r},$$

where $\binom{n}{r}$ is the binomial coefficient, and we write the sum in the form

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\sum_{r=0}^{[n/2]}(-1)^r\binom{n}{r}x^{2n-2r}.$$

When $[n/2] < r \leq n$, the term x^{2n-2r} has degree less than n, so its nth derivative is zero. Therefore we do not alter the sum if we allow r to run from 0 to n. This gives us

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\sum_{r=0}^{n}(-1)^r\binom{n}{r}x^{2n-2r}.$$

Now we recognize the sum on the right as the binomial expansion of $(x^2 - 1)^n$. Therefore we have

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n.$$

This is known as ***Rodrigues' formula***, in honor of Olinde Rodrigues (1794–1851), a French economist and reformer.

Using Rodrigues' formula and the differential equation, we can derive a number of important properties of the Legendre polynomials. Some of these properties are listed below. Their proofs are outlined in the next set of exercises.

For each $n \geq 0$ we have

$$P_n(1) = 1.$$

Moreover, $P_n(x)$ is the only polynomial which satisfies the Legendre equation

$$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$$

and has the value 1 when $x = 1$.

For each $n \geq 0$ we have

$$P_n(-x) = (-1)^n P_n(x).$$

This shows that P_n is an even function when n is even, and an odd function when n is odd.

We have already mentioned the orthogonality relation,

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = 0 \qquad \text{if} \quad m \neq n.$$

When $m = n$ we have the norm relation

$$\|P_n\|^2 = \int_{-1}^{1} [P_n(x)]^2\,dx = \frac{2}{2n + 1}.$$

Every polynomial of degree n can be expressed as a linear combination of the Legendre polynomials $P_0, P_1, \ldots, P_n$. In fact, if f is a polynomial of degree n we have

$$f(x) = \sum_{k=0}^{n} c_k P_k(x),$$

where

$$c_k = \frac{2k + 1}{2} \int_{-1}^{1} f(x)P_k(x)\,dx.$$

From the orthogonality relation it follows that

$$\int_{-1}^{1} g(x)P_n(x)\,dx = 0$$

for every polynomial g of degree less than n. This property can be used to prove that the Legendre polynomial P_n has n distinct real zeros and that they all lie in the open interval $(-1, 1)$.

6.21 Exercises

1. The Legendre equation (6.35) with $\alpha = 0$ has the polynomial solution $u_1(x) = 1$ and a solution u_2, not a polynomial, given by the series in Equation (6.41).

   (a) Show that the sum of the series for u_2 is given by

   $$u_2(x) = \frac{1}{2} \log \frac{1 + x}{1 - x} \qquad \text{for } |x| < 1.$$

   (b) Verify directly that the function u_2 in part (a) is a solution of the Legendre equation when $\alpha = 0$.

2. Show that the function f defined by the equation

$$f(x) = 1 - \frac{x}{2}\log\frac{1+x}{1-x}$$

for $|x| < 1$ satisfies the Legendre equation (6.35) with $\alpha = 1$. Express this function as a linear combination of the solutions u_1 and u_2 given in Equations (6.38) and (6.39).

3. The Legendre equation (6.35) can be written in the form

$$[(x^2 - 1)y']' - \alpha(\alpha + 1)y = 0.$$

(a) If a, b, c are constants with $a > b$ and $4c + 1 > 0$, show that a differential equation of the type

$$[(x - a)(x - b)y']' - cy = 0$$

can be transformed to a Legendre equation by a change of variable of the form $x = At + B$, with $A > 0$. Determine A and B in terms of a and b.

(b) Use the method suggested in part (a) to transform the equation

$$(x^2 - x)y'' + (2x - 1)y' - 2y = 0$$

to a Legendre equation.

4. Find two independent power-series solutions of the *Hermite equation*

$$y'' - 2xy' + 2\alpha y = 0$$

on an interval of the form $(-r, r)$. Show that one of these solutions is a polynomial when α is a nonnegative integer.

5. Find a power-series solution of the differential equation

$$xy'' + (3 + x^3)y' + 3x^2y = 0$$

valid for all x. Find a second solution of the form $y = x^{-2}\sum a_n x^n$ valid for all $x \neq 0$.

6. Find a power-series solution of the differential equation

$$x^2y'' + x^2y' - (ax + 2)y = 0$$

valid on an interval of the form $(-r, r)$.

7. Given two functions A and B analytic on an interval $(x_0 - r, x_0 + r)$, say

$$A(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n, \qquad B(x) = \sum_{n=0}^{\infty} b_n(x - x_0)^n.$$

It can be shown that the product $C(x) = A(x)B(x)$ is also analytic on $(x_0 - r, x_0 + r)$. This exercise shows that C has the power-series expansion

$$C(x) = \sum_{n=0}^{\infty} c_n(x - x_0)^n, \qquad \text{where } c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

(a) Use Leibniz's rule for the nth derivative of a product to show that the nth derivative of C is given by

$$C^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} A^{(k)}(x) B^{(n-k)}(x).$$

(b) Now use the fact that $A^{(k)}(x_0) = k!\, a_k$ and $B^{(n-k)}(x_0) = (n-k)!\, b_{n-k}$ to obtain

$$C^{(n)}(x_0) = n! \sum_{k=0}^{n} a_k b_{n-k}.$$

Since $C^{(n)}(x_0) = n!\, c_n$, this proves the required formula for c_n.

In Exercises 8 through 14, $P_n(x)$ denotes the Legendre polynomial of degree n. These exercises outline proofs of the properties of the Legendre polynomials described in Section 6.20.

8. (a) Use Rodrigues' formula to show that

$$P_n(x) = \frac{1}{2^n}(x+1)^n + (x-1)Q_n(x),$$

where $Q_n(x)$ is a polynomial.
(b) Prove that $P_n(1) = 1$ and that $P_n(-1) = (-1)^n$.
(c) Prove that $P_n(x)$ is the only polynomial solution of Legendre's equation (with $\alpha = n$) having the value 1 when $x = 1$.

9. (a) Use the differential equations satisfied by P_n and P_m to show that

$$[(1-x^2)(P_nP_m' - P_n'P_m)]' = [n(n+1) - m(m+1)]P_nP_m.$$

(b) If $m \neq n$, integrate the equation in (a) from -1 to 1 to give an alternate proof of the orthogonality relation

$$\int_{-1}^{1} P_n(x)P_m(x)\, dx = 0.$$

10. (a) Let $f(x) = (x^2 - 1)^n$. Use integration by parts to show that

$$\int_{-1}^{1} f^{(n)}(x) f^{(n)}(x)\, dx = -\int_{-1}^{1} f^{(n+1)}(x) f^{(n-1)}(x)\, dx.$$

Apply this formula repeatedly to deduce that the integral on the left is equal to

$$2(2n)! \int_0^1 (1-x^2)^n\, dx.$$

(b) The substitution $x = \cos t$ transforms the integral $\int_0^1 (1-x^2)^n\, dx$ to $\int_0^{\pi/2} \sin^{2n+1} t\, dt$. Use the relation

$$\int_0^{\pi/2} \sin^{2n+1} t\, dt = \frac{2n(2n-2)\cdots 2}{(2n+1)(2n-1)\cdots 3\cdot 1}$$

and Rodrigues' formula to obtain

$$\int_{-1}^{1} [P_n(x)]^2\, dx = \frac{2}{2n+1}.$$

11. (a) Show that

$$P_n(x) = \frac{(2n)!}{2^n(n!)^2}\, x^n + Q_n(x),$$

where $Q_n(x)$ is a polynomial of degree less than n.
(b) Express the polynomial $f(x) = x^4$ as a linear combination of P_0, P_1, P_2, P_3, and P_4.
(c) Show that every polynomial f of degree n can be expressed as a linear combination of the Legendre polynomials $P_0, P_1, \ldots, P_n$.

12. (a) If f is a polynomial of degree n, write

$$f(x) = \sum_{k=0}^{n} c_k P_k(x).$$

[This is possible because of Exercise 11 (c).] For a fixed m, $0 \le m \le n$, multiply both sides of this equation by $P_m(x)$ and integrate from -1 to 1. Use Exercises 9(b) and 10(b) to deduce the relation

$$c_m = \frac{2m+1}{2}\int_{-1}^{1} f(x)P_m(x)\,dx.$$

13. Use Exercises 9 and 11 to show that $\int_{-1}^{1} g(x)P_n(x)\,dx = 0$ for every polynomial g of degree less than n.

14. (a) Use Rolle's theorem to show that P_n cannot have any multiple zeros in the open interval $(-1, 1)$. In other words, any zeros of P_n which lie in $(-1, 1)$ must be simple zeros.
(b) Assume P_n has m zeros in the interval $(-1, 1)$. If $m = 0$, let $Q_0(x) = 1$. If $m \ge 1$, let

$$Q_m(x) = (x - x_1)(x - x_2) \ldots (x - x_m),$$

where $x_1, x_2, \ldots, x_m$ are the m zeros of P_n in $(-1, 1)$. Show that, at each point x in $(-1, 1)$, $Q_m(x)$ has the same sign as $P_n(x)$.
(c) Use part (b), along with Exercise 13, to show that the inequality $m < n$ leads to a contradiction. This shows that P_n has n distinct real zeros, all of which lie in the open interval $(-1, 1)$.

15. (a) Show that the value of the integral $\int_{-1}^{1} P_n(x)P'_{n+1}(x)\,dx$ is independent of n.
(b) Evaluate the integral $\int_{-1}^{1} x\, P_n(x)P_{n-1}(x)\,dx$.

6.22 The method of Frobenius

In Section 6.17 we learned how to find power-series solutions of the differential equation

$$y'' + P_1(x)y' + P_2(x)y = 0 \tag{6.43}$$

in an interval about a point x_0 where the coefficients P_1 and P_2 are analytic. If either P_1 or P_2 is not analytic near x_0, power-series solutions valid near x_0 may or may not exist. For example, suppose we try to find a power-series solution of the differential equation

$$x^2y'' - y' - y = 0 \tag{6.44}$$

near $x_0 = 0$. If we assume that a solution $y = \sum a_k x^k$ exists and substitute this series in the differential equation we are led to the recursion formula

$$a_{n+1} = \frac{n^2 - n - 1}{n+1}\, a_n.$$