14.16 Exercises

1. Let X be a random variable which measures the lifetime (in hours) of a certain type of vacuum tube. Assume X has an exponential distribution with parameter $\lambda = 0.001$. Determine T so that $P(X > T)$ is (a) 0.90; (b) 0.99. You may use the approximate formula $-\log(1 - x) = x + x^2/2$ in your calculations.
2. A radioactive material obeys an exponential decay law with half-life 2 years. Consider the decay time X (in years) of a single atom and assume that X is a random variable with an exponential distribution. Calculate the probability that an atom disintegrates (a) in the interval $1 \le X \le 2$; (b) in the interval $2 \le X \le 3$; (c) in the interval $2 \le X \le 3$, given that it has not disintegrated in the interval $0 \le X \le 2$; (d) in the interval $2 \le X \le 3$, given that it has not disintegrated in the interval $1 \le X \le 2$.
3. The length of time (in minutes) of long distance telephone calls from Caltech is found to be a random phenomenon with probability density function

$$f(t) = \begin{cases} ce^{-t/3} & \text{for } t > 0, \\ 0 & \text{for } t \le 0. \end{cases}$$

   Determine the value of c and calculate the probability that a long distance call will last (a) less than 3 minutes; (b) more than 6 minutes; (c) between 3 and 6 minutes; (d) more than 9 minutes.
4. Given real constants $\lambda > 0$ and c. Let

$$f(t) = \begin{cases} \lambda e^{-\lambda(t-c)} & \text{if } t \ge c, \\ 0 & \text{if } t < c. \end{cases}$$

   Verify that $\int_{-\infty}^{\infty} f(t)\, dt = 1$, and determine a distribution function F having f as its density. This is called an exponential distribution with two parameters, a ***decay parameter*** λ and a ***location parameter*** c.
5. State and prove an extension of Theorem 14.9 for exponential distributions with two parameters λ and c.
6. A random variable X has an exponential distribution with two parameters λ and c. Let $Y = aX + b$, where $a > 0$. Prove that Y also has an exponential distribution with two parameters λ' and c', and determine these parameters in terms of a, b, c, and λ.
7. In Exercise 16 of Section 11.28 it was shown that $\int_0^\infty e^{-x^2}\, dx = \sqrt{\pi}/2$. Use this result to prove that for $\sigma > 0$ we have

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\left(\frac{u - m}{\sigma}\right)^2\right\} du = 1.$$

8. A random variable X has a standard normal distribution Φ. Prove that (a) $\Phi(-x) = 1 - \Phi(x)$; (b) $P(|X| < k) = 2\Phi(k) - 1$; (c) $P(|X| > k) = 2(1 - \Phi(k))$.
9. A random variable X has a standard normal distribution Φ. Use Table 14.1 to calculate each of the following probabilities: (a) $P(X > 0)$; (b) $P(1 < X < 2)$; (c) $P(|X| < 3)$; (d) $P(|X| > 2)$.
10. A random variable X has a standard normal distribution Φ. Use Table 14.1 to find a number c such that (a) $P(|X| > c) = \frac{1}{2}$; (b) $P(|X| > c) = 0.98$.
11. Assume X has a normal distribution function F with mean m and variance σ^2, and let Φ denote the standard normal distribution.

(a) Prove that

$$F(t) = \Phi\left(\frac{t - m}{\sigma}\right).$$

(b) Find a value of c such that $P(|X - m| > c) = \frac{1}{2}$.
(c) Find a value of c such that $P(|X - m| > c) = 0.98$.

12. A random variable X is normally distributed with mean $m = 1$ and variance $\sigma^2 = 4$. Calculate each of the following probabilities: (a) $P(-3 \leq X \leq 3)$; (b) $P(-5 \leq X \leq 3)$.
13. An architect is designing a doorway for a public building to be used by people whose heights are normally distributed, with mean m = 5 ft. 9 in., and variance σ^2 where σ = 3 in. How low can the doorway be so that no more than 1 % of the people bump their heads?
14. If X has a standard normal distribution, prove that the random variable $Y = aX + b$ is also normal if $a \neq 0$. Determine the mean and variance of Y.
15. Assume a random variable X has a standard normal distribution, and let $Y = X^2$.

(a) Show that $F_Y(t) = \dfrac{2}{\sqrt{2\pi}} \displaystyle\int_0^{\sqrt{t}} e^{-u^2/2}\, du$ if $t \geq 0$.

(b) Determine $F_Y(t)$ when $t < 0$ and describe the density function f_Y.

14.17 Distributions of functions of random variables

If φ is a real-valued function whose domain includes the range of the random variable X, we can construct a new random variable Y by the equation

$$Y = \varphi(X),$$

which means that $Y(\omega) = \varphi[X(\omega)]$ for each ω in the sample space. If we know the distribution function F_X of X, how do we find the distribution F_Y of Y? We begin with an important special case. Suppose that φ is continuous and strictly increasing on the whole real axis and takes on every real value. In this case φ has a continuous strictly increasing inverse ψ such that, for all x and y,

$$y = \varphi(x) \qquad \text{if and only if } x = \psi(y).$$

By the definition of F_Y we have

$$F_Y(t) = P(Y \leq t) = P[\varphi(X) \leq t].$$

Since φ is strictly increasing and continuous, the events "$\varphi(X) \leq t$" and "$X \leq \psi(t)$" are identical. Therefore $P[\varphi(X) \leq t] = P[X \leq \psi(t)] = F_X[\psi(t)]$. Hence the distributions F_Y and F_X are related by the equation

$$F_Y(t) = F_X[\psi(t)]. \tag{14.27}$$

When the distribution F_X and the function ψ have derivatives we can differentiate both sides of (14.27), using the chain rule on the right, to obtain

$$F_Y'(t) = F_X'[\psi(t)] \cdot \psi'(t).$$

This gives us the following equation relating the densities:

$$f_Y(t) = f_X[\psi(t)] \cdot \psi'(t).$$

EXAMPLE 1. $Y = aX + b$, $a > 0$. In this case we have

$$\varphi(x) = ax + b, \qquad \psi(y) = \frac{y - b}{a}, \qquad \psi'(y) = \frac{1}{a}.$$

Since φ is continuous and strictly increasing we may write

$$F_Y(t) = F_X\left(\frac{t-b}{a}\right) \quad \text{and} \quad f_Y(t) = \frac{1}{a} f_X\left(\frac{t-b}{a}\right).$$

EXAMPLE 2. $Y = X^2$. In this case $\varphi(x) = x^2$ and the foregoing discussion is not directly applicable because φ is not strictly increasing. However, we can use the same method of reasoning to determine F_Y and f_Y. By the definition of F_Y we have

$$F_Y(t) = P(X^2 \le t).$$

If $t < 0$ the event "$X^2 \le t$" is empty and hence $P(X^2 \le t) = 0$. Therefore $F_Y(t) = 0$ for $t < 0$. If $t > 0$ we have

$$P(X^2 \le t) = P(-\sqrt{t} \le X \le \sqrt{t}) = F_X(\sqrt{t}) - F_X(-\sqrt{t}) + P(X = -\sqrt{t}).$$

For a continuous distribution F_X we have $P(X = -\sqrt{t}) = 0$ and we obtain the following relation between F_Y and F_X:

$$F_Y(t) = \begin{cases} 0 & \text{if } t < 0, \\ F_X(\sqrt{t}) - F_X(-\sqrt{t}) & \text{if } t > 0. \end{cases}$$

For all $t < 0$ and for those $t > 0$ such that F_X is differentiable at $\sqrt{t}$ and at $-\sqrt{t}$ we have the following equation relating the densities:

$$f_Y(t) = \begin{cases} 0 & \text{if } t < 0, \\ \dfrac{f_X(\sqrt{t}) + f_X(-\sqrt{t})}{2\sqrt{t}} & \text{if } t > 0 \end{cases}$$

Further problems of this type will be discussed in Section 14.23 with the help of two-dimensional random variables.

14.18 Exercises

1. Assume X has a uniform distribution on the interval [0, 1]. Determine the distribution function F_Y and a probability density f_Y of the random variable Y if:
   (a) $Y = 3X + 1$,
   (b) $Y = -3X + 1$,
   (c) $Y = X^2$,
   (d) $Y = \log |X|$,
   (e) $Y = \log X^2$,
   (f) $Y = e^X$.

2. Let X be a random variable with a continuous distribution function F_X. If φ is continuous and strictly increasing on the whole real axis and if $\varphi(x) \to a$ as $x \to -\infty$ and $\varphi(x) \to b$ as $x \to +\infty$, determine the distribution function F_Y of the random variable $Y = \varphi(X)$. Also, compute a density f_Y, assuming that F_X and φ are differentiable.
3. Assume X has a standard normal distribution. Determine a probability density function of the random variable Y when
   (a) $Y = X^2$,
   (b) $Y = |X|^{1/2}$,
   (c) $Y = e^X$,
   (d) $Y = \arctan X$.

14.19 Distributions of two-dimensional random variables

The concept of a distribution may be generalized to n-dimensional random variables in a straightforward way. The treatment of the case $n = 2$ will indicate how the extension takes place.

If X and Y are two one-dimensional random variables defined on a common sample space S, (X, Y) will denote the two-dimensional random variable whose value at a typical point ω of S is given by the pair of real numbers $(X(\omega), Y(\omega))$. The notation

$$X \le a, \ Y \le b$$

is an abbreviation for the set of all elements ω in S such that $X(\omega) \le a$ and $Y(\omega) \le b$; the probability of this event is denoted by

$$P(X \le a, \ Y \le b).$$

Notations such as $a < X \le b$, $c < Y \le d$, and $P(a < X \le b, \ c < Y \le d)$ are similarly defined.

The set of points (x, y) such that $x \le a$ and $y \le b$ is the Cartesian product $A \times B$ of the two one-dimensional infinite intervals $A = \{x \mid x \le a\}$ and $B = \{y \mid y \le b\}$. The set $A \times B$ is represented geometrically by the infinite rectangular region shown in Figure 14.12. The number $P(X \le a, Y \le b)$ represents the probability that a point $(X(\omega), Y(\omega))$ lies in this region. These probabilities are the two-dimensional analogs of the one-dimensional probabilities $P(X \le a)$, and are used to define two-dimensional probability distributions.

DEFINITION. *The distribution function of the two-dimensional random variable (X, Y) is the real-valued function F defined for all real a and b by the equation*

$$F(a, b) = P(X \le a, \ Y \le b).$$

It is also known as the joint distribution of the two one-dimensional random variables X and Y.

To compute the probability that (X, Y) lies in a rectangle we use the following theorem, a generalization of Theorem 14.2(b).

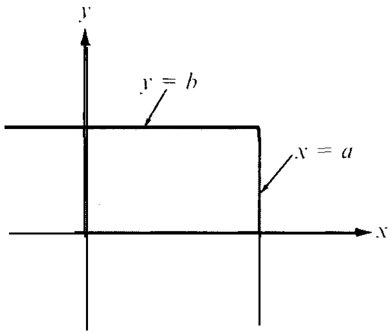

FIGURE 14.12 An infinite rectangular region $A \times B$, where $A = \{x \mid x \le a\}$ and $B = \{y \mid y \le b\}$.

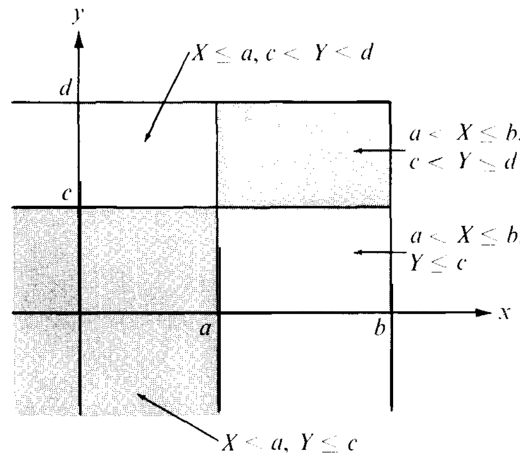

FIGURE 14.13 The event "$X \le b$, $Y \le d$" expressed as the union of four disjoint events.

THEOREM 14.10. *Let F be the distribution function of a two-dimensional random variable (X, Y). Then if $a < b$ and $c < d$ we have*

$$P(a < X \le b, c < Y \le d) = F(b, d) - F(a, d) - F(b, c) + F(a, c). \tag{14.28}$$

Proof. The two events "$X \le a$, $c < Y \le d$" and "$X \le a$, $Y \le c$" are disjoint, and their union is "$X \le a$, $Y \le d$." Adding probabilities we obtain $P(X \le a, c < Y \le d) + P(X \le a, Y \le c) = P(X \le a, Y \le d)$; hence

$$P(X \le a, c < Y \le d) = F(a, d) - F(a, c).$$

Similarly, we have

$$P(a < X \le b, Y \le c) = F(b, c) - F(a, c).$$

Now the four events

$$\text{"}X \le a, Y \le c,\text{"} \qquad \text{"}X \le a, c < Y \le d,\text{"}$$

$$\text{"}a < X \le b, Y \le c,\text{"} \qquad \text{"}a < X \le b, c < Y \le d\text{"}$$

are disjoint, and their union is "$X \le b$, $Y \le d$." (See Figure 14.13.) Adding the corresponding probabilities and using the two foregoing equations we obtain

$$F(a, c) + [F(a, d) - F(a, c)] + [F(b, c) - F(a, c)] + P(a < X \le b, c < Y \le d) = F(b,d),$$

which is equivalent to (14.28).

Formula (14.28) gives the probability that the random variable (X, Y) has a value in the rectangle $(a, b] \times (c, d]$. There are, of course, corresponding formulas for the rectangles, $[a, b] \times [c, d]$, $(a, b) \times (c, d)$, $[a, b) \times [c, d)$, and so forth.

Note: The analogy with mass may be extended to the two-dimensional case. Here the total mass 1 is distributed over a plane. The probability $P(a < X \le b, c < Y \le d)$ represents the total amount of mass located in the rectangle $(a, b] \times (c, d]$. The number $F(a, b)$ represents the amount in the infinite rectangular region $X \le a$, $Y \le b$. As in the one-dimensional case, the two most important types of distributions are those known as *discrete* and *continuous*. In the discrete case the entire mass is located in lumps concentrated at a finite or countably infinite number of points. In the continuous case the mass is smeared all over the plane with a uniform or varying thickness.

14.20 Two-dimensional discrete distributions

If a random variable (X, Y) is given we define a new function p, called the probability mass function of (X, Y), such that

$$p(x, y) = P(X = x, \ Y = y)$$

for every pair of real numbers (x, y). Let T denote the set of (x, y) for which $p(x, y) > 0$. It can be shown that T is either finite or countably infinite. If the sum of the $p(x, y)$ for all (x, y) in T is equal to 1, that is, if

$$\sum_{(x,y)\in T} p(x, y) = 1, \tag{14.29}$$

the random variable (X, Y) is said to be *discrete* (or jointly discrete). The points (x, y) in T are called the *mass points* of (X, Y).

Suppose that $x_1, x_2, x_3, \ldots$ and $y_1, y_2, y_3, \ldots$ are among the possible values of X and Y, respectively, and let

$$p_{ij} = P(X = x_i, Y = y_j).$$

If each p_{ij} is positive and if the sum of all the p_{ij} is 1, then the probability of an event "$(X, Y) \in E$" is the sum of all the p_{ij} taken over all x_i and y_j for which $(x_i, y_j) \in E$. We indicate this by writing

$$P[(X, Y) \in E] = \sum_{\substack{x_i \ y_j \\ (x_i, y_j)\in E}}\sum p_{ij}.$$

In particular, since $P(X \le x, Y \le y) = F(x, y)$, the joint distribution F (which is also called discrete) is given by the double sum

$$F(x, y) = \sum_{x_i \le x} \sum_{y_j \le y} p_{ij}. \tag{14.30}$$

The numbers p_{ij} can also be used to reconstruct the probability mass functions p_X and p_Y of the one-dimensional random variables X and Y. In fact, if E_{ij} denotes the event "$X = x_i$, $Y = y_j$," the events $E_{i1}, E_{i2}, E_{i3}, \ldots$ are disjoint and their union is the event "$X = x_i$." Hence, by countable additivity, we obtain

$$P(X = x_i) = \sum_{j=1}^{\infty} P(E_{ij}) = \sum_{j=1}^{\infty} p_{ij}. \tag{14.31}$$

Similarly, we have

$$P(Y = y_j) = \sum_{i=1}^{\infty} P(E_{ij}) = \sum_{i=1}^{\infty} p_{ij}. \tag{14.32}$$

Therefore, the corresponding one-dimensional distributions F_X and F_Y can be computed from the formulas

$$F_X(t) = \sum_{x_i \le t} P(X = x_i) = \sum_{x_i \le t} \sum_{j=1}^{\infty} p_{ij}$$

and

$$F_Y(t) = \sum_{y_j \le t} P(Y = y_j) = \sum_{y_j \le t} \sum_{i=1}^{\infty} p_{ij}.$$

For finite sample spaces, of course, the infinite series are actually finite sums.

14.21 Two-dimensional continuous distributions. Density functions

As might be expected, *continuous distributions* are those that are continuous over the whole plane. For the majority of continuous distributions F that occur in practice there exists a nonnegative function f (called the *probability density* of F) such that the probabilities of most events of interest can be computed by double integration of the density. That is, the probability of an event "$(X, Y) \in Q$" is given by the integral formula

$$P[(X, Y) \in Q] = \iint_Q f. \tag{14.33}$$

When such an f exists it is also called a probability density of the random variable (X, Y), or a joint density of X and Y. We shall not attempt to describe the class of regions Q for which (14.33) is to hold, except to mention that this class should be extensive enough to include all regions that arise in the ordinary applications of probability. For example, if a joint density exists we always have

$$P(a < X \le b, c < Y \le d) = \iint_R f(x, y)\, dx\, dy, \tag{14.34}$$

where $R = [a, b] \times [c, d]$. The integrand f is usually sufficiently well behaved for the double integral to be evaluated by iterated one-dimensional integration, in which case **(14.34)** becomes

$$P(a < X \le b, c < Y \le d) = \int_c^d \left[\int_a^b f(x, y)\, dx\right] dy = \int_a^b \left[\int_c^d f(x, y)\, dy\right] dx.$$

In all the examples we shall consider, this formula is also valid in the limiting cases in which a and c are replaced by $-\infty$ and in which b and d are replaced by $+\infty$. Thus we have

$$(14.35)\qquad F(b, d) = \int_{-\infty}^{d}\left[\int_{-\infty}^{b} f(x, y)\, dx\right] dy = \int_{-\infty}^{b}\left[\int_{-\infty}^{d} f(x, y)\, dy\right] dx$$

for all b and d, and

$$(14.36)\qquad \int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} f(x, y)\, dx\right] dy = \int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} f(x, y)\, dy\right] dx = 1.$$

Equations (14.35) and (14.36) are the continuous analogs of (14.30) and (14.29), respectively.

If a density exists it is not unique, since the integrand in (14.33) can be changed at a finite number of points without affecting the value of the integral. However, there is at most one continuous density function. In fact, at points of continuity of f we have the formulas

$$f(x, y) = D_{1,2}F(x, y) = D_{2,1}F(x, y),$$

obtained by differentiation of the integrals in (14.35).

As in the discrete case, the joint density f can be used to recover the one-dimensional densities f_X and f_Y. The formulas analogous to (14.31) and (14.32) are

$$f_X(x) = \int_{-\infty}^{+\infty} f(x, y)\,dy \qquad \text{and} \qquad f_Y(y) = \int_{-\infty}^{+\infty} f(x, y)\, dx.$$

The corresponding distributions $F_X(t)$ and $F_Y(t)$ are obtained, of course, by integrating the respective densities f_X and f_Y from $-\infty$ to t.

The random variables X and Y are called *independent* if the joint distribution $F(x, y)$ can be factored as follows,

$$F(x, y) = F_X(x)F_Y(y)$$

for all (x, y). Some consequences of independence are discussed in the next set of exercises.

EXAMPLE. Consider the function f that has the constant value 1 over the square $R = [0, 1] \times [0, 1]$, and the value 0 at all other points of the plane. A random variable (X, Y) having this density function is said to be uniformly distributed over R. The corresponding distributions function F is given by the following formulas:

$$F(x, y) = \begin{cases} xy & \text{if } (x, y) \in R, \\ x & \text{if } 0 < x < 1 \text{ and } y > 1, \\ y & \text{if } 0 < y < 1 \text{ and } x > 1, \\ 1 & \text{if } x \geq 1 \text{ and } y \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

The graph of F over R is part of the saddle-shaped surface $z = xy$. At all points (x, y) not on the boundary of R the mixed partial derivatives $D_{1,2}F(x, y)$ and $D_{2,1}F(x, y)$ exist and are equal to $f(x, y)$. This distribution is the product of two one-dimensional uniform distributions F_X and F_Y. Hence X and Y are independent.

14.22 Exercises

1. Let X and Y be one-dimensional random variables with distribution functions F_X and F_Y, and let F be the joint distribution of X and Y.
(a) Prove that X and Y are independent if and only if we have

$$P(a < X \le b, c < Y \le d) = P(a < X \le b)P(c < Y \le d)$$

for all a, b, c, d, with a < b and c < d.
(b) Consider the discrete case. Assume $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ are the mass points of X and Y, respectively. Let $a_i = P(X = x_i)$ and $b_j = P(Y = y_j)$. If $p_{ij} = P(X = x_i, Y = y_j)$, show that X and Y are independent if $p_{ij} = a_i b_j$ for all i and j.
(c) Let X and Y have continuous distributions with corresponding densities f_X and f_Y and let f denote the density of the joint distribution. Assume the continuity of all three densities. Show that the condition of independence is equivalent to the statement $f(x, y) = f_X(x)f_Y(y)$ for all (x, y). [*Hint:* Expressfas a derivative of the joint distribution F.]
2. Refer to Exercise 1. Suppose that $P(X = x_1, Y = y_1) = P(X = x_2, Y = y_2) = p/2$ and that $P(X = x_1, Y = y_2) = P(X = x_2, Y = y_1) = q/2$, where p and q are nonnegative with sum 1.
(a) Determine the one-dimensional probabilities $P(X = x_i)$ and $P(Y = y_j)$ for $i = 1, 2$ and $j = 1, 2$.
(b) For what value (or values) of p will X and Y be independent?
3. If a < b and c < d, definefas follows:

$$f(x, y) = \begin{cases} \dfrac{1}{(b-a)(d-c)} & \text{if } (x, y) \in [a, b] \times [c, d], \\ 0 & \text{otherwise.} \end{cases}$$

(a) Verify that this is the density of a continuous distribution F and determine F.
(b) Determine the one-dimensional distributions F_X and F_Y.
(c) Determine whether or not X and Y are independent.
4. If $P(Y \le b) \ne 0$, the conditional probability that X $\le$ a, given that Y $\le b$, is denoted by $P(X \le a \mid Y \le b)$, and is defined by the equation

$$P(X \le a \mid Y \le b) = \frac{P(X \le a, Y \le b)}{P(Y \le b)}.$$

If $P(Y \le b) = 0$, we define $P(X \le a \mid Y \le b) = P(X \le a)$. Similarly, if $P(X \le a) \ne 0$, we define $P(Y \le b \mid X \le a) = P(X \le a, Y \le b)/P(X \le a)$. If $P(X \le a) = 0$, we define $P(Y \le b \mid X \le a) = P(Y \le b)$.
(a) Refer to Exercise 1 and describe the independence of X and Y in terms of conditional probabilities.
(b) Consider the discrete case. Assume $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ are the mass points of X and Y, respectively. Show that

$$P(X = x_i) = \sum_{j=1}^{\infty} P(Y = y_j)P(X = x_i \mid Y = y_j)$$

and

$$P(Y = y_j) = \sum_{i=1}^{\infty} P(X = x_i)P(Y = y_j \mid X = x_i).$$

5. A gambling house offers its clients the following game: A coin is tossed. If the result of the first throw is tails, the player loses and the game is over. If the first throw is heads, a second throw is allowed. If heads occur the second time the player wins \$2, but if tails comes up the player wins \$1. Let X be the random variable which is equal to 1 or 0, according to whether heads or tails occurs on the first throw. Let Y be the random variable which counts the number of dollars won by the player. Use Exercise 4 (or some other method) to compute $P(Y = 0)$, $P(Y = 1)$, and $P(Y = 2)$.
6. Refer to Exercise 4. Derive the so-called Bayes' formulas :

$$P(X = x_k \mid Y = y_j) = \frac{P(X = x_k)P(Y = y_j \mid X = x_k)}{\sum_{i=1}^{\infty} P(X = x_i)P(Y = y_j \mid X = x_i)},$$

$$P(Y = y_k \mid X = x_i) = \frac{P(Y = y_k)P(X = x_i \mid Y = y_k)}{\sum_{j=1}^{\infty} P(Y = y_j)P(X = x_i \mid Y = y_j)}.$$

7. Given two urns A and B. Urn A contains one \$5 bill and two \$10 bills. Urn B contains three \$5 bills and one \$10 bill. Draw a bill from urn A and put it in urn B. Let Y be the random variable which counts the dollar value of the bill transferred. Now draw a bill from urn B and use the random variable Xto count its dollar value. Compute the conditional probabilities

$$P(Y = 5 \mid X = 10) \qquad \text{and} \qquad P(Y = 10 \mid X = 10).$$

[Hint: Use Bayes' formulas of Exercise 6.]
8. Given three identical boxes, each containing two drawers. Box number 1 has one gold piece in one drawer and one silver piece in the other. Box 2 has one gold piece in each drawer and Box 3 has one silver piece in each drawer. One drawer is opened at random and a gold piece is found. Compute the probability that the other drawer in the same box contains a silver piece. [Hint: Use Bayes' formulas of Exercise 6.]
9. Let Q be a plane region with positive area u(Q). A continuous two-dimensional random variable (X, Y) is said to have a *uniform distribution* over Q if its density functionfis given by the following formulas :

$$f(x, y) = \begin{cases} 1/a(Q) & \text{if } (x, y) \in Q, \\ 0 & \text{if } (x, y) \notin Q. \end{cases}$$

(a) If E is a subregion of Q with area $a(E)$, show that $a(E)/a(Q)$ is the probability of the event $(X, Y) \in E$.
(b) Raindrops fall at random on the square Q with vertices (1, 0), (0, 1), (-1, 0), (0, -1). An outcome is the point (x, y) in Q struck by a particular raindrop. Let $X(x, y) = x$ and $Y(x, y) = y$ and assume (X, Y) has a uniform distribution over Q. Determine the joint density function f and the one-dimensional densities f_X and f_Y. Are the random variables X and Y independent?
10. A two-dimensional random variable (X, Y) has the joint distribution function F. Let $U = X - a$, $V = Y - b$, where a and b are constants. If G denotes the joint distribution of (U, V) show that

$$G(u, v) = F(u + a, v + b).$$

Derive a similar relation connecting the density functionfof (X, Y) andg of (U, V) whenfis continuous.

14.23 Distributions of functions of two random variables

We turn now to the following problem: If X and Y are one-dimensional random variables with known distributions, how do we find the distributionof new random variables such as X + Y, XY, or $X^2 + Y^2$? This section describes a method that helps to answer questions like this. Two new random variables U and V are defined by equations of the form

$$U = M(X, Y), \qquad V = N(X, Y),$$

where $M(X, Y)$ or $N(X, Y)$ is the particular combination in which we are interested. From a knowledge of the joint distribution f of the two-dimensional random variable (X, Y) we calculate the joint distribution g of (U, V). Once g is known, the individual distributions of U and V are easily found.

To describe the method in detail, we consider a one-to-one mapping of the xy-plane onto the uv-plane defined by the pair of equations

$$u = M(x, y), \qquad v = N(x, y).$$

Let the inverse mapping be given by

$$x = Q(u, v), \qquad y = R(u, v),$$

and assume that Q and R have continuous partial derivatives. If T denotes a region in the xy-plane, let T' denote its image in the uv-plane, as suggested by Figure 14.14. Let X and Y be two one-dimensional continuous random variables having a continuous joint distribution and assume (X, Y) has a probability density function f. Define new random variables U and V by writing $U = M(X, Y)$, $V = N(X, Y)$. To determine a probability density g of the random variable (U, V) we proceed as follows:

The random variables X and Y are associated with a sample space S. For each ω in S we have $U(\omega) = M[X(\omega), Y(\omega)]$ and $V(\omega) = N[X(\omega), Y(\omega)]$. Since the mapping is one-to-one, the two sets

$$\{\omega \mid (U(\omega), V(\omega)) \in T'\} \qquad \text{and} \qquad \{\omega \mid (X(\omega), Y(\omega)) \in T\}$$

are equal. Therefore we have

$$P[(U, V) \in T'] = P[(X, Y) \in T]. \tag{14.37}$$

Since f is the density function of (X, Y) we can write

$$P[(X, Y) \in T] = \iint_T f(x, y)\, dx\, dy. \tag{14.38}$$

Using (14.37) and the formula for transforming a double integral we rewrite (14.38) as follows :

$$P[(U, V) \in T'] = \iint_{T'} f[Q(u, v), R(u, v)] \left| \frac{\partial(Q, R)}{\partial(u, v)} \right| du\, dv.$$

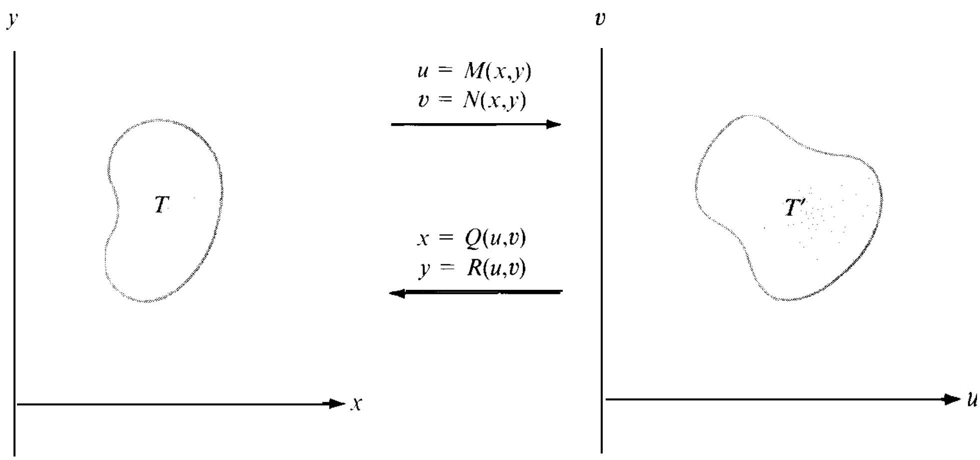

FIGURE 14.14 A one-to-one mapping of a region T in the xy-plane onto a region T' in the uv-plane.

Since this is valid for every region T' in the uu-plane a density g of (U, V) is given by the integrand on the right; that is, we have

(14.39) $$g(u, v) = f[Q(u, v), R(u, v)]\left|\frac{\partial(Q, R)}{\partial(u, v)}\right|.$$

The densities f_U and f_V can now be obtained by the integration formulas

$$f_U(u) = \int_{-\infty}^{\infty} g(u, v)\, dv, \qquad f_V(v) = \int_{-\infty}^{\infty} g(u, v)\, du.$$

EXAMPLE 1. ***The sum and difference of two random variables.*** Given two one-dimensional random variables X and Y with joint density f, determine density functions for the random variables $U = X + Y$ and $V = X - Y$.

Solution. We use the mapping given by $u = x + y$, $v = x - y$. This is a nonsingular linear transformation whose inverse is given by

$$x = \frac{u + v}{2} = Q(u, v), \qquad y = \frac{u - v}{2} = R(u, v).$$

The Jacobian determinant is

$$\frac{\partial(Q, R)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial Q}{\partial u} & \dfrac{\partial Q}{\partial v} \\ \dfrac{\partial R}{\partial u} & \dfrac{\partial R}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{2} & \dfrac{1}{2} \\ \dfrac{1}{2} & -\dfrac{1}{2} \end{vmatrix} = -\frac{1}{2}.$$