noted $O(n, R)$ and $O(n, C)$. Of course, the orthogonal group is also the group which preserves the quadratic form

$$q(x_1, \ldots, x_n) = x_1^2 + \cdots + x_n^2.$$

EXAMPLE 7. Let f be the symmetric bilinear form on R^n with quadratic form

$$q(x_1, \ldots, x_n) = \sum_{j=1}^{p} x_j^2 - \sum_{j=p+1}^{n} x_j^2.$$

Then f is non-degenerate and has signature $2p - n$. The group of matrices preserving a form of this type is called a **pseudo-orthogonal group.** When $p = n$, we obtain the orthogonal group $O(n, R)$ as a particular type of pseudo-orthogonal group. For each of the $n + 1$ values $p = 0, 1, 2, \ldots, n$, we obtain different bilinear forms f; however, for $p = k$ and $p = n - k$ the forms are negatives of one another and hence have the same associated group. Thus, when n is odd, we have $(n + 1)/2$ pseudo-orthogonal groups of $n \times n$ matrices, and when n is even, we have $(n + 2)/2$ such groups.

Theorem 8. *Let* V *be an* n*-dimensional vector space over the field of complex numbers, and let* f *be a non-degenerate symmetric bilinear form on* V. *Then the group preserving* f *is isomorphic to the complex orthogonal group* O(n, C).

Proof. Of course, by an isomorphism between two groups, we mean a one-one correspondence between their elements which 'preserves' the group operation. Let G be the group of linear operators on V which preserve the bilinear form f. Since f is both symmetric and non-degenerate, Theorem 4 tells us that there is an ordered basis $\mathcal{B}$ for V in which f is represented by the $n \times n$ identity matrix. Therefore, a linear operator T preserves f if and only if its matrix in the ordered basis $\mathcal{B}$ is a complex orthogonal matrix. Hence

$$T \to [T]_{\mathcal{B}}$$

is an isomorphism of G onto $O(n, C)$. ∎

Theorem 9. *Let* V *be an* n*-dimensional vector space over the field of real numbers, and let* f *be a non-degenerate symmetric bilinear form on* V. *Then the group preserving* f *is isomorphic to an* n × n *pseudo-orthogonal group.*

Proof. Repeat the proof of Theorem 8, using Theorem 5 instead of Theorem 4. ∎

EXAMPLE 8. Let f be the symmetric bilinear form on R^4 with quadratic form

$$q(x, y, z, t) = t^2 - x^2 - y^2 - z^2.$$

A linear operator T on R^4 which preserves this particular bilinear (or quadratic) form is called a **Lorentz transformation,** and the group preserving f is called the **Lorentz group.** We should like to give one method of describing some Lorentz transformations.

Let H be the real vector space of all 2×2 complex matrices A which are Hermitian, $A = A^*$. It is easy to verify that

$$\Phi(x, y, z, t) = \begin{bmatrix} t + x & y + iz \\ y - iz & t - x \end{bmatrix}$$

defines an isomorphism Φ of R^4 onto the space H. Under this isomorphism, the quadratic form q is carried onto the determinant function, that is

$$q(x, y, z, t) = \det \begin{bmatrix} t + x & y + iz \\ y - iz & t - x \end{bmatrix}$$

or

$$q(\alpha) = \det \Phi(\alpha).$$

This suggests that we might study Lorentz transformations on R^4 by studying linear operators on H which preserve determinants.

Let M be any complex 2×2 matrix and for a Hermitian matrix A define

$$U_M(A) = MAM^*.$$

Now MAM^* is also Hermitian. From this it is easy to see that U_M is a (real) linear operator on H. Let us ask when it is true that U_M 'preserves' determinants, i.e., $\det [U_M(A)] = \det A$ for each A in H. Since the determinant of M^* is the complex conjugate of the determinant of M, we see that

$$\det [U_M(A)] = |\det M|^2 \det A.$$

Thus U_M preserves determinants exactly when $\det M$ has absolute value 1.

So now let us select any 2×2 complex matrix M for which $|\det M| = 1$. Then U_M is a linear operator on H which preserves determinants. Define

$$T_M = \Phi^{-1} U_M \Phi.$$

Since Φ is an isomorphism, T_M is a linear operator on R^4. Also, T_M is a Lorentz transformation; for

$$\begin{aligned} q(T_M\alpha) &= q(\Phi^{-1}U_M\Phi\alpha) \\ &= \det (\Phi\Phi^{-1}U_M\Phi\alpha) \\ &= \det (U_M\Phi\alpha) \\ &= \det (\Phi\alpha) \\ &= q(\alpha) \end{aligned}$$

and so T_M preserves the quadratic form q.

By using specific 2×2 matrices M, one can use the method above to compute specific Lorentz transformations. There are two comments which we might make here; they are not difficult to verify.

(1) If M_1 and M_2 are invertible 2×2 matrices with complex entries, then $U_{M_1} = U_{M_2}$ if and only if M_2 is a scalar multiple of M_1. Thus, all of the Lorentz transformations exhibited above are obtainable from unimodular matrices M, that is, from matrices M satisfying $\det M = 1$. If M_1 and M_2 are unimodular matrices such that $M_1 \neq M_2$ and $M_1 \neq -M_2$, then $T_{M_1} \neq T_{M_2}$.

(2) Not every Lorentz transformation is obtainable by the above method.

Exercises

1. Let M be a member of the complex orthogonal group, $O(n, C)$. Show that M^t, $\overline{M}$, and $M^* = \overline{M}^t$ also belong to $O(n, C)$.

2. Suppose M belongs to $O(n, C)$ and that M' is similar to M. Does M' also belong to $O(n, C)$?

3. Let

$$y_j = \sum_{k=1}^{n} M_{jk}x_k$$

where M is a member of $O(n, C)$. Show that

$$\sum_j y_j^2 = \sum_j x_j^2.$$

4. Let M be an $n \times n$ matrix over C with columns $M_1, M_2, \ldots, M_n$. Show that M belongs to $O(n, C)$ if and only if

$$M_j^t M_k = \delta_{jk}.$$

5. Let X be an $n \times 1$ matrix over C. Under what conditions does $O(n, C)$ contain a matrix M whose first column is X?

6. Find a matrix in $O(3, C)$ whose first row is $(2i, 2i, 3)$.

7. Let V be the space of all $n \times 1$ matrices over C and f the bilinear form on V given by $f(X, Y) = X^tY$. Let M belong to $O(n, C)$. What is the matrix of f in the basis of V consisting of the columns $M_1, M_2, \ldots, M_n$ of M?

8. Let X be an $n \times 1$ matrix over C such that $X^tX = 1$, and I_j be the jth column of the identity matrix. Show there is a matrix M in $O(n, C)$ such that $MX = I_j$. If X has real entries, show there is an M in $O(n, R)$ with the property that $MX = I_j$.

9. Let V be the space of all $n \times 1$ matrices over C, A an $n \times n$ matrix over C, and f the bilinear form on V given by $f(X, Y) = X^tAY$. Show that f is invariant under $O(n, C)$, i.e., $f(MX, MY) = f(X, Y)$ for all X, Y in V and M in $O(n, C)$, if and only if A commutes with each member of $O(n, C)$.

10. Let S be any set of $n \times n$ matrices over C and S' the set of all $n \times n$ matrices over C which commute with each element of S. Show that S' is an algebra over C.

11. Let F be a subfield of C, V a finite-dimensional vector space over F, and f a non-singular bilinear form on V. If T is a linear operator on V preserving f, prove that $\det T = \pm 1$.

12. Let F be a subfield of C, V the space of $n \times 1$ matrices over F, A an invertible $n \times n$ matrix over F, and f the bilinear form on V given by $f(X, Y) = X^tAY$. If M is an $n \times n$ matrix over F, show that M preserves f if and only if $A^{-1}M^tA = M^{-1}$.

13. Let g be a non-singular bilinear form on a finite-dimensional vector space V. Suppose T is an invertible linear operator on V and that f is the bilinear form on V given by $f(\alpha, \beta) = g(\alpha, T\beta)$. If U is a linear operator on V, find necessary and sufficient conditions for U to preserve f.

14. Let T be a linear operator on C^2 which preserves the quadratic form $x_1^2 - x_2^2$. Show that

(a) $\det (T) = \pm 1$.

(b) If M is the matrix of T in the standard basis, then $M_{22} = \pm M_{11}$, $M_{21} = \pm M_{12}$, $M_{11}^2 - M_{12}^2 = 1$.

(c) If $\det M = 1$, then there is a non-zero complex number c such that

$$M = \frac{1}{2}\begin{bmatrix} c + \frac{1}{c} & c - \frac{1}{c} \\ c - \frac{1}{c} & c + \frac{1}{c} \end{bmatrix}.$$

(d) If $\det M = -1$ then there is a complex number c such that

$$M = \frac{1}{2}\begin{bmatrix} c + \frac{1}{c} & c - \frac{1}{c} \\ -c + \frac{1}{c} & -c - \frac{1}{c} \end{bmatrix}.$$

15. Let f be the bilinear form on C^2 defined by

$$f((x_1, x_2), (y_1, y_2)) = x_1y_2 - x_2y_1.$$

Show that

(a) if T is a linear operator on C^2, then $f(T\alpha, T\beta) = (\det T)f(\alpha, \beta)$ for all α, β in C^2.

(b) T preserves f if and only if $\det T = +1$.

(c) What does (b) say about the group of 2×2 matrices M such that $M^tAM = A$ where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}?$$

16. Let n be a positive integer, I the $n \times n$ identity matrix over C, and J the $2n \times 2n$ matrix given by

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

Let M be a $2n \times 2n$ matrix over C of the form

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where A, B, C, D are $n \times n$ matrices over C. Find necessary and sufficient conditions on A, B, C, D in order that $M^tJM = J$.

17. Find all bilinear forms on the space of $n \times 1$ matrices over R which are invariant under $O(n, R)$.

18. Find all bilinear forms on the space of $n \times 1$ matrices over C which are invariant under $O(n, C)$.

Appendix

This Appendix separates logically into two parts. The first part, comprising the first three sections, contains certain fundamental concepts which occur throughout the book (indeed, throughout mathematics). It is more in the nature of an introduction for the book than an appendix. The second part is more genuinely an appendix to the text.

Section 1 contains a discussion of sets, their unions and intersections. Section 2 discusses the concept of function, and the related ideas of range, domain, inverse function, and the restriction of a function to a subset of its domain. Section 3 treats equivalence relations. The material in these three sections, especially that in Sections 1 and 2, is presented in a rather concise manner. It is treated more as an agreement upon terminology than as a detailed exposition. In a strict logical sense, this material constitutes a portion of the prerequisites for reading the book; however, the reader should not be discouraged if he does not completely grasp the significance of the ideas on his first reading. These ideas are important, but the reader who is not too familiar with them should find it easier to absorb them if he reviews the discussion from time to time while reading the text proper.

Sections 4 and 5 deal with equivalence relations in the context of linear algebra. Section 4 contains a brief discussion of quotient spaces. It can be read at any time after the first two or three chapters of the book. Section 5 takes a look at some of the equivalence relations which arise in the book, attempting to indicate how some of the results in the book might be interpreted from the point of view of equivalence relations. Section 6 describes the Axiom of choice and its implications for linear algebra.