Chapter 9

Continuous functions on R

In previous chapters we have been focusing primarily on *sequences*. A sequence $(a_n)_{n=0}^\infty$ can be viewed as a function from $\mathbf{N}$ to $\mathbf{R}$, i.e., an object which assigns a real number a_n to each natural number n. We then did various things with these functions from $\mathbf{N}$ to $\mathbf{R}$, such as take their limit at infinity (if the function was convergent), or form suprema, infima, etc., or computed the sum of all the elements in the sequence (again, assuming the series was convergent).

Now we will look at functions not on the natural numbers $\mathbf{N}$, which are "discrete", but instead look at functions on a *continuum*[1] such as the real line $\mathbf{R}$, or perhaps on an interval such as $\{x \in \mathbf{R} : a \leq x \leq b\}$. Eventually we will perform a number of operations on these functions, including taking limits, computing derivatives, and evaluating integrals. In this chapter we will focus primarily on limits of functions, and on the closely related concept of a *continuous function.*

Before we discuss functions, though, we must first set out some notation for subsets of the real line.

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[1]We will not rigourously define the notion of a discrete set or a continuum in this text, but roughly speaking a set is discrete if each element is separated from the rest of the set by some non-zero distance, whereas a set is a *continuum* if it is connected and contains no "holes".

9.1 Subsets of the real line

Very often in analysis we do not work on the whole real line $\mathbf{R}$, but on certain subsets of the real line, such as the positive real axis $\{x \in \mathbf{R} : x > 0\}$. Also, we occasionally work with the extended real line $\mathbf{R}^*$ defined in Section 6.2, or in subsets of that extended real line.

There are of course infinitely many subsets of the real line; indeed, Cantor's theorem (Theorem 8.3.1; see also Exercise 8.3.4) shows that there are even more such sets than there are real numbers. However, there are certain special subsets of the real line (and the extended real line) which arise quite often. One such family of sets are the *intervals*.

Definition 9.1.1 (Intervals). Let $a, b \in \mathbf{R}^*$ be extended real numbers. We define the *closed interval* $[a, b]$ by

$$[a, b] := \{x \in \mathbf{R}^* : a \leq x \leq b\},$$

the *half-open intervals* $[a, b)$ and $(a, b]$ by

$$[a, b) := \{x \in \mathbf{R}^* : a \leq x < b\}; \quad (a, b] := \{x \in \mathbf{R}^* : a < x \leq b\},$$

and the *open intervals* (a, b) by

$$(a, b) := \{x \in \mathbf{R}^* : a < x < b\}.$$

We call a the *left endpoint* of these intervals, and b the *right endpoint*.

Remark 9.1.2. Once again, we are overloading the parenthesis notation; for instance, we are now using $(2, 3)$ to denote both an open interval from 2 to 3, as well as an ordered pair in the Cartesian plane $\mathbf{R}^2 := \mathbf{R} \times \mathbf{R}$. This can cause some genuine ambiguity, but the reader should still be able to resolve which meaning of the parentheses is intended from context. In some texts, this issue is resolved by using reversed brackets instead of parenthesis, thus for instance $[a, b)$ would now be $[a, b[$, $(a, b]$ would be $]a, b]$, and (a, b) would be $]a, b[$.

Examples 9.1.3. If a and b are real numbers (i.e., not equal to $+\infty$ or $-\infty$) then all of the above intervals are subsets of the real line, for instance $[2,3) = \{x \in \mathbf{R} : 2 \leq x < 3\}$. The positive real axis $\{x \in \mathbf{R} : x > 0\}$ is the open interval $(0, +\infty)$, while the non-negative real axis $\{x \in \mathbf{R} : x \geq 0\}$ is the half-open interval $[0, +\infty)$. Similarly, the negative real axis $\{x \in \mathbf{R} : x < 0\}$ is $(-\infty, 0)$, and the non-positive real axis $\{x \in \mathbf{R} : x \leq 0\}$ is $(-\infty, 0]$. Finally, the real line $\mathbf{R}$ itself is the open interval $(-\infty, +\infty)$, while the extended real line $\mathbf{R}^*$ is the closed interval $[-\infty, +\infty]$. We sometimes refer to an interval in which one endpoint is infinite (either $+\infty$ or $-\infty$) as *half-infinite* intervals, and intervals in which both endpoints are infinite as *doubly-infinite* intervals; all other intervals are *bounded intervals.* Thus $[2,3)$ is a bounded interval, the positive and negative real axes are half-infinite intervals, and $\mathbf{R}$ and $\mathbf{R}^*$ are infinite intervals.

Example 9.1.4. If $a > b$ then all four of the intervals $[a,b]$, $[a,b)$, $(a,b]$, and (a,b) are the empty set (why?). If $a = b$, then the three intervals $[a,b)$, $(a,b]$, and (a,b) are the empty set, while $[a,b]$ is just the singleton set $\{a\}$ (why?). Because of this, we call these intervals *degenerate*; most (but not all) of our analysis will be restricted to non-degenerate intervals.

Of course intervals are not the only interesting subsets of the real line. Other important examples include the *natural numbers* $\mathbf{N}$, the *integers* $\mathbf{Z}$, and the *rationals* $\mathbf{Q}$. One can form additional sets using such operations as union and intersection (see Section 3.1), for instance one could have a disconnected union of two intervals such as $(1,2) \cup [3,4]$, or one could consider the set $[-1,1] \cap \mathbf{Q}$ of rational numbers between -1 and 1 inclusive. Clearly there are infinitely many possibilities of sets one could create by such operations.

Just as sequences of real numbers have limit points, sets of real numbers have *adherent points*, which we now define.

Definition 9.1.5 (ε-adherent points). Let X be a subset of $\mathbf{R}$, let $\varepsilon > 0$, and let $x \in \mathbf{R}$. We say that x is *ε-adherent to X* iff there exists a $y \in X$ which is ε-close to x (i.e., $|x - y| \leq \varepsilon$).

Remark 9.1.6. The terminology "ε-adherent" is not standard in the literature. However, we shall shortly use it to define the notion of an adherent point, which is standard.

Example 9.1.7. The point 1.1 is 0.5-adherent to the open interval $(0, 1)$, but is not 0.1-adherent to this interval (why?). The point 1.1 is 0.5-adherent to the finite set $\{1, 2, 3\}$. The point 1 is 0.5-adherent to $\{1, 2, 3\}$ (why?).

Definition 9.1.8 (Adherent points). Let X be a subset of $\mathbf{R}$, and let $x \in \mathbf{R}$. We say that x is an *adherent point* of X iff it is ε-adherent to X for every $\varepsilon > 0$.

Example 9.1.9. The number 1 is ε-adherent to the open interval $(0, 1)$ for every $\varepsilon > 0$ (why?), and is thus an adherent point of $(0, 1)$. The point 0.5 is similarly an adherent point of $(0, 1)$. However, the number 2 is not 0.5-adherent (for instance) to $(0, 1)$, and is thus not an adherent point to $(0, 1)$.

Definition 9.1.10 (Closure). Let X be a subset of $\mathbf{R}$. The *closure* of X, sometimes denoted $\overline{X}$ is defined to be the set of all the adherent points of X.

Lemma 9.1.11 (Elementary properties of closures). *Let X and Y be arbitrary subsets of $\mathbf{R}$. Then $X \subseteq \overline{X}$, $\overline{X \cup Y} = \overline{X} \cup \overline{Y}$, and $\overline{X \cap Y} \subseteq \overline{X} \cap \overline{Y}$. If $X \subseteq Y$, then $\overline{X} \subseteq \overline{Y}$.*

Proof. See Exercise 9.1.2. □

We now compute some closures.

Lemma 9.1.12 (Closures of intervals). *Let $a < b$ be real numbers, and let I be any one of the four intervals (a, b), $(a, b]$, $[a, b)$, or $[a, b]$. Then the closure of I is $[a, b]$. Similarly, the closure of (a, ∞) or $[a, \infty)$ is $[a, \infty)$, while the closure of $(-\infty, a)$ or $(-\infty, a]$ is $(-\infty, a]$. Finally, the closure of $(-\infty, \infty)$ is $(-\infty, \infty)$.*

Proof. We will just show one of these facts, namely that the closure of (a, b) is $[a, b]$; the other results are proven similarly (or one can use Exercise 9.1.1).

First let us show that every element of $[a, b]$ is adherent to (a, b). Let $x \in [a, b]$. If $x \in (a, b)$ then it is definitely adherent to (a, b). If $x = b$ then x is also adherent to (a, b) (why?). Similarly when $x = a$. Thus every point in $[a, b]$ is adherent to (a, b).

Now we show that every point x that is adherent to (a, b) lies in $[a, b]$. Suppose for sake of contradiction that x does not lie in $[a, b]$, then either $x > b$ or $x < a$. If $x > b$ then x is not $(x - b)$-adherent to (a, b) (why?), and is hence not an adherent point to (a, b). Similarly, if $x < a$, then x is not $(a - x)$-adherent to $(a - b)$, and is hence not an adherent point to (a, b). This contradiction shows that x is in fact in $[a, b]$ as claimed. $\square$

Lemma 9.1.13. *The closure of* $\mathbf{N}$ *is* $\mathbf{N}$. *The closure of* $\mathbf{Z}$ *is* $\mathbf{Z}$. *The closure of* $\mathbf{Q}$ *is* $\mathbf{R}$, *and the closure of* $\mathbf{R}$ *is* $\mathbf{R}$. *The closure of the empty set* $\emptyset$ *is* $\emptyset$.

Proof. See Exercise 9.1.3. $\square$

The following lemma shows that adherent points of a set X can be obtained as the limit of elements in X:

Lemma 9.1.14. *Let* X *be a subset of* $\mathbf{R}$, *and let* $x \in \mathbf{R}$. *Then* x *is an adherent point of* X *if and only if there exists a sequence* $(a_n)_{n=0}^{\infty}$, *consisting entirely of elements in* X, *which converges to* x.

Proof. See Exercise 9.1.5. $\square$

Definition 9.1.15. A subset $E \subseteq \mathbf{R}$ is said to be *closed* if $\overline{E} = E$, or in other words that E contains all of its adherent points.

Examples 9.1.16. From Lemma 9.1.12 we see that if $a < b$ are real numbers, then $[a, b]$, $[a, +\infty)$, $(-\infty, a]$, and $(-\infty, +\infty)$ are closed, while (a, b), $(a, b]$, $[a, b)$, $(a, +\infty)$, and $(-\infty, a)$ are not. From Lemma 9.1.13 we see that $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{R}$, $\emptyset$ are closed, while $\mathbf{Q}$ is not.

From Lemma 9.1.14 we can define closure in terms of sequences:

Corollary 9.1.17. *Let X be a subset of* $\mathbf{R}$. *If X is closed, and $(a_n)_{n=0}^{\infty}$ is a convergent sequence consisting of elements in X, then $\lim_{n\to\infty} a_n$ also lies in X. Conversely, if it is true that every convergent sequence $(a_n)_{n=0}^{\infty}$ of elements in X has its limit in X as well, then X is necessarily closed.*

When we study differentiation in the next chapter, we shall need to replace the concept of an adherent point by the closely related notion of a *limit point.*

Definition 9.1.18 (Limit points). Let X be a subset of the real line. We say that x is a *limit point* (or a *cluster point*) of X iff it is an adherent point of $X\backslash\{x\}$. We say that x is an *isolated point* of X if $x \in X$ and there exists some $\varepsilon > 0$ such that $|x - y| > \varepsilon$ for all $y \in X\backslash\{x\}$.

Example 9.1.19. Let X be the set $X = (1, 2) \cup \{3\}$. Then 3 is an adherent point of X, but it is not a limit point of X, since 3 is not adherent to $X - \{3\} = (1, 2)$; instead, 3 is an isolated point of X. On the other hand, 2 is still a limit point of X, since 2 is adherent to $X - \{2\} = X$; but it is not isolated (why?).

Remark 9.1.20. From Lemma 9.1.14 we see that x is a limit point of X iff there exists a sequence $(a_n)_{n=0}^{\infty}$, consisting entirely of elements in X that are distinct from x, and such that $(a_n)_{n=0}^{\infty}$ converges to x. It turns out that the set of adherent points splits into the set of limit points and the set of isolated points (Exercise 9.1.9).

Lemma 9.1.21. *Let I be an interval (possibly infinite), i.e., I is a set of the form (a, b), $(a, b]$, $[a, b)$, $[a, b]$, $(a, +\infty)$, $[a, +\infty)$, $(-\infty, a)$, or $(-\infty, a]$. Then every element of I is a limit point of I.*

Proof. We show this for the case $I = [a, b]$; the other cases are similar and are left to the reader. Let $x \in I$; we have to show that x is a limit point of I. There are three cases: $x = a$, $a < x < b$, and $x = b$. If $x = a$, then consider the sequence $(x + \frac{1}{n})_{n=N}^{\infty}$. This sequence converges to x, and will lie inside $I - \{a\} = (a, b]$ if N is chosen large enough (why?). Thus by Remark 9.1.20 we see that $x = a$ is a limit point of $[a, b]$. A similar argument works when $a < x < b$. When $x = b$ one has to use the sequence $(x - \frac{1}{n})_{n=N}^{\infty}$ instead (why?) but the argument is otherwise the same. □

Next, we define the concept of a bounded set.

Definition 9.1.22 (Bounded sets). A subset X of the real line is said to be *bounded* if we have $X \subset [-M, M]$ for some real number $M > 0$.

Example 9.1.23. For any real numbers a, b, the interval $[a, b]$ is bounded, because it is contained inside $[-M, M]$, where $M := \max(|a|, |b|)$. However, the half-infinite interval $[0, +\infty)$ is unbounded (why?). In fact, no half-infinite interval or doubly infinite interval can be bounded. The sets $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{Q}$, and $\mathbf{R}$ are all unbounded (why?).

A basic property of closed and bounded sets is the following.

Theorem 9.1.24 (Heine-Borel theorem for the line). *Let X be a subset of $\mathbf{R}$. Then the following two statements are equivalent:*

(a) *X is closed and bounded.*

(b) *Given any sequence $(a_n)_{n=0}^{\infty}$ of real numbers which takes values in X (i.e., $a_n \in X$ for all n), there exists a subsequence $(a_{n_j})_{j=0}^{\infty}$ of the original sequence, which converges to some number L in X.*

Proof. See Exercise 9.1.13. □

Remark 9.1.25. This theorem shall play a key rôle in subsequent sections of this chapter. In the language of metric space topology, it asserts that every subset of the real line which is closed and bounded, is also compact; see Section 12.5. A more general version of this theorem, due to Eduard Heine (1821–1881) and Emile Borel (1871–1956), can be found in Theorem 12.5.7.

Exercise 9.1.1. Let X be any subset of the real line, and let Y be a set such that $X \subseteq Y \subseteq \overline{X}$. Show that $\overline{Y} = \overline{X}$.

Exercise 9.1.2. Prove Lemma 9.1.11.

Exercise 9.1.3. Prove Lemma 9.1.13. (Hint: for computing the closure of $\mathbf{Q}$, you will need Proposition 5.4.14.)

Exercise 9.1.4. Give an example of two subsets X, Y of the real line such that $\overline{X \cap Y} \neq \overline{X} \cap \overline{Y}$.

Exercise 9.1.5. Prove Lemma 9.1.14. (Hint: in order to prove one of the two implications here you will need axiom of choice, as in Lemma 8.4.5.)

Exercise 9.1.6. Let X be a subset of $\mathbf{R}$. Show that $\overline{X}$ is closed (i.e., $\overline{\overline{X}} = \overline{X}$). Furthermore, show that if Y is any closed set that contains X, then Y also contains $\overline{X}$. Thus the closure $\overline{X}$ of X is the smallest closed set which contains X.

Exercise 9.1.7. Let $n \geq 1$ be a positive integer, and let $X_1, \ldots, X_n$ be closed subsets of $\mathbf{R}$. Show that $X_1 \cup X_2 \cup \ldots \cup X_n$ is also closed.

Exercise 9.1.8. Let I be a set (possibly infinite), and for each $\alpha \in I$ let X_α be a closed subset of $\mathbf{R}$. Show that the intersection $\bigcap_{\alpha \in I} X_\alpha$ (defined in (3.3)) is also closed.

Exercise 9.1.9. Let X be a subset of the real line, and x be a real number. Show that every adherent point of X is either a limit point or an isolated point of X, but cannot be both. Conversely, show that every limit point and every isolated point of X is an adherent point of X.

Exercise 9.1.10. If X is a non-empty subset of $\mathbf{R}$, show that X is bounded if and only if $\inf(X)$ and $\sup(X)$ are finite.

Exercise 9.1.11. Show that if X is a bounded subset of $\mathbf{R}$, then the closure $\overline{X}$ is also bounded.

Exercise 9.1.12. Show that the union of any finite collection of bounded subsets of $\mathbf{R}$ is still a bounded set. Is this conclusion still true if one takes an infinite collection of bounded subsets of $\mathbf{R}$?

Exercise 9.1.13. Prove Theorem 9.1.24. (Hint: to show (a) implies (b), use the Bolzano-Weierstrass theorem (Theorem 6.6.8) and Corollary 9.1.17. To show (b) implies (a), argue by contradiction, using Corollary 9.1.17 to establish that X is closed. You will need the axiom of choice to show that X is bounded, as in Lemma 8.4.5.)

Exercise 9.1.14. Show that any finite subset of $\mathbf{R}$ is closed and bounded.

Exercise 9.1.15. Let E be a bounded subset of $\mathbf{R}$, and let $S := \sup(E)$ be the least upper bound of E. (Note from the least upper bound principle, Theorem 5.5.9, that S is a real number.) Show that S is an adherent point of E, and is also an adherent point of $\mathbf{R}\backslash E$.

9.2 The algebra of real-valued functions

You are familiar with many functions $f : \mathbf{R} \to \mathbf{R}$ from the real line to the real line. Some examples are: $f(x) := x^2 + 3x + 5$; $f(x) := 2^x/(x^2+1)$; $f(x) := \sin(x)\exp(x)$ (we will define sin and exp formally in Chapter 15). These are functions from $\mathbf{R}$ to $\mathbf{R}$ since to every real number x they assign a single real number $f(x)$. We can also consider more exotic functions, e.g.

$$f(x) := \begin{cases} 1 & \text{if } x \in \mathbf{Q} \\ 0 & \text{if } x \notin \mathbf{Q}. \end{cases}$$

This function is not *algebraic* (i.e., it cannot be expressed in terms of x purely by using the standard algebraic operations of $+$, $-$, $\times$, $/$, $\surd$, etc.; we will not need this notion in this text), but it is still a function from $\mathbf{R}$ to $\mathbf{R}$, because it still assigns a real number $f(x)$ to each $x \in \mathbf{R}$.

We can take any one of the previous functions $f : \mathbf{R} \to \mathbf{R}$ defined on all of $\mathbf{R}$, and *restrict* the domain to a smaller set $X \subseteq \mathbf{R}$, creating a new function, sometimes called $f|_X$, from X to $\mathbf{R}$. This is the same function as the original function f, but is only defined on a smaller domain. (Thus $f|_X(x) := f(x)$ when $x \in X$, and $f|_X(x)$ is undefined when $x \notin X$.) For instance, we can restrict the function $f(x) := x^2$, which is initially defined from $\mathbf{R}$ to $\mathbf{R}$, to the interval $[1, 2]$, thus creating a new function $f|_{[1,2]} : [1, 2] \to \mathbf{R}$, which is defined as $f|_{[1,2]}(x) = x^2$ when $x \in [1, 2]$ but is undefined elsewhere.

One could also restrict the range from $\mathbf{R}$ to some smaller subset Y of $\mathbf{R}$, provided of course that all the values of $f(x)$ lie inside Y. For instance, the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) := x^2$ could also be thought of as a function from $\mathbf{R}$ to $[0, \infty)$, instead of a function from $\mathbf{R}$ to $\mathbf{R}$. Formally, these two functions are different functions, but the distinction between them is so minor that we shall often be careless about the range of a function in our discussion.

Strictly speaking, there is a distinction between a *function* f, and its *value* $f(x)$ at a point x. f is a function; but $f(x)$ is a number (which depends on some free variable x). This distinction is rather subtle and we will not stress it too much, but there are times when one has to distinguish between the two. For instance, if $f : \mathbf{R} \to \mathbf{R}$ is the function $f(x) := x^2$, and $g := f|_{[1,2]}$ is the restriction of f to the interval $[1, 2]$, then f and g both perform the operation of squaring, i.e., $f(x) = x^2$ and $g(x) = x^2$, but the two functions f and g are not considered the same function, $f \neq g$, because they have different domains. Despite this distinction, we shall often be careless, and say things like "consider the function x^2+2x+3" when really we should be saying "consider the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) := x^2+2x+3$". (This distinction makes more of a difference when we start doing things like differentiation. For instance, if $f : \mathbf{R} \to \mathbf{R}$ is the function $f(x) = x^2$, then of course $f(3) = 9$, but the derivative of f at 3 is 6, whereas the derivative of 9 is of course 0, so we cannot simply "differentiate both sides" of $f(3) = 9$ and conclude that $6 = 0$.)

If X is a subset of $\mathbf{R}$, and $f : X \to \mathbf{R}$ is a function, we can form the *graph* $\{(x, f(x)) : x \in X\}$ of the function f; this is a subset of $X \times \mathbf{R}$, and hence a subset of the Euclidean plane $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$. One can certainly study a function through its graph, by using the geometry of the plane $\mathbf{R}^2$ (e.g., employing such concepts as tangent lines, area, and so forth). We however will pursue a more "analytic" approach, in which we rely instead on the properties of the real numbers to analyze these functions. The two approaches are complementary; the geometric approach offers more visual intuition, while the analytic approach offers rigour and precision. Both the geometric intuition and the analytic formalism become useful when extending analysis of functions of one variable to functions of many variables (or possibly even infinitely many variables).

Just as numbers can be manipulated arithmetically, so can functions: the sum of two functions is a function, the product of two functions is a function, and so forth.

Definition 9.2.1 (Arithmetic operations on functions). Given two functions $f : X \to \mathbf{R}$ and $g : X \to \mathbf{R}$, we can define their sum $f + g : X \to \mathbf{R}$ by the formula

$$(f + g)(x) := f(x) + g(x),$$

their difference $f - g : X \to \mathbf{R}$ by the formula

$$(f - g)(x) := f(x) - g(x),$$

their maximum $\max(f, g) : X \to \mathbf{R}$ by

$$\max(f, g)(x) := \max(f(x), g(x)),$$

their minimum $\min(f, g) : X \to \mathbf{R}$ by

$$\min(f, g)(x) := \min(f(x), g(x)),$$

their product $fg : X \to \mathbf{R}$ (or $f \cdot g : X \to \mathbf{R}$) by the formula

$$(fg)(x) := f(x)g(x),$$

and (provided that $g(x) \neq 0$ for all $x \in X$) the quotient $f/g : X \to \mathbf{R}$ by the formula

$$(f/g)(x) := f(x)/g(x).$$

Finally, if c is a real number, we can define the function $cf : X \to \mathbf{R}$ (or $c \cdot f : X \to \mathbf{R}$) by the formula

$$(cf)(x) := c \times f(x).$$

Example 9.2.2. If $f : \mathbf{R} \to \mathbf{R}$ is the function $f(x) := x^2$, and $g : \mathbf{R} \to \mathbf{R}$ is the function $g(x) := 2x$, then $f + g : \mathbf{R} \to \mathbf{R}$ is the function $(f + g)(x) := x^2 + 2x$, while $fg : \mathbf{R} \to \mathbf{R}$ is the function $fg(x) = 2x^3$. Similarly $f - g : \mathbf{R} \to \mathbf{R}$ is the function $(f - g)(x) := x^2 - 2x$, while $6f : \mathbf{R} \to \mathbf{R}$ is the function $(6f)(x) = 6x^2$. Observe that fg is not the same function as $f \circ g$, which maps $x \mapsto 4x^2$, nor is it the same as $g \circ f$, which maps $x \mapsto 2x^2$ (why?). Thus multiplication of functions and composition of functions are two different operations.

Exercise 9.2.1. Let $f : \mathbf{R} \to \mathbf{R}$, $g : \mathbf{R} \to \mathbf{R}$, $h : \mathbf{R} \to \mathbf{R}$. Which of the following identities are true, and which ones are false? In the former case, give a proof; in the latter case, give a counterexample.

$$(f + g) \circ h = (f \circ h) + (g \circ h)$$
$$f \circ (g + h) = (f \circ g) + (f \circ h)$$
$$(f + g) \cdot h = (f \cdot h) + (g \cdot h)$$
$$f \cdot (g + h) = (f \cdot g) + (f \cdot h)$$

9.3 Limiting values of functions

In Chapter 6 we defined what it means for a sequence $(a_n)_{n=0}^{\infty}$ to converge to a limit L. We now define a similar notion for what it means for a function f defined on the real line, or on some subset of the real line, to converge to some value at a point. Just as we used the notions of ε-closeness and eventual ε-closeness to deal with limits of sequences, we shall need a notion of ε-closeness and local ε-closeness to deal with limits of functions.

Definition 9.3.1 (ε-closeness). Let X be a subset of $\mathbf{R}$, let $f : X \to \mathbf{R}$ be a function, let L be a real number, and let $\varepsilon > 0$ be a real number. We say that the function f is *ε-close to L* iff $f(x)$ is ε-close to L for every $x \in X$.

Example 9.3.2. When the function $f(x) := x^2$ is restricted to the interval $[1, 3]$, then it is 5-close to 4, since when $x \in [1, 3]$ then $1 \leq f(x) \leq 9$, and hence $|f(x) - 4| \leq 5$. When instead it is restricted to the smaller interval $[1.9, 2.1]$, then it is 0.41-close to 4, since if $x \in [1.9, 2.1]$, then $3.61 \leq f(x) \leq 4.41$, and hence $|f(x) - 4| \leq 0.41$.

Definition 9.3.3 (Local ε-closeness). Let X be a subset of $\mathbf{R}$, let $f : X \to \mathbf{R}$ be a function, let L be a real number, x_0 be an adherent point of X, and $\varepsilon > 0$ be a real number. We say that f is *ε-close to L near x_0* iff there exists a $\delta > 0$ such that f becomes ε-close to L when restricted to the set $\{x \in X : |x - x_0| < \delta\}$.

Example 9.3.4. Let $f : [1, 3] \to \mathbf{R}$ be the function $f(x) := x^2$, restricted to the interval $[1, 3]$. This function is not 0.1-close to 4, since for instance $f(1)$ is not 0.1-close to 4. However, f is 0.1-close to 4 near 2, since when restricted to the set $\{x \in [1, 3] : |x - 2| < 0.01\}$, the function f is indeed 0.1-close to 4. This is because when $|x - 2| < 0.01$, we have $1.99 < x < 2.01$, and hence $3.9601 < f(x) < 4.0401$, and in particular $f(x)$ is 0.1-close to 4.

Example 9.3.5. Continuing with the same function f used in the previous example, we observe that f is not 0.1-close to 9, since for instance $f(1)$ is not 0.1-close to 9. However, f is 0.1-close to 9 near 3, since when restricted to the set $\{x \in [1, 3] : |x - 3| < 0.01\}$ - which is the same as the half-open interval $(2.99, 3]$ (why?), the function f becomes 0.1-close to 9 (since if $2.99 < x \leq 3$, then $8.9401 < f(x) \leq 9$, and hence $f(x)$ is 0.1-close to 9).

Definition 9.3.6 (Convergence of functions at a point). Let X be a subset of $\mathbf{R}$, let $f : X \to \mathbf{R}$ be a function, let E be a subset of X, x_0 be an adherent point of E, and let L be a real number. We say that f *converges to* L *at* x_0 *in* E, and write $\lim_{x \to x_0; x \in E} f(x) = L$, iff f is ε-close to L near x_0 for every $\varepsilon > 0$. If f does not converge to any number L at x_0, we say that f *diverges* at x_0, and leave $\lim_{x \to x_0; x \in E} f(x)$ undefined.

In other words, we have $\lim_{x \to x_0; x \in E} f(x) = L$ iff for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - L| \leq \varepsilon$ for all $x \in E$ such that $|x - x_0| < \delta$. (Why is this definition equivalent to the one given above?)

Remark 9.3.7. In many cases we will omit the set E from the above notation (i.e., we will just say that f converges to L at x_0, or that $\lim_{x \to x_0} f(x) = L$), although this is slightly dangerous. For instance, it sometimes makes a difference whether E actually contains x_0 or not. To give an example, if $f : \mathbf{R} \to \mathbf{R}$ is the function defined by setting $f(x) = 1$ when $x = 0$ and $f(x) = 0$ when $x \neq 0$, then one has $\lim_{x \to 0; x \in \mathbf{R} \setminus \{0\}} f(x) = 0$, but $\lim_{x \to 0; x \in \mathbf{R}} f(x)$ is undefined. Some authors only define the limit $\lim_{x \to x_0; x \in E} f(x)$ when E does not contain x_0 (so that x_0 is now a limit point of E