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Quaternions

It was William Rowan Hamilton (1805–1865) who first conceived complex numbers as ordered pairs of reals, subject to the multiplication rule $(a, b)(c, d) = (ac - bd,\ ad + bc)$. His greatest contribution to pure mathematics, however, was his creation of an algebraic system in which the commutative law of multiplication does not hold. Just as many people before Bolyai and Lobachevsky had thought of Euclid's fifth postulate as a necessary and sacred truth, so many people before Hamilton believed that the law of commutativity for the multiplication of numbers was decreed by heaven. Hamilton discovered that there are consistent algebraic systems for which this law does not not hold. (Matrix algebra came fourteen years later.)

What Hamilton discovered were 'quaternions'. The idea dawned on him while he was strolling along the Royal Canal in Dublin in 1843.

Quaternions are numbers of the form

$$a + bi + cj + dk,$$

where a, b, c, and d are real, and $i^2 = j^2 = k^2 = ijk = -1$.

If quaternion multiplication were commutative, it would be easy to derive $ij = -ij$ and from this it would follow that $i = j = 0$, and the whole system would collapse. We must have noncommutative multiplication. In particular, $ij \neq ji$. How do we know that such entities as i, j, and k exist? There are in fact three complex matrices:

$$i_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad i_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad i_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

such that $i_1^2 = i_2^2 = i_3^2 = i_1 i_2 i_3 = -1$, where 1 is interpreted as the identity matrix. We may define the quaternion $a = a_0 1 + a_1 i_1 + a_2 i_2 + a_3 i_3$ as the complex matrix

$$\begin{pmatrix} a_0 + ia_3 & a_1 + ia_2 \\ -a_1 + ia_2 & a_0 - ia_3 \end{pmatrix},$$

namely, $\begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix}$, where $u = a_0 + ia_3$ and $v = a_1 + ia_2$, the a_k being real numbers. Addition and multiplication of quaternions is just the usual matrix arithmetic. The *conjugate* of a quaternion, as distinguished from the conjugate of the complex matrix which represents it, is the quaternion

$$\bar{a} = a_0 1 - a_1 i_1 - a_2 i_2 - a_3 i_3.$$

Note that we often identify the real number a_0 with the matrix $a_0 1$. Using this identification, we have

$$a\bar{a} = a_0^2 + a_1^2 + a_2^2 + a_3^2 = \bar{a}a.$$

The *norm* $N(a)$ of a quaternion a is the product $a\bar{a}$ viewed as a real number, namely, $a_0^2 + a_1^2 + a_2^2 + a_3^2$. Note that the norm is 0 just in case the quaternion is the zero matrix. For a nonzero quaternion a, we can write

$$a(\bar{a}/N(a)) = 1 = (\bar{a}/N(a))a.$$

(Here $N(a)$ is the real number, not the matrix.) This means that nonzero quaternions have multiplicative inverses.

A *division ring* is a system of elements closed under two binary operations, addition and multiplication, such that

1. under addition, it is a commutative group,

2. under multiplication, the nonzero elements form a group, and

3. multiplication distributes over addition (on both sides).

A division ring is thus a ring (Chapter 3) in which every nonzero element has an inverse under multiplication. It satisfies all the axioms of a field, exept the commutative law of multiplication. It is sometimes called a *skew-field.* A key fact about quaternions is that they form a division ring.

It is also possible to represent quaternions by 4×4 real matrices . A cheap way to obtain such a representation is to replace i by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and 1 by $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ in the complex 2×2 matrix representing the quaternion. Instead we shall look at a more interesting way to obtain such a representation, which has wider implications.

The quaternion $x = x_0 + x_1 i_1 + x_2 i_2 + x_3 i_3$ gives rise to a column vector

$$[x] = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Given a quaternion a, consider the mapping which assigns to every such vector $[x]$ the vector $[ax]$. Clearly, this is a linear transformation of the four-dimensional vector space over $\mathbf{R}$. Hence there will be a 4×4 matrix $A = L(a)$ such that $A[x] = [ax]$. We say that $L(a)$ *represents* the quaternion a. In particular, $I = L(1)$ is the identity matrix. For example, the matrix $I_1 = L(i_1)$ is obtained as follows: the equation

$$i_1(x_0 + x_1 i_1 + x_2 i_2 + x_3 i_3) = -x_1 + x_0 i_1 - x_3 i_2 + x_2 i_3$$

may be interpreted thus:

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_1 \\ x_0 \\ -x_3 \\ x_2 \end{pmatrix}.$$

Hence $I_1 = L(i_1)$ is the 4×4 matrix appearing above. The matrices $I_2 = L(i_2)$ and $I_3 = L(i_3)$ are obtained similarly. Note that

$$L(ab)[x] = [(ab)x] = [a(bx)] = L(a)[bx] = L(a)L(b)[x]$$

for all vectors $[x]$. Therefore

$$L(ab) = L(a)L(b),$$

from which it easily follows that L is an injective homomorphism from the division ring of quaternions into the ring of all 4×4 matrices over $\mathbf{R}$.

Exercises

1. Show that the conjugate of ab is $\bar{b}\bar{a}$.

2. Show that $N(\bar{a}) = N(a)$ and $N(ab) = N(a)N(b)$.

3. Prove that every quaternion satisfies a quadratic equation with real coefficients.

4. Find all quaternions whose square is 1.

5. Prove that $L(\bar{a}) = A^t$ is the transpose of $A = L(a)$ and that $(N(a))^2 = \det(A)$, the determinant of A.

6. Show that with every quaternion a one may associate another 4×4 real matrix $R(a)$ such that $R(a)[x] = [xa]$ for all quaternions x.

7. Prove that $R(ab) = R(b)R(a)$ and $L(a)R(b) = R(b)L(a)$ for all quaternions a and b.