11. Let T be a linear operator on F^n. Define

$$D_T(\alpha_1, \ldots, \alpha_n) = \det (T\alpha_1, \ldots, T\alpha_n).$$

(a) Show that D_T is an alternating n-linear function.

(b) If

$$c = \det (T\epsilon_1, \ldots, T\epsilon_n)$$

show that for any n vectors $\alpha_1, \ldots, \alpha_n$ we have

$$\det (T\alpha_1, \ldots, T\alpha_n) = c \det (\alpha_1, \ldots, \alpha_n).$$

(c) If $\mathcal{B}$ is any ordered basis for F^n and A is the matrix of T in the ordered basis $\mathcal{B}$, show that $\det A = c$.

(d) What do you think is a reasonable name for the scalar c?

12. If σ is a permutation of degree n and A is an $n \times n$ matrix over the field F with row vectors $\alpha_1, \ldots, \alpha_n$, let $\sigma(A)$ denote the $n \times n$ matrix with row vectors $\alpha_{\sigma 1}, \ldots, \alpha_{\sigma n}$.

(a) Prove that $\sigma(AB) = \sigma(A)B$, and in particular that $\sigma(A) = \sigma(I)A$.

(b) If T is the linear operator of Exercise 9, prove that the matrix of T in the standard ordered basis is $\sigma(I)$.

(c) Is $\sigma^{-1}(I)$ the inverse matrix of $\sigma(I)$?

(d) Is it true that $\sigma(A)$ is similar to A?

13. Prove that the sign function on permutations is unique in the following sense. If f is any function which assigns to each permutation of degree n an integer, and if $f(\sigma\tau) = f(\sigma)f(\tau)$, then f is identically 0, or f is identically 1, or f is the sign function.

5.4. Additional Properties of Determinants

In this section we shall relate some of the useful properties of the determinant function on $n \times n$ matrices. Perhaps the first thing we should point out is the following. In our discussion of $\det A$, the rows of A have played a privileged role. Since there is no fundamental difference between rows and columns, one might very well expect that $\det A$ is an alternating n-linear function of the columns of A. This is the case, and to prove it, it suffices to show that

$$\det (A^t) = \det (A) \tag{5-17}$$

where A^t denotes the transpose of A.

If σ is a permutation of degree n,

$$A^t(i, \sigma i) = A(\sigma i, i).$$

From the expression (5-15) one then has

$$\det (A^t) = \sum_{\sigma} (\operatorname{sgn} \sigma) A(\sigma 1, 1) \cdots A(\sigma n, n).$$

When $i = \sigma^{-1}j$, $A(\sigma i, i) = A(j, \sigma^{-1}j)$. Thus

$$A(\sigma 1, 1) \cdots A(\sigma n, n) = A(1, \sigma^{-1}1) \cdots A(n, \sigma^{-1}n).$$

Since $\sigma\sigma^{-1}$ is the identity permutation,

$$(\operatorname{sgn} \sigma)(\operatorname{sgn} \sigma^{-1}) = 1 \quad \text{or} \quad \operatorname{sgn} (\sigma^{-1}) = \operatorname{sgn} (\sigma).$$

Furthermore, as σ varies over all permutations of degree n, so does σ^{-1}. Therefore

$$\begin{aligned} \det (A^t) &= \sum_\sigma (\operatorname{sgn} \sigma^{-1}) A(1, \sigma^{-1}1) \cdots A(n, \sigma^{-1}n) \\ &= \det A \end{aligned}$$

proving (5-17).

On certain occasions one needs to compute specific determinants. When this is necessary, it is frequently useful to take advantage of the following fact. *If B is obtained from A by adding a multiple of one row of A to another* (*or a multiple of one column to another*), *then*

(5-18) $$\det B = \det A.$$

We shall prove the statement about rows. Let B be obtained from A by adding $c\alpha_j$ to α_i, where $i < j$. Since det is linear as a function of the ith row

$$\begin{aligned} \det B &= \det A + c \det (\alpha_1, \ldots, \alpha_j, \ldots, \alpha_j, \ldots, \alpha_n) \\ &= \det A. \end{aligned}$$

Another useful fact is the following. Suppose we have an $n \times n$ matrix of the block form

$$\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$$

where A is an $r \times r$ matrix, C is an $s \times s$ matrix, B is $r \times s$, and 0 denotes the $s \times r$ zero matrix. Then

(5-19) $$\det \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} = (\det A)(\det C).$$

To prove this, define

$$D(A, B, C) = \det \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}.$$

If we fix A and B, then D is alternating and s-linear as a function of the rows of C. Thus, by Theorem 2

$$D(A, B, C) = (\det C) D(A, B, I)$$

where I is the $s \times s$ identity matrix. By subtracting multiples of the rows of I from the rows of B and using the statement above (5-18), we obtain

$$D(A, B, I) = D(A, 0, I).$$

Now $D(A, 0, I)$ is clearly alternating and r-linear as a function of the rows of A. Thus

$$D(A, 0, I) = (\det A) D(I, 0, I).$$

But $D(I, 0, I) = 1$, so

$$\begin{aligned} D(A, B, C) &= (\det C)D(A, B, I) \\ &= (\det C)D(A, 0, I) \\ &= (\det C)(\det A). \end{aligned}$$

By the same sort of argument, or by taking transposes

$$\det \begin{bmatrix} A & 0 \\ B & C \end{bmatrix} = (\det A)(\det C). \tag{5-20}$$

EXAMPLE 6. Suppose K is the field of rational numbers and we wish to compute the determinant of the 4×4 matrix

$$A = \begin{bmatrix} 1 & -1 & 2 & 3 \\ 2 & 2 & 0 & 2 \\ 4 & 1 & -1 & -1 \\ 1 & 2 & 3 & 0 \end{bmatrix}.$$

By subtracting suitable multiples of row 1 from rows 2, 3, and 4, we obtain the matrix

$$\begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 4 & -4 & -4 \\ 0 & 5 & -9 & -13 \\ 0 & 3 & 1 & -3 \end{bmatrix}$$

which we know by (5-18) will have the same determinant as A. If we subtract $\frac{5}{4}$ of row 2 from row 3 and then subtract $\frac{3}{4}$ of row 2 from row 4, we obtain

$$B = \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 4 & -4 & -4 \\ 0 & 0 & -4 & -8 \\ 0 & 0 & 4 & 0 \end{bmatrix}$$

and again $\det B = \det A$. The block form of B tells us that

$$\det A = \det B = \begin{vmatrix} 1 & -1 \\ 0 & 4 \end{vmatrix} \begin{vmatrix} -4 & -8 \\ 4 & 0 \end{vmatrix} = 4(32) = 128.$$

Now let $n > 1$ and let A be an $n \times n$ matrix over K. In Theorem 1, we showed how to construct a determinant function on $n \times n$ matrices, given one on $(n-1) \times (n-1)$ matrices. Now that we have proved the uniqueness of the determinant function, the formula (5-4) tells us the following. If we fix any column index j,

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} A_{ij} \det A(i|j).$$

The scalar $(-1)^{i+j} \det A(i|j)$ is usually called the i, j **cofactor** of A or the cofactor of the i, j entry of A. The above formula for $\det A$ is then called the expansion of det A by cofactors of the jth column (or sometimes the expansion by minors of the jth column). If we set

$$C_{ij} = (-1)^{i+j} \det A(i|j)$$

then the above formula says that for each j

$$\det A = \sum_{i=1}^{n} A_{ij} C_{ij}$$

where the cofactor C_{ij} is $(-1)^{i+j}$ times the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting the ith row and jth column of A.

If $j \neq k$, then

$$\sum_{i=1}^{n} A_{ik} C_{ij} = 0.$$

For, replace the jth column of A by its kth column, and call the resulting matrix B. Then B has two equal columns and so $\det B = 0$. Since $B(i|j) = A(i|j)$, we have

$$\begin{aligned} 0 &= \det B \\ &= \sum_{i=1}^{n} (-1)^{i+j} B_{ij} \det B(i|j) \\ &= \sum_{i=1}^{n} (-1)^{i+j} A_{ik} \det A(i|j) \\ &= \sum_{i=1}^{n} A_{ik} C_{ij}. \end{aligned}$$

These properties of the cofactors can be summarized by

$$\sum_{i=1}^{n} A_{ik} C_{ij} = \delta_{jk} \det A. \tag{5-21}$$

The $n \times n$ matrix adj A, which is the transpose of the matrix of cofactors of A, is called the **classical adjoint** of A. Thus

$$(\operatorname{adj} A)_{ij} = C_{ji} = (-1)^{i+j} \det A(j|i). \tag{5-22}$$

The formulas (5-21) can be summarized in the matrix equation

$$(\operatorname{adj} A)A = (\det A)I. \tag{5-23}$$

We wish to see that $A(\operatorname{adj} A) = (\det A)I$ also. Since $A^t(i|j) = A(j|i)^t$, we have

$$(-1)^{i+j} \det A^t(i|j) = (-1)^{j+i} \det A(j|i)$$

which simply says that the i, j cofactor of A^t is the j, i cofactor of A. Thus

$$\operatorname{adj} (A^t) = (\operatorname{adj} A)^t \tag{5-24}$$

By applying (5-23) to A^t, we obtain

$$(\operatorname{adj} A^t)A^t = (\det A^t)I = (\det A)I$$

and transposing

$$A(\operatorname{adj} A^t)^t = (\det A)I.$$

Using (5-24), we have what we want:

$$A(\operatorname{adj} A) = (\det A)I. \tag{5-25}$$

As for matrices over a field, an $n \times n$ matrix A over K is called **invertible over** K if there is an $n \times n$ matrix A^{-1} with entries in K such that $AA^{-1} = A^{-1}A = I$. If such an inverse matrix exists it is unique; for the same argument used in Chapter 1 shows that when $BA = AC = I$ we have $B = C$. The formulas (5-23) and (5-25) tell us the following about invertibility of matrices over K. If the element $\det A$ has a multiplicative inverse in K, then A is invertible and $A^{-1} = (\det A)^{-1} \operatorname{adj} A$ is the unique inverse of A. Conversely, it is easy to see that if A is invertible over K, the element $\det A$ is invertible in K. For, if $BA = I$ we have

$$1 = \det I = \det (AB) = (\det A)(\det B).$$

What we have proved is the following.

Theorem 4. *Let* A *be an* n $\times$ n *matrix over* K. *Then* A *is invertible over* K *if and only if det* A *is invertible in* K. *When* A *is invertible, the unique inverse for* A *is*

$$A^{-1} = (\det A)^{-1} \operatorname{adj} A.$$

In particular, an n $\times$ n *matrix over a field is invertible if and only if its determinant is different from zero.*

We should point out that this determinant criterion for invertibility proves that an $n \times n$ matrix with either a left or right inverse is invertible. This proof is completely independent of the proof which we gave in Chapter 1 for matrices over a field. We should also like to point out what invertibility means for matrices with polynomial entries. If K is the polynomial ring $F[x]$, the only elements of K which are invertible are the non-zero scalar polynomials. For if f and g are polynomials and $fg = 1$, we have $\deg f + \deg g = 0$ so that $\deg f = \deg g = 0$, i.e., f and g are scalar polynomials. So an $n \times n$ matrix over the polynomial ring $F[x]$ is invertible over $F[x]$ if and only if its determinant is a non-zero scalar polynomial.

Example 7. Let $K = R[x]$, the ring of polynomials over the field of real numbers. Let

$$A = \begin{bmatrix} x^2 + x & x + 1 \\ x - 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} x^2 - 1 & x + 2 \\ x^2 - 2x + 3 & x \end{bmatrix}.$$

Then, by a short computation, $\det A = x + 1$ and $\det B = -6$. Thus A is not invertible over K, whereas B is invertible over K. Note that

$$\operatorname{adj} A = \begin{bmatrix} 1 & -x - 1 \\ -x + 1 & x^2 + x \end{bmatrix}, \qquad \operatorname{adj} B = \begin{bmatrix} x & -x - 2 \\ -x^2 + 2x - 3 & x^2 - 1 \end{bmatrix}$$

and $(\text{adj } A)A = (x + 1)I$, $(\text{adj } B)B = -6I$. Of course,

$$B^{-1} = -\frac{1}{6}\begin{bmatrix} x & -x - 2 \\ -x^2 + 2x - 3 & 1 - x^2 \end{bmatrix}.$$

EXAMPLE 8. Let K be the ring of integers and

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Then $\det A = -2$ and

$$\text{adj } A = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}.$$

Thus A is not invertible as a matrix over the ring of integers; however, we can also regard A as a matrix over the field of rational numbers. If we do, then A is invertible and

$$A^{-1} = -\frac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & \frac{1}{2} \end{bmatrix}.$$

In connection with invertible matrices, we should like to mention one further elementary fact. Similar matrices have the same determinant, that is, if P is invertible over K and $B = P^{-1}AP$, then $\det B = \det A$. This is clear since

$$\det (P^{-1}AP) = (\det P^{-1})(\det A)(\det P) = \det A.$$

This simple observation makes it possible to define the determinant of a linear operator on a finite dimensional vector space. If T is a linear operator on V, we define the determinant of T to be the determinant of any $n \times n$ matrix which represents T in an ordered basis for V. Since all such matrices are similar, they have the same determinant and our definition makes sense. In this connection, see Exercise 11 of section 5.3.

We should like now to discuss **Cramer's rule** for solving systems of linear equations. Suppose A is an $n \times n$ matrix over the field F and we wish to solve the system of linear equations $AX = Y$ for some given n-tuple $(y_1, \ldots, y_n)$. If $AX = Y$, then

$$(\text{adj } A)AX = (\text{adj } A)Y$$

and so

$$(\det A)X = (\text{adj } A)Y.$$

Thus

$$\begin{aligned}(\det A)x_j &= \sum_{i=1}^{n} (\text{adj } A)_{ji}y_i \\ &= \sum_{i=1}^{n} (-1)^{i+j}y_i \det A(i|j).\end{aligned}$$

This last expression is the determinant of the $n \times n$ matrix obtained by replacing the jth column of A by Y. If $\det A = 0$, all this tells us nothing; however, if $\det A \neq 0$, we have what is known as Cramer's rule. Let A