We close this section by listing some miscellaneous properties of compact sets.

**Theorem 12.5.10.** *Let $(X, d)$ be a metric space.*

(a) *If $Y$ is a compact subset of $X$, and $Z \subseteq Y$, then $Z$ is compact if and only if $Z$ is closed.*

(b) *If $Y_1, \ldots, Y_n$ are a finite collection of compact subsets of $X$, then their union $Y_1 \cup \ldots \cup Y_n$ is also compact.*

(c) *Every finite subset of $X$ (including the empty set) is compact.*

*Proof.* See Exercise 12.5.7. □

*Exercise* 12.5.1. Show that Definitions 9.1.22 and 12.5.3 match when talking about subsets of the real line with the standard metric.

*Exercise* 12.5.2. Prove Proposition 12.5.5. (Hint: prove the completeness and boundedness separately. For both claims, use proof by contradiction. You will need the axiom of choice, as in Lemma 8.4.5.)

*Exercise* 12.5.3. Prove Theorem 12.5.7. (Hint: use Proposition 12.1.18 and Theorem 9.1.24.)

*Exercise* 12.5.4. Let $(\mathbf{R}, d)$ be the real line with the standard metric. Give an example of a continuous function $f : \mathbf{R} \to \mathbf{R}$, and an open set $V \subseteq \mathbf{R}$, such that the image $f(V) := \{f(x) : x \in V\}$ of $V$ is *not* open.

*Exercise* 12.5.5. Let $(\mathbf{R}, d)$ be the real line with the standard metric. Give an example of a continuous function $f : \mathbf{R} \to \mathbf{R}$, and a closed set $F \subseteq \mathbf{R}$, such that $f(F)$ is *not* closed.

*Exercise* 12.5.6. Prove Corollary 12.5.9. (Hint: work in the compact metric space $(K_1, d|_{K_1 \times K_1})$, and consider the sets $V_n := K_1 \backslash K_n$, which are open on $K_1$. Assume for sake of contradiction that $\bigcap_{n=1}^{\infty} K_n = \emptyset$, and then apply Theorem 12.5.8.)

*Exercise* 12.5.7. Prove Theorem 12.5.10. (Hint: for part (c), you may wish to use (b), and first prove that every singleton set is compact.)

*Exercise* 12.5.8. Let $(X, d_{l^1})$ be the metric space from Exercise 12.1.15. For each natural number $n$, let $e^{(n)} = (e_j^{(n)})_{j=0}^\infty$ be the sequence in $X$ such that $e_j^{(n)} := 1$ when $n = j$ and $e_j^{(n)} := 0$ when $n \neq j$. Show that the set $\{e^{(n)} : n \in \mathbf{N}\}$ is a closed and bounded subset of $X$, but is not compact. (This is despite the fact that $(X, d_{l^1})$ is even a complete metric space - a fact which we will not prove here. The problem is that not that $X$ is incomplete, but rather that it is "infinite-dimensional", in a sense that we will not discuss here.)

*Exercise* 12.5.9. Show that a metric space $(X, d)$ is compact if and only if every sequence in $X$ has at least one limit point.

*Exercise* 12.5.10. A metric space $(X, d)$ is called *totally bounded* if for every $\varepsilon > 0$, there exists a positive integer $n$ and a finite number of balls $B(x^{(1)}, \varepsilon), \ldots, B(x^{(n)}, \varepsilon)$ which cover $X$ (i.e., $X = \bigcup_{i=1}^n B(x^{(i)}, \varepsilon)$.

(a) Show that every totally bounded space is bounded.

(b) Show the following stronger version of Proposition 12.5.5: if $(X, d)$ is compact, then complete and totally bounded. (Hint: if $X$ is not totally bounded, then there is some $\varepsilon > 0$ such that $X$ cannot be covered by finitely many $\varepsilon$-balls. Then use Exercise 8.5.20 to find an infinite sequence of balls $B(x^{(n)}, \varepsilon/2)$ which are disjoint from each other. Use this to then construct a sequence which has no convergent subsequence.)

(c) Conversely, show that if $X$ is complete and totally bounded, then $X$ is compact. (Hint: if $(x^{(n)})_{n=1}^\infty$ is a sequence in $X$, use the total boundedness hypothesis to recursively construct a sequence of subsequences $(x^{(n;j)})_{n=1}^\infty$ of $(x^{(n)})_{n=1}^\infty$ for each positive integer $j$, such that for each $j$, the elements of the sequence $(x^{(n;j)})_{n=1}^\infty$ are contained in a single ball of radius $1/j$, and also that each sequence $(x^{(n;j+1)})_{n=1}^\infty$ is a subsequence of the previous one $(x^{(n;j)})_{n=1}^\infty$. Then show that the "diagonal" sequence $(x^{(n;n)})_{n=1}^\infty$ is a Cauchy sequence, and then use the completeness hypothesis.)

*Exercise* 12.5.11. Let $(X, d)$ have the property that every open cover of $X$ has a finite subcover. Show that $X$ is compact. (Hint: if $X$ is not compact, then by Exercise 12.5.9, there is a sequence $(x^{(n)})_{n=1}^\infty$ with no limit points. Then for every $x \in X$ there exists a ball $B(x, \varepsilon)$ containing $x$ which contains at most finitely many elements of this sequence. Now use the hypothesis.)

*Exercise* 12.5.12. Let $(X, d_{disc})$ be a metric space with the discrete metric $d_{disc}$.

(a) Show that $X$ is always complete.

(b) When is $X$ compact, and when is $X$ not compact? Prove your claim. (Hint: the Heine-Borel theorem will be useless here since that only applies to Euclidean spaces with the Euclidean metric.)

*Exercise* 12.5.13. Let $E$ and $F$ be two compact subsets of $\mathbf{R}$ (with the standard metric $d(x,y) = |x-y|$). Show that the Cartesian product $E \times F := \{(x,y) : x \in E, y \in F\}$ is a compact subset of $\mathbf{R}^2$ (with the Euclidean metric $d_{l^2}$).

*Exercise* 12.5.14. Let $(X,d)$ be a metric space, let $E$ be a non-empty compact subset of $X$, and let $x_0$ be a point in $X$. Show that there exists a point $x \in E$ such that

$$d(x_0, x) = \inf\{d(x_0, y) : y \in E\},$$

i.e., $x$ is the closest point in $E$ to $x_0$. (Hint: let $R$ be the quantity $R := \inf\{d(x_0,y) : y \in E\}$. Construct a sequence $(x^{(n)})_{n=1}^\infty$ in $E$ such that $d(x_0, x^{(n)}) \leq R + \frac{1}{n}$, and then use the compactness of $E$.)

*Exercise* 12.5.15. Let $(X,d)$ be a compact metric space. Suppose that $(K_\alpha)_{\alpha \in I}$ is a collection of closed sets in $X$ with the property that any finite subcollection of these sets necessarily has non-empty intersection, thus $\bigcap_{\alpha \in F} K_\alpha \neq \emptyset$ for all finite $F \subseteq I$. (This property is known as the *finite intersection property*.) Show that the *entire* collection has non-empty intersection, thus $\bigcap_{\alpha \in I} K_\alpha = \emptyset$. Show that by counterexample this statement fails if $X$ is not compact.

# Chapter 13

# Continuous functions on metric spaces

## 13.1 Continuous functions

In the previous chapter we studied a single metric space $(X, d)$, and the various types of sets one could find in that space. While this is already quite a rich subject, the theory of metric spaces becomes even richer, and of more importance to analysis, when one considers not just a single metric space, but rather *pairs* $(X, d_X)$ and $(Y, d_Y)$ of metric spaces, as well as *continuous functions* $f : X \to Y$ between such spaces. To define this concept, we generalize Definition 9.4.1 as follows:

**Definition 13.1.1** (Continuous functions). Let $(X, d_X)$ be a metric space, and let $(Y, d_Y)$ be another metric space, and let $f : X \to Y$ be a function. If $x_0 \in X$, we say that $f$ is *continuous at* $x_0$ iff for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d_Y(f(x), f(x_0)) < \varepsilon$ whenever $d_X(x, x_0) < \delta$. We say that $f$ is *continuous* iff it is continuous at every point $x \in X$.

**Remark 13.1.2.** Continuous functions are also sometimes called *continuous maps.* Mathematically, there is no distinction between the two terminologies.

**Remark 13.1.3.** If $f : X \to Y$ is continuous, and $K$ is any subset of $X$, then the restriction $f|_K : K \to Y$ of $f$ to $K$ is also continuous (why?).

We now generalize much of the discussion in Chapter 9. We first observe that continuous functions preserve convergence:

**Theorem 13.1.4** (Continuity preserves convergence). *Suppose that $(X, d_X)$ and $(Y, d_Y)$ are metric spaces. Let $f : X \to Y$ be a function, and let $x_0 \in X$ be a point in $X$. Then the following three statements are logically equivalent:*

(a) *$f$ is continuous at $x_0$.*

(b) *Whenever $(x^{(n)})_{n=1}^{\infty}$ is a sequence in $X$ which converges to $x_0$ with respect to the metric $d_X$, the sequence $(f(x^{(n)}))_{n=1}^{\infty}$ converges to $f(x_0)$ with respect to the metric $d_Y$.*

(c) *For every open set $V \subset Y$ that contains $f(x_0)$, there exists an open set $U \subset X$ containing $x_0$ such that $f(U) \subseteq V$.*

*Proof.* See Exercise 13.1.1. □

Another important characterization of continuous functions involves open sets.

**Theorem 13.1.5.** *Let $(X, d_X)$ be a metric space, and let $(Y, d_Y)$ be another metric space. Let $f : X \to Y$ be a function. Then the following four statements are equivalent:*

(a) *$f$ is continuous.*

(b) *Whenever $(x^{(n)})_{n=1}^{\infty}$ is a sequence in $X$ which converges to some point $x_0 \in X$ with respect to the metric $d_X$, the sequence $(f(x^{(n)}))_{n=1}^{\infty}$ converges to $f(x_0)$ with respect to the metric $d_Y$.*

(c) *Whenever $V$ is an open set in $Y$, the set $f^{-1}(V) := \{x \in X : f(x) \in V\}$ is an open set in $X$.*

(d) *Whenever $F$ is a closed set in $Y$, the set $f^{-1}(F) := \{x \in X : f(x) \in F\}$ is a closed set in $X$.*

*Proof.* See Exercise 13.1.2. □

**Remark 13.1.6.** It may seem strange that continuity ensures that the *inverse* image of an open set is open. One may guess instead that the reverse should be true, that the *forward* image of an open set is open; but this is not true; see Exercises 12.5.4, 12.5.5.

As a quick corollary of the above two Theorems we obtain

**Corollary 13.1.7** (Continuity preserved by composition). *Let $(X, d_X)$, $(Y, d_Y)$, and $(Z, d_Z)$ be metric spaces.*

(a) *If $f : X \to Y$ is continuous at a point $x_0 \in X$, and $g : Y \to Z$ is continuous at $f(x_0)$, then the composition $g \circ f : X \to Z$, defined by $g \circ f(x) := g(f(x))$, is continuous at $x_0$.*

(b) *If $f : X \to Y$ is continuous, and $g : Y \to Z$ is continuous, then $g \circ f : X \to Z$ is also continuous.*

*Proof.* See Exercise 13.1.3. □

**Example 13.1.8.** If $f : X \to \mathbf{R}$ is a continuous function, then the function $f^2 : X \to \mathbf{R}$ defined by $f^2(x) := f(x)^2$ is automatically continuous also. This is because we have $f^2 = g \circ f$, where $g : \mathbf{R} \to \mathbf{R}$ is the squaring function $g(x) := x^2$, and $g$ is a continuous function.

*Exercise* 13.1.1. Prove Theorem 13.1.4. (Hint: review your proof of Proposition 9.4.7.)

*Exercise* 13.1.2. Prove Theorem 13.1.5. (Hint: Theorem 13.1.4 already shows that (a) and (b) are equivalent.)

*Exercise* 13.1.3. Use Theorem 13.1.4 and Theorem 13.1.5 to prove Corollary 13.1.7.

*Exercise* 13.1.4. Give an example of functions $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ such that

(a) $f$ is not continuous, but $g$ and $g \circ f$ are continuous;

(b) $g$ is not continuous, but $f$ and $g \circ f$ are continuous;

(c) $f$ and $g$ are not continuous, but $g \circ f$ is continuous.

Explain briefly why these examples do not contradict Corollary 13.1.7.

*Exercise* 13.1.5. Let $(X, d)$ be a metric space, and let $(E, d|_{E\times E})$ be a subspace of $(X, d)$. Let $\iota_{E\to X} : E \to X$ be the inclusion map, defined by setting $\iota_{E\to X}(x) := x$ for all $x \in E$. Show that $\iota_{E\to X}$ is continuous.

*Exercise* 13.1.6. Let $f : X \to Y$ be a function from one metric space $(X, d_X)$ to another $(Y, d_Y)$. Let $E$ be a subset of $X$ (which we give the induced metric $d_X|_{E\times E}$), and let $f|_E : E \to Y$ be the restriction of $f$ to $E$, thus $f|_E(x) := f(x)$ when $x \in E$. If $x_0 \in E$ and $f$ is continuous at $x_0$, show that $f|_E$ is also continuous at $x_0$. (Is the converse of this statement true? Explain.) Conclude that if $f$ is continuous, then $f|_E$ is continuous. Thus restriction of the domain of a function does not destroy continuity. (Hint: use Exercise 13.1.5.)

*Exercise* 13.1.7. Let $f : X \to Y$ be a function from one metric space $(X, d_X)$ to another $(Y, d_Y)$. Suppose that the image $f(X)$ of $X$ is contained in some subset $E \subset Y$ of $Y$. Let $g : X \to E$ be the function which is the same as $f$ but with the range restricted from $Y$ to $E$, thus $g(x) = f(x)$ for all $x \in X$. We give $E$ the metric $d_Y|_{E\times E}$ induced from $Y$. Show that for any $x_0 \in X$, that $f$ is continuous at $x_0$ if and only if $g$ is continuous at $x_0$. Conclude that $f$ is continuous if and only if $g$ is continuous. (Thus the notion of continuity is not affected if one restricts the range of the function.)

## 13.2 Continuity and product spaces

Given two functions $f : X \to Y$ and $g : X \to Z$, one can define their *direct sum* $f \oplus g : X \to Y \times Z$ defined by $f \oplus g(x) := (f(x), g(x))$, i.e., this is the function taking values in the Cartesian product $Y \times Z$ whose first co-ordinate is $f(x)$ and whose second co-ordinate is $g(x)$ (cf. Exercise 3.5.7). For instance, if $f : \mathbf{R} \to \mathbf{R}$ is the function $f(x) := x^2 + 3$, and $g : \mathbf{R} \to \mathbf{R}$ is the function $g(x) = 4x$, then $f \oplus g : \mathbf{R} \to \mathbf{R}^2$ is the function $f \oplus g(x) := (x^2 + 3, 4x)$. The direct sum operation preserves continuity:

**Lemma 13.2.1.** *Let $f : X \to \mathbf{R}$ and $g : X \to \mathbf{R}$ be functions, and let $f \oplus g : X \to \mathbf{R}^2$ be their direct sum. We give $\mathbf{R}^2$ the Euclidean metric.*

(a) *If $x_0 \in X$, then $f$ and $g$ are both continuous at $x_0$ if and only if $f \oplus g$ is continuous at $x_0$.*

(b) *$f$ and $g$ are both continuous if and only if $f \oplus g$ is continuous.*

*Proof.* See Exercise 13.2.1. □

To use this, we first need another continuity result:

**Lemma 13.2.2.** *The addition function $(x, y) \mapsto x+y$, the subtraction function $(x, y) \mapsto x - y$, the multiplication function $(x, y) \mapsto xy$, the maximum function $(x, y) \mapsto \max(x, y)$, and the minimum function $(x, y) \mapsto \min(x, y)$, are all continuous functions from $\mathbf{R}^2$ to $\mathbf{R}$. The division function $(x, y) \mapsto x/y$ is a continuous function from $\mathbf{R} \times (\mathbf{R}\backslash\{0\}) = \{(x, y) \in \mathbf{R}^2 : y \neq 0\}$ to $\mathbf{R}$. For any real number $c$, the function $x \mapsto cx$ is a continuous function from $\mathbf{R}$ to $\mathbf{R}$.*

*Proof.* See Exercise 13.2.2. □

Combining these lemmas we obtain

**Corollary 13.2.3.** *Let $(X, d)$ be a metric space, let $f : X \to \mathbf{R}$ and $g : X \to \mathbf{R}$ be functions. Let $c$ be a real number.*

(a) *If $x_0 \in X$ and $f$ and $g$ are continuous at $x_0$, then the functions $f + g : X \to \mathbf{R}$, $f - g : X \to \mathbf{R}$, $fg : X \to \mathbf{R}$, $\max(f, g) : X \to \mathbf{R}$, $\min(f, g) : X \to \mathbf{R}$, and $cf : X \to \mathbf{R}$ (see Definition 9.2.1 for definitions) are also continuous at $x_0$. If $g(x) \neq 0$ for all $x \in X$, then $f/g : X \to \mathbf{R}$ is also continuous at $x_0$.*

(b) *If $f$ and $g$ are continuous, then the functions $f + g : X \to \mathbf{R}$, $f - g : X \to \mathbf{R}$, $fg : X \to \mathbf{R}$, $\max(f, g) : X \to \mathbf{R}$, $\min(f, g) : X \to \mathbf{R}$, and $cf : X \to \mathbf{R}$ are also continuous at $x_0$. If $g(x) \neq 0$ for all $x \in X$, then $f/g : X \to \mathbf{R}$ is also continuous at $x_0$.*

*Proof.* We first prove (a). Since $f$ and $g$ are continuous at $x_0$, then by Lemma 13.2.1 $f \oplus g : X \to \mathbf{R}^2$ is also continuous at $x_0$. On the other hand, from Lemma 13.2.2 the function $(x, y) \mapsto x + y$ is continuous at every point in $\mathbf{R}^2$, and in particular is continuous at $f \oplus g(x_0)$. If we then compose these two functions using Corollary

13.1.7 we conclude that $f+g : X \to \mathbf{R}$ is continuous. A similar argument gives the continuity of $f-g$, $fg$, $\max(f,g)$, $\min(f,g)$ and $cf$. To prove the claim for $f/g$, we first use Exercise 13.1.7 to restrict the range of $g$ from $\mathbf{R}$ to $\mathbf{R}\backslash\{0\}$, and then one can argue as before. The claim (b) follows immediately from (a). □

This corollary allows us to demonstrate the continuity of a large class of functions; we give some examples below.

*Exercise* 13.2.1. Prove Lemma 13.2.1. (Hint: use Proposition 12.1.18 and Theorem 13.1.4.)

*Exercise* 13.2.2. Prove Lemma 13.2.2. (Hint: use Theorem 13.1.5 and limit laws (Theorem 6.1.19).)

*Exercise* 13.2.3. Show that if $f : X \to \mathbf{R}$ is a continuous function, so is the function $|f| : X \to \mathbf{R}$ defined by $|f|(x) := |f(x)|$.

*Exercise* 13.2.4. Let $\pi_1 : \mathbf{R}^2 \to \mathbf{R}$ and $\pi_2 : \mathbf{R}^2 \to \mathbf{R}$ be the functions $\pi_1(x,y) := x$ and $\pi_2(x,y) := y$ (these two functions are sometimes called the *co-ordinate functions* on $\mathbf{R}^2$). Show that $\pi_1$ and $\pi_2$ are continuous. Conclude that if $f : \mathbf{R} \to X$ is any continuous function into a metric space $(X,d)$, then the functions $g_1 : \mathbf{R}^2 \to X$ and $g_2 : \mathbf{R}^2 \to X$ defined by $g_1(x,y) := f(x)$ and $g_2(x,y) := f(y)$ are also continuous.

*Exercise* 13.2.5. Let $n, m \geq 0$ be integers. Suppose that for every $0 \leq i \leq n$ and $0 \leq j \leq m$ we have a real number $c_{ij}$. Form the function $P : \mathbf{R}^2 \to \mathbf{R}$ defined by

$$P(x,y) := \sum_{i=0}^{n}\sum_{j=0}^{m} c_{ij}x^i y^j.$$

(Such a function is known as a *polynomial of two variables*; a typical example of such a polynomial is $P(x,y) = x^3 + 2xy^2 - x^2 + 3y + 6$.) Show that $P$ is continuous. (Hint: use Exercise 13.2.4 and Corollary 13.2.3.) Conclude that if $f : X \to \mathbf{R}$ and $g : X \to \mathbf{R}$ are continuous functions, then the function $P(f,g) : X \to \mathbf{R}$ defined by $P(f,g)(x) := P(f(x), g(x))$ is also continuous.

*Exercise* 13.2.6. Let $\mathbf{R}^m$ and $\mathbf{R}^n$ be Euclidean spaces. If $f : X \to \mathbf{R}^m$ and $g : X \to \mathbf{R}^n$ are continuous functions, show that $f \oplus g : X \to \mathbf{R}^{m+n}$ is also continuous, where we have identified $\mathbf{R}^m \times \mathbf{R}^n$ with $\mathbf{R}^{m+n}$ in the obvious manner. Is the converse statement true?

*Exercise* 13.2.7. Let $k \geq 1$, let $I$ be a finite subset of $\mathbf{N}^k$, and let $c : I \to \mathbf{R}$ be a function. Form the function $P : \mathbf{R}^k \to \mathbf{R}$ defined by

$$P(x_1, \ldots, x_k) := \sum_{(i_1,\ldots,i_k)\in I} c(i_1, \ldots, i_k) x_1^{i_1} \ldots x_k^{i_k}.$$

(Such a function is known as a *polynomial of $k$ variables*; a typical example of such a polynomial is $P(x_1, x_2, x_3) = 3x_1^3 x_2 x_3^2 - x_2 x_3^2 + x_1 + 5$.) Show that $P$ is continuous. (Hint: use induction on $k$, Exercise 13.2.6, and either Exercise 13.2.5 or Lemma 13.2.2.)

*Exercise* 13.2.8. Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. Define the metric $d_{X\times Y} : (X \times Y) \times (X \times Y) \to [0, \infty)$ by the formula

$$d_{X\times Y}((x, y), (x', y')) := d_X(x, x') + d_Y(y, y').$$

Show that $(X \times Y, d_{X\times Y})$ is a metric space, and deduce an analogue of Proposition 12.1.18 and Lemma 13.2.1.

*Exercise* 13.2.9. Let $f : \mathbf{R}^2 \to X$ be a function from $\mathbf{R}^2$ to a metric space $X$. Let $(x_0, y_0)$ be a point in $\mathbf{R}^2$. If $f$ is continuous at $(x_0, y_0)$, show that

$$\lim_{x\to x_0} \limsup_{y\to y_0} f(x, y) = \lim_{y\to y_0} \limsup_{x\to x_0} f(x, y) = f(x_0, y_0)$$

and

$$\lim_{x\to x_0} \liminf_{y\to y_0} f(x, y) = \lim_{y\to y_0} \liminf_{x\to x_0} f(x, y) = f(x_0, y_0).$$

In particular, we have

$$\lim_{x\to x_0} \lim_{y\to y_0} f(x, y) = \lim_{y\to y_0} \lim_{x\to x_0} f(x, y)$$

whenever the limits on both sides exist. (Note that the limits do not necessarily exist in general; consider for instance the function $f : \mathbf{R}^2 \to \mathbf{R}$ such that $f(x, y) = y \sin \frac{1}{x}$ when $xy \neq 0$ and $f(x, y) = 0$ otherwise.) Discuss the comparison between this result and Example 1.2.7.

*Exercise* 13.2.10. Let $f : \mathbf{R}^2 \to \mathbf{R}$ be a continuous function. Show that for each $x \in \mathbf{R}$, the function $y \mapsto f(x, y)$ is continuous on $\mathbf{R}$, and for each $y \in \mathbf{R}$, the function $x \mapsto f(x, y)$ is continuous on $\mathbf{R}$. Thus a function $f(x, y)$ which is jointly continuous in $(x, y)$ is also continuous in each variable $x, y$ separately.

*Exercise* 13.2.11. Let $f : \mathbf{R}^2 \to \mathbf{R}$ be the function defined by $f(x, y) := \frac{xy}{x^2+y^2}$ when $(x, y) \neq (0, 0)$, and $f(x, y) = 0$ otherwise. Show that for each fixed $x \in \mathbf{R}$, the function $y \mapsto f(x, y)$ is continuous on $\mathbf{R}$, and that for each fixed $y \in \mathbf{R}$, the function $x \mapsto f(x, y)$ is continuous on $\mathbf{R}$, but that the function $f : \mathbf{R}^2 \to \mathbf{R}$ is not continuous on $\mathbf{R}^2$. This shows that the converse to Exercise 13.2.10 fails; it is possible to be continuous in each variable separately without being jointly continuous.

## 13.3 Continuity and compactness

Continuous functions interact well with the concept of compact sets defined in Definition 12.5.1.

**Theorem 13.3.1** (Continuous maps preserve compactness). *Let $f : X \to Y$ be a continuous map from one metric space $(X, d_X)$ to another $(Y, d_Y)$. Let $K \subseteq X$ be any compact subset of $X$. Then the image $f(K) := \{f(x) : x \in K\}$ of $K$ is also compact.*

*Proof.* See Exercise 13.3.1. □

This theorem has an important consequence. Recall from Definition 9.6.5 the notion of a function $f : X \to \mathbf{R}$ attaining a maximum or minimum at a point. We may generalize Proposition 9.6.7 as follows:

**Proposition 13.3.2** (Maximum principle). *Let $(X, d)$ be a compact metric space, and let $f : X \to \mathbf{R}$ be a continuous function. Then $f$ is bounded. Furthermore, $f$ attains its maximum at some point $x_{max} \in X$, and also attains its minimum at some point $x_{min} \in X$.*

*Proof.* See Exercise 13.3.2. □

**Remark 13.3.3.** As was already noted in Exercise 9.6.1, this principle can fail if $X$ is not compact. This proposition should be compared with Lemma 9.6.3 and Proposition 9.6.7.

Another advantage of continuous functions on compact sets is that they are *uniformly continuous*. We generalize Definition 9.9.2 as follows:

**Definition 13.3.4** (Uniform continuity). Let $f : X \to Y$ be a map from one metric space $(X, d_X)$ to another $(Y, d_Y)$. We say that $f$ is *uniformly continuous* if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d_Y(f(x), f(x')) < \varepsilon$ whenever $x, x' \in X$ are such that $d_X(x, x') < \delta$.

Every uniformly continuous function is continuous, but not conversely (Exercise 13.3.3). But if the domain $X$ is compact, then the two notions are equivalent:

**Theorem 13.3.5.** *Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces, and suppose that $(X, d_X)$ is compact. If $f : X \to Y$ is function, then $f$ is continuous if and only if it is uniformly continuous.*

*Proof.* If $f$ is uniformly continuous then it is also continuous by Exercise 13.3.3. Now suppose that $f$ is continuous. Fix $\varepsilon > 0$. For every $x_0 \in X$, the function $f$ is continuous at $x_0$. Thus there exists a $\delta(x_0) > 0$, depending on $x_0$, such that $d_Y(f(x), f(x_0)) < \varepsilon/2$ whenever $d_X(x, x_0) < \delta(x_0)$. In particular, by the triangle inequality this implies that $d_Y(f(x), f(x')) < \varepsilon$ whenever $x \in B_{(X,d_X)}(x_0, \delta(x_0)/2)$ and $d_X(x', x) < \delta(x_0)/2$ (why?).

Now consider the (possibly infinite) collection of balls

$$\{B_{(X,d_X)}(x_0, \delta(x_0)/2) : x_0 \in X\}.$$

Each ball in this collection is of course open, and the union of all these balls covers $X$, since each point $x_0$ in $X$ is contained in its own ball $B_{(X,d_X)}(x_0, \delta(x_0)/2)$. Hence, by Theorem 12.5.8, there exist a finite number of points $x_1, \ldots, x_n$ such that the balls $B_{(X,d_X)}(x_j, \delta(x_j)/2)$ for $j = 1, \ldots, n$ cover $X$:

$$X \subseteq \bigcup_{j=1}^{n} B_{(X,d_X)}(x_j, \delta(x_j)/2).$$

Now let $\delta := \min_{j=1}^{n} \delta(x_j)/2$. Since each of the $\delta(x_j)$ are positive, and there are only a finite number of $j$, we see that $\delta > 0$. Now let $x, x'$ be any two points in $X$ such that $d_X(x, x') < \delta$. Since the balls $B_{(X,d_X)}(x_j, \delta(x_j)/2)$ cover $X$, we see that there must exist

$1 \le j \le n$ such that $x \in B_{(X,d_X)}(x_j, \delta(x_j)/2)$. Since $d_X(x, x') < \delta$, we have $d_X(x, x') < \delta(x_j)/2$, and so by the previous discussion we have $d_Y(f(x), f(x')) < \varepsilon$. We have thus found a $\delta$ such that $d_Y(f(x), f(x')) < \varepsilon$ whenever $d(x, x') < \delta$, and this proves uniform continuity as desired. □

*Exercise* 13.3.1. Prove Theorem 13.3.1.

*Exercise* 13.3.2. Prove Proposition 13.3.2. (Hint: modify the proof of Proposition 9.6.7.)

*Exercise* 13.3.3. Show that every uniformly continuous function is continuous, but give an example that shows that not every continuous function is uniformly continuous.

*Exercise* 13.3.4. Let $(X, d_X)$, $(Y, d_Y)$, $(Z, d_Z)$ be metric spaces, and let $f : X \to Y$ and $g : Y \to Z$ be two uniformly continuous functions. Show that $g \circ f : X \to Z$ is also uniformly continuous.

*Exercise* 13.3.5. Let $(X, d_X)$ be a metric space, and let $f : X \to \mathbf{R}$ and $g : X \to \mathbf{R}$ be uniformly continuous functions. Show that the direct sum $f \oplus g : X \to \mathbf{R}^2$ defined by $f \oplus g(x) := (f(x), g(x))$ is uniformly continuous.

*Exercise* 13.3.6. Show that the addition function $(x, y) \mapsto x + y$ and the subtraction function $(x, y) \mapsto x - y$ are uniformly continuous from $\mathbf{R}^2$ to $\mathbf{R}$, but the multiplication function $(x, y) \mapsto xy$ is not. Conclude that if $f : X \to \mathbf{R}$ and $g : X \to \mathbf{R}$ are uniformly continuous functions on a metric space $(X, d)$, then $f + g : X \to \mathbf{R}$ and $f - g : X \to \mathbf{R}$ are also uniformly continuous. Give an example to show that $fg : X \to \mathbf{R}$ need not be uniformly continuous. What is the situation for $\max(f, g)$, $\min(f, g)$, $f/g$, and $cf$ for a real number $c$?

## 13.4 Continuity and connectedness

We now describe another important concept in metric spaces, that of *connectedness.*

**Definition 13.4.1** (Connected spaces). Let $(X, d)$ be a metric space. We say that $X$ is *disconnected* iff there exist disjoint non-empty open sets $V$ and $W$ in $X$ such that $V \cup W = X$. (Equivalently, $X$ is disconnected if and only if $X$ contains a non-empty