this function, called the *Riemann hypothesis*, but to discuss it further is far beyond the scope of this text. I will mention however that there is a US$ 1 million prize - and instant fame among all mathematicians - attached to the solution to this problem.

Exercise 7.3.1. Use Proposition 7.3.1 to prove Corollary 7.3.2.

Exercise 7.3.2. Prove Lemma 7.3.3. (Hint: for the first part, use the zero test. For the second part, first use induction to establish the *geometric series formula*

$$\sum_{n=0}^{N} x^n = (1 - x^{N+1})/(1 - x)$$

and then apply Lemma 6.5.2.)

Exercise 7.3.3. Let $\sum_{n=0}^{\infty} a_n$ be an absolutely convergent series of real numbers such that $\sum_{n=0}^{\infty} |a_n| = 0$. Show that $a_n = 0$ for every natural number n.

7.4 Rearrangement of series

One feature of finite sums is that no matter how one rearranges the terms in a sequence, the total sum is the same. For instance,

$$a_1 + a_2 + a_3 + a_4 + a_5 = a_4 + a_3 + a_5 + a_1 + a_2.$$

A more rigourous statement of this, involving bijections, has already appeared earlier, see Remark 7.1.12.

One can ask whether the same thing is true for infinite series. If all the terms are non-negative, the answer is yes:

Proposition 7.4.1. *Let $\sum_{n=0}^{\infty} a_n$ be a convergent series of non-negative real numbers, and let $f : \mathbf{N} \to \mathbf{N}$ be a bijection. Then $\sum_{m=0}^{\infty} a_{f(m)}$ is also convergent, and has the same sum:*

$$\sum_{n=0}^{\infty} a_n = \sum_{m=0}^{\infty} a_{f(m)}.$$

Proof. We introduce the partial sums $S_N := \sum_{n=0}^{N} a_n$ and $T_M := \sum_{m=0}^{M} a_{f(m)}$. We know that the sequences $(S_N)_{n=0}^{\infty}$ and $(T_M)_{m=0}^{\infty}$ are increasing. Write $L := \sup(S_N)_{n=0}^{\infty}$ and $L' := \sup(T_M)_{m=0}^{\infty}$. By Proposition 6.3.8 we know that L is finite, and in fact $L = \sum_{n=0}^{\infty} a_n$; by Proposition 6.3.8 again we see that we will thus be done as soon as we can show that $L' = L$.

Fix M, and let Y be the set $Y := \{m \in \mathbf{N} : m \leq M\}$. Note that f is a bijection between Y and $f(Y)$. By Proposition 7.1.11, we have

$$T_M = \sum_{m=0}^{M} a_{f(m)} = \sum_{m \in Y} a_{f(m)} = \sum_{n \in f(Y)} a_n.$$

The sequence $(f(m))_{m=0}^{M}$ is finite, hence bounded, i.e., there exists an N such that $f(m) \leq N$ for all $m \leq N$. In particular $f(Y)$ is a subset of $\{n \in \mathbf{N} : n \leq N\}$, and so by Proposition 7.1.11 again (and the assumption that all the a_n are non-negative)

$$T_M = \sum_{n \in f(Y)} a_n \leq \sum_{n \in \{n \in \mathbf{N} : n \leq N\}} a_n = \sum_{n=0}^{N} a_n = S_N.$$

But since $(S_N)_{N=0}^{\infty}$ has a supremum of L, we thus see that $S_N \leq L$, and hence that $T_M \leq L$ for all M. Since L' is the least upper bound of $(T_M)_{M=0}^{\infty}$, this implies that $L' \leq L$.

A very similar argument (using the inverse f^{-1} instead of f) shows that every S_N is bounded above by L', and hence $L \leq L'$. Combining these two inequalities we obtain $L = L'$, as desired. □

Example 7.4.2. From Corollary 7.3.7 we know that the series

$$\sum_{n=1}^{\infty} 1/n^2 = 1 + 1/4 + 1/9 + 1/16 + 1/25 + 1/36 + \ldots$$

is convergent. Thus, if we interchange every pair of terms, to obtain

$$1/4 + 1 + 1/16 + 1/9 + 1/36 + 1/25 + \ldots$$

we know that this series is also convergent, and has the same sum. (It turns out that the value of this sum is $\zeta(2) \doteq \pi^2/6$, a fact which we shall prove in Exercise 16.5.2.)

Now we ask what happens when the series is not non-negative. Then as long as the series is *absolutely* convergent, we can still do rearrangements:

Proposition 7.4.3 (Rearrangement of series). *Let $\sum_{n=0}^{\infty} a_n$ be an absolutely convergent series of real numbers, and let $f : \mathbf{N} \to \mathbf{N}$ be a bijection. Then $\sum_{m=0}^{\infty} a_{f(m)}$ is also absolutely convergent, and has the same sum:*

$$\sum_{n=0}^{\infty} a_n = \sum_{m=0}^{\infty} a_{f(m)}.$$

Proof. (Optional) We apply Proposition 7.4.1 to the infinite series $\sum_{n=0}^{\infty} |a_n|$, which by hypothesis is a convergent series of non-negative numbers. If we write $L := \sum_{n=0}^{\infty} |a_n|$, then by Proposition 7.4.1 we know that $\sum_{m=0}^{\infty} |a_{f(m)}|$ also converges to L.

Now write $L' := \sum_{n=0}^{\infty} a_n$. We have to show that $\sum_{m=0}^{\infty} a_{f(m)}$ also converges to L'. In other words, given any $\varepsilon > 0$, we have to find an M such that $\sum_{m=0}^{M'} a_{f(m)}$ is ε-close to L' for every $M' \geq M$.

Since $\sum_{n=0}^{\infty} |a_n|$ is convergent, we can use Proposition 7.2.5 and find an N_1 such that $\sum_{n=p}^{q} |a_n| \leq \varepsilon/2$ for all $p, q \geq N_1$. Since $\sum_{n=0}^{\infty} a_n$ converges to L', the partial sums $\sum_{n=0}^{N} a_n$ also converge to L', and so there exists $N \geq N_1$ such that $\sum_{n=0}^{N} a_n$ is $\varepsilon/2$-close to L'.

Now the sequence $(f^{-1}(n))_{n=0}^{N}$ is finite, hence bounded, so there exists an M such that $f^{-1}(n) \leq M$ for all $0 \leq n \leq N$. In particular, for any $M' \geq M$, the set $\{f(m) : m \in \mathbf{N}; m \leq M'\}$ contains $\{n \in \mathbf{N} : n \leq N\}$(why?). So by Proposition 7.1.11, for any $M' \geq M$

$$\sum_{m=0}^{M'} a_{f(m)} = \sum_{n \in \{f(m) : m \in \mathbf{N}; m \leq M'\}} a_n = \sum_{n=0}^{N} a_n + \sum_{n \in X} a_n$$

where X is the set

$$X = \{f(m) : m \in \mathbf{N}; m \leq M'\} \backslash \{n \in \mathbf{N} : n \leq N\}.$$

The set X is finite, and is therefore bounded by some natural number q; we must therefore have

$$X \subseteq \{n \in \mathbf{N} : N + 1 \leq n \leq q\}$$

(why?). Thus

$$|\sum_{n \in X} a_n| \leq \sum_{n \in X} |a_n| \leq \sum_{n=N+1}^{q} |a_n| \leq \varepsilon/2$$

by our choice of N. Thus $\sum_{m=0}^{M'} a_{f(m)}$ is $\varepsilon/2$-close to $\sum_{n=0}^{N} a_n$, which as mentioned before is $\varepsilon/2$-close to L'. Thus $\sum_{m=0}^{M'} a_{f(m)}$ is ε-close to L for all $M' \geq M$, as desired. $\square$

Surprisingly, when the series is not absolutely convergent, then the rearrangements are very badly behaved.

Example 7.4.4. Consider the series

$$1/3 - 1/4 + 1/5 - 1/6 + 1/7 - 1/8 + \ldots.$$

This series is not absolutely convergent (why?), but is conditionally convergent by the alternating series test, and in fact the sum can be seen to converge to a positive number (in fact, it converges to $\ln(2) - 1/2 = 0.193147\ldots$, see Example 15.5.7). Basically, the reason why the sum is positive is because the quantities $(1/3 - 1/4)$, $(1/5 - 1/6)$, $(1/7 - 1/8)$ are all positive, which can then be used to show that every partial sum is positive. (Why? you have to break into two cases, depending on whether there are an even or odd number of terms in the partial sum.)

If, however, we rearrange the series to have two negative terms to each positive term, thus

$$1/3 - 1/4 - 1/6 + 1/5 - 1/8 - 1/10 + 1/7 - 1/12 - 1/14 + \ldots$$

then the partial sums quickly become negative (this is because $(1/3 - 1/4 - 1/6)$, $(1/5 - 1/8 - 1/9)$, and more generally $(1/(2n+1) - 1/4n - 1/(4n+2))$ are all negative), and so this series converges to a negative quantity; in fact, it converges to

$$(\ln(2) - 1)/2 = -.153426\ldots.$$

There is in fact a surprising result of Riemann, which shows that a series which is conditionally convergent but not absolutely convergent can in fact be rearranged to converge to *any* value (or rearranged to diverge, in fact - see Exercise 8.2.6); see Theorem 8.2.8.

To summarize, rearranging series is safe when the series is absolutely convergent, but is somewhat dangerous otherwise. (This is not to say that rearranging an absolutely divergent series necessarily gives you the wrong answer - for instance, in theoretical physics one often performs similar maneuvres, and one still (usually) obtains a correct answer at the end - but doing so is risky, unless it is backed by a rigourous result such as Proposition 7.4.3.)

Exercise 7.4.1. Let $\sum_{n=0}^{\infty} a_n$ be an absolutely convergent series of real numbers. Let $f : \mathbf{N} \to \mathbf{N}$ be an increasing function (i.e., $f(n+1) > f(n)$ for all $n \in \mathbf{N}$). Show that $\sum_{n=0}^{\infty} a_{f(n)}$ is also an absolutely convergent series. (Hint: try to compare each partial sum of $\sum_{n=0}^{\infty} a_{f(n)}$ with a (slightly different) partial sum of $\sum_{n=0}^{\infty} a_n$.)

7.5 The root and ratio tests

Now we can state and prove the famous root and ratio tests for convergence.

Theorem 7.5.1 (Root test). *Let $\sum_{n=m}^{\infty} a_n$ be a series of real numbers, and let $\alpha := \limsup_{n\to\infty} |a_n|^{1/n}$.*

(a) *If $\alpha < 1$, then the series $\sum_{n=m}^{\infty} a_n$ is absolutely convergent (and hence conditionally convergent).*

(b) *If $\alpha > 1$, then the series $\sum_{n=m}^{\infty} a_n$ is conditionally divergent (and hence cannot be absolutely convergent).*

(c) *If $\alpha = 1$, we cannot assert any conclusion.*

Proof. First suppose that $\alpha < 1$. Note that we must have $\alpha \geq 0$, since $|a_n|^{1/n} \geq 0$ for every n. Then we can find an $\varepsilon > 0$ such that $0 < \alpha+\varepsilon < 1$ (for instance, we can set $\varepsilon := (1-\alpha)/2$). By Proposition 6.4.12(a), there exists an $N \geq m$ such that $|a_n|^{1/n} \leq \alpha+\varepsilon$ for all $n \geq N$. In other words, we have $|a_n| \leq (\alpha+\varepsilon)^n$ for all $n \geq N$. But from the geometric series we have that $\sum_{n=N}^{\infty}(\alpha+\varepsilon)^n$ is absolutely convergent, since $0 < \alpha+\varepsilon < 1$ (note that the fact that we start from N is irrelevant by Proposition 7.2.14(c)). Thus by the comparison test, we see that $\sum_{n=N}^{\infty} a_n$ is absolutely convergent, and thus $\sum_{n=m}^{\infty} a_n$ is absolutely convergent, by Proposition 7.2.14(c) again.

Now suppose that $\alpha > 1$. Then by Proposition 6.4.12(b), we see that for every $N \geq m$ there exists an $n \geq N$ such that $|a_n|^{1/n} \geq 1$, and hence that $|a_n| \geq 1$. In particular, $(a_n)_{n=N}^{\infty}$ is not 1-close to 0 for any N, and hence $(a_n)_{n=m}^{\infty}$ is not eventually 1-close to 0. In particular, $(a_n)_{n=m}^{\infty}$ does not converge to zero. Thus by the zero test, $\sum_{n=m}^{\infty} a_n$ is conditionally divergent.

For $\alpha = 1$, see Exercise 7.5.3. □

The root test is phrased using the limit superior, but of course if $\lim_{n\to\infty} |a_n|^{1/n}$ converges then the limit is the same as the limit superior. Thus one can phrase the root test using the limit instead of the limit superior, but *only when the limit exists.*

The root test is sometimes difficult to use; however we can replace roots by ratios using the following lemma.

Lemma 7.5.2. *Let $(c_n)_{n=m}^{\infty}$ be a sequence of positive numbers. Then we have*

$$\liminf_{n\to\infty} \frac{c_{n+1}}{c_n} \leq \liminf_{n\to\infty} c_n^{1/n} \leq \limsup_{n\to\infty} c_n^{1/n} \leq \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}.$$

Proof. There are three inequalities to prove here. The middle inequality follows from Proposition 6.4.12(c). We shall prove the last inequality, and leave the first one to Exercise 7.5.1.

Write $L := \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}$. If $L = +\infty$ then there is nothing to prove (since $x \leq +\infty$ for every extended real number x), so we may assume that L is a finite real number. (Note that L cannot equal $-\infty$; why?). Since $\frac{c_{n+1}}{c_n}$ is always positive, we know that $L \geq 0$.

Let $\varepsilon > 0$. By Proposition 6.4.12(a), we know that there exists an $N \geq m$ such that $\frac{c_{n+1}}{c_n} \leq L+\varepsilon$ for all $n \geq N$. This implies that $c_{n+1} \leq c_n(L+\varepsilon)$ for all $n \geq N$. By induction this implies that

$$c_n \leq c_N(L+\varepsilon)^{n-N} \text{ for all } n \geq N$$

(why?). If we write $A := c_N(L+\varepsilon)^{-N}$, then we have

$$c_n \leq A(L+\varepsilon)^n$$

and thus

$$c_n^{1/n} \leq A^{1/n}(L+\varepsilon)$$

for all $n \geq N$. But we have

$$\lim_{n\to\infty} A^{1/n}(L+\varepsilon) = L+\varepsilon$$

by the limit laws (Theorem 6.1.19) and Lemma 6.5.3. Thus by the comparison principle (Lemma 6.4.13) we have

$$\limsup_{n\to\infty} c_n^{1/n} \leq L+\varepsilon.$$

But this is true for all $\varepsilon > 0$, so this must imply that

$$\limsup_{n\to\infty} c_n^{1/n} \leq L$$

(why? prove by contradiction), as desired. □

From Theorem 7.5.1 and Lemma 7.5.2 (and Exercise 7.5.3) we have

Corollary 7.5.3 (Ratio test). *Let $\sum_{n=m}^{\infty} a_n$ be a series of non-zero numbers. (The non-zero hypothesis is required so that the ratios $|a_{n+1}|/|a_n|$ appearing below are well-defined.)*

- *If* $\limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1$, *then the series* $\sum_{n=m}^{\infty} a_n$ *is absolutely convergent (hence conditionally convergent).*
- *If* $\liminf_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} > 1$, *then the series* $\sum_{n=m}^{\infty} a_n$ *is conditionally divergent (and thus cannot be absolutely convergent).*
- *In the remaining cases, we cannot assert any conclusion.*

Another consequence of Lemma 7.5.2 is the following limit:

Proposition 7.5.4. *We have* $\lim_{n\to\infty} n^{1/n} = 1$.

Proof. By Lemma 7.5.2 we have

$$\limsup_{n\to\infty} n^{1/n} \leq \limsup_{n\to\infty} (n+1)/n = \limsup_{n\to\infty} 1 + 1/n = 1$$

by Proposition 6.1.11 and limit laws (Theorem 6.1.19). Similarly we have

$$\liminf_{n\to\infty} n^{1/n} \geq \liminf_{n\to\infty} (n+1)/n = \liminf_{n\to\infty} 1 + 1/n = 1.$$

The claim then follows from Proposition 6.4.12(c) and (f). □

Remark 7.5.5. In addition to the ratio and root tests, another very useful convergence test is the *integral test*, which we will cover in Proposition 11.6.4.

Exercise 7.5.1. Prove the first inequality in Lemma 7.5.2.

Exercise 7.5.2. Let x be a real number with $|x| < 1$, and q be a real number. Show that the series $\sum_{n=1}^{\infty} n^q x^n$ is absolutely convergent, and that $\lim_{n\to\infty} n^q x^n = 0$.

Exercise 7.5.3. Give an example of a divergent series $\sum_{n=1}^{\infty} a_n$ of positive numbers a_n such that $\lim_{n\to\infty} a_{n+1}/a_n = \lim_{n\to\infty} a_n^{1/n} = 1$, and give an example of a convergent series $\sum_{n=1}^{\infty} b_n$ of positive numbers b_n such that $\lim_{n\to\infty} b_{n+1}/b_n = \lim_{n\to\infty} b_n^{1/n} = 1$. (Hint: use Corollary 7.3.7.) This shows that the ratio and root tests can be inconclusive even when the summands are positive and all the limits converge.

Chapter 8

Infinite sets

We now return to the study of set theory, and specifically to the study of cardinality of sets which are infinite (i.e., sets which do not have cardinality n for any natural number n), a topic which was initiated in Section 3.6.

8.1 Countability

From Proposition 3.6.14(c) we know that if X is a finite set, and Y is a proper subset of X, then Y does not have equal cardinality with X. However, this is not the case for infinite sets. For instance, from Theorem 3.6.12 we know that the set $\mathbf{N}$ of natural numbers is infinite. The set $\mathbf{N} - \{0\}$ is also infinite, thanks to Proposition 3.6.14(a) (why?), and is a proper subset of $\mathbf{N}$. However, the set $\mathbf{N} - \{0\}$, despite being "smaller" than $\mathbf{N}$, still has the same cardinality as $\mathbf{N}$, because the function $f : \mathbf{N} \to \mathbf{N} - \{0\}$ defined by $f(n) := n+1$, is a bijection from $\mathbf{N}$ to $\mathbf{N} - \{0\}$. (Why?) This is one characteristic of infinite sets; see Exercise 8.1.1.

We now distinguish two types of infinite sets: the countable sets and the uncountable sets.

Definition 8.1.1 (Countable sets). A set X is said to be *countably infinite* (or just *countable*) iff it has equal cardinality with the natural numbers $\mathbf{N}$. A set X is said to be *at most countable* iff it is either countable or finite. We say that a set is *uncountable* if it is infinite but not countable.

Remark 8.1.2. Countably infinite sets are also called *denumerable* sets.

Examples 8.1.3. From the preceding discussion we see that $\mathbf{N}$ is countable, and so is $\mathbf{N} - \{0\}$. Another example of a countable set is the even natural numbers $\{2n : n \in \mathbf{N}\}$, since the function $f(n) := 2n$ provides a bijection between $\mathbf{N}$ and the even natural numbers (why?).

Let X be a countable set. Then, by definition, we know that there exists a bijection $f : \mathbf{N} \to X$. Thus, every element of X can be written in the form $f(n)$ for exactly one natural number n. Informally, we thus have

$$X = \{f(0), f(1), f(2), f(3), \ldots\}.$$

Thus, a countable set can be arranged in a sequence, so that we have a zeroth element $f(0)$, followed by a first element $f(1)$, then a second element $f(2)$, and so forth, in such a way that all these elements $f(0), f(1), f(2), \ldots$ are all distinct, and together they fill out all of X. (This is why these sets are called *countable*; because we can literally count them one by one, starting from $f(0)$, then $f(1)$, and so forth.)

Viewed in this way, it is clear why the natural numbers

$$\mathbf{N} = \{0, 1, 2, 3, \ldots\},$$

the positive integers

$$\mathbf{N} - \{0\} = \{1, 2, 3, \ldots\},$$

and the even natural numbers

$$\{0, 2, 4, 6, 8, \ldots\}$$

are countable. However, it is not as obvious whether the integers

$$\mathbf{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

or the rationals

$$\mathbf{Q} = \{0, 1/4, -2/3, \ldots\}$$

or the reals

$$\mathbf{R} = \{0, \sqrt{2}, -\pi, 2.5, \ldots\}$$

are countable or not; for instance, it is not yet clear whether we can arrange the real numbers in a sequence $f(0), f(1), f(2), \ldots$. We will answer these questions shortly.

From Proposition 3.6.4 and Theorem 3.6.12, we know that countable sets are infinite; however it is not so clear whether all infinite sets are countable. Again, we will answer those questions shortly. We first need the following important principle.

Proposition 8.1.4 (Well ordering principle). *Let X be a non-empty subset of the natural numbers $\mathbf{N}$. Then there exists exactly one element $n \in X$ such that $n \leq m$ for all $m \in X$. In other words, every non-empty set of natural numbers has a minimum element.*

Proof. See Exercise 8.1.2. □

We will refer to the element n given by the well-ordering principle as the *minimum* of X, and write it as $\min(X)$. Thus for instance the minimum of the set $\{2, 4, 6, 8, \ldots\}$ is 2. This minimum is clearly the same as the infimum of X, as defined in Definition 5.5.10 (why?).

Proposition 8.1.5. *Let X be an infinite subset of the natural numbers $\mathbf{N}$. Then there exists a unique bijection $f : \mathbf{N} \to X$ which is increasing, in the sense that $f(n+1) > f(n)$ for all $n \in \mathbf{N}$. In particular, X has equal cardinality with $\mathbf{N}$ and is hence countable.*

Proof. We will give an incomplete sketch of the proof, with some gaps marked by a question mark (?); these gaps will be filled in Exercise 8.1.3.

We now define a sequence $a_0, a_1, a_2, \ldots$ of natural numbers recursively by the formula

$$a_n := \min\{x \in X : x \neq a_m \text{ for all } m < n\}.$$

Intuitively speaking, a_0 is the smallest element of X; a_1 is the second smallest element of X, i.e., the smallest element of X once

a_0 is removed; a_2 is the third smallest element of X; and so forth. Observe that in order to define a_n, one only needs to know the values of a_m for all $m < n$, so this definition is recursive. Also, since X is infinite, the set $\{x \in X : x \neq a_m \text{ for all } m < n\}$ is infinite(?), hence non-empty. Thus by the well-ordering principle, the minimum, $\min\{x \in X : x \neq a_m \text{ for all } m < n\}$ is always well-defined.

One can show(?) that a_n is an increasing sequence, i.e.

$$a_0 < a_1 < a_2 < \ldots$$

and in particular that(?) $a_n \neq a_m$ for all $n \neq m$. Also, we have(?) $a_n \in X$ for each natural number n.

Now define the function $f : \mathbf{N} \to X$ by $f(n) := a_n$. From the previous paragraph we know that f is one-to-one. Now we show that f is onto. In other words, we claim that for every $x \in X$, there exists an n such that $a_n = x$.

Let $x \in X$. Suppose for sake of contradiction that $a_n \neq x$ for every natural number n. Then this implies(?) that x is an element of the set $\{x \in X : x \neq a_m \text{ for all } m < n\}$ for all n. By definition of a_n, this implies that $x \geq a_n$ for every natural number n. However, since a_n is an increasing sequence, we have $a_n \geq n$ (?), and hence $x \geq n$ for every natural number n. In particular we have $x \geq x + 1$, which is a contradiction. Thus we must have $a_n = x$ for some natural number n, and hence f is onto.

Since $f : \mathbf{N} \to X$ is both one-to-one and onto, it is a bijection. We have thus found at least one increasing bijection f from $\mathbf{N}$ to X. Now suppose for sake of contradiction that there was at least one other increasing bijection g from $\mathbf{N}$ to X which was not equal to f. Then the set $\{n \in \mathbf{N} : g(n) \neq f(n)\}$ is non-empty, and define $m := \min\{n \in \mathbf{N} : g(n) \neq f(n)\}$, thus in particular $g(m) \neq f(m) = a_m$, and $g(n) = f(n) = a_n$ for all $n < m$. But we then must have(?)

$$g(m) = \min\{x \in X : x \neq a_t \text{ for all } t < m\} = a_m,$$

a contradiction. Thus there is no other increasing bijection from $\mathbf{N}$ to X other than f. $\square$

Since finite sets are at most countable by definition, we thus have

Corollary 8.1.6. *All subsets of the natural numbers are at most countable.*

Corollary 8.1.7. *If X is an at most countable set, and Y is a subset of X, then Y is at most countable.*

Proof. If X is finite then this follows from Proposition 3.6.14(c), so assume X is countable. Then there is a bijection $f : X \to \mathbf{N}$ between X and $\mathbf{N}$. Since Y is a subset of X, and f is a bijection from X and $\mathbf{N}$, then when we restrict f to Y, we obtain a bijection between Y and $f(Y)$. (Why is this a bijection?) Thus $f(Y)$ has equal cardinality with Y. But $f(Y)$ is a subset of $\mathbf{N}$, and hence at most countable by Corollary 8.1.6. Hence Y is also at most countable. □

Proposition 8.1.8. *Let Y be a set, and let $f : \mathbf{N} \to Y$ be a function. Then $f(\mathbf{N})$ is at most countable.*

Proof. See Exercise 8.1.4. □

Corollary 8.1.9. *Let X be a countable set, and let $f : X \to Y$ be a function. Then $f(X)$ is at most countable.*

Proof. See Exercise 8.1.5. □

Proposition 8.1.10. *Let X be a countable set, and let Y be a countable set. Then $X \cup Y$ is a countable set.*

Proof. See Exercise 8.1.7. □

To summarize, any subset or image of a countable set is at most countable, and any finite union of countable sets is still countable. We can now establish countability of the integers.

Corollary 8.1.11. *The integers $\mathbf{Z}$ are countable.*

Proof. We already know that the set $\mathbf{N} = \{0, 1, 2, 3, \ldots\}$ of natural numbers are countable. The set $-\mathbf{N}$ defined by

$$-\mathbf{N} := \{-n : n \in \mathbf{N}\} = \{0, -1, -2, -3, \ldots\}$$

is also countable, since the map $f(n) := -n$ is a bijection between $\mathbf{N}$ and this set. Since the integers are the union of $\mathbf{N}$ and $-\mathbf{N}$, the claim follows from Proposition 8.1.10 $\square$

To establish countability of the rationals, we need to relate countability with Cartesian products. In particular, we need to show that the set $\mathbf{N} \times \mathbf{N}$ is countable. We first need a preliminary lemma:

Lemma 8.1.12. *The set*

$$A := \{(n, m) \in \mathbf{N} \times \mathbf{N} : 0 \leq m \leq n\}$$

is countable.

Proof. Define the sequence a_0, a_1, $a_2, \ldots$ recursively by setting $a_0 := 0$, and $a_{n+1} := a_n + n + 1$ for all natural numbers n. Thus

$$a_0 = 0; a_1 = 0 + 1; a_2 = 0 + 1 + 2; a_3 = 0 + 1 + 2 + 3; \ldots.$$

By induction one can show that a_n is increasing, i.e., that $a_n > a_m$ whenever $n > m$ (why?).

Now define the function $f : A \to \mathbf{N}$ by

$$f(n, m) := a_n + m.$$

We claim that f is one-to-one. In other words, if (n, m) and (n', m') are any two distinct elements of A, then we claim that $f(n, m) \neq f(n', m')$.

To prove this claim, let (n, m) and (n', m') be two distinct elements of A. There are three cases: $n' = n$, $n' > n$, and $n' < n$. First suppose that $n' = n$. Then we must have $m \neq m'$, otherwise (n, m) and (n', m') would not be distinct. Thus $a_n + m \neq a_n + m'$, and hence $f(n, m) \neq f(n', m')$, as desired.

Now suppose that $n' > n$. Then $n' \geq n + 1$, and hence

$$f(n', m') = a_{n'} + m' \geq a_{n'} \geq a_{n+1} = a_n + n + 1.$$

But since $(n, m) \in A$, we have $m \leq n < n + 1$, and hence

$$f(n', m') \geq a_n + n + 1 > a_n + m = f(n, m),$$

and thus $f(n', m') \neq f(n, m)$.

The case $n' < n$ is proven similarly, by switching the rôles of n and n' in the previous argument. Thus we have shown that f is one-to-one. Thus f is a bijection from A to $f(A)$, and so A has equal cardinality with $f(A)$. But $f(A)$ is a subset of $\mathbf{N}$, and hence by Corollary 8.1.6 $f(A)$ is at most countable. Therefore A is at most countable. But, A is clearly not finite. (Why? Hint: if A was finite, then every subset of A would be finite, and in particular $\{(n, 0) : n \in \mathbf{N}\}$ would be finite, but this is clearly countably infinite, a contradiction.) Thus, A must be countable. □

Corollary 8.1.13. *The set* $\mathbf{N} \times \mathbf{N}$ *is countable.*

Proof. We already know that the set

$$A := \{(n, m) \in \mathbf{N} \times \mathbf{N} : 0 \leq m \leq n\}$$

is countable. This implies that the set

$$B := \{(n, m) \in \mathbf{N} \times \mathbf{N} : 0 \leq n \leq m\}$$

is also countable, since the map $f : A \to B$ given by $f(n, m) := (m, n)$ is a bijection from A to B (why?). But since $\mathbf{N} \times \mathbf{N}$ is the union of A and B (why?), the claim then follows from Proposition 8.1.10. □

Corollary 8.1.14. *If* X *and* Y *are countable, then* $X \times Y$ *is countable.*

Proof. See Exercise 8.1.8. □

Corollary 8.1.15. *The rationals* $\mathbf{Q}$ *are countable.*