8.1. The Greeks knew just the first four Mersenne primes and Mersenne discovered eight more. Before 1950, only the first 12 Mersenne primes were known. Then, with the help of ever more powerful computers, 18 more came to light. (Even after writing this, we learned of three more, corresponding to $m = 110,503$, $m = 756,839$, and $m = 858,433$. The last of these Mersenne primes has 258,716 digits.) We still do not know whether there are infinitely many Mersenne primes. Nor do we know if there are any odd perfect numbers. What we do know is that every even perfect number has the form given by Euclid. This was first proved by Leonhard Euler (1707–1783). His proof was as follows.

Proof that every even perfect number has Euclid's form:

Suppose n is an even perfect number. Then we can write it in the form $2^{m-1}q$ with q odd and $m, q > 1$. Each divisor of n has the form $2^r d$ where $0 \leq r \leq m-1$ and d is a divisor of q. Therefore

$$\sigma(n) = (1 + 2 + ... + 2^{m-1})\sigma(q) = (2^m - 1)\sigma(q).$$

Since n is perfect, $2^m q = \sigma(n) = (2^m - 1)\sigma(q)$.

Since $2^m - 1$ is odd, 2^m must divide $\sigma(q)$, say $\sigma(q) = 2^m k$, hence $q = (2^m - 1)k$. Among the divisors of q are q itself and k. These are different, since $m > 1$, and their sum is $2^m k$, which is the sum of all the divisors of q. Therefore, q has exactly two divisors and so is prime, hence $k = 1$ and $q = 2^m - 1$.

Perfect numbers are of interest not only as a challenge to computer programmers, they also play role in religious mysticism. For example, following Philo of Alexandria, Augustine writes in the *City of God*:

> Six is a number perfect in itself, and not because God created all things in six days; rather, the converse is true. God created all things in six days because this number is perfect, and it would have been perfect even if the work of six days did not exist.

In a recent book on Sufi mysticism, it is stated that 6 is the first 'complete' number and 28 is the second. Evidently, 'complete' here means 'perfect'.

Apparently, the Pythagoreans knew only one amicable pair of numbers. Although Euler found 60 such pairs, the second smallest pair $(1184, 1210)$ was only discovered in 1866 by Nicolo Paganini.

The Arabic mathematician Thabit ibn Qurra (826–901) gave a general procedure for discovering many amicable pairs, analogous to Euclid's procedure for discovering perfect numbers. See Exercise 5.

Exercises

1. Prove that every even perfect number, except 6, is the sum of the first 2^k odd cubes, for some k.

2. Show that, if m and n are relatively prime positive integers, then $\sigma(mn) = \sigma(m)\sigma(n)$.

3. Show that, if p is prime, $\sigma(p^k) = (p^{k+1} - 1)/(p - 1)$.

4. Obtain a formula for $\sigma(n)$ in terms of the prime factorizations of n.

5. Prove the result of Thabit ibn Qurra: if $p = 3 \times 2^{t+1} - 1, q = 3 \times 2^t - 1$ and $r = 9 \times 2^{2t+1} - 1$ are odd prime numbers, then $m = 2^{t+1}pq$ and $n = 2^{t+1}r$ are amicable.

6. Find two amicable pairs with the help of the above procedure.

9

Regular Polyhedra

The Pythagoreans knew that there are three ways to tile a plane (e.g., a bathroom floor) using congruent regular polygons. Indeed, since once can dissect a polygon with p sides into $p-2$ triangles, the sum of the angles of such a polygon is $(p-2)180°$. Thus each angle of a regular 'p-gon' is $(p-2)180°/p$. If q such angles meet at a point, then

$$q(p-2)180°/p = 360° ,$$

which may be simplified to yield the Egyptian problem: $1/2 = 1/p + 1/q$. Since p and q are integers greater then 2, we must have one of the following three possibilities:

p	q
3	6
4	4
6	3

The first of these gives the tiling with equilateral triangles, the second the tiling with squares, and the last the tiling with regular hexagons. No other regular polygon can be used to tile the plane.

A polyhedron is *regular* if its faces are congruent regular polygons and if the same number of faces meet at each vertex. Five regular polyhedra are the following:

- the cube, bounded by 6 squares, with 3 edges at each vertex,
- the tetrahedron, bounded by 4 equilateral triangles, with 3 edges meeting at each vertex,

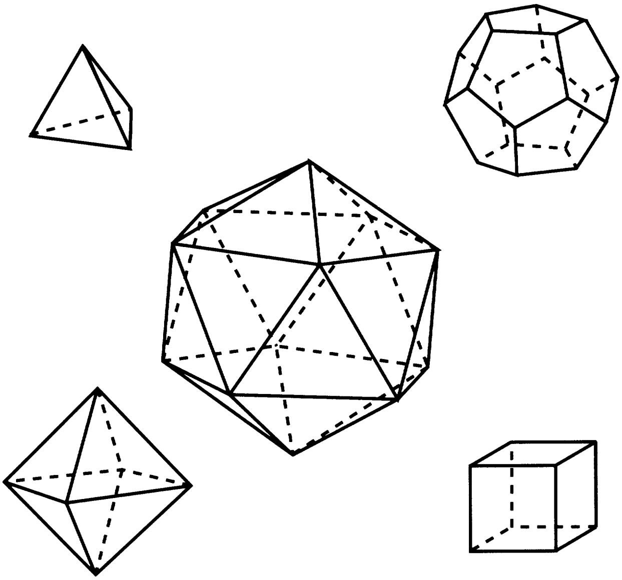

FIGURE 9.1. The Pythagorean solids

- the octahedron, bounded by 8 equilateral triangles, with 4 edges at each vertex,
- the icosahedron, bounded by 20 equilateral triangles, with 5 edges at each vertex,
- the dodecahedron, bounded by 12 regular pentagons, with 3 edges at each vertex (see Figure 9.1).

In the *Timaeus* (53-58), Plato explains the composition of the physical universe in terms of these five regular polyhedra. The cube is associated with earth, the tetrahedron with fire, the octahedron with air, the icosahedron with water and the dodecahedron with the whole cosmos. Plato explains the boiling of water by means of a 'chemical equation', which we might write as

$$F_4 + W_{20} \rightarrow 2A_8 + 2F_4.$$

That is, fire, with 4 faces, combines with water (20 faces) to produce 2 air atoms (each with 8 faces) and 2 fire atoms (each with 4 faces). Note that the numbers balance.

Such theorizing is very much in the spirit of the Pythagorean teaching that 'all is number'. Indeed, historians attribute the theory of the *Timaeus* to Pythagoras himself (Guthrie [1987]). It seems, however, that Pythagoras himself may not have known about the dodecahedron. According to one account, Hippasus (470 BC) was expelled from the Pythagorean or-