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Quaternions in Quantum Mechanics

What about quantum mechanics? If we adopt the representation of quaternions by 2×2 complex matrices (Chapter 8), the matrices ii_1, ii_2 and ii_3 are known as *Pauli spin matrices*, except for sign.

Now let us consider the relativistic form of Schrödinger's wave equation for the electron. This is known as the *Klein-Gordon equation* and is usually written

$$\left(\frac{\partial^2}{c^2 \partial t^2} - \nabla \circ \nabla\right) \phi = -\mu^2 \phi,$$

where $\mu = 2\pi m_0/h$ is proportional to the rest-mass m_0 of the electron, h being Planck's constant. Using biquaternion notation, we write this

$$D^c D\phi = -\mu^2 \phi.$$

It is assumed that $\phi = \phi_0 + i\phi_1$ is a complex valued function of the position x in Minkowski space.

The second order Klein-Gordon equation may be replaced by two first order equations as follows. Putting $D\phi = \mu\chi$, where $\chi = \chi_0 + i\chi_1$, we obtain

$$\mu D^c \chi = D^c D\phi = -\mu^2 \phi.$$

Hence the Klein-Gordon equation is equivalent to the following pair of equations:

$$D\phi = \mu\chi, \qquad D^c \chi = -\mu\phi.$$

Can these be combined into one first order equation?

Assume for the moment that there is an entity j such that $j^2 = -1$, $ji = -ij$, and $ji_k = i_k j$ for $k = 1, 2$ and 3. Then we have

$$\begin{aligned} D(\phi + j\chi) &= D\phi + Dj\chi \\ &= \mu\chi + jD^c\chi \\ &= \mu(\chi - j\phi) \\ &= -j\mu(\phi + j\chi). \end{aligned}$$

There is certainly no complex 4×4 matrix j which anticommutes with the complex number i. But let us pass to real 4×4 matrices and identify i_k with its first representation $L(i_k)$ (Chapter 8). We shall write j_k for $R(i_k)$, the second representation of i_k, as in Chapter 8, Exercise 7. Then

$$j_1^2 = j_2^2 = j_3^2 = j_3 j_2 j_1 = -1, \qquad j_k i_l = i_l j_k \qquad (k, l = 1, 2, 3).$$

Now replace the complex number i by the real matrix j_1 and write j_2 for j. Then the assumption made above is justified. Putting

$$\psi = \phi + j_2\chi = \phi_0 + j_1\phi_1 + j_2\chi_0 + j_3\chi_1,$$

we may write the above equation as follows:

$$D\psi + j_2\mu\psi = 0.$$

This is essentially Dirac's equation for the electron.

It can be shown that the sixteen matrices 1, i_k, j_l, $i_k j_l$ $(k, l = 1, 2, 3)$ are linearly independent (e.g., Jacobson [1980] p. 218, Theorem 4.6). Thus ψ is just an arbitrary real 4×4 matrix. However, as we may multiply Dirac's equation by the column vector $(1\,0\,0\,0)^t$, we may assume, without loss of generality, that ψ itself is a column vector $(\psi_0\, \psi_1\, \psi_2\, \psi_3)^t$ with real components. Nothing prevents us from allowing the ψ_k to be complex numbers, but, as far as the present analysis is concerned, there is no compelling reason for doing so. (However, complex values are forced upon us, as soon as we look at the electron in an electromagnetic field.)

Since a Lorentz transformation sends D onto $p^c D p^t$, we want ψ to be transformed to $p\psi$, hence $D\psi$ to $p^c D\psi$ and $j_2\mu\psi$ to $j_2\mu p\psi = p^c j_2\mu\psi$, thus making the Dirac equation Lorentz invariant.

It is important to note that the biquaternions of norm 1, p and $-p$, while yielding the same Lorentz transformation, both sending x onto pxp^{ct}, induce distinct transformations on ψ, sending it to $p\psi$ and $-p\psi$, respectively. This is the mathematical reason for saying that the electron has *spin* $\frac{1}{2}$.

Exercises

1. If the biquaternion a is viewed as a real 4×4 matrix, show that its transpose is not a^t, but a^{ct}, hence a point in Minkowski space is represented not by a Hermitian, but by a symmetric matrix. A general symmetric matrix has the form

$$x = x_0 + \sum_{k,l=1}^{3} x_{kl} i_k j_l,$$

but, for a point in Minkowski space $x_{kl} = 0$ unless $k = l = 1$, since Minkowski space has only four and not ten dimensions. (One version of *string theory* does allow for ten dimensions, presumably for quite different reasons.)

2. Maxwell predicted from his equations that electromagnetic energy is propagated in waves. Einstein used the equation $E = mc^2$ to predict that mass is convertible into energy. Dirac observed that in his equation μ might as well be replaced by $-\mu$ and predicted that the electron must have an anti-particle, now called the positron. Discuss to what extent such predictions from mathematical symbolism to physical reality are justified.