The homomorphisms α, β, γ in a homomorphism of short exact sequences are not independent. The next result gives some relations among these three homomorphisms.

Proposition 24. *(The Short Five Lemma)* Let α, β, γ be a homomorphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

(1) If α and γ are injective then so is β.
(2) If α and γ are surjective then so is β.
(3) If α and γ are isomorphisms then so is β (and then the two sequences are isomorphic).

Remark: These results hold also for short exact sequences of (possibly non-abelian) groups (as the proof demonstrates).

Proof: We shall prove (1), leaving the proof of (2) as an exercise (and (3) follows immediately from (1) and (2)). Suppose then that α and γ are injective and suppose $b \in B$ with $\beta(b) = 0$. Let $\psi : A \to B$ and $\varphi : B \to C$ denote the homomorphisms in the first short exact sequence. Since $\beta(b) = 0$, it follows in particular that the image of $\beta(b)$ in the quotient C' is also 0. By the commutativity of the diagram this implies that $\gamma(\varphi(b)) = 0$, and since γ is assumed injective, we obtain $\varphi(b) = 0$, i.e., b is in the kernel of φ. By the exactness of the first sequence, this means that b is in the image of ψ, i.e., $b = \psi(a)$ for some $a \in A$. Then, again by the commutativity of the diagram, the image of $\alpha(a)$ in B' is the same as $\beta(\psi(a)) = \beta(b) = 0$. But α and the map from A' to B' are injective by assumption, and it follows that $a = 0$. Finally, $b = \psi(a) = \psi(0) = 0$ and we see that β is indeed injective.

We have already seen that there is always at least one extension of a module C by A, namely the direct sum $B = A \oplus C$. In this case the module B contains a submodule C' isomorphic to C (namely $C' = 0 \oplus C$) as well as the submodule A, and this submodule complement to A "splits" B into a direct sum. In the case of groups the existence of a subgroup complement C' to a normal subgroup in B implies that B is a semidirect product (cf. Section 5 in Chapter 5). The fact that B is a direct sum in the context of modules is a reflection of the fact that the underlying group structure in this case is *abelian*; for abelian groups semidirect products are direct products. In either case the corresponding short exact sequence is said to "split":

Definition.

(1) Let R be a ring and let $0 \to A \xrightarrow{\psi} B \xrightarrow{\varphi} C \to 0$ be a short exact sequence of R-modules. The sequence is said to be *split* if there is an R-module complement to $\psi(A)$ in B. In this case, up to isomorphism, $B = A \oplus C$ (more precisely, $B = \psi(A) \oplus C'$ for some submodule C', and C' is mapped isomorphically onto C by φ: $\varphi(C') \cong C$).

(2) If $1 \to A \xrightarrow{\psi} B \xrightarrow{\varphi} C \to 1$ is a short exact sequence of groups, then the sequence is said to be *split* if there is a subgroup complement to $\psi(A)$ in B. In this case, up to isomorphism, $B = A \rtimes C$ (more precisely, $B = \psi(A) \rtimes C'$ for some subgroup C', and C' is mapped isomorphically onto C by φ: $\varphi(C') \cong C$).

In either case the extension B is said to be a *split extension* of C by A.

The question of whether an extension splits is the question of the existence of a complement to $\psi(A)$ in B isomorphic (by φ) to C, so the notion of a split extension may equivalently be phrased in the language of homomorphisms:

Proposition 25. The short exact sequence $0 \to A \xrightarrow{\psi} B \xrightarrow{\varphi} C \to 0$ of R-modules is split if and only if there is an R-module homomorphism $\mu : C \to B$ such that $\varphi \circ \mu$ is the identity map on C. Similarly, the short exact sequence $1 \to A \xrightarrow{\psi} B \xrightarrow{\varphi} C \to 1$ of groups is split if and only if there is a group homomorphism $\mu : C \to B$ such that $\varphi \circ \mu$ is the identity map on C.

Proof: This follows directly from the definitions: if μ is given define $C' = \mu(C) \subseteq B$ and if C' is given define $\mu = \varphi^{-1} : C \cong C' \subseteq B$.

Definition. With notation as in Proposition 25, any set map $\mu : C \to B$ such that $\varphi \circ \mu = \text{id}$ is called a *section* of φ. If μ is a *homomorphism* as in Proposition 25 then μ is called a *splitting homomorphism* for the sequence.

Note that a section of φ is nothing more than a choice of coset representatives in B for the quotient $B/\ker\varphi \cong C$. A section is a (splitting) homomorphism if this set of coset representatives forms a *submodule* (respectively, *subgroup*) in B, in which case this submodule (respectively, subgroup) gives a complement to $\psi(A)$ in B.

Examples

(1) The split short exact sequence $0 \to A \xrightarrow{\iota} A \oplus C \xrightarrow{\pi} C \to 0$ has the evident splitting homomorphism $\mu(c) = (0, c)$.

(2) The extension $0 \to \mathbb{Z} \xrightarrow{\iota} \mathbb{Z} \oplus (\mathbb{Z}/n\mathbb{Z}) \xrightarrow{\varphi} \mathbb{Z}/n\mathbb{Z} \to 0$, of $\mathbb{Z}/n\mathbb{Z}$ by $\mathbb{Z}$ is split (with splitting homomorphism μ mapping $\mathbb{Z}/n\mathbb{Z}$ isomorphically onto the second factor of the direct sum). On the other hand, the exact sequence of $\mathbb{Z}$-modules $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/n\mathbb{Z} \to 0$ is not split since there is no nonzero homomorphism of $\mathbb{Z}/n\mathbb{Z}$ into $\mathbb{Z}$.

(3) Neither D_8 nor Q_8 is a split extension of $Z_2 \times Z_2$ by Z_2 because in neither group is there a subgroup complement to the center (Section 2.5 gives the subgroup structures of these groups).

(4) The group D_8 *is* a split extension of Z_2 by Z_4, i.e., there is a split short exact sequence

$$1 \longrightarrow Z_4 \xrightarrow{\ \iota\ } D_8 \xrightarrow{\ \pi\ } Z_2 \longrightarrow 1,$$

namely,

$$1 \longrightarrow \langle r \rangle \xrightarrow{\ \iota\ } D_8 \xrightarrow{\ \pi\ } \langle \bar{s} \rangle \longrightarrow 1,$$

using our usual set of generators for D_8. Here ι is the inclusion map and $\pi : r^a s^b \mapsto \bar{s}^b$ is the projection onto the quotient $D_8/\langle r \rangle \cong Z_2$. The splitting homomorphism μ

maps $\langle \bar{s} \rangle$ isomorphically onto the complement $\langle s \rangle$ for $\langle r \rangle$ in D_8. Equivalently, D_8 is the semidirect product of the normal subgroup $\langle r \rangle$ (isomorphic to Z_4) with $\langle s \rangle$ (isomorphic to Z_2).

On the other hand, while Q_8 is also an extension of Z_2 by Z_4 (for example, $\langle i \rangle \cong Z_4$ has quotient isomorphic to Z_2), Q_8 is *not* a split extension of Z_2 by Z_4: no cyclic subgroup of Q_8 of order 4 has a complement in Q_8.

Section 5.5 contains many more examples of split extensions of groups.

Proposition 25 shows that an extension B of C by A is a split extension if and only if there is a splitting homomorphism μ of the projection map $\varphi : B \to C$ from B to the quotient C. The next proposition shows in particular that for modules this is equivalent to the existence of a splitting homomorphism for ψ at the other end of the sequence.

Proposition 26. Let $0 \to A \xrightarrow{\psi} B \xrightarrow{\varphi} C \to 0$ be a short exact sequence of modules (respectively, $1 \to A \xrightarrow{\psi} B \xrightarrow{\varphi} C \to 1$ a short exact sequence of groups). Then $B = \psi(A) \oplus C'$ for some submodule C' of B with $\varphi(C') \cong C$ (respectively, $B = \psi(A) \times C'$ for some subgroup C' of B with $\varphi(C') \cong C$) if and only if there is a homomorphism $\lambda : B \to A$ such that $\lambda \circ \psi$ is the identity map on A.

Proof: This is similar to the proof of Proposition 25. If λ is given, define $C' = \ker \lambda \subseteq B$ and if C' is given define $\lambda : B = \psi(A) \oplus C' \to A$ by $\lambda((\psi(a), c') = a$. Note that in this case $C' = \ker \lambda$ is *normal* in B, so that C' is a *normal* complement to $\psi(A)$ in B, which in turn implies that B is the *direct sum* of $\psi(A)$ and C' (cf. Theorem 9 of Section 5.4).

Proposition 26 shows that for general group extensions, the existence of a splitting homomorphism λ on the *left* end of the sequence is stronger than the condition that the extension splits: in this case the extension group is a *direct* product, and not just a *semidirect* product. The fact that these two notions are equivalent in the context of modules is again a reflection of the abelian nature of the underlying groups, where semidirect products are always direct products.

Modules and $Hom_R(D, \_\_)$

Let R be a ring with 1 and suppose the R-module M is an extension of N by L, with

$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow 0$$

the corresponding short exact sequence of R-modules. It is natural to ask whether properties for L and N imply related properties for the extension M. The first situation we shall consider is whether an R-module homomorphism from some fixed R-module D to either L or N implies there is also an R-module homomorphism from D to M.

The question of obtaining a homomorphism from D to M given a homomorphism from D to L is easily disposed of: if $f \in \mathrm{Hom}_R(D, L)$ is an R-module homomorphism from D to L then the composite $f' = \psi \circ f$ is an R-module homomorphism from D to

M. The relation between these maps can be indicated pictorially by the commutative diagram

$$\begin{tikzcd} D \arrow[d, "f"'] \arrow[dr, dashed, "f'"] & \\ L \arrow[r, "\psi"] & M \end{tikzcd}$$

Put another way, composition with ψ induces a map

$$\begin{aligned} \psi' : \operatorname{Hom}_R(D, L) &\longrightarrow \operatorname{Hom}_R(D, M) \\ f &\longmapsto f' = \psi \circ f. \end{aligned}$$

Recall that, by Proposition 2, $\operatorname{Hom}_R(D, L)$ and $\operatorname{Hom}_R(D, M)$ are abelian groups.

Proposition 27. Let D, L and M be R-modules and let $\psi : L \to M$ be an R-module homomorphism. Then the map

$$\begin{aligned} \psi' : \operatorname{Hom}_R(D, L) &\longrightarrow \operatorname{Hom}_R(D, M) \\ f &\longmapsto f' = \psi \circ f \end{aligned}$$

is a homomorphism of abelian groups. If ψ is injective, then ψ' is also injective, i.e.,

$$\text{if} \quad 0 \longrightarrow L \xrightarrow{\psi} M \quad \text{is exact,}$$

$$\text{then} \quad 0 \longrightarrow \operatorname{Hom}_R(D, L) \xrightarrow{\psi'} \operatorname{Hom}_R(D, M) \quad \text{is also exact.}$$

Proof: The fact that ψ' is a homomorphism is immediate. If ψ is injective, then distinct homomorphisms f and g from D into L give distinct homomorphisms $\psi \circ f$ and $\psi \circ g$ from D into M, which is to say that ψ' is also injective.

While obtaining homomorphisms into M from homomorphisms into the submodule L is straightforward, the situation for homomorphisms into the quotient N is much less evident. More precisely, given an R-module homomorphism $f : D \to N$ the question is whether there exists an R-module homomorphism $F : D \to M$ that *extends* or *lifts* f to M, i.e., that makes the following diagram commute:

$$\begin{tikzcd} & D \arrow[d, "f"] \arrow[dl, dashed, "F"'] \\ M \arrow[r, "\varphi"] & N \end{tikzcd}$$

As before, composition with the homomorphism φ induces a homomorphism of abelian groups

$$\begin{aligned} \varphi' : \operatorname{Hom}_R(D, M) &\longrightarrow \operatorname{Hom}_R(D, N) \\ F &\longmapsto F' = \varphi \circ F. \end{aligned}$$

In terms of φ', the homomorphism f to N lifts to a homomorphism to M if and only if f is in the image of φ' (namely, f is the image of the lift F).