céleste of Poincaré (1892, 1893, 1899). In this work Poincaré directed attention toward asymptotic behavior, in a sense complementing Newton's infinitesimal view with a view toward infinity, and his methods have become highly influential in twentieth-century dynamics.

13.3 Mechanical Curves

When Descartes gave his reasons for restricting *La Géométrie* to algebraic curves (which he called "geometric"; see Section 7.3), he explicitly excluded certain classical curves on the rather vague grounds that they

> belong only to mechanics, and are not among those curves that I think should be included here, since they must be conceived of as described by two separate movements whose relation does not admit of exact determination.
>
> [Descartes (1637), p. 44]

The curves that Descartes relegated "to mechanics" were those the Greeks had defined by certain hypothetical mechanisms, for example, the epicycles (described by rolling one circle on another) and the spiral of Archimedes (described by a point moving at constant speed along a uniformly rotating line). He was probably aware that the spiral is transcendental by virtue of the fact that it meets a straight line in infinitely many points. This is contrary to the behavior of an algebraic curve $p(x,y)=0$, which meets a straight line $y=mx+c$ in only finitely many points, corresponding to the finitely many solutions of $p(x,mx+c)=0$. This proof that there are transcendental curves was given explicitly by Newton (1687), Lemma XXVIII.

We do not know whether Descartes distinguished, say, the algebraic epicycles from the transcendental ones; nevertheless, it is broadly true that his "mechanical" curves were transcendental. This remained true with the great expansion of mechanics and calculus in the seventeenth century, and indeed most of the new transcendental curves originated in mechanics. In this section we shall look at three of the most important of them: the catenary, the cycloid, and the elastica.

The *catenary* is the shape of a hanging cord, assumed to be perfectly flexible and with mass uniformly distributed along its length. In practice, the flexibility and uniformity of mass are realized better by a hanging chain, hence the name "catenary," which comes from the Latin *catena* for chain. Hooke (1675) observed that the same curve occurs as the shape of an arch of infinitesimal stones. The catenary looks very much like a parabola and was at first conjectured to be one by Galileo. This was disproved by the 17-year-old Huygens (1646), though at the time he was unable to determine the correct curve. He did show, however, that the parabola was the shape assumed by a flexible cord loaded by weights that are uniformly distributed in the horizontal direction (as is approximately the case for the cable of a suspension bridge).

The problem of the catenary was finally solved independently by Johann Bernoulli (1691), Huygens (1691), and Leibniz (1691), in response to a challenge from Jakob Bernoulli in 1690. Johann Bernoulli showed that the curve satisfied the differential equation

$$\frac{dy}{dx} = \frac{s}{a},$$

where a is constant and $s =$ arc length OP (Figure 13.2). He derived this equation by replacing the portion OP of the chain, which is held in equilibrium by the tangential force F_1 at P and the horizontal force F_0, which is independent of P, by a point mass W equal to the weight of OP (hence proportional to s) held in equilibrium by the same forces. Comparing the directions and magnitudes of the forces gives

$$\frac{dy}{dx} = \frac{W}{F_0} = \frac{s}{a}.$$

By ingenious transformations Bernoulli reduced the equation to

$$dx = \frac{a\,dy}{\sqrt{y^2 - a^2}},$$

in other words, to an integral. This solution was as simple as could be stated at the time, since x is a transcendental function of y and hence can be expressed, at best, as an integral. Today, of course, we recognize the function as one of the "standard" ones and abbreviate the solution as

$$y = a\cosh\frac{x}{a} - a.$$

The *cycloid* is the curve generated by a point on the circumference of a circle rolling on a straight line. Despite being a natural limiting case in

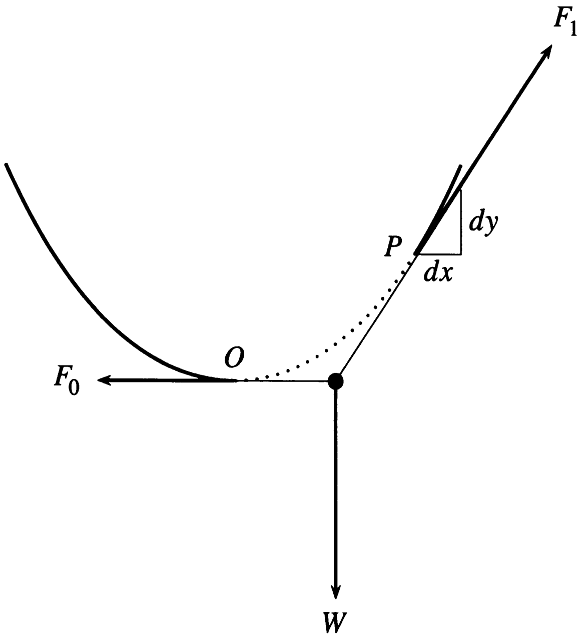

Figure 13.2: The catenary

the epicyclic family, the cycloid does not seem to have been investigated until the seventeenth century, when it became a favorite curve with mathematicians. It has many beautiful geometric properties, and even more remarkable mechanical properties. The first of these, discovered by Huygens (1659b), is that the cycloid is the *tautochrone* (equal-time curve). A particle constrained to slide along an inverted cycloid takes the same time to descend to the lowest point, regardless of its starting point.

Huygens (1673) made a classic application of this property to pendulum clocks, using a geometric property of the cycloid (Huygens, 1659c). If the pendulum, taken to be a weightless cord with a point mass at the end, is constrained to swing between two cycloidal "cheeks," as Huygens called them (Figure 13.3), then the point mass will travel along a cycloid. Consequently, the period of the cycloidal pendulum is independent of amplitude. This makes it theoretically superior to the ordinary pendulum whose period, though approximately constant for small amplitudes, actually involves an elliptic function. In practice, problems such as friction make the cycloidal pendulum no more accurate than the ordinary pendulum, but its theoretical superiority shut the ordinary pendulum out of mechanics for some time. Newton's *Principia*, for example, often mentions the cycloidal pendulum but never the simple pendulum.

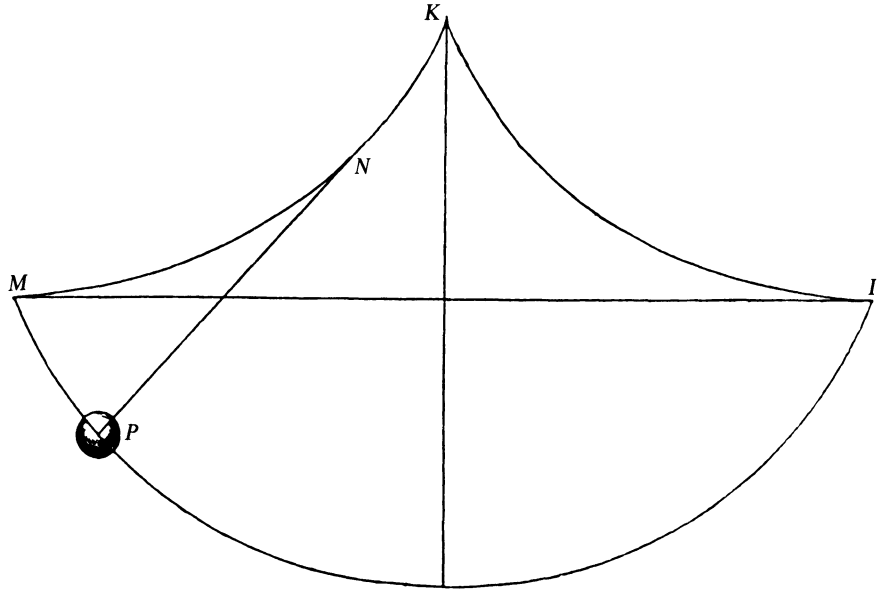

Figure 13.3: The cycloidal pendulum

The second remarkable property of the cycloid is that it is the *brachistochrone*, the curve of shortest time. Johann Bernoulli (1696) posed the problem of finding the curve, between given points A and B, along which a point mass descends in the shortest time. He already knew that the solution was a cycloid, and solutions were found independently by Jakob Bernoulli (1697), l'Hôpital (1697), Leibniz (1697), and Newton (1697). The problem is deeper than that of the tautochrone, because the cycloid has to be singled out from *all possible* curves between A and B. Jakob Bernoulli's solution was the most profound because it recognized the "variable curve" aspect of the problem, and it is now considered to be the first major step in the development of the calculus of variations.

The *elastica* was another of Jakob Bernoulli's discoveries, and likewise important in the development of another field—the theory of elliptic functions. The elastica is the curve assumed by a thin elastic rod compressed at the ends. Jakob Bernoulli (1694) showed that the curve satisfied a differential equation that he reduced to the form

$$ds = \frac{dx}{\sqrt{1-x^4}}.$$

To interpret this integral geometrically, he introduced the lemniscate and showed that its arc length was expressed by precisely the same integral. This was the beginning of the investigations of the lemniscatic integral, which included the important discoveries of Fagnano and Gauss mentioned in the last chapter. Euler's investigations of elliptic integrals were also stimulated by the elastica. Euler (1743) gave pictures of elastica that show they have periodic forms (Figure 13.4). These drawings were the first to show the real period of elliptic functions, though of course periodicity was implicit in the first elliptic integral, the arc length of the ellipse (the real period being the circumference of the ellipse).

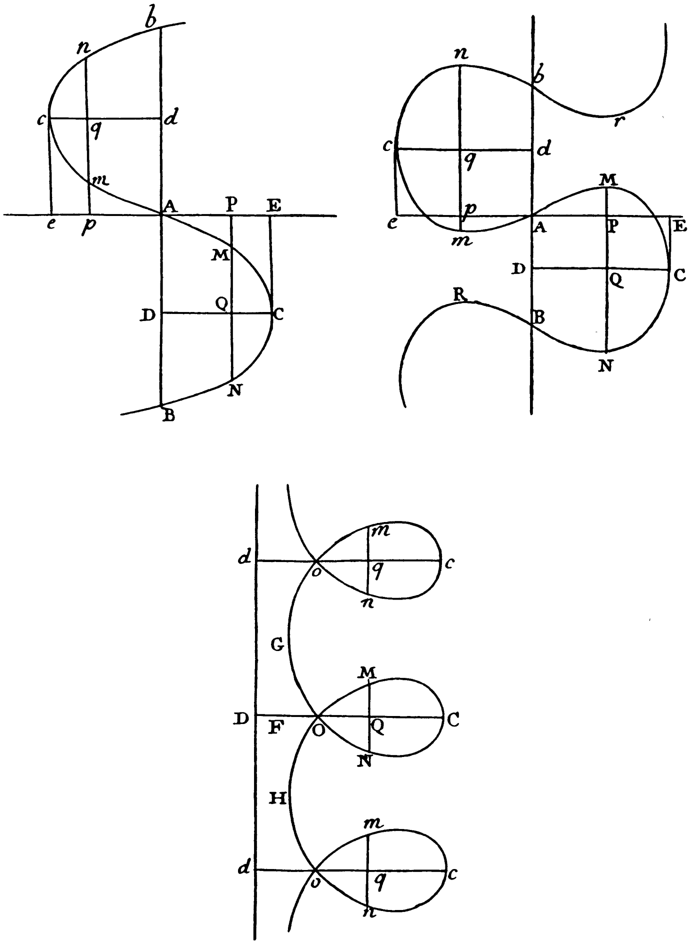

Figure 13.4: Forms of the elastica

EXERCISES

The derivation of the cosh function from the catenary equation is helped by a tricky formula for $\frac{d^2y}{dx^2}$, which you should verify first if it is not familiar to you.

13.3.1 Use

$$ds = \sqrt{dx^2 + dy^2} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{d}{dy}\frac{1}{2}\left(\frac{dy}{dx}\right)^2$$

to transform the differential equation

$$\frac{dy}{dx} = \frac{s}{a}$$

to

$$\frac{dx}{dz} = \frac{a}{\sqrt{1+z^2}}, \tag{1}$$

where $z = dy/dx$.

13.3.2 Solve (1) for x and hence show that the original equation has solution

$$y = a\cosh\frac{x}{a} + \text{const.}$$

It is considerably easier to solve the suspension bridge equation, which perhaps is why Huygens was able to do it at age 17, and before much calculus was known.

13.3.3 How should the formula $\frac{dy}{dx} = \frac{s}{a}$ be modified if the load is uniformly distributed in the horizontal direction (as in a suspension bridge)?

13.3.4 Solve the modified equation from Exercise 13.3.3, and hence show that the solution is a parabola.

Finally, we can verify that the catenary is indeed a transcendental curve.

13.3.5 Show that the functions sin and cos, and hence the functions sinh and cosh, are transcendental. *Hint*: You may need to use complex numbers.

13.4 The Vibrating String

The problem of the vibrating string is one of the most fertile in mathematics, being the source of such diverse fields as partial differential equations, Fourier series, and set theory. It is also remarkable in being perhaps the only setting in which the sense of hearing led to important mathematical discoveries. As we saw in Section 1.5, the Pythagoreans discovered the relationship between pitch and length by hearing the harmonious tones produced by two strings whose lengths were in a simple whole-number ratio. Thus in a sense it was possible to "hear the length of the string," and some later discoveries of mathematically significant properties of the strings—overtones, for example—were initially prompted by hearing [see Dostrovsky (1975)].

Various authors in ancient times suggested that the physical basis of pitch was frequency of vibration, but it was not until the seventeenth century that the precise relationship between frequency and length was discovered, by Descartes' mentor Isaac Beeckman. In 1615 Beeckman gave a simple geometric argument to show that frequency is inversely proportional to length; hence the Pythagorean ratios of lengths can also be interpreted as (reciprocal) ratios of frequencies. The latter interpretation is more fundamental because frequency alone determines pitch, whereas length determines pitch only when the material, cross section, and tension of the string are fixed. The relation between frequency ν, and tension T, cross-sectional area A, and length l was discovered experimentally by Mersenne (1625) to be

$$\nu \propto \frac{1}{l}\sqrt{\frac{T}{A}}.$$

The first derivation of Mersenne's law from mathematical assumptions was given by Taylor (1713), in a paper that marks the beginning of the modern theory of the vibrating string. In it he discovered the simplest possibility for the instantaneous shape of the string, the half sine wave

$$y = k \sin \frac{\pi x}{l}$$

and established generally that the force on an element was proportional to d^2y/dx^2.

The latter result was the starting point for a dramatic advance in the theory by d'Alembert (1747). Taking into account the dependence of y on time t as well as x, d'Alembert realized that acceleration should be expressed by $\partial^2 y/\partial t^2$ and the force found by Taylor by $\partial^2 y/\partial x^2$, hence *partial* derivatives are involved. Newton's second law then gives what is now called the *wave equation*,

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2},$$

writing the constant of proportionality as $1/c^2$. Undeterred by the novelty of this partial differential equation, d'Alembert forged ahead to a general solution as follows. The equation may be simplified by a change of time scale $s = ct$ to

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial s^2}. \tag{1}$$

The chain rule gives

$$\begin{aligned} d\left(\frac{\partial y}{\partial x} \pm \frac{\partial y}{\partial s}\right) &= \frac{\partial^2 y}{\partial x^2}dx + \frac{\partial^2 y}{\partial x \partial s}(ds \pm dx) \pm \frac{\partial^2 y}{\partial s^2}ds \\ &= \left(\frac{\partial^2 y}{\partial s^2} \pm \frac{\partial^2 y}{\partial x \partial s}\right)(ds \pm dx) \end{aligned}$$

from which d'Alembert concluded that

$$\frac{\partial^2 y}{\partial s^2} + \frac{\partial^2 y}{\partial x \partial s}$$

is a function of $s + x$ and

$$\frac{\partial^2 y}{\partial s^2} - \frac{\partial^2 y}{\partial x \partial s}$$

is a function of $s - x$, whence, say

$$\frac{\partial y}{\partial x} + \frac{\partial y}{\partial s} = \int \left(\frac{\partial^2 y}{\partial s^2} + \frac{\partial^2 y}{\partial x \partial s}\right) d(s+x) = f(s+x)$$

and similarly

$$\frac{\partial y}{\partial x} - \frac{\partial y}{\partial s} = g(s-x).$$

This gives

$$\frac{\partial y}{\partial x} = \frac{1}{2}(f(s+x) + g(s-x)), \quad \frac{\partial y}{\partial s} = \frac{1}{2}(f(s+x) - g(s-x)),$$

and finally

$$\begin{aligned} y &= \int \left(\frac{\partial y}{\partial x}dx + \frac{\partial y}{\partial s}ds\right) \\ &= \int \frac{1}{2}(f(s+x)(ds+dx) - g(s-x)(ds-dx)) \\ &= \Phi(s+x) + \Psi(s-x). \end{aligned}$$

Reversing the argument, we see that the functions Φ and Ψ can be arbitrary, at least as long as they admit the various differentiations involved.

But how arbitrary *is* an arbitrary function? Is it as arbitrary as an arbitrarily shaped string? The vibrating string problem caught eighteenth-century mathematicians unprepared to answer these questions. They had understood a function to be something expressed by a formula, possibly an infinite series, and this had been thought to guarantee differentiability. Yet the most natural shape of the vibrating string was one with a nondifferentiable point—the triangle of the plucked string as it is released—so nature seemed to demand an extension of the concept of function beyond the world of formulas.

The confusion was heightened when Daniel Bernoulli (1753) claimed, on physical grounds, that a general solution of the wave equation *could* be expressed by a formula, the infinite trigonometric series

$$y = a_1 \sin\frac{\pi x}{l}\cos\frac{\pi ct}{l} + a_2 \sin\frac{2\pi x}{l}\cos\frac{2\pi ct}{l} + \cdots.$$

This amounts to claiming that any mode of vibration results from the superposition of simple modes, a fact he considered to be intuitively evident. The nth term in the series represents the nth mode, generalizing Taylor's formula for the fundamental mode and building in the time dependence; but Daniel Bernoulli gave no method for calculating the coefficient a_n.

We now know that his intuition was correct and that the triangular wave form, among others, is representable by a trigonometric series. However, it was well into the nineteenth century before anything like a clear understanding of trigonometric series was obtained. The fact that the triangular wave could be represented by a series made it a bona fide function by classical standards, hence mathematicians were brought to the realization that a series representation does not guarantee differentiability. Later, continuity was also called into question, and infinitely subtle problems concerning the convergence of trigonometric series led Cantor to develop the theory of sets (see Chapter 23).

These remarkably remote consequences of what seemed at first to be a purely physical question were of course not the only fruits of the vibrating string investigations. Trigonometric series proved to be valuable all over mathematics, from the theory of heat, where Fourier applied them with such success that they became known as *Fourier series*, to the theory of numbers. Their most famous application to number theory is probably the Dirichlet (1837) proof that any arithmetic progression $a, a+b, a+2b, \ldots,$ where $\gcd(a,b) = 1$, contains infinitely many primes. Pythagoras would surely have approved!