ancient Greeks realized (Euclid's *Elements* XII 2), one can use what we call 'mathematical induction' to show that an inscribed regular 2^n-gon takes up more than $1 - 1/2^{n-1}$ of the area of a circle.

If we inscribe a regular 2^n-gon in a circle, its longest diagonals are diameters of that circle. The Greeks knew that the area of a regular 2^n-gon is proportional to the square on its longest diagonal – a result which follows from the fact that the area of a triangle with given angles is proportional to the square of its longest side – and from this it follows that, insofar as a circle is like a regular 2^n-gon, its area is proportional to the square on its diameter. Indeed, somewhat later than Antiphon, Eudoxus (408 – ca. 355 BC) gave a rigorous proof that the area of a circle is, in fact, proportional to the square on its diameter.

Antiphon boldly claimed that a circle simply *is* a regular polygon (with a large number of sides). In making this claim, Antiphon entered a lively discussion, started by Zeno (450 BC) and others, about whether space is continuous or discrete. If space is discrete, then there is some minimum area e. If n is so large that $1/2^{n-1}$ of the area of the circle is less than e, then an inscribed regular 2^n-gon, in taking up more than $1 - 1/2^{n-1}$ of the area, actually takes up *all* the area.

Another early Athenian mathematician who worked on the geometry of the circle was Hippocrates, who came from the Greek island of Chios, near present-day Turkey. (He is not to be confused with the physician, famous for his oath, who came from Cos.) Hippocrates, it is said, had been swindled in business and came to Athens about 430 BC to recover his property through legal action. The case dragged on, and Hippocrates used the time to study philosophy and supported himself by teaching geometry.

Hippocrates was responsible for much of the material in Books III and IV of Euclid's *Elements.* He called the square of a quantity 'dynamos', hence our 'power'. He pioneered the custom of reducing one theorem to another and may have been one of the first to use the method of *reductio ad absurdum* in mathematics.

He was also the first to find the precise area of a region bounded by curves, as we shall now see. Construct semicircles on three sides of a right triangle. By the converse of the theorem of Thales, the semicircle on the hypotenuse passes through the vertex at the right angle. The semicircles on the other two sides of the right triangle are supposed to lie outside the triangle. (See Figure 12.1.) The areas included in the two smaller semicircles, but not in the semicircle on the hypotenuse, are called *lunes* (after the crescent moon).

Hippocrates argued as follows. If the vertices of the triangle are A, B and C, with the right angle at C, then $AC^2 + CB^2 = AB^2$ (by the theorem of Pythagoras). Since the area of a circle is proportional to the square on its diameter, the area of a semicircle is likewise proportional to the square on

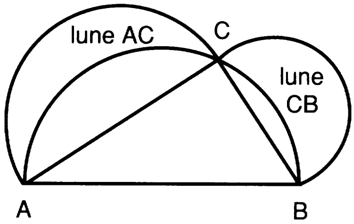

FIGURE 12.1. Lunes of Hippocrates

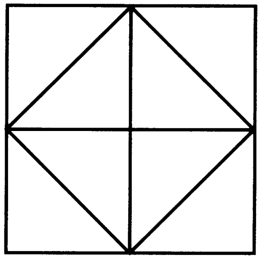

FIGURE 12.2. Doubling the square

the diameter. Therefore the sum of the areas of the semicircles on AC and CB equals the area of the semicircle on AB. Subtracting the areas where the semicircles overlap, we may conclude that the sum of the areas of the lunes equals the area of the right triangle. That is, since semicircle AC plus semicircle CB equals semicircle AB, it follows that

$$\text{lune } AC + \text{lune } CB = \text{triangle } ABC = \tfrac{1}{2}BC \times AC.$$

Hippocrates also contributed to the problem of 'doubling the cube'. This was the problem of determining the length x such that $x^3 = 2$, preferably using only the geometry of straight lines and circles.

Legend has it that, during the plague of 430 BC, the Athenians consulted the oracle of Delos for help. The oracle replied that they should double the altar of Apollo, a marble cube, in size. When the plague refused to abate, the oracle explained that the Athenians had doubled the edges of the cube, not its volume. Although the Athenians did not succeed with this task, at least not according to the methods acceptable to Plato, the plague seems to have stopped anyway.

Hippocrates noted that one could double the volume of a cube, with edge one unit in length, if one could find quantities x and y such that $1/x = x/y = y/2$; for then x^3 would equal 2. However, he did not succeed in constructing these quantities in a way that satisfied Plato (who wanted to use only straightedge and compass).