23. *(Localization and* Tor*)* Let $D^{-1}R$ be the localization of the commutative ring R with respect to the multiplicative subset D of R. Prove that localization commutes with Tor, i.e., $D^{-1}\operatorname{Tor}_n^R(A, B) \cong \operatorname{Tor}_n^{D^{-1}R}(D^{-1}A, D^{-1}B)$ for all R-modules A and B and all $n \geq 0$. [Use the previous exercise and the fact that $D^{-1}R$ is flat over R, cf. Proposition 42(6) in Section 15.4.]

24. *(Flatness is local)* Suppose R is a commutative ring. Prove that an R-module M is flat if and only if every localization M_P is a flat R_P-module for every maximal (hence also for every prime) ideal in R. [Use the previous exercise together with the characterization of flatness in terms of Tor.]

25. If R is an integral domain with field of fractions F, prove that $\operatorname{Tor}_1^R(F/R, B) \cong t(B)$ for any R-module B, where $t(B)$ denotes the R-torsion submodule of B.

An R-module M is said to be *finitely presented* if there is an exact sequence

$$R^s \longrightarrow R^t \longrightarrow M \longrightarrow 0$$

of R-modules for some integers s and t. Equivalently, M is finitely generated by t elements and the kernel of the corresponding R-module homomorphism $R^t \to M$ can be generated by s elements.

26. (a) Prove that every finitely generated module over a Noetherian ring R is finitely presented. [Use Exercise 8 in Section 15.1.]
    (b) Prove that an R-module M is finitely presented and projective if and only if M is a direct summand of R^n for some integer $n \geq 1$.

27. Suppose that M is a finitely presented R-module and that $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} M \to 0$ is an exact sequence of R-modules. This exercise proves that if B is a finitely generated R-module then A is also a finitely generated R-module.
    (a) Suppose $R^s \xrightarrow{\psi} R^t \xrightarrow{\varphi} M \to 0$ and $e_1, \ldots, e_t$ is an R-module basis for R^t. Show that there exist $b_1, \ldots, b_t \in B$ so that $\beta(b_i) = \varphi(e_i)$ for $i = 1, \ldots, t$.
    (b) If f is the R-module homomorphism from R^t to B defined by $f(e_i) = b_i$ for $i = 1, \ldots, t$, show that $f(\psi(R^s)) \subseteq \ker \beta$. [Use $\varphi \circ \psi = 0$.] Conclude that there is a commutative diagram

$$\begin{tikzcd}
 & R^s \arrow[r, "\psi"] \arrow[d, "g"] & R^t \arrow[r, "\varphi"] \arrow[d, "f"] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r, "\alpha"] & B \arrow[r, "\beta"] & M \arrow[r] & 0
\end{tikzcd}$$

    of R-modules with exact rows.
    (c) Prove that $A/\operatorname{image} g \cong B/\operatorname{image} f$ and use this to prove that A is finitely generated. [For the isomorphism, use the Snake Lemma in Exercise 3. Then show that image g and $A/\operatorname{image} g$ are both finitely generated and apply Exercise 7 of Section 10.3.]
    (d) If I is an ideal of R conclude that R/I is a finitely presented R-module if and only if I is a finitely generated ideal.

28. Suppose R is a local ring with unique maximal ideal $\mathfrak{m}$ and M is a finitely presented R-module. Suppose $m_1, \ldots, m_s$ are elements in M whose images in $M/\mathfrak{m}M$ form a basis for $M/\mathfrak{m}M$ as a vector space over the field $R/\mathfrak{m}$.
    (a) Prove that $m_1, \ldots, m_s$ generate M as an R-module. [Use Nakayama's Lemma.]
    (b) Conclude from (a) that there is an exact sequence $0 \to \ker \varphi \to R^s \xrightarrow{\varphi} M \to 0$ that maps a set of free generators of R^s to the elements $m_1, \ldots, m_s$. Deduce that there is

an exact sequence

$$\mathrm{Tor}_1^R(M, R/\mathfrak{m}) \longrightarrow (\ker\varphi)/\mathfrak{m}(\ker\varphi) \longrightarrow 0.$$

[Use the Tor long exact sequence with respect to tensoring with $R/\mathfrak{m}$, using the fact that $N \otimes R/\mathfrak{m} \cong N/\mathfrak{m}N$ for any R-module N (Example 8 following Corollary 12 in Section 10.4] and the fact that $\varphi : (R/\mathfrak{m})^s \cong M/\mathfrak{m}M$ is an isomorphism by the choice of $m_1, \ldots, m_s$.]

(c) Prove that if $\mathrm{Tor}_1^R(M, R/\mathfrak{m}) = 0$ then $m_1, \ldots, m_s$ are a set of *free* R-module generators for M. [Use the previous exercise and Nakayama's Lemma to show that $\ker\varphi = 0$.]

29. Suppose R is a local ring with unique maximal ideal $\mathfrak{m}$. This exercise proves that a finitely generated R-module is flat if and only if it is free.

(a) Prove that $M = F/K$ is the quotient of a finitely generated free module F by a submodule K with $K \subseteq \mathfrak{m}F$. [Let F be a free module with $F/\mathfrak{m}F \cong M/\mathfrak{m}M$.]

(b) Suppose $x \in K$ and write $x = a_1e_1 + \cdots + a_ne_n$ where $e_1, \ldots, e_n$ are an R-basis for F. Let $I = (a_1, \ldots, a_n)$ be the ideal of R generated by $a_1, \ldots, a_n$). Prove that if M is flat, then $I = \mathfrak{m}I$ and deduce that $K = 0$, so M is free. [Use Exercise 25(d) of Section 10.5 to see that $x \in IK \subseteq \mathfrak{m}IF$ and conclude that $I \subseteq \mathfrak{m}I$. Then apply Nakayama's Lemma to the finitely generated ideal I.]

30. Suppose R is a local ring with unique maximal ideal $\mathfrak{m}$, M is an R-module, and consider the following statements:

(i) M is a free R-module,
(ii) M is a projective R-module,
(iii) M is a flat R-module, and
(iv) $\mathrm{Tor}_1^R(M, R/\mathfrak{m}) = 0$.

(a) Prove that (i) implies (ii) implies (iii) implies (iv).

(b) Prove that (i) , (ii), and (iii) are equivalent if M is finitely generated. (Exercise 34 below shows (iii) need not imply (i) or (ii) if M is finitely generated but R is not local.) [Use the previous exercise.]

(c) Prove that (i), (ii), (iii), and (iv) are equivalent if M is finitely presented. (Exercise 35 below shows that (iv) need not imply (i), (ii) or (iii) if M is finitely generated but not finitely presented.) [Use Exercise 28.]

Remark: It is a theorem of Kaplansky (cf. *Projective Modules*, Annals of Mathematics, 68(1958), pp. 372-377) that (i) and (ii) are equivalent without the condition that M be finitely generated.

31. *(Localization and* Hom *for Finitely Presented Modules)* Suppose $D^{-1}R$ is the localization of the commutative ring R with respect to the multiplicative subset D of R, and let M be a finitely presented R-module.

(a) For any R-modules A and B prove there is a unique $D^{-1}R$-module homomorphism from $D^{-1}\mathrm{Hom}_R(A, B)$ to $\mathrm{Hom}_{D^{-1}R}(D^{-1}A, D^{-1}B)$ that maps $\varphi \in \mathrm{Hom}_R(A, B)$ to the homomorphism from $D^{-1}A$ to $D^{-1}B$ induced by φ.

(b) For any R-module N and any $m \geq 1$ show that $\mathrm{Hom}_R(R^m, N) \cong N^m$ as R-modules and deduce that $D^{-1}\mathrm{Hom}_R(R^m, N) \cong (D^{-1}N)^m$ as $D^{-1}R$-modules.

(c) Suppose $R^s \longrightarrow R^t \longrightarrow M \longrightarrow 0$ is exact. Prove there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & D^{-1}\mathrm{Hom}_R(M,N) \arrow[r] \arrow[d] & D^{-1}\mathrm{Hom}_R(R^t,N) \arrow[r] \arrow[d] & D^{-1}\mathrm{Hom}_R(R^s,N) \arrow[d] \\
0 \arrow[r] & \mathrm{Hom}_{D^{-1}R}(D^{-1}M, D^{-1}N) \arrow[r] & \mathrm{Hom}_{D^{-1}R}((D^{-1}R)^t, D^{-1}N) \arrow[r] & \mathrm{Hom}_{D^{-1}R}((D^{-1}R)^s, D^{-1}N)
\end{tikzcd}$$

of $D^{-1}R$-modules with exact rows. [For the first row first take R-module homomorphisms from the terms in the presentation for M into N using Theorem 33 of Section 10.5 (noting the first comment in the proof) and then tensor with the flat R-module $D^{-1}R$, cf. Propositions 41 and 42(6) in Section 15.4. For the second row first tensor the presentation with $D^{-1}R$ and then take $D^{-1}R$-module homomorphisms into $D^{-1}N$.]

(d) Use (b) to prove that localization commutes with taking homomorphisms when M is finitely presented, i.e., $D^{-1}\mathrm{Hom}_R(M, N) \cong \mathrm{Hom}_{D^{-1}R}(D^{-1}M, D^{-1}N)$ as $D^{-1}R$-modules. [Show the second two vertical maps in the diagram above are isomorphisms and deduce that the left vertical map is also an isomorphism.] (This result is not true in general if M is not finitely presented.)

32. *(Localization and* Ext *for Finitely Presented Modules)* Suppose $D^{-1}R$ is the localization of the commutative ring R with respect to the multiplicative subset D of R. Prove that if M is a finitely presented R-module then $D^{-1}\mathrm{Ext}^n_R(M, N) \cong \mathrm{Ext}^n_{D^{-1}R}(D^{-1}M, D^{-1}N)$ as $D^{-1}R$-modules for every R-module N and every $n \geq 0$. [Use a projective resolution of N and the previous exercise, noting that tensoring the resolution with $D^{-1}R$ gives a projective resolution for the $D^{-1}R$-module $D^{-1}N$.]

33. Suppose R is a commutative ring and M is a finitely presented R-module (for example a finitely generated module over a Noetherian ring, or a quotient, R/I, of R by a finitely generated ideal I, cf. Exercises 26 and 27). Prove that the following are equivalent:

(a) M is a projective R-module,

(b) M is a flat R-module,

(c) M is locally free, i.e., each localization M_P is a free R_P-module for every maximal (hence also for every prime) ideal P of R.

In particular show that finitely generated projective modules are the same as finitely presented flat modules. [Exercises 24 and 30 show that (b) is equivalent to (c). Use the Ext criterion for projectivity and Exercises 30 and 32 to see that (a) is equivalent to (c).]

34. **(a)** Prove that *every* R-module for the commutative ring R is flat if and only if every finitely generated ideal I of R is a direct summand of R, in which case every finitely generated ideal of R is principal and projective (such a ring is said to be *absolutely flat*). [Use Exercise 15, the previous exercise applied to the finitely presented R-module R/I, and the remarks following Proposition 16.]

(b) Prove that every Boolean ring is absolutely flat. [Use Exercise 24 in Section 7.4, noting that if $I = Rx$ then x is an idempotent so $R = Rx \oplus R(1 - x)$.]

(c) Let R be the direct product and I the direct sum of countably many copies of $\mathbb{Z}/2\mathbb{Z}$. Prove that I is an ideal of the Boolean ring R that is not finitely generated and that the cyclic R-module $M = R/I$ is flat but not projective (so finitely generated flat modules need not be projective).

35. Let R be the local ring obtained by localizing the ring of C^∞ functions on the open interval $(-1, 1)$ at the maximal ideal of functions that are 0 at $x = 0$ (cf. Exercise 45 of Section 15.2), let $\mathfrak{m} = (x)$ be the unique maximal ideal of R and let P be the prime ideal $\cap_{n\geq 1}\mathfrak{m}^n$. Set $M = R/P$.

(a) Prove that $\mathrm{Tor}_1^R(M, R/\mathfrak{m}) = 0$. [Use Exercise 19 applied with $r = x$, noting that R/P is an integral domain.]

(b) Prove that M is not flat (hence not projective). [Let F be as in Exercise 45 of Section 15.2. Show that the sequence $0 \to R \to R \to R/(F) \to 0$ induced by multiplication by F is exact, but is not exact after tensoring with M.]