Theorem 39. (*Schur's Theorem*) If E is any finite group containing a normal subgroup N whose order and index are relatively prime, then N has a complement in E.

Remark: Recall that a subgroup whose order and index are relatively prime is called a *Hall subgroup*, so Schur's Theorem says that every normal Hall subgroup has a complement that splits the group as a semidirect product.

Proof: We use induction on the order of E. Since we may assume $N \neq 1$, let p be a prime dividing $|N|$ and let P be a Sylow p-subgroup of N. Let E_0 be the normalizer in E of P and let $N_0 = N \cap E_0$. By Frattini's Argument (Proposition 6 in Section 6.1) $E = E_0 N$. It follows from the Second Isomorphism Theorem that N_0 is a (normal) Hall subgroup of E_0 and $|E_0 : N_0| = |E : N|$ (cf. Exercise 10 of Section 3.3).

If $E_0 < E$, then by induction applied to N_0 in E_0 we obtain that E_0 contains a complement K to N_0. Since $|K| = |E_0 : N_0|$, K is also a complement to N in E, as needed. Thus we may assume $E_0 = E$, i.e., P is normal in E.

Since the center of P, $Z(P)$, is characteristic in P, it is normal in E (cf. Section 4.4). If $Z(P) = N$, then N is abelian and the theorem follows from Corollary 38. Thus we may assume $Z(P) \neq N$. Let bars denote passage to the quotient group $E/Z(P)$. Then $\overline{N}$ is a normal Hall subgroup of $\overline{E}$. By induction it has a complement $\overline{K}$ in $\overline{E}$. Let E_1 be the complete preimage of $\overline{K}$ in E. Then $|E_1| = |\overline{K}||Z(P)| = |E/N||Z(P)|$, so $Z(P)$ is a normal Hall subgroup of E_1. By induction $Z(P)$ has a complement in E_1 which is seen by order considerations to also be a complement to N in E. This completes the proof.

Examples

(1) If $G = Z_2$ and $A = \mathbb{Z}/2\mathbb{Z}$ then G acts trivially on A and so $H^2(G, A) = A^G/NA = \mathbb{Z}/2\mathbb{Z}$ by the computation of the cohomology of cyclic groups in Section 2, so by Theorem 36 there are precisely two inequivalent extensions of G by A. These are the cyclic group of order 4 and the Klein 4-group, the latter being split and hence corresponding to the trivial class in H^2.

(2) If $G = \langle g \rangle \cong Z_2$ and $A = \langle a \rangle \cong \mathbb{Z}/4\mathbb{Z}$ is a group of order 4 on which G acts trivially, then $H^2(G, A) = A/2A \cong \mathbb{Z}/2\mathbb{Z}$ by the computation of the cohomology of cyclic groups. As in the previous example there are two inequivalent extensions of G by A; evidently these are the groups Z_8 and $Z_4 \times Z_2$, the latter split extension corresponding to the trivial cohomology class.

If $E = \langle r \rangle \times \langle s \rangle$ denotes the split extension of G by A, where $|r| = 4$ and $|s| = 2$, then $\mu_i(g) = r^i s$ for $i = 0, \dots, 3$ give the four normalized sections of G in E. The sections μ_0, μ_2 both give the zero factor set f. The sections μ_1, μ_3 both give the factor set f' with $f'(g, g) = a^2 \in A$. Both f and f' give normalized 2-cocycles lying in the trivial cohomology class of $H^2(G, A)$. The extension E_f corresponding to the zero 2-cocycle f is the group with the elements $(a, 1)$ and $(1, g)$ as the usual generators (of orders 4 and 2, respectively) for $Z_4 \times Z_2$. In $E_{f'}$, however, $(a, 1)$ has order 4 but so does $(1, g)$ since $(1, g)^2 = (f'(g, g), g^2) = (a^2, 1)$. The 2-cocycles f and f' differ by the coboundary f_1 with $f_1(1) = 1$ and $f_1(g) = r$. The isomorphism $\beta(a, g) = (a + f_1(g), g)$ from E_f to $E_{f'}$ maps the generators $(a, 1)$ and $(1, g)$ of E_f to the generators $(a, 1)$ and (a, g) of $E_{f'}$ and gives the explicit equivalence of these two extensions.

The situation where G acts on A by inversion is handled in Exercise 3.

(3) Suppose $G = Z_2$ and A is the Klein 4-group. If G acts nontrivially on A then G interchanges two of the nonidentity elements, say a and b, of A and fixes the third nonidentity element c. Then $A^G = NA = \{1, c\}$ and so $H^2(G, A) = 0$, and so every extension E of G by A splits. This can be seen directly, as follows. Since the action is nontrivial, such a group must be nonabelian, hence must be D_8. From the lattice of D_8 in Section 2.5 one sees that for each Klein 4-group there is a subgroup of order 2 in D_8 not contained in the 4-group and that subgroup splits the extension.

If G acts trivially on A then $H^2(G, A) = A/2A \cong A$, so there are 4 inequivalent extensions of G by A in this case. These are considered in Exercise 1.

Example: (Groups of Order 8 and $H^2(Z_2 \times Z_2, \mathbb{Z}/2\mathbb{Z})$)

Let $G = \{1, a, b, c\}$ be the Klein 4-group and let $A = \mathbb{Z}/2\mathbb{Z}$. The 2-group G must act trivially on A. The elements of $H^2(G, A)$ classify extensions E of order 8 which has a quotient group by some Z_2 subgroup that is isomorphic to the Klein 4-group. Although there are, up to group isomorphism, only four such groups, we shall see that there are *eight* inequivalent extensions.

Since $G \times G$ has 16 elements, we have $|C^2(G, A)| = 2^{16}$. The cocycle condition (26) here reduces to

$$f(g, h) + f(gh, k) = f(h, k) + f(g, hk) \qquad \text{for all } g, h, k \in G. \tag{17.38}$$

The following relations hold for the subgroup $Z^2(G, A)$ of cocycles:

(1) $f(g, 1) = f(1, g) = f(1, 1)$, for all $g \in G$

(2) $f(g, 1) + f(g, a) + f(g, b) + f(g, c) = 0$, for all $g \in G$

(3) $f(1, h) + f(a, h) + f(b, h) + f(c, h) = 0$, for all $h \in G$.

The first of these come from (38) by setting $h = k = 1$ and by setting $g = h = 1$. The other two relations come from (38) by setting $g = h$ and $h = k$, respectively, using relations (1) and (2). It follows that every 2-cocycle f can be represented by a vector $(\alpha, \beta, \gamma, \delta, \epsilon)$ in $\mathbb{F}_2$ where

$$\alpha = f(1, g) = f(g, 1), \text{ for all } g \in G,$$
$$\beta = f(a, a), \quad \gamma = f(a, b), \quad \delta = f(b, a), \quad \epsilon = f(b, b)$$

because the relations above then determine the remaining values of f:

$$f(a, c) = \alpha + \beta + \gamma \qquad f(b, c) = \alpha + \delta + \epsilon \qquad f(c, a) = \alpha + \beta + \delta$$
$$f(c, b) = \alpha + \gamma + \epsilon \qquad f(c, c) = \alpha + \beta + \gamma + \epsilon.$$

It follows that $|Z^2(G, A)| \le 2^5$. Although one could eventually show that every function satisfying these relations is a 2-cocycle (hence the order is exactly 32), this will follow from other considerations below.

A cocycle f is a coboundary if there is a function $f_1 : G \to A$ such that

$$f(g, h) = f_1(h) - f_1(gh) + f_1(g), \qquad \text{for all } g, h \in G.$$

This coboundary condition is easily seen to be equivalent to the conditions:

(i) $f(g, 1) = f(1, g) = f(g, g)$ for all $g \in G$, and

(ii) $f(g, h) = f(g', h')$ whenever g, h are distinct nonidentity elements and so are g', h'.

These relations are equivalent to $\alpha = \beta = \epsilon$ and $\gamma = \delta$. Thus $B^2(G, A)$ consists of the vectors $(\alpha, \alpha, \gamma, \gamma, \alpha)$, and so $H^2(G, A)$ has dimension at most 3 (i.e., order at most $2^3 = 8$). It is easy to see that $\{(0, \beta, \gamma, 0, \epsilon)\}$ with β, γ, and ϵ in $\mathbb{F}_2$ gives a set of representatives for $Z^2(G, A)/B^2(G, A)$, and each of these representative cocycles is normalized. We now prove $|H^2(G, A)| = 8$ (and also that $|Z^2(G, A)| = 2^5$) by explicitly exhibiting eight inequivalent group extensions,

Suppose E is an extension of G by A, where for simplicity we assume $A \le E$. If $\mu : G \to E$ is a section, the factor set for E associated to μ satisfies

$$\mu(g)\mu(h) = f(g, h)\mu(gh).$$

The group E is generated by $\mu(a)$, $\mu(b)$ and A, and A is contained in the center of E since G acts trivially on A. Hence E is abelian if and only if $\mu(a)\mu(b) = \mu(b)\mu(a)$, which by the relation above occurs if and only if $f(a, b) = f(b, a)$. If g is a nonidentity element in G, we also see from the relation above that $\mu(g)$ is an element of order 2 in E if and only if $f(g, g) = 0$. Because A is contained in the center of E, both elements in any nonidentity coset $A\mu(g)$ have the same order (either 2 or 4).

There are four groups of order 8 containing a normal subgroup of order 2 with quotient group isomorphic to the Klein 4-group: $Z_2 \times Z_2 \times Z_2$, $Z_4 \times Z_2$, D_8, and Q_8.

The group $E \cong Z_2 \times Z_2 \times Z_2$ is the split extension of G by A and has $f = 0$ as factor set.

When $E \cong Q_8$, in the usual notation for the quaternion group $A = \langle -1 \rangle$. In this (non-abelian) group every nonidentity coset consists of elements of order 4, and this property is unique to Q_8, so the resulting factor set f satisfies $f(g, g) \neq 0$ for all nonidentity elements in G.

When $E \cong Z_4 \times Z_2 = \langle x \rangle \times \langle y \rangle$ we must have $A = \langle x^2 \rangle$. The cosets Ax and Axy both consist of elements of order 4, and the coset Ay consists of elements of order 2, so exactly one of $\mu(a)$, $\mu(b)$ or $\mu(c)$ is an element of order 2 and the other two must be of order 4. This suggests three homomorphisms from E to G, defined on generators by

$$\begin{array}{ll} \pi_1(y) = a & \pi_1(x) = b \\ \pi_2(y) = b & \pi_2(x) = a\,. \\ \pi_3(y) = c & \pi_3(x) = a \end{array}$$

Each of these homomorphisms maps surjectively onto G, has A as kernel, and has $\mu(a)$ (respectively, $\mu(b)$, $\mu(c)$) an element of order 2 in E. Any isomorphism of E with itself that is the identity on A must take the unique nonidentity coset Ay of A consisting of elements of order 2 to itself. Hence any extension equivalent to the extension E_1 defined by π_1 also maps y to a (since the equivalence is the identity on G). It follows that the three extensions defined by π_1, π_2 and π_3 are inequivalent.

The situation when $E \cong D_8 = \langle r, s \rangle$ is similar. In this case $A = \langle r^2 \rangle$, the cosets As and Asr consist of elements of order 2, and the coset Ar consists of elements of order 4. In this case exactly one of $\mu(a)$, $\mu(b)$ or $\mu(c)$ is an element of order 4 and the other two are of order 2, suggesting the three homomorphisms defined on generators by

$$\begin{array}{ll} \pi_1(r) = a & \pi_1(s) = b \\ \pi_2(r) = b & \pi_2(s) = a\,. \\ \pi_3(r) = c & \pi_3(s) = a \end{array}$$

As before, the corresponding extensions are inequivalent.

The existence of 8 inequivalent extensions of G by A proves that $|H^2(G, A)| = 8$, and hence that these are a complete list of all the inequivalent extensions. In particular, the extension $E_1' \cong Z_4 \times Z_2$ defined by the homomorphism π_1' mapping y to a and x to c must be equivalent to the extension E_1 above (and similarly for the other two extensions isomorphic to $Z_4 \times Z_2$ and the three extensions for D_8). This proves the existence of certain outer automorphisms for these groups, cf. Exercise 9.