map $\psi : A \to A$ are compatible homomorphisms. The corresponding induced group homomorphism on cohomology is called the *restriction homomorphism*:

$$\text{Res} : H^n(G, A) \longrightarrow H^n(H, A), \quad n \geq 0.$$

The terminology comes from the fact that the map on cochains from $C^n(G, A)$ to $C^n(H, A)$ is simply restricting a map $f$ from $G^n$ to $A$ to the subgroup $H^n$ of $G^n$.

(3) *(The Inflation Homomorphism)* Suppose $H$ is a normal subgroup of $G$ and $A$ is a $G$-module. The elements $A^H$ of $A$ that are fixed by $H$ are naturally a module for the quotient group $G/H$ under the action defined by $(gH)\cdot a = g\cdot a$. It is then immediate that the projection $\varphi : G \to G/H$ and the inclusion $\psi : A^H \to A$ are compatible homomorphisms. The corresponding induced group homomorphism on cohomology is called the *inflation homomorphism*:

$$\text{Inf} : H^n(G/H, A^H) \longrightarrow H^n(G, A), \quad n \geq 0.$$

(4) *(The Corestriction Homomorphism)* Suppose that $H$ is a subgroup of $G$ of index $m$ and that $A$ is a $G$-module. Let $g_1, \dots, g_m$ be representatives for the left cosets of $H$ in $G$. Define a map

$$\psi : M_H^G(A) \longrightarrow A \qquad \text{by} \qquad f \longmapsto \sum_{i=1}^m g_i \cdot f(g_i^{-1}).$$

Note that if we change any coset representative $g_i$ by $g_i h$, then $(g_i h) f((g_i h)^{-1}) = g_i h f(h^{-1} g_i^{-1}) = g_i h h^{-1} f(g_i^{-1}) = g_i f(g_i^{-1})$ so the map $\psi$ is independent of the choice of coset representatives. It is easy to see that $\psi$ is a $G$-module homomorphism (and even that it is surjective), so we obtain a group homomorphism from $H^n(G, M_H^G(A))$ to $H^n(G, A)$, for all $n \geq 0$. Since $A$ is also an $H$-module, by Shapiro's Lemma we have an isomorphism $H^n(G, M_H^G(A)) \cong H^n(H, A)$. The composition of these two homomorphisms is called the *corestriction homomorphism:*

$$\text{Cor} : H^n(H, A) \longrightarrow H^n(G, A), \quad n \geq 0.$$

This homomorphism can be computed explicitly by composing the isomorphism $\Psi$ in the proof of Shapiro's Lemma for any resolution of $\mathbb{Z}$ by projective $G$-modules $P_n$ (note these are $G$-modules and not simply $H$-modules) with the map $\psi$, as follows. For a cocycle $f \in \text{Hom}_{\mathbb{Z}H}(P_n, A)$ representing a cohomology class $c \in H^n(H, A)$, a cocycle $\text{Cor}(f) \in \text{Hom}_{\mathbb{Z}G}(P_n, A)$ representing $\text{Cor}(c) \in H^n(G, A)$ is given by

$$\text{Cor}(f)(p) = \sum_{i=1}^m g_i \cdot \Psi(f)(p)(g_i^{-1}) = \sum_{i=1}^m g_i f(g_i^{-1} p),$$

for $p \in P_n$. When $n = 0$ this is particularly simple since we can take $P_0 = \mathbb{Z}G$. In this case $f \in \text{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, A) = M_H^G(A)$ is a cocycle if $f = a$ is constant for some $a \in A^H$ and then $\text{Cor}(f)$ is the constant function with value $\sum_{i=1}^m g_i \cdot a \in A^G$:

$$\text{Cor} : H^0(H, A) = A^H \longrightarrow A^G = H^0(G, A)$$
$$a \longmapsto \sum_{i=1}^m g_i \cdot a.$$

The next result establishes a fundamental relation between the restriction and corestriction homomorphisms.

**Proposition 26.** Suppose $H$ is a subgroup of $G$ of index $m$. Then $\mathrm{Cor} \circ \mathrm{Res} = m$, i.e., if $c$ is a cohomology class in $H^n(G, A)$ for some $G$-module $A$, then

$$\mathrm{Cor}(\mathrm{Res}(c)) = mc \in H^n(G, A) \quad \text{for all } n \geq 0.$$

*Proof:* This follows from the explicit formula for corestriction in Example 4 above, as follows. If $f \in \mathrm{Hom}_{\mathbb{Z}H}(P_n, A)$ were in $\mathrm{Hom}_{\mathbb{Z}G}(P_n, A)$, i.e., if $f$ were also a $G$-module homomorphism, then $g_i f(g_i^{-1} p) = g_i g_i^{-1} f(p) = f(p)$, for $1 \leq i \leq m$. Since restriction is the induced map on cohomology of the natural inclusion of $\mathrm{Hom}_{\mathbb{Z}G}(P_n, A)$ into $\mathrm{Hom}_{\mathbb{Z}H}(P_n, A)$, for such an $f$ we obtain

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathbb{Z}G}(P_n, A) & \xrightarrow{\mathrm{Res}} & \mathrm{Hom}_{\mathbb{Z}H}(P_n, A) & \xrightarrow{\mathrm{Cor}} & \mathrm{Hom}_{\mathbb{Z}G}(P_n, A) \\ f & \longmapsto & f & \longmapsto & mf. \end{array}$$

It follows that $\mathrm{Res} \circ \mathrm{Cor}$ is multiplication by $m$ on the cohomology groups as well.

**Corollary 27.** Suppose the finite group $G$ has order $m$. Then $mH^n(G, A) = 0$ for all $n \geq 1$ and any $G$-module $A$.

*Proof:* Let $H = 1$, so that $[G : H] = m$, in Proposition 26. Then for any class $c \in H^n(G, A)$ we have $mc = \mathrm{Cor}(\mathrm{Res}(c))$. Since $\mathrm{Res}(c) \in H^n(H, A) = H^n(1, A)$, we have $\mathrm{Res}(c) = 0$ for all $n \geq 1$ by the second example preceding Proposition 20. Hence $mc = 0$ for all $n \geq 1$, which is the corollary.

**Corollary 28.** If $G$ is a finite group then $H^n(G, A)$ is a torsion abelian group for all $n \geq 1$ and all $G$-modules $A$.

*Proof:* This is immediate from the previous corollary.

**Corollary 29.** Suppose $G$ is a finite group whose order is relatively prime to the exponent of the $G$-module $A$. Then $H^n(G, A) = 0$ for all $n \geq 1$. In particular, if $A$ is a finite abelian group with $(|G|, |A|) = 1$ then $H^n(G, A) = 0$ for all $n \geq 1$.

*Proof:* This follows since the abelian group $H^n(G, A)$ is annihilated by $|G|$ by the previous corollary and is annihilated by the exponent of $A$ by Proposition 20.

Note that the statements in the preceding corollaries are not in general true for $n = 0$, since then $H^0(G, A) = A^G$, which need not even be torsion.

We mention without proof the following result. Suppose that $H$ is a normal subgroup of $G$ and $A$ is a $G$-module. The cohomology groups $H^n(H, A)$ can be given the structure of $G/H$-modules (cf. Exercise 17). It can be shown that there is an exact sequence

$$0 \to H^1(G/H, A^H) \xrightarrow{\mathrm{Inf}} H^1(G, A) \xrightarrow{\mathrm{Res}} H^1(H, A)^{G/H} \xrightarrow{\mathrm{Tra}} H^2(G/H, A^H) \xrightarrow{\mathrm{Inf}} H^2(G, A)$$

where $H^1(H, A)^{G/H}$ denotes the fixed points of $H^1(H, A)$ under the action of $G/H$ and Tra is the so-called *transgression homomorphism*. This exact sequence relates the cohomology groups for $G$ to the cohomology groups for the normal subgroup $H$ and for the quotient group $G/H$. Put another way, the cohomology for $G$ is related to the cohomology for the factors in the filtration $1 \le H \le G$ for $G$. More generally, one could try to relate the cohomology for $G$ to the cohomology for the factors in a longer filtration for $G$. This is the theory of *spectral sequences* and is an important tool in homological algebra.

## Galois Cohomology and Profinite Groups

One important application of group cohomology occurs when the group $G$ is the Galois group of a field extension $K/F$. In this case there are many groups of interest on which $G$ acts, for example the additive group of $K$, the multiplicative group $K^\times$, etc. The Galois group $G = \mathrm{Gal}(K/F)$ is the inverse limit $\varprojlim \mathrm{Gal}(L/F)$ of the Galois groups of the finite extensions $L$ of $F$ contained in $K$ and is a compact topological group with respect to its Krull topology (i.e., the group operations on $G$ are continuous with respect to the topology defined by the subgroups $\mathrm{Gal}(K/L)$ of $G$ of finite index), cf. Section 14.9. In this situation it is useful (and often essential) to take advantage of the additional topological structure of $G$. For example the subfields of $K$ containing $F$ correspond bijectively with the *closed* subgroups of $G = \mathrm{Gal}(K/F)$, and the example of the composite of the quadratic extensions of $\mathbb{Q}$ discussed in Section 14.9 shows that in general there are many subgroups of $G$ that are not closed. Fortunately, the modifications necessary to define the cohomology groups in this context are relatively minor and apply to arbitrary inverse limits of finite groups (the *profinite* groups). If $G$ is a profinite group then $G = \varprojlim G/N$ where the inverse limit is taken over the open normal subgroups $N$ of $G$ (cf. Exercise 23).

**Definition.** If $G$ is a profinite group then a *discrete $G$-module* $A$ is a $G$-module $A$ with the discrete topology such that the action of $G$ on $A$ is continuous, i.e., the map $G \times A \to A$ mapping $(g, a)$ to $g \cdot a$ is continuous.

Since $A$ is given the discrete topology, every subset of $A$ is open, and in particular every element $a \in A$ is open. The continuity of the action of $G$ on $A$ is then equivalent to the statement that the stabilizer $G_a$ of $a$ in $G$ is an open subgroup of $G$, hence is of finite index since $G$ is compact (cf. Exercise 22). This in turn is equivalent to the statement that $A = \cup A^H$ where the union is over the open subgroups $H$ of $G$.

Some care must be taken in defining the cohomology groups $H^n(G, A)$ of a profinite group $G$ acting on a discrete $G$-module $A$ since there are not enough projectives in this category. For example, when $G$ is infinite, the free $G$-module $\mathbb{Z}G$ is not a discrete $G$-module ($G$ does not act continuously, cf. Exercise 25). Nevertheless, the explicit description of $H^n(G, A)$ given in this section (occasionally referred to as the *discrete* cohomology groups) can be easily modified — it is only necessary to require the cochains $C^n(G, A)$ to be *continuous* maps from $G^n$ to $A$. The definition of the coboundary maps $d_n$ in equation (18) is precisely the same, as is the definition of the groups of cocycles, coboundaries, and the corresponding cohomology groups. It is customary not to introduce a separate notation for these cohomology groups, but to specify which cohomology is meant in the terminology.