Definition. In a group G a sequence of subgroups

$$1 = N_0 \leq N_1 \leq N_2 \leq \cdots \leq N_{k-1} \leq N_k = G$$

is called a *composition series* if $N_i \trianglelefteq N_{i+1}$ and N_{i+1}/N_i a simple group, $0 \leq i \leq k-1$. If the above sequence is a composition series, the quotient groups N_{i+1}/N_i are called *composition factors* of G.

Keep in mind that it is not assumed that each $N_i \trianglelefteq G$, only that $N_i \trianglelefteq N_{i+1}$. Thus

$$1 \trianglelefteq \langle s \rangle \trianglelefteq \langle s, r^2 \rangle \trianglelefteq D_8 \quad \text{and} \quad 1 \trianglelefteq \langle r^2 \rangle \trianglelefteq \langle r \rangle \trianglelefteq D_8$$

are two composition series for D_8 and in each series there are 3 composition factors, each of which is isomorphic to (the simple group) Z_2.

Theorem 22. (Jordan–Hölder) Let G be a finite group with $G \neq 1$. Then

(1) G has a composition series and

(2) The composition factors in a composition series are unique, namely, if $1 = N_0 \leq N_1 \leq \cdots \leq N_r = G$ and $1 = M_0 \leq M_1 \leq \cdots \leq M_s = G$ are two composition series for G, then $r = s$ and there is some permutation, π, of $\{1, 2, \ldots, r\}$ such that

$$M_{\pi(i)}/M_{\pi(i)-1} \cong N_i/N_{i-1}, \qquad 1 \leq i \leq r.$$

Proof: This is fairly straightforward. Since we shall not explicitly use this theorem to prove others in the text we outline the proof in a series of exercises at the end of this section.

Thus every finite group has a "factorization" (i.e., composition series) and although the series itself need not be unique (as D_8 shows) the number of composition factors and their isomorphism types are uniquely determined. Furthermore, nonisomorphic groups may have the same (up to isomorphism) list of composition factors (see Exercise 2). This motivates a two-part program for classifying all finite groups up to isomorphism:

The Hölder Program

(1) Classify all finite simple groups.

(2) Find all ways of "putting simple groups together" to form other groups.

These two problems form part of an underlying motivation for much of the development of group theory. Analogues of these problems may also be found as recurring themes throughout mathematics. We include a few more comments on the current status of progress on these problems.

The classification of finite simple groups (part (1) of the Hölder Program) was completed in 1980, about 100 years after the formulation of the Hölder Program. Efforts by over 100 mathematicians covering between 5,000 and 10,000 journal pages (spread over some 300 to 500 individual papers) have resulted in the proof of the following result:

Theorem. There is a list consisting of 18 (infinite) families of simple groups and 26 simple groups not belonging to these families (the *sporadic* simple groups) such that every finite simple group is isomorphic to one of the groups in this list.

One example of a family of simple groups is $\{Z_p \mid p \text{ a prime}\}$. A second infinite family in the list of finite simple groups is:

$$\{SL_n(\mathbb{F})/Z(SL_n(\mathbb{F})) \mid n \in \mathbb{Z}^+, n \geq 2 \text{ and } \mathbb{F} \text{ a finite field }\}.$$

These groups are all simple except for $SL_2(\mathbb{F}_2)$ and $SL_2(\mathbb{F}_3)$ where $\mathbb{F}_2$ is the finite field with 2 elements and $\mathbb{F}_3$ is the finite field with 3 elements. This is a 2-parameter family (n and $\mathbb{F}$ being independent parameters). We shall not prove these groups are simple (although it is not technically beyond the scope of the text) but rather refer the reader to the book *Finite Group Theory* (by M. Aschbacher, Cambridge University Press, 1986) for proofs and an extensive discussion of the simple group problem. A third family of finite simple groups, the alternating groups, is discussed in the next section; we shall prove these groups are simple in the next chapter.

To gain some idea of the complexity of the classification of finite simple groups the reader may wish to peruse the proof of one of the cornerstones of the entire classification:

Theorem. (Feit–Thompson) If G is a simple group of odd order, then $G \cong Z_p$ for some prime p.

This proof takes 255 pages of hard mathematics.[2]

Part (2) of the Hölder Program, sometimes called the *extension problem*, was rather vaguely formulated. A more precise description of "putting two groups together" is: given groups A and B, describe how to obtain all groups G containing a normal subgroup N such that $N \cong B$ and $G/N \cong A$. For instance, if $A = B = Z_2$, there are precisely two possibilities for G, namely, Z_4 and V_4 (see Exercise 10 of Section 2.5) and the Hölder program seeks to describe how the two groups of order 4 could have been built from two Z_2's without a priori knowledge of the existence of the groups of order 4. This part of the Hölder Program is extremely difficult, even when the subgroups involved are of small order. For example, all composition factors of a group G have order 2 if and only if $|G| = 2^n$, for some n (one implication is easy and we shall prove both implications in Chapter 6). It is known, however, that the number of nonisomorphic groups of order 2^n grows (exponentially) as a function of 2^n, so the number of ways of putting groups of 2-power order together is not bounded. Nonetheless, there are a wealth of interesting and powerful techniques in this subtle area which serve to unravel the structure of large classes of groups. We shall discuss only a couple of ways of building larger groups from smaller ones (in the sense above) but even from this limited excursion into the area of group extensions we shall construct numerous new examples of groups and prove some classification theorems.

One class of groups which figures prominently in the theory of polynomial equations is the class of *solvable* groups:

[2] *Solvability of groups of odd order*, Pacific Journal of Mathematics, 13(1963), pp. 775–1029.

Definition. A group G is *solvable* if there is a chain of subgroups

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \dots \trianglelefteq G_s = G$$

such that G_{i+1}/G_i is abelian for $i = 0, 1, \dots, s-1$.

The terminology comes from the correspondence in Galois Theory between these groups and polynomials which can be solved by radicals (which essentially means there is an algebraic formula for the roots). Exercise 8 shows that finite solvable groups are precisely those groups whose composition factors are all of prime order.

One remarkable property of finite solvable groups is the following generalization of Sylow's Theorem due to Philip Hall (cf. Theorem 6.11 and Theorem 19.8).

Theorem. The finite group G is solvable if and only if for every divisor n of $|G|$ such that $(n, \frac{|G|}{n}) = 1$, G has a subgroup of order n.

As another illustration of how properties of a group G can be deduced from combined information from a normal subgroup N and the quotient group G/N we prove

if N and G/N are solvable, then G is solvable.

To see this let $\overline{G} = G/N$, let $1 = N_0 \trianglelefteq N_1 \trianglelefteq \dots \trianglelefteq N_n = N$ be a chain of subgroups of N such that N_{i+1}/N_i is abelian, $0 \le i < n$ and let $\overline{1} = \overline{G_0} \trianglelefteq \overline{G_1} \trianglelefteq \dots \trianglelefteq \overline{G_m} = \overline{G}$ be a chain of subgroups of $\overline{G}$ such that $\overline{G_{i+1}}/\overline{G_i}$ is abelian, $0 \le i < m$. By the Lattice Isomorphism Theorem there are subgroups G_i of G with $N \le G_i$ such that $G_i/N = \overline{G_i}$ and $G_i \trianglelefteq G_{i+1}$, $0 \le i < m$. By the Third Isomorphism Theorem

$$\overline{G_{i+1}}/\overline{G_i} = (G_{i+1}/N)/(G_i/N) \cong G_{i+1}/G_i.$$

Thus

$$1 = N_0 \trianglelefteq N_1 \trianglelefteq \dots \trianglelefteq N_n = N = G_0 \trianglelefteq G_1 \trianglelefteq \dots \trianglelefteq G_m = G$$

is a chain of subgroups of G all of whose successive quotient groups are abelian. This proves G is solvable.

It is inaccurate to say that finite group theory is concerned *only* with the Hölder Program. It *is* accurate to say that the Hölder Program suggests a large number of problems and motivates a number of algebraic techniques. For example, in the study of the extension problem where we are given groups A and B and wish to find G and $N \trianglelefteq G$ with $N \cong B$ and $G/N \cong A$, we shall see that (under certain conditions) we are led to an *action* of the group A on the set B. Such actions form the crux of the next chapter (and will result in information both about simple and non-simple groups) and this notion is a powerful one in mathematics not restricted to the theory of groups.

The final section of this chapter introduces another family of groups and although in line with our interest in simple groups, it will be of independent importance throughout the text, particularly in our study later of determinants and the solvability of polynomial equations.