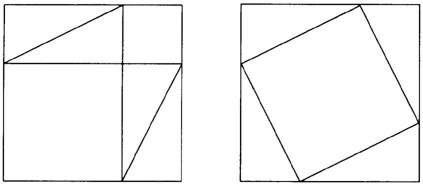

**FIGURE 2.1** A proof of the Pythagorean theorem.

triangles and squares do not exist, so the theorem has to be about some kind of ideal or abstract objects. And yet, we are surely using our experience with actual triangles and squares to draw conclusions about the abstract ones.

Thus the gift of geometric intuition is both a blessing and a curse. It gives us amazingly direct access to mathematical results; yet we cannot be satisfied with the results seen by our intuition until they have been validated by logic. The validation can be very hard work, and it would be disappointing if its *only* outcome was confirmation of results we already believe. A method of validating intuition is worthwhile only if it takes us further than intuition alone.

The most conservative solution to the problem of validating intuition is the so-called *synthetic geometry*. In this system, all theorems are derived by pure logic from a (rather long) list of visually plausible axioms about points, lines, circles, planes,and so on. Thus we can be sure that all theorems proved in synthetic geometry will be intuitively acceptable. This was the approach initiated in Book I of Euclid's *Elements* and perfected in David Hilbert's *Foundations of Geometry* (1899). Its advantages are that it is self-contained (no concepts from outside geometry) and close to intuition (the steps in a proof may imitate the way we "see" a theorem). However, it fails to explain the mysterious similarity between geometry and arithmetic; the fact that geometric quantities, like numbers, can be added, subtracted, and (in the case of lengths) even multiplied. It looks like geometry and arithmetic share a common ground, and mathematics should explain why.

The search for a common ground of arithmetic and geometry led to the so-called *analytic geometry*, initiated in René Descartes' *Geometry* (1637) and also perfected by Hilbert. It is more efficient as a way of making geometry rigorous, and history has shown it to be more fruitful than synthetic geometry in its consequences. It enriches both arithmetic and geometry with new concepts, and in fact with the whole new mathematical world of algebra and calculus. As we shall see in later chapters, the new world is not separate from the old, but it increases our understanding of it. Algebra not only throws new light on geometry, it also enables us to solve problems about the natural numbers that were previously beyond reach.

Analytic geometry will be developed in the next chapter. In the meantime, we will use intuition freely to gain a bird's eye view of the landmark results and concepts in geometry, to see how far arithmetic concepts apply to geometric quantities, and to see why the number concept needs to be extended to build a common foundation for arithmetic and geometry.

## Exercises

The Pythagorean theorem has been discovered many times, in different cultures, and proved in many different ways. The very immediate proof indicated in Figure 2.1 was given by Bhâskara II in 12th century India. Another way the theorem may have been discovered was suggested by Magnus (1974), p. 159. It comes from thinking about the tiled floor shown in Figure 2.2.

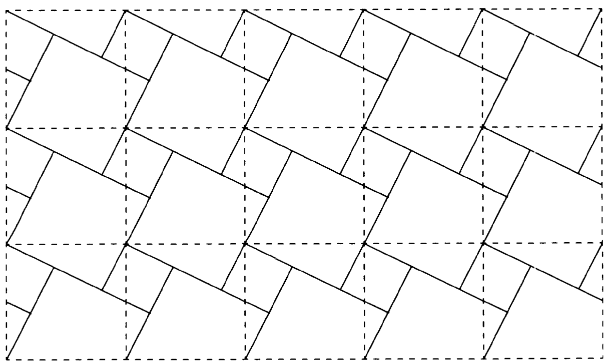

**FIGURE 2.2** Pythagorean theorem in a tiled floor.

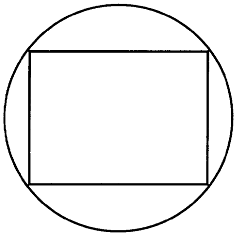

**FIGURE 2.3** Rectangle in a circle.

2.1.1. Explain how Figure 2.2 is related to the Pythagorean theorem. (The dotted squares are not the tiles; they are a hint.)

The converse Pythagorean theorem is also important: *if $a$, $b$, and $c$ are lengths such that $a^2 + b^2 = c^2$ then the triangle with sides $a$, $b$, and $c$ is right-angled* (with the right angle formed by the sides $a$ and $b$).

2.1.2. Deduce the converse Pythagorean theorem from the Pythagorean theorem itself.

2.1.3. Deduce the Pythagorean theorem from its converse.

Another very old theorem is that *any right-angled triangle fits in a semicircle, with the hypotenuse as diameter*. According to legend, synthetic geometry began with a proof of this theorem by Thales in the 6th century B.C.

2.1.4. Why might Figure 2.3 lead you to believe that any right-angled triangle fits in a semicircle?

## 2.2 Constructions

The aim of Euclid's geometry is to study the properties of the simplest curves, the straight line, and the circle. These are drawn by the simplest drawing instruments, the ruler and the compass; hence much of the *Elements* consists of so-called *ruler and compass constructions*. In fact, two of Euclid's axioms state that the following constructions are possible.

- To draw a straight line from any point to any other point.
- To draw a circle with any center and radius.

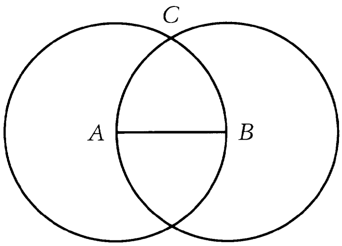


**FIGURE 2.4** Constructing an equilateral triangle.

His axioms do not state the existence of anything else, so all other figures are shown to exist by actually constructing them. Certain points, lines, and circles being given, new ones are constructed using ruler and compass. This creates new points, from which new lines and circles are constructed, and so on, until the required figure is obtained.

Euclid's first proposition is that it is possible to construct an equilateral triangle with a given side $AB$, and his first figure shows how it is done (Figure 2.4). Namely, draw the circle with center $A$ and radius $AB$, then the circle with center $B$ and radius $BA$, and connect $A$ and $B$ by straight lines to one intersection, $C$, of these circles.

Several other important constructions come from this.

1. **Bisecting an angle.**
   Drawing a circle with center at the apex $O$ of the angle marks off equal sides $OA$ and $OB$ (Figure 2.5). Then if we construct an equilateral triangle $ABC$, the line $OC$ will bisect the angle $AOB$.

2. **Bisecting a line segment.**
   Given the line segment $AB$, construct the equilateral triangle $ABC$. Then the bisector of the angle $ACB$ also bisects $AB$.

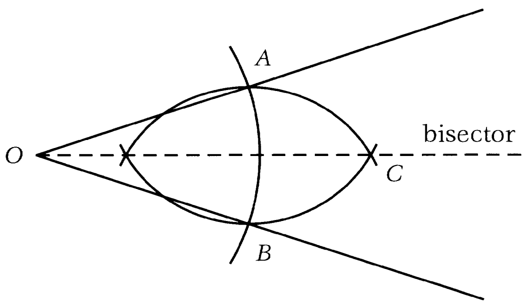


**FIGURE 2.5** Bisecting an angle.

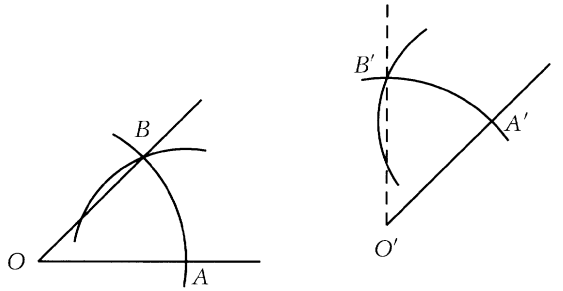


**FIGURE 2.6**
Replicating an angle.

3. **Constructing the perpendicular to a line through a point $O$ not on it.**
   Draw a circle with center $O$, large enough to cut the line at points $A$ and $B$. Then the bisector of angle $AOB$ will be the required perpendicular.

As one notices from these constructions, the compass gives an easy way to replicate a given line segment. It is also possible to replicate a given angle. For example, one can draw a circle with unit radius centered on the apex $O$ of the angle, then use the line $AB$ between its intersections as a second radius to find points $A'$, $B'$ so that angle $B'O'A'$ = angle $BOA$, with $O'$ a given point on the given line (Figure 2.6). Euclid uses angle replication to construct a parallel to a given line through a given point. He chooses a point $O$ at random on the line, joins it to the given point $O'$, then replicates the angle $O'OX$ as angle $OO'X'$ (Figure 2.7).

The construction of parallels is needed to divide a line segment $AB$ into $n$ equal parts, for any natural number $n$. (The special case

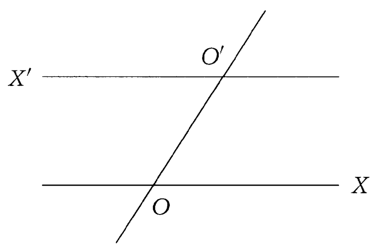


**FIGURE 2.7** Constructing a parallel.

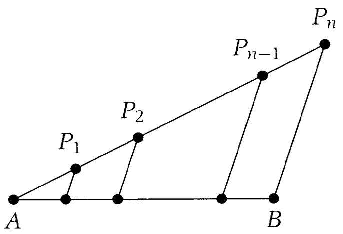


**FIGURE 2.8** Cutting a line segment into equal parts.

$n = 2$, or bisection, is not typical, because it does not require parallels.) Along any line through $A$ (other than $AB$) mark $n$ equally spaced points $P_1, P_2, \ldots, P_n$ by repeatedly replicating an arbitrary line segment $AP_1$. Then join $P_n$ to $B$, and draw parallels to $P_nB$ through $P_1, P_2, \ldots, P_{n-1}$ (Figure 2.8). These parallels cut $AB$ into $n$ equal parts.

## Exercises

Most of the following problems are variations on the theme of finding the perpendicular bisector of a line segment.

2.2.1. Describe the construction of the perpendicular to a line through a given point on the line.

2.2.2. Given a circle, but *not* its center, give a construction to find the center.

2.2.3. The perpendicular bisectors of the sides of a triangle meet at a single point. What property of the perpendicular bisector makes this obvious?

2.2.4. Use Exercise 2.2.3 to find a circle passing through the vertices of any triangle.

2.2.5. Describe the construction of a square and a regular hexagon.

The Greeks also found a construction of the regular pentagon, but no essentially new constructions were found until Gauss in 1796 found how to construct the regular 17-gon. This led to an algebraic theory of constructibility that explained why no constructions had been found for the regular 7-gon, 11-gon, and others. The astonishing result of Gauss's theory (completed by Pierre Wantzel in 1837) is that the regular $n$-gon is constructible if and only if $n$ is the product of a power of 2 by distinct Fermat primes. (Recall that these were defined in Exercise 1.2.7.)

2.2.6. If $\gcd(m, n) = 1$ and the regular $m$-gon and $n$-gon are constructible, show that the regular $mn$-gon is also constructible.

## 2.3 Parallels and Angles

The crucial assumption in Euclid's geometry—the one that makes the geometry "Euclidean"—is the *parallel axiom*. It can be stated in many different ways, the most concise of which is probably the one given by Playfair in 1795.

> **Parallel axiom.** *If $\mathcal{L}$ is a line and $P$ is a point not on $\mathcal{L}$, then there is exactly one line through $P$ that does not meet $\mathcal{L}$.*

The single line through $P$ that does not meet $\mathcal{L}$ is called the *parallel* to $\mathcal{L}$ through $P$. Euclid's statement is more complicated, and it involves the concept of angle, which is not mentioned in the Playfair version. This would ordinarily be regarded as inelegant mathematics, but in this case it is more informative, and it points us toward some important consequences of the parallel axiom.

> That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles. [From the edition of Euclid's *Elements* by Heath (1925), p. 202]

Figure 2.9 shows the situation described by Euclid, which is what happens with *non*parallel lines. If the angles $\alpha$ and $\beta$ have a sum less than two right angles, then $\mathcal{L}$ meets $\mathcal{M}$ on the side where $\alpha$ and $\beta$ are. To see why Euclid's statement is equivalent to Playfair's, one only needs to know that angles $\alpha$ and $\beta$ sum to two right angles if they can be moved so that together they form a straight line (Figure 2.10). The same figure shows that the *vertically opposite angles*, both marked $\alpha$, are equal, because each of them plus $\beta$ equals two right angles.

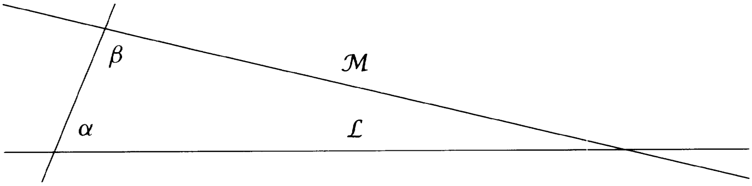


**FIGURE 2.9** Nonparallel lines.

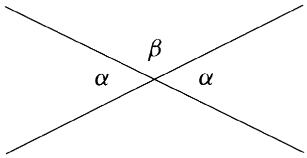


**FIGURE 2.10** Vertically opposite angles.

Euclid's statement of the parallel axiom tells us that a line $\mathcal{M}$ can fail to meet line $\mathcal{L}$ only when the two interior angles $\alpha$ and $\beta$ sum to two right angles. This follows from the facts about angles just mentioned. If $\alpha+\beta$ is greater than two right angles, then the interior angles on the *other* side of the transverse line $\mathcal{N}$ will sum to less than two right angles, and hence $\mathcal{M}$ will meet $\mathcal{L}$ on the other side. If $\alpha+\beta$ equals two right angles, the interior angles on the other side of the transverse line $\mathcal{N}$ are also $\alpha$ and $\beta$, hence also of sum equal to two right angles. In this case it follows by symmetry that $\mathcal{L}$ and $\mathcal{M}$ meet on both sides or neither side. The possibility of two meetings is ruled out by another axiom, that there is only one straight line through any two points. Hence there is exactly one line $\mathcal{M}$ through $P$ that does not meet $\mathcal{L}$: the line for which $\alpha + \beta$ equals two right angles (Figure 2.11). The most important property of angles that follows from the parallel axiom is that *the angle sum of a triangle is two right angles*. The proof is based on Figure 2.12, which shows an arbitrary triangle with a parallel to one side drawn through the opposite vertex. The angles of the triangle recur as shown and hence sum to a straight line.

It follows, by pasting triangles together, that the angle sum of any quadrilateral is four right angles. In particular, in a quadrilateral with equal angles, each angle is a right angle. This means that

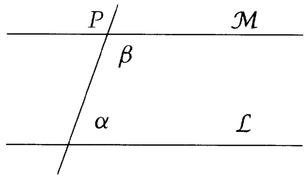


**FIGURE 2.11** Parallel lines and related angles.

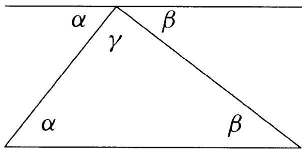


**FIGURE 2.12** Angle sum of a triangle.

rectangles and squares of any size exist. Of course, this is what we always thought, but we can now see that it follows from a small number of more basic statements about straight lines, among them the crucial parallel axiom. As we shall see in Section 2.5, the existence of rectangles is the key to the intuitive concept of area and to finding a common ground for geometry and arithmetic.

# Exercises

The crucial role of the parallel axiom can be seen from the number of important statements that are equivalent to it, and hence cannot be proved without it. Not only are the Playfair and Euclid versions equivalent to each other (Exercise 2.3.1 completes the proof of this), they are also equivalent to the statement about the angle sum of a triangle (Exercise 2.3.2).

2.3.1. Deduce Euclid's version of the parallel axiom from Playfair's.

2.3.2. Deduce Euclid's version of the parallel axiom from the statement that the angle sum of a triangle is two right angles.