APPEND. $DGC = \phi$, ac propterea $DG = \frac{f}{cos.\phi}$ & $CD = \frac{f.\,sin.\,\phi}{cos\,\phi}$. Ex sectionis quæsitæ puncto quovis M ducatur MT parallela ipsi DG: atque, ob $TQ = f - x$, & angulum $QTM = \phi$, erit $TM = \frac{f - x}{cos.\,\phi}$ & $QM = \frac{(f - x).\,sin\,\phi}{cos.\,\phi} = z$. Ducatur MS parallela ipsi TG, ideoque normalis in DG, erit $MS = TG = PQ = y$, & $DS = \frac{x}{cos.\,\phi}$.

56. Sumantur nunc rectæ DS & SM pro Coordinatis sectionis quæsitæ; sitque $DS = t$, & $SM = u$. Hinc erit $y = u$, $x = t.\,cos.\phi$: &, ob $z = \frac{(f - x).\,sin.\,\phi}{cos.\,\phi}$, erit $z = f.tang.\phi - t.sin.\phi$. Substituantur isti valores in æquatione pro Cylindro $aacc = aayy + ccxx$, atque resultabit pro sectione quæsita ista æquatio $aacc = aauu + cctt\,(cos.\,\phi)^2$: quæ indicat sectionem fore Ellipsin Centrum in puncto D habentem, cujus alter Axis principalis in rectam DG cadat, alter vero ad hunc sit normalis. Erit vero semiaxis in rectam DG cadens (facto $u = o$) $= \frac{a}{cos.\,\phi}$. Vel, ducatur recta BH parallela ipsi GD, erit $BH = \frac{a}{cos.\,\phi}$ alter semiaxis sectionis quæsitæ, alter vero conjugatus erit $= c = CE$.

57. Erit ergo sectio Cylindri hoc modo orta Ellipsis, cujus semiaxes conjugati erunt $\frac{a}{cos.\,\phi}$ & c. Quod si ergo in Basi $AEBF$ fuerit $AC = a$ semiaxis major; tum, ob $\frac{a}{cos.\,\phi}$ majorem quam a, sectiones erunt Ellipses magis oblongæ, quam Basis. Sin autem fuerit c minor quam a: seu, si intersectio GT fuerit Axi majori Basis parallela, tum fieri potest ut in sectione ambo Axes fiant inter se æquales, atque adeo sectio Circulus evadat. Eveniet hoc si fuerit $\frac{a}{cos.\,\phi} = c$, seu $cos.\phi = \frac{a}{c}$.

$\frac{a}{c}$. Cum igitur ſit in Triangulo BCH ad C rectangulo angulus $CBH = \phi$, erit $cof.\,\phi = \frac{BC}{BH} = \frac{a}{BH}$. Quare, ſi ſumatur $BH = CE$, ſectiones erunt Circuli, quod cum duplici modo fieri queat, rectam $BH = CE$ ſive ſupra ſive infra conſtituendo, binæ exiſtent ſectionum circularium ſeries, quæ ad Axem CD oblique erunt inclinatæ; ex quo hujuſmodi Cylindri ſcaleni appellantur. Cap. III.

58. Sit nunc recta GT, utcunque oblique poſita, interſectio plani ſecantis cum Baſi, ad quam ex Centro Baſis C demittatur perpendiculum $GC = f$; & ponatur angulus $BCG = \theta$; ſitque angulus inclinationis $CGD = \phi$, cui æqualis erit angulus QTM, ducta QT ad GT normali. Erit ergo $DG = \frac{f}{cof.\,\phi}$, & $CD = \frac{f.\,fin.\,\phi}{cof.\,\phi}$. Sit M punctum in ſectione quæſita, unde ad Baſin perpendiculum MQ hincque porro ad Axem QP demittatur; ita ut, vocatis $CP = x$, $PQ = y$ & $QM = z$, ſit $aacc = aayy + ccxx$. Ducantur porro ad interſectionem GT normales PV, QT; erit $GV = x.\,fin.\,\theta$, $PV = f - x.\,cof.\,\theta$; &, ob angulum $QPW = \theta$, fiet $QW = y.\,fin.\,\theta$, $PW = VT = y.\,cof.\,\theta$, & $QT = f - x.\,cof.\,\theta + y.\,fin.\,\theta$. Denique, ducta MT, ob angulum $MTQ = \phi$, erit $TM = \frac{z}{fin.\,\phi}$ & $QT = \frac{z.\,cof.\,\phi}{fin.\,\phi}$. Tab. XXXIV. Fig. 131.

59. Compleatur parallelogrammum rectangulum $GSMT$; & vocetur $DS = t$, $SM = GT = u$: eritque $u = GV + VT = x.\,fin.\,\theta + y.\,cof.\,\theta$. At, ob $QT = f - x.\,cof.\,\theta + y.\,fin.\,\theta$, erit $QT - CG = y.\,fin.\,\theta - x.\,cof.\,\theta$, ex quo fit $DS = TM - DG = \frac{y.\,fin.\,\theta - x.\,cof.\,\theta}{cof.\,\phi} = t$. Cum igitur ſit $x.\,fin.\,\theta + y.\,cof.\,\theta = u$, & $y.\,fin.\,\theta - x.\,cof.\,\theta = t.\,cof.\,\phi$, habebitur $y = u.\,cof.\,\theta + t.\,fin.\,\theta.\,cof.\,\phi$, & $x = u.\,fin.\,\theta - t.\,cof.\,\theta.\,cof.\,\phi$. Qui valores in æquatione $aacc = aayy + ccxx$ loco x & y ſubſtituti dabunt

$aacc =$

APPEND.

$$aacc = \frac{aauu\,(cof.\theta)^2 + 2aaut.fin.\theta.cof.\theta.cof.\phi + aatt\,(fin.\theta)^2\,(cof.\phi)^2}{ccuu\,(fin.\theta)^2 - 2ccut.fin.\theta.cof.\theta.cof.\phi + cctt\,(cof.\theta)^2\,(cof.\phi)^2}$$

quam æquationem patet eſſe ad Ellipſin, cujus Centrum ſit in D, at Coordinatæ DS & SM ad Axes principales non ſint normales, niſi ſit $a = c$ ſeu Cylindrus rectus.

TAB. XXXIV. Fig. 132.

60. Ad hanc ſectionem propius cognoſcendam, ſit $aMebf$ Curva, cujus æquatio eſt inventa inter Coordinatas $DS = t$ & $MS = u$; ſitque, brevitatis ergo iſta æquatio $aacc = \alpha uu + 2\beta tu + \gamma tt$; ita, ut pro caſu præſente, habeatur

$$\alpha = aa\,(cof.\theta)^2 + cc\,(fin.\,\theta)^2$$

&

$$\beta = (aa - cc).\,fin.\,\theta.\,cof.\,\theta.\,cof.\,\phi$$

atque

$$\gamma = aa\,(fin.\,\theta)^2\,(cof.\,\phi)^2 + cc\,(cof.\,\theta)^2\,(cof.\,\phi)^2.$$

Sint hujus ſectionis ab & ef Axes principales conjugati, ductaque ad eorum alterutrum Applicata Mp, vocetur $Dp = p$ & $Mp = q$; ac ponatur angulus $aDH = \zeta$; erit $u = p.fin.\zeta + q.cof.\zeta$ & $t = p.cof.\zeta - q\,fin.\zeta$, quibus valoribus ſubſtitutis, fiet

$$aacc = \begin{matrix} +\,\alpha(fin.\,\zeta)^2 \\ +\,2\beta.fin.\zeta.cof.\zeta \\ +\,\gamma.(cof.\,\zeta)^2 \end{matrix}\; pp \;\begin{matrix} +\,2\alpha.fin.\zeta.cof.\zeta \\ +\,2\beta.(cof.\zeta)^2 \\ -\,2\beta.(fin.\zeta)^2 \\ -\,2\gamma.fin.\zeta.cof.\zeta \end{matrix}\; pq \;\begin{matrix} +\,\alpha(cof.\,\zeta)^2 \\ -\,2\beta.fin.\zeta.cof.\zeta \\ +\,\gamma.(fin.\,\zeta)^2 \end{matrix}\; qq$$

61. Hæc jam æquatio cum referatur ad Diametrum orthogonalem, coëfficiens ipſius pq debet eſſe $= 0$: unde, ob $2.fin.\zeta.cof.\zeta = fin.\,2\zeta$, & $(cof.\,\zeta)^2 - (fin.\,\zeta)^2 = cof.\,2\,\zeta$, fiet $(\alpha - \gamma).\,fin.\,2\,\zeta + 2\,\beta.\,cof.\,2\,\zeta = 0$: ideoque $tang.\,2\,\zeta = \frac{2\beta}{\gamma - \alpha}$: unde angulus aDH, ac proinde poſitio Diametrorum principalium cognoſcitur. Hinc porro ipſi ſemiaxes definiuntur, hoc modo

$aD =$

$$aD = \frac{ac}{\sqrt{(\alpha(\sin.\zeta)^2 + 2\beta.\sin.\zeta.\cos.\zeta + \gamma(\cos.\zeta)^2)}}$$
&
$$eD = \frac{ac}{\sqrt{(\alpha(\cos.\zeta)^2 - 2\beta.\sin.\zeta.\cos.\zeta + \gamma(\sin.\zeta)^2)}}.$$

62. Quia est $2\beta = \frac{2(\gamma - \alpha).\sin.\zeta.\cos.\zeta}{\cos.\zeta^2 - \sin.\zeta^2}$, erit, valore hoc in expressionibus inventis substituto,

$$aD = \frac{ac\sqrt{(\cos.\zeta^2 - \sin.\zeta^2)}}{\sqrt{(\gamma.\cos.\zeta^2 - \alpha.\sin.\zeta^2)}} = \frac{ac\sqrt{2.\cos.2\zeta}}{\sqrt{((\alpha+\gamma).\cos.2\zeta - \alpha + \gamma)}}$$
&
$$eD = \frac{ac\sqrt{(\cos.\zeta^2 - \sin.\zeta^2)}}{\sqrt{(\alpha.\cos.\zeta^2 - \gamma.\sin.\zeta^2)}} = \frac{ac\sqrt{2.\cos.2\zeta}}{\sqrt{((\alpha+\gamma).\cos.2\zeta + \alpha - \gamma)}}.$$

Horum ergo semiaxium productum erit

$$aD.eD = \frac{2aacc.\cos.2\zeta}{\sqrt{(2\alpha\gamma(1 + (\cos.2\zeta)^2) - \alpha\alpha + \gamma\gamma)(\sin.2\zeta)^2)}}.$$

At, cum sit

$$(\gamma - \alpha).\sin.2\zeta = 2\beta.\cos.2\zeta$$

erit

$$(\alpha\alpha + \gamma\gamma)(\sin.2\zeta)^2 = 4\beta\beta(\cos.2\zeta)^2 + 2\alpha\gamma(\sin.2\zeta)^2$$

ideoque

$$aD.eD = \frac{2aacc.\cos.2\zeta}{\sqrt{(4\alpha\gamma(\cos.2\zeta)^2 - 4\beta\beta(\cos.2\zeta)^2)}} = \frac{aacc}{\sqrt{(\alpha\gamma - \beta\beta)}} = \frac{ac}{\cos.\Phi}.$$

63. Simili modo, cum sint quadrata

$$aD^2 = \frac{2aacc.\cos.2\zeta}{(\alpha+\gamma).\cos.2\zeta - \alpha + \gamma}$$
&
$$eD^2 = \frac{2aacc.\cos.2\zeta}{(\alpha+\gamma).\cos.2\zeta + \alpha - \gamma},$$

erit

$$aD^2 + eD^2 = \frac{4aacc.(\alpha+\gamma)(\cos.2\zeta)^2}{4\alpha\gamma(\cos.2\zeta)^2 - 4\beta\beta(\cos 2\zeta)^2} = \frac{(\alpha+\gamma)aacc}{\alpha\gamma - \beta\beta}.$$

Hincque elicitur

APPEND. $aD + eD = \frac{ac\sqrt{(\alpha + \gamma + 2\sqrt{(\alpha\gamma - \beta\beta)})}}{\sqrt{(\alpha\gamma - \beta\beta)}}$

&

$$aD - eD = \frac{ac\sqrt{(\alpha + \gamma - 2\sqrt{(\alpha\gamma - \beta\beta)})}}{\sqrt{(\alpha\gamma - \beta\beta)}}$$

Semiaxes ergo aD & eD erunt radices hujus æquationis

$$(\alpha\gamma - \beta\beta)x^4 - (\alpha + \gamma)\, aaccxx + a^4c^4 = 0,$$

at eſt

$$\sqrt{(\alpha\gamma - \beta\beta)} = ac.\, cof.\, \Phi.$$

64. Cum ſit $aD.eD = \frac{ac}{cof.\,\Phi}$, atque Φ ſit angulus quem planum ſecans cum plano baſis conſtituit, hinc ſequens elegans Theorema conſequimur.

THEOREMA.

„Si Cylindrus quicunque ſecetur plano quocunque, erit rectangulum Axium ſectionis ad rectangulum Axium Baſis Cylindri, „uti ſecans anguli, quem planum ſectionis cum plano Baſis conſtituit, ad ſinum totum".

Quare, cum omnia parallelogramma circa Diametros conjugatas deſcripta æqualia ſint rectangulis ex Axibus formatis, etiam parallelogramma iſta circa Baſin & ſectionem quamcunque Cylindri formata eandem inter ſe tenebunt rationem.

TAB. XXXV. Fig. 133. 65. Natura autem hujuſmodi ſectionum obliquarum Cylindri commodius ſequenti modo definiri poterit. Si fuerit Baſis Cylindri Ellipſis $AEBF$, cujus ſemiaxes $AC = BC = a$, $EC = CF = c$, atque recta CD ad Centrum Baſis C perpendicularis Axis Cylindri: ſecetur iſte Cylindrus plano, cujus cum plano Baſis interſectio ſit recta TH ad Axem AB productum utcunque oblique poſita, ad quam ex C perpendiculum demittatur CH, ſitque angulus $GCH = \theta$. Tranſeat planum ſecans per Axis Cylindri punctum D; erit, ducta DH, angulus CHD inclinatio plani ſecantis ad planum Baſis, qui angulus vocetur $= \Phi$. Poſita ergo $CG = f$, erit

$GH =$