Here x is expressed not only by means of the rational operations, but also requires a square root. Note that, if $B^2 - 4AC = 0$, the line is tangent to the circle and, if $B^2 - 4AC < 0$, it does not meet the circle at all.

(5) To find the intersection of two circles, we must solve the pair of equations:

$$x^2 + y^2 + dx + ey + f = 0,$$
$$x^2 + y^2 + d'x + e'y + f' = 0.$$

By subtracting, we may here replace the second equation by the linear equation

$$(d - d')x + (e - e')y + f - f' = 0,$$

so the situation is the same as that already treated under (4) above. The last equation represents the straight line passing through the two points of intersection of the given circles, or, if the given circles merely touch, their common tangent. Of course, it may also happen that the two circles have no common points at all.

We had seen earlier that geometric constructions with ruler and compass allow us to carry out the rational operations and to extract square roots. We have now shown the converse: combinations of rational operations and square roots are the only arithmetical operations which can be carried out in this way. That is, when we perform ruler and compass constructions (1) to (5), we only get lengths that can be expressed in terms of the operations $+$, $-$, $\times$, $/$, and $\sqrt{}$. To solve problems **I** to **IV** by ruler and compass constructions is thus equivalent to expressing the real numbers $\sqrt[3]{2}$, $\cos 20°$, $\sqrt{\pi}$ and $\cos(360°/7)$ by rational operations and square roots.

Exercises

1. Show how to carry out the constructions (a) to (d) in the text, using ruler and compass only.

2. Carry out a ruler and compass construction of the golden section $(-1 + \sqrt{5})/2$.

3. To construct a regular heptagon one has to find the angle $\theta = 360°/7$. Show that $u = 2\cos\theta$ satisfies the cubic equation

$$u^3 + u^2 - 2u - 1 = 0.$$

15

The Impossibility of Solving the Classical Problems

The ancient Greeks were unable to solve problems **I** to **IV** using ruler and compass constructions, for a good reason: it cannot be done. Concerning problem **III**, this was shown only in 1882 by C.L.F. Lindemann (1852–1939), who proved that π is not an algebraic number, which implies, in particular, that $\sqrt{\pi}$ cannot be constructed by rational operations and square roots. His method was based on an earlier proof by Hermite, who had shown that $e = \lim_{n\to\infty}(1+\frac{1}{n})^n$ is not algebraic.

Problems **I**, **II** and **IV** have one thing in common: they can all be expressed by cubic equations, namely,

$$u^3 - 2 = 0, \quad u^3 - 3u - 1 = 0, \quad u^3 + u^2 - 2u - 1 = 0.$$

(For the last see Exercise 3 of Chapter 14.) First let us make sure that they have no rational solutions. (For the first equation we already know this.)

Lemma 15.1. *If $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$ is a polynomial equation with integer coefficients, then any rational solution is of the form p/q, with p a factor of a_0 and q a factor of a_n. In particular, when $a_n = 1$, any rational solution will be an integer and a factor of a_0.*

Proof: Let p/q be a rational solution, where $q \neq 0$ and $\gcd(p,q) = 1$. Putting $x = p/q$ in the equation and multiplying by q^n, we obtain

$$a_n p^n + a_{n-1}p^{n-1}q + \cdots + a_0 q^n = 0.$$

Since p divides all the terms except possibly the last, it must divide the last term also. Since $\gcd(p,q) = 1$, it follows from the unique factorization into primes that p divides a_0. Similarly, q divides a_n.