When R is commutative there is a natural map from R into $\text{End}(M)$ given by $r \mapsto rI$, where the latter endomorphism of M is just multiplication by r on M (cf. Exercise 7). The image of R is contained in the center of $\text{End}(M)$ so if R has an identity, $\text{End}(M)$ is an R-algebra. The ring homomorphism (cf. Exercise 7) from R to $\text{End}_R(M)$ may not be injective since for some r we may have $rm = 0$ for all $m \in M$ (e.g., $R = \mathbb{Z}$, $M = \mathbb{Z}/2\mathbb{Z}$, and $r = 2$). When R is a field, however, this map is injective (in general, no unit is in the kernel of this map) and the copy of R in $\text{End}_R(M)$ is called the (subring of) *scalar transformations.*

Next we prove that every submodule N of an R-module M is "normal" in the sense that we can *always* form the quotient module M/N, and the natural projection $\pi : M \to M/N$ is an R-module homomorphism with kernel N. The proof of this fact and, more generally, the subsequent proofs of the isomorphism theorems for modules follow easily from the corresponding facts for groups. The reason for this is because a module is first of all an *abelian* group and so *every* submodule is automatically a normal subgroup and any module homomorphism is, in particular, a homomorphism of abelian groups, all of which we have already considered in Chapter 3. What remains to be proved in order to extend results on abelian groups to corresponding results on modules is to check that the action of R is compatible with these group quotients and homomorphisms. For example, the map π above was shown to be a group homomorphism in Chapter 3 but the abelian group M/N must be shown to be an R-module (i.e., to have an action by R) and property (b) in the definition of a module homomorphism must be checked for π.

Proposition 3. Let R be a ring, let M be an R-module and let N be a submodule of M. The (additive, abelian) quotient group M/N can be made into an R-module by defining an action of elements of R by

$$r(x + N) = (rx) + N, \qquad \text{for all } r \in R,\ x + N \in M/N.$$

The natural projection map $\pi : M \to M/N$ defined by $\pi(x) = x + N$ is an R-module homomorphism with kernel N.

Proof: Since M is an abelian group under $+$ the quotient group M/N is defined and is an abelian group. To see that the action of the ring element r on the coset $x + N$ is well defined, suppose $x + N = y + N$, i.e., $x - y \in N$. Since N is a (left) R-submodule, $r(x - y) \in N$. Thus $rx - ry \in N$ and $rx + N = ry + N$, as desired. Now since the operations in M/N are "compatible" with those of M, the axioms for an R-module are easily checked in the same way as was done for quotient groups. For example, axiom 2(b) holds as follows: for all $r_1, r_2 \in R$ and $x + N \in M/N$, by definition of the action of ring elements on elements of M/N

$$\begin{aligned}(r_1r_2)(x + N) &= (r_1r_2x) + N \\ &= r_1(r_2x + N) \\ &= r_1(r_2(x + N)).\end{aligned}$$

The other axioms are similarly checked — the details are left as an exercise. Finally, the natural projection map π described above is, in particular, the natural projection of the abelian group M onto the abelian group M/N hence is a group homomorphism with kernel N. The kernel of any module homomorphism is the same as its kernel when viewed as a homomorphism of the abelian group structures. It remains only to show π is a module homomorphism, i.e., $\pi(rm) = r\pi(m)$. But

$$\begin{aligned}\pi(rm) &= rm + N \\ &= r(m + N) \qquad \text{(by definition of the action of } R \text{ on } M/N) \\ &= r\pi(m).\end{aligned}$$

This completes the proof.

All the isomorphism theorems stated for groups also hold for R-modules. The proofs are similar to that of Proposition 3 above in that they begin by invoking the corresponding theorem for groups and then prove that the group homomorphisms are also R-module homomorphisms. To state the Second Isomorphism Theorem we need the following.

Definition. Let A, B be submodules of the R-module M. The *sum* of A and B is the set

$$A + B = \{a + b \mid a \in A, b \in B\}.$$

One can easily check that the sum of two submodules A and B is a submodule and is the smallest submodule which contains both A and B.

Theorem 4. (Isomorphism Theorems)

(1) *(The First Isomorphism Theorem for Modules)* Let M, N be R-modules and let $\varphi : M \to N$ be an R-module homomorphism. Then $\ker \varphi$ is a submodule of M and $M/\ker \varphi \cong \varphi(M)$.

(2) *(The Second Isomorphism Theorem)* Let A, B be submodules of the R-module M. Then $(A + B)/B \cong A/(A \cap B)$.

(3) *(The Third Isomorphism Theorem)* Let M be an R-module, and let A and B be submodules of M with $A \subseteq B$. Then $(M/A)/(B/A) \cong M/B$.

(4) *(The Fourth or Lattice Isomorphism Theorem)* Let N be a submodule of the R-module M. There is a bijection between the submodules of M which contain N and the submodules of M/N. The correspondence is given by $A \leftrightarrow A/N$, for all $A \supseteq N$. This correspondence commutes with the processes of taking sums and intersections (i.e., is a lattice isomorphism between the lattice of submodules of M/N and the lattice of submodules of M which contain N).

Proof: Exercise.

EXERCISES

In these exercises R is a ring with 1 and M is a left R-module.

1. Use the submodule criterion to show that kernels and images of R-module homomorphisms are submodules.
2. Show that the relation "is R-module isomorphic to" is an equivalence relation on any set of R-modules.
3. Give an explicit example of a map from one R-module to another which is a group homomorphism but not an R-module homomorphism.
4. Let A be any $\mathbb{Z}$-module, let a be any element of A and let n be a positive integer. Prove that the map $\varphi_a : \mathbb{Z}/n\mathbb{Z} \to A$ given by $\varphi(\overline{k}) = ka$ is a well defined $\mathbb{Z}$-module homomorphism if and only if $na = 0$. Prove that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, A) \cong A_n$, where $A_n = \{a \in A \mid na = 0\}$ (so A_n is the annihilator in A of the ideal (n) of $\mathbb{Z}$ — cf. Exercise 10, Section 1).
5. Exhibit all $\mathbb{Z}$-module homomorphisms from $\mathbb{Z}/30\mathbb{Z}$ to $\mathbb{Z}/21\mathbb{Z}$.
6. Prove that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/(n, m)\mathbb{Z}$.
7. Let z be a fixed element of the center of R. Prove that the map $m \mapsto zm$ is an R-module homomorphism from M to itself. Show that for a commutative ring R the map from R to $\mathrm{End}_R(M)$ given by $r \mapsto rI$ is a ring homomorphism (where I is the identity endomorphism).
8. Let $\varphi : M \to N$ be an R-module homomorphism. Prove that $\varphi(\mathrm{Tor}(M)) \subseteq \mathrm{Tor}(N)$ (cf. Exercise 8 in Section 1).
9. Let R be a commutative ring. Prove that $\mathrm{Hom}_R(R, M)$ and M are isomorphic as left R-modules. [Show that each element of $\mathrm{Hom}_R(R, M)$ is determined by its value on the identity of R.]
10. Let R be a commutative ring. Prove that $\mathrm{Hom}_R(R, R)$ and R are isomorphic as rings.
11. Let $A_1, A_2, \ldots, A_n$ be R-modules and let B_i be a submodule of A_i for each $i = 1, 2, \ldots, n$. Prove that
$$(A_1 \times \cdots \times A_n)/(B_1 \times \cdots \times B_n) \cong (A_1/B_1) \times \cdots \times (A_n/B_n).$$
[Recall Exercise 14 in Section 5.1.]
12. Let I be a left ideal of R and let n be a positive integer. Prove
$$R^n/IR^n \cong R/IR \times \cdots \times R/IR \quad (n \text{ times})$$
where IR^n is defined as in Exercise 5 of Section 1. [Use the preceding exercise.]
13. Let I be a nilpotent ideal in a commutative ring R (cf. Exercise 37, Section 7.3), let M and N be R-modules and let $\varphi : M \to N$ be an R-module homomorphism. Show that if the induced map $\overline{\varphi} : M/IM \to N/IN$ is surjective, then φ is surjective.
14. Let $R = \mathbb{Z}[x]$ be the ring of polynomials in x and let $A = \mathbb{Z}[t_1, t_2, \ldots]$ be the ring of polynomials in the independent indeterminates $t_1, t_2, \ldots$. Define an action of R on A as follows: 1) let $1 \in R$ act on A as the identity, 2) for $n \geq 1$ let $x^n \circ 1 = t_n$, let $x^n \circ t_i = t_{n+i}$ for $i = 1, 2, \ldots$, and let x^n act as 0 on monomials in A of (total) degree at least two, and 3) extend $\mathbb{Z}$-linearly, i.e., so that the module axioms 2(a) and 2(c) are satisfied.
    (a) Show that $x^{p+q} \circ t_i = x^p \circ (x^q \circ t_i) = t_{p+q+i}$ and use this to show that under this action the ring A is a (unital) R-module.
    (b) Show that the map $\varphi : R \to A$ defined by $\varphi(r) = r \circ 1_A$ is an R-module homomorphism of the ring R into the ring A mapping 1_R to 1_A, but is not a ring homomorphism from R to A.