

The following are easy consequences of the definition.

1. Any set which contains a linearly dependent set is linearly dependent.
2. Any subset of a linearly independent set is linearly independent.
3. Any set which contains the  $0$  vector is linearly dependent; for  $1 \cdot 0 = 0$ .
4. A set  $S$  of vectors is linearly independent if and only if each finite subset of  $S$  is linearly independent, i.e., if and only if for any distinct vectors  $\alpha_1, \dots, \alpha_n$  of  $S$ ,  $c_1\alpha_1 + \dots + c_n\alpha_n = 0$  implies each  $c_i = 0$ .

**Definition.** Let  $V$  be a vector space. A **basis** for  $V$  is a linearly independent set of vectors in  $V$  which spans the space  $V$ . The space  $V$  is **finite-dimensional** if it has a finite basis.

EXAMPLE 12. Let  $F$  be a subfield of the complex numbers. In  $F^3$  the vectors

$$\begin{aligned}\alpha_1 &= (3, 0, -3) \\ \alpha_2 &= (-1, 1, 2) \\ \alpha_3 &= (4, 2, -2) \\ \alpha_4 &= (2, 1, 1)\end{aligned}$$

are linearly dependent, since

$$2\alpha_1 + 2\alpha_2 - \alpha_3 + 0 \cdot \alpha_4 = 0.$$

The vectors

$$\begin{aligned}\epsilon_1 &= (1, 0, 0) \\ \epsilon_2 &= (0, 1, 0) \\ \epsilon_3 &= (0, 0, 1)\end{aligned}$$

are linearly independent

EXAMPLE 13. Let  $F$  be a field and in  $F^n$  let  $S$  be the subset consisting of the vectors  $\epsilon_1, \epsilon_2, \dots, \epsilon_n$  defined by

$$\begin{aligned}\epsilon_1 &= (1, 0, 0, \dots, 0) \\ \epsilon_2 &= (0, 1, 0, \dots, 0) \\ &\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \epsilon_n &= (0, 0, 0, \dots, 1).\end{aligned}$$

Let  $x_1, x_2, \dots, x_n$  be scalars in  $F$  and put  $\alpha = x_1\epsilon_1 + x_2\epsilon_2 + \dots + x_n\epsilon_n$ . Then

$$(2-12) \quad \alpha = (x_1, x_2, \dots, x_n).$$

This shows that  $\epsilon_1, \dots, \epsilon_n$  span  $F^n$ . Since  $\alpha = 0$  if and only if  $x_1 = x_2 = \dots = x_n = 0$ , the vectors  $\epsilon_1, \dots, \epsilon_n$  are linearly independent. The set  $S = \{\epsilon_1, \dots, \epsilon_n\}$  is accordingly a basis for  $F^n$ . We shall call this particular basis the **standard basis** of  $F^n$ .

EXAMPLE 14. Let  $P$  be an invertible  $n \times n$  matrix with entries in the field  $F$ . Then  $P_1, \dots, P_n$ , the columns of  $P$ , form a basis for the space of column matrices,  $F^{n \times 1}$ . We see that as follows. If  $X$  is a column matrix, then

$$PX = x_1P_1 + \dots + x_nP_n.$$

Since  $PX = 0$  has only the trivial solution  $X = 0$ , it follows that  $\{P_1, \dots, P_n\}$  is a linearly independent set. Why does it span  $F^{n \times 1}$ ? Let  $Y$  be any column matrix. If  $X = P^{-1}Y$ , then  $Y = PX$ , that is,

$$Y = x_1P_1 + \dots + x_nP_n.$$

So  $\{P_1, \dots, P_n\}$  is a basis for  $F^{n \times 1}$ .

EXAMPLE 15. Let  $A$  be an  $m \times n$  matrix and let  $S$  be the solution space for the homogeneous system  $AX = 0$  (Example 7). Let  $R$  be a row-reduced echelon matrix which is row-equivalent to  $A$ . Then  $S$  is also the solution space for the system  $RX = 0$ . If  $R$  has  $r$  non-zero rows, then the system of equations  $RX = 0$  simply expresses  $r$  of the unknowns  $x_1, \dots, x_n$  in terms of the remaining  $(n - r)$  unknowns  $x_j$ . Suppose that the leading non-zero entries of the non-zero rows occur in columns  $k_1, \dots, k_r$ . Let  $J$  be the set consisting of the  $n - r$  indices different from  $k_1, \dots, k_r$ :

$$J = \{1, \dots, n\} - \{k_1, \dots, k_r\}.$$

The system  $RX = 0$  has the form

$$\begin{array}{ccc} x_{k_1} + \sum_J c_{1j}x_j & = & 0 \\ \vdots & & \vdots \\ x_{k_r} + \sum_J c_{rj}x_j & = & 0 \end{array}$$

where the  $c_{ij}$  are certain scalars. All solutions are obtained by assigning (arbitrary) values to those  $x_j$ 's with  $j$  in  $J$  and computing the corresponding values of  $x_{k_1}, \dots, x_{k_r}$ . For each  $j$  in  $J$ , let  $E_j$  be the solution obtained by setting  $x_j = 1$  and  $x_i = 0$  for all other  $i$  in  $J$ . We assert that the  $(n - r)$  vectors  $E_j, j$  in  $J$ , form a basis for the solution space.

Since the column matrix  $E_j$  has a 1 in row  $j$  and zeros in the rows indexed by other elements of  $J$ , the reasoning of Example 13 shows us that the set of these vectors is linearly independent. That set spans the solution space, for this reason. If the column matrix  $T$ , with entries  $t_1, \dots, t_n$ , is in the solution space, the matrix

$$N = \sum_J t_j E_j$$

is also in the solution space and is a solution such that  $x_j = t_j$  for each  $j$  in  $J$ . The solution with that property is unique; hence,  $N = T$  and  $T$  is in the span of the vectors  $E_j$ .

EXAMPLE 16. We shall now give an example of an infinite basis. Let  $F$  be a subfield of the complex numbers and let  $V$  be the space of polynomial functions over  $F$ . Recall that these functions are the functions from  $F$  into  $F$  which have a rule of the form

$$f(x) = c_0 + c_1x + \cdots + c_nx^n.$$

Let  $f_k(x) = x_k$ ,  $k = 0, 1, 2, \dots$ . The (infinite) set  $\{f_0, f_1, f_2, \dots\}$  is a basis for  $V$ . Clearly the set spans  $V$ , because the function  $f$  (above) is

$$f = c_0f_0 + c_1f_1 + \cdots + c_nf_n.$$

The reader should see that this is virtually a repetition of the definition of polynomial function, that is, a function  $f$  from  $F$  into  $F$  is a polynomial function if and only if there exists an integer  $n$  and scalars  $c_0, \dots, c_n$  such that  $f = c_0f_0 + \cdots + c_nf_n$ . Why are the functions independent? To show that the set  $\{f_0, f_1, f_2, \dots\}$  is independent means to show that each finite subset of it is independent. It will suffice to show that, for each  $n$ , the set  $\{f_0, \dots, f_n\}$  is independent. Suppose that

$$c_0f_0 + \cdots + c_nf_n = 0.$$

This says that

$$c_0 + c_1x + \cdots + c_nx^n = 0$$

for every  $x$  in  $F$ ; in other words, every  $x$  in  $F$  is a root of the polynomial  $f(x) = c_0 + c_1x + \cdots + c_nx^n$ . We assume that the reader knows that a polynomial of degree  $n$  with complex coefficients cannot have more than  $n$  distinct roots. It follows that  $c_0 = c_1 = \cdots = c_n = 0$ .

We have exhibited an infinite basis for  $V$ . Does that mean that  $V$  is not finite-dimensional? As a matter of fact it does; however, that is not immediate from the definition, because for all we know  $V$  might also have a finite basis. That possibility is easily eliminated. (We shall eliminate it in general in the next theorem.) Suppose that we have a finite number of polynomial functions  $g_1, \dots, g_r$ . There will be a largest power of  $x$  which appears (with non-zero coefficient) in  $g_1(x), \dots, g_r(x)$ . If that power is  $k$ , clearly  $f_{k+1}(x) = x^{k+1}$  is not in the linear span of  $g_1, \dots, g_r$ . So  $V$  is not finite-dimensional.

A final remark about this example is in order. Infinite bases have nothing to do with 'infinite linear combinations.' The reader who feels an irresistible urge to inject power series

$$\sum_{k=0}^{\infty} c_kx^k$$

into this example should study the example carefully again. If that does not effect a cure, he should consider restricting his attention to finite-dimensional spaces from now on.

**Theorem 4.** Let  $V$  be a vector space which is spanned by a finite set of vectors  $\beta_1, \beta_2, \dots, \beta_m$ . Then any independent set of vectors in  $V$  is finite and contains no more than  $m$  elements.

*Proof.* To prove the theorem it suffices to show that every subset  $S$  of  $V$  which contains more than  $m$  vectors is linearly dependent. Let  $S$  be such a set. In  $S$  there are distinct vectors  $\alpha_1, \alpha_2, \dots, \alpha_n$  where  $n > m$ . Since  $\beta_1, \dots, \beta_m$  span  $V$ , there exist scalars  $A_{ij}$  in  $F$  such that

$$\alpha_j = \sum_{i=1}^m A_{ij} \beta_i.$$

For any  $n$  scalars  $x_1, x_2, \dots, x_n$  we have

$$\begin{aligned} x_1 \alpha_1 + \dots + x_n \alpha_n &= \sum_{j=1}^n x_j \alpha_j \\ &= \sum_{j=1}^n x_j \sum_{i=1}^m A_{ij} \beta_i \\ &= \sum_{j=1}^n \sum_{i=1}^m (A_{ij} x_j) \beta_i \\ &= \sum_{i=1}^m \left( \sum_{j=1}^n A_{ij} x_j \right) \beta_i. \end{aligned}$$

Since  $n > m$ , Theorem 6 of Chapter 1 implies that there exist scalars  $x_1, x_2, \dots, x_n$  not all 0 such that

$$\sum_{j=1}^n A_{ij} x_j = 0, \quad 1 \leq i \leq m.$$

Hence  $x_1 \alpha_1 + x_2 \alpha_2 + \dots + x_n \alpha_n = 0$ . This shows that  $S$  is a linearly dependent set. ■

**Corollary 1.** If  $V$  is a finite-dimensional vector space, then any two bases of  $V$  have the same (finite) number of elements.

*Proof.* Since  $V$  is finite-dimensional, it has a finite basis

$$\{\beta_1, \beta_2, \dots, \beta_m\}.$$

By Theorem 4 every basis of  $V$  is finite and contains no more than  $m$  elements. Thus if  $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$  is a basis,  $n \leq m$ . By the same argument,  $m \leq n$ . Hence  $m = n$ . ■

This corollary allows us to define the **dimension** of a finite-dimensional vector space as the number of elements in a basis for  $V$ . We shall denote the dimension of a finite-dimensional space  $V$  by  $\dim V$ . This allows us to reformulate Theorem 4 as follows.

**Corollary 2.** Let  $V$  be a finite-dimensional vector space and let  $n = \dim V$ . Then

- (a) any subset of  $V$  which contains more than  $n$  vectors is linearly dependent;  
 (b) no subset of  $V$  which contains fewer than  $n$  vectors can span  $V$ .

**EXAMPLE 17.** If  $F$  is a field, the dimension of  $F^n$  is  $n$ , because the standard basis for  $F^n$  contains  $n$  vectors. The matrix space  $F^{m \times n}$  has dimension  $mn$ . That should be clear by analogy with the case of  $F^n$ , because the  $mn$  matrices which have a 1 in the  $i, j$  place with zeros elsewhere form a basis for  $F^{m \times n}$ . If  $A$  is an  $m \times n$  matrix, then the solution space for  $A$  has dimension  $n - r$ , where  $r$  is the number of non-zero rows in a row-reduced echelon matrix which is row-equivalent to  $A$ . See Example 15.

If  $V$  is any vector space over  $F$ , the zero subspace of  $V$  is spanned by the vector 0, but  $\{0\}$  is a linearly dependent set and not a basis. For this reason, we shall agree that the zero subspace has dimension 0. Alternatively, we could reach the same conclusion by arguing that the empty set is a basis for the zero subspace. The empty set spans  $\{0\}$ , because the intersection of all subspaces containing the empty set is  $\{0\}$ , and the empty set is linearly independent because it contains no vectors.

**Lemma.** Let  $S$  be a linearly independent subset of a vector space  $V$ . Suppose  $\beta$  is a vector in  $V$  which is not in the subspace spanned by  $S$ . Then the set obtained by adjoining  $\beta$  to  $S$  is linearly independent.

*Proof.* Suppose  $\alpha_1, \dots, \alpha_m$  are distinct vectors in  $S$  and that

$$c_1\alpha_1 + \dots + c_m\alpha_m + b\beta = 0.$$

Then  $b = 0$ ; for otherwise,

$$\beta = \left(-\frac{c_1}{b}\right)\alpha_1 + \dots + \left(-\frac{c_m}{b}\right)\alpha_m$$

and  $\beta$  is in the subspace spanned by  $S$ . Thus  $c_1\alpha_1 + \dots + c_m\alpha_m = 0$ , and since  $S$  is a linearly independent set each  $c_i = 0$ . ■

**Theorem 5.** If  $W$  is a subspace of a finite-dimensional vector space  $V$ , every linearly independent subset of  $W$  is finite and is part of a (finite) basis for  $W$ .

*Proof.* Suppose  $S_0$  is a linearly independent subset of  $W$ . If  $S$  is a linearly independent subset of  $W$  containing  $S_0$ , then  $S$  is also a linearly independent subset of  $V$ ; since  $V$  is finite-dimensional,  $S$  contains no more than  $\dim V$  elements.

We extend  $S_0$  to a basis for  $W$ , as follows. If  $S_0$  spans  $W$ , then  $S_0$  is a basis for  $W$  and we are done. If  $S_0$  does not span  $W$ , we use the preceding lemma to find a vector  $\beta_1$  in  $W$  such that the set  $S_1 = S_0 \cup \{\beta_1\}$  is independent. If  $S_1$  spans  $W$ , fine. If not, apply the lemma to obtain a vector  $\beta_2$

in  $W$  such that  $S_2 = S_1 \cup \{\beta_2\}$  is independent. If we continue in this way, then (in not more than  $\dim V$  steps) we reach a set

$$S_m = S_0 \cup \{\beta_1, \dots, \beta_m\}$$

which is a basis for  $W$ . ■

**Corollary 1.** *If  $W$  is a proper subspace of a finite-dimensional vector space  $V$ , then  $W$  is finite-dimensional and  $\dim W < \dim V$ .*

*Proof.* We may suppose  $W$  contains a vector  $\alpha \neq 0$ . By Theorem 5 and its proof, there is a basis of  $W$  containing  $\alpha$  which contains no more than  $\dim V$  elements. Hence  $W$  is finite-dimensional, and  $\dim W \leq \dim V$ . Since  $W$  is a proper subspace, there is a vector  $\beta$  in  $V$  which is not in  $W$ . Adjoining  $\beta$  to any basis of  $W$ , we obtain a linearly independent subset of  $V$ . Thus  $\dim W < \dim V$ . ■

**Corollary 2.** *In a finite-dimensional vector space  $V$  every non-empty linearly independent set of vectors is part of a basis.*

**Corollary 3.** *Let  $A$  be an  $n \times n$  matrix over a field  $F$ , and suppose the row vectors of  $A$  form a linearly independent set of vectors in  $F^n$ . Then  $A$  is invertible.*

*Proof.* Let  $\alpha_1, \alpha_2, \dots, \alpha_n$  be the row vectors of  $A$ , and suppose  $W$  is the subspace of  $F^n$  spanned by  $\alpha_1, \alpha_2, \dots, \alpha_n$ . Since  $\alpha_1, \alpha_2, \dots, \alpha_n$  are linearly independent, the dimension of  $W$  is  $n$ . Corollary 1 now shows that  $W = F^n$ . Hence there exist scalars  $B_{ij}$  in  $F$  such that

$$\epsilon_i = \sum_{j=1}^n B_{ij} \alpha_j, \quad 1 \leq i \leq n$$

where  $\{\epsilon_1, \epsilon_2, \dots, \epsilon_n\}$  is the standard basis of  $F^n$ . Thus for the matrix  $B$  with entries  $B_{ij}$  we have

$$BA = I. \quad \blacksquare$$

**Theorem 6.** *If  $W_1$  and  $W_2$  are finite-dimensional subspaces of a vector space  $V$ , then  $W_1 + W_2$  is finite-dimensional and*

$$\dim W_1 + \dim W_2 = \dim (W_1 \cap W_2) + \dim (W_1 + W_2).$$

*Proof.* By Theorem 5 and its corollaries,  $W_1 \cap W_2$  has a finite basis  $\{\alpha_1, \dots, \alpha_k\}$  which is part of a basis

$$\{\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_m\} \quad \text{for } W_1$$

and part of a basis

$$\{\alpha_1, \dots, \alpha_k, \gamma_1, \dots, \gamma_n\} \quad \text{for } W_2.$$

The subspace  $W_1 + W_2$  is spanned by the vectors

$$\alpha_1, \dots, \alpha_k, \quad \beta_1, \dots, \beta_m, \quad \gamma_1, \dots, \gamma_n$$