taken to be complex. This was done in the nineteenth century, and on this basis a more rigorous derivation of Newton's series was given by Puiseux (1850). For this reason, the fractional power-series expansions of algebraic functions are now called *Puiseux expansions*.

EXERCISE

The impossibility of an ordinary power series for $x^{1/2}$ can be shown as follows.

10.5.1 Any ordinary power-series expansion of $x^{1/2}$ would have to be of the form

$$x^{1/2} = a_1x + a_2x^2 + a_3x^3 + \cdots$$

because $x^{1/2} = 0$ when $x = 0$. Now square both sides and deduce a contradiction.

10.6 Generating Functions

Fibonacci (1202) introduced a famous sequence now known as the *Fibonacci sequence*

$$1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

in which each term (after the first two) is the sum of two preceding terms. Despite this simple law of formation, there is no obvious formula for the nth term of the sequence. Such a formula was only discovered more than 500 years later, by de Moivre (1730), and in doing so de Moivre introduced a powerful new application of infinite series, the method of *generating functions*. This method, which is of great importance in combinatorics, probability, and number theory, will be illustrated using the Fibonacci sequence itself.

It is technically convenient to begin with $F_0 = 0$ and $F_1 = 1$, then take subsequent terms as above (so $F_2 = 1$, $F_3 = 2$, $F_4 = 3, \ldots$) by defining

$$F_{n+2} = F_{n+1} + F_n \quad \text{for} \quad n \geq 0.$$

This is an example of *linear recurrence relation*, and it was to solve such relations in probability theory that de Moivre introduced generating functions. The generating function for the Fibonacci sequence is

$$f(x) = F_0 + F_1x + F_2x^2 + F_3x^3 + \cdots.$$

We notice that

$$\begin{aligned} xf(x) &= F_0x + F_1x^2 + F_2x^3 + \cdots, \\ x^2 f(x) &= \qquad\quad F_0x^2 + F_1x^3 + \cdots. \end{aligned}$$

Hence

$$\begin{aligned} f(x) - xf(x) - x^2 f(x) = F_0 + F_1x - F_0x \\ + (F_2 - F_1 - F_0)x^2 \\ + (F_3 - F_2 - F_1)x^3 \\ + \cdots, \end{aligned}$$

that is, $f(x)(1-x-x^2) = F_0 + F_1x - F_0x = x$ because all the coefficients $F_{n+2} - F_{n+1} - F_n = 0$ by definition of the Fibonacci sequence. Thus

$$f(x) = \frac{x}{1-x-x^2},$$

and using the roots $(-1 \pm \sqrt{5})/2 = 2/(1 \pm \sqrt{5})$ of $1 - x - x^2 = 0$ to factorize the denominator we get

$$f(x) = \frac{x}{[1-((1+\sqrt{5})/2)x][1-((1-\sqrt{5})/2)x]}.$$

Then splitting into partial fractions

$$f(x) = \frac{1}{\sqrt{5}}\left[\frac{1}{1-((1+\sqrt{5})/2)x} - \frac{1}{1-((1-\sqrt{5})/2)x}\right],$$

and using the geometric series expansions

$$\begin{aligned} \frac{1}{1-((1+\sqrt{5})/2)x} &= 1 + \frac{1+\sqrt{5}}{2}x + \left(\frac{1+\sqrt{5}}{2}\right)^2 x^2 + \cdots, \\ \frac{1}{1-((1-\sqrt{5})/2)x} &= 1 + \frac{1-\sqrt{5}}{2}x + \left(\frac{1-\sqrt{5}}{2}\right)^2 x^2 + \cdots, \end{aligned}$$

we finally get

$$\begin{aligned} f(x) = &\frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right]x + \cdots \\ &+ \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]x^n + \cdots. \end{aligned}$$

Equating this with the definition $f(x) = F_0 + F_1x + F_2x^2 + \cdots$ gives

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]. \tag{1}$$

No wonder a formula for F_n was hard to find! One would not have expected the irrational $\sqrt{5}$ to be involved in the integer-valued function F_n. The explanation is that the Fibonacci sequence actually defines $\sqrt{5}$, because $F_{n+1}/F_n \to (1+\sqrt{5})/2$ (the golden ratio) as $n \to \infty$, so (1) in effect defines the individual terms of the Fibonacci sequence in terms of the sequence as a whole (or, if one prefers, in terms of the behavior of the sequence at infinity). The remarkable fact that the definition of F_n becomes explicit, rather than recursive, when expressed in terms of $(1+\sqrt{5})/2$ is due to the simplicity of the generating function $f(x)$, which encodes the whole sequence.

The recursive property of Fibonacci numbers used in de Moivre's proof is that they satisfy a linear recurrence relation; that is, F_n is expressed as a fixed linear combination of earlier terms in the sequence. The proof is easily generalized to show that the generating function $\sum a_n x^n$ of any sequence $\{a_n\}$ defined by a linear recurrence relation is rational. Also, the proof can be reversed to show that the power series of any rational function has coefficients that satisfy a linear recurrence relation. Thus rational functions can be characterized in terms of their power series, a fact that was noticed by Kronecker (1881), Section IX.

Exercises

The formula $F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$ gives several interesting limit and approximation properties of F_n. For example:

10.6.1 Show $\frac{F_{n+1}}{F_n} \to \frac{1+\sqrt{5}}{2}$ as $n \to \infty$.

10.6.2 Show that $F_n =$ nearest integer to $\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$.

10.6.3 Using $1/(1+F_n/F_{n+1}) = F_{n+1}/F_{n+2}$, or otherwise, show that

$$\frac{1+\sqrt{5}}{2} = 1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}.$$

10.7 The Zeta Function

The purpose of a generating function is to encode a complicated sequence by a function (of a real or complex variable) that is in some ways simpler. The method of encoding need not be as direct as taking the nth term of the sequence to be the coefficient of x^n. For example, a famous *product formula* of Euler (1748a), p. 288, encodes the sequence $2, 3, 5, 7, 11, \ldots$, of prime numbers as the following sum of powers of $1, 2, 3, 4, \ldots$

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots$$

(the *zeta function*). Euler's formula is

$$\frac{1}{(1-1/2^s)}\frac{1}{(1-1/3^s)}\frac{1}{(1-1/5^s)}\frac{1}{(1-1/7^s)}\frac{1}{(1-1/11^s)}\cdots$$
$$= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots.$$

The factors on the left-hand side are $(1 - 1/p_n^s)^{-1}$, where p_n is the nth prime. We expand each such factor as a geometric series

$$1 + \frac{1}{p_n^s} + \frac{1}{p_n^{2s}} + \frac{1}{p_n^{3s}} + \cdots.$$

Multiplying all these series together, we get the reciprocal of each possible product of primes, to the sth power, exactly once. That is, the left-hand side is the sum

$$1 + \sum \frac{1}{p_1^{m_1 s} p_2^{m_2 s} \cdots p_r^{m_r s}} = 1 + \sum \frac{1}{(p_1^{m_1} p_2^{m_2} \cdots p_r^{m_r})^s}$$

in which each product $p_1^{m_1} p_2^{m_2} \cdots p_r^{m_r}$ of primes occurs exactly once. But each natural number ≥ 2 is expressible in just one way as a product of primes (Section 3.3), hence the latter sum equals the right-hand side of Euler's formula

$$1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots.$$

Initially the exponent $s > 1$ was there only to ensure convergence. We saw in Section 10.1 that $\zeta(s)$ diverges when $s = 1$; it converges when $s > 1$. Riemann (1859) discovered that $\zeta(s)$ becomes much more powerful when s is taken to be a complex variable. In recognition of this, $\zeta(s)$ is often called the *Riemann* zeta function. Euler's result of Section 10.4 can be rephrased as $\zeta(2) = \pi^2/6$. The values of $\zeta(4)$, $\zeta(6)$, $\zeta(8)$, ... were also found by Euler and turn out to be rational multiples of π^4, π^6, π^8, ..., respectively. The values of $\zeta(3)$, $\zeta(5)$, ... have no known relationship to π or other standard constants, though Apéry (1981) showed that $\zeta(3)$ is irrational. The most famous conjecture about $\zeta(s)$, and one of the most sought-after results in mathematics today, is the so-called *Riemann hypothesis*: $\zeta(s) = 0$ only when $\mathrm{Re}(s) = \frac{1}{2}$.

EXERCISES

Although $\zeta(s)$ is not defined for $s = 1$ (because this gives the divergent series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$), this situation can be exploited to give a new proof that there are infinitely many primes. (Thus the Euler product formula encapsulates two apparently unrelated results—unique prime factorization, and the infinite number of primes.)

10.7.1 (Euler) Show that if there are only finitely many primes $p_1, \ldots, p_n$, then

$$\frac{1}{1-1/p_1} \cdot \frac{1}{1-1/p_2} \cdot \cdots \cdot \frac{1}{1-1/p_n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots.$$

Deduce that there are infinitely many primes.

The statement of the Riemann hypothesis needs some qualification, because $\zeta(s)$ can be defined for certain values of s for which the series $1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots$ is not meaningful. This follows from the formula

$$\zeta(1-s) = 2(2\pi)^{-s} \cos\frac{s\pi}{2} \Gamma(s)\zeta(s)$$

discovered by Riemann and called the *functional equation* for the zeta function. The functional equation enables us to define $\zeta(1-s)$ when $\zeta(s)$ is known, and it also shows that there are certain "trivial zeros" of $\zeta(1-s)$, namely, where $\cos\frac{s\pi}{2} = 0$. The trivial zeros are ignored in the statement of the Riemann hypothesis.

10.7.2 Which s give a trivial zero of $\zeta(1-s)$?

The function Γ in the functional equation is the *gamma function*, introduced by Euler to generalize the factorial function: $\Gamma(n) = (n-1)!$ for integer values of n. An amusing consequence of the functional equation is that we can assign values to certain divergent series, such as $1+2+3+4+\cdots$, by interpreting them as $\zeta(1-s)$, then reinterpreting $\zeta(1-s)$ by the functional equation.

10.7.3 By suitable reinterpretation, show that

$$1+2+3+4+\cdots = -\frac{1}{12}.$$

10.8 Biographical Notes: Gregory and Euler

James Gregory was born in 1638 in Drumoak, near Aberdeen, the youngest of three sons of John Gregory, the town's minister. He received his early education from his mother, Janet Anderson, whose uncle Alexander had been secretary to Viète and editor of Viète's posthumously published works. The middle brother, David, also had mathematical ability and after their father's death in 1651, he encouraged James in his subsequent studies at grammar school and Marischal College in Aberdeen. Marischal College now possesses the only known portrait of James Gregory (Figure 10.3).

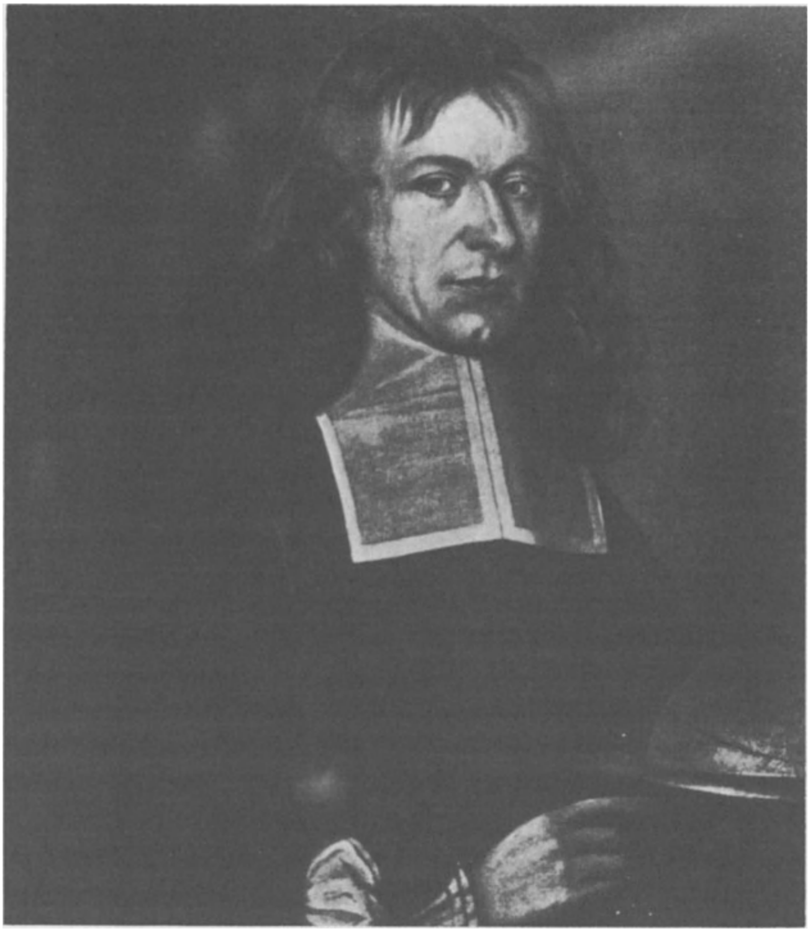

Figure 10.3: James Gregory (Marischal College)