Jyesthadeva credits Madhava (1340–1425). The proof given by Jyesthadeva is based on a geometric lemma and a limit calculation involving sums of powers of integers. Actually, a more general result is proved, which we would call the infinite series for the inverse tan function:

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots .$$

The formula for π results by substituting $x = 1$, because $\tan^{-1} 1 = \pi/4$. A reconstruction of the proof may be found in Katz (1993), pp. 452–453.

The formula was rediscovered in Europe by James Gregory and Leibniz around 1670, using calculus. For readers familiar with calculus, it should be mentioned that the fundamental theorem of calculus makes a dramatic simplification in the Indian proof, replacing the awkward problem of evaluating

$$\lim_{n\to\infty} \frac{1^{2i} + 2^{2i} + \cdots + (n-1)^{2i}}{n^{2i+1}}$$

by the integration of x^{2i}, which every beginner in calculus can do. Somewhat earlier, before the fundamental theorem of calculus was known, Wallis (1655) used some ingenious guesswork to discover an expression for π as an *infinite product*:

$$\frac{\pi}{4} = \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots .$$

This result can be obtained rigorously by the technique of integration by parts and is now a common exercise in calculus textbooks. Wallis' colleague Brouncker managed to transform the infinite product into the infinite *continued fraction*

$$\frac{4}{\pi} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \ddots}}}},$$

and this result was also reported by Wallis (1655). (We'll say more about continued fractions in Chapter 8, because they are also of great interest in the study of square roots.)

Brouncker's continued fraction is not of the standard type, which has all the numerators equal to 1, and in fact the standard continued fraction,

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \ddots}}}},$$

does not have any discernible pattern of denominators. It is nevertheless an interesting curiosity that the truncated fraction,

$$3 + \cfrac{1}{7 + \cfrac{1}{15 + \frac{1}{1}}},$$

is precisely Zǔ Chōngzhi's approximation to π, 355/113. The exceptional accuracy of this approximation is partly due to stopping just before the large denominator 292.

Euler (1748) linked Brouncker's continued fraction, and hence Wallis's infinite product, to the Indian series for $\pi/4$ by transforming the series into the continued fraction. Thus Wallis' product is in some sense a rediscovery of the formula

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots,$$

or at least confirmation of its fundamental nature.

Euler (1748) also found a whole family of formulas for even powers of π, starting with

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$
$$\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots$$
$$\frac{\pi^6}{945} = \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \cdots.$$

He transformed these into infinite products involving the prime numbers, for example,

$$\frac{\pi^2}{6} = \frac{1}{1-2^{-2}} \cdot \frac{1}{1-3^{-2}} \cdot \frac{1}{1-7^{-2}} \cdot \frac{1}{1-11^{-2}} \cdots,$$

using the wonderful *Euler product formula*

$$\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots = \frac{1}{1-2^{-s}} \cdot \frac{1}{1-3^{-s}} \cdot \frac{1}{1-7^{-s}} \cdot \frac{1}{1-11^{-s}} \cdots,$$

which is valid for any $s > 1$, and is essentially equivalent to unique prime factorization.

(This last remark is supposed to tempt you to look for an explanation of the Euler product formula. You can find it by expanding $\frac{1}{1-p^{-1}}$ as a geometric series

$$\frac{1}{1-p^{-s}} = 1 + p^{-s} + p^{-2s} + p^{-3s} + \cdots .$$

Then observe that the product of these series, for all primes p, includes each term n^{-s} exactly once, because each $n > 1$ equals exactly one product of primes. This formula shows, incidentally, that mathematicians were aware of unique prime factorization well before it was explicitly stated by Gauss in 1801.)

Additive Functions and the Axiom of Choice

The construction of additive functions in Section 5.7* is tailored to solve Hilbert's third problem and no more; it sidesteps the awkward question: *is there a nonconstant additive function f, defined on all of $\mathbb{R}$, such that $f(\pi) = 0$?* However, this is a very interesting question. We shall therefore explore it a little further, if only because a *yes* answer could simplify the solution of Hilbert's third problem.

Suppose that $f(\pi) = 0$ and f is additive, that is, $f(\alpha+\beta) = f(\alpha)+f(\beta)$ for all real numbers α and β. It follows that $f(\pi/2) = 0$ also, because

$$0 = f(\pi) = f(\pi/2) + f(\pi/2) = 2f(\pi/2).$$

It similarly follows that $f(\pi/n) = 0$ for any positive integer n, and hence

$$f(m\pi/n) = mf(\pi/n) = 0$$

for any integer m. Thus in fact $f(r\pi) = 0$ for any rational number r. But the number $r\pi$ can be made arbitrarily close to any real number we please, by suitable choice of the rational number r. Hence, if f is a continuous function, it can only be the constant function 0.

Thus we must certainly give up the idea of looking for continuous additive functions, and we cannot avoid f being zero at all rational multiples $r\pi$ of π. Fortunately, there are many numbers outside this set, for example, the nonzero rationals. We can choose $f(a_1)$ to be any value we like, say, 1, when a_1 is not of the form $r\pi$ for a rational r. The value of $f(a_1)$ determines the value $f(r'a_1) = r'f(a_1)$ for any rational r', and hence (by additivity again) the value of $f(r\pi + r'a_1)$ for any pair of rationals r, r'.

Now there are countably many pairs of rationals (by an argument like that used in Section 3.10 to prove there are countably many fractions), and hence countably many numbers $r\pi + r'a_1$. Numbers exist outside this set, by the uncountability of $\mathbb{R}$ proved in Section 3.10, so we are still free to choose the value of f on any one of them, say, a_2. This determines the value of f on the countably many numbers $r\pi + r'a_1 + r''a_2$ where r, r', and r'' are rational, and so on. As long as the values on which f is defined do not exhaust $\mathbb{R}$, we can choose a number a on which f is undefined, and give $f(a)$ any value we like.

The trouble is, doing this for infinitely many values $a_1, a_2, a_3, \ldots$ will *not* exhaust $\mathbb{R}$, precisely because $\mathbb{R}$ is uncountable. Our intuition balks at continuing the sequence of choices to uncountable length, and the best we can do is *assume* that it is possible. This assumption (in a more precise form, of course) is called the *axiom of choice*.

No matter how we approach the problem of additive functions on $\mathbb{R}$, the axiom of choice is needed. For example, we could approach the problem as we did for finite additive functions in Section 5.7*, by constructing a *basis over* $\mathbb{Q}$ for $\mathbb{R}$. Choose any real β_1 to be the first member of the basis, and more generally let β_{n+1} be any real that is not one of the countably many rational combinations of $\beta_1, \beta_2, \ldots, \beta_n$. The axiom of choice allows us to assume that $\mathbb{R}$ can be exhausted by extending the sequence of choices to uncountable length. If so, we then have a basis over $\mathbb{Q}$ for $\mathbb{R}$, and each real number x is a unique rational combination of basis members.

If β_i is any basis member, it follows as in Section 5.7* that the following function f_i is additive.

$$f_i(x) = \text{coefficient of } \beta_i \text{ in the expression for } x \\ \text{as a rational combination of basis elements.}$$

We can therefore use the functions f_i as before to solve Hilbert's third problem. This argument is more in the mathematical mainstream, because it is quite usual to assume the existence of bases.

On the other hand, one does not want to assume the axiom of choice unnecessarily, because it is a dubious axiom.

In fact, the situation of the axiom of choice is not unlike the situation of Euclid's parallel axiom before the discovery of non-Euclidean geometry. The axiom of choice is not as natural as the other axioms of set theory, and we know that it can be neither proved nor disproved from them. But the axioms of set theory are the most powerful axioms we know to settle mathematical questions; anything outside them, such as the axiom of choice, is currently a matter of blind faith. From this point of view, it is comforting that Hilbert's third problem can be solved without it.