(b) if $|\psi(g)| = \psi(1)$ and φ is faithful then $g \in Z(G)$ (where $|\psi(g)|$ is the complex absolute value of $\psi(g)$). [Use the method of proof of Proposition 14.]

13. Let $\varphi : G \to GL(V)$ be a representation and let $\chi : G \to \mathbb{C}^\times$ be a degree 1 representation. Prove that $\chi\varphi : G \to GL(V)$ defined by $\chi\varphi(g) = \chi(g)\varphi(g)$ is a representation (note that multiplication of the linear transformation $\varphi(g)$ by the complex number $\chi(g)$ is well defined). Show that $\chi\varphi$ is irreducible if and only if φ is irreducible. Show that if ψ is the character afforded by φ then $\chi\psi$ is the character afforded by $\chi\varphi$. Deduce that the product of any irreducible character with a character of degree 1 is also an irreducible character.

The next few exercises study the notion of *algebraically conjugate* characters. These exercises may be considered as extensions of Proposition 14 and some consequences of these extensions. In particular we obtain a group-theoretic characterization of the conditions under which all irreducible characters of a group take values in $\mathbb{Q}$.

Let F be the subfield of $\mathbb{C}$ of all elements that are algebraic over $\mathbb{Q}$ (the field of algebraic numbers). Thus F is the algebraic closure of $\mathbb{Q}$ contained in $\mathbb{C}$ and all the results established over $\mathbb{C}$ hold without change over F.

14. Note that since $F \subseteq \mathbb{C}$, every representation $\varphi : G \to GL_m(F)$ may also be considered as a complex representation. Prove that if φ is a representation over F that is irreducible over F, then φ is also irreducible when considered over the larger field $\mathbb{C}$ (note that this is not true if F is not algebraically closed — cf. Exercise 2(c) above). Show that the set of irreducible characters of G over F is the same as the set of irreducible characters over $\mathbb{C}$ (i.e., these are exactly the same set of class functions on G). Deduce that every complex representation is equivalent to a representation over F. [Since F is algebraically closed of characteristic 0, the irreducible characters over either F or $\mathbb{C}$ are characterized by the first orthogonality relation.]

Let $\varphi : G \to GL_m(F)$ be any representation with character ψ. Let $\mathbb{Q}(\varphi)$ denote the subfield of F generated by all the entries of the matrices $\varphi(g)$ for all $g \in G$.

15. Prove that $\mathbb{Q}(\varphi)$ is a finite extension of $\mathbb{Q}$.

Now let K be any Galois extension of $\mathbb{Q}$ containing $\mathbb{Q}(\varphi)$ and let $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$. In fact, since every automorphism of K extends to an automorphism of F, we may assume σ is any automorphism of F. The map $\varphi^\sigma : G \to GL_n(F)$ is defined by letting $\varphi^\sigma(g)$ be the $n \times n$ matrix whose entries are obtained by applying the field automorphism σ to the entries of the matrix $\varphi(g)$.

16. Prove that φ^σ is a representation. Prove also that the character of φ^σ is ψ^σ, where $\psi^\sigma(g) = \sigma(\psi(g))$.

17. Prove that φ is irreducible if and only if φ^σ is irreducible.

The representation φ^σ (or character ψ^σ) is called the *algebraic conjugate* of φ by σ (or of ψ, respectively); two representations φ_1 and φ_2 (or characters ψ_1 and ψ_2) are said to be *algebraically conjugate* if there is some automorphism σ of F such that $\varphi_1^\sigma = \varphi_2$ (or $\psi_1^\sigma = \psi_2$, respectively). Some care needs to be taken with this (standard) notation since the exponential notation usually denotes a right action whereas automorphisms of F act on the left on representations: $\varphi^{(\sigma\tau)} = (\varphi^\tau)^\sigma$.

Let $\mathbb{Q}(\psi)$ be the subfield of F generated by the numbers $\psi(g)$ for all $g \in G$. Let $|G| = n$ and let ϵ be a primitive n^{th} root of 1 in F.

18. Prove that $\mathbb{Q}(\psi) \subseteq \mathbb{Q}(\epsilon)$. Deduce that $\mathbb{Q}(\psi)$ is a Galois extension of $\mathbb{Q}$ with abelian Galois group. [See Proposition 14.]

Recall from Section 14.5 that $\mathrm{Gal}(\mathbb{Q}(\epsilon)/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^\times$, where the Galois automorphisms are given on the generator ϵ by $\sigma_a : \epsilon \mapsto \epsilon^a$, where a is an integer relatively prime to n.

19. Prove that if $\sigma_a \in \mathrm{Gal}(\mathbb{Q}(\epsilon)/\mathbb{Q})$ is the field automorphism defined above, then for all $g \in G$ we have $\psi^{\sigma_a}(g) = \psi(g^a)$. [Use the method of Proposition 14.]

20. Prove that if g is an element of G which is conjugate to g^a for all integers a relatively prime to n, then $\psi(g) \in \mathbb{Q}$, for every character ψ of G. [Use the preceding exercise and the fact that $\mathbb{Q}$ is the field fixed by all σ_a's.]

21. Prove that an element $g \in G$ is conjugate to g^a for all integers a relatively prime to $|G|$ if and only if g is conjugate to $g^{a'}$ for all integers a' relatively prime to $|g|$.

22. Show for any positive integer n that every character of the symmetric group S_n is rational valued (i.e., $\psi(g) \in \mathbb{Q}$ for all $g \in S_n$ and all characters ψ of S_n).

The next two exercises establish the converse to Exercise 20.

23. Prove that elements x and y are conjugate in a group G if and only if $\chi(x) = \chi(y)$ for all irreducible characters χ of G.

24. Let $g \in G$ and assume that every irreducible character of G is rational valued on g. Prove that g is conjugate to g^a for every integer a relatively prime to $|G|$. [If g is not conjugate to g^a for some a relatively prime to $|G|$ then by the preceding exercise there is an irreducible character χ such that $\chi(g) \neq \chi(g^a)$. Derive a contradiction from the hypothesis that $\chi(g) \in \mathbb{Q}$.]

25. Describe which irreducible characters of the cyclic group of order n are algebraically conjugate.

26. Prove that every irreducible character of both Q_8 and D_8 is rational valued. Prove that D_{10} has an irreducible character that is not rational valued.

27. Let $G = H \times K$ and let $\varphi : H \to GL(V)$ be an irreducible representation of H with character χ. Then $G \xrightarrow{\pi_H} H \xrightarrow{\varphi} GL(V)$ gives an irreducible representation of G, where π_H is the natural projection; the character, $\widetilde{\chi}$, of this representation is $\widetilde{\chi}((h,k)) = \chi(h)$. Likewise any irreducible character ψ of K gives an irreducible character $\widetilde{\psi}$ of G with $\widetilde{\psi}((h,k)) = \psi(k)$.

(a) Prove that the product $\widetilde{\chi}\widetilde{\psi}$ is an irreducible character of G. [Show it has norm 1.]

(b) Prove that every irreducible character of G is obtained from such products of irreducible characters of the direct factors. [Use Theorem 10, either (3) or (4).]

28. *(Finite subgroups of $GL_2(\mathbb{Q})$)* Let G be a finite subgroup of $GL_2(\mathbb{Q})$.

(a) Show that $GL_2(\mathbb{Q})$ does not contain an element of order n for $n = 5, 7$, or $n \geq 9$. Deduce that $|G| = 2^a 3^b$. [Use rational canonical forms.]

(b) Show that the Klein 4-group is the only noncyclic abelian subgroup of $GL_2(\mathbb{Q})$. Deduce from this and (a) that $|G| \mid 24$.

(c) Show that the only finite subgroups of $GL_2(\mathbb{Q})$ are the cyclic groups of order 1, 2, 3, 4, and 6, the Klein 4-group, and the dihedral groups of order 6, 8, and 12. [Use the classifications of groups of small order in Section 4.5 and Exercise 10 of Section 1 to restrict G to this list. Show conversely that each group listed has a 2-dimensional faithful rational representation.]

CHAPTER 19

Examples and Applications of Character Theory

19.1 CHARACTERS OF GROUPS OF SMALL ORDER

The *character table* of a finite group is the table of character values formatted as follows: list representatives of the r conjugacy classes along the top row and list the irreducible characters down the first column. The entry in the table in row χ_i and column g_j is $\chi_i(g_j)$. The character table of a finite group is unique up to a permutation of its rows and columns. It is customary to make the principal character the first row and the identity the first column and to list the characters in increasing order by degrees. In our examples we shall list the size of the conjugacy classes under each class so the entire table will have $r + 2$ rows and $r + 1$ columns (although strictly speaking, the character table is the $r \times r$ matrix of character values). This will enable one to easily check the "orthogonality of rows" using the first orthogonality relation: if the classes are represented by $g_1, \dots, g_r$ of sizes $d_1, \dots, d_r$ then

$$(\chi_i, \chi_j) = \frac{1}{|G|} \sum_{k=1}^{r} d_k \chi_i(g_k) \overline{\chi_j(g_k)}.$$

The second orthogonality relation says that the Hermitian product of any two distinct columns of a character table is zero (i.e., it gives an "orthogonality of columns").

A number of character tables are given in the *Atlas of Finite Groups* by Conway, Curtis, Norton, Parker and Wilson, Clarendon Press, 1985. These include the character table of the Monster simple group, M. The group M has 194 irreducible characters. The smallest degree of a nonprincipal irreducible character of M is 196883 and the largest degree is on the order of 2×10^{26}. Nonetheless, it is possible to compute the values of all these characters on all conjugacy classes of M.

For the first example of a character table let $G = \langle x \rangle$ be the cyclic group of order 2. Then G has 2 conjugacy classes and two irreducible characters:

classes:	1	x
sizes:	1	1
χ_1	1	1
χ_2	1	-1

Character Table of Z_2

The characters and representations of this abelian group are the same, and the irreducible representations of any abelian group are described in Example 1 at the end of Section 18.2.

Similarly, if $G = \langle x \rangle$ is cyclic of order 3, and ζ is a fixed primitive cube root of 1 (so $\zeta^2 = \overline{\zeta}$), then the character table of G is the following:

classes:	1	x	x^2
sizes:	1	1	1
χ_1	1	1	1
χ_2	1	ζ	ζ^2
χ_3	1	ζ^2	ζ

Character Table of Z_3

Next we construct the character table of S_3. Recall from Example 2 in Section 18.2 that S_3 has 3 irreducible characters whose values are described in that example and in Example 1 at the end of Section 18.3.

classes:	1	$(1\,2)$	$(1\,2\,3)$
sizes:	1	3	2
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

Character Table of S_3

Next we consider D_8, adopting the notation of Example 3 of Section 18.3. By Corollary 11, D_8 has four characters of degree 1. Also, in Example 3 we constructed an irreducible degree 2 representation. Since the sum of the squares of the degrees of these representations is 8, this accounts for all irreducible representations (or, since there are 5 conjugacy classes, there are 5 irreducible representations). If we let bars denote passage to the commutator quotient group (which is the Klein 4-group), then $\overline{1} = \overline{r^2}$. The degree 1 representations (= their characters) are computed by sending generators $\overline{s}$ and $\overline{r}$ to ± 1 (and the product class is mapped to the product of the values). Matrices for the degree 2 irreducible representation were computed in Example 3 of Section 18.3 and the character of this representation can be read directly from these matrices. The character table of D_8 is therefore the following:

classes:	1	r^2	s	r	sr
sizes:	1	1	2	2	2
χ_1	1	1	1	1	1
χ_2	1	1	-1	1	-1
χ_3	1	1	1	-1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

Character Table of D_8