Proof: Let $D = R - P$ so that D is a multiplicatively closed subset of both R and S. Then the following diagram commutes:

$$\begin{array}{ccc} R & \xrightarrow{\pi} & D^{-1}R = R_P \\ \downarrow{\scriptstyle\iota} & & \downarrow{\scriptstyle\iota} \\ S & \xrightarrow{\pi} & D^{-1}S \end{array}$$

where the vertical maps are inclusions. It is easy to see that $D^{-1}S$ is integral over R_P (Exercise 20). Let $\mathfrak{m}$ be any maximal ideal of $D^{-1}S$. Then $\mathfrak{m} \cap R_P$ is a maximal ideal in R_P by the second statement in Theorem 26(2) (note that the first part of Theorem 26(2) was not used in the proof of the second statement). By Proposition 38(1), $\mathfrak{m} \cap R_P$ is the extension of P to the local ring R_P, and the contraction of this ideal to R is just P. Put another way, the preimage of $\mathfrak{m}$ by the maps along the top and right of the diagram above is P. If $Q \subset S$ denotes the preimage of $\mathfrak{m}$ by the map along the bottom of the diagram, then Q is a prime ideal by Proposition 38(3). Since $Q \cap R$ is the pullback of Q by the map along the left of the diagram above, the commutativity of the diagram shows that $Q \cap R = P$.

Local Rings of Affine Algebraic Varieties

For the remainder of this section, let k be an algebraically closed field and let V be an affine variety over k with coordinate ring $k[V]$. Then $k[V]$ is an integral domain, so we may form its field of fractions:

$$k(V) = \{f/g \mid f, g \in k[V], \ g \neq 0\}.$$

The elements of $k(V)$ are called *rational functions* on V and $k(V)$ is called the *field of rational functions* on V. When $k[V]$ is a Unique Factorization Domain there is an essentially unique representative for f/g that is in "lowest terms," but in general each fraction $f/g \in k(V)$ has many representations as a ratio of two elements of $k[V]$. Since $k[V]$ is an integral domain, $f/g = f_1/g_1$ if and only if $fg_1 = f_1g$.

The elements of $k[V]$ can be considered as k-valued functions on V, and if the denominator doesn't vanish the same is true for an element of $k(V)$ (which helps to explain the terminology for this field). Since the same element of $k(V)$ may be written in the form f/g in several ways, we make the following definition:

Definition. We say f/g is *regular at v* or *defined at the point $v \in V$* if there is some $f_1, g_1 \in k[V]$ with $f/g = f_1/g_1$ and $g_1(v) \neq 0$.

If f_2, g_2 is another such pair with $g_2(v) \neq 0$, then $f_1(v)/g_1(v) = f_2(v)/g_2(v)$ as elements of k, so whenever f/g is regular at v there is a well defined way of specifying its value in k at v.

Example

The variety $V = \mathcal{Z}(xz - yw)$ in $\mathbb{A}^4$ has coordinate ring $k[V] = k[x, y, z, w]/(xz - yw)$. Consider the element $f = \bar{x}/\bar{y}$ in the quotient field $k(V)$ of $k[V]$. Since $\bar{x}\bar{z} = \bar{y}\bar{w}$ in $k[V]$, the element f can also be written as $\bar{w}/\bar{z}$. From the first expression for f it follows that f

is regular at all points of V where $\bar{y} \neq 0$, and from the second expression it follows that f is regular at all points of V where $\bar{z} \neq 0$. It is not too difficult to show that these are all the points of V where f is regular. Furthermore, there is no single expression $f = a/b$ for f with $a, b \in k[V]$ such that $b(v) \neq 0$ for every v where f is regular (cf. Exercise 25).

If $f/g \in k(V)$ is regular at the point v, say $f/g = f_1/g_1$ with $g_1(v) \neq 0$, then f/g is also regular at all the points v in the Zariski open neighborhood V_{g_1} of v where $g_1 \neq 0$. As a k-valued function on V this means that if f/g is defined at v, then it is also defined in a (Zariski open) neighborhood of v. Since any nonempty open set of an affine variety is Zariski dense (cf. Exercise 11 in Section 2), we see that every rational function on V is defined at a dense set of points in V (so "almost everywhere" in a suitable sense). Also, each pair f_1/g_1 and f_2/g_2 representing f/g agree as functions on the open neighborhood $V_{g_1} \cap V_{g_2}$ of v, but the "size" of this neighborhood depends on g_1 and g_2 — there is in general not a common open neighborhood of v where *all* representatives of f/g with nonzero denominator at v are simultaneously defined.

If v is a fixed point in V, then a rational function f/g is regular at v if and only if $f/g = f_1/g_1$ for some $f_1, g_1 \in k[V]$ with $g_1 \notin \mathcal{I}(v)$, the ideal of functions on V that are zero at v. This means that the set of rational functions that are defined at v is the same as the localization of $k[V]$ at the maximal ideal $\mathcal{I}(v)$:

Definition. For each point $v \in V$ the collection of rational functions on V that are defined at v,

$$\mathcal{O}_{v,V} = \{f/g \in k(V) \mid f/g \text{ is regular at } v\},$$

is called the *local ring of V at v*. Equivalently, the local ring of V at v is the localization of $k[V]$ at the maximal ideal $\mathcal{I}(v)$.

In particular, $\mathcal{O}_{v,V}$ is a local ring with unique maximal ideal $\mathfrak{m}_{v,V}$, where

$$\mathfrak{m}_{v,V} = \{f/g \in \mathcal{O}_{v,V} \mid f/g = f_1/g_1 \text{ with } f_1(v) = 0,\ g_1(v) \neq 0\}$$

is the set of rational functions on V that are defined and equal to 0 at v. Since $\mathcal{O}_{v,V}$ is a localization of the Noetherian integral domain $k[V]$ at a prime ideal, $\mathcal{O}_{v,V}$ is also a Noetherian integral domain. Note also that $\mathcal{O}_{v,V}/\mathfrak{m}_{v,V} \cong k[V]/\mathcal{I}(v) \cong k$ by Proposition 46(5).

Recall that the polynomial maps from V to k are also referred to as the *regular* maps of V to k. This is because these are precisely the rational functions on V that are regular everywhere:

Proposition 51. If V is an affine variety over an algebraically closed field k then the rational functions on V that are regular at all points of V are precisely the polynomial functions $k[V]$.

Proof: This follows from Proposition 48, which shows that the intersection (in $k(V)$) of all of the localizations of $k[V]$ at the maximal ideals of $k[V]$ is precisely $k[V]$.

Since the maximal ideals of $k[V]$ are in bijective correspondence with the points of V, the fact that the local ring $\mathcal{O}_{v,V}$ is the same as the localization of $k[V]$ at the maximal ideal corresponding to v shows that $\mathcal{O}_{v,V}$ depends intrinsically on the ring $k[V]$ and is independent of the embedding of V in a particular affine space.

Suppose $\varphi : V \to W$ is a morphism of affine varieties with associated k-algebra homomorphism $\widetilde{\varphi} : k[W] \to k[V]$. If $v \in V$ is mapped to $w \in W$ by φ, then it is straightforward to show that $\widetilde{\varphi}$ induces a homomorphism (also denoted by $\widetilde{\varphi}$) between the corresponding local rings:

$$\widetilde{\varphi} : \mathcal{O}_{w,W} \to \mathcal{O}_{v,V} \quad \text{where} \quad \widetilde{\varphi}(h/k) = \widetilde{\varphi}(h)/\widetilde{\varphi}(k),$$

and that under this homomorphism, $\widetilde{\varphi}^{-1}(\mathfrak{m}_{v,V}) = \mathfrak{m}_{w,W}$ (a homomorphism of local rings having this property is called a *local homomorphism*). Note that $\widetilde{\varphi}$ does not in general extend to a field homomorphism from *all* of $k(W)$ into $k(V)$ since elements of $k[W]$ lying in the kernel of $\widetilde{\varphi}$ do not map to invertible elements in $k(V)$. It is also easy to check that if $\psi \circ \varphi$ is a composition of morphisms then on the local rings $\widetilde{\psi \circ \varphi} = \widetilde{\varphi} \circ \widetilde{\psi}$.

The local ring $\mathcal{O}_{v,V}$ can be used to provide an algebraic definition of the "smoothness" (in the sense of the existence of tangents) of V at v, as we now indicate. Suppose first that $V = \mathcal{Z}(f)$ is the hypersurface variety in $\mathbb{A}^n$ defined by the zeros of an irreducible polynomial f in $k[x_1, \dots, x_n]$. For any point $v = (v_1, \dots, v_n)$ on V let $D_v(f)(x_1, \dots, x_n)$ be the linear polynomial:

$$D_v(f)(x_1, \dots, x_n) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(v)\, x_i,$$

where the partial derivative of f with respect to x_i is given by the usual formal rule for the derivative of a polynomial in x_i (with all other variables considered constant). The polynomial $D_v(f)(x_1 - v_1, \dots, x_n - v_n)$ is the first order Taylor polynomial of the function f at v, so gives the best linear approximation to $f(x_1, \dots, x_n) \in k[x_1, \dots, x_n]$ at v. It follows that if $\mathbf{T}$ is the linear variety $\mathcal{Z}(D_v(f)(x_1, \dots, x_n))$ consisting of those points where $D_v(f)$ is zero, then the translate $v + \mathbf{T}$ is "tangent" to the hypersurface $\mathcal{Z}(f)$ at v.

Example

Suppose $f = x^2 - y \in k[x, y]$, so that $V = \mathcal{Z}(f)$ is just the parabola $y = x^2$. We have $\partial f/\partial x = 2x$ and $\partial f/\partial y = -1$, which at $v = (3, 9)$ are equal to 6 and -1, respectively. Then

$$D_{(3,9)}(f)(x, y) = 6x - y,$$

and the corresponding linear variety $\mathbf{T}$ is the line $y = 6x$ through the origin. The translate $(3, 9) + \mathbf{T}$ is the usual tangent line to the parabola at $(3, 9)$. The Taylor expansion of $x^2 - y$ at $(3, 9)$ is $x^2 - y = [\,6(x-3) - (y-9)\,] + (x-3)^2$. The first order terms are $D_{(3,9)}(f)(x - 3, y - 9)$ and give the best linear approximation to $x^2 - y$ near (3,9).

It is straightforward to extend these notions to any affine variety V in $\mathbb{A}^n$.