The equation relating the series for $\pi/4$ to the continued fraction for $4/\pi$, namely

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \cfrac{1}{1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \cdots}}}}}$$

follows immediately from a more general equation

$$\frac{1}{A} - \frac{1}{B} + \frac{1}{C} - \frac{1}{D} + \cdots = \cfrac{1}{A + \cfrac{A^2}{B - A + \cfrac{B^2}{C - B + \cfrac{C^2}{D - C + \cdots}}}}$$

proved by Euler (1748a), p. 311. The following exercises give a proof of Euler's result.

9.4.3 Check that

$$\frac{1}{A} - \frac{1}{B} = \cfrac{1}{A + \cfrac{A^2}{B - A}}.$$

9.4.4 When $\frac{1}{B}$ on the left side in Exercise 9.4.3 is replaced by $\frac{1}{B} - \frac{1}{C}$, which equals $\cfrac{1}{B + \frac{B^2}{C-B}}$ by Exercise 9.4.3, show that B on the right should be replaced by $B + \frac{B^2}{C-B}$. Hence show that

$$\frac{1}{A} - \frac{1}{B} + \frac{1}{C} = \cfrac{1}{A + \cfrac{A^2}{B - A + \cfrac{B^2}{C - B}}}.$$

Thus when we modify the "tail end" of the series (replacing $\frac{1}{B}$ by $\frac{1}{B} - \frac{1}{C}$) only the "tail end" of the continued fraction is affected. This situation continues:

9.4.5 Generalize your argument in Exercise 9.4.4 to obtain a continued fraction for a series with n terms, and hence prove Euler's equation.

9.5   Newton's Calculus of Series

Newton made many of his most important discoveries in 1665/6, after studying the works of Descartes, Viète, and Wallis. In Schooten's edition of *La Géométrie* he encountered Hudde's rule for tangents to algebraic

curves, which was virtually a complete differential calculus from Newton's viewpoint. Although Newton made contributions to differentiation that are useful to *us*—the chain rule, for example—differentiation was a minor part of *his* calculus, which depended mainly on the manipulation of infinite series. Thus it is misleading to describe Newton as a founder of calculus unless one understands calculus, as he did, as an algebra of infinite series. In this calculus, differentiation and integration are carried out term by term on powers of x and hence are comparatively trivial.

At the beginning of his main work on calculus, *A Treatise of the Methods of Series and Fluxions* (also known by its abbreviated Latin name of *De methodis*), Newton clearly states his view of the role of infinite series:

> Since the operations of computing in numbers and with variables are closely similar ... I am amazed that it has occurred to no one (if you except N. Mercator with his quadrature of the hyperbola) to fit the doctrine recently established for decimal numbers in similar fashion to variables, especially since the way is then open to more striking consequences. For since this doctrine in species has the same relationship to Algebra that the doctrine in decimal numbers has to common Arithmetic, its operations of Addition, Subtraction, Multiplication, Division and Root extraction may be easily learnt from the latter's.
>
> [Newton (1671), pp. 33–35]

The quadrature (area determination) of the hyperbola mentioned by Newton was the result that we would write as

$$\int_0^x \frac{dt}{1+t} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

first published in Mercator (1668). Newton had discovered the same result in 1665, and it was partly his dismay in losing priority that led him to write *De methodis* and an earlier work *De analysi* [Newton (1669); the full title in English is *On Analysis by Equations Unlimited in Their Number of Terms*]. Newton also independently discovered the series for $\tan^{-1} x$, $\sin x$, and $\cos x$ in *De analysi*, without knowing that all three series had already been discovered by Indian mathematicians (see Section 10.1).

The Mercator and Indian results were both obtained by the method of

expanding a geometric series and integrating term by term. In our notation,

$$\int_0^x \frac{dt}{1+t} = \int_0^x (1 - t + t^2 - t^3 + \cdots) \, dt$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

and

$$\tan^{-1} x = \int_0^x \frac{dt}{1+t^2}$$

$$= \int_0^x (1 - t^2 + t^4 - t^6 + \cdots) \, dt$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots .$$

Newton routinely used these methods in *De analysi* and *De methodis*, but he greatly extended their scope by algebraic manipulation. Not only did he obtain sums, products, quotients, and roots, as foreshadowed in his introduction to *De methodis*, but his root extractions also extended to the general construction of *inverse functions* by the new idea of inverting infinite series. For example, after Newton [Newton (1671), p. 61] found the series $x - (x^2/2) + (x^3/3) - \cdots$, for $\int_0^x dt/(1+t)$, which of course is $\log(1+x)$, he set

$$y = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \tag{1}$$

and solved (1) for x (which we recognize to be the exponential function e^y, minus 1). His method is in tabular form like the arithmetic calculations of the time but equivalent to setting $x = a_0 + a_1 y + a_2 y^2 + \cdots$, substituting in the right-hand side of (1), and determining $a_0, a_1, a_2, \ldots$, successively by comparing with the coefficients on the left-hand side. Newton found the first few terms

$$x = y + \frac{1}{2}y^2 + \frac{1}{6}y^3 + \frac{1}{24}y^4 + \frac{1}{120}y^5 + \cdots$$

then confidently concluded that $a_n = 1/n!$ in the manner of Wallis. As he put it, "Now after the roots have been extracted to a suitable period, they may sometimes be extended at pleasure by observing the analogy of the series."

De Moivre (1698) gave a formula for inversion of series which justifies such conclusions; the impressive thing is that Newton could find such an

elegant result by such a forbidding method. His discovery of the series for $\sin x$ [Newton (1669), pp. 233, 237] is even more amazing. First he used the binomial series

$$(1+a)^p = 1 + pa + \frac{p(p-1)}{2!}a^2 + \frac{p(p-1)(p-2)}{3!}a^3 + \cdots$$

(though not with the natural choice $a = -x^2, p = -\frac{1}{2}$) to obtain

$$\sin^{-1} x = z = x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \cdots$$

by term-by-term integration then casually stated "I extract the root, which will be

$$x = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362,880}z^9 - \cdots,"$$

adding a few lines later, that the coefficient of z^{2n+1} is $1/(2n+1)!$.

EXERCISES

The kind of tabulation needed to perform inversion of series is the following, which shows the coefficients of $1, y, y^2, y^3, \ldots$ in x and its powers.

	1	y	y^2	y^3	$\cdots$
x	a_0	a_1	a_2	a_3	$\cdots$
x^2	a_0^2	$2a_0a_1$	$2a_0a_2 + a_1^2$	$2a_0a_3 + 2a_1a_2$	$\cdots$

9.5.1 Use the rows shown to substitute series in powers of y for x and x^2 in $y = x - \frac{x^2}{2} + \cdots$, and hence show that $a_0 = 0$, $a_1 = 1$, and $a_2 = 1/2$ in turn, by comparing coefficients on the two sides of the equation.

9.5.2 Compute the first few entries in the third row of the table (the coefficients of x^3), and hence show that $a_3 = 1/6$.

This shows why the inverse function $x = e^y - 1$ has a power series that begins

$$y + \frac{1}{2}y^2 + \frac{1}{6}y^3 + \cdots.$$

9.5.3 Show that the binomial series gives

$$\frac{1}{\sqrt{1-t^2}} = 1 + \frac{1}{2}t^2 + \frac{1\cdot 3}{2\cdot 4}t^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}t^6 + \cdots.$$

9.5.4 Use Exercise 9.5.3 and $\sin^{-1} x = \int_0^x dt/\sqrt{1-t^2}$ to derive Newton's series for $\sin^{-1} x$.

9.6   The Calculus of Leibniz

Newton's epoch-making works [Newton (1669, 1671)] were offered to the Royal Society and Cambridge University Press but, incredible as it now seems, were rejected for publication. Thus it happened that the first published paper on calculus was not by Newton but by Leibniz (1684). This led to Leibniz's initially receiving credit for the calculus and later to a bitter dispute with Newton and his followers over the question of priority for the discovery.

There is no doubt that Leibniz discovered calculus independently, that he had a better notation, and that his followers contributed more to the spread of calculus than did Newton's. Leibniz's work lacked the depth and virtuosity of Newton's, but then Leibniz was a librarian, a philosopher, and a diplomat with only a part-time interest in mathematics. His *Nova methodus* [Leibniz (1684)] is a relatively slight paper, though it does lay down some important fundamentals—the sum, product, and quotient rules for differentiation—and it introduces the dy/dx notation we now use. However, dy/dx was not just a symbol for Leibniz, as it is for us, but literally a quotient of *infinitesimals* dy and dx, which he viewed as differences (hence the symbol d) between neighboring values of y and x, respectively.

He also introduced the integral sign, $\int$, in his *De geometria* [Leibniz (1686)] and proved the fundamental theorem of calculus, that integration is the inverse of differentiation. This result was known to Newton and even, in a geometric form, to Newton's teacher Barrow, but it became more transparent in Leibniz's formalism. For Leibniz, $\int$ meant "sum," and $\int f(x)\,dx$ was literally a sum of terms $f(x)dx$, representing infinitesimal areas of height $f(x)$ and width dx. The difference operator d yields the last term $f(x)\,dx$ in the sum, and dividing by the infinitesimal dx yields $f(x)$. So voila!

$$\frac{d}{dx}\int f(x)\,dx = f(x)$$

—the fundamental theorem of calculus.

Leibniz's strength lay in the identification of important concepts, rather than in their technical development. He introduced the word "function" and was the first to begin thinking in function terms. He made the distinction between algebraic and transcendental functions and, in contrast to Newton, preferred "closed-form" expressions to infinite series. Thus the evaluation of $\int f(x)\,dx$ for Leibniz was the problem of finding a known

function whose derivative was $f(x)$, whereas for Newton it was the problem of expanding $f(x)$ in series, after which integration was trivial.

The search for closed forms was a wild goose chase but, like many efforts to solve intractable problems, it led to worthwhile results in other directions. Attempts to integrate rational functions raised the problem of factorization of polynomials and led ultimately to the fundamental theorem of algebra (see Chapter 14). Attempts to integrate $1/\sqrt{1-x^4}$ led to the theory of elliptic functions (Chapter 12). As mentioned in Section 9.1, the problem of deciding which algebraic functions may be integrated in closed form has been solved only recently, though not in a form suitable for calculus textbooks, which continue to remain oblivious to most of the developments since Leibniz. (One thing that has changed: it is now much easier to publish a calculus book than it was for Newton!)

EXERCISES

Leibniz (1702) was stymied by the integral $\int \frac{dx}{x^4+1}$, because he did not spot the factorization of $x^4 + 1$ into real quadratic factors.

9.6.1 Writing $x^4 + 1 = x^4 + 2x^2 + 1 - 2x^2$ or otherwise, split $x^4 + 1$ into real quadratic factors.

9.6.2 Use the factors in Exercise 9.6.1 to express $\frac{1}{x^4+1}$ in the partial fraction form

$$\frac{x+\sqrt{2}}{q_1(x)} + \frac{x-\sqrt{2}}{q_2(x)},$$

where $q_1(x)$ and $q_2(x)$ are real quadratic polynomials.

9.6.3 Without working out all the details, explain how the partial fractions in Exercise 9.6.2 can be integrated in terms of rational functions and the $\tan^{-1}$ function.

9.7   Biographical Notes: Wallis, Newton, and Leibniz

John Wallis (Figure 9.3) was born in 1616 in Ashford, Kent, and died in Oxford in 1703. He was one of five children of John Wallis, the rector of Ashford, and Joanna Chapman. He had two older sisters and two younger brothers. Young John Wallis was recognized as the academic talent of the family and at 14 was sent to Felsted, Essex, to attend the school of Martin Holbech, a famous teacher of the time. At school he learned Latin,

Greek, and Hebrew, but he did not meet mathematics until he was home on Christmas vacation in 1631. One of his brothers was learning arithmetic to prepare for a trade, and Wallis asked him to explain it. This turned out to be the only mathematical instruction Wallis ever received, even though he later studied at Emmanuel College in Cambridge.

Figure 9.3: John Wallis

As Wallis explained in his autobiography:

> Mathematicks were not, at that time, looked upon as Acca-
> demical Learning, but the business of Traders, Merchants, Sea-
> men, Carpenters, land-measurers, or the like; or perhaps some
> Almanak-makers in London. And of more than 200 at that
> time in our College, I do not know of any two that had more
> of Mathematicks than myself, which was but very little; hav-
> ing never made it my serious studie (otherwise than as a pleas-
> ant diversion) till some little time before I was designed for a
> Professor in it.
>
> [Wallis (1696), p. 27]

At Emmanuel College, Wallis studied divinity from 1632 to 1640,
when he gained a master of arts degree. College life evidently agreed with
him, and he would have stayed on as a fellow, had there been a place avail-
able. He did become a fellow of Queens College, Cambridge, for a year
but, since fellows had to remain unmarried, relinquished the fellowship
when he married in 1645. Thus it was that Wallis spent most of the 1640s
in the ministry.

The 1640s were a decisive decade in English history, with the rise of
the parliamentary opposition to Charles I and the king's execution in 1649.
Partly by luck and partly by adaptation to the new political conditions,
Wallis changed the direction of his life toward mathematics. Early in the
conflict he found he had the very valuable ability to decipher coded mes-
sages. To quote the autobiography again:

> About the beginning of our Civil Wars, in the year 1642, a
> Chaplain of S^r. William Waller showed me an intercepted
> Letter written in Cipher. ... He asked me (between jest and
> earnest) if I could make any thing of it. ... I judged it could
> be no more than a new Alphabet and, before I went to bed, I
> found it out, which was my first attempt upon Deciphering.
>
> [Wallis (1696), p. 37]

This was the first in a series of successes Wallis had in codebreaking for
the Parliamentarians, which gained him not only political favor but also a
reputation for mathematical skill. [For more information on Wallis' cryp-
tography see Kahn (1967), p. 166.] When the royalist Peter Turner was