The characterization of critical edges in partitionable graphs is implicit in the work of several authors.

8.1.44. Theorem. For an edge xy in a partitionable graph G, the following statements are equivalent.
A) xy is a critical edge.
B) $S \cup \{x\} \in \mathrm{X}(G - y)$.
C) xy belongs to $\omega - 1$ maximum cliques.

Proof: B $\Rightarrow$ A. $S \cup \{x, y\}$ is a stable set of size $\alpha + 1$ in $G - xy$.

A $\Rightarrow$ C. If xy is critical, then there is a set S such that $S \cup \{x\}$ and $S \cup \{y\}$ are maximum stable sets in G. Hence every maximum clique containing x but not y is disjoint from $S \cup \{y\}$. Since there are ω maximum cliques containing x and only one maximum clique disjoint from $S \cup \{y\}$, the remaining $\omega - 1$ maximum cliques containing x must also contain y.

C $\Rightarrow$ B. The stable sets in the unique coloring of $G - x$ are the mates of the cliques containing x. Since xy belongs to $\omega - 1$ maximum cliques, the mates of these $\omega - 1$ cliques belong to both $\mathrm{X}(G - x)$ and $\mathrm{X}(G - y)$. This leaves only $a + 1$ vertices in the graph, consisting of the vertices x, y and a stable set S such that $S \cup \{y\} \in \mathrm{X}(G - x)$ and $S \cup \{x\} \in \mathrm{X}(G - y)$. ■

8.1.45. Corollary. Let G be a partitionable graph. If xy is an edge appearing in no maximum clique, then $G - xy$ is partitionable. If x, y is a nonadjacent pair appearing in no maximum stable set, then $G + xy$ is partitionable.

Proof: By complementation, we need only prove the first statement. If we delete an edge appearing in no maximum clique, then by Theorem 8.1.44 it is not a critical edge, and we have $\omega(G - xy) = \omega(G)$ and $\alpha(G - xy) = \alpha(G)$. Since we have not destroyed any maximum clique and have not created a bigger stable set, we can use the optimal coloring and clique partition of $G - u$ to conclude that $\chi(G - xy - u) \leq \omega$ and $\theta(G - xy - u) \leq \alpha$. Hence $G - xy$ is partitionable, by Theorem 8.1.35. ■

The discussion in Example 8.1.37 suggests that edges appearing in no maximum clique are uninteresting "junk". Corollary 8.1.45 assures us that "junk is junk". The partitionable cycle-powers have no junk.

THE STRONG PERFECT GRAPH CONJECTURE

We have been proving properties of partitionable graphs in a "top down" approach to the SPGC, trying to find enough properties to eliminate all but odd cycles and their complements as p-critical graphs. The "bottom up" approach is to verify that the SPGC holds on larger and larger classes of graphs, until all are included.

8.1.46. Definition. An **odd hole** or **odd antihole** in G is an induced subgraph of G that is C_{2k+1} or $\overline{C}_{2k+1}$ (for some $k \geq 2$), respectively. A **graph having** no odd hole or antihole is a **Berge graph**.

One way to prove that a class **G** satisfies the SPGC is to prove that every Berge graph in **G** is perfect. A hereditary class **G** satisfies the SPGC if the odd cycles and their complements are the only p-critical graphs in **G**.

The SPGC holds for planar graphs (Tucker [1973]), toroidal graphs (Grinstead [1981]), graphs with $\Delta(G) \leq 6$ (Grinstead [1978]) or $\omega(G) \leq 3$ (Tucker [1977]), and for various classes defined by forbidding fixed small induced subgraphs (Meyniel [1976], Tucker [1977], Parthasarathy–Ravindra [1976, 1979], Chvátal–Sbihi [1988], Olariu [1989], Sun [1991]). We consider three families.

8.1.47. Definition. A **circular-arc graph** is the intersection graph of a family of arcs of a circle. A **circle graph** is the intersection graph of a family of chords of a circle. A $K_{1,3}$**-free** graph is a graph not having $K_{1,3}$ as an induced subgraph.

Every cycle is both a circle graph and a circular-arc graph, but neither of these classes contains the other (Exercise 47).

One way to prove the SPGC for a class **G** is to show that every partitionable graph in **G** belongs to another class **H** where the SPGC is known to hold. In this role we use the class $\{C_n^d\}$.

8.1.48. Theorem. (Chvátal [1976]) Cycle-powers satisfy the SPGC. In particular, the graph C_{aw+1}^{w-1} is p-critical if and only if $w = 2$ or $a = 2$, in which case the graph is an odd hole or antihole.

Proof: It suffices to consider the partitionable graph $G = C_{aw+1}^{w-1}$. This is p-critical when $a = 2$ or $w = 2$, so we may assume $a, w > 2$. Let the vertices be $\{v_0, \dots, v_{aw}\}$, and let $S = \{v_{iw+1}, v_{(i+1)w}\colon 0 \leq i \leq a-1\}$. The subgraph $G[S]$ is a cycle, since the indices of consecutive vertices in S are separated by 1 or $w-1$ (except that v_{aw} and v_1 are separated by 2), and indices of nonconsecutive vertices differ by at least w. To obtain C_{2a-1} as a proper induced subgraph, we replace $\{v_{(a-1)w}, v_{aw}, v_1, v_w\}$ with $\{v_{(a-1)w+1}, v_0, v_{w-1}\}$ in S. We conclude that G is not p-critical. ■

8.1.49. Theorem. (Tucker [1975]) The SPGC holds for circular-arc graphs.

Proof: Recall that $N[v]$ denotes $N(v) \cup \{v\}$, the closed neighborhood of v (Definition 3.1.29). When G is partitionable with distinct vertices x, y, we claim that $N[x] \not\subseteq N[y]$. Consider the clique Q containing x in $\Theta(G - y)$; we have $Q \subseteq N[x]$. If $N[y]$ contains $N[x]$, then $Q \cup \{y\}$ is a clique of size $\omega(G) + 1$.

Now, if G is a partitionable circular-arc graph, it suffices to show that $G = C_n^{\omega(G)-1}$, because the SPGC holds for cycle-powers (Theorem 8.1.48). Consider a circular-arc representation that assigns arc A_x to $x \in V$. Since $N[y]$ cannot contain $N[x]$, the arc A_x cannot lie within another arc A_y of the representation. If no arc contains another, then every arc that intersects A_x contains exactly one of its endpoints. Since the vertices corresponding to the arcs containing one point induce a clique, there are at most $\omega - 1$ other arcs containing each endpoint of A_x. Equality holds, since Theorem 8.1.42 requires $\delta(G) \geq 2\omega - 2$.

Starting from a given point p on the circle, let v_i be the vertex represented by the ith arc encountered moving clockwise from p. Since each arc meets exactly $\omega - 1$ others at each endpoint, v_i is adjacent to $v_{i+1}, \ldots, v_{i+\omega-1}$ (addition modulo n) for each i. Hence $G = C_n^{\omega-1}$. ■

The original proof of the SPGC for claw-free graphs (Parthasarathy–Ravindra [1976]) was quite intricate. Further study of p-critical graphs has shortened both it and the proof of the next theorem, which we will apply.

8.1.50. Theorem. (Giles–Trotter–Tucker [1984]) If a partitionable graph G has a cycle consisting of critical edges, then the subgraph G' obtained by deleting the edges belonging to no maximum clique is $C_n^{\omega-1}$

Proof: (Hartman [1995]) Suppose that G is a, w-partitionable. Deleting edges destroys no stable set. Deleting edges in no maximum clique destroys no maximum clique. Hence the coloring and clique covering of $G - x$ also yield $\chi(G' - x) \le w$ and $\theta(G' - x) \le a$ (regardless of whether $\alpha(G') > \alpha(G)$). By Theorem 8.1.35, G' is thus a, w-partitionable. Also, the clique coverings of $G' - x$ for various x force G' to be connected.

We next prove that if G has a u, v-path consisting of k critical edges, then u and v belong to at least $\omega - k$ common maximum cliques. We use induction on k, with Theorem 8.1.44 providing the basis step, $k = 1$. For $k > 1$, if y is the vertex before v on such a path, then the induction hypothesis puts u and y in $\omega - k + 1$ common maximum cliques. Since y belongs to exactly ω maximum cliques (by Theorem 8.1.39), and $\omega - 1$ of these contain v (by Theorem 8.1.44), at most one of the $\omega - k + 1$ cliques containing u and y can omit v.

Let C be a cycle of critical edges in G. Critical edges belong to maximum cliques, so C remains in G'. As shown above, ω vertices forming a path in G' induce a maximum clique in G'. If the length of C exceeds ω, then this establishes ω successive maximum cliques containing a given vertex x of C. By Theorem 8.1.39, these are all the maximum cliques of G containing x, and hence they include all the edges of G' incident to x. Hence C is a component of G', but G' is connected, so C contains all vertices of G'. This expresses G' as $C_n^{\omega-1}$.

If the length of C is at most ω, then $V(C)$ itself is a clique. If $x \in V(C)$, then the vertices of $C - x$ belong to distinct stable sets in the coloring $\mathrm{X}(G - x)$ defined by Theorem 8.1.41. Let $x_0, \ldots, x_k$ be the vertices of C in order. Let $S_1, \ldots, S_k$ be the stable sets in $G - V(C)$ such that $S_i \cup \{x_i\} \in \mathrm{X}(G - x_0)$. Because $x_i x_{i+1}$ is a critical edge, x_i and x_{i+1} belong to $\omega - 1$ common maximum cliques (Theorem 8.1.44), and hence by Theorem 8.1.41 the colorings $\mathrm{X}(G - x_i)$ and $\mathrm{X}(G - x_{i+1})$ have $\omega - 1$ common stable sets. The remaining set differs only in having x_i or x_{i+1}. Hence $\mathrm{X}(G - x_1)$ contains $S_i \cup \{x_i\}$ for $i \ge 2$, and it also contains $S_1 \cup \{x_0\}$.

Continuing these substitutions while following the edges of C, we find that $\mathrm{X}(G - x_k)$ contains $S_i \cup \{x_{i-1}\}$ for $1 \le i \le k$. Taking one more step to return to x_0, we find that $\mathrm{X}(G - x_0)$ contains $S_i \cup \{x_{i-1}\}$ for $2 \le i \le k$ and contains $S_1 \cup \{x_k\}$. Since $k \ge 2$ and $\alpha \ge 2$, these sets are different from our initial sets in $\mathrm{X}(G - x_0)$. Since the coloring $\mathrm{X}(G - x_0)$ is unique, we have obtained a contradiction, and the case $n(C) \le \omega$ does not arise. ■

8.1.51. Theorem. (Chvátal [1976]) If G is a p-critical graph such that the spanning subgraph G' obtained by deleting the edges of G belonging to no maximum clique is a cycle-power C_n^d, then G is an odd hole or odd antihole (and equals G').

Proof: A p-critical graph is partitionable. The stable sets and maximum cliques in G are stable sets and cliques in G', and by Theorem 8.1.35 we again conclude that G' is partitionable with $\alpha(G') = \alpha(G) = a$ and $\omega(G') = \omega(G) = w$. Hence $G' = C_{aw+1}^{w-1}$. We index the vertices so that the maximum cliques of G' (and G) consist of w cyclically consecutive vertices, and the maximum stable sets have the form $v_i, v_{i+w}, \ldots, v_{i+aw}$. In particular, vertices separated by a multiple of w on the cycle $v_0, \ldots, v_{aw}$ are nonadjacent in G' and in the full graph G.

If $G' = G$, then Theorem 8.1.48 implies that G is an odd hole or odd antihole. If $G' \neq G$, then $a, w > 2$, since otherwise deleting an edge increases the number of maximum stable sets or decreases the number of maximum cliques.

For $a, w \geq 3$, we exhibit an imperfect proper induced subgraph H of G (the induced odd cycle in G' obtained in Theorem 8.1.48 may have a chord in G). Let $S = \{v_{aw}, v_1, v_w, v_{w+2}\} \cup \{v_{iw+1}\colon 2 \leq i \leq a-1\}$, and let $T = \{v_{(a-1)w+1}, v_{aw}, v_1, v_w\} \cup \{v_{w+i}\colon 2 \leq i \leq w-1\}$. The sets S and T have sizes $a+2$ and $w+2$, and for $a, w \geq 3$ they share exactly the five vertices $\{v_{(a-1)w+}, v_{aw}, v_1, v_w, v_{w+2}\}$. Furthermore, S intersects every maximum clique of G' (and hence of G), and T intersects every maximum stable set of G' (and hence of G) (Exercise 49). Letting $H = G - (S \cup T)$, this yields $\alpha(H) = a - 1$ and $\omega(H) = w - 1$. Now imperfection follows from

$$n(H) \geq n(G) - (a + w + 4 - 5) > (a-1)(w-1). \qquad \blacksquare$$

8.1.52. Corollary. (Giles–Trotter–Tucker [1984]) If G is a p-critical graph and for each $v \in V(G)$ the minimum coloring $\mathrm{X}(G - v)$ has (at least) two sets that each contain exactly one neighbor of v, then G is an odd hole or an odd antihole.

Proof: When some set in $\mathrm{X}(G - v)$ has exactly one neighbor u of v, the edge uv is critical. Hence the hypothesis implies that the subgraph of critical edges has minimum degree at least 2 and therefore contains a cycle. By Theorem 8.1.50, the subgraph G' obtained by deleting the edges belonging to no maximum clique is $C_n^{\omega-1}$. By Theorem 8.1.51, G is an odd hole or an odd antihole. $\blacksquare$

8.1.53. Corollary. (Parthasarathy–Ravindra [1976]) The SPGC holds for $K_{1,3}$-free graphs.

Proof: (Giles–Trotter–Tucker [1984]) Let G be a p-critical $K_{1,3}$-free graph. For each $v \in V(G)$, $N(v)$ induces a perfect subgraph having no stable set of size 3. This means that $N(v)$ can be covered by two cliques, which implies $d(v) \leq 2\omega(G) - 2$. Each of the $\omega(G)$ stable sets in $\mathrm{X}(G - v)$ contains a neighbor of v, else adding v creates a larger stable set. With $d(v) \leq 2\omega(G) - 2$, at least two of these sets have exactly one neighbor of v. Hence G satisfies the hypothesis of Corollary 8.1.52, and G is an odd hole or antihole. $\blacksquare$

Corollary 8.1.53 also yields the SPGC for circle graphs (Exercise 50). The general SPGC remains open, but a result intermediate between it and the PGT is known (it is immediately implied by the SPGC and immediately implies the PGT). Chvátal conjectured that if G and H have the same vertex set and have the same 4-tuples of vertices that induce P_4, then G is perfect if and only if H is perfect. Reed [1987] proved this "Semi-Strong Perfect Graph Theorem".

EXERCISES

8.1.1. (−) Compute $\chi(G)$ and $\omega(G)$ for the complement of the odd cycle C_{2k+1}.

8.1.2. (−) Determine the smallest imperfect graph G such that $\chi(G) = \omega(G)$.

8.1.3. (!) P_4-free graphs are also called **cographs**, which stands for "complement reducible". A graph is **complement reducible** if it can be reduced to an empty graph by successively taking complements within components.

a) Prove that a graph G is P_4-free if and only if it is complement reducible.

b) Use part (a) and the Perfect Graph Theorem to prove that every P_4-free graph is perfect. (Seinsche [1974])

8.1.4. *Clique identification.* Suppose that $G = G_1 \cup G_2$, that $G_1 \cap G_2$ is a clique, and that G_1 and G_2 are perfect. Without using the Star-cutset Lemma, prove that G is perfect.

8.1.5. Find an imperfect graph G having a star-cutset C such that the C-lobes of G are perfect graphs. (Comment: Thus identification at star-cutsets does not preserve perfection, although no p-critical graph has a star-cutset.)

8.1.6. Let G be a cartesian product of complete graphs. Prove that $\alpha(G) = \theta(G)$. Prove that $K_2 \square K_2 \square K_3$ is not perfect.

8.1.7. Prove that $C_5 \vee K_1$ is the only color-critical 4-chromatic graph with six vertices.

8.1.8. (+) Prove that G is an odd cycle if and only if $\alpha(G) = (n(G) - 1)/2$ and $\alpha(G - u - v) = \alpha(G)$ for all $u, v \in V(G)$. (Melnikov–Vizing [1971], Greenwell [1978])

8.1.9. Let $v_1, \dots, v_n$ be a simplicial elimination ordering of G, and let $Q(v_i) = \{v_j \in N(v_i)\colon j > i\}$. Note that $Q(v_i)$ is the clique of neighbors of v_i at the time when v_i is deleted in the elimination ordering. Let $S = \{y_1, \dots, y_k\}$ be the stable set obtained "greedily" from the ordering $v_1, \dots, v_n$; that is, set $y_1 = v_1$, discard $N(y_1)$ from the remainder of the ordering, and proceed iteratively, at each step adding the least remaining element x to the stable set and discarding what remains of $Q(x)$.

a) Prove that applying the greedy coloring algorithm to the construction ordering $v_n, \dots, v_1$ yields an optimal coloring and that $\omega(G) = 1 + \max \sum_{x \in V(G)} |Q(x)|$. (Fulkerson–Gross [1965])

b) Prove that S is a maximum stable set and that the sets $\{y_i\} \cup Q(y_i)$ form a minimum clique covering. (Gavril [1972])

8.1.10. Add a test to the MCS algorithm to check whether the resulting ordering is a simplicial elimination ordering. (Tarjan–Yannakakis [1984])

8.1.11. Prove directly (without using a simplicial elimination ordering) that the intersection graph of a family of subtrees of a tree has no chordless cycle.

8.1.12. (−) Prove that every graph is the intersection graph of a family of subtrees of some graph.

8.1.13. Prove that every chordal graph has an intersection representation by subtrees of a host tree with maximum degree 3.

8.1.14. Let Q be a maximal clique in a connected chordal graph G. For all $x \in V(G)$, prove that Q has two vertices whose distances from x are different. (Voloshin [1982])

8.1.15. *Intersection graphs of subtrees of a graph.* A **fraternal orientation** of a graph is an orientation such that any pair of vertices with a common successor are adjacent.

a) (−) Prove that a graph is chordal if and only if it has an acyclic fraternal orientation.

b) (−) Obtain a graph with no fraternal orientation.

c) A family of trees in a graph is *rootable* if the trees can be assigned roots so that a pair of them intersects if and only if at least one of the two roots belongs to both subtrees. Prove that G has a fraternal orientation if and only if G is the intersection graph of a rootable family of subtrees of some graph. (Gavril–Urrutia [1994])

8.1.16. (!) Prove that a simple graph G is a forest if and only if every pairwise intersecting family of paths in G has a common vertex. (Hint: For sufficiency, use induction on the number of paths in the family.)

8.1.17. (!) *Forbidden subgraph characterization of split graphs.* A graph is a **split graph** if its vertices can be partitioned into a clique and a stable set.

a) Prove that if G is a split graph, then G and $\overline{G}$ are chordal graphs. Observe that if G and $\overline{G}$ are chordal graphs, then G has no induced subgraph in $\{C_4, 2K_2, C_5\}$.

b) Prove that if G is a simple graph with no induced subgraph in $\{C_4, 2K_2, C_5\}$, then G is a split graph. (Hint: Among the maximum-sized cliques, let Q be one such that $G - Q$ has the minimum number of edges. Prove that $G - Q$ is a stable set, using the choice of Q and the forbidden subgraph conditions.) (Hammer–Simeone [1981])

8.1.18. Let $d_1 \ge \cdots \ge d_n$ be the degree sequence of a simple graph G, and let m be the largest value of k such that $d_k \ge k - 1$. Prove that G is a split graph if and only if $\sum_{i=1}^{m} d_i = m(m-1) + \sum_{i=m+1}^{n} d_i$. (Comment: Compare with Exercise 3.3.28.) (Hammer–Simeone [1981])

8.1.19. (−) Determine the trees that are split graphs, and construct a pair of nonisomorphic split graphs with the same degree sequence.

8.1.20. The k**-trees** are the graphs that arise from a k-clique by 0 or more iterations of adding a new vertex joined to a k-clique in the old graph. Prove that G is a k-tree if and only if G satisfies the following three properties:

1) G is connected.
2) G has a k-clique but no $k+2$-clique.
3) Every minimal vertex separator of G is a k-clique.

8.1.21. Let G be an n-vertex chordal graph having no clique of order $k+2$. Prove that $e(G) \le kn - \binom{k+1}{2}$, with equality if and only if G is a k-tree.

8.1.22. (+) Generalize Theorem 2.2.3 (Cayley's Formula) by proving that the number of k-trees with vertex set $[n]$ is $\binom{n}{k}[k(n-k)+1]^{n-k-2}$. (Hint: Generalize the Prüfer code for *rooted* trees, which generates a list with $n-1$ entries and never deletes the root. In a k-tree, the vertices belonging to exactly one $k+1$-clique are the *leaves*. A k-tree can be grown using any k-clique as a root. The lists generated from k-trees with a fixed root have as symbols 0 and pairs ij, where i comes from some k-set and j from some $n-k$-set.) (Greene–Iba [1975]; other proofs in Beineke–Pippert [1969], Moon [1969])

8.1.23. Suppose that G is a chordal graph with $\omega(G) = r$. Prove that G has at most $\binom{r}{j} + \binom{r-1}{j-1}(n-r)$ cliques of size j, with equality (for all j simultaneously) if and only if G is an $r-1$-tree.

8.1.24. *The Helly property of the real line.* Suppose that $I_1, \ldots, I_k$ are pairwise intersecting real intervals. Prove that $I_1, \ldots, I_k$ have a common point.

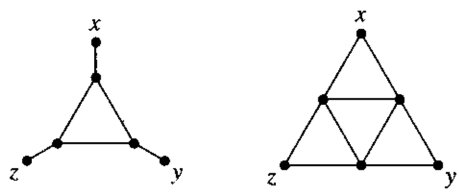
8.1.25. Prove directly that a tree is an interval graph if and only if it is a caterpillar (a tree having a path that contains at least one vertex of each edge).

8.1.26. (!) Let G be an interval graph. Prove that $\overline{G}$ is a comparability graph and that G is a chordal graph. (Hint: Establish a simplicial elimination ordering.)

8.1.27. Prove that a graph G has an interval representation if and only if the clique-vertex incidence matrix of G has the consecutive 1s property.

8.1.28. Prove that G is an interval graph if and only if the vertices of G can be ordered $v_1, \ldots, v_n$ such that $v_i \leftrightarrow v_k$ implies $v_j \leftrightarrow v_k$ whenever $i < j < k$. (Jacobson–McMorris–Mulder [1991], for example)

8.1.29. An **asteroidal triple** in a graph is a triple of vertices x, y, z such that between any two there exists a path avoiding the neighborhood of the third. Prove that no asteroidal triple occurs in an interval graph. (Comment: Interval graphs are precisely the chordal graphs that have no asteroidal triples) (Lekkerkerker–Boland [1962]))



8.1.30. Six professors visited the library on the day the rare book was stolen. Each entered once, stayed for some time, and then left. For any two of them that were in the library at the same time, at least one of them saw the other. Detectives questioned the professors and gathered the following testimony:

PROFESSOR	CLAIMED TO HAVE SEEN
Abe	Burt, Eddie
Burt	Abe, Ida
Charlotte	Desmond, Ida
Desmond	Abe, Ida
Eddie	Burt, Charlotte
Ida	Charlotte, Eddie

In this situation, "lying" means providing false information, not omitting information. Assume that the culprit tried to frame another suspect by lying. If one professor lied, who was it? (Golumbic [1980, p20])

8.1.31. (+) Prove that G is a unit interval graph (representable by intervals of the same length) if and only if $A(G) + I$ has the consecutive 1s property. (Roberts [1968])

8.1.32. (+) Prove that G is a proper interval graph (representable by intervals such that none properly contains another) if and only if the clique-vertex incidence matrix of G has the consecutive 1s property for both rows and columns. (Fishburn [1985]

8.1.33. (−) Prove that every P_4-free graph is a Meyniel graph.

8.1.34. (!) Prove that every chordal graph is o-triangulated.

8.1.35. Let C be an odd cycle in a graph with no induced odd cycle. Prove that $V(C)$ has three pairwise-adjacent vertices such that paths joining them in C all have odd length.

8.1.36. (+) Prove that the conditions below are equivalent.

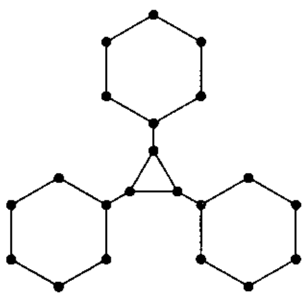
A) Every odd cycle of length at least 5 has a crossing pair of chords.

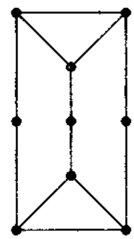
B) For every pair $x, y \in V(G)$, chordless x, y-paths are all even or all odd.

(Hint: For A $\Rightarrow$ B, consider a pair P_1, P_2 of x, y-paths with opposite parity such that the sum of their lengths is minimal.) (Burlet–Uhry [1984])

8.1.37. Prove that every perfectly orderable graph is strongly perfect. (Hint: Use Lemma 8.1.25) (Chvátal [1984])

8.1.38. (!) Prove that the graphs below are strongly perfect but not perfectly orderable.



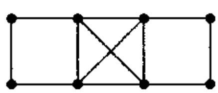


8.1.39. (−) Prove that the graph on the left above is a Meyniel graph but is not perfectly orderable. Prove that the graph $\overline{P_5}$ is perfectly orderable but is not a Meyniel graph.

8.1.40. (!) *Weakly chordal graphs.*

a) Prove that every chordal graph is weakly chordal.

b) Prove that the graph below is weakly chordal but not strongly perfect.



8.1.41. (−) A **skew partition** of G is a partition of $V(G)$ into two nonempty sets X, Y such that $G[X]$ is disconnected and $\overline{G}[Y]$ is disconnected. Chvátal [1985b] conjectured that no minimal imperfect graph has a skew partition. Prove that this implies the Star-Cutset Lemma and is implied by the SPGC.

8.1.42. Prove that the 10-vertex graph in Example 8.1.37 is 3, 3-partitionable. (Chvátal–Graham–Perold–Whitesides [1979])

8.1.43. (−) Let x and v be vertices of a partitionable graph G. Prove that if $x \not\leftrightarrow v$, then every maximum clique containing x consists of one vertex from each stable set that is the mate of a clique containing v. State the complementary assertion when $x \nleftrightarrow v$. (Buckingham–Golumbic [1983])

8.1.44. (+) Prove that no p-critical graph has **antitwins**, which are a pair of vertices such that every other vertex is adjacent to exactly one of them. (Hint: Given a p-critical graph with antitwins $\{x, y\}$, let S be the stable set containing y in the unique optimal coloring of $G - x$. Find among the vertices of the $\omega - 1$-colorable subgraph $G - x - S$ an $\omega - 1$ clique in $N(x)$ that doesn't extend into $N(y)$. Similarly, find a stable set in $N(y)$ that doesn't extend into $N(x)$. Now build an induced 5-cycle.) (Note: The partitionable graph of Example 8.1.37 has antitwins.) (Olariu [1988])

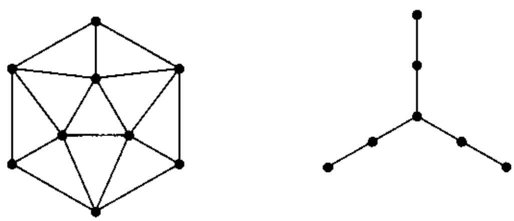
8.1.45. Vertices x, y form an **even pair** if every chordless x, y-path has even length (number of edges). **Twins** (nonadjacent vertices with the same neighborhood) are a special case.

a) Suppose that S_1, S_2 are maximum stable sets in a partitionable graph G. Prove that the subgraph of G induced by the symmetric difference of S_1 and S_2 is connected. (Bland–Huang–Trotter [1979])

b) Use part (a) to prove that no p-critical graph has an even pair. (Comment: Hence no p-critical graph has twins, which proves yet again that vertex duplication preserves perfection.) (Meyniel [1987], Bertschi–Reed [1988])

8.1.46. Let G be a partitionable graph, and let S_1, S_2 be stable sets in the optimal coloring of $G - x$. Use part (a) of the preceding problem to prove that the subgraph of G induced by $S_1 \cup S_2 \cup \{x\}$ is 2-connected. (Buckingham–Golumbic [1983])

8.1.47. Prove that one graph below is a circle graph but not a circular-arc graph, and prove that the other is a circular-arc graph but not a circle graph.



8.1.48. (!) The graph $K_{1,3} + e$ is the 4-vertex graph obtained by adding one edge to $K_{1,3}$. Using the perfection of Meyniel graphs, prove that $K_{1,3} + e$-free graphs satisfy the SPGC. (Meyniel [1976])

8.1.49. Let $G = C^{w-1}_{aw+1}$. Let $S = \{v_{aw}, v_1, v_w, v_{w+2}\} \cup \{v_{iw+1}\colon 2 \le i \le a-1\}$, and let $T = \{v_{(a-1)w+1}, v_{aw}, v_1, v_w\} \cup \{v_{w+i}\colon 2 \le i \le w-1\}$. Prove that S intersects every maximum clique of G and that T intersects every maximum stable set of G. (Chvátal [1976])

8.1.50. (!) *SPGC for circle graphs.* (Buckingham–Golumbic [1983])

a) Use Lemma 8.1.28 to prove that if x is a vertex in a partitionable graph G, then $G - N[x]$ is connected, where $N[x] = N(x) \cup \{x\}$.

b) Use part (a) to prove that partitionable circle graphs are $K_{1,3}$-free.

c) Conclude from part (b) and Corollary 8.1.53 that the SPGC holds for circle graphs.

8.2. Matroids

Many results of graph theory extend or simplify in the theory of matroids. These include the greedy algorithm for minimum spanning trees, the strong duality between maximum matching and minimum vertex cover in bipartite graphs, and the geometric duality relating planar graphs and their duals.

Matroids arise in many contexts but are special enough to have rich combinatorial structure. When a result from graph theory generalizes to matroids, it can then be interpreted in other special cases. Several difficult theorems about graphs have found easier proofs using matroids.

Matroids were introduced by Whitney [1935] to study planarity and algebraic aspects of graphs, by MacLane [1936] to study geometric lattices, and by van der Waerden [1937] to study independence in vector spaces. Most of the language comes from these contexts. Here we emphasize applications to graphs.

HEREDITARY SYSTEMS AND EXAMPLES

In many mathematical contexts, we study sets that avoid conflicts; often this is called "independence". Inherent in this notion is that subsets of independent sets are independent, and the empty set is independent.

8.2.1. Example. *Acyclic sets of edges.* Let E be the edge set of a graph G, and let $X \subseteq E$ be "independent" if it contains no cycle. Every subset of an independent set is independent, and the empty set is independent. The cycles are the minimal dependent sets.

Consider the kite $K_4 - e$, which has five edges. Since spanning trees of this graph have three edges, every set having more than three edges is dependent. Also the two triangles are dependent; this yields eight dependent sets and 24 independent sets among the subsets of E. There are three minimal dependent sets (the cycles) and eight maximal independent sets (the spanning trees). ■

8.2.2. Definition. A **hereditary family** or **ideal** is a collection of sets, $\mathbf{F}$, such that every subset of a set in $\mathbf{F}$ is also in $\mathbf{F}$. A **hereditary system** M on E consists of a nonempty ideal $\mathbf{I}_M$ of subsets of E and the various ways of specifying that ideal, called *aspects* of M.

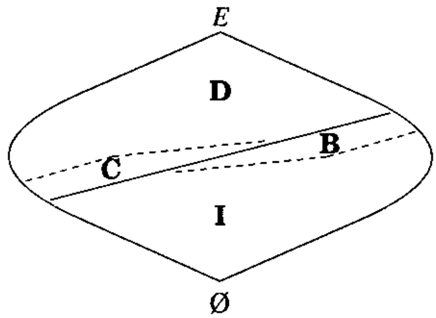
The elements of $\mathbf{I}_M$ are the **independent sets** of M. The other subsets of E (comprising the family $\mathbf{D}_M$) are **dependent**. The **bases** are the maximal independent sets, and the **circuits** are the minimal dependent sets; $\mathbf{B}_M$ and $\mathbf{C}_M$ denote these families of subsets of E.

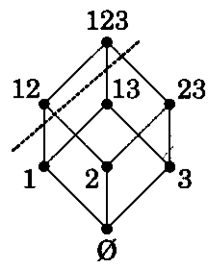
The **rank** of a subset of E is the maximum size of an independent set in it. The **rank function** r_M is defined by $r(X) = \max\{|Y| : Y \subseteq X,\ Y \in \mathbf{I}\}$.

8.2.3. Example. *Hereditary systems.* Label each vertex $a = (a_1, \dots, a_n)$ of the hypercube Q_n by the corresponding set $X_a = \{i : a_i = 1\}$. Draw Q_n in the plane so that the vertical coordinates of vertices are in order by the size of the sets labeling them.

The diagram below illustrates the relationships among the independent sets, bases, circuits, and dependent sets of a hereditary system. The bases are the maximal elements of the family **I** and the circuits are the minimal elements not in **I**. In every hereditary system, Ø belongs to **I**. If every set is independent, then there is no circuit, but there is always at least one base.

In the example on the right, the independent sets are the acyclic edge sets in a graph with three edges. The only dependent sets are $\{1, 2\}$ and $\{1, 2, 3\}$, the only circuit is $\{1, 2\}$, and the bases are $\{1, 3\}$ and $\{2, 3\}$. The rank of an independent set is its size. For the dependent sets, we have $r(\{1, 2\}) = 1$ and $r(\{1, 2, 3\}) = 2$. ■





8.2.4. Remark. *Aspects of hereditary systems.* A hereditary system M is determined by any of $\mathbf{I}_M$, $\mathbf{B}_M$, $\mathbf{C}_M$, r_M, etc., because each aspect specifies the others. We have expressed $\mathbf{B}_M, \mathbf{C}_M, r_M$ in terms of $\mathbf{I}_M$. Conversely, if we know $\mathbf{B}_M$, then $\mathbf{I}_M$ consists of the sets contained in members of $\mathbf{B}_M$. If we know $\mathbf{C}_M$, then $\mathbf{I}_M$ consists of the sets containing no member of $\mathbf{C}_M$. If we know r_M, then $\mathbf{I}_M = \{X \subseteq E\colon r_M(X) = |X|\}$. ■

Hereditary systems are too general to behave nicely. We restrict our attention to hereditary systems having an additional property, and these we call matroids. We can translate any restriction on $\mathbf{I}_M$ into a corresponding restriction on some other aspect of the hereditary system. Because hereditary systems can be specified in many ways, we have many equivalent definitions of matroids. Using various motivating examples, we state several of these properties that characterize matroids. Later we prove that they are equivalent. We begin with the fundamental example from graphs.

8.2.5. Definition. The **cycle matroid** $M(G)$ of a graph G is the hereditary system on $E(G)$ whose circuits are the cycles of G. A hereditary system that is $M(G)$ for some graph G is a **graphic matroid**.

8.2.6. Example. *Bases in cycle matroids.* The bases of the cycle matroid $M(G)$ are the edge sets of the maximal forests in G. Each maximal forest contains a spanning tree from each component, so they have the same size. Consider $B_1, B_2 \in \mathbf{B}$ with $e \in B_1 - B_2$. Deleting e from B_1 disconnects some component of B_1; since B_2 contains a tree spanning that component of G, some edge $f \in B_2 - B_1$ can be added to $B_1 - e$ to reconnect it.

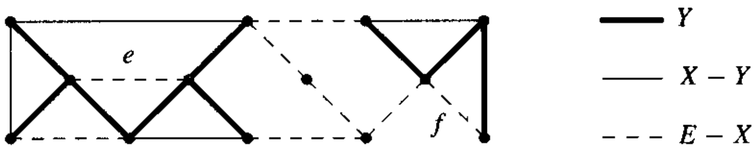
For a hereditary system M, the **base exchange property** is: if $B_1, B_2 \in \mathbf{B}_M$, then for all $e \in B_1 - B_2$ there exists $f \in B_2 - B_1$ such that $B_1 - e + f \in \mathbf{B}_M$. Matroids are the hereditary systems satisfying the base exchange property. ■

8.2.7. Remark. In this subject, we often discuss inclusion and omission of single elements from sets. For symmetry and simplicity, we use the symbols $+$ and $-$ instead of $\cup$ and $-$ for this, and we drop the set brackets on 1-element sets. ■

8.2.8. Example. *Rank function in cycle matroids.* Let G be a graph with n vertices. For $X \subseteq E(G)$, let G_X denote the spanning subgraph of G with edge set X. In $M(G)$, an independent subset of X is the edge set of a forest in G_X. When G_X has k components, the maximum size of such a forest is $n - k$. Hence $r(X) = n - k$. Below we show such a forest Y (bold) within X (bold and solid).

If $r(X+e) = r(X)$ for some $e \in E - X$, then the endpoints of e lie in a single component of G_X; adding e does not combine components. If we add two such edges, then again we do not combine components. Therefore, $r(X) = r(X+e) = r(X+f)$ implies $r(X) = r(X+e+f)$.

For a hereditary system M on E, the **(weak) absorption property** is: if $X \subseteq E$ and $e, f \in E$, then $r(X) = r(X+e) = r(X+f)$ implies $r(X+e+f) = r(X)$. Matroids are the hereditary systems satisfying the absorption property (name suggested by A. Kézdy). ■



Graphs may have loops and multiple edges. In cycle matroids, they lead to circuits of sizes 1 and 2. We use these terms for hereditary systems in general.

8.2.9. Definition. In a hereditary system, a **loop** is an element forming a circuit of size 1. **Parallel elements** are distinct non-loops forming a circuit of size 2. A hereditary system is **simple** if it has no loops or parallel elements.

8.2.10. Definition. The **vectorial matroid** on a set E of vectors in a vector space is the hereditary system whose independent sets are the linearly independent subsets of vectors in E. A matroid expressible in this way is a **linear matroid** (or **representable matroid**). The **column matroid** $M(A)$ of a matrix A is the vectorial matroid defined on its columns.

8.2.11. Example. *Circuits in vectorial matroids.* The set E may have repeated vectors; these would be parallel elements. The circuits are the minimal sets $\{x_1, \dots, x_k\} \subseteq E$ such that $\sum c_i x_i = 0$ using coefficients not all zero. Minimality forces all $c_i \neq 0$.

Let C_1, C_2 be distinct circuits containing x. Using the equations of dependence for C_1 and C_2, we can write x as a linear combination in terms of $C_1 - x$ and in terms of $C_2 - x$. Equating these expressions yields an equation of dependence for $C_1 \cup C_2 - x$; thus $C_1 \cup C_2 - x$ contains a circuit.

For a hereditary system M on E, the **(weak) elimination property** is: whenever C_1, C_2 are distinct circuits and $x \in C_1 \cap C_2$, another member of $\mathbf{C}_M$ is contained in $C_1 \cup C_2 - x$. Matroids are the hereditary systems satisfying the weak elimination property.

The column matroid of the matrix below is also the cycle matroid $M(K_4 - e)$.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$ ∎

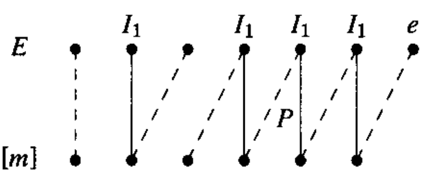
8.2.12. Definition. The **transversal matroid** induced by sets $A_1, \dots, A_m$ with union E is the hereditary system on E whose independent sets are the systems of distinct representatives of subsets of $\{A_1, \dots, A_m\}$. Equivalently, letting G be the $E, [m]$-bigraph defined by $e \leftrightarrow i$ if and only if $e \in A_i$, the independent sets are the subsets of E that are saturated by matchings in G.

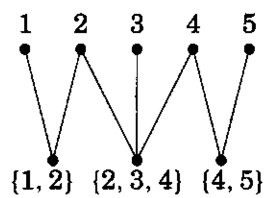
8.2.13. Example. *Independent sets in transversal matroids.* When M, M' are matchings in G and $|M'| > |M|$, the symmetric difference $M \triangle M'$ contains an M-augmenting path P (Theorem 3.1.10). Replacing $M \cap P$ with $M' \cap P$ yields a matching of size $|M| + 1$ that saturates all vertices of M plus the endpoints of P.

Consider independent sets I_1, I_2 in the transversal matroid generated by $A_1, \dots, A_m$. In the associated bipartite graph, let M_1, M_2 be matchings saturating I_1, I_2, respectively (on the left below, M_1 is solid and M_2 is dashed). If $|I_2| > |I_1|$, then the matching obtained from M_1 by using an M_1-augmenting path in $M_2 \triangle M_1$ saturates I_1 plus an element $e \in I_2 - I_1$; this "augments" I_2.

For a hereditary system on E, the **augmentation property** is: for distinct $I_1, I_2 \in \mathbf{I}$ with $|I_2| > |I_1|$, there exists $e \in I_2 - I_1$ such that $I_1 \cup \{e\} \in \mathbf{I}$. Matroids are the hereditary systems satisfying the augmentation property.

The transversal matroid of the family $\mathbf{A} = \{\{1, 2\}, \{2, 3, 4\}, \{4, 5\}\}$, illustrated by the bipartite graph on the right, is again $M(K_4 - e)$. ∎



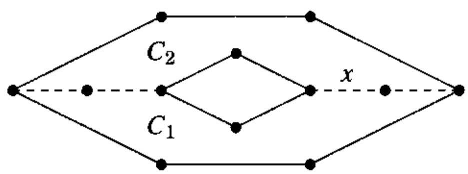


The name "transversal matroid" arises from the use of "transversal" in systems of distinct representatives. An SDR for a subset of $\{A_1, \ldots, A_m\}$ is a **partial transversal** for the full system. The independent sets of the transversal matroid on $\bigcup A_i$ are the partial transversals of $\{A_1, \ldots, A_m\}$. That these are matroids was discovered by Edmonds and Fulkerson [1965] and independently by Mirsky and Perfect [1967], who extended the result to infinite sets.

Every matroid must satisfy all properties of matroids. Once we show that the properties defined above are equivalent for hereditary systems, we need only verify one to use all. First we check that they all hold for cycle matroids.

8.2.14. Example. *Augmentation in cycle matroids.* Consider $I_1, I_2 \in \mathbf{I}_{M(G)}$. As in Example 8.2.8, the spanning subgraph G_{I_1} has $k = n - |I_1|$ components, and its largest forest has $n - k = |I_1|$ edges. Therefore, the forest I_2 has some edge with endpoints in two components of G_{I_1}. This edge can be added to I_1 to obtain a larger independent set. Hence the augmentation property holds. ■

8.2.15. Example. *Weak elimination in cycle matroids.* The circuits of $M(G)$ are the edge sets of cycles of G. Cycles have even degree at each vertex. If $C_1, C_2 \in \mathbf{C}$, then the symmetric difference $C_1 \triangle C_2$ also has even degree at each vertex. If $C_1 \neq C_2$, this implies that $C_1 \triangle C_2$ contains a cycle (see Proposition 1.2.27). This is stronger than the weak elimination property, since $C_1 \triangle C_2 \subseteq C_1 \cup C_2 - x$. In the figure below, C_1 and C_2 are face boundaries of length 9 sharing the dashed edges, and $C_1 \triangle C_2$ is the union of two disjoint cycles. ■



For transversal matroids, the base exchange property is similar to the augmentation property; Exercise 9 considers the weak elimination property. For linear matroids, directly verifying the augmentation or base exchange property requires the algebraic result that k linearly independent vectors cannot all be expressed as linear combinations of a smaller set. Instead, we can use Theorem 8.2.20. Since the weak elimination property holds for independent sets of vectors, many theorems of linear algebra follow from Theorem 8.2.20!

8.2.16. Remark. *Notational conventions*: Boldface $\mathbf{I}, \mathbf{B}, \mathbf{C}$ for families of subsets of E allows $I \in \mathbf{I}$, $B \in \mathbf{B}$, $C \in \mathbf{C}$ to denote members of the families. Roman letters I, B, C, R denote properties that yield matroids. We use e, f, x, y as elements of E, and we use X, Y, F as subsets of E. ■

Every hereditary family is the collection of independent sets of a hereditary system. A collection $\mathbf{B}$ is realizable as the set of bases of a hereditary system if and only if $\mathbf{B}$ is nonempty and no element of $\mathbf{B}$ contains another. A collection