Now let us replace this set of numbers by the set $\{1,\ 2,\ 4,\ 8,\ 16,\ 32\}$ and consider

$$(1+x)(1+x^2)(1+x^4)(1+x^8)(1+x^{16})(1+x^{32}) = \sum_{i=0}^{\infty} f(n)x^n.$$

Then $f(n)$ is the number of ways in which n can be written as the sum of distinct powers of 2, up to 32. Clearly, $f(n) = 0$ when $n \geq 64$. What if $n < 64$? The left-hand side can also be written

$$\frac{1-x^2}{1-x} \cdot \frac{1-x^4}{1-x^2} \cdot \frac{1-x^8}{1-x^4} \cdot \frac{1-x^{16}}{1-x^8} \cdot \frac{1-x^{32}}{1-x^{16}} \cdot \frac{1-x^{64}}{1-x^{32}} = \frac{1-x^{64}}{1-x}$$

$$= 1 + x + x^2 + \cdots + x^{63} = \sum_{n=0}^{63} x^n.$$

Hence $f(n) = 1$ if $n < 64$.

Suppose, instead of stopping with x^{32}, we form the *infinite* product $\prod_{n=0}^{\infty}(1 + x^{2^n})$. Then we can show similarly that *every* natural number can be written in the scale of 2.

Exercises

1. Write out a scale 7 multiplication table.

2. Show how to convert a scale 10 numeral to a scale 7 numeral.

3. Give a proof, in the spirit of the 18th century, that every natural number can be written uniquely in the scale of 3. (Hint: form the infinite product $(1+x+x^2)(1+x^3+x^6)(1+x^9+x^{18})\cdots$ and evaluate it in two different ways.)

4. Likewise, show that any integer can be written uniquely as $\sum_{k=0}^{n} a_k 3^k$, where $a_k = -1, 0$ or 1.

3
Prime Numbers

It would be impossible to write a history of mathematics without mentioning prime numbers, and it would be improper to give an account of prime numbers without going into the history of mathematics. Prime numbers enter into almost every branch of mathematics; they are as fundamental as they are ubiquitous. Their history can be used as a framework for a history of mathematics generally. In this chapter, we take a brief look at the fascinating subject of primes.

The Egyptians might have written

$$\frac{4}{5} = \frac{1}{2} + \frac{1}{4} + \frac{1}{20} \,.$$

From this, it follows that

$$\frac{4}{10} = \frac{1}{4} + \frac{1}{8} + \frac{1}{40}$$

and that

$$\frac{4}{15} = \frac{1}{6} + \frac{1}{12} + \frac{1}{60} \,.$$

The moral to be drawn from this is that, to express a/b as a sum of unit fractions, it suffices to consider the case when b cannot be factored into smaller numbers. An integer greater than 1 which cannot be factored into numbers, all of which are smaller than the original integer, is called *prime*. The first few primes are

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, \ldots.$$

Note that a positive integer is *prime* if and only if it has exactly two positive integer divisors.

Early on, people noticed that a pile of small stones can sometimes be arranged in a rectangle and sometimes it cannot. Thus, although we do not have any record of this, the Egyptians probably knew the difference between composite and prime numbers. Indeed, it is not impossible that some Egyptian scribe may have noticed that, if every proper fraction of the form $4/p$, with p prime, and greater than 3, can be expressed as a sum of three distinct unit fractions, then every proper fraction of the form $4/n$, with n any positive integer greater than 4, can be so expressed. (See the problem of Erdös, mentioned in Chapter 1.)

It was the Greeks who first *proved* that the number of primes is infinite. A proof is found in Euclid's *Elements* (300 BC).

Euclid's Lemma (Book VII Proposition 31):

Every integer $n > 1$ is divisible by some prime number.

Proof: Among the divisors of n which are greater than 1, let p be the smallest. Then p has no divisors other than 1 and p — any other divisor of p would be a divisor of n as well — and hence p is prime.

Euclid's Theorem (Book IX Proposition 20):

Given any finite list of primes $p_1, p_2 \ldots p_k$, there is a prime not on this list.

Proof: Consider the number $n = p_1 p_2 \cdots p_k + 1$. Clearly, n is not divisible by any of the primes on the list; for, upon dividing n by p_i, we get remainder 1. From the lemma we know that n does have a prime factor q (possibly n itself). Hence there is a prime, namely, q, which is not on the list. QED.

In Proposition 14 of Book IX, Euclid proved that, if n is a square-free positive integer (that is, one with no square factor other than 1), then n has a factorization into primes which is unique (if you list the prime factors in order of increasing size). However, it was not until 1801 that the unique factorization was formally proved for *any* positive integer n. This was done by Carl Friedrich Gauss (1777–1855) in his *Disquisitiones Arithmeticae*. Although mathematicians used the unique factorization theorem long before 1801, and although almost any one of them could have found a proof for it, Gauss was the first person actually to sit down and do so. Perhaps the other mathematicians considered the theorem too obvious to be worth proving. One way to prove that every positive integer greater than 1 has a unique factorization into primes is as follows.

Proof of the Unique Factorization Theorem:

Let n be the smallest positive integer, if there is one, which has 2 (or more) factorizations into primes:

$$n = pqr \ldots = p'q'r' \ldots.$$

We assume the primes are written in nondecreasing order. By minimality of n, $p \neq p'$ (or we could cancel off the p's and get a smaller number with