(6) In many situations it is easier to specify an explicit matrix representation of a group G rather than to exhibit an FG-module. For example, recall that the dihedral group D_{2n} has the presentation

$$D_{2n} = \langle r, s \mid r^n = s^2 = 1,\ rs = sr^{-1} \rangle.$$

If R and S are any matrices satisfying the relations $R^n = S^2 = I$ and $RS = SR^{-1}$ then the map $r \mapsto R$ and $s \mapsto S$ extends uniquely to a homomorphism from D_{2n} to the matrix group generated by R and S, hence gives a representation of D_{2n}. An explicit example of matrices $R, S \in M_2(\mathbb{R})$ may be obtained as follows. If a regular n-gon is drawn on the x, y plane centered at the origin with the line $y = x$ as one of its lines of symmetry then the matrix R that rotates the plane through $2\pi/n$ radians and the matrix S that reflects the plane about the line $y = x$ both send this n-gon onto itself. It follows that these matrices act as symmetries of the n-gon and so satisfy the above relations. These matrices are readily computed (cf. Exercise 25, Section 1.6) and so the maps

$$r \mapsto R = \begin{pmatrix} \cos 2\pi/n & -\sin 2\pi/n \\ \sin 2\pi/n & \cos 2\pi/n \end{pmatrix} \quad \text{and} \quad s \mapsto S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

extend uniquely to a (degree 2) representation of D_{2n} into $GL_2(\mathbb{R})$. Since the matrices R and S have orders n and 2 respectively, it follows that they generate a subgroup of $GL_2(\mathbb{R})$ of order $2n$ and hence this representation is faithful.

(7) By using the usual generators and relations for the quaternion group

$$Q_8 = \langle i, j \mid i^4 = j^4 = 1,\ i^2 = j^2,\ i^{-1}ji = j^{-1} \rangle$$

one may similarly obtain (cf. Exercise 26, Section 1.6) a representation φ from Q_8 to $GL_2(\mathbb{C})$ defined by

$$\varphi(i) = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix} \quad \text{and} \quad \varphi(j) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

This representation of Q_8 is faithful.

(8) A 4-dimensional representation of the quaternion group Q_8 may be obtained from the real Hamilton quaternions, $\mathbb{H}$ (cf. Section 7.1). The group Q_8 is a subgroup of the multiplicative group of units of $\mathbb{H}$ and each of the elements of Q_8 acts by left multiplication on the 4-dimensional real vector space $\mathbb{H}$. Since the real numbers are in the center of $\mathbb{H}$ (i.e., since $\mathbb{H}$ is an $\mathbb{R}$-algebra), left multiplication is $\mathbb{R}$-linear. This linear action thus gives a homomorphism from Q_8 into $GL_4(\mathbb{R})$. One can easily write out the explicit matrices of each of the elements of Q_8 with respect to the basis $1, i, j, k$ of $\mathbb{H}$. For example, left multiplication by i acts by $1 \mapsto i$, $i \mapsto -1$, $j \mapsto k$ and $k \mapsto -j$ and left multiplication by j acts by $1 \mapsto j$, $i \mapsto -k$, $j \mapsto -1$ and $k \mapsto i$ so

$$i \mapsto \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad j \mapsto \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

This representation of Q_8 is also faithful.

(9) Suppose that H is a normal subgroup of the group G and suppose that H is an elementary abelian p-group for some prime p. Then $V = H$ is a vector space over $\mathbb{F}_p$, where the scalar a acts on the vector v by $av = v^a$ (see Section 10.1). The action of each element of G by conjugation on V is $\mathbb{F}_p$-linear because $gv^ag^{-1} = (gvg^{-1})^a$ and this action of G on V makes V into an $\mathbb{F}_pG$-module (the automorphisms of elementary abelian p-groups were discussed in Sections 4.4 and 10.1). The kernel of

this representation is the set of elements of G that commute with every element of H, $C_G(H)$ (which always contains the abelian group H itself). Thus the action of a group on subsets of itself often affords linear representations over finite fields. Representations of groups over finite fields are called *modular representations* and these are fundamental to the study of the internal structure of groups.

(10) For an example of an FG-submodule, let $G = S_n$ and let V be the FS_n-module described in Example 3. Let N be the subspace of V consisting of vectors all of whose coordinates are equal, i.e.,

$$N = \{\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n \mid \alpha_1 = \alpha_2 = \cdots = \alpha_n\}$$

(this is a 1-dimensional S_n-stable subspace). Each $\sigma \in S_n$ fixes each vector in N so the submodule N affords the trivial representation of S_n. As an exercise, one may show that if $n \geq 3$ then N is the *unique* 1-dimensional subspace of V which is S_n-stable, i.e., N is the unique 1-dimensional FS_n-submodule (N is called the *trace* submodule of FS_n).

Another FS_n-submodule of V is the subspace I of all vectors whose coordinates sum to zero:

$$I = \{\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n \mid \alpha_1 + \alpha_2 + \cdots + \alpha_n = 0\}.$$

Again I is an S_n-stable subspace (since each $\sigma \in S_n$ permutes the coordinates of each vector in V, each σ leaves the sum of the coefficients unchanged). Since I is the kernel of the linear transformation from V onto F which sends a vector to the sum of its coefficients (called the augmentation map — cf. Section 7.3), I has dimension $n - 1$.

(11) If $V = FG$ is the regular representation of G described in Example 2 above, then V has FG-submodules of dimensions 1 and $|G| - 1$ as in the preceding example:

$$N = \{\alpha_1 g_1 + \alpha_2 g_2 + \cdots + \alpha_n g_n \mid \alpha_1 = \alpha_2 = \cdots = \alpha_n\}$$
$$I = \{\alpha_1 g_1 + \alpha_2 g_2 + \cdots + \alpha_n g_n \mid \alpha_1 + \alpha_2 + \cdots + \alpha_n = 0\}.$$

In fact N and I are 2-sided ideals of FG (not just left ideals — note that N is in the center of FG). The ideal I is called the *augmentation ideal* of FG and N is called the *trace ideal* of FG.

Recall that in the study of a linear transformation T of a vector space V to itself we made V into an $F[x]$-module (where x acted as T on V); our goal was to decompose V into a direct sum of cyclic submodules. In this way we were able to find a basis of V for which the matrix of T with respect to this basis was in some *canonical* form. Changing the basis of V did not change the module V but changed the matrix representation of T by similarity (i.e., changed the isomorphism between $GL(V)$ and $GL_n(F)$). We introduce the analogous terminology to describe when two FG-modules are the same up to a change of basis.

Definition. Two representations of G are *equivalent* (or *similar*) if the FG-modules affording them are isomorphic modules. Representations which are not equivalent are called *inequivalent*.

Suppose $\varphi : G \to GL(V)$ and $\psi : G \to GL(W)$ are equivalent representations (here V and W must be vector spaces over the same field F). Let $T : V \to W$ be an FG-module isomorphism between them. Since T is, in particular, an F-module isomorphism, T is a vector space isomorphism, so V and W must have the same dimension. Furthermore, for all $g \in G$, $v \in V$ we have $T(g \cdot v) = g \cdot (T(v))$, since T is an isomorphism of FG-modules. By definition of the action of ring elements this means $T(\varphi(g)v) = \psi(g)(T(v))$, that is

$$T \circ \varphi(g) = \psi(g) \circ T \qquad \text{for all } g \in G.$$

In particular, if we identify V and W as vector spaces, then two representations φ and ψ of G on a vector space V are equivalent if and only if there is some $T \in GL(V)$ such that $T \circ \varphi(g) \circ T^{-1} = \psi(g)$ for all $g \in G$. This T is a *simultaneous* change of basis for all $\varphi(g)$, $g \in G$.

In matrix terminology, two representations φ and ψ are equivalent if there is a fixed invertible matrix P such that

$$P\varphi(g)P^{-1} = \psi(g) \qquad \text{for all } g \in G.$$

The linear transformation T or the matrix P above is said to *intertwine* the representations φ and ψ (it gives the "rule" for changing φ into ψ).

In order to study the decomposition of an FG-module into (direct sums of) submodules we shall need some terminology. We state these definitions for arbitrary rings since we shall be discussing direct sum decompositions in greater generality in the next section.

Definition. Let R be a ring and let M be a nonzero R-module.

(1) The module M is said to be *irreducible* (or *simple*) if its only submodules are 0 and M; otherwise M is called *reducible*.
(2) The module M is said to be *indecomposable* if M cannot be written as $M_1 \oplus M_2$ for any nonzero submodules M_1 and M_2; otherwise M is called *decomposable*.
(3) The module M is said to be *completely reducible* if it is a direct sum of irreducible submodules.
(4) A representation is called *irreducible, reducible, indecomposable, decomposable* or *completely reducible* according to whether the FG-module affording it has the corresponding property.
(5) If M is a completely reducible R-module, any direct summand of M is called a *constituent* of M (i.e., N is a constituent of M if there is a submodule N' of M such that $M = N \oplus N'$).

An irreducible module is, by definition, both indecomposable and completely reducible. We shall shortly give examples of indecomposable modules that are not irreducible.

If $R = FG$, an irreducible FG-module V is a nonzero F-vector space with no nontrivial, proper G-invariant subspaces. For example, if $\dim_F V = 1$ then V is necessarily irreducible (its only subspaces are 0 and V).

Suppose V is a finite dimensional FG-module and V is reducible. Let U be a G-invariant subspace. Form a basis of V by taking a basis of U and enlarging it to a