Thus, we wish to solve the equation $z^5 = 1$, that is,

$$(z-1)(z^4+z^3+z^2+z+1)=0,$$

where $z = \cos\theta + i\sin\theta$. Note that $z^{-1} = \cos\theta - i\sin\theta$, so that

$$2\cos\theta = z + z^{-1} = u$$

say. Clearly, $z = 1$ is not a satisfactory solution, so our problem reduces to solving

$$z^4+z^3+z^2+z+1=0\,.$$

Dividing by z^2, we write this as

$$z^2+z+1+z^{-1}+z^{-2}=0\,.$$

Now $z+z^{-1}=u$, hence $z^2+2+z^{-2}=u^2$, so that $z^2+z^{-2}=u-2$. The equation to be solved then becomes

$$u^2+u-1=0\,.$$

Discarding the negative solution, we find

$$u=\frac{-1+\sqrt{5}}{2}\,,$$

a number which we recall from Chapter 12 as the golden section.

Of course this is not how the Greeks attacked the problem, as they did not know about complex numbers, and, at the time of Euclid, had not yet invented trigonometry. Nonetheless, our analysis shows that they could construct the angle θ, since $2\cos\theta = (-1+\sqrt{5})/2$ involves only rational operations and square roots. Indeed, when Euclid constructs a regular pentagon in Book IV, Proposition 11, by what looks to us like a rather complicated method, he makes use (via Proposition 10) of the earlier construction of the golden section in Book II, Proposition 11 (taken up once more in Book VI, Proposition 30). The Greeks could of course construct squares and regular hexagons, but one problem they did leave open was the following:

IV constructing a regular heptagon, that is, a seven sided figure.

By the same method we just employed for constructing a regular pentagon, we can see that this problem reduces to the following cubic equation in $u = 2\cos(360°/7)$:

$$u^3+u^2-2u-1=0\,.$$

Although the ancient Greeks did not know this, the only arithmetical operations that can be carried out with ruler and compass constructions are rational operations and square root extractions and, of course, combinations of such. To understand why this is so, we have to make use of analytic geometry, which was only developed in the seventeenth century by René Descartes.

His pioneering idea was to represent every point in the plane by a pair of real numbers (x, y) and to observe, conversely, that every such pair represents a point. Unlike the Greeks, we need not confine x and y to be positive: if we use the modern rectangular coordinate system, x is negative in the second and third quadrant, y is negative in the third and fourth quadrant. We can now say that a straight line consists of all points (x, y) satisfying an equation of the form

$$ax + by + c = 0,$$

where a, b and c are given real numbers, and a circle consists of all points (x, y) satisfying an equation of the form

$$x^2 + y^2 + dx + ey + f = 0.$$

Now what happens when we perform the following operations:

1. join two given points,
2. draw a circle with given center and radius,
3. intersect two straight lines,
4. intersect a circle and a straight line,
5. intersect two circles?

(1) Suppose the given points are (x_1, y_1) and (x_2, y_2). Then we easily see that the straight line has the equation

$$(y_1 - y_2)x + (x_2 - x_1)y + (x_1y_2 - x_2y_1) = 0,$$

in other words, an equation of the form

$$ax + by + c = 0,$$

where a, b and c are expressed in terms of the given quantities x_1, y_1, x_2, y_2 by means of the operations of addition, subtraction and multiplication.

(2) Suppose the center is (α, β) and the radius is ρ, then the equation of the circle is

$$(x - \alpha)^2 + (y - \beta)^2 = \rho^2,$$

that is,

$$x^2 + y^2 - 2\alpha x - 2\beta y + \alpha^2 + \beta^2 - \rho^2 = 0.$$

Thus the equation of the circle is of the form

$$x^2 + y^2 + dx + ey + f = 0,$$

where d, e and f are again expressed in terms of the given quantities α, β and ρ by means of addition, subtraction and multiplication.

(3) To find the intersection of two given straight lines, we must solve the pair of equations:

$$ax + by + c = 0,$$

$$a'x + b'y + c' = 0.$$

We obtain the solution

$$x = -\frac{cb' - c'b}{ab' - a'b}, \quad y = -\frac{ac' - a'c}{ab' - a'b}.$$

(It is assumed that $ab' - a'b \neq 0$, otherwise the two lines are parallel or even coincide.) Again we see that the new quantities x and y are obtained from the given quantities a, b, c, a', b' and c' by means of the rational operations, including division.

(4) To find the intersection of a circle and a straight line, we must solve the pair of equations:

$$ax + by + c = 0,$$

$$x^2 + y^2 + dx + ey + f = 0.$$

Assuming, for example, that $b \neq 0$, we get

$$y = -\frac{a}{b}x - \frac{c}{b}$$

from the first equation. When we substitute this into the second equation, we obtain a quadratic equation:

$$Ax^2 + Bx + C = 0,$$

where A, B and C are expressed by means of the rational operations in terms of the given quantities a, b, c, d, e and f. In particular, $A = 1 + a^2/b^2 > 0$. Finally, solving for x, we obtain:

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}.$$