16

Complex Numbers and Functions

16.1 Complex Functions

When Bombelli (1572) introduced complex numbers, he implicitly introduced complex functions as well. The solution y of the cubic equation $y^3 = py + q$,

$$y = \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}} + \sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}},$$

involves the cube root of a complex argument when $(q/2)^2 < (p/3)^3$. However, in this context complex functions were no more (or less) problematic than complex numbers. It was sometimes surprising to find that functions turned out to be equal, as when Leibniz and de Moivre showed (Section 6.6) that

$$x = \frac{1}{2}\sqrt[n]{y + \sqrt{y^2 - 1}} + \frac{1}{2}\sqrt[n]{y - \sqrt{y^2 - 1}},$$

where y is the polynomial in $x = \sin\theta$ that equals $\sin n\theta$, but one did not worry about the meaning of the functions as long as their equations could be checked by algebra.

Things became more puzzling with transcendental functions, in particular those defined by integration. A key example was the logarithm function, which comes from integrating $dz/(1+z)$. Once this function was understood, the reason for algebraic miracles like the Leibniz–de Moivre theorem became much clearer.

The story of the complex logarithm began when Johann Bernoulli (1702) noted that

$$\frac{dz}{1+z^2} = \frac{dz}{2(1+z\sqrt{-1})} + \frac{dz}{2(1-z\sqrt{-1})}$$

and drew the conclusion that "imaginary logarithms express real circular sectors." He did not actually perform the integration, but he could have got

$$\tan^{-1} z = \frac{1}{2i} \log \frac{i-z}{i+z}$$

since Euler gives him credit for a similar formula when writing to him in Euler (1728b). However, this may have been the young Euler's deference to his former teacher, because Johann Bernoulli showed poor understanding of logarithms as the correspondence continued. He persistently claimed that $\log(-x) = \log(x)$ on the grounds that

$$\frac{d}{dx}\log(-x) = \frac{1}{x} = \frac{d}{dx}\log(x)$$

despite a reminder from Euler (1728b) that equality of derivatives does not imply equality of integrals. Euler went on to suggest that the complex logarithm had infinitely many values.

In the meantime, Cotes (1714) had also discovered a relation between complex logarithms and circular functions:

$$\log(\cos x + i \sin x) = ix.$$

Recognizing the importance of this result, he entitled his work *Harmonia mensurarum* (Harmony of measures). The "measures" in question were the logarithm and inverse tangent functions, which "measured" the hyperbola and the circle, respectively, via the integrals $\int dx/(1+x)$ and $\int dx/(1+x^2)$. A wide class of integrals had been reduced to these two types, but it was not understood why two apparently unrelated "measures" should be required. Cotes' result was the first [apart from the near-miss of Johann Bernoulli (1702)] to relate the two, showing that in the wider domain of complex functions the logarithm and inverse circular functions are essentially the same.

The most compact statement of their relationship was attained around 1740, when Euler shifted attention from the logarithm function to its inverse, the exponential function. The definitive formula

$$e^{ix} = \cos x + i \sin x$$

was first published by Euler (1748a), who derived it by comparing series expansions of both sides. Euler's formulation in terms of the single-valued function e^{ix} gave a simple explanation of the many values of the logarithm (which Cotes had missed) as a consequence of the periodicity of cos and sin. A direct explanation, based on the definition of log as an integral, was not possible until Gauss (1811) clarified the meaning of complex integrals and pointed out their dependence on the path of integration (see Section 16.3).

Euler's formula also shows

$$(\cos x + i \sin x)^n = e^{inx} = \cos nx + i \sin nx$$

and hence gives a deeper explanation of the Leibniz–de Moivre formula. More generally, the addition theorems for cos and sin (Section 12.4) could be seen as consequences of the much simpler addition formula for the exponential function

$$e^{u+v} = e^u \cdot e^v.$$

The imaginary function e^{ix} was so much more coherent than its real constituents $\cos x$ and $\sin x$ that it was difficult to do without it, and Euler's formula gave mathematicians a strong push toward the eventual acceptance of complex numbers. A more detailed account of the role of the logarithm and exponential functions in the development of complex numbers may be found in Cajori (1913).

At almost the same time that Euler elucidated cos and sin, d'Alembert found many real functions occurring naturally in pairs as the real and imaginary parts of complex functions—in hydrodynamics. As mentioned in Section 13.5, d'Alembert (1752) discovered the equations

$$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = 0, \tag{1}$$

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 0 \tag{2}$$

relating the velocity components P, Q in two-dimensional steady irrotational fluid flow. Equations (1) and (2) come from the requirements that

$Qdx+Pdy$ and $Pdx-Qdy$ be complete differentials, in which case another complete differential is

$$Qdx+Pdy+i(Pdx-Qdy)=(Q+iP)\left(dx+\frac{dy}{i}\right)=(Q+iP)d\left(x+\frac{y}{i}\right).$$

D'Alembert concluded that this means $Q+iP$ is a function f of $x+y/i$, so that $Q=\mathrm{Re}(f)$ and $P=\mathrm{Im}(f)$.

To feel the force of this result, one has to forget the modern definition of function, under which $u(x,y)+iv(x,y)$ is a function of $x+iy$ for *any* functions u, v. In the eighteenth-century context, a "function" $f(x+iy)$ of $x+iy$ was calculable from $x+iy$ by elementary operations; at worst, $f(x+iy)$ was a power series in $x+iy$. This imposes a strong constraint on u, v, namely that

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}.$$

These were just the equations d'Alembert found in his hydrodynamical investigations, but they came to be named the Cauchy–Riemann equations, because these mathematicians stressed their key role in the study of complex functions. The concept of complex function was solidified when Cauchy (1837) showed that a function $f(z)$, where $z=x+iy$, only had to be differentiable in order to be expressible as a power series in z. Thus it suffices to define a complex function $f(z)$ to be one that is differentiable with respect to z in order to guarantee that f is defined with eighteenth-century strictness. It follows, in particular, that the first derivative of f entails derivatives of all orders and that the values of f in any neighborhood determine its values everywhere. This "rigidity" in the notion of complex function is enough of a constraint to enable nontrivial properties to be proved, but at the same time it leaves enough flexibility—one might say "fluidity"—to cover important general situations.

EXERCISES

Euler's derivation of $e^{ix}=\cos x+i\sin x$ is easy to explain using the power series

$$e^y=1+\frac{y}{1!}+\frac{y^2}{2!}+\frac{y^3}{3!}+\cdots$$

and

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$$

found in Section 9.5.