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Cardinal Numbers

The ideas in this chapter (and Chapter 13) are due to Georg Cantor (1845–1918). Many mathematicians at first rejected Cantor's work for ideological reasons, claiming that there could be no 'actual infinity' in mathematics. Cantor found it impossible to get a decent job and, in 1884, suffered a mental breakdown from which he never fully recovered.

The *cardinal numbers* include the natural numbers $0, 1, 2, \ldots$, but go beyond them by including various kinds of infinity. We have some of the same problems in defining a cardinal number as we had with the number 2. Roughly speaking, a cardinal number tells you how many elements there are in a given set. This notion is clear enough for finite sets, but for infinite sets we need some more discussion.

Two sets are said to have the *same cardinality* if and only if there is a one-to-one correspondence between them:

$A \cong B \iff$ there is some $f : A \to B$ such that f is one-to-one and onto (or *injective* and *surjective*).

For example, $\{3, 5, 7\}$ has the same cardinality as $\{0, 1, 2\}$. Note that $\cong$ is an equivalence relation.

If n is a positive integer, we say that a set A has *cardinality* n if and only if $A \cong \{0, 1, 2, \cdots, n-1\}$, and we write $|A| = n$. The cadinality of the empty set is 0: $|\emptyset| = 0$.

A set A is *finite* when $|A|$ is a natural number. Otherwise A is *infinite*. For example, $\mathbf{N}$ is infinite.

We say a set A has cardinality $\aleph_0$ if and only if $A \cong \mathbf{N}$. Such sets are called 'countably infinite' or just 'countable'. For example, $f(m) = 2m$ is a 1-to-1 function from the set $\mathbf{N}$ of natural numbers onto the set E of even numbers. Thus the cardinality of E is $\aleph_0$ and we write $|E| = \aleph_0$. The above example is a special case of what is usually called 'Galileo's Theorem':

If a set S has cardinality $\aleph_0$, then any infinite subset of S has cardinality $\aleph_0$.

Galileo's theorem might lead one to think that there is only one infinite cardinal. However, one of Cantor's achievements was to show that the power set of any set has a higher cardinality than the set itself. By the *power set* of a set S, we mean the set $P(S)$ whose members are the subsets of S.

We write $|A| < |B|$ (A has a lower cardinality than B) if and only if $A \ncong B$ but B has a proper subset C such that $A \cong C$.

Note that $P(S)$ always has a proper subset C, the set of singletons of S, such that $C \cong S$.

We now have Cantor's Theorem:

Theorem 12.1. *Any set A has a lower cardinality than $P(A)$.*

Proof: To obtain a contradiction, suppose there is a 1-to-1, onto function $f : A \to P(A)$. Then every subset of A has the form $f(a)$ for some $a \in A$.

Let $S = \{x \in A | x \notin f(x)\}$. This is a subset of A, so there is some $a \in A$ such that $S = f(a)$. If $a \in S$ then, by the defining property of S, $a \notin f(a) = S$, hence $a \notin S$. But then $a \notin f(a)$ and so $a \in S$. Contradiction.

If a finite set has n elements then its power set has 2^n elements. (This is because, in forming a subset, there are two choices for each of the elements in the original set: include it in the subset or leave it out.) In general, if a set has cardinality k, we use the symbol 2^k to denote the cardinality of its power set. For example, $|P(\mathbf{N})| = 2^{\aleph_0}$. By Cantor's Theorem,

$$\aleph_0 < 2^{\aleph_0} < 2^{2^{\aleph_0}} < \cdots.$$

In other words, there are an infinite number of infinities. (It was statements like these that got Cantor into trouble with Kronecker.)

Is there some subset of $P(N)$ whose cardinality is greater than $\aleph_0$ but not as great as $2^{\aleph_0}$? The hypothesis that the answer to this question is NO is called the *continuum hypothesis.* In 1940 Kurt Gödel showed that the continuum hypothesis is consistent with the usual axioms of set theory. In 1963 Paul Cohen showed that the negation of the continuum hypothesis is also consistent with the usual axioms of set theory. In other words, our basic notions about sets neither imply nor preclude the continuum hypothesis. We say that the continuum hypothesis is *independent* of the axioms of set theory.

Exercises

1. Suppose that $f : A \to B$, $g : B \to A$, $fg = 1_B$ and $gf = 1_A$ (where 1_S is the identity function on S). Prove that f is one-one and onto. Conversely, if f is one-one and onto, show that it has an inverse g.

2. Let S be the set of natural numbers of the form $2^a 3^b$ where a and b are positive integers whose gcd is 1. Use Galileo's theorem to show that $|S| = \aleph_0$.

3. Using Exercise 2, show that the set of positive rationals has cardinality $\aleph_0$.

4. Let T be the set of all sequences formed from 0's and 1's. Show that $|T| = 2^{\aleph_0}$.

5. Let T be the set of reals from 0 to 1. Show that $|T = 2^{\aleph_0}$.

6. Prove Galileo's Theorem.

7. Prove that the set of all functions from $\mathbf{N}$ to $\mathbf{N}$ has higher cardinality than $\mathbf{N}$.

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Cardinal Arithmetic

We begin by defining three binary operations for sets in general:

$$A \times B = \{(a, b) | a \in A \text{ and } b \in B\},$$

$$A^B = \{f | f : B \to A\},$$

$$A + B = (A \times 0) \cup (B \times 1).$$

Note that $A \times \emptyset = \emptyset$, and, if $B \neq \emptyset$, $\emptyset^B = \emptyset$. These definitions are motivated by the fact that, for finite sets A and B,

$$|A \times B| = |A| \times |B|,$$

$$|A^B| = |A|^{|B|},$$

$$|A + B| = |A| + |B|.$$

We also have the following theorem, which generalizes the results of Chapter 2. We have written 0 for $\emptyset$ and 1 for $\{\emptyset\}$.

Theorem 13.1.

1. $A + B \cong B + A$,
2. $(A + B) + C \cong A + (B + C)$,
3. $A \times B \cong B \times A$,
4. $(A \times B) \times C \cong A \times (B \times C)$,