Equation (3) is satisfied by all points $(X,Y) = (x/z, y/z)$ of C, together with other possible coordinate triples with $z = 0$. The latter form horizontal lines approached by the lines from O to points of C as X or $Y \to \infty$, so it is natural to regard them as *the points at infinity of* C. In particular, each line $ax + by + cz = 0$ has one point at infinity, with coordinates $(tb, -ta, 0)$ for all $t \neq 0$.

In geometric terms, we have enlarged the (X,Y)-plane $\mathbb{R}^2$ to the *real projective plane* $\mathbb{R}P^2$ by reinterpreting each point (X,Y) as the line from O to (X,Y) and completing this set of lines to the set of all lines through O. The horizontal lines, which have no interpretation in the (X,Y)-plane, are interpreted as points at infinity. In the process, each algebraic curve C in the (X,Y)-plane is enlarged to its *projective completion* $\bar{C}$ [with equation $\bar{p}(x,y,z) = 0$] by including the points at infinity that are the limits of its ordinary points. We can model $\mathbb{R}P^2$ by a surface if each line through O is replaced by its intersection with the unit sphere, namely, a pair of *antipodal* (diametrically opposite) points. The points at infinity then become antipodal pairs on the equator $z = 0$, which shows they are the same as all other points. A line L in $\mathbb{R}^2$, given by a linear equation $aX + bY + c = 0$, has as completion the *projective line* $\bar{L}$ with homogeneous linear equation $ax + by + cz = 0$, which represents a plane through O. Thus the points of $\bar{L}$ lie in a plane through O and hence are modelled by the antipodal pairs on a great circle. The *line at infinity*, $z = 0$, consists simply of the antipodal pairs on the equator, and hence is the same as any other projective line.

A projective line can be visualized as a great semicircle (which contains one representative from each antipodal pair) with its ends identified. This is a closed curve, so Kepler and Desargues were not far wrong in thinking of a projective line as a circle. The projective plane, however, is not a sphere but something more peculiar, as was noticed by Klein (1874). On a sphere, any simple closed curve separates the surface into two parts. A "small" closed curve in the projective plane $\mathbb{R}P^2$, that is, one strictly contained in a hemisphere of the model, also separates $\mathbb{R}P^2$, but a "large" one does not. The equator, for instance, does not separate the upper hemisphere from the lower, because the hemispheres are the *same place* under the antipodal point identification! A less paradoxical view of this is seen by going back to the model of $\mathbb{R}P^2$ whose elements are lines through O. The lines through the equator do not separate the lines through the upper hemisphere from the lines through the lower hemisphere, because these are the same lines.

EXERCISES

The model of $\mathbb{R}P^2$ whose points are lines through O and whose lines are planes through O also helps in visualizing other basic properties of projective lines:

8.5.1 Use this interpretation of projective lines to show that all lines in a family of parallels have the same point at infinity.

8.5.2 Likewise, show that any two projective lines meet in exactly one point.

In passing from points in a plane to lines through O, it is clear that there is nothing special about the plane $z = 1$ in which we started—the points in *any* plane in $\mathbb{R}^3$ not containing O correspond to distinct lines through O. Conversely, we can pass from lines through O to points in any plane not containing O. In fact, it is often convenient to view projective curves in *different* such planes. This corresponds to taking different projections of the same curve, and it enables us to show, for example, that $y = x^3$ and $y^2 = x^3$ are projectively the same.

8.5.3 Let X, Y denote the x, y coordinates in the plane $z = 1$ (as before), and let X', Z' denote the x, z coordinates in the plane $y = 1$. Show that the curves $Y = X^3$, $(Z')^2 = (X')^3$ have the same equation in the homogeneous coordinates x, y, z.

8.5.4 Deduce that $Y = X^3$ is mapped onto $(Z')^2 = (X')^3$ by projection from O of the plane $z = 1$ onto the plane $y = 1$.

Now let us return to the interpretation of the projective plane $\mathbb{R}P^2$ as a surface, the sphere with antipodal points identified. The following result shows another way in which $\mathbb{R}P^2$ differs from a sphere.

8.5.5 Show that a strip of $\mathbb{R}P^2$ surrounding a projective line is a Möbius band (Figure 8.18.)

8.6 Bézout's Theorem Revisited

As we saw in Section 7.5, a precise account of points at infinity is needed to obtain Bézout's theorem that a curve of degree m meets a curve of degree n in mn points. The projective completion does this. The preceding exercises show that lines (curves of degree 1) meet in $1 \times 1 = 1$ point. In general, if C_m is a curve with homogeneous equation of degree m,

$$p_m(x,y,z) = 0 \tag{1}$$

and if C_n is a curve with homogeneous equation of degree n,

$$p_n(x,y,z) = 0 \tag{2}$$

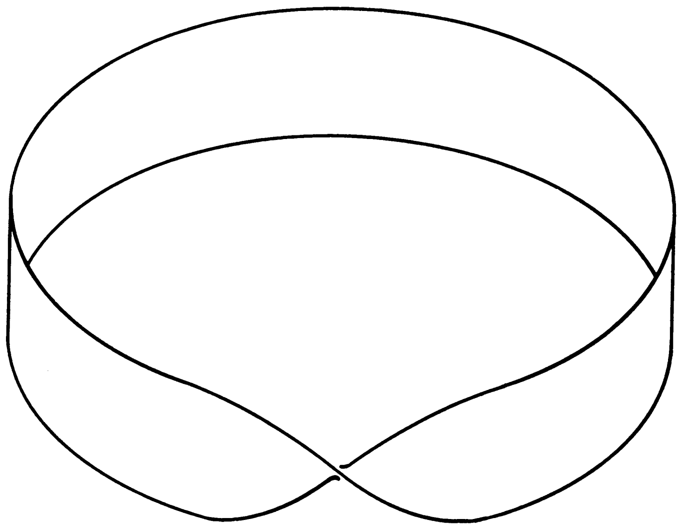

Figure 8.18: A Möbius band

one wishes to show that the equation

$$r_{mn}(x,y) = 0, \tag{3}$$

which results from eliminating z between (1) and (2), is homogeneous of degree mn. This is not hard to do (see exercises), but it seems that a homogeneous formulation of Bézout's theorem, with a rigorous proof that the resultant r_{mn} has degree mn, was not given until the late 1800s [according to Kline (1972), p. 553, the "proper count of multiplicities" was first made by Halphen in 1873].

An obvious condition must be included in the hypothesis of Bézout's theorem: that the curves C_m and C_n have no common component. The algebraic equivalent of this condition is that the polynomials p_m, p_n have no nonconstant common factor. Then the form of Bézout's theorem that can be proved with the help of homogeneous coordinates is *curves C_m, C_n with homogeneous equations $p_m(x,y,z) = 0$, $p_n(x,y,z) = 0$ of degrees m, n and no common component have intersections given by the solutions of a homogeneous equation $r_{mn}(x,y) = 0$ of degree mn.*

A useful consequence of Bézout's theorem is that curves C_m, C_n of degrees m, n with *more* than mn intersections have a common component.

Exercises

As the Chinese discovered (see Section 6.2), the problem of elimination belongs to linear algebra. In the case of Bézout's theorem, this includes the determinant criterion for a set of homogeneous equations to have a nonzero solution, and it leads to an expression for the resultant r_{mn} as a determinant.

8.6.1 Suppose that

$$p_m(x,y,z) = a_0 z^m + a_1 z^{m-1} + \cdots + a_m,$$
$$p_n(x,y,z) = b_0 z^n + b_1 z^{n-1} + \cdots + b_n$$

are homogeneous polynomials of degrees m, n. Thus $a_i(x,y)$ is homogeneous of degree i, $b_j(x,y)$ is homogeneous of degree j. By multiplying p_m and p_n by suitable powers of z, show that the equations

$$p_m = 0 \qquad \text{and} \qquad p_n = 0$$

are equivalent to a system of $m+n$ homogeneous linear equations in the variables $z^{m+n-1}, \ldots, z^2, z^1, z^0$, which in turn is equivalent to

$$r_{mn}(x,y) \equiv \begin{vmatrix}
a_0 & a_1 & \cdots & a_m & 0 & & \cdots & 0 \\
0 & a_0 & a_1 & \cdots & a_m & 0 & \cdots & 0 \\
\vdots & & \ddots & & & \ddots & \ddots & \\
 & & & & & & & 0 \\
0 & \cdots & 0 & a_0 & & \cdots & & a_m \\
b_0 & b_1 & \cdots & & b_n & 0 & \cdots & 0 \\
0 & b_0 & b_1 & \cdots & & b_n & & \vdots \\
\vdots & & \ddots & & & & \ddots & 0 \\
0 & \cdots & 0 & b_0 & & \cdots & & b_n
\end{vmatrix} = 0.$$

8.6.2 Show that a polynomial $p(x,y)$ is homogeneous of degree $k \Leftrightarrow p(tx,ty) = t^k p(x,y)$.

8.6.3 Show $r_{mn}(tx,ty) = t^{mn} r_{mn}(x,y)$. *Hint*: Multiply the rows of $r_{mn}(tx,ty)$ by suitable powers of t to arrange that each element in any column contains the same power of t. Then remove these factors from the columns so that $r_{mn}(x,y)$ remains.

8.7 Pascal's Theorem

Pascal's *Essay on Conics* [Pascal (1640)] was written in late 1639, when Pascal was 16. He probably had heard about projective geometry from his father, who was a friend of Desargues. The *Essay* contained the first statement of a famous result that became known as Pascal's theorem or the *mystic hexagram.* The theorem states that the pairs of opposite sides of a hexagon inscribed in a conic section meet in three collinear points. (The vertices of the hexagon can occur in any order on the curve. In Figure 8.19 the order was chosen to enable the three intersections to lie inside the curve.) Pascal's proof is not known, but he probably established the theorem for the circle first, then trivially extended it to arbitrary conics by projection.

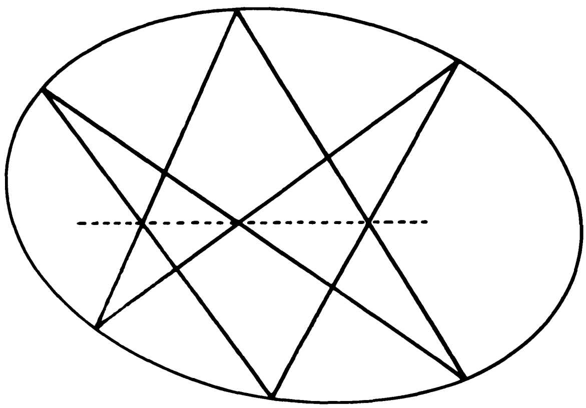

Figure 8.19: Pascal's theorem

Plücker (1847) threw new light on Pascal's theorem by showing it to be an easy consequence of Bézout's theorem. Plücker used an auxiliary theorem about cubics which can be bypassed, giving the following direct deduction from Bézout's theorem.

Let $L_1, L_2, \ldots, L_6$ be the successive sides of the hexagon. The unions of alternate sides, $L_1 \cup L_3 \cup L_5$ and $L_2 \cup L_4 \cup L_6$, can be regarded as cubic curves

$$l_{135}(x,y,z) = 0, \qquad l_{246}(x,y,z) = 0,$$

where each l is a product of three linear factors. These two curves meet in nine points: the six vertices of the hexagon and the three intersections of opposite sides. Let

$$c(x,y,z) = 0 \tag{1}$$

be the equation of the conic that contains the six vertices.

We can choose constants α, β so that the cubic curve

$$\alpha l_{135}(x,y,z) + \beta l_{246}(x,y,z) = 0 \tag{2}$$

passes through any given point P. Let P be a point on the conic, unequal to the six vertices. Then the curves (1), (2) of degrees 2, 3, have $7 > 2 \times 3$ points in common, and hence a common component by Bézout's theorem. Since c has no nonconstant factor, by hypothesis, this common component must be c itself. Hence

$$\alpha l_{135} + \beta l_{246} = cp \tag{3}$$

for some polynomial p, which must be linear since the left-hand side of (3) has degree 3 and c has degree 2. Since the curve $\alpha l_{135} + \beta l_{246} = 0$ passes through the nine points common to $l_{135} = 0$ and $l_{246} = 0$, while $c = 0$ passes through only six of them, the remaining three (the intersections of opposite sides) must be on the line $p = 0$.

EXERCISES

8.7.1 Generalize the preceding argument to show that if two degree n curves meet in n^2 points, nm of which lie on a curve of degree m, then the remaining $n(n-m)$ points lie on a curve of degree $n-m$.

An important special case of Pascal's theorem was discovered around 300 CE by Pappus, and it is called the *theorem of Pappus*. In this theorem, the conic is a "degenerate" conic section, consisting of two straight lines.

The usual statement of Pappus' theorem, like that of Pascal's theorem, says that the intersections of opposite sides of the hexagon are in a straight line. However, if we avail ourselves of the freedom to take this line to be at infinity, then Pappus' theorem takes a form that is easier to visualize and prove.

8.7.2 Interpret Figure 8.20 as an illustration of Pappus' theorem.

8.7.3 Write down a statement of the theorem corresponding to Figure 8.20, the conclusion of which is that P_1Q_3 and P_2Q_2 are parallel. (Equivalently, $OP_1/OP_2 = OQ_3/OQ_2$.)

8.7.4 Deduce the required equation from two other equations that express parallelism in Figure 8.20.

8.7.5 Also draw the figure and prove the theorem in the case where the two lines P_1P_2 and Q_1Q_2 do *not* meet at O, that is, when they too are parallel.

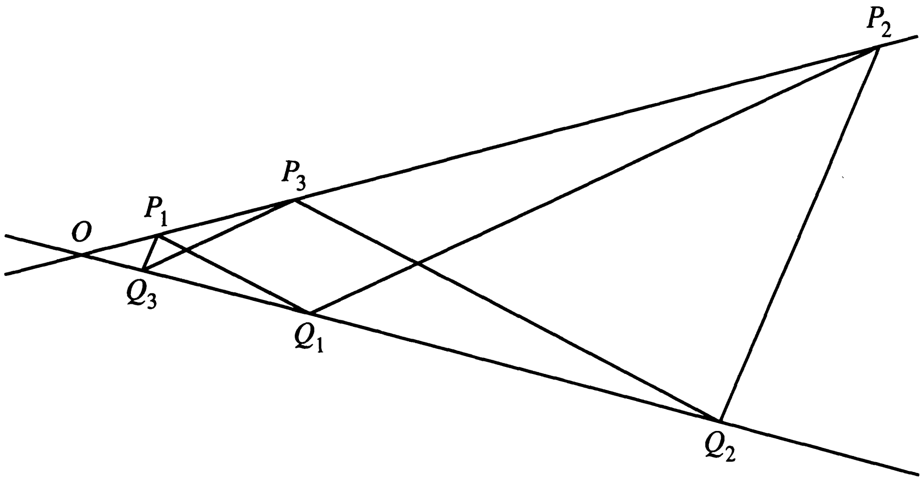

Figure 8.20: Illustration of Pappus' theorem

8.8 Biographical Notes: Desargues and Pascal

Girard Desargues was born in Lyons in 1591 and died in 1661. He was one of nine children of Girard Desargues, a tithe collector, and Jeanne Croppet. He was evidently brought up in Lyons, but information about his early life is lacking. By 1626 he was working as an engineer in Paris and may have used his expertise in the famous siege of La Rochelle in 1628, during which a dike was built across the harbor to prevent English ships from relieving the city.

In the 1630s he joined the circle of Marin Mersenne, which met regularly in Paris to discuss scientific topics, and in 1636 contributed a chapter to a book of Mersenne on music theory. In the same year he published a 12-page booklet on perspective, the first hint of his ideas in projective geometry. The *Brouillon projet* [Desargues (1639)] was published in an edition of only 50 copies and won very little support. In fact, its reception was generally hostile, and Desargues was engaged in a pamphleteering battle for years with his detractors [see Taton (1951), pp. 36–45]. At first his only supporters were Pascal, most of whose work on projective geometry is also lost, and the engraver Abraham Bosse, who expounded Desargues' perspective method [Bosse (1648)]. Desargues became discouraged by the attacks on his work and left the dissemination of his ideas up to Bosse, who was not really mathematically equipped for the task. Projective geometry secured a place in mathematics only with the publication of a book by Phillipe de la Hire [de la Hire (1673)], whose father, Laurent, had been a student of Desargues. It seems quite likely that la Hire's book influenced Newton. For this and more on Desargues' mathematical legacy, see Field and Gray (1987), Ch. 3.

Around 1645 Desargues turned his talents to architecture, perhaps to demonstrate to his critics the practicality of his graphical methods. He was responsible for various houses and public buildings in Paris and Lyons, excelling in complex structures such as staircases. His best-known achievement in engineering, a system for raising water at the château of Beaulieu, near Paris, is also interesting from the geometrical viewpoint. It makes the first use of epicyclic curves (Section 2.5) in cogwheels, as was noted by Huygens (1671). Huygens visited the château at the time when it was owned by Charles Perrault, the author of *Cinderella* and *Puss in Boots*.

Desargues apparently returned to scientific circles in Paris toward the end of his life—Huygens heard him give a talk on the existence of geometric points on November 9, 1660—but information about this period is scanty. His will was read in Lyons on October 8, 1661, but the date and place of his death are unknown.

Blaise Pascal (Figure 8.21) was born in Clermont-Ferrand in 1623 and died in Paris in 1662. His mother, Antoinette Bagon, died when he was three, and Blaise was brought up by his father, Etienne. Etienne Pascal was a lawyer with an interest in mathematics who belonged to Mersenne's circle and, as mentioned earlier, was a friend of Desargues. He has a curve named after him, the *limaçon of Pascal.* In 1631 Etienne took Blaise and his two sisters to Paris and gave up all official duties to devote himself to their education. Thus Blaise Pascal never went to school or university, but by the age of 16 he was learned in Latin, Greek, mathematics, and science. And of course he had written his *Essay on Conics* and discovered Pascal's theorem.

The *Essay on Conics* [Pascal (1640)] is a short pamphlet containing an outline of the great treatise on conics he had begun to prepare, and which is now lost. It includes a statement of Pascal's theorem for the circle. Pascal worked on his treatise until 1654, when it was nearly complete, but he never mentioned it thereafter. Leibniz saw the manuscript when he was in Paris in 1676, but no further sightings are known.

In 1640 Pascal and his sisters joined their father in Rouen, where he had become a tax official. Pascal got the idea of constructing a calculating machine to help his father in his work. He found a theoretical solution around