(2) Let R be a ring and let $\mathcal{F}$ be any functor from R–**Mod** to itself. The zero map is a natural transformation from $\mathcal{F}$ to itself: $\eta_A = 0_A$ for every R-module A, where 0_A is the zero map from A to itself. This is not a natural isomorphism.

(3) Let $\mathcal{F}$ be the identity functor from **Grp** to itself, and let $\mathcal{G}$ be the abelianizing functor (Example 3) considered here as a map from **Grp** to itself. For each group G let $\eta_G : G \to G/G'$ be the usual projection map onto the quotient group. Then η is a natural transformation (but not an isomorphism) with respect to these two functors. (We call the maps η_G the *natural projection* maps.)

(4) Let $\mathcal{G} = \mathcal{D}^2$ be the double dual functor from the category of finite dimensional vector spaces over a field K to itself (Example 7). Then there is a natural isomorphism η from the identity functor to $\mathcal{G}$ given by

$$\eta_V : V \to V^{**} \qquad \text{by} \qquad \eta_V(v) = E_v$$

where E_v is "evaluation at v" for every $v \in V$.

(5) Let $\mathcal{GL}_n$ be the functor from **CRing** to **Grp** defined as follows. Each object (commutative ring) R is mapped by $\mathcal{GL}_n$ to the group $GL_n(R)$ of $n \times n$ invertible matrices with entries from R. For each ring homomorphism $f : R \to S$ let $\mathcal{GL}_n(f)$ be the map of matrices that applies f to each matrix entry. Since f sends 1 to 1 it follows that $\mathcal{GL}_n(f)$ sends invertible matrices to invertible matrices (cf. Exercise 4 in Section 1). Let $\mathcal{G}$ be the functor from **CRing** to **Grp** that maps each ring R to its group of units $R^\times$, and each ring homomorphism f to its restriction to the groups of units (also denoted by f). The *determinant* is a natural transformation from $\mathcal{GL}_n$ to $\mathcal{G}$ because the determinant is defined by the same polynomial for all rings so that the following diagram commutes:

$$\begin{tikzcd}
GL_n(R) \arrow[r, "\det"] \arrow[d, "\mathcal{GL}_n(f)"'] & R^\times \arrow[d, "f"] \\
GL_n(S) \arrow[r, "\det"] & S^\times
\end{tikzcd}$$

Let **C**, **D** and **E** be categories, let $\mathcal{F}$ be a functor from **C** to **D**, and let $\mathcal{G}$ be a functor from **D** to **E**. There is an obvious notion of the composition of functors $\mathcal{GF}$ from **C** to **E**. When **E** $=$ **C** the composition $\mathcal{GF}$ maps **C** to itself and $\mathcal{FG}$ maps **D** to itself. We say **C** and **D** are *isomorphic* if for some $\mathcal{F}$ and $\mathcal{G}$ we have $\mathcal{GF}$ is the identity functor $\mathcal{I}_C$, and $\mathcal{FG} = \mathcal{I}_D$. By the discussion in Section 10.1 the categories $\mathbb{Z}$–**Mod** and **Ab** are isomorphic. It also follows from observations in Section 10.1 that the categories of elementary abelian p-groups and vector spaces over $\mathbb{F}_p$ are isomorphic. In practice we tend to identify such isomorphic categories. The following generalization of isomorphism between categories gives a broader and more useful notion of when two categories are "similar."

Definition. Categories **C** and **D** are said to be *equivalent* if there are functors $\mathcal{F}$ from **C** to **D** and $\mathcal{G}$ from **D** to **C** such that the functor $\mathcal{GF}$ is naturally isomorphic to $\mathcal{I}_C$ (the identity functor of **C**) and $\mathcal{FG}$ is naturally isomorphic to the identity functor $\mathcal{I}_D$.

It is an exercise that equivalence of categories is reflexive, symmetric and transitive. The example of Affine k-algebras in Section 15.5 is an equivalence of categories (where one needs to modify the direction of the arrows in the definition of a natural transformation to accommodate the contravariant functors in this example). Another example (which requires some proving) is that for R a commutative ring with 1 the categories of left modules R–**Mod** and $M_{n\times n}(R)$–**Mod** are equivalent.

Finally, we introduce the concepts of universal arrows and universal objects.

Definition.

(1) Let $\mathbf{C}$ and $\mathbf{D}$ be categories, let $\mathcal{F}$ be a functor from $\mathbf{C}$ to $\mathbf{D}$, and let X be an object in $\mathbf{D}$. A *universal arrow* from X to $\mathcal{F}$ is a pair $(U(X), \iota)$, where $U(X)$ is an object in $\mathbf{C}$ and $\iota : X \to \mathcal{F}U(X)$ is a morphism in $\mathbf{D}$ satisfying the following property: for any object A in $\mathbf{C}$ if φ is any morphism from X to $\mathcal{F}A$ in $\mathbf{D}$, then there exists a unique morphism $\Phi : U(X) \to A$ in $\mathbf{C}$ such that $\mathcal{F}(\Phi)\iota = \varphi$, i.e., the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[dr, "\varphi"'] & \mathcal{F}U(X) \arrow[d, "\mathcal{F}(\Phi)"] \\ & \mathcal{F}A \end{tikzcd}$$

(2) Let $\mathbf{C}$ be a category and let $\mathcal{F}$ be a functor from $\mathbf{C}$ to the category **Set** of all sets. A *universal element* of the functor $\mathcal{F}$ is a pair (U, ι), where U is an object in $\mathbf{C}$ and ι is an element of the set $\mathcal{F}U$ satisfying the following property: for any object A in $\mathbf{C}$ and any element g in the set $\mathcal{F}A$ there is a unique morphism $\varphi : U \to A$ in $\mathbf{C}$ such that $\mathcal{F}(\varphi)(\iota) = g$.

Examples

(1) (*Universal Arrow: Free Objects*) Let R be a ring with 1. We translate into the language of universal arrows the statement that if $U(X)$ is the free R-module on a set X then any set map from X to an R-module A extends uniquely by R-linearity to an R-module homomorphism from $U(X)$ to A (cf. Theorem 6, Section 10.3): Let $\mathcal{F}$ be the forgetful functor from R–**Mod** to **Set**, so that $\mathcal{F}$ maps an R-module A to the set A, i.e., $A = \mathcal{F}A$ as sets. Let X be any set (i.e., an object in **Set**), let $U(X)$ be the free R-module with basis X, and let $\iota : X \to \mathcal{F}U(X)$ be the set map which sends each $b \in X$ to the basis element b in $U(X)$. Then the universal property of free R-modules is precisely the result that $(U(X), \iota)$ is a universal arrow from X to the forgetful functor $\mathcal{F}$.

Similarly, free groups, vector spaces (which are free modules over a field), polynomial algebras (which are free R-algebras) and the like are all instances of universal arrows.

(2) (*Universal Arrow: Fields of Fractions*) Let $\mathcal{F}$ be the forgetful functor from the category of fields to the category of integral domains, where the morphisms in both categories are *injective* ring homomorphisms. For any integral domain X let $U(X)$ be its field of fractions and let ι be the inclusion of X into $U(X)$. Then $(U(X), \iota)$ is a universal arrow from X to the functor $\mathcal{F}$ (cf. Theorem 15(2) in Section 7.5).

(3) (*Universal Object: Tensor Products*) This example refers to the construction of the tensor product of two modules in Section 10.4. Let $\mathbf{C} = R$–**Mod** be the category of R-modules over the commutative ring R, and let M and N be R-modules. For each R-module A let $\text{Bilin}(M, N; A)$ denote the set of all R-bilinear functions from $M \times N$ to A. Define a functor from R–**Mod** to **Set** on objects by

$$\mathcal{F} : A \longrightarrow \text{Bilin}(M, N; A),$$

and if $\varphi : A \to B$ is an R-module homomorphism then

$$\mathcal{F}(\varphi)(h) = \varphi \circ h \qquad \text{for every } h \in \mathrm{Bilin}(M, N; A).$$

Let $U = M \otimes_R N$ and let ι be the bilinear function

$$\iota : M \times N \to M \otimes_R M \qquad \text{by} \qquad \iota(m, n) = m \otimes n,$$

so ι is an element of the set $\mathrm{Bilin}(M, N; M \otimes_R N) = \mathcal{F}U$. Then $(M \otimes_R N, \iota)$ is a universal element of $\mathcal{F}$ because for any R-module A and for any bilinear map $g : M \times N \to A$ (i.e., any element of $\mathcal{F}A$) there is a unique R-module homomorphism $\varphi : M \otimes_R N \to A$ such that $g = \varphi \circ \iota = \mathcal{F}(\varphi)(\iota)$.

EXERCISES

1. Let **Nor**–N be the category described in Exercise 1 of Section 1, and let $\mathcal{F}$ be the inclusion functor from **Nor**–N into **Grp**. Describe a functor $\mathcal{G}$ from **Nor**–N into **Grp** such that the transformation η defined by $\eta_G : G \to G/N$ is a natural transformation from $\mathcal{F}$ to $\mathcal{G}$.

2. Let H and K be groups and let $\mathcal{H}\times$ and $\mathcal{K}\times$ be functors from **Grp** to itself described in Exercise 2 of Section 1. Let $\varphi : H \to K$ be a group homomorphism.

(a) Show that the maps $\eta_A : H \times A \to K \times A$ by $\eta_A(h, a) = (\varphi(h), a)$ determine a natural transformation η from $\mathcal{H}\times$ to $\mathcal{K}\times$.

(b) Show that the transformation η is a natural isomorphism if and only if φ is a group isomorphism.

3. Express the universal property of the commutator quotient group — described in Proposition 7(5) of Section 5.4 — as a universal arrow for some functor $\mathcal{F}$.

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