Oresme called his coordinates "longitude" and "latitude," but he seems to have been the first to use them to represent functions such as velocity as a function of time. Setting up the coordinate system *before* determining the curve was Oresme's step beyond the Greeks, but he too lacked the algebra to go further.

The step that finally made analytic geometry feasible was the solution of equations and the improvement of notation in the sixteenth century, which we discussed in the previous chapter. This step made it possible to consider equations, and hence curves, in some generality and to have confidence in one's ability to manipulate them. As we shall see in the next section, the two founders of analytic geometry, Fermat and Descartes, were both strongly influenced by these developments.

For more details on the development of analytic geometry, the reader is referred to an excellent book by Boyer (1956).

EXERCISE

7.1.1 Generalize the idea of Menaechmus to show that any cubic equation

$$ax^3+bx^2+cx+d=0 \quad \text{with} \quad d \neq 0$$

may be solved by intersecting the hyperbola $xy=1$ with a parabola.

7.2 Fermat and Descartes

There have been several occasions in the history of mathematics when an important discovery was made independently and almost simultaneously by two individuals: noneuclidean geometry by Bolyai and Lobachevsky, elliptic functions by Abel and Jacobi, the calculus by Newton and Leibniz, for example. To the extent that we can rationally explain these remarkable events, it must be on the basis of ideas already "in the air," of conditions becoming favorable for their crystallization. As I tried to show in the previous section, conditions were favorable for analytic geometry at the beginning of the seventeenth century. Thus it is not completely surprising that the subject was independently discovered by Fermat (1629) and Descartes (1637). (Descartes' work *La Géométrie* may in fact have been started in the 1620s. In any case it is independent of Fermat, whose work was not published until 1679.)

It is a surprise to learn, however, that both Fermat and Descartes began with an analytic solution of the same classical geometric problem, the four-line problem of Apollonius, and that the main discovery of each was that second-degree equations correspond to conic sections. Up to this point Fermat was more systematic than Descartes, but that was as far as he went. He was content to leave his work in a "simple and crude" state, confident that it would grow in stature when nourished by new inventions.

Descartes, on the other hand, treated many higher-degree curves and clearly understood the power of algebraic methods in geometry. He wanted to withhold this power from his contemporaries, however, particularly the rival mathematician Roberval, as he admitted in a letter to Mersenne [see Boyer (1956), p. 104]. *La Géométrie* was written to boast about his discoveries, not to explain them. There is little systematic development, and proofs are frequently omitted with a sarcastic remark such as, "I shall not stop to explain this in more detail, because I should deprive you of the pleasure of mastering it yourself" (p. 10). Descartes' conceit is so great that it is a pleasure to see him come a cropper occasionally, as on p. 91: "The ratios between straight and curved lines are not known, and I believe cannot be discovered by human minds." He was referring to the then-unsolved problem of determining the length of curves, but he spoke too soon, for in 1657 Neil and van Heuraet found the length of an arc of the semicubical parabola $y^2 = x^3$, and the calculus soon made such problems routine. [A full and interesting account of the story of arc length may be found in Hofmann (1974), Ch. 8.]

EXERCISES

As we now know, all conic sections may be given by the following standard form equations (from Section 2.4):

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ (ellipse)}, \qquad y = ax^2 \text{ (parabola)}, \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ (hyperbola)}.$$

The reduction of an arbitrary quadratic equation in x and y to one of these forms depends on suitable choice of origin and axes, as Fermat and Descartes discovered. The main steps are outlined in the following exercises.

7.2.1 Show that a *quadratic form* $ax^2 + bxy + cy^2$ may be converted to a form $a'x'^2 + b'y'^2$ by suitable choice of θ in the substitution

$$x = x'\cos\theta - y'\sin\theta,$$
$$y = x'\sin\theta + y'\cos\theta,$$

by checking that the coefficient of $x'y'$ is $(c-a)\sin 2\theta + b\cos 2\theta$.

7.2.2 Deduce from Exercise 7.2.1 that, by a suitable rotation of axes, any quadratic curve may be expressed in the form $a'x'^2 + b'y'^2 + c'x' + d'y' + e' = 0$.

7.2.3 If $b' = 0$, but $a' \neq 0$, show that the substitution $x' = x'' + f$ gives either a standard-form parabola, or the "double line" $x''^2 = 0$.

(Why is this called a "double line," and is it a section of a cone?)

7.2.4 If both a' and b' are nonzero, show that a shift of origin gives the standard form for either an ellipse or a hyperbola, or else a pair of lines.

7.3 Algebraic Curves

> I could give here several other ways of tracing and conceiving a series of curved lines, each curve more complex than any preceding one, but I think the best way to group together all such curves and then classify them in order is by recognizing the fact that all points of those curves which we may call "geometric," that is, those which admit of precise and exact measurement, must bear a definite relation to all points of a straight line, and that this relation must be expressed by means of a single equation.
>
> [Descartes (1637), p. 48]

In this passage Descartes defines what we now call *algebraic curves*. The fact that he calls them "geometric" shows his attachment to the Greek idea that curves are the product of geometric constructions. He is using the notation of equations not to define curves directly but to restrict the notion of geometric construction more severely than the Greeks did, thereby restricting the concept of curve. As we saw in Section 2.5, the Greeks considered some constructions, such as rolling one circle on another, which are capable of producing transcendental curves. Descartes called such curves "mechanical" and found a way to exclude them by his restriction to curves "expressed by means of a single equation." It becomes clear in the lines following the preceding quotation that he means polynomial equations, since he gives a classification of equations by degree.

Descartes' rejection of transcendental curves was short-sighted, as the calculus soon provided techniques to handle them, but nevertheless it was fruitful to concentrate on algebraic curves. The notion of degree, in particular, was a useful measure of complexity. First-degree curves are the simplest possible, namely, straight lines; second-degree are the next simplest, conic sections. With third-degree curves one sees the new phenomena of inflections, double points, and cusps. Inflection and cusp are familiar from $y = x^3$ and $y^2 = x^3$, respectively; we also saw a cusp on the cissoid (Section 2.5). A classical example of a cubic with a double point is the *folium* (leaf) *of Descartes* (1638),

$$x^3 + y^3 = 3axy.$$

The "leaf" is the closed portion to the right of the double point; Descartes misunderstood the rest of the curve through neglect of negative coordinates. The true shape of the folium was first given by Huygens (1692). Figure 7.1 is Huygens' drawing, which also shows the asymptote to the curve.

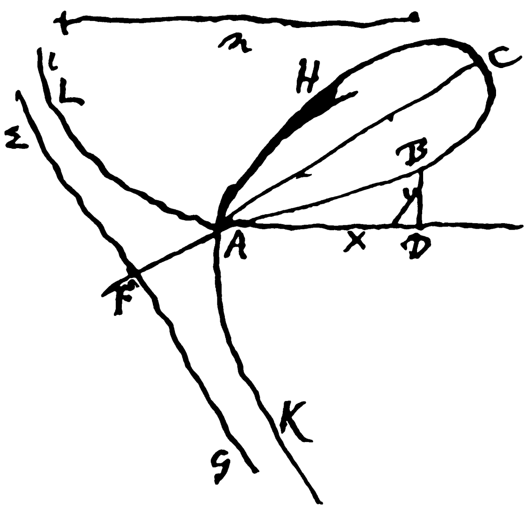

Figure 7.1: Huygens' drawing of the folium

An excellent account of the early history of curves can be found in Brieskorn and Knörrer (1981), Chapter 1. Many individual curves, with diagrams, equations, and historical notes, can be found in Gomes Teixeira (1995a,b,c). The development of Descartes' concept of curve has been studied by Bos (1981).

EXERCISES

The folium is a cubic curve to which Diophantus' chord method (Section 3.5) applies. One takes the line $y = tx$ through the "obvious" rational point $(0,0)$ on the curve, and finds its other point of intersection. This construction also enables us to express an arbitrary point (x,y) on the curve in terms of the parameter t.

7.3.1 Show that the folium of Descartes has parametric equations

$$x = \frac{3at}{1+t^3}, \qquad y = \frac{3at^2}{1+t^3}$$

and use these equations to show that it is tangential to the axes at 0.

7.3.2 Show that the equation $x^3 + y^3 = 3axy$ of the folium may be written in the form

$$x+y = \frac{3a}{\frac{x}{y}+\frac{y}{x}-1}.$$

7.3.3 Show that x/y and y/x tend to -1 as $x \to \pm\infty$ on the folium, and hence deduce the equation of its asymptote from Exercise 7.3.2.

A whole family of "multileaved" curves was studied by Grandi (1723):

7.3.4 The *roses of Grandi* are given by the polar equations

$$r = a\cos n\theta$$

for integer values of n. [Figure 7.2 shows some of these curves, as given by Grandi (1723)]. Show that the roses of Grandi are algebraic.

Figure 7.2: Roses of Grandi

7.3.5 Show that the "rose" for $n = 1$ is a circle and that the "rose" for $n = 2$ has cartesian equation

$$(x^2+y^2)^3 = a^2(x^2-y^2)^2.$$

7.4 Newton's Classification of Cubics

Since first- and second-degree curves are straight lines and conics, they were well understood before the advent of analytic geometry. Up to the end of the eighteenth century most mathematicians considered them not amenable to further clarification, and hence an unsuitable subject for the new methods. A famous example is the Greek-style treatment of planetary orbits in Newton's *Principia* [Newton (1687)]. The classical attitude to low-degree curves was summed up by d'Alembert in his article on geometry in the *Encylopédie* (1751):

> Algebraic calculation is not to be applied to the propositions of elementary geometry because it is not necessary to use this calculus to facilitate demonstrations, and it appears that there are no demonstrations which can really be facilitated by this calculus except for the solution of problems of second degree by the line and circle.

Thus the first new problem opened up by analytic geometry, and also the first considered properly to belong to the subject, was the investigation of cubic curves. These curves were classified, more or less completely, by Newton (1695) [see Ball (1890) for a commentary].

Newton (1667) began this work with the general cubic in x and y,

$$ay^3+bxy^2+cx^2y+dx^3+ey^2+fxy+gx^2+hy+kx+l=0,$$

making a general transformation of axes, leading to an equation with 84 terms, then showing that the latter equation could be reduced to one of the forms

$$\begin{aligned}
Axy^2+By &= Cx^3+Dx^2+Ex+F,\\
xy &= Ax^3+Bx^2+Cx+D,\\
y^2 &= Ax^3+Bx^2+Cx+D,\\
y &= Ax^3+Bx^2+Cx+D.
\end{aligned}$$

Newton then divided the curves into species according to the roots of the right-hand side, obtaining 72 species (and overlooking 6). His paper does not contain detailed proofs; these were supplied by Stirling (1717), along with four of the species Newton had missed. Newton's classification was criticized by some later mathematicians, such as Euler, for lacking a general principle. A unifying principle was certainly desirable, to reduce the complexity of the classification. And such a principle was already implicit in one of Newton's passing remarks, Section 29, "On the Genesis of Curves by Shadows." This principle, which will be explained in the next chapter, reduces cubics to the five types seen in Figure 7.3 [taken from an English translation of Newton's paper published in 1710; see Whiteside (1964)].

The reader may wonder where the most familiar cubic, $y = x^3$, appears among these five. The answer is that it is equivalent to the one with a cusp, in Newton's Figure 75. This is explained in the next chapter.

EXERCISES

The cubic curves that Newton called "cuspidate" and "nodated" are algebraically simpler than the others. In particular, they can be parameterized by rational functions.

7.4.1 Find a parameterization $x = p(t), y = q(t)$ of the semicubical parabola $y^2 = x^3$ by polynomials p and q, (i) by inspection, (ii) by finding the second intersection point of the line $y = tx$ through the cusp $(0,0)$.

7.4.2 Find rational functions $x = r(t), y = s(t)$ that parameterize $y^2 = x^2(x+1)$, by finding the second intersection of the line $y = tx$ through the double point of the curve.

7.5 Construction of Equations and Bézout's Theorem

In Sections 7.1, 7.2, and 7.3 the development of analytic geometry is outlined from the first observations of equations as properties of curves to the full realization that equations *defined* curves and that the concept of (polynomial) equation was the key to the concept of (algebraic) curve. With hindsight, we can say that Descartes' *La Géométrie* [Descartes (1637)] was a major step in the maturation of the subject, but the book does not conclusively establish what analytic geometry is. In fact, it is largely devoted to two transitional topics in the development of the subject: the sixteenth-century theory of equations and the now almost forgotten discipline called "construction of equations."

The paradigm construction of an equation was Menaechmus' construction of $\sqrt[3]{2}$ by intersecting a parabola and hyperbola. From a geometric

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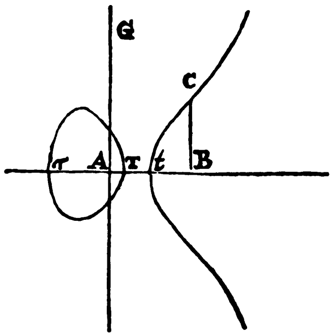
Fig. 71.



of the Form of a Bell, with an Oval at its Vertex. And this makes a *Sixty seventh Species*.

If two of the Roots are equal, a Parabola will be formed, either *Nodated* by touching an Oval,

Fig. 72.

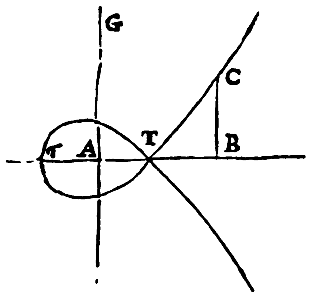
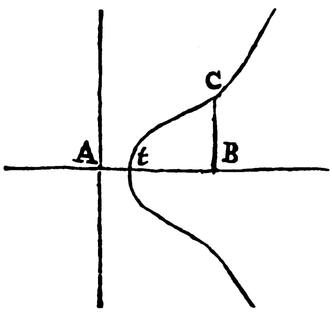


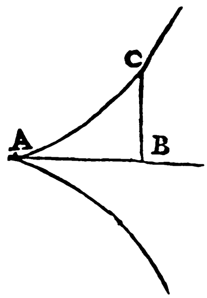
Fig. 73.



or *Punctate*, by having the Oval infinitely small. Which two *Species* are the *Sixty eighth* and *Sixty ninth*.

If three of the Roots are equal, the Parabola will be *Cuspidate* at the Vertex. And this is the

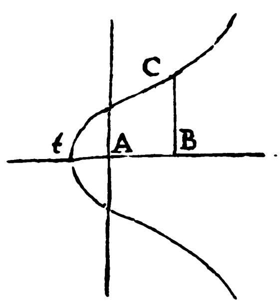
Fig. 75.



Neilian Parabola, commonly called Semi-cubical. Which makes the *Seventieth Species*.

If two of the Roots are impossible, there will (See *Fig*. 73.)

Fig. 73.



be a *Pure* Parabola of a Bell-like Form. And this makes the *Seventy first Species*.

Figure 7.3: Newton's classification of cubic curves