(a) Prove that W_i is the set of all vectors α in V such that $(T - c_iI)^m\alpha = 0$ for some positive integer m (which may depend upon α).

(b) Prove that the dimension of W_i is d_i. (*Hint:* If T_i is the operator induced on W_i by T, then $T_i - c_iI$ is nilpotent; thus the characteristic polynomial for $T_i - c_iI$ must be x^{e_i} where e_i is the dimension of W_i (proof?); thus the characteristic polynomial of T_i is $(x - c_i)^{e_i}$; now use the fact that the characteristic polynomial for T is the product of the characteristic polynomials of the T_i to show that $e_i = d_i$.)

5. Let V be a finite-dimensional vector space over the field of complex numbers. Let T be a linear operator on V and let D be the diagonalizable part of T. Prove that if g is any polynomial with complex coefficients, then the diagonalizable part of $g(T)$ is $g(D)$.

6. Let V be a finite-dimensional vector space over the field F, and let T be a linear operator on V such that rank $(T) = 1$. Prove that either T is diagonalizable or T is nilpotent, not both.

7. Let V be a finite-dimensional vector space over F, and let T be a linear operator on V. Suppose that T commutes with every diagonalizable linear operator on V. Prove that T is a scalar multiple of the identity operator.

8. Let V be the space of $n \times n$ matrices over a field F, and let A be a fixed $n \times n$ matrix over F. Define a linear operator T on V by $T(B) = AB - BA$. Prove that if A is a nilpotent matrix, then T is a nilpotent operator.

9. Give an example of two 4×4 nilpotent matrices which have the same minimal polynomial (they necessarily have the same characteristic polynomial) but which are not similar.

10. Let T be a linear operator on the finite-dimensional space V, let $p = p_1^{r_1} \cdots p_k^{r_k}$ be the minimal polynomial for T, and let $V = W_1 \oplus \cdots \oplus W_k$ be the primary decomposition for T, i.e., W_j is the null space of $p_j(T)^{r_j}$. Let W be any subspace of V which is invariant under T. Prove that

$$W = (W \cap W_1) \oplus (W \cap W_2) \oplus \cdots \oplus (W \cap W_k).$$

11. What's wrong with the following proof of Theorem 13? Suppose that the minimal polynomial for T is a product of linear factors. Then, by Theorem 5, T is triangulable. Let $\mathcal{B}$ be an ordered basis such that $A = [T]_{\mathcal{B}}$ is upper-triangular. Let D be the diagonal matrix with diagonal entries $a_{11}, \ldots, a_{nn}$. Then $A = D + N$, where N is strictly upper-triangular. Evidently N is nilpotent.

12. If you thought about Exercise 11, think about it again, after you observe what Theorem 7 tells you about the diagonalizable and nilpotent parts of T.

13. Let T be a linear operator on V with minimal polynomial of the form p^n, where p is irreducible over the scalar field. Show that there is a vector α in V such that the T-annihilator of α is p^n.

14. Use the primary decomposition theorem and the result of Exercise 13 to prove the following. If T is any linear operator on a finite-dimensional vector space V, then there is a vector α in V with T-annihilator equal to the minimal polynomial for T.

15. If N is a nilpotent linear operator on an n-dimensional vector space V, then the characteristic polynomial for N is x^n.

7. The Rational and Jordan Forms

7.1. Cyclic Subspaces and Annihilators

Once again V is a finite-dimensional vector space over the field F and T is a fixed (but arbitrary) linear operator on V. If α is any vector in V, there is a smallest subspace of V which is invariant under T and contains α. This subspace can be defined as the intersection of all T-invariant subspaces which contain α; however, it is more profitable at the moment for us to look at things this way. If W is any subspace of V which is invariant under T and contains α, then W must also contain the vector $T\alpha$; hence W must contain $T(T\alpha) = T^2\alpha$, $T(T^2\alpha) = T^3\alpha$, etc. In other words W must contain $g(T)\alpha$ for every polynomial g over F. The set of all vectors of the form $g(T)\alpha$, with g in $F[x]$, is clearly invariant under T, and is thus the smallest T-invariant subspace which contains α.

Definition. *If α is any vector in* V, *the* T-**cyclic subspace generated by** *α is the subspace* $Z(\alpha; T)$ *of all vectors of the form* $g(T)\alpha$, g *in* F[x]. *If* $Z(\alpha; T) = V$, *then α is called a* **cyclic vector** *for* T.

Another way of describing the subspace $Z(\alpha; T)$ is that $Z(\alpha; T)$ is the subspace spanned by the vectors $T^k\alpha$, $k \geq 0$, and thus α is a cyclic vector for T if and only if these vectors span V. We caution the reader that the general operator T has no cyclic vectors.

EXAMPLE 1. For any T, the T-cyclic subspace generated by the zero vector is the zero subspace. The space $Z(\alpha; T)$ is one-dimensional if and only if α is a characteristic vector for T. For the identity operator, every non-zero vector generates a one-dimensional cyclic subspace; thus, if $\dim V > 1$, the identity operator has no cyclic vector. An example of an operator which has a cyclic vector is the linear operator T on F^2 which is represented in the standard ordered basis by the matrix

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Here the cyclic vector (a cyclic vector) is ϵ_1; for, if $\beta = (a, b)$, then with $g = a + bx$ we have $\beta = g(T)\epsilon_1$. For this same operator T, the cyclic subspace generated by ϵ_2 is the one-dimensional space spanned by ϵ_2, because ϵ_2 is a characteristic vector of T.

For any T and α, we shall be interested in linear relations

$$c_0\alpha + c_1T\alpha + \cdots + c_kT^k\alpha = 0$$

between the vectors $T^j\alpha$, that is, we shall be interested in the polynomials $g = c_0 + c_1x + \cdots + c_kx^k$ which have the property that $g(T)\alpha = 0$. The set of all g in $F[x]$ such that $g(T)\alpha = 0$ is clearly an ideal in $F[x]$. It is also a non-zero ideal, because it contains the minimal polynomial p of the operator T ($p(T)\alpha = 0$ for every α in V).

Definition. *If α is any vector in V, the* **T-annihilator** *of α is the ideal $M(\alpha; T)$ in $F[x]$ consisting of all polynomials g over F such that $g(T)\alpha = 0$. The unique monic polynomial p_α which generates this ideal will also be called the* **T-annihilator** *of α.*

As we pointed out above, the T-annihilator p_α divides the minimal polynomial of the operator T. The reader should also note that $\deg(p_\alpha) > 0$ unless α is the zero vector.

Theorem 1. *Let α be any non-zero vector in V and let p_α be the T-annihilator of α.*

(i) *The degree of p_α is equal to the dimension of the cyclic subspace $Z(\alpha; T)$.*

(ii) *If the degree of p_α is k, then the vectors $\alpha, T\alpha, T^2\alpha, \ldots, T^{k-1}\alpha$ form a basis for $Z(\alpha; T)$.*

(iii) *If U is the linear operator on $Z(\alpha; T)$ induced by T, then the minimal polynomial for U is p_α.*

Proof. Let g be any polynomial over the field F. Write

$$g = p_\alpha q + r$$

where either $r = 0$ or $\deg(r) < \deg(p_\alpha) = k$. The polynomial $p_\alpha q$ is in the T-annihilator of α, and so

$$g(T)\alpha = r(T)\alpha.$$

Since $r = 0$ or $\deg(r) < k$, the vector $r(T)\alpha$ is a linear combination of the vectors $\alpha, T\alpha, \ldots, T^{k-1}\alpha$, and since $g(T)\alpha$ is a typical vector in

$Z(\alpha; T)$, this shows that these k vectors span $Z(\alpha; T)$. These vectors are certainly linearly independent, because any non-trivial linear relation between them would give us a non-zero polynomial g such that $g(T)\alpha = 0$ and $\deg(g) < \deg(p_\alpha)$, which is absurd. This proves (i) and (ii).

Let U be the linear operator on $Z(\alpha; T)$ obtained by restricting T to that subspace. If g is any polynomial over F, then

$$\begin{aligned} p_\alpha(U)g(T)\alpha &= p_\alpha(T)g(T)\alpha \\ &= g(T)p_\alpha(T)\alpha \\ &= g(T)0 \\ &= 0. \end{aligned}$$

Thus the operator $p_\alpha(U)$ sends every vector in $Z(\alpha; T)$ into 0 and is the zero operator on $Z(\alpha; T)$. Furthermore, if h is a polynomial of degree less than k, we cannot have $h(U) = 0$, for then $h(U)\alpha = h(T)\alpha = 0$, contradicting the definition of p_α. This shows that p_α is the minimal polynomial for U. ▮

A particular consequence of this theorem is the following: If α happens to be a cyclic vector for T, then the minimal polynomial for T must have degree equal to the dimension of the space V; hence, the Cayley-Hamilton theorem tells us that the minimal polynomial for T is the characteristic polynomial for T. We shall prove later that for any T there is a vector α in V which has the minimal polynomial of T for its annihilator. It will then follow that T has a cyclic vector if and only if the minimal and characteristic polynomials for T are identical. But it will take a little work for us to see this.

Our plan is to study the general T by using operators which have a cyclic vector. So, let us take a look at a linear operator U on a space W of dimension k which has a cyclic vector α. By Theorem 1, the vectors $\alpha, \dots, U^{k-1}\alpha$ form a basis for the space W, and the annihilator p_α of α is the minimal polynomial for U (and hence also the characteristic polynomial for U). If we let $\alpha_i = U^{i-1}\alpha$, $i = 1, \dots, k$, then the action of U on the ordered basis $\mathcal{B} = \{\alpha_1, \dots, \alpha_k\}$ is

$$\begin{aligned} U\alpha_i &= \alpha_{i+1}, \qquad i = 1, \dots, k-1 \\ U\alpha_k &= -c_0\alpha_1 - c_1\alpha_2 - \cdots - c_{k-1}\alpha_k \end{aligned} \tag{7-1}$$

where $p_\alpha = c_0 + c_1x + \cdots + c_{k-1}x^{k-1} + x^k$. The expression for $U\alpha_k$ follows from the fact that $p_\alpha(U)\alpha = 0$, i.e.,

$$U^k\alpha + c_{k-1}U^{k-1}\alpha + \cdots + c_1U\alpha + c_0\alpha = 0.$$

This says that the matrix of U in the ordered basis $\mathcal{B}$ is

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & 0 & \cdots & 0 & -c_1 \\ 0 & 1 & 0 & \cdots & 0 & -c_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -c_{k-1} \end{bmatrix}. \tag{7-2}$$

The matrix (7-2) is called the **companion matrix** of the monic polynomial p_α.

Theorem 2. *If* U *is a linear operator on the finite-dimensional space* W, *then* U *has a cyclic vector if and only if there is some ordered basis for* W *in which* U *is represented by the companion matrix of the minimal polynomial for* U.

Proof. We have just observed that if U has a cyclic vector, then there is such an ordered basis for W. Conversely, if we have some ordered basis $\{\alpha_1, \ldots, \alpha_k\}$ for W in which U is represented by the companion matrix of its minimal polynomial, it is obvious that α_1 is a cyclic vector for U. ▌

Corollary. *If* A *is the companion matrix of a monic polynomial* p, *then* p *is both the minimal and the characteristic polynomial of* A.

Proof. One way to see this is to let U be the linear operator on F^k which is represented by A in the standard ordered basis, and to apply Theorem 1 together with the Cayley-Hamilton theorem. Another method is to use Theorem 1 to see that p is the minimal polynomial for A and to verify by a direct calculation that p is the characteristic polynomial for A. ▌

One last comment—if T is any linear operator on the space V and α is any vector in V, then the operator U which T induces on the cyclic subspace $Z(\alpha; T)$ has a cyclic vector, namely, α. Thus $Z(\alpha; T)$ has an ordered basis in which U is represented by the companion matrix of p_α, the T-annihilator of α.

Exercises

1. Let T be a linear operator on F^2. Prove that any non-zero vector which is not a characteristic vector for T is a cyclic vector for T. Hence, prove that either T has a cyclic vector or T is a scalar multiple of the identity operator.

2. Let T be the linear operator on R^3 which is represented in the standard ordered basis by the matrix

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Prove that T has no cyclic vector. What is the T-cyclic subspace generated by the vector $(1, -1, 3)$?

3. Let T be the linear operator on C^3 which is represented in the standard ordered basis by the matrix

$$\begin{bmatrix} 1 & i & 0 \\ -1 & 2 & -i \\ 0 & 1 & 1 \end{bmatrix}.$$