Figure 8.10: Half plane version of Escher's *Circle Limit I*

By restricting Möbius transformations to the boundary of the half plane, half plane geometry can be compressed into the geometry of $\mathbb{RP}^1$, even though $\mathbb{RP}^1$ has no concepts of length or angle. Conversely, length and angle emerge when $\mathbb{RP}^1$ is expanded to the half plane.

Exercises

We can now confirm the impression given by Figure 8.7, that each non-Euclidean line is infinite in both directions, as demanded by Euclid's second axiom.

8.6.1 Show that the y-axis, and hence any non-Euclidean line, can be divided into infinitely many segments of equal non-Euclidean length.

8.6.2 Find a Möbius transformation sending $0, \infty$ to $-1, 1$, respectively, and hence mapping the y-axis onto the unit semicircle.

8.6.3 Using the transformation found in Exercise 8.6.2, find an infinite sequence of points on the unit semicircle that are equally spaced in the sense of non-Euclidean length.

Supposing that the equal faces shown in Figure 8.9 have non-Euclidean width ε, which can be as small as we please, we can draw some interesting conclusions about the non-Euclidean distance between non-Euclidean lines.

8.6.4 Show that the non-Euclidean distance between the lines $x = 0$ and $x = 1$ tends to zero as y tends to ∞.

8.6.5 Show that the Möbius transformation $z \mapsto 2/(1-z)$ sends the unit circle and the line $x = 1$ to the lines $x = 1$ and $x = 0$, respectively.

8.6.6 Deduce from Exercise 8.6.5 that the non-Euclidean distance between the unit circle and the line $x = 1$ tends to zero as these non-Euclidean lines approach the x-axis.

8.7 Non-Euclidean translations and rotations

Like the Euclidean plane, the half plane has isometries called *translations* and *rotations*, which are products of two reflections. Their nature depends on whether the lines of reflection meet or have a common end.

A translation is the product of reflections in non-Euclidean lines that do not meet and do not have a common end. A simple example is $z \mapsto 2z$, which is the product of reflections in the circles with center 0 and radii 1 and $\sqrt{2}$. This translation maps each face in Figure 8.9 to the one above it. Any non-Euclidean translation maps a unique non-Euclidean line, called the *translation axis*, into itself. Also mapped into themselves are the curves at constant non-Euclidean distance from the translation axis, which (for distance > 0) are *not* non-Euclidean lines. For $z \mapsto 2z$, the translation axis is the y-axis and the equidistant curves are the Euclidean lines $y = ax$. Each non-Euclidean line perpendicular to the translation axis is mapped onto another such line.

Figure 8.11 shows the translation axis, two equidistant curves (in gray), and some of their perpendiculars (on the left when the axis is vertical, and on the right when it is not). Notice that the equidistant curves in general are Euclidean circles passing through the two ends of the translation axis. The translation moves each non-Euclidean perpendicular to the next.

The product of reflections in two non-Euclidean lines that meet at a point P is a *non-Euclidean rotation* about P. The point P remains fixed and points at non-Euclidean distance r from P remain at non-Euclidean distance r from P, since reflection is a non-Euclidean isometry. Hence, these points move on a *non-Euclidean circle of radius* r. It turns out that a non-Euclidean circle is a Euclidean circle, although its non-Euclidean

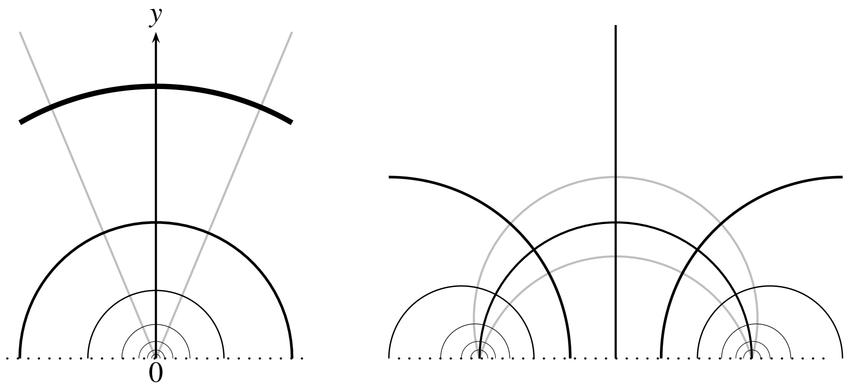

Figure 8.11: Non-Euclidean translations

center (the point at constant non-Euclidean distance from all its points) is not its Euclidean center.

For example, if we take the product of the reflection $z \mapsto -\overline{z}$ in the y-axis with the reflection $z \mapsto -1/\overline{z}$ in the unit circle, the result is a rotation through angle π about the point i where these two non-Euclidean lines meet. More generally, if we have two non-Euclidean lines through P meeting at angle θ, then the product of reflections in these lines is a rotation about P through angle 2θ. Figure 8.12 shows four non-Euclidean lines through i and two non-Euclidean circles (in gray) with non-Euclidean center at i. A rotation of $\pi/4$ about i moves each non-Euclidean line to the next and maps each circle into itself.

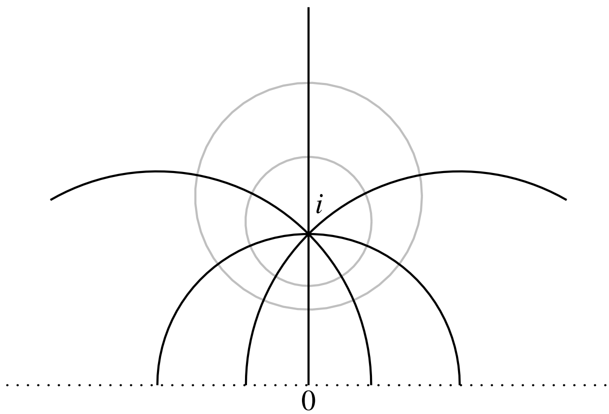

Figure 8.12: A non-Euclidean rotation about i

A limiting case of rotation is where the two lines of reflection do not meet in the half plane, but have a common end P on the boundary $\mathbb{R} \cup \{\infty\}$ at infinity. Here P is a fixed point, each non-Euclidean line ending at P is moved to another line ending at P, and each curve perpendicular to all these lines is mapped onto itself. This kind of isometry is called a *limit rotation*, and each curve mapped onto itself is called a *limit circle* or *horocycle*.

The simplest example is the Euclidean horizontal translation $z \mapsto z+1$, which is the product of reflections in the vertical lines $x = 0$ and $x = 1/2$. Each vertical line $x = a$ is mapped to the line $x = a+1$, and each horizontal line $y = b$ is mapped onto itself. Thus, the horizontal lines $y = b$, which we know are *not* non-Euclidean lines, are limit circles.

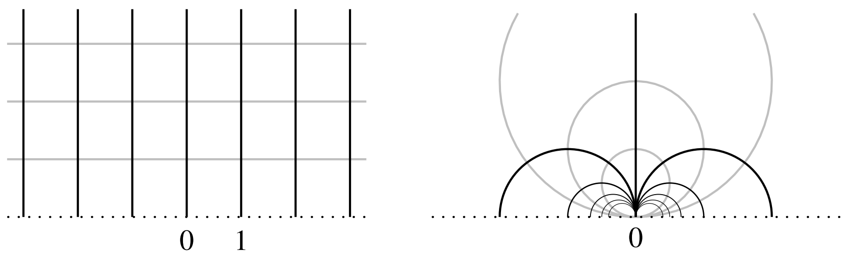

Figure 8.13: Limit rotations

Like equidistant curves, limit circles can be Euclidean lines, but generally they are Euclidean circles. Figure 8.13 shows the exceptional case $z \mapsto z+1$, where the limit circles are the Euclidean horizontal lines (in gray), and the typical case $z \mapsto z/(1-z)$, where the limit circles are the gray circles tangential to the boundary at the fixed point $z = 0$.

As in the previous pictures, the isometry moves each non-Euclidean line to the next, and maps each gray curve onto itself.

Exercises

8.7.1 Check that the product of reflections in the y-axis and the unit circle is $z \mapsto -1/z$, and that i is the fixed point of this map.

8.7.2 Show also that $z \mapsto -1/z$ maps each circle of the form $|z - ti| = \sqrt{t^2 - 1}$ onto itself.

The limit rotation $z \mapsto z/(1-z)$ above is obtained by moving the limit rotation $z \mapsto z+1$ about ∞ to a limit rotation about 0 with the help of the rotation $z \mapsto -1/z$ that exchanges 0 and ∞.

8.7.3 If $f(z) = z + 1$ and $g(z) = -1/z$, show that $gfg^{-1}(z) = z/(1-z)$.

8.7.4 Describe in words what g^{-1}, f, g in succession do to the half plane, and hence explain geometrically why gfg^{-1} has fixed point 0.

8.8 Three reflections or two involutions

It is possible to prove that each isometry of the half plane is the product of three reflections, following much the same approach as was used in Section 3.7 to prove the three reflections theorem for the Euclidean plane. The details of this approach are worked out in my book *Geometry of Surfaces*.

However, our approach to isometries of the Euclidean plane began with a definition of Euclidean distance; we then had to *find* the transformations that leave it invariant. Here we know the isometries of the half plane—the Möbius transformations—so the only problem is to express them as products in some simple way. To do this, we can interpret Möbius transformations on $\mathbb{RP}^1$, and exploit known theorems of projective geometry. Surprisingly, there is a theorem about $\mathbb{RP}^1$ that goes one better than the three reflections theorem, namely the *two involutions theorem* from Veblen and Young's 1910 book *Projective Geometry*, p. 223.

An *involution* is a transformation f such that f^2 is the identity. Thus, the involutions include the reflections, but some other transformations as well, such as the function $x \mapsto -1/x$, which (when extended to the half plane) represents a half turn about the point i. The name "involution" is one of many terms introduced into projective geometry by Desargues, and it is the only one that has stuck.

To pave the way for the two involutions theorem (and the three reflections theorem that follows from it), we first note three consequences of the results in Section 5.8 about transformations of $\mathbb{RP}^1$.

- *Any four points $p,q,r,s \in \mathbb{RP}^1$ can be mapped to q,p,s,r, respectively, by a linear fractional transformation.*

  Notice that $[p,q:r,s] = [q,p:s,r]$ because

  $$\frac{(r-p)(s-q)}{(r-q)(s-p)} = \frac{(s-q)(r-p)}{(s-p)(r-q)}.$$

  Hence, by the "criterion for four-point maps" in Section 5.8, there is a linear fractional f mapping p,q,r,s to q,p,s,r, respectively.

- *If g is a linear fractional transformation that exchanges two points, then g is an involution.*

  Suppose that p and q are two points with $g(p) = q$ and $g(q) = p$. Let r be another point, not fixed by g, and suppose that $g(r) = s$. Because any linear fractional function is one-to-one, it follows that p, q, r, s are different. Hence, by the previous result, there is a linear fractional f mapping p, q, r, s to q, p, s, r, respectively.

  Because f agrees with g on the three points p, q, r, the functions f and g are identical by the "uniqueness of three-point maps" in Section 5.8. For any nonfixed point r of g, we therefore have

  $$g^2(r) = g(s) = f(s) = r,$$

  and if r is a fixed point, then $g^2(r) = r$ obviously. Hence, $g^2(x) = x$ for any $x \in \mathbb{RP}^1$, and so g is an involution.

- *For any three points p, q, r, there is an involution that exchanges p, q and fixes r.*

  By "existence of three-point maps" from Section 5.8, there is a linear fractional function g that sends p, q, r to q, p, r, respectively. Thus, g fixes r, and because it exchanges p and q, it is an involution by the previous result.

Two involutions theorem. *Any linear fractional transformation h of $\mathbb{RP}^1$ is the product of two involutions.*

If $h =$ identity, then $h =$ identity $\cdot$ identity, which is the product of two involutions. If not, let p be a point not fixed by h, so

$$h(p) = r \neq p,$$

and let $h(r) = q$. Then $q \neq r$, because h^{-1} is also a linear fractional transformation and hence one-to-one. If $q = p$, then h exchanges p and r. Hence, h is itself an involution by the second result above.

We can therefore assume that p, q, r are three different points; in which case, the third result above gives a linear fractional involution f such that

$$f(p) = q, \quad f(q) = p, \quad f(r) = r.$$

Also, fh exchanges the two points p and r because

$$fh(p) = f(r) = r, \quad fh(r) = f(q) = p.$$