rather than an adherent point), or would use $\lim_{x\to x_0;x\in E} f(x)$ to denote what we would call $\lim_{x\in x_0;x\in E\backslash\{x_0\}} f(x)$, but we have chosen a slightly more general notation, which allows the possibility that E contains x_0.

Example 9.3.8. Let $f : [1,3] \to \mathbf{R}$ be the function $f(x) := x^2$. We have seen before that f is 0.1-close to 4 near 2. A similar argument shows that f is 0.01-close to 4 near 2 (one just has to pick a smaller value of δ).

Definition 9.3.6 is rather unwieldy. However, we can rewrite this definition in terms of a more familiar one, involving limits of sequences.

Proposition 9.3.9. *Let X be a subset of $\mathbf{R}$, let $f : X \to \mathbf{R}$ be a function, let E be a subset of X, let x_0 be an adherent point of E, and let L be a real number. Then the following two statements are logically equivalent:*

(a) *f converges to L at x_0 in E.*

(b) *For every sequence $(a_n)_{n=0}^{\infty}$ which consists entirely of elements of E, which converges to x_0, the sequence $(f(a_n))_{n=0}^{\infty}$ converges to $f(x_0)$.*

Proof. See Exercise 9.3.1. □

In view of the above proposition, we will sometimes write "$f(x) \to L$ as $x \to x_0$ in E" or "f has a limit L at x_0 in E" instead of "f converges to L at x_0", or "$\lim_{x\to x_0} f(x) = L$".

Remark 9.3.10. With the notation of Proposition 9.3.9, we have the following corollary: if $\lim_{x\to x_0;x\in E} f(x) = L$, and $\lim_{n\to\infty} a_n = x_0$, then $\lim_{n\to\infty} f(a_n) = L$.

Remark 9.3.11. We only consider limits of a function f at x_0 in the case when x_0 is an adherent point of E. When x_0 is not an adherent point then it is not worth it to define the concept of a limit. (Can you see why there will be problems?)

Remark 9.3.12. The variable x used to denote a limit is a dummy variable; we could replace it by any other variable and obtain exactly the same limit. For instance, if $\lim_{x \to x_0; x \in E} f(x) = L$, then $\lim_{y \to x_0; y \in E} f(y) = L$, and conversely (why?).

Proposition 9.3.9 has some immediate corollaries. For instance, we now know that a function can have at most one limit at each point:

Corollary 9.3.13. *Let X be a subset of $\mathbf{R}$, let E be a subset of X, let x_0 be an adherent point of E, and let $f : X \to \mathbf{R}$ be a function. Then f can have at most one limit at x_0 in E.*

Proof. Suppose for sake of contradiction that there are two distinct numbers L and L' such that f has a limit L at x_0 in E, and such that f also has a limit L' at x_0 in E. Since x_0 is an adherent point of E, we know by Lemma 9.1.14 that there is a sequence $(a_n)_{n=0}^{\infty}$ consisting of elements in E which converges to x_0. Since f has a limit L at x_0 in E, we thus see by Proposition 9.3.9, that $(f(a_n))_{n=0}^{\infty}$ converges to L. But since f also has a limit L' at x_0 in E, we see that $(f(a_n))_{n=0}^{\infty}$ also converges to L'. But this contradicts the uniqueness of limits of sequences (Proposition 6.1.7). □

Using the limit laws for sequences, one can now deduce the limit laws for functions:

Proposition 9.3.14 (Limit laws for functions). *Let X be a subset of $\mathbf{R}$, let E be a subset of X, let x_0 be an adherent point of E, and let $f : X \to \mathbf{R}$ and $g : X \to \mathbf{R}$ be functions. Suppose that f has a limit L at x_0 in E, and g has a limit M at x_0 in E. Then $f + g$ has a limit $L + M$ at x_0 in E, $f - g$ has a limit $L - M$ at x_0 in E, $\max(f, g)$ has a limit $\max(L, M)$ at x_0 in E, $\min(f, g)$ has a limit $\min(L, M)$ at x_0 in E and fg has a limit LM at x_0 in E. If c is a real number, then cf has a limit cL at x_0 in E. Finally, if g is non-zero on E (i.e., $g(x) \neq 0$ for all $x \in E$) and M is non-zero, then f/g has a limit L/M at x_0 in E.*

Proof. We just prove the first claim (that $f+g$ has a limit $L+M$); the others are very similar and are left to Exercise 9.3.2. Since x_0 is an adherent point of E, we know by Lemma 9.1.14 that there is a sequence $(a_n)_{n=0}^\infty$ consisting of elements in E, which converges to x_0. Since f has a limit L at x_0 in E, we thus see by Proposition 9.3.9, that $(f(a_n))_{n=0}^\infty$ converges to L. Similarly $(g(a_n))_{n=0}^\infty$ converges to M. By the limit laws for sequences (Theorem 6.1.19) we conclude that $((f+g)(a_n))_{n=0}^\infty$ converges to $L+M$. By Proposition 9.3.9 again, this implies that $f+g$ has a limit $L+M$ at x_0 in E as desired (since $(a_n)_{n=0}^\infty$ was an arbitrary sequence in E converging to x_0). □

Remark 9.3.15. One can phrase Proposition 9.3.14 more informally as saying that

$$\begin{aligned}
\lim_{x\to x_0}(f\pm g)(x) &= \lim_{x\to x_0} f(x) \pm \lim_{x\to x_0} g(x)\\
\lim_{x\to x_0}\max(f,g)(x) &= \max\left(\lim_{x\to x_0} f(x), \lim_{x\to x_0} g(x)\right)\\
\lim_{x\to x_0}\min(f,g)(x) &= \min\left(\lim_{x\to x_0} f(x), \lim_{x\to x_0} g(x)\right)\\
\lim_{x\to x_0}(fg)(x) &= \lim_{x\to x_0} f(x) \lim_{x\to x_0} g(x)\\
\lim_{x\to x_0}(f/g)(x) &= \frac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)}
\end{aligned}$$

(where we have dropped the restriction $x \in E$ for brevity) but bear in mind that these identities are only true when the right-hand side makes sense, and furthermore for the final identity we need g to be non-zero, and also $\lim_{x\to x_0} g(x)$ to be non-zero. (See Example 1.2.4 for some examples of what goes wrong when limits are manipulated carelessly.)

Using the limit laws in Proposition 9.3.14 we can already deduce several limits. First of all, it is easy to check the basic limits

$$\lim_{x\to x_0; x\in\mathbf{R}} c = c$$

and

$$\lim_{x \to x_0; x \in \mathbf{R}} x = x_0$$

for any real numbers x_0 and c. (Why? use Proposition 9.3.9.) By the limit laws we can thus conclude that

$$\lim_{x \to x_0; x \in \mathbf{R}} x^2 = x_0^2$$

$$\lim_{x \to x_0; x \in \mathbf{R}} cx = cx_0$$

$$\lim_{x \to x_0; x \in \mathbf{R}} x^2 + cx + d = x_0^2 + cx_0 + d$$

etc., where c, d are arbitrary real numbers.

If f converges to L at x_0 in X, and Y is any subset of X such that x_0 is still an adherent point of Y, then f will also converge to L at x_0 in Y (why?). Thus convergence on a large set implies convergence on a smaller set. The converse, however, is not true:

Example 9.3.16. Consider the *signum function* sgn : $\mathbf{R} \to \mathbf{R}$, defined by

$$\operatorname{sgn}(x) := \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Then $\lim_{x \to 0; x \in (0,\infty)} \operatorname{sgn}(x) = 1$ (why?), whereas $\lim_{x \to 0; x \in (-\infty,0)} = -1$ (why?) and $\lim_{x \to 0; x \in \mathbf{R}} \operatorname{sgn}(x)$ is undefined (why?). Thus it is sometimes dangerous to drop the set X from the notation of limit. However, in many cases it is safe to do so; for instance, since we know that $\lim_{x \to x_0; x \in \mathbf{R}} x^2 = x_0^2$, we know in fact that $\lim_{x \in x_0; x \in X} x^2 = x_0^2$ for any set X with x_0 as an adherent point (why?). Thus it is safe to write $\lim_{x \to x_0} x^2 = x_0^2$.

Example 9.3.17. Let $f(x)$ be the function

$$f(x) := \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0. \end{cases}$$

Then $\lim_{x\to 0;x\in\mathbf{R}-\{0\}} f(x) = 0$ (why?), but $\lim_{x\to 0;x\in\mathbf{R}} f(x)$ is undefined (why). (When this happens, we say that f has a "removable singularity" or "removable discontinuity" at 0. Because of such singularities, it is sometimes the convention when writing $\lim_{x\to x_0} f(x)$ to automatically exclude x_0 from the set; for instance, in the textbook, $\lim_{x\to x_0} f(x)$ is used as shorthand for $\lim_{x\to x_0;x\in X-\{x_0\}} f(x)$.)

On the other hand, the limit at x_0 should only depend on the values of the function near x_0; the values away from x_0 are not relevant. The following proposition reflects this intuition:

Proposition 9.3.18 (Limits are local). *Let X be a subset of $\mathbf{R}$, let E be a subset of X, let x_0 be an adherent point of E, let $f : X \to \mathbf{R}$ be a function, and let L be a real number. Let $\delta > 0$. Then we have*

$$\lim_{x\to x_0;x\in E} f(x) = L$$

if and only if

$$\lim_{x\to x_0;x\in E\cap(x_0-\delta,x_0+\delta)} f(x) = L.$$

Proof. See Exercise 9.3.3. □

Informally, the above proposition asserts that

$$\lim_{x\to x_0;x\in E} f(x) = \lim_{x\to x_0;x\in E\cap(x_0-\delta,x_0+\delta)} f(x).$$

Thus the limit of a function at x_0, if it exists, only depends on the values of f near x_0; the values far away do not actually influence the limit.

We now give a few more examples of limits.

Example 9.3.19. Consider the functions $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ defined by $f(x) := x + 2$ and $g(x) := x + 1$. Then $\lim_{x\to 2;x\in\mathbf{R}} f(x) = 4$ and $\lim_{x\to 2;x\in\mathbf{R}} g(x) = 3$. We would like to use the limit laws to conclude that $\lim_{x\to 2;x\in\mathbf{R}} f(x)/g(x) = 4/3$, or in other words that $\lim_{x\to 2;x\in\mathbf{R}} \frac{x+2}{x+1} = \frac{4}{3}$. Strictly speaking, we cannot use Proposition 9.3.14 to ensure this, because $x+1$ is zero at $x = -1$, and so $f(x)/g(x)$ is not defined. However, this is easily solved, by restricting the domain of f and g from $\mathbf{R}$ to a smaller domain, such as $\mathbf{R} - \{1\}$. Then Proposition 9.3.14 does apply, and we have $\lim_{x\to 2; x\in\mathbf{R}-\{1\}} \frac{x+2}{x+1} = \frac{4}{3}$.

Example 9.3.20. Consider the function $f : \mathbf{R} - \{1\} \to \mathbf{R}$ defined by $f(x) := (x^2-1)/(x-1)$. This function is well-defined for every real number except 1, so $f(1)$ is undefined. However, 1 is still an adherent point of $\mathbf{R} - \{1\}$ (why?), and the limit $\lim_{x\to 1; x\in\mathbf{R}-\{1\}} f(x)$ is still defined. This is because on the domain $\mathbf{R} - \{1\}$ we have the identity $(x^2-1)/(x-1) = (x+1)(x-1)/(x-1) = x+1$, and $\lim_{x\to 1; x\in\mathbf{R}-\{1\}} x+1 = 2$.

Example 9.3.21. Let $f : \mathbf{R} \to \mathbf{R}$ be the function

$$f(x) := \begin{cases} 1 & \text{if } x \in \mathbf{Q} \\ 0 & \text{if } x \notin \mathbf{Q}. \end{cases}$$

We will show that $f(x)$ has no limit at 0 in $\mathbf{R}$. Suppose for sake of contradiction that $f(x)$ had some limit L at 0 in $\mathbf{R}$. Then we would have $\lim_{n\to\infty} f(a_n) = L$ whenever $(a_n)_{n=0}^\infty$ is a sequence of non-zero numbers converging to 0. Since $(1/n)_{n=0}^\infty$ is such a sequence, we would have

$$L = \lim_{n\to\infty} f(1/n) = \lim_{n\to\infty} 1 = 1.$$

On the other hand, since $(\sqrt{2}/n)_{n=0}^\infty$ is another sequence of non-zero numbers converging to 0 - but now these numbers are irrational instead of rational - we have

$$L = \lim_{n\to\infty} f(\sqrt{2}/n) = \lim_{n\to\infty} 0 = 0.$$

Since $1 \neq 0$, we have a contradiction. Thus this function does not have a limit at 0.

Exercise 9.3.1. Prove Proposition 9.3.9.

Exercise 9.3.2. Prove the remaining claims in Proposition 9.3.14.

Exercise 9.3.3. Prove Lemma 9.3.18.

9.4 Continuous functions

We now introduce one of the most fundamental notions in the theory of functions - that of *continuity*.

Definition 9.4.1 (Continuity). Let X be a subset of $\mathbf{R}$, and let $f : X \to \mathbf{R}$ be a function. Let x_0 be an element of X. We say that f is *continuous at* x_0 iff we have

$$\lim_{x \to x_0; x \in X} f(x) = f(x_0);$$

in other words, the limit of $f(x)$ as x converges to x_0 in X exists and is equal to $f(x_0)$. We say that f is *continuous on* X (or simply *continuous*) iff f is continuous at x_0 for every $x_0 \in X$. We say that f is *discontinuous at* x_0 iff it is not continuous at x_0.

Example 9.4.2. Let c be a real number, and let $f : \mathbf{R} \to \mathbf{R}$ be the constant function $f(x) := c$. Then for every real number $x_0 \in \mathbf{R}$, we have

$$\lim_{x \to x_0; x \in \mathbf{R}} f(x) = \lim_{x \to x_0; x \in \mathbf{R}} c = c = f(x_0),$$

thus f is continuous at every point $x_0 \in \mathbf{R}$, or in other words f is continuous on $\mathbf{R}$.

Example 9.4.3. Let $f : \mathbf{R} \to \mathbf{R}$ be the identity function $f(x) := x$. Then for every real number $x_0 \in \mathbf{R}$, we have

$$\lim_{x \to x_0; x \in \mathbf{R}} f(x) = \lim_{x_0 \in x; x \in \mathbf{R}} x = x_0 = f(x_0),$$

thus f is continuous at every point $x_0 \in \mathbf{R}$, or in other words f is continuous on $\mathbf{R}$.

Example 9.4.4. Let sgn : $\mathbf{R} \to \mathbf{R}$ be the signum function defined in Example 9.3.16. Then $\text{sgn}(x)$ is continuous at every non-zero value of x; for instance, at 1, we have (using Proposition 9.3.18)

$$\begin{aligned}
\lim_{x\to 1; x\in\mathbf{R}} \operatorname{sgn}(x) &= \lim_{x\to 1; x\in(0.9,1.1)} \operatorname{sgn}(x) \\
&= \lim_{x\to 1; x\in(0.9,1.1)} 1 \\
&= 1 \\
&= \operatorname{sgn}(1).
\end{aligned}$$

On the other hand, sgn is not continuous at 0, since the limit $\lim_{x\to 0; x\in\mathbf{R}} \operatorname{sgn}(x)$ does not exist.

Example 9.4.5. Let $f : \mathbf{R} \to \mathbf{R}$ be the function

$$f(x) := \begin{cases} 1 & \text{if } x \in \mathbf{Q} \\ 0 & \text{if } x \notin \mathbf{Q}. \end{cases}$$

Then by the discussion in the previous section, f is not continuous at 0. In fact, it turns out that f is not continuous at any real number x_0 (can you see why?).

Example 9.4.6. Let $f : \mathbf{R} \to \mathbf{R}$ be the function

$$f(x) := \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

Then f is continuous at every non-zero real number (why?), but is not continuous at 0. However, if we restrict f to the right-hand line $[0, \infty)$, then the resulting function $f|_{[0,\infty)}$ now becomes continuous everywhere in its domain, including 0. Thus restricting the domain of a function can make a discontinuous function continuous again.

There are several ways to phrase the statement that "f is continuous at x_0":

Proposition 9.4.7 (Equivalent formulations of continuity). *Let X be a subset of $\mathbf{R}$, let $f : X \to \mathbf{R}$ be a function, and let x_0 be an element of X. Then the following three statements are logically equivalent:*

(a) *f is continuous at x_0.*

(b) *For every sequence $(a_n)_{n=0}^{\infty}$ consisting of elements of X with $\lim_{n\to\infty} a_n = x_0$, we have $\lim_{n\to\infty} f(a_n) = f(x_0)$.*

(c) *For every $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ for all $x \in X$ with $|x - x_0| < \delta$.*

Proof. See Exercise 9.4.1. □

Remark 9.4.8. A particularly useful consequence of Proposition 9.4.7 is the following: if f is continuous at x_0, and $a_n \to x_0$ as $n \to \infty$, then $f(a_n) \to f(x_0)$ as $n \to \infty$ (provided that all the elements of the sequence $(a_n)_{n=0}^{\infty}$ lie in the domain of f, of course). Thus continuous functions are very useful in computing limits.

The limit laws in Proposition 9.3.14, combined with the definition of continuity in Definition 9.4.1, immediately imply

Proposition 9.4.9 (Arithmetic preserves continuity). *Let X be a subset of $\mathbf{R}$, and let $f : X \to \mathbf{R}$ and $g : X \to \mathbf{R}$ be functions. Let $x_0 \in X$. Then if f and g are both continuous at x_0, then the functions $f + g$, $f - g$, $\max(f, g)$, $\min(f, g)$ and fg are also continuous at x_0. If g is non-zero on X, then f/g is also continuous at x_0.*

In particular, the sum, difference, maximum, minimum, and product of continuous functions are continuous; and the quotient of two continuous functions is continuous as long as the denominator does not become zero.

One can use Proposition 9.4.9 to show that a lot of functions are continuous. For instance, just by starting from the fact that constant functions are continuous, and the identity function $f(x) = x$ is continuous (Exercise 9.4.2), one can show that the function $g(x) := \max(x^3 + 4x^2 + x + 5, x^4 - x^3)/(x^2 - 4)$, for instance, is continuous at every point of $\mathbf{R}$ except the two points $x = +2$, $x = -2$ where the denominator vanishes.

Some other examples of continuous functions are given below.

Proposition 9.4.10 (Exponentiation is continuous, I). *Let $a > 0$ be a positive real number. Then the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) := a^x$ is continuous.*

Proof. See Exercise 9.4.3. □

Proposition 9.4.11 (Exponentiation is continuous, II). *Let p be a real number. Then the function $f : (0, \infty) \to \mathbf{R}$ defined by $f(x) := x^p$ is continuous.*

Proof. See Exercise 9.4.4. □

There is a stronger statement than Propositions 9.4.10, 9.4.11, namely that exponentiation is *jointly continuous* in both the exponent and the base, but this is harder to show; see Exercise 15.5.10.

Proposition 9.4.12 (Absolute value is continuous). *The function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) := |x|$ is continuous.*

Proof. This follows since $|x| = \max(x, -x)$ and the functions $x, -x$ are already continuous. □

The class of continuous functions is not only closed under addition, subtraction, multiplication, and division, but is also closed under composition:

Proposition 9.4.13 (Composition preserves continuity). *Let X and Y be subsets of $\mathbf{R}$, and let $f : X \to Y$ and $g : Y \to \mathbf{R}$ be functions. Let x_0 be a point in X. If f is continuous at x_0, and g is continuous at $f(x_0)$, then the composition $g \circ f : X \to \mathbf{R}$ is continuous at x_0.*

Proof. See Exercise 9.4.5. □

Example 9.4.14. Since the function $f(x) := 3x+1$ is continuous on all of $\mathbf{R}$, and the function $g(x) := 5^x$ is continuous on all of $\mathbf{R}$, the function $g \circ f(x) = 5^{3x+1}$ is continuous on all of $\mathbf{R}$. By several applications of the above propositions, one can show that far more complicated functions, e.g., $h(x) := |x^2 - 8x + 7|^{\sqrt{2}}/(x^2 + 1)$, are also continuous. (Why is this function continuous?) There are still a few functions though that are not yet easy to test for continuity, such as $k(x) := x^x$; this function can be dealt with more easily once we have the machinery of logarithms, which we will see in Section 15.5.

Exercise 9.4.1. Prove Proposition 9.4.7. (Hint: this can largely be done by applying the previous propositions and lemmas. Note that to prove (a),(b), and (c) are equivalent, you do not have to prove all six equivalences, but you do have to prove at least three; for instance, showing that (a) implies (b), (b) implies (c), and (c) implies (a) will suffice, although this is not necessarily the shortest or simplest way to do this question.)

Exercise 9.4.2. Let X be a subset of $\mathbf{R}$, and let $c \in \mathbf{R}$. Show that the constant function $f : X \to \mathbf{R}$ defined by $f(x) := c$ is continuous, and show that the identity function $g : X \to \mathbf{R}$ defined by $g(x) := x$ is also continuous.

Exercise 9.4.3. Prove Proposition 9.4.10. (Hint: you can use Lemma 6.5.3, combined with the squeeze test (Corollary 6.4.14) and Proposition 6.7.3.)

Exercise 9.4.4. Prove Proposition 9.4.11. (Hint: from limit laws (Proposition 9.3.14) one can show that $\lim_{x\to 1} x^n = 1$ for all integers n. From this and the squeeze test (Corollary 6.4.14) deduce that $\lim_{x\to 1} x^p = 1$ for all real numbers p. Finally, apply Proposition 6.7.3.)

Exercise 9.4.5. Prove Proposition 9.4.13.

Exercise 9.4.6. Let X be a subset of $\mathbf{R}$, and let $f : X \to \mathbf{R}$ be a continuous function. If Y is a subset of X, show that the restriction $f|_Y : Y \to \mathbf{R}$ of f to Y is also a continuous function. (Hint: this is a simple result, but it requires you to follow the definitions carefully.)

Exercise 9.4.7. Let $n \geq 0$ be an integer, and for each $0 \leq i \leq n$ let c_i be a real number. Let $P : \mathbf{R} \to \mathbf{R}$ be the function

$$P(x) := \sum_{i=0}^{n} c_i x^i;$$

such a function is known as a *polynomial of one variable*; a typical example is $P(x) = 6x^4 - 3x^2 + 4$. Show that P is continuous.

9.5 Left and right limits

We now introduce the notion of left and right limits, which can be thought of as two seperate "halves" of the complete limit $\lim_{x \to x_0; x \in X} f(x)$.

Definition 9.5.1 (Left and right limits). Let X be a subset of $\mathbf{R}$, $f : X \to \mathbf{R}$ be a function, and let x_0 be a real number. If x_0 is an adherent point of $X \cap (x_0, \infty)$, then we define the *right limit* $f(x_0+)$ of f at x_0 by the formula

$$f(x_0+) := \lim_{x \to x_0; x \in X \cap (x_0, \infty)} f(x),$$

provided of course that this limit exists. Similarly, if x_0 is an adherent point of $X \cap (-\infty, x_0)$, then we define the *left limit* $f(x_0-)$ of f at x_0 by the formula

$$f(x_0-) := \lim_{x \to x_0; x \in X \cap (-\infty, x_0)} f(x),$$

again provided that the limit exists. (Thus in many cases $f(x_0+)$ and $f(x_0-)$ will not be defined.)

Sometimes we use the shorthand notations

$$\lim_{x \to x_0+} f(x) := \lim_{x \to x_0; x \in X \cap (x_0, \infty)} f(x);$$
$$\lim_{x \to x_0-} f(x) := \lim_{x \to x_0; x \in X \cap (-\infty, x_0)} f(x)$$

when the domain X of f is clear from context.

Example 9.5.2. Consider the signum function $\mathrm{sgn} : \mathbf{R} \to \mathbf{R}$ defined in Example 9.3.16. We have

$$\mathrm{sgn}(0+) = \lim_{x \to x_0; x \in \mathbf{R} \cap (0, \infty)} \mathrm{sgn}(x) = \lim_{x \to x_0; x \in \mathbf{R} \cap (0, \infty)} 1 = 1$$

and

$$\mathrm{sgn}(0-) = \lim_{x \to x_0; x \in \mathbf{R} \cap (-\infty, 0)} \mathrm{sgn}(x) = \lim_{x \to x_0; x \in \mathbf{R} \cap (-\infty, 0)} -1 = -1,$$

while $\mathrm{sgn}(0) = 0$ by definition.

Note that f does not necessarily have to be defined at x_0 in order for $f(x_0+)$ or $f(x_0-)$ to be defined. For instance, if $f : \mathbf{R} - \{0\} \to \mathbf{R}$ is the function $f(x) := x/|x|$, then $f(0+) = 1$ and $f(0-) = -1$ (why?), even though $f(0)$ is undefined.

From Proposition 9.4.7 we see that if the right limit $f(x_0+)$ exists, and $(a_n)_{n=0}^\infty$ is a sequence in X converging to x_0 from the right (i.e., $a_n > x_0$ for all $n \in \mathbf{N}$), then $\lim_{n\to\infty} f(a_n) = f(x_0+)$. Similarly, if $(b_n)_{n=0}^\infty$ is a sequence converging to x_0 from the left (i.e., $a_n < x_0$ for all $n \in \mathbf{N}$) then $\lim_{n\to\infty} f(a_n) = f(x_0-)$.

Let x_0 be an adherent point of both $X \cap (x_0, \infty)$ and $X \cap (-\infty, x_0)$. If f is continuous at x_0, it is clear from Proposition 9.4.7 that $f(x_0+)$ and $f(x_0-)$ both exist and are equal to $f(x_0)$. (Can you see why?) A converse is also true (compare this with Proposition 6.4.12(f)):

Proposition 9.5.3. *Let X be a subset of $\mathbf{R}$ containing a real number x_0, and suppose that x_0 is an adherent point of both $X \cap (x_0, \infty)$ and $X \cap (-\infty, x_0)$. Let $f : X \to \mathbf{R}$ be a function. If $f(x_0+)$ and $f(x_0-)$ both exist and are both equal to $f(x_0)$, then f is continuous at x_0.*

Proof. Let us write $L := f(x_0)$. Then by hypothesis we have

$$\lim_{x\to x_0; x\in X\cap(x_0,\infty)} f(x) = L \tag{9.1}$$

and

$$\lim_{x\to x_0; x\in X\cap(-\infty,x_0)} f(x) = L. \tag{9.2}$$

Let $\varepsilon > 0$ be given. From (9.1) and Proposition 9.4.7, we know that there exists a $\delta_+ > 0$ such that $|f(x) - L| < \varepsilon$ for all $x \in X \cap (x_0, \infty)$ for which $|x - x_0| < \delta_+$. From (9.2) we similarly know that there exists a $\delta_- > 0$ such that $|f(x) - L| < \varepsilon$ for all $x \in X \cap (-\infty, x_0)$ for which $|x - x_0| < \delta_-$. Now let $\delta := \min(\delta_-, \delta_+)$; then $\delta > 0$ (why?), and suppose that $x \in X$ is such that $|x - x_0| < \delta$. Then there are three cases: $x > x_0$, $x = x_0$, and $x < x_0$, but in all three cases we know that $|f(x) - L| < \varepsilon$. (Why? the reason is different in each of the three cases.) By Proposition 9.4.7 we thus have that f is continuous at x_0, as desired. $\square$

As we saw with the signum function in Example 9.3.16, it is possible for the left and right limits $f(x_0-)$, $f(x_0+)$ of a function f at a point x_0 to both exist, but not be equal to each other; when this happens, we say that f has a *jump discontinuity* at x_0. Thus, for instance, the signum function has a jump discontinuity at zero. Also, it is possible for the left and right limits $f(x_0-)$, $f(x_0+)$ to exist and be equal each other, but not be equal to $f(x_0)$; when this happens we say that f has a *removable discontinuity* (or *removable singularity*) at x_0. For instance, if we take $f : \mathbf{R} \to \mathbf{R}$ to be the function

$$f(x) := \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0, \end{cases}$$

then $f(0+)$ and $f(0-)$ both exist and equal 0 (why?), but $f(0)$ equals 1; thus f has a removable discontinuity at 0.

Remark 9.5.4. Jump discontinuities and removable discontinuities are not the only way a function can be discontinuous. Another way is for a function to go to infinity at the discontinuity: for instance, the function $f : \mathbf{R} - \{0\} \to \mathbf{R}$ defined by $f(x) := 1/x$ has a discontinuity at 0 which is neither a jump discontinuity or a removable singularity; informally, $f(x)$ converges to $+\infty$ when x approaches 0 from the right, and converges to $-\infty$ when x approaches 0 from the left. These types of singularities are sometimes known as *asymptotic discontinuities.* There are also *oscillatory discontinuities*, where the function remains bounded but still does not have a limit near x_0. For instance, the function $f : \mathbf{R} \to \mathbf{R}$ defined by

$$f(x) := \begin{cases} 1 & \text{if } x \in \mathbf{Q} \\ 0 & \text{if } x \notin \mathbf{Q} \end{cases}$$

has an oscillatory discontinuity at 0 (and in fact at any other real number also). This is because the function does not have left or right limits at 0, despite the fact that the function is bounded.

The study of discontinuities (also called *singularities*) continues further, but is beyond the scope of this text. For instance, singularities play a key rôle in complex analysis.