Note that the groups $\operatorname{Ext}_R^n(A, D)$ are also the cohomology groups of the cochain complex obtained from (7) by replacing the term $\operatorname{Hom}_R(A, D)$ with zero (which does not effect the cochain property), i.e., they are the cohomology groups of the cochain complex $0 \to \operatorname{Hom}_R(P_0, D) \to \cdots$.

We shall show below that these cohomology groups do not depend on the choice of projective resolution of A. Before doing so we identify the 0^{th} cohomology group and give some examples.

Proposition 3. For any R-module A we have $\operatorname{Ext}_R^0(A, D) \cong \operatorname{Hom}_R(A, D)$.

Proof: Since the sequence $P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} A \to 0$ is exact, it follows that the corresponding sequence $0 \to \operatorname{Hom}_R(A, D) \xrightarrow{\epsilon} \operatorname{Hom}_R(P_0, D) \xrightarrow{d_1} \operatorname{Hom}_R(P_1, D)$ is also exact by Theorem 33 in Section 10.5 (noting the first comment in the proof). Hence $\operatorname{Ext}_R^0(A, D) = \ker d_1 = \operatorname{image} \epsilon \cong \operatorname{Hom}_R(A, D)$, as claimed.

Examples

(1) Let $R = \mathbb{Z}$ and let $A = \mathbb{Z}/m\mathbb{Z}$ for some $m \geq 2$. By the proposition we have $\operatorname{Ext}_{\mathbb{Z}}^0(\mathbb{Z}/m\mathbb{Z}, D) \cong \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, D)$, and it follows that $\operatorname{Ext}_{\mathbb{Z}}^0(\mathbb{Z}/m\mathbb{Z}, D) \cong {}_mD$, where ${}_mD = \{d \in D \mid md = 0\}$ are the elements of D that have order dividing m. For the higher cohomology groups, we use the simple projective resolution

$$0 \longrightarrow \mathbb{Z} \xrightarrow{m} \mathbb{Z} \longrightarrow \mathbb{Z}/m\mathbb{Z} \longrightarrow 0$$

for A given by multiplication by m on $\mathbb{Z}$. Taking homomorphisms into a fixed $\mathbb{Z}$-module D gives the cochain complex

$$0 \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, D) \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, D) \xrightarrow{m} \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, D) \longrightarrow 0 \longrightarrow \cdots.$$

We have $D \cong \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, D)$ (cf. Example 4 following Corollary 32 in Section 10.5) and under this isomorphism we have $\operatorname{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/m\mathbb{Z}, D) \cong D/mD$ for any abelian group D. It follows immediately from the definition and the cochain complex above that $\operatorname{Ext}_{\mathbb{Z}}^n(\mathbb{Z}/m\mathbb{Z}, D) = 0$ for all $n \geq 2$ and any abelian group D, which we summarize as

$$\begin{aligned} &\operatorname{Ext}_{\mathbb{Z}}^0(\mathbb{Z}/m\mathbb{Z}, D) \cong {}_mD \\ &\operatorname{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/m\mathbb{Z}, D) \cong D/mD \\ &\operatorname{Ext}_{\mathbb{Z}}^n(\mathbb{Z}/m\mathbb{Z}, D) = 0, \quad \text{for all } n \geq 2. \end{aligned}$$

(2) The same abelian groups may be modules over several different rings R and the Ext_R cohomology groups depend on R. For example, suppose $R = \mathbb{Z}/m\mathbb{Z}$ for some integer $m \geq 1$. An R-module D is the same as an abelian group D with exponent dividing m, i.e., $mD = 0$. In particular, for any divisor d of m, the group $\mathbb{Z}/d\mathbb{Z}$ is an R-module, and

$$\cdots \xrightarrow{m/d} \mathbb{Z}/m\mathbb{Z} \xrightarrow{d} \mathbb{Z}/m\mathbb{Z} \xrightarrow{m/d} \mathbb{Z}/m\mathbb{Z} \xrightarrow{d} \mathbb{Z}/m\mathbb{Z} \longrightarrow \mathbb{Z}/d\mathbb{Z} \longrightarrow 0$$

is a projective (in fact, free) resolution of $\mathbb{Z}/d\mathbb{Z}$ as a $\mathbb{Z}/m\mathbb{Z}$-module, where the final map is the natural projection mapping $x \bmod m$ to $x \bmod d$. Taking homomorphisms into the $\mathbb{Z}/m\mathbb{Z}$-module D, using the isomorphism $\operatorname{Hom}_{\mathbb{Z}/m\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, D) \cong D$, and removing the first term gives the cochain complex

$$0 \longrightarrow D \xrightarrow{d} D \xrightarrow{m/d} D \xrightarrow{d} D \xrightarrow{m/d} \cdots.$$

Hence

$$\operatorname{Ext}^0_{\mathbb{Z}/m\mathbb{Z}}(\mathbb{Z}/d\mathbb{Z}, D) \cong {}_dD,$$
$$\operatorname{Ext}^n_{\mathbb{Z}/m\mathbb{Z}}(\mathbb{Z}/d\mathbb{Z}, D) \cong {}_{(m/d)}D/dD, \quad n \text{ odd}, n \geq 1,$$
$$\operatorname{Ext}^n_{\mathbb{Z}/m\mathbb{Z}}(\mathbb{Z}/d\mathbb{Z}, D) \cong {}_dD/(m/d)D, \quad n \text{ even}, n \geq 2,$$

where ${}_kD = \{d \in D \mid kd = 0\}$ denotes the set of elements of D killed by k. In particular, $\operatorname{Ext}^n_{\mathbb{Z}/p^2\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$ for all $n \geq 0$, whereas, for example, $\operatorname{Ext}^n_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) = 0$ for all $n \geq 2$.

In order to show that the cohomology groups $\operatorname{Ext}^n_R(A, D)$ are independent of the choice of projective resolution of A we shall need to be able to "compare" resolutions. The next proposition shows that an R-module homomorphism from A to B lifts to a homomorphism from a projective resolution of A to a projective resolution of B — this lifting property is one instance where the projectivity of the modules in the resolution is important.

Proposition 4. Let $f : A \to A'$ be any homomorphism of R-modules and take projective resolutions of A and A', respectively. Then for each $n \geq 0$ there is a lift f_n of f such that the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r, "d_2"] & P_1 \arrow[r, "d_1"] \arrow[d, "f_1"] & P_0 \arrow[r, "\epsilon"] \arrow[d, "f_0"] & A \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r, "d_2'"] & P_1' \arrow[r, "d_1'"] & P_0' \arrow[r, "\epsilon'"] & A' \arrow[r] & 0
\end{tikzcd} \tag{17.8}$$

where the rows are the projective resolutions of A and A', respectively.

Proof: Given the two rows and map f in (8), then since P_0 is projective we may lift the map $f\epsilon : P_0 \to A'$ to a map $f_0 : P_0 \to P_0'$ in such a way that $\epsilon' f_0 = f\epsilon$ (Proposition 30(2) in Section 10.5). This gives the first lift of f. Proceeding inductively in this fashion, assume f_n has been defined to make the diagram commutative to that point. Thus image $f_n d_{n+1} \subseteq \ker d_n'$. The projectivity of P_{n+1} implies that we may lift the map $f_n d_{n+1} : P_{n+1} \to P_n'$ to a map $f_{n+1} : P_{n+1} \to P_{n+1}'$ to make the diagram commute at the next stage. This completes the proof.

The commutative diagram in Proposition 4 implies that the induced diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_R(A, D) \arrow[r] & \operatorname{Hom}_R(P_0, D) \arrow[r] & \operatorname{Hom}_R(P_1, D) \arrow[r] & \cdots \\
0 \arrow[r] & \operatorname{Hom}_R(A, D) \arrow[r] \arrow[u, "f"] & \operatorname{Hom}_R(P_0', D) \arrow[r] \arrow[u, "f_0"] & \operatorname{Hom}_R(P_1', D) \arrow[r] \arrow[u, "f_1"] & \cdots
\end{tikzcd} \tag{17.9}$$

is also commutative. The two rows of this diagram are cochain complexes, and this commutative diagram depicts a homomorphism of these cochain complexes. By Proposition 1 we have an induced map on their cohomology groups:

Proposition 5. Let $f : A \to A'$ be a homomorphism of R-modules and take projective resolutions of A and A' as in Proposition 4. Then for every n there is an induced group homomorphism $\varphi_n : \mathrm{Ext}^n_R(A', D) \to \mathrm{Ext}^n_R(A, D)$ on the cohomology groups obtained via these resolutions, and the maps φ_n depend only on f, not on the choice of lifts f_n in Proposition 4.

Proof: The existence of the map on the cohomology groups Ext^n_R follows from Proposition 1 applied to the homomorphism of cochain complexes (9). The more difficult part is showing these maps do not depend on the choice of lifts f_n in Proposition 4. This is easily seen to be equivalent to showing that if f is the zero map, then the induced maps on cohomology groups are also all zero. Assume then that $f = 0$. By the projectivity of the modules P_i one may inductively define R-module homomorphisms $s_n : P_n \to P'_{n+1}$ with the property that for all n,

$$f_n = d'_{n+1}s_n + s_{n-1}d_n \tag{17.10}$$

so the maps s_n give reverse downward diagonal arrows across the squares in (8). (The collection of maps $\{s_n\}$ is called a *chain homotopy* between the chain homomorphism given by the f_n and the zero chain homomorphism, cf. Exercise 4.) Taking homomorphisms into D gives diagram (9) with additional upward diagonal arrows from the homomorphisms induced by the s_n, and these induced homomorphisms satisfy the relations in (10) (i.e., they form a homotopy between cochain complex homomorphisms). It is now an easy exercise using the diagonal maps added to (9) to see that any element in $\mathrm{Hom}_R(P'_n, D)$ representing a coset in $\mathrm{Ext}^n_R(A', D)$ maps to the zero coset in $\mathrm{Ext}^n_R(A, D)$ (cf. Exercise 4). This completes the argument.

One may also check that the homomorphism $\varphi_0 : \mathrm{Ext}^0_R(A', D) \to \mathrm{Ext}^0_R(A, D)$ in Proposition 5 is the same as the map $f : \mathrm{Hom}_R(A', D) \to \mathrm{Hom}_R(A, D)$ defined in Section 10.5 once the corresponding groups have been identified via the isomorphism in Proposition 3.

Theorem 6. The groups $\mathrm{Ext}^n_R(A, D)$ depend only on A and D, i.e., they are independent of the choice of projective resolution of A.

Proof: In the notation of Proposition 4 let $A' = A$, let $f : A \to A'$ be the identity map and let the two rows of (8) be two projective resolutions of A. For any choice of lifts of the identity map, the resulting homomorphisms on cohomology groups $\varphi_n : \mathrm{Ext}^n_R(A', D) \to \mathrm{Ext}^n_R(A, D)$ are seen to be isomorphisms as follows. Add a third row to the diagram (8) by copying the projective resolution in the top row below the second row. Let g be the identity map from A' to A and lift g to maps $g_n : P'_n \to P_n$ by Proposition 4. Let $\psi_n : \mathrm{Ext}^n_R(A, D) \to \mathrm{Ext}^n_R(A', D)$ be the resulting map on cohomology groups. The maps $g_n \circ f_n : P_n \to P_n$ are now a lift of the identity map $g \circ f$, and they are seen to induce the homomorphisms $\varphi_n \circ \psi_n$ on the cohomology groups. However, since the first and third rows are identical, taking the identity map from P_n to itself for all n is a particular lift of $g \circ f$, and this choice clearly induces the identity map on cohomology groups. The last assertion of Proposition 5 then implies that $\varphi_n \circ \psi_n$ is also the identity on $\mathrm{Ext}^n_R(A, D)$. By a symmetric argument $\psi_n \circ \varphi_n$ is the identity on $\operatorname{Ext}^n_R(A', D)$. This shows the maps φ_n and ψ_n are isomorphisms, as needed to complete the proof.

For a fixed R-module D and fixed integer $n \geq 0$, Proposition 5 and Theorem 6 show that $\operatorname{Ext}^n_R(\_\_, D)$ defines a (contravariant) functor from the category of R-modules to the category of abelian groups.

The next result shows that projective resolutions for a submodule and corresponding quotient module of an R-module M can be fit together to give a projective resolution of M.

Proposition 7. *(Simultaneous Resolution)* Let $0 \to L \to M \to N \to 0$ be a short exact sequence of R-modules, let $L = A$ have a projective resolution as in (6) above, and let N have a similar projective resolution where the projective modules are denoted by $\overline{P}_n$. Then there is a resolution of M by the projective modules $P_n \oplus \overline{P}_n$ such that the following diagram commutes:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & P_1 \arrow[r] \arrow[d] & P_1 \oplus \overline{P}_1 \arrow[r] \arrow[d] & \overline{P}_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_0 \arrow[r] \arrow[d] & P_0 \oplus \overline{P}_0 \arrow[r] \arrow[d] & \overline{P}_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{17.11}$$

Moreover, the rows and columns of this diagram are exact and the rows are split.

Proof: The left and right nonzero columns of (11) are exact by hypothesis. The modules in the middle column are projective (cf. Exercise 3, Section 10.5) and the row maps are the obvious ones to make each row a split exact sequence. It remains then to define the vertical maps in the middle column in such a way as to make the diagram commute. This is accomplished in a straightforward manner, working inductively from the bottom upward — the first step in this process is outlined in Exercise 5.

Theorem 2 and Proposition 7 now yield the long exact sequence for Ext_R that extends the exact sequence (2).