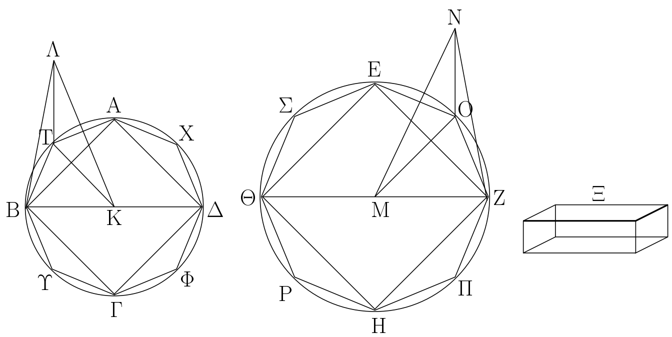
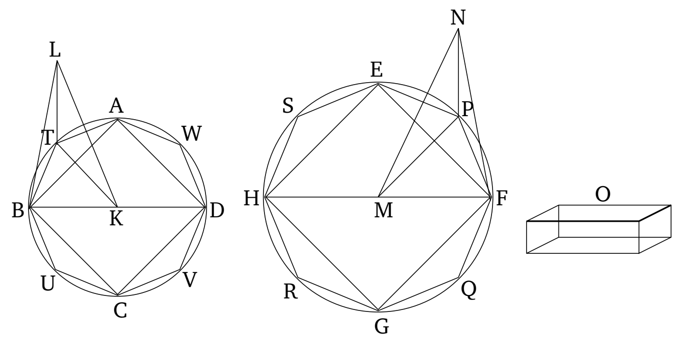
ὅτι ὁ κῶνος, οὗ βάσις μέν [ἐστιν] ὁ ΑΒΓΔ κύκλος, κορυφὴ δὲ τὸ Λ σημεῖον, πρὸς τὸν κῶνον, οὗ βάσις μέν [ἐστιν] ὁ ΕΖΗΘ κύκλος, κορυφὴ δὲ τὸ Ν σημεῖον, τριπλασίονα λόγον ἔχει ἤπερ ἡ ΒΔ πρὸς τὴν ΖΘ.




Εἰ γὰρ μὴ ἔχει ὁ ΑΒΓΔΛ κῶνος πρὸς τὸν ΕΖΗΘΝ κῶνον πριπλασίονα λόγον ἤπερ ἡ ΒΔ πρὸς τὴν ΖΘ, ἕξει ὁ ΑΒΓΔΛ κῶνος ἢ πρὸς ἔλασσόν τι τοῦ ΕΖΗΘΝ κώνου στερεὸν τριπλασίονα λόγον ἢ πρὸς μεῖζον. ἐχέτω πρότερον πρὸς ἔλασσον τὸ Ξ, καὶ ἐγγεγράφθω εἰς τὸν ΕΖΗΘ κύκλον τετράγωνον τὸ ΕΖΗΘ· τὸ ἄρα ΕΖΗΘ τετράγωνον μεῖζόν ἐστιν ἢ τὸ ἥμισυ τοῦ ΕΖΗΘ κύκλου. καὶ ἀνεστάτω ἐπὶ τοῦ ΕΖΗΘ τετραγώνου πυραμὶς τὴν αὐτὴν κορυφὴν ἔχουσα τῷ κώνῳ· ἡ ἄρα ἀνασταθεῖσα πυραμὶς μείζων ἐστὶν ἢ τὸ ἥμισυ μέρος τοῦ κώνου. τετμήσθωσαν δὴ αἱ ΕΖ, ΖΗ, ΗΘ, ΘΕ περιφέρειαι δίχα κατὰ τὰ Ο, Π, Ρ, Σ σημεῖα, καὶ ἐπεζεύχθωσαν αἱ ΕΟ, ΟΖ, ΖΠ, ΠΗ, ΗΡ, ΡΘ, ΘΣ, ΣΕ. καὶ ἕκαστον ἄρα τῶν ΕΟΖ, ΖΠΗ, ΗΡΘ, ΘΣΕ τριγώνων μεῖζόν ἐστιν ἢ τὸ ἥμισυ μέρος τοῦ καθ᾽ ἑαυτὸ τμήματος τοῦ ΕΖΗΘ κύκλου. καὶ ἀνεστάτω ἐφ᾽ ἑκάστου τῶν ΕΟΖ, ΖΠΗ, ΗΡΘ, ΘΣΕ τριγώνων πυραμὶς τὴν αὐτὴν κορυφὴν ἔχουσα τῷ κώνῳ· καὶ ἑκάστη ἄρα τῶν ἀνασταθεισῶν πυραμίδων μείζων ἐστὶν ἢ τὸ ἥμισυ μέρος τοῦ καθ᾽ ἑαυτὴν τμήματος τοῦ κώνου. τέμνοντες δὴ τὰς ὑπολειπομένας περιφερείας δίχα καὶ ἐπιζευγνύντες εὐθείας καὶ ἀνιστάντες ἐφ᾽ ἑκάστου τῶν τριγώνων πυραμίδας τὴν αὐτὴν κορυφὴν ἐχούσας τῷ κώνῳ καὶ τοῦτο ἀεὶ ποιοῦντες καταλείψομέν τινα ἀποτμήματα τοῦ κώνου, ἃ ἔσται ἐλάσσονα τῆς ὑπεροχῆς, ᾗ ὑπερέχει ὁ ΕΖΗΘΝ κῶνος τοῦ Ξ στερεοῦ. λελείφθω, καὶ ἔστω τὰ ἐπὶ τῶν ΕΟ, ΟΖ, ΖΠ, ΠΗ, ΗΡ, ΡΘ, ΘΣ, ΣΕ· λοιπὴ ἄρα ἡ πυραμίς, ἧς βάσις μέν ἐστι τὸ ΕΟΖΠΗΡΘΣ πολύγωνον, κορυφὴ δὲ τὸ Ν σημεῖον, μείζων ἐστὶ τοῦ Ξ στερεοῦ. ἐγγεγράφθω καὶ εἰς τὸν ΑΒΓΔ κύκλον τῷ ΕΟΖΠΗΡΘΣ πολυγώνῳ ὅμοιόν τε καὶ ὁμοίως κείμενον πολύγωνον τὸ ΑΤΒΥΓΦΔΧ, καὶ ἀνεστάτω ἐπὶ τοῦ ΑΤΒΥΓΦΔΧ πολυγώνου πυραμὶς τὴν αὐτὴν κορυφὴν ἔχουσα τῷ κώνῳ, καὶ τῶν μὲν περιεχόντων τὴν πυραμίδα, ἧς βάσις μέν ἐστι τὸ ΑΤΒΥΓΦΔΧ πολύγωνον, κορυφὴ δὲ τὸ Λ σημεῖον, ἓν τρίγωνον ἔστω τὸ ΛΒΤ, τῶν δὲ περειχόντων τὴν πυραμίδα, ἧς βάσις μέν ἐστι τὸ ΕΟΖΠΗΡΘΣ πολύγωνον,

and cylinders (are) $KL$ and $MN$ (respectively). I say that the cone whose base [is] circle $ABCD$, and apex the point $L$, has to the cone whose base [is] circle $EFGH$, and apex the point $N$, the cubed ratio that $BD$ (has) to $FH$.




For if cone $ABCDL$ does not have to cone $EFGHN$ the cubed ratio that $BD$ (has) to $FH$ then cone $ABCDL$ will have the cubed ratio to some solid either less than, or greater than, cone $EFGHN$. Let it, first of all, have (such a ratio) to (some) lesser (solid), $O$. And let the square $EFGH$ have been inscribed in circle $EFGH$ [Prop. 4.6]. Thus, square $EFGH$ is greater than half of circle $EFGH$ [Prop. 12.2]. And let a pyramid having the same apex as the cone have been set up on square $EFGH$. Thus, the pyramid set up is greater than the half part of the cone [Prop. 12.10]. So, let the circumferences $EF$, $FG$, $GH$, and $HE$ have been cut in half at points $P$, $Q$, $R$, and $S$ (respectively). And let $EP$, $PF$, $FQ$, $QG$, $GR$, $RH$, $HS$, and $SE$ have been joined. And, thus, each of the triangles $EPF$, $FQG$, $GRH$, and $HSE$ is greater than the half part of the segment of circle $EFGH$ about it [Prop. 12.2]. And let a pyramid having the same apex as the cone have been set up on each of the triangles $EPF$, $FQG$, $GRH$, and $HSE$. And thus each of the pyramids set up is greater than the half part of the segment of the cone about it [Prop. 12.10]. So, (if) the the remaining circumferences are cut in half, and straight-lines are joined, and pyramids having the same apex as the cone are set up on each of the triangles, and this is done continually, then we will (eventually) leave some segments of the cone whose (sum) is less than the excess by which cone $EFGHN$ exceeds solid $O$ [Prop. 10.1]. Let them have been left, and let them be the (segments) on $EP$, $PF$, $FQ$, $QG$, $GR$, $RH$, $HS$, and $SE$. Thus, the remaining pyramid whose base is polygon $EPFQGRHS$, and apex the point $N$, is greater than solid $O$. And let the polygon $ATBUCVDW$, similar, and similarly laid out, to polygon $EPFQGRHS$, have been inscribed in circle $ABCD$ [Prop. 6.18]. And let a pyramid having the same apex as the cone have been set up on polygon $ATBUCVDW$.

κορυφὴ δὲ τὸ Ν σημεῖον, ἓν τρίγωνον ἔστω τὸ ΝΖΟ, καὶ ἐπεζεύχθωσαν αἱ ΚΤ, ΜΟ. καὶ ἐπεὶ ὅμοιός ἐστιν ὁ ΑΒΓΔΛ κῶνος τῷ ΕΖΗΘΝ κώνῳ, ἔστιν ἄρα ὡς ἡ ΒΔ πρὸς τὴν ΖΘ, οὕτως ὁ ΚΛ ἄξων πρὸς τὸν ΜΝ ἄξονα. ὡς δὲ ἡ ΒΔ πρὸς τὴν ΖΘ, οὕτως ἡ ΒΚ πρὸς τὴν ΖΜ· καὶ ὡς ἄρα ἡ ΒΚ πρὸς τὴν ΖΜ, οὕτως ἡ ΚΛ πρὸς τὴν ΜΝ. καὶ ἐναλλὰξ ὡς ἡ ΒΚ πρὸς τὴν ΚΛ, οὕτως ἡ ΖΜ πρὸς τὴν ΜΝ. καὶ περὶ ἴσας γωνίας τὰς ὑπὸ ΒΚΛ, ΖΜΝ αἱ πλευραὶ ἀνάλογόν εἰσιν· ὅμοιον ἄρα ἐστὶ τὸ ΒΚΛ τρίγωνον τῷ ΖΜΝ τριγώνῳ. πάλιν, ἐπεί ἐστιν ὡς ἡ ΒΚ πρὸς τὴν ΚΤ, οὕτως ἡ ΖΜ πρὸς τὴν ΜΟ, καὶ περὶ ἴσας γωνίας τὰς ὑπὸ ΒΚΤ, ΖΜΟ, ἐπειδήπερ, ὃ μέρος ἐστὶν ἡ ὑπὸ ΒΚΤ γωνία τῶν πρὸς τῷ Κ κέντρῳ τεσσάρων ὀρθῶν, τὸ αὐτὸ μέρος ἐστὶ καὶ ἡ ὑπὸ ΖΜΟ γωνία τῶν πρὸς τῷ Μ κέντρῳ τεσσάρων ὀρθῶν· ἐπεὶ οὖν περὶ ἴσας γωνίας αἱ πλευραὶ ἀνάλογόν εἰσιν, ὅμοιον ἄρα ἐστι τὸ ΒΚΤ τρίγωνον τῷ ΖΜΟ τριγώνῳ. πάλιν, ἐπεὶ ἐδείχθη ὡς ἡ ΒΚ πρὸς τὴν ΚΛ, οὕτως ἡ ΖΜ πρὸς τὴν ΜΝ, ἴση δὲ ἡ μὲν ΒΚ τῇ ΚΤ, ἡ δὲ ΖΜ τῇ ΟΜ, ἔστιν ἄρα ὡς ἡ ΤΚ πρὸς τὴν ΚΛ, οὕτως ἡ ΟΜ πρὸς τὴν ΜΝ. καὶ περὶ ἴσας γωνίας τὰς ὑπὸ ΤΚΛ, ΟΜΝ· ὀρθαὶ γάρ· αἱ πλευραὶ ἀνάλογόν εἰσιν· ὅμοιον ἄρα ἐστὶ τὸ ΛΚΤ τρίγωνον τῷ ΝΜΟ τριγώνῳ. καὶ ἐπεὶ διὰ τὴν ὁμοιότητα τῶν ΛΚΒ, ΝΜΖ τριγώνων ἐστὶν ὡς ἡ ΛΒ πρὸς τὴν ΒΚ, οὕτως ἡ ΝΖ πρὸς τὴν ΖΜ, διὰ δὲ τὴν ὁμοιότητα τῶν ΒΚΤ, ΖΜΟ τριγώνων ἐστὶν ὡς ἡ ΚΒ πρὸς τὴν ΒΤ, οὕτως ἡ ΜΖ πρὸς τὴν ΖΟ, δι᾿ ἴσου ἄρα ὡς ἡ ΛΒ πρὸς τὴν ΒΤ, οὕτως ἡ ΝΖ πρὸς τὴν ΖΟ. πάλιν, ἐπεὶ διὰ τὴν ομοιότητα τῶν ΛΤΚ, ΝΟΜ τριγώνων ἐστὶν ὡς ἡ ΛΤ πρὸς τὴν ΤΚ, οὕτως ἡ ΝΟ πρὸς τὴν ΟΜ, διὰ δὲ τὴν ὁμοιότητα τῶν ΤΚΒ, ΟΜΖ τριγώνων ἐστὶν ὡς ἡ ΚΤ πρὸς τὴν ΤΒ, οὕτως ἡ ΜΟ πρὸς τὴν ΟΖ, δι᾿ ἴσου ἄρα ὡς ἡ ΛΤ πρὸς τὴν ΤΒ, οὕτως ἡ ΝΟ πρὸς τὴν ΟΖ. ἐδείχθη δὲ καὶ ὡς ἡ ΤΒ πρὸς τὴν ΒΛ, οὕτως ἡ ΟΖ πρὸς τὴν ΖΝ. δι᾿ ἴσου ἄρα ὡς ἡ ΤΛ πρὸς τὴν ΛΒ, οὕτως ἡ ΟΝ πρὸς τὴν ΝΖ. τῶν ΛΤΒ, ΝΟΖ ἄρα τριγώνων ἀνάλογόν εἰσιν αἱ πλευραί· ἰσογώνια ἄρα ἐστὶ τὰ ΛΤΒ, ΝΟΖ τρίγωνα· ὥστε καὶ ὅμοια. καὶ πυραμὶς ἄρα, ἧς βάσις μὲν τὸ ΒΚΤ τρίγωνον, κορυφὴ δὲ τὸ Λ σημεῖον, ὁμοία ἐστὶ πυραμίδι, ἧς βάσις μὲν τὸ ΖΜΟ τρίγωνον, κορυφὴ δὲ τὸ Ν σημεῖον· ὑπὸ γὰρ ὁμοίων ἐπιπέδων περιέχονται ἴσων τὸ πλῆθος. αἱ δὲ ὅμοιαι πυραμίδες καὶ τριγώνους ἔχουσαι βάσεις ἐν τριπλασίονι λόγῳ εἰσὶ τῶν ὁμολόγων πλευρῶν. ἡ ἄρα ΒΚΤΛ πυραμὶς πρὸς τὴν ΖΜΟΝ πυραμίδα τριπλασίονα λόγον ἔχει ἤπερ ἡ ΒΚ πρὸς τὴν ΖΜ. ὁμοίως δὴ ἐπιζευγνύντες ἀπὸ τῶν Α, Χ, Δ, Φ, Γ, Υ ἐπὶ τὸ Κ εὐθείας καὶ ἀπὸ τῶν Ε, Σ, Θ, Ρ, Η, Π ἐπὶ τὸ Μ καὶ ἀνιστάντες ἐφ᾿ ἑκάστου τῶν τριγώνων πυραμίδας τὴν αὐτὴν κορυφὴν ἐχούσας τοῖς κώνοις δείξομεν, ὅτι καὶ ἑκάστη τῶν ὁμοταγῶν πυραμίδων πρὸς ἑκάστην ὁμοταγῆ πυραμίδα τριπλασίονα λόγον ἕξει ἤπερ ἡ ΒΚ ὁμόλογος πλευρὰ πρὸς τὴν ΖΜ ὁμόλογον πλευράν, τουτέστιν ἤπερ ἡ ΒΔ πρὸς τὴν ΖΘ. καὶ ὡς ἓν τῶν ἡγουμένων πρὸς ἓν τῶν ἑπομένων, οὕτως ἅπαντα τὰ ἡγούμενα πρὸς ἅπαντα τὰ ἑπόμενα· ἔστιν ἄρα

And let $LBT$ be one of the triangles containing the pyramid whose base is polygon $ATBUCVDW$, and apex the point $L$. And let $NFP$ be one of the triangles containing the pyramid whose base is triangle $EPFQGRHS$, and apex the point $N$. And let $KT$ and $MP$ have been joined. And since cone $ABCDL$ is similar to cone $EFGHN$, thus as $BD$ is to $FH$, so axis $KL$ (is) to axis $MN$ [Def. 11.24]. And as $BD$ (is) to $FH$, so $BK$ (is) to $FM$. And, thus, as $BK$ (is) to $FM$, so $KL$ (is) to $MN$. And, alternately, as $BK$ (is) to $KL$, so $FM$ (is) to $MN$ [Prop. 5.16]. And the sides around the equal angles $BKL$ and $FMN$ are proportional. Thus, triangle $BKL$ is similar to triangle $FMN$ [Prop. 6.6]. Again, since as $BK$ (is) to $KT$, so $FM$ (is) to $MP$, and (they are) about the equal angles $BKT$ and $FMP$, inasmuch as whatever part angle $BKT$ is of the four right-angles at the center $K$, angle $FMP$ is also the same part of the four right-angles at the center $M$. Therefore, since the sides about equal angles are proportional, triangle $BKT$ is thus similar to traingle $FMP$ [Prop. 6.6]. Again, since it was shown that as $BK$ (is) to $KL$, so $FM$ (is) to $MN$, and $BK$ (is) equal to $KT$, and $FM$ to $PM$, thus as $TK$ (is) to $KL$, so $PM$ (is) to $MN$. And the sides about the equal angles $TKL$ and $PMN$—for (they are both) right-angles—are proportional. Thus, triangle $LKT$ (is) similar to triangle $NMP$ [Prop. 6.6]. And since, on account of the similarity of triangles $LKB$ and $NMF$, as $LB$ (is) to $BK$, so $NF$ (is) to $FM$, and, on account of the similarity of triangles $BKT$ and $FMP$, as $KB$ (is) to $BT$, so $MF$ (is) to $FP$ [Def. 6.1], thus, via equality, as $LB$ (is) to $BT$, so $NF$ (is) to $FP$ [Prop. 5.22]. Again, since, on account of the similarity of triangles $LTK$ and $NPM$, as $LT$ (is) to $TK$, so $NP$ (is) to $PM$, and, on account of the similarity of triangles $TKB$ and $PMF$, as $KT$ (is) to $TB$, so $MP$ (is) to $PF$, thus, via equality, as $LT$ (is) to $TB$, so $NP$ (is) to $PF$ [Prop. 5.22]. And it was shown that as $TB$ (is) to $BL$, so $PF$ (is) to $FN$. Thus, via equality, as $TL$ (is) to $LB$, so $PN$ (is) to $NF$ [Prop. 5.22]. Thus, the sides of triangles $LTB$ and $NPF$ are proportional. Thus, triangles $LTB$ and $NPF$ are equiangular [Prop. 6.5]. And, hence, (they are) similar [Def. 6.1]. And, thus, the pyramid whose base is triangle $BKT$, and apex the point $L$, is similar to the pyramid whose base is triangle $FMP$, and apex the point $N$. For they are contained by equal numbers of similar planes [Def. 11.9]. And similar pyramids which also have triangular bases are in the cubed ratio of corresponding sides [Prop. 12.8]. Thus, pyramid $BKTL$ has to pyramid $FMPN$ the cubed ratio that $BK$ (has) to $FM$. So, similarly, joining straight-lines from (points) $A$, $W$, $D$, $V$, $C$, and $U$ to (center) $K$, and from (points) $E$, $S$, $H$, $R$, $G$, and $Q$ to (center) $M$, and set-καὶ ὡς ἡ ΒΚΤΛ πυραμὶς πρὸς τὴν ΖΜΟΝ πυραμίδα, οὕτως ἡ ὅλη πυραμίς, ἧς βάσις τὸ ΑΤΒΥΓΦΔΧ πολύγωνον, κορυφὴ δὲ τὸ Λ σημεῖον, πρὸς τὴν ὅλην πυραμίδα, ἧς βάσις μὲν τὸ ΕΟΖΠΗΡΘΣ πολύγωνον, κορυφὴ δὲ τὸ Ν σημεῖον· ὥστε καὶ πυραμίς, ἧς βάσις μὲν τὸ ΑΤΒΥΓΦΔΧ, κορυφὴ δὲ τὸ Λ, πρὸς τὴν πυραμίδα, ἧς βάσις [μὲν] τὸ ΕΟΖΠΗΡΘΣ πολύγωνον, κορυφὴ δὲ τὸ Ν σημεῖον, τριπλασίονα λόγον ἔχει ἤπερ ἡ ΒΔ πρὸς τὴν ΖΘ. ὑπόκειται δὲ καὶ ὁ κῶνος, οὗ βάσις [μὲν] ὁ ΑΒΓΔ κύκλος, κορυφὴ δὲ τὸ Λ σημεῖον, πρὸς τὸ Ξ στερεὸν τριπλασίονα λόγον ἔχων ἤπερ ἡ ΒΔ πρὸς τὴν ΖΘ· ἔστιν ἄρα ὡς ὁ κῶνος, οὗ βάσις μέν ἐστιν ὁ ΑΒΓΔ κύκλος, κορυφὴ δὲ τὸ Λ, πρὸς τὸ Ξ στερεόν, οὕτως ἡ πυραμίς, ἧς βάσις μὲν τὸ ΑΤΒΥΓΦΔΧ [πολύγωνον], κορυφὴ δὲ τὸ Λ, πρὸς τὴν πυραμίδα, ἧς βάσις μέν ἐστι τὸ ΕΟΖΠΗΡΘΣ πολύγωνον, κορυφὴ δὲ τὸ Ν· ἐναλλὰξ ἄρα, ὡς ὁ κῶνος, οὗ βάσις μὲν ὁ ΑΒΓΔ κύκλος, κορυφὴ δὲ τὸ Λ, πρὸς τὴν ἐν αὐτῷ πυραμίδα, ἧς βάσις μὲν τὸ ΑΤΒΥΓΦΔΧ πολύγωνον, κορυφὴ δὲ τὸ Λ, οὕτως τὸ Ξ [στερεὸν] πρὸς τὴν πυραμίδα, ἧς βάσις μέν ἐστι τὸ ΕΟΖΠΗΡΘΣ πολύγωνον, κορυφὴ δὲ τὸ Ν. μείζων δὲ ὁ εἰρημένος κῶνος τῆς ἐν αὐτῷ πυραμίδος· ἐμπεριέχει γὰρ αὐτὴν. μεῖζον ἄρα καὶ τὸ Ξ στερεὸν τῆς πυραμίδος, ἧς βάσις μέν ἐστι τὸ ΕΟΖΠΗΡΘΣ πολύγωνον, κορυφὴ δὲ τὸ Ν. ἀλλὰ καὶ ἔλαττον· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ὁ κῶνος, οὗ βάσις ὁ ΑΒΓΔ κύκλος, κορυφὴ δὲ τὸ Λ [σημεῖον], πρὸς ἔλαττόν τι τοῦ κώνου στερεόν, οὗ βάσις μὲν ὁ ΕΖΗΘ κύκλος, κορυφὴ δὲ τὸ Ν σημεῖον, τριπλασίονα λόγον ἔχει ἤπερ ἡ ΒΔ πρὸς τὴν ΖΘ. ὁμοίως δὴ δείξομεν, ὅτι οὐδὲ ὁ ΕΖΗΘΝ κῶνος πρὸς ἔλαττόν τι τοῦ ΑΒΓΔΛ κώνου στερεὸν τριπλασίονα λόγον ἔχει ἤπερ ἡ ΖΘ πρὸς τὴν ΒΔ.

Λέγω δή, ὅτι οὐδὲ ὁ ΑΒΓΔΛ κῶνος πρὸς μεῖζόν τι τοῦ ΕΖΗΘΝ κώνου στερεὸν τριπλασίονα λόγον ἔχει ἤπερ ἡ ΒΔ πρὸς τὴν ΖΘ.

Εἰ γὰρ δυνατόν, ἐχέτω πρὸς μεῖζον τὸ Ξ. ἀνάπαλιν ἄρα τὸ Ξ στερεὸν πρὸς τὸν ΑΒΓΔΛ κῶνον τριπλασίονα λόγον ἔχει ἤπερ ἡ ΖΘ πρὸς τὴν ΒΔ. ὡς δὲ τὸ Ξ στερεὸν πρὸς τὸν ΑΒΓΔΛ κῶνον, οὕτως ὁ ΕΖΗΘΝ κῶνος πρὸς ἔλαττόν τι τοῦ ΑΒΓΔΛ κώνου στερεόν. καὶ ὁ ΕΖΗΘΝ ἄρα κῶνος πρὸς ἔλαττόν τι τοῦ ΑΒΓΔΛ κώνου στερεὸν τριπλασίονα λόγον ἔχει ἤπερ ἡ ΖΘ πρὸς τὴν ΒΔ· ὅπερ ἀδύνατον ἐδείχθη. οὐκ ἄρα ὁ ΑΒΓΔΛ κῶνος πρὸς μεῖζόν τι τοῦ ΕΖΗΘΝ κώνου στερεὸν τριπλασίονα λόγον ἔχει ἤπερ ἡ ΒΔ πρὸς τὴν ΖΘ. ἐδείχθη δέ, ὅτι οὐδὲ πρὸς ἔλαττον. ὁ ΑΒΓΔΛ ἄρα κῶνος πρὸς τὸν ΕΖΗΘΝ κῶνον τριπλασίονα λόγον ἔχει ἤπερ ἡ ΒΔ πρὸς τὴν ΖΘ.

Ὡς δὲ ὁ κῶνος πρὸς τὸν κῶνον, ὁ κύλινδρος πρὸς τὸν κύλινδρον· τριπλάσιος γὰρ ὁ κύλινδρος τοῦ κώνου ὁ ἐπὶ τῆς αὐτῆς βάσεως τῷ κώνῳ καὶ ἰσοϋψὴς αὐτῷ. καὶ ὁ κύλινδρος ἄρα πρὸς τὸν κύλινδρον τριπλασίονα λόγον ἔχει ἤπερ ἡ ΒΔ πρὸς τὴν ΖΘ.

Οἱ ἄρα ὅμοιοι κῶνοι καὶ κύλινδροι πρὸς ἀλλήλους ἐν

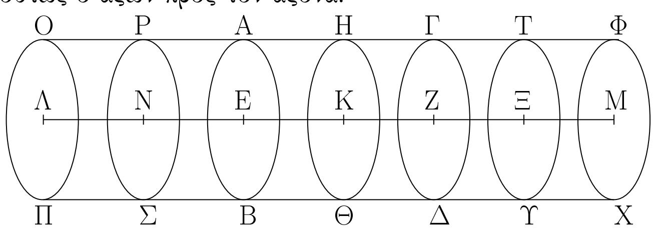
ting up pyramids having the same apexes as the cones on each of the triangles (so formed), we can also show that each of the pyramids (on base $ABCD$ taken) in order will have to each of the pyramids (on base $EFGH$ taken) in order the cubed ratio that the corresponding side $BK$ (has) to the corresponding side $FM$—that is to say, that $BD$ (has) to $FH$. And (for two sets of proportional magnitudes) as one of the leading (magnitudes is) to one of the following, so (the sum of) all of the leading (magnitudes is) to (the sum of) all of the following (magnitudes) [Prop. 5.12]. And, thus, as pyramid $BKTL$ (is) to pyramid $FMPN$, so the whole pyramid whose base is polygon $ATBUCVDW$, and apex the point $L$, (is) to the whole pyramid whose base is polygon $EPFQGRHS$, and apex the point $N$. And, hence, the pyramid whose base is polygon $ATBUCVDW$, and apex the point $L$, has to the pyramid whose base is polygon $EPFQGRHS$, and apex the point $N$, the cubed ratio that $BD$ (has) to $FH$. And it was also assumed that the cone whose base is circle $ABCD$, and apex the point $L$, has to solid $O$ the cubed ratio that $BD$ (has) to $FH$. Thus, as the cone whose base is circle $ABCD$, and apex the point $L$, is to solid $O$, so the pyramid whose base (is) [polygon] $ATBUCVDW$, and apex the point $L$, (is) to the pyramid whose base is polygon $EPFQGRHS$, and apex the point $N$. Thus, alternately, as the cone whose base (is) circle $ABCD$, and apex the point $L$, (is) to the pyramid within it whose base (is) the polygon $ATBUCVDW$, and apex the point $L$, so the [solid] $O$ (is) to the pyramid whose base is polygon $EPFQGRHS$, and apex the point $N$ [Prop. 5.16]. And the aforementioned cone (is) greater than the pyramid within it. For it encompasses it. Thus, solid $O$ (is) also greater than the pyramid whose base is polygon $EPFQGRHS$, and apex the point $N$. But, (it is) also less. The very thing is impossible. Thus, the cone whose base (is) circle $ABCD$, and apex the [point] $L$, does not have to some solid less than the cone whose base (is) circle $EFGH$, and apex the point $N$, the cubed ratio that $BD$ (has) to $EH$. So, similarly, we can show that neither does cone $EFGHN$ have to some solid less than cone $ABCDL$ the cubed ratio that $FH$ (has) to $BD$.

So, I say that neither does cone $ABCDL$ have to some solid greater than cone $EFGHN$ the cubed ratio that $BD$ (has) to $FH$.

For, if possible, let it have (such a ratio) to a greater (solid), $O$. Thus, inversely, solid $O$ has to cone $ABCDL$ the cubed ratio that $FH$ (has) to $BD$ [Prop. 5.7 corr.]. And as solid $O$ (is) to cone $ABCDL$, so cone $EFGHN$ (is) to some solid less than cone $ABCDL$ [12.2 lem.]. Thus, cone $EFGHN$ also has to some solid less than cone $ABCDL$ the cubed ratio that $FH$ (has) to $BD$. The very τριπλασίονι λόγῳ εἰσὶ τῶν ἐν ταῖς βάσεσι διαμέτρων· ὅπερ ἔδει δεῖξαι.

## ιγ΄.

Ἐὰν κύλινδρος ἐπιπέδῳ τμηθῇ παραλλήλῳ ὄντι τοῖς ἀπεναντίον ἐπιπέδοις, ἔσται ὡς ὁ κύλινδρος πρὸς τὸν κύλινδρον, οὕτως ὁ ἄξων πρὸς τὸν ἄξονα.




Κύλινδρος γὰρ ὁ ΑΔ ἐπιπέδῳ τῷ ΗΘ τετμήσθω παραλλήλῳ ὄντι τοῖς ἀπεναντίον ἐπιπέδοις τοῖς ΑΒ, ΓΔ, καὶ συμβαλλέτω τῷ ἄξονι τὸ ΗΘ ἐπίπεδον κατὰ τὸ Κ σημεῖον· λέγω, ὅτι ἐστὶν ὡς ὁ ΒΗ κύλινδρος πρὸς τὸν ΗΔ κύλινδρον, οὕτως ὁ ΕΚ ἄξων πρὸς τὸν ΚΖ ἄξονα.

Ἐκβεβλήσθω γὰρ ὁ ΕΖ ἄξων ἐφ᾽ ἑκάτερα τὰ μέρη ἐπὶ τὰ Λ, Μ σημεῖα, καὶ ἐκκείσθωσαν τῷ ΕΚ ἄξονι ἴσοι ὁσοιδηποτοῦν οἱ ΕΝ, ΝΛ, τῷ δὲ ΖΚ ἴσοι ὁσοιδηποτοῦν οἱ ΖΞ, ΞΜ, καὶ νοείσθω ὁ ἐπὶ τοῦ ΛΜ ἄξονος κύλινδρος ὁ ΟΧ, οὗ βάσεις οἱ ΟΠ, ΦΧ κύκλοι. καὶ ἐκβεβλήσθω διὰ τῶν Ν, Ξ σημείων ἐπίπεδα παράλληλα τοῖς ΑΒ, ΓΔ καὶ ταῖς βάσεσι τοῦ ΟΧ κυλίνδρου καὶ ποιείτωσαν τοὺς ΡΣ, ΤΥ κύκλους περὶ τὰ Ν, Ξ κέντρα. καὶ ἐπεὶ οἱ ΛΝ, ΝΕ, ΕΚ ἄξονες ἴσοι εἰσὶν ἀλλήλοις, οἱ ἄρα ΠΡ, ΡΒ, ΒΗ κύλινδροι πρὸς ἀλλήλους εἰσὶν ὡς αἱ βάσεις. ἴσαι δέ εἰσιν αἱ βάσεις· ἴσοι ἄρα καὶ οἱ ΠΡ, ΡΒ, ΒΗ κύλινδροι ἀλλήλοις. επεὶ οὖν οἱ ΛΝ, ΝΕ, ΕΚ ἄξονες ἴσοι εἰσὶν ἀλλήλοις, εἰσὶ δὲ καὶ οἱ ΠΡ, ΡΒ, ΒΗ κύλινδροι ἴσοι ἀλλήλοις, καί ἐστιν ἴσον τὸ πλῆθος τῷ πλήθει, ὁσαπλασίων ἄρα ὁ ΚΛ ἄξων τοῦ ΕΚ ἄξονος, τοσαυταπλασίων ἔσται καὶ ὁ ΠΗ κύλινδρος τοῦ ΗΒ κυλίνδρου. διὰ τὰ αὐτὰ δὴ καὶ ὁσαπλασίων ἐστὶν ὁ ΜΚ ἄξων τοῦ ΚΖ ἄξονος, τοσαυταπλασίων ἐστὶ καὶ ὁ ΧΗ κύλινδρος τοῦ ΗΔ κυλίνδρου. καὶ εἰ μὲν ἴσος ἐστὶν ὁ ΚΛ ἄξων τῷ ΚΜ ἄξονι, ἴσος ἔσται καὶ ὁ ΠΗ κύλινδρος τῷ ΗΧ κυλίνδρῳ,

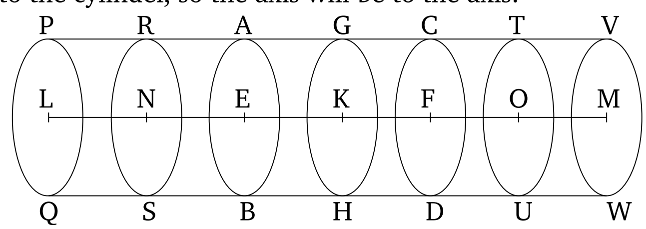
thing was shown (to be) impossible. Thus, cone $ABCDL$ does not have to some solid greater than cone $EFGHN$ the cubed ratio than $BD$ (has) to $FH$. And it was shown that neither (does it have such a ratio) to a lesser (solid). Thus, cone $ABCDL$ has to cone $EFGHN$ the cubed ratio that $BD$ (has) to $FG$.

And as the cone (is) to the cone, so the cylinder (is) to the cylinder. For a cylinder is three times a cone on the same base as the cone, and of the same height as it [Prop. 12.10]. Thus, the cylinder also has to the cylinder the cubed ratio that $BD$ (has) to $FH$.

Thus, similar cones and cylinders are in the cubed ratio of the diameters of their bases. (Which is) the very thing it was required to show.

## Proposition 13

If a cylinder is cut by a plane which is parallel to the opposite planes (of the cylinder) then as the cylinder (is) to the cylinder, so the axis will be to the axis.




For let the cylinder $AD$ have been cut by the plane $GH$ which is parallel to the opposite planes (of the cylinder), $AB$ and $CD$. And let the plane $GH$ have met the axis at point $K$. I say that as cylinder $BG$ is to cylinder $GD$, so axis $EK$ (is) to axis $KF$.

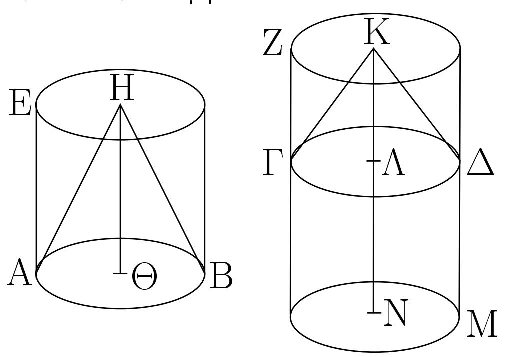
For let axis $EF$ have been produced in each direction to points $L$ and $M$. And let any number whatsoever (of lengths), $EN$ and $NL$, equal to axis $EK$, be set out (on the axis $EL$), and any number whatsoever (of lengths), $FO$ and $OM$, equal to (axis) $FK$, (on the axis $KM$). And let the cylinder $PW$, whose bases (are) the circles $PQ$ and $VW$, have been conceived on axis $LM$. And let planes parallel to $AB$, $CD$, and the bases of cylinder $PW$, have been produced through points $N$ and $O$, and let them have made the circles $RS$ and $TU$ around the centers $N$ and $O$ (respectively). And since axes $LN$, $NE$, and $EK$ are equal to one another, the cylinders $QR$, $RB$, and $BG$ are to one another as their bases [Prop. 12.11]. But the bases are equal. Thus, the cylinders $QR$, $RB$, and $BG$ (are) also equal to one another. Therefore, since the axes $LN$, $NE$, and $EK$ are equal to one another, and the cylinders $QR$, $RB$, and $BG$ are also equal to one another, and the number (of the former) is equal to the number (of the latter), thus as many multiples as axis $KL$

εἰ δὲ μείζων ὁ ἄξων τοῦ ἄξονος, μείζων καὶ ὁ κύλινδρος τοῦ κυλίνδρου, καὶ εἰ ἐλάσσων, ἐλάσσων. τεσσάρων δὴ μεγεθῶν ὄντων, ἀξόνων μὲν τῶν ΕΚ, ΚΖ, κυλίνδρων δὲ τῶν ΒΗ, ΗΔ, εἴληπται ἰσάκις πολλαπλάσια, τοῦ μὲν ΕΚ ἄξονος καὶ τοῦ ΒΗ κυλίνδρου ὅ τε ΛΚ ἄξων καὶ ὁ ΠΗ κύλινδρος, τοῦ δὲ ΚΖ ἄξονες καὶ τοῦ ΗΔ κυλίνδρου ὅ τε ΚΜ ἄξων καὶ ὁ ΗΧ κύλινδρος, καὶ δέδεικται, ὅτι εἰ ὑπερέχει ὁ ΚΛ ἄξων τοῦ ΚΜ ἄξονος, ὑπερέχει καὶ ὁ ΠΗ κύλινδρος τοῦ ΗΧ κυλίνδρου, καὶ εἰ ἴσος, ἴσος, καὶ εἰ ἐλάσσων, ἐλάσσων. ἔστιν ἄρα ὡς ὁ ΕΚ ἄξων πρὸς τὸν ΚΖ ἄξονα, οὕτως ὁ ΒΗ κύλινδρος πρὸς τὸν ΗΔ κύλινδρον· ὅπερ ἔδει δεῖξαι.

is of axis $EK$, so many multiples is cylinder $QG$ also of cylinder $GB$. And so, for the same (reasons), as many multiples as axis $MK$ is of axis $KF$, so many multiples is cylinder $WG$ also of cylinder $GD$. And if axis $KL$ is equal to axis $KM$ then cylinder $QG$ will also be equal to cylinder $GW$, and if the axis (is) greater than the axis then the cylinder (will also be) greater than the cylinder, and if (the axis is) less then (the cylinder will also be) less. So, there are four magnitudes—the axes $EK$ and $KF$, and the cylinders $BG$ and $GD$—and equal multiples have been taken of axis $EK$ and cylinder $BG$—(namely), axis $LK$ and cylinder $QG$—and of axis $KF$ and cylinder $GD$—(namely), axis $KM$ and cylinder $GW$. And it has been shown that if axis $KL$ exceeds axis $KM$ then cylinder $QG$ also exceeds cylinder $GW$, and if (the axes are) equal then (the cylinders are) equal, and if ($KL$ is) less then ($QG$ is) less. Thus, as axis $EK$ is to axis $KF$, so cylinder $BG$ (is) to cylinder $GD$ [Def. 5.5]. (Which is) the very thing it was required to show.

## ιδʹ.

Οἱ ἐπὶ ἴσων βάσεων ὄντες κῶνοι καὶ κύλινδροι πρὸς αλλήλους εἰσὶν ὡς τὰ ὕψη.

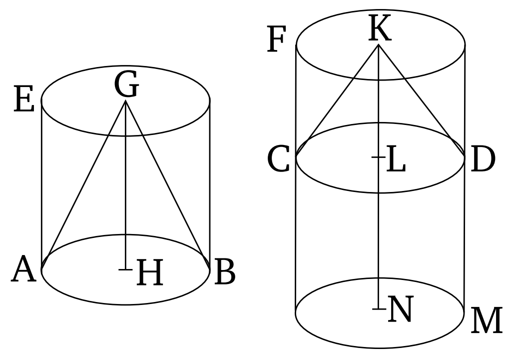



Ἔστωσαν γὰρ ἐπὶ ἴσων βάσεων τῶν ΑΒ, ΓΔ κύκλων κύλινδροι οἱ ΕΒ, ΖΔ· λέγω, ὅτι ἐστὶν ὡς ὁ ΕΒ κύλινδρος πρὸς τὸν ΖΔ κύλινδρον, οὕτως ὁ ΗΘ ἄξων πρὸς τὸν ΚΛ ἄξονα.

Ἐκβεβλήσθω γὰρ ὁ ΚΛ ἄξων ἐπὶ τὸ Ν σημεῖον, καὶ κείσθω τῷ ΗΘ ἄξονι ἴσος ὁ ΛΝ, καὶ περὶ ἄξονα τὸν ΛΝ κύλινδρος νενοήσθω ὁ ΓΜ. ἐπεὶ οὖν οἱ ΕΒ, ΓΜ κύλινδροι ὑπὸ τὸ αὐτὸ ὕψος εἰσίν, πρὸς ἀλλήλους εἰσὶν ὡς αἱ βάσεις. ἴσαι δέ εἰσίν αἱ βάσεις ἀλλήλαις· ἴσοι ἄρα εἰσὶ καὶ οἱ ΕΒ, ΓΜ κύλινδροι. καὶ ἐπεὶ κύλινδρος ὁ ΖΜ ἐπιπέδῳ τέτμηται τῷ ΓΔ παραλλήλῳ ὄντι τοῖς ἀπεναντίον ἐπιπέδοις, ἔστιν ἄρα ὡς ὁ ΓΜ κύλινδρος πρὸς τὸν ΖΔ κύλινδρον, οὕτως ὁ ΛΝ ἄξων πρὸς τὸν ΚΛ ἄξονα. ἴσος δέ ἐστιν ὁ μὲν ΓΜ κύλινδρος τῷ ΕΒ κυλίνδρῳ, ὁ δὲ ΛΝ ἄξων τῷ ΗΘ ἄξονι· ἔστιν ἄρα ὡς ὁ ΕΒ κύλινδρος πρὸς τὸν ΖΔ κύλινδρον, οὕτως ὁ ΗΘ ἄξων πρὸς τὸν ΚΛ ἄξονα. ὡς δὲ ὁ ΕΒ κύλινδρος πρὸς τὸν ΖΔ

## Proposition 14

Cones and cylinders which are on equal bases are to one another as their heights.

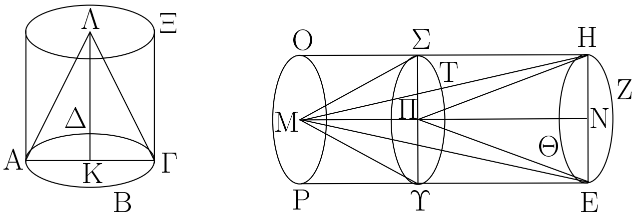



For let $EB$ and $FD$ be cylinders on equal bases, (namely) the circles $AB$ and $CD$ (respectively). I say that as cylinder $EB$ is to cylinder $FD$, so axis $GH$ (is) to axis $KL$.

For let the axis $KL$ have been produced to point $N$. And let $LN$ be made equal to axis $GH$. And let the cylinder $CM$ have been conceived about axis $LN$. Therefore, since cylinders $EB$ and $CM$ have the same height they are to one another as their bases [Prop. 12.11]. And the bases are equal to one another. Thus, cylinders $EB$ and $CM$ are also equal to one another. And since cylinder $FM$ has been cut by the plane $CD$, which is parallel to its opposite planes, thus as cylinder $CM$ is to cylinder $FD$, so axis $LN$ (is) to axis $KL$ [Prop. 12.13]. And cylinder $CM$ is equal to cylinder $EB$, and axis $LN$ to axis $GH$. Thus, as cylinder $EB$ is to cylinder $FD$, so axis $GH$ (is) κύλινδρον, οὕτως ὁ ΑΒΗ κῶνος πρὸς τὸν ΓΔΚ κῶνον. καὶ ὡς ἄρα ὁ ΗΘ ἄξων πρὸς τὸν ΚΛ ἄξονα, οὕτως ὁ ΑΒΗ κῶνος πρὸς τὸν ΓΔΚ κῶνον καὶ ὁ ΕΒ κύλινδρος πρὸς τὸν ΖΔ κύλινδρον· ὅπερ ἔδει δεῖξαι.

## ιεʹ.

Τῶν ἴσων κώνων καὶ κυλίνδρων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν· καὶ ὧν κώνων καὶ κυλίνδρων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν, ἴσοι εἰσὶν ἐκεῖνοι.



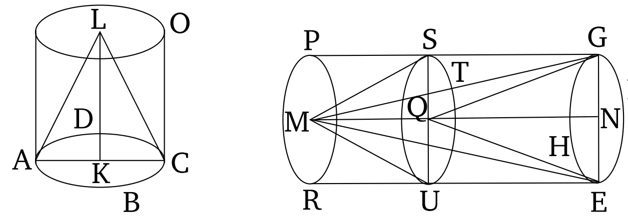

Ἔστωσαν ἴσοι κῶνοι καὶ κύλινδροι, ὧν βάσεις μὲν οἱ ΑΒΓΔ, ΕΖΗΘ κύκλοι, διάμετροι δὲ αὐτῶν αἱ ΑΓ, ΕΗ, ἄξονες δὲ οἱ ΚΛ, ΜΝ, οἵτινες καὶ ὕψη εἰσὶ τῶν κώνων ἢ κυλίνδρων, καὶ συμπεπληρώσθωσαν οἱ ΑΞ, ΕΟ κύλινδροι. λέγω, ὅτι τῶν ΑΞ, ΕΟ κυλίνδρων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν, καί ἐστιν ὡς ἡ ΑΒΓΔ βάσις πρὸς τὴν ΕΖΗΘ βάσιν, οὕτως τὸ ΜΝ ὕψος πρὸς τὸ ΚΛ ὕψος.

Τὸ γὰρ ΛΚ ὕψος τῷ ΜΝ ὕψει ἤτοι ἴσον ἐστὶν ἢ οὔ. ἔστω πρότερον ἴσον. ἔστι δὲ καὶ ὁ ΑΞ κύλινδρος τῷ ΕΟ κυλίνδρῳ ἴσος. οἱ δὲ ὑπὸ τὸ αὐτὸ ὕψος ὄντες κῶνοι καὶ κύλινδροι πρὸς ἀλλήλους εἰσὶν ὡς αἱ βάσεις· ἴση ἄρα καὶ ἡ ΑΒΓΔ βάσις τῇ ΕΖΗΘ βάσει. ὥστε καὶ ἀντιπέπονθεν, ὡς ἡ ΑΒΓΔ βάσις πρὸς τὴν ΕΖΗΘ βάσιν, οὕτως τὸ ΜΝ ὕψος πρὸς τὸ ΚΛ ὕψος. ἀλλὰ δὴ μὴ ἔστω τὸ ΛΚ ὕψος τῷ ΜΝ ἴσον, ἀλλ᾽ ἔστω μεῖζον τὸ ΜΝ, καὶ ἀφῃρήσθω ἀπὸ τοῦ ΜΝ ὕψους τῷ ΚΛ ἴσον τὸ ΠΝ, καὶ διὰ τοῦ Π σημείου τετμήσθω ὁ ΕΟ κύλινδρος ἐπιπέδῳ τῷ ΤΥΣ παραλλήλῳ τοῖς τῶν ΕΖΗΘ, ΡΟ κύκλων ἐπιπέδοις, καὶ ἀπὸ βάσεως μὲν τοῦ ΕΖΗΘ κύκλου, ὕψους δὲ τοῦ ΝΠ κύλινδρος νενοήσθω ὁ ΕΣ. καὶ ἐπεὶ ἴσος ἐστὶν ὁ ΑΞ κύλινδρος τῷ ΕΟ κυλίνδρῳ, ἔστιν ἄρα ὡς ὁ ΑΞ κύλινδρος πρὸς τὸν ΕΣ κυλίνδρον, οὕτως ὁ ΕΟ κύλινδρος πρὸς τὸν ΕΣ κύλινδρον. ἀλλ᾽ ὡς μὲν ὁ ΑΞ κύλινδρος πρὸς τὸν ΕΣ κύλινδρον, οὕτως ἡ ΑΒΓΔ βάσις πρὸς τὴν ΕΖΗΘ· ὑπὸ γὰρ τὸ αὐτὸ ὕψος εἰσὶν οἱ ΑΞ, ΕΣ κύλινδροι· ὡς δὲ ὁ ΕΟ κύλινδρος πρὸς τὸν ΕΣ, οὕτως τὸ ΜΝ ὕψος πρὸς τὸ ΠΝ ὕψος· ὁ γὰρ ΕΟ κύλινδρος ἐπιπέδῳ τέτμηται παραλλήλῳ ὄντι τοῖς ἀπεναντίον ἐπιπέδοις. ἔστιν ἄρα καὶ ὡς ἡ ΑΒΓΔ βάσις πρὸς τὴν ΕΖΗΘ βάσιν, οὕτως τὸ ΜΝ ὕψος πρὸς τὸ ΠΝ ὕψος. ἴσον δὲ τὸ ΠΝ ὕψος τῷ ΚΛ ὕψει· ἔστιν ἄρα ὡς ἡ ΑΒΓΔ βάσις πρὸς τὴν ΕΖΗΘ βάσιν, οὕτως τὸ ΜΝ ὕψος πρὸς τὸ ΚΛ ὕψος. τῶν ἄρα ΑΞ, ΕΟ κυλίνδρων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν.

to axis $KL$. And as cylinder $EB$ (is) to cylinder $FD$, so cone $ABG$ (is) to cone $CDK$ [Prop. 12.10]. Thus, also, as axis $GH$ (is) to axis $KL$, so cone $ABG$ (is) to cone $CDK$, and cylinder $EB$ to cylinder $FD$. (Which is) the very thing it was required to show.

## Proposition 15

The bases of equal cones and cylinders are reciprocally proportional to their heights. And, those cones and cylinders whose bases (are) reciprocally proportional to their heights are equal.




Let there be equal cones and cylinders whose bases are the circles $ABCD$ and $EFGH$, and the diameters of (the bases) $AC$ and $EG$, and (whose) axes (are) $KL$ and $MN$, which are also the heights of the cones and cylinders (respectively). And let the cylinders $AO$ and $EP$ have been completed. I say that the bases of cylinders $AO$ and $EP$ are reciprocally proportional to their heights, and (so) as base $ABCD$ is to base $EFGH$, so height $MN$ (is) to height $KL$.

For height $LK$ is either equal to height $MN$, or not. Let it, first of all, be equal. And cylinder $AO$ is also equal to cylinder $EP$. And cones and cylinders having the same height are to one another as their bases [Prop. 12.11]. Thus, base $ABCD$ (is) also equal to base $EFGH$. And, hence, reciprocally, as base $ABCD$ (is) to base $EFGH$, so height $MN$ (is) to height $KL$. And so, let height $LK$ not be equal to $MN$, but let $MN$ be greater. And let $QN$, equal to $KL$, have been cut off from height $MN$. And let the cylinder $EP$ have been cut, through point $Q$, by the plane $TUS$ (which is) parallel to the planes of the circles $EFGH$ and $RP$. And let cylinder $ES$ have been conceived, with base the circle $EFGH$, and height $NQ$. And since cylinder $AO$ is equal to cylinder $EP$, thus, as cylinder $AO$ (is) to cylinder $ES$, so cylinder $EP$ (is) to cylinder $ES$ [Prop. 5.7]. But, as cylinder $AO$ (is) to cylinder $ES$, so base $ABCD$ (is) to base $EFGH$. For cylinders $AO$ and $ES$ (have) the same height [Prop. 12.11]. And as cylinder $EP$ (is) to (cylinder) $ES$, so height $MN$ (is) to height $QN$. For cylinder $EP$ has been cut by a plane which is parallel to its opposite planes [Prop. 12.13]. And, thus, as base $ABCD$ is to base $EFGH$, so height $MN$ (is) to height $QN$ [Prop. 5.11]. And height $QN$

Ἀλλὰ δὴ τῶν ΑΞ, ΕΟ κυλίνδρων ἀντιπεπονθέτωσαν αἱ βάσεις τοῖς ὕψεσιν, καὶ ἔστω ὡς ἡ ΑΒΓΔ βάσις πρὸς τὴν ΕΖΗΘ βάσιν, οὕτως τὸ ΜΝ ὕψος πρὸς τὸ ΚΛ ὕψος· λέγω, ὅτι ἴσος ἐστὶν ὁ ΑΞ κύλινδρος τῷ ΕΟ κυλίνδρῳ.

Τῶν γὰρ αὐτῶν κατασκευασθέντων ἐπεί ἐστιν ὡς ἡ ΑΒΓΔ βάσις πρὸς τὴν ΕΖΗΘ βάσιν, οὕτως τὸ ΜΝ ὕψος πρὸς τὸ ΚΛ ὕψος, ἴσον δὲ τὸ ΚΛ ὕψος τῷ ΠΝ ὕψει, ἔσται ἄρα ὡς ἡ ΑΒΓΔ βάσις πρὸς τὴν ΕΖΗΘ βάσιν, οὕτως τὸ ΜΝ ὕψος πρὸς τὸ ΠΝ ὕψος. ἀλλ᾽ ὡς μὲν ἡ ΑΒΓΔ βάσις πρὸς τὴν ΕΖΗΘ βάσιν, οὕτως ὁ ΑΞ κύλινδρος πρὸς τὸν ΕΣ κύλινδρον· ὑπὸ γὰρ τὸ αὐτὸ ὕψος εἰσίν· ὡς δὲ τὸ ΜΝ ὕψος πρὸς τὸ ΠΝ [ὕψος], οὕτως ὁ ΕΟ κύλινδρος πρὸς τὸν ΕΣ κύλινδρον· ἔστιν ἄρα ὡς ὁ ΑΞ κύλινδρος πρὸς τὸν ΕΣ κύλινδρον, οὕτως ὁ ΕΟ κύλινδρος πρὸς τὸν ΕΣ. ἴσος ἄρα ὁ ΑΞ κύλινδρος τῷ ΕΟ κυλίνδρῳ. ὡσαύτως δὲ καὶ ἐπὶ τῶν κώνων· ὅπερ ἔδει δεῖξαι.

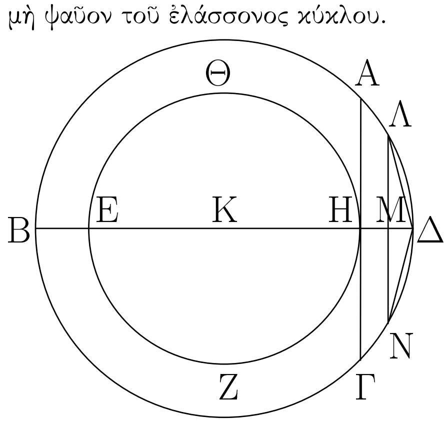
(is) equal to height $KL$. Thus, as base $ABCD$ is to base $EFGH$, so height $MN$ (is) to height $KL$. Thus, the bases of cylinders $AO$ and $EP$ are reciprocally proportional to their heights.

And, so, let the bases of cylinders $AO$ and $EP$ be reciprocally proportional to their heights, and (thus) let base $ABCD$ be to base $EFGH$, as height $MN$ (is) to height $KL$. I say that cylinder $AO$ is equal to cylinder $EP$.

For, with the same construction, since as base $ABCD$ is to base $EFGH$, so height $MN$ (is) to height $KL$, and height $KL$ (is) equal to height $QN$, thus, as base $ABCD$ (is) to base $EFGH$, so height $MN$ will be to height $QN$. But, as base $ABCD$ (is) to base $EFGH$, so cylinder $AO$ (is) to cylinder $ES$. For they are the same height [Prop. 12.11]. And as height $MN$ (is) to [height] $QN$, so cylinder $EP$ (is) to cylinder $ES$ [Prop. 12.13]. Thus, as cylinder $AO$ is to cylinder $ES$, so cylinder $EP$ (is) to (cylinder) $ES$ [Prop. 5.11]. Thus, cylinder $AO$ (is) equal to cylinder $EP$ [Prop. 5.9]. In the same manner, (the proposition can) also (be demonstrated) for the cones. (Which is) the very thing it was required to show.

## ιϛʹ.

Δύο κύκλων περὶ τὸ αὐτὸ κέντρον ὄντων εἰς τὸν μείζονα κύκλον πολύγωνον ἰσόπλευρόν τε καὶ ἀρτιόπλευρον ἐγγράψαι μὴ ψαῦον τοῦ ἐλάσσονος κύκλου.

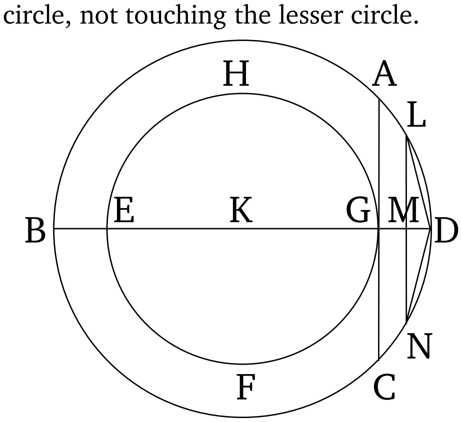



Ἔστωσαν οἱ δοθέντες δύο κύκλοι οἱ ΑΒΓΔ, ΕΖΗΘ περὶ τὸ αὐτὸ κέντρον τὸ Κ· δεῖ δὴ εἰς τὸν μείζονα κύκλον τὸν ΑΒΓΔ πολύγωνον ἰσόπλευρόν τε καὶ ἀρτιόπλευρον ἐγγράψαι μὴ ψαῦον τοῦ ΕΖΗΘ κύκλου.

Ἤχθω γὰρ διὰ τοῦ Κ κέντρου εὐθεῖα ἡ ΒΚΔ, καὶ ἀπὸ τοῦ Η σημείου τῇ ΒΔ εὐθείᾳ πρὸς ὀρθὰς ἤχθω ἡ ΗΑ καὶ διήχθω ἐπὶ τὸ Γ· ἡ ΑΓ ἄρα ἐφάπτεται τοῦ ΕΖΗΘ κύκλου. τέμνοντες δὴ τὴν ΒΑΔ περιφέρειαν δίχα καὶ τὴν ἡμίσειαν αὐτῆς δίχα καὶ τοῦτο ἀεὶ ποιοῦντες καταλείψομεν περιφέρειαν ἐλάσσονα τῆς ΑΔ. λελείφθω, καὶ ἔστω ἡ ΛΔ, καὶ ἀπὸ τοῦ Λ ἐπὶ τὴν ΒΔ κάθετος ἤχθω ἡ ΛΜ καὶ διήχθω

## Proposition 16

There being two circles about the same center, to inscribe an equilateral and even-sided polygon in the greater circle, not touching the lesser circle.




Let $ABCD$ and $EFGH$ be the given two circles, about the same center, $K$. So, it is necessary to inscribe an equilateral and even-sided polygon in the greater circle $ABCD$, not touching circle $EFGH$.

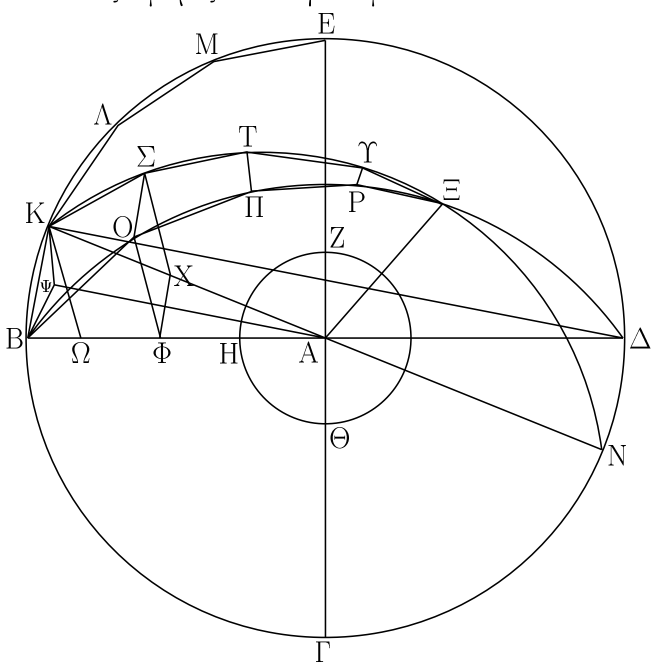
Let the straight-line $BKD$ have been drawn through the center $K$. And let $GA$ have been drawn, at right-angles to the straight-line $BD$, through point $G$, and let it have been drawn through to $C$. Thus, $AC$ touches circle $EFGH$ [Prop. 3.16 corr.]. So, (by) cutting circumference $BAD$ in half, and the half of it in half, and doing this continually, we will (eventually) leave a circumference less ἐπὶ τὸ Ν, καὶ ἐπεζεύχθωσαν αἱ ΛΔ, ΔΝ· ἴση ἄρα ἐστὶν ἡ ΛΔ τῇ ΔΝ. καὶ ἐπεὶ παράλληλός ἐστιν ἡ ΛΝ τῇ ΑΓ, ἡ δὲ ΑΓ ἐφάπτεται τοῦ ΕΖΗΘ κύκλου, ἡ ΛΝ ἄρα οὐκ ἐφάπτεται τοῦ ΕΖΗΘ κύκλου· πολλῷ ἄρα αἱ ΛΔ, ΔΝ οὐκ ἐφάπτονται τοῦ ΕΖΗΘ κύκλου. ἐὰν δὴ τῇ ΛΔ εὐθείᾳ ἴσας κατὰ τὸ συνεχὲς ἐναρμόσωμεν εἰς τὸν ΑΒΓΔ κύκλον, ἐγγραφήσεται εἰς τὸν ΑΒΓΔ κύκλον πολύγωνον ἰσόπλευρόν τε καὶ ἀρτιόπλευρον μὴ ψαῦον τοῦ ἐλάσσονος κύκλου τοῦ ΕΖΗΘ· ὅπερ ἔδει ποιῆσαι.

than $AD$ [Prop. 10.1]. Let it have been left, and let it be $LD$. And let $LM$ have been drawn, from $L$, perpendicular to $BD$, and let it have been drawn through to $N$. And let $LD$ and $DN$ have been joined. Thus, $LD$ is equal to $DN$ [Props. 3.3, 1.4]. And since $LN$ is parallel to $AC$ [Prop. 1.28], and $AC$ touches circle $EFGH$, $LN$ thus does not touch circle $EFGH$. Thus, even more so, $LD$ and $DN$ do not touch circle $EFGH$. And if we continuously insert (straight-lines) equal to straight-line $LD$ into circle $ABCD$ [Prop. 4.1] then an equilateral and even-sided polygon, not touching the lesser circle $EFGH$, will have been inscribed in circle $ABCD$.† (Which is) the very thing it was required to do.

† Note that the chord of the polygon, $LN$, does not touch the inner circle either.

## ιζʹ.

Δύο σφαιρῶν περὶ τὸ αὐτὸ κέντρον οὐσῶν εἰς τὴν μείζονα σφαῖραν στερεὸν πολύεδρον ἐγγράψαι μὴ ψαῦον τῆς ἐλάσσονος σφαίρας κατὰ τὴν ἐπιφάνειαν.

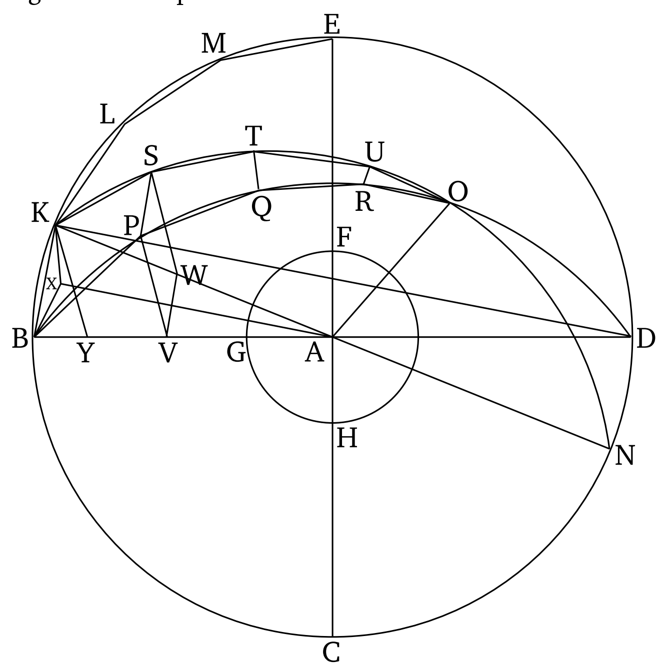



Νενοήσθωσαν δύο σφαῖραι περὶ τὸ αὐτὸ κέντρον τὸ Α· δεῖ δὴ εἰς τὴν μείζονα σφαῖραν στερεὸν πολύεδρον ἐγγράψαι μὴ ψαῦον τῆς ἐλάσσονος σφαίρας κατὰ τὴν ἐπιφάνειαν.

Τετμήσθωσαν αἱ σφαῖραι ἐπιπέδῳ τινὶ διὰ τοῦ κέντρου· ἔσονται δὴ αἱ τομαὶ κύκλοι, ἐπειδήπερ μενούσης τῆς διαμέτρου καὶ περιφερομένου τοῦ ἡμικυκλίου ἐγίγνετο ἡ σφαῖρα· ὥστε καὶ καθʼ οἵας ἂν θέσεως ἐπινοήσωμεν τὸ ἡμικύκλιον, τὸ διʼ αὐτοῦ ἐκβαλλόμενον ἐπίπεδον ποιήσει ἐπὶ τῆς ἐπιφανείας τῆς σφαίρας κύκλον. καὶ φανερόν, ὅτι καὶ μέγιστον, ἐπειδήπερ ἡ διάμετρος τῆς σφαίρας, ἥτις

## Proposition 17

There being two spheres about the same center, to inscribe a polyhedral solid in the greater sphere, not touching the lesser sphere on its surface.




Let two spheres have been conceived about the same center, $A$. So, it is necessary to inscribe a polyhedral solid in the greater sphere, not touching the lesser sphere on its surface.

Let the spheres have been cut by some plane through the center. So, the sections will be circles, inasmuch as a sphere is generated by the diameter remaining behind, and a semi-circle being carried around [Def. 11.14]. And, hence, whatever position we conceive (of for) the semi-circle, the plane produced through it will make a ἐστὶ καὶ τοῦ ἡμικυκλίου διάμετρος δηλαδὴ καὶ τοῦ κύκλου, μείζων ἐστὶ πασῶν τῶν εἰς τὸν κύκλον ἢ τὴν σφαῖραν διαγομένων [εὐθειῶν]. ἔστω οὖν ἐν μὲν τῇ μείζονι σφαίρᾳ κύκλος ὁ ΒΓΔΕ, ἐν δὲ τῇ ἐλάσσονι σφαίρᾳ κύκλος ὁ ΖΗΘ, καὶ ἤχθωσαν αὐτῶν δύο διάμετροι πρὸς ὀρθὰς ἀλλήλαις αἱ ΒΔ, ΓΕ, καὶ δύο κύκλων περὶ τὸ αὐτὸ κέντρον ὄντων τῶν ΒΓΔΕ, ΖΗΘ εἰς τὸν μείζονα κύκλον τὸν ΒΓΔΕ πολύγωνον ἰσόπλευρον καὶ ἀρτιόπλευρον ἐγγεγράφθω μὴ ψαῦον τοῦ ἐλάσσονος κύκλου τοῦ ΖΗΘ, οὗ πλευραὶ ἔστωσαν ἐν τῷ ΒΕ τεταρτημορίῳ αἱ ΒΚ, ΚΛ, ΛΜ, ΜΕ, καὶ ἐπιζευχθεῖσα ἡ ΚΑ διήχθω ἐπὶ τὸ Ν, καὶ ἀνεστάτω ἀπὸ τοῦ Α σημείου τῷ τοῦ ΒΓΔΕ κύκλου ἐπιπέδῳ πρὸς ὀρθὰς ἡ ΑΞ καὶ συμβαλλέτω τῇ ἐπιφανείᾳ τῆς σφαίρας κατὰ τὸ Ξ, καὶ διὰ τῆς ΑΞ καὶ ἑκατέρας τῶν ΒΔ, ΚΝ ἐπίπεδα ἐκβεβλήσθω· ποιήσουσι δὴ διὰ τὰ εἰρημένα ἐπὶ τῆς ἐπιφανείας τῆς σφαίρας μεγίστους κύκλους. ποιείτωσαν, ὧν ἡμικύκλια ἔστω ἐπὶ τῶν ΒΔ, ΚΝ διαμέτρων τὰ ΒΞΔ, ΚΞΝ. καὶ ἐπεὶ ἡ ΞΑ ὀρθή ἐστι πρὸς τὸ τοῦ ΒΓΔΕ κύκλου ἐπίπεδον, καὶ πάντα ἄρα τὰ διὰ τῆς ΞΑ ἐπίπεδά ἐστιν ὀρθὰ πρὸς τὸ τοῦ ΒΓΔΕ κύκλου ἐπίπεδον· ὥστε καὶ τὰ ΒΞΔ, ΚΞΝ ἡμικύκλια ὀρθά ἐστι πρὸς τὸ τοῦ ΒΓΔΕ κύκλου ἐπίπεδον. καὶ ἐπεὶ ἴσα ἐστὶ τὰ ΒΕΔ, ΒΞΔ, ΚΞΝ ἡμικύκλια· ἐπὶ γὰρ ἴσων εἰσὶ διαμέτρων τῶν ΒΔ, ΚΝ· ἴσα ἐστὶ καὶ τὰ ΒΕ, ΒΞ, ΚΞ τεταρτημόρια ἀλλήλοις. ὅσαι ἄρα εἰσὶν ἐν τῷ ΒΕ τεταρτημορίῳ πλευραὶ τοῦ πολυγώνου, τοσαῦταί εἰσι καὶ ἐν τοῖς ΒΞ, ΚΞ τεταρτημορίοις ἴσαι ταῖς ΒΚ, ΚΛ, ΛΜ, ΜΕ εὐθείαις. ἐγγεγράφθωσαν καὶ ἔστωσαν αἱ ΒΟ, ΟΠ, ΠΡ, ΡΞ, ΚΣ, ΣΤ, ΤΥ, ΥΞ, καὶ ἐπεζεύχθωσαν αἱ ΣΟ, ΤΠ, ΥΡ, καὶ ἀπὸ τῶν Ο, Σ ἐπὶ τὸ τοῦ ΒΓΔΕ κύκλου ἐπίπεδον κάθετοι ἤχθωσαν· πεσοῦνται δὴ ἐπὶ τὰς κοινὰς τομὰς τῶν ἐπιπέδων τὰς ΒΔ, ΚΝ, ἐπειδήπερ καὶ τὰ τῶν ΒΞΔ, ΚΞΝ ἐπίπεδα ὀρθά ἐστι πρὸς τὸ τοῦ ΒΓΔΕ κύκλου ἐπίπεδον. πιπτέτωσαν, καὶ ἔστωσαν αἱ ΟΦ, ΣΧ, καὶ ἐπεζεύχθω ἡ ΧΦ. καὶ ἐπεὶ ἐν ἴσοις ἡμικυκλίοις τοῖς ΒΞΔ, ΚΞΝ ἴσαι ἀπειλημμέναι εἰσὶν αἱ ΒΟ, ΚΣ, καὶ κάθετοι ἠγμέναι εἰσὶν αἱ ΟΦ, ΣΧ, ἴση [ἄρα] ἐστὶν ἡ μὲν ΟΦ τῇ ΣΧ, ἡ δὲ ΒΦ τῇ ΚΧ. ἔστι δὲ καὶ ὅλη ἡ ΒΑ ὅλῃ τῇ ΚΑ ἴση· καὶ λοιπὴ ἄρα ἡ ΦΑ λοιπῇ τῇ ΧΑ ἐστιν ἴση· ἔστιν ἄρα ὡς ἡ ΒΦ πρὸς τὴν ΦΑ, οὕτως ἡ ΚΧ πρὸς τὴν ΧΑ· παράλληλος ἄρα ἐστὶν ἡ ΧΦ τῇ ΚΒ. καὶ ἐπεὶ ἑκατέρα τῶν ΟΦ, ΣΧ ὀρθή ἐστι πρὸς τὸ τοῦ ΒΓΔΕ κύκλου ἐπίπεδον, παράλληλος ἄρα ἐστὶν ἡ ΟΦ τῇ ΣΧ. ἐδείχθη δὲ αὐτῇ καὶ ἴση· καὶ αἱ ΧΦ, ΣΟ ἄρα ἴσαι εἰσὶ καὶ παράλληλοι. καὶ ἐπεὶ παράλληλός ἐστιν ἡ ΧΦ τῇ ΣΟ, ἀλλὰ ἡ ΧΦ τῇ ΚΒ ἐστι παράλληλος, καὶ ἡ ΣΟ ἄρα τῇ ΚΒ ἐστι παράλληλος. καὶ ἐπιζευγνύουσιν αὐτὰς αἱ ΒΟ, ΚΣ· τὸ ΚΒΟΣ ἄρα τετράπλευρον ἐν ἑνί ἐστιν ἐπιπέδῳ, ἐπειδήπερ, ἐὰν ὦσι δύο εὐθεῖαι παράλληλοι, καὶ ἐφ᾽ ἑκατέρας αὐτῶν ληφθῇ τυχόντα σημεῖα, ἡ ἐπὶ τὰ σημεῖα ἐπιζευγνυμένη εὐθεῖα ἐν τῷ αὐτῷ ἐπιπέδῳ ἐστὶ ταῖς παραλλήλοις. διὰ τὰ αὐτὰ δὴ καὶ ἑκάτερον τῶν ΣΟΠΤ, ΤΠΡΥ τετραπλεύρων ἐν ἑνί ἐστιν ἐπιπέδῳ. ἔστι δὲ καὶ τὸ ΥΡΞ τρίγωνον ἐν ἑνὶ ἐπιπέδῳ. ἐὰν δὴ νοήσωμεν ἀπὸ

circle on the surface of the sphere. And (it is) clear that (it is) also a great (circle), inasmuch as the diameter of the sphere, which is also manifestly the diameter of the semi-circle and the circle, is greater than all of the (other) [straight-lines] drawn across in the circle or the sphere [Prop. 3.15]. Therefore, let $BCDE$ be the circle in the greater sphere, and $FGH$ the circle in the lesser sphere. And let two diameters of them have been drawn at right-angles to one another, (namely), $BD$ and $CE$. And there being two circles about the same center—(namely), $BCDE$ and $FGH$—let an equilateral and even-sided polygon have been inscribed in the greater circle, $BCDE$, not touching the lesser circle, $FGH$ [Prop. 12.16], of which let the sides in the quadrant $BE$ be $BK$, $KL$, $LM$, and $ME$. And, $KA$ being joined, let it have been drawn across to $N$. And let $AO$ have been set up at point $A$, at right-angles to the plane of circle $BCDE$. And let it meet the surface of the (greater) sphere at $O$. And let planes have been produced through $AO$ and each of $BD$ and $KN$. So, according to the aforementioned (discussion), they will make great circles on the surface of the (greater) sphere. Let them make (great circles), of which let $BOD$ and $KON$ be semi-circles on the diameters $BD$ and $KN$ (respectively). And since $OA$ is at right-angles to the plane of circle $BCDE$, all of the planes through $OA$ are thus also at right-angles to the plane of circle $BCDE$ [Prop. 11.18]. And, hence, the semi-circles $BOD$ and $KON$ are also at right-angles to the plane of circle $BCDE$. And since semi-circles $BED$, $BOD$, and $KON$ are equal—for (they are) on the equal diameters $BD$ and $KN$ [Def. 3.1]—the quadrants $BE$, $BO$, and $KO$ are also equal to one another. Thus, as many sides of the polygon as are in quadrant $BE$, so many are also in quadrants $BO$ and $KO$ equal to the straight-lines $BK$, $KL$, $LM$, and $ME$. Let them have been inscribed, and let them be $BP$, $PQ$, $QR$, $RO$, $KS$, $ST$, $TU$, and $UO$. And let $SP$, $TQ$, and $UR$ have been joined. And let perpendiculars have been drawn from $P$ and $S$ to the plane of circle $BCDE$ [Prop. 11.11]. So, they will fall on the common sections of the planes $BD$ and $KN$ (with $BCDE$), inasmuch as the planes of $BOD$ and $KON$ are also at right-angles to the plane of circle $BCDE$ [Def. 11.4]. Let them have fallen, and let them be $PV$ and $SW$. And let $WV$ have been joined. And since $BP$ and $KS$ are equal (circumferences) having been cut off in the equal semi-circles $BOD$ and $KON$ [Def. 3.28], and $PV$ and $SW$ are perpendiculars having been drawn (from them), $PV$ is [thus] equal to $SW$, and $BV$ to $KW$ [Props. 3.27, 1.26]. And the whole of $BA$ is also equal to the whole of $KA$. And, thus, as $BV$ is to $VA$, so $KW$ (is) to $WA$. $WV$ is thus parallel to $KB$ [Prop. 6.2]. And τῶν Ο, Σ, Π, Τ, Ρ, Υ σημείων ἐπὶ τὸ Α ἐπιζευγνυμένας εὐθείας, συσταθήσεταί τι σχῆμα στερεὸν πολύεδρον μεταξὺ τῶν ΒΞ, ΚΞ περιφερειῶν ἐκ πυραμίδων συγκείμενον, ὧν βάσεις μὲν τὰ ΚΒΟΣ, ΣΟΠΤ, ΤΠΡΥ τετράπλευρα καὶ τὸ ΥΡΞ τρίγωνον, κορυφὴ δὲ τὸ Α σημεῖον. ἐὰν δὲ καὶ ἐπὶ ἑκάστης τῶν ΚΛ, ΛΜ, ΜΕ πλευρῶν καθάπερ ἐπὶ τῆς ΒΚ τὰ αὐτὰ κατασκευάσωμεν καὶ ἔτι τῶν λοιπῶν τριῶν τεταρτημορίων, συσταθήσεταί τι σχῆμα πολύεδρον ἐγγεγραμμένον εἰς τὴν σφαῖραν πυραμίσι περιεχόμενον, ὧν βάσιες [μὲν] τὰ εἰρημένα τετράπλευρα καὶ τὸ ΥΡΞ τρίγωνον καὶ τὰ ὁμοταγῆ αὐτοῖς, κορυφὴ δὲ τὸ Α σημεῖον.

Λέγω ὅτι τὸ εἰρημένον πολύεδρον οὐκ ἐφάψεται τῆς ἐλάσσονος σφαίρας κατὰ τὴν ἐπιφάνειαν, ἐφ᾽ ἧς ἐστιν ὁ ΖΗΘ κύκλος.

Ἤχθω ἀπὸ τοῦ Α σημείου ἐπὶ τὸ τοῦ ΚΒΟΣ τετραπλεύρου ἐπίπεδον κάθετος ἡ ΑΨ καὶ συμβαλλέτω τῷ ἐπιπέδῳ κατὰ τὸ Ψ σημεῖον, καὶ ἐπεζεύχθωσαν αἱ ΨΒ, ΨΚ. καὶ ἐπεὶ ἡ ΑΨ ὀρθή ἐστι πρὸς τὸ τοῦ ΚΒΟΣ τετραπλεύρου ἐπίπεδον, καὶ πρὸς πάσας ἄρα τὰς ἁπτομένας αὐτῆς εὐθείας καὶ οὔσας ἐν τῷ τοῦ τετραπλεύρου ἐπιπέδῳ ὀρθή ἐστιν. ἡ ΑΨ ἄρα ὀρθή ἐστι πρὸς ἑκατέραν τῶν ΒΨ, ΨΚ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΒ τῇ ΑΚ, ἴσον ἐστὶ καὶ τὸ ἀπὸ τῆς ΑΒ τῷ ἀπὸ τῆς ΑΚ. καί ἐστι τῷ μὲν ἀπὸ τῆς ΑΒ ἴσα τὰ ἀπὸ τῶν ΑΨ, ΨΒ· ὀρθὴ γὰρ ἡ πρὸς τῷ Ψ· τῷ δὲ ἀπὸ τῆς ΑΚ ἴσα τὰ ἀπὸ τῶν ΑΨ, ΨΚ. τὰ ἄρα ἀπὸ τῶν ΑΨ, ΨΒ ἴσα ἐστὶ τοῖς ἀπὸ τῶν ΑΨ, ΨΚ. κοινὸν ἀφῃρήσθω τὸ ἀπὸ τῆς ΑΨ· λοιπὸν ἄρα τὸ ἀπὸ τῆς ΒΨ λοιπῷ τῷ ἀπὸ τῆς ΨΚ ἴσον ἐστίν· ἴση ἄρα ἡ ΒΨ τῇ ΨΚ. ὁμοίως δὴ δείξομεν, ὅτι καὶ αἱ ἀπὸ τοῦ Ψ ἐπὶ τὰ Ο, Σ ἐπιζευγνύμεναι εὐθεῖαι ἴσαι εἰσὶν ἑκατέρᾳ τῶν ΒΨ, ΨΚ. ὁ ἄρα κέντρῳ τῷ Ψ καὶ διαστήματι ἑνὶ τῶν ΨΒ, ΨΚ γραφόμενος κύκλος ἥξει καὶ διὰ τῶν Ο, Σ, καὶ ἔσται ἐν κύκλῳ τὸ ΚΒΟΣ τετράπλευρον.

Καὶ ἐπεὶ μείζων ἐστὶν ἡ ΚΒ τῆς ΧΦ, ἴση δὲ ἡ ΧΦ τῇ ΣΟ, μείζων ἄρα ἡ ΚΒ τῆς ΣΟ. ἴση δὲ ἡ ΚΒ ἑκατέρᾳ τῶν ΚΣ, ΒΟ· καὶ ἑκατέρα ἄρα τῶν ΚΣ, ΒΟ τῆς ΣΟ μείζων ἐστίν. καὶ ἐπεὶ ἐν κύκλῳ τετράπλευρόν ἐστι τὸ ΚΒΟΣ, καὶ ἴσαι αἱ ΚΒ, ΒΟ, ΚΣ, καὶ ἐλάττων ἡ ΟΣ, καὶ ἐκ τοῦ κέντρου τοῦ κύκλου ἐστὶν ἡ ΒΨ, τὸ ἄρα ἀπὸ τῆς ΚΒ τοῦ ἀπὸ τῆς ΒΨ μεῖζόν ἐστιν ἢ διπλάσιον. ἤχθω ἀπὸ τοῦ Κ ἐπὶ τὴν ΒΦ κάθετος ἡ ΚΩ. καὶ ἐπεὶ ἡ ΒΔ τῆς ΔΩ ἐλάττων ἐστὶν ἢ διπλῆ, καί ἐστιν ὡς ἡ ΒΔ πρὸς τὴν ΔΩ, οὕτως τὸ ὑπὸ τῶν ΔΒ, ΒΩ πρὸς τὸ ὑπὸ [τῶν] ΔΩ, ΩΒ, ἀναγραφομένου ἀπὸ τῆς ΒΩ τετραγώνου καὶ συμπληρουμένου τοῦ ἐπὶ τῆς ΩΔ παραλληλογράμμου καὶ τὸ ὑπὸ ΔΒ, ΒΩ ἄρα τοῦ ὑπὸ ΔΩ, ΩΒ ἔλαττόν ἐστιν ἢ διπλάσιον. καί ἐστι τῆς ΚΔ ἐπιζευγνυμένης τὸ μὲν ὑπὸ ΔΒ, ΒΩ ἴσον τῷ ἀπὸ τῆς ΒΚ, τὸ δὲ ὑπὸ τῶν ΔΩ, ΩΒ ἴσον τῷ ἀπὸ τῆς ΚΩ· τὸ ἄρα ἀπὸ τῆς ΚΒ τοῦ ἀπὸ τῆς ΚΩ ἔλασσόν ἐστιν ἢ διπλάσιον. ἀλλὰ τὸ ἀπὸ τῆς ΚΒ τοῦ ἀπὸ τῆς ΒΨ μεῖζόν ἐστιν ἢ διπλάσιον· μεῖζον ἄρα τὸ ἀπὸ τῆς ΚΩ τοῦ ἀπὸ τῆς ΒΨ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΒΑ τῇ ΚΑ, ἴσον ἐστὶ τὸ ἀπὸ τῆς ΒΑ τῷ ἀπὸ τῆς ΑΚ. καί

since $PV$ and $SW$ are each at right-angles to the plane of circle $BCDE$, $PV$ is thus parallel to $SW$ [Prop. 11.6]. And it was also shown (to be) equal to it. And, thus, $WV$ and $SP$ are equal and parallel [Prop. 1.33]. And since $WV$ is parallel to $SP$, but $WV$ is parallel to $KB$, $SP$ is thus also parallel to $KB$ [Prop. 11.1]. And $BP$ and $KS$ join them. Thus, the quadrilateral $KBPS$ is in one plane, inasmuch as if there are two parallel straight-lines, and a random point is taken on each of them, then the straight-line joining the points is in the same plane as the parallel (straight-lines) [Prop. 11.7]. So, for the same (reasons), each of the quadrilaterals $SPQT$ and $TQRU$ is also in one plane. And triangle $URO$ is also in one plane [Prop. 11.2]. So, if we conceive straight-lines joining points $P$, $S$, $Q$, $T$, $R$, and $U$ to $A$ then some solid polyhedral figure will have been constructed between the circumferences $BO$ and $KO$, being composed of pyramids whose bases (are) the quadrilaterals $KBPS$, $SPQT$, $TQRU$, and the triangle $URO$, and apex the point $A$. And if we also make the same construction on each of the sides $KL$, $LM$, and $ME$, just as on $BK$, and, further, (repeat the construction) in the remaining three quadrants, then some polyhedral figure which has been inscribed in the sphere will have been constructed, being contained by pyramids whose bases (are) the aforementioned quadrilaterals, and triangle $URO$, and the (quadrilaterals and triangles) similarly arranged to them, and apex the point $A$.

So, I say that the aforementioned polyhedron will not touch the lesser sphere on the surface on which the circle $FGH$ is (situated).

Let the perpendicular (straight-line) $AX$ have been drawn from point $A$ to the plane $KBPS$, and let it meet the plane at point $X$ [Prop. 11.11]. And let $XB$ and $XK$ have been joined. And since $AX$ is at right-angles to the plane of quadrilateral $KBPS$, it is thus also at right-angles to all of the straight-lines joined to it which are also in the plane of the quadrilateral [Def. 11.3]. Thus, $AX$ is at right-angles to each of $BX$ and $XK$. And since $AB$ is equal to $AK$, the (square) on $AB$ is also equal to the (square) on $AK$. And the (sum of the squares) on $AX$ and $XB$ is equal to the (square) on $AB$. For the angle at $X$ (is) a right-angle [Prop. 1.47]. And the (sum of the squares) on $AX$ and $XK$ is equal to the (square) on $AK$ [Prop. 1.47]. Thus, the (sum of the squares) on $AX$ and $XB$ is equal to the (sum of the squares) on $AX$ and $XK$. Let the (square) on $AX$ have been subtracted from both. Thus, the remaining (square) on $BX$ is equal to the remaining (square) on $XK$. Thus, $BX$ (is) equal to $XK$. So, similarly, we can show that the straight-lines joined from $X$ to $P$ and $S$ are equal to each of $BX$ and $XK$.

ἐστι τῷ μὲν ἀπὸ τῆς ΒΑ ἴσα τὰ ἀπὸ τῶν ΒΨ, ΨΑ, τῷ δὲ ἀπὸ τῆς ΚΑ ἴσα τὰ ἀπὸ τῶν ΚΩ, ΩΑ· τὰ ἄρα ἀπὸ τῶν ΒΨ, ΨΑ ἴσα ἐστὶ τοῖς ἀπὸ τῶν ΚΩ, ΩΑ, ὧν τὸ ἀπὸ τῆς ΚΩ μεῖζον τοῦ ἀπὸ τῆς ΒΨ· λοιπὸν ἄρα τὸ ἀπὸ τῆς ΩΑ ἔλασσόν ἐστι τοῦ ἀπὸ τῆς ΨΑ. μείζων ἄρα ἡ ΑΨ τῆς ΑΩ· πολλῷ ἄρα ἡ ΑΨ μείζων ἐστὶ τῆς ΑΗ. καί ἐστιν ἡ μὲν ΑΨ ἐπὶ μίαν τοῦ πολυέδρου βάσιν, ἡ δὲ ΑΗ ἐπὶ τὴν τῆς ἐλάσσονος σφαίρας ἐπιφάνειαν· ὥστε τὸ πολύεδρον οὐ ψαύσει τῆς ἐλάσσονος σφαίρας κατὰ τὴν ἐπιφάνειαν.

Δύο ἄρα σφαιρῶν περὶ τὸ αὐτὸ κέντρον οὐσῶν εἰς τὴν μείζονα σφαῖραν στερεὸν πολύεδρον ἐγγέγραπται μὴ ψαῦον τῆς ἐλάσσονος σφαίρας κατὰ τὴν ἐπιφάνειαν· ὅπερ ἔδει ποιῆσαι.

Thus, a circle drawn (in the plane of the quadrilateral) with center $X$, and radius one of $XB$ or $XK$, will also pass through $P$ and $S$, and the quadrilateral $KBPS$ will be inside the circle.

And since $KB$ is greater than $WV$, and $WV$ (is) equal to $SP$, $KB$ (is) thus greater than $SP$. And $KB$ (is) equal to each of $KS$ and $BP$. Thus, $KS$ and $BP$ are each greater than $SP$. And since quadrilateral $KBPS$ is in a circle, and $KB$, $BP$, and $KS$ are equal (to one another), and $PS$ (is) less (than them), and $BX$ is the radius of the circle, the (square) on $KB$ is thus greater than double the (square) on $BX$.[†] Let the perpendicular $KY$ have been drawn from $K$ to $BV$.[‡] And since $BD$ is less than double $DY$, and as $BD$ is to $DY$, so the (rectangle contained) by $DB$ and $BY$ (is) to the (rectangle contained) by $DY$ and $YB$—a square being described on $BY$, and a (rectangular) parallelogram (with short side equal to $BY$) completed on $YD$—the (rectangle contained) by $DB$ and $BY$ is thus also less than double the (rectangle contained) by $DY$ and $YB$. And, $KD$ being joined, the (rectangle contained) by $DB$ and $BY$ is equal to the (square) on $BK$, and the (rectangle contained) by $DY$ and $YB$ equal to the (square) on $KY$ [Props. 3.31, 6.8 corr.]. Thus, the (square) on $KB$ is less than double the (square) on $KY$. But, the (square) on $KB$ is greater than double the (square) on $BX$. Thus, the (square) on $KY$ (is) greater than the (square) on $BX$. And since $BA$ is equal to $KA$, the (square) on $BA$ is equal to the (square) on $AK$. And the (sum of the squares) on $BX$ and $XA$ is equal to the (square) on $BA$, and the (sum of the squares) on $KY$ and $YA$ (is) equal to the (square) on $KA$ [Prop. 1.47]. Thus, the (sum of the squares) on $BX$ and $XA$ is equal to the (sum of the squares) on $KY$ and $YA$, of which the (square) on $KY$ (is) greater than the (square) on $BX$. Thus, the remaining (square) on $YA$ is less than the (square) on $XA$. Thus, $AX$ (is) greater than $AY$. Thus, $AX$ is much greater than $AG$.[§] And $AX$ is (a perpendicular) on one of the bases of the polyhedron, and $AG$ (is a perpendicular) on the surface of the lesser sphere. Hence, the polyhedron will not touch the lesser sphere on its surface.

Thus, there being two spheres about the same center, a polyhedral solid has been inscribed in the greater sphere which does not touch the lesser sphere on its surface. (Which is) the very thing it was required to do.

† Since $KB$, $BP$, and $KS$ are greater than the sides of an inscribed square, which are each of length $\sqrt{2}\,BX$.

‡ Note that points $Y$ and $V$ are actually identical.

§ This conclusion depends on the fact that the chord of the polygon in proposition 12.16 does not touch the inner circle.

## Πόρισμα.

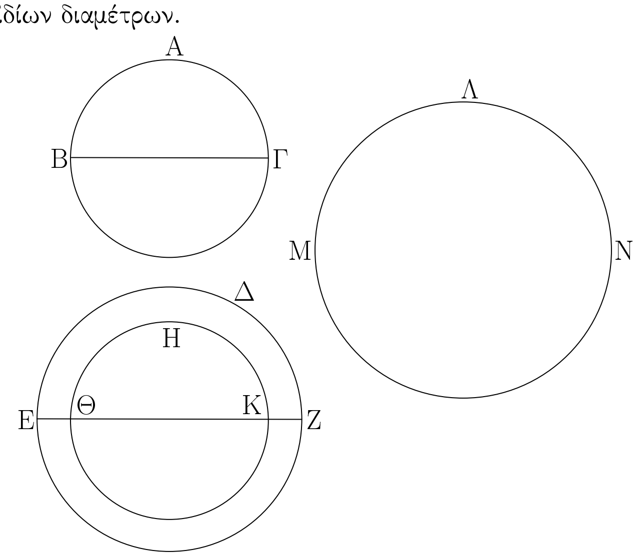
Ἐὰν δὲ καὶ εἰς ἑτάραν σφαῖραν τῷ ἐν τῇ ΒΓΔΕ σφαίρᾳ στερεῷ πολυέδρῳ ὅμοιον στερεὸν πολύεδρον ἐγγραφῇ, τὸ ἐν τῇ ΒΓΔΕ σφαίρᾳ στερεὸν πολύεδρον πρὸς τὸ ἐν τῇ ἑτέρᾳ σφαίρᾳ στερεὸν πολύεδρον τριπλασίονα λόγον ἔχει, ἤπερ ἡ τῆς ΒΓΔΕ σφαίρας διάμετρος πρὸς τὴν τῆς ἑτέρας σφαίρας διάμετρον. διαιρεθέντων γὰρ τῶν στερεῶν εἰς τὰς ὁμοιοπληθεῖς καὶ ὁμοιοταγεῖς πυραμίδας ἔσονται αἱ πυραμίδες ὅμοιαι. αἱ δὲ ὅμοιαι πυραμίδες πρὸς ἀλλήλας ἐν τριπλασίονι λόγῳ εἰσὶ τῶν ὁμολόγων πλευρῶν· ἡ ἄρα πυραμίς, ἧς βάσις μέν ἐστι τὸ ΚΒΟΣ τετράπλευρον, κορυφὴ δὲ τὸ Α σημεῖον, πρὸς τὴν ἐν τῇ ἑτέρᾳ σφαίρᾳ ὁμοιοταγῆ πυραμίδα τριπλασίονα λόγον ἔχει, ἤπερ ἡ ὁμόλογος πλευρὰ πρὸς τὴν ὁμόλογον πλευράν, τουτέστιν ἤπερ ἡ ΑΒ ἐκ τοῦ κέντρου τῆς σφαίρας τῆς περὶ κέντρον τὸ Α πρὸς τὴν ἐκ τοῦ κέντρου τῆς ἑτέρας σφαίρας. ὁμοίως καὶ ἑκάστη πυραμὶς τῶν ἐν τῇ περὶ κέντρον τὸ Α σφαίρᾳ πρὸς ἑκάστην ὁμοταγῆ πυραμίδα τῶν ἐν τῇ ἑτέρᾳ σφαίρᾳ τριπλασίονα λόγον ἕξει, ἤπερ ἡ ΑΒ πρὸς τὴν ἐκ τοῦ κέντρου τῆς ἑτέρας σφαίρας. καὶ ὡς ἓν τῶν ἡγουμένων πρὸς ἓν τῶν ἑπομένων, οὕτως ἅπαντα τὰ ἡγούμενα πρὸς ἅπαντα τὰ ἑπόμενα· ὥστε ὅλον τὸ ἐν τῇ περὶ κέντρον τὸ Α σφαίρᾳ στερεὸν πολύεδρον πρὸς ὅλον τὸ ἐν τῇ ἑτέρᾳ [σφαίρᾳ] στερεὸν πολύεδρον τριπλασίονα λόγον ἕξει, ἤπερ ἡ ΑΒ πρὸς τὴν ἐκ τοῦ κέντρου τῆς ἑτέρας σφαίρας, τουτέστιν ἤπερ ἡ ΒΔ διάμετρος πρὸς τὴν τῆς ἑτέρας σφαίρας διάμετρον· ὅπερ ἔδει δεῖξαι.

## Corollary

And, also, if a similar polyhedral solid to that in sphere $BCDE$ is inscribed in another sphere then the polyhedral solid in sphere $BCDE$ has to the polyhedral solid in the other sphere the cubed ratio that the diameter of sphere $BCDE$ has to the diameter of the other sphere. For if the solids are divided into similarly numbered, and similarly situated, pyramids, then the pyramids will be similar. And similar pyramids are in the cubed ratio of corresponding sides [Prop. 12.8 corr.]. Thus, the pyramid whose base is quadrilateral $KBPS$, and apex the point $A$, will have to the similarly situated pyramid in the other sphere the cubed ratio that a corresponding side (has) to a corresponding side. That is to say, that of radius $AB$ of the sphere about center $A$ to the radius of the other sphere. And, similarly, each pyramid in the sphere about center $A$ will have to each similarly situated pyramid in the other sphere the cubed ratio that $AB$ (has) to the radius of the other sphere. And as one of the leading (magnitudes is) to one of the following (in two sets of proportional magnitudes), so (the sum of) all the leading (magnitudes is) to (the sum of) all of the following (magnitudes) [Prop. 5.12]. Hence, the whole polyhedral solid in the sphere about center $A$ will have to the whole polyhedral solid in the other [sphere] the cubed ratio that (radius) $AB$ (has) to the radius of the other sphere. That is to say, that diameter $BD$ (has) to the diameter of the other sphere. (Which is) the very thing it was required to show.

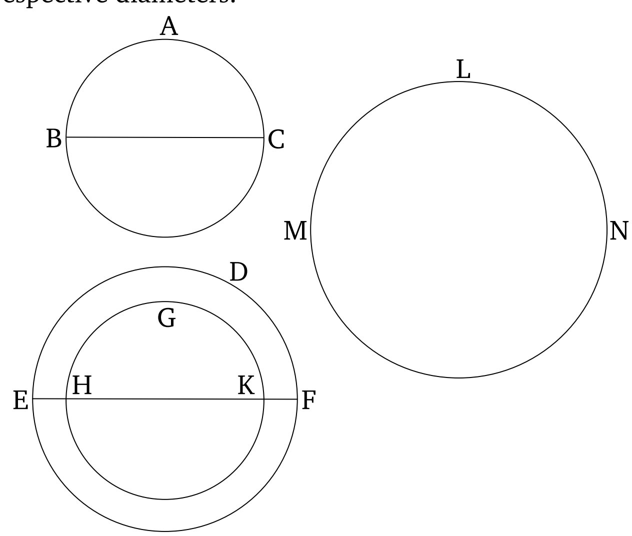
## ιηʹ.

Αἱ σφαῖραι πρὸς ἀλλήλας ἐν τριπλασίονι λόγῳ εἰσὶ τῶν ἰδίων διαμέτρων.




## Proposition 18

Spheres are to one another in the cubed ratio of their respective diameters.



Νενοήσθωσαν σφαῖραι αἱ ΑΒΓ, ΔΕΖ, διάμετροι δὲ αὐτῶν αἱ ΒΓ, ΕΖ· λέγω, ὅτι ἡ ΑΒΓ σφαῖρα πρὸς τὴν ΔΕΖ σφαῖραν τριπλασίονα λόγον ἔχει ἤπερ ἡ ΒΓ πρὸς τὴν ΕΖ.

Εἰ γὰρ μὴ ἡ ΑΒΓ σφαῖρα πρὸς τὴν ΔΕΖ σφαῖραν τριπλασίονα λόγον ἔχει ἤπερ ἡ ΒΓ πρὸς τὴν ΕΖ, ἕξει ἄρα ἡ ΑΒΓ σφαῖρα πρὸς ἐλάσσονά τινα τῆς ΔΕΖ σφαίρας τριπλασίονα λόγον ἢ πρὸς μείζονα ἤπερ ἡ ΒΓ πρὸς τὴν ΕΖ. ἐχέτω πρότερον πρὸς ἐλάσσονα τὴν ΗΘΚ, καὶ νενοήσθω ἡ ΔΕΖ τῇ ΗΘΚ περὶ τὸ αὐτὸ κέντρον, καὶ ἐγγεγράφθω εἰς τὴν μείζονα σφαῖραν τὴν ΔΕΖ στερεὸν πολύεδρον μὴ ψαῦον τῆς ἐλάσσονος σφαίρας τῆς ΗΘΚ κατὰ τὴν ἐπιφάνειαν, ἐγγεγράφθω δὲ καὶ εἰς τὴν ΑΒΓ σφαῖραν τῷ ἐν τῇ ΔΕΖ σφαίρᾳ στερεῷ πολυέδρῳ ὅμοιον στερεὸν πολύεδρον· τὸ ἄρα ἐν τῇ ΑΒΓ στερεὸν πολύεδρον πρὸς τὸ ἐν τῇ ΔΕΖ στερεὸν πολύεδρον τριπλασίονα λόγον ἔχει ἤπερ ἡ ΒΓ πρὸς τὴν ΕΖ. ἔχει δὲ καὶ ἡ ΑΒΓ σφαῖρα πρὸς τὴν ΗΘΚ σφαῖραν τριπλασίονα λόγον ἤπερ ἡ ΒΓ πρὸς τὴν ΕΖ· ἔστιν ἄρα ὡς ἡ ΑΒΓ σφαῖρα πρὸς τὴν ΗΘΚ σφαῖραν, οὕτως τὸ ἐν τῇ ΑΒΓ σφαίρᾳ στερεὸν πολύεδρον πρὸς τὸ ἐν τῇ ΔΕΖ σφαίρᾳ στερεὸν πολύεδρον· ἐναλλὰξ [ἄρα] ὡς ἡ ΑΒΓ σφαῖρα πρὸς τὸ ἐν αὐτῇ πολύεδρον, οὕτως ἡ ΗΘΚ σφαῖρα πρὸς τὸ ἐν τῇ ΔΕΖ σφαίρᾳ στερεὸν πολύεδρον. μείζων δὲ ἡ ΑΒΓ σφαῖρα τοῦ ἐν αὐτῇ πολυέδρου· μείζων ἄρα καὶ ἡ ΗΘΚ σφαῖρα τοῦ ἐν τῇ ΔΕΖ σφαίρᾳ πολυέδρου. ἀλλὰ καὶ ἐλάττων· ἐμπεριέχεται γὰρ ὑπ᾿ αὐτοῦ. οὐκ ἄρα ἡ ΑΒΓ σφαῖρα πρὸς ἐλάσσονα τῆς ΔΕΖ σφαίρας τριπλασίονα λόγον ἔχει ἤπερ ἡ ΒΓ διάμετρος πρὸς τὴν ΕΖ. ὁμοίως δὴ δείξομεν, ὅτι οὐδὲ ἡ ΔΕΖ σφαῖρα πρὸς ἐλάσσονα τῆς ΑΒΓ σφαίρας τριπλασίονα λόγον ἔχει ἤπερ ἡ ΕΖ πρὸς τὴν ΒΓ.

Λέγω δή, ὅτι οὐδὲ ἡ ΑΒΓ σφαῖρα πρὸς μείζονά τινα τῆς ΔΕΖ σφαίρας τριπλασίονα λόγον ἔχει ἤπερ ἡ ΒΓ πρὸς τὴν ΕΖ.

Εἰ γὰρ δυνατόν, ἐχέτω πρὸς μείζονα τὴν ΛΜΝ· ἀνάπαλιν ἄρα ἡ ΛΜΝ σφαῖρα πρὸς τὴν ΑΒΓ σφαῖραν τριπλασίονα λόγον ἔχει ἤπερ ἡ ΕΖ διάμετρος πρὸς τὴν ΒΓ διάμετρον. ὡς δὲ ἡ ΛΜΝ σφαῖρα πρὸς τὴν ΑΒΓ σφαῖραν, οὕτως ἡ ΔΕΖ σφαῖρα πρὸς ἐλάσσονά τινα τῆς ΑΒΓ σφαίρας, ἐπειδήπερ μείζων ἐστὶν ἡ ΛΜΝ τῆς ΔΕΖ, ὡς ἔμπροσθεν ἐδείχθη. καὶ ἡ ΔΕΖ ἄρα σφαῖρα πρὸς ἐλάσσονά τινα τῆς ΑΒΓ σφαίρας τριπλασίονα λόγον ἔχει ἤπερ ἡ ΕΖ πρὸς τὴν ΒΓ· ὅπερ ἀδύνατον ἐδείχθη. οὐκ ἄρα ἡ ΑΒΓ σφαῖρα πρὸς μείζονά τινα τῆς ΔΕΖ σφαίρας τριπλασίονα λόγον ἔχει ἤπερ ἡ ΒΓ πρὸς τὴν ΕΖ. ἐδείχθη δέ, ὅτι οὐδὲ πρὸς ἐλάσσονα. ἡ ἄρα ΑΒΓ σφαῖρα πρὸς τὴν ΔΕΖ σφαῖραν τριπλασίονα λόγον ἔχει ἤπερ ἡ ΒΓ πρὸς τὴν ΕΖ· ὅπερ ἔδει δεῖξαι.

Let the spheres $ABC$ and $DEF$ have been conceived, and (let) their diameters (be) $BC$ and $EF$ (respectively). I say that sphere $ABC$ has to sphere $DEF$ the cubed ratio that $BC$ (has) to $EF$.

For if sphere $ABC$ does not have to sphere $DEF$ the cubed ratio that $BC$ (has) to $EF$ then sphere $ABC$ will have to some (sphere) either less than, or greater than, sphere $DEF$ the cubed ratio that $BC$ (has) to $EF$. Let it, first of all, have (such a ratio) to a lesser (sphere), $GHK$. And let $DEF$ have been conceived about the same center as $GHK$. And let a polyhedral solid have been inscribed in the greater sphere $DEF$, not touching the lesser sphere $GHK$ on its surface [Prop. 12.17]. And let a polyhedral solid, similar to the polyhedral solid in sphere $DEF$, have also been inscribed in sphere $ABC$. Thus, the polyhedral solid in sphere $ABC$ has to the polyhedral solid in sphere $DEF$ the cubed ratio that $BC$ (has) to $EF$ [Prop. 12.17 corr.]. And sphere $ABC$ also has to sphere $GHK$ the cubed ratio that $BC$ (has) to $EF$. Thus, as sphere $ABC$ is to sphere $GHK$, so the polyhedral solid in sphere $ABC$ (is) to the polyhedral solid is sphere $DEF$. [Thus], alternately, as sphere $ABC$ (is) to the polygon within it, so sphere $GHK$ (is) to the polyhedral solid within sphere $DEF$ [Prop. 5.16]. And sphere $ABC$ (is) greater than the polyhedron within it. Thus, sphere $GHK$ (is) also greater than the polyhedron within sphere $DEF$ [Prop. 5.14]. But, (it is) also less. For it is encompassed by it. Thus, sphere $ABC$ does not have to (a sphere) less than sphere $DEF$ the cubed ratio that diameter $BC$ (has) to $EF$. So, similarly, we can show that sphere $DEF$ does not have to (a sphere) less than sphere $ABC$ the cubed ratio that $EF$ (has) to $BC$ either.

So, I say that sphere $ABC$ does not have to some (sphere) greater than sphere $DEF$ the cubed ratio that $BC$ (has) to $EF$ either.

For, if possible, let it have (the cubed ratio) to a greater (sphere), $LMN$. Thus, inversely, sphere $LMN$ (has) to sphere $ABC$ the cubed ratio that diameter $EF$ (has) to diameter $BC$ [Prop. 5.7 corr.]. And as sphere $LMN$ (is) to sphere $ABC$, so sphere $DEF$ (is) to some (sphere) less than sphere $ABC$, inasmuch as $LMN$ is greater than $DEF$, as was shown before [Prop. 12.2 lem.]. And, thus, sphere $DEF$ has to some (sphere) less than sphere $ABC$ the cubed ratio that $EF$ (has) to $BC$. The very thing was shown (to be) impossible. Thus, sphere $ABC$ does not have to some (sphere) greater than sphere $DEF$ the cubed ratio that $BC$ (has) to $EF$. And it was shown that neither (does it have such a ratio) to a lesser (sphere). Thus, sphere $ABC$ has to sphere $DEF$ the cubed ratio that $BC$ (has) to $EF$. (Which is) the very thing it was required to show.

# ELEMENTS BOOK 13

## *The Platonic Solids*†

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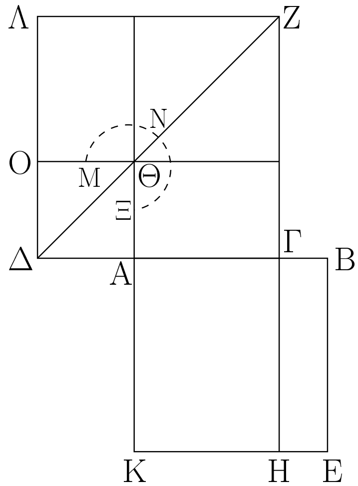
†The five regular solids—the cube, tetrahedron (*i.e.*, pyramid), octahedron, icosahedron, and dodecahedron—were problably discovered by the school of Pythagoras. They are generally termed “Platonic” solids because they feature prominently in Plato’s famous dialogue *Timaeus*. Many of the theorems contained in this book—particularly those which pertain to the last two solids—are ascribed to Theaetetus of Athens.

## αʹ.

Ἐὰν εὐθεῖα γραμμὴ ἄκρον καὶ μέσον λόγον τμηθῇ, τὸ μεῖζον τμῆμα προσλαβὸν τὴν ἡμίσειαν τῆς ὅλης πενταπλάσιον δύναται τοῦ ἀπὸ τῆς ἡμισείας τετραγώνου.



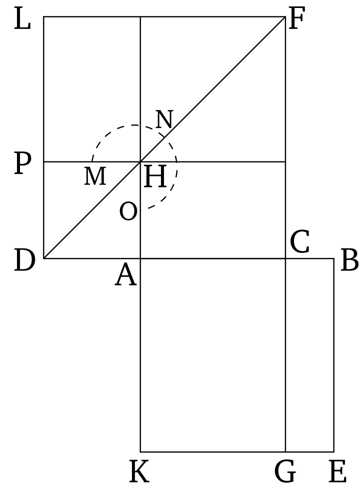

Εὐθεῖα γὰρ γραμμὴ ἡ ΑΒ ἄκρον καὶ μέσον λόγον τετμήσθω κατὰ τὸ Γ σημεῖον, καὶ ἔστω μεῖζον τμῆμα τὸ ΑΓ, καὶ ἐκβεβλήσθω ἐπ᾽ εὐθείας τῇ ΓΑ εὐθεῖα ἡ ΑΔ, καὶ κείσθω τῆς ΑΒ ἡμίσεια ἡ ΑΔ· λέγω, ὅτι πενταπλάσιόν ἐστι τὸ ἀπὸ τῆς ΓΔ τοῦ ἀπὸ τῆς ΔΑ.

Ἀναγεγράφθωσαν γὰρ ἀπὸ τῶν ΑΒ, ΔΓ τετράγωνα τὰ ΑΕ, ΔΖ, καὶ καταγεγράφθω ἐν τῷ ΔΖ τὸ σχῆμα, καὶ διήχθω ἡ ΖΓ ἐπὶ τὸ Η. καὶ ἐπεὶ ἡ ΑΒ ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Γ, τὸ ἄρα ὑπὸ τῶν ΑΒΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΓ. καί ἐστι τὸ μὲν ὑπὸ τῶν ΑΒΓ τὸ ΓΕ, τὸ δὲ ἀπὸ τῆς ΑΓ τὸ ΖΘ· ἴσον ἄρα τὸ ΓΕ τῷ ΖΘ. καὶ ἐπεὶ διπλῆ ἐστιν ἡ ΒΑ τῆς ΑΔ, ἴση δὲ ἡ μὲν ΒΑ τῇ ΚΑ, ἡ δὲ ΑΔ τῇ ΑΘ, διπλῆ ἄρα καὶ ἡ ΚΑ τῆς ΑΘ. ὡς δὲ ἡ ΚΑ πρὸς τὴν ΑΘ, οὕτως τὸ ΓΚ πρὸς τὸ ΓΘ· διπλάσιον ἄρα τὸ ΓΚ τοῦ ΓΘ. εἰσὶ δὲ καὶ τὰ ΛΘ, ΘΓ διπλάσια τοῦ ΓΘ. ἴσον ἄρα τὸ ΚΓ τοῖς ΛΘ, ΘΓ. ἐδείχθη δὲ καὶ τὸ ΓΕ τῷ ΘΖ ἴσον· ὅλον ἄρα τὸ ΑΕ τετράγωνον ἴσον ἐστὶ τῷ ΜΝΞ γνώμονι. καὶ ἐπεὶ διπλῆ ἐστιν ἡ ΒΑ τῆς ΑΔ, τετραπλάσιόν ἐστι τὸ ἀπὸ τῆς ΒΑ τοῦ ἀπὸ τῆς ΑΔ, τουτέστι τὸ ΑΕ τοῦ ΔΘ. ἴσον δὲ τὸ ΑΕ τῷ ΜΝΞ γνώμονι· καὶ ὁ ΜΝΞ ἄρα γνώμων τετραπλάσιός ἐστι τοῦ ΑΟ· ὅλον ἄρα τὸ ΔΖ πενταπλάσιόν ἐστι τοῦ ΑΟ. καί ἐστι τὸ μὲν ΔΖ τὸ ἀπὸ τῆς ΔΓ, τὸ δὲ ΑΟ τὸ ἀπὸ τῆς ΔΑ· τὸ ἄρα ἀπὸ τῆς ΓΔ πενταπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΔΑ.

Ἐὰν ἄρα εὐθεῖα ἄκρον καὶ μέσον λόγον τμηθῇ, τὸ μεῖζον τμῆμα προσλαβὸν τὴν ἡμίσειαν τῆς ὅλης πενταπλάσιον δύναται τοῦ ἀπὸ τῆς ἡμισείας τετραγώνου· ὅπερ ἔδει δεῖξαι.

## Proposition 1

If a straight-line is cut in extreme and mean ratio then the square on the greater piece, added to half of the whole, is five times the square on the half.




For let the straight-line $AB$ have been cut in extreme and mean ratio at point $C$, and let $AC$ be the greater piece. And let the straight-line $AD$ have been produced in a straight-line with $CA$. And let $AD$ be made (equal to) half of $AB$. I say that the (square) on $CD$ is five times the (square) on $DA$.

For let the squares $AE$ and $DF$ have been described on $AB$ and $DC$ (respectively). And let the figure in $DF$ have been drawn. And let $FC$ have been drawn across to $G$. And since $AB$ has been cut in extreme and mean ratio at $C$, the (rectangle contained) by $ABC$ is thus equal to the (square) on $AC$ [Def. 6.3, Prop. 6.17]. And $CE$ is the (rectangle contained) by $ABC$, and $FH$ the (square) on $AC$. Thus, $CE$ (is) equal to $FH$. And since $BA$ is double $AD$, and $BA$ (is) equal to $KA$, and $AD$ to $AH$, $KA$ (is) thus also double $AH$. And as $KA$ (is) to $AH$, so $CK$ (is) to $CH$ [Prop. 6.1]. Thus, $CK$ (is) double $CH$. And $LH$ plus $HC$ is also double $CH$ [Prop. 1.43]. Thus, $KC$ (is) equal to $LH$ plus $HC$. And $CE$ was also shown (to be) equal to $HF$. Thus, the whole square $AE$ is equal to the gnomon $MNO$. And since $BA$ is double $AD$, the (square) on $BA$ is four times the (square) on $AD$—that is to say, $AE$ (is four times) $DH$. And $AE$ (is) equal to gnomon $MNO$. And, thus, gnomon $MNO$ is also four times $AP$. Thus, the whole of $DF$ is five times $AP$. And $DF$ is the (square) on $DC$, and $AP$ the (square) on $DA$. Thus, the (square) on $CD$ is five times the (square) on $DA$.

Thus, if a straight-line is cut in extreme and mean ratio then the square on the greater piece, added to half of