8.2.52. Corollary. (Ford–Fulkerson [1958]) Families $\mathbf{A} = \{A_1, \ldots, A_m\}$ and $\mathbf{B} = \{B_1, \ldots, B_m\}$ have a common system of distinct representatives (CSDR) if and only if, for each $I, J \subseteq [m]$,

$$\left|(\bigcup_{i\in I} A_i) \cap (\bigcup_{j\in J} B_j)\right| \geq |I| + |J| - m.$$

Proof: A common partial SDR is a common independent set in the two transversal matroids M_1, M_2 induced on E by $\mathbf{A}$ and $\mathbf{B}$. To determine when there is a complete CSDR, we need only restate the condition $r_1(X) + r_2(\overline{X}) \geq m$ to find the appropriate condition on the set systems.

The rank formulas from Example 8.2.51 yield

$$r_1(X) + r_2(\overline{X}) = \min_{I\subseteq[m]}\{|A(I) \cap X| - |I| + m\} + \min_{J\subseteq[m]}\{\left|B(J) \cap \overline{X}\right| - |J| + m\}.$$

Hence $r_1(X) + r_2(\overline{X}) \geq m$ for all X if and only if

$$|A(I) \cap X| + \left|B(J) \cap \overline{X}\right| \geq |I| + |J| - m \text{ for all } X \subseteq E \text{ and } I, J \subseteq [m].$$

Given I, J, consider the contribution of an element of E to the left side. Each element of $A(I) \cap B(J)$ counts once whether it belongs to X or $\overline{X}$. Elements of $A(I) - B(J)$ count if and only if they belong to X, and elements of $B(J) - A(I)$ count if and only if they belong to $\overline{X}$. Hence the left side is minimized for I, J when $A(I) - B(J) \subseteq \overline{X}$ and $B(J) - A(I) \subseteq X$. In this case the left side equals $|A(I) \cap B(J)|$, which yields the Ford–Fulkerson condition. ∎

The augmenting path approach to maximum bipartite matching generalizes to matroid intersection. The algorithm yields a common independent set I of maximum size and a set X such that $r_1(X) + r_2(\overline{X}) = |I|$ (see Lawler [1976], Edmonds [1979], Faigle [1987]). Finding a maximum common independent set in three matroids is NP-complete (??s –).

MATROID UNION

The intersection of two matroids is seldom a matroid, but a natural concept of matroid union does always yield a matroid. Together with a useful min-max relation for the rank function, this is the content of the Matroid Union Theorem. The Matroid Intersection and Union Theorems are equivalent; they can be derived from each other. Welsh [1976] proves the Matroid Union Theorem first; here we obtain it from the Matroid Intersection Theorem.

8.2.53. Definition. The **union** $M_1 \cup \cdots \cup M_k$ of hereditary systems $M_1, \ldots, M_k$ on E is the hereditary system M on E defined by $\mathbf{I}_M = \{I_1 \cup \cdots \cup I_k \colon I_i \in \mathbf{I}_i\}$. The **direct sum** $M_1 \oplus \cdots \oplus M_k$ of hereditary systems $M_1, \ldots, M_k$ on disjoint sets $E_1, \ldots, E_k$ is the hereditary system M on $E_1 \cup \cdots \cup E_k$ defined by $\mathbf{I}_M = \{I_1 \cup \cdots \cup I_k : I_i \in \mathbf{I}_i\}$.

The direct sum $M_1 \oplus \cdots \oplus M_k$ on $E_1, \ldots, E_k$ can be expressed as the union of $M'_1, \ldots, M'_k$ on $E' = E_1 \cup \cdots \cup E_k$ by letting M'_i be a copy of M_i with the additional elements of $E' - E_i$ added as loops. When each M_i is a uniform matroid, the direct sum is a **generalized partition matroid**. Here $E_1, \ldots, E_k$ partition E, there are positive integers $r_1, \ldots, r_k$, and $X \in \mathbf{I}$ if $|X \cap E_i| \le r_i$. The partition matroids defined earlier arise when all $r_i = 1$.

8.2.54. Proposition. Given matroids $M_1, \ldots, M_k$ on disjoint sets $E_1, \ldots, E_k$, the direct sum $M = M_1 \oplus \cdots \oplus M_k$ is a matroid.

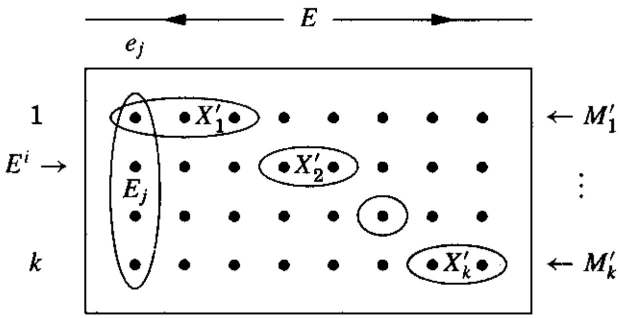
Proof: Since the $E_1, \ldots, E_k$ are pairwise disjoint, the intersection of any $I \in \mathbf{I}$ with each E_i is independent in M_i. If $I_1, I_2 \in \mathbf{I}$ with $|I_2| > |I_1|$, then $|I_2 \cap E_i| > |I_1 \cap E_i|$ for some i. Since both sets are independent in M_i, we can augment $I_1 \cap E_i$ from $I_2 \cap E_i$ and therefore I_1 from I_2. Hence $M_1 \oplus \cdots \oplus M_k$ satisfies the augmentation property. ■

Using a direct sum, we prove that the union of matroids is always a matroid, and we compute the rank function.

8.2.55. Theorem. (Matroid Union Theorem—Edmonds–Fulkerson [1965], Nash-Williams [1966]) If $M_1, \ldots, M_k$ are matroids on E with rank functions $r_1, \ldots, r_k$, then the union $M = M_1 \cup \cdots \cup M_k$ is a matroid with rank function $r(X) = \min_{Y \subseteq X}(|X - Y| + \sum r_i(Y))$.

Proof: (following Schrijver [to appear]). After proving the formula for the rank function, we will verify the submodularity property to prove that M is a matroid. First we reduce the computation of the rank function to the computation of $r(E)$. In the restriction of the hereditary system M to the set X, we have $\mathbf{I}_{M|X} = \{Y \subseteq X\colon Y \in \mathbf{I}_M\}$ and $r_{M|X}(Y) = r_M(Y)$ for $Y \subseteq X$. Thus $M|X = \cup_i (M_i|X)$, and applying the formula for the rank of the full union to $M|X$ yields $r_M(X)$.

Consider a k by $|E|$ grid of elements E' in which the jth column E_j consists of k copies of the element $e_j \in E$. We define two matroids N_1, N_2 on E' such that the maximum size of a set independent in both N_1 and N_2 equals the maximum size of a set independent in M. We then compute $r_M(E)$ by applying the Matroid Intersection Theorem to N_1 and N_2. Let M'_i be a copy of M_i defined on the elements E^i of row i in E'. Let N_1 be the direct sum matroid $M'_1 \oplus \cdots \oplus M'_k$, and let N_2 be the partition matroid induced on E' by the column partition $\{E_j\}$.



Each set $X \in \mathbf{I}_M$ has a decomposition as a disjoint union of subsets $X_i \in \mathbf{I}_i$, because $\mathbf{I}_i$ is a hereditary family. Given a decomposition $\{X_i\}$ of $X \in \mathbf{I}_M$, let X_i' be the copy of X_i in E^i. Since $\{X_i\}$ are disjoint, $\cup X_i'$ is independent in N_2, and $X_i \in \mathbf{I}_i$ implies that $\cup X_i'$ is also independent in N_1. From $X \in \mathbf{I}_M$, we have constructed $\cup X_i'$ of size $|X|$ in $\mathbf{I}_{N_1} \cap \mathbf{I}_{N_2}$. Conversely, any $X' \in \mathbf{I}_{N_1} \cap \mathbf{I}_{N_2}$ corresponds to a decomposition of a set in $\mathbf{I}_M$ of size $|X'|$ when the sets $X' \cap E^i$ are transferred back to E, because N_2 forbids multiple copies of elements.

Hence $r(E) = \max\{|I| : I \in \mathbf{I}_{N_1} \cap \mathbf{I}_{N_2}\}$. To compute this, let the rank functions of N_1, N_2 be q_1, q_2, and let r_i' be the rank function of the copy M_i' of M_i on E^i. We have $q_1(X') = \sum r_i'(X' \cap E^i)$, and $q_2(X')$ is the number of elements of E that have copies in X'. The Matroid Intersection Theorem yields $r(E) = \min_{X' \subseteq E'}\{q_1(X') + q_2(E' - X')\}$.

By Corollary 8.2.49, the minimum is achieved by a set X' such that $E' - X'$ is closed in N_2. The closed sets in the partition matroid N_2 are the sets that contain all or none of the copies of each element—the unions of full columns of E'. Given X' with $E' - X'$ closed in N_2, let $Y \subseteq E$ be the set of elements whose copies comprise X'. Then $q_2(E' - X') = |E - Y|$, and X' contains all copies of the elements of Y, so $q_1(X') = \sum r_i'(X' \cap E^i) = \sum r_i(Y)$. We conclude that $r(E) = \min_{Y \subseteq E}\{|E - Y| + \sum r_i(Y)\}$.

To show that M is a matroid, we verify submodularity for r. Given $X, Y \subseteq E$, the formula for r yields $U \subseteq X$ and $V \subseteq Y$ such that

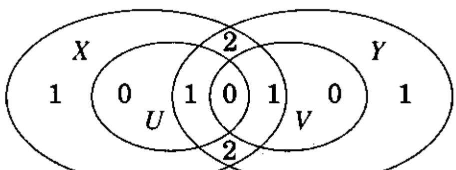
$$r(X) = |X - U| + \sum r_i(U); \qquad r(Y) = |Y - V| + \sum r_i(V).$$

Since $U \cap V \subseteq X \cap Y$ and $U \cup V \subseteq X \cup Y$, we also have

$$r(X \cap Y) \le |(X \cap Y) - (U \cap V)| + \sum r_i(U \cap V);$$

$$r(X \cup Y) \le |(X \cup Y) - (U \cup V)| + \sum r_i(U \cup V).$$

After applying the submodularity of each r_i and the diagram below, these inequalities yield $r(X \cap Y) + r(X \cup Y) \le r(X) + r(Y)$. ■



$$|(X \cap Y) - (U \cap V)| + |(X \cup Y) - (U \cup V)| = |X - U| + |Y - V|$$

In applying the Matroid Intersection Theorem, we needed N_1 to be a matroid, which required $\{M_i\}$ to be matroids. Hence this rank formula does not apply for unions of arbitrary hereditary systems.

The Matroid Union Theorem yields short proofs of min-max relations for packing and covering problems. In each formula below, the optimal subset is closed, since switching from X to $\sigma(X)$ improves the numerator without changing the denominator. The graph corollaries originally had difficult ad hoc proofs.

8.2.56. Corollary. (Matroid Covering Theorem—Edmonds [1965b]) In a loopless matroid M on E, the minimum number of independent sets whose union is E is $\max_{X\subseteq E}\left\lceil \frac{|X|}{r(X)} \right\rceil$.

Proof: Let $M_1, \ldots, M_k$ be copies of M on E. The set E is the union of k independent sets in M if and only if E is independent in $M' = M_1 \cup \cdots \cup M_k$. By the Matroid Union Theorem, $r'(E) \geq |E|$ is equivalent to $|E| - |Y| + \sum r_i(Y) \geq |E|$ for all $Y \subseteq E$. Since $r_i(Y) = r(Y)$ for all i, we conclude that E is the union of k independent sets if and only if $kr(Y) \geq |Y|$ for all $Y \subseteq E$. ∎

8.2.57. Corollary. (Nash-Williams [1964]) The minimum number of forests needed to cover the edges of a graph G (its **arboricity**) is $\max_{H\subseteq G}\left\lceil \frac{e(H)}{n(H)-1} \right\rceil$.

Proof: (Edmonds [1965b]) This follows immediately by applying Corollary 8.2.56 to $M(G)$. The best lower bound arises from a connected induced subgraph H (corresponding to a closed set in $M(G)$). ∎

8.2.58. Corollary. (Matroid Packing Theorem—Edmonds [1965c]) Given a matroid M on E, the maximum number of pairwise disjoint bases equals $\min_{X:\, r(X)<r(E)}\left\lfloor \frac{|E|-|X|}{r(E)-r(X)} \right\rfloor$.

Proof: The set E contains k disjoint bases if and only if $r'(E) \geq kr(E)$ in the union M' of k matroids $M_1, \ldots, M_k$ that are copies of M on E. By the Matroid Union Theorem, this requires $|E| - |Y| + \sum r_i(Y) \geq kr(E)$ for all $Y \subseteq E$. Since $r_i(Y) = r(Y)$ for all i, we conclude that k disjoint bases exist if and only if $|E| - |Y| \geq k(r(e) - r(Y))$ for all $Y \subseteq E$. ∎

8.2.59. Corollary. (Nash-Williams [1961], Tutte [1961a]) A graph G has k pairwise edge-disjoint spanning trees if and only if, for every vertex partition P, there are at least $k(|P| - 1)$ edges with endpoints in different sets of P.

Proof: (Edmonds [1965c]) We may assume that G is connected. By applying Corollary 8.2.58 to $M(G)$, we must determine when $|E| - |X| \geq k(r(E) - r(X))$ for each closed set X. The closed sets correspond to partitions of $V(G)$ into vertex sets inducing connected subgraphs. For each such partition $V_1, \ldots, V_p$, the corresponding closed set X is $\bigcup E(G[V_i])$ with rank $n - p$. Since $|E| - |X|$ counts the edges between sets of the partition and $r(E) - r(X) \geq p - 1$, the graph has k disjoint spanning trees if and only if the condition holds. ∎

EXERCISES

8.2.1. (−) Show that the stable sets of a graph need not be the independent sets of a matroid by finding vertex-weighted graphs where the ratio between the maximum weight of a stable set and the weight of a stable set found greedily is arbitrarily large.

8.2.2. (−) Characterize the graphs whose stable sets form the family of independent sets of a matroid on the set of vertices.

8.2.3. (−) Show that every partition matroid is a transversal matroid.

8.2.4. Modify the greedy algorithm to obtain (with proof) an algorithm for finding the maximum-weighted independent set in a matroid with arbitrary real weights (not necessarily nonnegative) on the elements.

8.2.5. Characterize the graphs whose matchings form the family of independent sets of a matroid on the set of edges.

8.2.6. (!) Determine which uniform matroids are graphic. Characterize the graphs whose cycle matroids are uniform matroids.

8.2.7. (!) Determine which partition matroids are graphic. Characterize the graphs whose cycle matroids are partition matroids.

8.2.8. Using only linear dependence, prove that vectorial matroids satisfy the induced circuit property: adding an element to a linearly independent set of vectors creates at most one minimal dependent set.

8.2.9. Describe the circuits of a transversal matroid M in terms of the corresponding bipartite graph G. Using only properties of bipartite graphs, prove that M satisfies the weak elimination property.

8.2.10. Let $M(G)$ be the cycle matroid of G. Let $k(X)$ be the number of components of the spanning subgraph G_X with edge set X; so $r(X) = n - k(X)$. Let U and V be the sets of components in G_X and G_Y, respectively. Let H be the U, V-bigraph with $u \leftrightarrow v$ when the components corresponding to u and v intersect.

a) Count the vertices and components of H in terms of the numbers $k(X)$, $k(Y)$, and $k(X \cap Y)$. Prove that $k(X \cup Y) \geq e(H)$.

b) Use part (a) to prove the submodularity property for $M(G)$ without using other properties of matroids. (Aigner [1979])

8.2.11. Use the König–Egerváry Theorem to prove directly that the rank function of a transversal matroid is submodular.

8.2.12. Let D be a digraph with distinguished source s and sink t. Let $E = V(D) - \{s, t\}$. For $X \subseteq E$, let $r(X)$ be the number of edges from $s \cup X$ to $\overline{X} \cup t$. Prove that r is submodular.

8.2.13. (−) For an element x in a hereditary system, prove that the following properties are equivalent and characterize loops.

a) $r(x) = 0$.
b) $x \in \sigma(\emptyset)$.
c) x is a circuit.
d) x belongs to no base.
e) Every set containing x is dependent.
f) x belongs to the span of every $X \subseteq E$.

8.2.14. (−) Prove equivalence of the following characterizations of parallel elements, assuming that $x \neq y$ and neither is a loop.

a) $r(x, y) = 1$.
b) $\{x, y\} \in \mathbf{C}$.
c) $x \in \sigma(y)$, $y \in \sigma(x)$, $r(x) = r(y) = 1$.

Furthermore, show that if x, y are parallel and $x \in \sigma(X)$, then $y \in \sigma(X)$.

8.2.15. (−) Suppose that $r(X) = r(X \cap Y)$ for some $X, Y \subseteq E$ in a matroid on E. Prove that $r(X \cup Y) = r(Y)$. Does the converse hold?

8.2.16. Let M be a hereditary system with nonnegative weights on E. Prove directly that if M satisfies the base exchange property (B), then the greedy algorithm always generates a maximum-weighted base.

8.2.17. *Alternative matroid axiomatics.* Let M be a hereditary system. Prove the following implications directly for M.

a) $(-)$ Submodularity (R) implies weak absorption (A).

b) Strong absorption (A′) implies submodularity (R) (without using uniformity). (Hint: Use induction on $|X \triangle Y|$.)

c) Base exchange (B) implies uniqueness of induced circuits (J).

d) $(-)$ Uniqueness of induced circuits (J) implies weak elimination (C).

e) Uniqueness of induced circuits (J) implies augmentation (I). (Hint: Use J and induction on $|I_1 - I_2|$ to obtain the augmentation.)

8.2.18. Prove that a hereditary system is a matroid if and only if it satisfies the "ultra-weak" augmentation property: If $I_1, I_2 \in \mathbf{I}$ with $|I_2| > |I_1|$ and $|I_1 - I_2| = 1$, then $I_1 + e \in \mathbf{I}$ for some $e \in I_2 - I_1$. (Chappell [1994a])

8.2.19. $(-)$ Let M be a matroid on E, and fix $A \subseteq E$. Obtain $\mathbf{I}'$ from $\mathbf{I}$ by deleting the sets that intersect A. Prove that $\mathbf{I}'$ is the family of independent sets of a matroid on E.

8.2.20. For a matroid on E with $e \notin B \in \mathbf{B}$, let $C(e, B)$ be the unique circuit in $B + e$.

a) For $e \notin B$, prove that $B - f + e$ is a base if and only if f belongs to $C(e, B)$.

b) For $e \in C \in \mathbf{C}$, prove that $C = C(e, B)$ for some base B.

8.2.21. $(-)$ Let B_1, B_2 be bases of a matroid such that $|B_1 \triangle B_2| = 2$. Prove that there is a unique circuit C such that $B_1 \triangle B_2 \subseteq C \subseteq B_1 \cup B_2$.

8.2.22. $(-)$ Let B_1, B_2 be bases in a matroid M. Given $X_1 \subseteq B_1$, prove that there exists $X_2 \subseteq B_2$ such that $(B_1 - X_1) \cup X_2$ and $(B_2 - X_2) \cup X_1$ are both bases of M. (Greene [1973])

8.2.23. (!) Let B_1, B_2 be distinct bases of a matroid M.

a) Let G be a B_1, B_2-bigraph with $e \in B_1$ adjacent to $f \in B_2$ when $B_2 + e - f \in \mathbf{B}$. Prove that G has a perfect matching.

b) Conclude from part (a) that there exists a bijection $\pi: B_1 \to B_2$ such that for each $e \in B_1$, the set $B_2 - \pi(e) + e$ is a base of M.

8.2.24. (!) Let B_1, B_2 be distinct bases of a matroid M.

a) Prove that for each $e \in B_1$, there is $f \in B_2$ such that $B_1 - e + f$ and $B_2 - f + e$ are bases. (Hint: Use the incorporation property. Note: This generalizes Exercise 2.1.34.)

b) Use the cycle matroid $M(K_4)$ to show that there may be no bijection $\pi: B_1 \to B_2$ such that e and $f = \pi(e)$ satisfy part (a) for all $e \in B_1$.

8.2.25. $(-)$ A collection of $|E| - r(E)$ circuits of a matroid on E form a **fundamental set of circuits** if it is possible to order the elements $e_1, \dots, e_n$ in such a way that C_i contains $e_{r(E)+i}$ but no higher-indexed element. Prove that every matroid has a fundamental set of circuits. (Whitney [1935])

8.2.26. $(-)$ Given k distinct circuits $\{C_i\}$ with none contained in the union of the others, and given a set X with $|X| < k$, prove that $\bigcup_{i=1}^k C_i - X$ contains a circuit. (Welsh [1976])

8.2.27. $(+)$ For a hereditary system, prove directly that the weak elimination property implies the strong elimination property, using induction on $|C_1 \cup C_2|$. (Lehman [1964])

8.2.28. (!) *Min-max relation for weighted independent set.* Let M be a matroid on E, with each $e \in E$ having nonnegative integer weight $w(e)$. Let $\mathbf{A}$ be the set of chains $X_1 \subseteq X_2 \subseteq \cdots$ such that each $e \in E$ appears in at least $w(e)$ sets in the chain (sets may repeat in the chain). Use the greedy algorithm to prove that

$$\max_{I \in \mathbf{I}} \sum_{e \in I} w(e) = \min_{\{X_i\} \in \mathbf{A}} \sum_i r(X_i).$$

8.2.29. (−) Let r and σ be the rank function and span function of a matroid. Prove that $r(X) = \min\{|Y|: Y \subseteq X, \sigma(Y) = \sigma(X)\}$.

8.2.30. Prove that a matroid of rank r has at least 2^r closed sets. (Lazarson [1957])

8.2.31. Prove that a matroid is simple if and only if 1) no element appears in every hyperplane, and 2) from every distinct pair of elements some hyperplane contains exactly one. Prove that these conditions also suffice for a family of sets to be the collection of hyperplanes of a simple matroid.

8.2.32. Prove that in a matroid, a set is a hypobase if and only if it is a hyperplane.

8.2.33. Use the weak elimination property to characterize when a family of sets is the family of hyperplanes of some matroid.

8.2.34. Prove that the closed sets of a matroid are the complements of the unions of cocircuits.

8.2.35. Let X be a closed set in a matroid M.

a) Let Y be a closed set contained in X such that $r(Y) = r(X) - 1$. Prove that M has a hyperplane H such that $Y = X \cap H$. (Hint: Given a maximal independent subset Z of Y, augment it by $e \in X$ and then to a base B, and let $H = \sigma(B - e)$.)

b) Prove that X is the intersection of $r(M) - r(X)$ distinct hyperplanes.

8.2.36. Prove the following properties of closed sets in a matroid.

a) The intersection of two closed sets is a closed set.

b) The span of a set is the intersection of all closed sets containing it. (Comment: Hence $\sigma(X)$ is the unique minimal closed set containing X.)

c) The union of two closed sets need not be a closed set.

8.2.37. Prove that $M.X$ has no loops if and only if $\overline{X}$ is closed.

8.2.38. (!) *Bases and cocircuits in matroids.*

a) Prove that when e belongs to a base B in a matroid M, there is exactly one cocircuit of M disjoint from $B - e$, and it contains e.

b) Use part (a) to prove that if C is a circuit of a matroid M and x, y are distinct elements of C, then there is a cocircuit $C^* \in \mathbf{C}^*$ with $C^* \cap C = \{x, y\}$. (Minty [1966])

c) Explain why part (b) is trivial for cycle matroids.

8.2.39. (−) Show that the dual of a simple matroid (no loops or parallel elements) need not be simple. Determine whether a set can be both a circuit and a cocircuit in a matroid.

8.2.40. (!) Use matroid duality to prove Euler's Formula for connected plane graphs.

8.2.41. Prove that any minor of a matroid obtained by restricting and then contracting can also be obtained by contracting and then restricting. In particular, if M is a matroid on E and $Y \subseteq X \subseteq E$, prove that $(M|X).Y = (M.\overline{X-Y})|Y$ and $(M.X)|Y = (M|\overline{X-Y}).Y$.

8.2.42. (!) Use duality and matroid restriction to prove that $r_{M.F}(X) = r_M(X \cup F) - r_M(F)$. Also derive the formula directly by proving that X is independent in $M.F$ if and only if adding X to $\overline{F}$ increases the rank by $|X|$.

8.2.43. Prove that the cycle matroid $M(G)$ is the column matroid over $\mathbb{Z}_2$ of the vertex-edge incidence matrix of G. (Hence every graphic matroid is binary.)

8.2.44. Tutte [1958] proved that a matroid if and only if it has no $U_{2,4}$-minor.

a) Prove that the matrix $\begin{pmatrix}1&0&1&1\\0&1&1&2\end{pmatrix}$ represents $U_{2,4}$ over $\mathbb{Z}_3$.

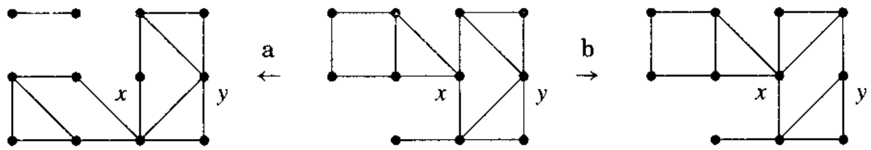
b) Prove that $U_{2,4}$ has no representation over $\mathbb{Z}_2$.

8.2.45. Prove that the three operations below preserve the cycle matroid of G.

a) Decompose G into its blocks $B_1, \dots, B_k$, and reassemble them to form another graph G' with blocks $B_1, \dots, B_k$.

b) In a block B of G that has a two-vertex cut $\{x, y\}$, interchange the neighbors of x and y in one of the components of $B - \{x, y\}$.

c) Add or delete isolated vertices.



(Comment: Whitney's 2-Isomorphism Theorem [1933b] states that G and H have the same cycle matroid if and only if some sequence of these operations turns G into H. Thus every 3-connected planar graph has only one dual graph, meaning essentially only one planar embedding. See also Kelmans [1980])

8.2.46. Construct a graph without isolated vertices that is an abstract dual of the graph below but is not a geometric dual of this graph. (Hint: Consider the operations of Exercise 8.2.45.) (Woodall, in Welsh [1976], p91–92)

8.2.47. The **matroid basis graph** is the graph having a vertex for each base of a matroid, with bases adjacent when their symmetric difference has size 2. Prove that every matroid basis graph has a spanning cycle, and interpret the result for graphic matroids and for uniform matroids. (Hint: Use contraction and restriction inductively to establish a spanning cycle through any edge.) (Holzmann–Harary [1972], Kung [1986, p72])

8.2.48. Use weak duality of linear programming to prove the weak duality property for matroid intersection: $|I| \le r_1(X) + r_2(\overline{X})$ for any $I \in \mathbf{I}_1 \cap \mathbf{I}_2$ and $X \subseteq E$. (Hint: Consider the discussion of dual pairs of linear programs in Remark 8.1.7.)

8.2.49. Let M_1, M_2 be two matroids on E.

a) Prove that the minimum size of a set in E that is spanning in both M_1 and M_2 is $\max_{X \subseteq E}(r_1(E) - r_1(X) + r_2(E) - r_2(\overline{X}))$.

b) Apply part (a) to prove that in a bipartite graph with no isolated vertices the minimum number of edges needed to cover all the vertices equals the maximum number of vertices with no edges among them. (König's "other" theorem)

c) From part (a), prove that the maximum size of a common independent set plus the minimum size of a common spanning set equals $r_1(E)+r_2(E)$. In particular, conclude Gallai's Theorem for bipartite graphs: in a bipartite graph with no isolated vertices, the maximum size of a matching plus the minimum number of edges needed to cover the vertices equals the number of vertices.

8.2.50. Use the Matroid Intersection Theorem to prove that in every acyclic orientation of G the vertices can be covered with at most $\alpha(G)$ pairwise-disjoint paths. (Chappell [1994b]) (Comment: This is the special case of Theorem 8.4.33 for acyclic digraphs.)

8.2.51. (−) Let M be the transversal matroid on $E = \cup A_i$ induced by sets $A_1, \dots, A_m$. Use Hall's Theorem for matchings in bipartite graphs to derive the rank function as $r(X) = \min_{Y \subseteq X}\{|X| - (|Y| - |N(Y)|)\}$.

8.2.52. Let G be an $E, [m]$-bigraph without isolated vertices. For $X \subseteq E$, let $r(X) = \min\{|N(J) \cap X| - |J| + m \colon J \subseteq [m]\}$. Prove that the following are equivalent for X.

A) Hall's Condition holds ($|N(S)| \ge |S|$ for all $S \subseteq X$).

B) $r(X) \ge |X|$.

C) X is saturated by some matching in G.

(Hint: The proof of B $\Rightarrow$ C uses paths from unsaturated vertices that alternate between edges outside and within a specified matching.)

8.2.53. (!) Let G be an $E, [m]$-bigraph without isolated vertices. For $X \subseteq E$ and $J \subseteq [m]$, let $g(X, J) = |N(J) \cap X| - |J|$, and let $r(X) = \min\{g(X, J) + m \colon J \subseteq [m]\}$. Say that J is *X-optimal* if $r(X) = g(X, J) + m$.

a) Prove that $r(\emptyset) = 0$ and that $r(X) \le r(X + e) \le r(X) + 1$.

b) Prove that r satisfies the weak absorption property.

8.2.54. Prove that restrictions and unions of transversal matroids are transversal matroids, but that contractions and duals of transversal matroids need not be.

8.2.55. *Gammoids.* Let D be a digraph, and let F, E be subsets of $V(D)$. The **gammoid** on E induced by D, F is the hereditary system given by $\mathbf{I} = \{X \subseteq E \colon$ there exist $|X|$ pairwise disjoint paths from F to $X\}$; equivalently, $r(X)$ is the maximum number of pairwise disjoint F, X-paths.

a) Verify that every transversal matroid is a gammoid.

b) (+) Prove that every gammoid is a matroid. (Hint: Use Menger's Theorem to verify the submodularity property. Verifying the augmentation property is also possible but somewhat longer.) (Mason [1972])

8.2.56. *Strict gammoids.* Let D be a directed graph, let F, E be subsets of the vertices of D, and let M be the gammoid on E induced by D, F (Exercise 8.2.55). When E consists of all vertices of D, the gammoid is a **strict gammoid**. Prove that a matroid is a strict gammoid if and only if it is the dual of a transversal matroid. (Hint: Use a natural correspondence between directed graphs on n vertices and bipartite graphs on $2n$ vertices.) (Ingleton–Piff [1973])

8.2.57. (−) Since the union of two matroids is a matroid, there should be a dual operation yielding its dual. Given matroids M_1, M_2 with spanning sets $\mathbf{S}_1, \mathbf{S}_1$, let $M_1 \wedge M_2$ be the hereditary system whose spanning sets are $\{X_1 \cap X_2 \colon X_1 \in \mathbf{S}_1, X_2 \in \mathbf{S}_2\}$. Prove that $M_1 \wedge M_2$ is the matroid $(M_1^* \cup M_2^*)^*$.

8.2.58. *Generalized transversal matroids.*

a) Let M be a matroid on E, and let $\mathbf{A} = \{A_1, \dots, A_m\}$ be a set system on E. Let M' be the hereditary system on $[m]$ whose independent sets are the subsets of $\mathbf{A}$ having transversals that belong to $\mathbf{I}_M$. Prove that M' is a matroid with rank function $r'(X) = \min_{Y \subseteq X}\{|X - Y| + r(A(Y))\}$.

b) Let E, F be finite sets, and let f be a function from E to F. For $X \subseteq E$, let $f(X)$ be the set of images of elements of X. Let M be a matroid on E. Let M' be the hereditary system on F defined by $\mathbf{I}_{M'} = \{f(X) \colon X \in \mathbf{I}_M\}$. Prove that M' is a matroid. Prove also that $r'(X) = \min_{Y \subseteq X}\{|X - Y| + r(f^{-1}(Y))\}$ when f is surjective.

8.2.59. Apply matroid sum and Exercise 8.2.58 to prove the Matroid Union Theorem.

8.2.60. (!) Prove that the maximum size of a common independent set in matroids M_1 and M_2 on E is $r_{M_1 \cup M_2^*}(E) - r_{M_2^*}(E)$. Use this to prove the Matroid Intersection Theorem by applying the Matroid Union Theorem to $M_1 \cup M_2^*$. (Comment: Thus these two theorems are equivalent.)

8.2.61. Let G be an n-vertex weighted graph, and let $E_1, \dots, E_{n-1}$ be a partition of $E(G)$ into $n-1$ sets. Is there a polynomial-time algorithm to compute a spanning tree of minimum weight among those that have exactly one edge in each subset E_i?

8.2.62. (!) Use the characterization of graphs having k pairwise edge-disjoint spanning trees (Corollary 8.2.59) to prove that every $2k$-edge-connected graph has k pairwise edge-disjoint spanning trees. Exhibit for each k a $2k$-edge-connected graph that does not have $k+1$ pairwise edge-disjoint spanning trees. (Nash-Williams [1961])

8.2.63. Given matroids $M_1, \dots, M_k$ on E, the **Matroid Partition Problem** is the problem of deciding whether an input set $X \subseteq E$ partitions into sets $I_1, \dots, I_k$ with $I_i \in \mathbf{I}_i$.

a) Use the Matroid Union Theorem to show that X is partitionable if and only if $|X - Y| + \sum r_i(Y) \geq |X|$ for all $Y \subseteq X$, and that all maximal partitionable sets are maximum partitionable sets.

b) Let M' be the union of k copies of a matroid M on E, and let X be a maximum partitionable set. Prove that there are disjoint sets $F_1, \dots, F_k \subseteq X$ such that $\{F_i\} \subseteq \mathbf{I}$ and $\overline{X} \subseteq \sigma(F_1) = \cdots = \sigma(F_k)$.

8.3. Ramsey Theory

"Ramsey theory" refers to the study of partitions of large structures. Typical results state that a special substructure must occur in some class of the partition. Motzkin described this by saying that "Complete disorder is impossible". The objects we consider are merely sets and numbers, and the techniques are little more than induction.

Ramsey's Theorem generalizes the pigeonhole principle, which itself concerns partitions of sets. We study applications of the pigeonhole principle, prove Ramsey's Theorem, and then focus on Ramsey-type questions for graphs. Finally, we discuss Sperner's Lemma about labelings of triangulations; like Ramsey's Theorem, it guarantees a special substructure.

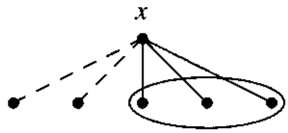
THE PIGEONHOLE PRINCIPLE REVISITED

The pigeonhole principle (Lemma A.57) states that if m objects are partitioned into n classes, then some class has at least $\lceil m/n \rceil$ objects (and some class has at most $\lfloor m/n \rfloor$ objects). This is a discrete version of the statement that every set of numbers contains a number at least as large as the average (and one at least as small). The concept is simple, but the applications can be quite subtle. The difficulty is how to define a partitioning problem relevant to the desired application. We illustrate this with four examples.

8.3.1. Proposition. Among six persons it is possible to find three mutual acquaintances or three mutual non-acquaintances.

Proof: (Exercise 1.1.29). In the language of graph theory, we are asked to show that for every simple graph G with six vertices, there is a triangle in G or in $\overline{G}$. The degrees of vertex x in G and $\overline{G}$ sum to 5, so the pigeonhole principle implies that one of them is at least 3.

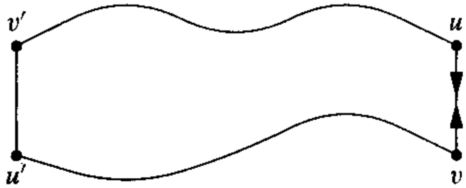
By symmetry, we may that assume $d_G(x) \geq 3$. If two neighbors of x are adjacent, then they form a triangle in G with x; otherwise, three neighbors of x form a triangle in $\overline{G}$. ∎



8.3.2. Theorem. (Graham–Entringer–Székely [1994]) If T is a spanning tree of the k-dimensional cube Q_k, then there is an edge of Q_k outside T whose addition to T creates a cycle of length at least $2k$.

Proof: For each vertex v of Q_k, expressed as a binary k-tuple, there is a complementary vertex v' that differs from v in each position. There is a unique v, v'-path in T; orient its first edge toward v'. Since $n(Q_k) = e(T) + 1$, doing this for each vertex orients some edge twice, by the pigeonhole principle.

Since this edge uv receives an orientation from u and from v, we have v on the u, u'-path and u on the v, v'-path in T. Hence the u, v'-path and the v, u'-path in T are disjoint. Each has length at least $k - 1$, since the distance in Q_k between a vertex and its complement is k. Finally, $u \leftrightarrow v$ in Q_k implies also $u' \leftrightarrow v'$, which completes a cycle of length at least $2k$. ∎



Theorem 8.3.2 implies that every spanning tree of Q_k has diameter at least $2k - 1$ (Graham–Harary [1992]).

8.3.3. Theorem. (Erdős–Szekeres [1935]) Every list of more than n^2 distinct numbers has a monotone sublist of length more than n.

Proof: Let $a = a_1, \ldots, a_{n^2+1}$ be the list. Assign position k the label (x_k, y_k), where x_k is the length of a longest increasing sublist ending at a_k, and y_k is the length of a longest decreasing sublist ending at a_k. If a has no monotone sublist of length $n + 1$, then x_k and y_k never exceed n, and there are only n^2 possible labels.

Since the list has length $n^2 + 1$, the pigeonhole principle now implies that two labels must be the same. This is impossible when the elements of a are distinct. When $i < j$ and $a_i < a_j$, we can append a_j to the longest increasing sequence ending at a_i. When $i < j$ and $a_i > a_j$, we can append a_j to the longest decreasing sequence ending at a_i. (See Exercise 5.1.43 for a generalization.) ∎

a:	7	4	1	8	5	2	9	6	3	0
x, y:	1, 1	1, 2	1, 3	2, 1	2, 2	2, 3	3, 1	3, 2	3, 3	4, 1

8.3.4. Theorem. (Graham–Kleitman [1973]) In every labeling of $E(K_n)$ using distinct integers, there is a trail of length at least $n-1$ along which the labels strictly increase.

Proof: We assign each vertex a weight equal to the length of the longest increasing trail ending there. If we can show that these n weights sum to at least $n(n-1)$, then the pigeonhole principle guarantees a vertex with a large enough weight. The problem is how to compute the weights and their sum.

We grow the graph from the trivial graph by adding the edges in order, updating the weights and their sum at each step. The vertex weights begin at 0. If the next edge joins two vertices whose weights were both i, then their weights both become $i+1$. If it joins two vertices of weights i and j with $i<j$, then their weights become $j+1$ and j.

In either case, each time an edge is added, the sum of the weights of the vertices increases by at least 2. Therefore, when the construction is finished, the sum of the vertex weights is at least $n(n-1)$. ∎

Finally, we note that the thresholds in the classes may differ.

8.3.5. Theorem. If $\sum p_i - k + 1$ objects are partitioned into k classes with quotas $\{p_i\}$, then some class must meet its quota.

Proof: If not, then at most $\sum(p_i - 1)$ objects can be accommodated. ∎

RAMSEY'S THEOREM

The pigeonhole principle guarantees a class with many objects when we partition objects into classes. The famous theorem of Ramsey [1930] makes a similar statement about partitioning the r-element subsets of objects into classes. Roughly put, Ramsey's Theorem says that whenever we partition the r-sets in a sufficiently large set S into k classes, there is a p-subset of S whose r-sets all lie in the same class.

A partition is a separation of a set into subsets, and the set we want to partition consists of subsets of another set, so for clarity we use the language of coloring instead of the language of partitioning. Recall that a **k-coloring** of a set is a partition of it into k classes. A class or its label is a **color**. Typically we use $[k]$ as the set of colors, in which case a k-coloring of X can be viewed as a function $f\colon X \to [k]$.

8.3.6. Definition. Let $\binom{S}{r}$ denote the set of r-element subsets (**r-sets**) of a set S. A set $T \subseteq S$ is **homogeneous** under a coloring of $\binom{S}{r}$ if all r-sets in T receive the same color; it is **i-homogeneous** if that color is i.

Let r and $p_1, \ldots, p_k$ be positive integers. If there is an integer N such that every k-coloring of $\binom{[N]}{r}$ yields an i-homogeneous set of size p_i for some i, then the smallest such integer is the **Ramsey number** $R(p_1, \ldots, p_k; r)$.

Ramsey's Theorem states that such an integer exists for every choice of r and $p_1, \dots, p_k$ (the latter are called **thresholds** or **quotas**). When the quotas all equal p, the theorem states that every k-coloring of the r-sets of a sufficiently large set has a p-set whose r-sets receive the same color. A thorough study of Ramsey's Theorem and other partitioning theorems appears in Graham–Rothschild–Spencer [1980, 1990].

Before proving the theorem, we consider the case $r = k = 2$, which is easy to describe in terms of edge-coloring of graphs. The proof for this case has the same structure as for the general case.

When $r = 2$, a k-partition of $\binom{S}{r}$ is merely a k-edge-coloring of the complete graph with vertex set S (not a proper edge-coloring). When $k = 2$, the time-honored tradition in Ramsey theory is that color 1 is "red" and color 2 is "blue".

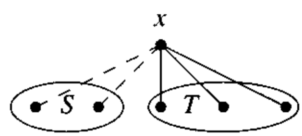
By Proposition 8.3.1, $R(3, 3; 2) \le 6$; we extend the argument to prove that

$$R(p_1, p_2; 2) \le R(p_1 - 1, p_2; 2) + R(p_1, p_2 - 1; 2).$$

Assuming that $R(p_1 - 1, p_2; 2)$ and $R(p_1, p_2 - 1; 2)$ exist, let N be their sum. Proving the bound for $R(p_1, p_2; 2)$ means showing that every red/blue-coloring of the edges of a complete graph with N vertices yields a p_1-set of vertices within which all edges are red or a p_2-set of vertices within which all edges are blue.

Consider a red/blue-coloring of K_N, and choose a vertex x. Let $s = R(p_1 - 1, p_2; 2)$ and $t = R(p_1, p_2 - 1; 2)$; there are $s + t - 1$ vertices other than x. Theorem 8.3.5 implies that x has at least s incident red edges or at least t incident blue edges.

By symmetry, we may assume that x has at least N incident red edges. By the definition of s, the complete subgraph induced by the neighbors of x along these edges has a blue p_2-clique or a red $p_1 - 1$-clique. The latter would combine with x to form a red p_1-clique. In either case, we obtain an i-homogeneous set of size p_i for some i. We postpone discussion of the resulting bound on $R(p_1, p_2; 2)$.



$|S| \ge R(p_1, p_2 - 1; 2)$ or $|T| \ge R(p_1 - 1, p_2; 2)$

8.3.7. Theorem. (Ramsey [1930]) Given positive integers r and $p_1, \dots, p_k$, there exists an integer N such that every k-coloring of $\binom{[N]}{r}$ yields an i-homogeneous set of size p_i for some i.

Proof: The proof is a "double" induction. We use induction on r, but the proof of the induction step itself uses induction on $\sum p_i$.

Basis step: $r = 1$. By Theorem 8.3.5, $R(p_1, \dots, p_k; 1)$ exists.

Induction step: $r > 1$. We assume that the claim in the theorem statement holds for k-colorings of the $r - 1$-subsets of a set, no matter what the thresholds are. We prove the same statement for k-colorings of the r-subsets of a set by induction on the sum of the quotas, $\sum p_i$.

Basis step: some quota p_i is less than r. In this case, a set of p_i objects contains no r-sets, so vacuously its r-sets all have color i. Hence $R(p_1, \ldots, p_k; r) = \min\{p_1, \ldots, p_k\}$ when $\min\{p_1, \ldots, p_k\} < r$.

For clarity, we state the induction step only for $k = 2$; the argument for general k is similar (Exercise 17). Write (p, q) for (p_1, p_2). Let

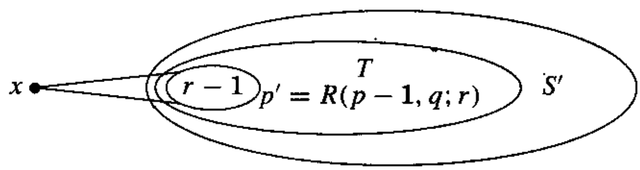
$$p' = R(p-1, q; r), \quad q' = R(p, q-1; r), \quad N = 1 + R(p', q'; r-1).$$

By the induction hypothesis of the inner induction, p' and q' exist. By the induction hypothesis of the outer induction, N also exists. Note that p' and q' may be very large; this is why we need the double induction.

Let S be a set of N elements, and choose $x \in S$. Consider a 2-coloring f of $\binom{S}{r}$. With colors (red,blue), we need to show that f has a red-homogeneous p-set or a blue-homogeneous q-set.

We use f to induce a 2-coloring f' of the $r-1$-sets of $S' = S - x$. This is the reason for our choice of $|S'|$ as a Ramsey number for $r-1$-sets. Define f' by assigning color i to an $(r-1)$-set in S' if its union with x has color i under f. Since $|S'| = R(p', q'; r-1)$, the induction hypothesis implies that some color meets its quota (p' or q') under f' (when $r = 2$, this step was the invocation of the pigeonhole principle). By symmetry, we may assume that the red quota is met. Let T be a p'-element subset of S' whose $r-1$-sets are red under f'.

We return to the original coloring f on the r-sets in T. Since $|T| = p' = R(p-1, q; r)$, under f there is a red-homogeneous $p-1$-set or a blue-homogeneous q-set in T. If there is a blue-homogeneous q-set, then we are done. If there is a red-homogeneous $p-1$-set P, then consider $P \cup \{x\}$. From the definition of T, the $(r-1)$-sets of P are all red under f', which means their unions with x are red under f. Hence $P \cup \{x\}$ is a red-homogeneous p-set under f. ∎



Like the pigeonhole principle, Ramsey's Theorem has subtle and fascinating applications. Ramsey's Theorem typically gives an elegant existence proof but a horribly large bound.

8.3.8. Theorem. (Erdős–Szekeres [1935]) Given an integer m, there exists a (least) integer $N(m)$ such that every set of at least $N(m)$ points in the plane with no three collinear contains an m-subset forming a convex m-gon.

Proof: We need two facts. *(1) Among five points in the plane, four determine a convex quadrilateral* (if no three are collinear). Construct the convex hull of the five points. If it is a pentagon or a quadrilateral, then the result follows