Since $y = x + a$ and $w = z + b$, we have

$$yw = (x + a)(z + b) = xz + az + xb + ab.$$

Thus

$$|yw - xz| = |az + bx + ab| \le |az| + |bx| + |ab| = |a||z| + |b||x| + |a||b|.$$

Since $|a| \le \varepsilon$ and $|b| \le \delta$, we thus have

$$|yw - xz| \le \varepsilon|z| + \delta|x| + \varepsilon\delta$$

and thus that yw and xz are $(\varepsilon|z| + \delta|x| + \varepsilon\delta)$-close. □

Remark 4.3.8. One should compare statements (a)-(c) of this proposition with the reflexive, symmetric, and transitive axioms of equality. It is often useful to think of the notion of "ε-close" as an approximate substitute for that of equality in analysis.

Now we recursively define exponentiation for natural number exponents, extending the previous definition in Definition 2.3.11.

Definition 4.3.9 (Exponentiation to a natural number). Let x be a rational number. To raise x to the power 0, we define $x^0 := 1$. Now suppose inductively that x^n has been defined for some natural number n, then we define $x^{n+1} := x^n \times x$.

Proposition 4.3.10 (Properties of exponentiation, I). *Let x, y be rational numbers, and let n, m be natural numbers.*

(a) *We have $x^n x^m = x^{n+m}$, $(x^n)^m = x^{nm}$, and $(xy)^n = x^n y^n$.*

(b) *We have $x^n = 0$ if and only if $x = 0$.*

(c) *If $x \ge y \ge 0$, then $x^n \ge y^n \ge 0$. If $x > y \ge 0$ and $n > 0$, then $x^n > y^n \ge 0$.*

(d) *We have $|x^n| = |x|^n$.*

Proof. See Exercise 4.3.3. □

Now we define exponentiation for negative integer exponents.

Definition 4.3.11 (Exponentiation to a negative number). Let x be a non-zero rational number. Then for any negative integer $-n$, we define $x^{-n} := 1/x^n$.

Thus for instance $x^{-3} = 1/x^3 = 1/(x \times x \times x)$. We now have x^n defined for any integer n, whether n is positive, negative, or zero. Exponentiation with integer exponents has the following properties (which supercede Proposition 4.3.10):

Proposition 4.3.12 (Properties of exponentiation, II). *Let x, y be non-zero rational numbers, and let n, m be integers.*

(a) *We have $x^n x^m = x^{n+m}$, $(x^n)^m = x^{nm}$, and $(xy)^n = x^n y^n$.*

(b) *If $x \geq y > 0$, then $x^n \geq y^n > 0$ if n is positive, and $0 < x^n \leq y^n$ if n is negative.*

(c) *If $x, y > 0$, $n \neq 0$, and $x^n = y^n$, then $x = y$.*

(d) *We have $|x^n| = |x|^n$.*

Proof. See Exercise 4.3.4. □

Exercise 4.3.1. Prove Proposition 4.3.3. (Hint: while all of these claims can be proven by dividing into cases, such as when x is positive, negative, or zero, several parts of the proposition can be proven without such a tedious division into cases. For instance one can use earlier parts of the proposition to prove later ones.)

Exercise 4.3.2. Prove the remaining claims in Proposition 4.3.7.

Exercise 4.3.3. Prove Proposition 4.3.10. (Hint: use induction.)

Exercise 4.3.4. Prove Proposition 4.3.12. (Hint: induction is not suitable here. Instead, use Proposition 4.3.10.)

Exercise 4.3.5. Prove that $2^N \geq N$ for all positive integers N. (Hint: use induction.)

4.4 Gaps in the rational numbers

Imagine that we arrange the rationals on a line, arranging x to the right of y if $x > y$. (This is a non-rigourous arrangement, since we have not yet defined the concept of a line, but this discussion is only intended to motivate the more rigourous propositions below.) Inside the rationals we have the integers, which are thus also arranged on the line. Now we work out how the rationals are arranged with respect to the integers.

Proposition 4.4.1 (Interspersing of integers by rationals). *Let x be a rational number. Then there exists an integer n such that $n \leq x < n+1$. In fact, this integer is unique (i.e., for each x there is only one n for which $n \leq x < n+1$). In particular, there exists a natural number N such that $N > x$ (i.e., there is no such thing as a rational number which is larger than all the natural numbers).*

Remark 4.4.2. The integer n for which $n \leq x < n+1$ is sometimes referred to as the *integer part* of x and is sometimes denoted $n = \lfloor x \rfloor$.

Proof. See Exercise 4.4.1. □

Also, between every two rational numbers there is at least one additional rational:

Proposition 4.4.3 (Interspersing of rationals by rationals). *If x and y are two rationals such that $x < y$, then there exists a third rational z such that $x < z < y$.*

Proof. We set $z := (x + y)/2$. Since $x < y$, and $1/2 = 1//2$ is positive, we have from Proposition 4.2.9 that $x/2 < y/2$. If we add $y/2$ to both sides using Proposition 4.2.9 we obtain $x/2 + y/2 < y/2 + y/2$, i.e., $z < y$. If we instead add $x/2$ to both sides we obtain $x/2 + x/2 < y/2 + x/2$, i.e., $x < z$. Thus $x < z < y$ as desired. □

Despite the rationals having this denseness property, they are still incomplete; there are still an infinite number of "gaps" or "holes" between the rationals, although this denseness property does ensure that these holes are in some sense infinitely small. For instance, we will now show that the rational numbers do not contain any square root of two.

Proposition 4.4.4. *There does not exist any rational number x for which $x^2 = 2$.*

Proof. We only give a sketch of a proof; the gaps will be filled in Exercise 4.4.3. Suppose for sake of contradiction that we had a rational number x for which $x^2 = 2$. Clearly x is not zero. We may assume that x is positive, for if x were negative then we could just replace x by $-x$ (since $x^2 = (-x)^2$). Thus $x = p/q$ for some positive integers p, q, so $(p/q)^2 = 2$, which we can rearrange as $p^2 = 2q^2$. Define a natural number p to be *even* if $p = 2k$ for some natural number k, and *odd* if $p = 2k + 1$ for some natural number k. Every natural number is either even or odd, but not both (why?). If p is odd, then p^2 is also odd (why?), which contradicts $p^2 = 2q^2$. Thus p is even, i.e., $p = 2k$ for some natural number k. Since p is positive, k must also be positive. Inserting $p = 2k$ into $p^2 = 2q^2$ we obtain $4k^2 = 2q^2$, so that $q^2 = 2k^2$.

To summarize, we started with a pair (p, q) of positive integers such that $p^2 = 2q^2$, and ended up with a pair (q, k) of positive integers such that $q^2 = 2k^2$. Since $p^2 = 2q^2$, we have $q < p$ (why?). If we rewrite $p' := q$ and $q' := k$, we thus can pass from one solution (p, q) to the equation $p^2 = 2q^2$ to a new solution (p', q') to the same equation which has a smaller value of p. But then we can repeat this procedure again and again, obtaining a sequence (p'', q''), (p''', q'''), etc. of solutions to $p^2 = 2q^2$, each one with a smaller value of p than the previous, and each one consisting of positive integers. But this contradicts the principle of infinite descent (see Exercise 4.4.2). This contradiction shows that we could not have had a rational x for which $x^2 = 2$. $\square$

On the other hand, we can get rational numbers which are arbitrarily close to a square root of 2:

Proposition 4.4.5. *For every rational number $\varepsilon > 0$, there exists a non-negative rational number x such that $x^2 < 2 < (x+\varepsilon)^2$.*

Proof. Let $\varepsilon > 0$ be rational. Suppose for sake of contradiction that there is no non-negative rational number x for which $x^2 < 2 < (x+\varepsilon)^2$. This means that whenever x is non-negative and $x^2 < 2$, we must also have $(x+\varepsilon)^2 < 2$ (note that $(x+\varepsilon)^2$ cannot equal 2, by Proposition 4.4.4). Since $0^2 < 2$, we thus have $\varepsilon^2 < 2$, which then implies $(2\varepsilon)^2 < 2$, and indeed a simple induction shows that $(n\varepsilon)^2 < 2$ for every natural number n. (Note that $n\varepsilon$ is non-negative for every natural number n - why?) But, by Proposition 4.4.1 we can find an integer n such that $n > 2/\varepsilon$, which implies that $n\varepsilon > 2$, which implies that $(n\varepsilon)^2 > 4 > 2$, contradicting the claim that $(n\varepsilon)^2 < 2$ for all natural numbers n. This contradiction gives the proof. $\square$

Example 4.4.6. If[3] $\varepsilon = 0.001$, we can take $x = 1.414$, since $x^2 = 1.999396$ and $(x+\varepsilon)^2 = 2.002225$.

Proposition 4.4.5 indicates that, while the set $\mathbf{Q}$ of rationals does not actually have $\sqrt{2}$ as a member, we can get as close as we wish to $\sqrt{2}$. For instance, the sequence of rationals

$$1.4, 1.41, 1.414, 1.4142, 1.41421, \ldots$$

seem to get closer and closer to $\sqrt{2}$, as their squares indicate:

$$1.96, 1.9881, 1.99396, 1.99996164, 1.9999899241, \ldots$$

Thus it seems that we can create a square root of 2 by taking a "limit" of a sequence of rationals. This is how we shall construct the real numbers in the next chapter. (There is another way to do so, using something called "Dedekind cuts", which we will not pursue here. One can also proceed using infinite decimal expansions, but there are some sticky issues when doing so, e.g., one

[3] We will use the decimal system for defining terminating decimals, for instance 1.414 is defined to equal the rational number 1414/1000. We defer the formal discussion on the decimal system to an Appendix (§B).

has to make $0.999\ldots$ equal to $1.000\ldots$, and this approach, despite being the most familiar, is actually *more* complicated than other approaches; see the Appendix §B.)

Exercise 4.4.1. Prove Proposition 4.4.1. (Hint: use Proposition 2.3.9.)

Exercise 4.4.2. A definition: a sequence $a_0, a_1, a_2, \ldots$ of numbers (natural numbers, integers, rationals, or reals) is said to be in *infinite descent* if we have $a_n > a_{n+1}$ for all natural numbers n (i.e., $a_0 > a_1 > a_2 > \ldots$).

(a) Prove the *principle of infinite descent*: that it is not possible to have a sequence of *natural numbers* which is in infinite descent. (Hint: assume for sake of contradiction that you can find a sequence of natural numbers which is in infinite descent. Since all the a_n are natural numbers, you know that $a_n \geq 0$ for all n. Now use induction to show in fact that $a_n \geq k$ for all $k \in \mathbf{N}$ and all $n \in \mathbf{N}$, and obtain a contradiction.)

(b) Does the principle of infinite descent work if the sequence a_1, a_2, $a_3, \ldots$ is allowed to take integer values instead of natural number values? What about if it is allowed to take positive rational values instead of natural numbers? Explain.

Exercise 4.4.3. Fill in the gaps marked (why?) in the proof of Proposition 4.4.4.

Chapter 5

The real numbers

To review our progress to date, we have rigourously constructed three fundamental number systems: the natural number system $\mathbf{N}$, the integers $\mathbf{Z}$, and the rationals $\mathbf{Q}$[1]. We defined the natural numbers using the five Peano axioms, and postulated that such a number system existed; this is plausible, since the natural numbers correspond to the very intuitive and fundamental notion of *sequential counting*. Using that number system one could then recursively define addition and multiplication, and verify that they obeyed the usual laws of algebra. We then constructed the integers by taking formal[2] differences of the natural numbers, a—b. We then constructed the rationals by taking formal quotients of the integers, $a//b$, although we need to exclude division by zero in order to keep the laws of algebra reasonable. (You are of course free to design your own number system, possibly including one where division by zero is permitted; but you will have to give up one

[1]The symbols $\mathbf{N}$, $\mathbf{Q}$, and $\mathbf{R}$ stand for "natural", "quotient", and "real" respectively. $\mathbf{Z}$ stands for "Zahlen", the German word for number. There is also the *complex numbers* $\mathbf{C}$, which obviously stands for "complex".

[2]*Formal* means "having the form of"; at the beginning of our construction the expression a—b did not actually *mean* the difference $a - b$, since the symbol — was meaningless. It only had the *form* of a difference. Later on we defined subtraction and verified that the formal difference was equal to the actual difference, so this eventually became a non-issue, and our symbol for formal differencing was discarded. Somewhat confusingly, this use of the term "formal" is unrelated to the notions of a formal argument and an informal argument.

or more of the field axioms from Proposition 4.2.4, among other things, and you will probably get a less useful number system in which to do any real-world problems.)

The rational system is already sufficient to do a lot of mathematics - much of high school algebra, for instance, works just fine if one only knows about the rationals. However, there is a fundamental area of mathematics where the rational number system does not suffice - that of *geometry* (the study of lengths, areas, etc.). For instance, a right-angled triangle with both sides equal to 1 gives a hypotenuse of $\sqrt{2}$, which is an *irrational* number, i.e., not a rational number; see Proposition 4.4.4. Things get even worse when one starts to deal with the sub-field of geometry known as *trigonometry*, when one sees numbers such as π or $\cos(1)$, which turn out to be in some sense "even more" irrational than $\sqrt{2}$. (These numbers are known as *transcendental numbers*, but to discuss this further would be far beyond the scope of this text.) Thus, in order to have a number system which can adequately describe geometry - or even something as simple as measuring lengths on a line - one needs to replace the rational number system with the real number system. Since differential and integral calculus is also intimately tied up with geometry - think of slopes of tangents, or areas under a curve - calculus also requires the real number system in order to function properly.

However, a rigourous way to construct the reals from the rationals turns out to be somewhat difficult - requiring a bit more machinery than what was needed to pass from the naturals to the integers, or the integers to the rationals. In those two constructions, the task was to introduce one more *algebraic* operation to the number system - e.g., one can get integers from naturals by introducing subtraction, and get the rationals from the integers by introducing division. But to get the reals from the rationals is to pass from a "discrete" system to a "continuous" one, and requires the introduction of a somewhat different notion - that of a *limit*. The limit is a concept which on one level is quite intuitive, but to pin down rigourously turns out to be quite difficult. (Even such great mathematicians as Euler and Newton had difficulty with this concept. It was only in the nineteenth century that mathematicians such as Cauchy and Dedekind figured out how to deal with limits rigourously.)

In Section 4.4 we explored the "gaps" in the rational numbers; now we shall fill in these gaps using limits to create the real numbers. The real number system will end up being a lot like the rational numbers, but will have some new operations - notably that of *supremum*, which can then be used to define limits and thence to everything else that calculus needs.

The procedure we give here of obtaining the real numbers as the limit of sequences of rational numbers may seem rather complicated. However, it is in fact an instance of a very general and useful procedure, that of *completing* one metric space to form another; see Exercise 12.4.8.

5.1 Cauchy sequences

Our construction of the real numbers shall rely on the concept of a *Cauchy sequence.* Before we define this notion formally, let us first define the concept of a sequence.

Definition 5.1.1 (Sequences). Let m be an integer. A *sequence* $(a_n)_{n=m}^\infty$ *of rational numbers* is any function from the set $\{n \in \mathbf{Z} : n \geq m\}$ to $\mathbf{Q}$, i.e., a mapping which assigns to each integer n greater than or equal to m, a rational number a_n. More informally, a sequence $(a_n)_{n=m}^\infty$ of rational numbers is a collection of rationals $a_m, a_{m+1}, a_{m+2}, \ldots$.

Example 5.1.2. The sequence $(n^2)_{n=0}^\infty$ is the collection 0, 1, 4, 9,... of natural numbers; the sequence $(3)_{n=0}^\infty$ is the collection 3, 3, 3,... of natural numbers. These sequences are indexed starting from 0, but we can of course make sequences starting from 1 or any other number; for instance, the sequence $(a_n)_{n=3}^\infty$ denotes the sequence $a_3, a_4, a_5, \ldots$, so $(n^2)_{n=3}^\infty$ is the collection $9, 16, 25, \ldots$ of natural numbers.

We want to define the real numbers as the limits of sequences of rational numbers. To do so, we have to distinguish which sequences of rationals are convergent and which ones are not. For instance, the sequence

$$1.4, 1.41, 1.414, 1.4142, 1.41421, \ldots$$

looks like it is trying to converge to something, as does

$$0.1, 0.01, 0.001, 0.0001, \ldots$$

while other sequences such as

$$1, 2, 4, 8, 16, \ldots$$

or

$$1, 0, 1, 0, 1, \ldots$$

do not. To do this we use the definition of ε-closeness defined earlier. Recall from Definition 4.3.4 that two rational numbers x, y are ε-close if $d(x, y) = |x - y| \leq \varepsilon$.

Definition 5.1.3 (ε-steadiness)**.** Let $\varepsilon > 0$. A sequence $(a_n)_{n=0}^{\infty}$ is said to be *ε-steady* iff each pair a_j, a_k of sequence elements is ε-close for every natural number j, k. In other words, the sequence $a_0, a_1, a_2, \ldots$ is ε-steady iff $d(a_j, a_k) \leq \varepsilon$ for all j, k.

Remark 5.1.4. This definition is not standard in the literature; we will not need it outside of this section; similarly for the concept of "eventual ε-steadiness" below. We have defined ε-steadiness for sequences whose index starts at 0, but clearly we can make a similar notion for sequences whose indices start from any other number: a sequence $a_N, a_{N+1}, \ldots$ is ε-steady if one has $d(a_j, a_k) \leq \varepsilon$ for all $j, k \geq N$.

Example 5.1.5. The sequence $1, 0, 1, 0, 1, \ldots$ is 1-steady, but is not 1/2-steady. The sequence $0.1, 0.01, 0.001, 0.0001, \ldots$ is 0.1-steady, but is not 0.01-steady (why?). The sequence 1, 2, 4, 8, 16, ... is not ε-steady for any ε (why?). The sequence $2, 2, 2, 2, \ldots$ is ε-steady for every $\varepsilon > 0$.

The notion of ε-steadiness of a sequence is simple, but does not really capture the *limiting* behavior of a sequence, because it is too sensitive to the initial members of the sequence. For instance, the sequence

$$10, 0, 0, 0, 0, 0, \ldots$$

is 10-steady, but is not ε-steady for any smaller value of ε, despite the sequence converging almost immediately to zero. So we need a more robust notion of steadiness that does not care about the initial members of a sequence.

Definition 5.1.6 (Eventual ε-steadiness). Let $\varepsilon > 0$. A sequence $(a_n)_{n=0}^{\infty}$ is said to be *eventually ε-steady* iff the sequence $a_N, a_{N+1}, a_{N+2}, \ldots$ is ε-steady for some natural number $N \geq 0$. In other words, the sequence $a_0, a_1, a_2, \ldots$ is eventually ε-steady iff there exists an $N \geq 0$ such that $d(a_j, a_k) \leq \varepsilon$ for all $j, k \geq N$.

Example 5.1.7. The sequence $a_1, a_2, \ldots$ defined by $a_n := 1/n$, (i.e., the sequence $1, 1/2, 1/3, 1/4, \ldots$) is not 0.1-steady, but is eventually 0.1-steady, because the sequence $a_{10}, a_{11}, a_{12}, \ldots$ (i.e., $1/10, 1/11, 1/12, \ldots$) is 0.1-steady. The sequence $10, 0, 0, 0, 0, \ldots$ is not ε-steady for any ε less than 10, but it is eventually ε-steady for every $\varepsilon > 0$ (why?).

Now we can finally define the correct notion of what it means for a sequence of rationals to "want" to converge.

Definition 5.1.8 (Cauchy sequences). A sequence $(a_n)_{n=0}^{\infty}$ of rational numbers is said to be a *Cauchy sequence* iff for every rational $\varepsilon > 0$, the sequence $(a_n)_{n=0}^{\infty}$ is eventually ε-steady. In other words, the sequence $a_0, a_1, a_2, \ldots$ is a Cauchy sequence iff for every $\varepsilon > 0$, there exists an $N \geq 0$ such that $d(a_j, a_k) \leq \varepsilon$ for all $j, k \geq N$.

Remark 5.1.9. At present, the parameter ε is restricted to be a positive rational; we cannot take ε to be an arbitrary positive real number, because the real numbers have not yet been constructed. However, once we do construct the real numbers, we shall see that the above definition will not change if we require ε to be real instead of rational (Proposition 6.1.4). In other words, we will eventually prove that a sequence is eventually ε-steady for every rational $\varepsilon > 0$ if and only if it is eventually ε-steady for every real $\varepsilon > 0$; see Proposition 6.1.4. This rather subtle distinction between a rational ε and a real ε turns out not to be very important in the long run, and the reader is advised not to pay too much attention as to what type of number ε should be.

Example 5.1.10. (Informal) Consider the sequence

$$1.4, 1.41, 1.414, 1.4142, \ldots$$

mentioned earlier. This sequence is already 1-steady. If one discards the first element 1.4, then the remaining sequence

$$1.41, 1.414, 1.4142, \ldots$$

is now 0.1-steady, which means that the original sequence was eventually 0.1-steady. Discarding the next element gives the 0.01-steady sequence $1.414, 1.4142, \ldots$; thus the original sequence was eventually 0.01-steady. Continuing in this way it seems plausible that this sequence is in fact ε-steady for every $\varepsilon > 0$, which seems to suggest that this is a Cauchy sequence. However, this discussion is not rigourous for several reasons, for instance we have not precisely defined what this sequence $1.4, 1.41, 1.414, \ldots$ really is. An example of a rigourous treatment follows next.

Proposition 5.1.11. *The sequence $a_1, a_2, a_3, \ldots$ defined by $a_n := 1/n$ (i.e., the sequence $1, 1/2, 1/3, \ldots$) is a Cauchy sequence.*

Proof. We have to show that for every $\varepsilon > 0$, the sequence a_1, $a_2, \ldots$ is eventually ε-steady. So let $\varepsilon > 0$ be arbitrary. We now have to find a number $N \geq 1$ such that the sequence a_N, $a_{N+1}, \ldots$ is ε-steady. Let us see what this means. This means that $d(a_j, a_k) \leq \varepsilon$ for every $j, k \geq N$, i.e.

$$|1/j - 1/k| \leq \varepsilon \text{ for every } j, k \geq N.$$

Now since $j, k \geq N$, we know that $0 < 1/j, 1/k \leq 1/N$, so that $|1/j - 1/k| \leq 1/N$. So in order to force $|1/j - 1/k|$ to be less than or equal to ε, it would be sufficient for $1/N$ to be less than ε. So all we need to do is choose an N such that $1/N$ is less than ε, or in other words that N is greater than $1/\varepsilon$. But this can be done thanks to Proposition 4.4.1. $\square$

As you can see, verifying from first principles (i.e., without using any of the machinery of limits, etc.) that a sequence is a Cauchy sequence requires some effort, even for a sequence as simple as $1/n$. The part about selecting an N can be particularly difficult for beginners - one has to think in reverse, working out what conditions on N would suffice to force the sequence $a_N, a_{N+1}, a_{N+2}, \ldots$ to be ε-steady, and then finding an N which obeys those conditions. Later we will develop some limit laws which allow us to determine when a sequence is Cauchy more easily.

We now relate the notion of a Cauchy sequence to another basic notion, that of a bounded sequence.

Definition 5.1.12 (Bounded sequences). Let $M \geq 0$ be rational. A finite sequence $a_1, a_2, \ldots, a_n$ is *bounded by M* iff $|a_i| \leq M$ for all $1 \leq i \leq n$. An infinite sequence $(a_n)_{n=1}^{\infty}$ is *bounded by M* iff $|a_i| \leq M$ for all $i \geq 1$. A sequence is said to be *bounded* iff it is bounded by M for some rational $M \geq 0$.

Example 5.1.13. The finite sequence $1, -2, 3, -4$ is bounded (in this case, it is bounded by 4, or indeed by any M greater than or equal to 4). But the infinite sequence $1, -2, 3, -4, 5, -6, \ldots$ is unbounded. (Can you prove this? Use Proposition 4.4.1.) The sequence $1, -1, 1, -1, \ldots$ is bounded (e.g., by 1), but is not a Cauchy sequence.

Lemma 5.1.14 (Finite sequences are bounded). *Every finite sequence $a_1, a_2, \ldots, a_n$ is bounded.*

Proof. We prove this by induction on n. When $n = 1$ the sequence a_1 is clearly bounded, for if we choose $M := |a_1|$ then clearly we have $|a_i| \leq M$ for all $1 \leq i \leq n$. Now suppose that we have already proved the lemma for some $n \geq 1$; we now prove it for $n+1$, i.e., we prove every sequence $a_1, a_2, \ldots, a_{n+1}$ is bounded. By the induction hypothesis we know that $a_1, a_2, \ldots, a_n$ is bounded by some $M \geq 0$; in particular, it must be bounded by $M + |a_{n+1}|$. On the other hand, a_{n+1} is also bounded by $M + |a_{n+1}|$. Thus $a_1, a_2, \ldots, a_n, a_{n++}$ is bounded by $M + |a_{n+1}|$, and is hence bounded. This closes the induction. □

Note that while this argument shows that every finite sequence is bounded, no matter how long the finite sequence is, it does not say anything about whether an infinite sequence is bounded or not; infinity is not a natural number. However, we have

Lemma 5.1.15 (Cauchy sequences are bounded). *Every Cauchy sequence $(a_n)_{n=1}^\infty$ is bounded.*

Proof. See Exercise 5.1.1. □

Exercise 5.1.1. Prove Lemma 5.1.15. (Hint: use the fact that a_n is eventually 1-steady, and thus can be split into a finite sequence and a 1-steady sequence. Then use Lemma 5.1.14 for the finite part. Note there is nothing special about the number 1 used here; any other positive number would have sufficed.)

5.2 Equivalent Cauchy sequences

Consider the two Cauchy sequences of rational numbers:

$$1.4, 1.41, 1.414, 1.4142, 1.41421, \ldots$$

and

$$1.5, 1.42, 1.415, 1.4143, 1.41422, \ldots$$

Informally, both of these sequences seem to be converging to the same number, the square root $\sqrt{2} = 1.41421\ldots$ (though this statement is not yet rigourous because we have not defined real numbers yet). If we are to define the real numbers from the rationals as limits of Cauchy sequences, we have to know when two Cauchy sequences of rationals give the same limit, without first defining