

The Jacobson radical is analogous to the Frattini subgroup of a group, and it enjoys some corresponding properties (cf. Exercise 24 in Section 6.1):

**Proposition 1.** Let  $\mathcal{J}$  be the Jacobson radical of the commutative ring  $R$ .

- (1) If  $I$  is a proper ideal of  $R$ , then so is  $(I, \mathcal{J})$ , the ideal generated by  $I$  and  $\mathcal{J}$ .
- (2) The Jacobson radical contains the nilradical of  $R$ :  $\text{rad } 0 \subseteq \text{Jac } R$ .
- (3) An element  $x$  belongs to  $\mathcal{J}$  if and only if  $1 - rx$  is a unit for all  $r \in R$ .
- (4) (*Nakayama's Lemma*) If  $M$  is any finitely generated  $R$ -module and  $\mathcal{J}M = M$ , then  $M = 0$ .

*Proof:* If  $I$  is a proper ideal in  $R$ , then  $I \subseteq M$  for some maximal ideal  $M$ . Since  $\mathcal{J} \subseteq M$ , also  $(I, \mathcal{J}) \subseteq M$ , which proves (1).

Part (2) follows from the definitions of the two radicals and Proposition 12 in Section 15.2 since maximal ideals are prime.

Suppose  $1 - rx$  is not a unit and let  $M$  be a maximal ideal containing  $1 - rx$ . Since  $1 \notin M$ ,  $rx \notin M$ , so  $x$  cannot belong to  $\mathcal{J}$  because  $\mathcal{J} \subseteq M$ . Conversely, suppose  $x \notin \mathcal{J}$ , i.e., there is a maximal ideal  $M$  with  $x \notin M$ . Then  $R = (x, M)$ , hence  $1 = rx + y$  for some  $y \in M$ . Thus  $1 - rx = y \in M$  and so  $1 - rx$  is not a unit, which proves (3).

To prove (4), assume  $M \neq 0$  and let  $n$  be the smallest integer such that  $M$  is generated by  $n$  elements, say  $m_1, \dots, m_n$ . Since  $M = \mathcal{J}M$  we have

$$m_n = r_1 m_1 + r_2 m_2 + \cdots + r_n m_n \quad \text{for some } r_1, r_2, \dots, r_n \in \mathcal{J}.$$

Thus  $(1 - r_n)m_n = r_1 m_1 + \cdots + r_{n-1} m_{n-1}$ . By (3),  $1 - r_n$  is a unit, so  $m_n$  lies in the module generated by  $m_1, \dots, m_{n-1}$ , contradicting the minimality of  $n$ . Hence  $M = 0$ , completing the proof.

**Definition.** A commutative ring  $R$  is said to be *Artinian* or to satisfy the *descending chain condition on ideals* (or *D.C.C. on ideals*) if there is no infinite decreasing chain of ideals in  $R$ , i.e., whenever  $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$  is a decreasing chain of ideals of  $R$ , then there is a positive integer  $m$  such that  $I_k = I_m$  for all  $k \geq m$ . Similarly, an  $R$ -module  $M$  is said to be Artinian if it satisfies D.C.C. on submodules.

It is immediate from the Lattice Isomorphism Theorem that every quotient  $R/I$  of an Artinian ring  $R$  by an ideal  $I$  is again an Artinian ring.

The following result for Artinian rings is parallel to results in Theorem 15.2. The proof is completely analogous, and so is left as an exercise.

**Proposition 2.** The following are equivalent:

- (1)  $R$  is an Artinian ring.
- (2) Every nonempty set of ideals of  $R$  contains a minimal element under inclusion.

The next result gives the main structure theorem for Artinian rings.

**Theorem 3.** Let  $R$  be an Artinian ring.

- (1) There are only finitely many maximal ideals in  $R$ .
- (2) The quotient  $R/(\text{Jac } R)$  is a direct product of a finite number of fields. More precisely, if  $M_1, \dots, M_n$  are the finitely many maximal ideals in  $R$  then

$$R/(\text{Jac } R) \cong k_1 \times \cdots \times k_n,$$

where  $k_i$  is the field  $R/M_i$  for  $1 \leq i \leq n$ .

- (3) Every prime ideal of  $R$  is maximal, i.e.,  $R$  has Krull dimension 0. The Jacobson radical of  $R$  equals the nilradical of  $R$  and is a nilpotent ideal:  $(\text{Jac } R)^m = 0$  for some  $m \geq 1$ .
- (4) The ring  $R$  is isomorphic to the direct product of a finite number of Artinian local rings.
- (5) Every Artinian ring is Noetherian.

*Proof:* To prove (1), let  $\mathcal{S}$  be the set of all ideals of  $R$  that are the intersection of a finite number of maximal ideals. By Proposition 2,  $\mathcal{S}$  has a minimal element, say  $M_1 \cap M_2 \cap \cdots \cap M_n$ . Then for any maximal ideal  $M$  we have

$$M \cap M_1 \cap M_2 \cap \cdots \cap M_n = M_1 \cap M_2 \cap \cdots \cap M_n,$$

so  $M \supseteq M_1 \cap M_2 \cap \cdots \cap M_n$ . By Exercise 11 in Section 7.4,  $M \supseteq M_i$  for some  $i$ . Thus  $M = M_i$  and so  $M_1, \dots, M_n$  are all the maximal ideals of  $R$ .

The proof of (2) is immediate from the Chinese Remainder Theorem (Section 7.6) applied to  $M_1, \dots, M_n$ , since these maximal ideals are clearly pairwise comaximal and their intersection is  $\text{Jac } R$ .

For (3), we first prove  $\mathcal{J} = \text{Jac } R$  is nilpotent. By D.C.C. there is some  $m > 0$  such that  $\mathcal{J}^m = \mathcal{J}^{m+i}$  for all positive  $i$ . By way of contradiction assume  $\mathcal{J}^m \neq 0$ . Let  $\mathcal{S}$  be the set of proper ideals  $I$  such that  $I\mathcal{J}^m \neq 0$ , so  $\mathcal{J} \in \mathcal{S}$ . Let  $I_0$  be a minimal element of  $\mathcal{S}$ . There is some  $x \in I_0$  such that  $x\mathcal{J}^m \neq 0$ , so by minimality we must have  $I_0 = (x)$ . But now  $((x)\mathcal{J})\mathcal{J}^m = x\mathcal{J}^{m+1} = x\mathcal{J}^m$ , so it follows by minimality of  $(x)$  that  $(x) = (x)\mathcal{J}$ . By Nakayama's Lemma above,  $(x) = 0$ , a contradiction. This proves  $\text{Jac } R$  is nilpotent.

Since  $\text{Jac } R$  is nilpotent, in particular  $\text{Jac } R \subseteq \text{rad } 0$ , so these two ideals are equal by the second statement in Proposition 1.

Every prime ideal  $P$  in  $R$  contains the nilradical of  $R$ , hence contains  $\text{Jac } R$  by what has already been proved,. The image of  $P$  is a prime ideal in the quotient ring  $R/(\text{Jac } R) = k_1 \times \cdots \times k_n$ . But in a direct product of rings  $R_1 \times R_2$  (where each  $R_i$  has a 1) every ideal is of the form  $I_1 \times I_2$ , where  $I_j$  is an ideal of  $R_j$  for  $j = 1, 2$  (cf. Exercise 3 in Section 7.6). It follows that a prime ideal in  $k_1 \times \cdots \times k_n$  consists of the elements that are 0 in one of the components. In particular, such a prime ideal is also a maximal ideal in  $k_1 \times \cdots \times k_n$  and it follows that  $P$  was a maximal ideal in  $R$ , which finishes the proof of (3).

Let  $M_1, \dots, M_n$  be all the distinct maximal ideals of  $R$  and let  $(\text{Jac } R)^m = 0$  as in (3). Then

$$\prod_{i=1}^n M_i^m \subseteq \left( \prod_{i=1}^n M_i \right)^m \subseteq (\text{Jac } R)^m = 0.$$

By the Chinese Remainder Theorem it follows that

$$R \cong (R/M_1^m) \times (R/M_2^m) \times \cdots \times (R/M_n^m),$$

and each  $R/M_i^m$  is an Artinian ring with unique maximal ideal  $M_i/M_i^m$ , proving (4).

To prove (5), it suffices by (4) to prove that an Artinian local ring is Noetherian, so assume  $R$  is Artinian with unique maximal ideal  $M$ . In this case we have  $M = \text{Jac } R$ , so  $M^m = (\text{Jac } R)^m = 0$  for some positive  $m$ . Then  $R \cong R/M^m$ , and in this case it is an exercise to see that  $R/M^m$  is Noetherian if and only if it is Artinian (cf. Exercise 8).

**Corollary 4.** The ring  $R$  is Artinian if and only if  $R$  is Noetherian and has Krull dimension 0.

*Proof:* The forward implication was proved in Theorem 3. Suppose now that  $R$  is Noetherian and that  $R$  has Krull dimension 0, i.e., that prime ideals of  $R$  are maximal. Since  $R$  is Noetherian, by Corollary 22(3) in Section 15.2, the ideal  $(0) = P_1 \cdots P_n$  is the product of (not necessarily distinct) prime ideals, and these prime ideals are then maximal since  $R$  has dimension 0. By the Chinese Remainder Theorem,  $R$  is isomorphic to the direct product of a finite number of Noetherian rings of the form  $R/M^m$  where  $M$  is a maximal ideal in  $R$ . As in the proof of (5) of the theorem,  $R/M^m$  is Artinian, and it follows that  $R$  is Artinian.

## Examples

- (1) Let  $n > 1$  be an integer. Since the ring  $R = \mathbb{Z}/n\mathbb{Z}$  is finite, it is Artinian. If  $n = p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s}$  is the unique factorization of  $n$  into distinct prime powers, then

$$\mathbb{Z}/n\mathbb{Z} \cong (\mathbb{Z}/p_1^{a_1}\mathbb{Z}) \times (\mathbb{Z}/p_2^{a_2}\mathbb{Z}) \times \cdots \times (\mathbb{Z}/p_s^{a_s}\mathbb{Z}).$$

Each  $\mathbb{Z}/p_i^{a_i}\mathbb{Z}$  is an Artinian local ring with unique maximal ideal  $(p_i)/(p_i^{a_i})$ , so this is the decomposition of  $\mathbb{Z}/n\mathbb{Z}$  given by Theorem 3(4). The Jacobson radical of  $R$  is the ideal generated by  $p_1 p_2 \cdots p_s$ , the squarefree part of  $n$  and  $R/(\text{Jac } R) \cong (\mathbb{Z}/p_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/p_s\mathbb{Z})$  is a direct product of fields. The ideals generated by  $p_i$  for  $i = 1, \dots, s$  are the maximal ideals of  $R$ .

- (2) For any field  $k$ , a  $k$ -algebra  $R$  that is finite dimensional as a vector space over  $k$  is Artinian because ideals in  $R$  are in particular  $k$ -subspaces of  $R$ , hence the length of any chain of ideals in  $R$  is bounded by  $\dim_k R$ .
- (3) Suppose  $f$  is a nonzero polynomial in  $k[x]$  where  $k$  is a field. Then the quotient ring  $R = k[x]/(f(x))$  is Artinian by the previous example. The decomposition of  $R$  as a direct product of Artinian local rings is given by

$$k[x]/(f(x)) \cong k[x]/(f_1(x)^{a_1}) \times \cdots \times k[x]/(f_s(x)^{a_s})$$

where  $f(x) = f_1(x)^{a_1} \cdots f_s(x)^{a_s}$  is the factorization of  $f(x)$  into powers of distinct irreducibles in  $k[x]$  (cf. Proposition 16 in Section 9.5). The Jacobson radical of  $R$  is the ideal generated by the squarefree part of  $f(x)$  and the maximal ideals of  $R$  are the ideals generated by the irreducible factors  $f_i(x)$  for  $i = 1, \dots, s$  similar to Example 1.