5. $A \times (B + C) \cong (A \times B) + (A \times C)$,
6. $(A^B)^C \cong A^{C \times B}$,
7. $A^B \times A^C \cong A^{B+C}$,
8. $(A \times B)^C \cong A^C \times B^C$,
9. $A + 0 \cong A$,
10. $A \times 1 \cong A$,
11. $A \times 0 \cong 0$,
12. $A^0 \cong 1$, $A^1 \cong A$ *and, if* $B \neq 0$, *then* $0^B \cong 0$.

We shall not give a complete proof of this theorem, but, as an example, we shall prove that $(A^B)^C \cong A^{C \times B}$.

Let $f$ be any member of $(A^B)^C$, that is, let $f$ be any function such that $f : C \longrightarrow A^B$. Then, if $c$ is a typical element of $C$, $f(c) \in A^B$. Hence, if $b$ is a typical element of $B$, $(f(c))(b) \in A$. Let $g$ be any member of $A^{C \times B}$, that is, let $g$ be any function such that $g : C \times B \rightarrow A$. Then, if $(c, b)$ is a typical element of $C \times B$ (with $c \in C$, $b \in B$), $g((c, b)) \in A$.

We define $F : A^{C \times B} \longrightarrow (A^B)^C$ such that $((F(g))(c))(b) = g((c, b))$. We define $G : (A^B)^C \longrightarrow A^{C \times B}$ such that $(G(f))((c, b)) = (f(c))(b)$.

Then $FG$ is just the identity function on $(A^B)^C$. For $(FG)(f) = F(G(f))$ and this equals $f$ just in case, for any $c \in C$, $(F(G(f)))(c) = f(c)$. Moreover, the last equation is true just in case, for any $b \in B$, $((F(G(f)))(c))(b) = (f(c))(b)$. Now $((F(G(f)))(c))(b) = (G(f))((c, b))$, by the definition of $F$. Furthermore, by the definition of G, $(G(f))((c, b)) = (f(c))(b)$ as required.

Again, $GF$ is just the identity function on $A^{C \times B}$. For $(GF)(g) = G(F(g))$ and this equals $g$ just in case, for any $(c, b) \in C \times B$, $(G(F(g)))((c, b)) = g((c, b))$. By the definition of $G$, $(G(F(g)))((c, b)) = ((F(g))(c))(b)$, and, by the definition of $F$ this equals $g((c, b))$ as required.

Hence $F$ is one-one and onto (Chapter 12, Exercise 1). Applying the function $F$ is called *currying* in Computer Science, after the logician Haskell B. Curry.

Here are some hints for proving the remaining 11 parts of the above theorem, skipping a few unnecessary parentheses.

(1) An element of the left-hand side must be of the form $(a, 0)$ or $(b, 1)$, where $a \in A$ and $b \in B$. Let $F(a, 0) = (a, 1)$ and $F(b, 1) = (b, 0)$. Find the inverse $G$ of $F$.

(3) An element of the left-hand side has the form $(a, b)$, where $a \in A$ and $b \in B$. Let $F(a, b) = (b, a)$ and find the inverse $G$ of $F$.

(5) An element of the left-hand side has the form $(a, (b, 0))$ or $(a, (c, 1))$, where $a \in A$, $b \in B$ and $c \in C$. Let $F(a, (b, 0)) = ((a, b), 0)$, $F(a, (c, 1)) = ((a, c), 1)$ and find the inverse $G$ of $F$.

(7) An element of the left-hand side has the form $(f, g)$, where $f : B \longrightarrow A$ and $g : C \longrightarrow A$. Define $F(f, g)$ by stipulating $F(f, g)(b, 0) = f(b)$ and $F(f, g)(c, 1) = g(c)$. An element of the right-hand side has the form $h : B+C \longrightarrow A$. Define $G(h)$ as the pair $(G(h)_0, G(h)_1)$, where $G(h)_0(b) = h(b, 0)$ and $G(h)_1(c) = h(c, 1)$. Show that $G$ is the inverse of $F$.

(8) An element of the left-hand side has the form $h : C \longrightarrow A \times B$. From this we obtain two functions $h_0 : C \longrightarrow A$ and $h_1 : C \longrightarrow B$ such that $h(c) = (h_0(c), h_1(c))$. Let $F(h) = (h_0, h_1)$. An element of the right-hand side is a pair $(f, g)$, where $f : C \longrightarrow A$ and $g : C \longrightarrow B$. Let $G(f, g)(c) = (f(c), g(c))$ and show that $G$ is the inverse of $F$.

(12) An element of $A^B$ is a function $f : B \longrightarrow A$, that is, a subset $f$ of $A \times B$ such that, for every $b \in B$, there exists a unique $a \in A$ such that $(a, b) \in f$. Show that there is exactly one function $\emptyset \longrightarrow A$, that the functions $\{\emptyset\} \longrightarrow A$ are in one-to-one correspondence with the elements of $A$, and, finally that, when $B \neq \emptyset$, there is no function $B \longrightarrow \emptyset$.

We can now define equivalent binary operations for cardinals, so that

$$
\begin{aligned}
|A| + |B| &= |A + B|, \\
|A| \times |B| &= |A \times B|, \\
|A|^{|B|} &= |A^B|.
\end{aligned}
$$

For example, $\aleph_0 + 2 = \aleph_0$ since $\mathbf{N} + \{0, 1\} \cong \mathbf{N}$.

# Exercises

1. Prove the remaining eleven parts of the above theorem.
2. Prove that $|A| \times (|B| + |C|) = (|A| \times |B|) + (|A| \times |C|)$.
3. Prove that $\aleph_0 + \aleph_0 = \aleph_0$.
4. Prove that $\aleph_0 \times 2 = \aleph_0$.
5. Prove that $\mathbf{N} + \mathbf{N}$ and $\mathbf{N} \times \mathbf{N}$ have the same cardinality as $\mathbf{N}$.
6. Prove that $\aleph_0^2 = \aleph_0$.
7. Prove that $|A| \times |A| \times |A| = |A|^3$.
8. Simplify $2^{\aleph_0} \times 2^{\aleph_0}$.
9. Simplify $8^{\aleph_0}$.

10. Prove that every infinite set has a subset which can be placed in one-one correspondence with **N**.

11. Project: Look up the Schroeder-Bernstein Theorem and prove:

$$\aleph_0^{\aleph_0^{\aleph_0}} = 2^{2^{\aleph_0}}.$$

# 14

# Continued Fractions

As Fowler shows in *The Mathematics of Plato's Academy*, continued fractions are implicit in ancient Greek mathematics. As far as we know, they were not explicitly defined before 1618, when Daniel Schwenter rediscovered them.

Continued fractions are implicit in Euclid's Algorithm (Book VII, Proposition 2) dating from 300 BC. This is a procedure for finding the greatest common divisor (gcd) of two positive integers. The *greatest common divisor* of two positive integers $a$ and $b$ is the positive integer $d$ whose divisors are precisely the common divisors of $a$ and $b$. In other words, $d$ divides $a$ and $b$ and any common divisor of $a$ and $b$ divides $d$. This notion can easily be extended to natural numbers or even integers, but one has to be careful. For example, $\gcd(0, 17) = 17$, but $\gcd(0, 0) = 0$, even though 17 is also a common divisor of 0 and 0, yet $17 > 0$. For the gcd of two nonzero integers, say 12 and $-15$, one has two candidates that meet the above definition, 3 and $-3$ in this case, and one usually chooses the positive one. The algorithm is best described with the help of an example. Suppose we want to find the gcd of 502 and 1604. We perform four divisions as follows.

$$
\begin{array}{rr}
 & 3 \\
502 & 1604 \\
 & 1506 \\
 & 98
\end{array}
\qquad
\begin{array}{rr}
 & 5 \\
98 & 502 \\
 & 490 \\
 & 12
\end{array}
\qquad
\begin{array}{rr}
 & 8 \\
12 & 98 \\
 & 96 \\
 & 2
\end{array}
\qquad
\begin{array}{rr}
 & 6 \\
2 & 12 \\
 & 12 \\
 & 0
\end{array}
$$

Since *dividend = (divisor × quotient) + remainder*, any number which divides dividend and divisor will also divide the remainder. Thus any factors common to the two original integers are carried through the calculations