Then $p_n - q_n = a_n, p_n + q_n = |a_n|, p_n \geq 0, q_n \geq 0$. The series Σp_n, Σq_n must both diverge.

For if both were convergent, then

$$\Sigma(p_n + q_n) = \Sigma|a_n|$$

would converge, contrary to hypothesis. Since

$$\sum_{n=1}^{N} a_n = \sum_{n=1}^{N} (p_n - q_n) = \sum_{n=1}^{N} p_n - \sum_{n=1}^{N} q_n,$$

divergence of Σp_n and convergence of Σq_n (or vice versa) implies divergence of Σa_n, again contrary to hypothesis.

Now let $P_1, P_2, P_3, \ldots$ denote the nonnegative terms of Σa_n, in the order in which they occur, and let $Q_1, Q_2, Q_3, \ldots$ be the absolute values of the negative terms of Σa_n, also in their original order.

The series $\Sigma P_n, \Sigma Q_n$ differ from $\Sigma p_n, \Sigma q_n$ only by zero terms, and are therefore divergent.

We shall construct sequences $\{m_n\}$, $\{k_n\}$, such that the series

$$(25)\quad P_1 + \cdots + P_{m_1} - Q_1 - \cdots - Q_{k_1} + P_{m_1+1} + \cdots + P_{m_2} - Q_{k_1+1} - \cdots - Q_{k_2} + \cdots,$$

which clearly is a rearrangement of Σa_n, satisfies (24).

Choose real-valued sequences $\{\alpha_n\}$, $\{\beta_n\}$ such that $\alpha_n \to \alpha$, $\beta_n \to \beta$, $\alpha_n < \beta_n$, $\beta_1 > 0$.

Let m_1, k_1 be the smallest integers such that

$$P_1 + \cdots + P_{m_1} > \beta_1,$$
$$P_1 + \cdots + P_{m_1} - Q_1 - \cdots - Q_{k_1} < \alpha_1;$$

let m_2, k_2 be the smallest integers such that

$$P_1 + \cdots + P_{m_1} - Q_1 - \cdots - Q_{k_1} + P_{m_1+1} + \cdots + P_{m_2} > \beta_2,$$
$$P_1 + \cdots + P_{m_1} - Q_1 - \cdots - Q_{k_1} + P_{m_1+1} + \cdots + P_{m_2} - Q_{k_1+1} - \cdots - Q_{k_2} < \alpha_2;$$

and continue in this way. This is possible since ΣP_n and ΣQ_n diverge.

If x_n, y_n denote the partial sums of (25) whose last terms are P_{m_n}, $-Q_{k_n}$, then

$$|x_n - \beta_n| \leq P_{m_n}, \qquad |y_n - \alpha_n| \leq Q_{k_n}.$$

Since $P_n \to 0$ and $Q_n \to 0$ as $n \to \infty$, we see that $x_n \to \beta$, $y_n \to \alpha$.

Finally, it is clear that no number less than α or greater than β can be a subsequential limit of the partial sums of (25).

3.55 Theorem *If Σa_n is a series of complex numbers which converges absolutely, then every rearrangement of Σa_n converges, and they all converge to the same sum.*

Proof Let $\Sigma a_n'$ be a rearrangement, with partial sums s_n'. Given $\varepsilon > 0$, there exists an integer N such that $m \geq n \geq N$ implies

$$\sum_{i=n}^{m} |a_i| \leq \varepsilon. \tag{26}$$

Now choose p such that the integers $1, 2, \ldots, N$ are all contained in the set $k_1, k_2, \ldots, k_p$ (we use the notation of Definition 3.52). Then if $n > p$, the numbers $a_1, \ldots, a_N$ will cancel in the difference $s_n - s_n'$, so that $|s_n - s_n'| \leq \varepsilon$, by (26). Hence $\{s_n'\}$ converges to the same sum as $\{s_n\}$.

EXERCISES

1. Prove that convergence of $\{s_n\}$ implies convergence of $\{|s_n|\}$. Is the converse true?
2. Calculate $\lim_{n \to \infty} (\sqrt{n^2 + n} - n)$.
3. If $s_1 = \sqrt{2}$, and

$$s_{n+1} = \sqrt{2 + \sqrt{s_n}} \qquad (n = 1, 2, 3, \ldots),$$

prove that $\{s_n\}$ converges, and that $s_n < 2$ for $n = 1, 2, 3, \ldots$.

4. Find the upper and lower limits of the sequence $\{s_n\}$ defined by

$$s_1 = 0; \qquad s_{2m} = \frac{s_{2m-1}}{2}; \qquad s_{2m+1} = \frac{1}{2} + s_{2m}.$$

5. For any two real sequences $\{a_n\}$, $\{b_n\}$, prove that

$$\limsup_{n \to \infty} (a_n + b_n) \leq \limsup_{n \to \infty} a_n + \limsup_{n \to \infty} b_n,$$

provided the sum on the right is not of the form $\infty - \infty$.

6. Investigate the behavior (convergence or divergence) of Σa_n if

(*a*) $a_n = \sqrt{n+1} - \sqrt{n}$;

(*b*) $a_n = \dfrac{\sqrt{n+1} - \sqrt{n}}{n}$;

(*c*) $a_n = (\sqrt[n]{n} - 1)^n$;

(*d*) $a_n = \dfrac{1}{1 + z^n}$, for complex values of z.

7. Prove that the convergence of Σa_n implies the convergence of

$$\Sigma \frac{\sqrt{a_n}}{n},$$

if $a_n \geq 0$.

8. If Σa_n converges, and if $\{b_n\}$ is monotonic and bounded, prove that $\Sigma a_n b_n$ converges.

9. Find the radius of convergence of each of the following power series:

(*a*) $\sum n^3 z^n$, (*b*) $\sum \frac{2^n}{n!} z^n$,

(*c*) $\sum \frac{2^n}{n^2} z^n$, (*d*) $\sum \frac{n^3}{3^n} z^n$.

10. Suppose that the coefficients of the power series $\sum a_n z^n$ are integers, infinitely many of which are distinct from zero. Prove that the radius of convergence is at most 1.

11. Suppose $a_n > 0$, $s_n = a_1 + \cdots + a_n$, and Σa_n diverges.

(*a*) Prove that $\sum \frac{a_n}{1 + a_n}$ diverges.

(*b*) Prove that

$$\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}}$$

and deduce that $\sum \frac{a_n}{s_n}$ diverges.

(*c*) Prove that

$$\frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n}$$

and deduce that $\sum \frac{a_n}{s_n^2}$ converges.

(*d*) What can be said about

$$\sum \frac{a_n}{1 + na_n} \quad \text{and} \quad \sum \frac{a_n}{1 + n^2 a_n}?$$

12. Suppose $a_n > 0$ and Σa_n converges. Put

$$r_n = \sum_{m=n}^{\infty} a_m.$$

(*a*) Prove that

$$\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} > 1 - \frac{r_n}{r_m}$$

if $m < n$, and deduce that $\sum \frac{a_n}{r_n}$ diverges.

(*b*) Prove that

$$\frac{a_n}{\sqrt{r_n}} < 2(\sqrt{r_n} - \sqrt{r_{n+1}})$$

and deduce that $\sum \frac{a_n}{\sqrt{r_n}}$ converges.

13. Prove that the Cauchy product of two absolutely convergent series converges absolutely.

14. If $\{s_n\}$ is a complex sequence, define its arithmetic means σ_n by

$$\sigma_n = \frac{s_0 + s_1 + \cdots + s_n}{n+1} \qquad (n = 0, 1, 2, \ldots).$$

(*a*) If $\lim s_n = s$, prove that $\lim \sigma_n = s$.
(*b*) Construct a sequence $\{s_n\}$ which does not converge, although $\lim \sigma_n = 0$.
(*c*) Can it happen that $s_n > 0$ for all n and that $\limsup s_n = \infty$, although $\lim \sigma_n = 0$?
(*d*) Put $a_n = s_n - s_{n-1}$, for $n \geq 1$. Show that

$$s_n - \sigma_n = \frac{1}{n+1} \sum_{k=1}^{n} k a_k .$$

Assume that $\lim (na_n) = 0$ and that $\{\sigma_n\}$ converges. Prove that $\{s_n\}$ converges. [This gives a converse of (*a*), but under the additional assumption that $na_n \to 0$.]
(*e*) Derive the last conclusion from a weaker hypothesis: Assume $M < \infty$, $|na_n| \leq M$ for all n, and $\lim \sigma_n = \sigma$. Prove that $\lim s_n = \sigma$, by completing the following outline:

If $m < n$, then

$$s_n - \sigma_n = \frac{m+1}{n-m}(\sigma_n - \sigma_m) + \frac{1}{n-m} \sum_{i=m+1}^{n} (s_n - s_i).$$

For these i,

$$|s_n - s_i| \leq \frac{(n-i)M}{i+1} \leq \frac{(n-m-1)M}{m+2}.$$

Fix $\varepsilon > 0$ and associate with each n the integer m that satisfies

$$m \leq \frac{n-\varepsilon}{1+\varepsilon} < m+1.$$

Then $(m+1)/(n-m) \leq 1/\varepsilon$ and $|s_n - s_i| < M\varepsilon$. Hence

$$\limsup_{n\to\infty} |s_n - \sigma| \leq M\varepsilon.$$

Since ε was arbitrary, $\lim s_n = \sigma$.

15. Definition 3.21 can be extended to the case in which the a_n lie in some fixed R^k. Absolute convergence is defined as convergence of $\Sigma|\mathbf{a}_n|$. Show that Theorems 3.22, 3.23, 3.25(*a*), 3.33, 3.34, 3.42, 3.45, 3.47, and 3.55 are true in this more general setting. (Only slight modifications are required in any of the proofs.)

16. Fix a positive number α. Choose $x_1 > \sqrt{\alpha}$, and define $x_2, x_3, x_4, \ldots,$ by the recursion formula

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right).$$

(*a*) Prove that $\{x_n\}$ decreases monotonically and that $\lim x_n = \sqrt{\alpha}$.

(*b*) Put $\varepsilon_n = x_n - \sqrt{\alpha}$, and show that

$$\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2x_n} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}}$$

so that, setting $\beta = 2\sqrt{\alpha}$,

$$\varepsilon_{n+1} < \beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^n} \qquad (n = 1, 2, 3, \ldots).$$

(*c*) This is a good algorithm for computing square roots, since the recursion formula is simple and the convergence is extremely rapid. For example, if $\alpha = 3$ and $x_1 = 2$, show that $\varepsilon_1/\beta < \frac{1}{10}$ and that therefore

$$\varepsilon_5 < 4 \cdot 10^{-16}, \qquad \varepsilon_6 < 4 \cdot 10^{-32}.$$

17. Fix $\alpha > 1$. Take $x_1 > \sqrt{\alpha}$, and define

$$x_{n+1} = \frac{\alpha + x_n}{1 + x_n} = x_n + \frac{\alpha - x_n^2}{1 + x_n}.$$

(*a*) Prove that $x_1 > x_3 > x_5 > \cdots$.

(*b*) Prove that $x_2 < x_4 < x_6 < \cdots$.

(*c*) Prove that $\lim x_n = \sqrt{\alpha}$.

(*d*) Compare the rapidity of convergence of this process with the one described in Exercise 16.

18. Replace the recursion formula of Exercise 16 by

$$x_{n+1} = \frac{p-1}{p}x_n + \frac{\alpha}{p}x_n^{-p+1}$$

where p is a fixed positive integer, and describe the behavior of the resulting sequences $\{x_n\}$.

19. Associate to each sequence $a = \{\alpha_n\}$, in which α_n is 0 or 2, the real number

$$x(a) = \sum_{n=1}^{\infty} \frac{\alpha_n}{3^n}.$$

Prove that the set of all $x(a)$ is precisely the Cantor set described in Sec. 2.44.

20. Suppose $\{p_n\}$ is a Cauchy sequence in a metric space X, and some subsequence $\{p_{n_i}\}$ converges to a point $p \in X$. Prove that the full sequence $\{p_n\}$ converges to p.

21. Prove the following analogue of Theorem 3.10(*b*): If $\{E_n\}$ is a sequence of closed nonempty and bounded sets in a *complete* metric space X, if $E_n \supset E_{n+1}$, and if

$$\lim_{n\to\infty} \operatorname{diam} E_n = 0,$$

then $\bigcap_1^\infty E_n$ consists of exactly one point.

22. Suppose X is a nonempty complete metric space, and $\{G_n\}$ is a sequence of dense open subsets of X. Prove Baire's theorem, namely, that $\bigcap_1^\infty G_n$ is not empty. (In fact, it is dense in X.) *Hint:* Find a shrinking sequence of neighborhoods E_n such that $\bar{E}_n \subset G_n$, and apply Exercise 21.

23. Suppose $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in a metric space X. Show that the sequence $\{d(p_n, q_n)\}$ converges. *Hint:* For any m, n,

$$d(p_n, q_n) \le d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n);$$

it follows that

$$|d(p_n, q_n) - d(p_m, q_m)|$$

is small if m and n are large.

24. Let X be a metric space.

(*a*) Call two Cauchy sequences $\{p_n\}$, $\{q_n\}$ in X *equivalent* if

$$\lim_{n\to\infty} d(p_n, q_n) = 0.$$

Prove that this is an equivalence relation.

(*b*) Let X^* be the set of all equivalence classes so obtained. If $P \in X^*$, $Q \in X^*$, $\{p_n\} \in P$, $\{q_n\} \in Q$, define

$$\Delta(P, Q) = \lim_{n\to\infty} d(p_n, q_n);$$

by Exercise 23, this limit exists. Show that the number $\Delta(P, Q)$ is unchanged if $\{p_n\}$ and $\{q_n\}$ are replaced by equivalent sequences, and hence that Δ is a distance function in X^*.

(*c*) Prove that the resulting metric space X^* is complete.

(*d*) For each $p \in X$, there is a Cauchy sequence all of whose terms are p; let P_p be the element of X^* which contains this sequence. Prove that

$$\Delta(P_p, P_q) = d(p, q)$$

for all $p, q \in X$. In other words, the mapping φ defined by $\varphi(p) = P_p$ is an isometry (i.e., a distance-preserving mapping) of X into X^*.

(*e*) Prove that $\varphi(X)$ is dense in X^*, and that $\varphi(X) = X^*$ if X is complete. By (*d*), we may identify X and $\varphi(X)$ and thus regard X as embedded in the complete metric space X^*. We call X^* the *completion* of X.

25. Let X be the metric space whose points are the rational numbers, with the metric $d(x, y) = |x - y|$. What is the completion of this space? (Compare Exercise 24.)

4

CONTINUITY

The function concept and some of the related terminology were introduced in Definitions 2.1 and 2.2. Although we shall (in later chapters) be mainly interested in real and complex functions (i.e., in functions whose values are real or complex numbers) we shall also discuss vector-valued functions (i.e., functions with values in R^k) and functions with values in an arbitrary metric space. The theorems we shall discuss in this general setting would not become any easier if we restricted ourselves to real functions, for instance, and it actually simplifies and clarifies the picture to discard unnecessary hypotheses and to state and prove theorems in an appropriately general context.

The domains of definition of our functions will also be metric spaces, suitably specialized in various instances.

LIMITS OF FUNCTIONS

4.1 Definition Let X and Y be metric spaces; suppose $E \subset X$, f maps E into Y, and p is a limit point of E. We write $f(x) \to q$ as $x \to p$, or

$$\lim_{x \to p} f(x) = q \tag{1}$$

if there is a point $q \in Y$ with the following property: For every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$d_Y(f(x), q) < \varepsilon \tag{2}$$

for all points $x \in E$ for which

$$0 < d_X(x, p) < \delta. \tag{3}$$

The symbols d_X and d_Y refer to the distances in X and Y, respectively.

If X and/or Y are replaced by the real line, the complex plane, or by some euclidean space R^k, the distances d_X, d_Y are of course replaced by absolute values, or by norms of differences (see Sec. 2.16).

It should be noted that $p \in X$, but that p need not be a point of E in the above definition. Moreover, even if $p \in E$, we may very well have $f(p) \neq \lim_{x \to p} f(x)$.

We can recast this definition in terms of limits of sequences:

4.2 Theorem *Let X, Y, E, f, and p be as in Definition* 4.1. *Then*

$$\lim_{x \to p} f(x) = q \tag{4}$$

if and only if

$$\lim_{n \to \infty} f(p_n) = q \tag{5}$$

for every sequence $\{p_n\}$ in E such that

$$p_n \neq p, \qquad \lim_{n \to \infty} p_n = p. \tag{6}$$

Proof Suppose (4) holds. Choose $\{p_n\}$ in E satisfying (6). Let $\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that $d_Y(f(x), q) < \varepsilon$ if $x \in E$ and $0 < d_X(x, p) < \delta$. Also, there exists N such that $n > N$ implies $0 < d_X(p_n, p) < \delta$. Thus, for $n > N$, we have $d_Y(f(p_n), q) < \varepsilon$, which shows that (5) holds.

Conversely, suppose (4) is false. Then there exists some $\varepsilon > 0$ such that for every $\delta > 0$ there exists a point $x \in E$ (depending on δ), for which $d_Y(f(x), q) \geq \varepsilon$ but $0 < d_X(x, p) < \delta$. Taking $\delta_n = 1/n$ ($n = 1, 2, 3, \ldots$), we thus find a sequence in E satisfying (6) for which (5) is false.

Corollary *If f has a limit at p, this limit is unique.*

This follows from Theorems 3.2(*b*) and 4.2.