CHAPTER 4 Rational Points

4.1 Pythagorean Triples

One of the most astonishing documents in the history of mathematics is a clay tablet in the Columbia University collection of Babylonian artifacts. Known as Plimpton 322, it dates from around 1800 B.C. and contains the two columns of numbers in Figure 4.1.

Few of the pairs (b, c) look at all familiar, and it is not obvious that they have any mathematical significance. However, they have a property that leaves no doubt what they are. *In every case, $c^2 - b^2$ is an integer square a^2*, hence the tablet is a virtual list of what we now call *Pythagorean triples* (a, b, c). In arithmetic terms, Pythagorean triples are simply triples of natural numbers with $a^2 + b^2 = c^2$, but by the converse Pythagorean theorem they are also *side lengths of right-angled triangles*. In fact, there is another column that shows the Babylonians were aware of this, and it explains why the pairs (b, c) are written in the given order. The column not included in the table here is a list of the values c^2/a^2, and they turn out to be in decreasing order, and roughly equally spaced. Thus the tablet is a kind of "database" of right-angled triangles, covering a range of shapes. Incidentally, it is an interesting question whether there are Pythagorean triples for which the angles of the corresponding triangles increase

b	c
119	169
3367	4825
4601	6649
12709	18541
65	97
319	481
2291	3541
799	1249
481	769
4961	8161
45	75
1679	2929
161	289
1771	3229
56	106

FIGURE 4.1 Pairs in Plimpton 322.

in *exactly* equal steps. We shall answer this question in Sections 5.4 and 5.8*.

The meaning of the pairs (b, c) was discovered by Otto Neugebauer and Abraham Sachs (1945), who also went on to speculate how the corresponding Pythagorean triples may have been found. This is a good question, because triples as large as (13500, 12709, 18541) were certainly not found by trial and error, but the answer is probably not straightforward. The *purpose* of Plimpton 322 can be guessed by constructing the corresponding triangles, and the preference for certain numbers can be explained by the Babylonian number system (see Exercises 4.1.1 and 4.1.3), but for enlightenment on *method* we must turn to ancient Greece.

Exercises

4.1.1. Check that $c^2 - b^2$ is a perfect square for each of the pairs (b, c) in the table. (Computer assistance is recommended.)

Computing the values $a = \sqrt{c^2 - b^2}$ is interesting because in many cases a turns out to be a "rounder" number than b or c. In fact, our base

10 notation fails to show how very round the numbers a really are. All but three of them are multiples of 60 and, of the remaining three, one is a multiple of 30 and the others are multiples of 12. The Babylonians wrote their numbers in base 60, so multiples of 60 were the roundest numbers in their notation, and the divisors 12 and 30 of 60 were also pretty round.

4.1.2. What is significant about the number 3456?

It is also interesting to compute the values of $b/c = \sin\theta$, where θ is the angle opposite b in the right-angled triangle, and see how the angles increase in roughly equal steps.

4.1.3. Show that the values of b/c in Plimpton 322 strictly increase, and find the corresponding values of θ.

4.2 Pythagorean Triples in Euclid

The easiest way to find big Pythagorean triples is to use a formula like

$$a = 2uv, \quad b = u^2 - v^2, \quad c = u^2 + v^2.$$

This formula gives $a^2 + b^2 = c^2$ for any values of u and v, so by substituting natural numbers for u and v it is possible to obtain arbitrarily large triples. Formulas like this, or perhaps more special ones like

$$a = 2u, \quad b = u^2 - 1, \quad c = u^2 + 1,$$

were probably known in ancient Babylon, Greece, India, and China.

However, the first rigorous treatment of Pythagorean triples occurs in Euclid. He actually set out to solve a simpler problem: *to find two square numbers such that their sum is also a square* (*Elements*, Book X, Lemma 1 to Proposition 28). He started with (a geometric form of) the identity

$$xy + \left(\frac{x-y}{2}\right)^2 = \left(\frac{x+y}{2}\right)^2,$$

which he had established earlier (Book II, Proposition 5). He then observed that it is enough to choose x and y so that xy is a square,