4.3 Definition Suppose we have two complex functions, f and g, both defined on E. By $f+g$ we mean the function which assigns to each point x of E the number $f(x)+g(x)$. Similarly we define the difference $f-g$, the product fg, and the quotient f/g of the two functions, with the understanding that the quotient is defined only at those points x of E at which $g(x) \neq 0$. If f assigns to each point x of E the same number c, then f is said to be a constant function, or simply a constant, and we write $f = c$. If f and g are real functions, and if $f(x) \geq g(x)$ for every $x \in E$, we shall sometimes write $f \geq g$, for brevity.

Similarly, if $\mathbf{f}$ and $\mathbf{g}$ map E into R^k, we define $\mathbf{f}+\mathbf{g}$ and $\mathbf{f}\cdot\mathbf{g}$ by

$$(\mathbf{f}+\mathbf{g})(x) = \mathbf{f}(x) + \mathbf{g}(x), \qquad (\mathbf{f}\cdot\mathbf{g})(x) = \mathbf{f}(x)\cdot\mathbf{g}(x);$$

and if λ is a real number, $(\lambda\mathbf{f})(x) = \lambda\mathbf{f}(x)$.

4.4 Theorem *Suppose $E \subset X$, a metric space, p is a limit point of E, f and g are complex functions on E, and*

$$\lim_{x\to p} f(x) = A, \qquad \lim_{x\to p} g(x) = B.$$

Then (*a*) $\lim_{x\to p} (f+g)(x) = A+B$;

(*b*) $\lim_{x\to p} (fg)(x) = AB$;

(*c*) $\lim_{x\to p} \left(\frac{f}{g}\right)(x) = \frac{A}{B}$, *if* $B \neq 0$.

Proof In view of Theorem 4.2, these assertions follow immediately from the analogous properties of sequences (Theorem 3.3).

Remark If $\mathbf{f}$ and $\mathbf{g}$ map E into R^k, then (*a*) remains true, and (*b*) becomes

(*b′*) $\lim_{x\to p} (\mathbf{f}\cdot\mathbf{g})(x) = \mathbf{A}\cdot\mathbf{B}$.

(Compare Theorem 3.4.)

CONTINUOUS FUNCTIONS

4.5 Definition Suppose X and Y are metric spaces, $E \subset X$, $p \in E$, and f maps E into Y. Then f is said to be *continuous at* p if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$d_Y(f(x), f(p)) < \varepsilon$$

for all points $x \in E$ for which $d_X(x, p) < \delta$.

If f is continuous at every point of E, then f is said to be *continuous on* E.

It should be noted that f has to be defined at the point p in order to be continuous at p. (Compare this with the remark following Definition 4.1.)

If p is an isolated point of E, then our definition implies that every function f which has E as its domain of definition is continuous at p. For, no matter which $\varepsilon > 0$ we choose, we can pick $\delta > 0$ so that the only point $x \in E$ for which $d_X(x, p) < \delta$ is $x = p$; then

$$d_Y(f(x), f(p)) = 0 < \varepsilon.$$

4.6 Theorem *In the situation given in Definition 4.5, assume also that p is a limit point of E. Then f is continuous at p if and only if $\lim_{x \to p} f(x) = f(p)$.*

Proof This is clear if we compare Definitions 4.1 and 4.5.

We now turn to compositions of functions. A brief statement of the following theorem is that a continuous function of a continuous function is continuous.

4.7 Theorem *Suppose X, Y, Z are metric spaces, $E \subset X$, f maps E into Y, g maps the range of f, $f(E)$, into Z, and h is the mapping of E into Z defined by*

$$h(x) = g(f(x)) \qquad (x \in E).$$

If f is continuous at a point $p \in E$ and if g is continuous at the point $f(p)$, then h is continuous at p.

This function h is called the *composition* or the *composite* of f and g. The notation

$$h = g \circ f$$

is frequently used in this context.

Proof Let $\varepsilon > 0$ be given. Since g is continuous at $f(p)$, there exists $\eta > 0$ such that

$$d_Z(g(y), g(f(p))) < \varepsilon \text{ if } d_Y(y, f(p)) < \eta \text{ and } y \in f(E).$$

Since f is continuous at p, there exists $\delta > 0$ such that

$$d_Y(f(x), f(p)) < \eta \text{ if } d_X(x, p) < \delta \text{ and } x \in E.$$

It follows that

$$d_Z(h(x), h(p)) = d_Z(g(f(x)), g(f(p))) < \varepsilon$$

if $d_X(x, p) < \delta$ and $x \in E$. Thus h is continuous at p.

4.8 Theorem *A mapping f of a metric space X into a metric space Y is continuous on X if and only if $f^{-1}(V)$ is open in X for every open set V in Y.*

(Inverse images are defined in Definition 2.2.) This is a very useful characterization of continuity.

Proof Suppose f is continuous on X and V is an open set in Y. We have to show that every point of $f^{-1}(V)$ is an interior point of $f^{-1}(V)$. So, suppose $p \in X$ and $f(p) \in V$. Since V is open, there exists $\varepsilon > 0$ such that $y \in V$ if $d_Y(f(p), y) < \varepsilon$; and since f is continuous at p, there exists $\delta > 0$ such that $d_Y(f(x), f(p)) < \varepsilon$ if $d_X(x, p) < \delta$. Thus $x \in f^{-1}(V)$ as soon as $d_X(x, p) < \delta$.

Conversely, suppose $f^{-1}(V)$ is open in X for every open set V in Y. Fix $p \in X$ and $\varepsilon > 0$, let V be the set of all $y \in Y$ such that $d_Y(y, f(p)) < \varepsilon$. Then V is open; hence $f^{-1}(V)$ is open; hence there exists $\delta > 0$ such that $x \in f^{-1}(V)$ as soon as $d_X(p, x) < \delta$. But if $x \in f^{-1}(V)$, then $f(x) \in V$, so that $d_Y(f(x), f(p)) < \varepsilon$.

This completes the proof.

Corollary *A mapping f of a metric space X into a metric space Y is continuous if and only if $f^{-1}(C)$ is closed in X for every closed set C in Y.*

This follows from the theorem, since a set is closed if and only if its complement is open, and since $f^{-1}(E^c) = [f^{-1}(E)]^c$ for every $E \subset Y$.

We now turn to complex-valued and vector-valued functions, and to functions defined on subsets of R^k.

4.9 Theorem *Let f and g be complex continuous functions on a metric space X. Then $f + g$, fg, and f/g are continuous on X.*

In the last case, we must of course assume that $g(x) \neq 0$, for all $x \in X$.

Proof At isolated points of X there is nothing to prove. At limit points, the statement follows from Theorems 4.4 and 4.6.

4.10 Theorem

(*a*) *Let $f_1, \ldots, f_k$ be real functions on a metric space X, and let $\mathbf{f}$ be the mapping of X into R^k defined by*

$$\mathbf{f}(x) = (f_1(x), \ldots, f_k(x)) \qquad (x \in X); \tag{7}$$

then $\mathbf{f}$ is continuous if and only if each of the functions $f_1, \ldots, f_k$ is continuous.
(*b*) *If $\mathbf{f}$ and $\mathbf{g}$ are continuous mappings of X into R^k, then $\mathbf{f} + \mathbf{g}$ and $\mathbf{f} \cdot \mathbf{g}$ are continuous on X.*

The functions $f_1, \ldots, f_k$ are called the *components* of $\mathbf{f}$. Note that $\mathbf{f} + \mathbf{g}$ is a mapping into R^k, whereas $\mathbf{f} \cdot \mathbf{g}$ is a *real* function on X.

Proof Part (*a*) follows from the inequalities

$$|f_j(x) - f_j(y)| \le |\mathbf{f}(x) - \mathbf{f}(y)| = \left\{ \sum_{i=1}^{k} |f_i(x) - f_i(y)|^2 \right\}^{\frac{1}{2}},$$

for $j = 1, \ldots, k$. Part (*b*) follows from (*a*) and Theorem 4.9.

4.11 Examples If $x_1, \ldots, x_k$ are the coordinates of the point $\mathbf{x} \in R^k$, the functions ϕ_i defined by

$$\phi_i(\mathbf{x}) = x_i \qquad (\mathbf{x} \in R^k) \tag{8}$$

are continuous on R^k, since the inequality

$$|\phi_i(\mathbf{x}) - \phi_i(\mathbf{y})| \le |\mathbf{x} - \mathbf{y}|$$

shows that we may take $\delta = \varepsilon$ in Definition 4.5. The functions ϕ_i are sometimes called the *coordinate functions*.

Repeated application of Theorem 4.9 then shows that every monomial

$$x_1^{n_1} x_2^{n_2} \ldots x_k^{n_k} \tag{9}$$

where $n_1, \ldots, n_k$ are nonnegative integers, is continuous on R^k. The same is true of constant multiples of (9), since constants are evidently continuous. It follows that every polynomial P, given by

$$P(\mathbf{x}) = \Sigma c_{n_1 \cdots n_k} x_1^{n_1} \ldots x_k^{n_k} \qquad (\mathbf{x} \in R^k), \tag{10}$$

is continuous on R^k. Here the coefficients $c_{n_1 \cdots n_k}$ are complex numbers, $n_1, \ldots, n_k$ are nonnegative integers, and the sum in (10) has finitely many terms.

Furthermore, every rational function in $x_1, \ldots, x_k$, that is, every quotient of two polynomials of the form (10), is continuous on R^k wherever the denominator is different from zero.

From the triangle inequality one sees easily that

$$\big| |\mathbf{x}| - |\mathbf{y}| \big| \le |\mathbf{x} - \mathbf{y}| \quad (\mathbf{x}, \mathbf{y} \in R^k). \tag{11}$$

Hence the mapping $\mathbf{x} \to |\mathbf{x}|$ is a continuous real function on R^k.

If now $\mathbf{f}$ is a continuous mapping from a metric space X into R^k, and if ϕ is defined on X by setting $\phi(p) = |\mathbf{f}(p)|$, it follows, by Theorem 4.7, that ϕ is a continuous real function on X.

4.12 Remark We defined the notion of continuity for functions defined on a *subset* E of a metric space X. However, the complement of E in X plays no role whatever in this definition (note that the situation was somewhat different for limits of functions). Accordingly, we lose nothing of interest by discarding the complement of the domain of f. This means that we may just as well talk only about continuous mappings of one metric space into another, rather than of mappings of subsets. This simplifies statements and proofs of some theorems. We have already made use of this principle in Theorems 4.8 to 4.10, and will continue to do so in the following section on compactness.

CONTINUITY AND COMPACTNESS

4.13 Definition A mapping $\mathbf{f}$ of a set E into R^k is said to be *bounded* if there is a real number M such that $|\mathbf{f}(x)| \le M$ for all $x \in E$.

4.14 Theorem *Suppose f is a continuous mapping of a compact metric space X into a metric space Y. Then $f(X)$ is compact.*

Proof Let $\{V_\alpha\}$ be an open cover of $f(X)$. Since f is continuous, Theorem 4.8 shows that each of the sets $f^{-1}(V_\alpha)$ is open. Since X is compact, there are finitely many indices, say $\alpha_1, \ldots, \alpha_n$, such that

$$X \subset f^{-1}(V_{\alpha_1}) \cup \cdots \cup f^{-1}(V_{\alpha_n}). \tag{12}$$

Since $f(f^{-1}(E)) \subset E$ for every $E \subset Y$, (12) implies that

$$f(X) \subset V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}. \tag{13}$$

This completes the proof.

Note: We have used the relation $f(f^{-1}(E)) \subset E$, valid for $E \subset Y$. If $E \subset X$, then $f^{-1}(f(E)) \supset E$; equality need not hold in either case.

We shall now deduce some consequences of Theorem 4.14.

4.15 Theorem *If $\mathbf{f}$ is a continuous mapping of a compact metric space X into R^k, then $\mathbf{f}(X)$ is closed and bounded. Thus, $\mathbf{f}$ is bounded.*

This follows from Theorem 2.41. The result is particularly important when f is real:

4.16 Theorem *Suppose f is a continuous real function on a compact metric space X, and*

$$M = \sup_{p \in X} f(p), \qquad m = \inf_{p \in X} f(p). \tag{14}$$

Then there exist points $p, q \in X$ such that $f(p) = M$ and $f(q) = m$.

The notation in (14) means that M is the least upper bound of the set of all numbers $f(p)$, where p ranges over X, and that m is the greatest lower bound of this set of numbers.

The conclusion may also be stated as follows: *There exist points p and q in X such that $f(q) \le f(x) \le f(p)$ for all $x \in X$*; that is, f attains its maximum (at p) and its minimum (at q).

Proof By Theorem 4.15, $f(X)$ is a closed and bounded set of real numbers; hence $f(X)$ contains

$$M = \sup f(X) \qquad \text{and} \qquad m = \inf f(X),$$

by Theorem 2.28.

4.17 Theorem *Suppose f is a continuous 1-1 mapping of a compact metric space X onto a metric space Y. Then the inverse mapping f^{-1} defined on Y by*

$$f^{-1}(f(x)) = x \qquad (x \in X)$$

is a continuous mapping of Y onto X.

Proof Applying Theorem 4.8 to f^{-1} in place of f, we see that it suffices to prove that $f(V)$ is an open set in Y for every open set V in X. Fix such a set V.

The complement V^c of V is closed in X, hence compact (Theorem 2.35); hence $f(V^c)$ is a compact subset of Y (Theorem 4.14) and so is closed in Y (Theorem 2.34). Since f is one-to-one and onto, $f(V)$ is the complement of $f(V^c)$. Hence $f(V)$ is open.

4.18 Definition Let f be a mapping of a metric space X into a metric space Y. We say that f is *uniformly continuous* on X if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d_Y(f(p), f(q)) < \varepsilon \tag{15}$$

for all p and q in X for which $d_X(p, q) < \delta$.

Let us consider the differences between the concepts of continuity and of uniform continuity. First, uniform continuity is a property of a function on a set, whereas continuity can be defined at a single point. To ask whether a given function is uniformly continuous at a certain point is meaningless. Second, if f is continuous on X, then it is possible to find, for each $\varepsilon > 0$ and for each point p of X, a number $\delta > 0$ having the property specified in Definition 4.5. This δ depends on ε *and* on p. If f is, however, uniformly continuous on X, then it is possible, for each $\varepsilon > 0$, to find *one* number $\delta > 0$ which will do for *all* points p of X.

Evidently, every uniformly continuous function is continuous. That the two concepts are equivalent on compact sets follows from the next theorem.