projective transformations could be used to define a "distance" $d(P,Q)$ in the unit disk—by saying $d(P,Q) = d(P',Q')$ if a transformation preserving the unit circle sends P to P' and Q to Q'—but he had not realized that the geometry obtained was that of Bolyai and Lobachevsky.

The pseudosphere is not entirely superseded by the projective model, since it remains the source of "real" distances and angles, whereas those in the projective model are necessarily distorted. One of the distinctive curves of the hyperbolic plane, the *horocycle*, or circle with center at infinity, is shown particularly clearly on the pseudosphere. If one imagines, following Beltrami (1868a), the pseudosphere wrapped by infinitely many turns of an infinitely thin covering, then the edge of this covering (along the rim of the pseudosphere) is a horocycle. The middle picture of Figure 18.6 shows the image of one turn of the covering, drawn solidly, and horocycles resulting from continued unwrapping are shown as dashed lines.

EXERCISES

Klein's three pictures illustrate the three types of *rigid motion* of the hyperbolic plane.

1. *Rotation*, in which one point of the plane is fixed and all other points move in *hyperbolic circles* about it. (A hyperbolic circle is the locus of a point moving at constant "distance" from a fixed point.)

2. *Limit rotation*, in which a point at infinity is fixed and all points of the plane move in horocycles centered on the fixed point at infinity.

3. *Translation*, in which a "line" moves along itself and the other points of the plane move along its *equidistant curves*. (An equidistant curve is the locus of a point moving at constant "distance" from a "line".)

18.4.1 Pick out *hyperbolic circles* and *equidistant curves* in the top and bottom pictures in Figure 18.6.

18.4.2 If the center of rotation in the top picture were not at the center of the disk, do you think the hyperbolic circles would be Euclidean circles?

18.4.3 Observe that equidistant curves at nonzero "distance" from the invariant "line" are *not* "lines." Does the translation move a point on an equidistant curve farther than a point on the invariant line?

18.4.4 Give an example of three points in the hyperbolic plane, not in a "line," that do not lie on a hyperbolic circle. (If this problem proves difficult, try it again after reading the next section.)

18.5 Beltrami's Conformal Models

The projective model of the hyperbolic plane distorts angles as well as lengths. One can see this with the asymptotic geodesics on the pseudosphere, which clearly tend to tangency at infinity yet are mapped onto lines meeting at a nonzero angle at the boundary of the unit disk (Figure 18.6). Beltrami (1868b) found that models with true angles—the so-called *conformal models*—could be obtained by sacrificing straightness of "lines." His basic conformal model is not, in fact, part of the plane but part of a hemisphere. It is erected over the projective model and its "lines" are vertical sections of the hemisphere (hence semicircles) over the "lines" of the projective model (Figure 18.8). The "distance" between points on the hemisphere is equal to the "distance" between the points beneath them in the projective model. Later we shall see that "distance" on the hemisphere also has a simple direct definition.

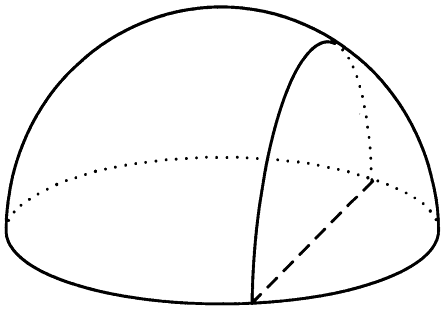

Figure 18.8: The hemisphere and the projective model

Two planar conformal models are obtained from the hemisphere model by stereographic projection, which, as we know from Section 16.2, preserves angles and sends circles to circles. The first of these is a disk (Figure 18.9) that, by change of scale, can again be taken as the unit disk. The second (Figure 18.10) is a half-plane, which we take to be the upper half-plane, $y > 0$. Since the "lines" in the hemisphere model are circular and orthogonal to the equator, "lines" in the planar conformal models are again circular, orthogonal to the boundary of the disk and half-plane, respectively, or straight lines in exceptional cases. To avoid continual mention of these exceptional cases—namely, line segments through the disk center and lines $x =$ constant in the half-plane—we consider lines to be circles of infinite radius.

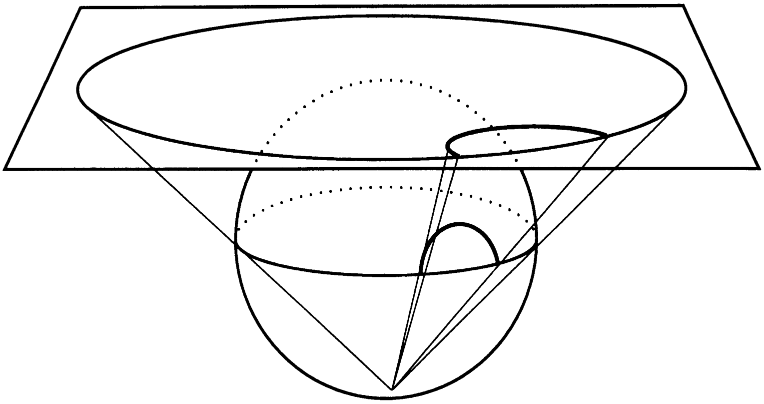

Figure 18.9: The conformal disk model

One of the beauties of the conformal models is that other important curves—hyperbolic "circles," horocycles, and equidistant curves—are also real circles. Each curve equidistant from a given "line" L is a circle through the end points of L on the boundary. Horocycles are circles tangential to the boundary and also, in the half-plane model, the lines $y =$ constant. A circle that does not meet the boundary is a hyperbolic "circle," but its "center," at equal "distance" from all its points, is not at the Euclidean center. Figure 18.11 shows some of these curves. Note also that asymptotic "lines" are tangential at "infinity" (the boundary) and that the boundary is their common perpendicular, thus resolving the situation that Saccheri (Section 18.1) thought to be contradictory.

"Distance" is particularly easy to express in the half-plane model. The "distance" ds between infinitesimally close points (x,y) and $(x+dx,y+dy)$ is

$$ds = \frac{\sqrt{dx^2+dy^2}}{y},$$

that is, the Euclidean distance divided by y. Thus "distance" $\to \infty$ as a point approaches the boundary $y = 0$ of the half-plane, as expected. Keeping x constant, we find by integration that "distance" along a vertical line increases exponentially with respect to Euclidean distance as y decreases. For example, the "distances" between the successive points at which $x = 0$ and $y = 1, \frac{1}{2}, \frac{1}{4}, \ldots$, are equal. The formula for ds was first obtained by Liouville (1850) by directly mapping the pseudosphere into the half-plane,

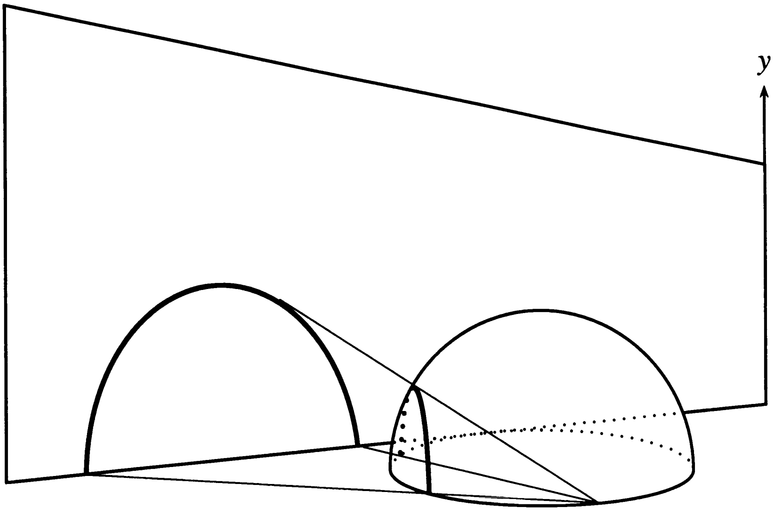

Figure 18.10: The conformal half-plane model

making simplifying changes of variable. However, Liouville did not realize that the half-plane with his "distance" formula was a model of hyperbolic geometry. The "distance" formula for the conformal disk had also been obtained before Beltrami, by Riemann (1854b), but again without noticing the hyperbolic geometry.

Beltrami (1868b) not only obtained these models, in a unified way, but he also extended the idea to n dimensions. For example, he gave a model of the three-dimensional space considered by Bolyai and Lobachevsky as the upper half, $z > 0$, of ordinary (x,y,z)-space, with "distance"

$$ds = \frac{\sqrt{dx^2 + dy^2 + dz^2}}{z}.$$

"Lines" are then semicircles orthogonal to $z = 0$ and "planes" are hemispheres orthogonal to $z = 0$. Restricting the "distance" function to such a hemisphere turns out to give Beltrami's hemisphere model. Thus the hemisphere model can be viewed as a hyperbolic plane lying in hyperbolic 3-space. The horospheres of the half-space model are spheres tangential to $z = 0$, together with the planes $z =$ constant. Beltrami (1868b) pointed out that on $z =$ constant we have

$$ds = \frac{\sqrt{dx^2 + dy^2 + dz^2}}{\text{constant}},$$

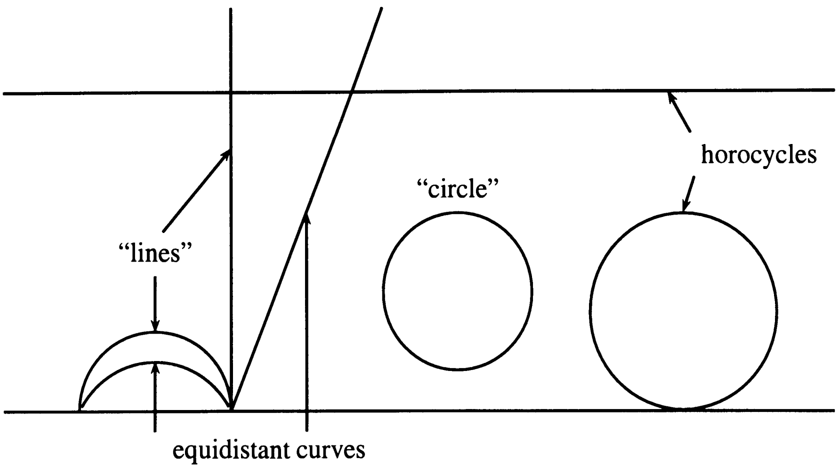

Figure 18.11: Some curves in the half-plane model

that is, "distance" is proportional to Euclidean distance. Thus he had an immediate proof of Wachter's wonderful theorem that the geometry of the horosphere is Euclidean.

EXERCISES

The mapping of the pseudosphere into the half-plane may be carried out as follows, using the parametric equations for the tractrix found in Exercise 17.5.2:

$$x = \sigma - \tanh\sigma, \quad y = \operatorname{sech}\sigma.$$

First we replace the parameter σ by the arc length τ along the tractrix.

18.5.1 Show that $\tau = \int_0^\sigma \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \log\cosh\sigma$, and hence $y = e^{-\tau}$.

Now take τ and the angle X of rotation as the coordinates on the pseudosphere obtained by rotating the tractrix about the x-axis.

18.5.2 Show that the length subtended by angle dX on a circular cross section of the pseudosphere is

$$y\,dX = e^{-\tau}dX,$$

and hence the distance between nearby points (X, τ) and $(X + dX, \tau + d\tau)$ on the pseudosphere is given by

$$ds^2 = e^{-2\tau}dX^2 + d\tau^2.$$

18.5.3 Finally, introduce the variable $Y = e^{\tau}$ and conclude that $ds = \dfrac{\sqrt{dX^2 + dY^2}}{Y}$.

Thus the pseudosphere is mapped into the (X,Y)-plane with preservation of distance, provided distance in the (X,Y) plane is defined by

$$ds = \frac{\sqrt{dX^2 + dY^2}}{Y}.$$

It follows, from what was said above, that geodesics on the pseudosphere correspond to semicircles with centers on the X-axis. This throws some light on the problem raised in Section 17.5—describing geodesics on the pseudosphere.

18.5.4 Explain why the region of the (X,Y)-plane corresponding to the pseudosphere is bounded by $X = 0$ and $X = 2\pi$ and it lies above some $Y =$ constant > 0.

18.5.5 By considering a semicircle crossing the region described in Exercise 18.5.4, show that there is no smooth closed geodesic on the pseudosphere.

18.6 The Complex Interpretations

One of the characteristics of the Euclidean plane is the existence of *regular tessellations*: tilings of the plane by regular polygons. There are of course three such tilings, based on the square, equilateral triangle, and regular hexagon (Figure 18.12). Associated with each tiling is a *group of rigid motions* of the plane that maps the tiling pattern onto itself. For example, the unit square pattern is mapped onto itself by unit translations parallel to the x and y axes and by the rotation of $\pi/2$ about the origin, and these three motions generate all motions of the tessellation onto itself. If we write $z = x + iy$, then these generating motions are given by the transformations

$$z \mapsto z+1, \qquad z \mapsto z+i, \qquad z \mapsto zi.$$

The triangle and hexagon tessellations have a similar group of motions, generated by

$$z \mapsto z+1, \quad z \mapsto z+\tau, \quad z \mapsto z\tau,$$

where $\tau = e^{i\pi/3}$ is the third vertex of the equilateral triangle whose other vertices are at 0, 1 (Figure 18.13). More generally, any motion of the Euclidean plane can be composed from translations $z \mapsto z + a$ and rotations $z \mapsto ze^{i\theta}$.

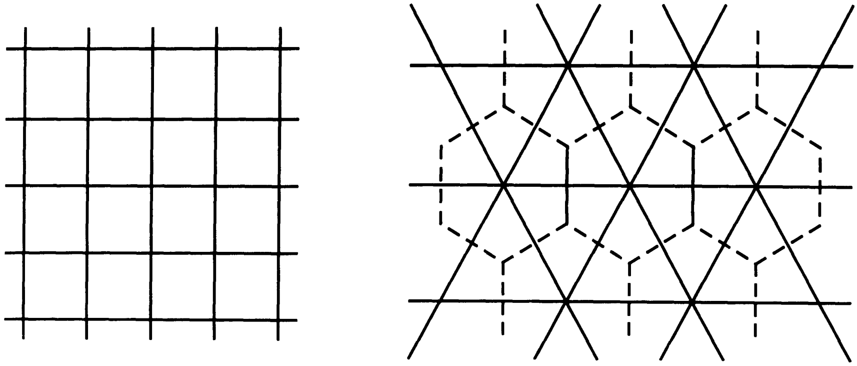

Figure 18.12: Tessellations of the Euclidean plane

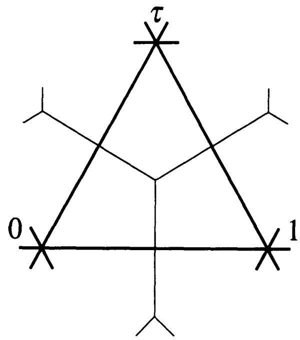

Figure 18.13: Relation between the triangle and hexagon tessellations

The sphere also admits a finite number of regular tessellations, obtained by central projections of the regular polyhedra (Section 2.2). Figure 18.14 shows the spherical tessellation corresponding to the icosahedron. (Each face has been further subdivided into six congruent triangles.) The motions that map such a tessellation onto itself can also be expressed as complex transformations by interpreting the sphere as $\mathbb{C} \cup \{\infty\}$ via stereographic projection (Section 16.2). Gauss (1819) found that any motion of the sphere can be expressed by a transformation of the form

$$z \mapsto \frac{az+b}{-\bar{b}z+\bar{a}},$$

where $a, b \in \mathbb{C}$ and an overbar denotes the complex conjugate.

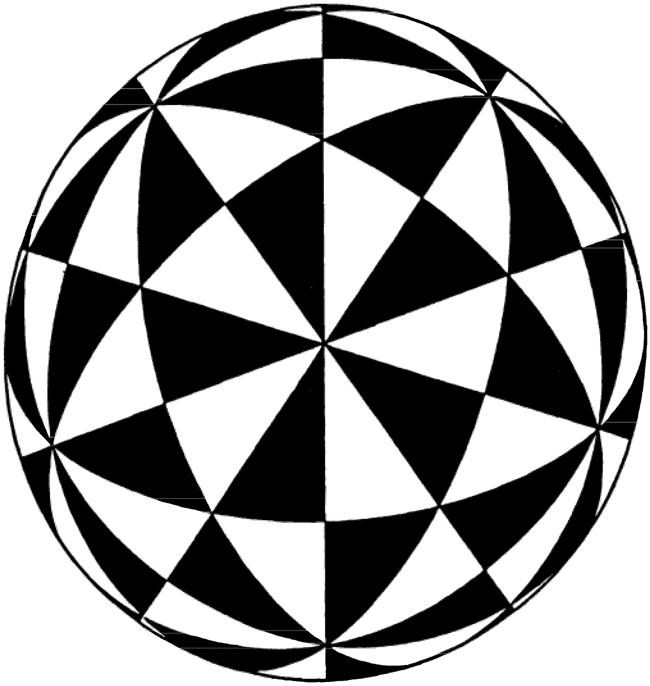

Figure 18.14: Icosahedral tessellation of the sphere

The conformal models of the hyperbolic plane can be regarded as parts of $\mathbb{C}$: the unit disk $\{z : |z| < 1\}$ and the half-plane $\{z : \mathrm{Im}(z) > 0\}$. Their rigid motions, being conformal transformations, are complex functions, and Poincaré (1882) made the beautiful discovery that they are of the form

$$z \mapsto \frac{az+b}{\overline{b}z+\overline{a}}$$

(for the disk) and

$$z \mapsto \frac{\alpha z+\beta}{\gamma z+\delta},$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ (for the half-plane). Infinitely many regular tessellations are possible, since the angles of a regular n-gon can be made arbitrarily small by increasing its area. For example, there are tessellations by equilateral triangles in which n triangles meet at each vertex, for each $n \geq 7$, and similar variety is possible for other polygons (see exercises).

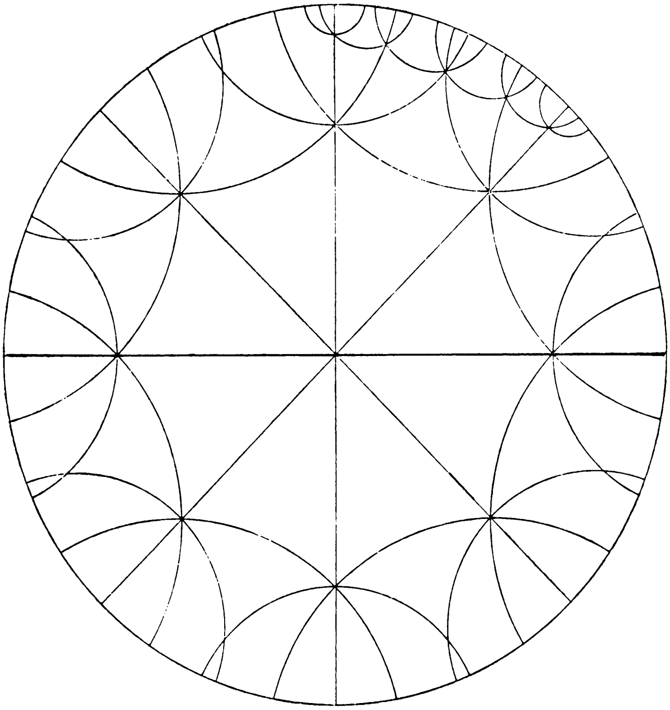

Figure 18.15: The Gauss tessellation

Some of these tessellations were known *before* Poincaré (1882) gave the complex interpretation of hyperbolic geometry, and even before any model of hyperbolic geometry was known at all. Figure 18.15 shows a tessellation by equilateral triangles of angle $\pi/4$ found in unpublished, and unfortunately undated, work of Gauss (*Werke*, vol. VIII, p. 104).

Others arise from the so-called hypergeometric differential equation and were rediscovered in the same context by Riemann (1858b) and Schwarz (1872) (the first published example, Figure 18.16). By explaining these tessellations in terms of hyperbolic geometry, Poincaré (1882) showed for the first time that hyperbolic geometry was part of preexisting mathematics, whose geometric nature had not previously been understood.

In a subsequent paper, Poincaré (1883) explained the geometric nature