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Scales of Notation

The ancient Egyptian language belongs to the Hamito-Semitic group of languages. Like the Indo-European group, it contains a system of counting by tens, undoubtably arising from the habit of counting using one's fingers. The notation used for writing numbers is also clearly based on the scale of ten. For some reason, standard French departs from decimal nomenclature; it expresses 97 as 4 times 20 plus 17. This seems odd, since it was the French who introduced the decimal system for weights and measures.

Some African languages express numbers in the scale of five. One may express natural numbers in any scale b, where b is an integer greater than 1, since every natural number is uniquely expressible in the form

$$a = a_0 + a_1 b + a_2 b^2 + a_3 b^3 + \cdots + a_n b^n,$$

where $0 \leq a_i < b$ (for $i = 0, 1, \ldots, n$). We write this more briefly as

$$a = (a_n a_{n-1} \cdots a_2 a_1 a_0)_b.$$

If there is no doubt which scale is in use, the subscript b may be dropped.

The Egyptians had a number system based on the scale of ten, but, as we saw above, they often worked with scale two: to multiply by 12, Ahmose expressed 12 as $4+8$, that is, $2^2 + 2^3$, or $12 = (1100)_2$. The Egyptians also took $b = 7$ in some of their calculations (Gillings, p. 227), since there are seven *palms* in a *cubit.* They had no symbol for zero; instead they used special symbols for different powers of ten.

The binary scale (with $b = 2$) shows up in the Chinese *Book of Changes* (1200 BC), a system of divination in which each six place binary number represents some concept. The digit 1 was associated with the male 'yang', and the digit 0 with the female 'yin'. The number $34 = (100010)_2$ was supposed to represent 'progress and success'. The binary scale also shows up in the Hindu classification of meters in verse, about 800 BC. Finally, it is of course used in the modern computer. The digit 1 is represented by a current, and the digit 0 by the absence of a current. Number scales are often found in recreational mathematics, as in the following three problems.

Six Weight Problem: A balance is a weighing apparatus with a central pivot, a beam, two scales and a set of counter-weights that are placed in one of the scales. Suppose we have some flour and we want to be able to put it into bags weighing anywhere from one to sixty-three kilograms. How can this be done using just six counter-weights?

Answer: Weights of 1, 2, 4, 8, 16 and 32 kilograms will allow you to weigh any integral load from 1 to $32 + 16 + 8 + 4 + 2 + 1 = 63$ kilograms.

Four Weight Problem: This time, suppose we are allowed to put weights on either scale. How can we weigh bags under 42 kilograms using only four weights?

Answer: Choose weights of 1, 3, 9 and 27 kilograms, since any integer a can be written uniquely in the form

$$a = a_0 + a_1 3 + a_2 3^2 + \cdots + a_n 3^n,$$

where each a_i is one of -1, 0 or 1.

The Game of Nim: This so-called Chinese game is played by two opponents, who take turns removing matches from several piles according to the following rules:

1. A player must remove at least one match in a turn.
2. A player may remove any number of matches from a single pile in a turn.

The player who removes the last match wins. Find a strategy for winning this game.

Answer: Express the number of matches in each pile in the scale of two and write these binary numbers one below the other. If, when it is your turn, you can arrange it so that each column adds up to an even number, then you can do the same in every subsequent turn and you will win the game.

As an example, suppose there are three piles containing 7, 5 and 3 matches. It is your turn. In binary notation, the piles contain the following number of matches:

$$\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 0 & 1 \\ & 1 & 1 \end{array}$$

To make the number of 1's in each column even, you take a match from the first pile, leading to

$$\begin{array}{ccc} 1 & 1 & 0 \\ 1 & 0 & 1 \\ & 1 & 1 \end{array}$$

Your opponent takes 2 matches, say, from the third pile, leaving

$$\begin{array}{ccc} 1 & 1 & 0 \\ 1 & 0 & 1 \\ & & 1 \end{array}$$

You take two matches from the first pile, yielding

$$\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 0 & 1 \\ & & 1 \end{array}$$

Your opponent then removes all the matches from the first pile, say, resulting in

$$\begin{array}{ccc} 1 & 0 & 1 \\ & & 1 \end{array}$$

You now take 4 matches from the first pile, leaving just one match in each pile. Your opponent has to take one of them, and you win by taking the last match.

What is going on in the Six Weight Problem? It is easily seen that the following three statements are equivalent:

1. Every integral load less than 64 kg can be weighed uniquely with 6 weights: 1, 2, 4, 8, 16 and 32 kg.

2. Every natural number less than 64 can be expressed uniquely as the sum of distinct powers of 2.

3. Every natural number less then 64 can be written uniquely in the scale of 2 with up to 6 digits, each 0 or 1.

A direct proof is quite easy, but a proof in the spirit of the 18th century is more interesting. In preparation for this proof, let us look at the following multiplication:

$$(1+x^2)(1+x^3)(1+x^5)(1+x^7) = 1+x^2+x^3+2x^5+2x^7+x^8+x^9+2x^{10}+\dots$$

Why are the coefficients of x^5, x^4 and x^3 equal to 2, 0 and 1, respectively? Because $5 = 2 + 3$ can be written as the sum of some of 2, 3, 5 and 7 in two ways (the first sum consists of one term only), 4 cannot be written as the sum of some of 2, 3, 5 and 7 at all, and 3 can only be written as the sum of these numbers in one way, the sum having one term. In general, the coefficient of x^n will be the number of ways in which n can be written as the sum of distinct members of the set $\{2, 3, 5, 7\}$.