EXERCISES

There is only one Dedekind cut $(L, U)$ corresponding to an irrational number $\alpha$, but there are two cuts corresponding to a rational number $a$:

$$L = \{r : r \le a\}, \quad U = \{r : r > a\}$$

and

$$L = \{r : r < a\}, \quad U = \{r : r \ge a\}.$$

To unify the theory of all reals we choose the latter cut, call it

$$L_a = \{r : r < a\}, \quad U_a = \{r : r \ge a\},$$

as the standard way to represent a rational $a$. We can then say, whether $x$ is rational or irrational, that the lower set for $x$ is

$$L_x = \{r : r < x\}.$$

Now we use lower sets to define $x + y$ and $xy$ for positive reals $x$ and $y$ as follows:

$$L_{x+y} = \{r + s : r < x \text{ and } s < y, \text{ where } r, s \text{ are rational}\}$$
$$L_{xy} = \{rs : r < x \text{ and } s < y, \text{ where } r, s \text{ are rational}\}.$$

**4.2.1** Show that these are valid definitions of $x + y$ and $xy$ when $x$ and $y$ are rational.

The true power of these definitions, as Dedekind realized, is that they allow rigorous proofs of results like $\sqrt{2}\sqrt{3} = \sqrt{6}$ that (in Dedekind's opinion) had never been rigorously proved before. Such proofs are possible, but still not trivial. Even to prove that $\sqrt{2}\sqrt{2} = 2$ one still has to prove the next two results.

**4.2.2** If $r^2 < 2$ and $s^2 < 2$, show that $rs < 2$.

**4.2.3** If a rational $t < 2$, show that $t = rs$ for some rational $r, s$ with $r^2 < 2, s^2 < 2$.

**4.2.4** Why do Exercises 4.2.2 and 4.2.3 show that $\sqrt{2}\sqrt{2} = 2$?

**4.2.5** Give a similar proof that $\sqrt{2}\sqrt{3} = \sqrt{6}$.

## 4.3 The Method of Exhaustion

The method of exhaustion, also credited to Eudoxus, is a generalization of his theory of proportions. Just as an irrational length is determined by the rational lengths on either side of it, more general unknown quantities become determined by arbitrarily close approximations using known figures. Examples given by Eudoxus (and expounded in Book XII of Euclid's

*Elements*) are an approximation of the circle by inner and outer polygons (Figure 4.1) and an approximation of a pyramid by stacks of prisms (Figure 4.2, which shows the most obvious approximation, not the cunning one actually used by Euclid). In both cases the approximating figures are known quantities, on the basis of the theory of proportions and the theorem that area of triangle $= 1/2$ base $\times$ height.

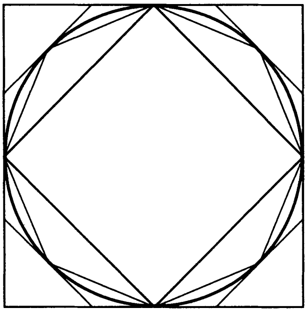

Figure 4.1: Approximating the circle

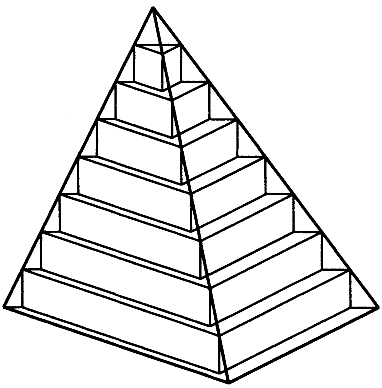

Figure 4.2: Approximating the pyramid

The polygonal approximations are used to show that the area of any circle is proportional to the square on its radius, as follows. Suppose $P_1 \subset P_2 \subset P_3 \subset \cdots$ are the inner polygons and $Q_1 \supset Q_2 \supset Q_3 \supset \cdots$ are the outer polygons. Each polygon is obtained from its predecessor by bisecting the arcs between its vertices, as shown in Figure 4.1. It can then be shown, by elementary geometry, that the area difference $Q_i - P_i$ can be made arbitrarily small, and hence $P_i$ approximates the area $C$ of the circle arbitrarily closely.

On the other hand, elementary geometry also shows that the area $P_i$ is proportional to the square, $R^2$, of the radius. Writing the area as $P_i(R)$ and using the theory of proportions to handle ratios of areas, we have

$$P_i(R) : P_i(R') = R^2 : R'^2. \tag{1}$$

Now let $C(R)$ denote the area of the circle of radius $R$, and suppose

$$C(R) : C(R') < R^2 : R'^2. \tag{2}$$

By choosing a $P_i$ that approximates $C$ sufficiently closely we also get

$$P_i(R) : P_i(R') < R^2 : R'^2,$$

which contradicts (1). Hence the $<$ sign in (2) is incorrect, and we can similarly show that $>$ is incorrect. Thus the only possibility is

$$C(R) : C(R') = R^2 : R'^2,$$

that is, the area of a circle is proportional to the square of its radius.

Notice that "exhaustion" does not mean using an infinite sequence of steps to show that area is proportional to the square of the radius. Rather, one shows that any *disproportionality* can be refuted in a *finite* number of steps (by going to a suitable $P_i$). This is typical of the way in which exhaustion arguments avoid mention of limits and infinity.

In the case of the pyramid, one uses elementary geometry again to show that stacks of prisms approximate the pyramid arbitrarily closely. Then exhaustion shows that the volume of a pyramid, like that of a prism, is proportional to base $\times$ height (see exercises below). Finally, there is a clever argument to show that the constant of proportionality is $1/3$. We can restrict to the case of triangular pyramids (since any pyramid can be cut into these), and Figure 4.3 shows how a triangular prism is cut into three triangular pyramids. Any two of these pyramids can be seen to have equal base and height—although which face is taken to be the base depends on which pyramids are being compared—hence all three are equal in volume. Each is therefore one-third of the prism, that is, $1/3$ base $\times$ height.

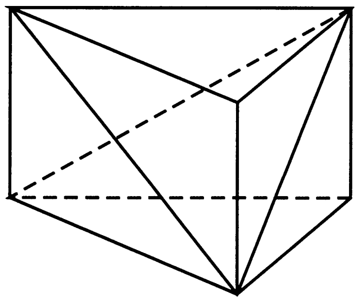

Figure 4.3: Cutting a prism into pyramids

It is interesting that Euclid does *not* need the method of exhaustion in the theory of area for polygons. All this can be done by dissection arguments such as that showing area of triangle $= 1/2$ base $\times$ height (Figure 4.4). In fact, it was shown by Farkas Bolyai (1832a) that any polygons $P$, $Q$ of equal area can be cut into polygonal pieces $P_1, \ldots, P_n$ and $Q_1, \ldots, Q_n$ such that $P_i$ is congruent to $Q_i$. Thus we can *define* polygons to be equal in area if they possess dissections into such correspondingly congruent pieces.

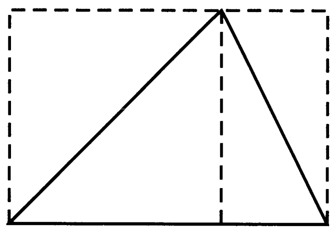

Figure 4.4: Area of a triangle

In Hilbert's famous list of mathematical problems [Hilbert (1900a)], the third was to decide whether an analogous definition was possible for polyhedra. Dehn (1900) showed that it was not; in fact, a tetrahedron and a cube of equal volume cannot be dissected into corresponding congruent polyhedral pieces. Hence infinite processes of some kind, such as the method of exhaustion, are needed to define equality of volume. A readable account of Dehn's theorem and related results may be found in Boltyansky (1978).

EXERCISES

Although the method of exhaustion is not needed for the area theory of polygons, it is nevertheless a helpful stepping stone toward cases where exhaustion *is* necessary, such as volumes of polyhedra or areas of curved regions.

**4.3.1** Show that the area of two triangles with the same base and height can be approximated arbitrarily closely by the same set of rectangles, differently stacked (Figure 4.5).

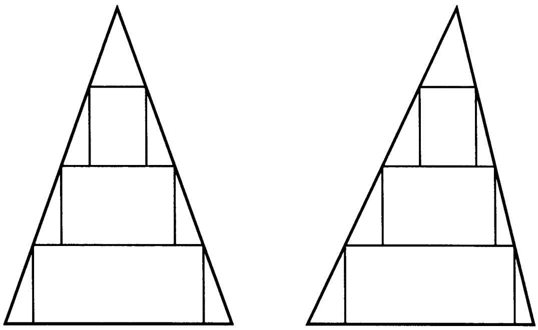

Figure 4.5: Approximations to triangles

**4.3.2** Show similarly that any two tetrahedra with the same base and height can be approximated arbitrarily closely by the same prisms, differently stacked (Figure 4.6).

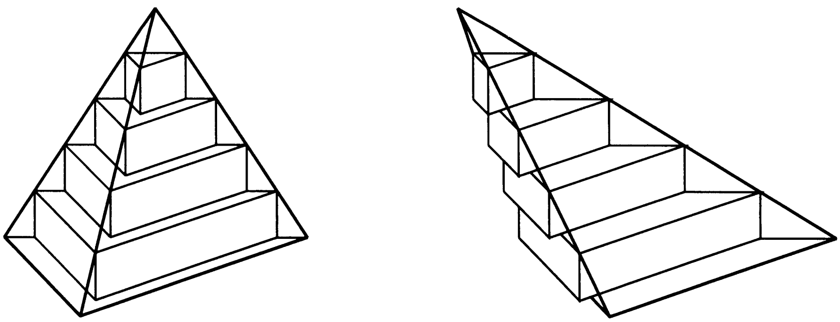

Figure 4.6: Approximations to tetrahedra

Around 1800, Legendre used the result of Exercise 4.3.2 to give another proof that the volume of a pyramid is 1/3 that of a prism with the same base and height [see Heath (1925), Book XII, Proposition 5]. He used the above dissection of a prism into three tetrahedra, pairwise of the same base and height, so he only had to do the following.

**4.3.3** Deduce from Exercise 4.3.2 that pyramids of equal base and height have equal volume.

Another interesting approach to the volume of the tetrahedron by exhaustion was given by Euclid [see Heath (1925), Book XII, Proposition 4]. He dissected the tetrahedron into two smaller tetrahedra and two prisms as shown in Figure 4.7, with vertices at the edge midpoints of the original tetrahedron.

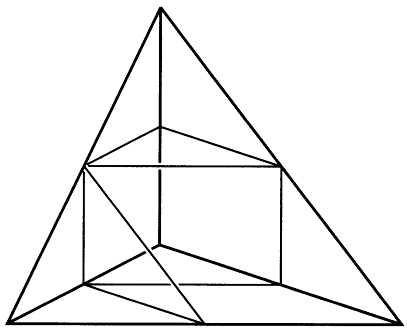

Figure 4.7: Euclid's dissection of the tetrahedron

**4.3.4** Show that the two prisms occupy more than half the volume of the tetrahedron. (Hence, by iterating the construction in the smaller tetrahedra, the volume of the tetrahedron may be approximated arbitrarily closely by prisms.)

**4.3.5** Show that the volume of the two prisms in Figure 4.7 is 1/4 base $\times$height (the base and height of the tetrahedron, that is).

By computing the volumes of the corresponding prisms in the smaller tetrahedra, we find the volume of the original tetrahedron as a sum of a geometric series.

**4.3.6** Show that the total volume of the prisms is

$$\left(\frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+\cdots\right) \text{base} \times \text{height} = 1/3 \text{ base} \times \text{height}.$$

In the next section we study a construction of Archimedes that is curiously similar to this one of Euclid. Each step cuts pieces out of the leftovers from the previous step and leads to a similar geometric series.

## 4.4 The Area of a Parabolic Segment

The method of exhaustion was brought to full maturity by Archimedes (287–212 BCE). Among his most famous results were the volume and surface area of the sphere and the area of a parabolic segment. As mentioned in Section 4.1, Archimedes first discovered these results by nonrigorous methods, later confirming them by the method of exhaustion. Perhaps the most interesting and natural of his exhaustion proofs is the one for the area of the parabolic segment. The segment is exhausted by polygons similarly to Eudoxus' exhaustion of the circle, but the area is obtained outright and not merely in proportion to another figure.

To simplify the construction slightly we assume that the segment is cut off by a chord perpendicular to the axis of the symmetry of the parabola. Archimedes divides the parabolic segment into triangles $\Delta_1, \Delta_2, \Delta_3, \ldots$, as shown in Figure 4.8 (labeled by their subscripts). The middle vertex of each triangle lies on the parabola halfway between the other two (measured horizontally). These triangles clearly exhaust the parabolic segment, and so it remains to compute their area. Quite surprisingly, this turns into a geometric series.

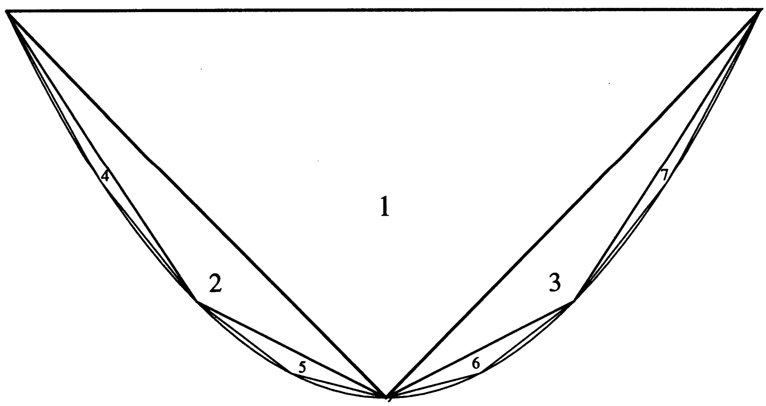


Figure 4.8: The parabolic segment

We shall briefly indicate how this comes about by studying $\Delta_3$ (Figure 4.9). Since $OP = \frac{1}{2}OX$, $PQ = \frac{1}{4}PS$ by definition of the parabola. On the other hand, $SR = \frac{1}{2}PS$, hence $QR = \frac{1}{4}PS$. Now $\Delta_3$ is the sum of the triangles $RQZ$ and $OQR$, which have the same base $RQ$ and "height" $OP = PX$, hence equal area. We have just seen that $RQZ$ has half the base of $SRZ$ and it has the same height, hence (calling figures equal when they have the same area)

$$\begin{aligned}\Delta_3 &= SRZ\\ &= \frac{1}{4}OYZ\\ &= \frac{1}{8}\Delta_1.\end{aligned}$$

By symmetry, $\Delta_2 = \Delta_3$, so $\Delta_2 + \Delta_3 = \frac{1}{4}\Delta_1$.

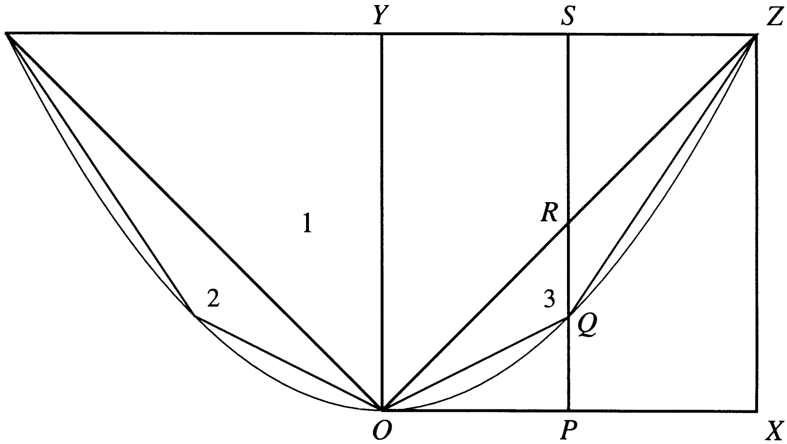


Figure 4.9: A triangle in the segment

A similar argument shows

$$\Delta_4 + \Delta_5 + \Delta_6 + \Delta_7 = \frac{1}{16}\Delta_1$$

and so on, each new chain of triangles having one-fourth the area of the previous chain. Consequently,

$$\begin{aligned}\text{area of parabolic segment} &= \Delta_1\left(1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \cdots\right)\\ &= \frac{4}{3}\Delta_1.\end{aligned}$$

Of course, Archimedes does not use the infinite series but uses exhaustion, showing that any area $< \frac{4}{3}\Delta_1$ can be exceeded by taking sufficiently many of the triangles $\Delta_i$. The sum of the *finite* geometric series needed for this was known from Euclid's *Elements*, Book IX, where Euclid used it for the theorem about perfect numbers (Section 3.2).

## EXERCISES

Archimedes' method of approximation by triangles was a brilliant success on the parabolic segment, but not suited to many other curves. A more generally useful method is approximation by rectangles, probably known to you from calculus. The area of a parabolic segment can also be computed this way, though less gracefully, and indeed Archimedes did this too. We look at other curved areas that can be evaluated by rectangle approximation in Section 9.2.

Probably the simplest area that *cannot* be found by this method is the area under the hyperbola $y = 1/x$, from $x = 1$ to $x = t$. This is because the area in question is $\log t$, and the logarithm function cannot be defined by elementary means. But if instead one takes the area to be $\log t$ *by definition*, then it is possible to derive the basic *property* of the logarithm—

$$\log ab = \log a + \log b$$

—and by means Archimedes would have understood.

**4.4.1** Suppose we approximate the area $\log a$ under $y = 1/x$ from 1 to $a$ by $n$ rectangles of equal width, as shown in Figure 4.10.

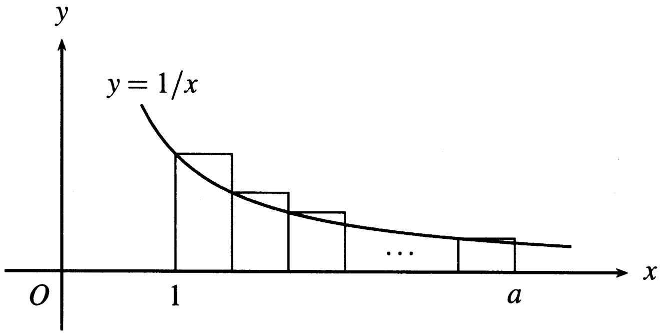


Figure 4.10: Rectangle approximation to $\log a$

Show that the corresponding approximation to the area under $y = 1/x$ from $b$ to $ab$ by $n$ rectangles has exactly the same area. (In fact, corresponding rectangles have equal area.)