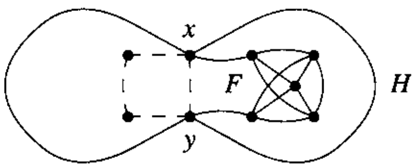
6.2.7. Lemma. If G is a graph with fewest edges among all nonplanar graphs without Kuratowski subgraphs, then G is 3-connected.

Proof: Deleting an edge of G cannot create a Kuratowski subgraph in G. The hypothesis thus guarantees that deleting one edge produces a planar subgraph, and hence G is a minimal nonplanar graph. By Lemma 6.2.5, G is 2-connected.

Suppose that G has a separating 2-set $S = \{x, y\}$. Since G is nonplanar, the union of xy with some S-lobe is nonplanar (Lemma 6.2.6); let H be such a graph. Since H has fewer edges than G, the minimality of G forces H to have a Kuratowski subgraph F. All of F appears in G except possibly the edge xy.

Since S is a minimal vertex cut, both x and y have neighbors in every S-lobe. Thus we can replace xy in F with an x, y-path through another S-lobe to obtain a Kuratowski subgraph of G. This contradicts the hypothesis that G has no Kuratowski subgraph, so G has no separating 2-set. ∎



CONVEX EMBEDDINGS

To complete the proof of Kuratowski's Theorem, it suffices to prove that 3-connected graphs without Kuratowski subgraphs are planar. We will use induction. In order to facilitate the proof of the induction step, it is helpful to prove a stronger statement.

6.2.8. Definition. A **convex embedding** of a graph is a planar embedding in which each face boundary is a convex polygon.

Tutte [1960, 1963] proved that every 3-connected planar graph has a convex embedding. This is best possible in terms of connectivity, since for $n \geq 4$ the 2-connected planar graph $K_{2,n}$ has no convex embedding. We follow Thomassen's approach to proving Kuratowski's Theorem by proving Tutte's stronger conclusion for 3-connected graphs without Kuratowski subgraphs. (Another proof of Tutte's result is based on ear decompositions—Kelmans [2000].)

We prove this theorem of Tutte by induction on $n(G)$. The paradigm for proving conditional statements by induction (Remark 1.3.25) tells us what lemmas we need. Our hypotheses are "3-connected" and "no Kuratowski subgraph"; our conclusion is "convex embedding". For a graph G satisfying the hypotheses, we need to find a smaller graph G' that satisfies *both* hypotheses in order to apply the induction hypothesis.

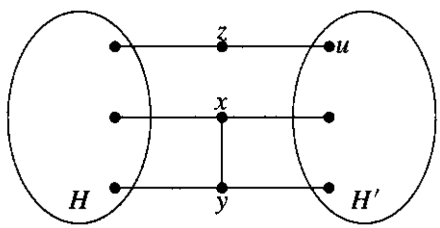
The first lemma allows us to obtain a smaller 3-connected graph G' by contracting some edge in G. The second shows that G' will also satisfy the hypothesis of having no Kuratowski subgraph. The proof will then be completed by obtaining a convex embedding of G from a convex embedding of G'.

6.2.9. Lemma. (Thomassen [1980]) Every 3-connected graph G with at least five vertices has an edge e such that $G \cdot e$ is 3-connected.

Proof: We use contradiction and extremality. Consider an edge e with endpoints x, y. If $G \cdot e$ is not 3-connected, then it has a separating 2-set S. Since G is 3-connected, S must include the vertex obtained by shrinking e. Let z denote the other vertex of S and call it the *mate* of the adjacent pair x, y. Note that $\{x, y, z\}$ is a separating 3-set in G.

Suppose that G has no edge whose contraction yields a 3-connected graph, so every adjacent pair has a mate. Among all the edges of G, choose $e = xy$ and their mate z so that the resulting disconnected graph $G - \{x, y, z\}$ has a component H with the largest order. Let H' be another component of $G - \{x, y, z\}$ (see the figure below). Since $\{x, y, z\}$ is a minimal separating set, each of x, y, z has a neighbor in each of H, H'. Let u be a neighbor of z in H', and let v be the mate of u, z.

By the definition of "mate", $G - \{z, u, v\}$ is disconnected. However, the subgraph of G induced by $V(H) \cup \{x, y\}$ is connected. Deleting v from this subgraph, if it occurs there, cannot disconnect it, since then $G - \{z, v\}$ would be disconnected. Therefore, $G_{V(H)\cup\{x,y\}} - v$ is contained in a component of $G - \{z, u, v\}$ that has more vertices than H, which contradicts the choice of x, y, z. ■



Next we need to show that edge contraction preserves the absence of Kuratowski subgraphs. We introduce a convenient term: the **branch vertices** in a subdivision H' of H are the vertices of degree at least 3 in H'.

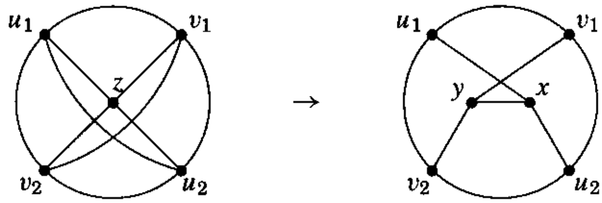
6.2.10. Lemma. If G has no Kuratowski subgraph, then also $G \cdot e$ has no Kuratowski subgraph.

Proof: We prove the contrapositive: If $G \cdot e$ contains a Kuratowski subgraph, then so does G. Let z be the vertex of $G \cdot e$ obtained by contracting $e = xy$. If z is not in H, then H itself is a Kuratowski subgraph of G. If $z \in V(H)$ but z is not a branch vertex of H, then we obtain a Kuratowski subgraph of G from H by replacing z with x or y or with the edge xy.

Similarly, if z is a branch vertex in H and at most one edge incident to z in

H is incident to x in G, then expanding z into xy lengthens that path, and y is the corresponding branch vertex for a Kuratowski subgraph in G.

In the remaining case (shown below), H is a subdivision of K_5 and z is a branch vertex, and the four edges incident to z in H consist of two incident to x and two incident to y in G. In this case, let u_1, u_2 be the branch vertices of H that are at the other ends of the paths leaving z on edges incident to x in G, and let v_1, v_2 be the branch vertices of H that are at the other ends of the paths leaving z on edges incident to y in G. By deleting the u_1, u_2-path and v_1, v_2-path from H, we obtain a subdivision of $K_{3,3}$ in G, in which y, u_1, u_2 are the branch vertices for one partite set and x, v_1, v_2 are the branch vertices of the other. ■



K_5 subdivision in G' $\quad$ $K_{3,3}$ subdivision in G

Now we can prove Tutte's Theorem.

6.2.11. Theorem. (Tutte [1960, 1963]) If G is a 3-connected graph with no subdivision of K_5 or $K_{3,3}$, then G has a convex embedding in the plane with no three vertices on a line.

Proof: (Thomassen [1980, 1981]) We use induction on $n(G)$.

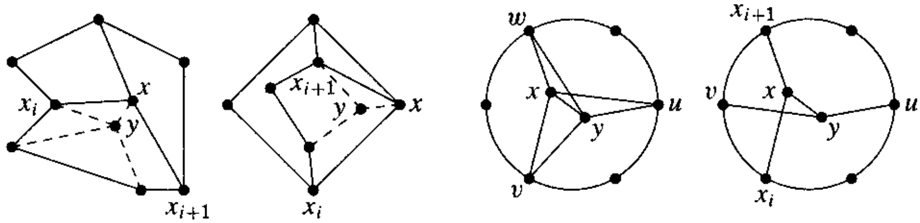
Basis step: $n(G) \leq 4$. The only 3-connected graph with at most four vertices is K_4, which has such an embedding.

Induction step: $n(G) \geq 5$. Let e be an edge such that $G \cdot e$ is 3-connected, as guaranteed by Lemma 6.2.9. Let z be the vertex obtained by contracting e. By Lemma 6.2.10, $G \cdot e$ has no Kuratowski subgraph. By the induction hypothesis, we obtain a convex embedding of $H = G \cdot e$ with no three vertices on a line.

In this embedding, the subgraph obtained by deleting the edges incident to z has a face containing z (perhaps unbounded). Since $H - z$ is 2-connected, the boundary of this face is a cycle C. All neighbors of z lie on C; they may be neighbors in G of x or y or both, where x and y are the original endpoints of e.

The convex embedding of H includes straight segments from z to all its neighbors. Let $x_1, \ldots, x_k$ be the neighbors of x in cyclic order on C. If all neighbors of y lie in the portion of C from x_i to x_{i+1}, then we obtain a convex embedding of G by putting x at z in H and putting y at a point close to z in the wedge formed by xx_i and xx_{i+1}, as shown in the diagrams for Case 0 below.

If this does not occur, then either 1) y shares three neighbors u, v, w with x, or 2) y has neighbors u, v that alternate on C with neighbors x_i, x_{i+1} of x. In Case 1, C together with xy and the edges from $\{x, y\}$ to $\{u, v, x\}$ form a subdivision of K_5. In Case 2, C together with the paths uyv, x_ixx_{i+1}, and xy form a subdivision of $K_{3,3}$. Since we are considering only graphs without Kuratowski subgraphs, in fact Case 0 must occur. ■



Case 0 Case 1 Case 2

Together, Lemma 6.2.7 and Theorem 6.2.11 imply Kuratowski's Theorem (Theorem 6.2.2). Fáry's Theorem can be obtained separately: if a graph has a planar embedding, then it has a straight-line planar embedding (Exercise 6).

For applications in computer science, we want more—a straight-line planar embedding in which the vertices are located at the integer points in a relatively small grid. Schnyder [1992] proved that every n-vertex planar graph has a straight-line embedding in which the vertices are located at integer points in the grid $[n-1] \times [n-1]$.

Many other characterizations of planar graphs have been proved; some are mentioned in the exercises. We describe two additional characterizations.

6.2.12.* Definition. A graph H is a **minor** of a graph G if a copy of H can be obtained from G by deleting and/or contracting edges of G.

For example, K_5 is a minor of the Petersen graph, although the Petersen graph does not contain a subdivision of K_5.

6.2.13.* Remark. Deletions and contractions can be performed in any order, as long as we keep track of which edge is which. Thus the minors of G can be described as "contractions of subgraphs of G".

If G contains a subdivision of H, say H', then H also is a minor of G, obtained by deleting the edges of G not in H' and then contracting edges incident to vertices of degree 2. If H has maximum degree at most 3, then H is a minor of G if and only if G contains a subdivision of H (Exercise 11).

Wagner [1937] proved that a graph G is planar if and only if neither K_5 nor $K_{3,3}$ is a minor of G. Exercise 12 obtains this from Kuratowski's Theorem. ■

6.2.14.* Remark. Some characterizations are more closely related to actual embeddings. For example, when a 3-connected graph is drawn in the plane, deleting the vertex set of a facial cycle leaves a connected subgraph.

We say that a cycle in a graph is **nonseparating** if its vertex set is not a separating set. Kelmans [1980, 1981b] proved that a subdivision of a 3-connected graph is planar if and only if every edge e lies in exactly two nonseparating cycles. Kelmans [1993] surveys related material. ■

PLANARITY TESTING (optional)

Dirac and Schuster [1954] gave the first short proof of Kuratowski's Theorem. Appearing in Harary [1969, 109–112], Bondy–Murty [1976, p153–156], and Chartrand–Lesniak [1986, p96–98], it uses special subgraphs of a graph.

6.2.15. Definition. When H is a subgraph of G, an H**-fragment** of G is either
1) an edge not in H whose endpoints are in H, or
2) a component of $G - V(H)$ together with the edges (and vertices of attachment) that connect it to H.

Together with the subgraph H itself, the H-fragments form a decomposition of G. The H-fragments are the "pieces" that must be added to an embedding of H to obtain an embedding of G. Historically, the term "H-bridge" was used; we use "H-fragment" to avoid confusion with other uses of "bridge".

An H-fragment differs from a $V(H)$-lobe because the H-fragment omits the edges of H. Also, an H-fragment may be a single edge not in H but joining vertices of H, since H need not be an induced subgraph.

For the 3-connected case of Kuratowski's Theorem, Dirac and Schuster considered a minimal nonplanar 3-connected graph G with no Kuratowski subgraph. Deleting an edge e yields a planar 2-connected graph. After choosing a cycle C through the endpoints of e, we can add e to the embedding unless there is a C-fragment embedded inside C and another embedded outside C that "conflict" with e. As in the proof of Theorem 6.2.11, this produces a Kuratowski subgraph of G. Tutte used the idea of conflicting C-fragments to obtain another characterization of planar graphs.

6.2.16. Definition. Let C be a cycle in a graph G. Two C-fragments A, B **conflict** if they have three common vertices of attachment to C or if there are four vertices v_1, v_2, v_3, v_4 in cyclic order on C such that v_1, v_3 are vertices of attachment of A and v_2, v_4 are vertices of attachment of B. The **conflict graph** of C is a graph whose vertices are the C-fragments of G, with conflicting C-fragments adjacent.

Tutte [1958] proved that G is planar if and only if the conflict graph of each cycle in G is bipartite (Exercise 13). We used this idea in our first proof that K_5 and $K_{3,3}$ are nonplanar (Proposition 6.1.2); the conflict graph of a spanning cycle in $K_{3,3}$ is C_3, and the conflict graph of a spanning cycle in K_5 is C_5.

Nonplanar 3-connected graphs have Kuratowski subgraphs of a special type. Kelmans [1984a] conjectured this extension of Kuratowski's Theorem, and it was proved independently by Kelmans [1983, 1984b] and Thomassen [1984]: Every 3-connected nonplanar graph with at least six vertices contains a cycle with three pairwise crossing chords.

Characterizations of planarity lead us to ask whether we can test quickly whether a graph is planar. There are linear-time algorithms due to Hopcroft and Tarjan [1974] and to Booth and Luecker [1976], but these are very complicated (Gould [1988, p177–185] discusses the ideas used in the Hopcroft–Tarjan algorithm). A simpler earlier algorithm is not linear but runs in polynomial time. Due to Demoucron, Malgrange, and Pertuiset [1964], it uses H-fragments.

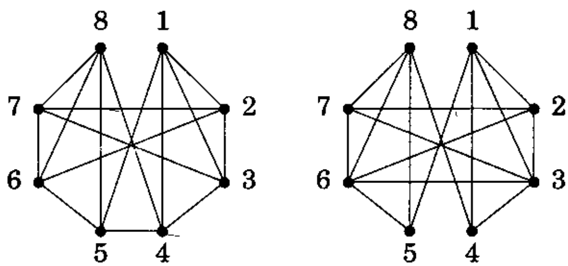
The idea is that if a planar embedding of H can be extended to a planar embedding of G, then in that extension every H-fragment of G appears inside a single face of H. We build increasingly larger plane subgraphs H of G that can be extended to an embedding of G if G is planar. We try to enlarge H by making small decisions that won't lead to trouble.

To enlarge H, we choose a face F that can accept an H-fragment B; the boundary of F must contain all vertices of attachment of B. Although we do not know the best way to embed B in F, a single path in B between vertices of attachment by itself has only one way to be added across F, so we add such a path. The details of choosing F and B appear below. Like the other algorithms mentioned, this algorithm produces an embedding if G is planar.

6.2.17. Algorithm. (Planarity Testing)
Input: A 2-connected graph. (Since G is planar if and only if each block of G is planar, and Algorithm 4.1.23 computes blocks, we may assume that G is a block with at least three vertices.)
Idea: Successively add paths from current fragments. Maintain the vertex sets forming face boundaries of the subgraph already embedded.
Initialization: G_0 is an arbitary cycle in G embedded in the plane, with two face boundaries consisting of its vertices.
Iteration: Having determined G_i, find G_{i+1} as follows.

1. Determine all G_i-fragments of the input block G.
2. For each G_i-fragment B, determine all faces of G_i that contain all vertices of attachment of B; call this set $F(B)$.
3. If $F(B)$ is empty for some B, return NONPLANAR. If $|F(B)| = 1$ for some B, select such a B. If $|F(B)| > 1$ for every B, select any B.
4. Choose a path P between two vertices of attachment of the selected B. Embed P across a face in $F(B)$. Call the resulting graph G_{i+1} and update the list of face boundaries.
5. If $G_{i+1} = G$, return PLANAR. Otherwise, augment i and return to Step 1.

6.2.18. Example. Consider the two graphs below (from Bondy–Murty [1976, p165–166]). Algorithm 6.2.17 produces a planar embedding of the graph on the left, but it terminates in Step 3 for the graph on the right. The cycle 12348765 has three pairwise crossing chords: 14, 27, 36. ■



6.2.19. Theorem. (Demoucron–Malgrange–Pertuiset [1964]) Algorithm 6.2.17 produces a planar embedding if G is planar.

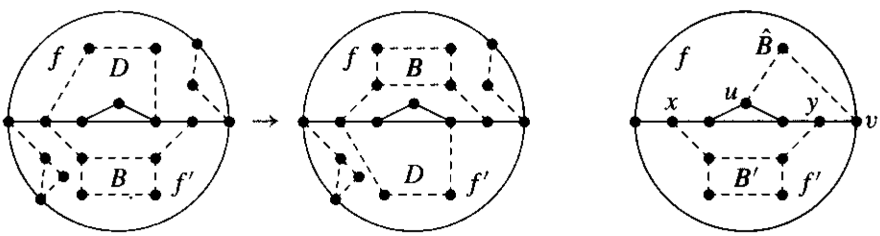
Proof: We may assume that G is 2-connected. A cycle appears as a simple closed curve in every planar embedding. Since we can reflect the plane, every embedding of a cycle in a planar graph G extends to an embedding of G.

Hence G_0 extends to a planar embedding of G if G is planar. It suffices to show that if the plane graph G_i is extendable to a planar embedding of G and the algorithm produces a plane graph G_{i+1} from G_i, then G_{i+1} also is extendable to a planar embedding of G. Note that every G_i-fragment has at least two vertices of attachment, since G is 2-connected,

If some G_i-fragment B has $|F(B)| = 1$, then there is only one face of G_i that can contain P in an extension of G_i to a planar embedding of G. The algorithm puts P in that face to obtain G_{i+1}, so in this case G_{i+1} is extendable.

Problems can arise only if $|F(B)| > 1$ for all B and we select the wrong face in which to embed a path P from the selected fragment. Suppose that (1) we embed P in face $f \in F(B)$, and (2) G_i can be extended to a planar embedding $\hat{G}$ of G in which P is inside face $f' \in F(B)$. We modify $\hat{G}$ to show that G_i can be extended to another embedding G' of G in which P is inside f. This shows that our choice causes no problem, and the constructed G_{i+1} is extendable.

Let C be the set of vertices in the boundaries of both f and f'; this includes the vertices of attachment of B. We draw G' by switching between f and f' all G_i-fragments that $\hat{G}$ places in f or f' and whose vertices of attachment lie in C. We show this on the left below, where edges of G not present in G_i are dashed.



$\hat{G}$ G'

The change switches B and produces the desired embedding G' unless some unswitched G_i-fragment $\hat{B}$ conflicts with a switched fragment. Since the switch is symmetric in f and f' and changes only their interiors, we may assume that $\hat{B}$ appears in f in $\hat{G}$. "Conflict" means that $\hat{G}$ has some B' in f', which we are trying to move to f, such that $\hat{B}$ and B' are adjacent in the conflict graph of f.

Let $\hat{A}, A'$ denote the vertex sets where $\hat{B}, B'$ attach to the boundary of f. Since $\hat{B}$ and B' conflict, $\hat{A}, A'$ have three common vertices or four alternating vertices on the boundary of f. Since $A' \subseteq C$ but $\hat{A} \not\subseteq C$, the first possibility implies the second. Let x, u, y, v be the alternation, with $x, y \in A' \subseteq C$ and $u, v \in \hat{A}$. We may assume that $u \notin C$, as shown on the right above; if there is no such alternation, then $\hat{B}, B'$ do not conflict or $\hat{B}$ can switch to f'.

Since $u \notin C$ and y is between u and v on f, no other face contains both u and v. Thus $\hat{B}$ fails to have its vertices of attachment contained in at least two faces, contradicting the hypothesis that $|F(\hat{B})| > 1$. ■

We can begin by checking that G has at most $3n - 6$ edges, maintain appropriate lists for the face boundaries, and perform the other operations via searches of linear size. Thus this algorithm runs in quadratic time. The proof of Kuratowski's Theorem by Klotz [1989] also gives a quadratic algorithm to test planarity, and it finds a Kuratowski subgraph when G is not planar.

EXERCISES

6.2.1. (−) Prove that the complement of the 3-dimensional cube Q_3 is nonplanar.

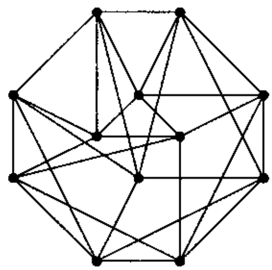
6.2.2. (−) Give three proofs that the Petersen graph is nonplanar.

a) Using Kuratowski's Theorem.

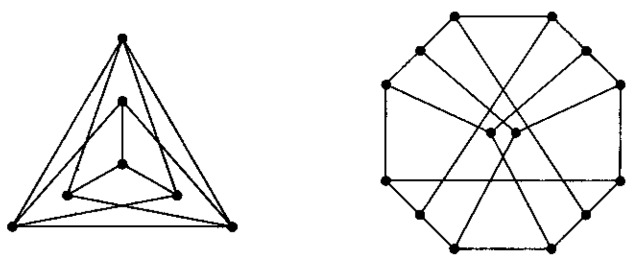
b) Using Euler's Formula and the fact that the Petersen graph has girth 5.

c) Using the planarity-testing algorithm of Demoucron–Malgrange–Pertuiset.

6.2.3. (−) Find a convex embedding in the plane for the graph below.



6.2.4. (−) For each graph below, prove nonplanarity or provide a convex embedding.



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6.2.5. Determine the minimum number of edges that must be deleted from the Petersen graph to obtain a planar subgraph.

6.2.6. (!) *Fáry's Theorem.* Let R be a region in the plane bounded by a simple polygon with at most five sides (**simple polygon** means the edges are line segments that do not cross). Prove there is a point x inside R that "sees" all of R, meaning that the segment from x to any point of R does not cross the boundary of R. Use this to prove inductively that every simple planar graph has a straight-line embedding.

6.2.7. (!) Use Kuratowski's Theorem to prove that G is outerplanar if and only if it has no subgraph that is a subdivision of K_4 or $K_{2,3}$. (Hint: To apply Kuratowski's Theorem, find an appropriate modification of G. This is much easier than trying to mimic a proof of Kuratowski's Theorem.)

6.2.8. (!) Prove that every 3-connected graph with at least six vertices that contains a subdivision of K_5 also contains a subdivision of $K_{3,3}$. (Wagner [1937])

6.2.9. (+) For $n \geq 5$, prove that the maximum number of edges in a simple planar n-vertex graph not having two disjoint cycles is $2n-1$. (Comment: Compare with Exercise 5.2.28.) (Markus [1999])

6.2.10. (!) Let $f(n)$ be the maximum number of edges in a simple n-vertex graph containing no $K_{3,3}$-subdivision.

a) Given that $n-2$ is divisible by 3, construct a graph to show that $f(n) \geq 3n-5$.

b) Prove that $f(n) = 3n-5$ when $n-2$ is divisible by 3 and that otherwise $f(n) = 3n-6$. (Hint: Use induction on n, invoking Exercise 6.2.8 for the 3-connected case.) (Thomassen [1984])

(Comment: Mader [1998] proved the more difficult result that $3n-6$ is the maximum number of edges in an n-vertex simple graph with no K_5-subdivision.)

6.2.11. (!) Let H be a graph with maximum degree at most 3. Prove that a graph G contains a subdivision of H if and only if G contains a subgraph contractible to H.

6.2.12. (!) Wagner [1937] proved that the following condition is necessary and sufficient for a graph G to be planar: neither K_5 nor $K_{3,3}$ can be obtained from G by performing deletions and contractions of edges.

a) Show that deletion and contraction of edges preserve planarity. Conclude from this that Wagner's condition is necessary.

b) Use Kuratowski's Theorem to prove that Wagner's condition is sufficient.

6.2.13. Prove that a graph G is planar if and only if for every cycle C in G, the conflict graph for C is bipartite. (Tutte [1958])

6.2.14. Let x and y be vertices of a planar graph G. Prove that G has a planar embedding with x and y on the same face unless $G-x-y$ has a cycle C with x and y in conflicting C-fragments in G. (Hint: Use Kuratowski's Theorem. Comment: Tutte proved this without Kuratowski's Theorem and used it to prove Kuratowski's Theorem.)

6.2.15. Let G be a 3-connected simple plane graph containing a cycle C. Prove that C is the boundary of a face in G if and only if G has exactly one C-fragment. (Comment: Tutte [1963] proved this to obtain Whitney's [1933b] result that 3-connected planar graphs have essentially only one planar embedding. See also Kelmans [1981a])

6.2.16. (+) Let G be an outerplanar graph with n vertices, and let P be a set of n points in the plane, no three of which lie on a line. The *extreme points* of P induce a convex polygon that contains the other points in its interior.

a) Let p_1, p_2 be consecutive extreme points of P. Prove that there is a point $p \in P - \{p_1, p_2\}$ such that 1) no point of P is inside $p_1 p_2 p$, and 2) some line l through p separates p_1 from p_2, meets P only at p, and has exactly $i-2$ points of P on the side of l containing p_2.

b) Prove that G has a straight-line embedding with its vertices mapped onto P. (Hint: Use part (a) to prove the stronger statement that if v_1, v_2 are two consecutive vertices of the unbounded face of a maximal outerplanar graph G, and p_1, p_2 are consecutive vertices of the convex hull of P, then G can be straight-line embedded on P such that $f(v_1) = p_1$ and $f(v_2) = p_2$.) (Gritzmann–Mohar–Pach–Pollack [1989])

6.3. Parameters of Planarity

Every property and parameter we have studied for general graphs can be studied for planar graphs. The problem of greatest historical interest is the maximum chromatic number of planar graphs. We will also study parameters that measure how far a graph is from being a planar graph.

COLORING OF PLANAR GRAPHS

Because every simple n-vertex planar graph has at most $3n-6$ edges, such a graph has a vertex of degree at most 5. This yields an inductive proof that planar graphs are 6-colorable (see Exercise 2). Heawood improved the bound.

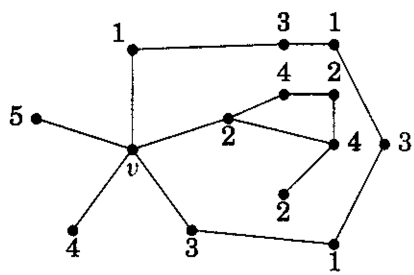
6.3.1. Theorem. (Five Color Theorem—Heawood [1890]) Every planar graph is 5-colorable.

Proof: We use induction on $n(G)$.

Basis step: $n(G) \le 5$. All such graphs are 5-colorable.

Induction step: $n(G) > 5$. The edge bound (Theorem 6.1.23) implies that G has a vertex v of degree at most 5. By the induction hypothesis, $G - v$ is 5-colorable. Let $f\colon V(G-v) \to [5]$ be a proper 5-coloring of $G - v$. If G is not 5-colorable, then f assigns each color to some neighbor of v, and hence $d(v) = 5$. Let v_1, v_2, v_3, v_4, v_5 be the neighbors of v in clockwise order around v. Name the colors so that $f(v_i) = i$.

Let $G_{i,j}$ denote the subgraph of $G - v$ induced by the vertices of colors i and j. Switching the two colors on any component of $G_{i,j}$ yields another proper 5-coloring of $G - v$. If the component of $G_{i,j}$ containing v_i does not contain v_j, then we can switch the colors on it to remove color i from $N(v)$. Now giving color i to v produces a proper 5-coloring of G. Thus G is 5-colorable unless, for each choice of i and j, the component of $G_{i,j}$ containing v_i also contains v_j. Let $P_{i,j}$ be a path in $G_{i,j}$ from v_i to v_j, illustrated below for $(i, j) = (1, 3)$.



Consider the cycle C completed with $P_{1,3}$ by v; this separates v_2 from v_4.

By the Jordan Curve Theorem, the path $P_{2,4}$ must cross C. Since G is planar, paths can cross only at shared vertices. The vertices of $P_{1,3}$ all have color 1 or 3, and the vertices of $P_{2,4}$ all have color 2 or 4, so they have no common vertex.

By this contradiction, G is 5-colorable. ■

Every planar graph is 5-colorable, but are five colors ever needed? The history of this infamous question is discussed in Aigner [1984, 1987], Ore [1967a], Saaty–Kainen [1977, 1986], Appel–Haken [1989], and Fritsch–Fritsch [1998]. The earliest known posing of the Four Color Problem is in a letter of October 23, 1852, from Augustus de Morgan to Sir William Hamilton. The question was asked by de Morgan's student Frederick Guthrie, who later attributed it to his brother Francis Guthrie. It was phrased in terms of map coloring.

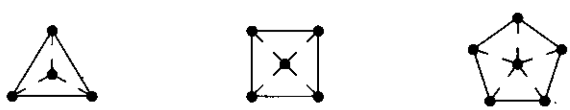
The problem's ease of statement and geometric subtleties invite fallacious proofs; some were published and remained unexposed for years. It does not suffice to forbid five pairwise-adjacent regions, since there are 5-chromatic graphs not containing K_5 (recall Mycielski's construction, for example).

Cayley announced the problem to the London Mathematical Society in 1878, and Kempe [1879] published a "solution". In 1890, Heawood published a refutation. Nevertheless, Kempe's idea of alternating paths, used by Heawood to prove the Five Color Theorem, led eventually to a proof by Appel and Haken [1976, 1977, 1986] (working with Koch). A path on which the colors alternate between two specified colors is a **Kempe chain**.

In proving the Five Color Theorem inductively, we argued that a minimal counterexample contains a vertex of degree at most 5 and that a planar graph with such a vertex cannot be a minimal counterexample. This suggests an approach to the Four Color Problem; we seek an unavoidable set of graphs that can't be present! We need only consider triangulations, since every simple planar graph is contained in a triangulation.

6.3.2. Definition. A **configuration** in a planar triangulation is a separating cycle C (the **ring**) together with the portion of the graph inside C. For the Four Color Problem, a set of configurations is **unavoidable** if a minimal counterexample must contain a member of it. A configuration is **reducible** if a planar graph containing it cannot be a minimal counterexample.

6.3.3. Example. *An unavoidable set.* We have remarked that $\delta(G) \leq 5$ for every simple planar graph. In a triangulation, every vertex has degree at least 3. Thus the set of three configurations below is unavoidable.



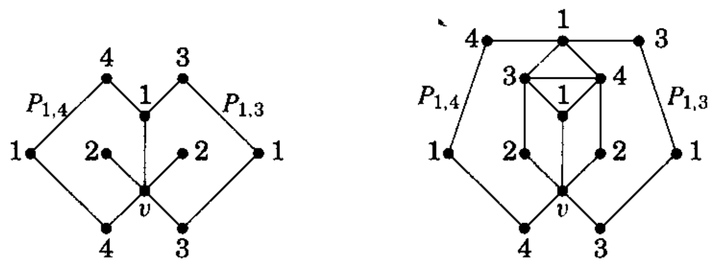
The edges from the ring to the interior are drawn with dashes because a configuration (in a triangulation) is completely determined if we state the degrees of the vertices adjacent to the ring and delete the ring (Exercise 7). Thus these configurations are written as "• 3", "• 4", and "• 5", respectively. ■

When we say that a configuration cannot be in a minimal counterexample, we mean that if it appears in a triangulation G, then it can be replaced to obtain a triangulation G' with fewer vertices such that every 4-coloring of G' can be manipulated to obtain a 4-coloring of G.

6.3.4. Remark. *Kempe's proof.* Let us try to prove the Four Color Theorem by induction using the unavoidable set $\{\bullet 3, \bullet 4, \bullet 5\}$. The approach is similar to Theorem 6.3.1. We can extend a 4-coloring of $G - v$ to complete a 4-coloring of G unless all four colors appear on $N(v)$. Thus "$\bullet$ 3" is reducible. If $d(v) = 4$, then the Kempe-chain argument works as in Theorem 6.3.1, and "$\bullet$ 4" is reducible.

Now consider "$\bullet$ 5". When $d(v) = 5$, the restriction to triangulations implies that the repeated color on $N(v)$ in the proper 4-coloring of $G - v$ appears on nonconsecutive neighbors of v. Let v_1, v_2, v_3, v_4, v_5 again be the neighbors of v in clockwise order. In the 4-coloring f of $G - v$, we may assume by symmetry that $f(v_5) = 2$ and that $f(v_i) = i$ for $1 \le i \le 4$.

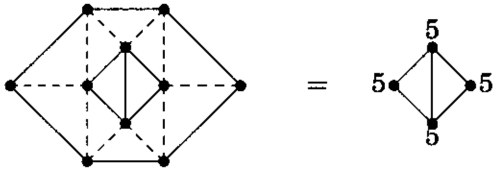
Define $G_{i,j}$ and $P_{i,j}$ as in Theorem 6.3.1. We can eliminate color 1 from $N(v)$ unless the chains $P_{1,3}$ and $P_{1,4}$ exist from v_1 to v_3 and v_4, respectively, as shown on the left below. The component H of $G_{2,4}$ containing v_2 is separated from v_4 and v_5 by the cycle completed by v with $P_{1,3}$. Also, the component H' of $G_{2,3}$ containing v_5 is separated from v_2 and v_3 by the cycle completed by v with $P_{1,4}$. We can eliminate color 2 from $N(v)$ by switching colors 2 and 4 in H and colors 2 and 3 in H'. Right? This was the final case in Kempe's proof.



The problem is that $P_{1,3}$ and $P_{1,4}$ can intertwine, intersecting at a vertex with color 1 as shown on the right above. We can make the switch in H or in H', but making them both creates a pair of adjacent vertices with color 2. ■

Because of this difficulty, we have not shown that "$\bullet$ 5" is reducible, and we must consider larger configurations. Heesch [1969] contributed the idea of seeking configurations with small ring size instead of few vertices inside. It is not hard to show that every configuration having ring size 3 or 4 is reducible (Exercise 9). This is equivalent to showing that no minimal 5-chromatic triangulation has a separating cycle of length at most 4.

6.3.5.* Example. Birkhoff [1913] pushed the idea farther. He proved that every configuration with ring size 5 that has more than one vertex inside is reducible. He also proved that the configuration with ring size 6 below, called the **Birkhoff diamond**, is reducible.



Proving that the Birkhoff diamond is reducible takes a full page of detailed analysis. One approach is to try to show that all proper 4-colorings of the ring extend to the interior. Although some cases can be combined, and some do extend, in some cases it is necessary to use Kempe chains to show that the coloring can be changed into one that extends. ■

The intricate analysis of this first nontrivial example suggests that we have barely begun. The detail remaining is enormous. From 1913 to 1950, additional reducible configurations were found, enough to prove that all planar graphs with at most 36 vertices are 4-colorable. This was slow progress. In the 1960s, Heesch focused attention on the size of the ring, gave heuristics for finding reducible configurations, and developed methods for generating unavoidable sets.

The first proof used configurations with ring size up to 14. A ring of size 13 has 66430 distinguishable 4-colorings. Reducibility requires showing that each leads to a 4-coloring of the full graph. Kempe-chain arguments and partial collapsing of the configuration may be needed, so reducibility proofs are not easy.

Appel and Haken, working with Koch, improved upon the heuristics of Heesch and others to restrict computer searches to "promising" configurations. Using 1000 hours of computer time on three computers in 1976, they found an unavoidable set of 1936 reducible configurations, all with ring size at most 14.

6.3.6. Theorem. (Four Color Theorem—Appel–Haken–Koch [1977]) Every planar graph is 4-colorable. ■

By 1983, refinements led to an unavoidable set of 1258 reducible configurations. The proof was revisited by Robertson, Sanders, Seymour, and Thomas [1996], using the same approach. They reduced the rules used for producing unavoidable sets to a set of 32 rules. Their simplifications yielded an unavoidable set of 633 reducible configurations. They made their computer code available on the Internet; in 1997, it would prove the Four Color Theorem on a desktop workstation in about three hours.

6.3.7.* Remark. *Discharging.* To generate unavoidable sets, we replace the problem case (vertex of degree 5) by larger configurations involving a vertex of degree 5; this can be viewed as a more detailed case analysis for the hard case. Systematic rules are needed to maintain a reasonably small exhaustive set.

In a triangulation, $\sum d(v) = 2e(G) = 6n - 12$. We rewrite this as $12 = \sum(6-d(v))$ and think of $6-d(v)$ as a **charge** on vertex v. Because 12 is positive, some vertices must have positive charge (degree 5). The rules for replacing bad cases involve moving the charge around; they are called **discharging rules**. Since positive charge must remain somewhere, we obtain new unavoidable sets. The next proposition describes the effect of the simplest discharging rule. ■

6.3.8.* Proposition. Every planar triangulation with minimum degree 5 contains a configuration in the set below.

$$5 \bullet\!\!-\!\!\!-\!\!\bullet\, 5 \qquad\qquad 5 \bullet\!\!-\!\!\!-\!\!\bullet\, 6$$

Proof: Start with charge defined by $6 - d(v)$. The first discharging rule takes the charge from each vertex of positive charge (degree 5) and distributes that charge equally among its neighbors.

A vertex of degree 5 or 6 now having positive charge must have a neighbor of degree 5. A vertex of degree 7 now having positive charge must have at least six neighbors of degree 5. Since G is a triangulation, this requires adjacent vertices of degree 5. No vertex of degree 8 or more can acquire positive charge from this discharging rule.

The total charge in the graph remains 12, so some vertex v has positive charge. For each case of $d(v)$, one of the specified configurations occurs. ■

Discharging methods are now being applied to attack other problems using computer-assisted analysis by cases.

The proof of the Four Color Theorem met with considerable uproar. Some objected in principle to the use of a computer. Others complained that the proof was too long to be verified. Others worried about computer error. A few errors were found in the original algorithms, but these were fixed (Appel–Haken [1986]). Those who have checked calculations by hand recognize that the probability of human error in a mathematical proof is much higher than the probability of computer error when the algorithm has been proved correct.

CROSSING NUMBER

In the remainder of this section, we consider parameters that measure a graph's deviation from planarity. One natural parameter is the number of planar graphs needed to form the graph; Exercises 16–20 consider this.

6.3.9. Definition. The **thickness** of a graph G is the minimum number of planar graphs in a decomposition of G into planar graphs.

6.3.10. Proposition. A simple graph G with n vertices and m edges has thickness at least $m/(3n-6)$. If G has no triangles, then it has thickness at least $m/(2n-4)$.

Proof: By Theorem 6.1.23, the denominator is the maximum size of each planar subgraph. The pigeonhole principle then yields the inequality. ■

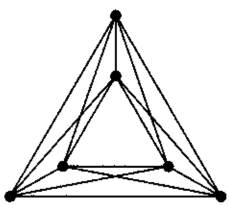
Sometimes we simply must draw a graph in the plane, even if it is not a planar graph. For example, a circuit laid out on a chip corresponds to a drawing of a graph. Since wire crossings lessen performance and cause potential problems, we try to minimize the number of crossings. We discuss the resulting parameter in the remainder of this subsection.

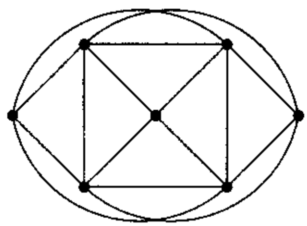
6.3.11. Definition. The **crossing number** $\nu(G)$ of a graph G is the minimum number of crossings in a drawing of G in the plane.

6.3.12. Example. *$\nu(K_6) = 3$ and $\nu(K_{3,2,2}) = 2$.* We can determine the crossing number of some small graphs by considering maximal planar subgraphs. Consider a drawing of G in the plane. If H is a maximal plane subgraph of this drawing, then every edge of G not in H crosses some edge of H, so the drawing has at least $e(G) - e(H)$ crossings. If G has n vertices, then $e(H) \le 3n - 6$. If also G has no triangles, then $e(H) \le 2n - 4$.

Since K_6 has 15 edges, and planar 6-vertex graphs have at most 12 edges, we have $\nu(K_6) \ge 3$. The drawing on the left below proves equality.

Since $K_{3,2,2}$ has 16 edges, and planar graphs with seven vertices have at most 15 edges, $\nu(K_{3,2,2}) \ge 1$. The best drawing we find has two crossings, as shown on the right below. To improve the lower bound, observe that $K_{3,2,2}$ contains $K_{3,4}$. Because $K_{3,4}$ is triangle-free, its planar subgraphs have at most $2 \cdot 7 - 4 = 10$ edges, and hence $\nu(K_{3,4}) \ge 2$. Every drawing of $K_{3,2,2}$ contains a drawing of $K_{3,4}$, so $\nu(K_{3,2,2}) \ge \nu(K_{3,4}) \ge 2$. ■





6.3.13. Proposition. Let G be an n-vertex graph with m edges. If k is the maximum number of edges in a planar subgraph of G, then $\nu(G) \ge m - k$. Furthermore, $\nu(G) \ge \frac{m^2}{2k} - \frac{m}{2}$.

Proof: Given a drawing of G in the plane, let H be a maximal subgraph of G whose edges do not cross in this drawing. Every edge not in H crosses at least one edge in H; otherwise, it could be added to H. Since H has at most k edges, we have at least $m - k$ crossings between edges of H and edges of $G - E(H)$.

After discarding $E(H)$, we have at least $m - k$ edges remaining. The same argument yields at least $(m - k) - k$ crossings in the drawing of the remaining graph. Iterating the argument yields at least $\sum_{i=1}^{t}(m - ik)$ crossings, where $t = \lfloor m/k \rfloor$. The value of the sum is $mt - kt(t+1)/2$.

We now write $m = tk + r$, where $0 \le r \le k - 1$. We substitute $t = (m - r)/k$ in the value of the sum and simplify to obtain $\nu(G) \ge \frac{m^2}{2k} - \frac{m}{2} + \frac{r(k-r)}{2k}$. ■