

Since every ideal I in the Noetherian ring $k[x_1, x_2, \dots, x_n]$ is finitely generated, say $I = (f_1, f_2, \dots, f_q)$, it follows from (3) that $\mathcal{Z}(I) = \mathcal{Z}(f_1) \cap \mathcal{Z}(f_2) \cap \dots \cap \mathcal{Z}(f_q)$, i.e., each affine algebraic set is the intersection of a finite number of hypersurfaces in \mathbb{A}^n . Note that this “geometric” property in affine n -space is a consequence of an “algebraic” property of the corresponding coordinate ring (namely, Hilbert’s Basis Theorem).

If V is an algebraic set in affine n -space, then there may be many ideals I such that $V = \mathcal{Z}(I)$. For example, in affine 2-space over \mathbb{R} the y -axis is the locus of the ideal (x) of $\mathbb{R}[x, y]$, and also the locus of (x^2) , (x^3) , etc. More generally, the zeros of any polynomial are the same as the zeros of all its positive powers, and it follows that $\mathcal{Z}(I) = \mathcal{Z}(I^k)$ for all $k \geq 1$. We shall study the relationship between ideals that determine the same affine algebraic set in the next section when we discuss radicals of ideals.

While the ideal whose locus determines a particular algebraic set V is not unique, there is a unique largest ideal that determines V , given by the set of *all* polynomials that vanish on V . In general, for any subset A of \mathbb{A}^n define

$$\mathcal{I}(A) = \{f \in k[x_1, \dots, x_n] \mid f(a_1, a_2, \dots, a_n) = 0 \text{ for all } (a_1, a_2, \dots, a_n) \in A\}.$$

It is immediate that $\mathcal{I}(A)$ is an *ideal*, and is the unique largest ideal of functions that are identically zero on A . This defines a correspondence

$$\mathcal{I} : \{\text{subsets in } \mathbb{A}^n\} \rightarrow \{\text{ideals of } k[\mathbb{A}^n]\}.$$

Examples

- (1) In the Euclidean plane, $\mathcal{I}(\text{the } x\text{-axis})$ is the ideal generated by y in the coordinate ring $\mathbb{R}[x, y]$.
- (2) Over any field k , the ideal of functions vanishing at $(a_1, a_2, \dots, a_n) \in \mathbb{A}^n$ is a maximal ideal since it is the kernel of the surjective ring homomorphism from $k[x_1, \dots, x_n]$ to the field k given by evaluation at (a_1, a_2, \dots, a_n) . It follows that

$$\mathcal{I}((a_1, a_2, \dots, a_n)) = (x_1 - a_1, x_2 - a_2, \dots, x_n - a_n).$$

- (3) Let $V = \mathcal{Z}(x^3 - y^2)$ in \mathbb{A}^2 . If $(a, b) \in \mathbb{A}^2$ is an element of V then $a^3 = b^2$. If $a \neq 0$, then also $b \neq 0$ and we can write $a = (b/a)^2$, $b = (b/a)^3$. It follows that V is the set $\{(a^2, a^3) \mid a \in k\}$. For any polynomial $f(x, y) \in k[x, y]$ we can write $f(x, y) = f_0(x) + f_1(x)y + (x^3 - y^2)g(x, y)$. For $f(x, y) \in \mathcal{I}(V)$, i.e., $f(a^2, a^3) = 0$ for all $a \in k$, it follows that $f_0(a^2) + f_1(a^2)a^3 = 0$ for all $a \in k$. If $f_0(x) = a_r x^r + \dots + a_0$ and $f_1(x) = b_s x^s + \dots + b_0$ then

$$f_0(x^2) + x^3 f_1(x^2) = (a_r x^{2r} + \dots + a_0) + (b_s x^{2s+3} + \dots + b_0 x^3)$$

and this polynomial is 0 for every $a \in k$. If k is infinite, this polynomial has infinitely many zeros, which can happen only if all of the coefficients are zero. The coefficients of the terms of even degree are the coefficients of $f_0(x)$ and the coefficients of the terms of odd degree are the coefficients of $f_1(x)$, so it follows that $f_0(x)$ and $f_1(x)$ are both 0. It follows that $f(x, y) = (x^3 - y^2)g(x, y)$, and so

$$\mathcal{I}(V) = (x^3 - y^2) \subset k[x, y].$$

If k is finite, however, there may be elements in $\mathcal{I}(V)$ not lying in the ideal $(x^3 - y^2)$. For example, if $k = \mathbb{F}_2$, then V is simply the set $\{(0, 0), (1, 1)\}$ and so $\mathcal{I}(V)$ contains the polynomial $x(x - 1)$ (cf. Exercise 15).

The following properties of the map \mathcal{I} are very easy exercises. Let A and B be subsets of \mathbb{A}^n .

- (6) If $A \subseteq B$ then $\mathcal{I}(B) \subseteq \mathcal{I}(A)$ (i.e., \mathcal{I} is also *contravariant*).
- (7) $\mathcal{I}(A \cup B) = \mathcal{I}(A) \cap \mathcal{I}(B)$.
- (8) $\mathcal{I}(\emptyset) = k[x_1, \dots, x_n]$ and, if k is infinite, $\mathcal{I}(\mathbb{A}^n) = 0$.

Moreover, there are easily verified relations between the maps \mathcal{Z} and \mathcal{I} :

- (9) If A is any subset of \mathbb{A}^n then $A \subseteq \mathcal{Z}(\mathcal{I}(A))$, and if I is any ideal then $I \subseteq \mathcal{I}(\mathcal{Z}(I))$.
- (10) If $V = \mathcal{Z}(I)$ is an affine algebraic set then $V = \mathcal{Z}(\mathcal{I}(V))$, and if $I = \mathcal{I}(A)$ then $\mathcal{I}(\mathcal{Z}(I)) = I$, i.e., $\mathcal{Z}(\mathcal{I}(\mathcal{Z}(I))) = \mathcal{Z}(I)$ and $\mathcal{I}(\mathcal{Z}(\mathcal{I}(A))) = \mathcal{I}(A)$.

The last relation shows that the maps \mathcal{Z} and \mathcal{I} act as inverses of each other provided one restricts to the collection of affine algebraic sets $V = \mathcal{Z}(I)$ in \mathbb{A}^n and to the set of ideals in $k[\mathbb{A}^n]$ of the form $\mathcal{I}(V)$. In the case where the field k is algebraically closed we shall (in the following two sections) characterize those ideals I that are of the form $\mathcal{I}(V)$ for some affine algebraic set V in terms of purely ring-theoretic properties of the ideal I (this is the famous “Zeros Theorem” of Hilbert, cf. Theorem 32).

Definition. If $V \subseteq \mathbb{A}^n$ is an affine algebraic set the quotient ring $k[\mathbb{A}^n]/\mathcal{I}(V)$ is called the *coordinate ring of V* , and is denoted by $k[V]$.

Note that for $V = \mathbb{A}^n$ and k infinite we have $\mathcal{I}(V) = 0$, so this definition extends the previous terminology. The polynomials in $k[\mathbb{A}^n]$ define k -valued functions on V simply by restricting these functions on \mathbb{A}^n to the subset V . Two such polynomial functions f and g define the *same* function on V if and only if $f - g$ is identically 0 on V , which is to say that $f - g \in \mathcal{I}(V)$. Hence the cosets $\bar{f} = f + \mathcal{I}(V)$ giving the elements of the quotient $k[V]$ are precisely the restrictions to V of ordinary polynomial functions f from \mathbb{A}^n to k (which helps to explain the notation $k[V]$). If x_i denotes the i^{th} coordinate function on \mathbb{A}^n (projecting an n -tuple onto its i^{th} component), then the restriction \bar{x}_i of x_i to V (which also just gives the i^{th} component of the elements in V viewed as a subset of \mathbb{A}^n) is an element of $k[V]$, and $k[V]$ is finitely generated as a k -algebra by $\bar{x}_1, \dots, \bar{x}_n$ (although this need not be a minimal generating set).

Example

If $V = \mathcal{Z}(xy - 1)$ is the hyperbola $y = 1/x$ in \mathbb{R}^2 , then $\mathbb{R}[V] = \mathbb{R}[x, y]/(xy - 1)$. The polynomials $f(x, y) = x$ (the x -coordinate function) and $g(x, y) = x + (xy - 1)$, which are different functions on \mathbb{R}^2 , define the same function on the subset V . On the point $(1/2, 2) \in V$, for example, both give the value $1/2$. In the quotient ring $\mathbb{R}[V]$ we have $\bar{x}\bar{y} = 1$, so $\mathbb{R}[V] \cong \mathbb{R}[x, 1/x]$. For any function $\bar{f} \in \mathbb{R}[V]$ and any $(a, b) \in V$ we have $\bar{f}(a, b) = f(a, 1/a)$ for any polynomial $f \in k[x, y]$ mapping to \bar{f} in the quotient.

Suppose now that $V \subseteq \mathbb{A}^n$ and $W \subseteq \mathbb{A}^m$ are two affine algebraic sets. Since V and W are defined by the vanishing of polynomials, the most natural algebraic maps between V and W are those defined by polynomials:

Definition. A map $\varphi : V \rightarrow W$ is called a *morphism* (or *polynomial map* or *regular map*) of algebraic sets if there are polynomials $\varphi_1, \dots, \varphi_m \in k[x_1, x_2, \dots, x_n]$ such that

$$\varphi((a_1, \dots, a_n)) = (\varphi_1(a_1, \dots, a_n), \dots, \varphi_m(a_1, \dots, a_n))$$

for all $(a_1, \dots, a_n) \in V$. The map $\varphi : V \rightarrow W$ is an *isomorphism* of algebraic sets if there is a morphism $\psi : W \rightarrow V$ with $\varphi \circ \psi = 1_W$ and $\psi \circ \varphi = 1_V$.

Note that in general $\varphi_1, \varphi_2, \dots, \varphi_m$ are not uniquely defined. For example, both $f = x$ and $g = x + (xy - 1)$ in the example above define the same morphism from $V = Z(xy - 1)$ to $W = \mathbb{A}^1$.

Suppose F is a polynomial in $k[x_1, \dots, x_m]$. Then $F \circ \varphi = F(\varphi_1, \varphi_2, \dots, \varphi_m)$ is a polynomial in $k[x_1, \dots, x_n]$ since $\varphi_1, \varphi_2, \dots, \varphi_m$ are polynomials in x_1, \dots, x_n . If $F \in \mathcal{I}(W)$, then $F \circ \varphi((a_1, a_2, \dots, a_n)) = 0$ for every $(a_1, a_2, \dots, a_n) \in V$ since $\varphi((a_1, a_2, \dots, a_n)) \in W$. Thus $F \circ \varphi \in \mathcal{I}(V)$. It follows that φ induces a well defined map from the quotient ring $k[x_1, \dots, x_m]/\mathcal{I}(W)$ to the quotient ring $k[x_1, \dots, x_n]/\mathcal{I}(V)$:

$$\begin{aligned}\tilde{\varphi} : k[W] &\rightarrow k[V] \\ f &\mapsto f \circ \varphi\end{aligned}$$

where $f \circ \varphi$ is given by $F \circ \varphi + \mathcal{I}(V)$ for any polynomial $F = F(x_1, \dots, x_m)$ with $f = F + \mathcal{I}(W)$. It is easy to check that $\tilde{\varphi}$ is a k -algebra homomorphism (for example, $\tilde{\varphi}(f + g) = (f + g) \circ \varphi = f \circ \varphi + g \circ \varphi = \tilde{\varphi}(f) + \tilde{\varphi}(g)$ shows that $\tilde{\varphi}$ is additive). Note also the contravariant nature of $\tilde{\varphi}$: the morphism from V to W induces a k -algebra homomorphism from $k[W]$ to $k[V]$.

Suppose conversely that Φ is any k -algebra homomorphism from the coordinate ring $k[W] = k[x_1, \dots, x_m]/\mathcal{I}(W)$ to $k[V] = k[x_1, \dots, x_n]/\mathcal{I}(V)$. Let F_i be a representative in $k[x_1, \dots, x_n]$ for the image under Φ of $\bar{x}_i \in k[W]$ (i.e., $\Phi(\bar{x}_i \bmod \mathcal{I}(W))$ is $F_i \bmod \mathcal{I}(V)$). Then $\varphi = (F_1, \dots, F_m)$ defines a polynomial map from \mathbb{A}^n to \mathbb{A}^m , and in fact φ is a morphism from V to W . To see this it suffices to check that φ maps a point of V to a point of W since by definition φ is already defined by polynomials. If $g \in \mathcal{I}(W) \subset k[x_1, \dots, x_m]$, then in $k[W]$ we have

$$g(x_1 + \mathcal{I}(W), \dots, x_m + \mathcal{I}(W)) = g(x_1, \dots, x_m) + \mathcal{I}(W) = \mathcal{I}(W) = 0 \in k[W],$$

and so

$$\Phi(g(x_1 + \mathcal{I}(W), \dots, x_m + \mathcal{I}(W))) = 0 \in k[V].$$

Since Φ is a k -algebra homomorphism, it follows that

$$g(\Phi(x_1 + \mathcal{I}(W)), \dots, \Phi(x_m + \mathcal{I}(W))) = 0 \in k[V].$$

By definition, $\Phi(x_i + \mathcal{I}(W)) = F_i \bmod \mathcal{I}(V)$, so

$$g(F_1 \bmod \mathcal{I}(V), \dots, F_m \bmod \mathcal{I}(V)) = 0 \in k[V],$$

i.e.,

$$g(F_1, \dots, F_m) \in \mathcal{I}(V).$$

It follows that $g(F_1(a_1, \dots, a_n), \dots, F_m(a_1, \dots, a_n)) = 0$ for every $(a_1, \dots, a_n) \in V$. This shows that if $(a_1, \dots, a_n) \in V$, then every polynomial in $\mathcal{I}(W)$ vanishes