

FIGURE 11.27 Mapping by a linear transformation.

Solving for  $x$  and  $y$  we find

$$x = \frac{v - u}{2} \quad \text{and} \quad y = \frac{v + u}{2}.$$

The Jacobian determinant is  $J(u, v) = -\frac{1}{2}$ . To find the image  $T$  of  $S$  in the  $uv$ -plane we note that the lines  $x = 0$  and  $y = 0$  map onto the lines  $u = v$  and  $u = -v$ , respectively; the line  $x + y = 2$  maps onto the line  $v = 2$ . Points inside  $S$  satisfy  $0 < x + y < 2$  and these are carried into points of  $T$  satisfying  $0 < v < 2$ . Therefore the new region of integration  $T$  is a triangular region, as shown in Figure 11.27. The double integral in question becomes

$$\iint_S e^{(y-x)/(y+x)} dx dy = \frac{1}{2} \iint_T e^{u/v} du dv.$$

Integrating first with respect to  $u$  we find

$$\frac{1}{2} \iint_T e^{u/v} du dv = \frac{1}{2} \int_0^2 \left[ \int_{-v}^v e^{u/v} du \right] dv = \frac{1}{2} \int_0^2 v \left( e - \frac{1}{e} \right) dv = e - \frac{1}{e}.$$

## 11.28 Exercises

In each of Exercises 1 through 5, make a sketch of the region  $S$  and express the double integral  $\iint_S f(x, y) dx dy$  as an iterated integral in polar coordinates.

1.  $S = \{(x, y) \mid x^2 + y^2 \leq a^2\}$ , where  $a > 0$ .
2.  $S = \{(x, y) \mid x^2 + y^2 \leq 2x\}$ .
3.  $S = \{(x, y) \mid a^2 \leq x^2 + y^2 \leq b^2\}$ , where  $0 < a < b$ .
4.  $S = \{(x, y) \mid 0 \leq y \leq 1 - x, 0 \leq x \leq 1\}$ .
5.  $S = \{(x, y) \mid x^2 \leq y \leq 1, -1 \leq x \leq 1\}$ .

In each of Exercises 6 through 9, transform the integral to polar coordinates and compute its value. (The letter  $a$  denotes a positive constant.)

$$6. \int_0^{2a} \left[ \int_0^{\sqrt{2ax-x^2}} (x^2 + y^2) dy \right] dx.$$

$$8. \int_0^1 I \int_{x^2}^x (x^2 + y^2)^{-1/2} dy dx.$$

$$7. \int_0^a \left[ \int_0^x \sqrt{x^2 + y^2} dy \right] dx.$$

$$9. \int_0^a \left[ \int_0^{\sqrt{a^2-y^2}} (x^2 + y^2) dx \right] dy.$$

In Exercises 10 through 13, transform each of the given integrals to one or more iterated integrals in polar coordinates.

$$10. \int_0^1 \left[ \int_0^{1-x} f(x, y) dy \right] dx.$$

$$12. \int_0^1 \left[ \int_{1-x}^{\sqrt{1-x^2}} f(x, y) dy \right] dx.$$

$$11. \int_0^2 \left[ \int_x^{\sqrt{x^3}} f(\sqrt{x^2 + y^2}) dy \right] dx.$$

$$13. \int_0^1 \left[ \int_0^{x^2} f(x, y) dy \right] dx.$$

14. Use a suitable linear transformation to evaluate the double integral

$$\iint_S (x - y)^2 \sin^2(x + y) dx dy$$

where  $S$  is the parallelogram with vertices  $(\pi, 0)$ ,  $(2\pi, \pi)$ ,  $(\pi, 2\pi)$ ,  $(0, \pi)$ .

15. A parallelogram  $S$  in the  $xy$ -plane has vertices  $(0, 0)$ ,  $(2, 10)$ ,  $(3, 17)$ , and  $(1, 7)$ .

(a) Find a linear transformation  $u = ax + by$ ,  $v = cx + dy$ , which maps  $S$  onto a rectangle  $R$  in the  $uv$ -plane with opposite vertices  $(0, 0)$  and  $(4, 2)$ . The vertex  $(2, 10)$  should map onto a point on the  $u$ -axis.

(b) Calculate the double integral  $\iint_S xy dx dy$  by transforming it into an equivalent integral over the rectangle  $R$  of part (a).

16. If  $r > 0$ , let  $Z(r) = \int_{-r}^r e^{-u^2} du$ .

(a) Show that  $I^2(r) = \iint_R e^{-(x^2+y^2)} dx dy$ , where  $R$  is the square  $R = [-r, r] \times [-r, r]$ .

(b) If  $C_1$  and  $C_2$  are the circular disks inscribing and circumscribing  $R$ , show that

$$\iint_{C_1} e^{-(x^2+y^2)} dx dy < I^2(r) < \iint_{C_2} e^{-(x^2+y^2)} dx dy.$$

(c) Express the integrals over  $C_1$  and  $C_2$  in polar coordinates and use (b) to deduce that  $Z(r) \rightarrow \sqrt{\pi}$  as  $r \rightarrow \infty$ . This proves that  $\int_0^\infty e^{-u^2} du = \sqrt{\pi}/2$ .

(d) Use part (c) to deduce that  $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ , where  $\Gamma$  is the gamma function.

17. Consider the mapping defined by the equations

$$x = u + v, \quad y = v - u^2.$$

(a) Compute the Jacobian determinant  $J(u, v)$ .

(b) A triangle  $T$  in the  $uv$ -plane has vertices  $(0, 0)$ ,  $(2, 0)$ ,  $(0, 2)$ . Describe, by means of a sketch, its image  $S$  in the  $xy$ -plane.

(c) Calculate the area of  $S$  by a double integral extended over  $S$  and also by a double integral extended over  $T$ .

(d) Evaluate  $\iint_S (x - y + 1)^{-2} dx dy$ .

18. Consider the mapping defined by the two equations  $x = u^2 - v^2$ ,  $y = 2uv$ .

(a) Compute the Jacobian determinant  $J(u, v)$ .

(b) Let  $T$  denote the rectangle in the  $uv$ -plane with vertices  $(1, 1)$ ,  $(2, 1)$ ,  $(2, 3)$ ,  $(1, 3)$ . Describe, by means of a sketch, the image  $S$  in the  $xy$ -plane.

(c) Evaluate the double integral  $\iint_C xy \, dx \, dy$  by making the change of variables  $x = u^2 - v^2$ ,

$y = 2uv$ , where  $C = \{(x, y) \mid x^2 + y^2 \leq 1\}$ .

19. Evaluate the double integral

$$I(p, r) = \iint_R \frac{dx \, dy}{(p'' + x^2 + y^2)^p}$$

over the circular disk  $R = \{(x, y) \mid x^2 + y^2 \leq r^2\}$ . Determine those values of  $p$  for which  $I(p, r)$  tends to a limit as  $r \rightarrow +\infty$ .

In Exercises 20 through 22, establish the given equations by introducing a suitable change of variables in each case.

$$20. \iint_S f(x+y) \, dx \, dy = \int_{-1}^1 f(u) \, du, \quad \text{where } S = \{(x, y) \mid |x| + |y| \leq 1\}.$$

$$21. \iint_S f(ax+by+c) \, dx \, dy = 2 \int_{-1}^1 \sqrt{1-u^2} f(u\sqrt{a^2+b^2}+c) \, du,$$

where  $S = \{(x, y) \mid x^2 + y^2 \leq 1\}$  and  $a^2 + b^2 \neq 0$ .

$$22. \iint_S f(xy) \, dx \, dy = \log 2 \int_1^2 f(u) \, du, \quad \text{where } S \text{ is the region in the first quadrant bounded by the curves } xy = 1, xy = 2, y = x, y = 4x.$$

### 11.29 Proof of the transformation formula in a special case

As mentioned earlier, the transformation formula

$$(11.35) \quad \iint_S f(x, y) \, dx \, dy = \iint_T f[X(u, v), Y(u, v)] |J(u, v)| \, du \, dv$$

can be deduced as a consequence of the special case in which  $S$  is a rectangle and  $f$  is identically 1. In this case the formula simplifies to

$$(11.36) \quad \iint_R dx \, dy = \iint_{R^*} |J(u, v)| \, du \, dv.$$

Here  $R$  denotes a rectangle in the  $xy$ -plane and  $R^*$  denotes its image in the  $uv$ -plane (see Figure 11.28) under a one-to-one mapping

$$u = U(x, y), \quad v = V(x, y).$$

The inverse mapping is given by

$$x = X(u, v), \quad y = Y(u, v),$$

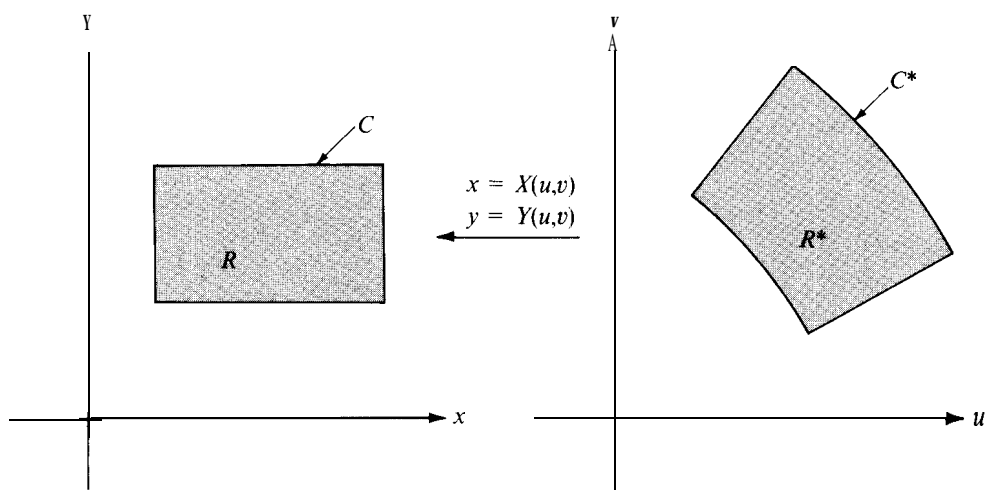


FIGURE 11.28 The transformation law for double integrals derived from Green's theorem.

and  $J(u, v)$  denotes the Jacobian determinant,

$$J(u, v) = \begin{vmatrix} \frac{\partial X}{\partial u} & \frac{\partial X}{\partial v} \\ \frac{\partial Y}{\partial u} & \frac{\partial Y}{\partial v} \end{vmatrix}$$

In this section we use Green's theorem to prove (11.36), and in the next section we deduce the more general formula (11.35) from the special case in (11.36).

For the proof we assume that the functions  $X$  and  $Y$  have continuous second-order partial derivatives and that the Jacobian is never 0 in  $R^*$ . Then  $J(u, v)$  is either positive everywhere or negative everywhere. The significance of the sign of  $J(u, v)$  is that when a point  $(x, y)$  traces out the boundary of  $R$  in the counterclockwise direction, the image point  $(u, v)$  traces out the boundary of  $R^*$  in the counterclockwise direction if  $J(u, v)$  is positive and in the opposite direction if  $J(u, v)$  is negative. In the proof we shall assume that  $J(u, v) > 0$ .

The idea of the proof is to express each double integral in (11.36) as a line integral, using Green's theorem. Then we verify equality of the two line integrals by expressing each in parametric form.

We begin with the double integral in the  $xy$ -plane, writing

$$\iint_R dx \, dy = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \, dy,$$

where  $Q(x, y) = x$  and  $P(x, y) = 0$ . By Green's theorem this double integral is equal to the line integral

$$\int_C P \, dx + Q \, dy = \int_C x \, dy.$$

Here  $C$  is the boundary of  $R$  traversed in a counterclockwise direction.

Similarly, we transform the double integral in the  $uv$ -plane into a line integral around the boundary  $C^*$  of  $R^*$ . The integrand,  $J(u, v)$ , can be written as follows:

$$\begin{aligned} J(u, v) &= \frac{\partial X}{\partial u} \frac{\partial Y}{\partial v} - \frac{\partial X}{\partial v} \frac{\partial Y}{\partial u} = \frac{\partial X}{\partial u} \frac{\partial Y}{\partial v} + X \frac{\partial^2 Y}{\partial u \partial v} - X \frac{\partial^2 Y}{\partial v \partial u} - \frac{\partial X}{\partial v} \frac{\partial Y}{\partial u} \\ &= \frac{\partial}{\partial u} \left( X \frac{\partial Y}{\partial v} \right) - \frac{\partial}{\partial v} \left( X \frac{\partial Y}{\partial u} \right). \end{aligned}$$

Applying Green's theorem to the double integral over  $R^*$  we find

$$\iint_{R^*} J(u, v) \, du \, dv = \int_{C^*} \left( X \frac{\partial Y}{\partial u} \, du + X \frac{\partial Y}{\partial v} \, dv \right).$$

Therefore, to complete the proof of (11.36) we need only verify that

$$(11.37) \quad \int_C x \, dy = \int_{C^*} \left( X \frac{\partial Y}{\partial u} \, du + X \frac{\partial Y}{\partial v} \, dv \right).$$

We introduce a parametrization of  $C^*$  and use this to find a representation of  $C$ . Suppose  $C^*$  is described by a function  $\mathbf{a}$  defined on an interval  $[a, b]$ , say

$$\mathbf{a}(t) = U(t)\mathbf{i} + V(t)\mathbf{j}.$$

Let

$$\boldsymbol{\beta}(t) = X[U(t), V(t)]\mathbf{i} + Y[U(t), V(t)]\mathbf{j}.$$

Then as  $t$  varies over the interval  $[a, b]$ , the vector  $\mathbf{a}(t)$  traces out the curve  $C^*$  and  $\boldsymbol{\beta}(t)$  traces out  $C$ . By the chain rule, the derivative of  $\boldsymbol{\beta}$  is given by

$$\boldsymbol{\beta}'(t) = \left[ \frac{\partial X}{\partial u} U'(t) + \frac{\partial X}{\partial v} V'(t) \right] \mathbf{i} + \left[ \frac{\partial Y}{\partial u} U'(t) + \frac{\partial Y}{\partial v} V'(t) \right] \mathbf{j}.$$

Hence

$$\int_C x \, dy = \int_a^b X[U(t), V(t)] \left( \frac{\partial Y}{\partial u} U'(t) + \frac{\partial Y}{\partial v} V'(t) \right) dt.$$

The last integral over  $[a, b]$  is also obtained by parametrizing the line integral over  $C^*$  in (11.37). Therefore the two line integrals in (11.37) are equal, which proves (11.36).

### 11.30 Proof of the transformation formula in the general case

In this section we deduce the general transformation formula

$$(11.38) \quad \iint_S f(x, y) \, dx \, dy = \iint_T f[X(u, v), Y(u, v)] |J(u, v)| \, du \, dv$$

from the special case treated in the foregoing section,

$$(11.39) \quad \iint_{\mathbf{R}} dx \, dy = \iint_{\mathbf{R}^*} |J(u, v)| \, du \, dv,$$

where  $\mathbf{R}$  is a rectangle and  $\mathbf{R}^*$  is its image in the  $uv$ -plane.

First we prove that we have

$$(11.40) \quad \iint_{\mathbf{R}} s(x, y) \, dx \, dy = \iint_{\mathbf{R}^*} s[X(u, v), Y(u, v)] |J(u, v)| \, du \, dv,$$

where  $s$  is any step function defined on  $\mathbf{R}$ . For this purpose, let  $\mathbf{P}$  be a partition of  $\mathbf{R}$  into  $mn$  subrectangles  $R_{ij}$  of dimensions  $\Delta x_i$  and  $\Delta y_j$ , and let  $c_{ij}$  be the constant value that  $s$  takes on the open subrectangle  $R_{ij}$ . Applying (11.39) to the rectangle  $R_{ij}$  we find

$$\Delta x_i \Delta y_j = \iint_{R_{ij}} dx \, dy = \iint_{R_{ij}^*} |J(u, v)| \, du \, dv.$$

Multiplying both sides by  $c_{ij}$  and summing on  $i$  and  $j$  we obtain

$$(11.41) \quad \sum_{i=1}^n \sum_{j=1}^m c_{ij} \Delta x_i \Delta y_j = \sum_{i=1}^n \sum_{j=1}^m c_{ij} \iint_{R_{ij}^*} |J(u, v)| \, du \, dv.$$

Since  $s$  is a step function, this is the same as

$$(11.42) \quad \sum_{i=1}^n \sum_{j=1}^m c_{ij} \Delta x_i \Delta y_j = \sum_{i=1}^n \sum_{j=1}^m \iint_{R_{ij}^*} s[X(u, v), Y(u, v)] |J(u, v)| \, du \, dv.$$

Using the additive property of double integrals we see that (11.42) is the same as (11.40). Thus, (11.40) is a consequence of (11.39).

Next we show that the step function  $s$  in (11.40) can be replaced by any function  $f$  for which both sides of (11.40) exist. Let  $f$  be integrable over a rectangle  $\mathbf{R}$  and choose step functions  $s$  and  $t$  satisfying the inequalities

$$(11.43) \quad s(x, y) \leq f(x, y) \leq t(x, y),$$

for all points  $(x, y)$  in  $\mathbf{R}$ . Then we also have

$$(11.44) \quad s[X(u, v), Y(u, v)] \leq f[X(u, v), Y(u, v)] \leq t[X(u, v), Y(u, v)]$$

for every point  $(u, v)$  in the image  $\mathbf{R}^*$ . For brevity, write  $S(u, v)$  for  $s[X(u, v), Y(u, v)]$  and define  $F(u, v)$  and  $T(u, v)$  similarly. Multiplying the inequalities in (11.44) by  $|J(u, v)|$  and integrating over  $\mathbf{R}^*$  we obtain

$$\iint_{\mathbf{R}^*} S(u, v) |J(u, v)| \, du \, dv \leq \iint_{\mathbf{R}^*} F(u, v) |J(u, v)| \, du \, dv \leq \iint_{\mathbf{R}^*} T(u, v) |J(u, v)| \, du \, dv.$$

Because of (11.40), the foregoing inequalities are the same as

$$\iint_R s(x, y) dx dy \leq \iint_{R^*} F(u, v) |J(u, v)| du dv \leq \iint_R t(x, y) dx dy.$$

Therefore  $\iint_{R^*} F(u, v) |J(u, v)| du dv$  is a number which lies between the integrals  $\iint_R s(x, y) dx dy$  and  $\iint_R t(x, y) dx dy$  for all choices of step functions  $s$  and  $t$  satisfying (11.43). Since  $f$  is integrable, this implies that

$$\iint_R f(x, y) dx dy = \int_{R^*} F(u, v) |J(u, v)| du dv$$

and hence (11.38) is valid for integrable functions defined over rectangles.

Once we know that (11.38) is valid for rectangles we can easily extend it to more general regions  $S$  by the usual procedure of enclosing  $S$  in a rectangle  $R$  and extending the function  $\mathbf{f}$  to a new function  $\tilde{f}$  which agrees with  $\mathbf{f}$  on  $S$  and has the value 0 outside  $S$ . Then we note that

$$\iint_S f = \iint_R \tilde{f} = \iint_{R^*} \tilde{f}[X(u, v), Y(u, v)] |J(u, v)| du dv = \int_T F(u, v) |J(u, v)| du dv$$

and this proves that (11.38) is, indeed, a consequence of (11.39).

### 11.31 Extensions to higher dimensions

The concept of multiple integral can be extended from 2-space to  $n$ -space for any  $n \geq 3$ . Since the development is entirely analogous to the case  $n = 2$  we merely sketch the principal results.

The integrand is a scalar field  $\mathbf{f}$  defined and bounded on a set  $S$  in  $n$ -space. The integral off over  $S$ , called an  $n$ -fold integral, is denoted by the symbols

$$\int \cdots \int_S f, \quad \text{or} \quad \int_S \cdots \int f(x_1, \dots, x_n) dx_1 \cdots dx_n,$$

with  $n$  integral signs, or more simply with one integral sign,  $\int_S \mathbf{f}(\mathbf{x}) d\mathbf{x}$ , where  $\mathbf{x} = (x_1, \dots, x_n)$ . When  $n = 3$  we write  $(x, y, z)$  instead of  $(x_1, x_2, x_3)$  and denote triple integrals by

$$\iiint_S f, \quad \text{or} \quad \iiint_S f(x, y, z) dx dy dz.$$

First we define the  $n$ -fold integral for a step function defined on an  $n$ -dimensional interval. We recall that an  $n$ -dimensional closed interval  $[a, \mathbf{b}]$  is the Cartesian product of  $n$  closed one-dimensional intervals  $[a, b_k]$ , where  $\mathbf{a} = (a_1, \dots, a_n)$  and  $\mathbf{b} = (b_1, \dots, b_n)$ . An  $n$ -dimensional open interval  $(a, \mathbf{b})$  is the Cartesian product of  $n$  open intervals  $(a_k, b_k)$ . The volume of  $[a, \mathbf{b}]$  or of  $(a, \mathbf{b})$  is defined to be the product of the lengths of the component intervals,

$$(b_1 - a_1) \cdots (b_n - a_n).$$

If  $P_1, \dots, P_n$  are partitions of  $[a, b_1], \dots, [a, b_n]$ , respectively, the Cartesian product  $P = P_1 \times \dots \times P_n$  is called a partition of  $[a, b]$ . A function defined on  $[a, b]$  is called a step function if it is constant on each of the open subintervals determined by some partition  $P$ . The  $n$ -fold integral of such a step function is defined by the formula

$$\int \cdots \int_{[a, b]} f = \sum_i c_i v_i,$$

where  $c_i$  is the constant value that takes on the  $i$ th open subinterval and  $v_i$  is the volume of the  $i$ th subinterval. The sum is a finite sum extended over all the subintervals of  $P$ .

Having defined the  $n$ -fold integral for step functions, we define the integral for more general bounded functions defined on intervals, following the usual procedure. Let  $s$  and  $t$  denote step functions such that  $s \leq f \leq t$  on  $[a, b]$ . If there is one and only one number  $Z$  such that

$$\int \cdots \int_{[a, b]} s \leq I \leq \int \cdots \int_{[a, b]} t$$

for all choices of  $s$  and  $t$  satisfying  $s \leq f \leq t$ , then  $f$  is said to be integrable on  $[a, b]$ , and the number  $Z$  is called the  $n$ -fold integral of  $f$ .

$$I = \int \cdots \int_{[a, b]} f.$$

As in the two-dimensional case, the integral exists if  $f$  is continuous on  $[a, b]$ . It also exists iff  $f$  is bounded on  $[a, b]$  and if the set of discontinuities of  $f$  has  $n$ -dimensional content 0. A bounded set  $S$  has  $n$ -dimensional content 0 if for every  $\epsilon > 0$  there is a finite collection of  $n$ -dimensional intervals whose union includes  $S$  and the sum of whose volumes does not exceed  $\epsilon$ .

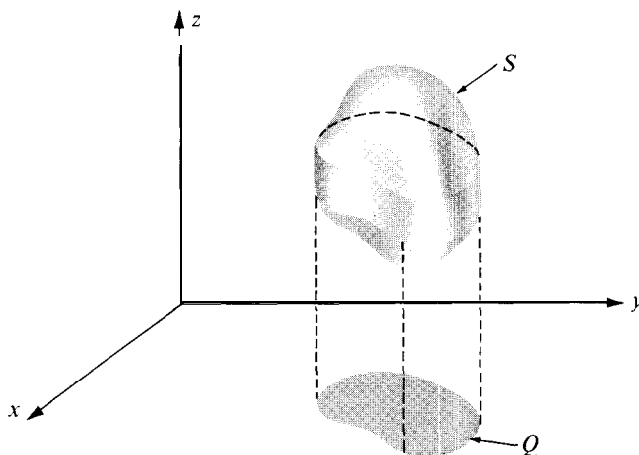
To define the  $n$ -fold integral of a bounded function  $f$  over a more general bounded set  $S$ , we extend  $f$  to a new function  $\tilde{f}$  which agrees with  $f$  on  $S$  and has the value zero outside  $S$ ; the integral of  $f$  over  $S$  is defined to be the integral of  $\tilde{f}$  over some interval containing  $S$ .

Some multiple integrals can be calculated by using iterated integrals of lower dimension. For example, suppose  $S$  is a set in 3-space described as follows:

$$(11.45) \quad S = \{(x, y, z) \mid (x, y) \in Q \text{ and } \varphi_1(x, y) \leq z \leq \varphi_2(x, y)\},$$

where  $Q$  is a two-dimensional region, called the projection of  $S$  on the  $xy$ -plane, and  $\varphi_1, \varphi_2$  are continuous on  $S$ . (An example is shown in Figure 11.29.) Sets of this type are bounded by two surfaces with Cartesian equations  $z = \varphi_1(x, y)$ , and  $z = \varphi_2(x, y)$  and (perhaps) a portion of the cylinder generated by a line moving parallel to the  $z$ -axis along the boundary of  $Q$ . Lines parallel to the  $z$ -axis intersect this set in line segments joining the lower surface to the upper one. If  $f$  is continuous on the interior of  $S$  we have the iteration formula

$$(11.46) \quad \iiint_S f(x, y, z) \, dx \, dy \, dz = \iint_Q \left[ \int_{\varphi_1(x, y)}^{\varphi_2(x, y)} f(x, y, z) \, dz \right] dx \, dy.$$



**FIGURE 11.29** A solid  $S$  and its projection  $Q$  in the  $xy$ -plane.

That is, for fixed  $x$  and  $y$ , the first integration is performed with respect to  $z$  from the lower boundary surface to the upper one. This reduces the calculation to a double integral over the projection  $Q$ , which can be treated by the methods discussed earlier.

There are two more types of sets analogous to those described by (11.45) in which the  $x$ - and  $y$ -axes play the role of the  $z$ -axis, with the projections taken in the  $yz$ - or  $xz$ -planes, respectively. Triple integrals over such sets can be computed by iteration, with formulas analogous to (11.46). Most 3-dimensional sets that we shall encounter are either of one of the three types just mentioned or they can be split into a finite number of pieces, each of which is of one of these types.

Many iteration formulas exist for  $n$ -fold integrals when  $n > 3$ . For example, if  $Q$  is a  $k$ -dimensional interval and  $R$  an  $m$ -dimensional interval, then an  $(m + k)$ -fold integral over  $Q \times R$  is the iteration of an  $m$ -fold integral with a  $k$ -fold integral,

$$\int_{Q \times R} \cdots \int f = \int_Q \cdots \int \left[ \int_R \cdots \int f dx_1 \cdots dx_m \right] dx_{m+1} \cdots dx_{m+k},$$

provided all the multiple integrals in question exist. Later in this chapter we illustrate the use of iterated multiple integrals in computing the volume of an  $n$ -dimensional sphere.

### 11.32 Change of variables in an $n$ -fold integral

The formula for making a change of variables in a double integral has a direct extension for  $n$ -fold integrals. We introduce new variables  $u_1, \dots, u_n$  related to  $x_1, \dots, x_n$  by  $n$  equations of the form

$$x_1 = X_1(u_1, \dots, u_n), \quad \dots \quad x_n = X_n(u_1, \dots, u_n).$$