where $\bar{x}$ is the x-coordinate of the centroid of the square. Since $\bar{x}$ is obviously $\frac{1}{2}$, the value of the line integral is $\frac{3}{2}$.

EXAMPLE 3. ***Area expressed as a line integral.*** The double integral for the area $a(R)$ of a region R can be expressed in the form

$$a(R) = \iint_R dx\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy,$$

where P and Q are such that $\partial Q/\partial x - \partial P/\partial y = 1$. For example, we can take $Q(x, y) = \frac{1}{2}x$ and $P(x, y) = -\frac{1}{2}y$. If R is the region enclosed by a Jordan curve C we can apply Green's theorem to express $a(R)$ as a line integral,

$$a(R) = \int_C P\,dx + Q\,dy = \frac{1}{2}\int_C -y\,dx + x\,dy.$$

If the boundary curve C is described by parametric equations, say

$$x = X(t), \qquad y = Y(t), \qquad a \le t \le b,$$

the line integral for area becomes

$$a(R) = \frac{1}{2}\int_a^b \{-Y(t)X'(t) + X(t)Y'(t)\}\,dt = \frac{1}{2}\int_a^b \begin{vmatrix} X(t) & Y(t) \\ X'(t) & Y'(t) \end{vmatrix}\,dt.$$

11.21 A necessary and sufficient condition for a two-dimensional vector field to be a gradient

Let $f(x, y) = P(x, y)i + Q(x, y)j$ be a vector field that is continuously differentiable on an open set S in the plane. Iff is a gradient on S we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \tag{11.22}$$

everywhere on S. In other words, the condition (11.22) is ***necessary*** for f to be a gradient. As we have already noted, this condition is not sufficient. For example, the vector field

$$f(x, y) = \frac{-y}{x^2 + y^2}\,i + \frac{x}{x^2 + y^2}\,j$$

satisfies (11.22) everywhere on the set $S = \mathbf{R}^2 - \{(0, 0)\}$, but f is not a gradient on S. In Theorem 10.9 we proved that condition (11.22) is both necessary and sufficient for f to be a gradient on S if the set S is ***convex.*** With the help of Green's theorem we can extend this result to a more general class of plane sets known as ***simply connected*** sets. They are defined as follows.

DEFINITION OF A SIMPLY CONNECTED PLANE SET. *Let S be an open connected set in the plane. Then S is called simply connected if, for every Jordan curve C which lies in S, the inner region of C is also a subset of* S.

An annulus (the set of points lying between two concentric circles) is not simply connected because the inner region of a circle concentric with the bounding circles and of radius between theirs is not a subset of the annulus. Intuitively speaking, we say that S is simply connected when it has no "holes." Another way to describe simple connectedness is to say that a curve C_1 in S connecting any two points may be continuously deformed into any other curve C_2 in S joining these two points, with all intermediate curves during the deformation lying completely in S. An alternative definition, which can be shown to be equivalent to the one given here, states that an open connected set S is simply connected if its complement (relative to the whole plane) is connected. For example, an annulus is not simply connected because its complement is disconnected. An open connected set that is not simply connected is called multiply connected.

THEOREM 11.11. *Let $f(x, y) = P(x, y)\boldsymbol{i} + Q(x, y)\boldsymbol{j}$ be a vectorjeld that is continuously differentiable on an open simply connected set S in the plane. Then f is a gradient on S if and only if we have*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \textit{everywhere on } S . \tag{11.23}$$

Proof. As already noted, condition (11.23) is necessary for f to be a gradient. We shall prove now that it is also sufficient.

It can be shown that in any open connected plane set S, every pair of points a and x can be joined by a simple step-polygon, that is, by a polygon whose edges are parallel to the coordinate axes and which has no self-intersections. If the line integral of f from a to x has the same value for every simple step-polygon in S joining a to x, then exactly the same argument used to prove Theorem 10.4 shows that f is a gradient on S. Therefore, we need

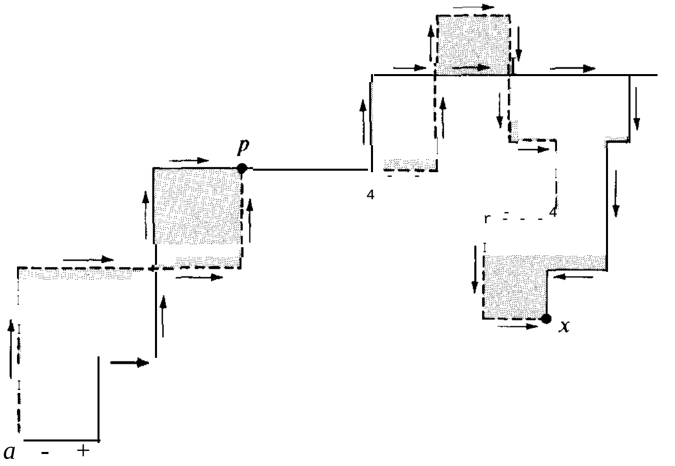


FIGURE 11.16 Independence of the path in a simply connected region.

only verify that the line integral **off** from a to x has the same value for every simple step-polygon in S joining a to x.

Let C_1 and C_2 be two simple step-polygons in S joining a to x. Portions of these polygons may coincide along certain line segments. The remaining portions will intersect at most a finite number of times, and will form the boundaries of a finite number of polygonal regions, say $R_1, \ldots, R$,. Since S is assumed to be simply connected, each of the regions R_k is a subset of S. An example is shown in Figure 11.16. The solid line represents C_1, the dotted line represents C_2, and the shaded regions represent $R_1, \ldots, R_m$. (These two particular polygons coincide along the segment pq.)

We observe next that the line integral **off** from a to x along C_1 plus the integral from x back to a along C_2 is zero because the integral along the closed path is a sum of integrals taken over the line segments common to C_1 and C_2 plus a sum of integrals taken around the boundaries of the regions R,. The integrals over the common segments cancel in pairs, since each common segment is traversed twice, in opposite directions, and their sum is zero. The integral over the boundary Γ_k of each region R_k is also zero because, by Green's theorem we may write

$$\int_{\Gamma_k} P\,dx + Q\,dy = \pm \iint_{R_k} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy,$$

and the integrand of the double integral is zero because of the hypothesis $\partial Q/\partial x = \partial P/\partial y$. It follows that the integral from a to x along C_1 is equal to that along C_2. As we have already noted, this implies that f is a gradient in S.

11.22 Exercises

1. Use Green's theorem to evaluate the line integral $\oint_C y^2\,dx + x\,dy$ when
   (a) C is the square with vertices (0, 0), $(2, 0), (2, 2), (0, 2)$.
   (b) C is the square with vertices ($\pm$ 1, $\pm$ 1).
   (c) C is the square with vertices $(\pm 2, 0)$, (0, f2).
   (d) C is the circle of radius 2 and center at the origin.
   (e) C has the vector equation $a(t) = 2\cos^3 t\, i + 2\sin^3 t\, j,\ 0 \le t \le 2\pi$.
2. If $P(x, y) = xe^{-y^2}$ and $Q(x, y) = -x^2 y\, e^{-y^2} + 1/(x^2 + y^2)$, evaluate the line integral $\oint P\,dx + Q\,dy$ around the boundary of the square of side $2a$ determined by the inequalities $|x| \le a$ and $|y| \le a$.
3. Let C be a simple closed curve in the xy-plane and let I_z denote the moment of inertia (about the z-axis) of the region enclosed by C. Show that an integer n exists such that

$$nI_z = \oint_C x^3\,dy - y^3\,dx.$$

4. Given two scalar fields u and v that are continuously differentiable on an open set containing the circular disk R whose boundary is the circle $x^2 + y^2 = 1$. Define two vector fieldsfand g as follows:

$$f(x, y) = v(x, y)i + u(x, y)j, \qquad g(x, y) = \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)i + \left(\frac{\partial v}{\partial x} - \frac{\partial v}{\partial y}\right)j.$$

Find the value of the double integral $\iint_R f \cdot g\,dx\,dy$ if it is known that on the boundary of R we have $u(x, y) = 1$ and $v(x, y) = y$.

5. If f and g are continuously differentiable in an open connected set S in the plane, show that $\oint_C f\nabla g \cdot d\boldsymbol{\alpha} = -\oint_C g\nabla f \cdot d\boldsymbol{\alpha}$ for every piecewise smooth Jordan curve C in S.
6. Let u and v be scalar fields having continuous first- and second-order partial derivatives in an open connected set Sin the plane. Let R be a region in S bounded by a piecewise smooth Jordan curve C. Show that:

(a) $$\oint_C uv\,dx + uv\,dy = \iint_R \left\{v\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right) + u\left(\frac{\partial v}{\partial x} - \frac{\partial v}{\partial y}\right)\right\} dx\,dy.$$

(b) $$\frac{1}{2}\oint_C \left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}\right) dx + \left(u\frac{\partial v}{\partial y} - v\frac{\partial u}{\partial y}\right) dy = \iint_R \left(u\frac{\partial^2 v}{\partial x\,\partial y} - v\frac{\partial^2 u}{\partial x\,\partial y}\right) dx\,dy.$$

Normalderivatives. In Section 10.7 we defined line integrals with respect to arc length in such a way that the following equation holds:

$$\int_C P\,dx + Q\,dy = \int_C \boldsymbol{f} \cdot \boldsymbol{T}\,ds,$$

where $\boldsymbol{f} = P\boldsymbol{i} + Q\boldsymbol{j}$ and $\boldsymbol{T}$ is the unit tangent vector to C. (The dot product $\boldsymbol{f} \cdot \boldsymbol{T}$ is called the tangential component off along C.) If C is a Jordan curve described by a continuously differentiable function a, say $\boldsymbol{\alpha}(t) = X(t)\boldsymbol{i} + Y(t)\boldsymbol{j}$, the unit ***outer normal*** $\boldsymbol{n}$ of C is defined by the equation

$$\boldsymbol{n}(t) = \frac{1}{\|\boldsymbol{\alpha}'(t)\|}\,(Y'(t)\boldsymbol{i} - X'(t)\boldsymbol{j})$$

whenever $\|\boldsymbol{\alpha}'(t)\| \neq 0$. If φ is a scalar field with gradient $\nabla\varphi$ on C, the ***normalderivative*** $\partial\varphi/\partial n$ is defined on C by the equation

$$\frac{\partial \varphi}{\partial n} = \nabla\varphi \cdot \boldsymbol{n}.$$

This is, of course, the directional derivative of φ in the direction of $\boldsymbol{n}$. These concepts occur in the remaining exercises of this section.

7. If $\boldsymbol{f} = Q\boldsymbol{i} - P\boldsymbol{j}$, show that

$$\int_C P\,dx + Q\,dy = \int_C \boldsymbol{f} \cdot \boldsymbol{n}\,ds.$$

(The dot product $\boldsymbol{f} \cdot \boldsymbol{n}$ is called the normal component off along C.)

8. Let f and g be scalar fields with continuous first- and second-order partial derivatives on an open set S in the plane. Let R denote a region (in S) whose boundary is a piecewise smooth Jordan curve C. Prove the following identities, where $\nabla^2 u = \partial^2 u/\partial x^2 + \partial^2 u/\partial y^2$.

(a) $$\oint_C \frac{\partial g}{\partial n}\,ds = \iint_R \nabla^2 g\,dx\,dy.$$

(b) $$\oint_C f\frac{\partial g}{\partial n}\,ds = \iint_R (f\nabla^2 g + \nabla f \cdot \nabla g)\,dx\,dy.$$

(c) $$\oint_C \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right) ds = \iint_R (f\nabla^2 g - g\nabla^2 f)\,dx\,dy.$$

The identity in (c) is known as *Green's formula;* it shows that

$$\oint_C f \frac{\partial g}{\partial n}\, ds = \oint_C g \frac{\partial f}{\partial n}\, ds$$

whenever f and g are both harmonic on R (that is, when $\nabla^2 f = \nabla^2 g = 0$ on R).

9. Suppose the differential equation

$$P(x, y)\, dx + Q(x, y)\, dy = 0$$

has an integrating factor $\mu(x, y)$ which leads to a one-parameter family of solutions of the form $\varphi(x, y) = C$. If the slope of the curve $\varphi(x, y) = C$ at (x, y) is $\tan\theta$, the unit normal vector $\boldsymbol{n}$ is taken to mean

$$\boldsymbol{n} = \sin\theta\, \boldsymbol{i} - \cos\theta\, \boldsymbol{j}.$$

There is a scalar field $g(x, y)$ such that the normal derivative of φ is given by the formula

$$\frac{\partial\varphi}{\partial n} = \mu(x, y) g(x, y),$$

where $\partial\varphi/\partial n = \nabla\varphi \cdot \boldsymbol{n}$. Find an explicit formula for $g(x, y)$ in terms of $P(x, y)$ and $Q(x, y)$.

★11.23 Green's theorem for multiply connected regions

Green's theorem can be generalized to apply to certain multiply connected regions.

THEOREM 11.12. GREEN'S THEOREM FOR MULTIPLY CONNECTED REGIONS. *Let $C_1, \ldots, C_n$ be n piecewise smooth Jordan curves having the following properties:*

(a) *No two of the curves intersect.*

(b) *The curves $C_2, \ldots, C_n$ all lie in the interior of C_1,*

(c) *Curve C_i lies in the exterior of curve C_j for each $i \neq j$, $i > 1$, $j > 1$.*

Let R denote the region which consists of the union of C_1 with that portion of the interior of C_1 that is not inside any of the curves $C_2, C_3, \ldots, C_n$. (An example of such a region is shown in Figure 11.17.) Let P and Q be continuously differentiable on an open set S containing R. Then we have the following identity.

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy = \oint_{C_1} (P\, dx + Q\, dy) - \sum_{k=2}^{n} \oint_{C_k} (P\, dx + Q\, dy). \tag{11.24}$$

The theorem can be proved by introducing crosscuts which transform R into a union of a finite number of simply connected regions bounded by Jordan curves. Green's theorem is applied to each part separately, and the results are added together. We shall illustrate how this proof may be carried out when $n = 2$. The more general case may be dealt with by using induction on the number n of curves.

The idea of the proof when $n = 2$ is illustrated by the example shown in Figure 11.18, where C_1 and C_2 are two circles, C_1 being the larger circle. Introduce the crosscuts AB and CD, as shown in the figure. Let K_1 denote the Jordan curve consisting of the upper

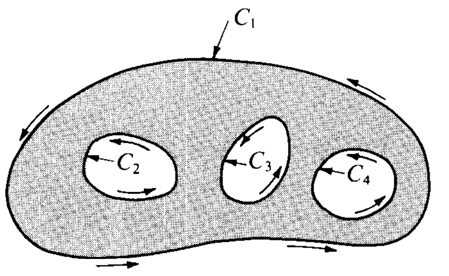


FIGURE 11.17 A multiply connected region.

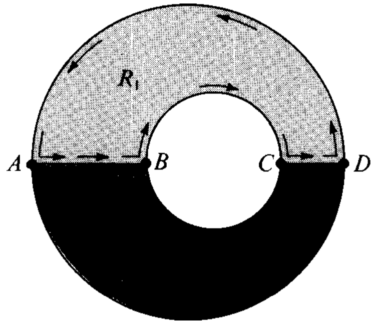


FIGURE 11.18 Proof of Green's theorem for a multiply connected region.

half of C_2, the upper half of C_1 , and the segments AB and CD. Let K_2 denote the Jordan curve consisting of the lower half of C_1, the lower half of C_2, and the two segments AB and CD. Now apply Green's theorem to each of the regions bounded by K_1 and K_2 and add the two identities so obtained. The line integrals along the crosscuts cancel (since each crosscut is traversed once in each direction), resulting in the equation

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy = \oint_{C_1} (P\, dx + Q\, dy) - \oint_{C_2} (P\, dx + Q\, dy).$$

The minus sign appears because of the direction in which C_2 is traversed. This is Equation (11.24) when $n = 2$.

For a simply connected region, the condition $\partial P/\partial y = \partial Q/\partial x$ implies that the line integral $\int P\, dx + Q\, dy$ is independent of the path (Theorem 11.11). As we have already noted, if S is ***not*** simply connected, the condition $\partial P/\partial y = \partial Q/\partial x$ does not necessarily imply independence of the path. However, in this case there is a substitute for independence that can be deduced from Theorem 11.12.

THEOREM 11.13. INVARIANCE OF A LINE INTEGRAL UNDER DEFORMATION OF THE PATH. *Let P and Q be continuously differentiable on an open connected set S in the plane, and assume that $\partial P/\partial y = \partial Q/\partial x$ everywhere on S. Let C_1 and C_2 be two piecewise smooth Jordan curves lying in S and satisfying the following conditions:*

(a) *C_2 lies in the interior of C_1.*

(b) *Those points inside C_1 which lie outside C_2 are in S. (Figure 11.19 shows an example.) Then we have*

$$\oint_{C_1} P\, dx + Q\, dy = \oint_{C_2} P\, dx + Q\, dy, \tag{11.25}$$

where both curves are traversed in the same direction.

Proof. Under the conditions stated, Equation (11.24) is applicable when $n = 2$. The region R consists of those points lying between the two curves C_1 and C_2 and the curves

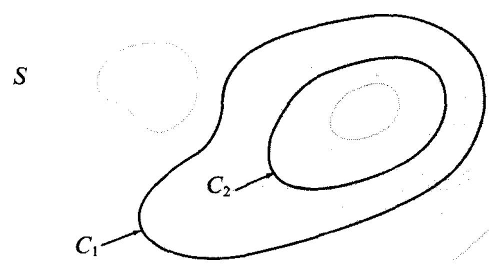


FIGURE 11.19. Invariance of a line integral under deformation of the path.

themselves. Since $\partial P/\partial y = \partial Q/\partial x$ in S, the left member of Equation (11.24) is zero and we obtain (11.25).

Theorem 11.13 is sometimes described by saying that if $\partial P/\partial y = \partial Q/\partial x$ in S the value of a line integral along a piecewise smooth simple closed curve in S is unaltered if the path is deformed to any other piecewise smooth simple closed curve in S, provided all intermediate curves remain within the set S during the deformation. The set S is assumed to be open and connected-it need not be simply connected.

★11.24 The winding number

We have seen that the value of a line integral often depends both on the curve along which the integration takes place and on the direction in which the curve is traversed. For example, the identity in Green's theorem requires the line integral to be taken in the counterclockwise direction. In a completely rigorous treatment of Green's theorem it would be necessary to describe analytically what it means to traverse a closed curve in the "counterclockwise direction." For some special curves this can be done by making specific statements about the vector-valued function **a** which describes the curve. For example, the vector-valued function **a** defined on the interval $[0, 2\pi]$ by the equation

$$\text{(11.26)} \qquad \mathbf{a}(t) = (a \cos t + x_0)\boldsymbol{i} + (a \sin t + y_0)\boldsymbol{j}$$

describes a circle of radius a with center at (x_0, y_0). This particular function is said to describe the circle in a *positive* or *counterclockwise* direction. On the other hand, if we replace t by $-t$ on the right of (11.26) we obtain a new function which is said to describe the circle in a *negative* or *clockwise* direction. In this way we have given a completely analytical description of "clockwise" and "counterclockwise" for a circle. However, it is not so simple to describe the same idea for an *arbitrary* closed curve. For piecewise smooth curves this may be done by introducing the concept of the *winding number,* an analytic device which gives us a mathematically precise way of counting the number of times a radius vector $\boldsymbol{a}$ "winds around" a given point as it traces out a given closed curve. In this section we shall describe briefly one method for introducing the winding number.

Then we shall indicate how it can be used to assign positive and negative directions to closed curves.

Let C be a piecewise smooth closed curve in the plane described by a vector-valued function $\boldsymbol{\alpha}$ defined on an interval *[a, b],* say

$$u(t) = X(t)\boldsymbol{i} + Y(t)j \qquad \text{if} \quad a \le t \le b.$$

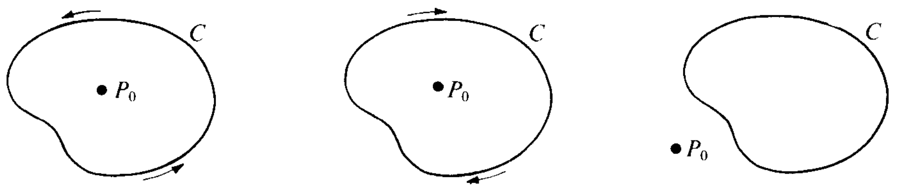
Let $P_0 = (x_0, y_0)$ be a point which does not lie on the curve C. Then the winding number of **a** with respect to the point P_0 is denoted by $W(\boldsymbol{\alpha}; P_0)$; it is defined to be the value of the following integral :

$$W(\boldsymbol{\alpha}; P_0) = \frac{1}{2\pi}\int_a^b \frac{[X(t) - x_0]Y'(t) - [Y(t) - y_0]X'(t)}{[X(t) - x_0]^2 + [Y(t) - y_0]^2}\,dt \tag{11.27}$$

This is the same as the line integral

$$\frac{1}{2\pi}\oint_C \frac{-(y - y_0)\,dx + (x - x_0)\,dy}{(x - x_0)^2 + (y - y_0)^2}. \tag{11.28}$$

It can be shown that the value of this integral is always an *integer,* positive, negative, or zero. Moreover, if C is a *Jordan* curve *(simple* closed curve) this integer is 0 if P_0 is *outside* C and has the value + 1 or $-$ 1 if P_0 is *inside* C. (See Figure 11.20.) Furthermore, *W(a;* P_0)



Winding number + 1 Winding number $-$ 1 Winding number 0

FIGURE 11.20 Illustrating the possible values of the winding number of a Jordan curve C with respect to P_0.

is either $+$ 1 for every point P_0 inside C or it is $-$ 1 for every such point. This enables us to define positive and negative orientations for C as follows: If the winding number $W(\boldsymbol{\alpha};$ *P,)* is + 1 for every point P_0 inside C we say that *a* traces out *C* in the *positive* or *counterclockwise direction.* If the winding number is -1 we say that *a* traces out C in the *negative* or *clockwise direction.* [An example of the integral in (11.28) with $x_0 = y_0 = 0$ was encountered earlier in Example 2 of Section 10.16.]

To prove that the integral for the winding number is always $+1$ or $-$ 1 for a simple closed curve enclosing (x_0, y_0) we use Theorem 11.13. Let S denote the open connected region consisting of all points in the plane except (x_0 , y,). Then the line integral in (11.28) may be written as $\int_C P\,dx + Q\,dy$, and it is easy to verify that $\partial P/\partial y = \partial Q/\partial x$ everywhere in S. Therefore, if (x_0 , y,,) is inside C, Theorem 11.13 tells us that we may replace the curve C by a circle with center at (x,, y_0) without changing the value of the integral. Now we verify that for a circle the integral for the winding number is either + 1 or $-$ 1, depending on whether the circle is positively or negatively oriented. For a positively oriented circle we may use the representation in Equation (11.26). In this case we have

$$X(t) = a\cos t + x_0, \qquad Y(t) = a\sin t + y_0,$$

and the integrand in (11.27) is identically equal to 1. Therefore we obtain

$$W(\boldsymbol{\alpha}; P_0) = \frac{1}{2\pi}\int_0^{2\pi} 1\, dt = 1.$$

By a similar argument we find that the intergal is -1 when C is negatively oriented. This proves that the winding number is either +1 or -1 for a simple closed curve enclosing the point (x_0, y_0).

★11.25 Exercises

1. Let $S = \{(x, y) \mid x^2 + y^2 > 0\}$, and let

$$P(x, y) = \frac{y}{x^2 + y^2}, \qquad Q(x, y) = \frac{-x}{x^2 + y^2}$$

if $(x, y) \in S$. Let C be a piecewise smooth Jordan curve lying in S.
(a) If $(0, 0)$ is inside C, show that the line integral $\int_C P\,dx + Q\,dy$ has the value $\pm 2\pi$, and explain when the plus sign occurs.
(b) Compute the value of the line integral $\int_C P\,dx + Q\,dy$ when $(0, 0)$ is outside C.

2. If $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j}$ and $r = \|\boldsymbol{r}\|$, let

$$\boldsymbol{f}(x, y) = \frac{\partial(\log r)}{\partial y}\boldsymbol{i} - \frac{\partial(\log r)}{\partial x}\boldsymbol{j}$$

for $r > 0$. Let C be a piecewise smooth Jordan curve lying in the annulus $1 < x^2 + y^2 < 25$, and find all possible values of the line integral of $\boldsymbol{f}$ along C.

3. A connected plane region with exactly one "hole" is called ***doubly connected.*** (The annulus $1 < x^2 + y^2 < 2$ is an example.) If P and Q are continuously differentiable on an open doubly connected region R, and if $\partial P/\partial y = \partial Q/\partial x$ everywhere in R, how many distinct values are possible for line integrals $\int_C P\,dx + Q\,dy$ taken around piecewise smooth Jordan curves in R?

4. Solve Exercise 3 for triply connected regions, that is, for connected plane regions with exactly two holes.

5. Let P and Q be two scalar fields which have continuous derivatives satisfying $\partial P/\partial y = \partial Q/\partial x$ everywhere in the plane except at three points. Let C_1, C_2, C_3 be three nonintersecting circles having centers at these three points, as shown in Figure 11.21, and let $I_k = \oint_{C_k} P\,dx + Q\,dy$. Assume that $I_1 = 12$, $I_2 = 10$, $I_3 = 15$.
(a) Find the value of $\int_C P\,dx + Q\,dy$, where C is the figure-eight curve shown.
(b) Draw another closed curve Γ along which $\int P\,dx + Q\,dy = 1$. Indicate on your drawing the direction in which Γ is traversed.
(c) If $I_1 = 12$, $I_2 = 9$, and $I_3 = 15$, show that there is no closed curve Γ along which $\int P\,dx + Q\,dy = 1$.

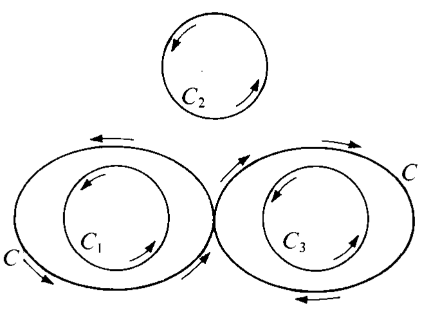


FIGURE 11.21 Exercise 5.

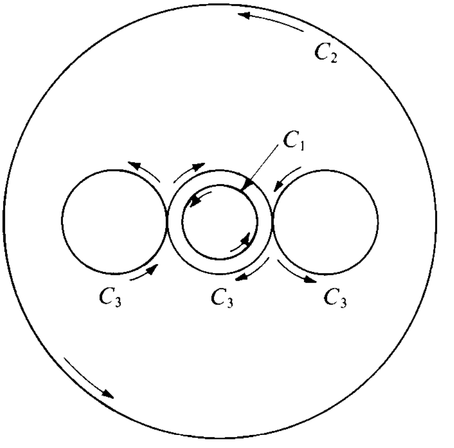


FIGURE 11.22 Exercise 6.

6. Let $I_k = \oint_{C_k} P\,dx + Q\,dy$, wheie

$$P(x, y) = -Y\left[\frac{1}{(x-1)^2+y^2} + \frac{1}{x^2+y^2} + \frac{1}{(x+1)^2+y^2}\right]$$

and

$$Q(x, y) = \frac{x-1}{(x-1)^2+y^2} + \frac{x}{x^2+y^2} + \frac{x+1}{(x+1)^2+y^2}.$$

In Figure 11.22, C_1 is the smallest circle, $x^2 + y^2 = \frac{1}{8}$ (traced counterclockwise), C_2 is the largest circle, $x^2 + y^2 = 4$ (traced counterclockwise), and C_3 is the curve made up of the three intermediate circles $(x-1)^2 + y^2 = \frac{1}{4}$, $x^2 + y^2 = \frac{1}{4}$, and $(x+1)^2 + y^2 = \frac{1}{4}$ traced out as shown. If $I_2 = 6\pi$ and $I_3 = 2\pi$, find the value of I_1.

11.26 Change of variables in a double integral

In one-dimensional integration theory the method of substitution often enables us to evaluate complicated integrals by transforming them into simpler ones or into types that can be more easily recognized. The method is based on the formula

$$\int_a^b f(x)\,dx = \int_c^d f[g(t)]g'(t)\,dt, \tag{11.29}$$

where $a = g(c)$ and $b = g(d)$. We proved this formula (in Volume I) under the assumptions that g has a continuous derivative on an interval $[c, d]$ and that f is continuous on the set of values taken by $g(t)$ as t runs through the interval $[c, d]$.

There is a two-dimensional analogue of (11.29) called the formula for making a change of variables in a double integral. It transforms an integral of the form $\iint_S f(x, y)\,dx\,dy$, extended over a region S in the xy-plane, into another double integral $\iint_T F(u, v)\,du\,dv$, extended over a new region *Tin* the uv-plane. The exact relationship between the regions S and T and the integrands $f(x, y)$ and $F(u, v)$ will be discussed presently. The method of