If G had an element of order 21 then the normalizer of a Sylow 3-subgroup of G would have order divisible by 7. Thus n_3 would be relatively prime to 7. Since then $n_3 \mid 8$ we would have $n_3 = 4$ contrary to (1). This proves:

(3) *G has no elements of order 21.*

By Sylow's Theorem $n_3 = 7$ or 28; we next rule out the former possibility. Assume $n_3 = 7$, let $P \in Syl_3(G)$ and let T be a Sylow 2-subgroup of the group $N_G(P)$ of order 24. Each Sylow 3-subgroup normalizes some Sylow 7-subgroup of G so P normalizes a Sylow 7-subgroup R of G. For every $t \in T$ we also have that $P = tPt^{-1}$ normalizes tRt^{-1}. The subgroup T acts by conjugation on the set of eight Sylow 7-subgroups of G and since no element of order 2 in G normalizes a Sylow 7-subgroup by (2), it follows that T acts transitively, i.e., every Sylow 7-subgroup of G is one of the tRt^{-1}. Hence P normalizes every Sylow 7-subgroup of G, i.e., P is contained in the intersection of the normalizers of all Sylow 7-subgroups. But this intersection is a proper normal subgroup of G, so it must be trivial. This contradiction proves:

(4) $n_3 = 28$ *and the normalizer of a Sylow 3-subgroup has order 6.*

Since $n_2 = 7$ or 21, we have $n_2 \not\equiv 1 \bmod 8$, so by Exercise 21 there is a pair of distinct Sylow 2-subgroups that have nontrivial intersection; over all such pairs let T_1 and T_2 be chosen with $U = T_1 \cap T_2$ of maximal order. We next prove

(5) *U is a Klein 4-group and $N_G(U) \cong S_4$.*

Let $N = N_G(U)$. Since $|U| = 2$ or 4 and N permutes the nonidentity elements of U by conjugation, a subgroup of order 7 in N would commute with some element of order 2 in U, contradicting (2). It follows that the order of N is not divisible by 7. By Exercise 13, N has more than one Sylow 2-subgroup, hence $|N| = 2^a \cdot 3$, where $a = 2$ or 3. Let $P \in Syl_3(N)$. Since P is a Sylow 3-subgroup of G, by (4) the group $N_N(P)$ has order 3 or 6 (with P as its unique subgroup of order 3). Thus by Sylow's Theorem N must have four Sylow 3-subgroups, and these are permuted transitively by N under conjugation. Since any group of order 12 must have either a normal Sylow 2-subgroup or a normal Sylow 3-subgroup (cf. Section 4.5), $|N| = 24$. Let K be the kernel of N acting by conjugation on its four Sylow 3-subgroups, so K is the intersection of the normalizers of the Sylow 3-subgroups of N. If $K = 1$ then $N \cong S_4$ as asserted; so consider when $K \neq 1$. Since $K \leq N_N(P)$, the group K has order dividing 6, and since P does not normalize another Sylow 3-subgroup, P is not contained in K. It follows that $|K| = 2$. But now N/K is a group of order 12 which is seen to have more than one Sylow 2-subgroup and four Sylow 3-subgroups, contrary to the property of groups of order 12 cited earlier. This proves $N \cong S_4$. Since S_4 has a unique nontrivial normal 2-subgroup, V_4, (5) holds. Since $N \cong S_4$, it follows that N contains a Sylow 2-subgroup of G and also that $N_N(P) \cong S_3$ (so also $N_G(P) \cong S_3$ by (4)). Hence we obtain

(6) *Sylow 2-subgroups of G are isomorphic to D_8*, and

(7) *the normalizer in G of a Sylow 3-subgroup is isomorphic to S_3 and so G has no elements of order 6.*

By (2) and (7), no element of order 2 commutes with an element of odd prime order. If $T \in Syl_2(G)$, then $T \cong D_8$ by (6), so $Z(T) = \langle z \rangle$ where z is an element of order 2. Then $T \leq C_G(z)$ and $|C_G(z)|$ has no odd prime factors by what was just said, so $C_G(z) = T$. Since any element normalizing T would normalize its center, hence commute with z, it follows that Sylow 2-subgroups of G are self-normalizing. This gives

(8) $\quad n_2 = 21$ *and* $C_G(z) = T$, *where* $T \in Syl_2(G)$ *and* $Z(T) = \langle z \rangle$.

Since $|C_G(z)| = 8$, the element z in (8) has 21 conjugates. By (6), G has one conjugacy class of elements of order 4, which by (6) and (8) contains 42 elements. By (2) there are 48 elements of order 7, and by (4) there are 56 elements of order 3. These account for all 167 nonidentity elements of G, and so every element of order 2 must be conjugate to z, i.e.,

(9) $\quad$ *G has a unique conjugacy class of elements of order 2.*

Continuing with the same notation, let $T \in Syl_2(G)$ with $U \leq T$ and let W be the other Klein 4-group in T. It follows from Sylow's Theorem that U and W are not conjugate in G since they are not conjugate in $N_G(T) = T$ (cf. Exercise 50 in Section 4.5). We argue next that

(10) $\quad N_G(W) \cong S_4$.

To see this let $W = \langle z, w \rangle$ where, as before, $\langle z \rangle = Z(T)$. Since w is conjugate in G to z, $C_G(w) = T_0$ is another Sylow 2-subgroup of G containing W but different from T. Thus $W = T \cap T_0$. Since U was an arbitrary maximal intersection of Sylow 2-subgroups of G, the argument giving (5) implies (10).

We now record results which we have proved or which are easy consequences of (1) to (10).

Proposition 14. If G is a simple group of order 168, then the following hold:

(1) $n_2 = 21$, $n_3 = 7$ and $n_7 = 8$

(2) Sylow 2-subgroups of G are dihedral, Sylow 3- and 7-subgroups are cyclic

(3) G is isomorphic to a subgroup of A_7 and G has no subgroup of index ≤ 6

(4) the conjugacy classes of G are the following: the identity; two classes of elements of order 7 each of which contains 24 elements (represented by any element of order 7 and its inverse); one class of elements of order 3 containing 56 elements; one class of elements of order 4 containing 42 elements; one class of elements of order 2 containing 21 elements
(in particular, every element of G has order a power of a prime)

(5) if $T \in Syl_2(G)$ and U, W are the two Klein 4-groups in T, then U and W are not conjugate in G and $N_G(U) \cong N_G(W) \cong S_4$

(6) G has precisely three conjugacy classes of maximal subgroups, two of which are isomorphic to S_4 and one of which is isomorphic to the non-abelian group of order 21.

All of the calculations above were predicated on the assumption that there exists a simple group of order 168. The fact that none of these arguments leads to a contradiction does not *prove* the existence of such a group, but rather just gives strong evidence that there *may* be a simple group of this order. We next illustrate how the internal subgroup structure of G gives rise to a geometry on which G acts, and so leads to a proof that a simple group of order 168 is unique, if it exists (which we shall also show).

Continuing the above notation let $U_1, \ldots, U_7$ be the conjugates of U and let $W_1, \ldots, W_7$ be the conjugates of W. Call the U_i *points* and the W_j *lines*. Define an "incidence relation" by specifying that

the point U_i is on the line W_j if and only if U_i normalizes W_j.

Note that U_i normalizes W_j if and only if $U_i W_j \cong D_8$, which in turn occurs if and only if W_j normalizes U_i. In each point or line stabilizer—which is isomorphic to S_4—there is a unique normal 4-group, V, and precisely three other (nonnormal) 4-groups A_1, A_2, A_3. The groups VA_i are the three Sylow 2-subgroups of the S_4. We therefore have:

(11) *each line contains exactly 3 points and each point lies on exactly 3 lines.*

Since any two nonnormal 4-groups in an S_4 generate the S_4, hence uniquely determine the other two Klein groups in that S_4, we obtain

(12) *any 2 points on a line uniquely determine the line (and the third point on it).*

Since there are 7 points and 7 lines, elementary counting now shows that

(13) *each pair of points lies on a unique line, and each pair of lines intersects in a unique point.*

(This configuration of points and lines thus satisfies axioms for what is termed a *projective plane.*) It is now straightforward to show that the incidence geometry is uniquely determined and may be represented by the graph in Figure 1, where points are vertices and lines are the six sides and medians of the triangle together with the inscribed circle—see Exercise 27. This incidence geometry is called the *projective plane of order 2* or the *Fano Plane*, and will be denoted by $\mathcal{F}$. (Generally, a projective plane of "order" N has $N^2 + N + 1$ points, and the same number of lines.) Note that at this point the projective plane $\mathcal{F}$ *does* exist—we have explicitly exhibited points and lines satisfying (11) to (13)—even though the group G is not yet known to exist.

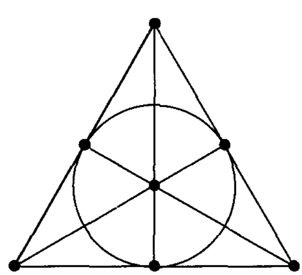

Figure 1

An *automorphism* of this plane is any permutation of points and lines that preserves the incidence relation. For example, any of the six symmetries of the triangle in Figure 1