the four variables:

$$\sigma \cdot p(x_1, x_2, x_3, x_4) = p(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)})$$

for all $\sigma \in S_4$.

(a) Find the polynomials in the orbit of S_4 on R containing $x_1 + x_2$ (recall from Exercise 12 in Section 2.2 that the stabilizer of this polynomial has order 4).

(b) Find the polynomials in the orbit of S_4 on R containing $x_1x_2 + x_3x_4$ (recall from Exercise 12 in Section 2.2 that the stabilizer of this polynomial has order 8).

(c) Find the polynomials in the orbit of S_4 on R containing $(x_1 + x_2)(x_3 + x_4)$.

7. Let G be a transitive permutation group on the finite set A. A *block* is a nonempty subset B of A such that for all $\sigma \in G$ either $\sigma(B) = B$ or $\sigma(B) \cap B = \emptyset$ (here $\sigma(B)$ is the set $\{\sigma(b) \mid b \in B\}$).

(a) Prove that if B is a block containing the element a of A, then the set G_B defined by $G_B = \{\sigma \in G \mid \sigma(B) = B\}$ is a subgroup of G containing G_a.

(b) Show that if B is a block and $\sigma_1(B), \sigma_2(B), \dots, \sigma_n(B)$ are all the distinct images of B under the elements of G, then these form a partition of A.

(c) A (transitive) group G on a set A is said to be *primitive* if the only blocks in A are the trivial ones: the sets of size 1 and A itself. Show that S_4 is primitive on $A = \{1, 2, 3, 4\}$. Show that D_8 is not primitive as a permutation group on the four vertices of a square.

(d) Prove that the transitive group G is primitive on A if and only if for each $a \in A$, the only subgroups of G containing G_a are G_a and G (i.e., G_a is a *maximal* subgroup of G, cf. Exercise 16, Section 2.4). [Use part (a).]

8. A transitive permutation group G on a set A is called *doubly transitive* if for any (hence all) $a \in A$ the subgroup G_a is transitive on the set $A - \{a\}$.

(a) Prove that S_n is doubly transitive on $\{1, 2, \dots, n\}$ for all $n \geq 2$.

(b) Prove that a doubly transitive group is primitive. Deduce that D_8 is not doubly transitive in its action on the 4 vertices of a square.

9. Assume G acts transitively on the finite set A and let H be a normal subgroup of G. Let $\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_r$ be the distinct orbits of H on A.

(a) Prove that G permutes the sets $\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_r$ in the sense that for each $g \in G$ and each $i \in \{1, \dots, r\}$ there is a j such that $g\mathcal{O}_i = \mathcal{O}_j$, where $g\mathcal{O} = \{g \cdot a \mid a \in \mathcal{O}\}$ (i.e., in the notation of Exercise 7 the sets $\mathcal{O}_1, \dots, \mathcal{O}_r$ are blocks). Prove that G is transitive on $\{\mathcal{O}_1, \dots, \mathcal{O}_r\}$. Deduce that all orbits of H on A have the same cardinality.

(b) Prove that if $a \in \mathcal{O}_1$ then $|\mathcal{O}_1| = |H : H \cap G_a|$ and prove that $r = |G : HG_a|$. [Draw the sublattice describing the Second Isomorphism Theorem for the subgroups H and G_a of G. Note that $H \cap G_a = H_a$.]

10. Let H and K be subgroups of the group G. For each $x \in G$ define the HK *double coset* of x in G to be the set

$$HxK = \{hxk \mid h \in H,\ k \in K\}.$$

(a) Prove that HxK is the union of the left cosets $x_1K, \dots, x_nK$ where $\{x_1K, \dots, x_nK\}$ is the orbit containing xK of H acting by left multiplication on the set of left cosets of K.

(b) Prove that HxK is a union of right cosets of H.

(c) Show that HxK and HyK are either the same set or are disjoint for all $x, y \in G$. Show that the set of HK double cosets partitions G.

(d) Prove that $|HxK| = |K| \cdot |H : H \cap xKx^{-1}|$.

(e) Prove that $|HxK| = |H| \cdot |K : K \cap x^{-1}Hx|$.

4.2 GROUPS ACTING ON THEMSELVES BY LEFT MULTIPLICATION — CAYLEY'S THEOREM

In this section G is any group and we first consider G *acting on itself* (i.e., $A = G$) *by left multiplication*:

$$g \cdot a = ga \qquad \text{for all } g \in G,\ a \in G$$

where ga denotes the product of the two group elements g and a in G (if G is written additively, the action will be written $g \cdot a = g + a$ and called left translation). We saw in Section 1.7 that this satisfies the two axioms of a group action.

When G is a finite group of order n it is convenient to label the elements of G with the integers $1, 2, \dots, n$ in order to describe the permutation representation afforded by this action. In this way the elements of G are listed as $g_1, g_2, \dots, g_n$ and for each $g \in G$ the permutation σ_g may be described as a permutation of the indices $1, 2, \dots, n$ as follows:

$$\sigma_g(i) = j \qquad \text{if and only if} \qquad gg_i = g_j.$$

A different labelling of the group elements will give a different description of σ_g as a permutation of $\{1, 2, \dots, n\}$ (cf. the exercises).

Example

Let $G = \{1, a, b, c\}$ be the Klein 4-group whose group table is written out in Section 2.5. Label the group elements $1, a, b, c$ with the integers 1,2,3,4, respectively. Under this labelling we compute the permutation σ_a induced by the action of left multiplication by the group element a:

$a \cdot 1 = a1 = a$ and so $\sigma_a(1) = 2$
$a \cdot a = aa = 1$ and so $\sigma_a(2) = 1$
$a \cdot b = ab = c$ and so $\sigma_a(3) = 4$ and
$a \cdot c = ac = b$ and so $\sigma_a(4) = 3$.

With this labelling of the elements of G we see that $\sigma_a = (1\ 2)(3\ 4)$. In the permutation representation associated to the action of the Klein 4-group on itself by left multiplication one similarly computes that

$$a \mapsto \sigma_a = (1\ 2)(3\ 4) \qquad b \mapsto \sigma_b = (1\ 3)(2\ 4) \qquad c \mapsto \sigma_c = (1\ 4)(2\ 3),$$

which explicitly gives the permutation representation $G \to S_4$ associated to this action under this labelling.

It is easy to see (and we shall prove this shortly in a more general setting) that the action of a group on itself by left multiplication is always transitive and faithful, and that the stabilizer of any point is the identity subgroup (these facts can be checked by inspection for the above example).

We now consider a generalization of the action of a group by left multiplication on the set of its elements. Let H be any subgroup of G and let A be the set of all left cosets of H in G. Define an action of G on A by

$$g \cdot aH = gaH \qquad \text{for all } g \in G,\ aH \in A$$

where gaH is the left coset with representative ga. One easily checks that this satisfies the two axioms for a group action, i.e., that G does act on the set of left cosets of H

by left multiplication. In the special case when H is the identity subgroup of G the coset aH is just $\{a\}$ and if we identify the element a with the set $\{a\}$, this action by left multiplication on left cosets of the identity subgroup is the same as the action of G on itself by left multiplication.

When H is of finite index m in G it is convenient to label the left cosets of H with the integers $1, 2, \dots, m$ in order to describe the permutation representation afforded by this action. In this way the distinct left cosets of H in G are listed as $a_1H, a_2H, \dots, a_mH$ and for each $g \in G$ the permutation σ_g may be described as a permutation of the indices $1, 2, \dots, m$ as follows:

$$\sigma_g(i) = j \qquad \text{if and only if} \qquad ga_iH = a_jH.$$

A different labelling of the group elements will give a different description of σ_g as a permutation of $\{1, 2, \dots, m\}$ (cf. the exercises).

Example

Let $G = D_8$ and let $H = \langle s \rangle$. Label the distinct left cosets $1H, rH, r^2H, r^3H$ with the integers 1,2,3,4 respectively. Under this labelling we compute the permutation σ_s induced by the action of left multiplication by the group element s on the left cosets of H:

$s \cdot 1H = sH = 1H$ and so $\sigma_s(1) = 1$
$s \cdot rH = srH = r^3H$ and so $\sigma_s(2) = 4$
$s \cdot r^2H = sr^2H = r^2H$ and so $\sigma_s(3) = 3$
$s \cdot r^3H = sr^3H = rH$ and so $\sigma_s(4) = 2$.

With this labelling of the left cosets of H we obtain $\sigma_s = (2\ 4)$. In the permutation representation associated to the action of D_8 on the left cosets of $\langle s \rangle$ by left multiplication one similarly computes that $\sigma_r = (1\ 2\ 3\ 4)$. Note that the permutation representation is a homomorphism, so once its value has been determined on generators for D_8 its value on any other element can be determined (e.g., $\sigma_{sr^2} = \sigma_s\sigma_r^2$).

Theorem 3. Let G be a group, let H be a subgroup of G and let G act by left multiplication on the set A of left cosets of H in G. Let π_H be the associated permutation representation afforded by this action. Then

(1) G acts transitively on A
(2) the stabilizer in G of the point $1H \in A$ is the subgroup H
(3) the kernel of the action (i.e., the kernel of π_H) is $\bigcap_{x\in G} xHx^{-1}$, and $\ker \pi_H$ is the largest normal subgroup of G contained in H.

Proof: To see that G acts transitively on A, let aH and bH be any two elements of A, and let $g = ba^{-1}$. Then $g \cdot aH = (ba^{-1})aH = bH$, and so the two arbitrary elements aH and bH of A lie in the same orbit, which proves (1). For (2), the stabilizer of the point $1H$ is, by definition, $\{g \in G \mid g \cdot 1H = 1H\}$, i.e., $\{g \in G \mid gH = H\} = H$.

By definition of π_H we have

$$\begin{aligned}\ker \pi_H &= \{g \in G \mid gxH = xH \text{ for all } x \in G\}\\ &= \{g \in G \mid (x^{-1}gx)H = H \text{ for all } x \in G\}\\ &= \{g \in G \mid x^{-1}gx \in H \text{ for all } x \in G\}\\ &= \{g \in G \mid g \in xHx^{-1} \text{ for all } x \in G\} = \bigcap_{x\in G} xHx^{-1},\end{aligned}$$