We shall immediately become less formal and say G is a group acting on a set A. The expression $g \cdot a$ will usually be written simply as ga when there is no danger of confusing this map with, say, the group operation (remember, $\cdot$ is not a binary operation and ga is always a member of A). Note that on the left hand side of the equation in property (1) $g_2 \cdot a$ is an element of A so it makes sense to act on this by g_1. On the right hand side of this equation the product $(g_1 g_2)$ is taken in G and the resulting group element acts on the set element a.

Before giving some examples of group actions we make some observations. Let the group G act on the set A. For each fixed $g \in G$ we get a map σ_g defined by

$$\begin{aligned} \sigma_g &: A \to A \\ \sigma_g(a) &= g \cdot a. \end{aligned}$$

We prove two important facts:

(i) for each fixed $g \in G$, σ_g is a *permutation* of A, and
(ii) the map from G to S_A defined by $g \mapsto \sigma_g$ is a homomorphism.

To see that σ_g is a permutation of A we show that as a set map from A to A it has a 2-sided inverse, namely $\sigma_{g^{-1}}$ (it is then a permutation by Proposition 1 of Section 0.1). For all $a \in A$

$$\begin{aligned} (\sigma_{g^{-1}} \circ \sigma_g)(a) &= \sigma_{g^{-1}}(\sigma_g(a)) && \text{(by definition of function composition)} \\ &= g^{-1} \cdot (g \cdot a) && \text{(by definition of } \sigma_{g^{-1}} \text{ and } \sigma_g) \\ &= (g^{-1} g) \cdot a && \text{(by property (1) of an action)} \\ &= 1 \cdot a = a && \text{(by property (2) of an action).} \end{aligned}$$

This proves $\sigma_{g^{-1}} \circ \sigma_g$ is the identity map from A to A. Since g was arbitrary, we may interchange the roles of g and g^{-1} to obtain $\sigma_g \circ \sigma_{g^{-1}}$ is also the identity map on A. Thus σ_g has a 2-sided inverse, hence is a permutation of A.

To check assertion (ii) above let $\varphi : G \to S_A$ be defined by $\varphi(g) = \sigma_g$. Note that part (i) shows that σ_g is indeed an element of S_A. To see that φ is a homomorphism we must prove $\varphi(g_1 g_2) = \varphi(g_1) \circ \varphi(g_2)$ (recall that S_A is a group under function composition). The permutations $\varphi(g_1 g_2)$ and $\varphi(g_1) \circ \varphi(g_2)$ are equal if and only if their values agree on every element $a \in A$. For all $a \in A$

$$\begin{aligned} \varphi(g_1 g_2)(a) &= \sigma_{g_1 g_2}(a) && \text{(by definition of } \varphi) \\ &= (g_1 g_2) \cdot a && \text{(by definition of } \sigma_{g_1 g_2}) \\ &= g_1 \cdot (g_2 \cdot a) && \text{(by property (1) of an action)} \\ &= \sigma_{g_1}(\sigma_{g_2}(a)) && \text{(by definition of } \sigma_{g_1} \text{ and } \sigma_{g_2}) \\ &= (\varphi(g_1) \circ \varphi(g_2))(a) && \text{(by definition of } \varphi). \end{aligned}$$

This proves assertion (ii) above.

Intuitively, a group action of G on a set A just means that every element g in G acts as a permutation on A in a manner consistent with the group operations in G; assertions (i) and (ii) above make this precise. The homomorphism from G to S_A given above is called the *permutation representation* associated to the given action. It is easy to see that this process is reversible in the sense that if $\varphi : G \to S_A$ is any homomorphism from a group G to the symmetric group on a set A, then the map from $G \times A$ to A defined by

$$g \cdot a = \varphi(g)(a) \qquad \text{for all } g \in G, \text{ and all } a \in A$$

satisfies the properties of a group action of G on A. Thus actions of a group G on a set A and the homomorphisms from G into the symmetric group S_A are in bijective correspondence (i.e., are essentially the same notion, phrased in different terminology).

We should also note that the definition of an action might have been more precisely named a *left* action since the group elements appear on the left of the set elements. We could similarly define the notion of a *right* action.

Examples

Let G be a group and A a nonempty set. In each of the following examples the check of properties (1) and (2) of an action are left as exercises.

(1) Let $ga = a$, for all $g \in G$, $a \in A$. Properties (1) and (2) of a group action follow immediately. This action is called the *trivial action* and G is said to *act trivially* on A. Note that *distinct* elements of G induce the *same* permutation on A (in this case the identity permutation). The associated permutation representation $G \to S_A$ is the trivial homomorphism which maps every element of G to the identity.

If G acts on a set B and distinct elements of G induce *distinct* permutations of B, the action is said to be *faithful*. A faithful action is therefore one in which the associated permutation representation is injective.

The *kernel* of the action of G on B is defined to be $\{g \in G \mid gb = b \text{ for all } b \in B\}$, namely the elements of G which fix *all* the elements of B. For the trivial action, the kernel of the action is all of G and this action is not faithful when $|G| > 1$.

(2) The axioms for a vector space V over a field F include the two axioms that the multiplicative group $F^{\times}$ act on the set V. Thus vector spaces are familiar examples of actions of multiplicative groups of fields where there is even more structure (in particular, V must be an abelian group) which can be exploited. In the special case when $V = \mathbb{R}^n$ and $F = \mathbb{R}$ the action is specified by

$$\alpha(r_1, r_2, \dots, r_n) = (\alpha r_1, \alpha r_2, \dots, \alpha r_n)$$

for all $\alpha \in \mathbb{R}$, $(r_1, r_2, \dots, r_n) \in \mathbb{R}^n$, where αr_i is just multiplication of two real numbers.

(3) For any nonempty set A the symmetric group S_A acts on A by $\sigma \cdot a = \sigma(a)$, for all $\sigma \in S_A$, $a \in A$. The associated permutation representation is the identity map from S_A to itself.

(4) If we fix a labelling of the vertices of a regular n-gon, each element α of D_{2n} gives rise to a permutation σ_α of $\{1, 2, \dots, n\}$ by the way the symmetry α permutes the corresponding vertices. The map of $D_{2n} \times \{1, 2, \dots, n\}$ onto $\{1, 2, \dots, n\}$ defined by $(\alpha, i) \to \sigma_\alpha(i)$ defines a group action of D_{2n} on $\{1, 2, \dots, n\}$. In keeping with our notation for group actions we can now dispense with the formal and cumbersome notation $\sigma_\alpha(i)$ and write αi in its place. Note that this action is faithful: distinct symmetries of a regular n-gon induce distinct permutations of the vertices.

When $n = 3$ the action of D_6 on the three (labelled) vertices of a triangle gives an injective homomorphism from D_6 to S_3. Since these groups have the same order, this map must also be surjective, i.e., is an isomorphism: $D_6 \cong S_3$. This is another proof of the same fact we established via generators and relations in the preceding section. Geometrically it says that any permutation of the vertices of a triangle is a symmetry. The analogous statement is not true for any n-gon with $n \geq 4$ (just by order considerations we cannot have D_{2n} isomorphic to S_n for any $n \geq 4$).

(5) Let G be any group and let $A = G$. Define a map from $G \times A$ to A by $g \cdot a = ga$, for each $g \in G$ and $a \in A$, where ga on the right hand side is the product of g and a in the group G. This gives a group action of G on itself, where each (fixed) $g \in G$ permutes the elements of G by *left multiplication*:

$$g : a \mapsto ga \qquad \text{for all } a \in G$$

(or, if G is written additively, we get $a \mapsto g + a$ and call this *left translation*). This action is called the *left regular action* of G on itself. By the cancellation laws, this action is faithful (check this).

Other examples of actions are given in the exercises.

EXERCISES

1. Let F be a field. Show that the multiplicative group of nonzero elements of F (denoted by $F^\times$) acts on the set F by $g \cdot a = ga$, where $g \in F^\times$, $a \in F$ and ga is the usual product in F of the two field elements (state clearly which axioms in the definition of a field are used).

2. Show that the additive group $\mathbb{Z}$ acts on itself by $z \cdot a = z + a$ for all $z, a \in \mathbb{Z}$.

3. Show that the additive group $\mathbb{R}$ acts on the x, y plane $\mathbb{R} \times \mathbb{R}$ by $r \cdot (x, y) = (x + ry, y)$.

4. Let G be a group acting on a set A and fix some $a \in A$. Show that the following sets are subgroups of G (cf. Exercise 26 of Section 1):
 (a) the kernel of the action,
 (b) $\{g \in G \mid ga = a\}$ — this subgroup is called the *stabilizer* of a in G.

5. Prove that the kernel of an action of the group G on the set A is the same as the kernel of the corresponding permutation representation $G \to S_A$ (cf. Exercise 14 in Section 6).

6. Prove that a group G acts faithfully on a set A if and only if the kernel of the action is the set consisting only of the identity.

7. Prove that in Example 2 in this section the action is faithful.

8. Let A be a nonempty set and let k be a positive integer with $k \leq |A|$. The symmetric group S_A acts on the set B consisting of all subsets of A of cardinality k by $\sigma \cdot \{a_1, \dots, a_k\} = \{\sigma(a_1), \dots, \sigma(a_k)\}$.
 (a) Prove that this is a group action.
 (b) Describe explicitly how the elements $(1\ 2)$ and $(1\ 2\ 3)$ act on the six 2-element subsets of $\{1, 2, 3, 4\}$.

9. Do both parts of the preceding exercise with "ordered k-tuples" in place of "k-element subsets," where the action on k-tuples is defined as above but with set braces replaced by parentheses (note that, for example, the 2-tuples (1,2) and (2,1) are different even though the sets $\{1, 2\}$ and $\{2, 1\}$ are the same, so the sets being acted upon are different).

10. With reference to the preceding two exercises determine:
 (a) for which values of k the action of S_n on k-element subsets is faithful, and
 (b) for which values of k the action of S_n on ordered k-tuples is faithful.