In general it may not be possible to lift a homomorphism f from D to N to a homomorphism from D to M. For example, consider the nonsplit exact sequence $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \to 0$ from the previous set of examples. Let $D = \mathbb{Z}/2\mathbb{Z}$ and let f be the identity map from D into N. Any homomorphism F of D into $M = \mathbb{Z}$ must map D to 0 (since $\mathbb{Z}$ has no elements of order 2), hence $\pi \circ F$ maps D to 0 in N, and in particular, $\pi \circ F \neq f$. Phrased in terms of the map φ', this shows that

$$\text{if} \quad M \xrightarrow{\ \varphi\ } N \longrightarrow 0 \quad \text{is exact,}$$

$$\text{then} \quad \mathrm{Hom}_R(D, M) \xrightarrow{\ \varphi'\ } \mathrm{Hom}_R(D, N) \longrightarrow 0 \quad \text{is } \textit{not necessarily} \text{ exact.}$$

These results relating the homomorphisms into L and N to the homomorphisms into M can be neatly summarized as part of the following theorem.

Theorem 28. Let D, L, M, and N be R-modules. If

$$0 \longrightarrow L \xrightarrow{\ \psi\ } M \xrightarrow{\ \varphi\ } N \longrightarrow 0 \quad \text{is exact,}$$

then the associated sequence

$$0 \to \mathrm{Hom}_R(D, L) \xrightarrow{\psi'} \mathrm{Hom}_R(D, M) \xrightarrow{\varphi'} \mathrm{Hom}_R(D, N) \quad \text{is exact.} \tag{10.10}$$

A homomorphism $f : D \to N$ lifts to a homomorphism $F : D \to M$ if and only if $f \in \mathrm{Hom}_R(D, N)$ is in the image of φ'. In general $\varphi' : \mathrm{Hom}_R(D, M) \to \mathrm{Hom}_R(D, N)$ need not be surjective; the map φ' is surjective if and only if every homomorphism from D to N lifts to a homomorphism from D to M, in which case the sequence (10) can be extended to a short exact sequence.

The sequence (10) is exact for *all* R-modules D if and only if the sequence

$$0 \to L \xrightarrow{\psi} M \xrightarrow{\varphi} N \quad \text{is exact.}$$

Proof: The only item in the first statement that has not already been proved is the exactness of (10) at $\mathrm{Hom}_R(D, M)$, i.e., $\ker \varphi' = \mathrm{image}\, \psi'$. Suppose $F : D \to M$ is an element of $\mathrm{Hom}_R(D, M)$ lying in the kernel of φ', i.e., with $\varphi \circ F = 0$ as homomorphisms from D to N. If $d \in D$ is any element of D, this implies that $\varphi(F(d)) = 0$ and $F(d) \in \ker \varphi$. By the exactness of the sequence defining the extension M we have $\ker \varphi = \mathrm{image}\, \psi$, so there is some element $l \in L$ with $F(d) = \psi(l)$. Since ψ is injective, the element l is unique, so this gives a well defined map $F' : D \to L$ given by $F'(d) = l$. It is an easy check to verify that F' is a homomorphism, i.e., $F' \in \mathrm{Hom}_R(D, L)$. Since $\psi \circ F'(d) = \psi(l) = F(d)$, we have $F = \psi'(F')$ which shows that F is in the image of ψ', proving that $\ker \varphi' \subseteq \mathrm{image}\, \psi'$. Conversely, if F is in the image of ψ' then $F = \psi'(F')$ for some $F' \in \mathrm{Hom}_R(D, L)$ and so $\varphi(F(d)) = \varphi(\psi(F'(d)))$ for any $d \in D$. Since $\ker \varphi = \mathrm{image}\, \psi$ we have $\varphi \circ \psi = 0$, and it follows that $\varphi(F(d)) = 0$ for any $d \in D$, i.e., $\varphi'(F) = 0$. Hence F is in the kernel of φ', proving the reverse containment: $\mathrm{image}\, \psi' \subseteq \ker \varphi'$.

For the last statement in the theorem, note first that the surjectivity of φ was not required for the proof that (10) is exact, so the "if" portion of the statement has already been proved. For the converse, suppose that the sequence (10) is exact for all R-modules D. In general, $\mathrm{Hom}_R(R, X) \cong X$ for any left R-module X, the isomorphism being given by mapping a homomorphism to its value on the element $1 \in R$ (cf. Exercise 10(b)). Taking $D = R$ in (10), the exactness of the sequence $0 \to L \xrightarrow{\psi} M \xrightarrow{\varphi} N$ follows easily.

By Theorem 28, the sequence

$$0 \longrightarrow \mathrm{Hom}_R(D, L) \xrightarrow{\psi'} \mathrm{Hom}_R(D, M) \xrightarrow{\varphi'} \mathrm{Hom}_R(D, N) \longrightarrow 0 \qquad (10.11)$$

is in general *not* a short exact sequence since the homomorphism φ' need not be surjective. The question of whether this sequence is exact precisely measures the extent to which the homomorphisms from D into M are uniquely determined by pairs of homomorphisms from D into L and D into N. More precisely, this sequence is exact if and only if there is a bijection $F \leftrightarrow (g, f)$ between homomorphisms $F : D \to M$ and pairs of homomorphisms $g : D \to L$ and $f : D \to N$ given by $F|_{\psi(L)} = \psi'(g)$ and $f = \varphi'(F)$.

One situation in which the sequence (11) is exact occurs when the original sequence $0 \to L \to M \to N \to 0$ is a *split* exact sequence, i.e., when $M = L \oplus N$. In this case the sequence (11) is also a split exact sequence, as the first part of the following proposition shows.

Proposition 29. Let D, L and N be R-modules. Then

(1) $\mathrm{Hom}_R(D, L \oplus N) \cong \mathrm{Hom}_R(D, L) \oplus \mathrm{Hom}_R(D, N)$, and

(2) $\mathrm{Hom}_R(L \oplus N, D) \cong \mathrm{Hom}_R(L, D) \oplus \mathrm{Hom}_R(N, D)$.

Proof: Let $\pi_1 : L \oplus N \to L$ be the natural projection from $L \oplus N$ to L and similarly let π_2 be the natural projection to N. If $f \in \mathrm{Hom}_R(D, L \oplus N)$ then the compositions $\pi_1 \circ f$ and $\pi_2 \circ f$ give elements in $\mathrm{Hom}_R(D, L)$ and $\mathrm{Hom}_R(D, N)$, respectively. This defines a map from $\mathrm{Hom}_R(D, L \oplus N)$ to $\mathrm{Hom}_R(D, L) \oplus \mathrm{Hom}_R(D, N)$ which is easily seen to be a homomorphism. Conversely, given $f_1 \in \mathrm{Hom}_R(D, L)$ and $f_2 \in \mathrm{Hom}_R(D, N)$, define the map $f \in \mathrm{Hom}_R(D, L \oplus N)$ by $f(d) = (f_1(d), f_2(d))$. This defines a map from $\mathrm{Hom}_R(D, L) \oplus \mathrm{Hom}_R(D, N)$ to $\mathrm{Hom}_R(D, L \oplus N)$ that is easily checked to be a homomorphism inverse to the map above, proving the isomorphism in (1). The proof of (2) is similar and is left as an exercise.

The results in Proposition 29 extend immediately by induction to any finite direct sum of R-modules. These results are referred to by saying that Hom *commutes with finite direct sums in either variable* (compare to Theorem 17 for a corresponding result for tensor products). For infinite direct sums the situation is more complicated. Part (1) remains true if $L \oplus N$ is replaced by an arbitrary direct sum and the direct sum on the right hand side is replaced by a direct product (Exercise 13 shows that the direct product is necessary). Part (2) remains true if the direct sums on both sides are replaced by direct products.

This proposition shows that if the sequence

$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow 0$$

is a split short exact sequence of R-modules, then

$$0 \longrightarrow \operatorname{Hom}_R(D, L) \xrightarrow{\psi'} \operatorname{Hom}_R(D, M) \xrightarrow{\varphi'} \operatorname{Hom}_R(D, N) \longrightarrow 0$$

is also a split short exact sequence of abelian groups for every R-module D. Exercise 14 shows that a converse holds: if $0 \to \operatorname{Hom}_R(D, L) \xrightarrow{\psi'} \operatorname{Hom}_R(D, M) \xrightarrow{\varphi'} \operatorname{Hom}_R(D, N) \to 0$ is exact for every R-module D then $0 \to L \xrightarrow{\psi} M \xrightarrow{\varphi} N \to 0$ is a split short exact sequence (which then implies that if the original Hom sequence is exact for every D, then in fact it is split exact for every D).

Proposition 29 identifies a situation in which the sequence (11) is exact in terms of the modules L, M, and N. The next result adopts a slightly different perspective, characterizing instead the modules D having the property that the sequence (10) in Theorem 28 can *always* be extended to a short exact sequence:

Proposition 30. Let P be an R-module. Then the following are equivalent:

(1) For any R-modules L, M, and N, if

$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow 0$$

is a short exact sequence, then

$$0 \longrightarrow \operatorname{Hom}_R(P, L) \xrightarrow{\psi'} \operatorname{Hom}_R(P, M) \xrightarrow{\varphi'} \operatorname{Hom}_R(P, N) \longrightarrow 0$$

is also a short exact sequence.

(2) For any R-modules M and N, if $M \xrightarrow{\varphi} N \to 0$ is exact, then every R-module homomorphism from P into N lifts to an R-module homomorphism into M, i.e., given $f \in \operatorname{Hom}_R(P, N)$ there is a lift $F \in \operatorname{Hom}_R(P, M)$ making the following diagram commute:

$$\begin{tikzcd} & P \arrow[dl, dashed, "F"'] \arrow[d, "f"] & \\ M \arrow[r, "\varphi"] & N \arrow[r] & 0 \end{tikzcd}$$

(3) If P is a quotient of the R-module M then P is isomorphic to a direct summand of M, i.e., every short exact sequence $0 \to L \to M \to P \to 0$ splits.

(4) P is a direct summand of a free R-module.

Proof: The equivalence of (1) and (2) is a restatement of a result in Theorem 28. Suppose now that (2) is satisfied, and let $0 \to L \xrightarrow{\psi} M \xrightarrow{\varphi} P \to 0$ be exact. By (2), the identity map from P to P lifts to a homomorphism μ making the following diagram commute:

$$\begin{tikzcd} & P \arrow[dl, dashed, "\mu"'] \arrow[d, "id"] & \\ M \arrow[r, "\varphi"] & P \arrow[r] & 0 \end{tikzcd}$$

Then $\varphi \circ \mu = 1$, so μ is a splitting homomorphism for the sequence, which proves (3).

Every module P is the quotient of a free module (for example, the free module on the set of elements in P), so there is always an exact sequence $0 \to \ker\varphi \to \mathcal{F} \xrightarrow{\varphi} P \to 0$ where $\mathcal{F}$ is a free R-module (cf. Example 4 following Corollary 23). If (3) is satisfied, then this sequence splits, so $\mathcal{F}$ is isomorphic to the direct sum of $\ker\varphi$ and P, which proves (4).

Finally, to prove (4) implies (2), suppose that P is a direct summand of a free R-module on some set S, say $\mathcal{F}(S) = P \oplus K$, and that we are given a homomorphism f from P to N as in (2). Let π denote the natural projection from $\mathcal{F}(S)$ to P, so that $f \circ \pi$ is a homomorphism from $\mathcal{F}(S)$ to N. For any $s \in S$ define $n_s = f \circ \pi(s) \in N$ and let $m_s \in M$ be any element of M with $\varphi(m_s) = n_s$ (which exists because φ is surjective). By the universal property for free modules (Theorem 6 of Section 3), there is a unique R-module homomorphism F' from $\mathcal{F}(S)$ to M with $F'(s) = m_s$. The diagram is the following:

$$\begin{tikzcd}
 & \mathcal{F}(S) = P \oplus K \arrow[d, "\pi"] \arrow[ddl, dashed, "F'"'] & \\
 & P \arrow[d, "f"] & \\
M \arrow[r, "\varphi"] & N \arrow[r] & 0
\end{tikzcd}$$

By definition of the homomorphism F' we have $\varphi \circ F'(s) = \varphi(m_s) = n_s = f \circ \pi(s)$, from which it follows that $\varphi \circ F' = f \circ \pi$ on $\mathcal{F}(S)$, i.e., the diagram above is commutative. Now define a map $F : P \to M$ by $F(d) = F'((d, 0))$. Since F is the composite of the injection $P \to \mathcal{F}(S)$ with the homomorphism F', it follows that F is an R-module homomorphism. Then

$$\varphi \circ F(d) = \varphi \circ F'((d, 0)) = f \circ \pi((d, 0)) = f(d)$$

i.e., $\varphi \circ F = f$, so the diagram

$$\begin{tikzcd}
 & P \arrow[d, "f"] \arrow[dl, dashed, "F"'] & \\
M \arrow[r, "\varphi"] & N \arrow[r] & 0
\end{tikzcd}$$

commutes, which proves that (4) implies (2) and completes the proof.

Definition. An R-module P is called *projective* if it satisfies any of the equivalent conditions of Proposition 30.

The third statement in Proposition 30 can be rephrased as saying that any module M that projects onto P has (an isomorphic copy of) P as a direct summand, which explains the terminology.

The following result is immediate from Proposition 30 (and its proof):

Corollary 31. Free modules are projective. A finitely generated module is projective if and only if it is a direct summand of a finitely generated free module. Every module is a quotient of a projective module.