to specify what f does to any input. Nevertheless, just as the empty set is a set, the empty function is a function, albeit not a particularly interesting one. Note that for each set X, there is only one function from $\emptyset$ to X, since Definition 3.3.7 asserts that all functions from $\emptyset$ to X are equal (why?).

This notion of equality obeys the usual axioms (Exercise 3.3.1). A fundamental operation available for functions is *composition.*

Definition 3.3.10 (Composition). Let $f : X \to Y$ and $g : Y \to Z$ be two functions, such that the range of f is the same set as the domain of g. We then define the *composition* $g \circ f : X \to Z$ of the two functions g and f to be the function defined explicitly by the formula

$$(g \circ f)(x) := g(f(x)).$$

If the range of f does not match the domain of g, we leave the composition $g \circ f$ undefined.

It is easy to check that composition obeys the axiom of substitution (Exercise 3.3.1).

Example 3.3.11. Let $f : \mathbf{N} \to \mathbf{N}$ be the function $f(n) := 2n$, and let $g : \mathbf{N} \to \mathbf{N}$ be the function $g(n) := n + 3$. Then $g \circ f$ is the function

$$g \circ f(n) = g(f(n)) = g(2n) = 2n + 3,$$

thus for instance $g \circ f(1) = 5$, $g \circ f(2) = 7$, and so forth. Meanwhile, $f \circ g$ is the function

$$f \circ g(n) = f(g(n)) = f(n + 3) = 2(n + 3) = 2n + 6,$$

thus for instance $f \circ g(1) = 8$, $f \circ g(2) = 10$, and so forth.

The above example shows that composition is not commutative: $f \circ g$ and $g \circ f$ are not necessarily the same function. However, composition is still associative:

Lemma 3.3.12 (Composition is associative). *Let $f : X \to Y$, $g : Y \to Z$, and $h : Z \to W$ be functions. Then $f \circ (g \circ h) = (f \circ g) \circ h$.*

Proof. Since $g \circ h$ is a function from Y to W, $f \circ (g \circ h)$ is a function from X to W. Similarly $f \circ g$ is a function from X to Z, and hence $(f \circ g) \circ h$ is a function from X to W. Thus $f \circ (g \circ h)$ and $(f \circ g) \circ h$ have the same domain and range. In order to check that they are equal, we see from Definition 3.3.7 that we have to verify that $(f \circ (g \circ h))(x) = ((f \circ g) \circ h)(x)$ for all $x \in X$. But by Definition 3.3.10

$$\begin{aligned}(f \circ (g \circ h))(x) &= f((g \circ h)(x)) \\ &= f(g(h(x)) \\ &= (f \circ g)(h(x)) \\ &= ((f \circ g) \circ h)(x)\end{aligned}$$

as desired. □

Remark 3.3.13. Note that while g appears to the left of f in the expression $g \circ f$, the function $g \circ f$ applies the right-most function f first, before applying g. This is often confusing at first; it arises because we traditionally place a function f to the left of its input x rather than to the right. (There are some alternate mathematical notations in which the function is placed to the right of the input, thus we would write xf instead of $f(x)$, but this notation has often proven to be more confusing than clarifying, and has not as yet become particularly popular.)

We now describe certain special types of functions: *one-to-one* functions, *onto* functions, and *invertible* functions.

Definition 3.3.14 (One-to-one functions). A function f is *one-to-one* (or *injective*) if different elements map to different elements:

$$x \neq x' \quad \Longrightarrow \quad f(x) \neq f(x').$$

Equivalently, a function is one-to-one if

$$f(x) = f(x') \quad \Longrightarrow \quad x = x'.$$

Example 3.3.15. (Informal) The function $f : \mathbf{Z} \to \mathbf{Z}$ defined by $f(n) := n^2$ is not one-to-one because the distinct elements -1, 1 map to the same element 1. On the other hand, if we restrict this function to the natural numbers, defining the function $g : \mathbf{N} \to \mathbf{Z}$ by $g(n) := n^2$, then g is now a one-to-one function. Thus the notion of a one-to-one function depends not just on what the function does, but also what its domain is.

Remark 3.3.16. If a function $f : X \to Y$ is not one-to-one, then one can find distinct x and x' in the domain X such that $f(x) = f(x')$, thus one can find two inputs which map to one output. Because of this, we say that f is *two-to-one* instead of *one-to-one.*

Definition 3.3.17 (Onto functions). A function f is *onto* (or *surjective*) if $f(X) = Y$, i.e., every element in Y comes from applying f to some element in X:

$$\text{For every } y \in Y, \text{ there exists } x \in X \text{ such that } f(x) = y.$$

Example 3.3.18. (Informal) The function $f : \mathbf{Z} \to \mathbf{Z}$ defined by $f(n) := n^2$ is not onto because the negative numbers are not in the image of f. However, if we restrict the range $\mathbf{Z}$ to the set $A := \{n^2 : n \in \mathbf{Z}\}$ of square numbers, then the function $g : \mathbf{Z} \to A$ defined by $g(n) := n^2$ is now onto. Thus the notion of an onto function depends not just on what the function does, but also what its range is.

Remark 3.3.19. The concepts of injectivity and surjectivity are in many ways dual to each other; see Exercises 3.3.2, 3.3.4, 3.3.5 for some evidence of this.

Definition 3.3.20 (Bijective functions). Functions $f : X \to Y$ which are both one-to-one and onto are also called *bijective* or *invertible.*

Example 3.3.21. Let $f : \{0, 1, 2\} \to \{3, 4\}$ be the function $f(0) := 3$, $f(1) := 3$, $f(2) := 4$. This function is not bijective because if we set $y = 3$, then there is more than one x in

$\{0, 1, 2\}$ such that $f(x) = y$ (this is a failure of injectivity). Now let $g : \{0, 1\} \to \{2, 3, 4\}$ be the function $g(0) := 2$, $g(1) := 3$; then g is not bijective because if we set $y = 4$, then there is no x for which $g(x) = y$ (this is a failure of surjectivity). Now let $h : \{0, 1, 2\} \to \{3, 4, 5\}$ be the function $h(0) := 3$, $h(1) := 4$, $h(2) := 5$. Then h is bijective, because each of the elements 3, 4, 5 comes from exactly one element from 0, 1, 2.

Example 3.3.22. The function $f : \mathbf{N} \to \mathbf{N}\backslash\{0\}$ defined by $f(n) := n{+}{+}$ is a bijection (in fact, this fact is simply restating Axioms 2.2, 2.3, 2.4). On the other hand, the function $g : \mathbf{N} \to \mathbf{N}$ defined by the same definition $g(n) := n{+}{+}$ is not a bijection. Thus the notion of a bijective function depends not just on what the function does, but also what its range (and domain) are.

Remark 3.3.23. If a function $x \mapsto f(x)$ is bijective, then we sometimes call f a *perfect matching* or a *one-to-one correspondence* (not to be confused with the notion of a one-to-one function), and denote the action of f using the notation $x \leftrightarrow f(x)$ instead of $x \mapsto f(x)$. Thus for instance the function h in the above example is the one-to-one correspondence $0 \leftrightarrow 3$, $1 \leftrightarrow 4$, $2 \leftrightarrow 5$.

Remark 3.3.24. A common error is to say that a function $f : X \to Y$ is bijective iff "for every x in X, there is exactly one y in Y such that $y = f(x)$." This is not what it means for f to be bijective; rather, this is merely stating what it means for f to be a *function*. A function cannot map one element to two different elements, for instance one cannot have a function f for which $f(0) = 1$ and also $f(0) = 2$. The functions f, g given in the previous example are not bijective, but they are still functions, since each input still gives exactly one output.

If f is bijective, then for every $y \in Y$, there is exactly one x such that $f(x) = y$ (there is at least one because of surjectivity, and at most one because of injectivity). This value of x is denoted $f^{-1}(y)$; thus f^{-1} is a function from Y to X. We call f^{-1} the *inverse* of f.

Exercise 3.3.1. Show that the definition of equality in Definition 3.3.7 is reflexive, symmetric, and transitive. Also verify the substitution property: if $f, \tilde{f} : X \to Y$ and $g, \tilde{g} : Y \to Z$ are functions such that $f = \tilde{f}$ and $g = \tilde{g}$, then $f \circ g = \tilde{f} \circ \tilde{g}$.

Exercise 3.3.2. Let $f : X \to Y$ and $g : Y \to Z$ be functions. Show that if f and g are both injective, then so is $g \circ f$; similarly, show that if f and g are both surjective, then so is $g \circ f$.

Exercise 3.3.3. When is the empty function injective? surjective? bijective?

Exercise 3.3.4. In this section we give some cancellation laws for composition. Let $f : X \to Y$, $\tilde{f} : X \to Y$, $g : Y \to Z$, and $\tilde{g} : Y \to Z$ be functions. Show that if $g \circ f = g \circ \tilde{f}$ and g is injective, then $f = \tilde{f}$. Is the same statement true if g is not injective? Show that if $g \circ f = \tilde{g} \circ f$ and f is surjective, then $g = \tilde{g}$. Is the same statement true if f is not surjective?

Exercise 3.3.5. Let $f : X \to Y$ and $g : Y \to Z$ be functions. Show that if $g \circ f$ is injective, then f must be injective. Is it true that g must also be injective? Show that if $g \circ f$ is surjective, then g must be surjective. Is it true that f must also be surjective?

Exercise 3.3.6. Let $f : X \to Y$ be a bijective function, and let $f^{-1} : Y \to X$ be its inverse. Verify the cancellation laws $f^{-1}(f(x)) = x$ for all $x \in X$ and $f(f^{-1}(y)) = y$ for all $y \in Y$. Conclude that f^{-1} is also invertible, and has f as its inverse (thus $(f^{-1})^{-1} = f$).

Exercise 3.3.7. Let $f : X \to Y$ and $g : Y \to Z$ be functions. Show that if f and g are bijective, then so is $g \circ f$, and we have $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Exercise 3.3.8. If X is a subset of Y, let $\iota_{X \to Y} : X \to Y$ be the *inclusion map from X to Y*, defined by mapping $x \mapsto x$ for all $x \in X$, i.e., $\iota_{X \to Y}(x) := x$ for all $x \in X$. The map $\iota_{X \to X}$ is in particular called the *identity map* on X.

(a) Show that if $X \subseteq Y \subseteq Z$ then $\iota_{Y \to Z} \circ \iota_{X \to Y} = \iota_{X \to Z}$.

(b) Show that if $f : A \to B$ is any function, then $f = f \circ \iota_{A \to A} = \iota_{B \to B} \circ f$.

(c) Show that, if $f : A \to B$ is a bijective function, then $f \circ f^{-1} = \iota_{B \to B}$ and $f^{-1} \circ f = \iota_{A \to A}$.

(d) Show that if X and Y are disjoint sets, and $f : X \to Z$ and $g : Y \to Z$ are functions, then there is a unique function $h : X \cup Y \to Z$ such that $h \circ \iota_{X \to X \cup Y} = f$ and $h \circ \iota_{Y \to X \cup Y} = g$.

3.4 Images and inverse images

We know that a function $f : X \to Y$ from a set X to a set Y can take individual elements $x \in X$ to elements $f(x) \in Y$. Functions can also take subsets in X to subsets in Y:

Definition 3.4.1 (Images of sets). If $f : X \to Y$ is a function from X to Y, and S is a set in X, we define $f(S)$ to be the set

$$f(S) := \{f(x) : x \in S\};$$

this set is a subset of Y, and is sometimes called the *image* of S under the map f. We sometimes call $f(S)$ the *forward image* of S to distinguish it from the concept of the *inverse image* $f^{-1}(S)$ of S, which is defined below.

Note that the set $f(S)$ is well-defined thanks to the axiom of replacement (Axiom 3.6). One can also define $f(S)$ using the axiom of specification (Axiom 3.5) instead of replacement, but we leave this as a challenge to the reader.

Example 3.4.2. If $f : \mathbf{N} \to \mathbf{N}$ is the map $f(x) = 2x$, then the forward image of $\{1, 2, 3\}$ is $\{2, 4, 6\}$:

$$f(\{1, 2, 3\}) = \{2, 4, 6\}.$$

More informally, to compute $f(S)$, we take every element x of S, and apply f to each element individually, and then put all the resulting objects together to form a new set.

In the above example, the image had the same size as the original set. But sometimes the image can be smaller, because f is not one-to-one (see Definition 3.3.14):

Example 3.4.3. (Informal) Let $\mathbf{Z}$ be the set of integers (which we will define rigourously in the next section) and let $f : \mathbf{Z} \to \mathbf{Z}$ be the map $f(x) = x^2$, then

$$f(\{-1, 0, 1, 2\}) = \{0, 1, 4\}.$$

Note that f is not one-to-one because $f(-1) = f(1)$.

Note that

$$x \in S \implies f(x) \in f(S)$$

but in general

$$f(x) \in f(S) \not\Rightarrow x \in S;$$

for instance in the above informal example, $f(-2)$ lies in the set $f(\{-1, 0, 1, 2\})$, but -2 is not in $\{-1, 0, 1, 2\}$. The correct statement is

$$y \in f(S) \iff y = f(x) \text{ for some } x \in S$$

(why?).

Definition 3.4.4 (Inverse images). If U is a subset of Y, we define the set $f^{-1}(U)$ to be the set

$$f^{-1}(U) := \{x \in X : f(x) \in U\}.$$

In other words, $f^{-1}(U)$ consists of all the elements of X which map into U:

$$f(x) \in U \iff x \in f^{-1}(U).$$

We call $f^{-1}(U)$ the *inverse image* of U.

Example 3.4.5. If $f : \mathbf{N} \to \mathbf{N}$ is the map $f(x) = 2x$, then $f(\{1, 2, 3\}) = \{2, 4, 6\}$, but $f^{-1}(\{1, 2, 3\}) = \{1\}$. Thus the forward image of $\{1, 2, 3\}$ and the backwards image of $\{1, 2, 3\}$ are quite different sets. Also note that

$$f(f^{-1}(\{1, 2, 3\})) \neq \{1, 2, 3\}$$

(why?).

Example 3.4.6. (Informal) If $f : \mathbf{Z} \to \mathbf{Z}$ is the map $f(x) = x^2$, then

$$f^{-1}(\{0, 1, 4\}) = \{-2, -1, 0, 1, 2\}.$$

Note that f does not have to be invertible in order for $f^{-1}(U)$ to make sense. Also note that images and inverse images do not quite invert each other, for instance we have

$$f^{-1}(f(\{-1, 0, 1, 2\})) \neq \{-1, 0, 1, 2\}$$

(why?).

Remark 3.4.7. If f is a bijective function, then we have defined f^{-1} in two slightly different ways, but this is not an issue because both definitions are equivalent (Exercise 3.4.1).

As remarked earlier, functions are not sets. However, we do consider functions to be a type of object, and in particular we should be able to consider sets of functions. In particular, we should be able to consider the set of *all* functions from a set X to a set Y. To do this we need to introduce another axiom to set theory:

Axiom 3.10 (Power set axiom). *Let X and Y be sets. Then there exists a set, denoted Y^X, which consists of all the functions from X to Y, thus*

$$f \in Y^X \iff (f \textit{ is a function with domain } X \textit{ and range } Y).$$

Example 3.4.8. Let $X = \{4, 7\}$ and $Y = \{0, 1\}$. Then the set Y^X consists of four functions: the function that maps $4 \mapsto 0$ and $7 \mapsto 0$; the function that maps $4 \mapsto 0$ and $7 \mapsto 1$; the function that maps $4 \mapsto 1$ and $7 \mapsto 0$; and the function that maps $4 \mapsto 1$ and $7 \mapsto 1$. The reason we use the notation Y^X to denote this set is that if Y has n elements and X has m elements, then one can show that Y^X has n^m elements; see Proposition 3.6.14(f).

One consequence of this axiom is

Lemma 3.4.9. *Let X be a set. Then the set*

$$\{Y : Y \textit{ is a subset of } X\}$$

is a set.

Proof. See Exercise 3.4.6. □

Remark 3.4.10. The set $\{Y : Y$ is a subset of X$\}$ is known as the *power set* of X and is denoted 2^X. For instance, if a, b, c are distinct objects, we have

$$2^{\{a,b,c\}} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}.$$

Note that while $\{a, b, c\}$ has 3 elements, $2^{\{a,b,c\}}$ has $2^3 = 8$ elements. This gives a hint as to why we refer to the power set of X as 2^X; we return to this issue in Chapter 8.

For sake of completeness, let us now add one further axiom to our set theory, in which we enhance the axiom of pairwise union to allow unions of much larger collections of sets.

Axiom 3.11 (Union). *Let A be a set, all of whose elements are themselves sets. Then there exists a set $\bigcup A$ whose elements are precisely those objects which are elements of the elements of A, thus for all objects x*

$$x \in \bigcup A \iff (x \in S \text{ for some } S \in A).$$

Example 3.4.11. If $A = \{\{2,3\},\{3,4\},\{4,5\}\}$, then $\bigcup A = \{2,3,4,5\}$ (why?).

The axiom of union, combined with the axiom of pair set, implies the axiom of pairwise union (Exercise 3.4.8). Another important consequence of this axiom is that if one has some set I, and for every element $\alpha \in I$ we have some set A_α, then we can form the union set $\bigcup_{\alpha \in I} A_\alpha$ by defining

$$\bigcup_{\alpha \in I} A_\alpha := \bigcup \{A_\alpha : \alpha \in I\},$$

which is a set thanks to the axiom of replacement and the axiom of union. Thus for instance, if $I = \{1,2,3\}$, $A_1 := \{2,3\}$, $A_2 := \{3,4\}$, and $A_3 := \{4,5\}$, then $\bigcup_{\alpha \in \{1,2,3\}} A_\alpha = \{2,3,4,5\}$. More generally, we see that for any object y,

$$y \in \bigcup_{\alpha \in I} A_\alpha \iff (y \in A_\alpha \text{ for some } \alpha \in I). \tag{3.2}$$

In situations like this, we often refer to I as an *index set*, and the elements α of this index set as *labels*; the sets A_α are then called a *family of sets*, and are *indexed* by the labels $\alpha \in A$. Note that if I was empty, then $\bigcup_{\alpha \in I} A_\alpha$ would automatically also be empty (why?).

We can similarly form intersections of families of sets, as long as the index set is non-empty. More specifically, given any non-empty set I, and given an assignment of a set A_α to each $\alpha \in I$, we can define the intersection $\bigcap_{\alpha \in I} A_\alpha$ by first choosing some element β of I (which we can do since I is non-empty), and setting

$$\bigcap_{\alpha \in I} A_\alpha := \{x \in A_\beta : x \in A_\alpha \text{ for all } \alpha \in I\}, \tag{3.3}$$

which is a set by the axiom of specification. This definition may look like it depends on the choice of β, but it does not (Exercise 3.4.9). Observe that for any object y,

$$y \in \bigcap_{\alpha \in I} A_\alpha \iff (y \in A_\alpha \text{ for all } \alpha \in I) \tag{3.4}$$

(compare with (3.2)).

Remark 3.4.12. The axioms of set theory that we have introduced (Axioms 3.1-3.11, excluding the dangerous Axiom 3.8) are known as the*Zermelo-Fraenkel axioms of set theory*[3], after Ernest Zermelo (1871–1953) and Abraham Fraenkel (1891–1965). There is one further axiom we will eventually need, the famous *axiom of choice* (see Section 8.4), giving rise to the *Zermelo-Fraenkel-Choice (ZFC) axioms of set theory*, but we will not need this axiom for some time.

Exercise 3.4.1. Let $f : X \to Y$ be a bijective function, and let $f^{-1} : Y \to X$ be its inverse. Let V be any subset of Y. Prove that the forward image of V under f^{-1} is the same set as the inverse image of V under f; thus the fact that both sets are denoted by $f^{-1}(V)$ will not lead to any inconsistency.

Exercise 3.4.2. Let $f : X \to Y$ be a function from one set X to another set Y, let S be a subset of X, and let U be a subset of Y. What, in general, can one say about $f^{-1}(f(S))$ and S? What about $f(f^{-1}(U))$ and U?

[3]These axioms are formulated slightly differently in other texts, but all the formulations can be shown to be equivalent to each other.

Exercise 3.4.3. Let A, B be two subsets of a set X, and let $f : X \to Y$ be a function. Show that $f(A \cap B) \subseteq f(A) \cap f(B)$, that $f(A) \backslash f(B) \subseteq f(A \backslash B)$, $f(A \cup B) = f(A) \cup f(B)$. For the first two statements, is it true that the $\subseteq$ relation can be improved to $=$?

Exercise 3.4.4. Let $f : X \to Y$ be a function from one set X to another set Y, and let U, V be subsets of Y. Show that $f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$, that $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$, and that $f^{-1}(U \backslash V) = f^{-1}(U) \backslash f^{-1}(V)$.

Exercise 3.4.5. Let $f : X \to Y$ be a function from one set X to another set Y. Show that $f(f^{-1}(S)) = S$ for every $S \subseteq Y$ if and only if f is surjective. Show that $f^{-1}(f(S)) = S$ for every $S \subseteq X$ if and only if f is injective.

Exercise 3.4.6. Prove Lemma 3.4.9. (Hint: start with the set $\{0, 1\}^X$ and apply the replacement axiom, replacing each function f with the object $f^{-1}(\{1\})$.) See also Exercise 3.5.11.

Exercise 3.4.7. Let X, Y be sets. Define a *partial function* from X to Y to be any function $f : X' \to Y'$ whose domain X' is a subset of X, and whose range Y' is a subset of Y. Show that the collection of all partial functions from X to Y is itself a set. (Hint: use Exercise 3.4.6, the power set axiom, the replacement axiom, and the union axiom.)

Exercise 3.4.8. Show that Axiom 3.4 can be deduced from Axiom 3.3 and Axiom 3.11.

Exercise 3.4.9. Show that if β and β' are two elements of a set I, and to each $\alpha \in I$ we assign a set A_α, then

$$\{x \in A_\beta : x \in A_\alpha \text{ for all } \alpha \in I\} = \{x \in A_{\beta'} : x \in A_\alpha \text{ for all } \alpha \in I\},$$

and so the definition of $\bigcap_{\alpha \in I} A_\alpha$ defined in (3.3) does not depend on β. Also explain why (3.4) is true.

Exercise 3.4.10. Suppose that I and J are two sets, and for all $\alpha \in I \cup J$ let A_α be a set. Show that $(\bigcup_{\alpha \in I} A_\alpha) \cup (\bigcup_{\alpha \in J} A_\alpha) = \bigcup_{\alpha \in I \cup J} A_\alpha$. If I and J are non-empty, show that $(\bigcap_{\alpha \in I} A_\alpha) \cap (\bigcap_{\alpha \in J} A_\alpha) = \bigcap_{\alpha \in I \cup J} A_\alpha$.

Exercise 3.4.11. Let X be a set, let I be a non-empty set, and for all $\alpha \in I$ let A_α be a subset of X. Show that

$$X \backslash \bigcup_{\alpha \in I} A_\alpha = \bigcap_{\alpha \in I} (X \backslash A_\alpha)$$

and

$$X \backslash \bigcap_{\alpha \in I} A_\alpha = \bigcup_{\alpha \in I} (X \backslash A_\alpha).$$

This should be compared with de Morgan's laws in Proposition 3.1.28 (although one cannot derive the above identities directly from de Morgan's laws, as I could be infinite).

3.5 Cartesian products

In addition to the basic operations of union, intersection, and differencing, another fundamental operation on sets is that of the *Cartesian product.*

Definition 3.5.1 (Ordered pair). If x and y are any objects (possibly equal), we define the *ordered pair* (x, y) to be a new object, consisting of x as its first component and y as its second component. Two ordered pairs (x, y) and (x', y') are considered equal if and only if both their components match, i.e.

$$(x, y) = (x', y') \iff (x = x' \text{ and } y = y'). \tag{3.5}$$

This obeys the usual axioms of equality (Exercise 3.5.3). Thus for instance, the pair $(3, 5)$ is equal to the pair $(2 + 1, 3 + 2)$, but is distinct from the pairs $(5, 3)$, $(3, 3)$, and $(2, 5)$. (This is in contrast to sets, where $\{3, 5\}$ and $\{5, 3\}$ are equal.)

Remark 3.5.2. Strictly speaking, this definition is partly an axiom, because we have simply postulated that given any two objects x and y, that an object of the form (x, y) exists. However, it is possible to define an ordered pair using the axioms of set theory in such a way that we do not need any further postulates (see Exercise 3.5.1).

Remark 3.5.3. We have now "overloaded" the parenthesis symbols () once again; they now are not only used to denote grouping of operators and arguments of functions, but also to enclose ordered pairs. This is usually not a problem in practice as one can still determine what usage the symbols () were intended for from context.

Definition 3.5.4 (Cartesian product). If X and Y are sets, then we define the *Cartesian product* $X \times Y$ to be the collection of ordered pairs, whose first component lies in X and second component lies in Y, thus

$$X \times Y = \{(x,y) : x \in X, y \in Y\}$$

or equivalently

$$a \in (X \times Y) \iff (a = (x,y) \text{ for some } x \in X \text{ and } y \in Y).$$

Remark 3.5.5. We shall simply assume that our notion of ordered pair is such that whenever X and Y are sets, the Cartesian product $X \times Y$ is also a set. This is however not a problem in practice; see Exercise 3.5.1.

Example 3.5.6. If $X := \{1,2\}$ and $Y := \{3,4,5\}$, then

$$X \times Y = \{(1,3),(1,4),(1,5),(2,3),(2,4),(2,5)\}$$

and

$$Y \times X = \{(3,1),(4,1),(5,1),(3,2),(4,2),(5,2)\}.$$

Thus, strictly speaking, $X \times Y$ and $Y \times X$ are different sets, although they are very similar. For instance, they always have the same number of elements (Exercise 3.6.5).

Let $f : X \times Y \to Z$ be a function whose domain $X \times Y$ is a Cartesian product of two other sets X and Y. Then f can either be thought of as a function of one variable, mapping the single input of an ordered pair (x,y) in $X \times Y$ to an output $f(x,y)$ in Z, or as a function of two variables, mapping an input $x \in X$ and another input $y \in Y$ to a single output $f(x,y)$ in Z. While the two notions are technically different, we will not bother to distinguish the two, and think of f simultaneously as a function of one variable with domain $X \times Y$ and as a function of two variables with domains X and Y. Thus for instance the addition operation $+$ on the natural numbers can now be re-interpreted as a function $+ : \mathbf{N} \times \mathbf{N} \to \mathbf{N}$, defined by $(x,y) \mapsto x + y$.

One can of course generalize the concept of ordered pairs to ordered triples, ordered quadruples, etc:

Definition 3.5.7 (Ordered n-tuple and n-fold Cartesian product). Let n be a natural number. An *ordered n-tuple* $(x_i)_{1\leq i\leq n}$ (also denoted $(x_1, \ldots, x_n)$) is a collection of objects x_i, one for every natural number i between 1 and n; we refer to x_i as the i^{th} *component* of the ordered n-tuple. Two ordered n-tuples $(x_i)_{1\leq i\leq n}$ and $(y_i)_{1\leq i\leq n}$ are said to be equal iff $x_i = y_i$ for all $1 \leq i \leq n$. If $(X_i)_{1\leq i\leq n}$ is an ordered n-tuple of sets, we define their *Cartesian product* $\prod_{1\leq i\leq n} X_i$ (also denoted $\prod_{i=1}^n X_i$ or $X_1 \times \ldots \times X_n$) by

$$\prod_{1\leq i\leq n} X_i := \{(x_i)_{1\leq i\leq n} : x_i \in X_i \text{ for all } 1 \leq i \leq n\}.$$

Again, this definition simply postulates that an ordered n-tuple and a Cartesian product always exist when needed, but using the axioms of set theory one can explicitly construct these objects (Exercise 3.5.2).

Remark 3.5.8. One can show that $\prod_{1\leq i\leq n} X_i$ is indeed a set. Indeed, from the power set axiom we can consider the set of all functions $i \mapsto x_i$ from the domain $\{1 \leq i \leq n\}$ to the range $\bigcup_{1\leq i\leq n} X_i$, and then we can restrict using the axiom of specification to restrict to those functions $i \mapsto x_i$ for which $x_i \in X_i$ for all $1 \leq i \leq n$. One can generalize this construction to infinite Cartesian products, see Definition 8.4.1.

Example 3.5.9. Let $a_1, b_1, a_2, b_2, a_3, b_3$ be objects, and let $X_1 := \{a_1, b_1\}$, $X_2 := \{a_2, b_2\}$, and $X_3 := \{a_3, b_3\}$. Then we have

$$\begin{aligned} X_1 \times X_2 \times X_3 = &\{(a_1, a_2, a_3), (a_1, a_2, b_3), (a_1, b_2, a_3), (a_1, b_2, b_3), \\ &(b_1, a_2, a_3), (b_1, a_2, b_3), (b_1, b_2, a_3), (b_1, b_2, b_3)\} \\ (X_1 \times X_2) \times X_3 = &\\ &\{((a_1, a_2), a_3), ((a_1, a_2), b_3), ((a_1, b_2), a_3), ((a_1, b_2), b_3), \\ &((b_1, a_2), a_3), ((b_1, a_2), b_3), ((b_1, b_2), a_3), ((b_1, b_2), b_3)\} \\ X_1 \times (X_2 \times X_3) = &\\ &\{(a_1, (a_2, a_3)), (a_1, (a_2, b_3)), (a_1, (b_2, a_3)), (a_1, (b_2, b_3)), \\ &(b_1, (a_2, a_3)), (b_1, (a_2, b_3)), (b_1, (b_2, a_3)), (b_1, (b_2, b_3))\}. \end{aligned}$$