12.19 The divergence theorem (Gauss' theorem)

Stokes' theorem expresses a relationship between an integral extended over a surface and a line integral taken over the one or more curves forming the boundary of this surface. The divergence theorem expresses a relationship between a triple integral extended over a solid and a surface integral taken over the boundary of this solid.

THEOREM **12.6.** DIVERGENCE THEOREM. ***Let V be a solid in 3-space bounded by an orientable closed surface S, and let n be the unit outer normal to S. If F is a continuously differentiable vector field defined on V, we have***

(12.53)
$$\iiint_V (\operatorname{div} \boldsymbol{F})\, dx\, dy\, dz = \iint_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS .$$

Note: If we express $\boldsymbol{F}$ and $\boldsymbol{n}$ in terms of their components, say

$$\boldsymbol{F}(x, y, z) = P(x, y, z)\boldsymbol{i} + Q(x, y, z)\boldsymbol{j} + R(x, y, z)\boldsymbol{k}$$

and

$$\boldsymbol{n} = \cos\alpha\, \boldsymbol{i} + \cos\beta\, \boldsymbol{j} + \cos\gamma\, \boldsymbol{k},$$

then Equation (12.53) can be written as

(12.54)
$$\iiint_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx\, dy\, dz = \iint_S (P\cos\alpha + Q\cos\beta + R\cos\gamma)\, dS.$$

Proof. It suffices to establish the three equations

$$\iiint_V \frac{\partial P}{\partial x}\, dx\, dy\, dz = \iint_S P\cos\alpha\, dS,$$

$$\iiint_V \frac{\partial Q}{\partial y}\, dx\, dy\, dz = \iint_S Q\cos\beta\, dS,$$

$$\iiint_V \frac{\partial R}{\partial z}\, dx\, dy\, dz = \iint_S R\cos\gamma\, dS,$$

and add the results to obtain (12.54). We begin with the third of these formulas and prove it for solids of a very special type.

Assume $\boldsymbol{V}$ is a set of points (x, y, z) satisfying a relation of the form

$$g(x, y) \le z \le f(x, y) \qquad \text{for } (x, y) \text{ in } T,$$

where T is a connected region in the xy-plane, and f and g are continuous functions on T, with $g(x, y) \le f(x, y)$ for each (x, y) in T. Geometrically, this means that T is the projection of V on the xy-plane. Every line through T parallel to the z-axis intersects the solid V along a line segment connecting the surface $z = g(x, y)$ 1.0 the surface $z = f(x, y)$. The boundary surface S consists of an upper cap S_1, given by the explicit formula $z = f(x, y)$; a lower part S_2, given by $z = g(x, y)$; and (possibly) a portion S_3 of the cylinder generated by a line moving parallel to the z-axis along the boundary of T. The outer normal to S has a nonnegative z-component on S_1, has a nonpositive component on S_2, and is parallel to the xy-plane on S_3. Solids of this type will be called "xy-projectable." (An example is shown in Figure 12.18.) They include all convex solids (for example, solid spheres, ellipsoids, cubes) and many solids that are not convex (for example, solid tori with axes parallel to the z-axis).

The idea of the proof is quite simple. We express the triple integral as a double integral extended over the projection T. Then we show that this double integral has the same value

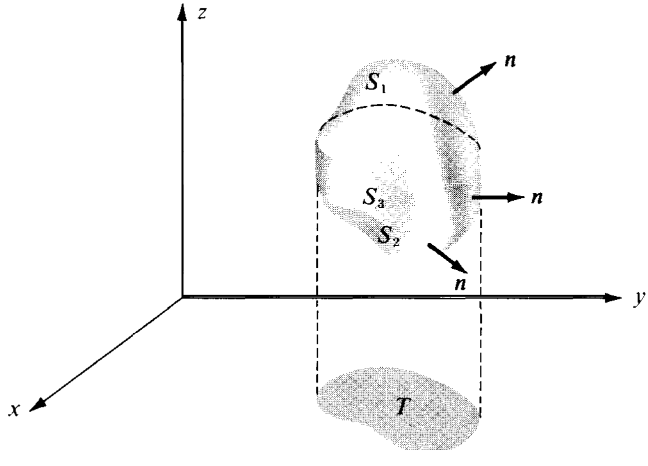

FIGURE 12.18 An example of a solid that is xy-projectable.

as the surface integral in question. We begin with the formula

$$\iiint_V \frac{\partial R}{\partial z}\,dx\,dy\,dz = \iint_T \left[\int_{g(x,y)}^{f(x,y)} \frac{\partial R}{\partial z}\,dz\right] dx\,dy\,.$$

The one-dimensional integral with respect to z may be evaluated by the second fundamental theorem of calculus, giving us

(12.55) $$\iiint_V \frac{\partial R}{\partial z}\,dx\,dy\,dz = \iint_T \{R[x, y, f(x, y)] - R[x, y, g(x, y)]\}\,dx\,dy\,.$$

For the surface integral we can write

(12.56) $$\iint_S R\cos\gamma\,dS = \iint_{S_1} R\cos\gamma\,dS + \iint_{S_2} R\cos\gamma\,dS + \iint_{S_3} R\cos\gamma\,dS\,.$$

On S_3 the normal $\boldsymbol{n}$ is parallel to the xy-plane, so $\cos\gamma = 0$ and the integral over S_3 is zero. On the surface S_1 we use the representation

$$\boldsymbol{r}(x, y) = x\boldsymbol{i} + y\boldsymbol{j} + f(x, y)\boldsymbol{k},$$

and on S_2 we use the representation

$$\boldsymbol{r}(x, y) = x\boldsymbol{i} + y\boldsymbol{j} + g(x, y)\boldsymbol{k}.$$

On S_1 the normal $\boldsymbol{n}$ has the same direction as the vector product $\partial\boldsymbol{r}/\partial x \times \partial\boldsymbol{r}/\partial y$, so we can write [see Equation (12.25), p. 436]

$$\iint_{S_1} R\cos\gamma\, dS = \iint_{S_1} R\, dx \wedge dy = \iint_T R[x, y, f(x, y)]\, dx\, dy.$$

On S_2 the normal $\boldsymbol{n}$ has the direction opposite to that of $\partial\boldsymbol{r}/\partial x \times \partial\boldsymbol{r}/\partial y$ so, by Equation (12.26), we have

$$\iint_{S_2} R\cos\gamma\, dS = -\iint_{S_2} R\, dx \wedge dy = -\iint_T R[x, y, g(x, y)]\, dx\, dy.$$

Therefore Equation (12.56) becomes

$$\iint_S R\cos\gamma\, dS = \iint_T \{R[x, y, f(x, y)] - R[x, y, g(x, y)]\}\, dx\, dy.$$

Comparing this with Equation (12.55) we see that

$$\iiint_V \frac{\partial R}{\partial z}\, dx\, dy\, dz = \iint_S R\cos\gamma\, dS.$$

In the foregoing proof the assumption that V is xy-projectable enabled us to express the triple integral over V as a double integral over its projection T in the xy-plane. It is clear that if V is yz-projectable we can use the same type of argument to prove the identity

$$\iiint_V \frac{\partial P}{\partial x}\, dx\, dy\, dz = \iint_S P\cos\alpha\, dS;$$

and if V is xz-projectable we obtain

$$\iiint_V \frac{\partial Q}{\partial y}\, dx\, dy\, dz = \iint_S Q\cos\beta\, dS.$$

Thus we see that the divergence theorem is valid for all solids projectable on all three coordinate planes. In particular the theorem holds for every convex solid.

A solid torus with its axis parallel to the z-axis is xy-projectable but not xz-projectable or yz-projectable. To extend the divergence theorem to such a solid we cut the torus into four equal parts by planes through its axis parallel to the xz- and yz-planes, respectively, and we apply the divergence theorem to each part. The triple integral over the whole torus is the sum of the triple integrals over the four parts. When we add the surface integrals over the four parts we find that the contributions from the faces common to adjacent parts cancel each other, since the outward normals have opposite directions on two such faces. Therefore the sum of the surface integrals over the four parts is equal to the surface integral over the entire torus. This example illustrates how the divergence theorem can be extended to certain nonconvex solids.

12.20 Applications of the divergence theorem

The concepts of curl and divergence of a vector field $\boldsymbol{F} = P\boldsymbol{i} + Q\boldsymbol{j} + R\boldsymbol{k}$ were introduced in Section 12.12 by the formulas

$$\text{div } \boldsymbol{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \tag{12.57}$$

and

$$\text{curl } \boldsymbol{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\boldsymbol{k}. \tag{12.58}$$

To compute div $\boldsymbol{F}$ and curl $\boldsymbol{F}$ from these formulas requires a knowledge of the components of $\boldsymbol{F}$. These components, in turn, depend on the choice of coordinate axes in 3-space. A change in the position of the coordinate axes would mean a change in the components of $\boldsymbol{F}$ and, presumably, a corresponding change in the functions div $\boldsymbol{F}$ and curl $\boldsymbol{F}$. With the help of Stokes' theorem and the divergence theorem we can obtain formulas for the divergence and curl that do not involve the components of $\boldsymbol{F}$. These formulas show that the curl and divergence represent intrinsic properties of the vector field $\boldsymbol{F}$ and do not depend on the particular choice of coordinate axes. We discuss first the formula for the divergence.

THEOREM 12.7. *Let $V(t)$ be a solid sphere of radius $t > 0$ with center at a point $\boldsymbol{a}$ in 3-space, and let $S(t)$ denote the boundary of $V(t)$. Let $\boldsymbol{F}$ be a vector field that is continuously differentiable on $V(t)$. Then if $|V(t)|$ denotes the volume of $V(t)$, and if $\boldsymbol{n}$ denotes the unit outer normal to S, we have*

$$\text{div } \boldsymbol{F}(\boldsymbol{a}) = \lim_{t\to 0} \frac{1}{|V(t)|} \iint_{S(t)} \boldsymbol{F}\cdot\boldsymbol{n}\, dS. \tag{12.59}$$

Proof. Let $\varphi = \text{div } \boldsymbol{F}$. If $\epsilon > 0$ is given we must find a $\delta > 0$ such that

$$\left|\varphi(\boldsymbol{a}) - \frac{1}{|V(t)|} \iint_{S(t)} \boldsymbol{F}\cdot\boldsymbol{n}\, dS\right| < \epsilon \qquad \text{whenever } 0 < t < \delta. \tag{12.60}$$

Since φ is continuous at $\boldsymbol{a}$, for the given ϵ there is a 3-ball $B(\boldsymbol{a}; h)$ such that

$$|\varphi(\boldsymbol{x}) - \varphi(\boldsymbol{a})| < \frac{\epsilon}{2} \qquad \text{whenever } \boldsymbol{x} \in B(\boldsymbol{a}; h).$$

Therefore, if we write $\varphi(\boldsymbol{a}) = \varphi(\boldsymbol{x}) + [\varphi(\boldsymbol{a}) - \varphi(\boldsymbol{x})]$ and integrate both sides of this equation over a solid sphere $V(t)$ of radius $t < h$, we find

$$\varphi(\boldsymbol{a})\,|V(t)| = \iiint_{V(t)} \varphi(\boldsymbol{x})\, dx\, dy\, dz + \iiint_{V(t)} [\varphi(\boldsymbol{a}) - \varphi(\boldsymbol{x})]\, dx\, dy\, dz\,.$$

If we apply the divergence theorem to the first triple integral on the right and then transpose this term to the left, we obtain the relation

$$\left|\varphi(\boldsymbol{a})\,|V(t)| - \iint_{S(t)} \boldsymbol{F}\cdot\boldsymbol{n}\,dS\right| \le \iiint_{V(t)} |\varphi(\boldsymbol{a}) - \varphi(\boldsymbol{x})|\;dx\;dy\,dz \le \frac{\epsilon}{2}|V(t)| < \epsilon|V(t)|.$$

When we divide this inequality by $|V(t)|$ we see that (12.60) holds with $\delta = \boldsymbol{h}.$ This proves the theorem.

In the foregoing proof we made no special use of the fact that V(t) was a sphere. The same theorem holds true if, instead of spheres, we use any set of solids V(t) for which the divergence theorem is valid, provided these solids contain the point $\boldsymbol{a}$ and shrink to a as $t \to 0$. For example, each V(t) could be a cube inscribed in a sphere of radius t about a; exactly the same proof would apply.

Theorem 12.7 can be used to give a physical interpretation of the divergence. Suppose $\boldsymbol{F}$ represents the flux density vector of a steady flow. Then the surface integral $\iint_{S(t)} \boldsymbol{F}\cdot\boldsymbol{n}\,dS$ measures the total mass of fluid flowing through S in unit time in the direction of $\boldsymbol{n}$. The quotient $\iint_{S(t)} \boldsymbol{F}\cdot\boldsymbol{n}\,dS/|V(t)|$ represents the mass per unit volume that flows through S in unit time in the direction of $\boldsymbol{n}$. As $t \to 0$, the limit of this quotient is the divergence of $\boldsymbol{F}$ at a. Hence the divergence at a can be interpreted as the time rate of change of mass per unit volume per unit time at $\boldsymbol{a}$.

In some books on vector analysis, Equation (12.59) is laken as the ***dejnition*** of divergence. This makes it possible to assign a physical meaning to the divergence immediately. Also, formula (12.59) does not involve the components of $\boldsymbol{F}$. Therefore it holds true in any system of coordinates. If we choose for $V(t)$ a cube with its edges parallel to the xyz-coordinate axes and center at a, we can use Equation (12.59) to deduce the formula in (12.57) which expresses div $\boldsymbol{F}$ in terms of the components of $\boldsymbol{F}$. This procedure is outlined in Exercise 14 of Section 12.21.

There is a formula analogous to (12.59) that is sometimes used as an alternative definition of the curl. It states that

(12.61) $$\operatorname{curl} \boldsymbol{F}(\boldsymbol{a}) = \lim_{t\to 0} \frac{1}{|V(t)|} \iint_{S(t)} \boldsymbol{n} \times \boldsymbol{F}\,dS,$$

where $V(t)$ and $S(t)$ have the same meanings as in Theorem 12.7. The surface integral that appears on the right has a vector-valued integrand. Such integrals can be defined in terms of components. The proof of (12.61) is analogous to that of Theorem 12.7.

There is another formula involving the curl that can be deduced from (12.61) or derived independently. It states that

(12.62) $$\boldsymbol{n}\cdot\operatorname{curl} \boldsymbol{F}(\boldsymbol{a}) = \lim_{t\to 0} \frac{1}{|S(t)|} \oint_{C(t)} \boldsymbol{F}\cdot d\boldsymbol{\alpha}.$$

In this formula, $S(t)$ is a circular disk of radius t and center at a, and $|S(t)|$ denotes its area. The vector $\boldsymbol{n}$ is a unit normal to $S(t)$, and $\boldsymbol{\alpha}$ is the function that traces out $C(t)$ in a direction that appears counterclockwise when viewed from the tip of n. The vector field F is assumed to be continuously differentiable on $S(t)$. A proof of (12.62) can be given by the same method we used to prove (12.59). We let $\varphi(x) = n \cdot \text{curl } F(x)$ and argue as before, except that we use surface integrals instead of triple integrals and Stokes' theorem instead of the divergence theorem.

If F is a velocity field, the line integral over $C(t)$ is called the circulation of F along $C(t)$; the limit in (12.62) represents the circulation per unit area at the point a. Thus, $n \cdot \text{curl } F(a)$ can be regarded as a "circulation density" of F at point a, with respect to a plane perpendicular to n at a.

When n takes the successive values i, j, and k, the dot products $i \cdot \text{curl } F$, $j \cdot \text{curl } F$, and $k \cdot \text{curl } F$ are the components of curl F in rectangular coordinates. When Equation (12.61) is taken as the starting point for the definition of curl, the formula in (12.58) for the rectangular components of curl F can be deduced from (12.62) in exactly this manner.

12.21 Exercises

1. Let S be the surface of the unit cube, $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$, and let n be the unit outer normal to S. If $F(x, y, z) = x^2 i + y^2 j + z^2 k$, use the divergence theorem to evaluate the surface integral $\iint_S F \cdot n \, dS$. Verify the result by evaluating the surface integral directly.

2. The sphere $x^2 + y^2 + z^2 = 25$ is intersected by the plane $z = 3$. The smaller portion forms a solid V bounded by a closed surface S_0 made up of two parts, a spherical part S_1 and a planar part S_2. If the unit outer normal of V is $\cos\alpha \, i + \cos\beta \, j + \cos\gamma \, k$, compute the value of the surface integral

$$\iint_S (xz \cos\alpha + yz \cos\beta + \cos\gamma) \, dS$$

if (a) S is the spherical cap S_1, (b) S is the planar base S_2, (c) S is the complete boundary S_0. Solve part (c) by use of the results of parts (a) and (b), and also by use of the divergence theorem.

3. Let $n = \cos\alpha \, i + \cos\beta \, j + \cos\gamma \, k$ be the unit outer normal to a closed surface S which bounds a homogeneous solid V of the type described in the divergence theorem. Assume that the center of mass $(\bar{x}, \bar{y}, \bar{z})$ and the volume $|V|$ of V are known. Evaluate the following surface integrals in terms of $|V|$ and $\bar{x}, \bar{y}, \bar{z}$.

(a) $\iint_S (x \cos\alpha + y \cos\beta + z \cos\gamma) \, dS$.

(b) $\iint_S (xz \cos\alpha + 2yz \cos\beta + 3z^2 \cos\gamma) \, dS$.

(c) $\iint_S (y^2 \cos\alpha + 2xy \cos\beta - xz \cos\gamma) \, dS$.

(d) Express $\iint_S (x^2 + y^2)(xi + yj) \cdot n \, dS$ in terms of the volume $|V|$ and a moment of inertia of the solid.

In Exercises 4 through 10, $\partial f/\partial n$ and $\partial g/\partial n$ denote directional derivatives of scalar fields f and g in the direction of the unit outer normal n to a closed surface S which bounds a solid V of the type described in the divergence theorem. That is, $\partial f/\partial n = \nabla f \cdot \boldsymbol{n}$ and $\partial g/\partial n = \nabla g \cdot \boldsymbol{n}$. In each of these exercises prove the given statement. You may assume continuity of all derivatives involved.

4. $\displaystyle\iint_S \frac{\partial f}{\partial n}\,dS = \iiint_V \nabla^2 f\,dx\,dy\,dz.$

5. $\displaystyle\iint_S \frac{\partial f}{\partial n}\,dS = 0$ whenever f is harmonic in V.

6. $\displaystyle\iint_S f\frac{\partial g}{\partial n}\,dS = \iiint_V f\nabla^2 g\,dx\,dy\,dz + \iiint_V \nabla f\cdot\nabla g\,dx\,dy\,dz.$

7. $\displaystyle\iint_S \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right)dS = \iiint_V (f\nabla^2 g - g\nabla^2 f)\,dx\,dy\,dz.$

8. $\displaystyle\iint_S f\frac{\partial g}{\partial n}\,dS = \iint_S g\frac{\partial f}{\partial n}\,dS$ if both f and g are harmonic in V.

9. $\displaystyle\iint_S f\frac{\partial f}{\partial n}\,dS = \iiint_V |\nabla f|^2\,dx\,dy\,dz$ if f is harmonic in V.

10. $\displaystyle\nabla^2 f(\boldsymbol{a}) = \lim_{t\to 0}\frac{1}{|V(t)|}\iint_{S(t)}\frac{\partial f}{\partial n}\,dS,$ where $V(t)$ is a solid sphere of radius t with center at $\boldsymbol{a}$, $S(t)$ is the surface of $V(t)$, and $|V(t)|$ is the volume of $V(t)$.

11. Let V be a convex region in 3-space whose boundary is a closed surface S and let $\boldsymbol{n}$ be the unit outer normal to S. Let $\boldsymbol{F}$ and $\boldsymbol{G}$ be two continuously differentiable vector fields such that

$$\text{curl } \boldsymbol{F} = \text{curl } \boldsymbol{G} \quad \text{and} \quad \text{div } \boldsymbol{F} = \text{div } \boldsymbol{G} \quad \text{everywhere in } V,$$

and such that

$$\boldsymbol{G}\cdot\boldsymbol{n} = \boldsymbol{F}\cdot\boldsymbol{n} \quad \text{everywhere on } S,$$

Prove that $\boldsymbol{F} = \boldsymbol{G}$ everywhere in V. [*Hint:* Let $\boldsymbol{H} = \boldsymbol{F} - \boldsymbol{G}$, find a scalar field f such that $\boldsymbol{H} = \nabla f$, and use a suitable identity to prove that $\iiint_V \|\nabla f\|^2\,dx\,dy\,dz = 0$. From this deduce that $\boldsymbol{H} = 0$ in V.]

12. Given a vector field $\boldsymbol{G}$ and two scalar fields f and g, each continuously differentiable on a convex solid V bounded by a closed surface S. Let $\boldsymbol{n}$ denote the unit outer normal to S. Prove that there is at most one vector field $\boldsymbol{F}$ satisfying the following three conditions:

$$\text{curl } \boldsymbol{F} = \boldsymbol{G} \quad \text{and} \quad \text{div } \boldsymbol{F} = g \quad \text{in } V, \qquad \boldsymbol{F}\cdot\boldsymbol{n} = f \quad \text{on } S.$$

13. Let S be a smooth parametric surface with the property that each line emanating from a point $\boldsymbol{P}$ intersects S once at most. Let $\Omega(S)$ denote the set of lines emanating from $\boldsymbol{P}$ and passing through S. (See Figure 12.19.) The set $\Omega(S)$ is called the ***solid angle*** with vertex $\boldsymbol{P}$ subtended by S. Let $\Sigma(a)$ denote the intersection of $\Omega(S)$ with the surface of the sphere of radius a centered at $\boldsymbol{P}$. The quotient

$$\frac{\text{area of } \Sigma(a)}{a^2}$$

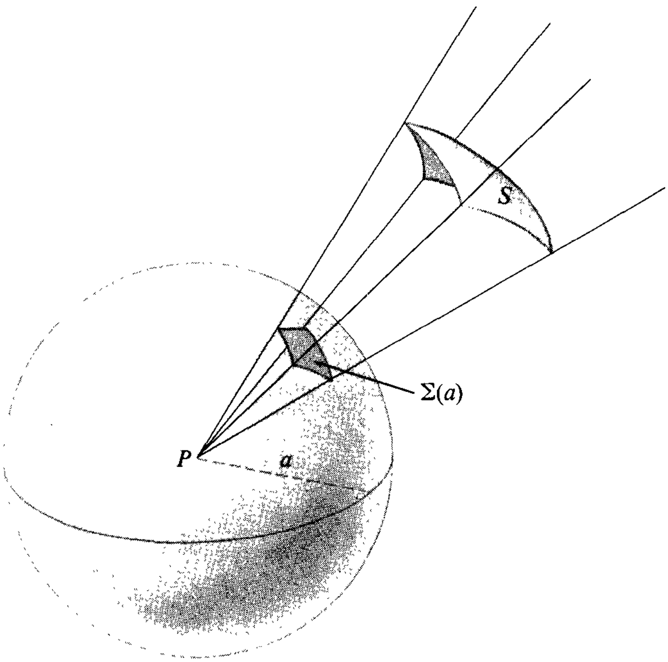

FIGURE 12.19 The solid angle $\Omega(S)$ with vertex P subtended by a surface S. It is measured by the quotient $|\Omega(S)| = \dfrac{\text{area of } \Sigma(a)}{a^2}$.

is denoted by $|\Omega(S)|$ and is used as a measure of the solid angle $\Omega(S)$.

(a) Prove that this quotient is equal to the surface integral

$$\iint_S \frac{\boldsymbol{r} \cdot \boldsymbol{n}}{r^3}\, dS,$$

where $\boldsymbol{r}$ is the radius vector from P to an arbitrary point of S, and $r = \|\boldsymbol{r}\|$. The vector $\boldsymbol{n}$ is the unit normal to S directed away from P. This shows that the quotient for $|\Omega(S)|$ is independent of the radius a. Therefore the solid angle can be measured by the area of the intersection of $\Omega(S)$ and the unit sphere about P. *[Hint:* Apply the divergence theorem to the portion of $\Omega(S)$ lying between S and C(a).]

(b) Two planes intersect along the diameter of a sphere with center at P. The angle of intersection is θ, where $0 < \theta < \pi$. Let S denote the smaller portion of the surface of the sphere intercepted by the two planes. Show that $\Omega(S)| = 2\%$.

14. Let k'(t) denote a cube of edge $2t$ and center at $\boldsymbol{a}$, and let S(t) denote the boundary of the cube. Let $\boldsymbol{n}$ be the unit outer normal of S(t) and let $|V(t)|$ denote the volume of the cube. For a given vector field $\boldsymbol{F}$ that is continuously differentiable at a, assume that the following limit exists :

$$\lim_{t\to 0} \frac{1}{|V(t)|} \iint_{S(t)} \boldsymbol{F} \cdot \boldsymbol{n}\, dS,$$

and use this limit as the definition of the divergence, div $\boldsymbol{F}(\boldsymbol{a})$. Choose xyz-coordinate axes parallel to the edges of $V(t)$ and let P, Q, and R be the components of $\boldsymbol{F}$ relative to this coordinate system. Prove that div $\boldsymbol{F}(\boldsymbol{a}) = D_1P(\boldsymbol{a}) + D_2Q(\boldsymbol{a}) + D_3R(\boldsymbol{a})$. [*Hint:* Express the surface integral as a sum of six double integrals taken over the faces of the cube. Then show that $1/|V(t)|$ times the sum of the two double integrals over the faces perpendicular to the z-axis approaches the limit $D_3R(a)$ as $t \to 0$. Argue in a similar way for the remaining terms.]

15. A scalar field φ which is never zero has the properties

$$\|\nabla\varphi\|^2 = 4\varphi \qquad \text{and} \qquad \operatorname{div}(\varphi\nabla\varphi) = 10\varphi.$$

Evaluate the surface integral

$$\iint_S \frac{\partial\varphi}{\partial n}\,dS,$$

where S is the surface of a unit sphere with center at the origin, and $\partial\varphi/\partial n$ is the directional derivative of φ in the direction of the unit outer normal to S.

PART 3

SPECIAL TOPICS

13

SET FUNCTIONS AND ELEMENTARY PROBABILITY

13.1 Historical introduction

A gambler's dispute in 1654 led to the creation of a mathematical theory of probability by two famous French mathematicians, Blaise Pascal and Pierre de Fermat. Antoine Gombaud, Chevalier de Méré, a French nobleman with an interest in gaming and gambling questions, called Pascal's attention to an apparent contradiction concerning a popular dice game. The game consisted in throwing a pair of dice 24 times; the problem was to decide whether or not to bet even money on the occurrence of at least one "double six" during the 24 throws. A seemingly well-established gambling rule led de Méré to believe that betting on a double six in 24 throws would be profitable, but his own calculations indicated just the opposite.

This problem and others posed by de Méré led to an exchange of letters between Pascal and Fermat in which the fundamental principles of probability theory were formulated for the first time. Although a few special problems on games of chance had been solved by some Italian mathematicians in the 15th and 16th centuries, no general theory was developed before this famous correspondence.

The Dutch scientist Christian Huygens, a teacher of Leibniz, learned of this correspondence and shortly thereafter (in 1657) published the first book on probability; entitled *De Ratiociniis* in *Ludo Aleae*, it was a treatise on problems associated with gambling. Because of the inherent appeal of games of chance, probability theory soon became popular, and the subject developed rapidly during the 18th century. The major contributors during this period were Jakob Bernoulli† (1654-1705) and Abraham de Moivre (1667-1754).

In 1812 Pierre de Laplace (1749-1827) introduced a host of new ideas and mathematical techniques in his book, *Théorie Analytique des Probabilités.* Before Laplace, probability theory was solely concerned with developing a mathematical analysis of games of chance. Laplace applied probabilistic ideas to many scientific and practical problems. The *theory of errors, actuarial mathematics,* and *statistical mechanics* are examples of some of the important applications of probability theory developed in the 19th century.

Like so many other branches of mathematics, the development of probability theory has been stimulated by the variety of its applications. Conversely, each advance in the theory has enlarged the scope of its influence. Mathematical statistics is one important branch of applied probability; other applications occur in such widely different fields as

† Sometimes referred to as James Bernoulli.

genetics, psychology, economics, and engineering. Many workers have contributed to the theory since Laplace's time; among the most important are Chebyshev, Markov, von Mises, and Kolmogorov.

One of the difficulties in developing a mathematical theory of probability has been to arrive at a definition of probability that is precise enough for use in mathematics, yet comprehensive enough to be applicable to a wide range of phenomena. The search for a widely acceptable definition took nearly three centuries and was marked by much controversy. The matter was finally resolved in the 20th century by treating probability theory on an axiomatic basis. In 1933 a monograph by a Russian mathematician A. Kolmogorov outlined an axiomatic approach that forms the basis for the modern theory. (Kolmogorov's monograph is available in English translation as ***Foundations of Probability*** Theory, Chelsea, New York, 1950.) Since then the ideas have been refined somewhat and probability theory is now part of a more general discipline known as measure theory.

This chapter presents the basic notions of modern elementary probability theory along with some of its connections to measure theory. Applications are also given, primarily to games of chance such as coin tossing, dice, and card games. This introductory account is intended to demonstrate the logical structure of the subject as a deductive science and to give the reader a feeling for probabilistic thinking.

13.2 Finitely additive set functions

The area of a region, the length of a curve, or the mass of a system of particles is a number which measures the size or content of a set. All these measures have certain properties in common. Stated abstractly, they lead to a general concept called a *finitely* ***additive set function.*** Later we shall introduce probability as another example of such a function. To prepare the way, we discuss first some properties common to all these functions.

A function $f\colon \mathscr{A} \to \mathbf{R}$ whose domain is a collection $\mathscr{A}$ of sets and whose function values are real numbers, is called a *set function.* If A is a set in class $\mathscr{A}$, the value of the function at A is denoted by f(A).

DEFINITION OF A FINITELY ADDITIVE SET FUNCTION. ***A set function $f\colon \mathscr{A} \to \mathbf{R}$ is said to be finitely additive if***

(13.1) $$f(A \cup B) = f(A) + f(B)$$

whenever A and B are disjoint sets in $\mathscr{A}$ such that $A \cup B$ is also in $\mathscr{A}$.

Area, length, and mass are all examples of finitely additive set functions. This section discusses further consequences of Equation (13.1).

In the usual applications, the sets in $\mathscr{A}$ are subsets of a given set S, called a universal set. It is often necessary to perform the operations of union, intersection, and complementation on sets in $\mathscr{A}$. To make certain that $\mathscr{A}$ is closed under these operations we restrict $\mathscr{A}$ to be a ***Boolean algebra,*** which is defined as follows.

DEFINITION OF A BOOLEAN ALGEBRA OF SETS. *A nonempty class $\mathscr{A}$ of subsets of a given universal set S is called a Boolean algebra if for every A and B in $\mathscr{A}$ we have*

$$A \cup B \in \mathscr{A} \quad \text{and} \quad A' \in \mathscr{A}.$$

Here $A' = S - A$, the complement of A relative to S.

A Boolean algebra $\mathscr{A}$ is also closed under intersections and differences, since we have

$$A \cap B = (A' \cup B')' \quad \text{and} \quad A - B = A \cap B'.$$

This implies that the empty set $\varnothing$ belongs to $\mathscr{A}$ since $\varnothing = A - A$ for some A in $\mathscr{A}$. Also, the universal set S is in $\mathscr{A}$ since $S = \varnothing'$.

Many Boolean algebras can be formed from the subsets of a given universal set S. The smallest such algebra is the class $\mathscr{A}_0 = \{\varnothing, S\}$, which consists of only two special subsets: $\varnothing$ and S. At the other extreme is the class $\mathscr{A}_1$, which consists of *all* subsets of S. Every Boolean algebra $\mathscr{A}$ consisting of subsets of S satisfies the inclusion relations $\mathscr{A}_0 \subseteq \mathscr{A} \subseteq \mathscr{A}_1$.

The property of finite additivity of set functions in Equation (13.1) requires A and B to be disjoint sets. The next theorem drops this requirement.

THEOREM 13.1. *Let $f: \mathscr{A} \to \mathbf{R}$ be a finitely additive set function defined on a Boolean algebra $\mathscr{A}$ of sets. Then for all A and B in $\mathscr{A}$ we have*

$$f(A \cup B) = f(A) + f(B - A), \tag{13.2}$$

and

$$f(A \cup B) = f(A) + f(B) - f(A \cap B). \tag{13.3}$$

Proof. The sets A and $B - A$ are disjoint and their union is $A \cup B$. Hence, by applying (13.1) to A and $B - A$ we obtain (13.2).

To prove (13.3) we first note that $A \cap B'$ and B are disjoint sets whose union is $A \cup B$. Hence by (13.1) we have

$$f(A \cup B) = f(A \cap B') + f(B). \tag{13.4}$$

Also, $A \cap B'$ and $A \cap B$ are disjoint sets whose union is A, so (13.1) gives us

$$f(A) = f(A \cap B') + f(A \cap B). \tag{13.5}$$

Subtracting (13.5) from (13.4) we obtain (13.3).

13.3 Finitely additive measures

The set functions which represent area, length, and mass have further properties in common. For example, they are all *nonnegative* set functions. That is,

$$f(A) \geq 0$$

for each A in the class $\mathscr{A}$ under consideration.

DEFINITION OF A FINITELY ADDITIVE MEASURE. *A nonnegative set function $f: \mathscr{A} \rightarrow$ R that is finitely additive is called a finitely additive measure, or simply a measure.*

Using Theorem 13.1 we immediately obtain the following further properties of measures.

THEOREM 13.2. *Let $f: \mathscr{A} \rightarrow$ R be a finitely additive measure defined on a Boolean algebra $\mathscr{A}$. Then for all sets A and B in $\mathscr{A}$ we have*

(a) $f(A \cup B) \leq f(A) + f(B)$.
(b) $f(B - A) = f(B) - f(A)$ if $A \subseteq B$.
(c) $f(A) \leq f(B)$ if $A \subseteq B$. (Monotone property)
(d) $f(\varnothing) = 0$.

Proof. Part (a) follows from (13.3), and part (b) follows from (13.2). Part (c) follows from (b), and part (d) follows by taking $A = B = \varnothing$ in (b).

EXAMPLE. ***The number of elements in a finite set.*** Let $S = \{a_1, a_2, \ldots, a_n\}$ be a set consisting of n (distinct) elements, and let $\mathscr{A}$ denote the class of all subsets of S. For each A in $\mathscr{A}$, let $\nu(A)$ denote the number of distinct elements in A (ν is the Greek letter "nu"). It is easy to verify that this function is finitely additive on $\mathscr{A}$. In fact, if A has k elements and if B has m elements, then $\nu(A) = k$ and $\nu(B) = m$. If A and B are disjoint it is clear that the union $A \cup B$ is a subset of S with $k + m$ elements, so

$$\nu(A \cup B) = k + m = \nu(A) + \nu(B).$$

This particular set function is nonnegative, so ν is a measure.

13.4 Exercises

1. Let $\mathscr{A}$ denote the class of all subsets of a given universal set and let A and B be arbitrary sets in $\mathscr{A}$. Prove that:
   (a) $A \cap B'$ and B are disjoint.
   (b) $A \cup B = (A \cap B') \cup B$. (This formula expresses $A \cup B$ as the union of disjoint sets.)
   (c) $A \cap B$ and $A \cap B'$ are disjoint.
   (d) $(A \cap B) \cup (A \cap B') = A$. (This formula expresses A as a union of two disjoint sets.)
2. Exercise 1(b) shows how to express the union of two sets as the union of two *disjoint* sets. Express in a similar way the union of three sets $A_1 \cup A_2 \cup A_3$ and, more generally, of n sets $A_1 \cup A_2 \cup \cdots \cup A_n$. Illustrate with a diagram when $n = 3$.
3. A study of a set S consisting of 1000 college graduates ten years after graduation revealed that the "successes" formed a subset A of 400 members, the Caltech graduates formed a subset B of 300 members, and the intersection $A \cap B$ consisted of 200 members.
   (a) For each of the following properties, use set notation to describe, in terms of unions and intersections of A, B, and their complements A' and B' relative to S, the subsets consisting of those persons in S that have the property:
      (i) Neither a "success" nor a Caltech graduate.
      (i) A "success" but not a Caltech graduate.
      (iii) A "success" or a Caltech graduate, or both.

(iv) Either a "success" or a Caltech graduate, but not both.
(v) Belongs to not more than one of A or B.

(b) Determine the exact number of individuals in each of the five subsets of part (a).

4. Letfbe a finitely additive set function defined on a class $\mathscr{A}$ of sets. Let $A,, \ldots, A,$ be n sets in $\mathscr{A}$ such that $A_i \cap A_j = \varnothing$ if $i \neq j$. (Such a collection is called a *disjoint collection* of sets.) If the union $\bigcup_{k=1}^{m} A_k$ is in class $\mathscr{A}$ for all $m \leq n$, use induction to prove that

$$f\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} f(A_k).$$

In Exercises 5, 6, and 7, S denotes a finite set consisting of n distinct elements, say S = $\{a_1, a_2, \ldots, a_n\}$.

5. Let $A, = \{a_1\}$, the subset consisting of a, alone. Show that the class $\mathscr{B}_1 = \{\varnothing, A,, A_1', S\}$ is the smallest Boolean algebra containing $A,$.
6. Let A, = $\{a_1\}$, $A, = \{a_2\}$. Describe, in a manner similar to that used in Exercise 5, the smallest Boolean algebra $\mathscr{B}_2$ containing both $A,$ and $A,$.
7. Do the same as in Exercise 6 for the subsets $A, = \{a_1\}$, $A_2 = \{a_2\}$, and A, = $\{a_3\}$.
8. If $\mathscr{B}_k$ denotes the smallest Boolean algebra which contains the k subsets $A, = \{a_1\}$, $A, = \{a_2\}, \ldots, A_k = \{a_k\}$, show that $\mathscr{B}_k$ contains 2^{k+1} subsets of S if $k < n$ and 2^n subsets if $k = n$.
9. Letfbe a finitely additive set function defined on the Boolean algebra of all subsets of a given universal set S. Suppose it is known that

$$f(A \cap B) = f(A)f(B)$$

for two particular subsets A and B of S. If $f(S) = 2$, prove that

$$f(A \cup B) = f(A') + f(B') - f(A')f(B').$$

10. If A and B are sets, their *symmetric difference* $A \triangle B$ is the set defined by the equation A A B = $(A - B) \cup (B - A)$. Prove each of the following properties of the symmetric difference.
(a) $A \triangle B = B \triangle A$.
(b) $A \triangle A = \varnothing$.
(c) $A \triangle B \subseteq (A \triangle C) \cup (C \triangle B)$.
(d) $A \triangle B$ is disjoint from each of A and B.
(e) $(A$ AB$) \triangle C = A \triangle (B \triangle C)$.
(f) Iffis a finitely additive set function defined on the Boolean algebra $\mathscr{A}$ of all subsets of a given set S, then for all A and B in $\mathscr{A}$ we have $f(A$ A $B) = f(A) + f(B) - 2f(A \cap B)$.

13.5 The definition of probability for finite sample spaces

In the language of set functions, probability is a specific kind of measure (to be denoted here by P) defined on a specific Boolean algebra $\mathscr{B}$ of sets. The elements of $\mathscr{B}$ are subsets of a universal set S. In probability theory the universal set S is called a *sample space.* We discuss the definition of probability first for finite sample spaces and later for infinite sample spaces.

DEFINITION OF PROBABILITY FOR FINITE SAMPLE SPACES. *Let $\mathscr{B}$ denote a Boolean algebra whose elements are subsets of a given finite set S. A set function P dejined on $\mathscr{B}$ is called a*