If I is an ideal of R then it is easy to see that every element of ${}^eI$ can be written in the form a/d for some $a \in I$ and $d \in D$, so the extension of I to $D^{-1}R$ is also frequently denoted by $D^{-1}I$.

Proposition 38. In the preceding notation we have

(1) For any ideal J of $D^{-1}R$ we have $J = {}^e({}^cJ)$. In particular, every ideal of $D^{-1}R$ is the extension of some ideal of R, and distinct ideals of $D^{-1}R$ have distinct contractions in R.

(2) For any ideal I of R we have

$$ {}^c({}^eI) = \{r \in R \mid dr \in I \text{ for some } d \in D\}. $$

Also, ${}^eI = D^{-1}R$ if and only if $I \cap D \neq \emptyset$.

(3) Extension and contraction give a bijective correspondence

$$ \left\{ \begin{array}{c} \text{prime ideals } P \text{ of } R \\ \text{with } P \cap D = \emptyset \end{array} \right\} \quad \underset{c}{\overset{e}{\rightleftarrows}} \quad \left\{ \text{prime ideals of } D^{-1}R \right\}. $$

(4) If R is Noetherian (or Artinian) then $D^{-1}R$ is Noetherian (Artinian, respectively).

Proof: We always have ${}^e({}^cJ) \subseteq J$. For the reverse inclusion let $a/d \in J$. Then $a/1 = d(a/d) \in J$, and so $a \in \pi^{-1}(J) = {}^cJ$. Thus $a/1 \in {}^e({}^cJ)$, so we also have $(a/1)(1/d) = a/d \in {}^e({}^cJ)$, hence $J = {}^e({}^cJ)$. This proves the first statement in (1) and the second statement follows immediately.

Let $I' = \{r \in R \mid dr \in I \text{ for some } d \in D\}$. We first show $I' \subseteq {}^c({}^eI)$. If $r \in I'$ then there is some $d \in D$ such that $dr = a \in I$. Then $r/1 = a/d \in {}^eI$, so $r \in {}^c({}^eI)$. To show the reverse containment ${}^c({}^eI) \subseteq I'$, let $r \in {}^c({}^eI)$ so that $r/1 = a/d$ for some $a \in I$ and $d \in D$. Then $x(dr - a) = 0$ for some $x \in D$, so $xdr = xa \in I$, and because $xd \in D$ it follows that $r \in I'$. This proves the first assertion of (2). Now ${}^eI = D^{-1}R$ if and only if $1/1 \in {}^eI$, if and only if $1 \in {}^c({}^eI) = I'$. The second assertion of (2) then follows from the definition of I'.

To prove (3) observe first that if Q is a prime ideal in $D^{-1}R$, then its preimage under any homomorphism sending 1 to 1 is a prime ideal (cf. Exercise 13, Section 7.4), so c maps prime ideals of $D^{-1}R$ to prime ideals of R disjoint from D. In the reverse direction, let P be a prime ideal of R disjoint from D and let $Q = {}^eP$ and suppose $(a/d_1)(b/d_2) \in Q$. Then $(ab)/(d_1d_2) \in Q$, so $ab/(d_1d_2) = c/d$ for some $c \in P$ and $d \in D$. Then $x(dab - d_1d_2c) = 0$ for some $x \in D$. Since $c \in P$ we have $xdab \in P$, and since P is a prime ideal disjoint from D we have $ab \in P$. Since P is prime, either $a \in P$ or $b \in P$, hence a/d_1 or b/d_2 is in Q. This proves Q is a prime ideal and shows that e maps prime ideals of R disjoint from D to prime ideals of $D^{-1}R$. Finally, it follows immediately from (2) that $P = {}^c({}^eP)$ for every prime ideal of R disjoint from D. Thus c and e are inverse correspondences, hence are bijections between these sets of prime ideals. This establishes (3).

By (1) every ascending (respectively, descending) chain of distinct ideals in $D^{-1}R$ contracts to an ascending (respectively, descending) chain of distinct ideals in R, giving (4) and completing the proof.

Because $1 \in D$, first localizing the ideal I and then contracting that localization as in (2) results in an ideal in R containing I: $I \subseteq {}^c({}^eI)$.

Definition. Suppose R is a commutative ring with 1 and D is a multiplicatively closed subset containing 1. The *saturation* of the ideal I in R with respect to D is the ideal ${}^c({}^eI)$ in R, where contraction and extension are computed with respect to $\pi : R \mapsto D^{-1}R$. If $I = {}^c({}^eI)$ then I is said to be *saturated* with respect to D.

Loosely speaking, (2) of Proposition 38 shows that the saturation of I consists of elements of R that would lie in I if we allowed denominators from D. The ideal is saturated with respect to D if we don't obtain any additional elements even if we allow denominators from D.

We can apply our results on localization to give an algorithm for determining whether an ideal P in the polynomial ring $k[x_1, \ldots, x_n]$ with coefficients in the field k is prime. The basic idea is to use the fact that $k[x_1, \ldots, x_i] = k[x_1, \ldots, x_{i-1}][x_i]$ to consider inductively whether the ideals $P_i = P \cap k[x_1, \ldots, x_i]$ are prime.

In general, suppose R is a commutative ring. If P is a prime ideal in $R[x]$ then $P \cap R$ is a prime ideal in R and so $S = R/(P \cap R)$ is an integral domain. Let F denote its quotient field. We then have two natural ring homomorphisms:

$$R[x] \longrightarrow (R/P \cap R)[x] = S[x] \longrightarrow F[x]$$

where the first is the natural projection homomorphism and the second is the natural inclusion induced by $S \subseteq F$. Note that $F[x]$ is the localization of $S[x]$ with respect to the multiplicatively closed set $D = S - \{0\}$. The next proposition shows that the image of P under the first homomorphism is a prime ideal in $S[x]$ that is saturated with respect to D and extends to a prime ideal in $F[x]$, and that, conversely, we can determine whether an ideal is prime in $R[x]$ by these properties.

Proposition 39. Suppose R is a commutative ring with 1 and I is an ideal in $R[x]$. Then I is a prime ideal in $R[x]$ if and only if
i. $J = I \cap R$ is a prime ideal in R, i.e., $S = R/J$ is an integral domain, and
ii. if $\bar{I}$ is the image of I in $S[x]$ then $\bar{I}F[x]$ is a prime ideal in $F[x]$ satisfying $\bar{I}F[x] \cap S[x] = \bar{I}$.

Proof: Suppose I is a prime ideal in $R[x]$, so that $J = I \cap R$ is a prime ideal in R and $S = R/J$ is an integral domain. By Proposition 2 in Chapter 9, the kernel of the reduction homomorphism $R[x] \mapsto S[x] = (R/J)[x]$ is $J[x]$, which is contained in $I[x]$, so we have a ring isomorphism $R[x]/I \cong S[x]/\bar{I}$. Since $R[x]/I$ is an integral domain, it follows that $\bar{I}$ is a prime ideal in the integral domain $S[x]$. The elements of $\bar{I} \cap S$ are the images of the elements in $R \cap I$, so $\bar{I} \cap S = 0$. Since the ring $F[x]$ is the localization of $S[x]$ with respect to the multiplicatively closed set $S - \{0\}$, condition (ii) follows by Proposition 38(3).

Conversely, if I is not prime, then either J is not prime in R or J is prime in R but $\bar{I}$ is not prime in $S[x]$. In the latter case either $\bar{I}F[x]$ is not prime in $F[x]$ or, again by Proposition 38(3), $\overline{I}$ is not saturated. Thus, if I is not prime, either (i) or (ii) fails, completing the proof.

Since $F[x]$ is a Euclidean Domain, the ideal $\overline{I}F[x] = (h(x))$ in Proposition 39 is principal, and is prime if and only if $h(x)$ is either 0 or is irreducible in $F[x]$. Suppose $h(x)$ is an element in I whose image in $S[x]$ has leading coefficient $a \in S$. The next proposition shows that a gives a bound on the denominators necessary for the saturation $\overline{I}F[x] \cap S[x]$ and can be used to compute this saturation.

Proposition 40. Let S be an integral domain with fraction field F and let A be a nonzero ideal in $S[x]$. Suppose $AF[x] = (h(x))$ where $h(x)$ is a polynomial in $S[x]$ with leading coefficient $a \in S$. Let S_a be the localization of S with respect to the powers of a. Then

(1) $AF[x] \cap S[x] = AS_a[x] \cap S[x]$, and

(2) if $\mathcal{A}$ denotes the ideal generated by A and $1 - at$ in the polynomial ring $S[x, t]$, then $AS_a[x] \cap S[x] = \mathcal{A} \cap S[x]$.

Proof: We first show $AF[x] \cap S_a[x] = AS_a[x]$. Since $S_a \subseteq F$, the containment $AS_a[x] \subseteq AF[x] \cap S_a[x]$ is immediate. Suppose now that $f(x) \in AF[x] \cap S_a[x]$. If the leading term of $f(x)$ is sx^N and the leading term of $h(x)$ is ax^m, then since $AF[x] = (h(x))$ we have $N \geq m$. Then the polynomial $f(x) - (s/a)x^{N-m}h(x)$ is again in $AF[x] \cap S_a[x]$ and is of lower degree than $f(x)$. Iterating, we see that $f(x)$ can be written as a polynomial in $S_a[x]$ times $h(x)$, so $f(x) \in AS_a[x]$. Intersecting both sides of $AF[x] \cap S_a[x] = AS_a[x]$ with $S[x]$ gives the first statement in the proposition.

To prove the second statement, suppose first that $f(x) \in \mathcal{A} \cap S[x]$. Then we can write $f(x) = f_1(x, t)b(x) + f_2(x, t)(1 - at)$ for some polynomials $b(x) \in A$ and $f_1, f_2 \in S[x, t]$. Substituting $t = 1/a$ gives $f(x) = f_1(x, 1/a)b(x)$, and since $f_1(x, 1/a) \in S_a[x]$, we obtain $f(x) \in AS_a[x] \cap S[x]$. Conversely, suppose that $f(x) = b(x)g(x) \in S[x]$ where $g(x) \in S_a(x)$ and $b(x) \in A$. If a^N is the largest power of a appearing in the denominators of the coefficients of $g(x)$ then $a^N g(x) \in S[x]$. Writing $f(x) = (at)^N f(x) + (1 - (at)^N)f(x) = b(x)t^N(a^N g(x)) + (1 - (at)^N)f(x)$ we see that $f(x) \in \mathcal{A} \cap S[x]$, giving the reverse containment and completing the proof.

Suppose now that P is an ideal in $k[x_1, \ldots, x_n]$. Let P_i for $i = 1, \ldots, n$ be the intersection of P with $k[x_1, \ldots, x_i]$. We use Propositions 39 and 40 to determine inductively whether $P_1, P_2, \ldots, P_n = P$ are prime ideals in their respective polynomial rings.

The ideal P_1 will be prime in the Euclidean Domain $k[x_1]$ if and only if it is 0 or is generated by an irreducible polynomial. Suppose now that $i \geq 2$ and we have already proved that P_{i-1} is a prime ideal in $k[x_1, \ldots, x_{i-1}]$, so that the quotient ring $S = k[x_1, \ldots, x_{i-1}]/P_{i-1}$ is an integral domain. If F denotes the quotient field of S, then by Proposition 39, P_i is a prime ideal in $k[x_1, \ldots, x_i]$ if and only if its image in $(k[x_1, \ldots, x_{i-1}]/P_{i-1})[x_i] = S[x_i]$ is a saturated ideal whose extension to the Euclidean Domain $F[x_i]$ is a prime ideal. Suppose $h(x_i) \in S[x_i]$ is a generator for this ideal and a is the leading coefficient of $h(x_i)$. Then $(h(x_i))$ is a prime ideal in $F[x_i]$ if and only if