1. Let V be a finite dimensional vector space. Prove that the map $\varphi \mapsto \varphi^*$ in Theorem 20 gives a ring isomorphism of $\mathrm{End}(V)$ with $\mathrm{End}(V^*)$.
2. Let V be the collection of polynomials with coefficients in $\mathbb{Q}$ in the variable x of degree at most 5 with $1, x, x^2, \ldots, x^5$ as basis. Prove that the following are elements of the dual space of V and express them as linear combinations of the dual basis:
   (a) $E : V \to \mathbb{Q}$ defined by $E(p(x)) = p(3)$ (i.e., evaluation at $x = 3$).
   (b) $\varphi : V \to \mathbb{Q}$ defined by $\varphi(p(x)) = \int_0^1 p(t)dt$.
   (c) $\varphi : V \to \mathbb{Q}$ defined by $\varphi(p(x)) = \int_0^1 t^2 p(t)dt$.
   (d) $\varphi : V \to \mathbb{Q}$ defined by $\varphi(p(x)) = p'(5)$ where $p'(x)$ denotes the usual derivative of the polynomial $p(x)$ with respect to x.
3. Let S be any subset of V^* for some finite dimensional space V. Define $\mathrm{Ann}(S) = \{v \in V \mid f(v) = 0 \text{ for all } f \in S\}$. ($\mathrm{Ann}(S)$ is called the *annihilator of S in V*).
   (a) Prove that $\mathrm{Ann}(S)$ is a subspace of V.
   (b) Let W_1 and W_2 be subspaces of V^*. Prove that $\mathrm{Ann}(W_1 + W_2) = \mathrm{Ann}(W_1) \cap \mathrm{Ann}(W_2)$ and $\mathrm{Ann}(W_1 \cap W_2) = \mathrm{Ann}(W_1) + \mathrm{Ann}(W_2)$.
   (c) Let W_1 and W_2 be subspaces of V^*. Prove that $W_1 = W_2$ if and only if $\mathrm{Ann}(W_1) = \mathrm{Ann}(W_2)$.
   (d) Prove that the annihilator of S is the same as the annihilator of the subspace of V^* spanned by S.
   (e) Assume V is finite dimensional with basis $v_1, \ldots, v_n$. Prove that if $S = \{v_1^*, \ldots, v_k^*\}$ for some $k \leq n$, then $\mathrm{Ann}(S)$ is the subspace spanned by $\{v_{k+1}, \ldots, v_n\}$.
   (f) Assume V is finite dimensional. Prove that if W^* is any subspace of V^* then $\dim \mathrm{Ann}(W^*) = \dim V - \dim W^*$.
4. If V is infinite dimensional with basis $\mathcal{A}$, prove that $\mathcal{A}^* = \{v^* \mid v \in \mathcal{A}\}$ does *not* span V^*.
5. If V is infinite dimensional with basis $\mathcal{A}$, prove that V^* is isomorphic to the direct product of copies of F indexed by $\mathcal{A}$. Deduce that $\dim V^* > \dim V$. [Use Exercise 14, Section 1.]

11.4 DETERMINANTS

Although we shall be using the theory primarily for vector spaces over a field, the theory of determinants can be developed with no extra effort over arbitrary commutative rings with 1. Thus in this section R is any commutative ring with 1 and $V_1, V_2, \ldots, V_n, V$ and W are R-modules. For convenience we repeat the definition of multilinear functions from Section 10.4.

Definition.

(1) A map $\varphi : V_1 \times V_2 \times \cdots \times V_n \to W$ is called *multilinear* if for each fixed i and fixed elements $v_j \in V_j$, $j \neq i$, the map

$$V_i \to W \qquad \text{defined by} \qquad x \mapsto \varphi(v_1, \ldots, v_{i-1}, x, v_{i+1}, \ldots, v_n)$$

is an R-module homomorphism. If $V_i = V$, $i = 1, 2, \ldots, n$, then φ is called an *n-multilinear function on V*, and if in addition $W = R$, φ is called an *n-multilinear form on V*.

(2) An n-multilinear function φ on V is called *alternating* if $\varphi(v_1, v_2, \ldots, v_n) = 0$ whenever $v_i = v_{i+1}$ for some $i \in \{1, 2, \ldots, n-1\}$ (i.e., φ is zero whenever two consecutive arguments are equal). The function φ is called *symmetric* if interchanging v_i and v_j for any i and j in $(v_1, v_2, \ldots, v_n)$ does not alter the value of φ on this n-tuple.

When $n = 2$ (respectively, 3) one says φ is ***bilinear*** (respectively, ***trilinear***) rather than 2-multilinear (respectively, 3-multilinear). Also, when n is clear from the context we shall simply say φ is multilinear.

Example

For any fixed $m \geq 0$ the usual dot product on $V = \mathbb{R}^m$ is a bilinear form (here the ring R is the field of real numbers).

Proposition 22. Let φ be an n-multilinear alternating function on V. Then

(1) $\varphi(v_1, \ldots, v_{i-1}, v_{i+1}, v_i, v_{i+2}, \ldots, v_n) = -\varphi(v_1, v_2, \ldots, v_n)$ for any $i \in \{1, 2, \ldots, n-1\}$, i.e., the value of φ on an n-tuple is negated if two adjacent components are interchanged.

(2) For each $\sigma \in S_n$, $\varphi(v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(n)}) = \epsilon(\sigma)\varphi(v_1, v_2, \ldots, v_n)$, where $\epsilon(\sigma)$ is the sign of the permutation σ (cf. Section 3.5).

(3) If $v_i = v_j$ for any pair of distinct $i, j \in \{1, 2, \ldots, n\}$ then $\varphi(v_1, v_2, \ldots, v_n) = 0$.

(4) If v_i is replaced by $v_i + \alpha v_j$ in $(v_1, \ldots, v_n)$ for any $j \neq i$ and any $\alpha \in R$, the value of φ on this n-tuple is not changed.

Proof: (1) Let $\psi(x, y)$ be the function φ with variable entries x and y in positions i and $i+1$ respectively and fixed entries v_j in position j, for all other j. Thus (1) is the same as showing $\psi(y, x) = -\psi(x, y)$. Since φ is alternating $\psi(x + y, x + y) = 0$. Expanding $x + y$ in each variable in turn gives $\psi(x+y, x+y) = \psi(x, x) + \psi(x, y) + \psi(y, x) + \psi(y, y)$. Again, by the alternating property of φ, the first and last terms on the right hand side of the latter equation are zero. Thus $0 = \psi(x, y) + \psi(y, x)$, which gives (1).

(2) Every permutation can be written as a product of transpositions (cf. Section 3.5). Furthermore, every transposition may be written as a product of transpositions which interchange two successive integers (cf. Exercise 3 of Section 3.5). Thus every permutation σ can be written as $\tau_1 \cdots \tau_m$, where τ_k is a transposition interchanging two successive integers, for all k. It follows from m applications of (1) that

$$\varphi(v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(n)}) = \epsilon(\tau_m) \cdots \epsilon(\tau_1)\varphi(v_1, v_2, \ldots, v_n).$$

Finally, since ϵ is a homomorphism into the abelian group ± 1 (so the order of the factors ± 1 does not matter), $\epsilon(\tau_1) \cdots \epsilon(\tau_m) = \epsilon(\tau_1 \cdots \tau_m) = \epsilon(\sigma)$. This proves (2).

(3) Choose σ to be any permutation which fixes i and moves j to $i + 1$. Thus $(v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(n)})$ has two equal adjacent components so φ is zero on this n-tuple. By (2), $\varphi(v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(n)}) = \pm\varphi(v_1, v_2, \ldots, v_n)$. This implies (3).

(4) This follows immediately from (3) on expanding by linearity in the i^{th} position.

Proposition 23. Assume φ is an n-multilinear alternating function on V and that for some $v_1, v_2, \ldots, v_n$ and $w_1, w_2, \ldots, w_n \in V$ and some $\alpha_{ij} \in R$ we have

$$\begin{aligned} w_1 &= \alpha_{11}v_1 + \alpha_{21}v_2 + \cdots + \alpha_{n1}v_n \\ w_2 &= \alpha_{12}v_1 + \alpha_{22}v_2 + \cdots + \alpha_{n2}v_n \\ &\vdots \\ w_n &= \alpha_{1n}v_1 + \alpha_{2n}v_2 + \cdots + \alpha_{nn}v_n \end{aligned}$$

(we have purposely written the indices of the α_{ij} in "column format"). Then

$$\varphi(w_1, w_2, \ldots, w_n) = \sum_{\sigma \in S_n} \epsilon(\sigma)\alpha_{\sigma(1)\,1}\alpha_{\sigma(2)\,2} \cdots \alpha_{\sigma(n)\,n}\varphi(v_1, v_2, \ldots, v_n).$$

Proof: If we expand $\varphi(w_1, w_2, \ldots, w_n)$ by multilinearity we obtain a sum of n^n terms of the form $\alpha_{i_1\,1}\alpha_{i_2\,2} \ldots \alpha_{i_n\,n}\varphi(v_{i_1}, v_{i_2}, \ldots, v_{i_n})$, where the indices $i_1, i_2, \ldots, i_n$ each run over $1, 2, \ldots, n$. By Proposition 22(3), φ is zero on the terms where two or more of the i_j's are equal. Thus in this expansion we need only consider the terms where $i_1, \ldots, i_n$ are distinct. Such sequences are in bijective correspondence with permutations in S_n, so each nonzero term may be written as $\alpha_{\sigma(1)\,1}\alpha_{\sigma(2)\,2} \cdots \alpha_{\sigma(n)\,n}\varphi(v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(n)})$, for some $\sigma \in S_n$. Applying (2) of the previous proposition to each of these terms in the expansion of $\varphi(w_1, w_2, \ldots, w_n)$ gives the expression in the proposition.

Definition. An $n \times n$ *determinant function* on R is any function

$$\det : M_{n\times n}(R) \to R$$

that satisfies the following two axioms:

(1) det is an n-multilinear alternating form on $R^n (= V)$, where the n-tuples are the n columns of the matrices in $M_{n\times n}(R)$

(2) $\det(I) = 1$, where I is the $n \times n$ identity matrix.

On occasion we shall write $\det(A_1, A_2, \ldots, A_n)$ for $\det A$, where $A_1, A_2, \ldots, A_n$ are the columns of A.

Theorem 24. There is a unique $n \times n$ determinant function on R and it can be computed for any $n \times n$ matrix (α_{ij}) by the formula:

$$\det(\alpha_{ij}) = \sum_{\sigma \in S_n} \epsilon(\sigma)\alpha_{\sigma(1)\,1}\alpha_{\sigma(2)\,2} \cdots \alpha_{\sigma(n)\,n}.$$

Proof: Let $A_1, A_2, \ldots, A_n$ be the column vectors in a general $n \times n$ matrix (α_{ij}). We leave it as an exercise to check that the formula given in the statement of the theorem does satisfy the axioms of a determinant function—this gives existence of a determinant function. To prove uniqueness let e_i be the column n-tuple with 1 in position i and zeros in all other positions. Then

$$\begin{aligned} A_1 &= \alpha_{11}e_1 + \alpha_{21}e_2 + \cdots + \alpha_{n1}e_n \\ A_2 &= \alpha_{12}e_1 + \alpha_{22}e_2 + \cdots + \alpha_{n2}e_n \\ &\vdots \\ A_n &= \alpha_{1n}e_1 + \alpha_{2n}e_2 + \cdots + \alpha_{nn}e_n. \end{aligned}$$

By Proposition 23, $\det A = \sum_{\sigma\in S_n} \epsilon(\sigma)\alpha_{\sigma(1)\,1}\alpha_{\sigma(2)\,2}\cdots\alpha_{\sigma(n)\,n} \det(e_1, e_2, \ldots, e_n)$. Since by axiom (2) of a determinant function $\det(e_1, e_2, \ldots, e_n) = 1$, the value of $\det A$ is as claimed.

Corollary 25. The determinant is an n-multilinear function of the rows of $M_{n\times n}(R)$ and for any $n \times n$ matrix A, $\det A = \det(A^t)$, where A^t is the transpose of A.

Proof: The first statement is an immediate consequence of the second, so it suffices to prove that a matrix and its transpose have the same determinant. For $A = (\alpha_{ij})$ one calculates that

$$\det A^t = \sum_{\sigma\in S_n} \epsilon(\sigma)\alpha_{1\,\sigma(1)}\alpha_{2\,\sigma(2)}\cdots\alpha_{n\,\sigma(n)}.$$

Each number from 1 to n appears exactly once among $\sigma(1), \ldots, \sigma(n)$ so we may rearrange the product $\alpha_{1\,\sigma(1)}\alpha_{2\,\sigma(2)}\cdots\alpha_{n\,\sigma(n)}$ as $\alpha_{\sigma^{-1}(1)\,1}\alpha_{\sigma^{-1}(2)\,2}\cdots\alpha_{\sigma^{-1}(n)\,n}$. Also, the homomorphism ϵ takes values in $\{\pm 1\}$ so $\epsilon(\sigma) = \epsilon(\sigma^{-1})$. Thus the sum for $\det A^t$ may be rewritten as

$$\sum_{\sigma\in S_n} \epsilon(\sigma^{-1})\alpha_{\sigma^{-1}(1)\,1}\alpha_{\sigma^{-1}(2)\,2}\cdots\alpha_{\sigma^{-1}(n)\,n}.$$

The latter sum is over all permutations, so the index σ^{-1} may be replaced by σ. The resulting expression is the sum for $\det A$. This completes the proof.

Theorem 26. *(Cramer's Rule)* If $A_1, A_2, \ldots, A_n$ are the columns of an $n \times n$ matrix A and $B = \beta_1 A_1 + \beta_2 A_2 + \cdots + \beta_n A_n$, for some $\beta_1, \ldots, \beta_n \in R$, then

$$\beta_i \det A = \det(A_1, \ldots, A_{i-1}, B, A_{i+1}, \ldots, A_n).$$

Proof: This follows immediately from Proposition 22(3) on replacing the given expression for B in the i^{th} position and expanding by multilinearity in that position.

Corollary 27. If R is an integral domain, then $\det A = 0$ for $A \in M_n(R)$ if and only if the columns of A are R-linearly dependent as elements of the free R-module of rank n. Also, $\det A = 0$ if and only if the rows of A are R-linearly dependent.

Proof: Since $\det A = \det A^t$ the first sentence implies the second.

Assume first that the columns of A are linearly dependent and

$$0 = \beta_1 A_1 + \beta_2 A_2 + \cdots + \beta_n A_n$$