f is measurable and g is continuous, then h is not necessarily measurable. (For the details, we refer to McShane, page 241.)

The reader may have noticed that measure has not been mentioned in our discussion of measurable functions. In fact, the class of measurable functions on X depends only on the σ-ring $\mathfrak{M}$ (using the notation of Definition 11.12). For instance, we may speak of *Borel-measurable functions* on R^p, that is, of function f for which

$$\{x \mid f(x) > a\}$$

is always a Borel set, without reference to any particular measure.

SIMPLE FUNCTIONS

11.19 Definition Let s be a real-valued function defined on X. If the range of s is finite, we say that s is a *simple function.*

Let $E \subset X$, and put

$$K_E(x) = \begin{cases} 1 & (x \in E), \\ 0 & (x \notin E). \end{cases} \tag{48}$$

K_E is called the *characteristic function* of E.

Suppose the range of s consists of the distinct numbers $c_1, \ldots, c_n$. Let

$$E_i = \{x \mid s(x) = c_i\} \qquad (i = 1, \ldots, n).$$

Then

$$s = \sum_{n=1}^{n} c_i K_{E_i}, \tag{49}$$

that is, every simple function is a finite linear combination of characteristic functions. It is clear that s is measurable if and only if the sets $E_1, \ldots, E_n$ are measurable.

It is of interest that every function can be approximated by simple functions:

11.20 Theorem *Let f be a real function on X. There exists a sequence $\{s_n\}$ of simple functions such that $s_n(x) \to f(x)$ as $n \to \infty$, for every $x \in X$. If f is measurable, $\{s_n\}$ may be chosen to be a sequence of measurable functions. If $f \geq 0$, $\{s_n\}$ may be chosen to be a monotonically increasing sequence.*

Proof If $f \geq 0$, define

$$E_{ni} = \left\{ x \,\middle|\, \frac{i-1}{2^n} \leq f(x) < \frac{i}{2^n} \right\}, \qquad F_n = \{x \mid f(x) \geq n\}$$

for $n = 1, 2, 3, \ldots, i = 1, 2, \ldots, n2^n$. Put

$$s_n = \sum_{i=1}^{n2^n} \frac{i-1}{2^n} K_{E_{ni}} + nK_{F_n}. \tag{50}$$

In the general case, let $f = f^+ - f^-$, and apply the preceding construction to f^+ and to f^-.

It may be noted that the sequence $\{s_n\}$ given by (50) converges uniformly to f if f is bounded.

INTEGRATION

We shall define integration on a measurable space X, in which $\mathfrak{M}$ is the σ-ring of measurable sets, and μ is the measure. The reader who wishes to visualize a more concrete situation may think of X as the real line, or an interval, and of μ as the Lebesgue measure m.

11.21 Definition Suppose

$$s(x) = \sum_{i=1}^{n} c_i K_{E_i}(x) \qquad (x \in X, c_i > 0) \tag{51}$$

is measurable, and suppose $E \in \mathfrak{M}$. We define

$$I_E(s) = \sum_{i=1}^{n} c_i \mu(E \cap E_i). \tag{52}$$

If f is measurable and nonnegative, we define

$$\int_E f\, d\mu = \sup I_E(s), \tag{53}$$

where the sup is taken over all measurable simple functions s such that $0 \le s \le f$.

The left member of (53) is called the *Lebesgue integral* of f, with respect to the measure μ, over the set E. It should be noted that the integral may have the value $+\infty$.

It is easily verified that

$$\int_E s\, d\mu = I_E(s) \tag{54}$$

for every nonnegative simple measurable function s.

11.22 Definition Let f be measurable, and consider the two integrals

$$\int_E f^+\, d\mu, \qquad \int_E f^-\, d\mu, \tag{55}$$

where f^+ and f^- are defined as in (47).

If at least one of the integrals (55) is finite, we define

$$\int_E f\,d\mu = \int_E f^+\,d\mu - \int_E f^-\,d\mu. \tag{56}$$

If both integrals in (55) are finite, then (56) is finite, and we say that f is *integrable* (or *summable*) on E in the Lebesgue sense, with respect to μ; we write $f \in \mathscr{L}(\mu)$ on E. If $\mu = m$, the usual notation is: $f \in \mathscr{L}$ on E.

This terminology may be a little confusing: If (56) is $+\infty$ or $-\infty$, then the integral of f over E is defined, although f is not integrable in the above sense of the word; f is integrable on E only if its integral over E is finite.

We shall be mainly interested in integrable functions, although in some cases it is desirable to deal with the more general situation.

11.23 Remarks The following properties are evident:

(*a*) If f is measurable and bounded on E, and if $\mu(E) < +\infty$, then $f \in \mathscr{L}(\mu)$ on E.

(*b*) If $a \le f(x) \le b$ for $x \in E$, and $\mu(E) < +\infty$, then

$$a\mu(E) \le \int_E f\,d\mu \le b\mu(E).$$

(*c*) If f and $g \in \mathscr{L}(\mu)$ on E, and if $f(x) \le g(x)$ for $x \in E$, then

$$\int_E f\,d\mu \le \int_E g\,d\mu.$$

(*d*) If $f \in \mathscr{L}(\mu)$ on E, then $cf \in \mathscr{L}(\mu)$ on E, for every finite constant c, and

$$\int_E cf\,d\mu = c\int_E f\,d\mu.$$

(*e*) If $\mu(E) = 0$, and f is measurable, then

$$\int_E f\,d\mu = 0.$$

(*f*) If $f \in \mathscr{L}(\mu)$ on E, $A \in \mathfrak{M}$, and $A \subset E$, then $f \in \mathscr{L}(\mu)$ on A.

11.24 Theorem

(*a*) *Suppose f is measurable and nonnegative on X. For $A \in \mathfrak{M}$, define*

$$\phi(A) = \int_A f\,d\mu. \tag{57}$$

Then ϕ is countably additive on $\mathfrak{M}$.

(*b*) *The same conclusion holds if $f \in \mathscr{L}(\mu)$ on X.*

Proof It is clear that (*b*) follows from (*a*) if we write $f = f^+ - f^-$ and apply (*a*) to f^+ and to f^-.

To prove (*a*), we have to show that

$$\phi(A) = \sum_{n=1}^{\infty} \phi(A_n) \tag{58}$$

if $A_n \in \mathfrak{M}$ $(n = 1, 2, 3, \ldots)$, $A_i \cap A_j = 0$ for $i \neq j$, and $A = \bigcup_1^\infty A_n$.

If f is a characteristic function, then the countable additivity of ϕ is precisely the same as the countable additivity of μ, since

$$\int_A K_E \, d\mu = \mu(A \cap E).$$

If f is simple, then f is of the form (51), and the conclusion again holds.

In the general case, we have, for every measurable simple function s such that $0 \leq s \leq f$,

$$\int_A s \, d\mu = \sum_{n=1}^{\infty} \int_{A_n} s \, d\mu \leq \sum_{n=1}^{\infty} \phi(A_n).$$

Therefore, by (53),

$$\phi(A) \leq \sum_{n=1}^{\infty} \phi(A_n). \tag{59}$$

Now if $\phi(A_n) = +\infty$ for some n, (58) is trivial, since $\phi(A) \geq \phi(A_n)$. Suppose $\phi(A_n) < +\infty$ for every n.

Given $\varepsilon > 0$, we can choose a measurable function s such that $0 \leq s \leq f$, and such that

$$\int_{A_1} s \, d\mu \geq \int_{A_1} f \, d\mu - \varepsilon, \qquad \int_{A_2} s \, d\mu \geq \int_{A_2} f \, d\mu - \varepsilon. \tag{60}$$

Hence

$$\phi(A_1 \cup A_2) \geq \int_{A_1 \cup A_2} s \, d\mu = \int_{A_1} s \, d\mu + \int_{A_2} s \, d\mu \geq \phi(A_1) + \phi(A_2) - 2\varepsilon,$$

so that

$$\phi(A_1 \cup A_2) \geq \phi(A_1) + \phi(A_2).$$

It follows that we have, for every n,

(61) $$\phi(A_1 \cup \cdots \cup A_n) \geq \phi(A_1) + \cdots + \phi(A_n).$$

Since $A \supset A_1 \cup \cdots \cup A_n$, (61) implies

(62) $$\phi(A) \geq \sum_{n=1}^{\infty} \phi(A_n),$$

and (58) follows from (59) and (62).

Corollary *If $A \in \mathfrak{M}$, $B \in \mathfrak{M}$, $B \subset A$, and $\mu(A - B) = 0$, then*

$$\int_A f\,d\mu = \int_B f\,d\mu.$$

Since $A = B \cup (A - B)$, this follows from Remark 11.23(e).

11.25 Remarks The preceding corollary shows that sets of measure zero are negligible in integration.

Let us write $f \sim g$ on E if the set

$$\{x \mid f(x) \neq g(x)\} \cap E$$

has measure zero.

Then $f \sim f$; $f \sim g$ implies $g \sim f$; and $f \sim g$, $g \sim h$ implies $f \sim h$. That is, the relation $\sim$ is an equivalence relation.

If $f \sim g$ on E, we clearly have

$$\int_A f\,d\mu = \int_A g\,d\mu,$$

provided the integrals exist, for every measurable subset A of E.

If a property P holds for every $x \in E - A$, and if $\mu(A) = 0$, it is customary to say that P holds for almost all $x \in E$, or that P holds almost everywhere on E. (This concept of "almost everywhere" depends of course on the particular measure under consideration. In the literature, unless something is said to the contrary, it usually refers to Lebesgue measure.)

If $f \in \mathscr{L}(\mu)$ on E, it is clear that $f(x)$ must be finite almost everywhere on E. In most cases we therefore do not lose any generality if we assume the given functions to be finite-valued from the outset.

11.26 Theorem *If $f \in \mathscr{L}(\mu)$ on E, then $|f| \in \mathscr{L}(\mu)$ on E, and*

(63) $$\left|\int_E f\,d\mu\right| \leq \int_E |f|\,d\mu.$$

Proof Write $E = A \cup B$, where $f(x) \geq 0$ on A and $f(x) < 0$ on B. By Theorem 11.24,

$$\int_E |f| \, d\mu = \int_A |f| \, d\mu + \int_B |f| \, d\mu = \int_A f^+ \, d\mu + \int_B f^- \, d\mu < +\infty,$$

so that $|f| \in \mathscr{L}(\mu)$. Since $f \leq |f|$ and $-f \leq |f|$, we see that

$$\int_E f \, d\mu \leq \int_E |f| \, d\mu, \qquad -\int_E f \, d\mu \leq \int_E |f| \, d\mu,$$

and (63) follows.

Since the integrability of f implies that of $|f|$, the Lebesgue integral is often called an absolutely convergent integral. It is of course possible to define nonabsolutely convergent integrals, and in the treatment of some problems it is essential to do so. But these integrals lack some of the most useful properties of the Lebesgue integral and play a somewhat less important role in analysis.

11.27 Theorem *Suppose f is measurable on E, $|f| \leq g$, and $g \in \mathscr{L}(\mu)$ on E. Then $f \in \mathscr{L}(\mu)$ on E.*

Proof We have $f^+ \leq g$ and $f^- \leq g$.

11.28 Lebesgue's monotone convergence theorem *Suppose $E \in \mathfrak{M}$. Let $\{f_n\}$ be a sequence of measurable functions such that*

$$0 \leq f_1(x) \leq f_2(x) \leq \cdots \qquad (x \in E). \tag{64}$$

Let f be defined by

$$f_n(x) \to f(x) \qquad (x \in E) \tag{65}$$

as $n \to \infty$. Then

$$\int_E f_n \, d\mu \to \int_E f \, d\mu \qquad (n \to \infty). \tag{66}$$

Proof By (64) it is clear that, as $n \to \infty$,

$$\int_E f_n \, d\mu \to \alpha \tag{67}$$

for some α; and since $\int f_n \leq \int f$, we have

$$\alpha \leq \int_E f \, d\mu. \tag{68}$$

Choose c such that $0 < c < 1$, and let s be a simple measurable function such that $0 \le s \le f$. Put

$$E_n = \{x \mid f_n(x) \ge cs(x)\} \qquad (n = 1, 2, 3, \ldots).$$

By (64), $E_1 \subset E_2 \subset E_3 \subset \cdots$; and by (65),

$$E = \bigcup_{n=1}^{\infty} E_n. \tag{69}$$

For every n,

$$\int_E f_n \, d\mu \ge \int_{E_n} f_n \, d\mu \ge c \int_{E_n} s \, d\mu. \tag{70}$$

We let $n \to \infty$ in (70). Since the integral is a countably additive set function (Theorem 11.24), (69) shows that we may apply Theorem 11.3 to the last integral in (70), and we obtain

$$\alpha \ge c \int_E s \, d\mu. \tag{71}$$

Letting $c \to 1$, we see that

$$\alpha \ge \int_E s \, d\mu,$$

and (53) implies

$$\alpha \ge \int_E f \, d\mu. \tag{72}$$

The theorem follows from (67), (68), and (72).

11.29 Theorem *Suppose $f = f_1 + f_2$, where $f_i \in \mathscr{L}(\mu)$ on E $(i = 1, 2)$. Then $f \in \mathscr{L}(\mu)$ on E, and*

$$\int_E f \, d\mu = \int_E f_1 \, d\mu + \int_E f_2 \, d\mu. \tag{73}$$

Proof First, suppose $f_1 \ge 0, f_2 \ge 0$. If f_1 and f_2 are simple, (73) follows trivially from (52) and (54). Otherwise, choose monotonically increasing sequences $\{s_n'\}$, $\{s_n''\}$ of nonnegative measurable simple functions which converge to f_1, f_2. Theorem 11.20 shows that this is possible. Put $s_n = s_n' + s_n''$. Then

$$\int_E s_n \, d\mu = \int_E s_n' \, d\mu + \int_E s_n'' \, d\mu,$$

and (73) follows if we let $n \to \infty$ and appeal to Theorem 11.28.

Next, suppose $f_1 \geq 0, f_2 \leq 0$. Put

$$A = \{x \mid f(x) \geq 0\}, \qquad B = \{x \mid f(x) < 0\}.$$

Then f, f_1, and $-f_2$ are nonnegative on A. Hence

$$\int_A f_1 d\mu = \int_A f\, d\mu + \int_A (-f_2)\, d\mu = \int_A f\, d\mu - \int_A f_2\, d\mu. \tag{74}$$

Similarly, $-f, f_1$, and $-f_2$ are nonnegative on B, so that

$$\int_B (-f_2)\, d\mu = \int_B f_1\, d\mu + \int_B (-f)\, d\mu,$$

or

$$\int_B f_1\, d\mu = \int_B f\, d\mu - \int_B f_2\, d\mu, \tag{75}$$

and (73) follows if we add (74) and (75).

In the general case, E can be decomposed into four sets E_i on each of which $f_1(x)$ and $f_2(x)$ are of constant sign. The two cases we have proved so far imply

$$\int_{E_i} f\, d\mu = \int_{E_i} f_1\, d\mu + \int_{E_i} f_2\, d\mu \qquad (i = 1, 2, 3, 4),$$

and (73) follows by adding these four equations.

We are now in a position to reformulate Theorem 11.28 for series.

11.30 Theorem *Suppose $E \in \mathfrak{M}$. If $\{f_n\}$ is a sequence of nonnegative measurable functions and*

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \qquad (x \in E), \tag{76}$$

then

$$\int_E f\, d\mu = \sum_{n=1}^{\infty} \int_E f_n\, d\mu.$$

Proof The partial sums of (76) form a monotonically increasing sequence.

11.31 Fatou's theorem *Suppose $E \in \mathfrak{M}$. If $\{f_n\}$ is a sequence of nonnegative measurable functions and*

$$f(x) = \liminf_{n \to \infty} f_n(x) \qquad (x \in E),$$

then

$$\int_E f\, d\mu \leq \liminf_{n \to \infty} \int_E f_n\, d\mu. \tag{77}$$

Strict inequality may hold in (77). An example is given in Exercise 5.

Proof For $n = 1, 2, 3, \ldots$ and $x \in E$, put

$$g_n(x) = \inf f_i(x) \qquad (i \geq n).$$

Then g_n is measurable on E, and

$$0 \leq g_1(x) \leq g_2(x) \leq \cdots, \tag{78}$$

$$g_n(x) \leq f_n(x), \tag{79}$$

$$g_n(x) \to f(x) \qquad (n \to \infty). \tag{80}$$

By (78), (80), and Theorem 11.28,

$$\int_E g_n \, d\mu \to \int_E f \, d\mu, \tag{81}$$

so that (77) follows from (79) and (81).

11.32 Lebesgue's dominated convergence theorem *Suppose $E \in \mathfrak{M}$. Let $\{f_n\}$ be a sequence of measurable functions such that*

$$f_n(x) \to f(x) \qquad (x \in E) \tag{82}$$

as $n \to \infty$. If there exists a function $g \in \mathscr{L}(\mu)$ on E, such that

$$|f_n(x)| \leq g(x) \qquad (n = 1, 2, 3, \ldots, x \in E), \tag{83}$$

then

$$\lim_{n \to \infty} \int_E f_n \, d\mu = \int_E f \, d\mu. \tag{84}$$

Because of (83), $\{f_n\}$ is said to be dominated by g, and we talk about dominated convergence. By Remark 11.25, the conclusion is the same if (82) holds almost everywhere on E.

Proof First, (83) and Theorem 11.27 imply that $f_n \in \mathscr{L}(\mu)$ and $f \in \mathscr{L}(\mu)$ on E.

Since $f_n + g \geq 0$, Fatou's theorem shows that

$$\int_E (f + g) \, d\mu \leq \liminf_{n \to \infty} \int_E (f_n + g) \, d\mu,$$

or

$$\int_E f \, d\mu \leq \liminf_{n \to \infty} \int_E f_n \, d\mu. \tag{85}$$