κορυφὴ δὲ τὸ Κ σημεῖον.

Δεῖ δὴ αὐτὴν καὶ σφαίρᾳ περιλαβεῖν τῇ δοθείσῃ καὶ δεῖξαι, ὅτι ἡ τῆς σφαίρας διάμετρος ἡμιολία ἐστὶ δυνάμει τῆς πλευρᾶς τῆς πυραμίδος.

Ἐκβεβλήσθω γὰρ ἐπ᾽ εὐθείας τῇ ΚΘ εὐθεῖα ἡ ΘΛ, καὶ κείσθω τῇ ΓΒ ἴση ἡ ΘΛ. καὶ ἐπεί ἐστιν ὡς ἡ ΑΓ πρὸς τὴν ΓΔ, οὕτως ἡ ΓΔ πρὸς τὴν ΓΒ, ἴση δὲ ἡ μὲν ΑΓ τῇ ΚΘ, ἡ δὲ ΓΔ τῇ ΘΕ, ἡ δὲ ΓΒ τῇ ΘΛ, ἔστιν ἄρα ὡς ἡ ΚΘ πρὸς τὴν ΘΕ, οὕτως ἡ ΕΘ πρὸς τὴν ΘΛ· τὸ ἄρα ὑπὸ τῶν ΚΘ, ΘΛ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΕΘ. καί ἐστιν ὀρθὴ ἑκατέρα τῶν ὑπὸ ΚΘΕ, ΕΘΛ γωνιῶν· τὸ ἄρα ἐπὶ τῆς ΚΛ γραφόμενον ἡμικύκλιον ἥξει καὶ διὰ τοῦ Ε [ἐπειδήπερ ἐὰν ἐπιζεύξωμεν τὴν ΕΛ, ὀρθὴ γίνεται ἡ ὑπὸ ΛΕΚ γωνία διὰ τὸ ἰσογώνιον γίνεσθαι τὸ ΕΛΚ τρίγωνον ἑκατέρῳ τῶν ΕΛΘ, ΕΘΚ τριγώνων]. ἐὰν δὴ μενούσης τῆς ΚΛ περιενεχθὲν τὸ ἡμικύκλιον εἰς τὸ αὐτὸ πάλιν ἀποκατασταθῇ, ὅθεν ἤρξατο φέρεσθαι, ἥξει καὶ διὰ τῶν Ζ, Η σημείων ἐπιζευγνυμένων τῶν ΖΛ, ΛΗ καὶ ὀρθῶν ὁμοίως γινομένων τῶν πρὸς τοῖς Ζ, Η γωνιῶν· καὶ ἔσται ἡ πυραμὶς σφαίρᾳ περιειλημμένη τῇ δοθείσῃ. ἡ γὰρ ΚΛ τῆς σφαίρας διάμετρος ἴση ἐστὶ τῇ τῆς δοθείσης σφαίρας διαμετρῳ τῇ ΑΒ, ἐπειδήπερ τῇ μὲν ΑΓ ἴση κεῖται ἡ ΚΘ, τῇ δὲ ΓΒ ἡ ΘΛ.

Λέγω δή, ὅτι ἡ τῆς σφαίρας διάμετρος ἡμιολία ἐστὶ δυνάμει τῆς πλευρᾶς τῆς πυραμίδος.

Ἐπεὶ γὰρ διπλῆ ἐστιν ἡ ΑΓ τῆς ΓΒ, τριπλῆ ἄρα ἐστὶν ἡ ΑΒ τῆς ΒΓ· ἀναστρέψαντι ἡμιολία ἄρα ἐστὶν ἡ ΒΑ τῆς ΑΓ. ὡς δὲ ἡ ΒΑ πρὸς τὴν ΑΓ, οὕτως τὸ ἀπὸ τῆς ΒΑ πρὸς τὸ ἀπὸ τῆς ΑΔ [ἐπειδήπερ ἐπιζευγνμένης τῆς ΔΒ ἐστιν ὡς ἡ ΒΑ πρὸς τὴν ΑΔ, οὕτως ἡ ΔΑ πρὸς τὴν ΑΓ διὰ τὴν ὁμοιότητα τῶν ΔΑΒ, ΔΑΓ τριγώνων, καὶ εἶναι ὡς τὴν πρώτην πρὸς τὴν τρίτην, οὕτως τὸ ἀπὸ τῆς τρώτης πρὸς τὸ ἀπὸ τῆς δευτέρας]. ἡμιόλιον ἄρα καὶ τὸ ἀπὸ τῆς ΒΑ τοῦ ἀπὸ τῆς ΑΔ. καί ἐστιν ἡ μὲν ΒΑ ἡ τῆς δοθείσης σφαίρας διάμετρος, ἡ δὲ ΑΔ ἴση τῇ πλευρᾷ τῆς πυραμίδος.

Ἡ ἄρα τῆς σφαίρας διάμετρος ἡμιολία ἐστὶ τῆς πλευρᾶς τῆς πυραμίδος· ὅπερ ἔδει δεῖξαι.

also equal to EF. But, DA was shown (to be) equal to each of KE, KF, and KG. Thus, EF, FG, and GE are equal to KE, KF, and KG, respectively. Thus, the four triangles EFG, KEF, KFG, and KEG are equilateral. Thus, a pyramid, whose base is triangle EFG, and apex the point K, has been constructed from four equilateral triangles.

So, it is also necessary to enclose it in the given sphere, and to show that the square on the diameter of the sphere is one and a half times the (square) on the side of the pyramid.

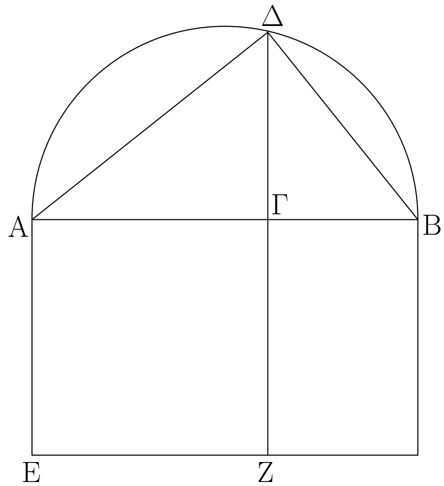
For let the straight-line HL have been produced in a straight-line with KH, and let HL be made equal to CB. And since as AC (is) to CD, so CD (is) to CB [Prop. 6.8 corr.], and AC (is) equal to KH, and CD to HE, and CB to HL, thus as KH is to HE, so EH (is) to HL. Thus, the (rectangle contained) by KH and HL is equal to the (square) on EH [Prop. 6.17]. And each of the angles KHE and EHL is a right-angle. Thus, the semi-circle drawn on KL will also pass through E [inasmuch as if we join EL then the angle LEK becomes a right-angle, on account of triangle ELK becoming equiangular to each of the triangles ELH and EHK [Props. 6.8, 3.31]]. So, if KL remains (fixed), and the semi-circle is carried around, and again established at the same (position) from which it began to be moved, it will also pass through points F and G, (because) if FL and LG are joined, the angles at F and G will similarly become right-angles. And the pyramid will have been enclosed by the given sphere. For the diameter, KL, of the sphere is equal to the diameter, AB, of the given sphere—inasmuch as KH was made equal to AC, and HL to CB.

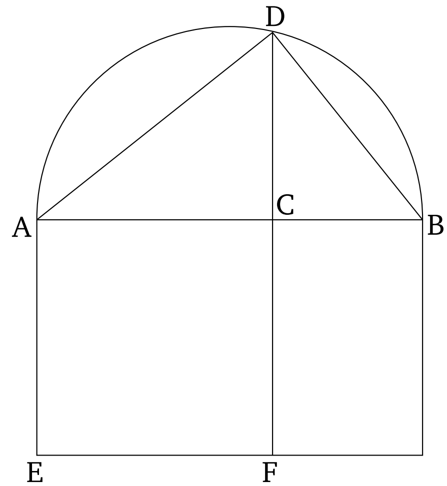
So, I say that the square on the diameter of the sphere is one and a half times the (square) on the side of the pyramid.

For since AC is double CB, AB is thus triple BC. Thus, via conversion, BA is one and a half times AC. And as BA (is) to AC, so the (square) on BA (is) to the (square) on AD [inasmuch as if DB is joined then as BA is to AD, so DA (is) to AC, on account of the similarity of triangles DAB and DAC. And as the first is to the third (of four proportional magnitudes), so the (square) on the first (is) to the (square) on the second.] Thus, the (square) on BA (is) also one and a half times the (square) on AD. And BA is the diameter of the given sphere, and AD (is) equal to the side of the pyramid.

Thus, the square on the diameter of the sphere is one and a half times the (square) on the side of the pyramid.† (Which is) the very thing it was required to show.

† If the radius of the sphere is unity then the side of the pyramid (*i.e.*, tetrahedron) is $\sqrt{8/3}$.





Λῆμμα.

Δεικτέον, ὅτι ἐστὶν ὡς ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως τὸ ἀπὸ τῆς ΑΔ πρὸς τὸ ἀπὸ τῆς ΔΓ.

Ἐκκείσθω γὰρ ἡ τοῦ ἡμικυκλίου καταγραφή, καὶ ἐπεζεύχθω ἡ ΔΒ, καὶ ἀναγεγράφθω ἀπὸ τῆς ΑΓ τετράγωνον τὸ ΕΓ, καὶ συμπεπληρώσθω τὸ ΖΒ παραλληλόγραμμον. ἐπεὶ οὖν διὰ τὸ ἰσογώνιον εἶναι τὸ ΔΑΒ τρίγωνον τῷ ΔΑΓ τριγώνῳ ἐστὶν ὡς ἡ ΒΑ πρὸς τὴν ΑΔ, οὕτως ἡ ΔΑ πρὸς τὴν ΑΓ, τὸ ἄρα ὑπὸ τῶν ΒΑ, ΑΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΔ. καὶ ἐπεί ἐστιν ὡς ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως τὸ ΕΒ πρὸς τὸ ΒΖ, καί ἐστι τὸ μὲν ΕΒ τὸ ὑπὸ τῶν ΒΑ, ΑΓ· ἴση γὰρ ἡ ΕΑ τῇ ΑΓ· τὸ δὲ ΒΖ τὸ ὑπὸ τῶν ΑΓ, ΓΒ, ὡς ἄρα ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως τὸ ὑπὸ τῶν ΒΑ, ΑΓ πρὸ τὸ ὑπὸ τῶν ΑΓ, ΓΒ. καί ἐστι τὸ μὲν ὑπὸ τῶν ΒΑ, ΑΓ ἴσον τῷ ἀπὸ τῆς ΑΔ, τὸ δὲ ὑπὸ τῶν ΑΓΒ ἴσον τῷ ἀπὸ τῆς ΔΓ· ἡ γὰρ ΔΓ κάθετος τῶν τῆς βάσεως τμημάτων τῶν ΑΓ, ΓΒ μέση ἀνάλογόν ἐστι διὰ τὸ ὀρθὴν εἶναι τὴν ὑπὸ ΑΔΒ. ὡς ἄρα ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως τὸ ἀπὸ τῆς ΑΔ πρὸς τὸ ἀπὸ τῆς ΔΓ· ὅπερ ἔδει δεῖξαι.

Lemma

It must be shown that as AB is to BC, so the (square) on AD (is) to the (square) on DC.

For, let the figure of the semi-circle have been set out, and let DB have been joined. And let the square EC have been described on AC. And let the parallelogram FB have been completed. Therefore, since, on account of triangle DAB being equiangular to triangle DAC [Props. 6.8, 6.4], (proportionally) as BA is to AD, so DA (is) to AC, the (rectangle contained) by BA and AC is thus equal to the (square) on AD [Prop. 6.17]. And since as AB is to BC, so EB (is) to BF [Prop. 6.1]. And EB is the (rectangle contained) by BA and AC—for EA (is) equal to AC. And BF the (rectangle contained) by AC and CB. Thus, as AB (is) to BC, so the (rectangle contained) by BA and AC (is) to the (rectangle contained) by AC and CB. And the (rectangle contained) by BA and AC is equal to the (square) on AD, and the (rectangle contained) by ACB (is) equal to the (square) on DC. For the perpendicular DC is the mean proportional to the pieces of the base, AC and CB, on account of ADB being a right-angle [Prop. 6.8 corr.]. Thus, as AB (is) to BC, so the (square) on AD (is) to the (square) on DC. (Which is) the very thing it was required to show.

ιδʹ.

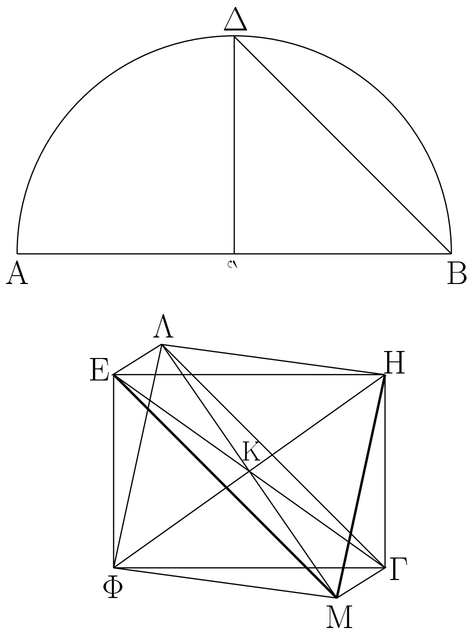
Ὀκτάεδρον συστήσασθαι καὶ σφαίρᾳ περιλαβεῖν, ᾗ καὶ τὰ πρότερα, καὶ δεῖξαι, ὅτι ἡ τῆς σφαίρας διάμετρος δυνάμει διπλασία ἐστὶ τῆς πλευρᾶς τοῦ ὀκταέδρου.

Ἐκκείσθω ἡ τῆς δοθείσης σφαίρας διάμετρος ἡ ΑΒ, καὶ τετμήσθω δίχα κατὰ τὸ Γ, καὶ γεγράφθω ἐπὶ τῆς ΑΒ ἡμικύκλιον τὸ ΑΔΒ, καὶ ἤχθω ἀπὸ τοῦ Γ τῇ ΑΒ πρὸς ὀρθὰς ἡ ΓΔ, καὶ ἐπεζεύχθω ἡ ΔΒ, καὶ ἐκκείσθω τετράγωνον τὸ ΕΖΗΘ ἴσην ἔχον ἑκάστην τῶν πλευρῶν τῇ ΔΒ, καὶ

Proposition 14

To construct an octahedron, and to enclose (it) in a (given) sphere, like in the preceding (proposition), and to show that the square on the diameter of the sphere is double the (square) on the side of the octahedron.

Let the diameter AB of the given sphere be laid out, and let it have been cut in half at C. And let the semi-circle ADB have been drawn on AB. And let CD be drawn from C at right-angles to AB. And let DB have ἐπεζεύχθωσαν αἱ ΘΖ, ΕΗ, καὶ ἀνεστάτω ἀπὸ τοῦ Κ σημείου τῷ τοῦ ΕΖΗΘ τετραγώνου ἐπιπέδῳ πρὸς ὀρθὰς εὐθεῖα ἡ ΚΛ καὶ διήχθω ἐπὶ τὰ ἕτερα μέρη τοῦ ἐπιπέδου ὡς ἡ ΚΜ, καὶ ἀφῃρήσθω ἀφ᾽ ἑκατέρας τῶν ΚΛ, ΚΜ μιᾷ τῶν ΕΚ, ΖΚ, ΗΚ, ΘΚ ἴση ἑκατέρα τῶν ΚΛ, ΚΜ, καὶ ἐπεζεύχθωσαν αἱ ΛΕ, ΛΖ, ΛΗ, ΛΘ, ΜΕ, ΜΖ, ΜΗ, ΜΘ.

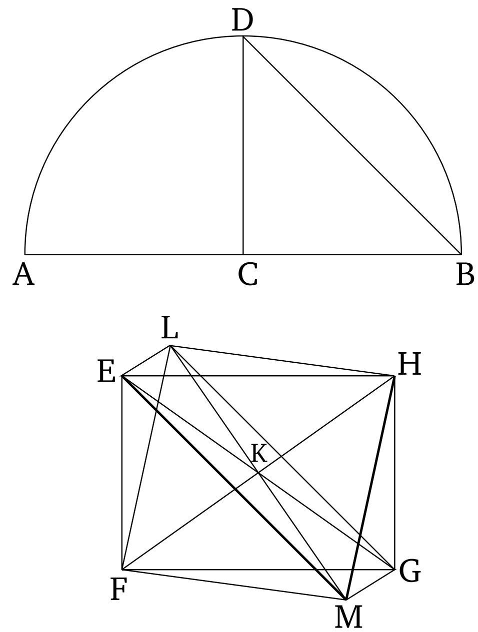


Καὶ ἐπεὶ ἴση ἐστὶν ἡ ΚΕ τῇ ΚΘ, καί ἐστιν ὀρθὴ ἡ ὑπὸ ΕΚΘ γωνία, τὸ ἄρα ἀπὸ τῆς ΘΕ διπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΕΚ. πάλιν, ἐπεὶ ἴση ἐστὶν ἡ ΛΚ τῇ ΚΕ, καί ἐστιν ὀρθὴ ἡ ὑπὸ ΛΚΕ γωνία, τὸ ἄρα ἀπὸ τῆς ΕΛ διπλάσιόν ἐστι τοῦ ἀπὸ ΕΚ. ἐδείχθη δὲ καὶ τὸ ἀπὸ τῆς ΘΕ διπλάσιον τοῦ ἀπὸ τῆς ΕΚ· τὸ ἄρα ἀπὸ τῆς ΛΕ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΕΘ· ἴση ἄρα ἐστὶν ἡ ΛΕ τῇ ΕΘ. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΛΘ τῇ ΘΕ ἐστιν ἴση· ἰσόπλευρον ἄρα ἐστὶ τὸ ΛΕΘ τρίγωνον. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἕκαστον τῶν λοιπῶν τριγώνων, ὧν βάσεις μέν εἰσιν αἱ τοῦ ΕΖΗΘ τετραγώνου πλευραί, κορυφαὶ δὲ τὰ Λ, Μ σημεῖα, ἰσόπλευρόν ἐστιν· ὀκτάεδρον ἄρα συνέσταται ὑπὸ ὀκτὼ τριγώνων ἰσοπλεύρων περιεχόμενον.

Δεῖ δὴ αὐτὸ καὶ σφαίρᾳ περιλαβεῖν τῇ δοθείσῃ καὶ δεῖξαι, ὅτι ἡ τῆς σφαίρας διάμετρος δυνάμει διπλασίων ἐστὶ τῆς τοῦ ὀκταέδρου πλευρᾶς.

Ἐπεὶ γὰρ αἱ τρεῖς αἱ ΛΚ, ΚΜ, ΚΕ ἴσαι ἀλλήλαις εἰσίν, τὸ ἄρα ἐπὶ τῆς ΛΜ γραφόμενον ἡμικύκλιον ἥξει καὶ διὰ τοῦ Ε. καὶ διὰ τὰ αὐτά, ἐὰν μενούσης τῆς ΛΜ περιενεχθὲν τὸ ἡμικύκλιον εἰς τὸ αὐτὸ ἀποκατασταθῇ, ὅθεν ἤρξατο φέρεσθαι, ἥξει καὶ διὰ τῶν Ζ, Η, Θ σημείων, καὶ ἔσται σφαίρᾳ περιειλημμένον τὸ ὀκτάεδρον. λέγω δή, ὅτι καὶ τῇ δοθείσῃ. ἐπεὶ γὰρ ἴση ἐστὶν ἡ ΛΚ τῇ ΚΜ, κοινὴ δὲ ἡ ΚΕ,

been joined. And let the square $EFGH$, having each of its sides equal to DB, be laid out. And let HF and EG have been joined. And let the straight-line KL have been set up, at point K, at right-angles to the plane of square $EFGH$ [Prop. 11.12]. And let it have been drawn across on the other side of the plane, like KM. And let KL and KM, equal to one of EK, FK, GK, and HK, have been cut off from KL and KM, respectively. And let LE, LF, LG, LH, ME, MF, MG, and MH have been joined.



And since KE is equal to KH, and angle EKH is a right-angle, the (square) on the HE is thus double the (square) on EK [Prop. 1.47]. Again, since LK is equal to KE, and angle LKE is a right-angle, the (square) on EL is thus double the (square) on EK [Prop. 1.47]. And the (square) on HE was also shown (to be) double the (square) on EK. Thus, the (square) on LE is equal to the (square) on EH. Thus, LE is equal to EH. So, for the same (reasons), LH is also equal to HE. Triangle LEH is thus equilateral. So, similarly, we can show that each of the remaining triangles, whose bases are the sides of the square $EFGH$, and apexes the points L and M, are equilateral. Thus, an octahedron contained by eight equilateral triangles has been constructed.

So, it is also necessary to enclose it by the given sphere, and to show that the square on the diameter of the sphere is double the (square) on the side of the octahedron.

For since the three (straight-lines) LK, KM, and KE are equal to one another, the semi-circle drawn on LM will thus also pass through E. And, for the same (reasons), if LM remains (fixed), and the semi-circle is carκαὶ γωνίας ὀρθὰς περιέχουσιν, βάσις ἄρα ἡ ΛΕ βάσει τῇ ΕΜ ἐστιν ἴση. καὶ ἐπεὶ ὀρθή ἐστιν ἡ ὑπὸ ΛΕΜ γωνία· ἐν ἡμικυκλίῳ γάρ· τὸ ἄρα ἀπὸ τῆς ΛΜ διπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΛΕ. πάλιν, ἐπεὶ ἴση ἐστὶν ἡ ΑΓ τῇ ΓΒ, διπλασία ἐστὶν ἡ ΑΒ τῆς ΒΓ. ὡς δὲ ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως τὸ ἀπὸ τῆς ΑΒ πρὸς τὸ ἀπὸ τῆς ΒΔ· διπλάσιον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΑΒ τοῦ ἀπὸ τῆς ΒΔ. ἐδείχθη δὲ καὶ τὸ ἀπὸ τῆς ΛΜ διπλάσιον τοῦ ἀπὸ τῆς ΛΕ. καί ἐστιν ἴσον τὸ ἀπὸ τῆς ΔΒ τῷ ἀπὸ τῆς ΛΕ· ἴση γὰρ κεῖται ἡ ΕΘ τῇ ΔΒ. ἴσον ἄρα καὶ τὸ ἀπὸ τῆς ΑΒ τῷ ἀπὸ τῆς ΛΜ· ἴση ἄρα ἡ ΑΒ τῇ ΛΜ. καί ἐστιν ἡ ΑΒ ἡ τῆς δοθείσης σφαίρας διάμετρος· ἡ ΛΜ ἄρα ἴση ἐστὶ τῇ τῆς δοθείσης σφαίρας διαμέτρῳ.

Περιείληπται ἄρα τὸ ὀκτάεδρον τῇ δοθείσῃ σφαίρᾳ. καὶ συναποδέδεικται, ὅτι ἡ τῆς σφαίρας διάμετρος δυνάμει διπλασίων ἐστὶ τῆς τοῦ ὀκταέδρου πλευρᾶς· ὅπερ ἔδει δεῖξαι.

ried around, and again established at the same (position) from which it began to be moved, then it will also pass through points F, G, and H, and the octahedron will have been enclosed by a sphere. So, I say that (it is) also (enclosed) by the given (sphere). For since LK is equal to KM, and KE (is) common, and they contain right-angles, the base LE is thus equal to the base EM [Prop. 1.4]. And since angle LEM is a right-angle—for (it is) in a semi-circle [Prop. 3.31]—the (square) on LM is thus double the (square) on LE [Prop. 1.47]. Again, since AC is equal to CB, AB is double BC. And as AB (is) to BC, so the (square) on AB (is) to the (square) on BD [Prop. 6.8, Def. 5.9]. Thus, the (square) on AB is double the (square) on BD. And the (square) on LM was also shown (to be) double the (square) on LE. And the (square) on DB is equal to the (square) on LE. For EH was made equal to DB. Thus, the (square) on AB (is) also equal to the (square) on LM. Thus, AB (is) equal to LM. And AB is the diameter of the given sphere. Thus, LM is equal to the diameter of the given sphere.

Thus, the octahedron has been enclosed by the given sphere, and it has been simultaneously proved that the square on the diameter of the sphere is double the (square) on the side of the octahedron.† (Which is) the very thing it was required to show.

† If the radius of the sphere is unity then the side of octahedron is $\sqrt{2}$.

ιεʹ.

Κύβον συστήσασθαι καὶ σφαίρᾳ περιλαβεῖν, ᾗ καὶ τὴν πυραμίδα, καὶ δεῖξαι, ὅτι ἡ τῆς σφαίρας διάμετρος δυνάμει τριπλασίων ἐστὶ τῆς τοῦ κύβου πλευρᾶς.

Ἐκκείσθω ἡ τῆς δοθείσης σφαίρας διάμετρος ἡ ΑΒ καὶ τετμήσθω κατὰ τὸ Γ ὥστε διπλῆν εἶναι τὴν ΑΓ τῆς ΓΒ, καὶ γεγράφθω ἐπὶ τῆς ΑΒ ἡμικύκλιον τὸ ΑΔΒ, καὶ ἀπὸ τοῦ Γ τῇ ΑΒ πρὸς ὀρθὰς ἤχθω ἡ ΓΔ, καὶ ἐπεζεύχθω ἡ ΔΒ, καὶ ἐκκείσθω τετράγωνον τὸ ΕΖΗΘ ἴσην ἔχον τὴν πλευρὰν τῇ ΔΒ, καὶ ἀπὸ τῶν Ε, Ζ, Η, Θ τῷ τοῦ ΕΖΗΘ τετραγώνου ἐπιπέδῳ πρὸς ὀρθὰς ἤχθωσαν αἱ ΕΚ, ΖΛ, ΗΜ, ΘΝ, καὶ ἀφῃρήσθω ἀπὸ ἑκάστης τῶν ΕΚ, ΖΛ, ΗΜ, ΘΝ μιᾷ τῶν ΕΖ, ΖΗ, ΗΘ, ΘΕ ἴση ἑκάστη τῶν ΕΚ, ΖΛ, ΗΜ, ΘΝ, καὶ ἐπεζεύχθωσαν αἱ ΚΛ, ΛΜ, ΜΝ, ΝΚ· κύβος ἄρα συνέσταται ὁ ΖΝ ὑπὸ ἓξ τετραγώνων ἴσων περιεχόμενος.

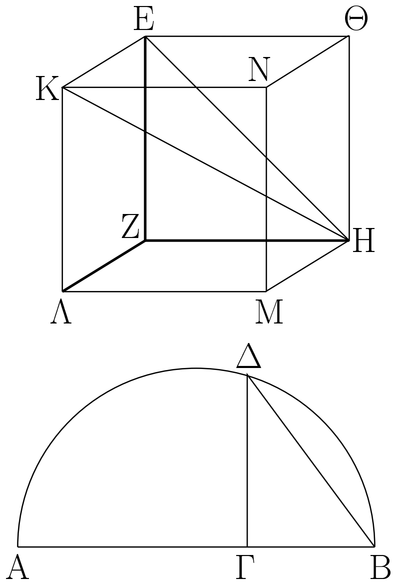
Δεῖ δὴ αὐτὸν καὶ σφαίρᾳ περιλαβεῖν τῇ δοθείσῃ καὶ δεῖξαι, ὅτι ἡ τῆς σφαίρας διάμετρος δυνάμει τριπλασία ἐστὶ τῆς πλευρᾶς τοῦ κύβου.

Proposition 15

To construct a cube, and to enclose (it) in a sphere, like in the (case of the) pyramid, and to show that the square on the diameter of the sphere is three times the (square) on the side of the cube.

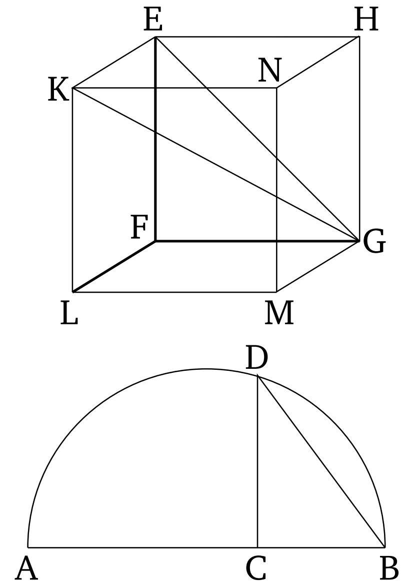
Let the diameter AB of the given sphere be laid out, and let it have been cut at C such that AC is double CB. And let the semi-circle ADB have been drawn on AB. And let CD have been drawn from C at right-angles to AB. And let DB have been joined. And let the square $EFGH$, having (its) side equal to DB, be laid out. And let EK, FL, GM, and HN have been drawn from (points) E, F, G, and H, (respectively), at right-angles to the plane of square $EFGH$. And let EK, FL, GM, and HN, equal to one of EF, FG, GH, and HE, have been cut off from EK, FL, GM, and HN, respectively. And let KL, LM, MN, and NK have been joined. Thus, a cube contained by six equal squares has been constructed.

So, it is also necessary to enclose it by the given sphere, and to show that the square on the diameter of the sphere is three times the (square) on the side of the cube.



Ἐπεζεύχθωσαν γὰρ αἱ ΚΗ, ΕΗ. καὶ ἐπεὶ ὀρθή ἐστιν ἡ ὑπὸ ΚΕΗ γωνία διὰ τὸ καὶ τὴν ΚΕ ὀρθὴν εἶναι πρὸς τὸ ΕΗ ἐπίπεδον δηλαδὴ καὶ πρὸς τὴν ΕΗ εὐθεῖαν, τὸ ἄρα ἐπὶ τῆς ΚΗ γραφόμενον ἡμικύκλιον ἥξει καὶ διὰ τοῦ Ε σημείου. πάλιν, ἐπεὶ ἡ ΗΖ ὀρθή ἐστι πρὸς ἑκατέραν τῶν ΖΛ, ΖΕ, καὶ πρὸς τὸ ΖΚ ἄρα ἐπίπεδον ὀρθή ἐστιν ἡ ΗΖ· ὥστε καὶ ἐὰν ἐπιζεύξωμεν τὴν ΖΚ, ἡ ΗΖ ὀρθὴ ἔσται καὶ πρὸς τὴν ΖΚ· καὶ δια τοῦτο πάλιν τὸ ἐπὶ τῆς ΗΚ γραφόμενον ἡμικύκλιον ἥξει καὶ διὰ τοῦ Ζ. ὁμοίως καὶ δια τῶν λοιπῶν τοῦ κύβου σημείων ἥξει. ἐὰν δὴ μενούσης τῆς ΚΗ περιενεχθὲν τὸ ἡμικύκλιον εἰς τὸ αὐτὸ ἀποκατασταθῇ, ὅθεν ἤρξατο φέρεσθαι, ἔσται σφαίρᾳ περιειλημμένος ὁ κύβος. λέγω δή, ὅτι καὶ τῇ δοθείσῃ. ἐπεὶ γὰρ ἴση ἐστὶν ἡ ΗΖ τῇ ΖΕ, καί ἐστιν ὀρθὴ ἡ πρὸς τῷ Ζ γωνία, τὸ ἄρα ἀπὸ τῆς ΕΗ διπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΕΖ. ἴση δὲ ἡ ΕΖ τῇ ΕΚ· τὸ ἄρα ἀπὸ τῆς ΕΗ διπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΕΚ· ὥστε τὰ ἀπὸ τῶν ΗΕ, ΕΚ, τουτέστι τὸ ἀπὸ τῆς ΗΚ, τριπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΕΚ. καὶ ἐπεὶ τριπλασίων ἐστὶν ἡ ΑΒ τῆς ΒΓ, ὡς δὲ ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως τὸ ἀπὸ τῆς ΑΒ πρὸς τὸ ἀπὸ τῆς ΒΔ, τριπλάσιον ἄρα τὸ ἀπὸ τῆς ΑΒ τοῦ ἀπὸ τῆς ΒΔ. ἐδείχθη δὲ καὶ τὸ ἀπὸ τῆς ΗΚ τοῦ ἀπὸ τῆς ΚΕ τριπλάσιον. καὶ κεῖται ἴση ἡ ΚΕ τῇ ΔΒ· ἴση ἄρα καὶ ἡ ΚΗ τῇ ΑΒ. καί ἐστιν ἡ ΑΒ τῆς δοθείσης σφαίρας διάμετρος· καὶ ἡ ΚΗ ἄρα ἴση ἐστὶ τῇ τῆς δοθείσης σφαίρας διαμέτρῳ.

Τῇ δοθείσῃ ἄρα σφαίρα περιείληπται ὁ κύβος· καὶ συναποδέδεικται, ὅτι ἡ τῆς σφαίρας διάμετρος δυνάμει τριπλασίων ἐστὶ τῆς τοῦ κύβου πλευρᾶς· ὅπερ ἔδει δεῖξαι.



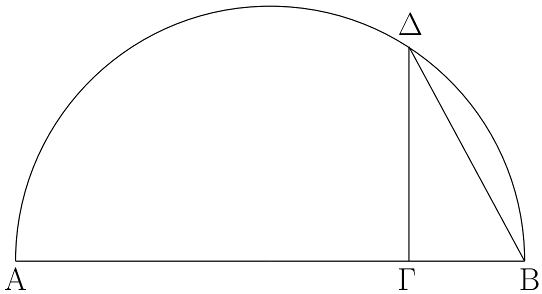
For let KG and EG have been joined. And since angle KEG is a right-angle—on account of KE also being at right-angles to the plane EG, and manifestly also to the straight-line EG [Def. 11.3]—the semi-circle drawn on KG will thus also pass through point E. Again, since GF is at right-angles to each of FL and FE, GF is thus also at right-angles to the plane FK [Prop. 11.4]. Hence, if we also join FK then GF will also be at right-angles to FK. And, again, on account of this, the semi-circle drawn on GK will also pass through point F. Similarly, it will also pass through the remaining (angular) points of the cube. So, if KG remains (fixed), and the semi-circle is carried around, and again established at the same (position) from which it began to be moved, then the cube will have been enclosed by a sphere. So, I say that (it is) also (enclosed) by the given (sphere). For since GF is equal to FE, and the angle at F is a right-angle, the (square) on EG is thus double the (square) on EF [Prop. 1.47]. And EF (is) equal to EK. Thus, the (square) on EG is double the (square) on EK. Hence, the (sum of the squares) on GE and EK—that is to say, the (square) on GK [Prop. 1.47]—is three times the (square) on EK. And since AB is three times BC, and as AB (is) to BC, so the (square) on AB (is) to the (square) on BD [Prop. 6.8, Def. 5.9], the (square) on AB (is) thus three times the (square) on BD. And the (square) on GK was also shown (to be) three times the (square) on KE. And KE was made equal to DB. Thus, KG (is) also equal to AB. And AB is the radius of the given sphere. Thus, KG is also equal to the diameter of the given sphere.

Thus, the cube has been enclosed by the given sphere. And it has simultaneously been shown that the square on the diameter of the sphere is three times the (square) on the side of the cube.† (Which is) the very thing it was required to show.

† If the radius of the sphere is unity then the side of the cube is $\sqrt{4/3}$.

ιϛ΄.

Εἰκοσάεδρον συστήσασθαι καὶ σφαίρᾳ περιλαβεῖν, ᾗ καὶ τὰ προειρημένα σχήματα, καὶ δεῖξαι, ὅτι ἡ τοῦ εἰκοσαέδρου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἐλάττων.

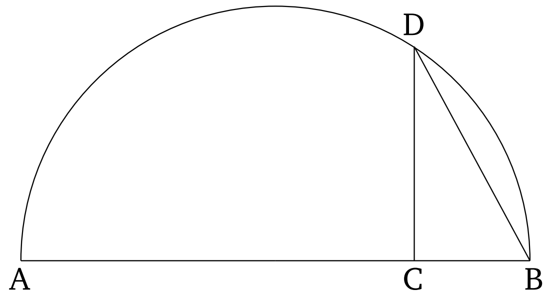


Ἐκκείσθω ἡ τῆς δοθείσης σφαίρας διάμετρος ἡ ΑΒ καὶ τετμήσθω κατὰ τὸ Γ ὥστε τετραπλῆν εἶναι τὴν ΑΓ τῆς ΓΒ, καὶ γεγράφθω ἐπὶ τῆς ΑΒ ἡμικύκλιον τὸ ΑΔΒ, καὶ ἤχθω ἀπὸ τοῦ Γ τῇ ΑΒ πρὸς ορθὰς γωνίας εὐθεῖα γραμμὴ ἡ ΓΔ, καί ἐπεζεύχθω ἡ ΔΒ, καὶ ἐκκείσθω κύκλος ὁ ΕΖΗΘΚ, οὗ ἡ ἐν τοῦ κέντρου ἴση ἔστω τῇ ΔΒ, καὶ ἐγγεγράφθω εἰς τὸν ΕΖΗΘΚ κύκλον πεντάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον τὸ ΕΖΗΘΚ, καὶ τετμήσθωσαν αἱ ΕΖ, ΖΗ, ΗΘ, ΘΚ, ΚΕ περιφέρειαι δίχα κατὰ τὸ Λ, Μ, Ν, Ξ, Ο σημεῖα, καὶ ἐπεζεύχθωσαν αἱ ΛΜ, ΜΝ, ΝΞ, ΞΟ, ΟΛ, ΕΟ. ἰσόπλευρον ἄρα ἐστὶ καὶ τὸ ΛΜΝΞΟ πεντάγωνον, καὶ δεκαγώνου ἡ ΕΟ εὐθεῖα. καὶ ἀνεστάτωσαν ἄπὸ τῶν Ε, Ζ, Η, Θ, Κ σημείων τῷ τοῦ κύκλου ἐπιπέδῳ πρὸς ὀρθὰς γωνίας εὐθεῖαι αἱ ΕΠ, ΖΡ, ΗΣ, ΘΤ, ΚΥ ἴσαι οὖσαι τῇ ἐκ τοῦ κέντρου τοῦ ΕΖΗΘΚ κύκλου, καὶ ἐπεζεύχθωσαν αἱ ΠΡ, ΡΣ, ΣΤ, ΤΥ, ΥΠ, ΠΛ, ΛΡ, ΡΜ, ΜΣ, ΣΝ, ΝΤ, ΤΞ, ΞΥ, ΥΟ, ΟΠ.

Καὶ ἐπεὶ ἑκατέρα τῶν ΕΠ, ΚΥ τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν, παράλληλος ἄρα ἐστὶν ἡ ΕΠ τῇ ΚΥ. ἔστι δὲ αὐτῇ καὶ ἴση· αἱ δὲ τὰς ἴσας τε καὶ παραλλήλους ἐπιζευγνύουσαι ἐπὶ τὰ αὐτὰ μέρη εὐθεῖαι ἴσαι τε καὶ παράλληλοί εἰσιν. ἡ ΠΥ ἄρα τῇ ΕΚ ἴση τε καὶ παράλληλός ἐστιν. πενταγώνου δὲ ἰσοπλεύρου ἡ ΕΚ· πενταγώνου ἄρα ἰσοπλεύρου καὶ ἡ ΠΥ τοῦ εἰς τὸν ΕΖΗΘΚ κύκλον ἐγγραφομένου. διὰ τὰ αὐτὰ δὴ καὶ ἑκάστη τῶν ΠΡ, ΡΣ, ΣΤ, ΤΥ πενταγώνου ἐστὶν ἰσοπλεύρου τοῦ εἰς τὸν ΕΖΗΘΚ κύκλον ἐγγραφομένου· ἰσόπλευρον ἄρα τὸ ΠΡΣΤΥ πεντάγωνον. καὶ ἐπεὶ ἑξαγώνου μέν ἐστιν ἡ ΠΕ, δεκαγώνου δὲ ἡ ΕΟ, καί ἐστιν ὀρθὴ ἡ ὑπὸ ΠΕΟ, πενταγώνου ἄρα ἐστὶν ἡ ΠΟ· ἡ γὰρ τοῦ πενταγώνου πλευρὰ δύναται τήν τε τοῦ ἑξαγώνου καὶ τὴν τοῦ δεκαγώνου τῶν εἰς τὸν αὐτὸν κύκλον ἐγγραφομένων. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΟΥ πενταγώνου ἐστὶ

Proposition 16

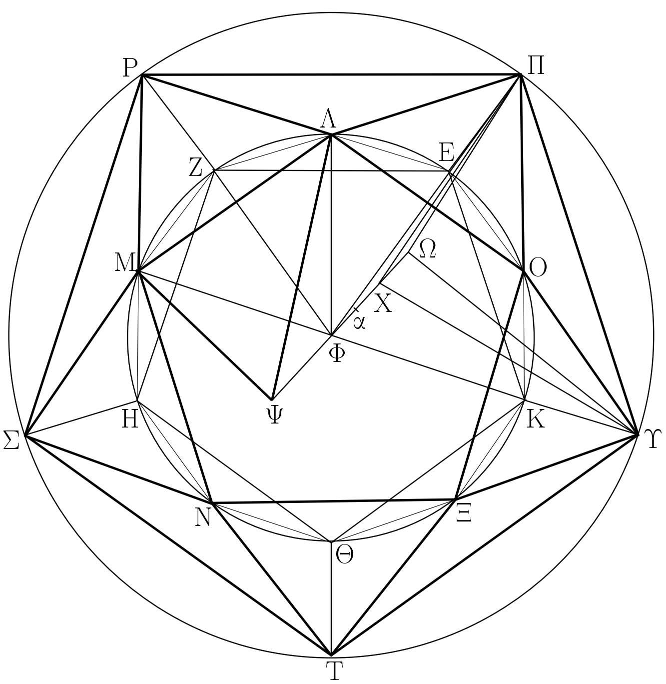
To construct an icosahedron, and to enclose (it) in a sphere, like the aforementioned figures, and to show that the side of the icosahedron is that irrational (straight-line) called minor.

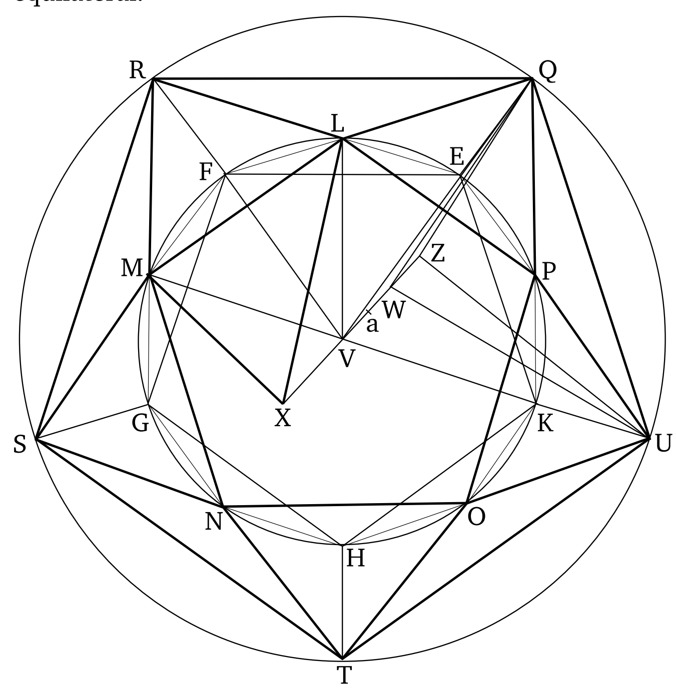


Let the diameter AB of the given sphere be laid out, and let it have been cut at C such that AC is four times CB [Prop. 6.10]. And let the semi-circle ADB have been drawn on AB. And let the straight-line CD have been drawn from C at right-angles to AB. And let DB have been joined. And let the circle $EFGHK$ be set down, and let its radius be equal to DB. And let the equilateral and equiangular pentagon $EFGHK$ have been inscribed in circle $EFGHK$ [Prop. 4.11]. And let the circumferences EF, FG, GH, HK, and KE have been cut in half at points L, M, N, O, and P (respectively). And let LM, MN, NO, OP, PL, and EP have been joined. Thus, pentagon $LMNOP$ is also equilateral, and EP (is) the side of the decagon (inscribed in the circle). And let the straight-lines EQ, FR, GS, HT, and KU, which are equal to the radius of circle $EFGHK$, have been set up at right-angles to the plane of the circle, at points E, F, G, H, and K (respectively). And let QR, RS, ST, TU, UQ, QL, LR, RM, MS, SN, NT, TO, OU, UP, and PQ have been joined.

And since EQ and KU are each at right-angles to the same plane, EQ is thus parallel to KU [Prop. 11.6]. And it is also equal to it. And straight-lines joining equal and parallel (straight-lines) on the same side are (themselves) equal and parallel [Prop. 1.33]. Thus, QU is equal and parallel to EK. And EK (is the side) of an equilateral pentagon (inscribed in circle $EFGHK$). Thus, QU (is) also the side of an equilateral pentagon inscribed in circle $EFGHK$. So, for the same (reasons), QR, RS, ST, and TU are also the sides of an equilateral pentagon inscribed in circle $EFGHK$. Pentagon $QRSTU$ (is) thus equilatπλευρά. ἔστι δὲ καὶ ἡ ΠΥ πενταγώνου· ἰσόπλευρον ἄρα ἐστὶ τὸ ΠΟΥ τρίγωνον. διὰ τὰ αὐτὰ δὴ καὶ ἕκαστον τῶν ΠΛΡ, ΡΜΣ, ΣΝΤ, ΤΞΥ ἰσόπλευρόν ἐστιν. καὶ ἐπεὶ πενταγώνου ἐδείχθη ἑκατέρα τῶν ΠΛ, ΠΟ, ἔστι δὲ καὶ ἡ ΛΟ πενταγώνου, ἰσόπλευρον ἄρα ἐστὶ τὸ ΠΛΟ τρίγωνον. διὰ τὰ αὐτὰ δὴ καὶ ἕκαστον τῶν ΛΡΜ, ΜΣΝ, ΝΤΞ, ΞΥΟ τριγώνων ἰσόπλευρόν ἐστιν.

eral. And side QE is (the side) of a hexagon (inscribed in circle $EFGHK$), and EP (the side) of a decagon, and (angle) QEP is a right-angle, thus QP is (the side) of a pentagon (inscribed in the same circle). For the square on the side of a pentagon is (equal to the sum of) the (squares) on (the sides of) a hexagon and a decagon inscribed in the same circle [Prop. 13.10]. So, for the same (reasons), PU is also the side of a pentagon. And QU is also (the side) of a pentagon. Thus, triangle QPU is equilateral. So, for the same (reasons), (triangles) QLR, RMS, SNT, and TOU are each also equilateral. And since QL and QP were each shown (to be the sides) of a pentagon, and LP is also (the side) of a pentagon, triangle QLP is thus equilateral. So, for the same (reasons), triangles LRM, MSN, NTO, and OUP are each also equilateral.





Εἰλήφθω τὸ κέντρον τοῦ ΕΖΗΘΚ κύκλου τὸ Φ σημεῖον· καὶ ἀπὸ τοῦ Φ τῷ τοῦ κύκλου ἐπιπέδῳ πρὸς ὀρθὰς ἀνεστάτω ἡ ΦΩ, καὶ ἐκβεβλήσθω ἐπὶ τὰ ἕτερα μέρη ὡς ἡ ΦΨ, καὶ ἀφῃρήσθω ἑξαγώνου μὲν ἡ ΦΧ, δεκαγώνου δὲ ἑκατέρα τῶν ΦΨ, ΧΩ, καὶ ἐπεζεύχθωσαν αἱ ΠΩ, ΠΧ, ΥΩ, ΕΦ, ΛΦ, ΛΨ, ΨΜ.

Καὶ ἐπεὶ ἑκατέρα τῶν ΦΧ, ΠΕ τῷ τοῦ κύκλου ἐπιπέδῳ πρὸς ὀρθάς ἐστιν, παράλληλος ἄρα ἐστὶν ἡ ΦΧ τῇ ΠΕ. εἰσὶ δὲ καὶ ἴσαι· καὶ αἱ ΕΦ, ΠΧ ἄρα ἴσαι τε καὶ παράλληλοί εἰσιν. ἑξαγώνου δὲ ἡ ΕΦ· ἑξαγώνου ἄρα καὶ ἡ ΠΧ. καὶ ἐπεὶ ἑξαγώνου μέν ἐστιν ἡ ΠΧ, δεκαγώνου δὲ ἡ ΧΩ, καὶ ὀρθή ἐστιν ἡ ὑπὸ ΠΧΩ γωνία, πενταγώνου ἄρα ἐστὶν ἡ ΠΩ. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΥΩ πενταγώνου ἐστίν, ἐπειδήπερ,

Let the center, point V, of circle $EFGHK$ have been found [Prop. 3.1]. And let VZ have been set up, at (point) V, at right-angles to the plane of the circle. And let it have been produced on the other side (of the circle), like VX. And let VW have been cut off (from XZ so as to be equal to the side) of a hexagon, and each of VX and WZ (so as to be equal to the side) of a decagon. And let QZ, QW, UZ, EV, LV, LX, and XM have been joined.

And since VW and QE are each at right-angles to the plane of the circle, VW is thus parallel to QE [Prop. 11.6]. And they are also equal. EV and QW are thus equal and parallel (to one another) [Prop. 1.33].

ἐὰν ἐπιζεύξωμεν τὰς ΦΚ, ΧΥ, ἴσαι καὶ ἀπεναντίον ἔσονται, καί ἐστιν ἡ ΦΚ ἐκ τοῦ κέντρου οὖσα ἑξαγώνου. ἑξαγώνου ἄρα καὶ ἡ ΧΥ. δεκαγώνου δὲ ἡ ΧΩ, καὶ ὀρθὴ ἡ ὑπὸ ΥΧΩ· πενταγώνου ἄρα ἡ ΥΩ. ἔστι δὲ καὶ ἡ ΠΥ πενταγώνου· ἰσόπλευρον ἄρα ἐστὶ τὸ ΠΥΩ τρίγωνον. διὰ τὰ αὐτὰ δὴ καὶ ἕκαστον τῶν λοιπῶν τριγώνων, ὧν βάσεις μέν εἰσιν αἱ ΠΡ, ΡΣ, ΣΤ, ΤΥ εὐθεῖαι, κορυφὴ δὲ τὸ Ω σημεῖον, ἰσόπλευρόν ἐστιν. πάλιν, ἐπεὶ ἑξαγώνου μὲν ἡ ΦΛ, δεκαγώνου δὲ ἡ ΦΨ, καὶ ὀρθή ἐστιν ἡ ὑπὸ ΛΦΨ γωνία, πενταγώνου ἄρα ἐστὶν ἡ ΛΨ. διὰ τὰ αὐτὰ δὴ ἐὰν ἐπιζεύξωμεν τὴν ΜΦ οὖσαν ἑξαγώνου, συνάγεται καὶ ἡ ΜΨ πενταγώνου. ἔστι δὲ καὶ ἡ ΛΜ πενταγώνου· ἰσόπλευρον ἄρα ἐστὶ τὸ ΛΜΨ τρίγωνον. ὁμοίως δὴ δειχθήσεται, ὅτι καὶ ἕκαστον τῶν λοιπῶν τριγώνων, ὧν βάσεις μέν εἰσιν αἱ ΜΝ, ΝΞ, ΞΟ, ΟΛ, κορυφὴ δὲ τὸ Ψ σημεῖον, ἰσόπλευρόν ἐστιν. συνέσταται ἄρα εἰκοσάεδρον ὑπὸ εἴκοσι τριγώνων ἰσοπλεύρων περιεχόμενον.

Δεῖ δὴ αὐτὸ καὶ σφαίρᾳ περιλαβεῖν τῇ δοθείσῃ καὶ δεῖξαι, ὅτι ἡ τοῦ εἰκοσαέδρου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἐλάσσων.

Ἐπεὶ γὰρ ἑξαγώνου ἐστὶν ἡ ΦΧ, δεκαγώνου δὲ ἡ ΧΩ, ἡ ΦΩ ἄρα ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Χ, καὶ τὸ μεῖζον αὐτῆς τμῆμά ἐστιν ἡ ΦΧ· ἔστιν ἄρα ὡς ἡ ΩΦ πρὸς τὴν ΦΧ, οὕτως ἡ ΦΧ πρὸς τὴν ΧΩ. ἴση δὲ ἡ μὲν ΦΧ τῇ ΦΕ, ἡ δὲ ΧΩ τῇ ΦΨ· ἔστιν ἄρα ὡς ἡ ΩΦ πρὸς τὴν ΦΕ, οὕτως ἡ ΕΦ πρὸς τὴν ΦΨ. καί εἰσιν ὀρθαὶ αἱ ὑπὸ ΩΦΕ, ΕΦΨ γωνίαι· ἐὰν ἄρα ἐπιζεύξωμεν τὴν ΕΩ εὐθεῖαν, ὀρθὴ ἔσται ἡ ὑπὸ ΨΕΩ γωνία διὰ τὴν ὁμοιότητα τῶν ΨΕΩ, ΦΕΩ τριγώνων. διὰ τὰ αὐτὰ δὴ ἐπεί ἐστιν ὡς ἡ ΩΦ πρὸς τὴν ΦΧ, οὕτως ἡ ΦΧ πρὸς τὴν ΧΩ, ἴση δὲ ἡ μὲν ΩΦ τῇ ΨΧ, ἡ δὲ ΦΧ τῇ ΧΠ, ἔστιν ἄρα ὡς ἡ ΨΧ πρὸς τὴν ΧΠ, οὕτως ἡ ΠΧ πρὸς τὴν ΧΩ. καὶ διὰ τοῦτο πάλιν ἐὰν ἐπιζεύξωμεν τὴν ΠΨ, ὀρθὴ ἔσται ἡ πρὸς τῷ Π γωνία· τὸ ἄρα ἐπὶ τῆς ΨΩ γραφόμενον ἡμικύκλιον ἥξει καὶ διὰ τοῦ Π. καὶ ἐὰν μενούσης τῆς ΨΩ περιενεχθὲν τὸ ἡμικύκλιον εἰς τὸ αὐτὸ πάλιν ἀποκατασταθῇ, ὅθεν ἤρξατο φέρεσθαι, ἥξει καὶ διὰ τοῦ Π καὶ τῶν λοιπῶν σημείων τοῦ εἰκοσαέδρου, καὶ ἔσται σφαίρᾳ περιειλημμένον τὸ εἰκοσάεδρον. λέγω δή, ὅτι καὶ τῇ δοθείσῃ. τετμήσθω γὰρ ἡ ΦΧ δίχα κατὰ τὸ α. καὶ ἐπεὶ εὐθεῖα γραμμὴ ἡ ΦΩ ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Χ, καὶ τὸ ἔλασσον αὐτῆς τμῆμά ἐστιν ἡ ΩΧ, ἡ ἄρα ΩΧ προσλαβοῦσα τὴν ἡμίσειαν τοῦ μείζονος τμήματος τὴν Χα πενταπλάσιον δύναται τοῦ ἀπὸ τῆς ἡμισείας τοῦ μείζονος τμήματος· πενταπλάσιον ἄρα ἐστὶ τὸ ἀπὸ τῆς Ωα τοῦ ἀπὸ τῆς αΧ. καί ἐστι τῆς μὲν Ωα διπλῆ ἡ ΩΨ, τῆς δὲ αΧ διπλῆ ἡ ΦΧ· πενταπλάσιον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΩΨ τοῦ ἀπὸ τῆς ΧΦ. καὶ ἐπεὶ τετραπλῆ ἐστιν ἡ ΑΓ τῆς ΓΒ, πενταπλῆ ἄρα ἐστὶν ἡ ΑΒ τῆς ΒΓ. ὡς δὲ ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως τὸ ἀπὸ τῆς ΑΒ πρὸς τὸ ἀπὸ τῆς ΒΔ· πενταπλάσιον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΑΒ τοῦ ἀπὸ τῆς ΒΔ. ἐδείχθη δὲ καὶ τὸ ἀπὸ τῆς ΩΨ πενταπλάσιον τοῦ ἀπὸ τῆς ΦΧ. καί ἐστιν ἴση ἡ ΔΒ τῇ

And EV (is the side) of a hexagon. Thus, QW (is) also (the side) of a hexagon. And since QW is (the side) of a hexagon, and WZ (the side) of a decagon, and angle QWZ is a right-angle [Def. 11.3, Prop. 1.29], QZ is thus (the side) of a pentagon [Prop. 13.10]. So, for the same (reasons), UZ is also (the side) of a pentagon—inasmuch as, if we join VK and WU then they will be equal and opposite. And VK, being (equal) to the radius (of the circle), is (the side) of a hexagon [Prop. 4.15 corr.]. Thus, WU (is) also the side of a hexagon. And WZ (is the side) of a decagon, and (angle) UWZ (is) a right-angle. Thus, UZ (is the side) of a pentagon [Prop. 13.10]. And QU is also (the side) of a pentagon. Triangle QUZ is thus equilateral. So, for the same (reasons), each of the remaining triangles, whose bases are the straight-lines QR, RS, ST, and TU, and apexes the point Z, are also equilateral. Again, since VL (is the side) of a hexagon, and VX (the side) of a decagon, and angle LVX is a right-angle, LX is thus (the side) of a pentagon [Prop. 13.10]. So, for the same (reasons), if we join MV, which is (the side) of a hexagon, MX is also inferred (to be the side) of a pentagon. And LM is also (the side) of a pentagon. Thus, triangle LMX is equilateral. So, similarly, it can be shown that each of the remaining triangles, whose bases are the (straight-lines) MN, NO, OP, and PL, and apexes the point X, are also equilateral. Thus, an icosahedron contained by twenty equilateral triangles has been constructed.

So, it is also necessary to enclose it in the given sphere, and to show that the side of the icosahedron is that irrational (straight-line) called minor.

For, since VW is (the side) of a hexagon, and WZ (the side) of a decagon, VZ has thus been cut in extreme and mean ratio at W, and VW is its greater piece [Prop. 13.9]. Thus, as ZV is to VW, so VW (is) to WZ. And VW (is) equal to VE, and WZ to VX. Thus, as ZV is to VE, so EV (is) to VX. And angles ZVE and EVX are right-angles. Thus, if we join straight-line EZ then angle XEZ will be a right-angle, on account of the similarity of triangles XEZ and VEZ. [Prop. 6.8]. So, for the same (reasons), since as ZV is to VW, so VW (is) to WZ, and ZV (is) equal to XW, and VW to WQ, thus as XW is to WQ, so QW (is) to WZ. And, again, on account of this, if we join QX then the angle at Q will be a right-angle [Prop. 6.8]. Thus, the semi-circle drawn on XZ will also pass through Q [Prop. 3.31]. And if XZ remains fixed, and the semi-circle is carried around, and again established at the same (position) from which it began to be moved, then it will also pass through (point) Q, and (through) the remaining (angular) points of the icosahedron. And the icosahedron will have been en-

ΦΧ· ἑκατέρα γὰρ αὐτῶν ἴση ἐστὶ τῇ ἐκ τοῦ κέντρου τοῦ ΕΖΗΘΚ κύκλου· ἴση ἄρα καὶ ἡ ΑΒ τῇ ΨΩ. καί ἐστιν ἡ ΑΒ ἡ τῆς δοθείσης σφαίρας διάμετρος· καὶ ἡ ΨΩ ἄρα ἴση ἐστὶ τῇ τῆς δοθείσης σφαίρας διαμέτρῳ· τῇ ἄρα δοθείσῃ σφαίρᾳ περιείληπται τὸ εἰκοσάεδρον.

Λέγω δή, ὅτι ἡ τοῦ εἰκοσαέδρου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἐλάττων. ἐπεὶ γὰρ ῥητή ἐστιν ἡ τῆς σφαίρας διάμετρος, καί ἐστι δυνάμει πενταπλασίων τῆς ἐκ τοῦ κέντρου τοῦ ΕΖΗΘΚ κύκλου, ῥητὴ ἄρα ἐστὶ καὶ ἡ ἐκ τοῦ κέντρου τοῦ ΕΖΗΘΚ κύκλου· ὥστε καὶ ἡ διάμετρος αὐτοῦ ῥητή ἐστιν. ἐὰν δὲ εἰς κύκλον ῥητὴν ἔχοντα τὴν διάμετρον πεντάγωνον ἰσόπλευρον ἐγγραφῇ, ἡ τοῦ πενταγώνου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἐλάττων. ἡ δὲ τοῦ ΕΖΗΘΚ πενταγώνου πλευρὰ ἡ τοῦ εἰκοσαέδρου ἐστίν. ἡ ἄρα τοῦ είκοσαέδρου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἐλάττων.

closed by a sphere. So, I say that (it is) also (enclosed) by the given (sphere). For let VW have been cut in half at a. And since the straight-line VZ has been cut in extreme and mean ratio at W, and ZW is its lesser piece, then the square on ZW added to half of the greater piece, Wa, is five times the (square) on half of the greater piece [Prop. 13.3]. Thus, the (square) on Za is five times the (square) on aW. And ZX is double Za, and VW double aW. Thus, the (square) on ZX is five times the (square) on WV. And since AC is four times CB, AB is thus five times BC. And as AB (is) to BC, so the (square) on AB (is) to the (square) on BD [Prop. 6.8, Def. 5.9]. Thus, the (square) on AB is five times the (square) on BD. And the (square) on ZX was also shown (to be) five times the (square) on VW. And DB is equal to VW. For each of them is equal to the radius of circle $EFGHK$. Thus, AB (is) also equal to XZ. And AB is the diameter of the given sphere. Thus, XZ is equal to the diameter of the given sphere. Thus, the icosahedron has been enclosed by the given sphere.

So, I say that the side of the icosahedron is that irrational (straight-line) called minor. For since the diameter of the sphere is rational, and the square on it is five times the (square) on the radius of circle $EFGHK$, the radius of circle $EFGHK$ is thus also rational. Hence, its diameter is also rational. And if an equilateral pentagon is inscribed in a circle having a rational diameter then the side of the pentagon is that irrational (straight-line) called minor [Prop. 13.11]. And the side of pentagon $EFGHK$ is (the side) of the icosahedron. Thus, the side of the icosahedron is that irrational (straight-line) called minor.

Πόρισμα.

Ἐκ δὴ τούτου φανερόν, ὅτι ἡ τῆς σφαίρας διάμετρος δυνάμει πενταπλασίων ἐστὶ τῆς ἐκ τοῦ κέντρου τοῦ κύκλου, ἀφ᾽ οὗ τὸ εἰκοσάεδρον ἀναγέγραπται, καὶ ὅτι ἡ τῆς σφαίρας διάμετρος σύγκειται ἔκ τε τῆς τοῦ ἑξαγώνου καὶ δύο τῶν τοῦ δεκαγώνου τῶν εἰς τὸν αὐτὸν κύκλον ἐγγραφομένων. ὅπερ ἔδει δεῖξαι.

Corollary

So, (it is) clear, from this, that the square on the diameter of the sphere is five times the square on the radius of the circle from which the icosahedron has been described, and that the the diameter of the sphere is the sum of (the side) of the hexagon, and two of (the sides) of the decagon, inscribed in the same circle.†

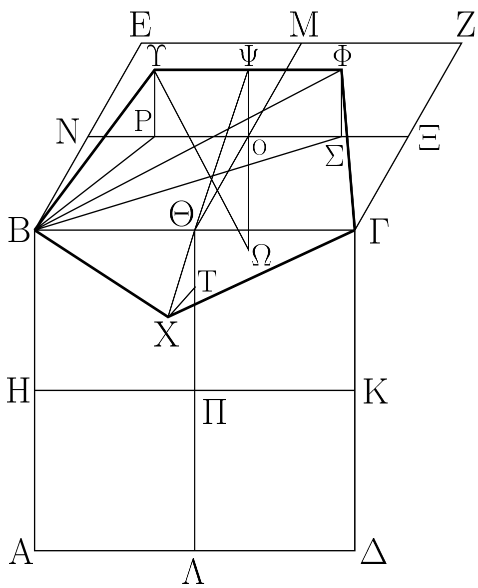
† If the radius of the sphere is unity then the radius of the circle is $2/\sqrt{5}$, and the sides of the hexagon, decagon, and pentagon/icosahedron are $2/\sqrt{5}$, $1-1/\sqrt{5}$, and $(1/\sqrt{5})\,\sqrt{10-2\,\sqrt{5}}$, respectively.

ιζ´.

Δωδεκάεδρον συστήσασθαι καὶ σφαίρᾳ περιλαβεῖν, ᾗ καὶ τὰ προειρημένα σχήματα, καὶ δεῖξαι, ὅτι ἡ τοῦ δωδεκαέδρου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἀποτομή.

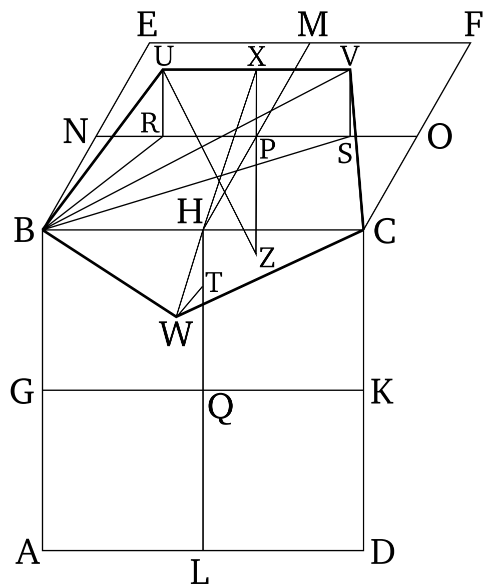
Proposition 17

To construct a dodecahedron, and to enclose (it) in a sphere, like the aforementioned figures, and to show that the side of the dodecahedron is that irrational (straight-line) called an apotome.



Ἐκκείσθωσαν τοῦ προειρημένου κύβου δύο ἐπίπεδα πρὸς ὀρθὰς ἀλλήλοις τὰ ΑΒΓΔ, ΓΒΕΖ, καὶ τετμήσθω ἑκάστη τῶν ΑΒ, ΒΓ, ΓΔ, ΔΑ, ΕΖ, ΕΒ, ΖΓ πλευρῶν δίχα κατὰ τὰ Η, Θ, Κ, Λ, Μ, Ν, Ξ, καὶ ἐπεζεύχθωσαν αἱ ΗΚ, ΘΛ, ΜΘ, ΝΞ, καὶ τετηήσθω ἑκάστη τῶν ΝΟ, ΟΞ, ΘΠ ἄκρον καὶ μέσον λόγον κατὰ τὰ Ρ, Σ, Τ σημεῖα, καὶ ἔστω αὐτῶν μείζονα τμήματα τὰ ΡΟ, ΟΣ, ΤΠ, καὶ ἀνεστάτωσαν ἀπὸ τῶν Ρ, Σ, Τ σημείων τοῖς τοῦ κύβου ἐπιπέδοις πρὸς ὀρθὰς ἐπὶ τὰ ἐκτὸς μέρη τοῦ κύβου αἱ ΡΥ, ΣΦ, ΤΧ, καὶ κείσθωσαν ἴσαι ταῖς ΡΟ, ΟΣ, ΤΠ, καὶ ἐπεζεύχθωσαν αἱ ΥΒ, ΒΧ, ΧΓ, ΓΦ, ΦΥ.

Λέγω, ὅτι τὸ ΥΒΧΓΦ πεντάγωνον ἰσόπλευρόν τε καὶ ἐν ἑνὶ ἐπιπέδῳ καὶ ἔτι ἰσογώνιόν ἐστιν. ἐπεζεύχθωσαν γὰρ αἱ ΡΒ, ΣΒ, ΦΒ. καὶ ἐπεὶ εὐθεῖα ἡ ΝΟ ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Ρ, καὶ τὸ μεῖζον τμῆμά ἐστιν ἡ ΡΟ, τὰ ἄρα ἀπὸ τῶν ΟΝ, ΝΡ τριπλάσιά ἐστι τοῦ ἀπὸ τῆς ΡΟ. ἴση δὲ ἡ μὲν ΟΝ τῇ ΝΒ, ἡ δὲ ΟΡ τῇ ΡΥ· τὰ ἄρα ἀπὸ τῶν ΒΝ, ΝΡ τριπλάσιά ἐστι τοῦ ἀπὸ τῆς ΡΥ. τοῖς δὲ ἀπὸ τῶν ΒΝ, ΝΡ τὸ ἀπὸ τῆς ΒΡ ἐστιν ἴσον· τὸ ἄρα ἀπὸ τῆς ΒΡ τριπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΡΥ· ὥστε τὰ ἀπὸ τῶν ΒΡ, ΡΥ τετραπλάσιά ἐστι τοῦ ἀπὸ τῆς ΡΥ. τοῖς δὲ ἀπὸ τῶν ΒΡ, ΡΥ ἴσον ἐστι τὸ ἀπὸ τῆς ΒΥ· τὸ ἄρα ἄπὸ τῆς ΒΥ τετραπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΥΡ· διπλῆ ἄρα ἐστὶν ἡ ΒΥ τῆς ΡΥ. ἔστι δὲ καὶ ἡ ΦΥ τῆς ΥΡ διπλῆ, ἐπειδήπερ καὶ ἡ ΣΡ τῆς ΟΡ, τουτέστι τῆς ΡΥ, ἐστι διπλῆ· ἴση ἄρα ἡ ΒΥ τῇ ΥΦ. ὁμοίως δὴ δειχθήσεται, ὅτι καὶ ἑκάστη τῶν ΒΧ, ΧΓ, ΓΦ ἑκατέρᾳ τῶν ΒΥ, ΥΦ ἐστιν ἴση. ἰσόπλευρον ἄρα ἐστὶ τὸ ΒΥΦΓΧ πεντάγωνον. λέγω δή, ὅτι καὶ ἐν ἑνί ἐστιν ἐπιπέδῳ. ἤχθω γὰρ ἀπὸ τοῦ Ο ἑκατέρᾳ τῶν ΡΥ, ΣΦ παράλληλος ἐπὶ τὰ ἐκτὸς τοῦ κύβου μέρη ἡ ΟΨ, καὶ ἐπεζεύχθωσαν αἱ ΨΘ, ΘΧ· λέγω, ὅτι ἡ ΨΘΧ εὐθεῖά ἐστιν. ἐπεὶ γὰρ ἡ ΘΠ ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Τ, καὶ τὸ μεῖζον αὐτῆς τμῆμά ἐστιν ἡ ΠΤ, ἔστιν ἄρα ὡς ἡ ΘΠ πρὸς τὴν ΠΤ, οὕτως ἡ ΠΤ πρὸς τὴν



Let two planes of the aforementioned cube [Prop. 13.15], $ABCD$ and $CBEF$, (which are) at right-angles to one another, be laid out. And let the sides AB, BC, CD, DA, EF, EB, and FC have each been cut in half at points G, H, K, L, M, N, and O (respectively). And let GK, HL, MH, and NO have been joined. And let NP, PO, and HQ have each been cut in extreme and mean ratio at points R, S, and T (respectively). And let their greater pieces be RP, PS, and TQ (respectively). And let RU, SV, and TW have been set up on the exterior side of the cube, at points R, S, and T (respectively), at right-angles to the planes of the cube. And let them be made equal to RP, PS, and TQ. And let UB, BW, WC, CV, and VU have been joined.

I say that the pentagon $UBWCV$ is equilateral, and in one plane, and, further, equiangular. For let RB, SB, and VB have been joined. And since the straight-line NP has been cut in extreme and mean ratio at R, and RP is the greater piece, the (sum of the squares) on PN and NR is thus three times the (square) on RP [Prop. 13.4]. And PN (is) equal to NB, and PR to RU. Thus, the (sum of the squares) on BN and NR is three times the (square) on RU. And the (square) on BR is equal to the (sum of the squares) on BN and NR [Prop. 1.47]. Thus, the (square) on BR is three times the (square) on RU. Hence, the (sum of the squares) on BR and RU is four times the (square) on RU. And the (square) on BU is equal to the (sum of the squares) on BR and RU [Prop. 1.47]. Thus, the (square) on BU is four times the (square) on UR. Thus, BU is double RU. And VU is also double UR, inasmuch as SR is also double PR—that is to say, RU. Thus, BU (is) equal to UV. So, similarly, it can be shown that each of BW, WC, CV is equal to each

ΤΘ. ἴση δὲ ἡ μὲν ΘΠ τῇ ΘΟ, ἡ δὲ ΠΤ ἑκατέρᾳ τῶν ΤΧ, ΟΨ· ἔστιν ἄρα ὡς ἡ ΘΟ πρὸς τὴν ΟΨ, οὕτως ἡ ΧΤ πρὸς τὴν ΤΘ. καί ἐστι παράλληλος ἡ μὲν ΘΟ τῇ ΤΧ· ἑκατέρα γὰρ αὐτῶν τῷ ΒΔ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν· ἡ δὲ ΤΘ τῇ ΟΨ· ἑκατέρα γὰρ αὐτῶν τῷ ΒΖ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν. ἐὰν δὲ δύο τρίγωνα συντεθῇ κατὰ μίαν γωνίαν, ὡς τὰ ΨΟΘ, ΘΤΧ, τὰς δύο πλευρὰς ταῖς δυσὶν ἀνάλογον ἔχοντα, ὥστε τὰς ὁμολόγους αὐτῶν πλευρὰς καὶ παραλλήλους εἶναι, αἱ λοιπαὶ εὐθεῖαι ἐπ᾽ εὐθείας ἔσονται· ἐπ᾽ εὐθείας ἄρα ἐστὶν ἡ ΨΘ τῇ ΘΧ. πᾶσα δὲ εὐθεῖα ἐν ἑνί ἐστιν ἐπιπέδῳ· ἐν ἑνὶ ἄρα ἐπιπέδῳ ἐστὶ τὸ ΥΒΧΓΦ πεντάγωνον.

Λέγω δή, ὅτι καὶ ἰσογώνιόν ἐστιν.

Ἐπεὶ γὰρ εὐθεῖα γραμμὴ ἡ ΝΟ ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Ρ, καὶ τὸ μεῖζον τμῆμά ἐστιν ἡ ΟΡ [ἔστιν ἄρα ὡς συναμφότερος ἡ ΝΟ, ΟΡ πρὸς τὴν ΟΝ, οὕτως ἡ ΝΟ πρὸς τὴν ΟΡ], ἴση δὲ ἡ ΟΡ τῇ ΟΣ [ἔστιν ἄρα ὡς ἡ ΣΝ πρὸς τὴν ΝΟ, οὕτως ἡ ΝΟ πρὸς τὴν ΟΣ], ἡ ΝΣ ἄρα ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Ο, καὶ τὸ μεῖζον τμῆμά ἐστιν ἡ ΝΟ· τὰ ἄρα ἀπὸ τῶν ΝΣ, ΣΟ τριπλάσιά ἐστι τοῦ ἀπὸ τῆς ΝΟ. ἴση δὲ ἡ μὲν ΝΟ τῇ ΝΒ, ἡ δὲ ΟΣ τῇ ΣΦ· τὰ ἄρα ἀπὸ τῶν ΝΣ, ΣΦ τετράγωνα τριπλάσιά ἐστι τοῦ ἀπὸ τῆς ΝΒ· ὥστε τὰ ἀπὸ τῶν ΦΣ, ΣΝ, ΝΒ τετραπλάσιά ἐστι τοῦ ἀπὸ τῆς ΝΒ. τοῖς δὲ ἀπὸ τῶν ΣΝ, ΝΒ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΣΒ· τὰ ἄρα ἀπὸ τῶν ΒΣ, ΣΦ, τουτέστι τὸ ἀπὸ τῆς ΒΦ [ὀρθὴ γὰρ ἡ ὑπὸ ΦΣΒ γωνία], τετραπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΝΒ· διπλῆ ἄρα ἐστὶν ἡ ΦΒ τῆς ΒΝ. ἔστι δὲ καὶ ἡ ΒΓ τῆς ΒΝ διπλῆ· ἴση ἄρα ἐστὶν ἡ ΒΦ τῇ ΒΓ. καὶ ἐπεὶ δύο αἱ ΒΥ, ΥΦ δυσὶ ταῖς ΒΧ, ΧΓ ἴσαι εἰσίν, καὶ βάσις ἡ ΒΦ βάσει τῇ ΒΓ ἴση, γωνία ἄρα ἡ ὑπὸ ΒΥΦ γωνίᾳ τῇ ὑπὸ ΒΧΓ ἐστιν ἴση. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἡ ὑπὸ ΥΦΓ γωνία ἴση ἐστὶ τῇ ὑπὸ ΒΧΓ· αἱ ἄρα ὑπὸ ΒΧΓ, ΒΥΦ, ΥΦΓ τρεῖς γωνίαι ἴσαι ἀλλήλαις εἰσίν. ἐὰν δὲ πενταγώνου ἰσοπλεύρου αἱ τρεῖς γωνίαι ἴσαι ἀλλήλαις ὦσιν, ἰσογώνιον ἔσται τὸ πεντάγωνον· ἰσογώνιον ἄρα ἐστὶ τὸ ΒΥΦΓΧ πεντάγωνον. ἐδείχθη δὲ καὶ ἰσόπλευρον· τὸ ἄρα ΒΥΦΓΧ πεντάγωνον ἰσόπλευρόν ἐστι καὶ ἰσογώνιον, καί ἐστιν ἐπὶ μιᾶς τοῦ κύβου πλευρᾶς τῆς ΒΓ. ἐὰν ἄρα ἐφ᾽ ἑκάστης τῶν τοῦ κύβου δώδεκα πλευρῶν τὰ αὐτὰ κατασκευάσωμεν, συσταθήσεταί τι σχῆμα στερεὸν ὑπὸ δώδεκα πενταγώνων ἰσοπλεύρων τε καὶ ἰσογωνίων περιεχόμενον, ὃ καλεῖται δωδεκάεδρον.

Δεῖ δὴ αὐτὸ καὶ σφαίρᾳ περιλαβεῖν τῇ δοθείσῃ καὶ δεῖξαι, ὅτι ἡ τοῦ δωδεκαέδρου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἀποτομή.

Ἐκβεβλήσθω γὰρ ἡ ΨΟ, καὶ ἔστω ἡ ΨΩ· συμβάλλει ἄρα ἡ ΟΩ τῇ τοῦ κύβου διαμέτρῳ, καὶ δίχα τέμνουσιν ἀλλήλας· τοῦτο γὰρ δέδεικται ἐν τῷ παρατελεύτῳ θεωρήματι τοῦ ἑνδεκάτου βιβλίου. τεμνέτωσαν κατὰ τὸ Ω· τὸ Ω ἄρα κέντρον ἐστὶ τῆς σφαίρας τῆς περιλαμβανούσης τὸν κύβον, καὶ ἡ ΩΟ ἡμίσεια τῆς πλευρᾶς τοῦ κύβου. ἐπεζεύχθω δὴ ἡ ΥΩ. καὶ ἐπεὶ εὐθεῖα γραμμὴ ἡ ΝΣ ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Ο, καὶ τὸ μεῖζον αὐτῆς τμῆμά ἐστιν ἡ ΝΟ,

of BU and UV. Thus, pentagon $BUVCW$ is equilateral. So, I say that it is also in one plane. For let PX have been drawn from P, parallel to each of RU and SV, on the exterior side of the cube. And let XH and HW have been joined. I say that XHW is a straight-line. For since HQ has been cut in extreme and mean ratio at T, and QT is its greater piece, thus as HQ is to QT, so QT (is) to TH. And HQ (is) equal to HP, and QT to each of TW and PX. Thus, as HP is to PX, so WT (is) to TH. And HP is parallel to TW. For of each of them is at right-angles to the plane BD [Prop. 11.6]. And TH (is parallel) to PX. For each of them is at right-angles to the plane BF [Prop. 11.6]. And if two triangles, like XPH and HTW, having two sides proportional to two sides, are placed together at a single angle such that their corresponding sides are also parallel then the remaining sides will be straight-on (to one another) [Prop. 6.32]. Thus, XH is straight-on to HW. And every straight-line is in one plane [Prop. 11.1]. Thus, pentagon $UBWCV$ is in one plane.

So, I say that it is also equiangular.

For since the straight-line NP has been cut in extreme and mean ratio at R, and PR is the greater piece [thus as the sum of NP and PR is to PN, so NP (is) to PR], and PR (is) equal to PS [thus as SN is to NP, so NP (is) to PS], NS has thus also been cut in extreme and mean ratio at P, and NP is the greater piece [Prop. 13.5]. Thus, the (sum of the squares) on NS and SP is three times the (square) on NP [Prop. 13.4]. And NP (is) equal to NB, and PS to SV. Thus, the (sum of the) squares on NS and SV is three times the (square) on NB. Hence, the (sum of the squares) on VS, SN, and NB is four times the (square) on NB. And the (square) on SB is equal to the (sum of the squares) on SN and NB [Prop. 1.47]. Thus, the (sum of the squares) on BS and SV—that is to say, the (square) on BV [for angle VSB (is) a right-angle]—is four times the (square) on NB [Def. 11.3, Prop. 1.47]. Thus, VB is double BN. And BC (is) also double BN. Thus, BV is equal to BC. And since the two (straight-lines) BU and UV are equal to the two (straight-lines) BW and WC (respectively), and the base BV (is) equal to the base BC, angle BUV is thus equal to angle BWC [Prop. 1.8]. So, similarly, we can show that angle UVC is equal to angle BWC. Thus, the three angles BWC, BUV, and UVC are equal to one another. And if three angles of an equilateral pentagon are equal to one another then the pentagon is equiangular [Prop. 13.7]. Thus, pentagon $BUVCW$ is equiangular. And it was also shown (to be) equilateral. Thus, pentagon $BUVCW$ is equilateral and equiangular, and it is on one of the sides, BC, of the cube. Thus, if we make the τὰ ἄρα ἀπὸ τῶν ΝΣ, ΣΟ τριπλάσιά ἐστι τοῦ ἀπὸ τῆς ΝΟ. ἴση δὲ ἡ μὲν ΝΣ τῇ ΨΩ, ἐπειδήπερ καὶ ἡ μὲν ΝΟ τῇ ΟΩ ἐστιν ἴση, ἡ δὲ ΨΟ τῇ ΟΣ. ἀλλὰ μὴν καὶ ἡ ΟΣ τῇ ΨΥ, ἐπεὶ καὶ τῇ ΡΟ· τὰ ἄρα ἀπὸ τῶν ΩΨ, ΨΥ τριπλάσιά ἐστι τοῦ ἀπὸ τῆς ΝΟ. τοῖς δὲ ἀπὸ τῶν ΩΨ, ΨΥ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΥΩ· τὸ ἄρα ἀπὸ τῆς ΥΩ τριπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΝΟ. ἔστι δὲ καὶ ἡ ἐκ τοῦ κέντρου τῆς σφαίρας τῆς περιλαμβανούσης τὸν κύβον δυνάμει τριπλασίων τῆς ἡμισείας τῆς τοῦ κύβου πλευρᾶς· προδέδεικται γὰρ κύβον συστήσασθαι καὶ σφαίρᾳ περιλαβεῖν καὶ δεῖξαι, ὅτι ἡ τῆς σφαίρας διάμετρος δυνάμει τριπλασίων ἐστὶ τῆς πλευρᾶς τοῦ κύβου. εἰ δὲ ὅλη τῆς ὅλης, καὶ [ἡ] ἡμίσεια τῆς ἡμισείας· καί ἐστιν ἡ ΝΟ ἡμίσεια τῆς τοῦ κύβου πλευρᾶς· ἡ ἄρα ΥΩ ἴση ἐστὶ τῇ ἐκ τοῦ κέντρου τῆς σφαίρας τῆς περιλαμβανούσης τὸν κύβον. καί ἐστι τὸ Ω κέντρον τῆς σφαίρας τῆς περιλαμβανούσης τὸν κύβον· τὸ Υ ἄρα σημεῖον πρὸς τῇ ἐπιφανείᾳ ἐστι τῆς σφαίρας. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἑκάστη τῶν λοιπῶν γωνιῶν τοῦ δωδεκαέδρου πρὸς τῇ ἐπιφανείᾳ ἐστὶ τῆς σφαίρας· περιείληπται ἄρα τὸ δωδεκάεδρον τῇ δοθείσῃ σφαίρᾳ.

Λέγω δή, ὅτι ἡ τοῦ δωδεκαέδρου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἀποτομή.

Ἐπεὶ γὰρ τῆς ΝΟ ἄκρον καὶ μέσον λόγον τετμημένης τὸ μεῖζον τμῆμά ἐστιν ὁ ΡΟ, τῆς δὲ ΟΞ ἄκρον καὶ μέσον λόγον τετμημένης τὸ μεῖζον τμῆμά ἐστιν ἡ ΟΣ, ὅλης ἄρα τῆς ΝΞ ἄκρον καὶ μέσον λόγον τεμνομένης τὸ μεῖζον τμῆμά ἐστιν ἡ ΡΣ. [οἷον ἐπεί ἐστιν ὡς ἡ ΝΟ πρὸς τὴν ΟΡ, ἡ ΟΡ πρὸς τὴν ΡΝ, καὶ τὰ διπλάσια· τὰ γὰρ μέρη τοῖς ἰσάκις πολλαπλασίοις τὸν αὐτὸν ἔχει λόγον· ὡς ἄρα ἡ ΝΞ πρὸς τὴν ΡΣ, οὕτως ἡ ΡΣ πρὸς συναμφότερον τὴν ΝΡ, ΣΞ. μείζων δὲ ἡ ΝΞ τῆς ΡΣ· μείζων ἄρα καὶ ἡ ΡΣ συναμφοτέρου τῆς ΝΡ, ΣΞ· ἡ ΝΞ ἄρα ἄκρον καὶ μέσον λόγον τέτμηται, καὶ τὸ μεῖζον αὐτῆς τμῆμά ἐστιν ἡ ΡΣ.] ἴση δὲ ἡ ΡΣ τῇ ΥΦ· τῆς ἄρα ΝΞ ἄκρον καὶ μέσον λόγον τεμνομένης τὸ μεῖζον τμῆμά ἐστιν ἡ ΥΦ. καὶ ἐπεὶ ῥητή ἐστιν τῆς σφαίρας διάμετρος καί ἐστι δυνάμει τριπλασίων τῆς τοῦ κύβου πλευρᾶς, ῥητὴ ἄρα ἐστὶν ἡ ΝΞ πλευρὰ οὖσα τοῦ κύβου. ἐὰν δὲ ῥητὴ γραμμὴ ἄκρον καὶ μέσον λόγον τμηθῇ, ἑκάτερον τῶν τμημάτων ἄλογός ἐστιν ἀποτομή.

Ἡ ΥΦ ἄρα πλευρὰ οὖσα τοῦ δωδεκαέδρου ἄλογός ἐστιν ἀποτομή.

same construction on each of the twelve sides of the cube then some solid figure contained by twelve equilateral and equiangular pentagons will have been constructed, which is called a dodecahedron.

So, it is necessary to enclose it in the given sphere, and to show that the side of the dodecahedron is that irrational (straight-line) called an apotome.

For let XP have been produced, and let (the produced straight-line) be XZ. Thus, PZ meets the diameter of the cube, and they cut one another in half. For, this has been proved in the penultimate theorem of the eleventh book [Prop. 11.38]. Let them cut (one another) at Z. Thus, Z is the center of the sphere enclosing the cube, and ZP (is) half the side of the cube. So, let UZ have been joined. And since the straight-line NS has been cut in extreme and mean ratio at P, and its greater piece is NP, the (sum of the squares) on NS and SP is thus three times the (square) on NP [Prop. 13.4]. And NS (is) equal to XZ, inasmuch as NP is also equal to PZ, and XP to PS. But, indeed, PS (is) also (equal) to XU, since (it is) also (equal) to RP. Thus, the (sum of the squares) on ZX and XU is three times the (square) on NP. And the (square) on UZ is equal to the (sum of the squares) on ZX and XU [Prop. 1.47]. Thus, the (square) on UZ is three times the (square) on NP. And the square on the radius of the sphere enclosing the cube is also three times the (square) on half the side of the cube. For it has previously been demonstrated (how to) construct the cube, and to enclose (it) in a sphere, and to show that the square on the diameter of the sphere is three times the (square) on the side of the cube [Prop. 13.15]. And if the (square on the) whole (is three times) the (square on the) whole, then the (square on the) half (is) also (three times) the (square on the) half. And NP is half of the side of the cube. Thus, UZ is equal to the radius of the sphere enclosing the cube. And Z is the center of the sphere enclosing the cube. Thus, point U is on the surface of the sphere. So, similarly, we can show that each of the remaining angles of the dodecahedron is also on the surface of the sphere. Thus, the dodecahedron has been enclosed by the given sphere.

So, I say that the side of the dodecahedron is that irrational straight-line called an apotome.

For since RP is the greater piece of NP, which has been cut in extreme and mean ratio, and PS is the greater piece of PO, which has been cut in extreme and mean ratio, RS is thus the greater piece of the whole of NO, which has been cut in extreme and mean ratio. [Thus, since as NP is to PR, (so) PR (is) to RN, and (the same is also true) of the doubles. For parts have the same ratio as similar multiples (taken in corresponding order) [Prop. 5.15]. Thus, as NO (is) to RS, so RS (is) to the sum of NR and SO. And NO (is) greater than RS. Thus, RS (is) also greater than the sum of NR and SO [Prop. 5.14]. Thus, NO has been cut in extreme and mean ratio, and RS is its greater piece.] And RS (is) equal to UV. Thus, UV is the greater piece of NO, which has been cut in extreme and mean ratio. And since the diameter of the sphere is rational, and the square on it is three times the (square) on the side of the cube, NO, which is the side of the cube, is thus rational. And if a rational (straight)-line is cut in extreme and mean ratio then each of the pieces is the irrational (straight-line called) an apotome.

Thus, UV, which is the side of the dodecahedron, is the irrational (straight-line called) an apotome [Prop. 13.6].

Πόρισμα.

Ἐκ δὴ τούτου φανερόν, ὅτι τῆς τοῦ κύβου πλευρᾶς ἄκρον καὶ μέσον λόγον τεμνομένης τὸ μεῖζον τμῆμά ἐστιν ἡ τοῦ δωδεκαέδρου πλευρά. ὅπερ ἔδει δεῖξαι.

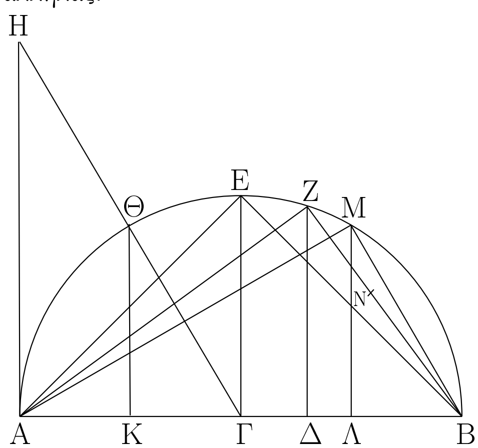
Corollary

So, (it is) clear, from this, that the side of the dodecahedron is the greater piece of the side of the cube, when it is cut in extreme and mean ratio.† (Which is) the very thing it was required to show.

† If the radius of the circumscribed sphere is unity then the side of the cube is $\sqrt{4/3}$, and the side of the dodecahedron is $(1/3)\,(\sqrt{15}-\sqrt{3})$.

ιηʹ.

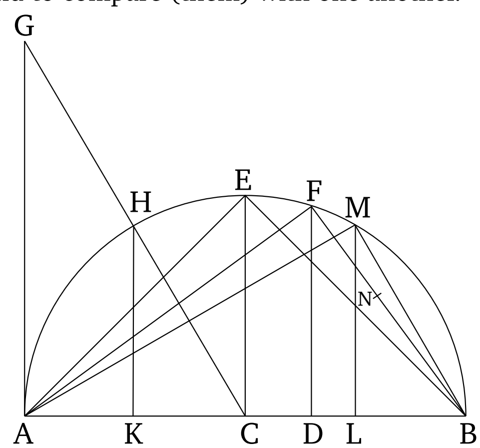
Τὰς πλευρὰς τῶν πέντε σχημάτων ἐκθέσθαι καὶ συγκρῖναι πρὸς ἀλλήλας.



Ἐκκείσθω ἡ τῆς δοθείσης σφαίρας διάμετρος ἡ ΑΒ, καὶ τετμήσθω κατὰ τὸ Γ ὥστε ἴσην εἶναι τὴν ΑΓ τῇ ΓΒ, κατὰ δὲ τὸ Δ ὥστε διπλασίονα εἶναι τὴν ΑΔ τῆς ΔΒ, καὶ γεγράφθω ἐπὶ τῆς ΑΒ ἡμικύκλιον τὸ ΑΕΒ, καὶ ἀπὸ τῶν Γ, Δ τῇ ΑΒ πρὸς ὀρθὰς ἤχθωσαν αἱ ΓΕ, ΔΖ, καὶ ἐπεζεύχθωσαν αἱ ΑΖ, ΖΒ, ΕΒ. καὶ ἐπεὶ διπλῆ ἐστιν ἡ ΑΔ τῆς ΔΒ, τριπλῆ ἄρα ἐστὶν ἡ ΑΒ τῆς ΒΔ. ἀναστρέψαντι ἡμιολία ἄρα ἐστὶν ἡ ΒΑ τῆς ΑΔ. ὡς δὲ ἡ ΒΑ πρὸς τὴν ΑΔ, οὕτως τὸ ἀπὸ τῆς ΒΑ

Proposition 18

To set out the sides of the five (aforementioned) figures, and to compare (them) with one another.†



Let the diameter, AB, of the given sphere be laid out. And let it have been cut at C, such that AC is equal to CB, and at D, such that AD is double DB. And let the semi-circle AEB have been drawn on AB. And let CE and DF have been drawn from C and D (respectively), at right-angles to AB. And let AF, FB, and EB have been joined. And since AD is double DB, AB is thus triple BD. Thus, via conversion, BA is one and a half πρὸς τὸ ἀπὸ τῆς ΑΖ· ἰσογώνιον γάρ ἐστι τὸ ΑΖΒ τρίγωνον τῷ ΑΖΔ τριγώνῳ· ἡμιόλιον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΒΑ τοῦ ἀπὸ τῆς ΑΖ. ἔστι δὲ καὶ ἡ τῆς σφαίρας διάμετρος δυνάμει ἡμιολία τῆς πλευρᾶς τῆς πυραμίδος. καί ἐστιν ἡ ΑΒ ἡ τῆς σφαίρας διάμετρος· ἡ ΑΖ ἄρα ἴση ἐστὶ τῇ πλευρᾷ τῆς πυραμίδος.

Πάλιν, ἐπεὶ διπλασίων ἐστὶν ἡ ΑΔ τῆς ΔΒ, τριπλῆ ἄρα ἐστὶν ἡ ΑΒ τῆς ΒΔ. ὡς δὲ ἡ ΑΒ πρὸς τὴν ΒΔ, οὕτως τὸ ἀπὸ τῆς ΑΒ πρὸς τὸ ἀπὸ τῆς ΒΖ· τριπλάσιον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΑΒ τοῦ ἀπὸ τῆς ΒΖ. ἔστι δὲ καὶ ἡ τῆς σφαίρας διάμετρος δυνάμει τριπλασίων τῆς τοῦ κύβου πλευρᾶς. καί ἐστιν ἡ ΑΒ ἡ τῆς σφαίρας διάμετρος· ἡ ΒΖ ἄρα τοῦ κύβου ἐστὶ πλευρά.

Καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΓ τῇ ΓΒ, διπλῆ ἄρα ἐστὶν ἡ ΑΒ τῆς ΒΓ. ὡς δὲ ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως τὸ ἀπὸ τῆς ΑΒ πρὸς τὸ ἀπὸ τῆς ΒΕ· διπλάσιον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΑΒ τοῦ ἀπὸ τῆς ΒΕ. ἔστι δὲ καὶ ἡ τῆς σφαίρας διάμετρος δυνάμει διπλασίων τῆς τοῦ ὀκταέδρου πλευρᾶς. καὶ ἐστιν ἡ ΑΒ ἡ τῆς δοθείσης σφαίρας διάμετρος· ἡ ΒΕ ἄρα τοῦ ὀκταέδρου ἐστὶ πλευρά.

Ἤχθω δὴ ἀπὸ τοῦ Α σημείου τῇ ΑΒ εὐθείᾳ πρὸς ὀρθὰς ἡ ΑΗ, καὶ κείσθω ἡ ΑΗ ἴση τῇ ΑΒ, καὶ ἐπεζεύχθω ἡ ΗΓ, καὶ ἀπὸ τοῦ Θ ἐπὶ τὴν ΑΒ κάθετος ἤχθω ἡ ΘΚ. καὶ ἐπεὶ διπλῆ ἐστιν ἡ ΗΑ τῆς ΑΓ· ἴση γὰρ ἡ ΗΑ τῇ ΑΒ· ὡς δὲ ἡ ΗΑ πρὸς τὴν ΑΓ, οὕτως ἡ ΘΚ πρὸς τὴν ΚΓ, διπλῆ ἄρα καὶ ἡ ΘΚ τῆς ΚΓ. τετραπλάσιον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΘΚ τοῦ ἀπὸ τῆς ΚΓ· τὰ ἄρα ἀπὸ τῶν ΘΚ, ΚΓ, ὅπερ ἐστὶ τὸ ἀπὸ τῆς ΘΓ, πενταπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΚΓ. ἴση δὲ ἡ ΘΓ τῇ ΓΒ· πενταπλάσιον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΒΓ τοῦ ἀπὸ τῆς ΓΚ. καὶ ἐπεὶ διπλῆ ἐστιν ἡ ΑΒ τῆς ΓΒ, ὧν ἡ ΑΔ τῆς ΔΒ ἐστι διπλῆ, λοιπὴ ἄρα ἡ ΒΔ λοιπῆς τῆς ΔΓ ἐστι διπλῆ. τριπλῆ ἄρα ἡ ΒΓ τῆς ΓΔ· ἐνναπλάσιον ἄρα τὸ ἀπὸ τῆς ΒΓ τοῦ ἀπὸ τῆς ΓΔ. πενταπλάσιον δὲ τὸ ἀπὸ τῆς ΒΓ τοῦ ἀπὸ τῆς ΓΚ· μεῖζον ἄρα τὸ ἀπὸ τῆς ΓΚ τοῦ ἀπὸ τῆς ΓΔ. μείζων ἄρα ἐστὶν ἡ ΓΚ τῆς ΓΔ. κείσθω τῇ ΓΚ ἴση ἡ ΓΛ, καὶ ἀπὸ τοῦ Λ τῇ ΑΒ πρὸς ὀρθὰς ἤχθω ἡ ΛΜ, καὶ ἐπεζεύχθω ἡ ΜΒ. καὶ ἐπεὶ πενταπλάσιόν ἐστι τὸ ἀπὸ τῆς ΒΓ τοῦ ἀπὸ τῆς ΓΚ, καί ἐστι τῆς μὲν ΒΓ διπλῆ ἡ ΑΒ, τῆς δὲ ΓΚ διπλῆ ἡ ΚΛ, πενταπλάσιον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΑΒ τοῦ ἀπὸ τῆς ΚΛ. ἔστι δὲ καὶ ἡ τῆς σφαίρας διάμετρος δυνάμει πενταπλασίων τῆς ἐκ τοῦ κέντρου τοῦ κύκλου, ἀφ᾽ οὗ τὸ εἰκοσάεδρον ἀναγέγραπται. καί ἐστιν ἡ ΑΒ ἡ τῆς σφαίρας διάμετρος· ἡ ΚΛ ἄρα ἐκ τοῦ κέντρου ἐστὶ τοῦ κύκλου, ἀφ᾽ οὗ τὸ εἰκοσάεδρον ἀναγέγραπται· ἡ ΚΛ ἄρα ἑξαγώνου ἐστὶ πλευρὰ τοῦ εἰρημένου κύκλου. καὶ ἐπεὶ ἡ τῆς σφαίρας διάμετρος σύγκειται ἔκ τε τῆς τοῦ ἑξαγώνου καὶ δύο τῶν τοῦ δεκαγώνου τῶν εἰς τὸν εἰρημένον κύκλον ἐγγραφομένων, καί ἐστιν ἡ μὲν ΑΒ ἡ τῆς σφαίρας διάμετρος, ἡ δὲ ΚΛ ἑξαγώνου πλευρά, καὶ ἴση ἡ ΑΚ τῇ ΛΒ, ἑκατέρα ἄρα τῶν ΑΚ, ΛΒ δεκαγώνου ἐστὶ πλευρὰ τοῦ ἐγγραφομένου εἰς τὸν κύκλον, ἀφ᾽ οὗ τὸ εἰκοσάεδρον ἀναγέγραπται. καὶ ἐπεὶ δεκαγώνου μὲν ἡ ΛΒ, ἑξαγώνου

times AD. And as BA (is) to AD, so the (square) on BA (is) to the (square) on AF [Def. 5.9]. For triangle AFB is equiangular to triangle AFD [Prop. 6.8]. Thus, the (square) on BA is one and a half times the (square) on AF. And the square on the diameter of the sphere is also one and a half times the (square) on the side of the pyramid [Prop. 13.13]. And AB is the diameter of the sphere. Thus, AF is equal to the side of the pyramid.

Again, since AD is double DB, AB is thus triple BD. And as AB (is) to BD, so the (square) on AB (is) to the (square) on BF [Prop. 6.8, Def. 5.9]. Thus, the (square) on AB is three times the (square) on BF. And the square on the diameter of the sphere is also three times the (square) on the side of the cube [Prop. 13.15]. And AB is the diameter of the sphere. Thus, BF is the side of the cube.

And since AC is equal to CB, AB is thus double BC. And as AB (is) to BC, so the (square) on AB (is) to the (square) on BE [Prop. 6.8, Def. 5.9]. Thus, the (square) on AB is double the (square) on BE. And the square on the diameter of the sphere is also double the (square) on the side of the octagon [Prop. 13.14]. And AB is the diameter of the given sphere. Thus, BE is the side of the octagon.

So let AG have been drawn from point A at right-angles to the straight-line AB. And let AG be made equal to AB. And let GC have been joined. And let HK have been drawn from H, perpendicular to AB. And since GA is double AC. For GA (is) equal to AB. And as GA (is) to AC, so HK (is) to KC [Prop. 6.4]. HK (is) thus also double KC. Thus, the (square) on HK is four times the (square) on KC. Thus, the (sum of the squares) on HK and KC, which is the (square) on HC [Prop. 1.47], is five times the (square) on KC. And HC (is) equal to CB. Thus, the (square) on BC (is) five times the (square) on CK. And since AB is double CB, of which AD is double DB, the remainder BD is thus double the remainder DC. BC (is) thus triple CD. The (square) on BC (is) thus nine times the (square) on CD. And the (square) on BC (is) five times the (square) on CK. Thus, the (square) on CK (is) greater than the (square) on CD. CK is thus greater than CD. Let CL be made equal to CK. And let LM have been drawn from L at right-angles to AB. And let MB have been joined. And since the (square) on BC is five times the (square) on CK, and AB is double BC, and KL double CK, the (square) on AB is thus five times the (square) on KL. And the square on the diameter of the sphere is also five times the (square) on the radius of the circle from which the icosahedron has been described [Prop. 13.16 corr.]. And AB is the diameter of the sphere. Thus, KL is the radius of the circle from δὲ ἡ ΜΛ· ἴση γάρ ἐστι τῇ ΚΛ, ἐπεὶ καὶ τῇ ΘΚ· ἴσον γὰρ ἀπέχουσιν ἀπὸ τοῦ κέντρου· καί ἐστιν ἑκατέρα τῶν ΘΚ, ΚΛ διπλασίων τῆς ΚΓ· πενταγώνου ἄρα ἐστὶν ἡ ΜΒ. ἡ δὲ τοῦ πενταγώνου ἐστὶν ἡ τοῦ εἰκοσαέδρου· εἰκοσαέδρου ἄρα ἐστὶν ἡ ΜΒ.

Καὶ ἐπεὶ ἡ ΖΒ κύβου ἐστὶ πλευρά, τετμήσθω ἄκρον καὶ μέσον λόγον κατὰ τὸ Ν, καὶ ἔστω μεῖζον τμῆμα τὸ ΝΒ· ἡ ΝΒ ἄρα δωδεκαέδρου ἐστὶ πλευρά.

Καὶ ἐπεὶ ἡ τῆς σφαίρας διάμετρος ἐδείχθη τῆς μὲν ΑΖ πλευρᾶς τῆς πυραμίδος δυνάμει ἡμιολία, τῆς δὲ τοῦ ὀκταέδρου τῆς ΒΕ δυνάμει διπλασίων, τῆς δὲ τοῦ κύβου τῆς ΖΒ δυνάμει τριπλασίων, οἵων ἄρα ἡ τῆς σφαίρας διάμετρος δυνάμει ἕξ, τοιούτων ἡ μὲν τῆς πυραμίδος τεσσάρων, ἡ δὲ τοῦ ὀκταέδρου τριῶν, ἡ δὲ τοῦ κύβου δύο. ἡ μὲν ἄρα τῆς πυραμίδος πλευρὰ τῆς μὲν τοῦ ὀκταέδρου πλευρᾶς δυνάμει ἐστὶν ἐπίτριτος, τῆς δὲ τοῦ κύβου δυνάμει διπλῆ, ἡ δὲ τοῦ ὀκταέδρου τῆς τοῦ κύβου δυνάμει ἡμιολία. αἱ μὲν οὖν εἰρημέναι τῶν τριῶν σχημάτων πλευραί, λέγω δὴ πυραμίδος καὶ ὀκταέδρου καὶ κύβου, πρὸς ἀλλήλας εἰσὶν ἐν λόγοις ῥητοῖς. αἱ δὲ λοιπαὶ δύο, λέγω δὴ ἥ τε τοῦ εἰκοσαέδρου καὶ ἡ τοῦ δωδεκαέδρου, οὔτε πρὸς ἀλλήλας οὔτε πρὸς τὰς προειρημένας εἰσὶν ἐν λόγοις ῥητοῖς· ἄλογοι γάρ εἰσιν, ἡ μὲν ἐλάττων, ἡ δὲ ἀποτομή.

Ὅτι μείζων ἐστὶν ἡ τοῦ εἰκοσαέδρου πλευρὰ ἡ ΜΒ τῆς τοῦ δωδεκαέδρου τῆς ΝΒ, δείξομεν οὕτως.

Ἐπεὶ γὰρ ἰσογώνιόν ἐστι τὸ ΖΔΒ τρίγωνον τῷ ΖΑΒ τριγώνῳ, ἀνάλογόν ἐστιν ὡς ἡ ΔΒ πρὸς τὴν ΒΖ, οὕτως ἡ ΒΖ πρὸς τὴν ΒΑ. καὶ ἐπεὶ τρεῖς εὐθεῖαι ἀνάλογόν εἰσιν, ἔστιν ὡς ἡ πρώτη πρὸς τὴν τρίτην, οὕτως τὸ ἀπὸ τῆς πρώτης πρὸς τὸ ἀπὸ τῆς δευτέρας· ἔστιν ἄρα ὡς ἡ ΔΒ πρὸς τὴν ΒΑ, οὕτως τὸ ἀπὸ τῆς ΔΒ πρὸς τὸ ἀπὸ τῆς ΒΖ· ἀνάπαλιν ἄρα ὡς ἡ ΑΒ πρὸς τὴν ΒΔ, οὕτως τὸ ἀπὸ τῆς ΖΒ πρὸς τὸ ἀπὸ τῆς ΒΔ. τριπλῆ δὲ ἡ ΑΒ τῆς ΒΔ· τριπλάσιον ἄρα τὸ ἀπὸ τῆς ΖΒ τοῦ ἀπὸ τῆς ΒΔ. ἔστι δὲ καὶ τὸ ἀπὸ τῆς ΑΔ τοῦ ἀπὸ τῆς ΔΒ τετραπλάσιον· διπλῆ γὰρ ἡ ΑΔ τῆς ΔΒ· μεῖζον ἄρα τὸ ἀπὸ τῆς ΑΔ τοῦ ἀπὸ τῆς ΖΒ· μείζων ἄρα ἡ ΑΔ τῆς ΖΒ· πολλῷ ἄρα ἡ ΑΛ τῆς ΖΒ μείζων ἐστίν. καὶ τῆς μὲν ΑΛ ἄκρον καὶ μέσον λόγον τεμνομένης τὸ μεῖζον τμῆμά ἐστιν ἡ ΚΛ, ἐπειδήπερ ἡ μὲν ΛΚ ἑξαγώνου ἐστίν, ἡ δὲ ΚΑ δεκαγώνου· τῆς δὲ ΖΒ ἄκρον καὶ μέσον λόγον τεμνομένης τὸ μεῖζον τμῆμά ἐστιν ἡ ΝΒ· μείζων ἄρα ἡ ΚΛ τῆς ΝΒ. ἴση δὲ ἡ ΚΛ τῇ ΛΜ· μείζων ἄρα ἡ ΛΜ τῆς ΝΒ [τῆς δὲ ΛΜ μείζων ἐστὶν ἡ ΜΒ]. πολλῷ ἄρα ἡ ΜΒ πλευρὰ οὖσα τοῦ εἰκοσαέδρου μείζων ἐστὶ τῆς ΝΒ πλευρᾶς οὔσης τοῦ δωδεκαέδρου· ὅπερ ἔδει δεῖξαι.

which the icosahedron has been described. Thus, KL is (the side) of the hexagon (inscribed) in the aforementioned circle [Prop. 4.15 corr.]. And since the diameter of the sphere is composed of (the side) of the hexagon, and two of (the sides) of the decagon, inscribed in the aforementioned circle, and AB is the diameter of the sphere, and KL the side of the hexagon, and AK (is) equal to LB, thus AK and LB are each sides of the decagon inscribed in the circle from which the icosahedron has been described. And since LB is (the side) of the decagon. And ML (is the side) of the hexagon—for (it is) equal to KL, since (it is) also (equal) to HK, for they are equally far from the center. And HK and KL are each double KC. MB is thus (the side) of the pentagon (inscribed in the circle) [Props. 13.10, 1.47]. And (the side) of the pentagon is (the side) of the icosahedron [Prop. 13.16]. Thus, MB is (the side) of the icosahedron.

And since FB is the side of the cube, let it have been cut in extreme and mean ratio at N, and let NB be the greater piece. Thus, NB is the side of the dodecahedron [Prop. 13.17 corr.].

And since the (square) on the diameter of the sphere was shown (to be) one and a half times the square on the side, AF, of the pyramid, and twice the square on (the side), BE, of the octagon, and three times the square on (the side), FB, of the cube, thus, of whatever (parts) the (square) on the diameter of the sphere (makes) six, of such (parts) the (square) on (the side) of the pyramid (makes) four, and (the square) on (the side) of the octagon three, and (the square) on (the side) of the cube two. Thus, the (square) on the side of the pyramid is one and a third times the square on the side of the octagon, and double the square on (the side) of the cube. And the (square) on (the side) of the octahedron is one and a half times the square on (the side) of the cube. Therefore, the aforementioned sides of the three figures—I mean, of the pyramid, and of the octahedron, and of the cube—are in rational ratios to one another. And (the sides of) the remaining two (figures)—I mean, of the icosahedron, and of the dodecahedron—are neither in rational ratios to one another, nor to the (sides) of the aforementioned (three figures). For they are irrational (straight-lines): (namely), a minor [Prop. 13.16], and an apotome [Prop. 13.17].

(And), we can show that the side, MB, of the icosahedron is greater that the (side), NB, or the dodecahedron, as follows.

For, since triangle FDB is equiangular to triangle FAB [Prop. 6.8], proportionally, as DB is to BF, so BF (is) to BA [Prop. 6.4]. And since three straight-lines are (continually) proportional, as the first (is) to the third,