If we take the cubic in the form

$$x^3 + ax + b = 0,$$

we can reduce it to an equation

$$4y^3 - 3y = c$$

with just one parameter, by setting $x = ky$ and choosing k so that

$$\frac{k^3}{ak} = \frac{-4}{3} \qquad \text{or} \qquad k = \sqrt{\frac{-4a}{3}}.$$

The point of the expression $4y^3 - 3y$ is that

$$4\cos^3\theta - 3\cos\theta = \cos 3\theta;$$

hence by setting $y = \cos\theta$ we obtain

$$\cos 3\theta = c.$$

If we are given c, then we can construct a triangle with angle $\cos^{-1} c = 3\theta$. Trisection of this angle gives us the solution $y = \cos\theta$ of the equation. Conversely, the problem of trisecting an angle with cosine c is equivalent to solving the cubic equation $4y^3 - 3y = c$.

[Of course, there is a problem with trigonometric interpretation when $|c| > 1$, which requires complex numbers for its resolution. Complex numbers are also involved in Cardano's formula, since the expression under the square root sign, $(q/2)^2 - (p/3)^3$, can be negative. It so happens that Viète's method requires complex numbers only when Cardano's does not, so between the two of them, complex numbers are avoided. Nevertheless, cubic equations are the birthplace of complex numbers, as we shall see when we study complex numbers in more detail later.]

Astonishingly, the problem of dividing an angle into any odd number of equal parts turns out to have an algebraic solution analogous to the algebraic solution of the cubic. Viète (1579) himself took the problem as far as finding expressions for $\cos n\theta$ and $\sin n\theta$ as polynomials in $\cos\theta$, $\sin\theta$, at least for certain values of n. Newton read Viète in 1663–4 and found the equation

$$y = nx - \frac{n(n^2-1)}{3!}x^3 + \frac{n(n^2-1)(n^2-3^2)}{5!}x^5 + \cdots$$

relating $y = \sin n\theta$ and $x = \sin\theta$ [see Newton (1676a) in Turnbull (1960)]. He asserted this result for arbitrary n, but we are interested in the case of odd integral n, when it reduces to a polynomial equation. The surprise is that Newton's equation then has a solution by nth roots analogous to the Cardano formula for cubics,

$$x = \frac{1}{2}\sqrt[n]{y+\sqrt{y^2-1}} + \frac{1}{2}\sqrt[n]{y-\sqrt{y^2-1}} \tag{1}$$

although only for n of the form $4m+1$. This formula appears out of the blue in de Moivre (1707). [It also appears in the unpublished Leibniz (1675), though without the restriction on n. See Schneider (1968), pp. 224–228.] He does not explain how he found it, but it is comprehensible to us as

$$\sin\theta = \frac{1}{2}\sqrt[n]{\sin n\theta + i\cos n\theta} + \frac{1}{2}\sqrt[n]{\sin n\theta - i\cos n\theta}, \tag{2}$$

a consequence of *our* version of de Moivre's formula

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \tag{3}$$

when $n = 4m+1$. (See Exercises 6.6.1 and 6.6.2.)

Viète himself came remarkably close to (3) in a posthumously published work [Viète (1615)]. He observed that the products of $\sin\theta$, $\cos\theta$ that occur in $\cos n\theta$, $\sin n\theta$ are the alternate terms in the expansion of $(\cos\theta + \sin\theta)^n$, except for certain minus signs. He failed only to notice that the signs could be explained by giving $\sin\theta$ the coefficient i. In any case, such an explanation would not have seemed natural to his contemporaries, who were far more comfortable with Cardano's formula than they were with i. In Section 14.5 we shall see how the perception of de Moivre's formula changed with the development of complex numbers.

Exercises

The reasons why (1) and (2) hold only for certain integer values, while (3) holds for all, can be understood by actually working out $(\sin\theta + i\cos\theta)^n$.

6.6.1 Use (3) and $\sin\alpha = \cos(\pi/2-\alpha)$, $\cos\alpha = \sin(\pi/2-\alpha)$ to show that

$$(\sin\theta + i\cos\theta)^n = \begin{cases} \sin n\theta + i\cos n\theta & \text{when } n = 4m+1 \\ -\sin n\theta - i\cos n\theta & \text{when } n = 4m+3. \end{cases}$$

6.6.2 Deduce from Exercise 6.6.1 that (2) is correct for $n = 4m+1$ and false for $n = 4m+3$, and hence that (1) is a correct relation between $y = \sin n\theta$ and $x = \sin\theta$ only when $n = 4m+1$.

6.6.3 Show that (1) is a correct relation between $y = \cos n\theta$ and $x = \cos\theta$ for *all* n [de Moivre (1730)].

6.7 Higher-Degree Equations

The general fourth degree, or *quartic*, equation

$$x^4 + ax^3 + bx^2 + cx + d = 0$$

was solved by Cardano's student Ferrari, and the solution was published in Cardano (1545), p. 237. A linear transformation reduces the equation to the form

$$x^4 + px^2 + qx + r = 0$$

or

$$(x^2 + p)^2 = px^2 - qx + p^2 - r$$

Then for any y

$$\begin{aligned}(x^2 + p + y)^2 &= (px^2 - qx + p^2 - r) + 2y(x^2 + p) + y^2 \\ &= (p + 2y)x^2 - qx + (p^2 - r + 2py + y^2).\end{aligned}$$

The quadratic $Ax^2 + Bx + C$ on the right-hand side will be a square if $B^2 - 4AC = 0$, which is a cubic equation for y. We can therefore solve for y and take the square root of both sides of the equation for x, which then becomes quadratic and hence also solvable. The final result is a formula for x using just square and cube roots of rational functions of the coefficients.

This impressive bonus to the solution of cubic equations raised hopes that higher-degree equations could also be solved by formulas built from the coefficients by rational operations and roots, and *solution by radicals*, as it was called, became a major goal of algebra for the next 250 years. However, all such efforts to solve the general equation of fifth degree (quintic) failed. The most that could be done was to reduce it to the form

$$x^5 - x - A = 0$$

with only one parameter. This was done by Bring (1786), and a sketch of his method may be seen in Pierpont (1895). Bring's result appeared in a very obscure publication and went unnoticed for 50 years, or it might have rekindled hopes for the solution of the quintic by radicals. As it happened, Ruffini (1799) offered the first proof that this is impossible. Ruffini's proof was not completely convincing; however, he was vindicated when a satisfactory proof was given by Abel (1826), and again with the beautiful general theory of equations of Galois (1831b).

A positive outcome of Bring's result was the nonalgebraic solution of the quintic by Hermite (1858). The reduction to an equation with one parameter opened the way to a solution by transcendental functions, analogous to Viète's solution of the cubic by circular functions. The appropriate functions, the elliptic modular functions, had been discovered by Gauss, Abel, and Jacobi, and Galois (1831a) had hinted at their relation to quintic equations. This extraordinary convergence of mathematical ideas was the subject of Klein (1884).

In view of the difficulties with the quintic, there was naturally very little progress with the general equation of degree n. However, two simple but important contributions were made by Descartes (1637). The first was the superscript notation for powers we now use: x^3, x^4, x^5, and so on. (Though not x^2, oddly enough. The square of x continued to be written xx until well into the next century.) The second was the theorem [Descartes (1637), p. 159] that a polynomial $p(x)$ with value 0 when $x = a$ has a factor $(x-a)$. Since division of a polynomial $p(x)$ of degree n by $(x-a)$ leaves a polynomial of degree $n-1$, Descartes' theorem raised the hope of factorizing each nth-degree polynomial into n linear factors. As Chapter 14 shows, this hope was fulfilled with the development of complex numbers.

EXERCISES

The main steps in the proof of Descartes' theorem go as follows. If the first step does not seem sufficiently easy, begin with $a = 1$.

6.7.1 Show that $x^n - a^n$ has a factor $x-a$. What is the quotient $(x^n - a^n)/(x-a)$? (And what does this have to do with geometric series?)

6.7.2 If $p(x) = a_k x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0$, use Exercise 6.7.1 to show that $p(x) - p(a)$ has a factor $x - a$.

6.7.3 Deduce Descartes' theorem from Exercise 6.7.2.

6.8 Biographical Notes: Tartaglia, Cardano, and Viète

Little is known about Scipione del Ferro, the discoverer of the first solution to cubic equations, other than his dates (1465–1526) and the fact that he was a professor of arithmetic and geometry at Bologna from 1496. This has possibly resulted in Tartaglia and Cardano receiving more mathematical credit than they deserve. On the other hand, there is no denying that

Tartaglia's and Cardano's personalities, their contrasting lives, and their quarrel make a story that is fascinating in its own right.

Niccolò Tartaglia (Figure 6.2) was born in Brescia in 1499 or 1500 and died in Venice in 1557. The name "Tartaglia" (meaning "stutterer") was actually a nickname; his real name is believed to have been Fontana.

Figure 6.2: Tartaglia

Tartaglia's childhood was scarred by poverty, following the death of his father, a mail courier, around 1506, and injuries suffered when Brescia was sacked by the French in 1512. Despite taking refuge in a cathedral, Tartaglia received five serious head wounds, including one to the mouth, which left him with his stutter. His life was saved only by the devoted nursing of his mother, who literally licked his wounds. Around the age of 14 he went to a teacher to learn the alphabet, but he ran out of money for his lessons by the letter K. This much is in Tartaglia's own sketch of his life [Tartaglia (1546), p. 69]. After that, the story goes, he stole a copybook and taught himself to read and write, sometimes using tombstones as slates for want of paper.

By 1534 he had a family and, still short of money, he moved to Venice. There he gave public mathematics lessons in the church of San Zanipolo and published scientific works. The famous disclosure of his method for solving cubic equations occurred on a visit to Cardano's house in Milan on March 25, 1539. When Cardano published it in 1545, Tartaglia angrily accused him of dishonesty. Tartaglia (1546), p. 120, claimed that Cardano had solemnly sworn never to publish the solution and to write it down only in cipher. Ferrari, who had been an 18-year-old servant of Cardano at the time, came to Cardano's defense, declaring that he had been present and there had been no promise of secrecy. In a series of 12 printed pamphlets, known as the *Cartelli* [reprinted by Masotti (1960)], Ferrari and Tartaglia traded insults and mathematical challenges; the two finally squared off in a public contest in the church of Santa Maria del Giardino, Milan, in 1548. It seems that Ferrari got the better of the exchange, as there was little subsequent improvement in Tartaglia's fortunes. He died alone, still impoverished, nine years later.

Apart from his solution of the cubic, Tartaglia is remembered for other contributions to science. It was he who discovered that a projectile should be fired at 45° to achieve maximum range [Tartaglia (1546), p. 6]. His conclusion was based on incorrect theory, however, as is clear from Tartaglia's diagrams of trajectories [for example, Figure 6.3; Tartaglia (1546), p. 16].

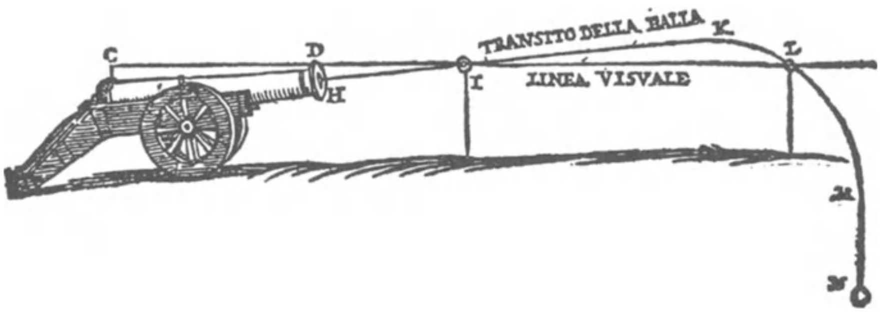

Figure 6.3: Tartaglia's trajectory of a cannonball

Tartaglia's Italian translation of the *Elements* was the first printed translation of Euclid in a modern language, and he also published an Italian translation of some of Archimedes' works. For information on these, and Tartaglia's mechanics, see Rose (1976), pp. 151–154.

Girolamo Cardano (Figure 6.4), often described in older English books by the anglicized name Jerome Cardan, was born in Pavia in 1501 and died in Rome in 1576. His father, Fazio, was a lawyer and physician who encouraged Girolamo's studies but otherwise seems to have treated him rather harshly, as did his mother, Chiara Micheri, whom Cardano described as "easily provoked, quick of memory and wit, and a fat, devout little woman." Cardano entered the University of Pavia in 1520 and completed a doctorate of medicine at Padua in 1526.

Figure 6.4: Cardano

He married in 1531 and, after struggling until 1539 for acceptance, became a successful physician in Milan—so successful, in fact, that his fame spread all over Europe. He evidently had a remarkable skill in diagnosis, though his contributions to medical knowledge were slight in comparison with those of his contemporaries Andreas Vesalius and Ambroise Paré.