3. Suppose

$$\begin{tikzcd} & A \arrow[r, "\alpha"] \arrow[d, "f"'] & B \arrow[r, "\beta"] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"'] & 0 \\ 0 \arrow[r] & A' \arrow[r, "\alpha'"] & B' \arrow[r, "\beta'"] & C' & \end{tikzcd}$$

is a commutative diagram of R-modules with exact rows.

(a) If $c \in \ker h$ and $\beta(b) = c$ prove that $g(b) \in \ker \beta'$ and conclude that $g(b) = \alpha'(a')$ for some $a' \in A'$. [Use the commutativity of the diagram.]

(b) Show that $\delta(c) = a'$ mod image f is a well defined R-module homomorphism from $\ker h$ to the quotient $A'/$ image f.

(c) *(The Snake Lemma)* Prove there is an exact sequence

$$\ker f \longrightarrow \ker g \longrightarrow \ker h \xrightarrow{\delta} \operatorname{coker} f \longrightarrow \operatorname{coker} g \longrightarrow \operatorname{coker} h$$

where $\operatorname{coker} f$ (the *cokernel* of f) is $A'/(\text{image } f)$ and similarly for $\operatorname{coker} g$ and $\operatorname{coker} h$.

(d) Show that if α is injective and β' is surjective (i.e., the two rows in the commutative diagram above can be extended to short exact sequences) then the exact sequence in (c) can be extended to the exact sequence

$$0 \longrightarrow \ker f \longrightarrow \ker g \longrightarrow \ker h \xrightarrow{\delta} \operatorname{coker} f \longrightarrow \operatorname{coker} g \longrightarrow \operatorname{coker} h \longrightarrow 0$$

4. Let $\mathcal{A} = \{A^n\}$ and $\mathcal{B} = \{B^n\}$ be cochain complexes, where the maps $A^n \to A^{n+1}$ and $B^n \to B^{n+1}$ in both complexes are denoted by d_{n+1} for all n. Cochain complex homomorphisms α and β from $\mathcal{A}$ to $\mathcal{B}$ are said to be *homotopic* if for all n there are module homomorphisms $s_n : A^{n+1} \to B^n$ such that the maps $\alpha_n - \beta_n$ from A^n to B^n satisfy

$$\alpha_n - \beta_n = d_n s_{n-1} + s_n d_{n+1}.$$

The collection of maps $\{s_n\}$ is called a *cochain homotopy* from α to β. One may similarly define chain homotopies between chain complexes.

(a) Prove that homotopic maps of cochain complexes induce the same maps on cohomology, i.e., if α and β are homotopic homomorphisms of cochain complexes then the induced group homomorphisms from $H^n(\mathcal{A})$ to $H^n(\mathcal{B})$ are equal for every $n \geq 0$. (Thus "homotopy" gives a sufficient condition for two maps of complexes to induce the same maps on cohomology or homology; this condition is not in general necessary.) [Use the definition of homotopy to show $(\alpha_n - \beta_n)(z) \in \text{image } d_n$ for every $z \in \ker d_{n+1}$.]

(b) Prove that the relation $\alpha \sim \beta$ if α and β are homotopic is an equivalence relation on any set of cochain complex homomorphisms.

5. Establish the first step in the Simultaneous Resolution result of Proposition 7 as follows: assume the first two nonzero rows in diagram (11) are given, except for the map from $P_0 \oplus \overline{P}_0$ to M (where the maps along the row of projective modules are the obvious injection and projection for this split exact sequence). Let $\mu : \overline{P}_0 \to M$ be a lifting to $\overline{P}_0$ of the map $\overline{P}_0 \to N$ (which exists because $\overline{P}_0$ is projective). Let λ be the composition $P_0 \to L \to M$ in the diagram. Define

$$\pi : P_0 \oplus \overline{P}_0 \to M \qquad \text{by} \qquad \pi(x, y) = \lambda(x) + \mu(y).$$

Show that with this definition the first two nonzero rows of (11) form a commutative diagram.

6. Let $0 \to \mathcal{A} \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\beta} \mathcal{C} \to 0$ be a short exact sequence of cochain complexes. Prove that if any two of $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are exact, then so is the third. [Use Theorem 2.]
7. Prove that a finitely generated abelian group A is free if and only if $\mathrm{Ext}^1(A, \mathbb{Z}) = 0$.
8. Prove that if $0 \to L \to M \to N \to 0$ is a split short exact sequence of R-modules, then for every $n \geq 0$ the sequence $0 \to \mathrm{Ext}_R^n(N, D) \to \mathrm{Ext}_R^n(M, D) \to \mathrm{Ext}_R^n(L, D) \to 0$ is also short exact and split. [Use a splitting homomorphism and Proposition 5.]
9. Show that

$$0 \longrightarrow \mathbb{Z}/d\mathbb{Z} \longrightarrow \mathbb{Z}/m\mathbb{Z} \xrightarrow{d} \mathbb{Z}/m\mathbb{Z} \xrightarrow{m/d} \mathbb{Z}/m\mathbb{Z} \xrightarrow{d} \mathbb{Z}/m\mathbb{Z} \xrightarrow{m/d} \cdots$$

is an injective resolution of $\mathbb{Z}/d\mathbb{Z}$ as a $\mathbb{Z}/m\mathbb{Z}$-module. [Use Proposition 36 in Section 10.5.] Use this to compute the groups $\mathrm{Ext}^n_{\mathbb{Z}/m\mathbb{Z}}(A, \mathbb{Z}/d\mathbb{Z})$ in terms of the dual group $\mathrm{Hom}_{\mathbb{Z}/m\mathbb{Z}}(A, \mathbb{Z}/m\mathbb{Z})$. In particular, if $m = p^2$ and $d = p$, give another derivation of the result $\mathrm{Ext}^n_{\mathbb{Z}/p^2\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$.
10. (a) Prove that an arbitrary direct sum $\oplus_{i \in I} P_i$ of projective modules P_i is projective and that an arbitrary direct product $\prod_{j \in J} Q_j$ of injective modules Q_j is injective.
    (b) Prove that an arbitrary direct sum of projective resolutions is again projective and use this to show $\mathrm{Ext}_R^n(\oplus_{i \in I} A_i, B) \cong \prod_{i \in I} \mathrm{Ext}_R^n(A_i, B)$ for any collection of R-modules A_i $(i \in I)$. [cf. Exercise 12 in Section 10.5.]
    (c) Prove that an arbitrary direct product of injective resolutions is an injective resolution and use this to show $\mathrm{Ext}_R^n(A, \prod_{j \in J} B_j) \cong \prod_{j \in J} \mathrm{Ext}_R^n(A, B_j)$ for any collection of R-modules B_j $(j \in J)$. [cf. Exercise 12 in Section 10.5.]
    (d) Prove that $\mathrm{Tor}_n^R(A, \oplus_{j \in J} B_j) \cong \oplus_{j \in J} \mathrm{Tor}_n^R(A, B_j)$ for any collection of R-modules B_j $(j \in J)$.
11. *(Bass' Characterization of Noetherian Rings)* Suppose R is a commutative ring.
    (a) If R is Noetherian, and I is any nonzero ideal in R show that the image of any R-module homomorphism $f : I \to \oplus_{j \in \mathcal{J}} Q_j$ from I into a direct sum of injective R-modules Q_j $(j \in \mathcal{J})$ is contained in some finite direct sum of the Q_j.
    (b) If R is Noetherian, prove that an arbitrary direct sum $\oplus_{j \in \mathcal{J}} Q_j$ of injective R-modules is again injective. [Use Baer's Criterion (Proposition 36) and Exercise 4 in Section 10.5 together with (a).]
    (c) Let $I_1 \subseteq I_2 \subseteq \dots$ be an ascending chain of ideals of R with union I and let $I/I_i \to Q_i$ for $i = 1, 2, \dots$ be an injection of the quotient I/I_i into an injective R-module Q_i (by Theorem 38 in Section 10.5). Prove that the composition of these injections with the product of the canonical projection maps $I \to I_i$ gives an R-module homomorphism $f : I \to \oplus_{i=1,2,\dots} Q_i$.
    (d) Prove the converse of (b): if an arbitrary direct sum $\oplus_{j \in \mathcal{J}} Q_j$ of injective R-modules is again injective then R is Noetherian. [If the direct sum in (c) is injective, use Baer's Criterion to lift f to a homomorphism $F : R \to \oplus_{i=1,2,\dots} Q_i$. If the component of $F(1)$ in Q_i is 0 for $i \geq n$ prove that $I = I_n$ and the ascending chain of ideals is finite.]
12. Prove Proposition 13: $\mathrm{Tor}_0^R(D, A) \cong D \otimes_R A$. [Follow the proof of Proposition 3.]
13. Prove Proposition 16 characterizing flat modules.
14. Suppose $0 \to A \to B \to C \to 0$ is a short exact sequence of R-modules. Prove that if C is a flat R-module, then A is flat if and only if B is also flat. [Use the Tor long exact sequence.] Give an example to show that if A and B are flat then C need not be flat.

15. (a) If I is an ideal in R and M is an R-module, prove that $\mathrm{Tor}_1^R(M, R/I)$ is isomorphic to the kernel of the map $M \otimes_R I \to M$ that maps $m \otimes i$ to mi for $i \in I$ and $m \in M$. [Use the Tor long exact sequence associated to $0 \to I \to R \to R/I \to 0$ noting that R is flat.]
(b) *(A Flatness Criterion using* Tor*)* Prove that the R-module M is flat if and only if $\mathrm{Tor}_1^R(M, R/I) = 0$ for every finitely generated ideal I of R. [Use Exercise 25 in Section 10.5.]

16. Suppose R is a P.I.D. and A and B are R-modules. If $t(B)$ denotes the torsion submodule of B show that $\mathrm{Tor}_1^R(A, t(B)) \cong \mathrm{Tor}_1^R(A, B)$ and deduce that $\mathrm{Tor}_1^R(A, B)$ is isomorphic to $\mathrm{Tor}_1^R(t(A), t(B))$. [Use Exercise 26 in Section 10.5 to show that $B/t(B)$ is flat over R, then use the Tor long exact sequence with $D = A$ applied to the short exact sequence $0 \to t(B) \to B \to B/t(B) \to 0$ and the remarks following Proposition 16.]

17. Let $A = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \cdots$. Prove that $\mathrm{Ext}^1(A, B) \cong (B/2B) \times (B/3B) \times (B/4B) \times \cdots$ for any abelian group A. [Use Exercise 10.] Prove that $\mathrm{Ext}^1(A, B) = 0$ if and only if B is divisible.

18. Prove that $\mathbb{Z}/2\mathbb{Z}$ is a projective $\mathbb{Z}/6\mathbb{Z}$-module and deduce that $\mathrm{Tor}_1^{\mathbb{Z}/6\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = 0$.

19. Suppose $r \neq 0$ is not a zero divisor in the commutative ring R.
(a) Prove that multiplication by r gives a free resolution $0 \to R \xrightarrow{r} R \to R/rR \to 0$ of the quotient R/rR.
(b) Prove that $\mathrm{Ext}_R^0(R/rR, B) = {}_rB$ is the set of elements $b \in B$ with $rb = 0$, that $\mathrm{Ext}_R^1(R/rR, B) \cong B/rB$, and that $\mathrm{Ext}_R^n(R/rR, B) = 0$ for $n \geq 2$ for every R-module B.
(c) Prove that $\mathrm{Tor}_0^R(A, R/rR) = A/rA$, that $\mathrm{Tor}_1^R(A, R/rR) = {}_rA$ is the set of elements $a \in A$ with $ra = 0$, and that $\mathrm{Tor}_n^R(A, R/rR) = 0$ for $n \geq 2$ for every R-module A.

20. Prove that $\mathrm{Tor}_0^{\mathbb{Z}/m\mathbb{Z}}(A, \mathbb{Z}/d\mathbb{Z}) \cong A/dA$, that $\mathrm{Tor}_n^{\mathbb{Z}/m\mathbb{Z}}(A, \mathbb{Z}/d\mathbb{Z}) \cong {}_dA/(m/d)A$ for n odd, $n \geq 1$, and that $\mathrm{Tor}_n^{\mathbb{Z}/m\mathbb{Z}}(A, \mathbb{Z}/d\mathbb{Z}) \cong {}_{(m/d)}A/dA$ for n even, $n \geq 2$. [Use the projective resolution in Example 2 following Proposition 3.]

21. Let $R = k[x, y]$ where k is a field, and let I be the ideal (x, y) in R.
(a) Let $\alpha : R \to R^2$ be the map $\alpha(r) = (yr, -xr)$ and let $\beta : R^2 \to R$ be the map $\beta((r_1, r_2)) = r_1 x + r_2 y$. Show that

$$0 \longrightarrow R \xrightarrow{\alpha} R^2 \xrightarrow{\beta} R \longrightarrow k \longrightarrow 0$$

where the map $R \to R/I = k$ is the canonical projection, gives a free resolution of k as an R-module.
(b) Use the resolution in (a) to show that $\mathrm{Tor}_2^R(k, k) \cong k$.
(c) Prove that $\mathrm{Tor}_1^R(k, I) \cong k$. [Use the long exact sequence corresponding to the short exact sequence $0 \to I \to R \to k \to 0$ and (b).]
(d) Conclude from (c) that the torsion free R-module I is not flat (compare to Exercise 26 in Section 10.5).

22. *(Flat Base Change for* Tor*)* Suppose R and S are commutative rings and $f : R \to S$ is a ring homomorphism making S into an R-module as in Example 6 following Corollary 12 in Section 10.4. Prove that if S is flat as an R-module, then $\mathrm{Tor}_n^R(A, B) \cong \mathrm{Tor}_n^S(S \otimes_R A, B)$ for all R-modules A and all S-modules B. [Show that since S is flat, tensoring an R-module projective resolution for A with S gives an S-module projective resolution of $S \otimes_R A$.]