EXERCISES

1. Show that the center of a direct product is the direct product of the centers:
$$Z(G_1 \times G_2 \times \cdots \times G_n) = Z(G_1) \times Z(G_2) \times \cdots \times Z(G_n).$$
Deduce that a direct product of groups is abelian if and only if each of the factors is abelian.
2. Let $G_1, G_2, \dots, G_n$ be groups and let $G = G_1 \times \cdots \times G_n$. Let I be a proper, nonempty subset of $\{1, \dots, n\}$ and let $J = \{1, \dots, n\} - I$. Define G_I to be the set of elements of G that have the identity of G_j in position j for all $j \in J$.
   (a) Prove that G_I is isomorphic to the direct product of the groups $G_i, i \in I$.
   (b) Prove that G_I is a normal subgroup of G and $G/G_I \cong G_J$.
   (c) Prove that $G \cong G_I \times G_J$.
3. Under the notation of the preceding exercise let I and K be any disjoint nonempty subsets of $\{1, 2, \dots, n\}$ and let G_I and G_K be the subgroups of G defined above. Prove that $xy = yx$ for all $x \in G_I$ and all $y \in G_K$.
4. Let A and B be finite groups and let p be a prime. Prove that any Sylow p-subgroup of $A \times B$ is of the form $P \times Q$, where $P \in Syl_p(A)$ and $Q \in Syl_p(B)$. Prove that $n_p(A \times B) = n_p(A)n_p(B)$. Generalize both of these results to a direct product of any finite number of finite groups (so that the number of Sylow p-subgroups of a direct product is the product of the numbers of Sylow p-subgroups of the factors).
5. Exhibit a nonnormal subgroup of $Q_8 \times Z_4$ (note that every subgroup of each factor is normal).
6. Show that all subgroups of $Q_8 \times E_{2^n}$ are normal.
7. Let $G_1, G_2, \dots, G_n$ be groups and let π be a fixed element of S_n. Prove that the map
$$\varphi_\pi : G_1 \times G_2 \times \cdots \times G_n \to G_{\pi^{-1}(1)} \times G_{\pi^{-1}(2)} \times \cdots \times G_{\pi^{-1}(n)}$$
defined by
$$\varphi_\pi(g_1, g_2, \dots, g_n) = (g_{\pi^{-1}(1)}, g_{\pi^{-1}(2)}, \dots, g_{\pi^{-1}(n)})$$
is an isomorphism (so that changing the order of the factors in a direct product does not change the isomorphism type).
8. Let $G_1 = G_2 = \cdots = G_n$ and let $G = G_1 \times \cdots \times G_n$. Under the notation of the preceding exercise show that $\varphi_\pi \in \text{Aut}(G)$. Show also that the map $\pi \mapsto \varphi_\pi$ is an injective homomorphism of S_n into $\text{Aut}(G)$. (In particular, $\varphi_{\pi_1} \circ \varphi_{\pi_2} = \varphi_{\pi_1\pi_2}$. It is at this point that the π^{-1}'s in the definition of φ_π are needed. The underlying reason for this is because if e_i is the n-tuple with 1 in position i and zeros elsewhere, $1 \le i \le n$, then S_n acts on $\{e_1, \dots, e_n\}$ by $\pi \cdot e_i = e_{\pi(i)}$; this is a left group action. If the n-tuple $(g_1, \dots, g_n)$ is represented by $g_1e_1 + \cdots + g_ne_n$, then this left group action on $\{e_1, \dots, e_n\}$ extends to a left group action on sums by
$$\pi \cdot (g_1e_1 + g_2e_2 + \cdots + g_ne_n) = g_1e_{\pi(1)} + g_2e_{\pi(2)} + \cdots + g_ne_{\pi(n)}.$$
The coefficient of $e_{\pi(i)}$ on the right hand side is g_i, so the coefficient of e_i is $g_{\pi^{-1}(i)}$. Thus the right hand side may be rewritten as $g_{\pi^{-1}(1)}e_1 + g_{\pi^{-1}(2)}e_2 + \cdots + g_{\pi^{-1}(n)}e_n$, which is precisely the sum attached to the n-tuple $(g_{\pi^{-1}(1)}, g_{\pi^{-1}(2)}, \dots, g_{\pi^{-1}(n)})$. In other words, any permutation of the "position vectors" $e_1, \dots, e_n$ (which fixes their coefficients) is the same as the inverse permutation on the coefficients (fixing the e_i's). If one uses π's in place of π^{-1}'s in the definition of φ_π then the map $\pi \mapsto \varphi_\pi$ is not necessarily a homomorphism — it corresponds to a *right* group action.)

9. Let G_i be a field F for all i and use the preceding exercise to show that the set of $n \times n$ matrices with one 1 in each row and each column is a subgroup of $GL_n(F)$ isomorphic to S_n (these matrices are called *permutation matrices* since they simply permute the standard basis $e_1, \ldots, e_n$ (as above) of the n-dimensional vector space $F \times F \times \cdots \times F$).

10. Let p be a prime. Let A and B be two cyclic groups of order p with generators x and y, respectively. Set $E = A \times B$ so that E is the elementary abelian group of order p^2: E_{p^2}. Prove that the distinct subgroups of E of order p are

$$\langle x \rangle, \quad \langle xy \rangle, \quad \langle xy^2 \rangle, \quad \ldots, \quad \langle xy^{p-1} \rangle, \quad \langle y \rangle$$

(note that there are $p + 1$ of them).

11. Let p be a prime and let $n \in \mathbb{Z}^+$. Find a formula for the number of subgroups of order p in the elementary abelian group E_{p^n}.

12. Let A and B be groups. Assume $Z(A)$ contains a subgroup Z_1 and $Z(B)$ contains a subgroup Z_2 with $Z_1 \cong Z_2$. Let this isomorphism be given by the map $x_i \mapsto y_i$ for all $x_i \in Z_1$. A *central product* of A and B is a quotient

$$(A \times B)/Z \quad \text{where} \quad Z = \{(x_i, y_i^{-1}) \mid x_i \in Z_1\}$$

and is denoted by $A * B$ — it is not unique since it depends on Z_1, Z_2 and the isomorphism between them. (Think of $A * B$ as the direct product of A and B "collapsed" by identifying each element $x_i \in Z_1$ with its corresponding element $y_i \in Z_2$.)

(a) Prove that the images of A and B in the quotient group $A * B$ are isomorphic to A and B, respectively, and that these images intersect in a central subgroup isomorphic to Z_1. Find $|A * B|$.

(b) Let $Z_4 = \langle x \rangle$. Let $D_8 = \langle r, s \rangle$ and $Q_8 = \langle i, j \rangle$ be given by their usual generators and relations. Let $Z_4 * D_8$ be the central product of Z_4 and D_8 which identifies x^2 and r^2 (i.e., $Z_1 = \langle x^2 \rangle$, $Z_2 = \langle r^2 \rangle$ and the isomorphism is $x^2 \mapsto r^2$) and let $Z_4 * Q_8$ be the central product of Z_4 and Q_8 which identifies x^2 and -1. Prove that $Z_4 * D_8 \cong Z_4 * Q_8$.

13. Give presentations for the groups $Z_4 * D_8$ and $Z_4 * Q_8$ constructed in the preceding exercise.

14. Let $G = A_1 \times A_2 \times \cdots \times A_n$ and for each i let B_i be a normal subgroup of A_i. Prove that $B_1 \times B_2 \times \cdots \times B_n \trianglelefteq G$ and that

$$(A_1 \times A_2 \times \cdots \times A_n)/(B_1 \times B_2 \times \cdots \times B_n) \cong (A_1/B_1) \times (A_2/B_2) \times \cdots \times (A_n/B_n).$$

The following exercise describes the direct product of an arbitrary collection of groups. The terminology for the Cartesian product of an arbitrary collection of sets may be found in the Appendix.

15. Let I be any nonempty index set and let $(G_i, \star_i)$ be a group for each $i \in I$. The *direct product* of the groups $G_i, i \in I$ is the set $G = \prod_{i \in I} G_i$ (the Cartesian product of the G_i's) with a binary operation defined as follows: if $\prod a_i$ and $\prod b_i$ are elements of G, then their product in G is given by

$$\Big(\prod_{i \in I} a_i\Big)\Big(\prod_{i \in I} b_i\Big) = \prod_{i \in I} (a_i \star_i b_i)$$

(i.e., the group operation in the direct product is defined componentwise).

(a) Show that this binary operation is well defined and associative.

(b) Show that the element $\prod 1_i$ satisfies the axiom for the identity of G, where 1_i is the identity of G_i for all i.

(c) Show that the element $\prod a_i^{-1}$ is the inverse of $\prod a_i$, where the inverse of each component element a_i is taken in the group G_i.

Conclude that the direct product is a group.
(Note that if $I = \{1, 2, \ldots, n\}$, this definition of the direct product is the same as the n-tuple definition in the text.)

16. State and prove the generalization of Proposition 2 to arbitrary direct products.

17. Let I be any nonempty index set and let G_i be a group for each $i \in I$. The *restricted direct product* or *direct sum* of the groups G_i is the set of elements of the direct product which are the identity in all but finitely many components, that is, the set of all elements $\prod a_i \in \prod_{i \in I} G_i$ such that $a_i = 1_i$ for all but a finite number of $i \in I$.

(a) Prove that the restricted direct product is a subgroup of the direct product.

(b) Prove that the restricted direct product is normal in the direct product.

(c) Let $I = \mathbb{Z}^+$ and let p_i be the i^{th} integer prime. Show that if $G_i = \mathbb{Z}/p_i\mathbb{Z}$ for all $i \in \mathbb{Z}^+$, then every element of the restricted direct product of the G_i's has finite order but $\prod_{i \in \mathbb{Z}^+} G_i$ has elements of infinite order. Show that in this example the restricted direct product is the torsion subgroup of the direct product (cf. Exercise 6, Section 2.1).

18. In each of (a) to (e) give an example of a group with the specified properties:

(a) an infinite group in which every element has order 1 or 2

(b) an infinite group in which every element has finite order but for each positive integer n there is an element of order n

(c) a group with an element of infinite order and an element of order 2

(d) a group G such that every finite group is isomorphic to some subgroup of G

(e) a nontrivial group G such that $G \cong G \times G$.

5.2 THE FUNDAMENTAL THEOREM OF FINITELY GENERATED ABELIAN GROUPS

Definition.

(1) A group G is *finitely generated* if there is a finite subset A of G such that $G = \langle A \rangle$.

(2) For each $r \in \mathbb{Z}$ with $r \geq 0$, let $\mathbb{Z}^r = \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$ be the direct product of r copies of the group $\mathbb{Z}$, where $\mathbb{Z}^0 = 1$. The group $\mathbb{Z}^r$ is called the *free abelian group of rank* r.

Note that any finite group G is, a fortiori, finitely generated: simply take $A = G$ as a set of generators. Also, $\mathbb{Z}^r$ is finitely generated by $e_1, e_2, \ldots, e_r$, where e_i is the n-tuple with 1 in position i and zeros elsewhere. We can now state the fundamental classification theorem for (finitely generated) abelian groups.

Theorem 3. *(Fundamental Theorem of Finitely Generated Abelian Groups)* Let G be a finitely generated abelian group. Then

(1)

$$G \cong \mathbb{Z}^r \times Z_{n_1} \times Z_{n_2} \times \cdots \times Z_{n_s},$$

for some integers $r, n_1, n_2, \ldots, n_s$ satisfying the following conditions: