35. Prove that $SL_n(F) \trianglelefteq GL_n(F)$ and describe the isomorphism type of the quotient group (cf. Exercise 9, Section 2.1).

36. Prove that if $G/Z(G)$ is cyclic then G is abelian. [If $G/Z(G)$ is cyclic with generator $xZ(G)$, show that every element of G can be written in the form $x^a z$ for some integer $a \in \mathbb{Z}$ and some element $z \in Z(G)$.]

37. Let A and B be groups. Show that $\{(a, 1) \mid a \in A\}$ is a normal subgroup of $A \times B$ and the quotient of $A \times B$ by this subgroup is isomorphic to B.

38. Let A be an abelian group and let D be the (diagonal) subgroup $\{(a, a) \mid a \in A\}$ of $A \times A$. Prove that D is a normal subgroup of $A \times A$ and $(A \times A)/D \cong A$.

39. Suppose A is the non-abelian group S_3 and D is the diagonal subgroup $\{(a, a) \mid a \in A\}$ of $A \times A$. Prove that D is not normal in $A \times A$.

40. Let G be a group, let N be a normal subgroup of G and let $\overline{G} = G/N$. Prove that $\overline{x}$ and $\overline{y}$ commute in $\overline{G}$ if and only if $x^{-1}y^{-1}xy \in N$. (The element $x^{-1}y^{-1}xy$ is called the *commutator* of x and y and is denoted by $[x, y]$.)

41. Let G be a group. Prove that $N = \langle x^{-1}y^{-1}xy \mid x, y \in G \rangle$ is a normal subgroup of G and G/N is abelian (N is called the *commutator subgroup* of G).

42. Assume both H and K are normal subgroups of G with $H \cap K = 1$. Prove that $xy = yx$ for all $x \in H$ and $y \in K$. [Show $x^{-1}y^{-1}xy \in H \cap K$.]

43. Assume $\mathcal{P} = \{A_i \mid i \in I\}$ is any partition of G with the property that $\mathcal{P}$ is a group under the "quotient operation" defined as follows: to compute the product of A_i with A_j take any element a_i of A_i and any element a_j of A_j and let $A_i A_j$ be the element of $\mathcal{P}$ containing $a_i a_j$ (this operation is assumed to be well defined). Prove that the element of $\mathcal{P}$ that contains the identity of G is a normal subgroup of G and the elements of $\mathcal{P}$ are the cosets of this subgroup (so $\mathcal{P}$ is just a quotient group of G in the usual sense).

3.2 MORE ON COSETS AND LAGRANGE'S THEOREM

In this section we continue the study of quotient groups. Since for finite groups one of the most important invariants of a group is its order we first prove that the order of a quotient group of a finite group can be readily computed: $|G/N| = \dfrac{|G|}{|N|}$. In fact we derive this as a consequence of a more general result, Lagrange's Theorem (see Exercise 19, Section 1.7). This theorem is one of the most important combinatorial results in finite group theory and will be used repeatedly. After indicating some easy consequences of Lagrange's Theorem we study more subtle questions concerning cosets of non-normal subgroups.

The proof of Lagrange's Theorem is straightforward and important. It is the same line of reasoning we used in Example 3 of the preceding section to compute $|D_8/Z(D_8)|$.

Theorem 8. *(Lagrange's Theorem)* If G is a finite group and H is a subgroup of G, then the order of H divides the order of G (i.e., $|H| \mid |G|$) and the number of left cosets of H in G equals $\dfrac{|G|}{|H|}$.

Proof: Let $|H| = n$ and let the number of left cosets of H in G equal k. By

Proposition 4 the set of left cosets of H in G partition G. By definition of a left coset the map:
$$H \to gH \qquad \text{defined by} \qquad h \mapsto gh$$
is a surjection from H to the left coset gH. The left cancellation law implies this map is injective since $gh_1 = gh_2$ implies $h_1 = h_2$. This proves that H and gH have the same order:
$$|gH| = |H| = n.$$
Since G is partitioned into k disjoint subsets each of which has cardinality n, $|G| = kn$. Thus $k = \dfrac{|G|}{n} = \dfrac{|G|}{|H|}$, completing the proof.

Definition. If G is a group (possibly infinite) and $H \le G$, the number of left cosets of H in G is called the *index* of H in G and is denoted by $|G : H|$.

In the case of finite groups the index of H in G is $\dfrac{|G|}{|H|}$. For G an infinite group the quotient $\dfrac{|G|}{|H|}$ does not make sense. Infinite groups may have subgroups of finite or infinite index (e.g., $\{0\}$ is of infinite index in $\mathbb{Z}$ and $\langle n \rangle$ is of index n in $\mathbb{Z}$ for every $n > 0$).

We now derive some easy consequences of Lagrange's Theorem.

Corollary 9. If G is a finite group and $x \in G$, then the order of x divides the order of G. In particular $x^{|G|} = 1$ for all x in G.

Proof: By Proposition 2.2, $|x| = |\langle x \rangle|$. The first part of the corollary follows from Lagrange's Theorem applied to $H = \langle x \rangle$. The second statement is clear since now $|G|$ is a multiple of the order of x.

Corollary 10. If G is a group of prime order p, then G is cyclic, hence $G \cong Z_p$.

Proof: Let $x \in G$, $x \ne 1$. Thus $|\langle x \rangle| > 1$ and $|\langle x \rangle|$ divides $|G|$. Since $|G|$ is prime we must have $|\langle x \rangle| = |G|$, hence $G = \langle x \rangle$ is cyclic (with any nonidentity element x as generator). Theorem 2.4 completes the proof.

With Lagrange's Theorem in hand we examine some additional examples of normal subgroups.

Examples

(1) Let $H = \langle (1\ 2\ 3) \rangle \le S_3$ and let $G = S_3$. We show $H \trianglelefteq S_3$. As noted in Section 2.2,
$$H \le N_G(H) \le G.$$
By Lagrange's Theorem, the order of H divides the order of $N_G(H)$ and the order of $N_G(H)$ divides the order of G. Since G has order 6 and H has order 3, the only possibilities for $N_G(H)$ are H or G. A direct computation gives
$$(1\ 2)(1\ 2\ 3)(1\ 2) = (1\ 3\ 2) = (1\ 2\ 3)^{-1}.$$

Since $(1\ 2) = (1\ 2)^{-1}$, this calculation shows that $(1\ 2)$ conjugates a generator of H to another generator of H. By Exercise 24 of Section 2.3 this is sufficient to prove that $(1\ 2) \in N_G(H)$. Thus $N_G(H) \neq H$ so $N_G(H) = G$, i.e., $H \trianglelefteq S_3$, as claimed. This argument illustrates that checking normality of a subgroup can often be reduced to a small number of calculations. A generalization of this example is given in the next example.

(2) Let G be any group containing a subgroup H of index 2. We prove $H \trianglelefteq G$. Let $g \in G - H$ so, by hypothesis, the two left cosets of H in G are $1H$ and gH. Since $1H = H$ and the cosets partition G, we must have $gH = G - H$. Now the two right cosets of H in G are $H1$ and Hg. Since $H1 = H$, we again must have $Hg = G - H$. Combining these gives $gH = Hg$, so every left coset of H in G is a right coset. By Theorem 6, $H \trianglelefteq G$. By definition of index, $|G/H| = 2$, so that $G/H \cong Z_2$. One must be careful to appreciate that the reason H is normal in this case is not because we can choose the same coset representatives 1 and g for both the left and right cosets of H but that there is a type of pigeon-hole principle at work: since $1H = H = H1$ for any subgroup H of any group G, the index assumption forces the remaining elements to comprise the remaining coset (either left or right). We shall see that this result is itself a special case of a result we shall prove in the next chapter.

Note that this result proves that $\langle i \rangle$, $\langle j \rangle$ and $\langle k \rangle$ are normal subgroups of Q_8 and that $\langle s, r^2 \rangle$, $\langle r \rangle$ and $\langle sr, r^2 \rangle$ are normal subgroups of D_8.

(3) The property "is a normal subgroup of" is not transitive. For example,

$$\langle s \rangle \trianglelefteq \langle s, r^2 \rangle \trianglelefteq D_8$$

(each subgroup is of index 2 in the next), however, $\langle s \rangle$ is not normal in D_8 because $rsr^{-1} = sr^2 \notin \langle s \rangle$.

We now examine some examples of non-normal subgroups. Although in abelian groups every subgroup is normal, this is not the case in non-abelian groups (in some sense Q_8 is the unique exception to this). In fact, there are groups G in which the only normal subgroups are the trivial ones: 1 and G. Such groups are called *simple groups* (simple does not mean easy, however). Simple groups play an important role in the study of general groups and this role will be described in Section 4. For now we emphasize that not every subgroup of a group G is normal in G; indeed, normal subgroups may be quite rare in G. The search for normal subgroups of a given group is in general a highly nontrivial problem.

Examples

(1) Let $H = \langle (1\ 2) \rangle \leq S_3$. Since H is of prime index 3 in S_3, by Lagrange's Theorem the only possibilities for $N_{S_3}(H)$ are H or S_3. Direct computation shows

$$(1\ 3)(1\ 2)(1\ 3)^{-1} = (1\ 3)(1\ 2)(1\ 3) = (2\ 3) \notin H$$

so $N_{S_3}(H) \neq S_3$, that is, H is not a normal subgroup of S_3. One can also see this by considering the left and right cosets of H; for instance

$$(1\ 3)H = \{(1\ 3), (1\ 2\ 3)\} \qquad \text{and} \qquad H(1\ 3) = \{(1\ 3), (1\ 3\ 2)\}.$$

Since the left coset $(1\ 3)H$ is the unique left coset of H containing $(1\ 3)$, the right coset $H(1\ 3)$ cannot be a left coset (see also Exercise 6). Note also that the "group operation" on the left cosets of H in S_3 defined by multiplying representatives is not