14. Let R be a commutative ring and let F be the free R-module of rank n. Prove that $\text{Hom}_R(F, M) \cong M \times \cdots \times M$ (n times). [Use Exercise 9 in Section 2 and Exercise 12.]

15. An element $e \in R$ is called a *central idempotent* if $e^2 = e$ and $er = re$ for all $r \in R$. If e is a central idempotent in R, prove that $M = eM \oplus (1-e)M$. [Recall Exercise 14 in Section 1.]

The next two exercises establish the Chinese Remainder Theorem for modules (cf. Section 7.6).

16. For any ideal I of R let IM be the submodule defined in Exercise 5 of Section 1. Let $A_1, \dots, A_k$ be any ideals in the ring R. Prove that the map

$$M \to M/A_1M \times \cdots \times M/A_kM \quad \text{defined by} \quad m \mapsto (m + A_1M, \dots, m + A_kM)$$

is an R-module homomorphism with kernel $A_1M \cap A_2M \cap \cdots \cap A_kM$.

17. In the notation of the preceding exercise, assume further that the ideals $A_1, \dots, A_k$ are pairwise comaximal (i.e., $A_i + A_j = R$ for all $i \neq j$). Prove that

$$M/(A_1 \cdots A_k)M \cong M/A_1M \times \cdots \times M/A_kM.$$

[See the proof of the Chinese Remainder Theorem for rings in Section 7.6.]

18. Let R be a Principal Ideal Domain and let M be an R-module that is annihilated by the nonzero, proper ideal (a). Let $a = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the unique factorization of a into distinct prime powers in R. Let M_i be the annihilator of $p_i^{\alpha_i}$ in M, i.e., M_i is the set $\{m \in M \mid p_i^{\alpha_i} m = 0\}$ — called the p_i-*primary component of* M. Prove that

$$M = M_1 \oplus M_2 \oplus \cdots \oplus M_k.$$

19. Show that if M is a finite abelian group of order $a = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ then, considered as a $\mathbb{Z}$-module, M is annihilated by (a), the p_i-primary component of M is the unique Sylow p_i-subgroup of M and M is isomorphic to the direct product of its Sylow subgroups.

20. Let I be a nonempty index set and for each $i \in I$ let M_i be an R-module. The *direct product* of the modules M_i is defined to be their direct product as abelian groups (cf. Exercise 15 in Section 5.1) with the action of R componentwise multiplication. The *direct sum* of the modules M_i is defined to be the restricted direct product of the abelian groups M_i (cf. Exercise 17 in Section 5.1) with the action of R componentwise multiplication. In other words, the direct sum of the M_i's is the subset of the direct product, $\prod_{i \in I} M_i$, which consists of all elements $\prod_{i \in I} m_i$ such that only finitely many of the components m_i are nonzero; the action of R on the direct product or direct sum is given by $r \prod_{i \in I} m_i = \prod_{i \in I} rm_i$ (cf. Appendix I for the definition of Cartesian products of infinitely many sets). The direct sum will be denoted by $\oplus_{i \in I} M_i$.
   (a) Prove that the direct product of the M_i's is an R-module and the direct sum of the M_i's is a submodule of their direct product.
   (b) Show that if $R = \mathbb{Z}$, $I = \mathbb{Z}^+$ and M_i is the cyclic group of order i for each i, then the direct sum of the M_i's is not isomorphic to their direct product. [Look at torsion.]

21. Let I be a nonempty index set and for each $i \in I$ let N_i be a submodule of M. Prove that the following are equivalent:
   (i) the submodule of M generated by all the N_i's is isomorphic to the direct sum of the N_i's
   (ii) if $\{i_1, i_2, \dots, i_k\}$ is any finite subset of I then $N_{i_1} \cap (N_{i_2} + \cdots + N_{i_k}) = 0$
   (iii) if $\{i_1, i_2, \dots, i_k\}$ is any finite subset of I then $N_1 + \cdots + N_k = N_1 \oplus \cdots \oplus N_k$
   (iv) for every element x of the submodule of M generated by the N_i's there are unique elements $a_i \in N_i$ for all $i \in I$ such that all but a finite number of the a_i are zero and x is the (finite) sum of the a_i.

22. Let R be a Principal Ideal Domain, let M be a torsion R-module (cf. Exercise 4) and let p be a prime in R (do not assume M is finitely generated, hence it need not have a nonzero annihilator — cf. Exercise 5). The *p-primary component* of M is the set of all elements of M that are annihilated by some positive power of p.
   (a) Prove that the p-primary component is a submodule. [See Exercise 13 in Section 1.]
   (b) Prove that this definition of p-primary component agrees with the one given in Exercise 18 when M has a nonzero annihilator.
   (c) Prove that M is the (possibly infinite) direct sum of its p-primary components, as p runs over all primes of R.

23. Show that any direct sum of free R-modules is free.

24. *(An arbitrary direct product of free modules need not be free)* For each positive integer i let M_i be the free $\mathbb{Z}$-module $\mathbb{Z}$, and let M be the direct product $\prod_{i\in\mathbb{Z}^+} M_i$ (cf. Exercise 20). Each element of M can be written uniquely in the form $(a_1, a_2, a_3, \ldots)$ with $a_i \in \mathbb{Z}$ for all i. Let N be the submodule of M consisting of all such tuples with only finitely many nonzero a_i. Assume M is a free $\mathbb{Z}$-module with basis $\mathcal{B}$.
   (a) Show that N is countable.
   (b) Show that there is some countable subset $\mathcal{B}_1$ of $\mathcal{B}$ such that N is contained in the submodule, N_1, generated by $\mathcal{B}_1$. Show also that N_1 is countable.
   (c) Let $\overline{M} = M/N_1$. Show that $\overline{M}$ is a free $\mathbb{Z}$-module. Deduce that if $\overline{x}$ is any nonzero element of $\overline{M}$ then there are only finitely many distinct positive integers k such that $\overline{x} = k\overline{m}$ for some $m \in M$ (depending on k).
   (d) Let $\mathcal{S} = \{(b_1, b_2, b_3, \ldots) \mid b_i = \pm i! \text{ for all } i\}$. Prove that $\mathcal{S}$ is uncountable. Deduce that there is some $s \in \mathcal{S}$ with $s \notin N_1$.
   (e) Show that the assumption M is free leads to a contradiction: By (d) we may choose $s \in \mathcal{S}$ with $s \notin N_1$. Show that for each positive integer k there is some $m \in M$ with $\overline{s} = k\overline{m}$, contrary to (c). [Use the fact that $N \subseteq N_1$.]

25. In the construction of direct limits, Exercise 8 of Section 7.6, show that if all A_i are R-modules and the maps ρ_{ij} are R-module homomorphisms, then the direct limit $A = \varinjlim A_i$ may be given the structure of an R-module in a natural way such that the maps $\rho_i : A_i \to A$ are all R-module homomorphisms. Verify the corresponding universal property (part (e)) for R-module homomorphisms $\varphi_i : A_i \to C$ commuting with the ρ_{ij}.

26. Carry out the analysis of the preceding exercise corresponding to inverse limits to show that an inverse limit of R-modules is an R-module satisfying the appropriate universal property (cf. Exercise 10 of Section 7.6).

27. *(Free modules over noncommutative rings need not have a unique rank)* Let M be the $\mathbb{Z}$-module $\mathbb{Z} \times \mathbb{Z} \times \cdots$ of Exercise 24 and let R be its endomorphism ring, $R = \mathrm{End}_{\mathbb{Z}}(M)$ (cf. Exercises 29 and 30 in Section 7.1). Define $\varphi_1, \varphi_2 \in R$ by

$$\varphi_1(a_1, a_2, a_3, \ldots) = (a_1, a_3, a_5, \ldots)$$
$$\varphi_2(a_1, a_2, a_3, \ldots) = (a_2, a_4, a_6, \ldots)$$

   (a) Prove that $\{\varphi_1, \varphi_2\}$ is a free basis of the left R-module R. [Define the maps ψ_1 and ψ_2 by $\psi_1(a_1, a_2, \ldots) = (a_1, 0, a_2, 0, \ldots)$ and $\psi_2(a_1, a_2, \ldots) = (0, a_1, 0, a_2, \ldots)$. Verify that $\varphi_i\psi_i = 1$, $\varphi_1\psi_2 = 0 = \varphi_2\psi_1$ and $\psi_1\varphi_1 + \psi_2\varphi_2 = 1$. Use these relations to prove that φ_1, φ_2 are independent and generate R as a left R-module.]
   (b) Use (a) to prove that $R \cong R^2$ and deduce that $R \cong R^n$ for all $n \in \mathbb{Z}^+$.

10.4 TENSOR PRODUCTS OF MODULES

In this section we study the tensor product of two modules M and N over a ring (not necessarily commutative) containing 1. Formation of the tensor product is a general construction that, loosely speaking, enables one to form another module in which one can take "products" mn of elements $m \in M$ and $n \in N$. The general construction involves various left- and right- module actions, and it is instructive, by way of motivation, to first consider an important special case: the question of "extending scalars" or "changing the base."

Suppose that the ring R is a subring of the ring S. Throughout this section, we always assume that $1_R = 1_S$ (this ensures that S is a unital R-module).

If N is a left S-module, then N can also be naturally considered as a left R-module since the elements of R (being elements of S) act on N by assumption. The S-module axioms for N include the relations

$$(s_1 + s_2)n = s_1 n + s_2 n \quad \text{and} \quad s(n_1 + n_2) = sn_1 + sn_2 \tag{10.1}$$

for all $s, s_1, s_2 \in S$ and all $n, n_1, n_2 \in N$, and the relation

$$(s_1 s_2)n = s_1(s_2 n) \quad \text{for all } s_1, s_2 \in S, \text{ and all } n \in N. \tag{10.2}$$

A particular case of the latter relation is

$$(sr)n = s(rn) \quad \text{for all } s \in S, r \in R \text{ and } n \in N. \tag{10.2$'$}$$

More generally, if $f : R \to S$ is a ring homomorphism from R into S with $f(1_R) = 1_S$ (for example the injection map if R is a subring of S as above) then it is easy to see that N can be considered as an R-module with $rn = f(r)n$ for $r \in R$ and $n \in N$. In this situation S can be considered as an *extension* of the ring R and the resulting R-module is said to be obtained from N by *restriction of scalars* from S to R.

Suppose now that R is a subring of S and we try to reverse this, namely we start with an R-module N and attempt to define an S-module structure on N that extends the action of R on N to an action of S on N (hence "extending the scalars" from R to S). In general this is impossible, even in the simplest situation: the ring R itself is an R-module but is usually not an S-module for the larger ring S. For example, $\mathbb{Z}$ is a $\mathbb{Z}$-module but it cannot be made into a $\mathbb{Q}$-module (if it could, then $\frac{1}{2} \circ 1 = z$ would be an element of $\mathbb{Z}$ with $z + z = 1$, which is impossible). Although $\mathbb{Z}$ itself cannot be made into a $\mathbb{Q}$-module it is *contained* in a $\mathbb{Q}$-module, namely $\mathbb{Q}$ itself. Put another way, there is an injection (also called an *embedding*) of the $\mathbb{Z}$-module $\mathbb{Z}$ into the $\mathbb{Q}$-module $\mathbb{Q}$ (and similarly the ring R can always be embedded as an R-submodule of the S-module S). This raises the question of whether an arbitrary R-module N can be embedded as an R-submodule of some S-module, or more generally, the question of what R-module homomorphisms exist from N to S-modules. For example, suppose N is a nontrivial *finite* abelian group, say $N = \mathbb{Z}/2\mathbb{Z}$, and consider possible $\mathbb{Z}$-module homomorphisms (i.e., abelian group homomorphisms) of N into some $\mathbb{Q}$-module. A $\mathbb{Q}$-module is just a vector space over $\mathbb{Q}$ and every nonzero element in a vector space over $\mathbb{Q}$ has infinite (additive) order. Since every element of N has finite order, every element of N must map to 0 under such a homomorphism. In other words there are *no* nonzero $\mathbb{Z}$-module homomorphisms from this N to *any* $\mathbb{Q}$-module, much less embeddings of N identifying