4.1.19. Proposition. Two blocks in a graph share at most one vertex.

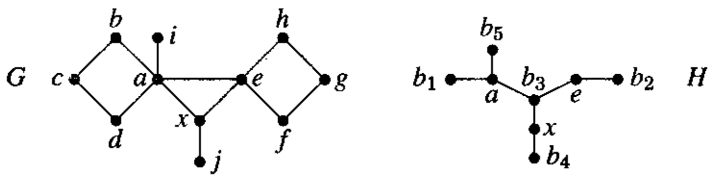
Proof: We use contradiction. Suppose that blocks B_1, B_2 have at least two common vertices. We show that $B_1 \cup B_2$ is a connected subgraph with no cut-vertex, which contradicts the maximality of B_1 and B_2.

When we delete one vertex from B_i, what remains is connected. Hence we retain a path in B_i from every vertex that remains to every vertex of $V(B_1) \cap V(B_2)$ that remains. Since the blocks have at least two common vertices, deleting a single vertex leaves a vertex in the intersection. We retain paths from all vertices to that vertex, so $B_1 \cup B_2$ cannot be disconnected by deleting one vertex. ■

Every edge by itself is a subgraph with no cut-vertex and hence is in a block. We conclude that the blocks of a graph decompose the graph. Blocks in a graph behave somewhat like strong components of a digraph (Definition 1.4.12), but strong components share no vertices (Exercise 1.4.13a). Thus although blocks in a graph decompose the edge set, strong components in a digraph merely partition the vertex set and usually omit edges.

When two blocks of G share a vertex, it must be a cut-vertex of G. The interaction between blocks and cut-vertices is described by a special graph.

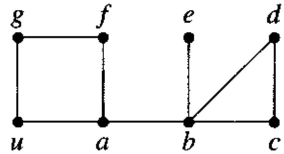
4.1.20.* Definition. The **block-cutpoint graph** of a graph G is a bipartite graph H in which one partite set consists of the cut-vertices of G, and the other has a vertex b_i for each block B_i of G. We include vb_i as an edge of H if and only if $v \in B_i$.



When G is connected, its block-cutpoint graph is a tree (Exercise 34) whose leaves are blocks of G. Thus a graph G that is not a single block has at least two blocks (**leaf blocks**) that each contain exactly one cut-vertex of G.

Blocks can be found using a technique for searching graphs. In **Depth-First Search** (DFS), we explore always from the most recently discovered vertex that has unexplored edges (also called **backtracking**). In contrast, Breadth-First Search (Algorithm 2.3.8) explores from the oldest vertex, so the difference between DFS and BFS is that in DFS we maintain the list of vertices to be searched as a Last-In First-Out "stack" rather than a queue.

4.1.21.* Example. *Depth-First Search.* In the graph below, one depth-first search from u finds the vertices in the order u, a, b, c, d, e, f, g. For both BFS and DFS, the order of discovery depends on the order of exploring edges from a searched vertex. ■



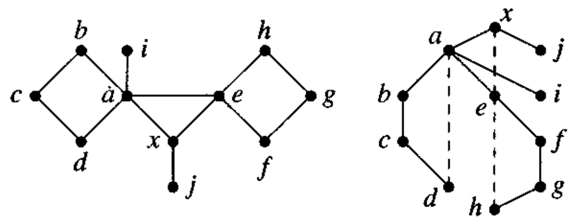
A breath-first or depth-first search from u generates a tree rooted at u; each time exploring a vertex x yields a new vertex v, we include the edge xv. This grows a tree that becomes a spanning tree of the component containing u. Applications of depth-first search rely on a fundamental property of the resulting spanning tree.

4.1.22.* Lemma. If T is a spanning tree of a connected graph G grown by DFS from u, then every edge of G not in T consists of two vertices v, w such that v lies on the u, w-path in T.

Proof: Let vw be an edge of G, with v encountered before w in the depth-first search. Because vw is an edge, we cannot finish v before w is added to T. Hence w appears somewhere in the subtree formed before finishing v, and the path from w to u contains v. ∎

4.1.23.* Algorithm. (Computing the blocks of a graph)
Input: A connected graph G. (The blocks of a graph are the blocks of its components, which can be found by depth-first search, so we may assume that G is connected.)
Idea: Build a depth-first search tree T of G, discarding portions of T as blocks are identified. Maintain one vertex called ACTIVE.
Initialization: Pick a root $x \in V(H)$; make x ACTIVE; set $T = \{x\}$.
Iteration: Let v denote the current active vertex.
1) If v has an unexplored incident edge vw, then
 1A) If $w \notin V(T)$, then add vw to T, mark vw explored, make w ACTIVE.
 1B) If $w \in V(T)$, then w is an ancestor of v; mark vw explored.
2) If v has no more unexplored incident edges, then
 2A) If $v \neq x$, and w is the parent of v, make w ACTIVE. If no vertex in the current subtree T' rooted at v has an explored edge to an ancestor above w, then $V(T') \cup \{w\}$ is the vertex set of a block; record this information and delete $V(T')$ from T.
 2B) If $v = x$, terminate. ∎

4.1.24.* Example. *Finding blocks.* For the graph below, one depth-first traversal from x visits the other vertices in the order $a, b, c, d, e, f, g, h, i, j$. We find blocks in the order $\{a, b, c, d\}$, $\{e, f, g, h\}$, $\{a, i\}$, $\{x, a, e\}$, $\{x, j\}$. After finding each block, we delete the vertices other than the highest. Exercise 36 requests a proof of correctness. ∎



EXERCISES

4.1.1. (−) Give a proof or a counterexample for each statement below.
a) Every graph with connectivity 4 is 2-connected.
b) Every 3-connected graph has connectivity 3.
c) Every k-connected graph is k-edge-connected.
d) Every k-edge-connected graph is k-connected.

4.1.2. (−) Give a counterexample to the following statement, add a hypothesis to correct it, and prove the corrected statement: If e is a cut-edge of G, then at least one vertex of e is a cut-vertex of G.

4.1.3. (−) Let G be an n-vertex simple graph other than K_n. Prove that if G is not k-connected, then G has a separating set of size $k-1$.

4.1.4. (−) Prove that a graph G is k-connected if and only if $G \vee K_r$ (Definition 3.3.6) is $k+r$-connected.

4.1.5. (−) Let G be a connected graph with at least three vertices. Form G' from G by adding an edge with endpoints x, y whenever $d_G(x, y) = 2$. Prove that G' is 2-connected.

4.1.6. (−) For a graph G with blocks $B_1, \ldots, B_k$, prove that $n(G) = (\sum_{i=1}^{k} n(B_i)) - k + 1$.

4.1.7. (−) Obtain a formula for the number of spanning trees of a connected graph in terms of the numbers of spanning trees of its blocks.

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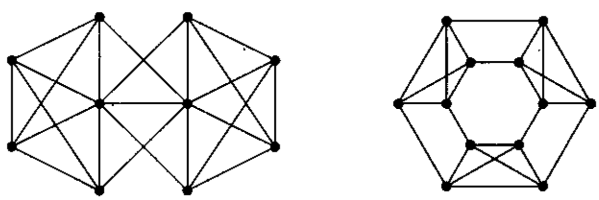
4.1.8. Determine $\kappa(G)$, $\kappa'(G)$, and $\delta(G)$ for each graph G drawn below.



4.1.9. For each choice of integers k, l, m with $0 < k \le l \le m$, construct a simple graph G with $\kappa(G) = k$, $\kappa'(G) = l$, and $\delta(G) = m$. (Chartrand–Harary [1968])

4.1.10. (!) Find (with proof) the smallest 3-regular simple graph having connectivity 1.

4.1.11. Prove that $\kappa'(G) = \kappa(G)$ when G is a simple graph with $\Delta(G) \le 3$.

4.1.12. Let n, k be positive integers with n even, k odd, and $n > k > 1$. Let G be the k-regular simple graph formed by placing n vertices on a circle and making each vertex adjacent to the opposite vertex and to the $(k-1)/2$ nearest vertices in each direction. Prove that $\kappa(G) = k$. (Harary [1962a])

4.1.13. In $K_{m,n}$, let S consist of a vertices from one partite set and b from the other.

a) Compute $\left|[S, \overline{S}]\right|$ in terms of a, b, m, n.

b) Use part (a) to prove numerically that $\kappa'(K_{m,n}) = \min\{m, n\}$.

c) Prove that every set of seven edges in $K_{3,3}$ is a disconnecting set, but no set of seven edges is an edge cut.

4.1.14. (!) Let G be a connected graph in which for every edge e, there are cycles C_1 and C_2 containing e whose only common edge is e. Prove that G is 3-edge-connected. Use this to show that the Petersen graph is 3-edge-connected.

4.1.15. (!) Use Proposition 4.1.12 and Theorem 4.1.11 to prove that the Petersen graph is 3-connected.

4.1.16. Use Proposition 4.1.12 to prove that the Petersen graph has an edge cut of size m if and only if $3 \le m \le 12$. (Hint: Consider $\left|[S, \overline{S}]\right|$ for $1 \le |S| \le 5$.)

4.1.17. Prove that deleting an edge cut of size 3 in the Petersen graph isolates a vertex.

4.1.18. Let G be a triangle-free graph with minimum degree at least 3. Prove that if $n(G) \le 11$, then G is 3-edge-connected. Show that this inequality is sharp by finding a 3-regular bipartite graph with 12 vertices that is not 3-edge-connected. (Galvin)

4.1.19. Prove that $\kappa(G) = \delta(G)$ if G is simple and $\delta(G) \ge n(G) - 2$. Prove that this is best possible for each $n \ge 4$ by constructing a simple n-vertex graph with minimum degree $n - 3$ and connectivity less than $n - 3$.

4.1.20. (!) Let G be a simple n-vertex graph with $n/2 - 1 \le \delta(G) \le n - 2$. Prove that G is k-connected for all k with $k \le 2\delta(G) + 2 - n$. Prove that this is best possible for all $\delta \ge n/2 - 1$ by constructing a simple n-vertex graph with minimum degree δ that is not k-connected for $k = 2\delta + 3 - n$. (Comment: Proposition 1.3.15 is the special case of this when $\delta(G) = (n-1)/2$.)

4.1.21. (+) Let G be a simple n-vertex graph with $n \ge k + l$ and $\delta(G) \ge \frac{n+l(k-2)}{l+1}$. Prove that if $G - S$ has more than l components, then $|S| \ge k$. Prove that the hypothesis on $\delta(G)$ is best possible for $n \ge k + l$ by constructing an appropriate n-vertex graph with minimum degree $\left\lfloor \frac{n+l(k-2)-1}{l+1} \right\rfloor$. (Comment: This generalizes Exercise 4.1.20.)

4.1.22. (!) *Sufficient condition for $k+1$-connected graphs.* (Bondy [1969])

a) Let G be a simple n-vertex graph with vertex degrees $d_1 \le \cdots \le d_n$. Prove that if $d_j \ge j + k$ whenever $j \le n - 1 - d_{n-k}$, then G is $k+1$-connected. (Comment: Exercise 1.3.64 is the special case of this when $k = 0$.)

b) Suppose that $0 \le j + k \le n$. Construct an n-vertex graph G such that $\kappa(G) \le k$ and G has j vertices of degree $j + k - 1$, has $n - j - k$ vertices of degree $n - j - 1$, and has k vertices of degree $n - 1$. In what sense does this show that part (a) is best possible?

4.1.23. (!) Let G be an r-connected graph of even order having no $K_{1,r+1}$ as an induced subgraph. Prove that G has a 1-factor. (Sumner [1974b])

4.1.24. (!) *Degree conditions for $\kappa' = \delta$.* Let G be a simple n-vertex graph. Use Corollary 4.1.13 to prove the following statements.

a) If $\delta(G) \ge \lfloor n/2 \rfloor$, then $\kappa'(G) = \delta(G)$. Prove this best possible by constructing for each $n \ge 3$ a simple n-vertex graph with $\delta(G) = \lfloor n/2 \rfloor - 1$ and $\kappa'(G) < \delta(G)$.

b) If $d(x) + d(y) \geq n - 1$ whenever $x \not\leftrightarrow y$, then $\kappa'(G) = \delta(G)$. Prove that this is best possible by constructing for each $n \geq 4$ and $\delta(G) = m \leq n/2 - 1$ an n-vertex graph G with $\kappa'(G) < \delta(G) = m$ in which $d(x) + d(y) \geq n - 2$ whenever $x \not\leftrightarrow y$.

4.1.25. (!) $\kappa'(G) = \delta(G)$ *for diameter 2.* Let G be a simple graph with diameter 2, and let $[S, \overline{S}]$ be a minimum edge cut with $|S| \leq |\overline{S}|$.

a) Prove that every vertex of S has a neighbor in $\overline{S}$.

b) Use part (a) and Corollary 4.1.13 to prove that $\kappa'(G) = \delta(G)$. (Plesník [1975])

4.1.26. (!) Let F be a set of edges in G. Prove that F is an edge cut if and only if F contains an even number of edges from every cycle in G. For example, when $G = C_n$, every even subset of the edges is an edge cut, but no odd subset is an edge cut. (Hint: For sufficiency, the task is to show that the components of $G - F$ can be grouped into two nonempty collections so that every edge of F has an endpoint in each collection.)

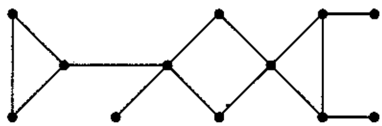
4.1.27. (!) Let $[S, \overline{S}]$ be an edge cut. Prove that there is a set of pairwise edge-disjoint bonds whose union (as edge sets) is $[S, \overline{S}]$. (Note: This is trivial if $[S, \overline{S}]$ is itself a bond.)

4.1.28. (!) Prove that the symmetric difference of two different edge cuts is an edge cut. (Hint: Draw a picture illustrating the two edge cuts and use it to guide the proof.)

4.1.29. (!) Let H be a spanning subgraph of a connected graph G. Prove that H is a spanning tree if and only if the subgraph $H^* = G - E(H)$ is a maximal subgraph that contains no bond. (Comment: See Section 8.2 for a more general context.)

4.1.30. (−) Let G be the simple graph with vertex set $\{1, \ldots, 11\}$ defined by $i \leftrightarrow j$ if and only if i, j have a common factor bigger than 1. Determine the blocks of G.

4.1.31. A **cactus** is a connected graph in which every block is an edge or a cycle. Prove that the maximum number of edges in a simple n-vertex cactus is $\lfloor 3(n-1)/2 \rfloor$. (Hint: $\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor$.)



4.1.32. Prove that every vertex of a graph has even degree if and only if every block is Eulerian.

4.1.33. Prove that a connected graph is k-edge-connected if and only if each of its blocks is k-edge-connected.

4.1.34. (!) *The block-cutpoint graph* (see Definition 4.1.20). Let H be the block-cutpoint graph of a graph G that has a cut-vertex. (Harary–Prins [1966])

a) Prove that H is a forest.

b) Prove that G has at least two blocks each of which contains exactly one cut-vertex of G.

c) Prove that a graph G with k components has exactly $k + \sum_{v \in V(G)} (b(v) - 1)$ blocks, where $b(v)$ is the number of blocks containing v.

d) Prove that every graph has fewer cut-vertices than blocks.

4.1.35. Let H and H' be two maximal k-connected subgraphs of a graph G. Prove that they have at most $k - 1$ common vertices. (Harary–Kodama [1964])

4.1.36. Prove that Algorithm 4.1.23 correctly computes blocks of graphs.

4.1.37. Develop an algorithm to compute the strong components of a digraph. Prove that it works. (Hint: Model the algorithm on Algorithm 4.1.23).

4.2. k-Connected Graphs

A communication network is fault-tolerant if it has alternative paths between vertices: the more disjoint paths, the better. In this section, we prove that this alternative measure of connection is essentially the same as k-connectedness. When $k = 1$, the definition already states that a graph G is 1-connected if and only if each pair of vertices is connected by a path. For larger k the equivalence is more subtle.

2-CONNECTED GRAPHS

We begin by characterizing 2-connected graphs.

4.2.1. Definition. Two paths from u to v are **internally disjoint** if they have no common internal vertex.

4.2.2. Theorem. (Whitney [1932a]) A graph G having at least three vertices is 2-connected if and only if for each pair $u, v \in V(G)$ there exist internally disjoint u, v-paths in G.

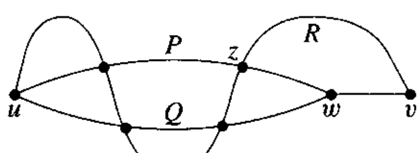
Proof: *Sufficiency.* When G has internally disjoint u, v-paths, deletion of one vertex cannot separate u from v. Since this condition is given for every pair u, v, deletion of one vertex cannot make any vertex unreachable from any other. We conclude that G is 2-connected.

Necessity. Suppose that G is 2-connected. We prove by induction on $d(u, v)$ that G has internally disjoint u, v-paths.

Basis step ($d(u, v) = 1$). When $d(u, v) = 1$, the graph $G - uv$ is connected, since $\kappa'(G) \geq \kappa(G) \geq 2$. A u, v-path in $G - uv$ is internally disjoint in G from the u, v-path formed by the edge uv itself.

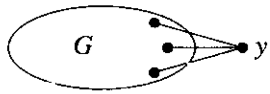
Induction step ($d(u, v) > 1$). Let $k = d(u, v)$. Let w be the vertex before v on a shortest u, v-path; we have $d(u, w) = k - 1$. By the induction hypothesis, G has internally disjoint u, w-paths P and Q. If $v \in V(P) \cup V(Q)$, then we find the desired paths in the cycle $P \cup Q$. Suppose not.

Since G is 2-connected, $G - w$ is connected and contains a u, v-path R. If R avoids P or Q, we are done, but R may share internal vertices with both P and Q. Let z be the last vertex of R (before v) belonging to $P \cup Q$. By symmetry, we may assume that $z \in P$. We combine the u, z-subpath of P with the z, v-subpath of R to obtain a u, v-path internally disjoint from $Q \cup wv$. ■



4.2.3. Lemma. (Expansion Lemma) If G is a k-connected graph, and G' is obtained from G by adding a new vertex y with at least k neighbors in G, then G' is k-connected.

Proof: We prove that a separating set S of G' must have size at least k. If $y \in S$, then $S - \{y\}$ separates G, so $|S| \geq k + 1$. If $y \notin S$ and $N(y) \subseteq S$, then $|S| \geq k$. Otherwise, y and $N(y) - S$ lie in a single component of $G' - S$. Thus again S must separate G and $|S| \geq k$. ■



4.2.4. Theorem. For a graph G with at least three vertices, the following conditions are equivalent (and characterize 2-connected graphs).
A) G is connected and has no cut-vertex.
B) For all $x, y \in V(G)$, there are internally disjoint x, y-paths.
C) For all $x, y \in V(G)$, there is a cycle through x and y.
D) $\delta(G) \geq 1$, and every pair of edges in G lies on a common cycle.

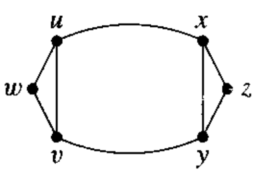
Proof: Theorem 4.2.2 proves A$\Leftrightarrow$B.

For B$\Leftrightarrow$C, note that cycles containing x and y correspond to pairs of internally disjoint x, y-paths.

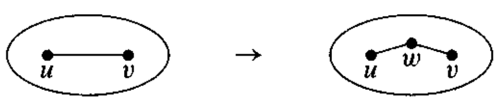
For D$\Rightarrow$C, the condition $\delta(G) \geq 1$ implies that vertices x and y are not isolated; we then apply the last part of D to edges incident to x and y. If there is only one such edge, then we use it and any edge incident to a third vertex.

To complete the proof, we assume that G satisfies the equivalent properties A and C and then derive D. Since G is connected, $\delta(G) \geq 1$. Now consider two edges uv and xy. Add to G the vertices w with neighborhood $\{u, v\}$ and z with neighborhood $\{x, y\}$. Since G is 2-connected, the Expansion Lemma (Lemma 4.2.3) implies that the resulting graph G' is 2-connected.

Hence condition C holds in G', so w and z lie on a cycle C in G'. Since w, z each have degree 2, C must contain the paths u, w, v and x, z, y but not the edges uv or xy. Replacing the paths u, w, v and x, z, y in C with the edges uv and xy yields the desired cycle through uv and xy in G. ■



4.2.5. Definition. In a graph G, **subdivision** of an edge uv is the operation of replacing uv with a path u, w, v through a new vertex w.



4.2.6. Corollary. If G is 2-connected, then the graph G' obtained by subdividing an edge of G is 2-connected.

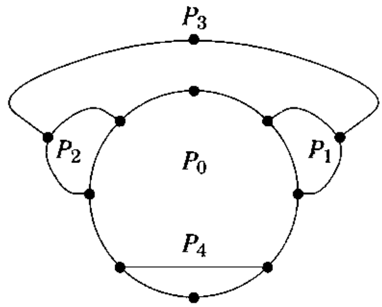
Proof: Let G' be formed from G by adding vertex w to subdivide uv. To show that G' is 2-connected, it suffices to find a cycle through arbitrary edges e, f of G' (by Theorem 4.2.4D).

Since G is 2-connected, any two edges of G lie on a common cycle (Theorem 4.2.4D). When our given edges e, f of G' lie in G, a cycle through them in G is also in G', unless it uses uv, in which case we modify the cycle. Here "modify the cycle" means "replace the edge uv with the u, v-path of length 2 through w".

When $e \in E(G)$ and $f \in \{uw, wv\}$, we modify a cycle passing through e and uv in G. When $\{e, f\} = \{uw, wv\}$, we modify a cycle through uv. ■

The class of 2-connected graphs has a characterization that expresses the construction of each such graph from a cycle and paths.

4.2.7. Definition. An **ear** of a graph G is a maximal path whose internal vertices have degree 2 in G. An **ear decomposition** of G is a decomposition $P_0, \ldots, P_k$ such that P_0 is a cycle and P_i for $i \geq 1$ is an ear of $P_0 \cup \cdots \cup P_i$.

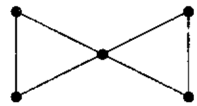


4.2.8. Theorem. (Whitney [1932a]) A graph is 2-connected if and only if it has an ear decomposition. Furthermore, every cycle in a 2-connected graph is the initial cycle in some ear decomposition.

Proof: *Sufficiency.* Since cycles are 2-connected, it suffices to show that adding an ear preserves 2-connectedness. Let u, v be the endpoints of an ear P to be added to a 2-connected graph G. Adding an edge cannot reduce connectivity, so $G + uv$ is 2-connected. A succession of edge subdivisions converts $G + uv$ into the graph $G \cup P$ in which P is an ear; by Corollary 4.2.6, each subdivision preserves 2-connectedness.

Necessity. Given a 2-connected graph G, we build an ear decomposition of G from a cycle C in G. Let $G_0 = C$. Let G_i be a subgraph obtained by successively adding i ears. If $G_i \neq G$, then we can choose an edge uv of $G - E(G_i)$ and an edge $xy \in E(G_i)$. Because G is 2-connected, uv and xy lie on a common cycle C'. Let P be the path in C' that contains uv and exactly two vertices of G_i, one at each end of P. Now P can be added to G_i to obtain a larger subgraph G_{i+1} in which P is an ear. The process ends only by absorbing all of G. ■

Every 2-connected graph is 2-edge-connected, but the converse does not hold. Recall that the bowtie is the graph consisting of two triangles sharing one common vertex; it is 2-edge-connected but not 2-connected. Since more graphs are 2-edge-connected, decomposition of 2-edge-connected graphs needs a more general operation. The proof is like that of Theorem 4.2.8.



4.2.9. Definition. A **closed ear** in a graph G is a cycle C such that all vertices of C except one have degree 2 in G. A **closed-ear decomposition** of G is a decomposition $P_0, \ldots, P_k$ such that P_0 is a cycle and P_i for $i \geq 1$ is either an (open) ear or a closed ear in G.

4.2.10. Theorem. A graph is 2-edge-connected if and only if it has a closed-ear decomposition, and every cycle in a 2-edge-connected graph is the initial cycle in some such decomposition.

Proof: *Sufficiency.* Cut-edges are the edges not on cycles (Theorem 1.2.14), so a connected graph is 2-edge-connected if and only if every edge lies on a cycle. The initial cycle is 2-edge-connected. When we add a closed ear, its edges form a cycle. When we add an open ear P to a connected graph G, a path in G connecting the endpoints of P completes a cycle containing all edges of P. In each case, the new graph also is connected. Thus adding an open or closed ear preserves 2-edge-connectedness.

Necessity. Given a 2-edge-connected graph G, let P_0 be a cycle in G. Consider a closed-ear decomposition $P_0, \ldots, P_i$ of a subgraph G_i of G. When $G_i \neq G$, we find an ear to add. Since G is connected, there is an edge $uv \in E(G) - E(G_i)$ with $u \in V(G_i)$. Since G is 2-edge-connected, uv lies on a cycle C. Follow C until it returns to $V(G_i)$, forming up to this point a path or cycle P. Adding P to G_i yields a larger subgraph G_{i+1} in which P is an open or closed ear. The process ends only by absorbing all of G. ■

CONNECTIVITY OF DIGRAPHS

Our results about k-connected and k-edge-connected graphs will apply as well for digraphs, where we use analogous terminology.

4.2.11. Definition. A **separating set** or **vertex cut** of a digraph D is a set $S \subseteq V(D)$ such that $D - S$ is not strongly connected. A digraph is **k-connected** if every vertex cut has at least k vertices. The minimum size of a vertex cut is the **connectivity** $\kappa(D)$.

For vertex sets S, T in a digraph D, let $[S, T]$ denote the set of edges with tail in S and head in T. An **edge cut** is the set $[S, \overline{S}]$ for some $\emptyset \neq$

$S \subset V(D)$. A digraph is **k-edge-connected** if every edge cut has at least k edges. The minimum size of an edge cut is the **edge-connectivity** $\kappa'(D)$.

4.2.12. Remark. Because $|[S, \overline{S}]|$ is the number of edges leaving S, we can restate the definition of edge-connectivity as follows: A graph or digraph G is k-edge-connected if and only if for every nonempty proper vertex subset S, there are at least k edges in G leaving S.

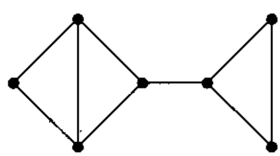
Note that $[S, T]$ is the set of edge from S to T. The meaning of this depends on whether we are discussing a graph or a digraph. In a graph, we take all edges that have endpoints in both sets. In a digraph, we take only the edges with tail in S and head in T. ∎

Strong digraphs are similar to 2-edge-connected graphs.

4.2.13. Proposition. Adding a (directed) ear to a strong digraph produces a larger strong digraph.

Proof: By Remark 4.2.12, a digraph is strong if and only if for every nonempty vertex subset there is a departing edge. If we add an open ear or closed ear P to a strong digraph D, then for every set S with $\emptyset \subset S \subset V(D)$ we already have an edge from S to $V(D) - S$. We need only consider sets that don't intersect $V(D)$ and sets that contain all of $V(D)$ but not all of $V(P)$. For every such set, there is an edge leaving it along P. ∎

When can the streets in a road network all be made one-way without making any location unreachable from some other location? In other words, when does a graph have a strong orientation? The graph below does not. The obvious necessary conditions are sufficient.



4.2.14. Theorem. (Robbins [1939]) A graph has a strong orientation if and only if it is 2-edge-connected.

Proof: *Necessity.* If a graph G is disconnected, then some vertices cannot reach others in any orientation. If G has a cut-edge xy oriented from x to y in an orientation D, then y cannot reach x in D. Hence G must be connected and have no cut-edge.

Sufficiency. When G is 2-edge-connected, it has a closed-ear decomposition. We orient the initial cycle consistently to obtain a strong digraph. As we add each new ear and direct it consistently, Proposition 4.2.13 guarantees that we still have a strong digraph. ∎

Robbins' Theorem generalizes for all k. When G has a k-edge-connected orientation, Remark 4.2.12 implies that G must be $2k$-edge-connected. Nash-Williams [1960] proved that this obvious necessary condition is also sufficient: a graph has a k-edge-connected orientation if and only if it is $2k$-edge-connected. This is easy when G is Eulerian (Exercise 21), but the general case is difficult (see Exercises 36–38). A thorough discussion of this and other orientation theorems appears in Frank [1993].

k-CONNECTED AND k-EDGE-CONNECTED GRAPHS

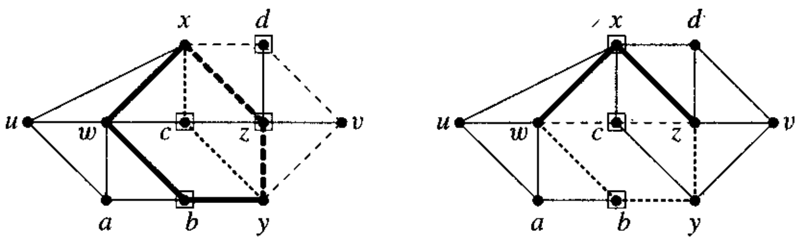
We have introduced two measures of good connection: invulnerability to deletions and multiplicity of alternative paths. Extending Whitney's Theorem, we show that these two notions are the same, for both vertex deletions and edge deletions, and for both graphs and digraphs.

We first discuss the "local" problem of x, y-paths for a fixed pair $x, y \in V(G)$. These definitions hold both for graphs and for digraphs.

4.2.15. Definition. Given $x, y \in V(G)$, a set $S \subseteq V(G) - \{x, y\}$ is an x, y-**separator** or x, y-**cut** if $G - S$ has no x, y-path. Let $\kappa(x, y)$ be the minimum size of an x, y-cut. Let $\lambda(x, y)$ be the maximum size of a set of pairwise internally disjoint x, y-paths. For $X, Y \subseteq V(G)$, an X, Y-**path** is a path having first vertex in X, last vertex in Y, and no other vertex in $X \cup Y$.

An x, y-cut must contain an internal vertex of every x, y-path, and no vertex can cut two internally disjoint x, y-paths. Therefore, always $\kappa(x, y) \geq \lambda(x, y)$. Thus the problems of finding the smallest cut and the largest set of paths are dual problems, like the duality between matching and covering in Chapter 3.

4.2.16. Example. In the graph G below, the set $S = \{b, c, z, d\}$ is an x, y-cut of size 4; thus $\kappa(x, y) \leq 4$. As shown on the left, G has four pairwise internally disjoint x, y-paths; thus $\lambda(x, y) \geq 4$. Since $\kappa(x, y) \geq \lambda(x, y)$ always, we have $\kappa(x, y) = \lambda(x, y) = 4$.



Consider also the pair w, z. As shown on the right, $\kappa(w, z) = \lambda(w, z) = 3$, with $\{b, c, x\}$ being a minimum w, z-cut. The graph G is 3-connected; for every pair $u, v \in V(G)$, we can find three pairwise internally disjoint u, v-paths.

From the equality for internally disjoint paths, we will obtain an analogous equality for edge-disjoint paths. Although $\kappa(w, z) = 3$ above, it takes four edges to break all w, z-paths, and there are four pairwise edge-disjoint w, z-paths. ■

What we call Menger's Theorem states that the local equality $\kappa(x, y) = \lambda(x, y)$ always holds. The global statement for connectivity and analogous results for edge-connectivity and digraphs were observed by others. All are considered forms of Menger's Theorem. More than 15 proofs of Menger's Theorem have been published, some yielding stronger results, some incorrect. (A gap in Menger's original argument was later repaired by König.)

4.2.17. Theorem. (Menger [1927]) If x, y are vertices of a graph G and $xy \notin E(G)$, then the minimum size of an x, y-cut equals the maximum number of pairwise internally disjoint x, y-paths.

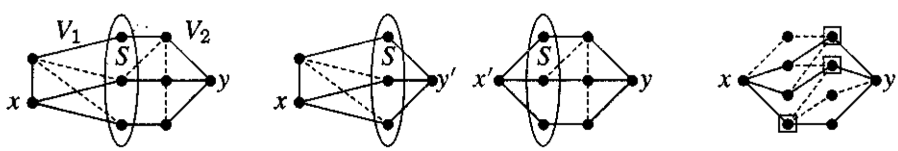
Proof: An x, y-cut must contain an internal vertex from each path in a set of pairwise internally disjoint x, y-paths. These vertices must be distinct, so $\kappa(x, y) \geq \lambda(x, y)$.

To prove equality, we use induction on $n(G)$. Basis step: $n(G) = 2$. Here $xy \notin E(G)$ yields $\kappa(x, y) = \lambda(x, y) = 0$. Induction step: $n(G) > 2$. Let $k = \kappa_G(x, y)$. We construct k pairwise internally disjoint x, y-paths. Note that since $N(x)$ and $N(y)$ are x, y-cuts, no minimum cut properly contains $N(x)$ or $N(y)$.

Case 1: G has a minimum x, y-cut S other than $N(x)$ or $N(y)$. To obtain the k desired paths, we combine x, S-paths and S, y-paths obtained from the induction hypothesis (as formed by solid edges shown below). Let V_1 be the set of vertices on x, S-paths, and let V_2 be the set of vertices on S, y-paths. We claim that $S = V_1 \cap V_2$. Since S is a minimal x, y-cut, every vertex of S lies on an x, y-path, and hence $S \subseteq V_1 \cap V_2$. If $v \in (V_1 \cap V_2)$, then following the x, v-portion of some x, S-path and then the v, y-portion of some S, y-path yields an x, y-path that avoids the x, y-cut S. This is impossible, so $S = V(G_1) \cap V(G_2)$. By the same argument, V_1 omits $N(y) - S$ and V_2 omits $N(x) - S$.

Form H_1 by adding to $G[V_1]$ a vertex y' with edges from S. Form H_2 by adding to $G[V_2]$ a vertex x' with edges to S. Every x, y-path in G starts with an x, S-path (contained in H_1), so every x, y'-cut in H_1 is an x, y-cut in G. Therefore, $\kappa_{H_1}(x, y') = k$, and similarly $\kappa_{H_2}(x', y) = k$.

Since V_1 omits $N(y) - S$ and V_2 omits $N(x) - S$, both H_1 and H_2 are smaller than G. Hence the induction hypothesis yields $\lambda_{H_1}(x, y') = k = \lambda_{H_2}(x', y)$. Since $V_1 \cap V_2 = S$, deleting y' from the k paths in H_1 and x' from the k paths in H_2 yields the desired x, S-paths and S, y-paths in G that combine to form k pairwise internally disjoint x, y-paths in G.



G H_1 H_2 G'

Case 1 Case 2

Case 2. Every minimum x, y-cut is $N(x)$ or $N(y)$. Again we construct the k desired paths. In this case, every vertex outside $\{x \cup N(x) \cup N(y) \cup y\}$ is in no minimum x, y-cut. If G has such a vertex v, then $\kappa_{G-v}(x, y) = k$, and applying the induction hypothesis to $G - v$ yields the desired x, y-paths in G. Also, if there exists $u \in N(x) \cap N(y)$, then u appears in every x, y-cut, and $\kappa_{G-u}(x, y) = k - 1$. Now applying the induction hypothesis to $G - v$ yields $k - 1$ paths to combine with the path x, v, y.

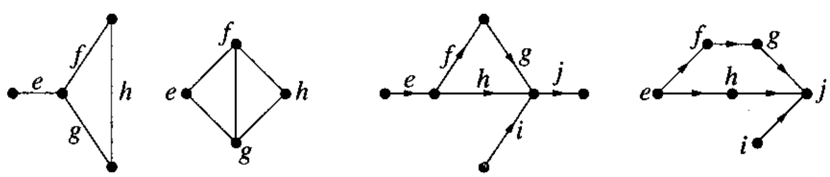
We may thus assume that $N(x)$ and $N(y)$ partition $V(G) - \{x, y\}$. Let G' be the bipartite graph with bipartition $N(x), N(y)$ and edge set $[N(x), N(y)]$. Every x, y-path in G uses some edge from $N(x)$ to $N(y)$, so the x, y-cuts in G are precisely the vertex covers of G'. Hence $\beta(G') = k$. By the König–Egerváry Theorem, G' has a matching of size k. These k edges yield k pairwise internally disjoint x, y-paths of length 3. ■

Case 2 is needed in the proof because when $S = N(x)$, the induction hypothesis cannot be used to obtain the S, y-paths.

The statement of Theorem 4.2.17 makes sense also for digraphs. The proof of the digraph version is exactly the same; we only need to replace $N(x)$ and $N(y)$ with $N^+(x)$ and $N^-(y)$ throughout.

We next develop the analogue of Theorem 4.2.17 for edge-disjoint paths, which we prove by applying Theorem 4.2.17 to a transformed graph. The main part of the transformation is an operation that we will use again in Chapter 7.

4.2.18. Definition. The **line graph** of a graph G, written $L(G)$, is the graph whose vertices are the edges of G, with $ef \in E(L(G))$ when $e = uv$ and $f = vw$ in G. Substituting "digraph" for "graph" in this sentence yields the definition of **line digraph**. For graphs, e and f share a vertex; for digraphs, the head of e must be the tail of f.



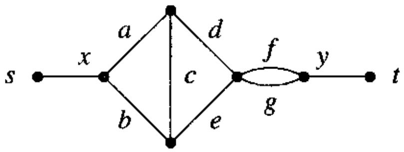
G $L(G)$ H $L(H)$

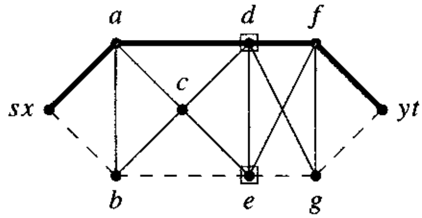
When disconnecting y from x by deleting edges, we use notation analogous to that of Definition 4.2.15: $\lambda'(x, y)$ is the maximum size of a set of pairwise edge-disjoint x, y-paths, and $\kappa'(x, y)$ is the minimum number of edges whose deletion makes y unreachable from x. Elias–Feinstein–Shannon [1956] and Ford–Fulkerson [1956] proved that always $\lambda'(x, y) = \kappa'(x, y)$ (using the methods of Section 4.3). We allow multiple edges and allow $xy \in E(G)$.

4.2.19. Theorem. If x and y are distinct vertices of a graph or digraph G, then the minimum size of an x, y-disconnecting set of edges equals the maximum number of pairwise edge-disjoint x, y-paths.

Proof: Modify G to obtain G' by adding two new vertices s, t and two new edges sx and yt. This does not change $\kappa'(x, y)$ or $\lambda'(x, y)$, and we can think of each path as starting from the edge sx and ending with the edge yt. A set of edges disconnects y from x in G if and only if the corresponding vertices of $L(G')$ form an sx, yt-cut. Similarly, edge-disjoint x, y-paths in G become internally disjoint sx, yt-paths in $L(G')$, and vice versa. Since $x \neq y$, we have no edge from sx to ty in $L(G')$. Applying Theorem 4.2.17 to $L(G')$ yields

$$\kappa'_G(x, y) = \kappa_{L(G')}(sx, yt) = \lambda_{L(G')}(sx, yt) = \lambda'_G(x, y). \quad \blacksquare$$





The global version for k-connected graphs, observed first by Whitney [1932a], is also commonly called Menger's Theorem. The global versions for edges and digraphs appeared in Ford–Fulkerson [1956].

4.2.20. Lemma. Deletion of an edge reduces connectivity by at most 1.

Proof: We discuss only graphs; the argument for digraphs is similar (Exercise 7). Since every separating set of G is a separating set of $G - xy$, we have $\kappa(G - xy) \leq \kappa(G)$. Equality holds unless $G - xy$ has a separating set S that is smaller than $\kappa(G)$ and hence is not a separating set of G. Since $G - S$ is connected, $G - xy - S$ has two components $G[X]$ and $G[Y]$, with $x \in X$ and $y \in Y$. In $G - S$, the only edge joining X and Y is xy.

If $|X| \geq 2$, then $S \cup \{x\}$ is a separating set of G, and $\kappa(G) \leq \kappa(G - xy) + 1$. If $|Y| \geq 2$, then again the inequality holds. In the remaining case, $|S| = n(G) - 2$. Since we have assumed that $|S| < \kappa(G)$, $|S| = n(G) - 2$ implies that $\kappa(G) \geq n(G) - 1$, which holds only for a complete graph. Thus $\kappa(G - xy) = n(G) - 2 = \kappa(G) - 1$, as desired. $\blacksquare$

4.2.21. Theorem. The connectivity of G equals the maximum k such that $\lambda(x, y) \geq k$ for all $x, y \in V(G)$. The edge-connectivity of G equals the maximum k such that $\lambda'(x, y) \geq k$ for all $x, y \in V(G)$. Both statements hold for graphs and for digraphs.

Proof: Since $\kappa'(G) = \min_{x,y \in V(G)} \kappa'(x, y)$, Theorem 4.2.19 immediately yields the claim for edge-connectivity.

For connectivity, we have $\kappa(x, y) = \lambda(x, y)$ for $xy \notin E(G)$, and $\kappa(G)$ is the minimum of these values. We need only show that $\lambda(x, y)$ cannot be smaller than $\kappa(G)$ when $xy \in E(G)$. Certainly deletion of xy reduces $\lambda(x, y)$ by 1, since xy itself is an x, y-path and cannot contribute to any other x, y-path. With this, Theorem 4.2.17, and Lemma 4.2.20, we have

$$\lambda_G(x, y) = 1 + \lambda_{G-xy}(x, y) = 1 + \kappa_{G-xy}(x, y) \geq 1 + \kappa(G - xy) \geq \kappa(G). \quad \blacksquare$$

APPLICATIONS OF MENGER'S THEOREM

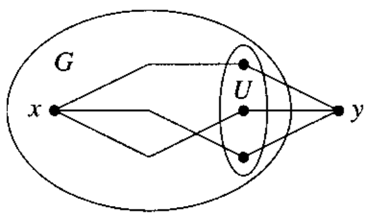
Dirac extended Menger's Theorem to other families of paths.

4.2.22. Definition. Given a vertex x and a set U of vertices, an x, U**-fan** is a set of paths from x to U such that any two of them share only the vertex x.

4.2.23. Theorem. (Fan Lemma, Dirac [1960]). A graph is k-connected if and only if it has at least $k+1$ vertices and, for every choice of x, U with $|U| \geq k$, it has an x, U-fan of size k.

Proof: *Necessity.* Given k-connected graph G, we construct G' from G by adding a new vertex y adjacent to all of U. The Expansion Lemma (Lemma 4.2.3) implies that G' also is k-connected, and then Menger's Theorem yields k pairwise internally disjoint x, y-paths in G'. Deleting y from these paths produces an x, U-fan of size k in G.

Sufficiency. Suppose that G satisfies the fan condition. For $v \in V(G)$ and $U = V(G) - \{v\}$, there is a v, U-fan of size k; thus $\delta(G) \geq k$. Given $w, z \in V(G)$, let $U = N(z)$. Since $|U| \geq k$, we have an w, U-fan of size k; extend each path by adding an edge to z. We obtain k pairwise internally disjoint w, z-paths, so $\lambda(w, z) \geq k$. This holds for all $w, z \in V(G)$, so G is k-connected. ■



The Fan Lemma generalizes considerably. Whenever X and Y are disjoint sets of vertices in a k-connected graph G and we specify integers at X and Y summing to k in each set, there are k pairwise internally disjoint X, Y-paths with the specified number ending at each point (Exercise 28). The Fan Lemma also yields the next result.

4.2.24.* Theorem. (Dirac [1960]) If G is a k-connected graph (with $k \geq 2$), and S is a set of k vertices in G, then G has a cycle including S in its vertex set.

Proof: We use induction on k. Basis step ($k = 2$): Theorem 4.2.2 (or Theorem 4.2.21) implies that any two vertices are connected by two internally disjoint paths, which form a cycle containing them.

Induction step ($k > 2$): With G and S as specified, choose $x \in S$. Since G is also $k-1$-connected, the induction hypothesis implies that all of $S - \{x\}$ lies on a cycle C. Suppose first that $n(C) = k - 1$. Since G is $k-1$-connected, we have an $x, V(C)$-fan of size $k-1$, and the paths of the fan to two consecutive vertices of C enlarge the cycle to include x.

Hence we may assume that $n(C) \geq k$. Since G is k-connected, G has an