face Φ to which (*c*) can be applied to show that $\int_\Phi \zeta = 0$. The same thing holds when u is fixed. By (*a*) and Stokes' theorem,

$$\int_{\partial\Psi} \zeta = \int_\Psi d\zeta = 0.$$

(*e*) Put $\lambda = -(z/r)\eta$, where

$$\eta = \frac{x\,dy - y\,dx}{x^2 + y^2},$$

as in Exercise 21. Then λ is a 1-form in the open set $V \subset R^3$ in which $x^2 + y^2 > 0$. Show that ζ is *exact in* V by showing that

$$\zeta = d\lambda.$$

(*f*) Derive (*d*) from (*e*), without using (*c*).

Hint: To begin with, assume $0 < u < \pi$ on E. By (*e*),

$$\int_\Omega \zeta = \int_{\partial\Omega} \lambda \quad \text{and} \quad \int_S \zeta = \int_{\partial S} \lambda.$$

Show that the two integrals of λ are equal, by using part (*d*) of Exercise 21, and by noting that z/r is the same at $\Sigma(u, v)$ as at $\Omega(u, v)$.

(*g*) Is ζ exact in the complement of every line through the origin?

23. Fix n. Define $r_k = (x_1^2 + \cdots + x_k^2)^{1/2}$ for $1 \le k \le n$, let E_k be the set of all $\mathbf{x} \in R^n$ at which $r_k > 0$, and let ω_k be the $(k-1)$-form defined in E_k by

$$\omega_k = (r_k)^{-k} \sum_{i=1}^{k} (-1)^{i-1} x_i\, dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_k.$$

Note that $\omega_2 = \eta$, $\omega_3 = \zeta$, in the terminology of Exercises 21 and 22. Note also that

$$E_1 \subset E_2 \subset \cdots \subset E_n = R^n - \{\mathbf{0}\}.$$

(*a*) Prove that $d\omega_k = 0$ in E_k.

(*b*) For $k = 2, \ldots, n$, prove that ω_k is exact in E_{k-1}, by showing that

$$\omega_k = d(f_k \omega_{k-1}) = (df_k) \wedge \omega_{k-1},$$

where $f_k(\mathbf{x}) = (-1)^k g_k(x_k/r_k)$ and

$$g_k(t) = \int_{-1}^{t} (1 - s^2)^{(k-3)/2}\, ds \qquad (-1 < t < 1).$$

Hint: f_k satisfies the differential equations

$$\mathbf{x} \cdot (\nabla f_k)(\mathbf{x}) = 0$$

and

$$(D_k f_k)(\mathbf{x}) = \frac{(-1)^k (r_{k-1})^{k-1}}{(r_k)^k}.$$

(*c*) Is ω_n exact in E_n?

(*d*) Note that (*b*) is a generalization of part (*e*) of Exercise 22. Try to extend some of the other assertions of Exercises 21 and 22 to ω_n, for arbitrary n.

24. Let $\omega = \Sigma a_i(\mathbf{x})\, dx_i$ be a 1-form of class $\mathscr{C}''$ in a convex open set $E \subset R^n$. Assume $d\omega = 0$ and prove that ω is exact in E, by completing the following outline:

Fix $\mathbf{p} \in E$. Define

$$f(\mathbf{x}) = \int_{[\mathbf{p},\mathbf{x}]} \omega \qquad (\mathbf{x} \in E).$$

Apply Stokes' theorem to affine-oriented 2-simplexes $[\mathbf{p}, \mathbf{x}, \mathbf{y}]$ in E. Deduce that

$$f(\mathbf{y}) - f(\mathbf{x}) = \sum_{i=1}^{n} (y_i - x_i) \int_0^1 a_i((1-t)\mathbf{x} + t\mathbf{y})\, dt$$

for $\mathbf{x} \in E$, $\mathbf{y} \in E$. Hence $(D_i f)(\mathbf{x}) = a_i(\mathbf{x})$.

25. Assume that ω is a 1-form in an open set $E \subset R^n$ such that

$$\int_\gamma \omega = 0$$

for every closed curve γ in E, of class $\mathscr{C}'$. Prove that ω is exact in E, by imitating part of the argument sketched in Exercise 24.

26. Assume ω is a 1-form in $R^3 - \{\mathbf{0}\}$, of class $\mathscr{C}'$ and $d\omega = 0$. Prove that ω is exact in $R^3 - \{\mathbf{0}\}$.

Hint: Every closed continuously differentiable curve in $R^3 - \{\mathbf{0}\}$ is the boundary of a 2-surface in $R^3 - \{\mathbf{0}\}$. Apply Stokes' theorem and Exercise 25.

27. Let E be an open 3-cell in R^3, with edges parallel to the coordinate axes. Suppose $(a, b, c) \in E$, $f_i \in \mathscr{C}'(E)$ for $i = 1, 2, 3$,

$$\omega = f_1\, dy \wedge dz + f_2\, dz \wedge dx + f_3\, dx \wedge dy,$$

and assume that $d\omega = 0$ in E. Define

$$\lambda = g_1\, dx + g_2\, dy$$

where

$$g_1(x, y, z) = \int_c^z f_2(x, y, s)\, ds - \int_b^y f_3(x, t, c)\, dt$$

$$g_2(x, y, z) = -\int_c^z f_1(x, y, s)\, ds,$$

for $(x, y, z) \in E$. Prove that $d\lambda = \omega$ in E.

Evaluate these integrals when $\omega = \zeta$ and thus find the form λ that occurs in part (*e*) of Exercise 22.

28. Fix $b > a > 0$, define

$$\Phi(r, \theta) = (r \cos \theta, r \sin \theta)$$

for $a \leq r \leq b$, $0 \leq \theta \leq 2\pi$. (The range of Φ is an annulus in R^2.) Put $\omega = x^3\, dy$, and compute both

$$\int_{\Phi} d\omega \quad \text{and} \quad \int_{\partial\Phi} \omega$$

to verify that they are equal.

29. Prove the existence of a function α with the properties needed in the proof of Theorem 10.38, and prove that the resulting function F is of class $\mathscr{C}'$. (Both assertions become trivial if E is an open cell or an open ball, since α can then be taken to be a constant. Refer to Theorem 9.42.)

30. If $\mathbf{N}$ is the vector given by (135), prove that

$$\det \begin{bmatrix} \alpha_1 & \beta_1 & \alpha_2\beta_3 - \alpha_3\beta_2 \\ \alpha_2 & \beta_2 & \alpha_3\beta_1 - \alpha_1\beta_3 \\ \alpha_3 & \beta_3 & \alpha_1\beta_2 - \alpha_2\beta_1 \end{bmatrix} = |\mathbf{N}|^2.$$

Also, verify Eq. (137).

31. Let $E \subset R^3$ be open, suppose $g \in \mathscr{C}''(E)$, $h \in \mathscr{C}''(E)$, and consider the vector field

$$\mathbf{F} = g \nabla h.$$

(*a*) Prove that

$$\nabla \cdot F = g \nabla^2 h + (\nabla g) \cdot (\nabla h)$$

where $\nabla^2 h = \nabla \cdot (\nabla h) = \Sigma \partial^2 h/\partial x_i^2$ is the so-called "Laplacian" of h.

(*b*) If Ω is a closed subset of E with positively oriented boundary $\partial\Omega$ (as in Theorem 10.51), prove that

$$\int_{\Omega} [g \nabla^2 h + (\nabla g) \cdot (\nabla h)] dV = \int_{\partial\Omega} g \frac{\partial h}{\partial n} dA$$

where (as is customary) we have written $\partial h/\partial n$ in place of $(\nabla h) \cdot \mathbf{n}$. (Thus $\partial h/\partial n$ is the directional derivative of h in the direction of the outward normal to $\partial\Omega$, the so-called *normal derivative* of h.) Interchange g and h, subtract the resulting formula from the first one, to obtain

$$\int_{\Omega} (g \nabla^2 h - h \nabla^2 g)\, dV = \int_{\partial\Omega} \left(g \frac{\partial h}{\partial n} - h \frac{\partial g}{\partial n} \right) dA.$$

These two formulas are usually called *Green's identities.*

(*c*) Assume that h is *harmonic* in E; this means that $\nabla^2 h = 0$. Take $g = 1$ and conclude that

$$\int_{\partial\Omega} \frac{\partial h}{\partial n} dA = 0.$$

Take $g = h$, and conclude that $h = 0$ in Ω if $h = 0$ on $\partial\Omega$.

(*d*) Show that Green's identities are also valid in R^2.

32. Fix δ, $0 < \delta < 1$. Let D be the set of all $(\theta, t) \in R^2$ such that $0 \leq \theta \leq \pi$, $-\delta \leq t \leq \delta$. Let Φ be the 2-surface in R^3, with parameter domain D, given by

$$\begin{aligned} x &= (1 - t \sin\theta)\cos 2\theta \\ y &= (1 - t\sin\theta)\sin 2\theta \\ z &= t\cos\theta \end{aligned}$$

where $(x, y, z) = \Phi(\theta, t)$. Note that $\Phi(\pi, t) = \Phi(0, -t)$, and that Φ is one-to-one on the rest of D.

The range $M = \Phi(D)$ of Φ is known as a *Möbius band*. It is the simplest example of a nonorientable surface.

Prove the various assertions made in the following description: Put $\mathbf{p}_1 = (0, -\delta)$, $\mathbf{p}_2 = (\pi, -\delta)$, $\mathbf{p}_3 = (\pi, \delta)$, $\mathbf{p}_4 = (0, \delta)$, $\mathbf{p}_5 = \mathbf{p}_1$. Put $\gamma_i = [\mathbf{p}_i, \mathbf{p}_{i+1}]$, $i = 1, \ldots, 4$, and put $\Gamma_i = \Phi \circ \gamma_i$. Then

$$\partial\Phi = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 .$$

Put $\mathbf{a} = (1, 0, -\delta)$, $\mathbf{b} = (1, 0, \delta)$. Then

$$\Phi(\mathbf{p}_1) = \Phi(\mathbf{p}_3) = \mathbf{a}, \qquad \Phi(\mathbf{p}_2) = \Phi(\mathbf{p}_4) = \mathbf{b},$$

and $\partial\Phi$ can be described as follows.

Γ_1 spirals up from $\mathbf{a}$ to $\mathbf{b}$; its projection into the (x, y)-plane has winding number $+1$ around the origin. (See Exercise 23, Chap. 8.)

$\Gamma_2 = [\mathbf{b}, \mathbf{a}]$.

Γ_3 spirals up from $\mathbf{a}$ to $\mathbf{b}$; its projection into the (x, y) plane has winding number -1 around the origin.

$\Gamma_4 = [\mathbf{b}, \mathbf{a}]$.

Thus $\partial\Phi = \Gamma_1 + \Gamma_3 + 2\Gamma_2$.

If we go from $\mathbf{a}$ to $\mathbf{b}$ along Γ_1 and continue along the "edge" of M until we return to $\mathbf{a}$, the curve traced out is

$$\Gamma = \Gamma_1 - \Gamma_3 ,$$

which may also be represented on the parameter interval $[0, 2\pi]$ by the equations

$$\begin{aligned} x &= (1 + \delta\sin\theta)\cos 2\theta \\ y &= (1 + \delta\sin\theta)\sin 2\theta \\ z &= -\delta\cos\theta. \end{aligned}$$

It should be emphasized that $\Gamma \neq \partial\Phi$: Let η be the 1-form discussed in Exercises 21 and 22. Since $d\eta = 0$, Stokes' theorem shows that

$$\int_{\partial\Phi} \eta = 0.$$

But although Γ is the "geometric" boundary of M, we have

$$\int_{\Gamma} \eta = 4\pi.$$

In order to avoid this possible source of confusion, Stokes' formula (Theorem 10.50) is frequently stated only for orientable surfaces Φ.

11

THE LEBESGUE THEORY

It is the purpose of this chapter to present the fundamental concepts of the Lebesgue theory of measure and integration and to prove some of the crucial theorems in a rather general setting, without obscuring the main lines of the development by a mass of comparatively trivial detail. Therefore proofs are only sketched in some cases, and some of the easier propositions are stated without proof. However, the reader who has become familiar with the techniques used in the preceding chapters will certainly find no difficulty in supplying the missing steps.

The theory of the Lebesgue integral can be developed in several distinct ways. Only one of these methods will be discussed here. For alternative procedures we refer to the more specialized treatises on integration listed in the Bibliography.

SET FUNCTIONS

If A and B are any two sets, we write $A - B$ for the set of all elements x such that $x \in A$, $x \notin B$. The notation $A - B$ does not imply that $B \subset A$. We denote the empty set by 0, and say that A and B are disjoint if $A \cap B = 0$.

11.1 Definition A family $\mathscr{R}$ of sets is called a *ring* if $A \in \mathscr{R}$ and $B \in \mathscr{R}$ implies

(1) $$A \cup B \in \mathscr{R}, \qquad A - B \in \mathscr{R}.$$

Since $A \cap B = A - (A - B)$, we also have $A \cap B \in \mathscr{R}$ if $\mathscr{R}$ is a ring.

A ring $\mathscr{R}$ is called a *σ-ring* if

(2) $$\bigcup_{n=1}^{\infty} A_n \in \mathscr{R}$$

whenever $A_n \in \mathscr{R}$ $(n = 1, 2, 3, \ldots)$. Since

$$\bigcap_{n=1}^{\infty} A_n = A_1 - \bigcup_{n=1}^{\infty} (A_1 - A_n),$$

we also have

$$\bigcap_{n=1}^{\infty} A_n \in \mathscr{R}$$

if $\mathscr{R}$ is a σ-ring.

11.2 Definition We say that ϕ is a set function defined on $\mathscr{R}$ if ϕ assigns to every $A \in \mathscr{R}$ a number $\phi(A)$ of the extended real number system. ϕ is *additive* if $A \cap B = 0$ implies

(3) $$\phi(A \cup B) = \phi(A) + \phi(B),$$

and ϕ is *countably additive* if $A_i \cap A_j = 0$ $(i \neq j)$ implies

(4) $$\phi\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \phi(A_n).$$

We shall always assume that the range of ϕ does not contain both $+\infty$ and $-\infty$; for if it did, the right side of (3) could become meaningless. Also, we exclude set functions whose only value is $+\infty$ or $-\infty$.

It is interesting to note that the left side of (4) is independent of the order in which the A_n's are arranged. Hence the rearrangement theorem shows that the right side of (4) converges absolutely if it converges at all; if it does not converge, the partial sums tend to $+\infty$, or to $-\infty$.

If ϕ is additive, the following properties are easily verified:

(5) $$\phi(0) = 0.$$

(6) $$\phi(A_1 \cup \cdots \cup A_n) = \phi(A_1) + \cdots + \phi(A_n)$$

if $A_i \cap A_j = 0$ whenever $i \neq j$.

$$\phi(A_1 \cup A_2) + \phi(A_1 \cap A_2) = \phi(A_1) + \phi(A_2). \tag{7}$$

If $\phi(A) \geq 0$ for all A, and $A_1 \subset A_2$, then

$$\phi(A_1) \leq \phi(A_2). \tag{8}$$

Because of (8), nonnegative additive set functions are often called monotonic.

$$\phi(A - B) = \phi(A) - \phi(B) \tag{9}$$

if $B \subset A$, and $|(\phi B)| < +\infty$.

11.3 Theorem *Suppose ϕ is countably additive on a ring $\mathscr{R}$. Suppose $A_n \in \mathscr{R}$ $(n = 1, 2, 3, \ldots)$, $A_1 \subset A_2 \subset A_3 \subset \cdots$, $A \in \mathscr{R}$, and*

$$A = \bigcup_{n=1}^{\infty} A_n.$$

Then, as $n \to \infty$,

$$\phi(A_n) \to \phi(A).$$

Proof Put $B_1 = A_1$, and

$$B_n = A_n - A_{n-1} \qquad (n = 2, 3, \ldots).$$

Then $B_i \cap B_j = 0$ for $i \neq j$, $A_n = B_1 \cup \cdots \cup B_n$, and $A = \bigcup B_n$. Hence

$$\phi(A_n) = \sum_{i=1}^{n} \phi(B_i)$$

and

$$\phi(A) = \sum_{i=1}^{\infty} \phi(B_i).$$

CONSTRUCTION OF THE LEBESGUE MEASURE

11.4 Definition Let R^p denote p-dimensional euclidean space. By an *interval* in R^p we mean the set of points $\mathbf{x} = (x_1, \ldots, x_p)$ such that

$$a_i \leq x_i \leq b_i \qquad (i = 1, \ldots, p), \tag{10}$$

or the set of points which is characterized by (10) with any or all of the $\leq$ signs replaced by $<$. The possibility that $a_i = b_i$ for any value of i is not ruled out; in particular, the empty set is included among the intervals.

If A is the union of a finite number of intervals, A is said to be an *elementary set.*

If I is an interval, we define

$$m(I) = \prod_{i=1}^{p} (b_i - a_i),$$

no matter whether equality is included or excluded in any of the inequalities (10).

If $A = I_1 \cup \cdots \cup I_n$, and if these intervals are pairwise disjoint, we set

$$m(A) = m(I_1) + \cdots + m(I_n). \tag{11}$$

We let $\mathscr{E}$ denote the family of all elementary subsets of R^p.

At this point, the following properties should be verified:

(12) $\mathscr{E}$ is a ring, but not a σ-ring.

(13) If $A \in \mathscr{E}$, then A is the union of a finite number of *disjoint* intervals.

(14) If $A \in \mathscr{E}$, $m(A)$ is well defined by (11); that is, if two different decompositions of A into disjoint intervals are used, each gives rise to the same value of $m(A)$.

(15) m is additive on $\mathscr{E}$.

Note that if $p = 1, 2, 3$, then m is length, area, and volume, respectively.

11.5 Definition A nonnegative additive set function ϕ defined on $\mathscr{E}$ is said to be *regular* if the following is true: To every $A \in \mathscr{E}$ and to every $\varepsilon > 0$ there exist sets $F \in \mathscr{E}$, $G \in \mathscr{E}$ such that F is closed, G is open, $F \subset A \subset G$, and

$$\phi(G) - \varepsilon \le \phi(A) \le \phi(F) + \varepsilon. \tag{16}$$

11.6 Examples

(*a*) *The set function m is regular.*

If A is an interval, it is trivial that the requirements of Definition 11.5 are satisfied. The general case follows from (13).

(*b*) Take $R^p = R^1$, and let α be a monotonically increasing function, defined for all real x. Put

$$\mu([a, b)) = \alpha(b-) - \alpha(a-),$$
$$\mu([a, b]) = \alpha(b+) - \alpha(a-),$$
$$\mu((a, b]) = \alpha(b+) - \alpha(a+),$$
$$\mu((a, b)) = \alpha(b-) - \alpha(a+).$$

Here $[a, b)$ is the set $a \le x < b$, etc. Because of the possible discontinuities of α, these cases have to be distinguished. If μ is defined for elementary sets as in (11), μ is regular on $\mathscr{E}$. The proof is just like that of (*a*).

Our next objective is to show that every regular set function on $\mathscr{E}$ can be extended to a countably additive set function on a σ-ring which contains $\mathscr{E}$.

11.7 Definition Let μ be additive, regular, nonnegative, and finite on $\mathscr{E}$. Consider countable coverings of any set $E \subset R^p$ by open elementary sets A_n:

$$E \subset \bigcup_{n=1}^{\infty} A_n.$$

Define

$$\mu^*(E) = \inf \sum_{n=1}^{\infty} \mu(A_n), \tag{17}$$

the inf being taken over all countable coverings of E by open elementary sets. $\mu^*(E)$ is called the *outer measure* of E, corresponding to μ.

It is clear that $\mu^*(E) \geq 0$ for all E and that

$$\mu^*(E_1) \leq \mu^*(E_2) \tag{18}$$

if $E_1 \subset E_2$.

11.8 Theorem

(*a*) *For every* $A \in \mathscr{E}$, $\mu^*(A) = \mu(A)$.

(*b*) *If* $E = \bigcup_{1}^{\infty} E_n$, *then*

$$\mu^*(E) \leq \sum_{n=1}^{\infty} \mu^*(E_n). \tag{19}$$

Note that (*a*) asserts that μ^* is an extension of μ from $\mathscr{E}$ to the family of *all* subsets of R^p. The property (19) is called *subadditivity*.

Proof Choose $A \in \mathscr{E}$ and $\varepsilon > 0$.

The regularity of μ shows that A is contained in an open elementary set G such that $\mu(G) \leq \mu(A) + \varepsilon$. Since $\mu^*(A) \leq \mu(G)$ and since ε was arbitrary, we have

$$\mu^*(A) \leq \mu(A). \tag{20}$$

The definition of μ^* shows that there is a sequence $\{A_n\}$ of open elementary sets whose union contains A, such that

$$\sum_{n=1}^{\infty} \mu(A_n) \leq \mu^*(A) + \varepsilon.$$