β non $= 0$. Tandem 2) si esset $\delta = 0$, ex $\alpha\delta - \beta\gamma = \pm 1$ fit $\beta = \pm 1$, $\gamma = \pm 1$, adeoque ex [2] $- A' = a$. Hinc $\sqrt{(D - \frac{a'A}{aa})} > \sqrt{D} > b$, quare in [5] signum superius accipiendum. Hinc $\frac{\gamma}{\alpha} > \frac{\sqrt{D}+b}{a'} > 1$, *Q. E. A.* — Quare theorema in omni sua extensione est demonstratum.

Quum differentia inter $\frac{\alpha}{\gamma}$ et $\frac{\beta}{\delta}$ sit $= \frac{1}{\gamma\delta}$: differentia inter $\frac{\pm\sqrt{D} - b}{a}$ et $\frac{\alpha}{\gamma}$ vel $\frac{\beta}{\delta}$ erit $<\frac{1}{\gamma\delta}$; inter $\frac{\pm\sqrt{D} - b}{a}$ autem et $\frac{\alpha}{\gamma}$, vel inter illam quantitatem et $\frac{\beta}{\delta}$ nulla fractio iacere poterit, cuius denominator non sit maior quam γ aut δ (*lemma praec.*). — Eodem modo differentia quantitatis $\frac{\pm\sqrt{D} + b}{a}$ a fractione $\frac{\gamma}{\alpha}$ vel hac $\frac{\delta}{\beta}$ erit minor quam $\frac{1}{\alpha\beta}$, et inter illam quantitatem et neutram harum fractionum iacere potest fractio cuius denominator non sit maior quam α et β.

192. Ex applicatione theor. praec. ad algorithmum art. 188 sequitur, quantitatem $\frac{\sqrt{D} - b}{a}$ quam per L designabimus, iacere inter $\frac{\alpha'}{\gamma'}$ et $\frac{\beta'}{\delta'}$; inter $\frac{\alpha''}{\gamma''}$ et $\frac{\beta''}{\delta''}$; inter $\frac{\alpha'''}{\gamma'''}$ et $\frac{\beta'''}{\delta'''}$ etc. (facile enim ex art. 189, 3 fin. deducitur, nullum horum limitum habere signum oppositum

signo ipsius a; quare quantitati radicali $\sqrt{D}$ signum positiuum tribui debet) siue inter $\frac{\alpha'}{\gamma'}$ et $\frac{\alpha''}{\gamma''}$; inter $\frac{\alpha''}{\gamma''}$ et $\frac{\alpha'''}{\gamma'''}$ etc. Omnes itaque fractiones $\frac{\alpha'}{\gamma'}$, $\frac{\alpha'''}{\gamma'''}$, $\frac{\alpha^{V}}{\gamma^{V}}$ etc. ipsi L ab eadem parte iacebunt, omnesque $\frac{\alpha''}{\gamma''}$, $\frac{\alpha^{IV}}{\gamma^{IV}}$, $\frac{\alpha^{VI}}{\gamma^{VI}}$ etc. a parte altera. Quoniam vero $\gamma' < \gamma'''$, $\frac{\alpha'}{\gamma'}$ iacebit *extra* $\frac{\alpha'''}{\gamma'''}$ et L, similique ratione $\frac{\alpha''}{\gamma''}$ extra L et $\frac{\alpha^{IV}}{\alpha^{IV}}$; $\frac{\alpha'''}{\gamma'''}$ extra L et $\frac{\alpha^{V}}{\gamma^{V}}$ etc. Vnde manifestum est, has quantitates iacere sequenti ordine: $\frac{\alpha'}{\gamma'}$, $\frac{\alpha'''}{\gamma'''}$, $\frac{\alpha^{V}}{\gamma^{V}} \ldots L \ldots \frac{\alpha^{VI}}{\gamma^{VI}}$, $\frac{\alpha^{IV}}{\gamma^{IV}}$, $\frac{\alpha''}{\gamma''}$. Differentia autem inter $\frac{\alpha'}{\gamma'}$ et L erit minor quam differentia inter $\frac{\alpha'}{\gamma'}$ et $\frac{\alpha''}{\gamma''}$ *i. e.* $< \frac{1}{\gamma'\gamma''}$, similique ratione differentia inter $\frac{\alpha''}{\gamma''}$ et L erit $< \frac{1}{\gamma''\gamma'''}$ etc. Quamobrem fractiones $\frac{\alpha'}{\gamma'}$, $\frac{\alpha''}{\gamma''}$, $\frac{\alpha'''}{\gamma'''}$ etc. continuo proprius ad limitem L accedunt, et quoniam γ', γ'', γ''' continuo in infinitum crescunt, differentia fractionum a limite quauis quantitate data minor fieri potest.

Ex art. 189 nulla quantitatum $\frac{\gamma}{\alpha}$, $\frac{'\gamma}{'\alpha}$, $\frac{''\gamma}{''\alpha}$ etc. signum idem habebit vt a; hinc per ratiocinia praecedentibus omnino similia sequitur, illas et hanc $\frac{-\sqrt{D}+b}{a'}$, quam per L designabimus, iacere sequenti ordine: $\frac{\gamma}{\alpha}$, $\frac{''\gamma}{''\alpha}$, $\frac{^{IV}\gamma}{^{IV}\alpha}$.... L'.... $\frac{^{V}\gamma}{^{V}\alpha}$, $\frac{'''\gamma}{'''\alpha}$, $\frac{'\gamma}{'\alpha}$. Differentia autem inter $\frac{\gamma}{\alpha}$ et L' minor erit quam $\frac{1}{'\alpha\alpha}$, differentia inter $\frac{'\gamma}{'\alpha}$ et L minor quam $\frac{1}{''\alpha'\alpha}$ etc. Quare fractiones $\frac{\gamma}{\alpha}$, $\frac{'\gamma}{'\alpha}$ etc. continuo proprius ad L' accedent, et differentia quauis quantitate data minor fieri poterit.

In ex. *art.* 188. fit $L = \frac{\sqrt{79}-8}{3} = 0{,}2960648$ et fractiones appropinquantes $\frac{0}{1}$, $\frac{1}{3}$, $\frac{2}{7}$, $\frac{3}{10}$, $\frac{8}{27}$, $\frac{45}{152}$, $\frac{143}{483}$ etc. Est autem $\frac{143}{483} = 0{,}2960662$. — Ibidem fit $L' = \frac{-\sqrt{79}+8}{5} = -0{,}1776388$. fractionesque approximantes $\frac{0}{1}$, $-\frac{1}{5}$, $-\frac{1}{6}$, $-\frac{2}{11}$, $-\frac{3}{17}$ $-\frac{8}{45}$, $-\frac{27}{152}$ $-\frac{143}{805}$ etc. Est vero $\frac{143}{805} = 0{,}1776397$.

193. Theorema. *Si formae reductae f, F proprie aequiualentes sunt: altera in alterius periodo contenta erit.*

Sit $f = (a, b, -a')$, $F = (A, B, -A')$, determinans harum formarum D, transeatque illa in hanc per substitutionem propriam $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$. Tum dico, si periodus formae f quaeratur progressioque vtrimque infinita formarum reductarum atque transformationum for-