than one way to represent a rational as such a ratio, e.g., $6/4 = 3/2$.

Since ten $= 10$, we will now use 10 instead of ten throughout, as is customary.

*Exercise* B.1.1. The purpose of this exercise is to demonstrate that the procedure of long addition taught to you in elementary school is actually valid. Let $A = a_n \ldots a_0$ and $B = b_m \ldots b_0$ be positive integer decimals. Let us adopt the convention that $a_i = 0$ when $i > n$, and $b_i = 0$ when $i > m$; for instance, if $A = 372$, then $a_0 = 2$, $a_1 = 7$, $a_2 = 3$, $a_3 = 0$, $a_4 = 0$, and so forth. Define the numbers $c_0, c_1, \ldots$ and $\varepsilon_0, \varepsilon_1, \ldots$ recursively by the following *long addition algorithm*.

- We set $\varepsilon_0 := 0$.
- Now suppose that $\varepsilon_i$ has already been defined for some $i \geq 0$. If $a_i + b_i + \varepsilon_i < 10$, we set $c_i := a_i + b_i + \varepsilon_i$ and $\varepsilon_{i+1} := 0$; otherwise, if $a_i + b_i + \varepsilon_i \geq 10$, we set $c_i := a_i + b_i + \varepsilon_i - 10$ and $\varepsilon_{i+1} = 1$. (The number $\varepsilon_{i+1}$ is the "carry digit" from the $i^{th}$ decimal place to the $(i+1)^{th}$ decimal place.)

Prove that the numbers $c_0, c_1, \ldots$ are all digits, and that there exists an $l$ such that $c_l \neq 0$ and $c_i = 0$ for all $i > l$. Then show that $c_l c_{l-1} \ldots c_1 c_0$ is the decimal representation of $A + B$.

Note that one could in fact use this algorithm to *define* addition, but it would look extremely complicated, and to prove even such simple facts as $(a + b) + c = a + (b + c)$ would be rather difficult. This is one of the reasons why we have avoided the decimal system in our construction of the natural numbers. The procedure for long multiplication (or long subtraction, or long division) is even worse to lay out rigourously; we will not do so here.

## B.2 The decimal representation of real numbers

We need a new symbol: the *decimal point* ".".

**Definition B.2.1** (Real decimals). A *real decimal* is any sequence of digits, and a decimal point, arranged as

$$\pm a_n \ldots a_0 . a_{-1} a_{-2} \ldots$$

which is finite to the left of the decimal point (so $n$ is a natural number), but infinite to the right of the decimal point, where $\pm$ is either $+$ or $-$, and $a_n \dots a_0$ is a natural number decimal (i.e., either a positive integer decimal, or 0). This decimal is equated to the real number

$$\pm a_n \dots a_0.a_{-1}a_{-2}\dots \equiv \pm 1 \times \sum_{i=-\infty}^{n} a_i \times 10^i.$$

The series is always convergent (Exercise B.2.1). Next, we show that every real number has at least one decimal representation:

**Theorem B.2.2** (Existence of decimal representations). *Every real number $x$ has at least one decimal representation*

$$x = \pm a_n \dots a_0.a_{-1}a_{-2}\dots.$$

*Proof.* We first note that $x = 0$ has the decimal representation $0.000\dots$. Also, once we find a decimal representation for $x$, we automatically get a decimal representation for $-x$ by changing the sign $\pm$. Thus it suffices to prove the theorem for positive real numbers $x$ (by Proposition 5.4.4).

Let $n \geq 0$ be any natural number. From the Archimedean property (Corollary 5.4.13) we know that there is a natural number $M$ such that $M \times 10^{-n} > x$. Since $0 \times 10^{-n} \leq x$, we thus see that there must exist a natural number $s_n$ such that $s_n \times 10^{-n} \leq x$ and $s_n{+}{+} \times 10^{-n} > x$. (If no such natural number existed, one could use induction to conclude that $s \times 10^{-n} \leq x$ for all natural numbers $s$, contradicting the Archimedean property.)

Now consider the sequence $s_0, s_1, s_2, \dots$. Since we have

$$s_n \times 10^{-n} \leq x < (s_n + 1) \times 10^{-n}$$

we thus have

$$(10 \times s_n) \times 10^{-(n{+}{+})} \leq x < (10 \times s_n + 10) \times 10^{-(n{+}{+})}.$$

On the other hand, we have

$$s_{n+1} \times 10^{-(n+1)} \leq x < (s_{n+1} + 1) \times 10^{-(n+1)}$$

and hence we have

$$10 \times s_n < s_{n+1} + 1 \text{ and } s_{n+1} < 10 \times s_n + 10.$$

From these two inequalities we see that we have

$$10 \times s_n \leq s_{n+1} \leq 10 \times s_n + 9$$

and hence we can find a digit $a_{n+1}$ such that

$$s_{n+1} = 10 \times s_n + a_n$$

and hence

$$s_{n+1} \times 10^{-(n+1)} = s_n \times 10^{-n} + a_{n+1} \times 10^{-(n+1)}.$$

From this identity and induction, we can obtain the formula

$$s_n \times 10^{-n} = s_0 + \sum_{i=0}^{n} a_i \times 10^{-i}.$$

Now we take limits of both sides (using Exercise B.2.1) to obtain

$$\lim_{n \to \infty} s_n \times 10^{-n} = s_0 + \sum_{i=0}^{\infty} a_i \times 10^{-i}.$$

On the other hand, we have

$$x - 10^{-n} \leq s_n \times 10^{-n} \leq x$$

for all $n$, so by the squeeze test (Corollary 6.4.14) we have

$$\lim_{n \to \infty} s_n \times 10^{-n} = x.$$

Thus we have

$$x = s_0 + \sum_{i=0}^{\infty} a_i \times 10^{-i}.$$

Since $s_0$ already has a positive integer decimal representation by Theorem B.1.4, we thus see that $x$ has a decimal representation as desired. $\square$

There is however one slight flaw with the decimal system: it is possible for one real number to have two decimal representations.

**Proposition B.2.3** (Failure of uniqueness of decimal representations). *The number 1 has two different decimal representations:* $1.000\ldots$ *and* $0.999\ldots$.

*Proof.* The representation $1 = 1.000\ldots$ is clear. Now let's compute $0.999\ldots$. By definition, this is the limit of the Cauchy sequence

$$0.9, 0.99, 0.999, 0.9999, \ldots.$$

But this sequence has 1 as a formal limit by Proposition 5.2.8. □

It turns out that these are the only two decimal representations of 1 (Exercise B.2.2). In fact, as it turns out, all real numbers have either one or two decimal representations - two if the real is a terminating decimal, and one otherwise (Exercise B.2.3).

*Exercise* B.2.1. If $a_n \ldots a_0.a_{-1}a_{-2}\ldots$ is a real decimal, show that the series $\sum_{i=-\infty}^{n} a_i \times 10^i$ is absolutely convergent.

*Exercise* B.2.2. Show that the only decimal representations

$$1 = \pm a_n \ldots a_0.a_{-1}a_{-2}\ldots$$

of 1 are $1 = 1.000\ldots$ and $1 = 0.999\ldots$.

*Exercise* B.2.3. A real number $x$ is said to be a *terminating decimal* if we have $x = n/10^{-m}$ for some integers $n, m$. Show that if $x$ is a terminating decimal, then $x$ has exactly two decimal representations, while if $x$ is not at terminating decimal, then $x$ has exactly one decimal representation.

*Exercise* B.2.4. Rewrite the proof of Corollary 8.3.4 using the decimal system.

# Index