26. Let V be the linear space of all real functions continuous on $[a, b]$. If $f \in V$, $g = T(f)$ means that

$$g(x) = \int_a^b f(t) \sin (x - t)\, dt \qquad \text{for } a \le x \le b.$$

27. Let V be the space of all real functions twice differentiable on an open interval (a, b). If $y \in V$, define $T(y) = y'' + Py' + Qy$, where P and Q are fixed constants.
28. Let V be the linear space of all real convergent sequences $\{x_n\}$. Define a transformation $T: V \to V$ as follows: If $x = \{x_n\}$ is a convergent sequence with limit a, let $T(x) = \{y_n\}$, where $y_n = a - x_n$ for $n \ge 1$. Prove that T is linear and describe the null space and range of T.
29. Let V denote the linear space of all real functions continuous on the interval $[-\pi, \pi]$. Let S be that subset of V consisting of all f satisfying the three equations

$$\int_{-\pi}^{\pi} f(t)\, dt = 0, \qquad \int_{-\pi}^{\pi} f(t) \cos t\, dt = 0, \qquad \int_{-\pi}^{\pi} f(t) \sin t\, dt = 0.$$

(a) Prove that S is a subspace of V.
(b) Prove that S contains the functions $f(x) = \cos nx$ and $f(x) = \sin nx$ for each $n = 2, 3, \ldots$.
(c) Prove that S is infinite-dimensional.
Let $T: V \to V$ be the linear transformation defined as follows: If $f \in V$, $g = T(f)$ means that

$$g(x) = \int_{-\pi}^{\pi} \{1 + \cos (x - t)\} f(t)\, dt.$$

(d) Prove that $T(V)$, the range of T, is finite-dimensional and find a basis for $T(V)$.
(e) Determine the null space of T.
(f) Find all real $c \ne 0$ and all nonzero f in V such that $T(f) = cf$. (Note that such an f lies in the range of T.)

30. Let $T: V \to W$ be a linear transformation of a linear space V into a linear space W. If V is infinite-dimensional, prove that at least one of $T(V)$ or $N(T)$ is infinite-dimensional.

[Hint: Assume $\dim N(T) = k$, $\dim T(V) = r$, let $e_1, \ldots, e_k$ be a basis for $N(T)$ and let $e_1, \ldots, e_k, e_{k+1}, \ldots, e_{k+n}$ be independent elements in V, where $n > r$. The elements $T(e_{k+1}), \ldots, T(e_{k+n})$ are dependent since $n > r$. Use this fact to obtain a contradiction.]

2.5 Algebraic operations on linear transformations

Functions whose values lie in a given linear space W can be added to each other and can be multiplied by the scalars in W according to the following definition.

DEFINITION. *Let $S: V \to W$ and $T: V \to W$ be two functions with a common domain V and with values in a linear space W. If c is any scalar in W, we define the sum $S + T$ and the product cT by the equations*

$$(2.4) \qquad (S + T)(x) = S(x) + T(x), \qquad (cT)(x) = cT(x)$$

for all x in V.

We are especially interested in the case where V is also a linear space having the same scalars as W. In this case we denote by $\mathscr{L}(V, W)$ the set of all linear transformations of V into W.

If S and T are two linear transformations in $\mathscr{L}(V, W)$, it is an easy exercise to verify that $S + T$ and cT are also linear transformations in $\mathscr{L}(V, W)$. More than this is true. With the operations just defined, the set $\mathscr{L}(V, W)$ itself becomes a new linear space. The zero transformation serves as the zero element of this space, and the transformation $(-1)T$ is the negative of T. It is a straightforward matter to verify that all ten axioms for a linear space are satisfied. Therefore, we have the following.

THEOREM 2.4. *The set $\mathscr{L}(V, W)$ of all linear transformations of V into W is a linear space with the operations of addition and multiplication by scalars defined as in (2.4).*

A more interesting algebraic operation on linear transformations is *composition* or *multiplication* of transformations. This operation makes no use of the algebraic structure of a linear space and can be defined quite generally as follows.

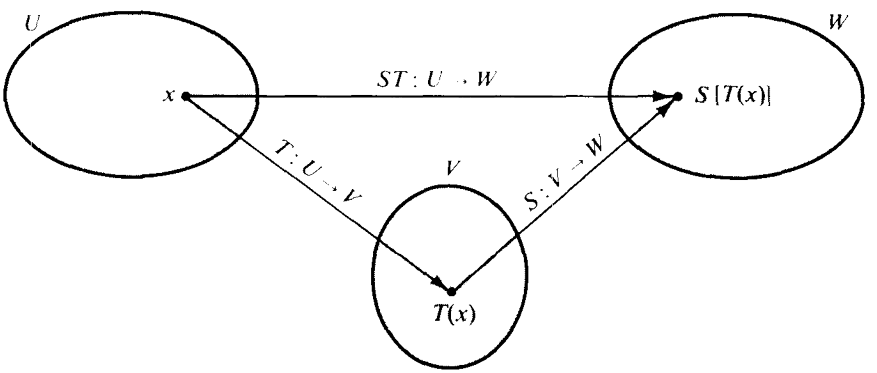

FIGURE 2.1 Illustrating the composition of two transformations.

DEFINITION. *Let U, V, W be sets. Let $T: U \to V$ be a function with domain U and values in V, and let $S: V \to W$ be another function with domain V and values in W. Then the composition ST is the function $ST: U \to W$ defined by the equation*

$$(ST)(x) = S[T(x)] \qquad \text{for every } x \text{ in } U.$$

Thus, to map x by the composition ST, we first map x by T and then map $T(x)$ by S. This is illustrated in Figure 2.1.

Composition of real-valued functions has been encountered repeatedly in our study of calculus, and we have seen that the operation is, in general, not commutative. However, as in the case of real-valued functions, composition does satisfy an associative law.

THEOREM 2.5. *If $T: U \to V$, $S: V \to W$, and $R: W \to X$ are three functions, then we have*

$$R(ST) = (RS)T.$$

Proof. Both functions $R(ST)$ and $(RS)T$ have domain U and values in X. For each x in U, we have

$$[R(ST)](x) = R[(ST)(x)] = R[S[T(x)]] \quad \text{and} \quad [(RS)T](x) = (RS)[T(x)] = R[S[T(x)]],$$

which proves that $R(ST) = (RS)T$.

DEFINITION. *Let $T: V \to V$ be a function which maps V into itself. We define integral powers of T inductively as follows:*

$$T^0 = I, \qquad T^n = TT^{n-1} \quad \text{for} \quad n \geq 1.$$

Here I is the identity transformation. The reader may verify that the associative law implies the law of exponents $T^mT^n = T^{m+n}$ for all nonnegative integers m and n.

The next theorem shows that the composition of *linear* transformations is again linear.

THEOREM 2.6. *If U, V, W are linear spaces with the same scalars, and if $T: U \to V$ and $S: V \to W$ are linear transformations, then the composition $ST: U \to W$ is linear.*

Proof. For all x, y in U and all scalars a and b, we have

$$(ST)(ax + by) = S[T(ax + by)] = S[aT(x) + bT(y)] = aST(x) + bST(y).$$

Composition can be combined with the algebraic operations of addition and multiplication of scalars in $\mathscr{L}(V, W)$ to give us the following.

THEOREM 2.7. *Let U, V, W be linear spaces with the same scalars, assume S and T are in $\mathscr{L}(V, W)$, and let c be any scalar.*

(a) *For any function R with values in V, we have*

$$(S + T)R = SR + TR \quad \text{and} \quad (cS)R = c(SR).$$

(b) *For any linear transformation $R: W \to U$, we have*

$$R(S + T) = RS + RT \quad \text{and} \quad R(cS) = c(RS).$$

The proof is a straightforward application of the definition of composition and is left as an exercise.

2.6 Inverses

In our study of real-valued functions we learned how to construct new functions by inversion of monotonic functions. Now we wish to extend the process of inversion to a more general class of functions.

Given a function T, our goal is to find, if possible, another function S whose composition with T is the identity transformation. Since composition is in general not commutative, we have to distinguish between ST and TS. Therefore we introduce two kinds of inverses which we call left and right inverses.

DEFINITION. *Given two sets V and W and a function $T: V \to W$. A function $S: T(V) \to V$ is called a left inverse of T if $S[T(x)] = x$ for all x in V, that is, if*

$$ST = I_V,$$

where I_V is the identity transformation on V. A function $R: T(V) \to V$ is called a right inverse of T if $T[R(y)] = y$ for all y in $T(V)$, that is, if

$$TR = I_{T(V)},$$

where $I_{T(V)}$ is the identity transformation on $T(V)$.

EXAMPLE. *A function with no left inverse but with two right inverses.* Let $V = (1, 2)$ and let $W = \{0\}$. Define $T: V \to W$ as follows: $T(1) = T(2) = 0$. This function has two right inverses $R: W \to V$ and $R' : W \to V$ given by

$$R(0) = 1, \qquad R'(0) = 2.$$

It cannot have a left inverse S since this would require

$$1 = S[T(1)] = S(0) \quad \text{and} \quad 2 = S[T(2)] = S(0).$$

This simple example shows that left inverses need not exist and that right inverses need not be unique.

Every function $T: V \to W$ has at least one right inverse. In fact, each y in $T(V)$ has the form $y = T(x)$ for at least one x in V. If we select one such x and define $R(y) = x$, then $T[R(y)] = T(x) = y$ for each y in $T(V)$, so R is a right inverse. Nonuniqueness may occur because there may be more than one x in V which maps onto a given y in $T(V)$. We shall prove presently (in Theorem 2.9) that if each y in $T(V)$ is the image of *exactly one* x in V, then right inverses are unique.

First we prove that if a left inverse exists it is unique and, at the same time, is a right inverse.

THEOREM 2.8. *A function $T: V \to W$ can have at most one left inverse. If T has a left inverse S, then S is also a right inverse.*

Proof. Assume T has two left inverses, $S: T(V) \to V$ and $S': T(V) \to V$. Choose any y in $T(V)$. We shall prove that $S(y) = S'(y)$. Now $y = T(x)$ for some x in V, so we have

$$S[T(x)] = x \quad \text{and} \quad S'[T(x)] = x,$$

since both S and S' are left inverses. Therefore $S(y) = x$ and $S'(y) = x$, so $S(y) = S'(y)$ for all y in $T(V)$. Therefore $S = S'$ which proves that left inverses are unique.

Now we prove that every left inverse S is also a right inverse. Choose any element y in $T(V)$. We shall prove that $T[S(y)] = y$. Since $y \in T(V)$, we have $y = T(x)$ for some x in V. But S is a left inverse, so

$$x = S[T(x)] = S(y).$$

Applying T, we get $T(x) = T[S(y)]$. But $y = T(x)$, so $y = T[S(y)]$, which completes the proof.

The next theorem characterizes all functions having left inverses.

THEOREM 2.9. *A function $T: V \rightarrow W$ has a left inverse if and only if T maps distinct elements of V onto distinct elements of W; that is, if and only if, for all x and y in V,*

$$(2.5) \qquad x \neq y \quad \text{implies} \quad T(x) \neq T(y).$$

Note: Condition (2.5) is equivalent to the statement

$$(2.6) \qquad T(x) = T(y) \quad \text{implies} \quad x = y.$$

A function T satisfying (2.5) or (2.6) for all x and y in V is said to be *one-to-one* on V.

Proof. Assume T has a left inverse S, and assume that $T(x) = T(y)$. We wish to prove that $x = y$. Applying S, we find $S[T(x)] = S[T(y)]$ Since $S[T(x)] = x$ and $S[T(y)] = y$, this implies $x = y$. This proves that a function with a left inverse is one-to-one on its domain.

Now we prove the converse. Assume T is one-to-one on V. We shall exhibit a function $S: T(V) \rightarrow V$ which is a left inverse of T. If $y \in T(V)$, then $y = T(x)$ for some x in V. By (2.6), there is *exactly one* x in V for which $y = T(x)$. Define S(y) to be this x. That is, we define S on $T(V)$ as follows:

$$S(y) = x \quad \text{means that} \quad T(x) = y.$$

Then we have $S[T(x)] = x$ for each x in V, so $ST = I_V$. Therefore, the function S so defined is a left inverse of T.

DEFINITION. *Let $T: V \rightarrow W$ be one-to-one on V. The unique left inverse of T (which we know is also a right inverse) is denoted by T^{-1}. We say that T is invertible, and we call T^{-1} the inverse of T.*

The results of this section refer to arbitrary functions. Now we apply these ideas to linear transformations.

2.7 One-to-one linear transformations

In this section, V and W denote linear spaces with the same scalars, and $T: V \to W$ denotes a linear transformation in $\mathscr{L}(V, W)$. The linearity of T enables us to express the one-to-one property in several equivalent forms.

THEOREM 2.10. *Let $T: V \to W$ be a linear transformation in $\mathscr{L}(V, W)$. Then the following statements are equivalent.*

(a) *T is one-to-one on V.*
(b) *T is invertible and its inverse $T^{-1}: T(V) \to V$ is linear.*
(c) *For all x in V, $T(x) = 0$ implies $x = 0$. That is, the null space $N(T)$ contains only the zero element of V.*

Proof. We shall prove that (a) implies (b), (b) implies (c), and (c) implies (a). First assume (a) holds. Then T has an inverse (by Theorem 2.9), and we must show that T^{-1} is linear. Take any two elements u and v in $T(V)$. Then $u = T(x)$ and $v = T(y)$ for some x and y in V. For any scalars a and b, we have

$$au + bv = aT(x) + bT(y) = T(ax + by),$$

since T is linear. Hence, applying T^{-1}, we have

$$T^{-1}(au + bv) = ax + by = aT^{-1}(u) + bT^{-1}(v),$$

so T^{-1} is linear.. Therefore (a) implies (b).

Next assume that (b) holds. Take any x in V for which $T(x) = 0$. Applying T^{-1}, we find that $x = T^{-1}(O) = O$, since T^{-1} is linear. Therefore, (b) implies (c).

Finally, assume (c) holds. Take any two elements u and v in V with $T(u) = T(v)$. By linearity, we have $T(u - v) = T(u) - T(v) = 0$, so $u - v = 0$. Therefore, Tis one-to-one on V, and the proof of the theorem is complete.

When V is finite-dimensional, the one-to-one property can be formulated in terms of independence and dimensionality, as indicated by the next theorem.

THEOREM 2.11. *Let $T: V \to W$ be a linear transformation in $\mathscr{L}(V, W)$ and assume that V is finite-dimensional, say* $\dim V = n$. *Then the following statements are equivalent.*

(a) *T is one-to-one on V.*
(b) *If $e_1, \ldots, e_p$ are independent elements in V, then $T(e_1), \ldots, T(e_p)$ are independent elements in $T(V)$.*
(c) $\dim T(V) = n$.
(d) *If $\{e_1, \ldots, e_n\}$ is a basis for V, then $\{T(e_1), \ldots, T(e_n)\}$ is a basis for $T(V)$.*

Proof. We shall prove that (a) implies (b), (b) implies (c), (c) implies (d), and (d) implies (a). Assume (a) holds. Let $e_1, \ldots, e_p$ be independent elements of V and consider the elements $T(e_1), \ldots, T(e_p)$ in T(V). Suppose that

$$\sum_{i=1}^{p} c_i T(e_i) = O$$

for certain scalars $c_1, \ldots,$ c,. By linearity, we obtain

$$T\left(\sum_{i=1}^{p} c_i e_i\right) = O, \quad \text{and hence} \quad \sum_{i=1}^{p} c_i e_i = O$$

since T is one-to-one. But $e_1, \ldots, e_p$ are independent, so $c_1 = \cdots = c_p = 0$. Therefore (a) implies (b).

Now assume (b) holds. Let {e,, . . . , e,} be a basis for V. By (b), the n elements $T(e_1), \ldots, T(e_n)$ in $T(V)$ are independent. Therefore, dim T(V) $\geq n$. But, by Theorem 2.3, we have dim T(V) $\leq n$. Therefore dim $T(V) = n$, so (b) implies (c).

Next, assume (c) holds and let {e,, . . . , e,} be a basis for V. Take any element y in T(V). Then y = T(x) for some x in V, so we have

$$x = \sum_{i=1}^{n} c_i e_i, \quad \text{and hence} \quad y = T(x) = \sum_{i=1}^{n} c_i T(e_i).$$

Therefore $\{T(e_1), \ldots, T(e_n)\}$ spans T(V). But we are assuming dim T(V) = n, so $\{T(e_1), \ldots, T(e_n)\}$ is a basis for T(V). Therefore (c) implies (d).

Finally, assume (d) holds. We will prove that T(x) = 0 implies x = 0. Let $\{e_1, \ldots,$ e,} be a basis for V. If x $\in V$, we may write

$$x = \sum_{i=1}^{n} c_i e_i, \quad \text{and hence} \quad T(x) = \sum_{i=1}^{n} c_i T(e_i).$$

If $T(x) = O$, then $c_1 = \cdots = c_n = 0$, since the elements $T(e_1), \ldots, T(e_n)$ are independent. Therefore x = O, so Tis one-to-one on V. Thus, (d) implies (a) and the proof is complete.

2.8 Exercises

1. Let $V = \{0, 1\}$. Describe all functions $T: V \to V$. There are four altogether. Label them as T_1, T_2, T_3, T_4 and make a multiplication table showing the composition of each pair. Indicate which functions are one-to-one on V and give their inverses.
2. Let $V = \{0, 1, 2\}$. Describe all functions $T: V \to V$ for which $T(V) = V$. There are six altogether. Label them as $T_1, \ldots, T_6$ and make a multiplication table showing the composition of each pair. Indicate which functions are one-to-one on V, and give their inverses.

In each of Exercises 3 through 12, a function $T: V_2 \to V_2$ is defined by the formula given for $T(x, y)$, where (x, y) is an arbitrary point in V_2. In each case determine whether T is one-to-one on V_2. If it is, describe its range $T(V_2)$; for each point (u, v) in $T(V_2)$, let (x, y) = $T^{-1}(u, v)$ and give formulas for determining x and y in terms of u and v.

3. $T(x, y) = (y, x)$.
4. $T(x, y) = (x, -y)$.
5. $T(x, y) = (x, 0)$.
6. $T(x, y) = (x, x)$.
7. $T(x, y) = (x^2, y^2)$.
8. $T(x, y) = (e^x, e^y)$.
9. $T(x, y) = (x, 1)$.
10. $T(x, y) = (x + 1, y + 1)$.
11. $T(x, y) = (x - y, x + y)$.
12. $T(x, y) = (2x - y, x + y)$.

In each of Exercises 13 through 20, a function $T: V_3 \to V_3$ is defined by the formula given for $T(x, y, z)$, where (x, y, z) is an arbitrary point in V_3. In each case, determine whether T is one-to-one on V_3. If it is, describe its range $T(V_3)$; for each point (u, v, w) in $T(V_3)$, let $(x, y, z) = T^{-1}(u, v, w)$ and give formulas for determining x, y, and z in terms of u, v, and w.

13. $T(x, y, z) = (z, y, x)$.
14. $T(x, y, z) = (x, y, 0)$.
15. $T(x, y, z) = (x, 2y, 3z)$.
16. $T(x, y, z) = (x, y, x + y + z)$.
17. $T(x, y, z) = (x + 1, y + 1, z - 1)$.
18. $T(x, y, z) = (x + 1, y + 2, z + 3)$.
19. $T(x, y, z) = (x, x + y, x + y + z)$.
20. $T(x, y, z) = (x + y, y + z, x + z)$.

21. Let $T: V \to V$ be a function which maps V into itself. Powers are defined inductively by the formulas $T^0 = I$, $T^n = TT^{n-1}$ for $n \geq 1$. Prove that the associative law for composition implies the law of exponents: $T^mT^n = T^{m+n}$. If T is invertible, prove that T^n is also invertible and that $(T^n)^{-1} = (T^{-1})^n$.

In Exercises, 22 through 25, S and T denote functions with domain V and values in V. In general, $ST \neq TS$. If $ST = TS$, we say that S and T ***commute***.

22. If S and T commute, prove that $(ST)^n = S^nT^n$ for all integers $n \geq 0$.
23. If S and T are invertible, prove that ST is also invertible and that $(ST)^{-1} = T^{-1}S^{-1}$. In other words, the inverse of ST is the composition of inverses, taken in reverse order.
24. If S and T are invertible and commute, prove that their inverses also commute.
25. Let V be a linear space. If S and T commute, prove that

$$(S + T)^2 = S^2 + 2ST + T^2 \quad \text{and} \quad (S + T)^3 = S^3 + 3S^2T + 3ST^2 + T^3.$$

Indicate how these formulas must be altered if $ST \neq TS$.
26. Let S and T be the linear transformations of V_3 into V_3 defined by the formulas $S(x, y, z) = (z, y, x)$ and $T(x, y, z) = (x, x + y, x + y + z)$, where (x, y, z) is an arbitrary point of V_3.
(a) Determine the image of (x, y, z) under each of the following transformations : ST, TS, $ST - TS$, S^2, T^2, $(ST)^2$, $(TS)^2$, $(ST - TS)^2$.
(b) Prove that S and T are one-to-one on V_3 and find the image of (u, v, w) under each of the following transformations : S^{-1}, T^{-1}, $(ST)^{-1}$, $(TS)^{-1}$.
(c) Find the image of (x, y, z) under $(T - I)^n$ for each $n \geq 1$.
27. Let V be the linear space of all real polynomials $p(x)$. Let D denote the differentiation operator and let T denote the integration operator which maps each polynomial p onto the polynomial q given by $q(x) = \int_0^x p(t)\, dt$. Prove that $DT = I_V$ but that $TD \neq I_V$. Describe the null space and range of TD.
28. Let V be the linear space of all real polynomials $p(x)$. Let D denote the differentiation operator and let T be the linear transformation that maps $p(x)$ onto $xp'(x)$.
(a) Let $p(x) = 2 + 3x - x^2 + 4x^3$ and determine the image of p under each of the following transformations: D, T, DT, TD, $DT - TD$, $T^2D^2 - D^2T^2$.
(b) Determine those p in V for which $T(p) = p$.
(c) Determine those p in V for which $(DT - 2D)(p) = 0$.
(d) Determine those p in V for which $(DT - TD)^n(p) = D^n(p)$.
29. Let V and D be as in Exercise 28 but let T be the linear transformation that maps $p(x)$ onto $xp(x)$. Prove that $DT - TD = I$ and that $DT^n - T^nD = nT^{n-1}$ for $n \geq 2$.
30. Let S and T be in $\mathscr{L}(V, V)$ and assume that $ST - TS = I$. Prove that $ST^n - T^nS = nT^{n-1}$ for all $n \geq 1$.
31. Let V be the linear space of all real polynomials $p(x)$. Let R, S, T be the functions which map an arbitrary polynomial $p(x) = c_0 + c_1x + \ldots + c_nx^n$ in V onto the polynomials $u(x)$, $s(x)$,

and t(x), respectively, where

$$r(x) = p(0), \qquad s(x) = \sum_{k=1}^{n} c_k x^{k-1}, \qquad t(x) = \sum_{k=0}^{n} c_k x^{k+1}.$$

(a) Let $p(x) = 2 + 3x - x^2 + x^3$ and determine the image of p under each of the following transformations: R, S, T, ST, TS, $(TS)^2$, T^2S^2, S^2T^2, TRS, RST.
(b) Prove that R, S, and T are linear and determine the null space and range of each.
(c) Prove that T is one-to-one on V and determine its inverse.
(d) If $n \geq 1$, express $(TS)^n$ and S^nT^n in terms of I and R.

32. Refer to Exercise 28 of Section 2.4. Determine whether T is one-to-one on V. If it is, describe its inverse.

2.9 Linear transformations with prescribed values

If V is finite-dimensional, we can always construct a linear transformation $T: V \to W$ with prescribed values at the basis elements of V, as described in the next theorem.

THEOREM 2.12. *Let $e_1, \ldots, e_n$ be a basis for an n-dimensional linear space V. Let $u_1, \ldots, u_n$ be n arbitrary elements in a linear space W. Then there is one and only one linear transformation $T: V \to W$ such that*

$$T(e_k) = u_k \quad \text{for} \quad k = 1, 2, \ldots, n. \tag{2.7}$$

This T maps an arbitrary element x in V as follows:

$$\text{If} \quad x = \sum_{k=1}^{n} x_k e_k, \quad \text{then} \quad T(x) = \sum_{k=1}^{n} x_k u_k. \tag{2.8}$$

Proof. Every x in V can be expressed uniquely as a linear combination of $e_1, \ldots, e_n$, the multipliers $x_1, \ldots, x_n$ being the components of x relative to the ordered basis $(e_1, \ldots, e_n)$. If we define T by (2.8), it is a straightforward matter to verify that T is linear. If $x = e_k$ for some k, then all components of x are 0 except the kth, which is 1, so (2.8) gives $T(e_k) = u_k$, are required.

To prove that there is only one linear transformation satisfying (2.7), let T' be another and compute $T'(x)$. We find that

$$T'(x) = T'\left(\sum_{k=1}^{n} x_k e_k\right) = \sum_{k=1}^{n} x_k T'(e_k) = \sum_{k=1}^{n} x_k u_k = T(x).$$

Since $T'(x) = T(x)$ for all x in V, we have $T' = T$, which completes the proof.

EXAMPLE. Determine the linear transformation $T: V_2 \to V_2$ which maps the basis elements $i = (1, 0)$ and $j = (0, 1)$ as follows:

$$T(i) = i + j, \qquad T(j) = 2i - j.$$

Solution. If $x = x_1 i + x_2 j$ is an arbitrary element of V_2, then $T(x)$ is given by

$$T(x) = x_1 T(i) + x_2 T(j) = x_1(i + j) + x_2(2i - j) = (x_1 + 2x_2)i + (x_1 - x_2)j.$$

2.10 Matrix representations of linear transformations

Theorem 2.12 shows that a linear transformation $T: V \to W$ of a finite-dimensional linear space V is completely determined by its action on a given set of basis elements $e_1, \ldots, e_n$. Now, suppose the space W is also finite-dimensional, say dim $W = m$, and let $w_1, \ldots, w_m$ be a basis for W. (The dimensions n and m may or may not be equal.) Since T has values in W, each element $T(e_k)$ can be expressed uniquely as a linear combination of the basis elements $w_1, \ldots, w_m$, say

$$T(e_k) = \sum_{i=1}^{m} t_{ik} w_i,$$

where $t_{1k}, \ldots, t_{mk}$ are the components of $T(e_k)$ relative to the ordered basis $(w_1, \ldots, w_m)$. We shall display the m-tuple $(t_{1k}, \ldots, t_{mk})$ vertically, as follows:

$$\begin{bmatrix} t_{1k} \\ t_{2k} \\ \cdot \\ \cdot \\ \cdot \\ t_{mk} \end{bmatrix}. \tag{2.9}$$

This array is called a *column vector* or a *column matrix*. We have such a column vector for each of the n elements $T(e_1), \ldots, T(e_n)$. We place them side by side and enclose them in one pair of brackets to obtain the following rectangular array:

$$\begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ t_{21} & t_{22} & \cdots & t_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ t_{m1} & t_{m2} & \cdots & t_{mn} \end{bmatrix}$$

This array is called *a matrix* consisting of *m rows* and *n columns*. We call it an *m* by *n* matrix, or an $m \times n$ matrix. The first row is the $1 \times n$ matrix $(t_{11}, t_{12}, \ldots, t_{1n})$. The $m \times 1$ matrix displayed in (2.9) is the kth column. The scalars t_{ik} are indexed so the first subscript i indicates the *row:*, and the second subscript k indicates the *column* in which t_{ik} occurs. We call t_{ik} the *ik-entry* or the *ik-element* of the matrix. The more compact notation

$$(t_{ik}), \qquad \text{or} \qquad (t_{ik})_{i,k=1}^{m,n},$$

is also used to denote the matrix whose ik-entry is t_{ik}.

Thus, every linear transformation T of an n-dimensional space V into an m-dimensional space W gives rise to an $m \times n$ matrix (t_{ik}) whose columns consist of the components of $T(e_1), \ldots, T(e_n)$ relative to the basis $(w_1, \ldots, w_m)$. We call this the *matrix representation* of T relative to the given choice of ordered bases $(e_1, \ldots, e_n)$ for V and $(w_1, \ldots, w_m)$ for W. Once we know the matrix (t_{ik}), the components of any element $T(x)$ relative to the basis $(w_1, \ldots, w_m)$ can be determined as described in the next theorem.

THEOREM 2.13. *Let T be a linear transformation in $\mathscr{L}(V, W)$, where* $\dim V = n$ *and* $\dim W = m$. *Let $(e_1, \ldots, e_n)$ and $(w_1, \ldots, w_m)$ be ordered bases for V and W, respectively, and let (t_{ik}) be the $m \times n$ matrix whose entries are determined by the equations*

$$T(e_k) = \sum_{i=1}^{m} t_{ik} w_i \,, \qquad \text{for} \quad k = 1, 2, \ldots, n. \tag{2.10}$$

Then an arbitrary element

$$x = \sum_{k=1}^{n} x_k e_k \tag{2.11}$$

in V with components $(x_1, \ldots, x_n)$ relative to $(e_1, \ldots, e_n)$ is mapped by T onto the element

$$T(x) = \sum_{i=1}^{m} y_i w_i \tag{2.12}$$

in W with components $(y_1, \ldots, y_m)$ relative to $(w_1, \ldots, w_m)$. The y_i are related to the components of x by the linear equations

$$y_i = \sum_{k=1}^{n} t_{ik} x_k \qquad \text{for} \quad i = 1, 2, \ldots, m. \tag{2.13}$$

Proof. Applying T to each member of (2.11) and using (2.10), we obtain

$$T(x) = \sum_{k=1}^{n} x_k T(e_k) = \sum_{k=1}^{n} x_k \sum_{i=1}^{m} t_{ik} w_i = \sum_{i=1}^{m} \left(\sum_{k=1}^{n} t_{ik} x_k \right) w_i = \sum_{i=1}^{m} y_i w_i \,,$$

where each y_i is given by (2.13). This completes the proof.

Having chosen a pair of bases $(e_1, \ldots, e_n)$ and $(w_1, \ldots, w_m)$ for V and W, respectively, every linear transformation $T: V \to W$ has a matrix representation (t_{ik}). Conversely, if we start with any mn scalars arranged as a rectangular matrix (t_{ik}) and choose a pair of ordered bases for V and W, then it is easy to prove that there is exactly one linear transformation $T: V \to W$ having this matrix representation. We simply define T at the basis elements of V by the equations in (2.10). Then, by Theorem 2.12, there is one and only one linear transformation $T: V \to W$ with these prescribed values. The image $T(x)$ of an arbitrary point x in V is then given by Equations (2.12) and (2.13).

EXAMPLE 1. *Construction of a linear transformation from a given matrix.* Suppose we start with the 2 x 3 matrix

$$\begin{bmatrix} 3 & 1 & -2 \\ 1 & 0 & 4 \end{bmatrix}.$$

Choose the usual bases of unit coordinate vectors for V_3 and V_2. Then the given matrix represents a linear transformation $T: V_3 \to V_2$ which maps an arbitrary vector (x_1, x_2, x_3) in V_3 onto the vector (y_1, y_2) in V_2 according to the linear equations

$$y_1 = 3x_1 + x_2 - 2x_3,$$

$$y_2 = x_1 + 0x_2 + 4x_3.$$

EXAMPLE 2. *Construction of a matrix representation of a given linear transformation.* Let V be the linear space of all real polynomials $p(x)$ of degree ≤ 3. This space has dimension 4, and we choose the basis $(1, x, x^2, x^3)$. Let D be the differentiation operator which maps each polynomial $p(x)$ in V onto its derivative $p'(x)$. We can regard D as a linear transformation of V into W, where W is the 3-dimensional space of all real polynomials of degree ≤ 2'. In W we choose the basis $(1, x, x^2)$. To find the matrix representation of D relative to this (choice of bases, we transform (differentiate) each basis element of V and express it as a linear combination of the basis elements of W. Thus, we find that

$$D(1) = 0 = 0 + 0x + 0x^2, \qquad D(x) = 1 = 1 + 0x + 0x^2,$$

$$D(x^2) = 2x = 0 + 2x + 0x^2, \qquad D(x^3) = 3x^2 = 0 + 0x + 3x^2.$$

The coefficients of these polynomials determine the *columns* of the matrix representation of D. Therefore, the required representation is given by the following 3 x 4 matrix:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

To emphasize that the matrix representation depends not only on the basis elements but also on their order, let us reverse the order of the basis elements in *Wand* use, instead, the ordered basis $(x^2, x, 1)$. Then the basis elements of V are transformed into the same polynomials obtained above, but the components of these polynomials relative to the new basis $(x^2, x, 1)$ appear in reversed order. Therefore, the matrix representation of D now becomes

$$\begin{bmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Let us compute a third matrix representation for D, using the basis $(1, 1 + x, 1 + x + x^2, 1 + x + x^2 + x^3)$ for V, and the basis $(1, x, x^2)$ for W. The basis elements of V are transformed as follows:

$$D(1) = 0, \qquad D(1 + x) = 1, \qquad D(1 + x + x^2) = 1 + 2x,$$

$$D(1 + x + x^2 + x^3) = 1 + 2x + 3x^2,$$

so the matrix representation in this case is

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

2.11 Construction of a matrix representation in diagonal form

Since it is possible to obtain different matrix representations of a given linear transformation by different choices of bases, it is natural to try to choose the bases so that the resulting matrix will have a particularly simple form. The next theorem shows that we can make all the entries 0 except possibly along the diagonal starting from the upper left-hand corner of the matrix. Along this diagonal there will be a string of ones followed by zeros, the number of ones being equal to the rank of the transformation. A matrix (t_{ik}) with all entries $t_{ik} = 0$ when $i \neq k$ is said to be a *diagonal matrix.*

THEOREM 2.14. *Let V and W be finite-dimensional linear spaces, with* $\dim V = n$ *and* $\dim W = m$. *Assume $T \in \mathscr{L}(V, W)$ and let* $r = \dim T(V)$ *denote the rank of T. Then there exists a basis $(e_1, \ldots, e_n)$ for V and a basis $(w_1, \ldots, w_m)$ for W such that*

(2.14) $$T(e_i) = w_i \quad \text{for} \quad i = 1, 2, \ldots, r,$$

and

(2.15) $$T(e_i) = O \quad \text{for} \quad i = r + 1, \ldots, n.$$

Therefore, the matrix (t_{ik}) of T relative to these bases has all entries zero except for the r diagonal entries

$$t_{11} = t_{22} = \cdots = t_{rr} = 1.$$

Proof. First we construct a basis for W. Since $T(V)$ is a subspace of W with $\dim T(V) = r$, the space $T(V)$ has a basis of r elements in W, say $w_1, \ldots, w_r$. By Theorem 1.7, these elements form a subset of some basis for W. Therefore we can adjoin elements $w_{r+1}, \ldots, w_m$ so that

(2.16) $$(w_1, \ldots, w_r, w_{r+1}, \ldots, w_m)$$

is a basis for W.

Now we construct a basis for V. Each of the first r elements w_i in (2.16) is the image of at least one element in V. Choose one such element in V and call it e_i. Then $T(e_i) = w_i$ for $i = 1, 2, \ldots, r$ so (2.14) is satisfied. Now let k be the dimension of the null space N(T). By Theorem 2.3 we have $n = k + r$. Since dim N(T) $= k$, the space N(T) has a basis consisting of k elements in V which we designate as $e_{r+1}, \ldots, e_{r+k}$. For each of these elements, Equation (2.15) is satisfied. Therefore, to complete the proof, we must show that the ordered set

$$(2.17) \qquad (e_1, \ldots, e_r, e_{r+1}, \ldots, e_{r+k})$$

is a basis for V. Since dim $V = n = r + k$, we need only show that these elements are independent. Suppose that some linear combination of them is zero, say

$$(2.18) \qquad \sum_{i=1}^{r+k} c_i e_i = O.$$

Applying T and using Equations (2.14) and (2.19, we find that

$$\sum_{i=1}^{r+k} c_i T(e_i) = \sum_{i=1}^{r} c_i w_i = O.$$

But $w_1, \ldots, w_r$ are independent, and hence $c_1 = \cdots = c_r = 0$. Therefore, the first r terms in (2.18) are zero, so (2.18) reduces to

$$\sum_{i=r+1}^{r+k} c_i e_i = O.$$

But $e_{r+1}, \ldots, e_{r+k}$ are independent since they form a basis for N(T), and hence $c_{r+1} = \cdots = c_{r+k} = 0$. Therefore, all the c_i in (2.18) are zero, so the elements in (2.17) form a basis for V. This completes the proof.

EXAMPLE. We refer to Example 2 of Section 2.10, where D is the differentiation operator which maps the space V of polynomials of degree ≤ 3 into the space W of polynomials of degree ≤ 2. In this example, the range $T(V) = W$, so T has rank 3. Applying the method used to prove Theorem 2.14, we choose any basis for W, for example the basis $(1, x, x^2)$. A set of polynomials in V which map onto these elements is given by $(x, \frac{1}{2}x^2, \frac{1}{3}x^3)$. We extend this set to get a basis for V by adjoining the constant polynomial 1, which is a basis for the null space of D. Therefore, if we use the basis $(x, \frac{1}{2}x^2, \frac{1}{3}x^3, 1)$ for V and the basis $(1, x, x^2)$ for W, the corresponding matrix representation for D has the diagonal form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$