Corollary 6. Let R be a Discrete Valuation Ring.

(1) The ring R is an integrally closed local ring with unique maximal ideal given by the elements with strictly positive valuation: $M = \{r \in R \mid \nu(r) > 0\}$. Every nonzero ideal in R is of the form M^n for some integer $n \geq 0$.

(2) The only prime ideals of R are M and 0, i.e., Spec $R = \{0, M\}$. In particular, a D.V.R. has Krull dimension 1.

Proof: Any U.F.D. is integrally closed in its fraction field (Example 3 in Section 15.3), so R is integrally closed. The remainder of the statements follow immediately from the description of the ideals of R in Proposition 5.

The definition of a Discrete Valuation Ring is extremely explicit in terms of a valuation on the fraction field, and as a result it appears that it might be difficult to recognize whether a given ring R is a D.V.R. from purely "internal" algebraic properties of R. In fact, the ring-theoretic properties in Proposition 5 and Corollary 6 characterize Discrete Valuation Rings. The following theorem gives several alternate algebraic descriptions of Discrete Valuation Rings in which there is no explicit mention of the valuation.

Theorem 7. The following properties of a ring R are equivalent:

(1) R is a Discrete Valuation Ring,

(2) R is a P.I.D. with a unique maximal ideal $P \neq 0$,

(3) R is a U.F.D. with a unique (up to associates) irreducible element t,

(4) R is a Noetherian integral domain that is also a local ring whose unique maximal ideal is nonzero and principal,

(5) R is a Noetherian, integrally closed, integral domain that is also a local ring of Krull dimension 1 i.e., R has a unique nonzero prime ideal: Spec $R = \{0, M\}$.

Proof: That (1) implies each of the other properties was proved above.

If (2) holds then (3) is immediate since irreducible elements generate prime ideals in a U.F.D. (Proposition 12, Section 8.3).

If (3) holds, then every nonzero element in R can be written uniquely in the form ut^n for some unit u and some $n \geq 0$. Then every nonzero element in the fraction field of R can be written uniquely in the form ut^n for some unit u and some $n \in \mathbb{Z}$. It is now straightforward to check that the map $\nu(ut^n) = n$ is a discrete valuation on the field of fractions of R, and R is the valuation ring of ν, and (1) holds.

Suppose (4) holds, let $M = (t)$ be the unique maximal ideal of R, and let $M_0 = \cap_{i=1}^{\infty} M^i$. Then $M_0 = MM_0$, and since R is Noetherian M_0 is finitely generated. By hypothesis $M = \text{Jac } R$, so by Nakayama's Lemma $M_0 = 0$. If I is any proper, nonzero ideal of R then there is some $n \geq 0$ such that $I \subseteq M^n$ but $I \not\subseteq M^{n+1}$. Let $a \in I - M^{n+1}$ and write $a = t^n u$ for some $u \in R$. Then $u \notin M$, and so u is a unit in the local ring R. Thus $(a) = (t^n) = M^n$ for every $a \in I - M^{n+1}$. This shows that $I = (t^n)$, and so every ideal of R is principal, which shows that (2) holds.

We have shown that (1), (2), (3) and (4) are equivalent, and that each of these implies (5). To complete the proof we show that (5) implies (4), which amounts to showing that the ideal M in (5) is a principal ideal. Since $0 \neq M = \text{Jac } R$ and M is finitely generated because R is Noetherian, by Nakayama's Lemma (Proposition 1(4)), $M \neq M^2$. Let $t \in M - M^2$. We argue that $M = (t)$. By Proposition 12 in Section 15.2, the assumption that M is the unique nonzero prime ideal in R implies that $M = \operatorname{rad}(t)$, and then Proposition 14 in Section 15.2 implies that some power of M is contained in (t). Proceeding by way of contradiction, assume $(t) \neq M$, so that $M^n \subseteq (t)$ but $M^{n-1} \not\subseteq (t)$ for some $n \geq 2$. Then there is an element $x \in M^{n-1} - (t)$ such that $xM \subseteq (t)$. Note that $t \neq 0$ so $y = x/t$ belongs to the field of fractions of R. Also, $y \notin R$ because $x = ty \notin (t)$. However, by choice of x we have $yM \subseteq R$, and then one checks that yM is an ideal in R. If $yM = R$ then $1 = ym$ for some $m \in M$. This leads to a contradiction because we would then have $t = xm \in M^2$, contrary to the choice of t. Thus yM is a proper ideal, hence is contained in the unique maximal ideal of R, namely $yM \subseteq M$. Now M is a finitely generated R-module on which y acts by left multiplication as an R-module homomorphism. By the same (determinant) method as in the proof of Proposition 23 in Section 15.3 there is a monic polynomial p with coefficients in R such that $p(y)m = 0$ for all $m \in M$. Since $p(y)$ is an element of a field containing R and M, we must have $p(y) = 0$. Hence y is integral over R. Since R is integrally closed by assumption, it follows that $y \in R$, a contradiction. Hence $M = (t)$ is principal, so (5) implies (4), completing the proof of the theorem.

Corollary 8. If R is any Noetherian, integrally closed, integral domain and P is a minimal nonzero prime ideal of R, then the localization R_P of R at P is a Discrete Valuation Ring.

Proof: By results in Section 15.4, the localization R_P is a Noetherian (Proposition 38(4)), integrally closed (Proposition 49), integral domain (Proposition 46(2)), that is a local ring with unique nonzero prime ideal (Proposition 46(4)), so R_P satisfies (5) in the theorem.

Examples

(1) If R is any Principal Ideal Domain then every localization R_P of R at a nonzero prime ideal $P = (p)$ is a Discrete Valuation Ring. This follows immediately from Corollary 8 since R is integrally closed (being a U.F.D., cf. Example 3 in Section 15.3) and nonzero prime ideals in a P.I.D. are maximal (Proposition 8.7). Note that the quotient field K of R_P is the same as the quotient field of R, so each nonzero prime p in R produces a valuation ν_p on K, given by the formula

$$\nu\left(p^n \frac{a}{b}\right) = n$$

where a and b are elements of R not divisible by p. This generalizes both Examples 1 and 2 above.

(2) The ring $\mathbb{Z}_p$ of p-adic integers is a Discrete Valuation Ring since it is a P.I.D. with unique maximal ideal $p\mathbb{Z}_p$ (cf. Exercise 11, Section 7.6). The fraction field of $\mathbb{Z}_p$ is called the *field of p-adic numbers* and is denoted $\mathbb{Q}_p$. The element p is a uniformizing parameter for $\mathbb{Z}_p$, so every nonzero element in $\mathbb{Q}_p$ can be written uniquely in the form $p^n u$ for some $n \in \mathbb{Z}$ and unit $u \in \mathbb{Z}_p^\times$, (where $u = a_0 + a_1 p + a_2 p^2 + \dots$ with $0 < a_0 < p$ as in Exercise 11(c), Section 7.6). The corresponding *p-adic valuation* ν_p on $\mathbb{Q}_p$ is then given by $\nu_p(p^n u) = n$.

A discrete valuation ν on a field K defines an associated *metric* (or "distance function"), d_ν, on K as follows: fix any real number $\beta > 1$ (the actual value of β does not matter for verifying the axioms of a metric), and for all $a, b \in K$ define

$$d_\nu(a, b) = \| a - b \|_\nu \qquad \text{where} \qquad \| a \|_\nu = \beta^{-\nu(a)}$$

and where we set $d_\nu(a, a) = 0$. It is easy to check that d_ν satisfies the three axioms for a metric:

(i) $d_\nu(a, b) \geq 0$, with equality holding if and only if $a = b$,
(ii) $d_\nu(a, b) = d_\nu(b, a)$, i.e., d_ν is symmetric,
(iii) $d_\nu(a, b) \leq d_\nu(a, c) + d_\nu(c, b)$, for all $a, b, c \in K$, i.e., d_ν satisfies the "triangle inequality."

The triangle inequality is a consequence of axiom (iii) of the discrete valuation. Indeed, a stronger version of the triangle inequality holds:

(iii)′ $d_\nu(a, b) \leq \max\{d_\nu(a, c), d_\nu(c, b)\}$, for all $a, b, c \in K$.

For this reason d_ν is sometimes called an *ultrametric*. One may now use Cauchy sequences to form the *completion* of K with respect to d_ν, denoted by K_ν, in the same way that the real numbers $\mathbb{R}$ are constructed from the rational numbers $\mathbb{Q}$. It is not difficult to show that K_ν is also a field with a discrete valuation that agrees with ν on the dense subset K of K_ν.

Examples

(1) Consider the p-adic valuation ν_p on $\mathbb{Q}$ and take $\beta = p$. Write $\| a \|_p$ for $\| a \|_{\nu_p}$, so that for a, b relatively prime to p,

$$\| p^n \frac{a}{b} \|_p = p^{-n}.$$

Note that integers (or rational numbers) have small p-adic absolute value if they are divisible by a large power of p. For example, the sequence $1, p, p^2, p^3, \ldots$ converges to zero in the p-adic metric.

It is not too difficult to see that the completion of $\mathbb{Q}$ with respect to the p-adic metric is the field $\mathbb{Q}_p$ of p-adic numbers, and the completion of $\mathbb{Z}$ is the ring $\mathbb{Z}_p$ of p-adic integers. One way to see this is to check that each element a of the completion may be represented as a *p-adic Laurent series*:

$$a = \sum_{n=n_0}^{\infty} a_i p^i \qquad \text{where } n_0 \in \mathbb{Z} \text{ and } a_i \in \{0, 1, \ldots, p-1\} \text{ for all } i,$$

and then use Example 2 previously. In terms of this expansion, the p-adic valuation is given by $\nu_p(a) = n_0$ (when $a_{n_0} \neq 0$).

(2) In a similar way, the completion of $F(x)$ with respect to the valuation ν_x in Example 2 at the beginning of this section gives the field $F((x))$ with corresponding valuation ring $F[[x]]$ in Example 3 in the same set of examples.

The completion of a field K with respect to a discrete valuation ν is a field K_ν in which the elements can be easily described in terms of a uniformizing parameter. In addition, K_ν is a topological space where the topology is defined by the metric d_ν. Furthermore, Cauchy sequences of elements in K_ν converge to elements of K_ν (i.e., K_ν