Since $T\beta_i = c_i\beta_i$, we have

$$f(T)\alpha = f(c_1)\beta_1 + \cdots + f(c_k)\beta_k \tag{7-20}$$

for every polynomial f. Given any scalars $t_1, \ldots, t_k$ there exists a polynomial f such that $f(c_i) = t_i$, $1 \leq i \leq k$. Therefore, $Z(\alpha; T)$ is just the subspace spanned by the vectors $\beta_1, \ldots, \beta_k$. What is the annihilator of α? According to (7-20), we have $f(T)\alpha = 0$ if and only if $f(c_i)\beta_i = 0$ for each i. In other words, $f(T)\alpha = 0$ provided $f(c_i) = 0$ for each i such that $\beta_i \neq 0$. Accordingly, the annihilator of α is the product

$$\prod_{\beta_i \neq 0} (x - c_i). \tag{7-21}$$

Now, let $\mathcal{B}_i = \{\beta_1^i, \ldots, \beta_{d_i}^i\}$ be an ordered basis for V_i. Let

$$r = \max_i d_i.$$

We define vectors $\alpha_1, \ldots, \alpha_r$ by

$$\alpha_j = \sum_{d_i \geq j} \beta_j^i, \qquad 1 \leq j \leq r. \tag{7-22}$$

The cyclic subspace $Z(\alpha_j; T)$ is the subspace spanned by the vectors β_j^i, as i runs over those indices for which $d_i \geq j$. The T-annihilator of α_j is

$$p_j = \prod_{d_i \geq j} (x - c_i). \tag{7-23}$$

We have

$$V = Z(\alpha_1; T) \oplus \cdots \oplus Z(\alpha_r; T)$$

because each β_j^i belongs to one and only one of the subspaces $Z(\alpha_1; T), \ldots, Z(\alpha_r; T)$ and $\mathcal{B} = (\mathcal{B}_1, \ldots, \mathcal{B}_k)$ is a basis for V. By (7-23), p_{j+1} divides p_j.

Exercises

1. Let T be the linear operator on F^2 which is represented in the standard ordered basis by the matrix

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Let $\alpha_1 = (0, 1)$. Show that $F^2 \neq Z(\alpha_1; T)$ and that there is no non-zero vector α_2 in F^2 with $Z(\alpha_2; T)$ disjoint from $Z(\alpha_1; T)$.

2. Let T be a linear operator on the finite-dimensional space V, and let R be the range of T.

(a) Prove that R has a complementary T-invariant subspace if and only if R is independent of the null space N of T.

(b) If R and N are independent, prove that N is the unique T-invariant subspace complementary to R.

3. Let T be the linear operator on R^3 which is represented in the standard ordered basis by the matrix

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Let W be the null space of $T - 2I$. Prove that W has no complementary T-invariant subspace. (*Hint:* Let $\beta = \epsilon_1$ and observe that $(T - 2I)\beta$ is in W. Prove there is no α in W with $(T - 2I)\beta = (T - 2I)\alpha$.)

4. Let T be the linear operator on F^4 which is represented in the standard ordered basis by the matrix

$$\begin{bmatrix} c & 0 & 0 & 0 \\ 1 & c & 0 & 0 \\ 0 & 1 & c & 0 \\ 0 & 0 & 1 & c \end{bmatrix}.$$

Let W be the null space of $T - cI$.

(a) Prove that W is the subspace spanned by ϵ_4.

(b) Find the monic generators of the ideals $S(\epsilon_4; W)$, $S(\epsilon_3; W)$, $S(\epsilon_2; W)$, $S(\epsilon_1; W)$.

5. Let T be a linear operator on the vector space V over the field F. If f is a polynomial over F and α is in V, let $f\alpha = f(T)\alpha$. If $V_1, \ldots, V_k$ are T-invariant subspaces and $V = V_1 \oplus \cdots \oplus V_k$, show that

$$fV = fV_1 \oplus \cdots \oplus fV_k.$$

6. Let T, V, and F be as in Exercise 5. Suppose α and β are vectors in V which have the same T-annihilator. Prove that, for any polynomial f, the vectors $f\alpha$ and $f\beta$ have the same T-annihilator.

7. Find the minimal polynomials and the rational forms of each of the following real matrices.

$$\begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} c & 0 & -1 \\ 0 & c & 1 \\ -1 & 1 & c \end{bmatrix}, \quad \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

8. Let T be the linear operator on R^3 which is represented in the standard ordered basis by

$$\begin{bmatrix} 3 & -4 & -4 \\ -1 & 3 & 2 \\ 2 & -4 & -3 \end{bmatrix}.$$

Find non-zero vectors $\alpha_1, \ldots, \alpha_r$ satisfying the conditions of Theorem 3.

9. Let A be the real matrix

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 3 & 1 & 3 \\ -3 & -3 & -5 \end{bmatrix}.$$

Find an invertible 3×3 real matrix P such that $P^{-1}AP$ is in rational form.

10. Let F be a subfield of the complex numbers and let T be the linear operator on F^4 which is represented in the standard ordered basis by the matrix

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & a & 2 & 0 \\ 0 & 0 & b & 2 \end{bmatrix}.$$

Find the characteristic polynomial for T. Consider the cases $a = b = 1$; $a = b = 0$; $a = 0$, $b = 1$. In each of these cases, find the minimal polynomial for T and non-zero vectors $\alpha_1, \ldots, \alpha_r$ which satisfy the conditions of Theorem 3.

11. Prove that if A and B are 3×3 matrices over the field F, a necessary and sufficient condition that A and B be similar over F is that they have the same characteristic polynomial and the same minimal polynomial. Give an example which shows that this is false for 4×4 matrices.

12. Let F be a subfield of the field of complex numbers, and let A and B be $n \times n$ matrices over F. Prove that if A and B are similar over the field of complex numbers, then they are similar over F. (*Hint:* Prove that the rational form of A is the same whether A is viewed as a matrix over F or a matrix over C; likewise for B.)

13. Let A be an $n \times n$ matrix with complex entries. Prove that if every characteristic value of A is real, then A is similar to a matrix with real entries.

14. Let T be a linear operator on the finite-dimensional space V. Prove that there exists a vector α in V with this property. If f is a polynomial and $f(T)\alpha = 0$, then $f(T) = 0$. (Such a vector α is called a **separating vector** for the algebra of polynomials in T.) When T has a cyclic vector, give a direct proof that any cyclic vector is a separating vector for the algebra of polynomials in T.

15. Let F be a subfield of the field of complex numbers, and let A be an $n \times n$ matrix over F. Let p be the minimal polynomial for A. If we regard A as a matrix over C, then A has a minimal polynomial f as an $n \times n$ matrix over C. Use a theorem on linear equations to prove $p = f$. Can you also see how this follows from the cyclic decomposition theorem?

16. Let A be an $n \times n$ matrix with *real* entries such that $A^2 + I = 0$. Prove that n is even, and if $n = 2k$, then A is similar over the field of real numbers to a matrix of the block form

$$\begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$$

where I is the $k \times k$ identity matrix.

17. Let T be a linear operator on a finite-dimensional vector space V. Suppose that
(a) the minimal polynomial for T is a power of an irreducible polynomial;
(b) the minimal polynomial is equal to the characteristic polynomial.
Show that no non-trivial T-invariant subspace has a complementary T-invariant subspace.

18. If T is a diagonalizable linear operator, then every T-invariant subspace has a complementary T-invariant subspace.

19. Let T be a linear operator on the finite-dimensional space V. Prove that T has a cyclic vector if and only if the following is true: Every linear operator U which commutes with T is a polynomial in T.

20. Let V be a finite-dimensional vector space over the field F, and let T be a linear operator on V. We ask when it is true that every non-zero vector in V is a cyclic vector for T. Prove that this is the case if and only if the characteristic polynomial for T is irreducible over F.

21. Let A be an $n \times n$ matrix with *real* entries. Let T be the linear operator on R^n which is represented by A in the standard ordered basis, and let U be the linear operator on C^n which is represented by A in the standard ordered basis. Use the result of Exercise 20 to prove the following: If the only subspaces invariant under T are R^n and the zero subspace, then U is diagonalizable.

7.3. The Jordan Form

Suppose that N is a nilpotent linear operator on the finite-dimensional space V. Let us look at the cyclic decomposition for N which we obtain from Theorem 3. We have a positive integer r and r non-zero vectors $\alpha_1, \ldots, \alpha_r$ in V with N-annihilators $p_1, \ldots, p_r$, such that

$$V = Z(\alpha_1; N) \oplus \cdots \oplus Z(\alpha_r; N)$$

and p_{i+1} divides p_i for $i = 1, \ldots, r-1$. Since N is nilpotent, the minimal polynomial is x^k for some $k \leq n$. Thus each p_i is of the form $p_i = x^{k_i}$, and the divisibility condition simply says that

$$k_1 \geq k_2 \geq \cdots \geq k_r.$$

Of course, $k_1 = k$ and $k_r \geq 1$. The companion matrix of x^{k_i} is the $k_i \times k_i$ matrix

$$A_i = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \tag{7-24}$$

Thus Theorem 3 gives us an ordered basis for V in which the matrix of N is the direct sum of the elementary nilpotent matrices (7-24), the sizes of which decrease as i increases. One sees from this that associated with a nilpotent $n \times n$ matrix is a positive integer r and r positive integers $k_1, \ldots, k_r$ such that $k_1 + \cdots + k_r = n$ and $k_i \geq k_{i+1}$, and these positive integers determine the rational form of the matrix, i.e., determine the matrix up to similarity.

Here is one thing we should like to point out about the nilpotent operator N above. The positive integer r is precisely the nullity of N; in fact, the null space has as a basis the r vectors

$$N^{k_i - 1}\alpha_i. \tag{7-25}$$

For, let α be in the null space of N. We write α in the form

$$\alpha = f_1\alpha_1 + \cdots + f_r\alpha_r$$

where f_i is a polynomial, the degree of which we may assume is less than k_i. Since $N\alpha = 0$, for each i we have

$$\begin{aligned} 0 &= N(f_i\alpha_i) \\ &= Nf_i(N)\alpha_i \\ &= (xf_i)\alpha_i. \end{aligned}$$

Thus xf_i is divisible by x^{k_i}, and since deg $(f_i) > k_i$ this means that

$$f_i = c_i x^{k_i - 1}$$

where c_i is some scalar. But then

$$\alpha = c_1(x^{k_1-1}\alpha_1) + \cdots + c_r(x^{k_r-1}\alpha_r)$$

which shows us that the vectors (7-25) form a basis for the null space of N. The reader should note that this fact is also quite clear from the matrix point of view.

Now what we wish to do is to combine our findings about nilpotent operators or matrices with the primary decomposition theorem of Chapter 6. The situation is this: Suppose that T is a linear operator on V and that the characteristic polynomial for T factors over F as follows:

$$f = (x - c_1)^{d_1} \cdots (x - c_k)^{d_k}$$

where $c_1, \ldots, c_k$ are distinct elements of F and $d_i \geq 1$. Then the minimal polynomial for T will be

$$p = (x - c_1)^{r_1} \cdots (x - c_k)^{r_k}$$

where $1 \leq r_i \leq d_i$. If W_i is the null space of $(T - c_iI)^{r_i}$, then the primary decomposition theorem tells us that

$$V = W_1 \oplus \cdots \oplus W_k$$

and that the operator T_i induced on W_i by T has minimal polynomial $(x - c_i)^{r_i}$. Let N_i be the linear operator on W_i defined by $N_i = T_i - c_iI$. Then N_i is nilpotent and has minimal polynomial x^{r_i}. On W_i, T acts like N_i plus the scalar c_i times the identity operator. Suppose we choose a basis for the subspace W_i corresponding to the cyclic decomposition for the nilpotent operator N_i. Then the matrix of T_i in this ordered basis will be the direct sum of matrices

$$\text{(7-26)} \qquad \begin{bmatrix} c & 0 & \cdots & 0 & 0 \\ 1 & c & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ & & & c & \\ 0 & 0 & \cdots & 1 & c \end{bmatrix}$$

each with $c = c_i$. Furthermore, the sizes of these matrices will decrease as one reads from left to right. A matrix of the form (7-26) is called an **elementary Jordan matrix with characteristic value** c. Now if we put all the bases for the W_i together, we obtain an ordered basis for V. Let us describe the matrix A of T in this ordered basis.