Gregory's first major achievement was the invention of the reflecting telescope, which he described in his book *Optica promota* of 1663. Unfortunately, he failed to get a satisfactory instrument constructed, and his design was overtaken by the simpler type invented by Newton. In the meantime, Gregory had decided to improve his scientific knowledge on the Continent, and he spent most of 1664 to 1668 studying mathematics in Italy. His teacher was Stefano degli Angeli (1623–1697) of Padua, from whom Gregory learned the methods of Cavalieri. The influence of the Italian school was evident in Gregory's geometric approach to integration problems in his first mathematical works, *Vera circuli et hyperbolae quadratura* [Gregory (1667)] and *Geometriae pars universalis* [Gregory (1668)], but so too was Gregory's originality. The books received glowing reviews in London and, when Gregory went there on his return from Italy, he was elected to the Royal Society.

The *Geometriae pars universalis* was mainly a systematization of the results in differentiation and integration then known, but it included the first published proof of the fundamental theorem of calculus. Important as this was, the theorem was not Gregory's alone, as Newton and Leibniz discovered it independently. What really set Gregory apart from other seventeenth-century mathematicians was the *Vera quadratura* (True Quadrature), an extraordinarily bold and imaginative attempt to prove that the numbers π and e are transcendental.

As mentioned in Section 2.3, transcendence of e and π was not proved until the nineteenth century, and certainly not by seventeenth-century methods, so it is understandable that Gregory's attempt fell short. Nevertheless, it is full of brilliant ideas: the unification of circular and hyperbolic functions (without the use of complex numbers), the concept of convergence, and the distinction between algebraic and transcendental functions. Gregory showed that areas cut off from both the circle and the hyperbola (giving π and various logarithms as special cases) could be obtained as the limit of alternate geometric and harmonic means:

$$\begin{aligned} i_{n+1} &= \sqrt{i_n I_n}, \\ \frac{1}{I_{n+1}} &= \frac{1}{2}\left(\frac{1}{i_{n+1}} + \frac{1}{I_n}\right), \\ \lim_{n\to\infty} i_n &= \lim_{n\to\infty} I_n = I. \end{aligned}$$

If $i_0 = 2$ and $I_0 = 4$, then I (the *geometric-harmonic mean* of 2 and 4) is π. If, on the other hand, $i_0 = 99/20$ and $I_0 = 18/11$, then I is $\log 10$. These examples of Gregory illustrate the way his geometric-harmonic mean embraces both circular and hyperbolic functions. The alternating procedure used to define the mean had an interesting echo in the work of Gauss, who investigated the analogously defined *arithmetic-geometric mean* in the 1790s, with far-reaching results (Section 12.6).

In 1669 Gregory returned to Scotland to take up the chair of mathematics at St. Andrew's. He married a young widow, Mary Burnet, the daughter of artist George Jameson, who was also descended from the Anderson family. James and Mary had two daughters and a son, who became professor of medicine in Aberdeen. The rather impressive Gregory family tree may be found in Turnbull's short biography of Gregory [Turnbull (1939)].

Gregory stayed at St. Andrew's for five years, during which he obtained his important results on series. However, his contact with other scientists was restricted to letters from London, and on hearing of Newton's related results he assumed that he had been anticipated and did not publish. The lack of contact, and hostility to mathematics at St. Andrew's, led him to accept the offer of a chair at Edinburgh in 1674. Alas, he had been in Edinburgh barely a year when he collapsed, apparently from a stroke, while showing the moons of Jupiter to a group of students. He died a few days later, in October 1675, too soon for the world to have understood the importance of his work.

Leonhard Euler was born in Basel in 1707 and died in St. Petersburg in 1783. His father, Paul, studied theology at the University of Basel, where he also attended the mathematics lectures of Jakob Bernoulli. After graduation he became a Protestant minister and married a minister's daughter, Margarete Bruckner. Leonhard was the first of their six children. The family was quite poor and, soon after Euler's birth, moved to a village outside Basel where they lived in a two-room house. Euler received his first mathematical instruction at home from his father. He later moved back to Basel to attend secondary school, but mathematics was not taught there, so he took some private lessons from a university student.

At 13, Euler entered the University of Basel, which had become the mathematical center of Europe under Johann Bernoulli, the younger brother and successor of Jakob. Bernoulli advised Euler to study mathematics on his own and made himself available on Saturday afternoons to help with any difficulties. Euler's official studies were in philosophy and law. After receiving his master's degree in philosophy in 1723, he followed his father's wish by entering the department of theology. However, he was falling increasingly under the spell of mathematics and realized he would have to drop the idea of becoming a minister.

There were few opportunities for mathematicians in Switzerland, and in 1727 Euler left Basel for St. Petersburg. Johann Bernoulli's sons, Daniel and Nicholas, had been appointed to the new Academy of Sciences there, and they persuaded the authorities to find a place for Euler. Euler had already shown promise with a couple of papers in *Acta Eruditorum* and an honorable mention in the Paris Academy competition of 1727, but in St. Petersburg he surpassed all expectations, producing top-quality work at a rate that has astounded mathematicians ever since. The early years in St. Petersburg with the Bernoullis must have been a young mathematician's dream. Yet it is equally true that Euler's productivity was unaffected by later setbacks, including the loss of his sight. He filled half the pages published by the St. Petersburg Academy from 1729 until over 50 years after his death (!), and he also accounted for half the production of the Berlin Academy between 1746 and 1771.

The first major changes in Euler's life in St. Petersburg occurred in 1733, when Daniel Bernoulli returned to Basel. Euler then became professor of mathematics but also had to take over the Department of Geography. In the same year, he married a compatriot, Katharina Gsell, the daughter of an artist who taught in St. Petersburg. They were eventually to have 13 children, 5 of whom reached maturity. Euler's duties in geography included the preparation of a map of Russia, a task that strained his eyes and perhaps led to the fever that destroyed the sight of his right eye in 1738. Figure 10.4 is a portrait from his good side.

By 1740 the political situation in St. Petersburg had become unsettled and Euler moved to Berlin, where Frederick the Great had just reorganized the Berlin Academy. Euler became director of the mathematical section and stayed in Berlin for 25 years. Some of his most famous works date from this period, in particular the *Introductio in analysin infinitorum* [Euler (1748a)] and the *Letters à une princesse d'Allemagne sur divers sujets de physique et de philosophie*, one of the classics of popular science. However, Euler was not comfortable in Berlin. There were quarrels over the leadership of the Academy, and the cynical Frederick tended to sneer at the pious and unassuming Euler. In 1762 Catherine the Great came to the throne in Russia, and the St. Petersburg Academy, with which Euler had maintained contact throughout, began to look attractive again.

In 1766 he moved back to St. Petersburg with his family (as a bonus,

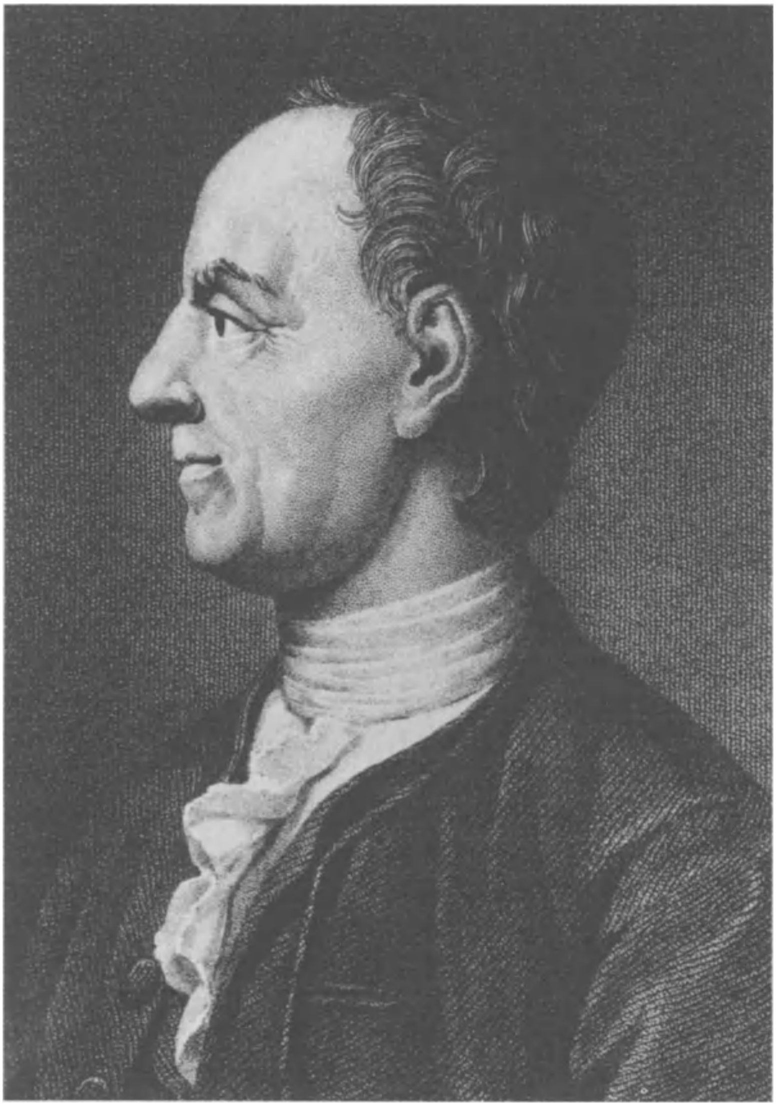

Figure 10.4: Leonhard Euler

his eldest son gained the chair of physics there). Soon after his arrival Euler suffered an illness that destroyed most of his remaining sight, and in 1771 he became completely blind. If anything, blindness concentrated Euler's mind more wonderfully. He had always had an extraordinary memory—knowing Virgil's *Aeneid* by heart, for example—and with assistance from two of his sons and other collaborators his flow of publications continued at a greater rate than ever. His *Algebra* [Euler (1770)] was dictated to his valet, yet it became the most successful mathematics textbook since Euclid's *Elements*.

One of Euler's most admirable qualities was a willingness to explain how his discoveries were made. Mathematicians of the eighteenth-century were less secretive than their sixteenth- and seventeenth-century predecessors, but Euler was unique in revealing his preliminary guesses, experiments, and partial proofs. Some of the most interesting of these exposés are presented in Pólya's book [Pólya (1954b)] on plausible reasoning. Chapter 6 of the book, for example, includes a translation of the memoir in which Euler announced the pentagonal number theorem. It is impossible to summarize all of Euler's contributions to mathematics here, though several of the highlights are presented in the chapters that follow. The best summary available is in Yushkevich's article on Euler in the *Dictionary of Scientific Biography*.