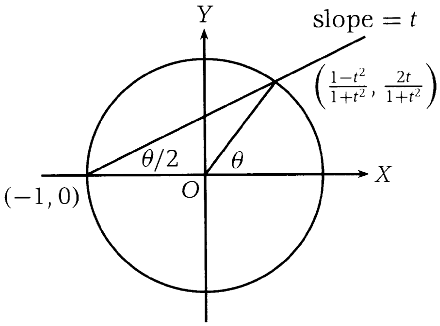

FIGURE 5.10 The half angle construction.

we need is the *addition formula for tan,* which is found as follows:

$$\begin{aligned}\tan(\theta+\phi) &= \frac{\sin(\theta+\phi)}{\cos(\theta+\phi)}\\ &= \frac{\sin\theta\cos\phi+\cos\theta\sin\phi}{\cos\theta\cos\phi-\sin\theta\sin\phi}\\ &= \frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi},\end{aligned}$$

dividing numerator and denominator by $\cos\theta\cos\phi$. This formula gives us what we want:

Rational addition formula. *If the line through O at angle θ has slope s and the line at angle ϕ has slope t, then the line at angle $\theta+\phi$ has slope* $(s+t)/(1-st)$. □

With this formula, we can finally solve the problem the Babylonians were trying to solve with their table of Pythagorean triples (Section 4.1). In effect, they were looking for equally spaced rational points on an arc of the unit circle. Any number of them can now be found, by starting with a point that has a small angle θ and a rational slope relative to O, and finding the points at angles 2θ, 3θ, and so on. By the rational addition formula, these points also have rational slopes relative to O, and hence they are rational points.

For example, suppose we start with the Pythagorean triple $(24, 7, 25)$. This corresponds to the rational point at slope $s = 7/24$ and angle θ of about 16°. By the addition formula for tan, the point at angle 2θ has slope

$$t = \frac{2s}{1-s^2} = \frac{2 \times 7/24}{1 - 7^2/24^2} = \frac{14 \times 24}{24^2 - 7^2} = \frac{336}{527},$$

and this point corresponds to the Pythagorean triple (527, 336, 625). The point at angle 3θ has slope

$$u = \frac{s+t}{1-st} = \frac{\frac{7}{24} + \frac{336}{527}}{1 - \frac{7}{24}\frac{336}{527}} = \frac{7 \times 527 + 336 \times 24}{24 \times 527 - 7 \times 336} = \frac{11753}{10296},$$

and this corresponds to the Pythagorean triple (10296, 11753, 15625). Clearly the process can be continued indefinitely, or at least until the computations become unmanageable.

We never return to the initial rational point on the circle by continuing this process. Hence, it is impossible to improve our solution of the Babylonian problem by subdividing a right angle, say, with equally spaced rational points. In fact, it is impossible to divide the circle into more than four equal parts by rational points. The exercises to Section 5.8* will explain why.

Exercises

Having seen how $i = \sqrt{-1}$ simplifies the addition formulas for cos and sin, we would also expect it to help with Pythagorean triples. This expectation is fulfilled. When one reflects on the connections between triples, points on the unit circle, and angles, the following procedure comes naturally to mind:

- Replace the triple (a, b, c) by the point $(a/c, b/c)$ on the unit circle.
- Think of the point $(a/c, b/c)$ as $(\cos\theta, \sin\theta)$, which in turn can be replaced by the number $\cos\theta + i\sin\theta$.
- Given two triples (a_1, b_1, c_1) and (a_2, b_2, c_2), form the corresponding numbers $a_1/c_1 + ib_1/c_1$ and $a_2/c_2 + ib_2/c_2$. The product $a_3/c_3 + ib_3/c_3$ of the latter numbers yields a new Pythagorean triple (a_3, b_3, c_3).

5.4.1. Show that if (a_1, b_1, c_1) and (a_2, b_2, c_2) are Pythagorean triples, then so is their "product" (a_3, b_3, c_3), where a_3 and b_3 are defined by

$$a_3 + ib_3 = (a_1 + ib_1)(a_2 + ib_2)$$

and $c_3 = \sqrt{a_3^2 + b_3^2}$.

5.4.2. Show that $c_3 = c_1 c_2$.

5.4.3. Show that the $(3, 4, 5)$ triple has "square" $(-7, 24, 25)$.

Conversely, one may "factorize" certain triples into products of simpler triples. Some of the triples (a, b, c) in Plimpton 322 "factorize" in this sense, though only a minority of them, because c is a prime number for most. A "factorization" of (a, b, c) implies that the angle with the rational slope b/a is a nontrivial sum of two angles with rational slope. In particular, if (a, b, c) is a "perfect square" then half its angle also has rational slope.

5.4.4. Show that the triples $(119, 120, 169)$ and $(161, 240, 289)$ from Plimpton 322 are "perfect squares" in this sense.

5.4.5. Can you suggest a method of "division" to find a second "factor" of a Pythagorean triple when one "factor" is known?

5.4.6. "Factorize" the following triples from Plimpton 322:

$$(319, 360, 481), \quad (1679, 2400, 2929), \quad (4601, 4800, 6649).$$

5.5* Hilbert's Third Problem

The rest of this chapter is concerned with a famous problem we met in Section 2.7: is it possible to cut a regular tetrahedron into finitely many pieces by planes and paste the pieces into a cube? As mentioned before, this problem is the main obstacle when we attempt to develop a theory of volume using only finite processes. Its importance was recognized by Hilbert, and he placed it at number 3 on the list of problems for 20th century mathematicians he announced in Paris in 1900. It was solved a few months later by Hilbert's student Max Dehn, with surprisingly simple methods. It is certainly the only one of Hilbert's problems whose solution can be described in a book such as this, and it happens to be relevant here, because trigonometry plays an important role in it.

Dehn's solution comes from focusing on the *dihedral angles* of a polyhedron, the angles between its faces. If a tetrahedron can be cut up and pasted into a cube, for example, then it looks like we have to build right angles (the dihedral angles of a cube) from the dihedral angles of a tetrahedron. It is not quite that simple, because the cuts can create dihedral angles in the interior, but one feels that these "cancel out" in some sense. Later we'll describe precisely how one keeps track of dihedral angles as a polyhedron is cut and pasted, but the first step is to actually find the dihedral angles of a tetrahedron. This is where trigonometry makes its first appearance in the problem; we have to measure the triangle ABC in which angle ABC is the dihedral angle of the tetrahedron (Figure 5.11).

It follows from Pythagoras' theorem that $AB = BC = \sqrt{3}/2$, and this in turn implies $BD = 1/\sqrt{2}$. Consequently, if α is the dihedral angle,

$$\cos\frac{\alpha}{2} = \frac{BD}{AB} = \frac{2}{\sqrt{2}\sqrt{3}},$$

and therefore

$$\cos\alpha = 2\cos^2\frac{\alpha}{2} - 1 = \frac{2\times 4}{6} - 1 = \frac{1}{3}.$$

Thus one of the things we have to understand is the relationship between $\pi/2$, the dihedral angle of the cube, and the angle whose cosine is $1/3$. The keys to this relationship turn out to be the addition formula and some basic number theory, as we shall see in Section

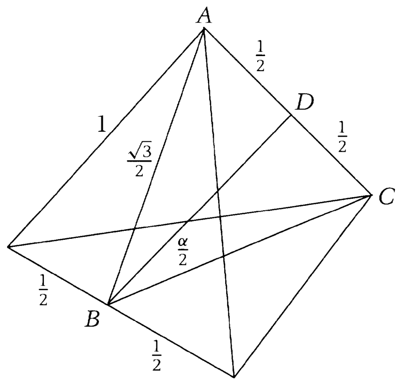

FIGURE 5.11 The dihedral angle of the regular tetrahedron.

5.8*. But first we need a better understanding of the behavior of a polyhedron under cutting and pasting, so that we can keep track of its dihedral angles.

Exercises

One of the nice properties of the angles of a polygon is that their sum is an integer multiple of π, in fact $(n-2)\pi$, where n is the number of vertices (Exercise 2.3.3). The dihedral angles of a polyhedron do not behave so nicely. Their sum is an integer multiple of π for some polyhedra, such as the cube, but for others it is not.

5.5.1. If α is the dihedral angle of the regular tetrahedron, show that $\cos 2\alpha = -7/9$ and $\cos 6\alpha = 329/729$.

5.5.2. Deduce from Exercise 5.5.1 that the dihedral angle sum of a regular tetrahedron is not an integer multiple of π.

5.6* The Dehn Invariant

Dehn solved Hilbert's third problem by a stroke of genius. He saw that volume is not the only thing conserved by cutting and pasting a polyhedron. Another is what might be called its *dihedral content*, an object that encodes the dihedral angles and ties them to the lengths of the corresponding edges.

An edge of length l and dihedral angle α makes a contribution to the dihedral content that is written $l \otimes \alpha$. The total dihedral content of a polyhedron (or a finite set S of polyhedra) is written

$$D(S) = l_1 \otimes \alpha_1 + l_2 \otimes \alpha_2 + \cdots + l_k \otimes \alpha_k,$$

where $l_1, l_2, \ldots, l_k$ are the lengths of the edges and $\alpha_1, \alpha_2, \ldots, \alpha_k$ are their respective dihedral angles. Because the grouping or order of the length $\otimes$ angle pairs does not matter, $+$ is an associative and commutative operation, and there is no harm in confusing it with ordinary addition.

However, $D(S)$ is so far just an expression containing some information about a *single* finite set S of polyhedra. If $D(S)$ is also to describe the various sets S', S'', ... obtainable from S by cutting and pasting, we shall need rules that transform $D(S)$ into $D(S')$, $D(S'')$, ... These rules are very easy to state:

$$l \otimes (\alpha + \beta) = l \otimes \alpha + l \otimes \beta \qquad \text{(Rule 1)}$$

$$(l + m) \otimes \alpha = l \otimes \alpha + m \otimes \alpha \qquad \text{(Rule 2)}$$

$$l \otimes \pi = 0 \qquad \text{(Rule 3)}$$

Rule 1 tells what to do when a cut is made along an edge, splitting its dihedral angle $\alpha + \beta$ into dihedral angles α and β (Figure 5.12). Conversely, it tells what to do when two dihedral angles are pasted into one.

Rule 2 tells what to do when a cut is made across an edge of length $l + m$, splitting it into edges of lengths l and m (Figure 5.13). Conversely, it tells what to do when two edges are joined end-to-end into one.

Rule 3 tells us that an edge with angle π can be ignored, as it should be, because it is not an actual edge. (One such spurious edge is produced, as in Figure 5.14, when we cut along an edge with dihedral angle $\pi + \alpha$, splitting off the dihedral angle α. This creates

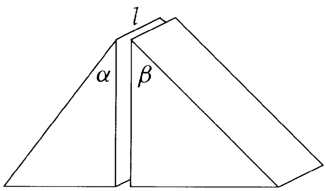

FIGURE 5.12 Why $l \otimes (\alpha + \beta) = l \otimes \alpha + l \otimes \beta$.

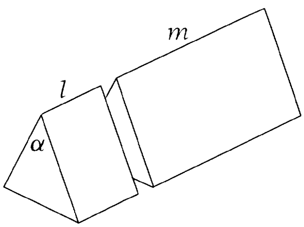

FIGURE 5.13 Why $(l + m) \otimes \alpha = l \otimes \alpha + m \otimes \alpha$.

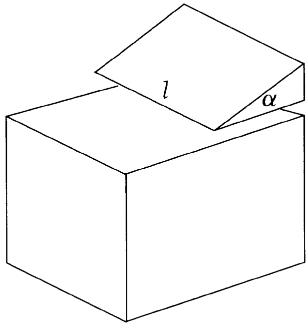

FIGURE 5.14 Why $l \otimes \pi = 0$.

an actual edge in one piece but not in the other, so $l \otimes (\pi + \alpha)$ is replaced by $l \otimes \alpha$.)

These figures are a little too simple, because they show the edge l perpendicular to the faces at its ends, so the dihedral angle is actually the angle visible at the end of l. But even if the dihedral angle is not visible, rules 1, 2, and 3 correctly express what happens to it under cutting and pasting.

When $D(S)$ is subjected to these rules it is called the *Dehn invariant*, because by definition it remains the same when S is cut or pasted. In particular, *if $\mathcal{P}$ and $\mathcal{Q}$ are equidecomposable polyhedra, then* $D(\mathcal{P}) = D(\mathcal{Q})$.

Example The Dehn invariant of the unit cube is 0.

This Dehn invariant is $12 \otimes \frac{\pi}{2}$, because the cube has 12 edges, each of length 1 and of dihedral angle $\pi/2$. But it follows from Rule 1 that

$$1 \otimes \frac{\pi}{2} + 1 \otimes \frac{\pi}{2} = 1 \otimes \left(\frac{\pi}{2} + \frac{\pi}{2}\right) = 1 \otimes \pi,$$

which equals 0 because $1 \otimes \pi = 0$ by Rule 3. □

Rules 1 and 2 are so simple that systems obeying them have a name—*tensor products*—and have been studied in modern algebra. A consequence of Rule 1, for example, is that $l \otimes 0 = 0$, because

$$l \otimes \alpha = l \otimes (\alpha + 0) = l \otimes \alpha + l \otimes 0.$$

Because of this, Rule 3 should be regarded not as a property of the $\otimes$ operation but as a property of the set of dihedral angles. This set is denoted by $\mathbb{R}/\pi\mathbb{Z}$ and, informally speaking, it is what $\mathbb{R}$ becomes when we pretend that $\pi = 0$. Its members are actually the sets $\{\ldots, \alpha - 2\pi, \alpha - \pi, \alpha, \alpha + \pi, \alpha + 2\pi, \ldots\}$ for each real number α. $\mathbb{R}/\pi\mathbb{Z}$ is very like the set of angles, which in fact is $\mathbb{R}/2\pi\mathbb{Z}$. For angles, we always want $2\pi = 0$, but with the Dehn invariant we also want $\pi = 0$, because an edge with dihedral angle π is not an edge at all.

The objects $l_1 \otimes \alpha_1 + l_2 \otimes \alpha_2 + \cdots + l_k \otimes \alpha_k$ that occur as values of the Dehn invariant are today called *tensors*. The set of them is denoted by $\mathbb{R} \otimes \mathbb{R}/\pi\mathbb{Z}$ and is called the *tensor product of* $\mathbb{R}$ *and* $\mathbb{R}/\pi\mathbb{Z}$. Tensor products are normally studied in advanced algebra courses, where more sophisticated methods are available. However, we shall be able to prove what we need about the Dehn invariant from first principles, as Dehn himself did.

Exercises

The simpler tensor product $\mathbb{R} \otimes \mathbb{R}$ is also related to a decomposition problem. Consider a set of rectangles with horizontal and vertical sides, and suppose that the rectangles can be cut and pasted along vertical and horizontal lines. We represent a single rectangle with horizontal side x and vertical side y by the tensor $x \otimes y$ and a set of them by a sum of such terms. If a vertical cut divides the $(x_1 + x_2) \otimes y$ rectangle into two, of width x_1 and x_2, respectively, then we have the rule

$$(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y.$$

Similarly, a horizontal cut yields the rule

$$x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2.$$

These rules define the tensor product $\mathbb{R} \otimes \mathbb{R}$. Thus each member of $\mathbb{R} \otimes \mathbb{R}$ may be interpreted as the set of all sums of rectangles that are equivalent under vertical and horizontal cut and paste. In particular, if $x \otimes y = x' \otimes y'$ it means that the rectangle with horizontal side x and vertical side y may be converted to the rectangle with horizontal side x' and vertical side y' in this way.