5.6.1. If x, y, x', y' are rational, and $xy = x'y'$, show that $x \otimes y = x' \otimes y'$.

This prompts the question: are rectangles of equal area always equivalent under vertical and horizontal cut and paste? In particular, can the rectangle $\sqrt{2} \otimes 1/\sqrt{2}$ be converted to the unit square $1 \otimes 1$ in this way? We shall answer this question in the next section, where it is revealed that tensors capture not only equivalence by cutting and pasting, but also the relations between rational and irrational numbers. The following exercise gives another clue to the role of rational numbers.

5.6.2. Show that $rx \otimes y = x \otimes ry$ for any rational r.

5.7* Additive Functions

The nearest thing to a tensor $l_1 \otimes \alpha_1 + l_2 \otimes \alpha_2 + \cdots + l_k \otimes \alpha_k$ we can build in ordinary algebra is a function $l_1 f(\alpha_1) + l_2 f(\alpha_2) + \cdots + l_k f(\alpha_k)$, where f is a function with the properties

$$lf(\alpha + \beta) = lf(\alpha) + lf(\beta)$$
$$(l + m)f(\alpha) = lf(\alpha) + mf(\alpha)$$
$$f(\pi) = 0$$

analogous to Rules 1, 2, and 3 for tensors. The second of these properties is true of all real functions (by the distributive law), so the relevant functions f are actually those with $f(\pi) = 0$ and the property $f(\alpha + \beta) = f(\alpha) + f(\beta)$, called *additivity*.

Admittedly, the only additive functions close at hand are the functions $f(x) = kx$, and the only one of these with $f(\pi) = 0$ is the constant function 0. This does not look promising, but luckily we do not need additive functions defined on all of $\mathbb{R}$. We only need additive functions defined on finite sets of reals, and enough of these can be obtained with the help of the following concept.

Definition A *basis over* $\mathbb{Q}$ for a finite set S of reals is a set $\{\beta_1, \beta_2, \ldots, \beta_n\}$ such that

1. Each x in S is expressible as $x = \beta_1 r_1 + \beta_2 r_2 + \cdots + \beta_n r_n$ for some rationals $r_1, r_2, \ldots, r_n$. (We call x a *rational combination* of $\beta_1, \beta_2, \ldots, \beta_n$).

2. The β_j are *rationally independent*, that is, $\beta_1 r_1 + \beta_2 r_2 + \cdots + \beta_n r_n = 0$ for rationals $r_1, r_2, \ldots, r_n$ only if all $r_j = 0$.

It follows that if $\{\beta_1, \beta_2, \ldots, \beta_n\}$ is a basis over $\mathbb{Q}$ for S then

- Each x in S is *uniquely* expressible in the form $x = \beta_1 r_1 + \beta_2 r_2 + \cdots + \beta_n r_n$ with rationals $r_1, r_2, \ldots, r_n$. Because if

  $$x = \beta_1 r_1 + \beta_2 r_2 + \cdots + \beta_n r_n = \beta_1 s_1 + \beta_2 s_2 + \cdots + \beta_n s_n$$

  are two different expressions for x with rational coefficients, we have

  $$\beta_1(r_1 - s_1) + \beta_2(r_2 - s_2) + \cdots + \beta_n(r_n - s_n) = 0$$

  with not all the rational coefficients $(r_j - s_j)$ zero, contrary to the rational independence of $\beta_1, \beta_2, \ldots, \beta_n$.
- The function

  $$f_i(x) = r_i, \quad \text{where} \quad x = \beta_1 r_1 + \beta_2 r_2 + \cdots + \beta_n r_n$$

  is well defined for all x in S and is additive because

  $$\begin{aligned} x' &= \beta_1 r_1' + \beta_2 r_2' + \cdots + \beta_n r_n' \\ \Rightarrow x + x' &= \beta_1(r_1 + r_1') + \beta_2(r_2 + r_2') + \cdots + \beta_n(r_n + r_n') \\ \Rightarrow f_i(x + x') &= r_i + r_i' = f_i(x) + f_i(x'). \end{aligned}$$

Thus a basis $\{\beta_1, \beta_2, \ldots, \beta_n\}$ over $\mathbb{Q}$ gives us functions f_i that are not only additive but are equal to 1 on β_i and 0 on other basis members. Such functions are just what we will need, so we would like a basis over $\mathbb{Q}$ for each finite set.

Construction of bases over $\mathbb{Q}$. *A finite set of reals has a basis over $\mathbb{Q}$.*

Proof Suppose $S = \{x_1, x_2, \ldots, x_m\}$ is a finite set of reals. Choose x_1 as the first basis element β_1. Then look at $x_2, x_3, x_4 \ldots$ in turn and let β_2 be the first x_j that is not a rational multiple of β_1, let β_3 be the next x_j that is not a rational combination of β_1 and β_2, and so on.

I claim that the set $\{\beta_1, \beta_2, \ldots, \beta_n\}$ obtained in this way is rationally independent. If not, there are rationals $r_1, r_2, \ldots, r_n$, not all zero, with $\beta_1 r_1 + \beta_2 r_2 + \cdots + \beta_n r_n = 0$. But if r_i is the last of them $\neq 0$ we have $\beta_i = -\beta_1 r_1/r_i - \beta_2 r_2/r_i - \cdots - \beta_{i-1} r_{i-1}/r_i$, contrary to the choice of β_i as some x_j that is not a rational combination of the previously chosen $\beta_1, \beta_2, \ldots$.

Also, each x_j in S is a rational combination of $\beta_1, \beta_2, \ldots, \beta_n$, either as a chosen basis element or as a rational combination of previously chosen basis elements. Hence $\{\beta_1, \beta_2, \ldots, \beta_n\}$ is a basis for S over $\mathbb{Q}$. □

This gives us enough additive functions. Now we use them to link rational independence with equidecomposability, in the following crucial theorem.

Rational independence theorem. *If $\alpha_1, \alpha_2, \ldots, \alpha_k$ and π are rationally independent, $l_1 \otimes \alpha_1 + l_2 \otimes \alpha_2 + \cdots + l_k \otimes \alpha_k = 0$ only if $l_1 = l_2 = \cdots = l_k = 0$.*

Proof By definition of tensors, $l_1 \otimes \alpha_1 + l_2 \otimes \alpha_2 + \cdots + l_k \otimes \alpha_k = 0$ means $l_1 \otimes \alpha_1 + l_2 \otimes \alpha_2 + \cdots + l_k \otimes \alpha_k$ can be converted to 0 by applying the rules

$$\begin{aligned} l \otimes (\alpha + \beta) &= l \otimes \alpha + l \otimes \beta, \\ (l + m) \otimes \alpha &= l \otimes \alpha + m \otimes \alpha, \\ l \otimes \pi &= 0. \end{aligned}$$

We can similarly convert $l_1 f(\alpha_1) + l_2 f(\alpha_2) + \cdots + l_k f(\alpha_k)$ to 0 for any additive function f with $f(\pi) = 0$, by applying the rules

$$\begin{aligned} lf(\alpha + \beta) &= lf(\alpha) + lf(\beta), \\ (l + m)f(\alpha) &= lf(\alpha) + mf(\alpha), \\ f(\pi) &= 0, \end{aligned}$$

provided f is defined on the finite set S of angles occurring in the proof that $l_1 \otimes \alpha_1 + l_2 \otimes \alpha_2 + \cdots + l_k \otimes \alpha_k = 0$.

Now if $\alpha_1, \alpha_2, \ldots, \alpha_k$ and π are rationally independent, they can be made members of a basis over $\mathbb{Q}$ of any finite set containing them, such as S. For example, put $\alpha_1, \alpha_2, \ldots, \alpha_k$ and π first on the list of members of S, and use the preceding construction. We therefore have an additive function f_i on S that is 1 on α_i and 0 on the other members of the basis.

Because f_i is defined on S, that is, on all angles occurring in the proof that $l_1 \otimes \alpha_1 + \cdots + l_k \otimes \alpha_k = 0$, we can similarly prove that $l_1 f_i(\alpha_1) + \cdots + l_k f_i(\alpha_k) = 0$. But, by definition of f_i, the latter equation is simply $l_i = 0$. And because i is arbitrary, this means that $l_1 = l_2 = \cdots = l_k = 0$. □

Exercises

5.7.1. There is a similar but simpler rational independence theorem for the tensor product $\mathbb{R} \otimes \mathbb{R}$ discussed in the previous set of exercises.

5.7.2. Show that if $y_1, y_2, \ldots, y_k$ are rationally independent, then $x_1 \otimes y_1 + x_2 \otimes y_2 + \cdots + x_k \otimes y_k = 0$ in $\mathbb{R} \otimes \mathbb{R}$ only if $x_1 = x_2 = \cdots = x_k = 0$.

5.7.3. Deduce from Exercise 5.7.1 that the rectangle with vertical side $\sqrt{2}$ and horizontal side $1/\sqrt{2}$ cannot be converted to the unit square by vertical and horizontal cut and paste.

The same idea may be used to show that the $\sqrt{2} \otimes \sqrt{3}$ rectangle cannot be converted to the $\sqrt{6} \otimes 1$ rectangle except by oblique cutting and pasting.

5.7.3. Show the rational independence of

- $\sqrt{2}$ and $\sqrt{3}$,
- $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{6}$,

and hence conclude that the $\sqrt{2} \otimes \sqrt{3}$ rectangle cannot be converted to the $\sqrt{6} \otimes 1$ rectangle by vertical and horizontal cut and paste.

5.8* The Tetrahedron and the Cube

The rational independence theorem tells us that $l \otimes \alpha \neq 0$ if $l \neq 0$ and α is rationally independent of π, that is, if α is not a rational multiple of π. Now the Dehn invariant of the regular tetrahedron with unit edges is $6 \otimes \alpha$, where $\cos \alpha = 1/3$ by Section 5.5*. We also know that the regular tetrahedron is equidecomposable with a cube only if its Dehn invariant equals the Dehn invariant of the cube, which is 0 by Section 5.6*.

Putting all this together, it remains to prove that the dihedral angle α of the regular tetrahedron is not a rational multiple of π. This is a pleasant exercise using the addition formula for cosine and some elementary number theory.

Dehn's theorem *The regular tetrahedron is not equidecomposable with the cube.*

Proof If α is a rational multiple of π then $n\alpha = m\pi$ for some integers m and n, in which case $\cos n\alpha = \pm 1$. We shall show that this is impossible for any natural number n (which is sufficient, because if $n\alpha = m\pi$ we can take n to be positive by changing the sign of m if necessary). In fact, we shall use induction on n to prove the stronger statement S_n:

$$\cos n\alpha = \frac{q_n}{3^n} \quad \text{for some integer } q_n \text{ not divisible by 3.}$$

S_1 is true because $\cos\alpha = 1/3$. Now suppose $S_1, S_2, \ldots, S_k$ are all true. We prove S_{k+1} by means of the identity

$$\cos(k+1)\alpha + \cos(k-1)\alpha = 2\cos k\alpha \cos\alpha,$$

which comes from adding the two addition formulas

$$\cos(k+1)\alpha = \cos k\alpha \cos\alpha - \sin k\alpha \sin\alpha,$$
$$\cos(k-1)\alpha = \cos k\alpha \cos\alpha + \sin k\alpha \sin\alpha.$$

The identity says that

$$\cos(k+1)\alpha = 2\cos k\alpha \cos\alpha - \cos(k-1)\alpha,$$

and by our induction hypothesis we have integers q_k and q_{k-1}, not divisible by 3, such that

$$\cos k\alpha = \frac{q_k}{3^k} \qquad \text{and} \qquad \cos(k-1)\alpha = \frac{q_{k-1}}{3^{k-1}}.$$

Because $\cos\alpha = 1/3$, it follows that

$$\cos(k+1)\alpha = \frac{(2/3)q_k}{3^k} - \frac{q_{k-1}}{3^{k-1}} = \frac{2q_k - 9q_{k-1}}{3^{k+1}} = \frac{q_{k+1}}{3^{k+1}},$$

where $q_{k+1} = 2q_k - 9q_{k-1}$ is also not divisible by 3 because $9q_{k-1}$ is and $2q_k$ is not.

This completes the induction step, and hence $\cos n\alpha \neq \pm 1$, for all natural numbers n, as required. □

Exercises

The proof of Dehn's theorem can be generalized to show that a rational multiple of π has irrational cosine, except when the angle is one of $\pi/3$,

$\pi/2$, π, or their integer multiples. The proof of this can be broken into a few easy stages. The first is to check what happens with $\pi/3$, $\pi/2$, and π.

5.8.1. Show that $\cos n\alpha$ is rational when $\alpha = \pi/3, \pi/2$, or π and n is an integer. Also show that the values of $\cos n\alpha$ in these cases are 0, ± 1, and $\pm\frac{1}{2}$.

Perhaps the fraction $\frac{1}{2} = \cos \pi/3$ is an exceptional value of $\cos\alpha$. What about fractions of the form $s/2^t$?

5.8.2. Suppose $\cos\alpha = u/2^v$, where u is an odd integer and v is an integer ≥ 2. Show by induction on n that

$$\cos n\alpha = \frac{u_n}{2^{nv-n+1}}, \quad \text{where } u_n \text{ is an odd integer.}$$

Hence deduce that α is not a rational multiple of π.

This disposes of the rational values of $\cos\alpha$ whose denominator is a power of 2. From now on, we can assume that $\cos\alpha$ has a denominator divisible by an odd prime p, so $\cos\alpha = r/p^v$, where $r = s/t$, s and t are integers not divisible by p, and v is an integer ≥ 1.

5.8.3. If $\cos\alpha = r/p^v$, with r and v as just described, show by induction on n that

$$\cos n\alpha = \frac{r_n}{p^{nv}},$$

where $r_n = s_n/t_n$, and s_n and t_n are integers not divisible by p. Hence deduce that α is not a rational multiple of π.

These results have interesting implications for Pythagorean triples and rational points on the unit circle. The Pythagorean triple (a, b, c) represents a triangle whose angle α (between the sides a and c) has rational cosine a/c. It follows from Exercises 5.8.1, 5.8.2, and 5.8.3 that α is not a rational multiple of π unless $\cos\alpha = \pm 1/2$, and in fact this is also impossible.

5.8.4. We cannot have $a/c = 1/2$ in a Pythagorean triple. Why?

5.8.5. Deduce that division of the unit circle into equal parts by rational points is possible only when the number of parts is 2 or 4.

5.9 Discussion

Formulas for π

Finding the value of π, the circumference of the circle of diameter 1, is one of the oldest and most fundamental problems in mathematics. Because the circle is the simplest curve, apart from the straight line, finding the length of the circle is surely the most obvious question in geometry once the basic questions about lines have been answered, as they are by the Pythagorean theorem. How surprising, then, that finding the length of the circle is *nothing* like finding the length of a line!

The Greeks were baffled by the problem and could only find approximations such as the one found by Archimedes,

$$3\frac{10}{71} < \pi < 3\frac{1}{7}.$$

The better approximation 355/113 (accurate to six decimal places) was found by the Chinese mathematician Zǔ Chōngzhi (429–500 A.D.) and was later rediscovered in Europe, along with approximations to more and more decimal places. However, finite rational approximations give little insight into the nature of π, because π is irrational. One would prefer an exact infinite description, provided it yields approximations in a uniform and comprehensible way.

Such a description, infinite yet miraculously simple, was first found in India around 1500 A.D. It expresses $\pi/4$ as an infinite sum of rational numbers:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

Like the Pythagorean theorem, this formula is one of the universal treasures of mathematics, which one might expect to be discovered by any advanced civilization. It was rediscovered in Europe around 1670, with a similar proof. One of its discoverers, Gottfried Wilhelm Leibniz, was so enchanted by the simplicity of the formula that he declared: "God loves odd numbers."

The original discoverer of the formula is not known for certain: the earliest surviving proof (around 1530) credits Nīlakaṇṭha, who flourished around 1500, but a slightly later manuscript by