be replaced by the left side of (90), i.e., by $\mathbf{0}$, without altering the determinant. But a matrix which has $\mathbf{0}$ for one column has determinant 0. Hence $\det [A] = 0$.

9.37 Remark Suppose $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ and $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ are bases in R^n. Every linear operator A on R^n determines matrices $[A]$ and $[A]_U$, with entries a_{ij} and α_{ij}, given by

$$A\mathbf{e}_j = \sum_i a_{ij}\mathbf{e}_i, \qquad A\mathbf{u}_j = \sum_i \alpha_{ij}\mathbf{u}_i.$$

If $\mathbf{u}_j = B\mathbf{e}_j = \Sigma b_{ij}\mathbf{e}_i$, then $A\mathbf{u}_j$ is equal to

$$\sum_k \alpha_{kj} B\mathbf{e}_k = \sum_k \alpha_{kj} \sum_i b_{ik}\mathbf{e}_i = \sum_i \Big(\sum_k b_{ik}\alpha_{kj}\Big)\mathbf{e}_i,$$

and also to

$$AB\mathbf{e}_j = A\sum_k b_{kj}\mathbf{e}_k = \sum_i \Big(\sum_k a_{ik}b_{kj}\Big)\mathbf{e}_i.$$

Thus $\Sigma b_{ik}\alpha_{kj} = \Sigma a_{ik}b_{kj}$, or

(91) $$[B][A]_U = [A][B].$$

Since B is invertible, $\det [B] \neq 0$. Hence (91), combined with Theorem 9.35, shows that

(92) $$\det [A]_U = \det [A].$$

The determinant of the matrix of a linear operator does therefore not depend on the basis which is used to construct the matrix. *It is thus meaningful to speak of the determinant of a linear operator, without having any basis in mind.*

9.38 Jacobians If $\mathbf{f}$ maps an open set $E \subset R^n$ into R^n, and if $\mathbf{f}$ is differentiable at a point $\mathbf{x} \in E$, the determinant of the linear operator $\mathbf{f}'(\mathbf{x})$ is called the *Jacobian of* $\mathbf{f}$ *at* $\mathbf{x}$. In symbols,

(93) $$J_{\mathbf{f}}(\mathbf{x}) = \det \mathbf{f}'(\mathbf{x}).$$

We shall also use the notation

(94) $$\frac{\partial(y_1, \ldots, y_n)}{\partial(x_1, \ldots, x_n)}$$

for $J_{\mathbf{f}}(\mathbf{x})$, if $(y_1, \ldots, y_n) = \mathbf{f}(x_1, \ldots, x_n)$.

In terms of Jacobians, the crucial hypothesis in the inverse function theorem is that $J_{\mathbf{f}}(\mathbf{a}) \neq 0$ (compare Theorem 9.36). If the implicit function theorem is stated in terms of the functions (59), the assumption made there on A amounts to

$$\frac{\partial(f_1, \ldots, f_n)}{\partial(x_1, \ldots, x_n)} \neq 0.$$

DERIVATIVES OF HIGHER ORDER

9.39 Definition Suppose f is a real function defined in an open set $E \subset R^n$, with partial derivatives $D_1 f, \ldots, D_n f$. If the functions $D_j f$ are themselves differentiable, then the *second-order partial derivatives* of f are defined by

$$D_{ij} f = D_i D_j f \qquad (i, j = 1, \ldots, n).$$

If all these functions $D_{ij} f$ are continuous in E, we say that f is of class $\mathscr{C}''$ in E, or that $f \in \mathscr{C}''(E)$.

A mapping $\mathbf{f}$ of E into R^m is said to be of class $\mathscr{C}''$ if each component of $\mathbf{f}$ is of class $\mathscr{C}''$.

It can happen that $D_{ij} f \neq D_{ji} f$ at some point, although both derivatives exist (see Exercise 27). However, we shall see below that $D_{ij} f = D_{ji} f$ whenever these derivatives are continuous.

For simplicity (and without loss of generality) we state our next two theorems for real functions of two variables. The first one is a mean value theorem.

9.40 Theorem *Suppose f is defined in an open set $E \subset R^2$, and $D_1 f$ and $D_{21} f$ exist at every point of E. Suppose $Q \subset E$ is a closed rectangle with sides parallel to the coordinate axes, having (a, b) and $(a + h, b + k)$ as opposite vertices $(h \neq 0,\ k \neq 0)$. Put*

$$\Delta(f, Q) = f(a + h, b + k) - f(a + h, b) - f(a, b + k) + f(a, b).$$

Then there is a point (x, y) in the interior of Q such that

$$\Delta(f, Q) = hk(D_{21} f)(x, y). \tag{95}$$

Note the analogy between (95) and Theorem 5.10; the area of Q is hk.

Proof Put $u(t) = f(t, b + k) - f(t, b)$. Two applications of Theorem 5.10 show that there is an x between a and $a + h$, and that there is a y between b and $b + k$, such that

$$\begin{aligned} \Delta(f, Q) &= u(a + h) - u(a) \\ &= hu'(x) \\ &= h[(D_1 f)(x, b + k) - (D_1 f)(x, b)] \\ &= hk(D_{21} f)(x, y). \end{aligned}$$

9.41 Theorem *Suppose f is defined in an open set $E \subset R^2$, suppose that $D_1 f$, $D_{21} f$, and $D_2 f$ exist at every point of E, and $D_{21} f$ is continuous at some point $(a, b) \in E$.*

Then $D_{12}f$ exists at (a, b) and

$$(D_{12}f)(a, b) = (D_{21}f)(a, b). \tag{96}$$

Corollary $D_{21}f = D_{12}f$ *if* $f \in \mathscr{C}''(E)$.

Proof Put $A = (D_{21}f)(a, b)$. Choose $\varepsilon > 0$. If Q is a rectangle as in Theorem 9.40, and if h and k are sufficiently small, we have

$$|A - (D_{21}f)(x, y)| < \varepsilon$$

for all $(x, y) \in Q$. Thus

$$\left| \frac{\Delta(f, Q)}{hk} - A \right| < \varepsilon,$$

by (95). Fix h, and let $k \to 0$. Since $D_2 f$ exists in E, the last inequality implies that

$$\left| \frac{(D_2 f)(a + h, b) - (D_2 f)(a, b)}{h} - A \right| \leq \varepsilon. \tag{97}$$

Since ε was arbitrary, and since (97) holds for all sufficiently small $h \neq 0$, it follows that $(D_{12}f)(a, b) = A$. This gives (96).

DIFFERENTIATION OF INTEGRALS

Suppose φ is a function of two variables which can be integrated with respect to one and which can be differentiated with respect to the other. Under what conditions will the result be the same if these two limit processes are carried out in the opposite order? To state the question more precisely: Under what conditions on φ can one prove that the equation

$$\frac{d}{dt} \int_a^b \varphi(x, t)\, dx = \int_a^b \frac{\partial \varphi}{\partial t}(x, t)\, dx \tag{98}$$

is true? (A counter example is furnished by Exercise 28.)

It will be convenient to use the notation

$$\varphi^t(x) = \varphi(x, t). \tag{99}$$

Thus φ^t is, for each t, a function of one variable.

9.42 Theorem *Suppose*

(*a*) *$\varphi(x, t)$ is defined for $a \leq x \leq b$, $c \leq t \leq d$;*
(*b*) *α is an increasing function on $[a, b]$;*

(*c*) $\varphi^t \in \mathscr{R}(\alpha)$ *for every* $t \in [c, d]$;

(*d*) $c < s < d$, *and to every* $\varepsilon > 0$ *corresponds a* $\delta > 0$ *such that*

$$|(D_2\varphi)(x, t) - (D_2\varphi)(x, s)| < \varepsilon$$

for all $x \in [a, b]$ *and for all* $t \in (s - \delta, s + \delta)$.

Define

$$f(t) = \int_a^b \varphi(x, t)\, d\alpha(x) \qquad (c \le t \le d). \tag{100}$$

Then $(D_2\varphi)^s \in \mathscr{R}(\alpha)$, $f'(s)$ *exists, and*

$$f'(s) = \int_a^b (D_2\varphi)(x, s)\, d\alpha(x). \tag{101}$$

Note that (*c*) simply asserts the existence of the integrals (100) for all $t \in [c, d]$. Note also that (*d*) certainly holds whenever $D_2\varphi$ is continuous on the rectangle on which φ is defined.

Proof Consider the difference quotients

$$\psi(x, t) = \frac{\varphi(x, t) - \varphi(x, s)}{t - s}$$

for $0 < |t - s| < \delta$. By Theorem 5.10 there corresponds to each (x, t) a number u between s and t such that

$$\psi(x, t) = (D_2\varphi)(x, u).$$

Hence (*d*) implies that

$$|\psi(x, t) - (D_2\varphi)(x, s)| < \varepsilon \qquad (a \le x \le b, \quad 0 < |t - s| < \delta). \tag{102}$$

Note that

$$\frac{f(t) - f(s)}{t - s} = \int_a^b \psi(x, t)\, d\alpha(x). \tag{103}$$

By (102), $\psi^t \to (D_2\varphi)^s$, uniformly on $[a, b]$, as $t \to s$. Since each $\psi^t \in \mathscr{R}(\alpha)$, the desired conclusion follows from (103) and Theorem 7.16.

9.43 Example One can of course prove analogues of Theorem 9.42 with $(-\infty, \infty)$ in place of $[a, b]$. Instead of doing this, let us simply look at an example. Define

$$f(t) = \int_{-\infty}^{\infty} e^{-x^2} \cos(xt)\, dx \tag{104}$$

and

$$g(t) = -\int_{-\infty}^{\infty} xe^{-x^2} \sin(xt)\, dx, \tag{105}$$

for $-\infty < t < \infty$. Both integrals exist (they converge absolutely) since the absolute values of the integrands are at most $\exp(-x^2)$ and $|x| \exp(-x^2)$, respectively.

Note that g is obtained from f by differentiating the integrand with respect to t. We claim that f is differentiable and that

$$f'(t) = g(t) \qquad (-\infty < t < \infty). \tag{106}$$

To prove this, let us first examine the difference quotients of the cosine: if $\beta > 0$, then

$$\frac{\cos(\alpha + \beta) - \cos\alpha}{\beta} + \sin\alpha = \frac{1}{\beta}\int_{\alpha}^{\alpha+\beta} (\sin\alpha - \sin t)\, dt. \tag{107}$$

Since $|\sin\alpha - \sin t| \le |t - \alpha|$, the right side of (107) is at most $\beta/2$ in absolute value; the case $\beta < 0$ is handled similarly. Thus

$$\left|\frac{\cos(\alpha + \beta) - \cos\alpha}{\beta} + \sin\alpha\right| \le |\beta| \tag{108}$$

for all β (if the left side is interpreted to be 0 when $\beta = 0$).

Now fix t, and fix $h \ne 0$. Apply (108) with $\alpha = xt$, $\beta = xh$; it follows from (104) and (105) that

$$\left|\frac{f(t+h) - f(t)}{h} - g(t)\right| \le |h| \int_{-\infty}^{\infty} x^2 e^{-x^2}\, dx.$$

When $h \to 0$, we thus obtain (106).

Let us go a step further: An integration by parts, applied to (104), shows that

$$f(t) = 2\int_{-\infty}^{\infty} xe^{-x^2} \frac{\sin(xt)}{t}\, dx. \tag{109}$$

Thus $tf(t) = -2g(t)$, and (106) implies now that f satisfies the differential equation

$$2f'(t) + tf(t) = 0. \tag{110}$$

If we solve this differential equation and use the fact that $f(0) = \sqrt{\pi}$ (see Sec. 8.21), we find that

$$f(t) = \sqrt{\pi} \exp\left(-\frac{t^2}{4}\right). \tag{111}$$

The integral (104) is thus explicitly determined.

EXERCISES

1. If S is a nonempty subset of a vector space X, prove (as asserted in Sec. 9.1) that the span of S is a vector space.

2. Prove (as asserted in Sec. 9.6) that BA is linear if A and B are linear transformations. Prove also that A^{-1} is linear and invertible.

3. Assume $A \in L(X, Y)$ and $A\mathbf{x} = \mathbf{0}$ only when $\mathbf{x} = \mathbf{0}$. Prove that A is then 1-1.

4. Prove (as asserted in Sec. 9.30) that null spaces and ranges of linear transformations are vector spaces.

5. Prove that to every $A \in L(R^n, R^1)$ corresponds a unique $\mathbf{y} \in R^n$ such that $A\mathbf{x} = \mathbf{x} \cdot \mathbf{y}$. Prove also that $\|A\| = |\mathbf{y}|$.

Hint: Under certain conditions, equality holds in the Schwarz inequality.

6. If $f(0, 0) = 0$ and

$$f(x, y) = \frac{xy}{x^2 + y^2} \qquad \text{if } (x, y) \neq (0, 0),$$

prove that $(D_1 f)(x, y)$ and $(D_2 f)(x, y)$ exist at every point of R^2, although f is not continuous at $(0, 0)$.

7. Suppose that f is a real-valued function defined in an open set $E \subset R^n$, and that the partial derivatives $D_1 f, \ldots, D_n f$ are bounded in E. Prove that f is continuous in E.

Hint: Proceed as in the proof of Theorem 9.21.

8. Suppose that f is a differentiable real function in an open set $E \subset R^n$, and that f has a local maximum at a point $\mathbf{x} \in E$. Prove that $f'(\mathbf{x}) = 0$.

9. If $\mathbf{f}$ is a differentiable mapping of a *connected* open set $E \subset R^n$ into R^m, and if $\mathbf{f}'(\mathbf{x}) = 0$ for every $\mathbf{x} \in E$, prove that $\mathbf{f}$ is constant in E.

10. If f is a real function defined in a convex open set $E \subset R^n$, such that $(D_1 f)(\mathbf{x}) = 0$ for every $\mathbf{x} \in E$, prove that $f(\mathbf{x})$ depends only on $x_2, \ldots, x_n$.

Show that the convexity of E can be replaced by a weaker condition, but that some condition is required. For example, if $n = 2$ and E is shaped like a horseshoe, the statement may be false.

11. If f and g are differentiable real functions in R^n, prove that

$$\nabla(fg) = f\,\nabla g + g\,\nabla f$$

and that $\nabla(1/f) = -f^{-2}\nabla f$ wherever $f \neq 0$.

12. Fix two real numbers a and b, $0 < a < b$. Define a mapping $\mathbf{f} = (f_1, f_2, f_3)$ of R^2 into R^3 by

$$\begin{aligned} f_1(s, t) &= (b + a \cos s) \cos t \\ f_2(s, t) &= (b + a \cos s) \sin t \\ f_3(s, t) &= a \sin s. \end{aligned}$$

Describe the range K of $\mathbf{f}$. (It is a certain compact subset of R^3.)
(*a*) Show that there are exactly 4 points $\mathbf{p} \in K$ such that

$$(\nabla f_1)(\mathbf{f}^{-1}(\mathbf{p})) = \mathbf{0}.$$

Find these points.
(*b*) Determine the set of all $\mathbf{q} \in K$ such that

$$(\nabla f_3)(\mathbf{f}^{-1}(\mathbf{q})) = \mathbf{0}.$$

(*c*) Show that one of the points $\mathbf{p}$ found in part (*a*) corresponds to a local maximum of f_1, one corresponds to a local minimum, and that the other two are neither (they are so-called "saddle points").

Which of the points $\mathbf{q}$ found in part (*b*) correspond to maxima or minima?
(*d*) Let λ be an irrational real number, and define $\mathbf{g}(t) = \mathbf{f}(t, \lambda t)$. Prove that $\mathbf{g}$ is a 1-1 mapping of R^1 onto a dense subset of K. Prove that

$$|\mathbf{g}'(t)|^2 = a^2 + \lambda^2(b + a\cos t)^2.$$

13. Suppose $\mathbf{f}$ is a differentiable mapping of R^1 into R^3 such that $|\mathbf{f}(t)| = 1$ for every t. Prove that $\mathbf{f}'(t)\cdot\mathbf{f}(t) = 0$.

Interpret this result geometrically.

14. Define $f(0, 0) = 0$ and

$$f(x, y) = \frac{x^3}{x^2 + y^2} \qquad \text{if } (x, y) \neq (0, 0).$$

(*a*) Prove that $D_1 f$ and $D_2 f$ are bounded functions in R^2. (Hence f is continuous.)
(*b*) Let $\mathbf{u}$ be any unit vector in R^2. Show that the directional derivative $(D_{\mathbf{u}} f)(0, 0)$ exists, and that its absolute value is at most 1.
(*c*) Let γ be a differentiable mapping of R^1 into R^2 (in other words, γ is a differentiable curve in R^2), with $\gamma(0) = (0, 0)$ and $|\gamma'(0)| > 0$. Put $g(t) = f(\gamma(t))$ and prove that g is differentiable for every $t \in R^1$.

If $\gamma \in \mathscr{C}'$, prove that $g \in \mathscr{C}'$.
(*d*) In spite of this, prove that f is not differentiable at $(0, 0)$.

Hint: Formula (40) fails.

15. Define $f(0, 0) = 0$, and put

$$f(x, y) = x^2 + y^2 - 2x^2 y - \frac{4x^6 y^2}{(x^4 + y^2)^2}$$

if $(x, y) \neq (0, 0)$.
(*a*) Prove, for all $(x, y) \in R^2$, that

$$4x^4 y^2 \leq (x^4 + y^2)^2.$$

Conclude that f is continuous.

(*b*) For $0 \le \theta \le 2\pi$, $-\infty < t < \infty$, define

$$g_\theta(t) = f(t \cos \theta, t \sin \theta).$$

Show that $g_\theta(0) = 0$, $g'_\theta(0) = 0$, $g''_\theta(0) = 2$. Each g_θ has therefore a strict local minimum at $t = 0$.

In other words, the restriction of f to each line through $(0, 0)$ has a strict local minimum at $(0, 0)$.

(*c*) Show that $(0, 0)$ is nevertheless not a local minimum for f, since $f(x, x^2) = -x^4$.

16. Show that the continuity of $\mathbf{f}'$ at the point $\mathbf{a}$ is needed in the inverse function theorem, even in the case $n = 1$: If

$$f(t) = t + 2t^2 \sin\left(\frac{1}{t}\right)$$

for $t \ne 0$, and $f(0) = 0$, then $f'(0) = 1$, f' is bounded in $(-1, 1)$, but f is not one-to-one in any neighborhood of 0.

17. Let $\mathbf{f} = (f_1, f_2)$ be the mapping of R^2 into R^2 given by

$$f_1(x, y) = e^x \cos y, \qquad f_2(x, y) = e^x \sin y.$$

(*a*) What is the range of f?

(*b*) Show that the Jacobian of f is not zero at any point of R^2. Thus every point of R^2 has a neighborhood in which f is one-to-one. Nevertheless, f is not one-to-one on R^2.

(*c*) Put $\mathbf{a} = (0, \pi/3)$, $\mathbf{b} = f(\mathbf{a})$, let $\mathbf{g}$ be the continuous inverse of $\mathbf{f}$, defined in a neighborhood of $\mathbf{b}$, such that $\mathbf{g}(\mathbf{b}) = \mathbf{a}$. Find an explicit formula for $\mathbf{g}$, compute $\mathbf{f}'(\mathbf{a})$ and $\mathbf{g}'(\mathbf{b})$, and verify the formula (52).

(*d*) What are the images under $\mathbf{f}$ of lines parallel to the coordinate axes?

18. Answer analogous questions for the mapping defined by

$$u = x^2 - y^2, \qquad v = 2xy.$$

19. Show that the system of equations

$$\begin{aligned} 3x + y - z + u^2 &= 0 \\ x - y + 2z + u &= 0 \\ 2x + 2y - 3z + 2u &= 0 \end{aligned}$$

can be solved for x, y, u in terms of z; for x, z, u in terms of y; for y, z, u in terms of x; but not for x, y, z in terms of u.

20. Take $n = m = 1$ in the implicit function theorem, and interpret the theorem (as well as its proof) graphically.

21. Define f in R^2 by

$$f(x, y) = 2x^3 - 3x^2 + 2y^3 + 3y^2.$$

(*a*) Find the four points in R^2 at which the gradient of f is zero. Show that f has exactly one local maximum and one local minimum in R^2.