really showed was that if you do not allow infinite processes, what we now call 'limits' in mathematics, then you cannot use mathematics to analyze motion. These arguments, found in Aristotle's *Physics* 239b5-240a18 and 233a21-31, are the following.

1. A point moving from 0 to 1 on the number line first covers a distance of 1/2, then a distance of 1/4, then a distance of 1/8, and so on. After n steps, it has covered a total distance of $1/2 + 1/4 + \ldots + 1/2^n = 1 - 1/2^n$. From the fact that there is no n such that $1 - 1/2^n = 1$, Zeno concluded that the point will never reach 1. In other words, motion from 0 to 1 is impossible.

   Today we get around Zeno's difficulty with the help of the notion of 'limit'. But this is a fairly sophisticated concept; according to nineteenth century mathematicians, the meaning of ' $\lim_{n \to \infty} f(n) = a$' is as follows: for every real $\epsilon > 0$, there exists a natural number k such that, for all $n > k$, $|f(n) - a| < \epsilon$. While the ancients may have had an intuitive notion of what is meant by a limit, the rigorous definition was surely beyond them.

2. Achilles and the tortoise are engaged in a race along a measured line. Achilles starts at 0, but the tortoise is given a head start, at 1, since Achilles runs twice as fast as the tortoise. Thus, when Achilles arrives at the point 1, the tortoise is at $1 + 1/2$; when Achilles arrives at the point $1 + 1/2$, the tortoise is at $1 + 1/2 + 1/4$, and so on. In general when Achilles gets to $2 - 1/2^{n-1}$, the tortoise is at $2 - 1/2^n$, just a little bit ahead. From this Zeno concludes that Achilles can never catch up with the tortoise. If it looks like Achilles catches up with the tortoise, that only means that motion is an illusion.

   The modern solution to this paradox is the same as the solution to paradox (1): $\lim_{n \to \infty}(2 - 1/2^{n-1}) = \lim_{n \to \infty}(2 - 1/2^n)$.

3. Since, at any instant, a flying arrow is in exactly one place, Zeno infers that, at that instant, it is motionless. (Indeed, if you took a high-speed photograph of the arrow, it would look as if it was perfectly still.) One is tempted to conclude that the speed of the arrow is 0, that it is not really moving.

   The argument seems to be that the speed $dx/dt = 0$ because $dx = 0$. But this would only follow if $dt \neq 0$. If we assume that an interval of time is an actually infinite set of (equal) instants, then each instant has zero duration, and $dt = 0$. But then we could equally well infer that $dx/dt = 17$, since $0 = 17 \times 0$.

   Zeno's argument works only if you assume that time is basically discrete, there being some smallest, finite 'quantum' of time (e.g., $1/2^{100}$ seconds). During each quantum of time, the arrow would be motionless, for if it moved, say, from 0 to 1, there would be a time before

it got to 1/2, and a time after, and the quantum of time would be divisible into two parts. Hence it really would not be moving. If there were some n such that every interval of time exceeds $1/2^n$ seconds, then Zeno would be right: motion is an illusion.

In writing the speed as dx/dt, we used the notation of the 17th century philosopher and mathematician Leibniz. He conceived of dx and dt as *infinitesimals* and thought of dx/dt as their actual ratio. The great Newton essentially shared this view, even though his notation was different. Infinitesimals were believed to be quantities which are infinitely small, yet unequal to zero. This idea was attacked by the 18th century philosopher Berkeley as being absurd, and 19th century mathematicians agreed with him. They redefined the ratio dx/dt as

$$\frac{dx}{dt} = \lim_{\delta t \to 0} \frac{\delta x}{\delta t} \, ,$$

where δx and δt are not necessarily small.

It was only in the middle of the 20th century that Abraham Robinson pointed out that infinitesimals may be introduced by fiat, just like the square root of -1, as *their existence does not lead to a contradiction.* Indeed, consider the following infinite collection of inequalities:

$$0 < dx, \ dx < 1, \ dx < 1/2, \ dx < 1/3, \ \ldots. \qquad (*)$$

Suppose we can derive a contradiction from this infinite collection. Now, it is generally agreed that a mathematical proof can have only a finite number of steps, this being part of the very definition of *proof.* Therefore, the proof of a contradiction from the assumptions $(*)$ can only mention a finite number of them, say the last being $dx < 1/n$. But this finite collection of assumptions does not lead to a contradiction, as $dx = 1/(n+1)$ satisfies all of them.

4. In Zeno's fourth argument there are three rows of people:

$$
\begin{array}{cccccc}
& A & A & A & A & \\
& B & B & B & B & \rightarrow \\
\leftarrow & C & C & C & C &
\end{array}
$$

The A's are stationary, the B's are moving to the right and the C's are moving to the left, at the same speed. Now suppose it takes a B one instant of time to pass an A, then it will take him half an instant of time to pass a C. It follows, that there is no such thing as an *instant*, if by that we mean an indivisible 'quantum of time': time is infinitely divisible, it is a substance.