

Thus, strictly speaking, the sets  $X_1 \times X_2 \times X_3$ ,  $(X_1 \times X_2) \times X_3$ , and  $X_1 \times (X_2 \times X_3)$  are distinct. However, they are clearly very related to each other (for instance, there are obvious bijections between any two of the three sets), and it is common in practice to neglect the minor distinctions between these sets and pretend that they are in fact equal. Thus a function  $f : X_1 \times X_2 \times X_3 \rightarrow Y$  can be thought of as a function of one variable  $(x_1, x_2, x_3) \in X_1 \times X_2 \times X_3$ , or as a function of three variables  $x_1 \in X_1$ ,  $x_2 \in X_2$ ,  $x_3 \in X_3$ , or as a function of two variables  $x_1 \in X_1$ ,  $(x_2, x_3) \in X_3$ , and so forth; we will not bother to distinguish between these different perspectives.

**Remark 3.5.10.** An ordered  $n$ -tuple  $x_1, \dots, x_n$  of objects is also called an *ordered sequence* of  $n$  elements, or a *finite sequence* for short. In Chapter 5 we shall also introduce the very useful concept of an *infinite sequence*.

**Example 3.5.11.** If  $x$  is an object, then  $(x)$  is a 1-tuple, which we shall identify with  $x$  itself (even though the two are, strictly speaking, not the same object). Then if  $X_1$  is any set, then the Cartesian product  $\prod_{1 \leq i \leq 1} X_i$  is just  $X_1$  (why?). Also, the *empty Cartesian product*  $\prod_{1 \leq i \leq 0} X_i$  gives, not the empty set  $\{\}$ , but rather the singleton set  $\{()\}$  whose only element is the 0-tuple  $()$ , also known as the *empty tuple*.

If  $n$  is a natural number, we often write  $X^n$  as shorthand for the  $n$ -fold Cartesian product  $X^n := \prod_{1 \leq i \leq n} X$ . Thus  $X^1$  is essentially the same set as  $X$  (if we ignore the distinction between an object  $x$  and the 1-tuple  $(x)$ ), while  $X^2$  is the Cartesian product  $X \times X$ . The set  $X^0$  is a singleton set  $\{()\}$  (why?).

We can now generalize the single choice lemma (Lemma 3.1.6) to allow for multiple (but finite) number of choices.

**Lemma 3.5.12** (Finite choice). *Let  $n \geq 1$  be a natural number, and for each natural number  $1 \leq i \leq n$ , let  $X_i$  be a non-empty set. Then there exists an  $n$ -tuple  $(x_i)_{1 \leq i \leq n}$  such that  $x_i \in X_i$  for all  $1 \leq i \leq n$ . In other words, if each  $X_i$  is non-empty, then the set  $\prod_{1 \leq i \leq n} X_i$  is also non-empty.*

*Proof.* We induct on  $n$  (starting with the base case  $n = 1$ ; the claim is also vacuously true with  $n = 0$  but is not particularly interesting in that case). When  $n = 1$  the claim follows from Lemma 3.1.6 (why?). Now suppose inductively that the claim has already been proven for some  $n$ ; we will now prove it for  $n++$ . Let  $X_1, \dots, X_{n++}$  be a collection of non-empty sets. By induction hypothesis, we can find an  $n$ -tuple  $(x_i)_{1 \leq i \leq n}$  such that  $x_i \in X_i$  for all  $1 \leq i \leq n$ . Also, since  $X_{n++}$  is non-empty, by Lemma 3.1.6 we may find an object  $a$  such that  $a \in X_{n++}$ . If we thus define the  $n++$ -tuple  $(y_i)_{1 \leq i \leq n++}$  by setting  $y_i := x_i$  when  $1 \leq i \leq n$  and  $y_i := a$  when  $i = n++$  it is clear that  $y_i \in X_i$  for all  $1 \leq i \leq n++$ , thus closing the induction.  $\square$

**Remark 3.5.13.** It is intuitively plausible that this lemma should be extended to allow for an infinite number of choices, but this cannot be done automatically; it requires an additional axiom, the *axiom of choice*. See Section 8.4.

*Exercise 3.5.1.* Suppose we *define* the ordered pair  $(x, y)$  for any objects  $x$  and  $y$  by the formula  $(x, y) := \{\{x\}, \{x, y\}\}$  (thus using several applications of Axiom 3.3). Thus for instance  $(1, 2)$  is the set  $\{\{1\}, \{1, 2\}\}$ ,  $(2, 1)$  is the set  $\{\{2\}, \{2, 1\}\}$ , and  $(1, 1)$  is the set  $\{\{1\}\}$ . Show that such a definition indeed obeys the property (3.5), and also whenever  $X$  and  $Y$  are sets, the Cartesian product  $X \times Y$  is also a set. Thus this definition can be validly used as a definition of an ordered pair. For an additional challenge, show that the alternate definition  $(x, y) := \{x, \{x, y\}\}$  also verifies (3.5) and is thus also an acceptable definition of ordered pair. (For this latter task one needs the axiom of regularity, and in particular Exercise 3.2.2.)

*Exercise 3.5.2.* Suppose we *define* an ordered  $n$ -tuple to be a surjective function  $x : \{i \in \mathbf{N} : 1 \leq i \leq n\} \rightarrow X$  whose range is some arbitrary set  $X$  (so different ordered  $n$ -tuples are allowed to have different ranges); we then write  $x_i$  for  $x(i)$ , and also write  $x$  as  $(x_i)_{1 \leq i \leq n}$ . Using this definition, verify that we have  $(x_i)_{1 \leq i \leq n} = (y_i)_{1 \leq i \leq n}$  if and only if  $x_i = y_i$  for all  $1 \leq i \leq n$ . Also, show that if  $(X_i)_{1 \leq i \leq n}$  are an ordered  $n$ -tuple of sets, then the Cartesian product, as defined in Definition 3.5.7, is indeed a set. (Hint: use Exercise 3.4.7 and the axiom of specification.)

*Exercise 3.5.3.* Show that the definitions of equality for ordered pair and ordered  $n$ -tuple obey the reflexivity, symmetry, and transitivity axioms.

*Exercise 3.5.4.* Let  $A, B, C$  be sets. Show that  $A \times (B \cup C) = (A \times B) \cup (A \times C)$ , that  $A \times (B \cap C) = (A \times B) \cap (A \times C)$ , and that  $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$ . (One can of course prove similar identities in which the rôles of the left and right factors of the Cartesian product are reversed.)

*Exercise 3.5.5.* Let  $A, B, C, D$  be sets. Show that  $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$ . Is it true that  $(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$ ? Is it true that  $(A \times B) \setminus (C \times D) = (A \setminus C) \times (B \setminus D)$ ?

*Exercise 3.5.6.* Let  $A, B, C, D$  be non-empty sets. Show that  $A \times B \subseteq C \times D$  if and only if  $A \subseteq C$  and  $B \subseteq D$ , and that  $A \times B = C \times D$  if and only if  $A = C$  and  $B = D$ . What happens if the hypotheses that the  $A, B, C, D$  are all non-empty are removed?

*Exercise 3.5.7.* Let  $X, Y$  be sets, and let  $\pi_{X \times Y \rightarrow X} : X \times Y \rightarrow X$  and  $\pi_{X \times Y \rightarrow Y} : X \times Y \rightarrow Y$  be the maps  $\pi_{X \times Y \rightarrow X}(x, y) := x$  and  $\pi_{X \times Y \rightarrow Y}(x, y) := y$ ; these maps are known as the *co-ordinate functions* on  $X \times Y$ . Show that for any functions  $f : Z \rightarrow X$  and  $g : Z \rightarrow Y$ , there exists a unique function  $h : Z \rightarrow X \times Y$  such that  $\pi_{X \times Y \rightarrow X} \circ h = f$  and  $\pi_{X \times Y \rightarrow Y} \circ h = g$ . (Compare this to the last part of Exercise 3.3.8, and to Exercise 3.1.7.) This function  $h$  is known as the *direct sum* of  $f$  and  $g$  and is denoted  $h = f \oplus g$ .

*Exercise 3.5.8.* Let  $X_1, \dots, X_n$  be sets. Show that the Cartesian product  $\prod_{i=1}^n X_i$  is empty if and only if at least one of the  $X_i$  is empty.

*Exercise 3.5.9.* Suppose that  $I$  and  $J$  are two sets, and for all  $\alpha \in I$  let  $A_\alpha$  be a set, and for all  $\beta \in J$  let  $B_\beta$  be a set. Show that  $(\bigcup_{\alpha \in I} A_\alpha) \cap (\bigcup_{\beta \in J} B_\beta) = \bigcup_{(\alpha, \beta) \in I \times J} (A_\alpha \cap B_\beta)$ .

*Exercise 3.5.10.* If  $f : X \rightarrow Y$  is a function, define the *graph* of  $f$  to be the subset of  $X \times Y$  defined by  $\{(x, f(x)) : x \in X\}$ . Show that two functions  $f : X \rightarrow Y$ ,  $\tilde{f} : X \rightarrow Y$  are equal if and only if they have the same graph. Conversely, if  $G$  is any subset of  $X \times Y$  with the property that for each  $x \in X$ , the set  $\{y \in Y : (x, y) \in G\}$  has exactly one element (or in other words,  $G$  obeys the *vertical line test*), show that there is exactly one function  $f : X \rightarrow Y$  whose graph is equal to  $G$ .

*Exercise 3.5.11.* Show that Axiom 3.10 can in fact be deduced from Lemma 3.4.9 and the other axioms of set theory, and thus Lemma 3.4.9 can be used as an alternate formulation of the power set axiom. (Hint: for any two sets  $X$  and  $Y$ , use Lemma 3.4.9 and the axiom of specification to construct the set of all subsets of  $X \times Y$  which obey the vertical line test. Then use Exercise 3.5.10 and the axiom of replacement.)

**Exercise 3.5.12.** This exercise will establish a rigorous version of Proposition 2.1.16. Let  $f : \mathbf{N} \times \mathbf{N} \rightarrow \mathbf{N}$  be a function, and let  $c$  be a natural number. Show that there exists a function  $a : \mathbf{N} \rightarrow \mathbf{N}$  such that

$$a(0) = c$$

and

$$a(n++) = f(n, a(n)) \text{ for all } n \in \mathbf{N},$$

and furthermore that this function is unique. (Hint: first show inductively, by a modification of the proof of Lemma 3.5.12, that for every natural number  $N \in \mathbf{N}$ , there exists a unique function  $a_N : \{n \in \mathbf{N} : n \leq N\} \rightarrow \mathbf{N}$  such that  $a_N(0) = c$  and  $a_N(n++) = f(n, a(n))$  for all  $n \in \mathbf{N}$  such that  $n < N$ .) For an additional challenge, prove this result without using any properties of the natural numbers other than the Peano axioms directly (in particular, without using the ordering of the natural numbers, and without appealing to Proposition 2.1.16). (Hint: first show inductively, using only the Peano axioms and basic set theory, that for every natural number  $N \in \mathbf{N}$ , there exists a unique pair  $A_N, B_N$  of subsets of  $\mathbf{N}$  which obeys the following properties: (a)  $A_N \cap B_N = \emptyset$ , (b)  $A_N \cup B_N = \mathbf{N}$ , (c)  $0 \in A_N$ , (d)  $N++ \in B_N$ , (e) Whenever  $n \in B_N$ , we have  $n++ \in B_N$ . (f) Whenever  $n \in A_N$  and  $n \neq N$ , we have  $n++ \in A_N$ . Once one obtains these sets, use  $A_N$  as a substitute for  $\{n \in \mathbf{N} : n \leq N\}$  in the previous argument.)

**Exercise 3.5.13.** The purpose of this exercise is to show that there is essentially only one version of the natural number system in set theory (cf. the discussion in Remark 2.1.12). Suppose we have a set  $\mathbf{N}'$  of “alternative natural numbers”, an “alternative zero”  $0'$ , and an “alternative increment operation” which takes any alternative natural number  $n' \in \mathbf{N}'$  and returns another alternative natural number  $n'++' \in \mathbf{N}'$ , such that the Peano axioms (Axioms 2.1-2.5) all hold with the natural numbers, zero, and increment replaced by their alternative counterparts. Show that there exists a bijection  $f : \mathbf{N} \rightarrow \mathbf{N}'$  from the natural numbers to the alternative natural numbers such that  $f(0) = 0'$ , and such that for any  $n \in \mathbf{N}$  and  $n' \in \mathbf{N}'$ , we have  $f(n) = n'$  if and only if  $f(n++) = n'++'$ . (Hint: use Exercise 3.5.12.)

### 3.6 Cardinality of sets

In the previous chapter we defined the natural numbers axiomatically, assuming that they were equipped with a 0 and an increment

operation, and assuming five axioms on these numbers. Philosophically, this is quite different from one of our main conceptualizations of natural numbers - that of *cardinality*, or measuring *how many* elements there are in a set. Indeed, the Peano axiom approach treats natural numbers more like *ordinals* than *cardinals*. (The cardinals are One, Two, Three, ..., and are used to count how many things there are in a set. The *ordinals* are First, Second, Third, ..., and are used to order a sequence of objects. There is a subtle difference between the two, especially when comparing infinite cardinals with infinite ordinals, but this is beyond the scope of this text). We paid a lot of attention to what number came *next* after a given number  $n$  - which is an operation which is quite natural for ordinals, but less so for cardinals - but did not address the issue of whether these numbers could be used to *count* sets. The purpose of this section is to address this issue by noting that the natural numbers *can* be used to count the cardinality of sets, as long as the set is finite.

The first thing is to work out when two sets have the same size: it seems clear that the sets  $\{1, 2, 3\}$  and  $\{4, 5, 6\}$  have the same size, but that both have a different size from  $\{8, 9\}$ . One way to define this is to say that two sets have the same size if they have the same number of elements, but we have not yet defined what the “number of elements” in a set is. Besides, this runs into problems when a set is infinite.

The right way to define the concept of “two sets having the same size” is not immediately obvious, but can be worked out with some thought. One intuitive reason why the sets  $\{1, 2, 3\}$  and  $\{4, 5, 6\}$  have the same size is that one can match the elements of the first set with the elements in the second set in a one-to-one correspondence:  $1 \leftrightarrow 4$ ,  $2 \leftrightarrow 5$ ,  $3 \leftrightarrow 6$ . (Indeed, this is how we first learn to count a set: we correspond the set we are trying to count with another set, such as a set of fingers on your hand). We will use this intuitive understanding as our rigorous basis for “having the same size”.

**Definition 3.6.1** (Equal cardinality). We say that two sets  $X$  and  $Y$  have *equal cardinality* iff there exists a bijection  $f : X \rightarrow Y$

from  $X$  to  $Y$ .

**Example 3.6.2.** The sets  $\{0, 1, 2\}$  and  $\{3, 4, 5\}$  have equal cardinality, since we can find a bijection between the two sets. Note that we do not yet know whether  $\{0, 1, 2\}$  and  $\{3, 4\}$  have equal cardinality; we know that one of the functions  $f$  from  $\{0, 1, 2\}$  to  $\{3, 4\}$  is not a bijection, but we have not proven yet that there might still be some other bijection from one set to the other. (It turns out that they do not have equal cardinality, but we will prove this a little later). Note that this definition makes sense regardless of whether  $X$  is finite or infinite (in fact, we haven't even defined what finite means yet).

**Remark 3.6.3.** The fact that two sets have equal cardinality does not preclude one of the sets from containing the other. For instance, if  $X$  is the set of natural numbers and  $Y$  is the set of even natural numbers, then the map  $f : X \rightarrow Y$  defined by  $f(n) := 2n$  is a bijection from  $X$  to  $Y$  (why?), and so  $X$  and  $Y$  have equal cardinality, despite  $Y$  being a subset of  $X$  and seeming intuitively as if it should only have "half" of the elements of  $X$ .

The notion of having equal cardinality is an equivalence relation:

**Proposition 3.6.4.** *Let  $X, Y, Z$  be sets. Then  $X$  has equal cardinality with  $X$ . If  $X$  has equal cardinality with  $Y$ , then  $Y$  has equal cardinality with  $X$ . If  $X$  has equal cardinality with  $Y$  and  $Y$  has equal cardinality with  $Z$ , then  $X$  has equal cardinality with  $Z$ .*

*Proof.* See Exercise 3.6.1. □

Let  $n$  be a natural number. Now we want to say when a set  $X$  has  $n$  elements. Certainly we want the set  $\{i \in \mathbf{N} : 1 \leq i \leq n\} = \{1, 2, \dots, n\}$  to have  $n$  elements. (This is true even when  $n = 0$ ; the set  $\{i \in \mathbf{N} : 1 \leq i \leq 0\}$  is just the empty set.) Using our notion of equal cardinality, we thus define:

**Definition 3.6.5.** Let  $n$  be a natural number. A set  $X$  is said to have *cardinality*  $n$ , iff it has equal cardinality with  $\{i \in \mathbf{N} : 1 \leq$

$i \leq n\}$ . We also say that  $X$  has  $n$  elements iff it has cardinality  $n$ .

**Remark 3.6.6.** One can use the set  $\{i \in \mathbf{N} : i < n\}$  instead of  $\{i \in \mathbf{N} : 1 \leq i \leq n\}$ , since these two sets clearly have equal cardinality. (Why? What is the bijection?)

**Example 3.6.7.** Let  $a, b, c, d$  be distinct objects. Then  $\{a, b, c, d\}$  has the same cardinality as  $\{i \in \mathbf{N} : i < 4\} = \{0, 1, 2, 3\}$  or  $\{i \in \mathbf{N} : 1 \leq i \leq 4\} = \{1, 2, 3, 4\}$  and thus has cardinality 4. Similarly, the set  $\{a\}$  has cardinality 1.

There might be one problem with this definition: a set might have two different cardinalities. But this is not possible:

**Proposition 3.6.8** (Uniqueness of cardinality). *Let  $X$  be a set with some cardinality  $n$ . Then  $X$  cannot have any other cardinality, i.e.,  $X$  cannot have cardinality  $m$  for any  $m \neq n$ .*

Before we prove this proposition, we need a lemma.

**Lemma 3.6.9.** *Suppose that  $n \geq 1$ , and  $X$  has cardinality  $n$ . Then  $X$  is non-empty, and if  $x$  is any element of  $X$ , then the set  $X - \{x\}$  (i.e.,  $X$  with the element  $x$  removed) has cardinality  $n - 1$ .*

*Proof.* If  $X$  is empty then it clearly cannot have the same cardinality as the non-empty set  $\{i \in \mathbf{N} : 1 \leq i \leq n\}$ , as there is no bijection from the empty set to a non-empty set (why?). Now let  $x$  be an element of  $X$ . Since  $X$  has the same cardinality as  $\{i \in \mathbf{N} : 1 \leq i \leq n\}$ , we thus have a bijection  $f$  from  $X$  to  $\{i \in \mathbf{N} : 1 \leq i \leq n\}$ . In particular,  $f(x)$  is a natural number between 1 and  $n$ . Now define the function  $g : X - \{x\}$  to  $\{i \in \mathbf{N} : 1 \leq i \leq n - 1\}$  by the following rule: for any  $y \in X - \{x\}$ , we define  $g(y) := f(y)$  if  $f(y) < f(x)$ , and define  $g(y) := f(y) - 1$  if  $f(y) > f(x)$ . (Note that  $f(y)$  cannot equal  $f(x)$  since  $y \neq x$  and  $f$  is a bijection.) It is easy to check that this map is also a bijection (why?), and so  $X - \{x\}$  has equal cardinality with  $\{i \in \mathbf{N} : 1 \leq i \leq n - 1\}$ . In particular  $X - \{x\}$  has cardinality  $n - 1$ , as desired.  $\square$

Now we prove the proposition.

*Proof of Proposition 3.6.8.* We induct on  $n$ . First suppose that  $n = 0$ . Then  $X$  must be empty, and so  $X$  cannot have any non-zero cardinality. Now suppose that the proposition is already proven for some  $n$ ; we now prove it for  $n++$ . Let  $X$  have cardinality  $n++$ ; and suppose that  $X$  also has some other cardinality  $m \neq n++$ . By Proposition 3.6.4,  $X$  is non-empty, and if  $x$  is any element of  $X$ , then  $X - \{x\}$  has cardinality  $n$  and also has cardinality  $m - 1$ , by Lemma 3.6.9. By induction hypothesis, this means that  $n = m - 1$ , which implies that  $m = n++$ , a contradiction. This closes the induction.  $\square$

Thus, for instance, we now know, thanks to Propositions 3.6.4 and 3.6.8, that the sets  $\{0, 1, 2\}$  and  $\{3, 4\}$  do not have equal cardinality, since the first set has cardinality 3 and the second set has cardinality 2.

**Definition 3.6.10** (Finite sets). A set is *finite* iff it has cardinality  $n$  for some natural number  $n$ ; otherwise, the set is called *infinite*. If  $X$  is a finite set, we use  $\#(X)$  to denote the cardinality of  $X$ .

**Example 3.6.11.** The sets  $\{0, 1, 2\}$  and  $\{3, 4\}$  are finite, as is the empty set (0 is a natural number), and  $\#(\{0, 1, 2\}) = 3$ ,  $\#(\{3, 4\}) = 2$ , and  $\#(\emptyset) = 0$ .

Now we give an example of an infinite set.

**Theorem 3.6.12.** *The set of natural numbers  $\mathbf{N}$  is infinite.*

*Proof.* Suppose for sake of contradiction that the set of natural numbers  $\mathbf{N}$  was finite, so it had some cardinality  $\#(\mathbf{N}) = n$ . Then there is a bijection  $f$  from  $\{i \in \mathbf{N} : 1 \leq i \leq n\}$  to  $\mathbf{N}$ . One can show that the sequence  $f(1), f(2), \dots, f(n)$  is bounded, or more precisely that there exists a natural number  $M$  such that  $f(i) \leq M$  for all  $1 \leq i \leq n$  (Exercise 3.6.3). But then the natural number  $M + 1$  is not equal to any of the  $f(i)$ , contradicting the hypothesis that  $f$  is a bijection.  $\square$

**Remark 3.6.13.** One can also use similar arguments to show that any unbounded set is infinite; for instance the rationals  $\mathbf{Q}$  and the reals  $\mathbf{R}$  (which we will construct in later chapters) are infinite. However, it is possible for some sets to be “more” infinite than others; see Section 8.3.

Now we relate cardinality with the arithmetic of natural numbers.

**Proposition 3.6.14** (Cardinal arithmetic).

- (a) *Let  $X$  be a finite set, and let  $x$  be an object which is not an element of  $X$ . Then  $X \cup \{x\}$  is finite and  $\#(X \cup \{x\}) = \#(X) + 1$ .*
- (b) *Let  $X$  and  $Y$  be finite sets. Then  $X \cup Y$  is finite and  $\#(X \cup Y) \leq \#(X) + \#(Y)$ . If in addition  $X$  and  $Y$  are disjoint (i.e.,  $X \cap Y = \emptyset$ ), then  $\#(X \cup Y) = \#(X) + \#(Y)$ .*
- (c) *Let  $X$  be a finite set, and let  $Y$  be a subset of  $X$ . Then  $Y$  is finite, and  $\#(Y) \leq \#(X)$ . If in addition  $Y \neq X$  (i.e.,  $Y$  is a proper subset of  $X$ ), then we have  $\#(Y) < \#(X)$ .*
- (d) *If  $X$  is a finite set, and  $f : X \rightarrow Y$  is a function, then  $f(X)$  is a finite set with  $\#(f(X)) \leq \#(X)$ . If in addition  $f$  is one-to-one, then  $\#(f(X)) = \#(X)$ .*
- (e) *Let  $X$  and  $Y$  be finite sets. Then Cartesian product  $X \times Y$  is finite and  $\#(X \times Y) = \#(X) \times \#(Y)$ .*
- (f) *Let  $X$  and  $Y$  be finite sets. Then the set  $Y^X$  (defined in Axiom 3.10) is finite and  $\#(Y^X) = \#(Y)^{\#(X)}$ .*

*Proof.* See Exercise 3.6.4. □

**Remark 3.6.15.** Proposition 3.6.14 suggests that there is another way to define the arithmetic operations of natural numbers; not defined recursively as in Definitions 2.2.1, 2.3.1, 2.3.11, but instead using the notions of union, Cartesian product, and power

set. This is the basis of *cardinal arithmetic*, which is an alternative foundation to arithmetic than the Peano arithmetic we have developed here; we will not develop this arithmetic in this text, but we give some examples of how one would work with this arithmetic in Exercises 3.6.5, 3.6.6.

This concludes our discussion of finite sets. We shall discuss infinite sets in Chapter 8, once we have constructed a few more examples of infinite sets (such as the integers, rationals and reals).

*Exercise 3.6.1.* Prove Proposition 3.6.4.

*Exercise 3.6.2.* Show that a set  $X$  has cardinality 0 if and only if  $X$  is the empty set.

*Exercise 3.6.3.* Let  $n$  be a natural number, and let  $f : \{i \in \mathbb{N} : 1 \leq i \leq n\} \rightarrow \mathbb{N}$  be a function. Show that there exists a natural number  $M$  such that  $f(i) \leq M$  for all  $1 \leq i \leq n$ . (Hint: induct on  $n$ . You may also want to peek at Lemma 5.1.14.) Thus finite subsets of the natural numbers are bounded.

*Exercise 3.6.4.* Prove Proposition 3.6.14.

*Exercise 3.6.5.* Let  $A$  and  $B$  be sets. Show that  $A \times B$  and  $B \times A$  have equal cardinality by constructing an explicit bijection between the two sets. Then use Proposition 3.6.14 to conclude an alternate proof of Lemma 2.3.2.

*Exercise 3.6.6.* Let  $A, B, C$  be sets. Show that the sets  $(A^B)^C$  and  $A^{B \times C}$  have equal cardinality by constructing an explicit bijection between the two sets. Conclude that  $(a^b)^c = a^{bc}$  for any natural numbers  $a, b, c$ . Use a similar argument to also conclude  $a^b \times a^c = a^{b+c}$ .

*Exercise 3.6.7.* Let  $A$  and  $B$  be sets. Let us say that  $A$  has *lesser or equal* cardinality to  $B$  if there exists an injection  $f : A \rightarrow B$  from  $A$  to  $B$ . Show that if  $A$  and  $B$  are finite sets, then  $A$  has lesser or equal cardinality to  $B$  if and only if  $\#(A) \leq \#(B)$ .

*Exercise 3.6.8.* Let  $A$  and  $B$  be sets such that there exists an injection  $f : A \rightarrow B$  from  $A$  to  $B$  (i.e.,  $A$  has lesser or equal cardinality to  $B$ ). Show that there then exists a surjection  $g : B \rightarrow A$  from  $B$  to  $A$ . (The converse to this statement requires the axiom of choice; see Exercise 8.4.3.)

*Exercise 3.6.9.* Let  $A$  and  $B$  be finite sets. Show that  $A \cup B$  and  $A \cap B$  are also finite sets, and that  $\#(A) + \#(B) = \#(A \cup B) + \#(A \cap B)$ .

### 3.6. Cardinality of sets

*Exercise 3.6.10.* Let  $A_1, \dots, A_n$  be finite sets such that  $\#(\bigcup_{i \in n} A_i) \leq \#(A_i)$  for all  $i \in n$ . Show that there exists  $i \in \{1, \dots, n\}$  such that  $\#(A_i) \geq \#(\bigcup_{i \in n} A_i)$ . (This is known as the *pigeonhole principle*.)

## Chapter 4

### Integers and rationals

#### 4.1 The integers

In Chapter 2 we built up most of the basic properties of the natural number system, but we have reached the limits of what one can do with just addition and multiplication. We would now like to introduce a new operation, that of subtraction, but to do that properly we will have to pass from the natural number system to a larger number system, that of the *integers*.

Informally, the integers are what you can get by subtracting two natural numbers; for instance,  $3 - 5$  should be an integer, as should  $6 - 2$ . This is not a complete definition of the integers, because (a) it doesn't say when two differences are equal (for instance we should know why  $3 - 5$  is equal to  $2 - 4$ , but is not equal to  $1 - 6$ ), and (b) it doesn't say how to do arithmetic on these differences (how does one add  $3 - 5$  to  $6 - 2$ ?). Furthermore, (c) this definition is circular because it requires a notion of subtraction, which we can only adequately define once the integers are constructed. Fortunately, because of our prior experience with integers we know what the answers to these questions should be. To answer (a), we know from our advanced knowledge in algebra that  $a - b = c - d$  happens exactly when  $a + d = c + b$ , so we can characterize equality of differences using only the concept of addition. Similarly, to answer (b) we know from algebra that  $(a - b) + (c - d) = (a + c) - (b + d)$  and that  $(a - b)(c - d) = (ac + bd) - (ad + bc)$ . So we will take advan-

tage of our foreknowledge by building all this into the *definition* of the integers, as we shall do shortly.

We still have to resolve (c). To get around this problem we will use the following work-around: we will temporarily write integers not as a difference  $a - b$ , but instead use a new notation  $a \text{---} b$  to define integers, where the  $\text{---}$  is a meaningless place-holder, similar to the comma in the Cartesian co-ordinate notation  $(x, y)$  for points in the plane. Later when we define subtraction we will see that  $a \text{---} b$  is in fact equal to  $a - b$ , and so we can discard the notation  $\text{---}$ ; it is only needed right now to avoid circularity. (These devices are similar to the scaffolding used to construct a building; they are temporarily essential to make sure the building is built correctly, but once the building is completed they are thrown away and never used again.) This may seem unnecessarily complicated in order to define something that we already are very familiar with, but we will use this device again to construct the rationals, and knowing these kinds of constructions will be very helpful in later chapters.

**Definition 4.1.1 (Integers).** An *integer* is an expression<sup>1</sup> of the form  $a \text{---} b$ , where  $a$  and  $b$  are natural numbers. Two integers are considered to be equal,  $a \text{---} b = c \text{---} d$ , if and only if  $a + d = c + b$ . We let  $\mathbf{Z}$  denote the set of all integers.

Thus for instance  $3 \text{---} 5$  is an integer, and is equal to  $2 \text{---} 4$ , because  $3 + 4 = 2 + 5$ . On the other hand,  $3 \text{---} 5$  is not equal to  $2 \text{---} 3$  because  $3 + 3 \neq 2 + 5$ . This notation is strange looking, and has a few deficiencies; for instance, 3 is not yet an integer, because it is not of the form  $a \text{---} b$ ! We will rectify these problems later.

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<sup>1</sup>In the language of set theory, what we are doing here is starting with the space  $\mathbf{N} \times \mathbf{N}$  of ordered pairs  $(a, b)$  of natural numbers. Then we place an *equivalence relation*  $\sim$  on these pairs by declaring  $(a, b) \sim (c, d)$  iff  $a + d = c + b$ . The set-theoretic interpretation of the symbol  $a \text{---} b$  is that it is the space of all pairs equivalent to  $(a, b)$ :  $a \text{---} b := \{(c, d) \in \mathbf{N} \times \mathbf{N} : (a, b) \sim (c, d)\}$ . However, this interpretation plays no rôle in how we manipulate the integers and we will not refer to it again. A similar set-theoretic interpretation can be given to the construction of the rational numbers later in this chapter, or the real numbers in the next chapter.