(2) $\varphi(G^{(i)}) = K^{(i)}$. In particular, homomorphic images and quotient groups of solvable groups are solvable (of solvable length less than or equal to that of the domain group).
(3) If $N$ is normal in $G$ and both $N$ and $G/N$ are solvable then so is $G$.

*Proof:* Part 1 follows from the observation that since $H \leq G$, by definition of commutator subgroups, $[H, H] \leq [G, G]$, i.e., $H^{(1)} \leq G^{(1)}$. Then, by induction,

$$H^{(i)} \leq G^{(i)} \quad \text{for all } i \in \mathbb{Z}^+.$$

In particular, if $G^{(n)} = 1$ for some $n$, then also $H^{(n)} = 1$. This establishes (1).

To prove (2) note that by definition of commutators,

$$\varphi([x, y]) = [\varphi(x), \varphi(y)]$$

so by induction $\varphi(G^{(i)}) \leq K^{(i)}$. Since $\varphi$ is surjective, every commutator in $K$ is the image of a commutator in $G$, hence again by induction we obtain equality for all $i$. Again, if $G^{(n)} = 1$ for some $n$ then $K^{(n)} = 1$. This proves (2).

Finally, if $G/N$ and $N$ are solvable, of lengths $n$ and $m$ respectively then by (2) applied to the natural projection $\varphi : G \to G/N$ we obtain

$$\varphi(G^{(n)}) = (G/N)^{(n)} = 1N$$

i.e., $G^{(n)} \leq N$. Thus $G^{(n+m)} = (G^{(n)})^{(m)} \leq N^{(m)} = 1$. Theorem 9 shows that $G$ is solvable, which completes the proof.

Some additional conditions under which finite groups are solvable are the following:

**Theorem 11.** Let $G$ be a finite group.
(1) (Burnside) If $|G| = p^a q^b$ for some primes $p$ and $q$, then $G$ is solvable.
(2) (Philip Hall) If for every prime $p$ dividing $|G|$ we factor the order of $G$ as $|G| = p^a m$ where $(p, m) = 1$, and $G$ has a subgroup of order $m$, then $G$ is solvable (i.e., if for all primes $p$, $G$ has a subgroup whose index equals the order of a Sylow $p$-subgroup, then $G$ is solvable — such subgroups are called Sylow $p$-complements).
(3) (Feit–Thompson) If $|G|$ is odd then $G$ is solvable.
(4) (Thompson) If for every pair of elements $x, y \in G$, $\langle x, y \rangle$ is a solvable group, then $G$ is solvable.

We shall prove Burnside's Theorem in Chapter 19 and deduce Philip Hall's generalization of it. As mentioned in Section 3.5, the proof of the Feit–Thompson Theorem takes 255 pages. Thompson's Theorem was first proved as a consequence of a 475 page paper (that in turn relies ultimately on the Feit–Thompson Theorem).

## A Proof of the Fundamental Theorem of Finite Abelian Groups

We sketch a group-theoretic proof of the result that every finite abelian group is a direct product of cyclic groups (i.e., Parts 1 and 2 of Theorem 5, Section 5.2) — the Classification of Finitely Generated Abelian Groups (Theorem 3, Section 5.2) will be derived as a consequence of a more general theorem in Chapter 12.

By Corollary 4 it suffices to prove that for $p$ a prime, any abelian $p$-group is a direct product of cyclic groups (the divisibility condition in Theorem 5.5 is trivially achieved by reordering factors). Let $A$ be an abelian $p$-group. We proceed by induction on $|A|$.

If $E$ is an elementary abelian $p$-group (i.e., $x^p = 1$ for all $x \in E$), we first prove the following result:

$$\text{for any } x \in E, \text{ there exists } M \leq E \text{ with } E = M \times \langle x \rangle.$$

If $x = 1$, let $M = E$. Otherwise let $M$ be a subgroup of $E$ of maximal order subject to the condition that $x$ not be an element of $M$. If $M$ is not of index $p$ in $E$, let $\overline{E} = E/M$. Then $\overline{E}$ is elementary abelian and there exists $\overline{y} \in \overline{E} - \langle \overline{x} \rangle$. Since $\overline{y}$ has order $p$, we also have $\overline{x} \notin \langle \overline{y} \rangle$. The complete preimage of $\langle \overline{y} \rangle$ in $E$ is a subgroup of $E$ that does not contain $x$ and whose order is larger than the order of $M$, contrary to the choice of $M$. This proves $|E : M| = p$, hence

$$E = M\langle x \rangle \quad \text{and} \quad M \cap \langle x \rangle = 1.$$

By the recognition theorem for direct products, Theorem 5.9, $E = M \times \langle x \rangle$, as asserted.

Now let $\varphi : A \to A$ be defined by $\varphi(x) = x^p$ (see Exercise 7, Section 5.2). Then $\varphi$ is a homomorphism since $A$ is abelian. Denote the kernel of $\varphi$ by $K$ and denote the image of $\varphi$ by $H$. By definition $K = \{x \in A \mid x^p = 1\}$ and $H$ is the subgroup of $A$ consisting of $p^{\text{th}}$ powers. Note that both $K$ and $A/H$ are elementary abelian. By the First Isomorphism Theorem

$$|A : H| = |K|.$$

By induction,

$$\begin{aligned} H &= \langle h_1 \rangle \times \cdots \times \langle h_r \rangle \\ &\cong Z_{p^{\alpha_1}} \times \cdots \times Z_{p^{\alpha_r}} \quad \alpha_i \geq 1, \ i = 1, 2, \ldots, r. \end{aligned}$$

By definition of $\varphi$, there exist elements $g_i \in A$ such that $g_i^p = h_i$, $1 \leq i \leq r$. Let $A_0 = \langle g_1, \cdots, g_r \rangle$. It is an exercise to see that

**(a)** $A_0 = \langle g_1 \rangle \times \cdots \times \langle g_r \rangle$,
**(b)** $A_0/H = \langle g_1 H \rangle \times \cdots \times \langle g_r H \rangle$ is elementary abelian of order $p^r$, and
**(c)** $H \cap K = \langle h_1^{p^{\alpha_1 - 1}} \rangle \times \cdots \times \langle h_r^{p^{\alpha_r - 1}} \rangle$ is elementary abelian of order $p^r$.

If $K$ is contained in $H$, then $|K| = |K \cap H| = p^r = |A_0 : H|$. In this case by comparing orders we see that $A_0 = A$ and the theorem is proved. Assume therefore that $K$ is not a subgroup of $H$ and use the bar notation to denote passage to the quotient group $A/H$. Let $x \in K - H$, so $|\overline{x}| = |x| = p$. By the initial remark of the proof applied to the elementary abelian $p$-group $E = \overline{A}$, there is a subgroup $\overline{M}$ of $\overline{A}$ such that

$$\overline{A} = \overline{M} \times \langle \overline{x} \rangle.$$

If $M$ is the complete preimage in $A$ of $\overline{M}$, then since $x$ has order $p$ and $x \notin M$, we have $\langle x \rangle \cap M = 1$. By the recognition theorem for direct products,

$$A = M \times \langle x \rangle.$$

By induction, $M$ is a direct product of cyclic groups, hence so is $A$. This completes the proof.

The uniqueness of the decomposition of a finite abelian group into a direct product of cyclic groups (Part 3 of Theorem 5.5) can also be proved by induction using the $p^{\text{th}}$-power map (i.e., using Exercise 7, Section 5.2). This is essentially the procedure we follow in Section 12.1 for the uniqueness part of the proof of the Fundamental Theorem of Finitely Generated Abelian Groups.

## EXERCISES

1. Prove that $Z_i(G)$ is a characteristic subgroup of $G$ for all $i$.
2. Prove Parts 2 and 4 of Theorem 1 for $G$ a finite nilpotent group, not necessarily a $p$-group.
3. If $G$ is finite prove that $G$ is nilpotent if and only if it has a normal subgroup of each order dividing $|G|$, and is cyclic if and only if it has a unique subgroup of each order dividing $|G|$.
4. Prove that a maximal subgroup of a finite nilpotent group has prime index.
5. Prove Parts 2 and 4 of Theorem 1 for $G$ an infinite nilpotent group.
6. Show that if $G/Z(G)$ is nilpotent then $G$ is nilpotent.
7. Prove that subgroups and quotient groups of nilpotent groups are nilpotent (your proof should work for infinite groups). Give an explicit example of a group $G$ which possesses a normal subgroup $H$ such that both $H$ and $G/H$ are nilpotent but $G$ is not nilpotent.
8. Prove that if $p$ is a prime and $P$ is a non-abelian group of order $p^3$ then $|Z(P)| = p$ and $P/Z(P) \cong Z_p \times Z_p$.
9. Prove that a finite group $G$ is nilpotent if and only if whenever $a,\ b \in G$ with $(|a|, |b|) = 1$ then $ab = ba$. [Use Part 4 of Theorem 3.]
10. Prove that $D_{2n}$ is nilpotent if and only if $n$ is a power of 2. [Use Exercise 9.]
11. Give another proof of Proposition 5 under the additional assumption that $G$ is abelian by invoking the Fundamental Theorem of Finite Abelian Groups.
12. Find the upper and lower central series for $A_4$ and $S_4$.
13. Find the upper and lower central series for $A_n$ and $S_n$, $n \geq 5$.
14. Prove that $G^i$ is a characteristic subgroup of $G$ for all $i$.
15. Prove that $Z_i(D_{2^n}) = D_{2^n}^{n-1-i}$.
16. Prove that $\mathbb{Q}$ has no maximal subgroups. [Recall Exercise 21, Section 3.2.]
17. Prove that $G^{(i)}$ is a characteristic subgroup of $G$ for all $i$.
18. Show that if $G'/G''$ and $G''/G'''$ are both cyclic then $G'' = 1$. [You may assume $G''' = 1$. Then $G/G''$ acts by conjugation on the cyclic group $G''$.]
19. Show that there is no group whose commutator subgroup is isomorphic to $S_4$. [Use the preceding exercise.]
20. Let $p$ be a prime, let $P$ be a $p$-subgroup of the finite group $G$, let $N$ be a normal subgroup of $G$ whose order is relatively prime to $p$ and let $\overline{G} = G/N$. Prove the following:
    (a) $N_{\overline{G}}(\overline{P}) = \overline{N_G(P)}$ [Use Frattini's Argument.]
    (b) $C_{\overline{G}}(\overline{P}) = \overline{C_G(P)}$. [Use part (a).]

For any group $G$ the *Frattini subgroup* of $G$ (denoted by $\Phi(G)$) is defined to be the intersection of all the maximal subgroups of $G$ (if $G$ has no maximal subgroups, set $\Phi(G) = G$). The next