Sheet 4

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Task 1

a) Bayes: $p(C=1|x) = \frac{p(x|C=1)\pi_1}{p(x|C=1)\pi_1 + p(x|C=0)\pi_0}$

$$= \frac{1}{1 + \frac{p(x|C=0)\pi_0}{p(x|C=1)\pi_1}}$$

log-odds: $g(x) = \ln\left(\frac{p(x|C=0)\pi_0}{p(x|C=1)\pi_1}\right)$

$\Rightarrow p(C=1|x) = \frac{1}{1+e^{-g(x)}}$

b) Gauß: $p(x|C=k) = \mathcal{N}(x|\mu_k, \Sigma_k) = \frac{1}{(2\pi)^{d}|\Sigma_k|^{1/2}} \exp\left[-\frac{1}{2}(x-\mu_k)^T \Sigma_k^{-1}(x-\mu_k)\right]$

log-odds: $g(x) = \ln\frac{\pi_1}{\pi_0} + \ln\frac{\mathcal{N}(x|\mu_1,\Sigma_1)}{\mathcal{N}(x|\mu_2,\Sigma_2)}$

$$= \ln\frac{\pi_1}{\pi_0} + \frac{1}{2}\ln\frac{|\Sigma_1|}{|\Sigma_0|} - \frac{1}{2}\left[(x-\mu_1)^T\Sigma_1^{-1}(x-\mu_1) - (x-\mu_2)^T\Sigma_2^{-1}(x-\mu_2)\right]$$

$\Sigma_0 = \Sigma_1 =: \Sigma$: $g(x) = \ln\frac{\pi_1}{\pi_0} - \frac{1}{2}\left[x^T\Sigma^{-1}x - 2x^T\Sigma^{-1}\mu_1 + \mu_1^T\Sigma^{-1}\mu_1\right.$ (I)

$\left. - (x^T\Sigma^{-1}x - 2x^T\Sigma^{-1}\mu_0 + \mu_0^T\Sigma^{-1}\mu_0)\right]$

$$= \underbrace{\ln\frac{\pi_1}{\pi_0}}_{①} + \underbrace{x^T\Sigma^{-1}(\mu_1-\mu_0)}_{:= w} \underbrace{- \frac{1}{2}(\mu_1^T\Sigma^{-1}\mu_1 - \mu_0\Sigma^{-1}\mu_0)}_{②}$$

$w_0 = ① + ②$

$$= x^T w + w_0$$

$$\Rightarrow p(C=1|x) = \frac{1}{1+\exp(x^T w + w_0)}$$

$\Sigma_0 \neq \Sigma_1$: (I) $\Rightarrow g(x) = \ln\frac{\pi_1}{\pi_0} - \frac{1}{2}\ln\frac{|\Sigma_1|}{|\Sigma_0|} - \frac{1}{2}x^T(\Sigma_0^{-1} - \Sigma_1^{-1})x$

$+ x^T(\Sigma_1^{-1}\mu_1 - \Sigma_0^{-1}\mu_0) + \text{const})$

$\Rightarrow$ QDA (Murphy 9.2.1)

c) $p(C=1|x)$ is not Gauß $\Leftarrow p(C=1|x) = \sigma(\text{lin})$, $\sigma(\text{quad})$ sigmoid-fct, ($\Sigma_0 = \Sigma_1$, $\Sigma_0 \neq \Sigma_1$)

which is not Gauß.

Task 2

a) **Split A**

$p_0 = 3/4$, $p_1 = 1/4$

misclassification: $1 - \max(p_0, p_1) = 1/4$

$G_1 = 1 - [(3/4)^2 + (1/4)^2] = 3/8$

$H = -\ 3/4 \ln 3/4 - 1/4 \ln 1/4 \approx .56$

} For both nodes

Split B

$(200, 0)$: $p_0 = 1$, $p_1 = 0$

mcf: $1 - \max(1, 0) = 0$

$G_1 = 1 - (1^2 + 0^2) = 0$

$H = -1 \cdot \ln 1 - 0 \cdot \ln 0 = 0$

$(200, 400)$: $p_0 = 1/3$, $p_1 = 2/3$

mcf $= 1/3$

$G_1 = 1 - [(1/3)^2 + (2/3)^2] = 4/9$

$H = -1/3 \ln 1/3 - 2/3 \ln 2/3 = .636513$

weighted sum: $w_{(200,0)} = 1/4$, $w_{(200,400)} = 3/4$

mcf: $1/4 \cdot 0 + 3/4 \cdot 1/3 = 1/4$

$G_1 = 1/4 \cdot 0 + 3/4 \cdot 4/9 = 1/3$

$H = 1/4 \cdot 0 + 3/4 \cdot .636513 \approx .48$

Comparison

mcf: tie

G_1: Split B

H: Split B