

2013-2014 春学期常微分试卷

1. $x \frac{dy}{dx} + (x+1)y = 3x^2 e^{-x} (x > 0).$

2. $2xy^3 dx = (1 - x^2 y^2) dy.$

3. $(y + xy^2) dx + (x - x^2 y) dy = 0.$

4. $2y \frac{d^2 y}{dx^2} - \left(\frac{dy}{dx} \right)^2 = 0, y(1) = 1, \frac{dy}{dx} \Big|_{x=1} = 2.$

5. $\frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} + 4y = e^{2x}.$

6. $(1 - x^2) \frac{d^2 y}{dx^2} + 2x \frac{dy}{dx} - 2y = 0.$

7. $x^2 \frac{d^2 y}{dx^2} + 4x \frac{dy}{dx} + 2y = \frac{1}{x+1}.$

8. 函数 $f(x)$ 具有二阶导数且二阶导数连续, 试求满足 $\int_0^x (x-t) \cdot f(t) dt = \cos x - f(x)$ 的函数 $f(x)$.

9.
$$\begin{cases} \frac{dx}{dt} = y + z \\ \frac{dy}{dt} = x + z. \\ \frac{dz}{dt} = x + y \end{cases}$$

2013-2014 春学期常微分试卷 参考答案

1. $x \frac{dy}{dx} + (x+1)y = 3x^2 e^{-x} (x > 0).$

解: $\frac{dy}{dx} = -\left(1 + \frac{1}{x}\right)y + 3xe^{-x}$

$$y = e^{\int -\left(1 + \frac{1}{x}\right)dx} \left(\int 3xe^{-x} e^{\int \left(1 + \frac{1}{x}\right)dx} dx + c \right)$$

$$= \frac{1}{xe^x} (\int 3xe^{-x} xe^x dx + c)$$

$$= \frac{1}{xe^x} (\int 3x^2 dx + c)$$

$$= \frac{1}{xe^x} (x^3 + c)$$

$$= \frac{x^2}{e^x} + \frac{c}{xe^x}.$$

2. $2xy^3 dx = (1 - x^2 y^2) dy.$

解: $2xy^3 dx + (x^2 y^2 - 1) dy = 0$

$$M = 2xy^3, N = x^2 y^2$$

$$\varphi(y) = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = -\frac{2}{y}$$

$$\mu = e^{\int \varphi(y) dy} = \frac{1}{y^2}$$

$$2xy dx + \left(x^2 - \frac{1}{y^2}\right) dy = 0$$

$$x^2 y + \frac{1}{y} = c$$

3. $(y + xy^2) dx + (x - x^2 y) dy = 0.$

解:

法一: $\frac{dy}{dx} = \frac{xy^2 + y}{x^2 y - x} = \frac{y + \frac{1}{x}}{x - \frac{1}{y}} = \frac{\frac{xy+1}{x}}{\frac{xy-1}{y}} = \frac{xy+1}{xy-1} \cdot \frac{y}{x}$

$$\text{令 } u = xy, \text{ 则 } \frac{dy}{dx} = \frac{d\left(\frac{u}{x}\right)}{dx} = \frac{1}{x} \cdot \frac{du}{dx} - \frac{1}{x^2} u = \frac{u+1}{u-1} \cdot \frac{u}{x^2}$$

$$\therefore \frac{du}{dx} = \frac{1}{x} \cdot \frac{2u^2}{u-1} \quad \therefore \left(\frac{1}{u} - \frac{1}{u^2}\right) du = 2 \frac{dx}{x}$$

$$\therefore \ln u + \frac{1}{u} = 2 \ln x + c$$

$$\therefore \ln \frac{y}{x} + \frac{1}{xy} = c$$

法二: 令 $u = \frac{y}{x}$, 则 $x \frac{du}{dx} + u = \frac{ux^2+1}{ux^2-1} \cdot u$

$$\text{则 } \frac{du}{dx} = \frac{2u}{ux^3-x} \text{ 则 } \frac{dx}{du} = \frac{1}{2} \left(x^3 - \frac{x}{u} \right)$$

$$\therefore \frac{2}{x^3} \frac{dx}{du} = 1 - \frac{1}{u} \frac{1}{x^2}$$

$$\text{令 } z = \frac{1}{x^2}, \text{ 则 } \frac{dz}{du} = \frac{1}{u} z - 1$$

$$\therefore z = e^{\int \frac{1}{u} du} \left(\int -e^{\int \frac{1}{u} du} du + c \right) = u(-\ln u + c)$$

$$\therefore \frac{1}{x^2} = \frac{y}{x} (-\ln \frac{y}{x} + c)$$

$$\therefore \frac{1}{xy} + \ln \frac{y}{x} = c$$

$$4. 2y \frac{d^2 y}{dx^2} - \left(\frac{dy}{dx} \right)^2 = 0, y(1) = 1, \left. \frac{dy}{dx} \right|_{x=1} = 2.$$

$$\text{解: 令 } p = \frac{dy}{dx}, \text{ 则 } \frac{dp}{dx} = \frac{dp}{dy} \cdot \frac{dy}{dx} = p \frac{dp}{dy}$$

$$\therefore 2yp \frac{dp}{dy} - p^2 = 0 \quad \because y \neq 0$$

$$\therefore \frac{dp}{dy} = \frac{1}{2y} p \quad \therefore p = ce^{\int \frac{1}{2y} dy} = c\sqrt{|y|}$$

$$\because y(1) = 1, \left. \frac{dy}{dx} \right|_{x=1} = 2 \quad \therefore c = 2$$

$$\therefore \frac{dy}{dx} = 2\sqrt{|y|} \quad \therefore \sqrt{|y|} = x + c$$

$$\because y(1) = 1 \quad \therefore c = 0$$

$$\therefore y = x^2$$

$$5. \frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} + 4y = e^{2x}.$$

$$\text{解: 特征方程 } \lambda^2 - 4\lambda + 4 = 0, \text{ 得 } \lambda = 2 \text{ (二重根)}$$

$$\therefore Y = c_1 e^{2x} + c_2 x e^{2x}$$

$$\text{令 } y^* = x^2 A e^{2x}, \text{ 代入得 } A = \frac{1}{2}$$

$$\therefore y = Y + y^* = c_1 e^{2x} + c_2 x e^{2x} + \frac{1}{2} x^2 e^{2x}$$

$$6. (1 - x^2) \frac{d^2 y}{dx^2} + 2x \frac{dy}{dx} - 2y = 0.$$

$$\text{解: 可知该方程特解 } y_1 = x$$

$$\text{则有 Liouville 公式, 得通解 } y = y_1 \left(c_1 + c_2 \int \frac{1}{y_1^2} e^{\int \frac{-2x}{1-x^2} dx} dx \right)$$

$$= x(c_1 + c_2 \int \frac{1}{x^2} (1 - x^2) dx)$$

$$= x \left(c_1 + c_2 \left(-\frac{1}{x} - x \right) \right)$$

$$= c_1 x - c_2 (1 + x^2)$$

$$7. x^2 \frac{d^2 y}{dx^2} + 4x \frac{dy}{dx} + 2y = \frac{1}{x+1}.$$

解: 令 $x = e^t$, 则 $\frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} = \frac{1}{x} \frac{dy}{dt}$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{1}{x} \frac{dy}{dt} \right) = \frac{1}{x^2} \left(\frac{d^2 y}{dt^2} - \frac{dy}{dt} \right)$$

$$\therefore \frac{d^2 y}{dt^2} + 3 \frac{dy}{dt} + 2y = \frac{1}{e^t + 1}$$

易知特征根 $\lambda_1 = -1$, $\lambda_2 = -2$, $Y = c_1 e^{-t} + c_2 e^{-2t} = c_1 y_1 + c_2 y_2$

命 $y = u_1(x)y_1 + u_2(x)y_2$ 则 $y' = u_1' y_1 + u_2' y_2 + y_1' u_1 + y_2' u_2$

命 $u_1' y_1 + u_2' y_2 = 0$ 则 $y'' = u_1'' y_1 + u_2'' y_2 + u_1' y_1' + u_2' y_2'$

带入原方程得 $u_1' y_1' + u_2' y_2' = \frac{1}{e^t + 1}$ 联立 $u_1' y_1 + u_2' y_2 = 0$

$$\text{得} \begin{cases} u_1' = \frac{e^t}{e^t + 1} \\ u_2' = -\frac{e^{2t}}{e^t + 1} \end{cases} \therefore \begin{cases} u_1 = \ln(e^t + 1) \\ u_2 = \ln(e^t + 1) - e^t \end{cases}$$

$$\therefore y = (u_1 + c_1)y_1 + (u_2 + c_2)y_2$$

$$= \ln(e^t + 1)(e^{-t} + e^{-2t}) + (c_1 + 1)e^{-t} + c_2 e^{-2t}$$

$$= \ln(x + 1) \left(\frac{1}{x} + \frac{1}{x^2} \right) + \frac{a_1}{x} + \frac{a_2}{x^2}$$

8. 函数 $f(x)$ 具有二阶导数且二阶导数连续, 试求满足 $\int_0^x (x-t) \cdot f(t) dt = \cos x -$

$f(x)$ 的函数 $f(x)$.

解: 当 $x = 0$ 时, $\int_0^0 (0-t) \cdot f(t) dt = \cos 0 - f(0) \therefore f(0) = 1$

对原方程两边求导得 $\int_0^x f(t) dt = -\sin x - f'(x)$

再对两边求导得 $f(x) = -\cos x - f''(x)$

$$\therefore f''(x) + f(x) = -\cos x$$

其次通解 $F = c_1 \cos x + c_2 \sin x$

$$\text{令 } f^* = x(A \cos x + B \sin x) \text{ 带入得 } \begin{cases} A = 0 \\ B = -\frac{1}{2} \end{cases}$$

$$\therefore f(x) = F + f^* = c_1 \cos x + c_2 \sin x - \frac{1}{2} x \sin x$$

$$9. \begin{cases} \frac{dx}{dt} = y + z \\ \frac{dy}{dt} = x + z. \\ \frac{dz}{dt} = x + y \end{cases}$$

$$\text{解: } D(\lambda) = \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = -\lambda^3 + 3\lambda + 2 = -(\lambda - 2)(\lambda + 1)^2$$

$$\therefore \lambda = 2 (\text{单根}), \lambda = -1 (\text{二重根})$$

$$\text{当 } \lambda = 2 \text{ 时, } (A - E\lambda) \cdot \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{vmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{vmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

$$\therefore \alpha : \beta : \gamma = \begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} : \begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} : \begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} = 3 : 3 : 3$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\text{当 } \lambda = -1 \text{ 时, } (A - E\lambda)^2 \cdot \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{vmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{vmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

$$\therefore v_0^{(1)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, v_0^{(2)} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\therefore v_1^{(1)} = (A - E\lambda) \cdot v_0^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, v_1^{(2)} = (A - E\lambda) \cdot v_0^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix}_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\therefore \text{综上得: } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} e^{-t}$$