Development of a Spectral-Element code

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Using Legendre polynomials on a Gauss-Lobatto quadrature.

$$P_n(x) = \frac{2n-1}{n} x P_{n-1}(x) - (n-1)P_{n-2}(x) \tag{1}$$

$$P_n'(x) = (2n-1)P_{n-1}(x) + P_{n-2}'(x) \tag{2}$$

where $P_0'(x) = 0$ and $P_1'(x) = 1$

Define a basis function

We need to define a basis function. We will define it as the Lagrange interpolation polynomial (Equation 5.46 in Hesthaven)

$$\phi_j(x) = \frac{-1}{n(n+1)} \frac{1-x^2}{x-x_j} \frac{P_n'(x)}{P_n(x_j)}. \tag{3}$$

Weak formulation

Refering to p.120 in Hestaven.

$$u_t + u_{xx} = f(x) \tag{4}$$

we take the weak form, so we take the inner product with respect to a test function $\phi_m(x)$,

$$\int_\Omega (u_t + u_{xx})\, \phi_m(x)\, d\Omega = \int_\Omega f(x)\phi_m(x)\, d\Omega \tag{5}$$

$$\int_\Omega u_t \phi_m(x)\, d\Omega + \int_\Omega u_{xx}\phi_m(x)\, d\Omega = \int_\Omega f(x)\phi_m(x)\, d\Omega. \tag{6}$$

We will discretize like

$$u = \sum_{j=0}^{N} u_j \, \phi_j. \tag{7}$$

The mass term $\sum_{j=0}^{N} \int_{-1}^{1} u_j' \phi_j \, \phi_m \, dx$ will be given as

$$M_{m,n} = \begin{cases} w(x) & m = n \\ 0 & \neq n \end{cases} \tag{8}$$

where $\int_{-1}^{1} u_n'(t) \, \phi_n(x) \, \phi_m(x) \, dx = u_n'(t) M_{m,n}$.

The symmetric stiffness term $\int_\Omega u_{xx} \phi_m(x) \, d\Omega$ will be integrated by parts to get to show how this works between two elements we will use the domain $-3 \leq x \leq 1$ where a unit element is $-1 \leq x \leq 1$.

$$\int_{-3}^{1} u_{xx} \phi_m \, dx \tag{9}$$

$$\int_{-3}^{-1} u_n(t) \, \phi_n''(x) \phi_m(x) \, dx + \int_{-1}^{1} u_n(t) \, \phi_n''(x) \phi_m(x) \, dx \tag{10}$$

integrating by parts,

$$\begin{aligned} & u_n^{(i)}(t) \, \phi_n'(x) \phi_m(x) \Big|_{-3}^{-1} - \int_{-3}^{-1} u_n^{(i)}(t) \, \phi_n'(x) \phi_m'(x) \, dx ... \\ & \quad + u_n^{(i+1)}(t) \, \phi_n'(x) \phi_m(x) \Big|_{-1}^{1} - \int_{-1}^{1} u_n^{(i+1)}(t) \, \phi_n'(x) \phi_m'(x) \, dx. \end{aligned} \tag{11}$$

Now evaluate with the basis being 1 at $x = -1$ and zero otherwise.

$$\begin{aligned} & u_n^{(i)}(t) \, \phi_n'(-1) \phi_m(-1) - \int_{-3}^{-1} u_n^{(i)}(t) \, \phi_n'(x) \phi_m'(x) \, dx ... \\ & \quad - u_n^{(i+1)}(t) \, \phi_n'(-1) \phi_m(-1) - \int_{-1}^{1} u_n^{(i+1)}(t) \, \phi_n'(x) \phi_m'(x) \, dx \end{aligned} \tag{12}$$

$$- \int_{-3}^{-1} u_n^{(i)}(t) \, \phi_n'(x) \phi_m'(x) \, dx - \int_{-1}^{1} u_n^{(i+1)}(t) \, \phi_n'(x) \phi_m'(x) \, dx \tag{13}$$

Its discretization time!!! and addition of the element counter i, where $i = 0, 1, 2, ..., K$. Replace j with n on the basis. blah.

$$u_n^{(i)} = \sum_{j=0}^{N} u_j^{(i)} \phi_j(x) \tag{14}$$

$$\sum_{j=0}^{N} u_j^{(i)} \left[-\int_{-3}^{-1} \phi_n'(x)\phi_m'(x)\,dx \right] + \sum_{j=0}^{N} u_j^{(i+1)} \left[-\int_{-1}^{1} \phi_n'(x)\phi_m'(x)\,dx \right]. \quad (15)$$

We see the term $\int_{-1}^{1} \phi_n'(x)\phi_m'(x)\,dx$ is repeating. The elemental stiffness matrix will take the form

$$K_{m,n} = -\int_{-1}^{1} \phi_n'(x)\phi_m'(x)\,dx. \quad (16)$$

From Hesthaven p. 95 the differentiation matrix takes the form

$$D_{ij} = \begin{cases} -\frac{N(N+1)}{4} & i = j = 0 \\ 0 & i = j \in [1, ..., N-1] \\ \frac{P_N(x_i)}{P_N(x_j)} \frac{1}{x_i - x_j} & i \neq j \\ \frac{N(N+1)}{4} & i = j = N \end{cases} \quad (17)$$

This means that we can approximate

$$\frac{d}{dx}\, u = D_1\, u \quad (18)$$

this also means that correct?

$$\frac{d^2}{dx^2}\, u = D_2\, u = D_1\, D_1\, u \quad (19)$$

This means the stiffness matrix is just $D_1\, D_1$??