

Development of a Spectral-Element code

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Using Legendre polynomials on a Gauss-Lobatto quadrature.

$$P_n(x) = \frac{2n-1}{n} x P_{n-1}(x) - (n-1) P_{n-2}(x) \quad (1)$$

$$P'_n(x) = (2n-1) P_{n-1}(x) + P'_{n-2}(x) \quad (2)$$

where $P'_0(x) = 0$ and $P'_1(x) = 1$

Define a basis function

We need to define a basis function. We will define it as the Lagrange interpolation polynomial (Equation 5.46 in Hesthaven)

$$\phi_j(x) = \frac{-1}{n(n+1)} \frac{1-x^2}{x-x_j} \frac{P'_n(x)}{P'_n(x_j)}. \quad (3)$$

Weak formulation

Referring to p.120 in Hestaven.

$$u_t + u_{xx} = f(x) \quad (4)$$

we take the weak form, so we take the inner product with respect to a test function $\phi_m(x)$,

$$\int_{\Omega} (u_t + u_{xx}) \phi_m(x) d\Omega = \int_{\Omega} f(x) \phi_m(x) d\Omega \quad (5)$$

$$\int_{\Omega} u_t \phi_m(x) d\Omega + \int_{\Omega} u_{xx} \phi_m(x) d\Omega = \int_{\Omega} f(x) \phi_m(x) d\Omega. \quad (6)$$

We will discretize like

$$u = \sum_{j=0}^N u_j \phi_j. \quad (7)$$

The mass term $\sum_{j=0}^N \int_{-1}^1 u'_j \phi_j \phi_m dx$ will be given as

$$M_{m,n} = \begin{cases} w(x) & m = n \\ 0 & \neq n \end{cases} \quad (8)$$

where $\int_{-1}^1 u'_n(t) \phi_n(x) \phi_m(x) dx = u'_n(t) M_{m,n}$.

The symmetric stiffness term $\int_{\Omega} u_{xx} \phi_m(x) d\Omega$ will be integrated by parts to get to show how this works between two elements we will use the domain $-3 \leq x \leq 1$ where a unit element is $-1 \leq x \leq 1$.

$$\int_{-3}^1 u_{xx} \phi_m dx \quad (9)$$

$$\int_{-3}^{-1} u_n(t) \phi_n''(x) \phi_m(x) dx + \int_{-1}^1 u_n(t) \phi_n''(x) \phi_m(x) dx \quad (10)$$

integrating by parts,

$$\begin{aligned} & u_n^{(i)}(t) \phi_n'(x) \phi_m(x) \Big|_{-3}^{-1} - \int_{-3}^{-1} u_n^{(i)}(t) \phi_n'(x) \phi_m'(x) dx \dots \\ & + u_n^{(i+1)}(t) \phi_n'(x) \phi_m(x) \Big|_{-1}^1 - \int_{-1}^1 u_n^{(i+1)}(t) \phi_n'(x) \phi_m'(x) dx. \end{aligned} \quad (11)$$

Now evaluate with the basis being 1 at $x = -1$ and zero otherwise.

$$u_n^{(i)}(t) \phi_n'(-1) \phi_m(-1) - \int_{-3}^{-1} u_n^{(i)}(t) \phi_n'(x) \phi_m'(x) dx \dots \quad (12)$$

$$\begin{aligned} & - u_n^{(i+1)}(t) \phi_n'(-1) \phi_m(-1) - \int_{-1}^1 u_n^{(i+1)}(t) \phi_n'(x) \phi_m'(x) dx \\ & - \int_{-3}^{-1} u_n^{(i)}(t) \phi_n'(x) \phi_m'(x) dx - \int_{-1}^1 u_n^{(i+1)}(t) \phi_n'(x) \phi_m'(x) dx \end{aligned} \quad (13)$$

Its discretization time!!! and addition of the element counter i , where $i = 0, 1, 2, \dots, K$. Replace j with n on the basis. blah.

$$u_n^{(i)} = \sum_{j=0}^N u_j^{(i)} \phi_j(x) \quad (14)$$

$$\sum_{j=0}^N u_j^{(i)} \left[- \int_{-3}^{-1} \phi'_n(x) \phi'_m(x) dx \right] + \sum_{j=0}^N u_j^{(i+1)} \left[- \int_{-1}^1 \phi'_n(x) \phi'_m(x) dx \right]. \quad (15)$$

We see the term $\int_{-1}^1 \phi'_n(x) \phi'_m(x) dx$ is repeating.

From Hesthaven p. 95 the differentiation matrix takes the form

$$D_{ij} = \begin{cases} -\frac{N(N+1)}{4} & i = j = 0 \\ 0 & i = j \in [1, \dots, N-1] \\ \frac{P_n(x_i)}{P_N(x_j)} \frac{1}{x_i - x_j} & i \neq j \\ \frac{N(N+1)}{4} & i = j = N \end{cases} \quad (16)$$

The elemental stiffness matrix will take the form

$$K_{m,n} = -D_{ij} D_{ij} \quad (17)$$