Development of a Spectral-Element code

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Using Legendre polynomials on a Gauss-Lobatto quadrature.

$$P_n(x) = \frac{2n-1}{n} x P_{n-1}(x) - (n-1)P_{n-2}(x)$$
$$P'_n(x) = (2n-1)P_{n-1}(x) + P'_{n-2}(x)$$

where $P'_0(x) = 0$ and $P'_1(x) = 1$

Define a basis function

We need to define a basis function. We will define it as the Lagrange interpolation polynomial (Equation 5.46 in Hestaven)

$$l_j(x) = \frac{-1}{N(N+1)} \frac{1-x^2}{x-x_j} \frac{P'_N(x)}{P_N(x_j)}.$$

This will cause a problem, $\lim x \to x_j = \infty$ Have to use L'hopital.
The numerator:

$$-\frac{d}{dx}(1-x^2)P'_N(x) = -P''(x) + (x^2 P''(x) + 2xP'(x))$$

$$-P''(x)(1-x^2) + 2xP'(x)$$

The denominator:

$$\frac{d}{dx} N(N+1)(x-x_j)P_N(x_j) = N(N+1)P_N(x_j)$$

Now we have:

$$l_j = \frac{-P''(x)(1-x^2) + 2xP'(x)}{N(N+1)P_N(x_j)}$$

This will become our basis,

$$\phi_n = \frac{-P''(x)(1-x^2) + 2xP'(x)}{N(N+1)P_N(x_n)}$$

I'm confused here.

Weak formulation

Refering to p.120 in Hestaven.

This will be used in our approximation with coefficients $a_n(t_i)$ as,

$$u_N(x,t) = \sum_{n=1}^{N-1} a_n(t)\,\phi_n(x_i)$$

$$\frac{\partial}{\partial t} u_N(x,t) = \sum_{n=1}^{N-1} a_n'(t)\,\phi_n(x_i)$$

$$\frac{\partial^2}{\partial x^2} u_N(x,t) = \sum_{n=1}^{N-1} a_n(t)\,\phi_n''(x_i).$$

$$\frac{\partial}{\partial t} u_N + \frac{\partial^2}{\partial x^2} u_N = f(x)$$

we take use the weak, so we take the inner product with respect to a test function $\phi_m(x)$,

$$\int_{-1}^{1} (a_n'(t)\,\phi_n(x_i) + a_n(t)\,\phi_n''(x_i))\,\phi_m(x_i)\,dx = \int_{-1}^{1} f(x_i)\phi_m(x)\,dx$$

$$\int_{-1}^{1} a_n'(t)\,\phi_n(x_i)\,\phi_m(x_i)\,dx + \int_{-1}^{1} a_n(t)\,\phi_n''(x_i)\phi_m(x_i)\,dx = \int_{-1}^{1} f(x_i)\phi_m(x)\,dx.$$

The mass term $\int_{-1}^{1} a_n'(t)\,\phi_n(x_i)\,\phi_m(x_i)\,dx$ will be given as

$$M_{m,n} = \begin{cases} w(x_i) & m = n \\ 0 & \neq n \end{cases}$$

where $\int_{-1}^{1} a_n'(t)\, \phi_n(x_i)\, \phi_m(x_i)\, dx = a_n'(t) M_{m,n}$.

The symmetric stiffness term $\int_{-1}^{1} a_n(t)\, \phi_n''(x_i) \phi_m(x_i)\, dx$ will be integrated by parts to get,

$$\phi_n(x_i)\phi_m'(x_i)\Big|_{-1}^{1} - \int_{-1}^{1} a_n(t)\, \phi_n'(x_i)\phi_m'(x_i)\, dx$$

The stiffness matrix term is

$$K_{m,n} = \begin{cases} -\sum_{i=1}^{N} \phi_n'(x_i)\phi_m'(x_i) w_i & 0 < n < N \\ ????? & n = 0 \\ ????? & n = N \end{cases}$$