

**MATH 4410**

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**Quiz #2; Due 11:59 pm, 1/19/2024**

1. Let  $S_n$  denote the maximum number of regions into which 3-dimensional space is partitioned by planes so that no three of the cutting planes are colinear and no four are concurrent (no four intersect in a point). Find a formula for  $S_n$  in terms of binomial coefficients or powers of  $n$ . *Hint/Suggestion:* Show that  $\Delta S_n = P_n$ , where  $P_n$  is the maximum number of regions into which 2-dimensional space is partitioned by  $n$  lines, no three concurrent.

An argument for  $\Delta S_n = P_n$  where  $P_n$  is the 2d version of this 3d problem: The statement means that 2d space can be thought to be the derivative of 3d space. Why? If you have a 2d function then the slope of the tangent line at any given point is the derivative. Thus for 2d space a 1d line is needed to describe the derivative. If you have a 3d function then the slope of the tangent plane at any given point is the derivative. Thus for 3d space a 2d plane is needed to describe the derivative. Thus 2d space is the derivative of 3d space. For this particular problem it also intuitively makes sense.  $P_n$  is the maximum number of regions into which 2d space is partitioned by lines so that no two lines are linear and no three intersect in a point. Taking the integral of  $P_n$  means to sum up layers to get 3d space. Thus you are turning the pizza into many layered pizzas and turning all the cutting lines into cutting planes.

$$\Delta S_n = P_n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} = 1 + \frac{n^1}{1!} + \frac{n^2}{2!} = 1 + n^1 + \frac{1}{2}n^2$$

$$\Delta^{(1)}f(n) = 1 + n^1 + \frac{1}{2}n^2$$

| n                  | 0 | 1 | 2 | 3 | 4  | 5  | 6  | 7  |
|--------------------|---|---|---|---|----|----|----|----|
| $\Delta^{(1)}f(n)$ | 1 | 2 | 4 | 7 | 11 | 16 | 22 | 29 |
| $\Delta^{(2)}f(n)$ | 1 | 2 | 3 | 4 | 5  | 6  | 7  |    |
| $\Delta^{(3)}f(n)$ | 1 | 1 | 1 | 1 | 1  | 1  |    |    |
| $\Delta^{(4)}f(n)$ | 0 | 0 | 0 | 0 | 0  |    |    |    |
| $\Delta^{(5)}f(n)$ | 0 | 0 | 0 | 0 |    |    |    |    |
| $\Delta^{(6)}f(n)$ | 0 | 0 | 0 |   |    |    |    |    |

$$\begin{aligned}
S_n &= \sum_{k=0}^n (1 + n^1 + \frac{1}{2}n^2) = \sum_{k=0}^n 1 + \sum_{k=0}^n k^1 + \frac{1}{2} \sum_{k=0}^n k^2 \\
&= (n+1) + \left[ \frac{k^2}{2} \right]_{k=0}^{k=n+1} + \frac{1}{2} \left[ \frac{k^3}{3} \right]_{k=0}^{k=n+1} \\
&= n+1 + \frac{1}{2}(n+1)^2 + \frac{1}{6}(n+1)^3 \\
&= n+1 + \frac{1}{2}(n+1)n + \frac{1}{6}(n+1)n(n-1) \\
&= \frac{1}{6}n^3 + \frac{1}{2}n^2 + \frac{4}{3}n + 1
\end{aligned}$$

There you have it

$$S_n = \frac{1}{6}n^3 + \frac{1}{2}n^2 + \frac{4}{3}n + 1$$

Let  $f(n) = S_n$

| n                  | -1 | 0 | 1 | 2 | 3  | 4  | 5  | 6  | 7  |
|--------------------|----|---|---|---|----|----|----|----|----|
| $f(n)$             | 0  | 1 | 3 | 7 | 14 | 25 | 41 | 63 | 92 |
| $\Delta^{(1)}f(n)$ | 1  | 2 | 4 | 7 | 11 | 16 | 22 | 29 |    |
| $\Delta^{(2)}f(n)$ | 1  | 2 | 3 | 4 | 5  | 6  | 7  |    |    |
| $\Delta^{(3)}f(n)$ | 1  | 1 | 1 | 1 | 1  | 1  |    |    |    |
| $\Delta^{(4)}f(n)$ | 0  | 0 | 0 | 0 | 0  |    |    |    |    |
| $\Delta^{(5)}f(n)$ | 0  | 0 | 0 | 0 |    |    |    |    |    |
| $\Delta^{(6)}f(n)$ | 0  | 0 | 0 |   |    |    |    |    |    |

Notice the first table is in the second table just shifted down and to the left.

2. Use the Discrete Taylor Theorem to determine a closed formula for the sum  $\sum_{k=0}^n k^3$ .

Given

$$f(n) = \sum_{k \geq 0} \Delta^{(k)} f(a) \binom{n-a}{k}$$

$$\binom{n}{k} = \frac{n^{\underline{k}}}{k!}$$

For  $k$  not equal to  $-1$

$$\sum_{n=a}^b n^{\underline{k}} = \frac{n^{\underline{k+1}}}{k+1} \Big|_{n=a}^{n=b+1}$$

Start

Let  $a = 0$

$$f(n) = \sum_{k \geq 0} \Delta^{(k)} f(0) \binom{n}{k}$$

$$k^3 = 0 \binom{k}{0} + 1 \binom{k}{1} + 6 \binom{k}{2} + 6 \binom{k}{3} + 0 \binom{k}{4} + 0 \binom{k}{5} + 0 \binom{k}{6} + 0 \binom{k}{7} + \dots$$

$$= 1 \frac{k^{\underline{1}}}{1!} + 6 \frac{k^{\underline{2}}}{2!} + 6 \frac{k^{\underline{3}}}{3!}$$

$$= k^{\underline{1}} + 3k^{\underline{2}} + k^{\underline{3}}$$

$$\sum_{k=0}^n k^3 = \sum_{k=0}^n (k^{\underline{1}} + 3k^{\underline{2}} + k^{\underline{3}})$$

$$= \sum_{k=0}^n k^{\underline{1}} + 3 \sum_{k=0}^n k^{\underline{2}} + \sum_{k=0}^n k^{\underline{3}}$$

$$= \frac{k^{\underline{2}}}{2} \Big|_{k=0}^{k=n+1} + 3 \left( \frac{k^{\underline{3}}}{3} \Big|_{k=0}^{k=n+1} \right) + \frac{k^{\underline{4}}}{4} \Big|_{k=0}^{k=n+1}$$

$$= \frac{(n+1)n}{2} + (n+1)n(n-1) + \frac{(n+1)n(n-1)(n-2)}{4}$$

$$= \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}$$

There you have it

$$\sum_{k=0}^n k^3 = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}$$

End

3. (a) Writing the result in terms of binomial coefficients, use the Discrete Taylor Theorem to determine a closed formula for the sum  $\sum_{k=-3}^n k^3(k+1)(k+2)(k+3)$  using center equal to 0.

Given

$$f(n) = \sum_{k \geq 0} \Delta^{(k)} f(a) \binom{n-a}{k}$$

Let  $a = 0$  as the prompt instructs

Start

$$\begin{aligned} k^3(k+1)(k+2)(k+3) &= 0 \binom{k}{0} + 24 \binom{k}{1} + 432 \binom{k}{2} + 1872 \binom{k}{3} + 3264 \binom{k}{4} + 2520 \binom{k}{5} + 720 \binom{k}{6} + 0 \binom{k}{7} + 0 \binom{k}{8} + \dots \\ &= 24 \frac{k^1}{1!} + 432 \frac{k^2}{2!} + 1872 \frac{k^3}{3!} + 3264 \frac{k^4}{4!} + 2520 \frac{k^5}{5!} + 720 \frac{k^6}{6!} \\ &= 24k^1 + 432 \frac{k^2}{2} + 1872 \frac{k^3}{6} + 3264 \frac{k^4}{24} + 2520 \frac{k^5}{120} + 720 \frac{k^6}{720} \\ &= 24k^1 + 216k^2 + 312k^3 + 136k^4 + 21k^5 + k^6 \end{aligned}$$

Plugging in

$$\begin{aligned} \sum_{k=-3}^n k^3(k+1)(k+2)(k+3) &= \sum_{k=-3}^n (24k^1 + 216k^2 + 312k^3 + 136k^4 + 21k^5 + k^6) \\ &= 24 \sum_{k=-3}^n k^1 + 216 \sum_{k=-3}^n k^2 + 312 \sum_{k=-3}^n k^3 + 136 \sum_{k=-3}^n k^4 + 21 \sum_{k=-3}^n k^5 + \sum_{k=-3}^n k^6 \end{aligned}$$

Using: for  $k$  not equal to -1  $\sum_{n=a}^b n^k = \frac{n^{k+1}}{k+1} \Big|_{n=a}^{n=b+1}$

$$\begin{aligned} &= 24 \left( \frac{k^2}{2} \Big|_{k=-3}^{k=n+1} \right) + 216 \left( \frac{k^3}{3} \Big|_{k=-3}^{k=n+1} \right) + 312 \left( \frac{k^4}{4} \Big|_{k=-3}^{k=n+1} \right) + 136 \left( \frac{k^5}{5} \Big|_{k=-3}^{k=n+1} \right) + 21 \left( \frac{k^6}{6} \Big|_{k=-3}^{k=n+1} \right) + \left( \frac{k^7}{7} \Big|_{k=-3}^{k=n+1} \right) \\ &= 24 \left( \frac{(n+1)^2}{2} - \frac{(-3)^2}{2} \right) + 216 \left( \frac{(n+1)^3}{3} - \frac{(-3)^3}{3} \right) + 312 \left( \frac{(n+1)^4}{4} - \frac{(-3)^4}{4} \right) + \\ &\quad 136 \left( \frac{(n+1)^5}{5} - \frac{(-3)^5}{5} \right) + 21 \left( \frac{(n+1)^6}{6} - \frac{(-3)^6}{6} \right) + \left( \frac{(n+1)^7}{7} - \frac{(-3)^7}{7} \right) \\ &= 12[(n+1)n - (-3)(-4)] + 72[(n+1)n(n-1) - (-3)(-4)(-5)] + \\ &\quad 78[(n+1)n(n-1)(n-2) - (-3)(-4)(-5)(-6)] + \frac{136}{5}[(n+1)n(n-1)(n-2)(n-3) - (-3)(-4)(-5)(-6)(-7)] + \\ &\quad \frac{21}{6}[(n+1)n(n-1)(n-2)(n-3)(n-4) - (-3)(-4)(-5)(-6)(-7)(-8)] + \\ &\quad \frac{1}{7}[(n+1)n(n-1)(n-2)(n-3)(n-4)(n-5) - (-3)(-4)(-5)(-6)(-7)(-8)(-9)] \end{aligned}$$

$$\begin{aligned}
&= [72n^3 + 12n^2 - 60n + 4176] + \\
&\quad [78n^4 - 156n^3 - 78n^2 + 156n - 28080] + \left[-\frac{136}{5}n^5 - 136n^4 + 136n^3 + 136n^2 - \frac{816}{5}n + 68544\right] + \\
&\quad \left[\frac{7}{2}n^6 - \frac{63}{2}n^5 + \frac{175}{2}n^4 - \frac{105}{2}n^3 - 91n^2 + 84n - 70560\right] + \\
&\quad \left[\frac{1}{7}n^7 - 2n^6 + 10n^5 - 20n^4 + 7n^3 + 22n^2 - \frac{120}{7}n + 25920\right] \\
&= 78n^4 - 84n^3 - 66n^2 + 96n - 23904 + \frac{7}{2}n^6 - \frac{43}{10}n^5 - \frac{97}{2}n^4 + \frac{167}{2}n^3 + 45n^2 - \frac{396}{5}n - 2016 + \\
&\quad \frac{1}{7}n^7 - 2n^6 + 10n^5 - 20n^4 + 7n^3 + 22n^2 - \frac{120}{7}n + 25920 \\
&= \frac{1}{7}n^7 + \frac{3}{2}n^6 + \frac{57}{10}n^5 + \frac{19}{2}n^4 + \frac{13}{2}n^3 + n^2 - \frac{12}{35}n
\end{aligned}$$

There you have it

$$\sum_{k=-3}^n k^3(k+1)(k+2)(k+3) = \frac{1}{7}n^7 + \frac{3}{2}n^6 + \frac{57}{10}n^5 + \frac{19}{2}n^4 + \frac{13}{2}n^3 + n^2 - \frac{12}{35}n$$

End

(b) Repeat part (a) but with center equal to  $-4$ .

Restate the problem: Writing the result in terms of binomial coefficients, use the Discrete Taylor Theorem to determine a closed formula for the sum  $\sum_{k=-3}^n k^3(k+1)(k+2)(k+3)$  using center equal to  $-4$ .

Given

$$f(n) = \sum_{k \geq 0} \Delta^{(k)} f(a) \binom{n-a}{k}$$

Let  $a = -4$  as the prompt instructs

$$f(n) = \sum_{k \geq 0} \Delta^{(k)} f(-4) \binom{n+4}{k}$$

Start

$$\begin{aligned} k^3(k+1)(k+2)(k+3) &= 384 \binom{(k+4)}{0} - 384 \binom{(k+4)}{1} + 384 \binom{(k+4)}{2} - \\ &\quad 384 \binom{(k+4)}{3} + 384 \binom{(k+4)}{4} - 360 \binom{(k+4)}{5} + 720 \binom{(k+4)}{6} + 0 \binom{(k+4)}{7} + 0 \binom{(k+4)}{8} + \dots \\ &= 384 - 384 \frac{(k+4)^1}{1!} + 384 \frac{(k+4)^2}{2!} - 384 \frac{(k+4)^3}{3!} + 384 \frac{(k+4)^4}{4!} - 360 \frac{(k+4)^5}{5!} + 720 \frac{(k+4)^6}{6!} \\ &= 384 - 384(k+4)^1 + 192(k+4)^2 - 64(k+4)^3 + 16(k+4)^4 - 3(k+4)^5 + (k+4)^6 \end{aligned}$$

Plugging in

$$\begin{aligned} \sum_{k=-3}^n k^3(k+1)(k+2)(k+3) &= \sum_{k=-3}^n (384 - 384(k+4)^1 + 192(k+4)^2 - 64(k+4)^3 + 16(k+4)^4 - 3(k+4)^5 + (k+4)^6) \\ &= 384 \sum_{k=-3}^n 1 - 384 \sum_{k=-3}^n (k+4)^1 + 192 \sum_{k=-3}^n (k+4)^2 - 64 \sum_{k=-3}^n (k+4)^3 + 16 \sum_{k=-3}^n (k+4)^4 - 3 \sum_{k=-3}^n (k+4)^5 + \sum_{k=-3}^n (k+4)^6 \\ &= 384(n+4) - 384 \left( \frac{(k+4)^2}{2} \right) \Big|_{k=-3}^{k=n+1} + 192 \left( \frac{(k+4)^3}{3} \right) \Big|_{k=-3}^{k=n+1} - 64 \left( \frac{(k+4)^4}{4} \right) \Big|_{k=-3}^{k=n+1} + \\ &\quad 16 \left( \frac{(k+4)^5}{5} \right) \Big|_{k=-3}^{k=n+1} - 3 \left( \frac{(k+4)^6}{6} \right) \Big|_{k=-3}^{k=n+1} + \left( \frac{(k+4)^7}{7} \right) \Big|_{k=-3}^{k=n+1} \\ &= 384[n+4] - 384 \left[ \frac{((n+1)+4)^2}{2} - \frac{((-3)+4)^2}{2} \right] + 192 \left[ \frac{((n+1)+4)^3}{3} - \frac{((-3)+4)^3}{3} \right] - \\ &\quad 64 \left[ \frac{((n+1)+4)^4}{4} - \frac{((-3)+4)^4}{4} \right] + 16 \left[ \frac{((n+1)+4)^5}{5} - \frac{((-3)+4)^5}{5} \right] - \\ &\quad 3 \left[ \frac{((n+1)+4)^6}{6} - \frac{((-3)+4)^6}{6} \right] + \left[ \frac{((n+1)+4)^7}{7} - \frac{((-3)+4)^7}{7} \right] \end{aligned}$$

$$= 384n + 1536 - 192[(n+5)^2 - (1)^2] + 64[(n+5)^3 - (1)^3] - \\ 16[(n+5)^4 - (1)^4] + \frac{16}{5}[(n+5)^5 - (1)^5] - \frac{1}{2}[(n+5)^6 - (1)^6] + \frac{1}{7}[(n+5)^7 - (1)^7]$$

$$= 384n + 1536 - 192(n+5)(n+4) + 64(n+5)(n+4)(n+3) - \\ 16(n+5)(n+4)(n+3)(n+2) + \frac{16}{5}(n+5)(n+4)(n+3)(n+2)(n+1) - \\ \frac{1}{2}(n+5)(n+4)(n+3)(n+2)(n+1)n + \frac{1}{7}(n+5)(n+4)(n+3)(n+2)(n+1)n(n-1) \\ = \frac{1}{7}n^7 + \frac{3}{2}n^6 + \frac{57}{10}n^5 + \frac{19}{2}n^4 + \frac{13}{2}n^3 + n^2 - \frac{12}{35}n$$

There you have it

$$\sum_{k=-3}^n k^3(k+1)(k+2)(k+3) = \frac{1}{7}n^7 + \frac{3}{2}n^6 + \frac{57}{10}n^5 + \frac{19}{2}n^4 + \frac{13}{2}n^3 + n^2 - \frac{12}{35}n$$

End

- (c) The formulae from (a) and (b) should look different, but are they? Determine whether they are the same or not.

The formula do look different but the solution is the same.

Formula (a)

$$\sum_{k=-3}^n k^3(k+1)(k+2)(k+3) = \sum_{k=-3}^n (24k^1 + 216k^2 + 312k^3 + 136k^4 + 21k^5 + k^6)$$

Formula (b)

$$\begin{aligned} \sum_{k=-3}^n k^3(k+1)(k+2)(k+3) = \\ \sum_{k=-3}^n (384 - 384(k+4)^1 + 192(k+4)^2 - 64(k+4)^3 + 16(k+4)^4 - 3(k+4)^5 + (k+4)^6) \end{aligned}$$

So

$$\begin{aligned} \sum_{k=-3}^n (24k^1 + 216k^2 + 312k^3 + 136k^4 + 21k^5 + k^6) = \\ \sum_{k=-3}^n (384 - 384(k+4)^1 + 192(k+4)^2 - 64(k+4)^3 + 16(k+4)^4 - 3(k+4)^5 + (k+4)^6) \end{aligned}$$

They both come out to  $\frac{1}{7}n^7 + \frac{3}{2}n^6 + \frac{57}{10}n^5 + \frac{19}{2}n^4 + \frac{13}{2}n^3 + n^2 - \frac{12}{35}n$  so they are the same.

This means the coefficients for both Formula must make up for the difference of moving the center.



4. Use summation by parts to find a closed formula for  $\sum_{k=0}^n \frac{H_k}{(k+1)(k+2)}$ .

Given

$$\sum_{k=0}^n u(k)\Delta v(k) = u(k)v(k)\Big|_{k=0}^{k=n+1} - \sum_{k=0}^n Ev(k)\Delta u(k)$$

Restate Problem

$$\sum_{k=0}^n H_k \frac{1}{(k+1)(k+2)} = \sum_{k=0}^n H_k k^{-2}$$

Let

$$u(k) = H_k$$

$$\Delta u(k) = \frac{1}{k+1}$$

$$v(k) = \sum_{i=0}^{k-1} i^{-2} = \frac{i^{-1}}{-1} \Big|_{i=0}^{i=k} = -k^{-1} + 0^{-1} = 1 - \frac{1}{k+1}$$

$$\Delta v(k) = k^{-2} = \Delta(1 - \frac{1}{k+1}) = \Delta 1 - \Delta(\frac{1}{k+1}) = 0 - (\frac{1}{k+2} - \frac{1}{k+1}) = \frac{1}{k+1} - \frac{1}{k+2} = \frac{1}{(k+1)(k+2)}$$

There for

$$\begin{aligned} \sum_{k=0}^n H_k k^{-2} &= H_k(1 - \frac{1}{k+1}) \Big|_{k=0}^{k=n+1} - \sum_{k=0}^n (E(1 - \frac{1}{k+1})) \frac{1}{k+1} \\ &= H_{n+1}(1 - \frac{1}{n+2}) - \sum_{k=0}^n (E(1 - \frac{1}{k+1})) \frac{1}{k+1} \\ &= H_{n+1} - \frac{H_{n+1}}{n+2} - \sum_{k=0}^n (1 - \frac{1}{k+2}) \frac{1}{k+1} \\ &= H_{n+1} - \frac{H_{n+1}}{n+2} - \sum_{k=0}^n (\frac{1}{k+1} - \frac{1}{(k+1)(k+2)}) \\ &= H_{n+1} - \frac{H_{n+1}}{n+2} - (\sum_{k=0}^n \frac{1}{k+1} - \sum_{k=0}^n \frac{1}{(k+1)(k+2)}) \\ &= H_{n+1} - \frac{H_{n+1}}{n+2} - \sum_{k=0}^n k^{-1} + \sum_{k=0}^n k^{-2} \\ &= H_{n+1} - \frac{H_{n+1}}{n+2} - (H_k \Big|_{k=0}^{k=n+1}) + (\frac{k^{-1}}{-1} \Big|_{k=0}^{k=n+1}) \\ &= H_{n+1} - \frac{H_{n+1}}{n+2} - (H_{n+1} - H_0) + (\frac{(n+1)^{-1}}{-1} - \frac{0^{-1}}{-1}) \\ &= -\frac{H_{n+1}}{n+2} + H_0 - \frac{1}{n+2} + 1 \\ &= -\frac{H_{n+1} - 1}{n+2} + H_0 + 1 \end{aligned}$$

There you go

$$\sum_{k=0}^n \frac{H_k}{(k+1)(k+2)} = -\frac{H_{n+1} - 1}{n+2} + H_0 + 1$$


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