MATH 4410
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Quiz #2; Due 11:59 pm, 1/19/2024

1. Let S_n denote the maximum number of regions into which 3-dimensional space is partitioned by planes so that no three of the cutting planes are colinear and no four are concurrent (no four intersect in a point). Find a formula for S_n in terms of binomial coefficients or powers of n. *Hint/Suggestion:* Show that $\Delta S_n = P_n$, where P_n is the maximum number of regions into which 2-dimensional space is partitioned by n lines, no three concurrent.

$$\Delta S_n = P_n = \binom{n}{0} + \binom{n}{2} + \binom{n}{4} = 1 + \frac{n^{\underline{2}}}{2!} + \frac{n^{\underline{4}}}{4!} = 1 + \frac{1}{2}n^{\underline{2}} + \frac{1}{24}n^{\underline{4}}$$

$\Delta^{(1)}f(n) = 1 + \frac{1}{2}n^{\underline{2}} + \frac{1}{24}n^{\underline{4}}$

n	0	1	2	3	4	5	6	7
$\Delta^{(1)}f(n)$	1	1	2	4	8	16	31	57
$\Delta^{(2)}f(n)$	0	1	2	4	8	15	26	
$\Delta^{(3)}f(n)$	1	1	2	4	7	11		
$\Delta^{(4)}f(n)$	0	1	2	3	4			
$\Delta^{(5)}f(n)$	1	1	1	1				
$\Delta^{(6)}f(n)$	0	0	0					

$$S_n = \sum_{k=0}^{n}(1 + \frac{1}{2}k^{\underline{2}} + \frac{1}{24}k^{\underline{4}}) = \sum_{k=0}^{n} 1 + \frac{1}{2}\sum_{k=0}^{n} k^{\underline{2}} + \frac{1}{24}\sum_{k=0}^{n} k^{\underline{4}}$$

$$= (n+1) + \frac{1}{2}\left[\frac{k^{\underline{3}}}{3}\Big|_{k=0}^{k=n+1}\right] + \frac{1}{24}\left[\frac{k^{\underline{5}}}{5}\Big|_{k=0}^{k=n+1}\right]$$

$$= n + 1 + \frac{1}{6}(n+1)^{\underline{3}} + \frac{1}{120}(n+1)^{\underline{5}}$$

$$= n + 1 + \frac{1}{6}(n+1)n(n-1) + \frac{1}{120}(n+1)n(n-1)(n-2)(n-3)$$

$$= \frac{1}{120}n^5 - \frac{1}{24}n^4 + \frac{5}{24}n^3 + \frac{1}{24}n^2 + \frac{47}{60}n + 1$$

There you have it

$$S_n = \frac{1}{120}n^5 - \frac{1}{24}n^4 + \frac{5}{24}n^3 + \frac{1}{24}n^2 + \frac{47}{60}n + 1$$

Let $f(n) = \Delta^{(0)}f(n) = S_n$

n	-1	0	1	2	3	4	5	6	7
$\Delta^{(0)}f(n)$	0	1	2	4	8	16	32	63	120
$\Delta^{(1)}f(n)$	1	1	2	4	8	16	31	57	
$\Delta^{(2)}f(n)$	0	1	2	4	8	15	26		
$\Delta^{(3)}f(n)$	1	1	2	4	7	11			
$\Delta^{(4)}f(n)$	0	1	2	3	4				
$\Delta^{(5)}f(n)$	1	1	1	1					
$\Delta^{(6)}f(n)$	0	0	0						

Notice the first table is in the second table just shifted down and to the left.

2. Use the Discrete Taylor Theorem to determine a closed formula for the sum $\sum_{k=0}^{n} k^3$.

Given

$$f(n) = \sum_{k\geq 0} \Delta^{(k)} f(a) \binom{n-a}{k}$$

$$\binom{n}{k} = \frac{n^{\underline{k}}}{k!}$$

For k not equal to -1

$$\sum_{n=a}^{b} n^{\underline{k}} = \frac{n^{\underline{k+1}}}{k+1}\Bigg|_{n=a}^{n=b+1}$$

Start
Let $a = 0$

$$f(n) = \sum_{k\geq 0} \Delta^{(k)} f(0) \binom{n}{k}$$

$$k^3 = 0\binom{k}{0} + 1\binom{k}{1} + 6\binom{k}{2} + 6\binom{k}{3} + 0\binom{k}{4} + 0\binom{k}{5} + 0\binom{k}{6} + 0\binom{k}{7} + \ldots$$

$$= 1\frac{k^{\underline{1}}}{1!} + 6\frac{k^{\underline{2}}}{2!} + 6\frac{k^{\underline{3}}}{3!}$$

$$= k^{\underline{1}} + 3k^{\underline{2}} + k^{\underline{3}}$$

$$\sum_{k=0}^{n} k^3 = \sum_{k=0}^{n} (k^{\underline{1}} + 3k^{\underline{2}} + k^{\underline{3}})$$

$$= \sum_{k=0}^{n} k^{\underline{1}} + 3\sum_{k=0}^{n} k^{\underline{2}} + \sum_{k=0}^{n} k^{\underline{3}}$$

$$= \frac{k^{\underline{2}}}{2}\Bigg|_{k=0}^{k=n+1} + 3(\frac{k^{\underline{3}}}{3}\Bigg|_{k=0}^{k=n+1}) + \frac{k^{\underline{4}}}{4}\Bigg|_{k=0}^{k=n+1}$$

$$= \frac{(n+1)n}{2} + (n+1)n(n-1) + \frac{(n+1)n(n-1)(n-2)}{4}$$

$$= \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}$$

There you have it

$$\sum_{k=0}^{n} k^3 = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}$$

End

3. (a) Writing the result in terms of binomial coefficients, use the Discrete Taylor Theorem to determine a closed formula for the sum $\sum_{k=-3}^{n} k^3(k+1)(k+2)(k+3)$ using center equal to 0.

Given

$$f(n) = \sum_{k\geq 0} \Delta^{(k)} f(a) \binom{n-a}{k}$$

Let $a = 0$ as the prompt instructs

Start

$$k^3(k+1)(k+2)(k+3) = 0\binom{k}{0}+24\binom{k}{1}+432\binom{k}{2}+1872\binom{k}{3}+3264\binom{k}{4}+2520\binom{k}{5}+720\binom{k}{6}+0\binom{k}{7}+0\binom{k}{8}+\ldots$$

$$= 24\frac{k^{\underline{1}}}{1!} + 432\frac{k^{\underline{2}}}{2!} + 1872\frac{k^{\underline{3}}}{3!} + 3264\frac{k^{\underline{4}}}{4!} + 2520\frac{k^{\underline{5}}}{5!} + 720\frac{k^{\underline{6}}}{6!}$$

$$= 24k^{\underline{1}} + 432\frac{k^{\underline{2}}}{2} + 1872\frac{k^{\underline{3}}}{6} + 3264\frac{k^{\underline{4}}}{24} + 2520\frac{k^{\underline{5}}}{120} + 720\frac{k^{\underline{6}}}{720}$$

$$= 24k^{\underline{1}} + 216k^{\underline{2}} + 312k^{\underline{3}} + 136k^{\underline{4}} + 21k^{\underline{5}} + k^{\underline{6}}$$

Plugging in

$$\sum_{k=-3}^{n} k^3(k+1)(k+2)(k+3) = \sum_{k=-3}^{n} (24k^{\underline{1}} + 216k^{\underline{2}} + 312k^{\underline{3}} + 136k^{\underline{4}} + 21k^{\underline{5}} + k^{\underline{6}})$$

$$= 24\sum_{k=-3}^{n} k^{\underline{1}} + 216\sum_{k=-3}^{n} k^{\underline{2}} + 312\sum_{k=-3}^{n} k^{\underline{3}} + 136\sum_{k=-3}^{n} k^{\underline{4}} + 21\sum_{k=-3}^{n} k^{\underline{5}} + \sum_{k=-3}^{n} k^{\underline{6}}$$

Using: for k not equal to -1 $\sum_{n=a}^{b} n^{\underline{k}} = \left.\frac{n^{\underline{k+1}}}{k+1}\right|_{n=a}^{n=b+1}$

$$= 24\left(\left.\frac{k^{\underline{2}}}{2}\right|_{k=-3}^{k=n+1}\right) + 216\left(\left.\frac{k^{\underline{3}}}{3}\right|_{k=-3}^{k=n+1}\right) + 312\left(\left.\frac{k^{\underline{4}}}{4}\right|_{k=-3}^{k=n+1}\right) + 136\left(\left.\frac{k^{\underline{5}}}{5}\right|_{k=-3}^{k=n+1}\right) + 21\left(\left.\frac{k^{\underline{6}}}{6}\right|_{k=-3}^{k=n+1}\right) + \left(\left.\frac{k^{\underline{7}}}{7}\right|_{k=-3}^{k=n+1}\right)$$

$$= 24\left(\frac{(n+1)^{\underline{2}}}{2} - \frac{(-3)^{\underline{2}}}{2}\right) + 216\left(\frac{(n+1)^{\underline{3}}}{3} - \frac{(-3)^{\underline{3}}}{3}\right) + 312\left(\frac{(n+1)^{\underline{4}}}{4} - \frac{(-3)^{\underline{4}}}{4}\right) +$$
$$136\left(\frac{(n+1)^{\underline{5}}}{5} - \frac{(-3)^{\underline{5}}}{5}\right) + 21\left(\frac{(n+1)^{\underline{6}}}{6} - \frac{(-3)^{\underline{6}}}{6}\right) + \left(\frac{(n+1)^{\underline{7}}}{7} - \frac{(-3)^{\underline{7}}}{7}\right)$$

$$= 12[(n+1)n - (-3)(-4)] + 72[(n+1)n(n-1) - (-3)(-4)(-5)] +$$
$$78[(n+1)n(n-1)(n-2) - (-3)(-4)(-5)(-6)] + \frac{136}{5}[(n+1)n(n-1)(n-2)(n-3) - (-3)(-4)(-5)(-6)(-7)] +$$
$$\frac{21}{6}[(n+1)n(n-1)(n-2)(n-3)(n-4) - (-3)(-4)(-5)(-6)(-7)(-8)] +$$
$$\frac{1}{7}[(n+1)n(n-1)(n-2)(n-3)(n-4)(n-5) - (-3)(-4)(-5)(-6)(-7)(-8)(-9)]$$

$$= [72n^3 + 12n^2 - 60n + 4176] + [78n^4 - 156n^3 - 78n^2 + 156n - 28080] + [\frac{136}{5}n^5 - 136n^4 + 136n^3 + 136n^2 - \frac{816}{5}n + 68544] + [\frac{7}{2}n^6 - \frac{63}{2}n^5 + \frac{175}{2}n^4 - \frac{105}{2}n^3 - 91n^2 + 84n - 70560] + [\frac{1}{7}n^7 - 2n^6 + 10n^5 - 20n^4 + 7n^3 + 22n^2 - \frac{120}{7}n + 25920]$$

$$= 78n^4 - 84n^3 - 66n^2 + 96n - 23904 + \frac{7}{2}n^6 - \frac{43}{10}n^5 - \frac{97}{2}n^4 + \frac{167}{2}n^3 + 45n^2 - \frac{396}{5}n - 2016 + \frac{1}{7}n^7 - 2n^6 + 10n^5 - 20n^4 + 7n^3 + 22n^2 - \frac{120}{7}n + 25920$$

$$= \frac{1}{7}n^7 + \frac{3}{2}n^6 + \frac{57}{10}n^5 + \frac{19}{2}n^4 + \frac{13}{2}n^3 + n^2 - \frac{12}{35}n$$

There you have it

$$\sum_{k=-3}^{n} k^3(k+1)(k+2)(k+3) = \frac{1}{7}n^7 + \frac{3}{2}n^6 + \frac{57}{10}n^5 + \frac{19}{2}n^4 + \frac{13}{2}n^3 + n^2 - \frac{12}{35}n$$

End

(b) Repeat part (a) but with center equal to -4.

Restate the problem: Writing the result in terms of binomial coefficients, use the Discrete Taylor Theorem to determine a closed formula for the sum $\sum_{k=-3}^{n} k^3(k+1)(k+2)(k+3)$ using center equal to -4.

Given

$$f(n) = \sum_{k\geq 0} \Delta^{(k)} f(a)\binom{n-a}{k}$$

Let $a = -4$ as the prompt instructs

$$f(n) = \sum_{k\geq 0} \Delta^{(k)} f(-4)\binom{n+4}{k}$$

Start

$$\begin{aligned} k^3(k+1)(k+2)(k+3) &= 384\binom{(k+4)}{0} - 384\binom{(k+4)}{1} + 384\binom{(k+4)}{2} - \\ &\quad 384\binom{(k+4)}{3} + 384\binom{(k+4)}{4} - 360\binom{(k+4)}{5} + 720\binom{(k+4)}{6} + 0\binom{(k+4)}{7} + 0\binom{(k+4)}{8} + \ldots \\ &= 384 - 384\frac{(k+4)^{\underline{1}}}{1!} + 384\frac{(k+4)^{\underline{2}}}{2!} - 384\frac{(k+4)^{\underline{3}}}{3!} + 384\frac{(k+4)^{\underline{4}}}{4!} - 360\frac{(k+4)^{\underline{5}}}{5!} + 720\frac{(k+4)^{\underline{6}}}{6!} \\ &= 384 - 384(k+4)^{\underline{1}} + 192(k+4)^{\underline{2}} - 64(k+4)^{\underline{3}} + 16(k+4)^{\underline{4}} - 3(k+4)^{\underline{5}} + (k+4)^{\underline{6}} \end{aligned}$$

Plugging in

$$\begin{aligned} \sum_{k=-3}^{n} k^3(k+1)(k+2)(k+3) &= \\ &\sum_{k=-3}^{n} (384 - 384(k+4)^{\underline{1}} + 192(k+4)^{\underline{2}} - 64(k+4)^{\underline{3}} + 16(k+4)^{\underline{4}} - 3(k+4)^{\underline{5}} + (k+4)^{\underline{6}}) \end{aligned}$$

$$= 384\sum_{k=-3}^{n} 1 - 384\sum_{k=-3}^{n} (k+4)^{\underline{1}} + 192\sum_{k=-3}^{n} (k+4)^{\underline{2}} - 64\sum_{k=-3}^{n} (k+4)^{\underline{3}} + 16\sum_{k=-3}^{n} (k+4)^{\underline{4}} - 3\sum_{k=-3}^{n} (k+4)^{\underline{5}} + \sum_{k=-3}^{n} (k+4)^{\underline{6}}$$

$$\begin{aligned} &= 384(n+4) - 384\left(\frac{(k+4)^{\underline{2}}}{2}\Bigg|_{k=-3}^{k=n+1}\right) + 192\left(\frac{(k+4)^{\underline{3}}}{3}\Bigg|_{k=-3}^{k=n+1}\right) - 64\left(\frac{(k+4)^{\underline{4}}}{4}\Bigg|_{k=-3}^{k=n+1}\right) + \\ &\quad 16\left(\frac{(k+4)^{\underline{5}}}{5}\Bigg|_{k=-3}^{k=n+1}\right) - 3\left(\frac{(k+4)^{\underline{6}}}{6}\Bigg|_{k=-3}^{k=n+1}\right) + \left(\frac{(k+4)^{\underline{7}}}{7}\Bigg|_{k=-3}^{k=n+1}\right) \end{aligned}$$

$$\begin{aligned} &= 384[n+4] - 384\left[\frac{((n+1)+4)^{\underline{2}}}{2} - \frac{((-3)+4)^{\underline{2}}}{2}\right] + 192\left[\frac{((n+1)+4)^{\underline{3}}}{3} - \frac{((-3)+4)^{\underline{3}}}{3}\right] - \\ &\quad 64\left[\frac{((n+1)+4)^{\underline{4}}}{4} - \frac{((-3)+4)^{\underline{4}}}{4}\right] + 16\left[\frac{((n+1)+4)^{\underline{5}}}{5} - \frac{((-3)+4)^{\underline{5}}}{5}\right] - \\ &\quad 3\left[\frac{((n+1)+4)^{\underline{6}}}{6} - \frac{((-3)+4)^{\underline{6}}}{6}\right] + \left[\frac{((n+1)+4)^{\underline{7}}}{7} - \frac{((-3)+4)^{\underline{7}}}{7}\right] \end{aligned}$$

$$= 384n + 1536 - 192[(n+5)^{\underline{2}} - (1)^{\underline{2}}] + 64[(n+5)^{\underline{3}} - (1)^{\underline{3}}] - 16[(n+5)^{\underline{4}} - (1)^{\underline{4}}] + \frac{16}{5}[(n+5)^{\underline{5}} - (1)^{\underline{5}}] - \frac{1}{2}[(n+5)^{\underline{6}} - (1)^{\underline{6}}] + \frac{1}{7}[(n+5)^{\underline{7}} - (1)^{\underline{7}}]$$

$$= 384n + 1536 - 192(n+5)(n+4) + 64(n+5)(n+4)(n+3) - 16(n+5)(n+4)(n+3)(n+2) + \frac{16}{5}(n+5)(n+4)(n+3)(n+2)(n+1) - \frac{1}{2}(n+5)(n+4)(n+3)(n+2)(n+1)n + \frac{1}{7}(n+5)(n+4)(n+3)(n+2)(n+1)n(n-1)$$

$$= \frac{1}{7}n^7 + \frac{3}{2}n^6 + \frac{57}{10}n^5 + \frac{19}{2}n^4 + \frac{13}{2}n^3 + n^2 - \frac{12}{35}n$$

There you have it

$$\sum_{k=-3}^{n} k^3(k+1)(k+2)(k+3) = \frac{1}{7}n^7 + \frac{3}{2}n^6 + \frac{57}{10}n^5 + \frac{19}{2}n^4 + \frac{13}{2}n^3 + n^2 - \frac{12}{35}n$$

End

(c) The formulae from (a) and (b) should look different, but are they? Determine whether they are the same or not.

The formula do look different but the solution is the same.

Formula (a)

$$\sum_{k=-3}^{n} k^3(k+1)(k+2)(k+3) = \sum_{k=-3}^{n} (24k^{\underline{1}} + 216k^{\underline{2}} + 312k^{\underline{3}} + 136k^{\underline{4}} + 21k^{\underline{5}} + k^{\underline{6}})$$

Formula (b)

$$\sum_{k=-3}^{n} k^3(k+1)(k+2)(k+3) =$$
$$\sum_{k=-3}^{n} (384 - 384(k+4)^{\underline{1}} + 192(k+4)^{\underline{2}} - 64(k+4)^{\underline{3}} + 16(k+4)^{\underline{4}} - 3(k+4)^{\underline{5}} + (k+4)^{\underline{6}})$$

So

$$\sum_{k=-3}^{n} (24k^{\underline{1}} + 216k^{\underline{2}} + 312k^{\underline{3}} + 136k^{\underline{4}} + 21k^{\underline{5}} + k^{\underline{6}}) =$$
$$\sum_{k=-3}^{n} (384 - 384(k+4)^{\underline{1}} + 192(k+4)^{\underline{2}} - 64(k+4)^{\underline{3}} + 16(k+4)^{\underline{4}} - 3(k+4)^{\underline{5}} + (k+4)^{\underline{6}})$$

They both come out to $\frac{1}{7}n^7 + \frac{3}{2}n^6 + \frac{57}{10}n^5 + \frac{19}{2}n^4 + \frac{13}{2}n^3 + n^2 - \frac{12}{35}n$ so they are the same.

This means the coefficients for both Formula must make up for the difference of moving the center.

4. Use summation by parts to find a closed formula for $\sum_{k=0}^{n} \frac{H_k}{(k+1)(k+2)}$.

Given

$$\sum_{k=0}^{n} u(k)\Delta v(k) = u(k)v(k)|_{k=0}^{k=n+1} - \sum_{k=0}^{n} Ev(k)\Delta u(k)$$

Restate Problem

$$\sum_{k=0}^{n} H_k \frac{1}{(k+1)(k+2)} = \sum_{k=0}^{n} H_k k^{\underline{-2}}$$

Let

$$u(k) = H_k$$
$$\Delta u(k) = \frac{1}{k+1}$$

$$v(k) = \sum_{i=0}^{k-1} i^{\underline{-2}} = \left.\frac{i^{\underline{-1}}}{-1}\right|_{i=0}^{i=k} = -k^{\underline{-1}} + 0^{\underline{-1}} = 1 - \frac{1}{k+1}$$

$$\Delta v(k) = k^{\underline{-2}} = \Delta(1 - \frac{1}{k+1}) = \Delta 1 - \Delta\frac{1}{k+1}) = 0 - (\frac{1}{k+2} - \frac{1}{k+1}) = \frac{1}{k+1} - \frac{1}{k+2} = \frac{1}{(k+1)(k+2)}$$

There for

$$\sum_{k=0}^{n} H_k k^{\underline{-2}} = H_k(1 - \frac{1}{k+1})\Big|_{k=0}^{k=n+1} - \sum_{k=0}^{n}(E(1 - \frac{1}{k+1}))\frac{1}{k+1}$$
$$= H_{n+1}(1 - \frac{1}{n+2}) - \sum_{k=0}^{n}(E(1 - \frac{1}{k+1}))\frac{1}{k+1}$$
$$= H_{n+1} - \frac{H_{n+1}}{n+2} - \sum_{k=0}^{n}(1 - \frac{1}{k+2})\frac{1}{k+1}$$
$$= H_{n+1} - \frac{H_{n+1}}{n+2} - \sum_{k=0}^{n}(\frac{1}{k+1} - \frac{1}{(k+1)(k+2)})$$
$$= H_{n+1} - \frac{H_{n+1}}{n+2} - (\sum_{k=0}^{n}\frac{1}{k+1} - \sum_{k=0}^{n}\frac{1}{(k+1)(k+2)})$$
$$= H_{n+1} - \frac{H_{n+1}}{n+2} - \sum_{k=0}^{n} k^{\underline{-1}} + \sum_{k=0}^{n} k^{\underline{-2}}$$
$$= H_{n+1} - \frac{H_{n+1}}{n+2} - (H_k|_{k=0}^{k=n+1}) + (\left.\frac{k^{\underline{-1}}}{-1}\right|_{k=0}^{k=n+1})$$
$$= H_{n+1} - \frac{H_{n+1}}{n+2} - (H_{n+1} - H_0) + (\frac{(n+1)^{\underline{-1}}}{-1} - \frac{0^{\underline{-1}}}{-1})$$
$$= -\frac{H_{n+1}}{n+2} + H_0 - \frac{1}{n+2} + 1$$
$$= -\frac{H_{n+1} - 1}{n+2} + H_0 + 1$$

There you go

$$\sum_{k=0}^{n} \frac{H_k}{(k+1)(k+2)} = -\frac{H_{n+1} - 1}{n+2} + H_0 + 1$$