MATH 4410

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Quiz #3; Due 11:59 pm, 1/24/2024

1. Find a closed formula for this recurrence:

$$C_0 = 0;$$
$$C_n = n + 1 + \frac{2}{n}\sum_{k=0}^{n-1} C_k, \quad \text{for } n > 0.$$

Given

$$a_n T_n = b_n T_{n-1} + c_n$$
$$T_n = \frac{1}{s_n a_n}(s_0 a_0 T_0 + \sum_{k=1}^{n} s_k c_k)$$

(Got this idea from Katlyn in the class) Rewrite C_n so that it's in the form,

$$a_n C_n = b_n C_{n-1} + c_n$$

Find C_{n-1}

$$C_{n-1} = (n-1) + 1 + (\frac{2}{(n-1)})\sum_{k=0}^{(n-1)-1} C_k$$
$$= n + (\frac{2}{n-1})\sum_{k=0}^{n-2} C_k$$

Ok, so I need to get $n + (\frac{2}{n-1})\sum_{k=0}^{n-2} C_k$ in C_n

Start

$$C_n = n + 1 + (\frac{2}{n})\sum_{k=0}^{n-1} C_k$$

Pull the $n-1$ iteration off the summation

$$= (\frac{2}{n})C_{n-1} + 1 + n + (\frac{2}{n})\sum_{k=0}^{n-2} C_k$$

Change $\frac{2}{n}$ to $(\frac{n-1}{n})(\frac{2}{n-1})$

$$= (\frac{2}{n})C_{n-1} + 1 + n + (\frac{n-1}{n})(\frac{2}{n-1})\sum_{k=0}^{n-2} C_k$$

Add $-n + n$ to $\frac{2}{n-1}$

$$= (\frac{2}{n})C_{n-1} + 1 + n + (\frac{n-1}{n})(-n + n + \frac{2}{n-1})\sum_{k=0}^{n-2} C_k$$

Distribute

$$= (\frac{2}{n})C_{n-1} + 1 + n + \frac{n-1}{n}(-n) + \frac{n-1}{n}(n) + \frac{n-1}{n}(\frac{2}{n-1})\sum_{k=0}^{n-2} C_k$$

Simplify

$$= (\frac{2}{n})C_{n-1} + 1 + n - (n-1) + \frac{n-1}{n}(n + \frac{2}{n-1})\sum_{k=0}^{n-2} C_k$$

Bro, I can substitute $(n+\frac{2}{n-1})\sum_{k=0}^{n-2} C_k$ for C_{n-1}

$$= (\frac{2}{n})C_{n-1} + 1 + n - (n-1) + \frac{n-1}{n}C_{n-1}$$

Simplify

$$= (\frac{2}{n})C_{n-1} + \frac{n-1}{n}C_{n-1} + 2$$
$$= \frac{(2)C_{n-1}}{n} + \frac{(n-1)C_{n-1}}{n} + 2$$
$$= \frac{(2)C_{n-1} + (n-1)C_{n-1}}{n} + 2$$
$$= C_{n-1}\frac{(2+n-1)}{n} + 2$$
$$= C_{n-1}\frac{(n+1)}{n} + 2$$

Alright so

$$C_n = \frac{(n+1)}{n}C_{n-1} + 2$$

So now I can get

$$a_n = 1$$
$$b_n = \frac{(n+1)}{n}$$
$$c_n = 2$$
$$s_n = \frac{1}{\frac{(n+1)!}{n!}} = \frac{n!}{(n+1)!} = \frac{1}{n+1}$$

Multiply by s_n

$$\frac{1}{n+1}C_n = \frac{1}{n+1}\frac{(n+1)}{n}C_{n-1} + \frac{2}{n+1}$$
$$\frac{1}{n+1}C_n = \frac{1}{n}C_{n-1} + \frac{2}{n+1}$$

Summation time

$$C_n = \frac{1}{\frac{1}{n+1}}[(1)(1)(0) + \sum_{k=1}^{n}(\frac{2}{k+1})]$$
$$= (n+1)(2)(\sum_{k=1}^{n}(\frac{1}{k+1})$$

Focus in on $\sum_{k=1}^{n}\frac{1}{k+1}$

$$\sum_{k=1}^{n}\frac{1}{k+1}$$
$$= \sum_{k=1}^{n} k^{\underline{-1}}$$
$$= H_k|_{k=1}^{k=n+1}$$
$$= H_{n+1} - H_1$$
$$= H_{n+1} - 1$$

Alright back to where we came from

$$= (n+1)(2)(H_{n+1} - 1)$$
$$= 2nH_{n+1} - 2n + 2H_{n+1} - 2$$

Well there you have it

$$C_n = 2nH_{n+1} - 2n + 2H_{n+1} - 2$$

I checked with $n = 1, 2, 3$.

2. Find a closed formula for this recurrence:

$$T_0 = 5;$$
$$2T_n = nT_{n-1} + 3 \cdot n!, \quad \text{for } n > 0.$$

Given

$$T_n = \frac{1}{s_n a_n}(s_0 a_0 T_0 + \sum_{k=1}^{n} s_k c_k)$$

Find

$$a_n = 2$$
$$b_n = n$$
$$c_n = 3n!$$
$$S_n = \frac{2^{n-1}}{n!}$$

Multiply

$$\frac{2^{n-1}}{n!}(2T_n) = \frac{2^{n-1}}{n!}(nT_{n-1} + 3 \cdot n!)$$
$$\frac{2^n}{n!}T_n = \frac{2^{n-1}}{n!}nT_{n-1} + \frac{2^{n-1}}{n!}(3 \cdot n!)$$
$$\frac{2^n}{n!}T_n = \frac{2^{n-1}}{(n-1)!}T_{n-1} + 2^{n-1}(3)$$

Summation time

$$T_n = \frac{n!}{2^n}((\frac{1}{2})(2)(5) + \sum_{k=1}^{n} 3(2^{k-1}))$$
$$= \frac{n!}{2^n}(5 + 3\sum_{k=1}^{n} 2^{k-1})$$

Focus in on $\sum_{k=1}^{n} 2^{k-1}$

$$\sum_{k=1}^{n} 2^{k-1} = x = 2^0 + 2^1 + 2^2 + ... + 2^{n-1}$$
$$x = 1 + 2^1 + 2^2 + ... + 2^{n-1}$$
$$2x = 2 + 2^2 + 2^3 + ... + 2^n$$
$$2x - x = (2 + 2^2 + 2^3 + ... + 2^n) - (1 + 2^1 + 2^2 + ... + 2^{n-1})$$
$$x = 2^n - 1$$

Back to where we came from

$$= \frac{n!}{2^n}(5 + 3(2^n - 1))$$
$$= \frac{n!}{2^n}(5 + (3)2^n - 3)$$
$$= \frac{n!}{2^n}((3)2^n + 2)$$
$$= (\frac{n!}{2^n})(3)(2^n) + (\frac{n!}{2^n})(2)$$
$$= (n!)(3) + \frac{(2)(n!)}{2^n}$$
$$= (3)(n!) + \frac{n!}{2^{n-1}}$$

There you have it

$$T_n = (3)(n!) + \frac{n!}{2^{n-1}}$$

Checked for $n = 1, 2, 3$
