MATH 4410

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Quiz #4; Due 11:59 pm, 2/2/2024

1. Construct a (the) generating function for the sequence of odd positive integers; that is, construct a function $f(x)$ such that $f(x) = \sum_{n\geq 0}(2n+1)x^n$.

Given

$$\sum_{n\geq 0} x^n = \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \dots$$

$$\sum_{n\geq 0} nx^n = \frac{x}{(1-x)^2}$$

Start

$$\sum_{n\geq 0}(2n+1)x^n = 1 + 3x + 5x^2 + 7x^3 + 9x^4 + 11x^5 + \dots$$

$$= \sum_{n\geq 0}(2nx^n + x^n)$$

$$= \sum_{n\geq 0} 2nx^n + \sum_{n\geq 0} x^n$$

$$= 2\sum_{n\geq 0} nx^n + \frac{1}{1-x}$$

$$= \frac{2x}{(1-x)^2} + \frac{1}{1-x}$$

$$= \frac{2x}{(1-x)^2} + \frac{1-x}{(1-x)^2}$$

$$= \frac{2x+1-x}{(1-x)^2}$$

$$= \frac{x+1}{(1-x)^2}$$

End

$$\sum_{n\geq 0}(2n+1)x^n = \frac{x+1}{(1-x)^2}$$

2. Construct a (the) generating function $g(x)$ for the sequence $(a_n)_{n\geq 0}$, where $a_n = \sum_{i=0}^{n}(2i+1)$.

Given

$$\sum_{n\geq 0} x^n = \frac{1}{1-x}$$

$$\sum_{n\geq 0} nx^n = \frac{x}{(1-x)^2}$$

Start

$$g(x) = \sum_{n\geq 0}(\sum_{i=0}^{n}(2i+1))x^n = 1x^0 + 4x^1 + 9x^2 + 16x^3 + ...$$

$$\sum_{i=0}^{n}(2i+1) = (n+1)^2$$

$$\begin{aligned}
\sum_{n\geq 0}(\sum_{i=0}^{n}(2i+1))x^n &= \sum_{n\geq 0}(n+1)^2 x^n \\
&= \sum_{n\geq 0}(n^2+2n+1)x^n \\
&= \sum_{n\geq 0} n^2x^n + 2\sum_{n\geq 0} nx^n + \sum_{n\geq 0} x^n \\
&= \sum_{n\geq 0} n^2x^n + \frac{2x}{(1-x)^2} + \frac{1}{1-x} \\
&= \sum_{n\geq 0} n^2x^n + \frac{x+1}{(1-x)^2}
\end{aligned}$$

Focus

$$\begin{aligned}
x^n &= xx^{n-1} \\
x\frac{d}{dx}(x^n) &= xnx^{n-1} = nx^n \\
x\frac{d}{dx}(x\frac{d}{dx}(x^n)) &= n^2x^n
\end{aligned}$$

Back to it

$$\begin{aligned}
&= \sum_{n\geq 0}(x\frac{d}{dx}(x\frac{d}{dx}(x^n))) + \frac{x+1}{(1-x)^2} \\
&= x\frac{d}{dx}(x\frac{d}{dx}(\sum_{n\geq 0}(x^n))) + \frac{x+1}{(1-x)^2} \\
&= x\frac{d}{dx}(x\frac{d}{dx}(\frac{1}{1-x})) + \frac{x+1}{(1-x)^2} \\
&= \frac{x^2+x}{(1-x)^3} + \frac{x+1}{(1-x)^2} \\
&= \frac{x^2+x}{(1-x)^3} + \frac{x+1}{(1-x)^2} \\
&= \frac{x+1}{(1-x)^3}
\end{aligned}$$

End

$$g(x) = \sum_{n\geq 0}(\sum_{i=0}^{n}(2i+1))x^n = \frac{x+1}{(1-x)^3}$$

3. From $g(x)$ found above, extract a formula for the coefficient on x^n, and verify that it is indeed $\sum_{i=0}^{n}(2i+1)$.

Start

$$\frac{x+1}{(1-x)^3} = \frac{A}{(1-x)} + \frac{B}{(1-x)^2} + \frac{C}{(1-x)^3}$$

I could write down the math but ... $A = 0$, $B = -1$, $C = 2$

$$\frac{x+1}{(1-x)^3} = \frac{-1}{(1-x)^2} + \frac{2}{(1-x)^3}$$

$$= -(\frac{1}{(1-x)^2}) + 2(\frac{1}{(1-x)^3})$$

Using FGF2 to get back to the summations

$$\frac{1}{(1-x)^k} = \sum_{n\geq 0} \binom{n+k-1}{n} x^n$$

$$= -(\sum_{n\geq 0}(1+n)x^n) + 2(\sum_{n\geq 0}\frac{1}{2}(1+n)(2+n)x^n)$$

$$= -(\sum_{n\geq 0}(1+n)x^n) + (\sum_{n\geq 0}(1+n)(2+n)x^n)$$

Here is the part where I turn the summations into functions that give me a_n

$$a_n = -(1+n) + (1+n)(2+n)$$

$$a_n = n^2 + 2n + 1$$

End

Checked for n = 0, 1, 2, 3

4. Construct a generating function for $(b_n)_{n\geq 0}$, where $b_n = \sum_{i=0}^{n} i^2$, and obtain a closed formula for b_n from the generating function.

Start

$$g(x) = \sum_{n\geq 0}(\sum_{i=0}^{n}(i^2)x^n)$$

$$f(n) = \sum_{i=0}^{n}(i^2)$$

$$\Delta f(n) = (n+1)^2$$

Back to the Discrete Taylor Theorem

$$f(n) = \sum_{k\geq 0}\Delta^{(k)}f(a)\binom{n-a}{k}$$

n	0	1	2	3	4	5	6
$\Delta^{(0)}f(n)$	0	1	5	14	30	55	91
$\Delta^{(1)}f(n)$	1	4	9	16	25	36	
$\Delta^{(2)}f(n)$	3	5	7	9	11		
$\Delta^{(3)}f(n)$	2	2	2	2			
$\Delta^{(4)}f(n)$	0	0	0				

Let $a = 0$

$$f(n) = 0\binom{n}{0} + 1\binom{n}{1} + 3\binom{n}{2} + 2\binom{n}{3} + 0\binom{n}{4} + ...$$

$$f(n) = 1\binom{n}{1} + 3\binom{n}{2} + 2\binom{n}{3}$$

Let use this $\binom{n}{k} = \frac{n^{\underline{k}}}{k!}$

$$f(n) = 1\frac{n^{\underline{1}}}{1!} + 3\frac{n^{\underline{2}}}{2!} + 2\frac{n^{\underline{3}}}{3!}$$

$$f(n) = n + 3\frac{n(n-1)}{2} + 2\frac{n(n-1)(n-2)}{6}$$

$$f(n) = n + \frac{3n^2-3n}{2} + \frac{(n^2-n)(n-2)}{3}$$

$$f(n) = n + \frac{3n^2-3n}{2} + \frac{n^3-3n^2+2n}{3}$$

$$f(n) = \frac{6n}{6} + \frac{3(3n^2-3n)}{6} + \frac{2(n^3-3n^2+2n)}{6}$$

$$f(n) = \frac{6n+9n^2-9n+2n^3-6n^2+4n}{6}$$

$$f(n) = \frac{2n^3+3n^2+n}{6}$$

So

$$f(n) = \sum_{i=0}^{n}(i^2) = \frac{2n^3+3n^2+n}{6}$$

Checked for $n = 0, 1, 2, 3$
Back to the point of all this,

$$g(x) = \sum_{n\geq 0}((\frac{2n^3+3n^2+n}{6})x^n)$$

$$g(x) = \frac{1}{6}\sum_{n\geq 0}(2n^3x^n + 3n^2x^n + nx^n)$$

$$g(x) = \frac{1}{6}(2(\sum_{n\geq 0} n^3x^n) + 3(\sum_{n\geq 0} n^2x^n) + \sum_{n\geq 0} nx^n)$$

I need to find

$$\sum_{n\geq 0} n^3x^n$$

Focus

$$n^3x^n = x\frac{d}{dx}(x\frac{d}{dx}(x\frac{d}{dx}(x^n)))$$

So

$$\sum_{n\geq 0} n^3x^n = \sum_{n\geq 0}(x\frac{d}{dx}(x\frac{d}{dx}(x\frac{d}{dx}(x^n))))$$

$$\sum_{n\geq 0} n^3x^n = x\frac{d}{dx}(x\frac{d}{dx}(x\frac{d}{dx}(\sum_{n\geq 0} x^n)))$$

$$\sum_{n\geq 0} n^3x^n = \frac{(x^3+4x^2+x)}{(1-x)^4}$$

Alright

$$\sum_{n\geq 0} x^n = \frac{1}{1-x}$$

$$\sum_{n\geq 0} nx^n = \frac{x}{(1-x)^2}$$

$$\sum_{n\geq 0} n^2x^n = \frac{x^2+x}{(1-x)^3}$$

$$\sum_{n\geq 0} n^3x^n = \frac{(x^3+4x^2+x)}{(1-x)^4}$$

Back to the point

$$g(x) = \frac{1}{6}(2(\frac{(x^3+4x^2+x)}{(1-x)^4}) + 3(\frac{x^2+x}{(1-x)^3}) + \frac{x}{(1-x)^2})$$

$$g(x) = \frac{1}{6}(2(\frac{(x^3+4x^2+x)}{(1-x)^4}) + 3(\frac{(x^2+x)(1-x)}{(1-x)^3(1-x)}) + \frac{x(1-x)^2}{(1-x)^2(1-x)^2})$$

$$g(x) = \frac{1}{6}(\frac{2x^3+8x^2+2x}{(1-x)^4} + \frac{3x-3x^3}{(1-x)^4} + \frac{x^3-2x^2+x}{(1-x)^4})$$

$$g(x) = \frac{1}{6}(\frac{2x^3+8x^2+2x+3x-3x^3+x^3-2x^2+x}{(1-x)^4})$$

$$g(x) = \frac{1}{6}(\frac{6x^2+6x}{(1-x)^4})$$

$$g(x) = \frac{x^2+x}{(1-x)^4}$$

Ok so the generating function is

$$g(x) = \frac{x^2+x}{(1-x)^4}$$

Now it's time to get a closed formula for b_n out of $g(x)$
Partial fractions!

$$g(x) = \frac{x^2 + x}{(1-x)^4} = \frac{A}{(1-x)} + \frac{B}{(1-x)^2} + \frac{C}{(1-x)^3} + \frac{D}{(1-x)^4}$$

Here are the results $A = 0,\ B = 1,\ C = -3,\ D = 2$

$$g(x) = \frac{x^2 + x}{(1-x)^4} = \frac{1}{(1-x)^2} - \frac{3}{(1-x)^3} + \frac{2}{(1-x)^4}$$

$$= \frac{1}{(1-x)^2} - 3(\frac{1}{(1-x)^3}) + 2(\frac{1}{(1-x)^4})$$

Using FGF2 to get back to the summations,

$$\frac{1}{(1-x)^k} = \sum_{n \geq 0} \binom{n+k-1}{n} x^n$$

$$= \sum_{n \geq 0} ((n+1)x^n) - 3(\sum_{n \geq 0} (\frac{1}{2}(n+1)(n+2)x^n)) + 2(\sum_{n \geq 0} (\frac{1}{6}(n+1)(n+2)(n+3)x^n))$$

Here is the part where I drop the summations and x^n

$$b_n = (n+1) - \frac{3}{2}(n+1)(n+2) + \frac{1}{3}(n+1)(n+2)(n+3)$$

$$b_n = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$

And there is the closed form for b_n

$$b_n = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$

Checked for $n = 0, 1, 2, 3$
