MATH 4410

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Quiz #5; Due 11:59 pm, 2/9/2024

1. Please find closed formulae for the following recurrence relations.

   (a) $a_n = 4a_{n-1} - a_{n-2} - 6a_{n-3}$, for $n > 2$, and $a_0 = 1$, $a_1 = 4$, and $a_2 = 15$.

Start

The recurrence relation

$$a_0 = 1$$
$$a_1 = 4$$
$$a_2 = 15$$

Under the condition that $n > 2$

$$a_n = 4a_{n-1} - a_{n-2} - 6a_{n-3}$$

To make a_n work for any value of n I need to make a stipulation
Define $a_n = 0$ if $n < 0$
Define $[n = 0]$ will be 1 if true and 0 otherwise
Here is the new recurrence relation,

$$a_n = 4a_{n-1} - a_{n-2} - 6a_{n-3} + [n = 0]$$

Let's test it

$$a_0 = 4a_{-1} - a_{-2} - 6a_{-3} + [n = 0] = 1$$
$$a_1 = 4a_0 - a_{-1} - 6a_{-2} + [n = 0] = 4$$
$$a_2 = 4a_1 - a_0 - 6a_{-1} + [n = 0] = 15$$

Alright, so this works

$$a_n = 4a_{n-1} - a_{n-2} - 6a_{n-3} + [n = 0]$$

Define $A(x) = \sum_{n\geq 0} a_n x^n$

$$A(x) = \sum_{n\geq 0} a_n x^n = 4\sum_{n\geq 0} a_{n-1}x^n - \sum_{n\geq 0} a_{n-2}x^n - 6\sum_{n\geq 0} a_{n-3}x^n + \sum_{n\geq 0}[n = 0]x^n$$
$$= 4x\sum_{n\geq 0} a_{n-1}x^{n-1} - x^2\sum_{n\geq 0} a_{n-2}x^{n-2} - 6x^3\sum_{n\geq 0} a_{n-3}x^{n-3} + 1$$

I checked that I can replace $\sum_{n\geq 0} a_{n-c}x^{n-c}$ with $\sum_{n\geq 0} a_n x^n$, for $c < 0$

$$A(x) = 4xA(x) - x^2A(x) - 6x^3A(x) + 1$$
$$A(x) - 4xA(x) + x^2A(x) + 6x^3A(x) = 1$$
$$A(x)(1 - 4x + x^2 + 6x^3) = 1$$
$$A(x) = \frac{1}{(1 - 4x + x^2 + 6x^3)}$$

Partial fraction decomposition

$$A(x) = \frac{1}{(1 - 4x + x^2 + 6x^3)} = \frac{1}{(x+1)(2x-1)(3x-1)} = \frac{B}{(x+1)} + \frac{C}{(2x-1)} + \frac{D}{(3x-1)}$$

I did the math on a white board $B = \frac{1}{12}$, $C = \frac{4}{3}$, and $D = \frac{-9}{4}$

$$A(x) = \frac{1}{12}\frac{1}{(x+1)} + \frac{4}{3}\frac{1}{(2x-1)} - \frac{9}{4}\frac{1}{(3x-1)}$$

$$= \frac{1}{12}(\frac{1}{1-(-x)}) - \frac{4}{3}(\frac{1}{1-(2x)}) + \frac{9}{4}(\frac{1}{1-(3x)})$$

$$= \frac{1}{12}(\sum_{n\geq 0}(-1)^n x^n) - \frac{4}{3}(\sum_{n\geq 0}(2)^n x^n) + \frac{9}{4}(\sum_{n\geq 0}(3)^n x^n)$$

Dropping the sums, and x^n you get

$$a_n = \frac{1}{12}((-1)^n) - \frac{4}{3}((2)^n) + \frac{9}{4}((3)^n)$$

$$= \frac{(-1)^n}{12} - \frac{4(2)^n}{3} + \frac{9(3)^n}{4}$$

$$= \frac{(-1)^n}{12} - \frac{16(2)^n}{12} + \frac{27(3)^n}{12}$$

$$= \frac{(-1)^n - 16(2)^n + 27(3)^n}{12}$$

There you have it

$$a_n = \frac{(-1)^n - 16(2)^n + 27(3)^n}{12}$$

Checked with n = 0, 1, 2, 3

End

(b) $L_n = L_{n-1} + L_{n-2}$, for $n > 1$, and $L_0 = 2$, and $L_1 = 1$. (This recurrence defines the *Lucas numbers*, which are very closely related to the Fibonacci numbers.)

Start

The recurrence relation

$$L_0 = 2$$

$$L_1 = 1$$

Under the condition that $n > 1$

$$L_n = L_{n-1} + L_{n-2}$$

To allow n to be any integer
Define $[n = 0]$ will give 2 if true and 0 otherwise
Define $[n = 1]$ will give -1 if true and 0 otherwise

$$L_n = L_{n-1} + L_{n-2} + [n = 0] + [n = 1]$$

Test

$$L_0 = L_{-1} + L_{-2} + 2 + 0 = 2$$

$$L_1 = L_0 + L_{-1} + 0 + -1 = 1$$

So it works

$$L_n = L_{n-1} + L_{n-2} + [n = 0] + [n = 1]$$

Define $L(x) = \sum_{n\geq 0} L_n x^n$
Let's get the summation and x^n in there and get the generating function

$$\begin{aligned} L(x) = \sum_{n\geq 0} L_n x^n &= \sum_{n\geq 0} L_{n-1}x^n + \sum_{n\geq 0} L_{n-2}x^n + \sum_{n\geq 0}[n = 0]x^n + \sum_{n\geq 0}[n = 1]x^n \\ &= \sum_{n\geq 0} L_{n-1}x^n + \sum_{n\geq 0} L_{n-2}x^n + 2 - x \\ &= x\sum_{n\geq 0} L_{n-1}x^{n-1} + x^2\sum_{n\geq 0} L_{n-2}x^{n-2} + 2 - x \end{aligned}$$

I checked $\sum_{n\geq 0} L_{n-c}x^{n-c} = \sum_{n\geq 0} L_n x^n$ for $c < 0$

$$L(x) = xL(x) + x^2L(x) + 2 - x$$

$$L(x) - xL(x) - x^2L(x) = 2 - x$$

$$L(x)(1 - x - x^2) = 2 - x$$

$$L(x) = \frac{2 - x}{(1 - x - x^2)}$$

Alright so here is the generating function,

$$L(x) = \frac{2 - x}{(1 - x - x^2)}$$

Find the reverse polynomial of $1 - x - x^2$

$$p(x) = 1 - x - x^2$$

$$p^R(x) = x^2 - x - 1$$

The roots of $p^R(x)$ are $\frac{1-\sqrt{5}}{2}$ and $\frac{1+\sqrt{5}}{2}$
Thus

$$1-x-x^2 = (1-(\frac{1-\sqrt{5}}{2})x)(1-(\frac{1+\sqrt{5}}{2})x)$$

To clean things up, let $p = \frac{1-\sqrt{5}}{2}$ and $q = \frac{1+\sqrt{5}}{2}$

$$1-x-x^2 = (1-px)(1-qx)$$

So

$$L(x) = \frac{2-x}{(1-x-x^2)} = \frac{2-x}{(1-px)(1-qx)}$$

Partial fraction time,

$$\frac{2-x}{(1-px)(1-qx)} = \frac{A}{(1-px)} + \frac{B}{(1-qx)}$$

$$2-x = A(1-qx) + B(1-px)$$

$$2-x = A - Aqx + B - Bpx$$

Implying

$$2 = A + B$$

$$1 = Aq + Bp$$

Solve for A

$$2 - B = A$$

Plug in

$$1 = (2-B)q + Bp$$

$$1 = 2q - Bq + Bp$$

$$1 - 2q = -Bq + Bp$$

$$1 - 2q = B(-q+p)$$

$$\frac{1-2q}{(-q+p)} = B$$

$$2 - \frac{1-2q}{(-q+p)} = A$$

Check for $2 = A + B$

$$2 = (2 - \frac{1-2q}{(-q+p)}) + (\frac{1-2q}{(-q+p)})$$

$$2(-q+p) = 2(-q+p) - 1 - 2q + 1 - 2q$$

$$2(-q+p) = 2(-q+p)$$

$$2 = 2$$

It worked for the first equation!
Check for $1 = Aq + Bp$

$$1 = (2 - \frac{1-2q}{(-q+p)})q + (\frac{1-2q}{(-q+p)})p$$

$$1 = 2q - \frac{q(1-2q)}{(-q+p)} + \frac{p(1-2q)}{(-q+p)}$$

$$(-q+p) = 2q(-q+p) - q(1-2q) + p(1-2q)$$

$$-q + p = -2q^2 + 2qp - q + 2q^2 + p - 2qp$$

$$-q+p=-q+p$$

It worked for the second equation!
So

$$A = 2 - \frac{1-2q}{(-q+p)}$$

$$B = \frac{1-2q}{(-q+p)}$$

Plug in

$$L(x) = \frac{2-x}{(1-x-x^2)} = \frac{2-x}{(1-px)(1-qx)} = \frac{A}{(1-px)} + \frac{B}{(1-qx)} = \frac{(2-\frac{1-2q}{(-q+p)})}{(1-px)} + \frac{(\frac{1-2q}{(-q+p)})}{(1-qx)}$$

$$= (2-\frac{1-2q}{(-q+p)})\frac{1}{(1-px)} + (\frac{1-2q}{(-q+p)})\frac{1}{(1-qx)}$$

Changing the generating function to summations

$$= (2-\frac{1-2q}{(-q+p)})\sum_{n\geq 0}(p)^n x^n + (\frac{1-2q}{(-q+p)})\sum_{n\geq 0}(q)^n x^n$$

The indexes line up so now I can throw the summations and x^n off and get L_n

$$L_n = (2-\frac{1-2q}{(-q+p)})(p)^n + (\frac{1-2q}{(-q+p)})(q)^n$$

$$L_n = (2-\frac{1-2q}{(p-q)})(p)^n + (\frac{1-2q}{(p-q)})(q)^n$$

$$L_n = 2p^n - (p^n)\frac{1-2q}{p-q} + (q^n)\frac{1-2q}{p-q}$$

$$L_n = 2p^n + (q^n)\frac{1-2q}{p-q} - (p^n)\frac{1-2q}{p-q}$$

$$L_n = 2p^n + \frac{1-2q}{p-q}(q^n - p^n)$$

With $p = \frac{1-\sqrt{5}}{2}$ and $q = \frac{1+\sqrt{5}}{2}$ the closed formula for L_n is,

$$L_n = 2p^n + \frac{1-2q}{p-q}(q^n - p^n)$$

Checked for $n = 0, 1, 2, 3$

End

2. Use generating functions to prove the identities:

   (a) $5F_n + L_n = 2L_{n+1}$, for L_n as defined in #1 (b), and F_n is the n-th Fibonacci number.

Start

$$L_0 = 2$$
$$L_1 = 1$$

Under the condition that $n > 1$

$$L_n = L_{n-1} + L_{n-2}$$

$$F_0 = 0$$
$$F_1 = 1$$

Under the condition that $n > 1$

$$F_n = F_{n-1} + F_{n-2}$$

The generating function for these recurrences are

$$L(x) = \frac{2-x}{1-x-x^2}$$
$$F(x) = \frac{x}{1-x-x^2}$$

Prove this relation

$$5F_n + L_n = 2L_{n+1}$$

Manipulate a little

$$L_n = 2L_{n+1} - 5F_n$$
$$L_n - 2L_{n+1} = -5F_n$$
$$L_n - 2L_n - 2L_{n-1} = -5F_n$$
$$L_n(1-2) = -5F_n + 2L_{n-1}$$
$$-L_n = -5F_n + 2L_{n-1}$$
$$L_n = 5F_n - 2L_{n-1}$$

If I can get the generating function for $L_n = 5F_n - 2L_{n-1}$ and show that it's the same generating function as, $\frac{2-x}{1-x-x^2}$ then I will have proved the relation

find the generating function to

$$L_n = 5F_n - 2L_{n-1}$$

make the recurrence work for any n.
Define $n < 0$ to be 0
Define $[n = 0]$ will give 2 if true, otherwise 0

$$L_n = 5F_n - 2L_{n-1} + [n = 0]$$

Test it

$$L_0 = 5F_0 - 2L_{-1} + [n = 0] = 5(0) - 2(0) + 2 = 2$$
$$L_1 = 5F_1 - 2L_0 + [n = 0] = 5(1) - 2(2) + 0 = 5 - 4 = 1$$

It works
Define $G(x) = \sum_{n\geq 0} L_n x^n$
Multiply by the sum and x^n

$$G(x) = \sum_{n\geq 0} L_n x^n = 5\sum_{n\geq 0} F_n x^n - 2\sum_{n\geq 0} L_{n-1}x^n + \sum_{n\geq 0}[n=0]x^n$$

$$= 5\sum_{n\geq 0} F_n x^n - 2x\sum_{n\geq 0} L_{n-1}x^{n-1} + 2$$

$$G(x) = 5F(x) - 2xG(x) + 2$$

$$G(x) + 2xG(x) = 5F(x) + 2$$

$$G(x)(1+2x) = 5F(x) + 2$$

$$G(x) = \frac{5F(x)+2}{(1+2x)}$$

Alright so I need to show that $G(x) = L(x)$ with $L(x) = \frac{2-x}{1-x-x^2}$

$$G(x) = \frac{5(\frac{x}{1-x-x^2}) + 2}{(1+2x)}$$

$$G(x) = \frac{\frac{5x}{1-x-x^2} + \frac{2-2x-2x^2}{1-x-x^2}}{(1+2x)}$$

$$G(x) = \frac{\frac{2+3x-2x^2}{1-x-x^2}}{(1+2x)}$$

$$G(x) = \frac{2+3x-2x^2}{1-x-x^2}\frac{1}{(1+2x)}$$

$$G(x) = \frac{2+3x-2x^2}{(1-x-x^2)(1+2x)}$$

$$G(x) = \frac{-(2x+1)(x-2)}{(1-x-x^2)(2x+1)}$$

$$G(x) = \frac{-(x-2)}{(1-x-x^2)}$$

$$G(x) = \frac{2-x}{1-x-x^2}$$

Well

$$L(x) = \frac{2-x}{1-x-x^2}$$

$$G(x) = \frac{2-x}{1-x-x^2}$$

So they are equal

$$G(x) = \frac{2-x}{1-x-x^2} = L(x)$$

This means that $5F_n + L_n = 2L_{n+1}$ is a true relation. Because they have the same generating function. Which means that the coefficients are the same. Which means they have the same closed formulae.

End

(b) $F_n + L_n = 2F_{n+1}$, L_n and F_n as above.

Start

$$L_0 = 2$$
$$L_1 = 1$$

Under the condition that $n > 1$

$$L_n = L_{n-1} + L_{n-2}$$

$$F_0 = 0$$
$$F_1 = 1$$

Under the condition that $n > 1$

$$F_n = F_{n-1} + F_{n-2}$$

The generating function for these recurrences are

$$L(x) = \frac{2-x}{1-x-x^2}$$
$$F(x) = \frac{x}{1-x-x^2}$$

I need to prove this relation

$$F_n + L_n = 2F_{n+1}$$

Manipulate a little

$$F_n + L_n = 2F_{n+1}$$
$$F_n - 2F_{n+1} = -L_n$$
$$F_n - 2(F_n + F_{n-1}) = -L_n$$
$$F_n - 2F_n - 2F_{n-1} = -L_n$$
$$F_n - 2F_n = 2F_{n-1} - L_n$$
$$F_n(1-2) = 2F_{n-1} - L_n$$
$$-F_n = 2F_{n-1} - L_n$$
$$F_n = L_n - 2F_{n-1}$$

If I can get the generating function for $F_n = L_n - 2F_{n-1}$ and show that it's the same generating function as, $\frac{x}{1-x-x^2}$ then I will have proved the relation

find the generating function to

$$F_n = L_n - 2F_{n-1}$$

make the recurrence work for any n.
Define $n < 0$ to be 0
Define $[n = 0]$ will give -2 if true, otherwise 0

$$F_n = L_n - 2F_{n-1} + [n = 0]$$

Test it

$$F_0 = L_0 - 2F_{-1} + [n = 0] = 2 - 2(0) - 2 = 0$$

$$F_1 = L_1 - 2F_0 + [n = 0] = 1 + 2(0) + 0 = 1$$

It works
Define $H(x) = \sum_{n\geq 0} F_n x^n$
Multiply by the sum and x^n

$$H(x) = \sum_{n\geq 0} F_n x^n = \sum_{n\geq 0} L_n x^n - 2\sum_{n\geq 0} F_{n-1}x^n + \sum_{n\geq 0}[n = 0]x^n$$

$$= \sum_{n\geq 0} L_n x^n - 2x\sum_{n\geq 0} F_{n-1}x^{n-1} - 2$$

$$H(x) = L(x) - 2xH(x) - 2$$

$$H(x) + 2xH(x) = L(x) - 2$$

$$H(x)(1 + 2x) = L(x) - 2$$

$$H(x) = \frac{L(x) - 2}{1 + 2x}$$

$$H(x) = \frac{\frac{2-x}{1-x-x^2} - 2}{1 + 2x}$$

$$H(x) = \frac{\frac{2-x}{1-x-x^2} - \frac{2(1-x-x^2)}{1-x-x^2}}{1 + 2x}$$

$$H(x) = \frac{\frac{2-x}{1-x-x^2} + \frac{-2+2x+2x^2}{1-x-x^2}}{1 + 2x}$$

$$H(x) = \frac{\frac{2-x-2+2x+2x^2}{1-x-x^2}}{1 + 2x}$$

$$H(x) = \frac{2 - x - 2 + 2x + 2x^2}{(1 + 2x)(1 - x - x^2)}$$

$$H(x) = \frac{x + 2x^2}{(1 + 2x)(1 - x - x^2)}$$

$$H(x) = \frac{x(1 + 2x)}{(1 + 2x)(1 - x - x^2)}$$

$$H(x) = \frac{x}{(1 - x - x^2)}$$

Well

$$H(x) = \frac{x}{(1 - x - x^2)}$$

$$F(x) = \frac{x}{1 - x - x^2}$$

So they are equal,

$$H(x) = \frac{x}{1 - x - x^2} = F(x)$$

This means that $F_n + L_n = 2F_{n+1}$ is a true relation. Because they have the same generating function. Which means that the coefficients are the same. Which means they have the same closed formulae.

End

3. Use generating functions and a computer to determine the number of ways to make \$1.00 using pennies, nickels, dimes, quarters, and a new denomination called the *Euclid* which has value \$0.28.

p Pennies 0.01
n Nickels 0.05
d Dimes 0.10
q Quarters 0.25
u Euclid 0.28

$$1p + 5n + 10d + 25q + 28u = 100$$

Find a generating function such that $p(x)n(x)d(x)q(x)u(x)$ expanded out to the 100th term is the number of ways to make \$1.00

$$p(x) = \frac{1}{1-x} = x^0 + x^1 + x^2 + x^3 + ...$$

$$n(x) = \frac{1}{(1-x)^5} = x^0 + x^5 + x^{10} + x^{15} + ...$$

$$d(x) = \frac{1}{(1-x)^{10}} = x^0 + x^{10} + x^{20} + x^{30} + ...$$

$$q(x) = \frac{1}{(1-x)^{25}} = x^0 + x^{25} + x^{50} + x^{75} + ...$$

$$u(x) = \frac{1}{(1-x)^{28}} = x^0 + x^{28} + x^{56} + x^{84} + ...$$

Getting the generating function I am after

$$(\frac{1}{1-x})(\frac{1}{(1-x)^5})(\frac{1}{(1-x)^{10}})(\frac{1}{(1-x)^{25}})(\frac{1}{(1-x)^{28}})$$

Through that into the computer

$$taylor(1/((1-x)*(1-x^5)*(1-x^{10})*(1-x^{25})*(1-x^{28})), x, 0, 101);$$

And you get

$$... + 353 * x^{99} + 382 * x^{100} + 390 * x^{101} + ...$$

Thus there are 382 ways to make \$1.00 out of \$0.01, \$0.05, \$0.10, \$0.25, and \$0.28 denominations.
