**MATH 4410**
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**Quiz #6; Due 11:59 pm, 2/14/2024**

1. Count the number of $n$-piece fruit baskets satisfying the following totally sensible and real-world-inspired constraints:

   (a) At most 1 banana;

   (b) At most 3 guavas;

   (c) A multiple of 4 apples;

   (d) An odd number of oranges;

   (e) At least one kiwi.

Got help from recitation

$$a)1+x$$
$$b)1+x+x^2+x^3$$
$$c)\frac{1}{1-x^4}$$
$$d)\frac{x}{1-x^2}$$
$$e)\frac{x}{1-x}$$

Multiply them together

$$F(x) = (1+x)(1+x+x^2+x^3)(\frac{1}{1-x^4})(\frac{x}{1-x^2})(\frac{x}{1-x})$$
$$= \frac{(1+x)(1+x+x^2+x^3)(x^2)}{(1-x^4)(1-x^2)(1-x)}$$

Simplified with wolfram

$$= -\frac{x^2}{(x-1)^3}$$
$$= -\frac{x^2}{(-1-(-x))^3}$$
$$= \frac{x^2}{(1-x)^3}$$

Alright so

$$F(x) = \frac{x^2}{(1-x)^3}$$
$$= \sum_{n\geq 0} \binom{n+3-1}{n} x^n$$
$$= \sum_{n\geq 0} \binom{n+2}{2} x^{n+2}$$
$$= \sum_{n\geq 2} \binom{n}{2} x^n$$

The sum at 0 and 1 is 0 so its ok to move the index

$$= \sum_{n\geq 0} \binom{n}{2} x^n$$

$$= \sum_{n \geq 0} \frac{n(n-1)}{2} x^n$$

Dropping the sum and $x^n$

$$f(n) = \frac{n(n-1)}{2}$$

So the generating function containing the valid fruit basket sequence is

$$F(x) = \frac{x^2}{(1-x)^3}$$

The closed form formula is

$$f(n) = \frac{n(n-1)}{2}$$

2. count the number of alien mRNA sequences of length $n$ built from the proteins named $W$, $X$, $Y$, and $Z$ noting that there must be an even number of $W$s and an odd number of $Z$s.

Got help from recitation

$$X(x) = Y(x) = e^x = \sum_{n \geq 0} \frac{x^n}{n!}$$

$$W(x) = \frac{e^x + e^{-x}}{2} = \sum_{n \geq 0} \frac{x^{2n}}{2n!}$$

$$Z(x) = \frac{e^x - e^{-x}}{2} = \sum_{n \geq 0} \frac{x^{2n+1}}{(2n+1)!}$$

$$G(x) = X(x)Y(x)W(x)Z(x) = (e^x)(e^x)(\frac{e^x + e^{-x}}{2})(\frac{e^x - e^{-x}}{2})$$

Simplified with wolfram

$$= \frac{e^{4x}}{4} - \frac{1}{4}$$

Time to find the closed form solution

$$= -\frac{1}{4} + \frac{1}{4} \sum_{n \geq 0} \frac{4^n}{n!} x^n$$

Dropping the summation and $x^n$ and define $[n = 0]$ to give $\frac{1}{4}$ if true and 0 otherwise.

$$g(n) = \frac{1}{4}(4^n) - [n = 0]$$

$$= 4^{n-1} - [n = 0]$$

Alright so the generating function for the problem is

$$G(x) = \frac{e^{4x}}{4} - \frac{1}{4}$$

The closed formula is

$$g(n) = 4^{n-1} - [n = 0]$$

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