MATH 4410
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Quiz #7; Due 11:59 pm, 3/20/2024

What Would Euclid Not Do?

Measuring Problem. Suppose we have two irregularly-shaped and non-graduated jugs A and B with positive integer capacities a and b units, respectively. Suppose also that $a \leq b$. We must use these jugs to precisely measure N units of water, where $0 \leq N \leq b$; the desired quantity of water will be in jug B.

The goal is to prove the following theorem.

Willis-Jackson Theorem: *We can precisely measure* N *units of water with two jugs of capacities* a *and* b*, where* $0 \leq N \leq b$ *and* N *is a multiple of* d*, where* $d = \gcd(a, b)$.

The proof of the theorem will be facilitated by the following algorithm.

Willis-Jackson Algorithm. *Algorithm for Precisely Measuring Water with* A *and* B.

Step One. Fill A;

Step Two. Pour the contents of A into B and if B ever becomes full, empty it and continue pouring A into B.

Note: An iteration of the Willis-Jackson algorithm is completed at the moment when A becomes empty after being non-empty.

1. Please prove the following lemma:

   Lemma 1. *If* L *is an integer linear combination of* a *and* b*, then the coefficient on* a *can be chosen so that it is positive.*

   Start proof

$$L = ax + by$$

$$L = ax + by + ab - ab$$

$$L = ax + ab + by - ab$$

$$L = a(x + b) + b(y - a)$$

   Same logic would follow if we added $2ab - 2ab$

$$L = a(x + 2b) + b(y - 2a)$$

$$L = a(x + 3b) + b(y - 3a)$$

$$\dots$$

$$L = a(x + kb) + b(y - ka)$$

   As set up the coefficient on a will be positive if k is large enough

   End proof

While implementing the Willis-Jackson Algorithm, there are *iterations* and *states*. An iteration, as noted above, is complete when (after being filled) A is emptied (B may not be empty at the end of an iteration). A *state* is, informally, any condition the jugs A and B can be observed to be in while the Willis-Jackson algorithm is being executed. A state is concisely denoted as an ordered pair (x, y), where x is the amount of water in A and y is the amount of water in B at the moment the jugs are observed.

Below are states we may observe with arrows between them indicating what state can be had from another. Note that under the Willis-Jackson algorithm, not all the states are encountered; for example, the transitions at lines 3 and 5 never occur because the Willis-Jackson algorithm has us empty A *into* B (not into thin air), and we never fill a partially-filled A.

$(0,0) \to (a,0)$	Fill A
$(0,0) \to (0,b)$	Fill B
$(j_1,j_2) \to (0,j_2)$	A is emptied
$(j_1,j_2) \to (j_1,0)$	B is emptied
$(j_1,j_2) \to (a,j_2)$	A is filled with water already present
$(j_1,j_2) \to (j_1,b)$	B is filled with water already present
$(j_1,j_2) \to (0,j_1+j_2)$	(if $j_1+j_2 \leq b$) A is poured into B
$(j_1,j_2) \to (j_1-(b-j_2),b)$	(if $b \leq j_1+j_2$) pour what you can into B from A
$(j_1,j_2) \to (j_1+j_2,0)$	(if $j_1+j_2 \leq a$) pour B into A
$(j_1,j_2) \to (a,j_2-(a-j_1))$	(if $a \leq j_1+j_2$) pour what you can into A from B

2. Use the Principle of Mathematical induction to prove the following lemma. (You do not need to prove the corollary – it follows from the lemma.)

Lemma 2. *Let d be a positive integer, a and b the capacities of jugs A and B, and j_1 and j_2 are amounts of water found in jugs A and B at some state. If $d \mid a$ and $d \mid b$, then $d \mid j_1$ and $d \mid j_2$.*

Corollary. *$d \mid X$, where X is any amount that results from a sequence of the states and transitions described above; in particular, $d \mid X$, where X is any amount of water in jug B obtained from the Willis-Jackson Algorithm.*

Start proof

Let w, x, y, z be integers
The definition of divisibility

$$j_1 = wd$$

$$j_2 = xd$$

$$a = yd$$

$$b = zd$$

$j_1 + j_2 = wd + xd = (w + x)d$ Because I know that w and x are integers I know that the coefficient on d is an integer. This makes d a divisor of $j_1 + j_2$

$j_1 - (b - j_2) = wd - (zd - xd) = wd - zd + xd = (w - z + x)d$ Same logic as above

$j_2 - (a - j_1) = xd - (yd - wd) = xd - yd + wd = (x - y + w)d$ Same logic as above

End proof

3. Please prove the following lemma, and explain how the Willis-Jackson Theorem follows.

Lemma 3. *Suppose* $d = \gcd(a, b)$ *and that* $d \mid N$. *If* $N = ak + by$, *where* k *is a positive integer,* y *an integer, and* $0 \leq N \leq b$, *then jug* B *will contain* N *units of water after* k *iterations of the Willis-Jackson Algorithm.*

Start proof

i. Because $0 \leq N \leq b$ that means N has the same bounds as j_2.

ii. Because $d \mid N$ that means $N = ak + by$ will always have a solution where k is a positive integer, y an integer (see question 1).

iii. Because $d \mid j_2$ (see question 2) and $d \mid N$ that means j_2 and N are bound by the same set and are further restricted by numbers in that set that are also divisible by d. The math notation is something like $N, j_2 \in \{d \mid n : 0 \geq n \geq b => n \in \mathbb{Z}\}$.

iiii. Because N, and j_2 are in the same set and by the definition of the algorithm we stop when j_2 and N are equal this means that we guarantee that after k iterations j_2 and N will be the same.

iiiii. Because of i., ii., iii., and iiii. it follows that jug B will contain N units of water after k iterations of the Willis-Jackson Algorithm.

End proof