Algebra: Chapter 0
Book by Paolo Aluffi

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III. Rings and modules

Exercise 1. *(Section 1, 17) Explain in what sense $R[x]$ agrees with monoid ring $R[\mathbb{N}]$.*

Proof. We define homomorphism $f : R[x] \to R[\mathbb{N}]$ so that $f\left(\sum_{i=0}^{\infty} a_i x^i\right) = \sum_{i=0}^{\infty} a_i \cdot i$. It is easy to see that is indeed a homomorphism. Define $g : R[\mathbb{N}] \to R[x]$ as $g\left(\sum_{i=0}^{\infty} a_i i\right) = \sum_{i=0^{\infty}} a_i x^i$. Again, it is easy to see it is a homomorphism. We now prove that they are infact isomorphism. For any $\sum_{i=0}^{\infty} a_i x^i \in R[x]$, we have

$$\begin{aligned}
g \circ f\left(\sum_{i=0}^{\infty} a_i x^i\right) &= g\left(f\left(\sum_{i=0}^{\infty} a_i x^i\right)\right) \\
&= g\left(\sum_{i=0}^{\infty} a_i \cdot i\right) \\
&= \sum_{i=0}^{\infty} a_i x^i.
\end{aligned}$$

Thus, $g \circ f = 1_{R[x]}$. Similarly, we can show $f \circ g = 1_{R[\mathbb{N}]}$. Hence, $R[x] \simeq R[\mathbb{N}]$. □

Exercise 2. *(Section 2, 13) Verify that the 'componentwise' product $R_1 \times R_2$ of two rings satisfies the universal property for products in a category.*

Proof. In general, the following result holds.

Theorem. *Let I be an indexed set. Consider some family of rings $(R_i)_{i\in I}$. Then the componentwise product $\prod_{i\in I} R_i$ satisfy the ump of indexed product in* **Rings**.

Let's first define the ump of indexed product.
Unique mapping property of product of an indexed family. Let $\mathbf{C}$ be a category with binary products. The object $\prod_{i\in I} X_i$ is said to be product of indexed family $(X_i)_{i\in I}$ if there exists morphism $\pi_i : \prod_{i\in I} X_i \to X_i$ for each $i \in I$ and for any object A such that there are morphism $x_i : A \to X_i$ for each i, there exists unique arrow $f : A \to \prod_{i\in I} X_i$ with the property that $x_i = \pi_i f$ for any $i \in I$. In other, the diagram

$$\begin{tikzcd}
& A \arrow[ddl, "x_j"'] \arrow[d, dashed, "\exists! f"] \arrow[ddr, "x_k"] & \\
& \prod_{i\in I} X_i \arrow[dl, "\pi_j"] \arrow[dr, "\pi_k"'] & \\
X_j & & X_k
\end{tikzcd}$$

must commute for all $j, k \in I$.

Let $\pi_i : \prod_{i\in I} R_i \to R_i$ be the projection homo. Consider a ring S such that there are ring homo $s_i : S \to R_i$ for each i. Define $f : S \to \prod_{i\in I} R_i$ so that $x \mapsto (s_i(x))_{i\in I}$ for any $x \in S$. It is easy to see f is unique homo making

$$
\begin{tikzcd}
& S \arrow[d, dashed, "\exists! f"] \arrow[ddl, "s_j"'] \arrow[ddr, "s_k"] & \\
& \prod_{i\in I} R_i \arrow[dl, "\pi_j"] \arrow[dr, "\pi_k"'] & \\
R_j & & R_k
\end{tikzcd}
$$

commute for any $j, k \in I$. Hence, $\prod_{i\in I} R_i$ statisfy ump of product of indexed family. As an corollary of this result, we have $R_1 \times R_2$ satisfy ump of binary product. $\square$

Exercise 3. *(Section 2, 14) Verify that $\mathbb{Z}[x_1, x_2]$ satisfies the universal property for the coproducts of two copies of $\mathbb{Z}[x]$ in the category of commutative rings. Explain why it does not satisfy it in* **Rings**.

Proof. In general, we have the following result.

Theorem. *Let I be any index set and let $(\mathbb{Z}[x_i])_{i\in I}$ be indexed family of set of polynomial with integer coefficients. Then $\mathbb{Z}[x_i]_{i\in I}$ satisfy the appropriate unique mapping property of indexed family of coproduct. That is*

$$\coprod_{i\in I} \mathbb{Z}[x_i] \simeq \mathbb{Z}[x_i]_{i\in I}.$$

Before we get into the proof of it, let's first define the required unique mapping property!

Unique mapping property of coproduct of an idexed family Let $\mathbf{C}$ be a category with binary coproducts. The object $\coprod_{i\in I} X_i$ is said to be the product of indexed family $(X_i)_{i\in I}$ if there are inclusion morphism $e_i : X_i \to \coprod_{i\in I} X_i$ for each $i \in I$, and for any object A such that there are morphisms $x_i : X_i \to A$ for each i, then there exists an unique morphism $f : \coprod_{i\in I} X_i \to A$ with the property $f \circ e_i = x_i$. In other words, f make

$$
\begin{tikzcd}
& A & \\
& \coprod_{i\in I} X_i \arrow[u, dashed, "\exists! f"] & \\
X_i \arrow[uur, "x_i"] \arrow[ur, "e_i"] & & X_j \arrow[uul, "x_j"'] \arrow[ul, "e_j"']
\end{tikzcd}
$$

commute for any $i, j \in I$.
Now, we will proceed to show that $\mathbb{Z}[x_i]_{i\in I}$ posses this UMP. It is easy to see that for each i, we have inclusion $e_i : \mathbb{Z}[x_i] \to \mathbb{Z}[x_i]_{i\in I}$. Consider a ring S such that there are morphism $s_i : \mathbb{Z}[x_i] \to S$ for every i. Define $f : \mathbb{Z}[x_i]_{i\in I} \to S$ so that

$$
\begin{aligned}
f\left(\sum m_{j_i} x_i^{j_i}\right)_{i\in I} &= \sum f(m_{j_i} f(x_i)^{j_i}) \\
&= \sum k(m_{j_i}) s_i(x_i^{j_i})
\end{aligned}
$$

for any polynomial in $\mathbb{Z}[x_i]_{i\in I}$, where $k : \mathbb{Z} \to S$ is the unique ring homomorphism (as $\mathbb{Z}$ is initial in **Rings**). Note that the summation is necessarily finite. It is easy to see that f is ring morphism and also unique one at that to satisfy

$$
\begin{tikzcd}
& S & \\
& Z[x_i]_{i\in I} \arrow[u, dashed, "\exists! f"] & \\
\mathbb{Z}[x_i] \arrow[uur, "s_i"] \arrow[ur, "e_i"] & & \mathbb{Z}[x_j] \arrow[uul, "s_j"'] \arrow[ul, "e_j"']
\end{tikzcd}
$$

Hence, $\mathbb{Z}[x_i]_{i\in I}$ satisfy UMP of indexed family of coporducts of $(\mathbb{Z}[x_i])_{i\in I}$. As an easy corollary of the theorem, we have $\mathbb{Z}[x_1, x_2] \simeq \mathbb{Z}[x_1] \coprod \mathbb{Z}[x_2]$. $\square$