
ALGEBRA: CHAPTER 0

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III. Rings and modules

Exercise 1. (Section 1, 17) Explain in what sense $R[x]$ agrees with monoid ring $R[\mathbb{N}]$.

Proof. We define homomorphism $f : R[x] \rightarrow R[\mathbb{N}]$ so that $f(\sum_{i=0}^{\infty} a_i x^i) = \sum_{i=0}^{\infty} a_i \cdot i$. It is easy to see that is indeed a homomorphism. Define $g : R[\mathbb{N}] \rightarrow R[x]$ as $g(\sum_{i=0}^{\infty} a_i i) = \sum_{i=0}^{\infty} a_i x^i$. Again, it is easy to see it is a homomorphism. We now prove that they are infact isomorphism. For any $\sum_{i=0}^{\infty} a_i x^i \in R[x]$, we have

$$\begin{aligned} g \circ f \left(\sum_{i=0}^{\infty} a_i x^i \right) &= g \left(f \left(\sum_{i=0}^{\infty} a_i x^i \right) \right) \\ &= g \left(\sum_{i=0}^{\infty} a_i \cdot i \right) \\ &= \sum_{i=0}^{\infty} a_i x^i. \end{aligned}$$

Thus, $g \circ f = 1_{R[x]}$. Similarly, we can show $f \circ g = 1_{R[\mathbb{N}]}$. Hence, $R[x] \simeq R[\mathbb{N}]$. \square

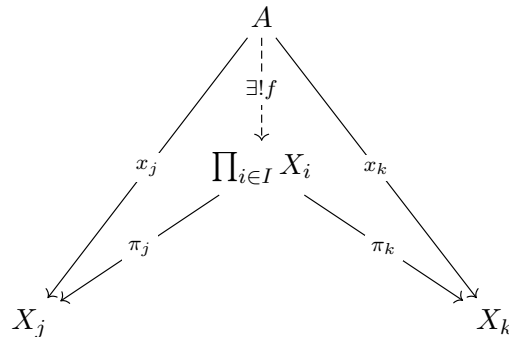
Exercise 2. (Section 2, 13) Verify that the 'componentwise' product $R_1 \times R_2$ of two rings satisfies the universal property for products in a category.

Proof. In general, the following result holds.

Theorem. Let I be an indexed set. Consider some family of rings $(R_i)_{i \in I}$. Then the componentwise product $\prod_{i \in I} R_i$ satisfy the ump of indexed product in **Rings**.

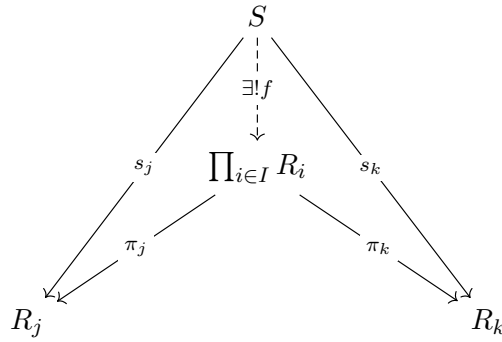
Let's first define the ump of indexed product.

Unique mapping property of product of an indexed family. Let \mathbf{C} be a category with binary products. The object $\prod_{i \in I} X_i$ is said to be product of indexed family $(X_i)_{i \in I}$ if there exists morphism $\pi_i : \prod_{i \in I} X_i \rightarrow X_i$ for each $i \in I$ and for any object A such that there are morphism $x_i : A \rightarrow X_i$ for each i , there exists unique arrow $f : A \rightarrow \prod_{i \in I} X_i$ with the property that $x_i = \pi_i f$ for any $i \in I$. In other, the diagram



must commute for all $j, k \in I$.

Let $\pi_i : \prod_{i \in I} R_i \rightarrow R_i$ be the projection homo. Consider a ring S such that there are ring homo $s_i : S \rightarrow R_i$ for each i . Define $f : S \rightarrow \prod_{i \in I} R_i$ so that $x \mapsto (s_i(x))_{i \in I}$ for any $x \in S$. It is easy to see f is unique homo making



commute for any $j, k \in I$. Hence, $\prod_{i \in I} R_i$ satisfy ump of product of indexed family. As an corollary of this result, we have $R_1 \times R_2$ satisfy ump of binary product. \square

Exercise 3. (Section 2, 14) Verify that $\mathbb{Z}[x_1, x_2]$ satisfies the universal property for the coproducts of two copies of $\mathbb{Z}[x]$ in the category of commutative rings. Explain why it does not satisfy it in **Rings**.

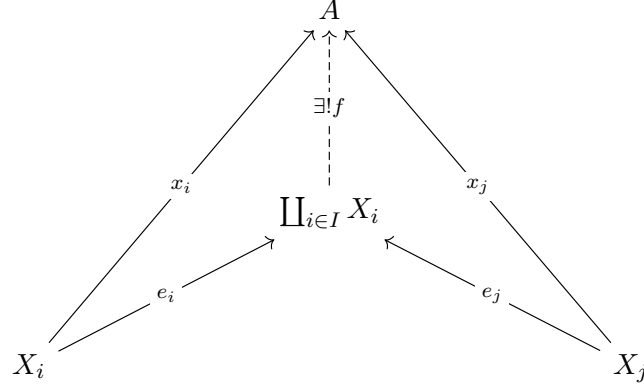
Proof. In general, we have the following result.

Theorem. Let I be any index set and let $(\mathbb{Z}[x_i])_{i \in I}$ be indexed family of set of polynomial with integer coefficients. Then $\mathbb{Z}[x_i]_{i \in I}$ satisfy the appropriate unique mapping property of indexed family of coproduct. That is

$$\coprod_{i \in I} \mathbb{Z}[x_i] \simeq \mathbb{Z}[x_i]_{i \in I}.$$

Before we get into the proof of it, let's first define the required unique mapping property!

Unique mapping property of coproduct of an indexed family Let \mathbf{C} be a category with binary coproducts. The object $\coprod_{i \in I} X_i$ is said to be the product of indexed family $(X_i)_{i \in I}$ if there are inclusion morphism $e_i : X_i \rightarrow \coprod_{i \in I} X_i$ for each $i \in I$, and for any object A such that there are morphisms $x_i : X_i \rightarrow A$ for each i , then there exists a unique morphism $f : \coprod_{i \in I} X_i \rightarrow A$ with the property $f \circ e_i = x_i$. In other words, f make

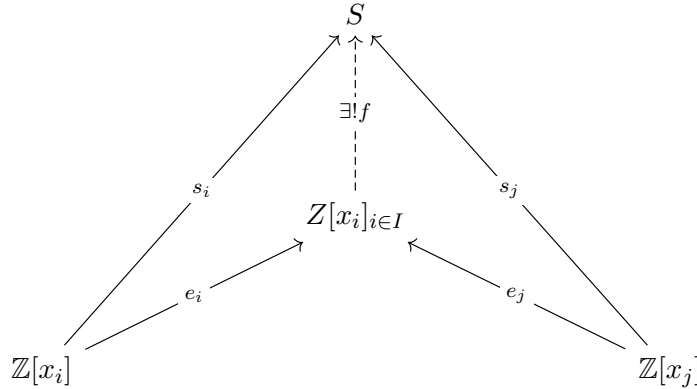


commute for any $i, j \in I$.

Now, we will proceed to show that $\mathbb{Z}[x_i]_{i \in I}$ posses this UMP. It is easy to see that for each i , we have inclusion $e_i : \mathbb{Z}[x_i] \rightarrow \mathbb{Z}[x_i]_{i \in I}$. Consider a ring S such that there are morphism $s_i : \mathbb{Z}[x_i] \rightarrow S$ for every i . Define $f : \mathbb{Z}[x_i]_{i \in I} \rightarrow S$ so that

$$\begin{aligned} f\left(\sum m_{j_i} x_i^{j_i}\right)_{i \in I} &= \sum f(m_{j_i} f(x_i)^{j_i}) \\ &= \sum k(m_{j_i}) s_i(x_i^{j_i}) \end{aligned}$$

for any polynomial in $\mathbb{Z}[x_i]_{i \in I}$, where $k : \mathbb{Z} \rightarrow S$ is the unique ring homomorphism (as \mathbb{Z} is initial in **Rings**). Note that the summation is necessarily finite. It is easy to see that f is ring morphism and also unique one at that to satisfy



Hence, $\mathbb{Z}[x_i]_{i \in I}$ satisfy UMP of indexed family of coproducts of $(\mathbb{Z}[x_i])_{i \in I}$. As an easy corollary of the theorem, we have $\mathbb{Z}[x_1, x_2] \simeq \mathbb{Z}[x_1] \coprod \mathbb{Z}[x_2]$. \square