
CATEGORY THEORY

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Chapter 1

Categories

Exercise 1.1. The object of **Rel** are sets, and an arrow $A \rightarrow B$ is a relation from A to B , that is, a subset $R \subseteq A \times B$. The equality relation $\{\langle a, a \rangle \in A \times A : a \in A\}$ is the identity arrow on a set A . Composition in **Rel** is to be given by

$$S \circ R = \{\langle a, c \rangle \in A \times C : \exists b (\langle a, b \rangle \in R \text{ and } \langle b, c \rangle \in S)\}$$

for $R \subseteq A \times B$ and $S \subseteq B \times C$.

(a) Show that **Rel** is a category.

(b) Show also that there is a functor $G : \mathbf{Sets} \rightarrow \mathbf{Rel}$ taking objects to themselves and each function $f : A \rightarrow B$ to its graph,

$$G(f) = \{\langle a, f(a) \rangle \in A \times B : a \in A\}.$$

(c) Finally, show that there is a functor $C : \mathbf{Rel}^{op} \rightarrow \mathbf{Rel}$ taking each relation $R \subseteq A \times B$ to its converse $R^c \subseteq B \times A$, where

$$\langle a, b \rangle \in R^c \leftrightarrow \langle b, a \rangle \in R.$$

Proof.

(a) First we prove that each object have identity arrow on it. Consider some set A . Let 1_A be the equality relationship on A . Let B be a set such that there is relation $R \subseteq A \times B$. Then if $(x, y) \in R \circ 1_A$, we must have $(x, y) \in R$ and if $(x, y) \in R$ then $(x, y) \in R \circ 1_A$. Therefore, $R \circ 1_A = R$. Similarly, we can prove that $1_A \circ B = B$ for some relation $B \subseteq C \times A$. Since we already know that composition of relationship is associative, it follows composition in **Rel** is associative.

(b) Since $1_A(x) = x$ for any set A , we have $G(1_A) = \{(x, 1_A(x)) : x \in A\} = \{(x, x) : x \in A\} = 1_{G(A)}$. Thus, identity is mapped to identity. Next we prove composition is preserved. Consider functions $f : A \rightarrow B$ and $g : B \rightarrow C$. We are going to prove that $G(g \circ f) = G(g) \circ G(f)$. Let $(x, g(f(x))) \in G(g \circ f)$. Then clearly $(x, f(x)) \in G(f)$ and since $f(x) \in B$, we have $(f(x), g(f(x))) \in G(g)$. Thus, $(x, g(f(x))) \in G(g) \circ G(f)$. Consider some $(x, g(f(x))) \in G(g \circ f)$. Now consider some $(x, y) \in G(g) \circ G(f)$. It follows that there exists some $z \in B$ such that $(x, z) \in G(f)$ and $(z, y) \in G(g)$. But then by the definition of the functor, it means $z = f(x)$ and $y = g(f(x))$. Thus, $(x, y) = (x, g(f(x))) \in G(g \circ f)$. Therefore, $G(g \circ f) = G(g) \circ G(f)$.

(c) Define functor $\mathcal{F} : \mathbf{Rel} \rightarrow \mathbf{Rel}^{op}$ so that $\mathcal{F}(A) = A$ for any set A and $\mathcal{F}(R) = R^c$ for any relationship (ie arrow). Since $1_A = 1_A^c$, we must have $\mathcal{F}(1_A) = 1_A = 1_{\mathcal{F}(A)}$. Now we prove composition is preserved. Clearly $\mathcal{F}(S \circ R) = (S \circ R)^c = S^c \circ R^c = \mathcal{F}(S) \circ \mathcal{F}(R)$. \square

Exercise 1.2. Consider the following isomorphisms of categories and determine which hold.

(a) $\mathbf{Rel} \cong \mathbf{Rel}^{op}$.

(b) $\mathbf{Sets} \cong \mathbf{Sets}^{op}$.

(c) For a fixed sets X with powerset $P(X)$, as poset categories $P(X) \cong P(X)^{op}$.

Proof.

(a) Define $\mathcal{F}_1 : \mathbf{Rel} \rightarrow \mathbf{Rel}^{op}$ so that $\mathcal{F}_1(A) = A$ for any set A and $\mathcal{F}_1(R) = R^c$ for any arrow $R \in \mathbf{Rel}$. Define $\mathcal{F}_2 : \mathbf{Rel}^{op} \rightarrow \mathbf{Rel}$ same way. We already proved it is a functor in last exercise. Then, since $(R^c)^c = R$,

we have $\mathcal{F}_1(\mathcal{F}_2(R)) = R$ and $\mathcal{F}_2(\mathcal{F}_1(R)) = R$. Hence, $\mathbf{Rel} \cong \mathbf{Rel}^{\text{op}}$.

(b) In **Sets**, there is an unique initial object i.e. emptyset. However, in $\mathbf{Sets}^{\text{op}}$, any singleton act as initial object ie it is only unique upto isomorphism. Therefore, **Sets** is not isomorphic to $\mathbf{Sets}^{\text{op}}$.

(c) Define functor $\mathcal{F}_1 : P(X) \rightarrow P(X)^{\text{op}}$ so that $\mathcal{F}(A) = X/A$ and $\mathcal{F}(A \subseteq B) = F(A) \supseteq F(B)$. Define $\mathcal{F}_2 : P(X)^{\text{op}} \rightarrow P(X)$ same way. Since $X/(X/A) = A$, we have $\mathcal{F}_1(\mathcal{F}_2(A)) = A$ and $\mathcal{F}_2(\mathcal{F}_1(A)) = A$. For arrows, we have

$$\begin{aligned}\mathcal{F}_1(\mathcal{F}_2(A \subseteq B)) &= \mathcal{F}_1(\mathcal{F}_2(A) \supseteq \mathcal{F}_2(B)) \\ &= F_1(\mathcal{F}_2(A)) \subseteq F_1(\mathcal{F}_2(B)) \\ &= A \subseteq B.\end{aligned}$$

Simillarly, $\mathcal{F}_2(\mathcal{F}_1(A \subseteq B)) = A \subseteq B$.

□

Exercise 1.3.

(a) Show that in **Sets**, the isomorphisms are exactly the bijections.

(b) Show that in **Monoids**, the isomorphism are exactly the bijective homomorphisms.

(c) Show that in **Posets**, the isomorphisms are not the same as the bijective homomorphisms.

Proof.

(a) First we prove the forward implication. Consider some isomorphic sets A and B . By the definition, we know that there exists $f : A \rightarrow B$ and $g : B \rightarrow A$ satisfying $g \circ f = 1_A$ and $f \circ g = 1_B$. Our goal is to prove f is bijection. Let $a_1, a_2 \in A$ and suppose $f(a_1) = f(a_2)$. Taking g both sides we have $a_1 = g(f(a_1)) = g(f(a_2)) = a_2$. Hence f is injective. Let $b \in B$. Then it follows $g(b) \in A$ but since $f(g(b)) = b$, we have f to be surjective. The converse implication is trivial.

(b) Since **Monoids** are concrete, we know objects are sets with structure and arrows are structure preserving function between the sets. It follows immediately, from previous exercises, that if M_1 and M_2 are isomorphic, we must have a bijective homomorphism $M_1 \rightarrow M_2$. Thus, we only need to prove converse implication. Suppose M_1 and M_2 are monoids such that there exists bijective homomorphism $f : M_1 \rightarrow M_2$. Note that our goal is to prove f^{-1} is a homomorphism since we already know from the previous exercise that $f \circ f^{-1} = 1_{M_2}$ and $f^{-1} \circ f = 1_{M_1}$. Consider some $a_1, a_2 \in M_2$. Since f is bijective, we know there are $b_1, b_2 \in M_1$ such that $f(b_1) = a_1$ and $f(b_2) = a_2$. We also must have $f(b_1 b_2) = f(b_1) f(b_2) = a_1 a_2$ because f is homomorphism. But that means $f^{-1}(a_1) f^{-1}(a_2) = f^{-1}(f(b_1)) f^{-1}(f(b_2)) = b_1 b_2 = f^{-1}(a_1 a_2)$.

(c) Consider posets, $P_1 = (\mathbb{Z}, |)$ and $P_2 = (\mathbb{Z}, \leq)$. It is easy to see that $id_{\mathbb{Z}}$ is order preserving bijective map since $a|b$ implies $a \leq b$. However the converse is not true, notice $2 \leq 3$ but 2 does not divide 3. □

Exercise 1.4.

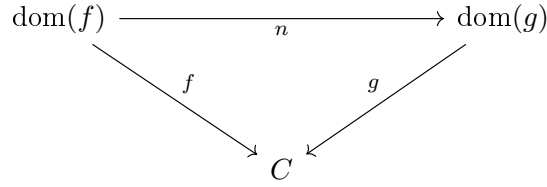
Let X be a topological space and preorder the points by specialization: $x \leq y$ iff y is contained in every open set that contains x . Show that this is a preorder, and that it is a poset if X is T_0 . Show that the ordering is trivial if X is T_1 .

Proof. First we prove that specialization relationship is preorder. It is easy to see why it is reflexive. For transitivity, consider $x \leq y$ and $y \leq z$ and Let A be arbitrary open set containing z . Since $y \leq z$, we have $y \in A$ but then since A is open set containing y , it must be the case that $x \in A$. Hence, $x \leq z$. As desired.

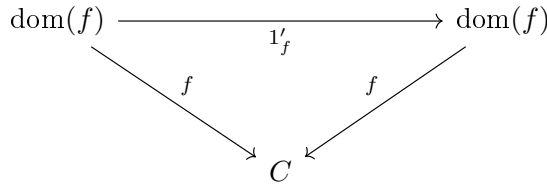
Since we already showed any topological space is preorder under specialization, we merely need to demonstrate antisymmetry in T_0 and T_1 to show they are poset. For the former, suppose $x \leq y$ in T_0 . It follows that there must exists an open set which contains y but not x , since y is contained in every open set containing x . That means T_0 has antisymmetry. In the case of T_1 , the ordering is trivial since no two distinct elements are related. □

Exercise 1.5. For any category \mathbf{C} , define a functor $\mathcal{U} : \mathbf{C}/C \rightarrow C$ from the slice category over an object C that "forgets about C ". Find a functor $F : \mathbf{C}/C \rightarrow \mathbf{C}^\rightarrow$ to the arrow category such that $\mathbf{dom} \circ F = \mathcal{U}$

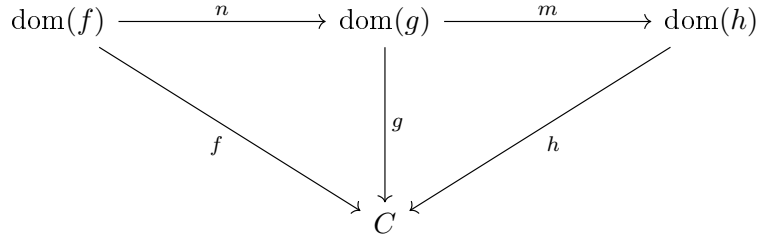
Proof. Recall that in slice category \mathbf{C}/C , an object is simply an arrow $\text{dom}(f) \xrightarrow{f} C$ and morphism $f \xrightarrow{f'} g$ is a morphism $\text{dom}(f) \xrightarrow{n} \text{dom}(g)$, in category \mathbf{C} , making



commutes. We define functor $\mathcal{U} : \mathbf{C}/C \rightarrow C$ so that $\mathcal{U}(f) = \text{dom}(f)$ for any object $f \in \text{obj}(\mathbf{C}/C)$ and $\mathcal{U}(f \xrightarrow{f'} g) = \text{dom}(f) \xrightarrow{n} \text{dom}(g)$ where n is the morphism $\text{dom}(f) \rightarrow \text{dom}(g)$ satisfying the above commutative diagram. We first check that identity is mapped to identity. Notice that $f \xrightarrow{1'_f} f$ is an arrow such that

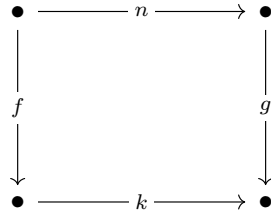


commutes i.e. $1'_f = \text{Id}_{\text{dom}(f)}$. Thus, $\mathcal{U}(f \xrightarrow{1'_f} f) = 1_{\text{dom}(f)}$ as desired. Next we prove that \mathcal{U} preserves the composition. Consider objects $\text{dom}(f) \xrightarrow{f} C$, $\text{dom}(g) \xrightarrow{g} C$, and $\text{dom}(h) \xrightarrow{h} C$ in $\text{obj}(\mathbf{C}/C)$ such that there is morphism $f' : f \rightarrow g$ and $g' : g \rightarrow h$. Let $n : \text{dom}(f) \rightarrow \text{dom}(g)$ and $m : \text{dom}(g) \rightarrow \text{dom}(h)$ making



commute. As apparent from diagram, $\mathcal{U}(g' \circ f') = m \circ n = \mathcal{U}(g') \circ \mathcal{U}(f')$.

We will now talk about constructing functor $\mathcal{F} : \mathbf{C}/C \rightarrow \mathbf{C}^\rightarrow$ and $\mathbf{dom} : \mathbf{C}^\rightarrow \rightarrow C$. Before we get into it, let us first discuss about \mathbf{C}^\rightarrow . In \mathbf{C}^\rightarrow , objects are morphisms of \mathbf{C} and morphism between $f, g \in \text{hom}(\mathbf{C})$ is $(n, k) \in \text{hom}(\mathbf{C})^2$ such that



commutes. Define $\mathcal{F} : \mathbf{C}/C \rightarrow \mathbf{C}^\rightarrow$ such that for any $f \in \text{obj}(\mathbf{C}/C)$, we have $\mathcal{F}(f) = f$ and $\mathcal{F}(f \xrightarrow{f'} g) = f \xrightarrow{(n, 1_C)} g$. It is easy to see it is a functor. Define $\mathbf{dom} : \mathbf{C}^\rightarrow \rightarrow C$ as $\mathbf{dom}(f) = \text{dom}(f)$ for any $f \in \text{obj}(\mathbf{C}^\rightarrow)$ and $\mathbf{dom}(f \xrightarrow{(n, k)} g) = \text{dom}(f) \xrightarrow{n} \text{dom}(g)$. Again, it's easy to verify that it is a functor.

Finally, we prove that $\mathbf{dom} \circ \mathcal{F} = \mathcal{U}$. For any $f \in \text{obj}(\mathbf{C}/C)$, we have $\mathbf{dom} \circ \mathcal{F}(f) = \mathbf{dom}(\mathcal{F}(f)) = \mathbf{dom}(f) = \text{dom}(f) = \mathcal{U}(f)$ and for any $f \xrightarrow{f'} g$ in \mathbf{C}/C , clearly $\mathbf{dom} \circ \mathcal{F}(f \xrightarrow{f'} g) = \mathbf{dom}(\mathcal{F}(f \xrightarrow{f'} g)) = \mathbf{dom}(f \xrightarrow{(n, 1_C)} g) = \text{dom}(f) \xrightarrow{n} \text{dom}(g) = \mathcal{U}(f \xrightarrow{f'} g)$. \square

Exercise 1.6. Construct the coslice category C/\mathbf{C} of a category \mathbf{C} from the dual category operation.

Proof.

Notation. In what follows, we refer morphisms of \mathbf{C} by f and corresponding arrow of \mathbf{C}^{op} by $f_{\mathbf{C}^{\text{op}}}$.

Let $f : C \rightarrow A$ and $g : C \rightarrow B$. Consider the slice of C in \mathbf{C}^{op} . The objects are $A \xrightarrow{f_{\mathbf{C}^{\text{op}}}} C$ and morphism $f_{\mathbf{C}^{\text{op}}} \xrightarrow{f'_{\mathbf{C}^{\text{op}}}} g_{\mathbf{C}^{\text{op}}}$ are $A \xrightarrow{k_{\mathbf{C}^{\text{op}}}} B$ of \mathbf{C}^{op} so that

$$\begin{array}{ccc} A & \xrightarrow{k_{\mathbf{C}^{\text{op}}}} & B \\ & \searrow f_{\mathbf{C}^{\text{op}}} & \swarrow g_{\mathbf{C}^{\text{op}}} \\ & C & \end{array}$$

commutes. Rewriting it in terms of \mathbf{C} , we see that \mathbf{C}^{op}/C has $C \xrightarrow{f} A$ as objects and $\text{cod}(g) \xrightarrow{k} \text{cod}(f)$ as morphism making

$$\begin{array}{ccc} A & \xleftarrow{k} & B \\ & \nwarrow f & \nearrow g \\ & C & \end{array}$$

commute. This is exactly the notion of coslice of \mathbf{C} with C . Define functor $\mathcal{W} : C/\mathbf{C} \rightarrow C$ so that $\mathcal{W}(f) = \text{cod}(f)$ and $\mathcal{W}(g \xrightarrow{f'} f) = \text{cod}(g) \xrightarrow{k} \text{cod}(f)$ where k is the morphism making

$$\begin{array}{ccc} \text{cod}(f) & \xleftarrow{k} & \text{cod}(g) \\ & \nwarrow f & \nearrow g \\ & C & \end{array}$$

Recall that in \mathbf{C}^{\rightarrow} , objects are morphisms of \mathbf{C} and morphism between $f, g \in \text{hom}(\mathbf{C})$ is $(n, k) \in \text{hom}(\mathbf{C})^2$ such that

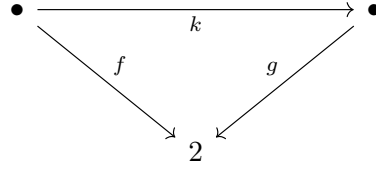
$$\begin{array}{ccc} \text{dom}(f) & \xleftarrow{n} & \text{dom}(g) \\ \downarrow f & & \downarrow g \\ \text{cod}(f) & \xleftarrow{k} & \text{cod}(g) \end{array}$$

commutes. Define functor $\mathbf{cod} : \mathbf{C}^{\rightarrow} \rightarrow \mathbf{C}$ so that $\mathbf{cod}(f) = \text{cod}(f)$ and $\mathbf{cod}(g \xrightarrow{(n,k)} f) = \text{cod}(g) \xrightarrow{k} \text{cod}(f)$ where k is the morphism between codomains making the last diagram commute. Define functor $\mathcal{G} : C/\mathbf{C} \rightarrow \mathbf{C}^{\rightarrow}$ so that $\mathcal{G}(g) = g$ for any $g \in \text{obj}(C/\mathbf{C})$ and $\mathcal{G}(g \xrightarrow{g'} f) = g \xrightarrow{(1_C, k)} f$. It is easy to verify all of them are functors.

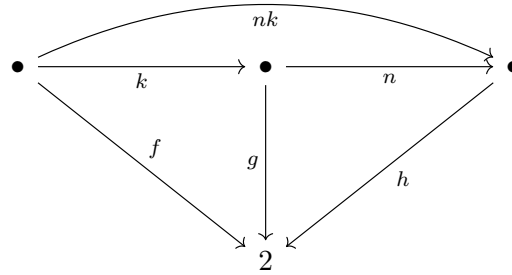
We will now prove that $\mathbf{cod} \circ \mathcal{G} = \mathcal{W}$. For any $g \in C/\mathbf{C}$, we have $\mathbf{cod} \circ \mathcal{G}(g) = \mathbf{cod}(\mathcal{G}(g)) = \mathbf{cod}(g) = \text{cod}(g) = \mathcal{W}(g)$ and for any $g \xrightarrow{g'} f$ in C/\mathbf{C} , clearly $\mathbf{cod} \circ \mathcal{G}(g \xrightarrow{g'} f) = \mathbf{cod}(\mathcal{G}(g \xrightarrow{g'} f)) = \mathbf{cod}(g \xrightarrow{(1_C, k)} f) = \text{cod}(g) \xrightarrow{k} \text{cod}(f) = \mathcal{W}(g \xrightarrow{g'} f)$. \square

Exercise 1.7. Let $2 = \{a, b\}$ be any set with exactly 2 elements a and b . Define a functor $F : \mathbf{Sets}/2 \rightarrow \mathbf{Sets} \times \mathbf{Sets}$ with $F(f : X \rightarrow 2) = (f^{-1}(a), f^{-1}(b))$. Is this an isomorphism of categories? What about the analogous situation with a one element set $1 = \{a\}$ instead of 2.

Proof. First we will discuss about **Sets**/2. Here we have arrows $X \xrightarrow{f} 2$ as objects, whereas morphism $f \xrightarrow{f'} g$ is a morphism $\text{dom}(f) \xrightarrow{k} \text{dom}(g)$ making



commute. Define $F : \mathbf{Sets}/2 \rightarrow \mathbf{Sets} \times \mathbf{Sets}$ so that $F(X \xrightarrow{f} 2) = (f^{-1}(a), f^{-1}(b))$ and $F(f \xrightarrow{f'} g) = (k_{f^{-1}(a)}, k_{f^{-1}(b)})$ where $k_{f^{-1}(a)}$ and $k_{f^{-1}(b)}$ are restriction of morphism k which makes the last diagram commute. The identity in **Sets**/2 for any object f is simply the identity on the domain of f . Therefore, we have $F(1_f) = (1_{f^{-1}(a)}, 1_{f^{-1}(b)})$ which is indeed identity on the object $(f^{-1}(a), f^{-1}(b))$. Thus, $F(1_f) = 1_{(F(f))}$. Now we will prove that composition is preserved. Consider $f \xrightarrow{f'} g$ and $g \xrightarrow{g'} h$ such that

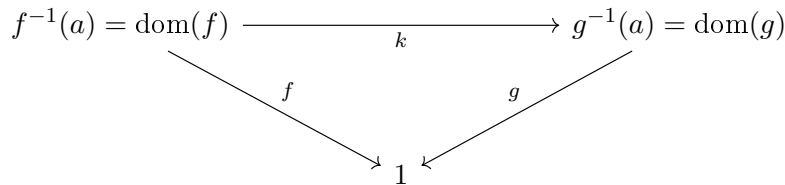


commutes. Then we have

$$\begin{aligned}
 F(g') \circ F(f') &= (n_{g^{-1}(a)}, n_{g^{-1}(b)}) \circ (k_{f^{-1}(a)}, k_{f^{-1}(b)}) \\
 &= (n_{g^{-1}(a)} \circ k_{f^{-1}(a)}, n_{g^{-1}(b)} \circ k_{f^{-1}(b)}) \\
 &= ((n \circ k)_{f^{-1}(a)}, (n \circ k)_{f^{-1}(b)}) \quad (\text{ran}(k_{f^{-1}(a)}) \subseteq g^{-1}(a) \text{ and } \text{ran}(k_{f^{-1}(b)}) \subseteq g^{-1}(b)) \\
 &= F(g' \circ f')
 \end{aligned}$$

Functor $F : \mathbf{Sets}/2 \rightarrow \mathbf{Sets}^2$ is not an isomorphism of category. Consider $A = \{c, d\}$ and $B = \{c, d, e\}$. There is no $f \in \text{obj}(\mathbf{Sets}/2)$ mapping to $(A, B) \in \text{obj}(\mathbf{Sets}^2)$ because if there was it would mean for any $x \in f^{-1}(a) = A \subseteq B = f^{-1}(b)$, we have $f(x) = a$ and $f(x) = b$ which is clearly not possible.

We now are going to prove that $\mathbf{Sets}/1 \simeq \mathbf{Sets}$ where $1 = \{a\}$ is any singleton set. Define functor $\mathcal{F} : \mathbf{Sets}/1 \rightarrow \mathbf{Sets}$ so that $\mathcal{F}(X \xrightarrow{f} 1) = f^{-1}(a)$ for any $f \in \text{obj}(\mathbf{Sets}/2)$, and $\mathcal{F}(f \xrightarrow{f'} g) = f^{-1}a \xrightarrow{k} g^{-1}(a)$ where k is an arrow making

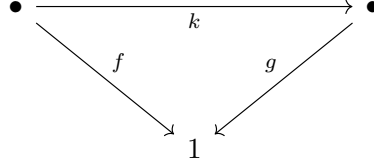


commute. Define functor $\mathcal{G} : \mathbf{Sets} \rightarrow \mathbf{Sets}/1$ so that $\mathcal{G}(A) = A \xrightarrow{f} 1$ for any $A \in \text{obj}(\mathbf{Sets})$ and $\mathcal{G}(A \xrightarrow{k} B) = f \xrightarrow{f'} g$ such that $f : A \rightarrow 1$ and $g : B \rightarrow 1$. It is easy to verify these are functor. Let $X \xrightarrow{f} 1 \in \text{obj}(\mathbf{Sets}/1)$. Then we have $\mathcal{G} \circ \mathcal{F}(f) = \mathcal{G}(\mathcal{F}(f)) = \mathcal{G}(f^{-1}(a)) = \mathcal{G}(X) = X \xrightarrow{f} 1$. Now consider some $f' : f \rightarrow g$ for some $X \xrightarrow{f} 1$ and $Y \xrightarrow{g} 1$, it follows $\mathcal{G} \circ \mathcal{F}(f') = \mathcal{G}(\mathcal{F}(f')) = \mathcal{G}(X \xrightarrow{k} Y) = f \xrightarrow{f'} g$. We know that X gets mapped to f under \mathcal{G} necessarily because it's an unique arrow $X \rightarrow 1$. Thus, $\mathcal{G} \circ \mathcal{F} = 1_{\mathbf{Sets}/1}$. Similarly, we can show $\mathcal{F} \circ \mathcal{G} = 1_{\mathbf{Sets}}$. \square

Remarks. In general, we have the following result.

Theorem. For any category \mathbf{C} having a terminal object, A , we must have $\mathbf{C}/A \simeq \mathbf{C}$.

Proof. Define functor $\mathcal{M} : \mathbf{C}/A \rightarrow \mathbf{C}$ so that $\mathcal{M}(f) = \text{dom}(f)$ for any $f \in \text{obj}(\mathbf{C}/A)$ and $\mathcal{M}(f \xrightarrow{f'} g) = \text{dom}(f) \xrightarrow{k} \text{dom}(g)$ such that



commutes. Define functor $\mathcal{N} : \mathbf{C} \rightarrow \mathbf{C}/A$ so that $\mathcal{N}(X) = X \xrightarrow{f} A$, we know such a f exists because A is terminal, and $\mathcal{N}(X \xrightarrow{k} Y) = f \xrightarrow{f'} g$ where $f : X \rightarrow A$ and $g : Y \rightarrow A$. Again, since A is terminal object, we know that such f and g are necessarily unique. Therefore, we have $\mathcal{N} \circ \mathcal{M}(f) = \mathcal{N}(\text{dom}(f)) = \mathcal{N}(\text{dom}(f)) = \text{dom}(f) \xrightarrow{f} A = f$ and $\mathcal{N}(\mathcal{M}(f \xrightarrow{f'} g)) = \mathcal{N}(\text{dom}(f) \xrightarrow{k} \text{dom}(g)) = f \xrightarrow{f'} g$. Thus, $\mathcal{N} \circ \mathcal{M} = 1_{\mathbf{C}/A}$. Similarly, we can show $\mathcal{M} \circ \mathcal{N} = 1_{\mathbf{C}}$. Hence, $\mathbf{C}/A \simeq \mathbf{C}$. \square

We now have the following result as the corollary of the theorem we just proved.

Corollary. For any singleton set $\{*\}$, $\mathbf{Sets} \simeq \mathbf{Sets}/\{*\}$.

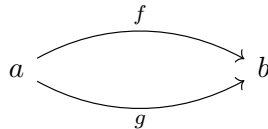
Exercise 1.8. Any category \mathbf{C} determines a preorder $P(\mathbf{C})$ by defining a binary relation \leq on the objects by

$$A \leq B \text{ if and only if there is an arrow } A \rightarrow B$$

Show that P determines a functor from categories to preorders. Also show that P is a (one-sided) inverse to the evident inclusion functor of preorder into categories (Note that in this problem, we are working in category of small categories).

Proof. Define functor $P : \mathbf{Cats} \rightarrow \mathbf{Preord}$ so that $P(\mathbf{C}) = (\text{obj}(\mathbf{C}), \leq)$, where $A \leq B$ if and only if $\text{hom}(A, B) \neq \emptyset$, and $P(\mathbf{C} \xrightarrow{\mathcal{F}} \mathbf{D}) = P(\mathbf{C}) \xrightarrow{k} P(\mathbf{D})$ where k is a function induced by \mathcal{F} i.e. $k(A) = \mathcal{F}(A) \in \text{obj}(\mathbf{D})$ for all $A \in \text{obj}(\mathbf{C})$. Notice that if $A \leq B$ then there is some $f \in \text{hom}(A, B)$ but then it means $\mathcal{F}(f) \in \text{hom}(\mathcal{F}(A), \mathcal{F}(B))$ which implies $\mathcal{F}(A) \leq \mathcal{F}(B)$. Thus, k is a monotone function. We now prove that P is a functor. Clearly, $P(\mathbf{C} \xrightarrow{1} \mathbf{C}) = P(\mathbf{C}) \xrightarrow{k} P(\mathbf{C}) = 1_{P(\mathbf{C})}$ because k is identity on $\text{obj}(\mathbf{C})$. Consider some categories C, D , and E such that $C \xrightarrow{F} D \xrightarrow{G} E$. Let $P(F) = k$ and $P(G) = m$. Then we have $P(G) \circ P(F) = m \circ k = P(G \circ F)$ because $m \circ k$ is function induced by $G \circ F$. Therefore, P is a functor. Since we already know that any preorder is a category, there must exist an inclusion functor, $I : \mathbf{Preord} \rightarrow \mathbf{cats}$, so that $I(A) = A$ and $I(f : A \rightarrow B) = A \xrightarrow{f} B$.

Notice that $P \circ I(A) = P(I(A)) = P(A) = (\text{obj}(A), \leq) = A$ and $P \circ I(A \xrightarrow{f} B) = P(I(A \xrightarrow{f} B)) = P(A \xrightarrow{f} B) = A \xrightarrow{f} B$. Hence, $P \circ I(A) = 1_{\mathbf{Preord}}$. However, P is not an isomorphism. To see why consider a small category C defined as



In this case, $P(C) = (\{a, b\}, \{(a, b)\})$. Applying I on it, we get the following category



Thus, $I(P(C)) \neq C$. \square

Remarks. The failure of isomorphism is primarily based on the fact that $P(C)$ identifies all morphism of $\text{hom}(A, B)$ as one morphism, for any object A and B . Therefore, we must have the following theorem to be true.

Theorem. Define \mathbf{Cats}_1 as the category of all categories such that $\text{hom}(A, B)$ is at most singleton for any object A and B . Then $\mathbf{Cats}_1 \simeq \mathbf{Posets}$.

Proof. Defining $P : \mathbf{Cats}_1 \rightarrow \mathbf{Preord}$ and $I : \mathbf{Preord} \rightarrow \mathbf{cats}$ same as the above proof, it's easy to see $P \circ I = 1_{\mathbf{Preord}}$. Now consider $I \circ P$. Clearly, $I \circ P(C) = I(P(C)) = C$. Notice that $P(C \xrightarrow{F} D) = C \xrightarrow{F} D$ because F is equal to the monotone function induced on C and D . Therefore, $I \circ P(C \xrightarrow{F} D) = I(P(C \xrightarrow{F} D)) = I(C \xrightarrow{F} D) = C \xrightarrow{F} D$. Hence, $\mathbf{Cats}_1 \simeq \mathbf{Posets}$. \square

Exercise 1.9. Show that the free monoid functor

$$M : \mathbf{Sets} \rightarrow \mathbf{Mon}$$

exists, in two different ways:

(a) Assuming the particular choice $M(X) = X^*$ and define its effect

$$M(f) : M(A) \rightarrow M(B)$$

on a function $f : A \rightarrow B$ to be

$$M(f)(a_1 \cdots a_k) = f(a_1) \cdots f(a_k), a_1, \dots, a_k \in A.$$

(b) Assume only the UMP of the free monoid and use it to determine M on functions, showing the result to be a functor.

Proof.

(a) First we prove identity is preserved. Consider some set A and let $w \in M(A) = A^*$ ie $w = a_1 a_2 \cdots a_k$ for $a_i \in A$. Then clearly,

$$\begin{aligned} M(1_A)(w) &= 1_A(a_1) \cdots 1_A(a_k) \\ &= a_1 \cdots a_k \\ &= w. \end{aligned}$$

Next we prove that composition is preserved. Consider $f : A \rightarrow B$ and $g : B \rightarrow C$. Then for any $w \in M(A)$, we have

$$\begin{aligned} (M(g) \circ M(f))(w) &= M(g)(f(a_1) \cdots f(a_k)) \\ &= (g \circ f)(a_1) \cdots (g \circ f)(a_k) \\ &= M(g \circ f)(w) \end{aligned}$$

(b) Consider $f : A \rightarrow B$ for some sets A and B . By proposition 1.9, we know free monoids has UMP. Applying UMP on free monoid of A and B , we see that there exists an unique monoidal homomorphism $\bar{f} : M(A) \rightarrow M(B)$ making

$$\begin{array}{ccc} |M(A)| & \xrightarrow{|\bar{f}|} & |M(B)| \\ & \nwarrow i_A \quad \nearrow I_B \circ f & \\ & A & \end{array}$$

commute in \mathbf{Sets} , where $i_A : A \rightarrow |M(A)|$ and $i_B : B \rightarrow |M(B)|$ are inclusion functions. Thus, $|\bar{f}| \circ i_A = i_B \circ f$, that is to say, for all $a \in A$, $\bar{f}(a) = f(a)$ but since \bar{f} is monoidal homomorphism, it would mean for any $a_1 \cdots a_n \in M(A)$, we must have

$$\begin{aligned} \bar{f}(a_1 \cdots a_n) &= \bar{f}(a_1) \cdots \bar{f}(a_n) \\ &= f(a_1) \cdots f(a_n) \end{aligned}$$

Define $M(f) = \bar{f}$. Since $M : \mathbf{Sets} \rightarrow \mathbf{Mon}$ so that $M(A) = A^*$ and $M(f)(a_1 \cdots a_n) = f(a_1) \cdots f(a_n)$, we can directly apply part (a) to see M is functor. \square

Exercise 1.10. Verify the UMP for free categories on graphs, defined as above with arrows being sequences of edges. Specficially, let $\mathbf{C}(G)$ be the free category on the graph G , so defined, and $i : G \rightarrow U(\mathbf{C}(G))$ the graph homomorphism taking vertices and edges to themselves, regarded as objects and arrows in $\mathbf{C}(G)$. Show that for any category \mathbf{D} and graph homomorphism $h : G \rightarrow U(\mathbf{D})$, there is a unique functor

$$\bar{h} : \mathbf{C}(G) \rightarrow \mathbf{D}$$

with

$$U(\bar{h}) \circ i = h$$

where $U : \mathbf{Cat} \rightarrow \mathbf{Graph}$ is the underlying forgetful functor.

Proof. Define $\bar{h} : \mathbf{C}(G) \rightarrow \mathbf{D}$ so that $\bar{h}(e_1 \cdots e_n) = h(e_1) \cdots h(e_n)$ for any path $e_1 \cdots e_n \in \mathbf{C}(G)$, $\bar{h}(1_a) = 1_{h(a)}$, and $\bar{h}(a) = h(a)$ for any object $a \in \text{obj}(\mathbf{C}(G))$. It is easy to see

$$\begin{array}{ccc} U(\mathbf{C}(G)) & \xrightarrow{U(\bar{h})} & U(\mathbf{D}) \\ & \swarrow i \quad \searrow h & \\ & G & \end{array}$$

commutes, by the way we defined \bar{h} . We now prove it is unique. Consider some functor $F : \mathbf{C}(G) \rightarrow \mathbf{D}$ satisfying $U(F) \circ i = h$. Then clearly

$$\begin{aligned} F(e_1 \cdots e_n) &= F(e_1) \cdots F(e_n) \\ &= h(e_1) \cdots h(e_n) \\ &= \bar{h}(e_1) \cdots \bar{h}(e_n) \\ &= \bar{h}(e_1 \cdots e_n) \end{aligned}$$

Thus, $F = \bar{h}$. \square

Remarks. We construct functor $N : \mathbf{Graphs} \rightarrow \mathbf{Cats}$. Consider some graphs G and H with $G \xrightarrow{h} H$. Applying UMP on $\mathbf{C}(G)$ and $\mathbf{C}(H)$, we see that there exists an unique functor $\bar{h} : \mathbf{C}(G) \rightarrow \mathbf{C}(H)$ such that

$$\begin{array}{ccc} U(\mathbf{C}(G)) & \xrightarrow{U(\bar{h})} & U(\mathbf{C}(H)) \\ \uparrow i_G & & \uparrow i_H \\ G & \xrightarrow{h} & H \end{array}$$

commutes in \mathbf{Graphs} . But that means for any edge e and vertex v in G , we must have $\bar{h}(v) = h(v)$ and $\bar{h}(e) = h(e)$. However, since \bar{h} is a functor, we have

$$\begin{aligned} \bar{h}(e_1 \cdots e_n) &= \bar{h}(e_1) \cdots \bar{h}(e_n) \\ &= h(e_1) \cdots h(e_n) \end{aligned}$$

Now define functor $N : \mathbf{Graphs} \rightarrow \mathbf{Cats}$ so that $N(G) = \mathbf{C}(G)$ and $N(G \xrightarrow{h} H) = \mathbf{C}(G) \xrightarrow{\bar{h}} \mathbf{C}(H)$. First we prove identity is preserved. Consider some $e_1 \cdots e_n \in \mathbf{C}(G)$, then we have

$$\begin{aligned} N(1_G)(e_1 \cdots e_n) &= 1_G(e_1) \cdots 1_G(e_n) \\ &= e_1 \cdots e_n. \end{aligned}$$

where $e_i \in G$ is an edge. Next we prove composition is preserved. Consider the graph homomorphism $h : G \rightarrow H$ and $k : H \rightarrow K$. Then

$$\begin{aligned}
 (N(k) \circ N(h))(e_1 \cdots e_n) &= N(k)(N(h)(e_1) \cdots N(h)(e_n)) \\
 &= N(k)(N(h)(e_1)) \cdots N(k)(N(h)(e_n)) \\
 &= (N(k) \circ N(h))(e_1) \cdots (N(k) \circ N(h))(e_n) \\
 &= N(k \circ h)(e_1 \cdots e_n).
 \end{aligned}$$

Exercise 1.11. Use the cayley representation to show that every small category is isomorphic to a "concrete" one, that is, one in which objects are sets and the arrows are functions between them.

Proof. Recall that cayley representation, $\bar{\mathbf{C}}$ of \mathbf{C} is a category with collection of objects as $\{\mathbf{C}/C : C \in \text{obj}(\mathbf{C})\}$ and arrows $\bar{g} : \mathbf{C}/C \rightarrow \mathbf{C}/D$, for some $g : C \rightarrow D$ in \mathbf{C} and any $f : X \rightarrow C$ in $\bar{\mathbf{C}}$ as $\bar{g} \circ f$ making

$$\begin{array}{ccc}
 X & & \\
 \downarrow f & \searrow g \circ f & \\
 C & \xrightarrow{g} & D
 \end{array}$$

commute in \mathbf{C} . It is easy to see that all objects are sets and arrows are thus set functions. We now prove $\bar{\mathbf{C}}$ is isomorphic to \mathbf{C} . Define functor $\mathcal{F} : \mathbf{C} \rightarrow \bar{\mathbf{C}}$ so that $\mathcal{F}(C) = \mathbf{C}/C$ and $\mathcal{F}(C \xrightarrow{f} D) = \mathbf{C}/C \xrightarrow{\bar{f}} \mathbf{C}/D$. It is easy to see that \mathcal{F} is bijective on objects and arrows. \square

Chapter 2

Abstract Structures

Exercise 2.1. Show that a function between sets is an epimorphism if and only if it is surjective. Conclude that the isos in **Sets** are exactly epi-monos.

Proof. The converse implication is trivial, so we only prove the forward direction. Let $f : A \rightarrow B$. Suppose f is not surjective. Then there is some $b \in B$ which is not a f -image. Define $i, j : B \rightarrow \{x, y\}$ such that $i(c) = x$ for all $c \in B$ and $j(c) = \begin{cases} x & \text{if } c \in f(A) \\ y & \text{if } c \notin f(A) \end{cases}$. This contradicts that f was epic. \square

Exercise 2.2. Show that in a poset category, all arrows are epi-monos.

Proof. Trivial. \square

Exercise 2.3. (Inverses are unique) If an arrow $f : A \rightarrow B$ has inverses $g, g' : B \rightarrow A$ (i.e., $g \circ f = 1_A$ and $f \circ g = 1_B$ and similarly for g'), then $g = g'$.

Proof. Consider the following commutative diagram.

$$\begin{array}{ccc} B & \xrightarrow{g'} & A \\ \downarrow g & \searrow 1_B & \downarrow f \\ A & \xrightarrow{f} & B \end{array}$$

But then since f is mono, $g = g'$. \square

Exercise 2.4. With regard to a commutative triangle,

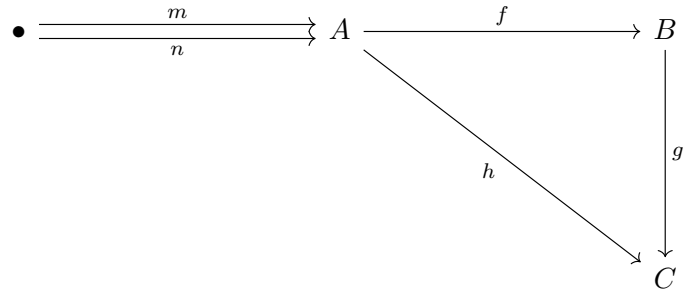
$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow h & \downarrow g \\ & & C \end{array}$$

- (a) if f and g are isos (resp. monos, resp. epis), so is h .
- (b) if h is monic then so is f .
- (c) if h is epic, so is g .

Proof.

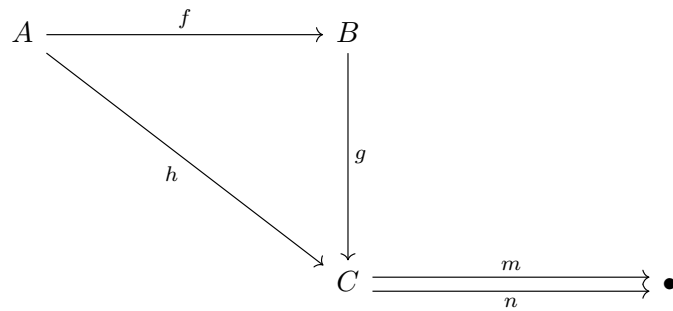
(a) If f and g are isos, then clearly $h^{-1} = f^{-1} \circ g^{-1}$. If f and g are both monos and $h \circ m = h \circ n$ then we have $g \circ (f \circ m) = g \circ (f \circ n)$ implies $f \circ m = f \circ n$ which in turns implies $m = n$ since f and g are monos. Similarly, if f and g are both epis then so is h .

(b) Consider the following commutative diagram. Consider the following commutative diagram.



Suppose $fm = fn$. Taking g both side, we see $hm = gfm = gfn = hn$, however, since h is mono, we have $m = n$.

(c) Consider the following commutative diagram.



Suppose $mg = ng$. That implies $mh = (mg)f = (ng)f = nh$ but since h is epic, we have $m = n$.

(d) Consider $f : \{1, 2\} \rightarrow \{3, 4, 5\}$ defined as $f = \{(1, 3), (2, 4)\}$ and $g : \{3, 4, 5\} \rightarrow \{6, 7\}$ as $g = \{(3, 6), (4, 7), (5, 7)\}$. Clearly, $g \circ f$ is injective while g is not. However, since in **Sets** mono is equivalent to injectivity, we are done. \square