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CATEGORY THEORY

BOOK BY STEVE AWODEY

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# Chapter 1

## Categories

**Exercise 1.1.** The object of **Rel** are sets, and an arrow  $A \rightarrow B$  is a relation from  $A$  to  $B$ , that is, a subset  $R \subseteq A \times B$ . The equality relation  $\{\langle a, a \rangle \in A \times A : a \in A\}$  is the identity arrow on a set  $A$ . Composition in **Rel** is to be given by

$$S \circ R = \{\langle a, c \rangle \in A \times C : \exists b (\langle a, b \rangle \in R \text{ and } \langle b, c \rangle \in S)\}$$

for  $R \subseteq A \times B$  and  $S \subseteq B \times C$ .

(a) Show that **Rel** is a category.

(b) Show also that there is a functor  $G : \mathbf{Sets} \rightarrow \mathbf{Rel}$  taking objects to themselves and each function  $f : A \rightarrow B$  to its graph,

$$G(f) = \{\langle a, f(a) \rangle \in A \times B : a \in A\}.$$

(c) Finally, show that there is a functor  $C : \mathbf{Rel}^{op} \rightarrow \mathbf{Rel}$  taking each relation  $R \subseteq A \times B$  to its converse  $R^c \subseteq B \times A$ , where

$$\langle a, b \rangle \in R^c \leftrightarrow \langle b, a \rangle \in R.$$

*Proof.*

(a) First we prove that each object have identity arrow on it. Consider some set  $A$ . Let  $1_A$  be the equality relationship on  $A$ . Let  $B$  be a set such that there is relation  $R \subseteq A \times B$ . Then if  $(x, y) \in R \circ 1_A$ , we must have  $(x, y) \in R$  and if  $(x, y) \in R$  then  $(x, y) \in R \circ 1_A$ . Therefore,  $R \circ 1_A = R$ . Similarly, we can prove that  $1_A \circ B = B$  for some relation  $B \subseteq C \times A$ . Since we already know that composition of relationship is associative, it follows composition in **Rel** is associative.

(b) Since  $1_A(x) = x$  for any set  $A$ , we have  $G(1_A) = \{(x, 1_A(x)) : x \in A\} = \{(x, x) : x \in A\} = 1_{G(A)}$ . Thus, identity is mapped to identity. Next we prove composition is preserved. Consider functions  $f : A \rightarrow B$  and  $g : B \rightarrow C$ . We are going to prove that  $G(g \circ f) = G(g) \circ G(f)$ . Let  $(x, g(f(x))) \in G(g \circ f)$ . Then clearly  $(x, f(x)) \in G(f)$  and since  $f(x) \in B$ , we have  $(f(x), g(f(x))) \in G(g)$ . Thus,  $(x, g(f(x))) \in G(g) \circ G(f)$ . Consider some  $(x, g(f(x))) \in G(g \circ f)$ . Now consider some  $(x, y) \in G(g) \circ G(f)$ . It follows that there exists some  $z \in B$  such that  $(x, z) \in G(f)$  and  $(z, y) \in G(g)$ . But then by the definition of the functor, it means  $z = f(x)$  and  $y = g(f(x))$ . Thus,  $(x, y) = (x, g(f(x))) \in G(g \circ f)$ . Therefore,  $G(g \circ f) = G(g) \circ G(f)$ .

(c) Define functor  $\mathcal{F} : \mathbf{Rel} \rightarrow \mathbf{Rel}^{op}$  so that  $\mathcal{F}(A) = A$  for any set  $A$  and  $\mathcal{F}(R) = R^c$  for any relationship (ie arrow). Since  $1_A = 1_A^c$ , we must have  $\mathcal{F}(1_A) = 1_A = 1_{\mathcal{F}(A)}$ . Now we prove composition is preserved. Clearly  $\mathcal{F}(S \circ R) = (S \circ R)^c = S^c \circ R^c = \mathcal{F}(S) \circ \mathcal{F}(R)$ .  $\square$

**Exercise 1.2.** Consider the following isomorphisms of categories and determine which hold.

(a)  $\mathbf{Rel} \cong \mathbf{Rel}^{op}$ .

(b)  $\mathbf{Sets} \cong \mathbf{Sets}^{op}$ .

(c) For a fixed sets  $X$  with powerset  $P(X)$ , as poset categories  $P(X) \cong P(X)^{op}$ .

*Proof.*

(a) Define  $\mathcal{F}_1 : \mathbf{Rel} \rightarrow \mathbf{Rel}^{op}$  so that  $\mathcal{F}_1(A) = A$  for any set  $A$  and  $\mathcal{F}_1(R) = R^c$  for any arrow  $R \in \mathbf{Rel}$ . Define  $\mathcal{F}_2 : \mathbf{Rel}^{op} \rightarrow \mathbf{Rel}$  same way. We already proved it is a functor in last exercise. Then, since  $(R^c)^c = R$ ,

we have  $\mathcal{F}_1(\mathcal{F}_2(R)) = R$  and  $\mathcal{F}_2(\mathcal{F}_1(R)) = R$ . Hence,  $\mathbf{Rel} \cong \mathbf{Rel}^{\text{op}}$ .

(b) In **Sets**, there is an unique initial object i.e. emptyset. However, in  $\mathbf{Sets}^{\text{op}}$ , any singleton act as initial object ie it is only unique upto isomorphism. Therefore, **Sets** is not isomorphic to  $\mathbf{Sets}^{\text{op}}$ .

(c) Define functor  $\mathcal{F}_1 : P(X) \rightarrow P(X)^{\text{op}}$  so that  $\mathcal{F}(A) = X/A$  and  $\mathcal{F}(A \subseteq B) = F(A) \supseteq F(B)$ . Define  $\mathcal{F}_2 : P(X)^{\text{op}} \rightarrow P(X)$  same way. Since  $X/(X/A) = A$ , we have  $\mathcal{F}_1(\mathcal{F}_2(A)) = A$  and  $\mathcal{F}_2(\mathcal{F}_1(A)) = A$ . For arrows, we have

$$\begin{aligned}\mathcal{F}_1(\mathcal{F}_2(A \subseteq B)) &= \mathcal{F}_1(\mathcal{F}_2(A) \supseteq \mathcal{F}_2(B)) \\ &= F_1(\mathcal{F}_2(A)) \subseteq F_1(\mathcal{F}_2(B)) \\ &= A \subseteq B.\end{aligned}$$

Simillarly,  $\mathcal{F}_2(\mathcal{F}_1(A \subseteq B)) = A \subseteq B$ .

□

### Exercise 1.3.

(a) Show that in **Sets**, the isomorphisms are exactly the bijections.

(b) Show that in **Monoids**, the isomorphism are exactly the bijective homomorphisms.

(c) Show that in **Posets**, the isomorphisms are not the same as the bijective homomorphisms.

*Proof.*

(a) First we prove the forward implication. Consider some isomorphic sets  $A$  and  $B$ . By the definition, we know that there exists  $f : A \rightarrow B$  and  $g : B \rightarrow A$  satisfying  $g \circ f = 1_A$  and  $f \circ g = 1_B$ . Our goal is to prove  $f$  is bijection. Let  $a_1, a_2 \in A$  and suppose  $f(a_1) = f(a_2)$ . Taking  $g$  both sides we have  $a_1 = g(f(a_1)) = g(f(a_2)) = a_2$ . Hence  $f$  is injective. Let  $b \in B$ . Then it follows  $g(b) \in A$  but since  $f(g(b)) = b$ , we have  $f$  to be surjective. The converse implication is trivial.

(b) Since **Monoids** are concrete, we know objects are sets with structure and arrows are structure preserving function between the sets. It follows immediately, from previous exercises, that if  $M_1$  and  $M_2$  are isomorphic, we must have a bijective homomorphism  $M_1 \rightarrow M_2$ . Thus, we only need to prove converse implication. Suppose  $M_1$  and  $M_2$  are monoids such that there exists bijective homomorphism  $f : M_1 \rightarrow M_2$ . Note that our goal is to prove  $f^{-1}$  is a homomorphism since we already know from the previous exercise that  $f \circ f^{-1} = 1_{M_2}$  and  $f^{-1} \circ f = 1_{M_1}$ . Consider some  $a_1, a_2 \in M_2$ . Since  $f$  is bijective, we know there are  $b_1, b_2 \in M_1$  such that  $f(b_1) = a_1$  and  $f(b_2) = a_2$ . We also must have  $f(b_1 b_2) = f(b_1) f(b_2) = a_1 a_2$  because  $f$  is homomorphism. But that means  $f^{-1}(a_1) f^{-1}(a_2) = f^{-1}(f(b_1)) f^{-1}(f(b_2)) = b_1 b_2 = f^{-1}(a_1 a_2)$ .

(c) Consider posets,  $P_1 = (\mathbb{Z}, |)$  and  $P_2 = (\mathbb{Z}, \leq)$ . It is easy to see that  $id_{\mathbb{Z}}$  is order preserving bijective map since  $a|b$  implies  $a \leq b$ . However the converse is not true, notice  $2 \leq 3$  but 2 does not divide 3. □

### Exercise 1.4.

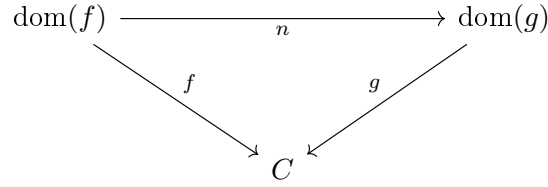
Let  $X$  be a topological space and preorder the points by specialization:  $x \leq y$  iff  $y$  is contained in every open set that contains  $x$ . Show that this is a preorder, and that it is a poset if  $X$  is  $T_0$ . Show that the ordering is trivial if  $X$  is  $T_1$ .

*Proof.* First we prove that specialization relationship is preorder. It is easy to see why it is reflexive. For transitivity, consider  $x \leq y$  and  $y \leq z$  and Let  $A$  be arbitrary open set containing  $z$ . Since  $y \leq z$ , we have  $y \in A$  but then since  $A$  is open set containing  $y$ , it must be the case that  $x \in A$ . Hence,  $x \leq z$ . As desired.

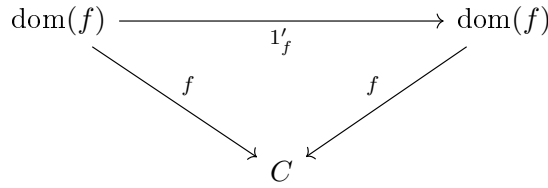
Since we already showed any topological space is preorder under specialization, we merely need to demonstrate antisymmetry in  $T_0$  and  $T_1$  to show they are poset. For the former, suppose  $x \leq y$  in  $T_0$ . It follows that there must exists an open set which contains  $y$  but not  $x$ , since  $y$  is contained in every open set containing  $x$ . That means  $T_0$  has antisymmetry. In the case of  $T_1$ , the ordering is trivial since no two distinct elements are related. □

**Exercise 1.5.** For any category  $\mathbf{C}$ , define a functor  $\mathcal{U} : \mathbf{C}/C \rightarrow C$  from the slice category over an object  $C$  that "forgets about  $C$ ". Find a functor  $F : \mathbf{C}/C \rightarrow \mathbf{C}^\rightarrow$  to the arrow category such that  $\mathbf{dom} \circ F = \mathcal{U}$

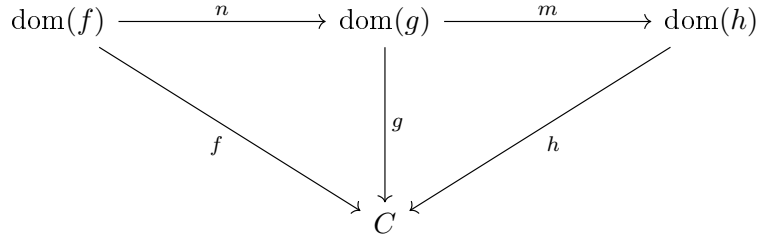
*Proof.* Recall that in slice category  $\mathbf{C}/C$ , an object is simply an arrow  $\text{dom}(f) \xrightarrow{f} C$  and morphism  $f \xrightarrow{f'} g$  is a morphism  $\text{dom}(f) \xrightarrow{n} \text{dom}(g)$ , in category  $\mathbf{C}$ , making



commutes. We define functor  $\mathcal{U} : \mathbf{C}/C \rightarrow C$  so that  $\mathcal{U}(f) = \text{dom}(f)$  for any object  $f \in \text{obj}(\mathbf{C}/C)$  and  $\mathcal{U}(f \xrightarrow{f'} g) = \text{dom}(f) \xrightarrow{n} \text{dom}(g)$  where  $n$  is the morphism  $\text{dom}(f) \rightarrow \text{dom}(g)$  satisfying the above commutative diagram. We first check that identity is mapped to identity. Notice that  $f \xrightarrow{1'_f} f$  is an arrow such that

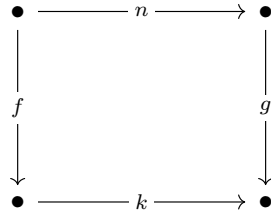


commutes i.e.  $1'_f = \text{Id}_{\text{dom}(f)}$ . Thus,  $\mathcal{U}(f \xrightarrow{1'_f} f) = 1_{\text{dom}(f)}$  as desired. Next we prove that  $\mathcal{U}$  preserves the composition. Consider objects  $\text{dom}(f) \xrightarrow{f} C$ ,  $\text{dom}(g) \xrightarrow{g} C$ , and  $\text{dom}(h) \xrightarrow{h} C$  in  $\text{obj}(\mathbf{C}/C)$  such that there is morphism  $f' : f \rightarrow g$  and  $g' : g \rightarrow h$ . Let  $n : \text{dom}(f) \rightarrow \text{dom}(g)$  and  $m : \text{dom}(g) \rightarrow \text{dom}(h)$  making



commute. As apparent from diagram,  $\mathcal{U}(g' \circ f') = m \circ n = \mathcal{U}(g') \circ \mathcal{U}(f')$ .

We will now talk about constructing functor  $\mathcal{F} : \mathbf{C}/C \rightarrow \mathbf{C}^\rightarrow$  and  $\mathbf{dom} : \mathbf{C}^\rightarrow \rightarrow C$ . Before we get into it, let us first discuss about  $\mathbf{C}^\rightarrow$ . In  $\mathbf{C}^\rightarrow$ , objects are morphisms of  $\mathbf{C}$  and morphism between  $f, g \in \text{hom}(\mathbf{C})$  is  $(n, k) \in \text{hom}(\mathbf{C})^2$  such that



commutes. Define  $\mathcal{F} : \mathbf{C}/C \rightarrow \mathbf{C}^\rightarrow$  such that for any  $f \in \text{obj}(\mathbf{C}/C)$ , we have  $\mathcal{F}(f) = f$  and  $\mathcal{F}(f \xrightarrow{f'} g) = f \xrightarrow{(n, 1_C)} g$ . It is easy to see it is a functor. Define  $\mathbf{dom} : \mathbf{C}^\rightarrow \rightarrow C$  as  $\mathbf{dom}(f) = \text{dom}(f)$  for any  $f \in \text{obj}(\mathbf{C}^\rightarrow)$  and  $\mathbf{dom}(f \xrightarrow{(n, k)} g) = \text{dom}(f) \xrightarrow{n} \text{dom}(g)$ . Again, it's easy to verify that it is a functor.

Finally, we prove that  $\mathbf{dom} \circ \mathcal{F} = \mathcal{U}$ . For any  $f \in \text{obj}(\mathbf{C}/C)$ , we have  $\mathbf{dom} \circ \mathcal{F}(f) = \mathbf{dom}(\mathcal{F}(f)) = \mathbf{dom}(f) = \text{dom}(f) = \mathcal{U}(f)$  and for any  $f \xrightarrow{f'} g$  in  $\mathbf{C}/C$ , clearly  $\mathbf{dom} \circ \mathcal{F}(f \xrightarrow{f'} g) = \mathbf{dom}(\mathcal{F}(f \xrightarrow{f'} g)) = \mathbf{dom}(f \xrightarrow{(n, 1_C)} g) = \text{dom}(f) \xrightarrow{n} \text{dom}(g) = \mathcal{U}(f \xrightarrow{f'} g)$ .  $\square$

**Exercise 1.6.** Construct the coslice category  $C/\mathbf{C}$  of a category  $\mathbf{C}$  from the dual category operation.

*Proof.*

**Notation.** In what follows, we refer morphisms of  $\mathbf{C}$  by  $f$  and corresponding arrow of  $\mathbf{C}^{\text{op}}$  by  $f_{\mathbf{C}^{\text{op}}}$ .

Let  $f : C \rightarrow A$  and  $g : C \rightarrow B$ . Consider the slice of  $C$  in  $\mathbf{C}^{\text{op}}$ . The objects are  $A \xrightarrow{f_{\mathbf{C}^{\text{op}}}} C$  and morphism  $f_{\mathbf{C}^{\text{op}}} \xrightarrow{f'_{\mathbf{C}^{\text{op}}}} g_{\mathbf{C}^{\text{op}}}$  are  $A \xrightarrow{k_{\mathbf{C}^{\text{op}}}} B$  of  $\mathbf{C}^{\text{op}}$  so that

$$\begin{array}{ccc} A & \xrightarrow{k_{\mathbf{C}^{\text{op}}}} & B \\ & \searrow f_{\mathbf{C}^{\text{op}}} & \swarrow g_{\mathbf{C}^{\text{op}}} \\ & C & \end{array}$$

commutes. Rewriting it in terms of  $\mathbf{C}$ , we see that  $\mathbf{C}^{\text{op}}/C$  has  $C \xrightarrow{f} A$  as objects and  $\text{cod}(g) \xrightarrow{k} \text{cod}(f)$  as morphism making

$$\begin{array}{ccc} A & \xleftarrow{k} & B \\ & \swarrow f & \searrow g \\ & C & \end{array}$$

commute. This is exactly the notion of coslice of  $\mathbf{C}$  with  $C$ . Define functor  $\mathcal{W} : C/\mathbf{C} \rightarrow \mathbf{C}$  so that  $\mathcal{W}(f) = \text{cod}(f)$  and  $\mathcal{W}(g \xrightarrow{f'} f) = \text{cod}(g) \xrightarrow{k} \text{cod}(f)$  where  $k$  is the morphism making

$$\begin{array}{ccc} \text{cod}(f) & \xleftarrow{k} & \text{cod}(g) \\ & \swarrow f & \searrow g \\ & C & \end{array}$$

Recall that in  $\mathbf{C}^{\rightarrow}$ , objects are morphisms of  $\mathbf{C}$  and morphism between  $f, g \in \text{hom}(\mathbf{C})$  is  $(n, k) \in \text{hom}(\mathbf{C})^2$  such that

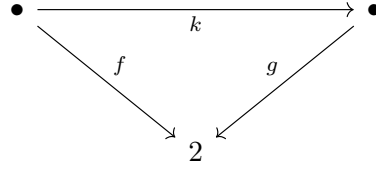
$$\begin{array}{ccc} \text{dom}(f) & \xleftarrow{n} & \text{dom}(g) \\ \downarrow f & & \downarrow g \\ \text{cod}(f) & \xleftarrow{k} & \text{cod}(g) \end{array}$$

commutes. Define functor  $\mathbf{cod} : \mathbf{C}^{\rightarrow} \rightarrow \mathbf{C}$  so that  $\mathbf{cod}(f) = \text{cod}(f)$  and  $\mathbf{cod}(g \xrightarrow{(n,k)} f) = \text{cod}(g) \xrightarrow{k} \text{cod}(f)$  where  $k$  is the morphism between codomains making the last diagram commute. Define functor  $\mathcal{G} : C/\mathbf{C} \rightarrow \mathbf{C}^{\rightarrow}$  so that  $\mathcal{G}(g) = g$  for any  $g \in \text{obj}(C/\mathbf{C})$  and  $\mathcal{G}(g \xrightarrow{g'} f) = g \xrightarrow{(1_C, k)} f$ . It is easy to verify all of them are functors.

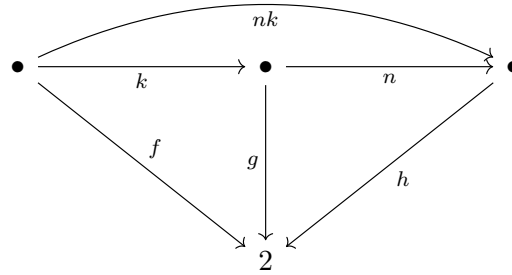
We will now prove that  $\mathbf{cod} \circ \mathcal{G} = \mathcal{W}$ . For any  $g \in C/\mathbf{C}$ , we have  $\mathbf{cod} \circ \mathcal{G}(g) = \mathbf{cod}(\mathcal{G}(g)) = \mathbf{cod}(g) = \text{cod}(g) = \mathcal{W}(g)$  and for any  $g \xrightarrow{g'} f$  in  $C/\mathbf{C}$ , clearly  $\mathbf{cod} \circ \mathcal{G}(g \xrightarrow{g'} f) = \mathbf{cod}(\mathcal{G}(g \xrightarrow{g'} f)) = \mathbf{cod}(g \xrightarrow{(1_C, k)} f) = \text{cod}(g) \xrightarrow{k} \text{cod}(f) = \mathcal{W}(g \xrightarrow{g'} f)$ .  $\square$

**Exercise 1.7.** Let  $2 = \{a, b\}$  be any set with exactly 2 elements  $a$  and  $b$ . Define a functor  $F : \mathbf{Sets}/2 \rightarrow \mathbf{Sets} \times \mathbf{Sets}$  with  $F(f : X \rightarrow 2) = (f^{-1}(a), f^{-1}(b))$ . Is this an isomorphism of categories? What about the analogous situation with a one element set  $1 = \{a\}$  instead of 2.

*Proof.* First we will discuss about **Sets**/2. Here we have arrows  $X \xrightarrow{f} 2$  as objects, whereas morphism  $f \xrightarrow{f'} g$  is a morphism  $\text{dom}(f) \xrightarrow{k} \text{dom}(g)$  making



commute. Define  $F : \mathbf{Sets}/2 \rightarrow \mathbf{Sets} \times \mathbf{Sets}$  so that  $F(X \xrightarrow{f} 2) = (f^{-1}(a), f^{-1}(b))$  and  $F(f \xrightarrow{f'} g) = (k_{f^{-1}(a)}, k_{f^{-1}(b)})$  where  $k_{f^{-1}(a)}$  and  $k_{f^{-1}(b)}$  are restriction of morphism  $k$  which makes the last diagram commute. The identity in **Sets**/2 for any object  $f$  is simply the identity on the domain of  $f$ . Therefore, we have  $F(1_f) = (1_{f^{-1}(a)}, 1_{f^{-1}(b)})$  which is indeed identity on the object  $(f^{-1}(a), f^{-1}(b))$ . Thus,  $F(1_f) = 1_{(F(f))}$ . Now we will prove that composition is preserved. Consider  $f \xrightarrow{f'} g$  and  $g \xrightarrow{g'} h$  such that

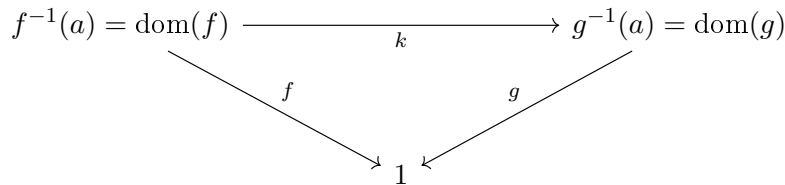


commutes. Then we have

$$\begin{aligned}
 F(g') \circ F(f') &= (n_{g^{-1}(a)}, n_{g^{-1}(b)}) \circ (k_{f^{-1}(a)}, k_{f^{-1}(b)}) \\
 &= (n_{g^{-1}(a)} \circ k_{f^{-1}(a)}, n_{g^{-1}(b)} \circ k_{f^{-1}(b)}) \\
 &= ((n \circ k)_{f^{-1}(a)}, (n \circ k)_{f^{-1}(b)}) \quad (\text{ran}(k_{f^{-1}(a)}) \subseteq g^{-1}(a) \text{ and } \text{ran}(k_{f^{-1}(b)}) \subseteq g^{-1}(b)) \\
 &= F(g' \circ f')
 \end{aligned}$$

Functor  $F : \mathbf{Sets}/2 \rightarrow \mathbf{Sets}^2$  is not an isomorphism of category. Consider  $A = \{c, d\}$  and  $B = \{c, d, e\}$ . There is no  $f \in \text{obj}(\mathbf{Sets}/2)$  mapping to  $(A, B) \in \text{obj}(\mathbf{Sets}^2)$  because if there was it would mean for any  $x \in f^{-1}(a) = A \subseteq B = f^{-1}(b)$ , we have  $f(x) = a$  and  $f(x) = b$  which is clearly not possible.

We now are going to prove that  $\mathbf{Sets}/1 \simeq \mathbf{Sets}$  where  $1 = \{a\}$  is any singleton set. Define functor  $\mathcal{F} : \mathbf{Sets}/1 \rightarrow \mathbf{Sets}$  so that  $\mathcal{F}(X \xrightarrow{f} 1) = f^{-1}(a)$  for any  $f \in \text{obj}(\mathbf{Sets}/2)$ , and  $\mathcal{F}(f \xrightarrow{f'} g) = f^{-1}a \xrightarrow{k} g^{-1}(a)$  where  $k$  is an arrow making

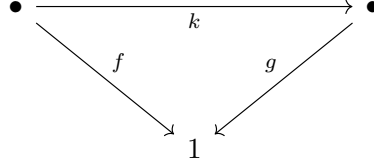


commute. Define functor  $\mathcal{G} : \mathbf{Sets} \rightarrow \mathbf{Sets}/1$  so that  $\mathcal{G}(A) = A \xrightarrow{f} 1$  for any  $A \in \text{obj}(\mathbf{Sets})$  and  $\mathcal{G}(A \xrightarrow{k} B) = f \xrightarrow{f'} g$  such that  $f : A \rightarrow 1$  and  $g : B \rightarrow 1$ . It is easy to verify these are functor. Let  $X \xrightarrow{f} 1 \in \text{obj}(\mathbf{Sets}/1)$ . Then we have  $\mathcal{G} \circ \mathcal{F}(f) = \mathcal{G}(\mathcal{F}(f)) = \mathcal{G}(f^{-1}(a)) = \mathcal{G}(X) = X \xrightarrow{f} 1$ . Now consider some  $f' : f \rightarrow g$  for some  $X \xrightarrow{f} 1$  and  $Y \xrightarrow{g} 1$ , it follows  $\mathcal{G} \circ \mathcal{F}(f') = \mathcal{G}(\mathcal{F}(f')) = \mathcal{G}(X \xrightarrow{k} Y) = f \xrightarrow{f'} g$ . We know that  $X$  gets mapped to  $f$  under  $\mathcal{G}$  necessarily because it's an unique arrow  $X \rightarrow 1$ . Thus,  $\mathcal{G} \circ \mathcal{F} = 1_{\mathbf{Sets}/1}$ . Similarly, we can show  $\mathcal{F} \circ \mathcal{G} = 1_{\mathbf{Sets}}$ .  $\square$

**Remarks.** In general, we have the following result.

**Theorem.** For any category  $\mathbf{C}$  having a terminal object,  $A$ , we must have  $\mathbf{C}/A \simeq \mathbf{C}$ .

*Proof.* Define functor  $\mathcal{M} : \mathbf{C}/A \rightarrow \mathbf{C}$  so that  $\mathcal{M}(f) = \text{dom}(f)$  for any  $f \in \text{obj}(\mathbf{C}/A)$  and  $\mathcal{M}(f \xrightarrow{f'} g) = \text{dom}(f) \xrightarrow{k} \text{dom}(g)$  such that



commutes. Define functor  $\mathcal{N} : \mathbf{C} \rightarrow \mathbf{C}/A$  so that  $\mathcal{N}(X) = X \xrightarrow{f} A$ , we know such a  $f$  exists because  $A$  is terminal, and  $\mathcal{N}(X \xrightarrow{k} Y) = f \xrightarrow{f'} g$  where  $f : X \rightarrow A$  and  $g : Y \rightarrow A$ . Again, since  $A$  is terminal object, we know that such  $f$  and  $g$  are necessarily unique. Therefore, we have  $\mathcal{N} \circ \mathcal{M}(f) = \mathcal{N}(\text{dom}(f)) = \mathcal{N}(\text{dom}(f)) = \text{dom}(f) \xrightarrow{f} A = f$  and  $\mathcal{N}(\mathcal{M}(f \xrightarrow{f'} g)) = \mathcal{N}(\text{dom}(f) \xrightarrow{k} \text{dom}(g)) = f \xrightarrow{f'} g$ . Thus,  $\mathcal{N} \circ \mathcal{M} = 1_{\mathbf{C}/A}$ . Similarly, we can show  $\mathcal{M} \circ \mathcal{N} = 1_{\mathbf{C}}$ . Hence,  $\mathbf{C}/A \simeq \mathbf{C}$ .  $\square$

We now have the following result as the corollary of the theorem we just proved.

**Corollary.** For any singleton set  $\{*\}$ ,  $\mathbf{Sets} \simeq \mathbf{Sets}/\{*\}$ .

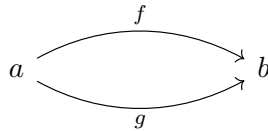
**Exercise 1.8.** Any category  $\mathbf{C}$  determines a preorder  $P(\mathbf{C})$  by defining a binary relation  $\leq$  on the objects by

$$A \leq B \text{ if and only if there is an arrow } A \rightarrow B$$

Show that  $P$  determines a functor from categories to preorders. Also show that  $P$  is a (one-sided) inverse to the evident inclusion functor of preorder into categories (Note that in this problem, we are working in category of small categories).

*Proof.* Define functor  $P : \mathbf{Cats} \rightarrow \mathbf{Preord}$  so that  $P(\mathbf{C}) = (\text{obj}(\mathbf{C}), \leq)$ , where  $A \leq B$  if and only if  $\text{hom}(A, B) \neq \emptyset$ , and  $P(\mathbf{C} \xrightarrow{\mathcal{F}} \mathbf{D}) = P(\mathbf{C}) \xrightarrow{k} P(\mathbf{D})$  where  $k$  is a function induced by  $\mathcal{F}$  i.e.  $k(A) = \mathcal{F}(A) \in \text{obj}(\mathbf{D})$  for all  $A \in \text{obj}(\mathbf{C})$ . Notice that if  $A \leq B$  then there is some  $f \in \text{hom}(A, B)$  but then it means  $\mathcal{F}(f) \in \text{hom}(\mathcal{F}(A), \mathcal{F}(B))$  which implies  $\mathcal{F}(A) \leq \mathcal{F}(B)$ . Thus,  $k$  is a monotone function. We now prove that  $P$  is a functor. Clearly,  $P(\mathbf{C} \xrightarrow{1} \mathbf{C}) = P(\mathbf{C}) \xrightarrow{k} P(\mathbf{C}) = 1_{P(\mathbf{C})}$  because  $k$  is identity on  $\text{obj}(\mathbf{C})$ . Consider some categories  $C, D$ , and  $E$  such that  $C \xrightarrow{F} D \xrightarrow{G} E$ . Let  $P(F) = k$  and  $P(G) = m$ . Then we have  $P(G) \circ P(F) = m \circ k = P(G \circ F)$  because  $m \circ k$  is function induced by  $G \circ F$ . Therefore,  $P$  is a functor. Since we already know that any preorder is a category, there must exist an inclusion functor,  $I : \mathbf{Preord} \rightarrow \mathbf{cats}$ , so that  $I(A) = A$  and  $I(f : A \rightarrow B) = A \xrightarrow{f} B$ .

Notice that  $P \circ I(A) = P(I(A)) = P(A) = (\text{obj}(A), \leq) = A$  and  $P \circ I(A \xrightarrow{f} B) = P(I(A \xrightarrow{f} B)) = P(A \xrightarrow{f} B) = A \xrightarrow{f} B$ . Hence,  $P \circ I(A) = 1_{\mathbf{Preord}}$ . However,  $P$  is not an isomorphism. To see why consider a small category  $C$  defined as



In this case,  $P(C) = (\{a, b\}, \{(a, b)\})$ . Applying  $I$  on it, we get the following category



Thus,  $I(P(C)) \neq C$ .  $\square$



**Remarks.** The failure of isomorphism is primarily based on the fact that  $P(C)$  identifies all morphism of  $\text{hom}(A, B)$  as one morphism, for any object  $A$  and  $B$ . Therefore, we must have the following theorem to be true.

**Theorem.** Define  $\mathbf{Cats}_1$  as the category of all categories such that  $\text{hom}(A, B)$  is at most singleton for any object  $A$  and  $B$ . Then  $\mathbf{Cats}_1 \simeq \mathbf{Posets}$ .

*Proof.* Defining  $P : \mathbf{Cats}_1 \rightarrow \mathbf{Preord}$  and  $I : \mathbf{Preord} \rightarrow \mathbf{cats}$  same as the above proof, it's easy to see  $P \circ I = 1_{\mathbf{Preord}}$ . Now consider  $I \circ P$ . Clearly,  $I \circ P(C) = I(P(C)) = C$ . Notice that  $P(C \xrightarrow{F} D) = C \xrightarrow{F} D$  because  $F$  is equal to the monotone function induced on  $C$  and  $D$ . Therefore,  $I \circ P(C \xrightarrow{F} D) = I(P(C \xrightarrow{F} D)) = I(C \xrightarrow{F} D) = C \xrightarrow{F} D$ . Hence,  $\mathbf{Cats}_1 \simeq \mathbf{Posets}$ .  $\square$

**Exercise 1.9.** Show that the free monoid functor

$$M : \mathbf{Sets} \rightarrow \mathbf{Mon}$$

exists, in two different ways:

(a) Assuming the particular choice  $M(X) = X^*$  and define its effect

$$M(f) : M(A) \rightarrow M(B)$$

on a function  $f : A \rightarrow B$  to be

$$M(f)(a_1 \cdots a_k) = f(a_1) \cdots f(a_k), a_1, \dots, a_k \in A.$$

(b) Assume only the UMP of the free monoid and use it to determine  $M$  on functions, showing the result to be a functor.

*Proof.*

(a) First we prove identity is preserved. Consider some set  $A$  and let  $w \in M(A) = A^*$  ie  $w = a_1 a_2 \cdots a_k$  for  $a_i \in A$ . Then clearly,

$$\begin{aligned} M(1_A)(w) &= 1_A(a_1) \cdots 1_A(a_k) \\ &= a_1 \cdots a_k \\ &= w. \end{aligned}$$

Next we prove that composition is preserved. Consider  $f : A \rightarrow B$  and  $g : B \rightarrow C$ . Then for any  $w \in M(A)$ , we have

$$\begin{aligned} (M(g) \circ M(f))(w) &= M(g)(f(a_1) \cdots f(a_k)) \\ &= (g \circ f)(a_1) \cdots (g \circ f)(a_k) \\ &= M(g \circ f)(w) \end{aligned}$$

(b) Consider  $f : A \rightarrow B$  for some sets  $A$  and  $B$ . By proposition 1.9, we know free monoids has UMP. Applying UMP on free monoid of  $A$  and  $B$ , we see that there exists an unique monoidal homomorphism  $\bar{f} : M(A) \rightarrow M(B)$  making

$$\begin{array}{ccc} |M(A)| & \xrightarrow{|\bar{f}|} & |M(B)| \\ & \nwarrow i_A \quad \nearrow I_B \circ f & \\ & A & \end{array}$$

commute in  $\mathbf{Sets}$ , where  $i_A : A \rightarrow |M(A)|$  and  $i_B : B \rightarrow |M(B)|$  are inclusion functions. Thus,  $|\bar{f}| \circ i_A = i_B \circ f$ , that is to say, for all  $a \in A$ ,  $\bar{f}(a) = f(a)$  but since  $\bar{f}$  is monoidal homomorphism, it would mean for any  $a_1 \cdots a_n \in M(A)$ , we must have

$$\begin{aligned} \bar{f}(a_1 \cdots a_n) &= \bar{f}(a_1) \cdots \bar{f}(a_n) \\ &= f(a_1) \cdots f(a_n) \end{aligned}$$

Define  $M(f) = \bar{f}$ . Since  $M : \mathbf{Sets} \rightarrow \mathbf{Mon}$  so that  $M(A) = A^*$  and  $M(f)(a_1 \cdots a_n) = f(a_1) \cdots f(a_n)$ , we can directly apply part (a) to see  $M$  is functor.  $\square$

**Exercise 1.10.** Verify the UMP for free categories on graphs, defined as above with arrows being sequences of edges. Specficially, let  $\mathbf{C}(G)$  be the free category on the graph  $G$ , so defined, and  $i : G \rightarrow U(\mathbf{C}(G))$  the graph homomorphism taking vertices and edges to themselves, regarded as objects and arrows in  $\mathbf{C}(G)$ . Show that for any category  $\mathbf{D}$  and graph homomorphism  $h : G \rightarrow U(\mathbf{D})$ , there is a unique functor

$$\bar{h} : \mathbf{C}(G) \rightarrow \mathbf{D}$$

with

$$U(\bar{h}) \circ i = h$$

where  $U : \mathbf{Cat} \rightarrow \mathbf{Graph}$  is the underlying forgetful functor.

*Proof.* Define  $\bar{h} : \mathbf{C}(G) \rightarrow \mathbf{D}$  so that  $\bar{h}(e_1 \cdots e_n) = h(e_1) \cdots h(e_n)$  for any path  $e_1 \cdots e_n \in \mathbf{C}(G)$ ,  $\bar{h}(1_a) = 1_{h(a)}$ , and  $\bar{h}(a) = h(a)$  for any object  $a \in \text{obj}(\mathbf{C}(G))$ . It is easy to see

$$\begin{array}{ccc} U(\mathbf{C}(G)) & \xrightarrow{U(\bar{h})} & U(\mathbf{D}) \\ & \swarrow i \quad \searrow h & \\ & G & \end{array}$$

commutes, by the way we defined  $\bar{h}$ . We now prove it is unique. Consider some functor  $F : \mathbf{C}(G) \rightarrow \mathbf{D}$  satisfying  $U(F) \circ i = h$ . Then clearly

$$\begin{aligned} F(e_1 \cdots e_n) &= F(e_1) \cdots F(e_n) \\ &= h(e_1) \cdots h(e_n) \\ &= \bar{h}(e_1) \cdots \bar{h}(e_n) \\ &= \bar{h}(e_1 \cdots e_n) \end{aligned}$$

Thus,  $F = \bar{h}$ .  $\square$

**Remarks.** We construct functor  $N : \mathbf{Graphs} \rightarrow \mathbf{Cats}$ . Consider some graphs  $G$  and  $H$  with  $G \xrightarrow{h} H$ . Applying UMP on  $\mathbf{C}(G)$  and  $\mathbf{C}(H)$ , we see that there exists an unique functor  $\bar{h} : \mathbf{C}(G) \rightarrow \mathbf{C}(H)$  such that

$$\begin{array}{ccc} U(\mathbf{C}(G)) & \xrightarrow{U(\bar{h})} & U(\mathbf{C}(H)) \\ \uparrow i_G & & \uparrow i_H \\ G & \xrightarrow{h} & H \end{array}$$

commutes in  $\mathbf{Graphs}$ . But that means for any edge  $e$  and vertex  $v$  in  $G$ , we must have  $\bar{h}(v) = h(v)$  and  $\bar{h}(e) = h(e)$ . However, since  $\bar{h}$  is a functor, we have

$$\begin{aligned} \bar{h}(e_1 \cdots e_n) &= \bar{h}(e_1) \cdots \bar{h}(e_n) \\ &= h(e_1) \cdots h(e_n) \end{aligned}$$

Now define functor  $N : \mathbf{Graphs} \rightarrow \mathbf{Cats}$  so that  $N(G) = \mathbf{C}(G)$  and  $N(G \xrightarrow{h} H) = \mathbf{C}(G) \xrightarrow{\bar{h}} \mathbf{C}(H)$ . First we prove identity is preserved. Consider some  $e_1 \cdots e_n \in \mathbf{C}(G)$ , then we have

$$\begin{aligned} N(1_G)(e_1 \cdots e_n) &= 1_G(e_1) \cdots 1_G(e_n) \\ &= e_1 \cdots e_n. \end{aligned}$$

where  $e_i \in G$  is an edge. Next we prove composition is preserved. Consider the graph homomorphism  $h : G \rightarrow H$  and  $k : H \rightarrow K$ . Then

$$\begin{aligned}
(N(k) \circ N(h))(e_1 \cdots e_n) &= N(k)(N(h)(e_1) \cdots N(h)(e_n)) \\
&= N(k)(N(h)(e_1)) \cdots N(k)(N(h)(e_n)) \\
&= (N(k) \circ N(h))(e_1) \cdots (N(k) \circ N(h))(e_n) \\
&= N(k \circ h)(e_1 \cdots e_n).
\end{aligned}$$

**Exercise 1.11.** Use the cayley representation to show that every small category is isomorphic to a "concrete" one, that is, one in which objects are sets and the arrows are functions between them.

*Proof.* Recall that cayley representation,  $\bar{\mathbf{C}}$  of  $\mathbf{C}$  is a category with collection of objects as  $\{\mathbf{C}/C : C \in \text{obj}(\mathbf{C})\}$  and arrows  $\bar{g} : \mathbf{C}/C \rightarrow \mathbf{C}/D$ , for some  $g : C \rightarrow D$  in  $\mathbf{C}$  and any  $f : X \rightarrow C$  in  $\bar{\mathbf{C}}$  as  $\bar{g} \circ f$  making

$$\begin{array}{ccc}
X & & \\
\downarrow f & \searrow g \circ f & \\
C & \xrightarrow{g} & D
\end{array}$$

commute in  $\mathbf{C}$ . It is easy to see that all objects are sets and arrows are thus set functions. We now prove  $\bar{\mathbf{C}}$  is isomorphic to  $\mathbf{C}$ . Define functor  $\mathcal{F} : \mathbf{C} \rightarrow \bar{\mathbf{C}}$  so that  $\mathcal{F}(C) = \mathbf{C}/C$  and  $\mathcal{F}(C \xrightarrow{f} D) = \mathbf{C}/C \xrightarrow{\bar{f}} \mathbf{C}/D$ . It is easy to see that  $\mathcal{F}$  is bijective on objects and arrows.  $\square$



## Chapter 2

# Abstract Structures

**Exercise 2.1.** Show that a function between sets is an epimorphism if and only if it is surjective. Conclude that the isos in **Sets** are exactly epi-monos.

*Proof.* The converse implication is trivial, so we only prove the forward direction. Let  $f : A \rightarrow B$ . Suppose  $f$  is not surjective. Then there is some  $b \in B$  which is not a  $f$ -image. Define  $i, j : B \rightarrow \{x, y\}$  such that  $i(c) = x$  for all  $c \in B$  and  $j(c) = \begin{cases} x & \text{if } c \in f(A) \\ y & \text{if } c \notin f(A) \end{cases}$ . This contradicts that  $f$  was epic.  $\square$

**Exercise 2.2.** Show that in a poset category, all arrows are epi-monos.

*Proof.* Trivial.  $\square$

**Exercise 2.3.** (Inverses are unique) If an arrow  $f : A \rightarrow B$  has inverses  $g, g' : B \rightarrow A$  (i.e.,  $g \circ f = 1_A$  and  $f \circ g = 1_B$  and similarly for  $g'$ ), then  $g = g'$ .

*Proof.* Consider the following commutative diagram.

$$\begin{array}{ccc} B & \xrightarrow{g'} & A \\ \downarrow g & \searrow 1_B & \downarrow f \\ A & \xrightarrow{f} & B \end{array}$$

But then since  $f$  is mono,  $g = g'$ .  $\square$

**Exercise 2.4.** With regard to a commutative triangle,

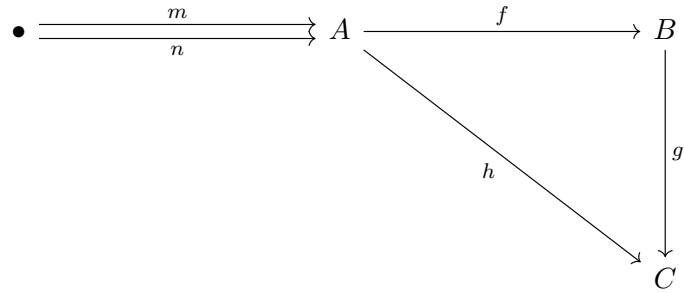
$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow h & \downarrow g \\ & & C \end{array}$$

- (a) if  $f$  and  $g$  are isos (resp. monos, resp. epis), so is  $h$ .
- (b) if  $h$  is monic then so is  $f$ .
- (c) if  $h$  is epic, so is  $g$ .

*Proof.*

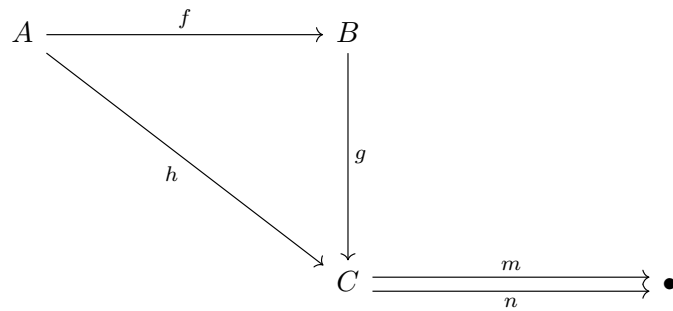
(a) If  $f$  and  $g$  are isos, then clearly  $h^{-1} = f^{-1} \circ g^{-1}$ . If  $f$  and  $g$  are both monos and  $h \circ m = h \circ n$  then we have  $g \circ (f \circ m) = g \circ (f \circ n)$  implies  $f \circ m = f \circ n$  which in turns implies  $m = n$  since  $f$  and  $g$  are monos. Similarly, if  $f$  and  $g$  are both epis then so is  $h$ .

(b) Consider the following commutative diagram. Consider the following commutative diagram.



Suppose  $fm = fn$ . Taking  $g$  both side, we see  $hm = gfm = gfn = hn$ , however, since  $h$  is mono, we have  $m = n$ .

(c) Consider the following commutative diagram.



Suppose  $mg = ng$ . That implies  $mh = (mg)f = (ng)f = nh$  but since  $h$  is epic, we have  $m = n$ .

(d) Consider  $f : \{1, 2\} \rightarrow \{3, 4, 5\}$  defined as  $f = \{(1, 3), (2, 4)\}$  and  $g : \{3, 4, 5\} \rightarrow \{6, 7\}$  as  $g = \{(3, 6), (4, 7), (5, 7)\}$ . Clearly,  $g \circ f$  is injective while  $g$  is not. However, since in **Sets** mono is equivalent to injectivity, we are done.  $\square$

**Exercise 2.5.** Show that the following are equivalent for an arrow

$$f : A \rightarrow B$$

in any category:

- (a)  $f$  is an iso.
- (b)  $f$  is both a mono and split epi.
- (c)  $f$  is both a split mono and an epi.
- (d)  $f$  is both a split mono and a split epi.

*Proof.* We are going to prove (b), (c), (d) are equivalent to (a). The forward implication for all of them is trivial, so we only prove the converse.

(b) Let  $f_R$  be the right inverse of  $f$ . Consider

$$\begin{aligned} f(f_R f) &= (f f_R) f \\ &= 1 \circ f \\ &= f \\ &= f \circ 1 \end{aligned}$$

Thus,  $f(f_R f) = f \circ 1$ , however since  $f$  is mono, we have  $f_R f = 1$ . As desired.

(c) Let  $f_L$  be the left inverse of  $f$ . Consider

$$\begin{aligned} (f f_L) f &= f(f_L f) \\ &= f \circ 1 \\ &= f \\ &= 1 \circ f. \end{aligned}$$

However, since  $f$  is epi, we have  $f f_L = 1$ .

(d) Clearly since split mono implies mono, by applying part (b), we have  $f$  to be iso. □