Category Theory

Book by Steve Awodey

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Chapter 1

Categories

Exercise 1.1. *The object of* **Rel** *are sets, and an arrow* $A \to B$ *is a relation from* A *to* B*, that is, a subset* $R \subseteq A \times B$*. The equality relation* $\{\langle a, a\rangle \in A \times A : a \in A\}$ *is the identity arrow on a set* A*. Composition in* **Rel** *is to be given by*

$$S \circ R = \{\langle a, c\rangle \in A \times C : \exists b(\langle a, b\rangle \in R \text{ and } \langle b, c\rangle \in S)\}$$

for $R \subseteq A \times B$ *and* $S \subseteq B \times C$*.*
(a) Show that **Rel** *is a category.*
(b) Show also that there is a functor $G : \textbf{Sets} \to \textbf{Rel}$ *taking objects to themselves and each function* $f : A \to B$ *to its graph,*

$$G(f) = \{\langle a, f(a)\rangle \in A \times B : a \in A\}.$$

(c) Finally, show that there is a functor $C : \textbf{Rel}^{op} \to \textbf{Rel}$ *taking each relation* $R \subseteq A \times B$ *to its converse* $R^c \subseteq B \times A$*, where*

$$\langle a, b\rangle \in R^c \leftrightarrow \langle b, a\rangle \in R.$$

Proof.
(a) First we prove that each object have identity arrow on it. Consider some set A. Let 1_A be the equality relationship on A. Let B be a set such that there is relation $R \subseteq A \times B$. Then if $(x, y) \in R \circ 1_A$, we must have $(x, y) \in R$ and if $(x, y) \in R$ then $(x, y) \in R \circ 1_A$. Therefore, $R \circ 1_A = R$. Similarly, we can prove that $1_A \circ B = B$ for some relation $B \subseteq C \times A$. Since we already know that composition of relationship is associative, it follows composition in **Rel** is assoicative.

(b) Since $1_A(x) = x$ for any set A, we have $G(1_A) = \{(x, 1_A(x)) : x \in A\} = \{(x, x) : x \in A\} = 1_{G(A)}$. Thus, identity is mapped to identity. Next we prove composition is preserved. Consider functions $f : A \to B$ and $g : B \to C$. We are going to prove that $G(g \circ f) = G(g) \circ G(f)$. Let $(x, g(f(x))) \in G(g \circ f)$. Then clearly $(x, f(x)) \in G(f)$ and since $f(x) \in B$, we have $(f(x), g(f(x))) \in G(g)$. Thus, $(x, g(f(x))) \in G(g) \circ G(f)$. Consider some $(x, g(f(x))) \in G(g \circ f)$. Now consider some $(x, y) \in G(g) \circ G(f)$. It follows that there exists some $z \in B$ such that $(x, z) \in G(f)$ and $(z, y) \in G(g)$. But then by the definition of the functor, it means $z = f(x)$ and $y = g(f(x))$. Thus, $(x, y) = (x, g(f(x))) \in G(g \circ f)$. Therefore, $G(g \circ f) = G(g) \circ G(f)$.

(c) Define functor $\mathcal{F} : \textbf{Rel} \to \textbf{Rel}^{\text{op}}$ so that $\mathcal{F}(A) = A$ for any set A and $\mathcal{F}(R) = R^c$ for any relationship (ie arrow). Since $1_A = 1_A^c$, we must have $\mathcal{F}(1_A) = 1_A = 1_{\mathcal{F}(A)}$. Now we prove composition is preserved. Clearly $\mathcal{F}(S \circ R) = (S \circ R)^c = S^c \circ R^c = \mathcal{F}(S) \circ \mathcal{F}(R)$. □

Exercise 1.2. *Consider the following isomorphisms of categories and determine which hold.*
(a) $\textbf{Rel} \cong \textbf{Rel}^{op}$*.*
(b) $\textbf{Sets} \cong \textbf{Sets}^{op}$*.*
(c) For a fixed sets X *with powerset* $P(X)$*, as poset categories* $P(X) \cong P(X)^{op}$*.*

Proof.
(a) Define $\mathcal{F}_1 : \textbf{Rel} \to \textbf{Rel}^{\text{op}}$ so that $\mathcal{F}_1(A) = A$ for any set A and $\mathcal{F}_1(R) = R^c$ for any arrow $R \in \textbf{Rel}$. Define $F_2 : \textbf{Rel}^{\text{op}} \to \textbf{Rel}$ same way. We already proved it is a functor in last exercise. Then, since $(R^c)^c = R$,

we have $\mathcal{F}_1(\mathcal{F}_2(R)) = R$ and $F_2(\mathcal{F}_1(R)) = R$. Hence, $\mathbf{Rel} \cong \mathbf{Rel}^{\mathrm{op}}$.

(b) In **Sets**, there is an unique initial object i.e. emptyset. However, in $\mathbf{Sets}^{\mathrm{op}}$, any singleton act as initial object ie it is only unique upto isomorphism. Therefore, **Sets** is not isomorphic to $\mathbf{Sets}^{\mathrm{op}}$.

(c) Define functor $\mathcal{F}_1 : P(X) \to P(X)^{\mathrm{op}}$ so that $\mathcal{F}(A) = X/A$ and $\mathcal{F}(A \subseteq B) = F(A) \supseteq F(B)$. Define $F_2 : P(X)^{\mathrm{op}} \to P(X)$ same way. Since $X/(X/A) = A$, we have $\mathcal{F}_1(\mathcal{F}_2(A)) = A$ and $\mathcal{F}_2(\mathcal{F}_1(A)) = A$. For arrows, we have

$$\begin{aligned}\mathcal{F}_1(F_2(A \subseteq B)) &= \mathcal{F}_1(\mathcal{F}_2(A) \supseteq F(B)) \\ &= F_1(F_2(A)) \subseteq F_1(F_2(B)) \\ &= A \subseteq B.\end{aligned}$$

Simillarly, $\mathcal{F}_2(F_1(A \subseteq B)) = A \subseteq B$.

□

Exercise 1.3.
(a) Show that in **Sets**, *the isomorphisms are exactly the bijections.*
(b) Show that in **Monoids**, *the isomorphism are exactly the bijective homomorphisms.*
(c) Show that in **Posets**, *the isomorphisms are not the same as the bijective homomorphisms.*

Proof.
(a) First we prove the forward implication. Consider some isomorphic sets A and B. By the definition, we know that there exists $f : A \to B$ and $g : B \to A$ satisfying $g \circ f = 1_A$ and $f \circ g = 1_B$. Our goal is to prove f is bijection. Let $a_1, a_2 \in A$ and suppose $f(a_1) = f(a_2)$. Taking g both sides we have $a_1 = g(f(a_1)) = g(f(a_2)) = a_2$. Hence f is injective. Let $b \in B$. Then it follows $g(b) \in A$ but since $f(g(b)) = b$, we have f to be surjective. The converse implication is trivial.

(b) Since **Monoids** are concrete, we know objects are sets with structure and arrows are structure preserving function between the sets. It follows immediately, from previous exercises, that if M_1 and M_2 are isomorphic, we must have a bijective homomorphism $M_1 \to M_2$. Thus, we only need to prove converse implication. Suppose M_1 and M_2 are monoids such that there exists bijective homomorphism $f : M_1 \to M_2$. Note that our goal is to prove f^{-1} is a homomorphism since we already know from the previous exercise that $f \circ f^{-1} = 1_{M_2}$ and $f^{-1} \circ f = 1_{M_1}$. Consider some $a_1, a_2 \in M_2$. Since f is bijective, we know there are $b_1, b_2 \in M_1$ such that $f(b_1) = a_1$ and $f(b_2) = a_2$. We also must have $f(b_1 b_2) = f(b_1)f(b_2) = a_1 a_2$ because f is homomorphism. But that means $f^{-1}(a_1)f^{-1}(a_2) = f^{-1}(f(b_1))f^{-1}(f(b_2)) = b_1 b_2 = f^{-1}(a_1 a_2)$.

(c) Consider posets, $P_1 = (\mathbb{Z}, |)$ and $P_2 = (\mathbb{Z}, \leq)$. It is easy to see that $id_{\mathbb{Z}}$ is order preserving bijective map since $a|b$ implies $a \leq b$. However the converse is not true, notice $2 \leq 3$ but 2 does not divide 3. □

Exercise 1.4.
Let X be a topological space and preorder the points by specialization: $x \leq y$ iff y is contained in every open set that contains x. Show that this is a preorder, and that it is a poset if X is T_0. Show that the ordering is trivial if X is T_1.

Proof. First we prove that specialization relationship is preorder. It is easy to see why it is reflexive. For transitivity, consider $x \leq y$ and $y \leq z$ and Let A be arbitrary open set containing z. Since $y \leq z$, we have $y \in A$ but then since A is open set containing y, it must be the case that $x \in A$. Hence, $x \leq z$. As desired.

Since we already showed any topological space is preorder under specialization, we merely need to demonstrate antisymmetry in T_0 and T_1 to show they are poset. For the former, suppose $x \leq y$ in T_0. It follows that there must exists an open set which contains y but not x, since y is contained in every open set containing x. That means T_0 has antisymmetry. In the case of T_1, the ordering is trivial since no two distinct elements are related. □

Exercise 1.5. *For any category $\mathbf{C}$, define a functor $\mathcal{U} : \mathbf{C}/C \to C$ frome the slice category over an object C that "forgets about C". Find a functor $F : \mathbf{C}/C \to \mathbf{C}^{\to}$ to the arrow category such that $\mathbf{dom} \circ F = U$*

Proof. Recall that in slice category $\mathbf{C}/C$, an object is simply an arrow $\mathrm{dom}(f) \xrightarrow{f} C$ and morphism $f \xrightarrow{f'} g$ is a morphism $\mathrm{dom}(f) \xrightarrow{n} \mathrm{dom}(g)$, in category $\mathbf{C}$, making

$$\begin{tikzcd}
\mathrm{dom}(f) \arrow[rr, "n"'] \arrow[dr, "f"] & & \mathrm{dom}(g) \arrow[dl, "g"'] \\
& C &
\end{tikzcd}$$

commutes. We define functor $\mathcal{U} : \mathbf{C}/C \to C$ so that $\mathcal{U}(f) = \mathrm{dom}(f)$ for any object $f \in \mathrm{obj}(\mathbf{C}/C)$ and $\mathcal{U}(f \xrightarrow{f'} g) = \mathrm{dom}(f) \xrightarrow{n} \mathrm{dom}(g)$ where n is the morphism $\mathrm{dom}(f) \to \mathrm{dom}(g)$ satisfying the above commutative diagram. We first check that identity is mapped to identity. Notice that $f \xrightarrow{1'_f} f$ is an arrow such that

$$\begin{tikzcd}
\mathrm{dom}(f) \arrow[rr, "1'_f"'] \arrow[dr, "f"] & & \mathrm{dom}(f) \arrow[dl, "f"'] \\
& C &
\end{tikzcd}$$

commutes i.e. $1'_f = \mathrm{Id}_{\mathrm{dom}(f)}$. Thus, $\mathcal{U}(f \xrightarrow{1'_f} f) = 1_{\mathrm{dom}(f)}$ as desired. Next we prove that $\mathcal{U}$ preserves the composition. Consider objects $\mathrm{dom}(f) \xrightarrow{f} C, \mathrm{dom}(g) \xrightarrow{g} C$, and $\mathrm{dom}(h) \xrightarrow{h} C$ in $\mathrm{obj}(\mathbf{C}/C)$ such that there is morphism $f' : f \to g$ and $g' : g \to h$. Let $n : \mathrm{dom}(f) \to \mathrm{dom}(g)$ and $m : \mathrm{dom}(g) \to \mathrm{dom}(h)$ making

$$\begin{tikzcd}
\mathrm{dom}(f) \arrow[r, "n"] \arrow[dr, "f"'] & \mathrm{dom}(g) \arrow[r, "m"] \arrow[d, "g"] & \mathrm{dom}(h) \arrow[dl, "h"] \\
& C &
\end{tikzcd}$$

commute. As apparent from diagram, $\mathcal{U}(g' \circ f') = m \circ n = \mathcal{U}(g') \circ \mathcal{U}(f')$.

We will now talk about constructing functor $\mathcal{F} : \mathbf{C}/C \to \mathbf{C}^{\to}$ and $\mathbf{dom} : \mathbf{C}^{\to} \to \mathbf{C}$. Before we get into it, let us first discuss about $\mathbf{C}^{\to}$. In $\mathbf{C}^{\to}$, objects are morphisms of $\mathbf{C}$ and morphism between $f, g \in \hom(\mathbf{C})$ is $(n, k) \in \hom(\mathbf{C})^2$ such that

$$\begin{tikzcd}
\bullet \arrow[r, "n"] \arrow[d, "f"] & \bullet \arrow[d, "g"] \\
\bullet \arrow[r, "k"] & \bullet
\end{tikzcd}$$

commutes. Define $\mathcal{F} : \mathbf{C}/C \to \mathbf{C}^{\to}$ such that for any $f \in \mathrm{obj}(\mathcal{C}/C)$, we have $\mathcal{F}(f) = f$ and $\mathcal{F}(f \xrightarrow{f'} g) = f \xrightarrow{(n, 1_C)} g$. It is easy to see it is a functor. Define $\mathbf{dom} : \mathbf{C}^{\to} \to C$ as $\mathbf{dom}(f) = \mathrm{dom}(f)$ for any $f \in \mathrm{obj}(\mathbf{C}^{\to})$ and $\mathbf{dom}(f \xrightarrow{(n,k)} g) = \mathrm{dom}(f) \xrightarrow{n} \mathrm{dom}(g)$. Again, it's easy to verify that it is a functor.

Finally, we prove that $\mathbf{dom} \circ \mathcal{F} = \mathcal{U}$. For any $f \in \mathrm{obj}(\mathbf{C}/C)$, we have $\mathbf{dom} \circ \mathcal{F}(f) = \mathbf{dom}(\mathcal{F}(f)) = \mathbf{dom}(f) = \mathrm{dom}(f) = \mathcal{U}(f)$ and for any $f \xrightarrow{f'} g$ in $\mathbf{C}/C$, clearly $\mathbf{dom} \circ \mathcal{F}(f \xrightarrow{f'} g) = \mathbf{dom}(\mathcal{F}(f \xrightarrow{f'} g)) = \mathbf{dom}(f \xrightarrow{(n,1_C)} g) = \mathrm{dom}(f) \xrightarrow{n} \mathrm{dom}(g) = \mathcal{U}(f \xrightarrow{f'} g)$. $\square$

Exercise 1.6. *Construct the coslice category $C/\mathbf{C}$ of a category $\mathbf{C}$ from the dual category operation.*

Proof.

Notation. *In what follows, we refer morphisms of $\mathbf{C}$ by f and corresponding arrow of $\mathbf{C}^{op}$ by $f_{C^{op}}$.*
Let $f : C \to A$ and $g : C \to B$. Consider the slice of C in $\mathbf{C}^{\mathrm{op}}$. The objects are $A \xrightarrow{f_{C^{\mathrm{op}}}} C$ and morphism $f_{C^{\mathrm{op}}} \xrightarrow{f'_{C^{\mathrm{op}}}} g_{C^{\mathrm{op}}}$ are $A \xrightarrow{k_{C^{\mathrm{op}}}} B$ of $\mathbf{C}^{\mathrm{op}}$ so that

$$\begin{tikzcd}
\mathrm{A} \arrow[rr, "k_{C^{\mathrm{op}}}"] \arrow[dr, "f_{C^{\mathrm{op}}}"'] & & \mathrm{B} \arrow[dl, "g_{C^{\mathrm{op}}}"] \\
& C &
\end{tikzcd}$$

commutes. Rewriting it in terms of $\mathbf{C}$, we see that $\mathbf{C}^{\mathrm{op}}/C$ has $C \xrightarrow{f} A$ as objects and $\mathrm{cod}(g) \xrightarrow{k} \mathrm{cod}(f)$ as morphism making

$$\begin{tikzcd}
A & & B \arrow[ll, "k"'] \\
& C \arrow[ul, "f"] \arrow[ur, "g"'] &
\end{tikzcd}$$

commute. This is exactly the notion of coslice of $\mathbf{C}$ with C. Define functor $\mathcal{W} : C/\mathbf{C} \to C$ so that $\mathcal{W}(f) = \mathrm{cod}(f)$ and $\mathcal{W}(g \xrightarrow{f'} f) = \mathrm{cod}(g) \xrightarrow{k} \mathrm{cod}(f)$ where k is the morphism making

$$\begin{tikzcd}
\mathrm{cod}(f) & & \mathrm{cod}(g) \arrow[ll, "k"'] \\
& C \arrow[ul, "f"] \arrow[ur, "g"'] &
\end{tikzcd}$$

Recall that in $\mathbf{C}^{\to}$, objects are morphisms of $\mathbf{C}$ and morphism between $f, g \in \hom(\mathbf{C})$ is $(n, k) \in \hom(\mathbf{C})^2$ such that

$$\begin{tikzcd}
\mathrm{dom}(f) \arrow[d, "f"] & \mathrm{dom}(g) \arrow[l, "n"'] \arrow[d] \\
\mathrm{cod}(f) & \mathrm{cod}(g) \arrow[l, "k"']
\end{tikzcd}$$

commutes. Define functor $\mathbf{cod} : \mathbf{C}^{\to} \to \mathbf{C}$ so that $\mathbf{cod}(f) = \mathrm{cod}(f)$ and $\mathbf{cod}(g \xrightarrow{(n,k)} f) = \mathrm{cod}(g) \xrightarrow{k} \mathrm{cod}(f)$ where k is the morphism between codomains making the last diagram commute. Define functor $\mathcal{G} : C/\mathbf{C} \to \mathbf{C}^{\to}$ so that $\mathcal{G}(g) = g$ for any $g \in \mathrm{obj}(C/\mathbf{C})$ and $\mathcal{G}(g \xrightarrow{g'} f) = g \xrightarrow{(1_C, k)} f$. It is easy to verify all of them are functors.

We will now prove that $\mathbf{cod} \circ \mathcal{G} = \mathcal{W}$. For any $g \in C/\mathbf{C}$, we have $\mathbf{cod} \circ \mathcal{G}(g) = \mathbf{cod}(\mathcal{G}(g)) = \mathbf{cod}(g) = \mathrm{cod}(g) = \mathcal{W}(g)$ and for any $g \xrightarrow{g'} f$ in $C/\mathbf{C}$, clearly $\mathbf{cod} \circ \mathcal{G}(g \xrightarrow{g'} f) = \mathbf{cod}(\mathcal{G}(g \xrightarrow{g'} f)) = \mathbf{cod}(g \xrightarrow{(1_C, k)} f) = \mathrm{cod}(g) \xrightarrow{k} \mathrm{cod}(f) = \mathcal{W}(g \xrightarrow{g'} f)$. □

Exercise 1.7. *Let $2 = \{a, b\}$ be any set with exactly 2 elements a and b. Define a functor $F : \mathbf{Sets}/2 \to \mathbf{Sets} \times \mathbf{Sets}$ with $F(f : X \to 2) = (f^{-1}(a), f^{-1}(b))$. Is this an isomorphism of categories? What about the analogous situation with a one element set $1 = \{a\}$ instead of 2.*

Proof. First we will discuss about **Sets**/2. Here we have arrows $X \xrightarrow{f} 2$ as objects, whereas morphism $f \xrightarrow{f'} g$ is a morphism $\mathrm{dom}(f) \xrightarrow{k} \mathrm{dom}(g)$ making

$$\begin{tikzcd}
\bullet \arrow[rr, "k"] \arrow[dr, "f"'] & & \bullet \arrow[dl, "g"] \\
& 2 &
\end{tikzcd}$$

commute. Define $F : \mathbf{Sets}/2 \to \mathbf{Sets} \times \mathbf{Sets}$ so that $F(X \xrightarrow{f} 2) = (f^{-1}(a), f^{-1}(b))$ and $F(f \xrightarrow{f'} g) = (k_{f^{-1}(a)}, k_{f^{-1}(b)})$ where $k_{f^{-1}(a)}$ and $k_{f^{-1}(b)}$ are restriction of morphism k which makes the last diagram commute. The identity in **Sets**/2 for any object f is simply the identity on the domain of f. Therefore, we have $F(1_f) = (1_{f^{-1}(a)}, 1_{f^{-1}(b)})$ which is indeed identity on the object $(f^{-1}(a), f^{-1}(b))$. Thus, $F(1_f) = 1_{(F(f))}$. Now we will prove that composition is preserved. Consider $f \xrightarrow{f'} g$ and $g \xrightarrow{g'} h$ such that

$$\begin{tikzcd}
\bullet \arrow[r, "k"] \arrow[rr, bend left, "nk"] \arrow[dr, "f"'] & \bullet \arrow[r, "n"] \arrow[d, "g"] & \bullet \arrow[dl, "h"] \\
& 2 &
\end{tikzcd}$$

commutes. Then we have

$$\begin{aligned}
F(g') \circ F(f') &= (n_{g^{-1}(a)}, n_{g^{-1}(b)}) \circ (k_{f^{-1}(a)}, k_{f^{-1}(b)}) \\
&= (n_{g^{-1}(a)} \circ k_{f^{-1}(a)}, n_{g^{-1}(b)} \circ k_{f^{-1}(b)}) \\
&= ((n \circ k)_{f^{-1}(a)}, (n \circ k)_{f^{-1}(b)}) && (\mathrm{ran}(k_{f^{-1}(a)}) \subseteq g^{-1}(a) \text{ and } \mathrm{ran}(k_{f^{-1}(b)}) \subseteq g^{-1}(b)) \\
&= F(g' \circ f')
\end{aligned}$$

Functor $F : \mathbf{Sets}/2 \to \mathbf{Sets}^2$ is not an isomorphism of category. Consider $A = \{c, d\}$ and $B = \{c, d, e\}$. There is no $f \in \mathrm{obj}\mathbf{Sets}/2$ mapping to $(A, B) \in \mathrm{obj}(\mathbf{Sets}^2)$ because if there was it would mean for any $x \in f^{-1}(a) = A \subseteq B = f^{-1}(b)$, we have $f(x) = a$ and $f(x) = b$ which is clearly not possible.

We now are going to prove that $\mathbf{Sets}/1 \simeq \mathbf{Sets}$ where $1 = \{a\}$ is any singleton set. Define functor $\mathcal{F} : \mathbf{Sets}/1 \to \mathbf{Sets}$ so that $\mathcal{F}(X \xrightarrow{f} 1) = f^{-1}(a)$ for any $f \in \mathrm{obj}(\mathbf{Sets}/2)$, and $\mathcal{F}(f \xrightarrow{f'} g) = f^{-1}a \xrightarrow{k} g^{-1}(a)$ where k is an arrow making

$$\begin{tikzcd}
f^{-1}(a) = \mathrm{dom}(f) \arrow[rr, "k"] \arrow[dr, "f"'] & & g^{-1}(a) = \mathrm{dom}(g) \arrow[dl, "g"] \\
& 1 &
\end{tikzcd}$$

commute. Define functor $\mathcal{G} : \mathbf{Sets} \to \mathbf{Sets}/1$ so that $\mathcal{G}(A) = A \xrightarrow{f} 1$ for any $A \in \mathrm{obj}(\mathbf{Sets})$ and $\mathcal{G}(A \xrightarrow{k} B) = f \xrightarrow{f'} g$ such that $f : A \to 1$ and $g : B \to 1$. It is easy to verify these are functor. Let $X \xrightarrow{f} 1 \in \mathrm{obj}(\mathbf{Sets}/1)$. Then we have $\mathcal{G} \circ \mathcal{F}(f) = \mathcal{G}(\mathcal{F}(f)) = \mathcal{G}(f^{-1}(a)) = \mathcal{G}(X) = X \xrightarrow{f} 1$. Now consider some $f' : f \to g$ for some $X \xrightarrow{f} 1$ and $Y \xrightarrow{g} 1$, it follows $\mathcal{G} \circ \mathcal{F}(f') = \mathcal{G}(\mathcal{F}(f')) = \mathcal{G}(X \xrightarrow{k} Y) = f \xrightarrow{f'} g$. We know that X gets mapped to f under $\mathcal{G}$ necessarily because it's an unique arrow $X \to 1$. Thus, $\mathcal{G} \circ \mathcal{F} = 1_{\mathbf{Sets}/1}$. Simillarly, we can show $\mathcal{F} \circ \mathcal{G} = 1_{\mathbf{Sets}}$. $\square$

Remarks. *In general, we have the following result.*

Theorem. *For any category* $\mathbf{C}$ *having a terminal object,* A*, we must have* $\mathbf{C}/A \simeq \mathbf{C}$.

Proof. Define functor $\mathcal{M} : \mathbf{C}/A \to \mathbf{C}$ so that $\mathcal{M}(f) = \mathrm{dom}(f)$ for any $f \in \mathrm{obj}(\mathbf{C}/A)$ and $\mathcal{M}(f \xrightarrow{f'} g) = \mathrm{dom}(f) \xrightarrow{k} \mathrm{dom}(g)$ such that

$$\begin{tikzcd} \bullet \arrow[rr, "k"] \arrow[dr, "f"'] & & \bullet \arrow[dl, "g"] \\ & 1 & \end{tikzcd}$$

commutes. Define functor $\mathcal{N} : \mathbf{C} \to \mathbf{C}/A$ so that $\mathcal{N}(X) = X \xrightarrow{f} A$, we know such a f exists because A is terminal, and $\mathcal{N}(X \xrightarrow{k} Y) = f \xrightarrow{f'} g$ where $f : X \to A$ and $g : Y \to A$. Again, since A is terminal object, we know that such f and g are necessarily unique. Therefore, we have $\mathcal{N} \circ \mathcal{M}(f) = \mathcal{N}(\mathcal{M}(f)) = \mathcal{N}(\mathrm{dom}(f)) = \mathrm{dom}(f) \xrightarrow{f} A = f$ and $\mathcal{N}(\mathcal{M}(f \xrightarrow{f'} g)) = \mathcal{N}(\mathrm{dom}(f) \xrightarrow{k} \mathrm{dom}(g)) = f \xrightarrow{f'} g$. Thus, $\mathcal{N} \circ \mathcal{M} = 1_{\mathbf{C}/A}$. Simillarly, we can show $\mathcal{M} \circ \mathcal{N} = 1_C$. Hence, $\mathbf{C}/A \simeq \mathbf{C}$. $\square$

We now have the following result as the corollary of the theorem we just proved.

Corollary. *For any singleton set* $\{*\}$, $\mathbf{Sets} \simeq \mathbf{Sets}/\{*\}$.

Exercise 1.8. *Any category* $\mathbf{C}$ *determines a preorder* $P(\mathbf{C})$ *by defining a binary relation* $\leq$ *on the objects by*

$$A \leq B \textit{ if and only if there is an arrow } A \to B$$

Show that P *determines a functor from categories to preoders. Also show that* P *is a (one-sided) inverse to the evident inclusion functor of preorder into categories (Note that in this problem, we are working in category of small categories).*

Proof. Define functor $P : \mathbf{Cats} \to \mathbf{Preord}$ so that $P(\mathbf{C}) = (\mathrm{obj}(\mathbf{C}), \leq)$, where $A \leq B$ if and only if $\hom(A, B) \neq \emptyset$, and $P(\mathbf{C} \xrightarrow{\mathcal{F}} D) = P(\mathbf{C}) \xrightarrow{k} P(D)$ where k is a function induced by $\mathcal{F}$ i.e. $k(A) = \mathcal{F}(A) \in \mathrm{obj}(D)$ for all $A \in \mathrm{obj}(\mathbf{C})$. Notice that if $A \leq B$ then there is some $f \in \hom(A, B)$ but then it means $\mathcal{F}(f) \in \hom(\mathcal{F}(A), \mathcal{F}(B))$ which implies $\mathcal{F}(A) \leq \mathcal{F}(B)$. Thus, k is a montone function. We now prove that P is a functor. Clearly, $P(\mathbf{C} \xrightarrow{1} \mathbf{C}) = P(\mathbf{C}) \xrightarrow{k} P(\mathbf{C}) = 1_{P(C)}$ because k is identity on $\mathrm{obj}(C)$. Consider some categories C, D, and E such that $C \xrightarrow{F} D \xrightarrow{G} E$. Let $P(F) = k$ and $P(G) = m$. Then we have $P(G) \circ P(F) = m \circ k = P(G \circ F)$ because $m \circ k$ is function induced by $G \circ F$. Therefore, P is a functor. Since we already know that any preorder is a category, there must exists an inclusion functor, $I : \mathbf{Preord} \to \mathbf{cats}$, so that $I(A) = A$ and $I(f : A \to B) = A \xrightarrow{f} B$.

Notice that $P \circ I(A) = P(I(A)) = P(A) = (\mathrm{obj}(A), \leq) = A$ and $P \circ I(A \xrightarrow{f} B) = P(I(A \xrightarrow{f} B)) = P(A \xrightarrow{f} B) = A \xrightarrow{f} B$. Hence, $P \circ I(A) = 1_{\mathbf{Preord}}$. However, P is not an isomorphism. To see why consider a small category C defined as

$$\begin{tikzcd} a \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & b \end{tikzcd}$$

In this case, $P(C) = (\{a, b\}, \{(a, b)\})$. Applying I on it, we get the following category

$$\begin{tikzcd} a \arrow[r] & b \end{tikzcd}$$

Thus, $I(P(C)) \neq C$. $\square$

Remarks. *The failure of isomorphism is primarily based on the fact that $P(C)$ identifies all morphism of* $\hom(A, B)$ *as one morphism, for any object A and B. Therefore, we must have the following theorem to be true.*

Theorem. *Define* $\mathbf{Cats}_1$ *as the category of all categories such that* $\hom(A, B)$ *is at most singleton for any object A and B. Then* $\mathbf{Cats}_1 \simeq \mathbf{Posets}$*.*

Proof. Defining $P : \mathbf{Cats}_1 \to \mathbf{Preord}$ and $I : \mathbf{Preord} \to \mathbf{cats}$ same as the above proof, it's easy to see $P \circ I = 1_{\mathbf{Preord}}$. Now consider $I \circ P$. Clearly, $I \circ P(C) = I(P(C)) = C$. Notice that $P(C \xrightarrow{F} D) = C \xrightarrow{F} D$ because F is equal to the monotone function induced on C and D. Therefore, $I \circ P(C \xrightarrow{F} D) = I(P(C \xrightarrow{F} D)) = I(C \xrightarrow{F} D) = C \xrightarrow{F} D$. Hence, $\mathbf{Cats}_1 \simeq \mathbf{Posets}$. □

Exercise 1.9. *Show that the free monoid functor*

$$M : \mathbf{Sets} \to \mathbf{Mon}$$

exists, in two different ways:
(a) Assuming the particular choice $M(X) = X^*$ and define its effect

$$M(f) : M(A) \to M(B)$$

on a function $f : A \to B$ to be

$$M(f)(a_1 \cdots a_k) = f(a_1) \cdots f(a_k), a_1, \cdots, a_k \in A.$$

(b) Assume only the UMP of the free monoid and use it to determine M on functions, showing the result to be a functor.

Proof.
(a) First we prove identity is preseved. Consider some set A and let $w \in M(A) = A^*$ ie $w = a_1 a_2 \cdots a_k$ for $a_i \in A$. Then clearly,

$$\begin{aligned} M(1_A)(w) &= 1_A(a_1) \cdots 1_A(a_k) \\ &= a_1 \cdots a_k \\ &= w. \end{aligned}$$

Next we prove that composition is preserved. Consider $f : A \to B$ and $g : B \to C$. Then for any $w \in M(A)$, we have

$$\begin{aligned} (M(g) \circ M(f))(w) &= M(g)(f(a_1) \cdots f(a_k)) \\ &= (g \circ f)(a_1) \cdots (g \circ f)(a_k) \\ &= M(g \circ f)(w) \end{aligned}$$

(b) Consider $f : A \to B$ for some sets A and B. By proposition 1.9, we know free monoids has UMP. Applying UMP on free monoid of A and B, we see that there exists an unique monoidal homomorphism $\bar{f} : M(A) \to M(B)$ making

$$\begin{tikzcd} {|M(A)|} \arrow[rr, "|\bar{f}|"] & & {|M(B)|} \\ & A \arrow[ul, "i_A"] \arrow[ur, "I_B \circ f"'] & \end{tikzcd}$$

commute in **Sets**, where $i_A : A \to |M(A)|$ and $i_B : B \to |M(B)|$ are inclusion functions. Thus, $|\bar{f}| \circ i_A = i_B \circ f$, that is to say, for all $a \in A$, $\bar{f}(a) = f(a)$ but since $\bar{f}$ is monoidal homomorphism, it would mean for any $a_1 \cdots a_n \in M(A)$, we must have

$$\begin{aligned} \bar{f}(a_1 \cdots a_n) &= \bar{f}(a_1) \cdots \bar{f}(a_n) \\ &= f(a_1) \cdots f_(a_n) \end{aligned}$$

Define $M(f) = \bar{f}$. Since $M : \mathbf{Sets} \to \mathbf{Mon}$ so that $M(A) = A^*$ and $M(f)(a_1 \cdots a_n) = f(a_1) \cdots f(a_n)$, we can directly apply part (a) to see M is functor. $\square$

Exercise 1.10. *Verify the UMP for free categories on graphs, defined as above with arrows being sequences of edges. Specficially, let $\mathbf{C}(G)$ be the free category on the graph G, so defined, and $i : G \to U(\mathbf{C}(G))$ the graph homomorphism taking vertices and edges to themselves, regarded as objects and arrows in $\mathbf{C}(G)$. Show that for any category $\mathbf{D}$ and graph homomorphism $h : G \to U(\mathbf{D})$, there is a unique functor*

$$\bar{h} : \mathbf{C}(G) \to \mathbf{D}$$

with

$$U(\bar{h}) \circ i = h$$

where $U : \mathbf{Cat} \to \mathbf{Graph}$ is the underlying forgetful functor.

Proof. Define $\bar{h} : \mathbf{C}(G) \to \mathbf{D}$ so that $\bar{h}(e_1 \cdots e_n) = h(e_1) \cdots h(e_n)$ for any path $e_1 \cdots e_n \in \mathbf{C}(G)$, $\bar{h}(1_a) = 1_{h(a)}$, and $\bar{h}(a) = h(a)$ for any object $a \in \mathrm{obj}(\mathbf{C}(G))$. It is easy to see

$$\begin{tikzcd} U(C(G)) \arrow[rr, "U(\bar{h})"] & & U(D) \\ & G \arrow[ul, "i"] \arrow[ur, "h"'] & \end{tikzcd}$$

commutes, by the way we defined $\bar{h}$. We now prove it is unique. Consider some functor $F : \mathbf{C}(G) \to D$ satisfying $U(F) \circ i = h$. Then clearly

$$\begin{aligned} F(e_1 \cdots e_n) &= F(e_1) \cdots F(e_n) \\ &= h(e_1) \cdots h(e_n) \\ &= \bar{h}(e_1) \cdots \bar{h}(e_n) \\ &= \bar{h}(e_n \cdots e_n) \end{aligned}$$

Thus, $F = \bar{h}$. $\square$

Remarks. *We construct functor $N : \mathbf{Graphs} \to \mathbf{Cats}$. Consider some graphs G and H with $G \xrightarrow{h} H$. Applying UMP on $\mathbf{C}(G)$ and $\mathbf{C}(H)$, we see that there exists an unique functor $\bar{h} : \mathbf{C}(G) \to \mathbf{C}(H)$ such that*

$$\begin{tikzcd} U(\mathbf{C}(G)) \arrow[r, "U(\bar{h})"'] & U(\mathbf{C}(H)) \\ G \arrow[u, "i_G"] \arrow[r, "h"] & H \arrow[u, "i_H"'] \end{tikzcd}$$

commutes in **Graphs**. *But that means for any edge e and vertex v in G, we must have $\bar{h}(v) = h(v)$ and $\bar{h}(e) = h(e)$. However, since $\bar{h}$ is a functor, we have*

$$\begin{aligned} \bar{h}(e_1 \cdots e_n) &= \bar{h}(e_1) \cdots \bar{h}(e_n) \\ &= h(e_1) \cdots h(e_n) \end{aligned}$$

Now define functor $N : \mathbf{Graphs} \to \mathbf{Cats}$ so that $N(G) = \mathbf{C}(G)$ and $N(G \xrightarrow{h} H) = \mathbf{C}(G) \xrightarrow{\bar{h}} \mathbf{C}(H)$. First we prove identity is preserved. Consider some $e_1 \cdots e_n \in \mathbf{C}(G)$, then we have

$$\begin{aligned} N(1_G)(e_1 \cdots e_n) &= 1_G(e_1) \cdots 1_G(e_n) \\ & e_1 \cdots e_n. \end{aligned}$$

where $e_i \in G$ is an edge. Next we prove composition is preserved. Consider the graph homomorphism $h : G \to H$ and $k : H \to K$. Then

$$\begin{aligned}(N(k) \circ N(h))(e_1 \cdots e_n) &= N(k)(N(h)(e_1) \cdots N(h)(e_n)) \\ &= N(k)(N(h)(e_1)) \cdots N(k)(N(h)(e_n)) \\ &= (N(k) \circ N(h))(e_1) \cdots (N(k) \circ N(h))(e_n) \\ &= N(k \circ h)(e_1 \cdots e_n).\end{aligned}$$

Exercise 1.11. *Use the cayley representation to show that every small category is isomorphic to a "concrete" one, that is, one in which objects are sets and the arrows are functions between them.*

Proof. Recall that cayley representation, $\bar{\mathbf{C}}$ of $\mathbf{C}$ is a category with collection of objects as $\{\mathbf{C}/C : C \in \mathrm{obj}(C)\}$ and and arrows $\bar{g} : \mathbf{C}/C \to \mathbf{C}/D$, for some $g : C \to D$ in $\mathbf{C}$ and any $f : X \to C$ in $\bar{C}$ as $g \circ f$ making

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, "g\circ f"] & \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

commute in $\mathbf{C}$. It is easy to see that all objects are are sets and arrows are thus set functions. We now prove $\bar{\mathbf{C}}$ is isomorphic to $\mathbf{C}$. Define functor $\mathcal{F} : \mathbf{C} \to \bar{\mathbf{C}}$ so that $F(C) = \mathbf{C}/C$ and $F(C \xrightarrow{f} D) = \mathbf{C}/C \xrightarrow{\bar{f}} \mathbf{C}/D$. It is easy to see that $\mathcal{F}$ is bijective on objects and arrows. $\square$

Chapter 2

Abstract Structures

Exercise 2.1. *Show that a function between sets is an epimorphism if and only if it is surjective. Conclude that the isos in* **Sets** *are exactly epi-monos.*

Proof. The converse implication is trivial, so we only prove the forward direction. Let $f : A \twoheadrightarrow B$. Suppose f is not surjective. Then there is some $b \in B$ which is not a f-image. Define $i, j : B \to \{x, y\}$ such that $i(c) = x$ for all $c \in B$ and $j(c) = \begin{cases} x & \text{if}\, c \in f(A) \\ y & \text{if}\, c \notin f(A) \end{cases}$. This contradicts that f was epic. $\square$

Exercise 2.2. *Show that in a poset category, all arrows are epi-monos.*

Proof. Trivial. $\square$

Exercise 2.3. *(Inverses are unique) If an arrow $f : A \to B$ has inverses $g, g' : B \to A$ (i.e., $g \circ f = 1_A$ and $f \circ g = 1_B$ and similarly for g'), then $g = g'$.*

Proof. Consider the following commutative diagram.

$$\begin{tikzcd}
B \arrow[r, "g'"] \arrow[d, "g"] \arrow[dr, "1_B"] & A \arrow[d, "f"] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

But then since f is mono, $g = g'$. $\square$

Exercise 2.4. *With regard to a commutative triangle,*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, "g"] \\
& C
\end{tikzcd}$$

(a) if f and g are isos (resp. monos, resp. epis), so is h.
(b) if h is monic then so is f.
(c) iif h is epic, so is g.

Proof.
(a) If f and g are isos, then clearly $h^{-1} = f^{-1} \circ g^{-1}$. If f and g are both monos and $h \circ m = h \circ n$ then we have $g \circ (f \circ m) = g \circ (f \circ n)$ implies $f \circ m = f \circ n$ which in turns implies $m = n$ since f and g are monos. Simillarly, if f and g are both epis then so is h.

(b) Consider the following commutative diagram. Consider the following commutative diagram.

$$
\begin{tikzcd}
\bullet \arrow[r, shift left, "m"] \arrow[r, shift right, "n"'] & A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, "g"] \\
 & & C
\end{tikzcd}
$$

Suppose $fm = fn$. Taking g both side, we see $hm = gfm = gfn = hn$, however, since h is mono, we have $m = n$.

(c) Consider the following commutative diagram.

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, "g"] & \\
 & C \arrow[r, shift left, "m"] \arrow[r, shift right, "n"'] & \bullet
\end{tikzcd}
$$

Suppose $mg = ng$. That implies $mh = (mg)f = (ng)f = nh$ but since h is epic, we have $m = n$.

(d) Consider $f : \{1, 2\} \to \{3, 4, 5\}$ defined as $f = \{(1, 3), (2, 4)\}$ and $g : \{3, 4, 5\} \to \{6, 7\}$ as $g = \{(3, 6), (4, 7), (5, 7)\}$. Clearly, $g \circ f$ is injective while g is not. However, since in **Sets** mono is equivalent to injectivity, we are done. $\square$