Category Theory

Book by Steve Awodey

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Contents

Chapter 1

Categories

Exercise 1.1. *The object of* **Rel** *are sets, and an arrow* $A \to B$ *is a relation from* A *to* B*, that is, a subset* $R \subseteq A \times B$*. The equality relation* $\{\langle a, a\rangle \in A \times A : a \in A\}$ *is the identity arrow on a set* A*. Composition in* **Rel** *is to be given by*

$$S \circ R = \{\langle a, c\rangle \in A \times C : \exists b(\langle a, b\rangle \in R \text{ and } \langle b, c\rangle \in S)\}$$

for $R \subseteq A \times B$ *and* $S \subseteq B \times C$*.*
(a) Show that **Rel** *is a category.*
(b) Show also that there is a functor $G : \mathbf{Sets} \to \mathbf{Rel}$ *taking objects to themselves and each function* $f : A \to B$ *to its graph,*

$$G(f) = \{\langle a, f(a)\rangle \in A \times B : a \in A\}.$$

(c) Finally, show that there is a functor $C : \mathbf{Rel}^{op} \to \mathbf{Rel}$ *taking each relation* $R \subseteq A \times B$ *to its converse* $R^c \subseteq B \times A$*, where*

$$\langle a, b\rangle \in R^c \leftrightarrow \langle b, a\rangle \in R.$$

Proof.
(a) First we prove that each object have identity arrow on it. Consider some set A. Let 1_A be the equality relationship on A. Let B be a set such that there is relation $R \subseteq A \times B$. Then if $(x, y) \in R \circ 1_A$, we must have $(x, y) \in R$ and if $(x, y) \in R$ then $(x, y) \in R \circ 1_A$. Therefore, $R \circ 1_A = R$. Similarly, we can prove that $1_A \circ B = B$ for some relation $B \subseteq C \times A$. Since we already know that composition of relationship is associative, it follows composition in **Rel** is assoicative.

(b) Since $1_A(x) = x$ for any set A, we have $G(1_A) = \{(x, 1_A(x)) : x \in A\} = \{(x, x) : x \in A\} = 1_{G(A)}$. Thus, identity is mapped to identity. Next we prove composition is preserved. Consider functions $f : A \to B$ and $g : B \to C$. We are going to prove that $G(g \circ f) = G(g) \circ G(f)$. Let $(x, g(f(x))) \in G(g \circ f)$. Then clearly $(x, f(x)) \in G(f)$ and since $f(x) \in B$, we have $(f(x), g(f(x))) \in G(g)$. Thus, $(x, g(f(x))) \in G(g) \circ G(f)$. Consider some $(x, g(f(x))) \in G(g \circ f)$. Now consider some $(x, y) \in G(g) \circ G(f)$. It follows that there exists some $z \in B$ such that $(x, z) \in G(f)$ and $(z, y) \in G(g)$. But then by the definition of the functor, it means $z = f(x)$ and $y = g(f(x))$. Thus, $(x, y) = (x, g(f(x))) \in G(g \circ f)$. Therefore, $G(g \circ f) = G(g) \circ G(f)$.

(c) Define functor $\mathcal{F} : \mathbf{Rel} \to \mathbf{Rel}^{\mathrm{op}}$ so that $\mathcal{F}(A) = A$ for any set A and $\mathcal{F}(R) = R^c$ for any relationship (ie arrow). Since $1_A = 1_A^c$, we must have $\mathcal{F}(1_A) = 1_A = 1_{\mathcal{F}(A)}$. Now we prove composition is preserved. Clearly $\mathcal{F}(S \circ R) = (S \circ R)^c = S^c \circ R^c = \mathcal{F}(S) \circ \mathcal{F}(R)$. □

Exercise 1.2. *Consider the following isomorphisms of categories and determine which hold.*
(a) $\mathbf{Rel} \cong \mathbf{Rel}^{op}$*.*
(b) $\mathbf{Sets} \cong \mathbf{Sets}^{op}$*.*
(c) For a fixed sets X *with powerset* $P(X)$*, as poset categories* $P(X) \cong P(X)^{op}$*.*

Proof.
(a) Define $\mathcal{F}_1 : \mathbf{Rel} \to \mathbf{Rel}^{\mathrm{op}}$ so that $\mathcal{F}_1(A) = A$ for any set A and $\mathcal{F}_1(R) = R^c$ for any arrow $R \in \mathbf{Rel}$. Define $F_2 : \mathbf{Rel}^{\mathrm{op}} \to \mathbf{Rel}$ same way. We already proved it is a functor in last exercise. Then, since $(R^c)^c = R$,

we have $\mathcal{F}_1(\mathcal{F}_2(R)) = R$ and $F_2(\mathcal{F}_1(R)) = R$. Hence, $\mathbf{Rel} \cong \mathbf{Rel}^{\mathrm{op}}$.

(b) In **Sets**, there is an unique initial object i.e. emptyset. However, in $\mathbf{Sets}^{\mathrm{op}}$, any singleton act as initial object ie it is only unique upto isomorphism. Therefore, **Sets** is not isomorphic to $\mathbf{Sets}^{\mathrm{op}}$.

(c) Define functor $\mathcal{F}_1 : P(X) \to P(X)^{\mathrm{op}}$ so that $\mathcal{F}(A) = X/A$ and $\mathcal{F}(A \subseteq B) = F(A) \supseteq F(B)$. Define $F_2 : P(X)^{\mathrm{op}} \to P(X)$ same way. Since $X/(X/A) = A$, we have $\mathcal{F}_1(\mathcal{F}_2(A)) = A$ and $\mathcal{F}_2(\mathcal{F}_1(A)) = A$. For arrows, we have

$$\begin{aligned} \mathcal{F}_1(F_2(A \subseteq B)) &= \mathcal{F}_1(\mathcal{F}_2(A) \supseteq F(B)) \\ &= F_1(F_2(A)) \subseteq F_1(F_2(B)) \\ &= A \subseteq B. \end{aligned}$$

Simillarly, $\mathcal{F}_2(F_1(A \subseteq B)) = A \subseteq B$.

□

Exercise 1.3.
(a) Show that in **Sets***, the isomorphisms are exactly the bijections.*
(b) Show that in **Monoids***, the isomorphism are exactly the bijective homomorphisms.*
(c) Show that in **Posets***, the isomorphisms are not the same as the bijective homomorphisms.*

Proof.
(a) First we prove the forward implication. Consider some isomorphic sets A and B. By the definition, we know that there exists $f : A \to B$ and $g : B \to A$ satisfying $g \circ f = 1_A$ and $f \circ g = 1_B$. Our goal is to prove f is bijection. Let $a_1, a_2 \in A$ and suppose $f(a_1) = f(a_2)$. Taking g both sides we have $a_1 = g(f(a_1)) = g(f(a_2)) = a_2$. Hence f is injective. Let $b \in B$. Then it follows $g(b) \in A$ but since $f(g(b)) = b$, we have f to be surjective. The converse implication is trivial.

(b) Since **Monoids** are concrete, we know objects are sets with structure and arrows are structure preserving function between the sets. It follows immediately, from previous exercises, that if M_1 and M_2 are isomorphic, we must have a bijective homomorphism $M_1 \to M_2$. Thus, we only need to prove converse implication. Suppose M_1 and M_2 are monoids such that there exists bijective homomorphism $f : M_1 \to M_2$. Note that our goal is to prove f^{-1} is a homomorphism since we already know from the previous exercise that $f \circ f^{-1} = 1_{M_2}$ and $f^{-1} \circ f = 1_{M_1}$. Consider some $a_1, a_2 \in M_2$. Since f is bijective, we know there are $b_1, b_2 \in M_1$ such that $f(b_1) = a_1$ and $f(b_2) = a_2$. We also must have $f(b_1 b_2) = f(b_1)f(b_2) = a_1 a_2$ because f is homomorphism. But that means $f^{-1}(a_1)f^{-1}(a_2) = f^{-1}(f(b_1))f^{-1}(f(b_2)) = b_1 b_2 = f^{-1}(a_1 a_2)$.

(c) Consider posets, $P_1 = (\mathbb{Z}, |)$ and $P_2 = (\mathbb{Z}, \leq)$. It is easy to see that $id_{\mathbb{Z}}$ is order preserving bijective map since $a|b$ implies $a \leq b$. However the converse is not true, notice $2 \leq 3$ but 2 does not divide 3. □

Exercise 1.4.
Let X be a topological space and preorder the points by specialization: $x \leq y$ iff y is contained in every open set that contains x. Show that this is a preorder, and that it is a poset if X is T_0. Show that the ordering is trivial if X is T_1.

Proof. First we prove that specialization relationship is preorder. It is easy to see why it is reflexive. For transitivity, consider $x \leq y$ and $y \leq z$ and Let A be arbitrary open set containing z. Since $y \leq z$, we have $y \in A$ but then since A is open set containing y, it must be the case that $x \in A$. Hence, $x \leq z$. As desired.

Since we already showed any topological space is preorder under specialization, we merely need to demonstrate antisymmetry in T_0 and T_1 to show they are poset. For the former, suppose $x \leq y$ in T_0. It follows that there must exists an open set which contains y but not x, since y is contained in every open set containing x. That means T_0 has antisymmetry. In the case of T_1, the ordering is trivial since no two distinct elements are related. □

Exercise 1.5. *For any category* $\mathbf{C}$*, define a functor* $\mathcal{U} : \mathbf{C}/C \to C$ *frome the slice category over an object* C *that "forgets about* C*". Find a functor* $F : \mathbf{C}/C \to \mathbf{C}^{\to}$ *to the arrow category such that* $\mathbf{dom} \circ F = U$

Proof. Recall that in slice category $\mathbf{C}/C$, an object is simply an arrow $\mathrm{dom}(f) \xrightarrow{f} C$ and morphism $f \xrightarrow{f'} g$ is a morphism $\mathrm{dom}(f) \xrightarrow{n} \mathrm{dom}(g)$, in category $\mathbf{C}$, making

$$\begin{tikzcd}
\mathrm{dom}(f) \arrow[rr, "n"'] \arrow[dr, "f"] & & \mathrm{dom}(g) \arrow[dl, "g"'] \\
& C &
\end{tikzcd}$$

commutes. We define functor $\mathcal{U} : \mathbf{C}/C \to C$ so that $\mathcal{U}(f) = \mathrm{dom}(f)$ for any object $f \in \mathrm{obj}(\mathbf{C}/C)$ and $\mathcal{U}(f \xrightarrow{f'} g) = \mathrm{dom}(f) \xrightarrow{n} \mathrm{dom}(g)$ where n is the morphism $\mathrm{dom}(f) \to \mathrm{dom}(g)$ satisfying the above commutative diagram. We first check that identity is mapped to identity. Notice that $f \xrightarrow{1'_f} f$ is an arrow such that

$$\begin{tikzcd}
\mathrm{dom}(f) \arrow[rr, "1'_f"'] \arrow[dr, "f"] & & \mathrm{dom}(f) \arrow[dl, "f"'] \\
& C &
\end{tikzcd}$$

commutes i.e. $1'_f = \mathrm{Id}_{\mathrm{dom}(f)}$. Thus, $\mathcal{U}(f \xrightarrow{1'_f} f) = 1_{\mathrm{dom}(f)}$ as desired. Next we prove that $\mathcal{U}$ preserves the composition. Consider objects $\mathrm{dom}(f) \xrightarrow{f} C, \mathrm{dom}(g) \xrightarrow{g} C$, and $\mathrm{dom}(h) \xrightarrow{h} C$ in $\mathrm{obj}(\mathbf{C}/C)$ such that there is morphism $f' : f \to g$ and $g' : g \to h$. Let $n : \mathrm{dom}(f) \to \mathrm{dom}(g)$ and $m : \mathrm{dom}(g) \to \mathrm{dom}(h)$ making

$$\begin{tikzcd}
\mathrm{dom}(f) \arrow[r, "n"] \arrow[dr, "f"'] & \mathrm{dom}(g) \arrow[r, "m"] \arrow[d, "g"] & \mathrm{dom}(h) \arrow[dl, "h"] \\
& C &
\end{tikzcd}$$

commute. As apparent from diagram, $\mathcal{U}(g' \circ f') = m \circ n = \mathcal{U}(g') \circ \mathcal{U}(f')$.

We will now talk about constructing functor $\mathcal{F} : \mathbf{C}/C \to \mathbf{C}^{\to}$ and $\mathbf{dom} : \mathbf{C}^{\to} \to \mathbf{C}$. Before we get into it, let us first discuss about $\mathbf{C}^{\to}$. In $\mathbf{C}^{\to}$, objects are morphisms of $\mathbf{C}$ and morphism between $f, g \in \mathrm{hom}(\mathbf{C})$ is $(n, k) \in \mathrm{hom}(\mathbf{C})^2$ such that

$$\begin{tikzcd}
\bullet \arrow[r, "n"] \arrow[d, "f"] & \bullet \arrow[d, "g"] \\
\bullet \arrow[r, "k"] & \bullet
\end{tikzcd}$$

commutes. Define $\mathcal{F} : \mathbf{C}/C \to \mathbf{C}^{\to}$ such that for any $f \in \mathrm{obj}(\mathcal{C}/C)$, we have $\mathcal{F}(f) = f$ and $\mathcal{F}(f \xrightarrow{f'} g) = f \xrightarrow{(n, 1_C)} g$. It is easy to see it is a functor. Define $\mathbf{dom} : \mathbf{C}^{\to} \to C$ as $\mathbf{dom}(f) = \mathrm{dom}(f)$ for any $f \in \mathrm{obj}(\mathbf{C}^{\to})$ and $\mathbf{dom}(f \xrightarrow{(n,k)} g) = \mathrm{dom}(f) \xrightarrow{n} \mathrm{dom}(g)$. Again, it's easy to verify that it is a functor.

Finally, we prove that $\mathbf{dom} \circ \mathcal{F} = \mathcal{U}$. For any $f \in \mathrm{obj}(\mathbf{C}/C)$, we have $\mathbf{dom} \circ \mathcal{F}(f) = \mathbf{dom}(\mathcal{F}(f)) = \mathbf{dom}(f) = \mathrm{dom}(f) = \mathcal{U}(f)$ and for any $f \xrightarrow{f'} g$ in $\mathbf{C}/C$, clearly $\mathbf{dom} \circ \mathcal{F}(f \xrightarrow{f'} g) = \mathbf{dom}(\mathcal{F}(f \xrightarrow{f'} g)) = \mathbf{dom}(f \xrightarrow{(n, 1_C)} g) = \mathrm{dom}(f) \xrightarrow{n} \mathrm{dom}(g) = \mathcal{U}(f \xrightarrow{f'} g)$. □

Exercise 1.6. *Construct the coslice category $C/\mathbf{C}$ of a category $\mathbf{C}$ from the dual category operation.*

Proof.

Notation. *In what follows, we refer morphisms of $\mathbf{C}$ by f and corresponding arrow of $\mathbf{C}^{op}$ by $f_{C^{\mathrm{op}}}$.*

Let $f : C \to A$ and $g : C \to B$. Consider the slice of C in $\mathbf{C}^{\mathrm{op}}$. The objects are $A \xrightarrow{f_{C^{\mathrm{op}}}} C$ and morphism $f_{C^{\mathrm{op}}} \xrightarrow{f'_{C^{\mathrm{op}}}} g_{C^{\mathrm{op}}}$ are $A \xrightarrow{k_{C^{\mathrm{op}}}} B$ of $\mathbf{C}^{\mathrm{op}}$ so that

$$\begin{tikzcd}
\mathrm{A} \arrow[rr, "k_{C^{\mathrm{op}}}"'] \arrow[dr, "f_{C^{\mathrm{op}}}"] & & \mathrm{B} \arrow[dl, "g_{C^{\mathrm{op}}}"'] \\
& C &
\end{tikzcd}$$

commutes. Rewriting it in terms of $\mathbf{C}$, we see that $\mathbf{C}^{\mathrm{op}}/C$ has $C \xrightarrow{f} A$ as objects and $\mathrm{cod}(g) \xrightarrow{k} \mathrm{cod}(f)$ as morphism making

$$\begin{tikzcd}
A & & B \arrow[ll, "k"] \\
& C \arrow[ul, "f"] \arrow[ur, "g"'] &
\end{tikzcd}$$

commute. This is exactly the notion of coslice of $\mathbf{C}$ with C. Define functor $\mathcal{W} : C/\mathbf{C} \to C$ so that $\mathcal{W}(f) = \mathrm{cod}(f)$ and $\mathcal{W}(g \xrightarrow{f'} f) = \mathrm{cod}(g) \xrightarrow{k} \mathrm{cod}(f)$ where k is the morphism making

$$\begin{tikzcd}
\mathrm{cod}(f) & & \mathrm{cod}(g) \arrow[ll, "k"] \\
& C \arrow[ul, "f"] \arrow[ur, "g"'] &
\end{tikzcd}$$

Recall that in $\mathbf{C}^{\to}$, objects are morphisms of $\mathbf{C}$ and morphism between $f, g \in \hom(\mathbf{C})$ is $(n, k) \in \hom(\mathbf{C})^2$ such that

$$\begin{tikzcd}
\mathrm{dom}(f) \arrow[d, "f"] & \mathrm{dom}(g) \arrow[l, "n"] \arrow[d] \\
\mathrm{cod}(f) & \mathrm{cod}(g) \arrow[l, "k"]
\end{tikzcd}$$

commutes. Define functor $\mathbf{cod} : \mathbf{C}^{\to} \to \mathbf{C}$ so that $\mathbf{cod}(f) = \mathrm{cod}(f)$ and $\mathbf{cod}(g \xrightarrow{(n,k)} f) = \mathrm{cod}(g) \xrightarrow{k} \mathrm{cod}(f)$ where k is the morphism between codomains making the last diagram commute. Define functor $\mathcal{G} : C/\mathbf{C} \to \mathbf{C}^{\to}$ so that $\mathcal{G}(g) = g$ for any $g \in \mathrm{obj}(C/\mathbf{C})$ and $\mathcal{G}(g \xrightarrow{g'} f) = g \xrightarrow{(1_C,k)} f$. It is easy to verify all of them are functors.

We will now prove that $\mathbf{cod} \circ \mathcal{G} = \mathcal{W}$. For any $g \in C/\mathbf{C}$, we have $\mathbf{cod} \circ \mathcal{G}(g) = \mathbf{cod}(\mathcal{G}(g)) = \mathbf{cod}(g) = \mathrm{cod}(g) = \mathcal{W}(g)$ and for any $g \xrightarrow{g'} f$ in $C/\mathbf{C}$, clearly $\mathbf{cod} \circ \mathcal{G}(g \xrightarrow{g'} f) = \mathbf{cod}(\mathcal{G}(g \xrightarrow{g'} f)) = \mathbf{cod}(g \xrightarrow{(1_C,k)} f) = \mathrm{cod}(g) \xrightarrow{k} \mathrm{cod}(f) = \mathcal{W}(g \xrightarrow{g'} f)$. □

Exercise 1.7. *Let $2 = \{a, b\}$ be any set with exactly 2 elements a and b. Define a functor $F : \mathbf{Sets}/2 \to \mathbf{Sets} \times \mathbf{Sets}$ with $F(f : X \to 2) = (f^{-1}(a), f^{-1}(b))$. Is this an isomorphism of categories? What about the analogous situation with a one element set $1 = \{a\}$ instead of 2.*

Proof. First we will discuss about **Sets**/2. Here we have arrows $X \xrightarrow{f} 2$ as objects, whereas morphism $f \xrightarrow{f'} g$ is a morphism $\text{dom}(f) \xrightarrow{k} \text{dom}(g)$ making

$$\begin{tikzcd} \bullet \arrow[rr, "k"] \arrow[dr, "f"'] & & \bullet \arrow[dl, "g"] \\ & 2 & \end{tikzcd}$$

commute. Define $F : \mathbf{Sets}/2 \to \mathbf{Sets} \times \mathbf{Sets}$ so that $F(X \xrightarrow{f} 2) = (f^{-1}(a), f^{-1}(b))$ and $F(f \xrightarrow{f'} g) = (k_{f^{-1}(a)}, k_{f^{-1}(b)})$ where $k_{f^{-1}(a)}$ and $k_{f^{-1}(b)}$ are restriction of morphism k which makes the last diagram commute. The identity in **Sets**/2 for any object f is simply the identity on the domain of f. Therefore, we have $F(1_f) = (1_{f^{-1}(a)}, 1_{f^{-1}(b)})$ which is indeed identity on the object $(f^{-1}(a), f^{-1}(b))$. Thus, $F(1_f) = 1_{(F(f))}$. Now we will prove that composition is preserved. Consider $f \xrightarrow{f'} g$ and $g \xrightarrow{g'} h$ such that

$$\begin{tikzcd} \bullet \arrow[r, "k"] \arrow[rr, bend left, "nk"] \arrow[dr, "f"'] & \bullet \arrow[r, "n"] \arrow[d, "g"] & \bullet \arrow[dl, "h"] \\ & 2 & \end{tikzcd}$$

commutes. Then we have

$$\begin{aligned} F(g') \circ F(f') &= (n_{g^{-1}(a)}, n_{g^{-1}(b)}) \circ (k_{f^{-1}(a)}, k_{f^{-1}(b)}) \\ &= (n_{g^{-1}(a)} \circ k_{f^{-1}(a)}, n_{g^{-1}(b)} \circ k_{f^{-1}(b)}) \\ &= ((n \circ k)_{f^{-1}(a)}, (n \circ k)_{f^{-1}(b)}) && (\text{ran}(k_{f^{-1}(a)}) \subseteq g^{-1}(a) \text{ and } \text{ran}(k_{f^{-1}(b)}) \subseteq g^{-1}(b)) \\ &= F(g' \circ f') \end{aligned}$$

Functor $F : \mathbf{Sets}/2 \to \mathbf{Sets}^2$ is not an isomorphism of category. Consider $A = \{c, d\}$ and $B = \{c, d, e\}$. There is no $f \in \text{obj}\mathbf{Sets}/2$ mapping to $(A, B) \in \text{obj}(\mathbf{Sets}^2)$ because if there was it would mean for any $x \in f^{-1}(a) = A \subseteq B = f^{-1}(b)$, we have $f(x) = a$ and $f(x) = b$ which is clearly not possible.

We now are going to prove that $\mathbf{Sets}/1 \simeq \mathbf{Sets}$ where $1 = \{a\}$ is any singleton set. Define functor $\mathcal{F} : \mathbf{Sets}/1 \to \mathbf{Sets}$ so that $\mathcal{F}(X \xrightarrow{f} 1) = f^{-1}(a)$ for any $f \in \text{obj}(\mathbf{Sets}/2)$, and $\mathcal{F}(f \xrightarrow{f'} g) = f^{-1}a \xrightarrow{k} g^{-1}(a)$ where k is an arrow making

$$\begin{tikzcd} f^{-1}(a) = \text{dom}(f) \arrow[rr, "k"] \arrow[dr, "f"'] & & g^{-1}(a) = \text{dom}(g) \arrow[dl, "g"] \\ & 1 & \end{tikzcd}$$

commute. Define functor $\mathcal{G} : \mathbf{Sets} \to \mathbf{Sets}/1$ so that $\mathcal{G}(A) = A \xrightarrow{f} 1$ for any $A \in \text{obj}(\mathbf{Sets})$ and $\mathcal{G}(A \xrightarrow{k} B) = f \xrightarrow{f'} g$ such that $f : A \to 1$ and $g : B \to 1$. It is easy to verify these are functor. Let $X \xrightarrow{f} 1 \in \text{obj}(\mathbf{Sets}/1)$. Then we have $\mathcal{G} \circ \mathcal{F}(f) = \mathcal{G}(\mathcal{F}(f)) = \mathcal{G}(f^{-1}(a)) = \mathcal{G}(X) = X \xrightarrow{f} 1$. Now consider some $f' : f \to g$ for some $X \xrightarrow{f} 1$ and $Y \xrightarrow{g} 1$, it follows $\mathcal{G} \circ \mathcal{F}(f') = \mathcal{G}(\mathcal{F}(f')) = \mathcal{G}(X \xrightarrow{k} Y) = f \xrightarrow{f'} g$. We know that X gets mapped to f under $\mathcal{G}$ necessarily because it's an unique arrow $X \to 1$. Thus, $\mathcal{G} \circ \mathcal{F} = 1_{\mathbf{Sets}/1}$. Simillarly, we can show $\mathcal{F} \circ \mathcal{G} = 1_{\mathbf{Sets}}$. $\square$

Remarks. *In general, we have the following result.*

Theorem. *For any category* $\mathbf{C}$ *having a terminal object, A, we must have* $\mathbf{C}/A \simeq \mathbf{C}$.

Proof. Define functor $\mathcal{M} : \mathbf{C}/A \to \mathbf{C}$ so that $\mathcal{M}(f) = \mathrm{dom}(f)$ for any $f \in \mathrm{obj}(\mathbf{C}/A)$ and $\mathcal{M}(f \xrightarrow{f'} g) = \mathrm{dom}(f) \xrightarrow{k} \mathrm{dom}(g)$ such that

$$\begin{tikzcd} \bullet \arrow[rr, "k"] \arrow[dr, "f"'] & & \bullet \arrow[dl, "g"] \\ & 1 & \end{tikzcd}$$

commutes. Define functor $\mathcal{N} : \mathbf{C} \to \mathbf{C}/A$ so that $\mathcal{N}(X) = X \xrightarrow{f} A$, we know such a f exists because A is terminal, and $\mathcal{N}(X \xrightarrow{k} Y) = f \xrightarrow{f'} g$ where $f : X \to A$ and $g : Y \to A$. Again, since A is terminal object, we know that such f and g are necessarily unique. Therefore, we have $\mathcal{N} \circ \mathcal{M}(f) = \mathcal{N}(\mathcal{M}(f)) = \mathcal{N}(\mathrm{dom}(f)) = \mathrm{dom}(f) \xrightarrow{f} A = f$ and $\mathcal{N}(\mathcal{M}(f \xrightarrow{f'} g)) = \mathcal{N}(\mathrm{dom}(f) \xrightarrow{k} \mathrm{dom}(g)) = f \xrightarrow{f'} g$. Thus, $\mathcal{N} \circ \mathcal{M} = 1_{\mathbf{C}/A}$. Simillarly, we can show $\mathcal{M} \circ \mathcal{N} = 1_C$. Hence, $\mathbf{C}/A \simeq \mathbf{C}$. $\square$

We now have the following result as the corollary of the theorem we just proved.

Corollary. *For any singleton set* $\{*\}$, $\mathbf{Sets} \simeq \mathbf{Sets}/\{*\}$.

Exercise 1.8. *Any category* $\mathbf{C}$ *determines a preorder* $P(\mathbf{C})$ *by defining a binary relation $\leq$ on the objects by*

$$A \leq B \textit{ if and only if there is an arrow } A \to B$$

Show that P determines a functor from categories to preoders. Also show that P is a (one-sided) inverse to the evident inclusion functor of preorder into categories (Note that in this problem, we are working in category of small categories).

Proof. Define functor $P : \mathbf{Cats} \to \mathbf{Preord}$ so that $P(\mathbf{C}) = (\mathrm{obj}(\mathbf{C}), \leq)$, where $A \leq B$ if and only if $\hom(A, B) \neq \emptyset$, and $P(\mathbf{C} \xrightarrow{\mathcal{F}} D) = P(\mathbf{C}) \xrightarrow{k} P(D)$ where k is a function induced by $\mathcal{F}$ i.e. $k(A) = \mathcal{F}(A) \in \mathrm{obj}(D)$ for all $A \in \mathrm{obj}(\mathbf{C})$. Notice that if $A \leq B$ then there is some $f \in \hom(A, B)$ but then it means $\mathcal{F}(f) \in \hom(\mathcal{F}(A), \mathcal{F}(B))$ which implies $\mathcal{F}(A) \leq \mathcal{F}(B)$. Thus, k is a montone function. We now prove that P is a functor. Clearly, $P(\mathbf{C} \xrightarrow{1} \mathbf{C}) = P(\mathbf{C}) \xrightarrow{k} P(\mathbf{C}) = 1_{P(C)}$ because k is identity on $\mathrm{obj}(C)$. Consider some categories C, D, and E such that $C \xrightarrow{F} D \xrightarrow{G} E$. Let $P(F) = k$ and $P(G) = m$. Then we have $P(G) \circ P(F) = m \circ k = P(G \circ F)$ because $m \circ k$ is function induced by $G \circ F$. Therefore, P is a functor. Since we already know that any preorder is a category, there must exists an inclusion functor, $I : \mathbf{Preord} \to \mathbf{cats}$, so that $I(A) = A$ and $I(f : A \to B) = A \xrightarrow{f} B$.

Notice that $P \circ I(A) = P(I(A)) = P(A) = (\mathrm{obj}(A), \leq) = A$ and $P \circ I(A \xrightarrow{f} B) = P(I(A \xrightarrow{f} B)) = P(A \xrightarrow{f} B) = A \xrightarrow{f} B$. Hence, $P \circ I(A) = 1_{\mathbf{Preord}}$. However, P is not an isomorphism. To see why consider a small category C defined as

$$\begin{tikzcd} a \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & b \end{tikzcd}$$

In this case, $P(C) = (\{a, b\}, \{(a, b)\})$. Applying I on it, we get the following category

$$a \longrightarrow b$$

Thus, $I(P(C)) \neq C$. $\square$

Remarks. *The failure of isomorphism is primarily based on the fact that $P(C)$ identifies all morphism of* $\hom(A, B)$ *as one morphism, for any object A and B. Therefore, we must have the following theorem to be true.*

Theorem. *Define* $\mathbf{Cats}_1$ *as the category of all categories such that* $\hom(A, B)$ *is at most singleton for any object A and B. Then* $\mathbf{Cats}_1 \simeq \mathbf{Posets}$.

Proof. Defining $P : \mathbf{Cats}_1 \to \mathbf{Preord}$ and $I : \mathbf{Preord} \to \mathbf{cats}$ same as the above proof, it's easy to see $P \circ I = 1_{\mathbf{Preord}}$. Now consider $I \circ P$. Clearly, $I \circ P(C) = I(P(C)) = C$. Notice that $P(C \xrightarrow{F} D) = C \xrightarrow{F} D$ because F is equal to the monotone function induced on C and D. Therefore, $I \circ P(C \xrightarrow{F} D) = I(P(C \xrightarrow{F} D)) = I(C \xrightarrow{F} D) = C \xrightarrow{F} D$. Hence, $\mathbf{Cats}_1 \simeq \mathbf{Posets}$. $\square$

Exercise 1.9. *Show that the free monoid functor*

$$M : \mathbf{Sets} \to \mathbf{Mon}$$

exists, in two different ways:
(a) Assuming the particular choice $M(X) = X^*$ and define its effect

$$M(f) : M(A) \to M(B)$$

on a function $f : A \to B$ to be

$$M(f)(a_1 \cdots a_k) = f(a_1) \cdots f(a_k), a_1, \cdots, a_k \in A.$$

(b) Assume only the UMP of the free monoid and use it to determine M on functions, showing the result to be a functor.

Proof.
(a) First we prove identity is preseved. Consider some set A and let $w \in M(A) = A^*$ ie $w = a_1 a_2 \cdots a_k$ for $a_i \in A$. Then clearly,

$$\begin{aligned} M(1_A)(w) &= 1_A(a_1) \cdots 1_A(a_k) \\ &= a_1 \cdots a_k \\ &= w. \end{aligned}$$

Next we prove that composition is preserved. Consider $f : A \to B$ and $g : B \to C$. Then for any $w \in M(A)$, we have

$$\begin{aligned} (M(g) \circ M(f))(w) &= M(g)(f(a_1) \cdots f(a_k)) \\ &= (g \circ f)(a_1) \cdots (g \circ f)(a_k) \\ &= M(g \circ f)(w) \end{aligned}$$

(b) Consider $f : A \to B$ for some sets A and B. By proposition 1.9, we know free monoids has UMP. Applying UMP on free monoid of A and B, we see that there exists an unique monoidal homomorphism $\bar{f} : M(A) \to M(B)$ making

$$\begin{tikzcd} {|M(A)|} \arrow[rr, "|\bar{f}|"] & & {|M(B)|} \\ & A \arrow[ul, "i_A"] \arrow[ur, "I_B \circ f"'] & \end{tikzcd}$$

commute in **Sets**, where $i_A : A \to |M(A)|$ and $i_B : B \to |M(B)|$ are inclusion functions. Thus, $|\bar{f}| \circ i_A = i_B \circ f$, that is to say, for all $a \in A$, $\bar{f}(a) = f(a)$ but since $\bar{f}$ is monoidal homomorphism, it would mean for any $a_1 \cdots a_n \in M(A)$, we must have

$$\begin{aligned} \bar{f}(a_1 \cdots a_n) &= \bar{f}(a_1) \cdots \bar{f}(a_n) \\ &= f(a_1) \cdots f_{(}a_n) \end{aligned}$$

Define $M(f) = \bar{f}$. Since $M : \mathbf{Sets} \to \mathbf{Mon}$ so that $M(A) = A^*$ and $M(f)(a_1 \cdots a_n) = f(a_1) \cdots f(a_n)$, we can directly apply part (a) to see M is functor. $\square$

Exercise 1.10. *Verify the UMP for free categories on graphs, defined as above with arrows being sequences of edges. Specficially, let* $\mathbf{C}(G)$ *be the free category on the graph* G*, so defined, and* $i : G \to U(\mathbf{C}(G))$ *the graph homomorphism taking vertices and edges to themselves, regarded as objects and arrows in* $\mathbf{C}(G)$*. Show that for any category* $\mathbf{D}$ *and graph homomorphism* $h : G \to U(\mathbf{D})$*, there is a unique functor*

$$\bar{h} : \mathbf{C}(G) \to \mathbf{D}$$

with

$$U(\bar{h}) \circ i = h$$

where $U : \mathbf{Cat} \to \mathbf{Graph}$ *is the underlying forgetful functor.*

Proof. Define $\bar{h} : \mathbf{C}(G) \to \mathbf{D}$ so that $\bar{h}(e_1 \cdots e_n) = h(e_1) \cdots h(e_n)$ for any path $e_1 \cdots e_n \in \mathbf{C}(G)$, $\bar{h}(1_a) = 1_{h(a)}$, and $\bar{h}(a) = h(a)$ for any object $a \in \mathrm{obj}(\mathbf{C}(G))$. It is easy to see

$$\begin{tikzcd}
U(C(G)) \arrow[rr, "U(\bar{h})"] & & U(D) \\
& G \arrow[ul, "i"] \arrow[ur, "h"'] &
\end{tikzcd}$$

commutes, by the way we defined $\bar{h}$. We now prove it is unique. Consider some functor $F : \mathbf{C}(G) \to D$ satisfying $U(F) \circ i = h$. Then clearly

$$\begin{aligned}
F(e_1 \cdots e_n) &= F(e_1) \cdots F(e_n) \\
&= h(e_1) \cdots h(e_n) \\
&= \bar{h}(e_1) \cdots \bar{h}(e_n) \\
&= \bar{h}(e_n \cdots e_n)
\end{aligned}$$

Thus, $F = \bar{h}$. $\square$

Remarks. *We construct functor* $N : \mathbf{Graphs} \to \mathbf{Cats}$*. Consider some graphs* G *and* H *with* $G \xrightarrow{h} H$*. Applying UMP on* $\mathbf{C}(G)$ *and* $\mathbf{C}(H)$*, we see that there exists an unique functor* $\bar{h} : \mathbf{C}(G) \to \mathbf{C}(H)$ *such that*

$$\begin{tikzcd}
U(\mathbf{C}(G)) \arrow[r, "U(\bar{h})"'] & U(\mathbf{C}(H)) \\
G \arrow[u, "i_G"] \arrow[r, "h"] & H \arrow[u, "i_H"']
\end{tikzcd}$$

commutes in $\mathbf{Graphs}$*. But that means for any edge* e *and vertex* v *in* G*, we must have* $\bar{h}(v) = h(v)$ *and* $\bar{h}(e) = h(e)$*. However, since* $\bar{h}$ *is a functor, we have*

$$\begin{aligned}
\bar{h}(e_1 \cdots e_n) &= \bar{h}(e_1) \cdots \bar{h}(e_n) \\
&= h(e_1) \cdots h(e_n)
\end{aligned}$$

Now define functor $N : \mathbf{Graphs} \to \mathbf{Cats}$ *so that* $N(G) = \mathbf{C}(G)$ *and* $N(G \xrightarrow{h} H) = \mathbf{C}(G) \xrightarrow{\bar{h}} \mathbf{C}(H)$*. First we prove identity is preserved. Consider some* $e_1 \cdots e_n \in \mathbf{C}(G)$*, then we have*

$$\begin{aligned}
N(1_G)(e_1 \cdots e_n) &= 1_G(e_1) \cdots 1_G(e_n) \\
& e_1 \cdots e_n.
\end{aligned}$$

where $e_i \in G$ is an edge. Next we prove composition is preserved. Consider the graph homomorphism $h : G \to H$ and $k : H \to K$. Then

$$\begin{aligned}(N(k) \circ N(h))(e_1 \cdots e_n) &= N(k)(N(h)(e_1) \cdots N(h)(e_n)) \\ &= N(k)(N(h)(e_1)) \cdots N(k)(N(h)(e_n)) \\ &= (N(k) \circ N(h))(e_1) \cdots (N(k) \circ N(h))(e_n) \\ &= N(k \circ h)(e_1 \cdots e_n).\end{aligned}$$

Exercise 1.11. *Use the cayley representation to show that every small category is isomorphic to a "concrete" one, that is, one in which objects are sets and the arrows are functions between them.*

Proof. Recall that cayley representation, $\bar{\mathbf{C}}$ of $\mathbf{C}$ is a category with collection of objects as $\{\mathbf{C}/C : C \in \text{obj}(C)\}$ and and arrows $\bar{g} : \mathbf{C}/C \to \mathbf{C}/D$, for some $g : C \to D$ in $\mathbf{C}$ and any $f : X \to C$ in $\bar{C}$ as $g \circ f$ making

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, "g \circ f"] & \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

commute in $\mathbf{C}$. It is easy to see that all objects are are sets and arrows are thus set functions. We now prove $\bar{\mathbf{C}}$ is isomorphic to $\mathbf{C}$. Define functor $\mathcal{F} : \mathbf{C} \to \bar{\mathbf{C}}$ so that $F(C) = \mathbf{C}/C$ and $F(C \xrightarrow{f} D) = \mathbf{C}/C \xrightarrow{\bar{f}} \mathbf{C}/D$. It is easy to see that $\mathcal{F}$ is bijective on objects and arrows. $\square$

Chapter 2

Abstract Structures

Exercise 2.1. *Show that a function between sets is an epimorphism if and only if it is surjective. Conclude that the isos in* **Sets** *are exactly epi-monos.*

Proof. The converse implication is trivial, so we only prove the forward direction. Let $f : A \twoheadrightarrow B$. Suppose f is not surjective. Then there is some $b \in B$ which is not a f-image. Define $i, j : B \to \{x, y\}$ such that $i(c) = x$ for all $c \in B$ and $j(c) = \begin{cases} x & \text{if } c \in f(A) \\ y & \text{if } c \notin f(A) \end{cases}$. This contradicts that f was epic. □

Exercise 2.2. *Show that in a poset category, all arrows are epi-monos.*

Proof. Trivial. □

Exercise 2.3. *(Inverses are unique) If an arrow $f : A \to B$ has inverses $g, g' : B \to A$ (i.e., $g \circ f = 1_A$ and $f \circ g = 1_B$ and similarly for g'), then $g = g'$.*

Proof. Consider the following commutative diagram.

$$\begin{tikzcd} B \arrow[r, "g'"] \arrow[d, "g"] \arrow[dr, "1_B"] & A \arrow[d, "f"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

But then since f is mono, $g = g'$. □

Exercise 2.4. *With regard to a commutative triangle,*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

(a) if f and g are isos (resp. monos, resp. epis), so is h.
(b) if h is monic then so is f.
(c) iif h is epic, so is g.

Proof.
(a) If f and g are isos, then clearly $h^{-1} = f^{-1} \circ g^{-1}$. If f and g are both monos and $h \circ m = h \circ n$ then we have $g \circ (f \circ m) = g \circ (f \circ n)$ implies $f \circ m = f \circ n$ which in turns implies $m = n$ since f and g are monos. Simillarly, if f and g are both epis then so is h.

(b) Consider the following commutative diagram. Consider the following commutative diagram.

$$\begin{tikzcd}
\bullet \arrow[r, shift left, "m"] \arrow[r, shift right, "n"'] & A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, "g"] \\
& & C
\end{tikzcd}$$

Suppose $fm = fn$. Taking g both side, we see $hm = gfm = gfn = hn$, however, since h is mono, we have $m = n$.

(c) Consider the following commutative diagram.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, "g"] & \\
& C \arrow[r, shift left, "m"] \arrow[r, shift right, "n"'] & \bullet
\end{tikzcd}$$

Suppose $mg = ng$. That implies $mh = (mg)f = (ng)f = nh$ but since h is epic, we have $m = n$.

(d) Consider $f : \{1, 2\} \to \{3, 4, 5\}$ defined as $f = \{(1, 3), (2, 4)\}$ and $g : \{3, 4, 5\} \to \{6, 7\}$ as $g = \{(3, 6), (4, 7), (5, 7)\}$. Clearly, $g \circ f$ is injective while g is not. However, since in **Sets** mono is equivalent to injectivity, we are done. $\square$

Exercise 2.5. *Show that the following are equivalent for an arrow*

$$f : A \to B$$

in any category:
(a) f is an iso.
(b) f is both a mono and split epi.
(c) f is both a split mono and an epi.
(d) f is both a split mono and a split epi.

Proof. We are going to prove (b), (c), (d) are equivalent to (a). The forward implication for all of them is trivial, so we only prove the converse.

(b) Let f_R be the right inverse of f. Consider

$$\begin{aligned} f(f_R f) &= (f f_R) f \\ &= 1 \circ f \\ &= f \\ &= f \circ 1 \end{aligned}$$

Thus, $f(f_R f) = f \circ 1$, however since f is mono, we have $f_R f = 1$. As desired.

(c) Let f_L be the left inverse of f. Consider

$$\begin{aligned}(ff_L)f &= f(f_Lf)\\ &= f \circ 1\\ &= f\\ &= 1 \circ f.\end{aligned}$$

However, since f is epi, we have $ff_L = 1$.

(d) Clearly since split mono implies mono, by applying part (b), we have f to be iso. □

Exercise 2.6. *Show that a homomorphism $h : G \to H$ of graph is monic just if it is injective on both edges and vertices.*

Proof. Recall that a directed graph is a structure $G = (G_0, G_1, s, t)$ where G_1 (edges) and G_0 (vertices) are sets, and $s : E \to V$ (source) and $t : E \to V$ (target) are set theoretic functions. Furthermore, graph homomorphism $h : G \to H$ is the ordered pair (h_0, h_1) making

$$\begin{tikzcd}
G_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[d, "h_1"] & G_0 \arrow[d, "h_0"'] \\
H_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & H_0
\end{tikzcd}$$

commute.
Suppose h_1 and h_0 are injective and consider morphisms $g, k : K \to G$ such that $hg = hk$. It means that $(h_1g_1, h_0g_0) = (h_1k_1, h_1k_0)$ which implies $h_1g_1 = h_1k_1$. However, since h_1 is mono (injective), we have $g_1 = k_1$. Similarly, $g_0 = k_0$. Hence, $(g_1, g_0) = (k_1, k_0)$.
Conversely, suppose $h : G \to H$ is monic. Consider the graph $K = (\{x\}, \{\})$. Define $g, k : K \to G$ so that $g_0(x) \neq k_0(x)$. Thus, $g = (g_0, g_1) \neq (k_0, k_1) = k$. Since h is monic, it follows $(h_0g_0, h_1g_1) = hg \neq hk = (h_0k_0, h_1k_1)$, however because $g_1 = k_1$ vacously, we must have $h_0g_0 \neq h_0k_0$. This proves that h_0 is monic and hence injective. For proving h_1 is injective, consider the following cases;
Case-1. There is no pair of vertices, in G, having more than one edge between them. In this case, clearly injectivity of h_0 implies injectivity of h_1.
Case-2. There is atleast one pair of vertices, in G, with more than one edge between them. Let $e_2, e_3 : c \to d$ be such edges in G. Consider the graph $M = (\{x, y\}, \{e\})$. Define $g, h : M \to G$ so that $g_0(x) = c = k_0(x)$ and $g_0(y) = d = k_0(y)$, and $g_1(e) = e_2$ and $k_1(e) = e_3$. Clearly, $g \neq k$. Since h is monic we have $(h_0g_0, h_1g_1) = hg \neq hk = (h_0k_0, h_1k_1)$, however because $g_0 = k_0$, it follows $h_1g_1 \neq h_1k_1$. This proves h_1 is monic and so injective. □

Exercise 2.7. *Show that in any category, any retract of a projective object is also projective.*

Proof. Let P be a projective object and A be a retract of P. That is to say, there are maps $e : P \to A$ and $s : A \to P$ with the property $es = 1_A$. Consider object E and X such that there are maps $g : A \to X$ and an epi $e_2 : E \to X$. Our goal is to define a map $\bar{g} : A \to E$ so that $e_2\,\bar{g} = g$. Rewriting g as $g = g(es) = (ge)s$, we see that ge is a map $P \to X$. Therefore, there exists arrow $\bar{f}$ making

$$\begin{tikzcd}
 & & E \arrow[d, "e_2"] \\
A \arrow[r, shift left, "s"] \arrow[rr, bend right, "g"'] & P \arrow[l, shift left, "e"] \arrow[ur, dashed, "\bar{f}"] & X
\end{tikzcd}$$

commute. Defining $\bar{g}$ as $\bar{g} = \bar{f}s$, we have the desired result. $\square$

Exercise 2.8. *Show that all sets are projective.*

Proof. Consider arbitrary sets P, X, and E such that there is map $f : P \to X$ and epi $e : E \to X$. Our goal is to define a map $\bar{f} : P \to E$ such that

$$\begin{tikzcd}
 & E \arrow[d, two heads, "e"] \\
P \arrow[r, "f"'] \arrow[ur, dashed, "\bar{f}"] & X
\end{tikzcd}$$

commutes. Consider some element $x \in P$, we have $e^{-1}(f(x)) \in E$. Since e is epi, we have $e^{-1}(f(x))$ to be non empty for all $x \in X$. Clearly, there exists a function $z : \{e^{-1}(f(x)) : x \in X\} \to \bigcup\{e^{-1}(f(x)) : x \in X\}$ such that $z(e^{-1}(f(x))) \in e^{-1}(f(x))$. Define $\bar{f} : X \to E$ to be $x \mapsto z(e^{-1}(f(x)))$. $\square$

Exercise 2.9. *Show that the epis among posets are the surjections (on elements), and that the one element poset* **1** *is projective.*

Proof. Trivial. $\square$

Exercise 2.10. *Show that sets, regarded as discrete posets, are projective in the category of posets. Give an example of a poset that is not projective. Show that every projective poset is discrete, that is, a set. Conclude that* **Sets** *is (isomorphic to) the full subcategory of projectives in* **Pos**, *consisting of all projective posets and all monotone maps between them.*

Proof. It is hard not to see that sets, regarded as discrete posets, are easily projective in the category of posets. Consider some non-discrete poset P and X, and a discrete poset E such that there is montone function $f : P \to X$ and an epi $e : E \to X$. Clearly, we can never define $\bar{f}$ which would make

$$\begin{tikzcd}
 & E \arrow[d, "e"] \\
P \arrow[r, "f"'] \arrow[ur, dashed, "\bar{f}"] & X
\end{tikzcd}$$

commute. Consequently, all projective objects in **Posets** must be discrete. Thus, we trivially have the isomorphism from **Sets** to projective objects in **Posets**. $\square$

Exercise 2.11. *Let A be a set. Define an $A-$monoid to be a monoid M equipped with a function $m : A \to \mathcal{U}(M)$ (to the underlying set of M). A morphism $h : (M, m) \to (N, n)$ of $A-$monoids is to be a monoid homomorphism $h : M \to N$ such that $\mathcal{U}(h) \circ m = n$ (a commutative triangle). Together with the evident identities and composites, this defines a category **A-Mon** of A-monoids.*
Show that an initial object in $tbA - Mon$ is the same thing as a free monoid $M(A)$ on A.

Proof. Recall that UMP of the free monoid of A asserts that given any monoid N and any function $A \xrightarrow{f} |N|$, there is an unique monoid homomorphism $M(A) \xrightarrow{\overline{f}} N$ such that $|\overline{f}| \circ i = f$, that is to say, the following diagram commutes

$$\begin{tikzcd} |M(A)| \arrow[rr, "|\overline{f}|"] & & |N| \\ A \arrow[u, "i"] \arrow[urr, "f"'] & & \end{tikzcd}$$

in **Sets**. We are now ready to justify why the free monoid $M(A)$ is intial object of $A-$mon. To attain this goal, we need to show that for any $(N, n) \in \text{obj}(\textbf{A-Mon})$, there is unique morphism $(M(A), i) \to (N, n)$. Trivially, enacting UMP of free monoid get us that. $\square$

Exercise 2.12. *Show that for any Boolean algebra B, Boolean homomorphism $h : B \to \mathbf{2}$ corresponds exactly to ultrafilters in B.*

Proof. First we prove the forward implication. Suppose B is a boolean algebra and consider boolean homomorphism $h : B \to \mathbf{2}$. We hope to prove that $h^{-1}(1)$ and $h^{-1}(0)$ are both ultrafilters. Clearly h^{-1} and $h^{-1}(0)$ are filters since otherwise it would contradict h being a homomorphism. Furthermore because $h : B \to \mathbf{2}$, for all $x \in B$, either $x \in h^{-1}(1)$ or $x \in h^{-1}$ which shows they are indeed ultrafilters. Conversely, suppose U is some ultrafilter on B. Define $g : B \to \mathbf{2}$ as

$$g(x) = \begin{cases} 0, & \text{if } x \notin U \\ 1, & \text{if } x \in U \end{cases}$$

Clearly $g(0)$ must be 0 because $0 \in U$ would mean that U is not maximal filter. Similarly U being a filter assure us that $1 \in U$ and so $g(1) = 1$. It is easy to see $g(a \wedge b) = g(a) \wedge g(b)$ and $g(a \vee b) = g(a) \vee g(b)$ considering the properties of $\mathbf{2}$. $\square$

Exercise 2.13. *In any category with binary products, show directly that*

$$A \times (B \times C) \simeq (A \times B) \times C$$

Proof. First we define UMP of trenary product. In any category $\mathbf{C}$, define trenary product of the objects A, B, and C to be an object P (commonly written as $A \times B \times C$) such that P has the form

$$\begin{tikzcd} & P \arrow[dl, "\pi_a"'] \arrow[d, "\pi_b"] \arrow[dr, "\pi_c"] & \\ A & B & C \end{tikzcd}$$

and for any other object X of the form

$$\begin{tikzcd} & X \arrow[dl, "x_a"'] \arrow[d, "x_b"] \arrow[dr, "x_c"] & \\ A & B & C \end{tikzcd}$$

there exists a unique $f : X \to P$ making the diagram

$$\begin{tikzcd}
& X \arrow[ddl, "x_a"'] \arrow[d, dashed, "\exists! f"] \arrow[dd, bend left, "x_b"] \arrow[ddr, "x_c"] & \\
& P \arrow[dl, "\pi_1"] \arrow[d, "\pi_2"] \arrow[dr, "\pi_3"] & \\
A & B & C
\end{tikzcd}$$

commute, that is, such that $x_i = \pi_i f$. It is easy to see trenary product are unique upto isomorphism. The idea we are employing is to show both $A \times (B \times C)$ and $(A \times B) \times C$ satisfy UMP of trenary product. This would imply they are isomorphic. For what follow we are assuming that it is valid to take ternary and binary product of any objects in the category. Consider some object X such that there are arrows $x_a : X \to A$, $x_b : X \to B$, and $x_c : X \to C$. Taking binary product of A and B first, we see there must be unique arrow $f_1 : X \to A \times B$ making

$$\begin{tikzcd}
& X \arrow[ddl, "x_a"'] \arrow[d, dashed, "\exists! f_1"] \arrow[ddr, "x_b"] & \\
& A \times B \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"'] & \\
A & & B
\end{tikzcd}$$

commute. Subsequently, taking binary product of $A \times B$ and C, we see that there must exists unique morphism $f_2 : X \to (A \times B) \times C$ making

$$\begin{tikzcd}
& X \arrow[ddl, "f_1"'] \arrow[d, dashed, "\exists! f_2"] \arrow[ddr, "x_c"] & \\
& (A \times B) \times C \arrow[dl, "\pi_4"] \arrow[dr, "\pi_3"'] & \\
A \times B & & C
\end{tikzcd}$$

commute, where π_4 is the projection $\pi_4 : (A \times B) \times C \to A \times B$. Summarizing it, we have the following commutative diagram

$$
\begin{tikzcd}
 & X \arrow[ddl, "x_a"'] \arrow[d, dashed, "\exists! f_2"] \arrow[dd, bend left, "x_b"] \arrow[ddr, "x_c"] & \\
 & (A \times B) \times C \arrow[dl, "\pi_1\pi_4"'] \arrow[d, "\pi_2\pi_4"] \arrow[dr, "\pi_3"] & \\
A & B & C
\end{tikzcd}
$$

Therefore, $(A \times B) \times C$ satisfy the UMP of trenary product of A, B and C. By a similar construction, taking binary product of B and C first and then the resultant with A, we would see $A \times (B \times C)$ satisfy UMP of trenary product. Hence, $(A \times B) \times C \simeq A \times (B \times C)$. $\square$

Exercise 2.14.
(a) For any index set I, define the product $\pi_{i \in I} X_i$ of an I-indexed family of objects $(X_i)_{i \in I}$ in a category, by giving a UMP generalizing that for binary products.
(b) Show that in **Sets**, *for any set X the set X^I of all functions $f : I \to X$ has this UMP, with respect to the constant family where $X_i = X$ for all $i \in I$, and thus*

$$X^I \simeq \prod_{i \in I} X.$$

Proof.
(a) The object $\prod_{i \in I} X_i$ is said to be product of indexed family $(X_i)_{i \in I}$ if there exists morphism $\pi_i : \prod_{i \in I} X_i \to X_i$ for each $i \in I$ and for any object A such that there are morphism $x_i : A \to X_i$ for each i, there exists unique arrow $f : A \to \prod_{i \in I} X_i$ with the property that $x_i = \pi_i f$ for any $i \in I$. In other, the diagram

$$
\begin{tikzcd}
 & A \arrow[ddl, "x_j"'] \arrow[d, dashed, "\exists! f"] \arrow[ddr, "x_k"] & \\
 & \prod_{i \in I} X_i \arrow[dl, "\pi_j"'] \arrow[dr, "\pi_k"] & \\
X_j & & X_k
\end{tikzcd}
$$

must commute for all $j, k \in I$.

(b) For every $i \in I$, we have projection morphism $\pi_i : X^I \to X$ which sends $f \mapsto f(i)$. Consider some object A with morphisms $g_i : A \to X$ for every $i \in I$. Define $f : A \to X^I$ such that $a \mapsto (I \xrightarrow{g} X)$ where g maps $i \mapsto g_i(a)$. Thus, for any $a \in A$ and $i \in I$, we have $\pi_i \circ f(a) = \pi_i(f(a)) = \pi_i(g) = g_i(a)$. Hence, f

makes

$$\begin{tikzcd}
& A \arrow[d, dashed, "\exists! f"'] \arrow[ddl, "g_i"'] \arrow[ddr, "g_j"] & \\
& X^I \arrow[dl, "\pi_i"] \arrow[dr, "\pi_j"'] & \\
X & & X
\end{tikzcd}$$

commute for all $i, j \in I$. For uniqueness, assume there is $h : A \to X^I$ making the last diagram commute. Consider some $a \in A$. For simplicity, Let $h(a) = I \xrightarrow{h_1} X$ and $f(a) = I \xrightarrow{h_2} X$. Clearly, we must have $\pi_i \circ h(a) = g_i(a) = \pi_i \circ f(a)$, that is to say, $h_1(i) = h_2(i)$ for all $i \in I$. Hence, $h_1 = h_2$ and consequently $f = h$. Therefore, X^I satisfy UMP of indexed product, and so $X^I \cong \Pi_{i \in I} X$. $\square$

Exercise 2.15. *Given a category* $\mathbf{C}$ *with objects* A *and* B*, define the category* $\mathbf{C}_{A,B}$ *to have objects* (X, x_1, x_2) *where* $x_1 : X \to A$ *and* $x_2 : X \to B$*, and with arrows* $f : (X, x_1, x_2) \to (Y, y_1, y_2)$ *being arrows* $f : X \to Y$ *with* $y_1 \circ f = x_1$ *and* $y_2 \circ f = x_2$*. Show that* $\mathbf{C}_{A,B}$ *has a terminal object just in case* A *and* B *have product in* $\mathbf{C}$*.*

Proof. In general, we have the following result.

Theorem. *Let* I *be an index set. Define category* $C_{(A_i)_{i \in I}}$ *to have objects* $(X, (x_i)_{i \in I})$*, where* $x_i : X \to A_i$*, and with arrows* $f : (X, (x_i)_{i \in I}) \to (Y, (y_i)_{i \in I})$ *being arrow* $f : X \to Y$ *with* $y_i \circ f = x_i$ *for all* $i \in I$*. Then* $C_{(A_i)_{i \in I}}$ *has a terminal object just in case the product* $\prod_{i \in I} A_i$ *exists.*

Clearly, $(\prod_{i \in I} A_i, (\pi_i)_{i \in I})$ is an object of $\mathbf{C}_{(A_i)_{i \in I}}$ and because of UMP of indexed product, we know for any object $(X, (x_i)_{i \in I})$ of $\mathbf{C}_{(A_i)_{i \in I}}$, there is unique morphism $f : (X, (x_i)_{i \in I}) \to (\prod_{i \in I} A_i, (\pi_i)_{i \in I})$ in $\mathbf{C}_{(A_i)_{i \in I}}$. Thus, $\prod_{i \in I} A_i$ is an terminal object of the category. Therefore, we have the desired result as corollary of our theorem. $\square$

Remarks. *It is easy to construct isomorphism* $\mathcal{F} : \mathbf{C} \to \mathbf{C}_{\emptyset}$ *and as a consequent of it, we must have* $\prod_{i \in \emptyset} A_i$ *to be isomorphic to terminal object of* $\mathbf{C}$*. This means that empty product acts as a terminal object in any category.*

Exercise 2.16. *In the category of types* $\mathbf{C}(\lambda)$ *of the* λ*-calculus, determine the product functor* $A, B \mapsto A \times B$ *explicitly. Also, show that for any fixed type* A*, there is a functor* $A \to (-) : \mathbf{C}(\lambda) \to \mathbf{C}(\lambda)$*, taking any type* X *to* $A \to X$*.*

Proof. Define functor $\mathcal{F} : \mathbf{C}(\lambda) \times \mathbf{C}(\lambda) \to \mathbf{C}(\lambda)$ so that $F(A, B) = A \times B$, for any types A and B, and $F(f, g) = \lambda x^{A \times B}.(f\pi_1\ x, g\pi_2\ x)$ for any $A \xrightarrow{f} C$ and $B \xrightarrow{g} D$. We now prove $\mathcal{F}$ is a functor. For any $(1_A, 1_B) \in \mathrm{Arr}(\mathbf{C}(\lambda))$, we have

$$\begin{aligned}
\mathcal{F}(1_A, 1_B)x &= \lambda x^{A \times B}.(1_A\pi_1\ x, 1_B\pi_2\ x) \\
&= \lambda x^{A \times B}.(1_A(\pi_1 x), 1_B(\pi_2 x)) \\
&= \lambda x^{A \times B}.(1_A((\lambda y.\mathrm{fst}(y))x), 1_B((\lambda y.\mathrm{snd}(y))x)) \\
&= \lambda x^{A \times B}.(1_A\ \mathrm{fst}(x), 1_B\ \mathrm{snd}(x)) \\
&= \lambda x^{A \times B}.(\mathrm{fst}(x), \mathrm{snd}(x)) \\
&= \lambda x^{A \times B}.x \\
&= 1_{A \times B} \\
&= 1_{\mathcal{F}(A,B)}.
\end{aligned}$$

Thus, $\mathcal{F}$ preserves identity. Next we prove that it preserves composition. In what follows, for the sake of simplicity, we would defer from specifiying the type of binding variable. Consider some $(f,g),(h,j) \in \mathrm{Arr}(\mathbf{C}(\lambda))$. Then

$$\begin{aligned}
\mathcal{F}(f,g)\circ\mathcal{F}(h,j) &= \lambda x.F(f,g)\,(\mathcal{F}(h,j)\ x)\\
&= \lambda x.\mathcal{F}(f,g)\,((\lambda y.(h\pi_1\ y, j\pi_2\ y))x)\\
&= \lambda x.\mathcal{F}(f,g)(h\pi_1\ x, j\pi_2\ x)\\
&= \lambda x.\mathcal{F}(f,g)(h(\lambda z.\mathrm{fst}(z)\ x), j(\lambda z.\mathrm{snd}(z)\ x))\\
&= \lambda x.\mathcal{F}(f,g)(h\ \mathrm{fst}(x), j\ \mathrm{snd}(x))\\
&= \lambda x.(\lambda y.(f\pi_1\ y, g\pi_2\ y))(h\ \mathrm{fst}(x), j\ \mathrm{snd}(x))\\
&= \lambda x.(\lambda y.(f\mathrm{fst}(y), g\ \mathrm{snd}y))(h\ \mathrm{fst}(x), j\ \mathrm{snd}(x))\\
&= \lambda x.(f(\mathrm{fst}((h\ \mathrm{fst}(x), j\ \mathrm{snd}(x)))), g(\mathrm{snd}((h\ \mathrm{fst}(x), j\ \mathrm{snd}(x)))))\\
&= \lambda x.(f(h\ \mathrm{fst}(x)), g(j\ \mathrm{snd}(x)))\\
&= \lambda x.((f\circ h)\mathrm{fst}(x), (g\circ j)\mathrm{snd}x)\\
&= \lambda x.((f\circ h)\pi_1\ x, (g\circ j)\pi_2\ x)\\
&= \mathcal{F}((f\circ h, g\circ j))\\
&= \mathcal{F}((f,g)\circ(h,j)).
\end{aligned}$$

Hence, $\mathcal{F} : \mathbf{C}(\lambda)\times\mathbf{C}(\lambda)\to\mathbf{C}(\lambda)$ is a functor. Next we define functor $(A\to(-)) : \mathbf{C}(\lambda)\to\mathbf{C}(\lambda)$. Define $\mathcal{G} : \mathbf{C}(\lambda)\to\mathbf{C}(\lambda)$ so that $G(X) = A\to X$ and $G(X\xrightarrow{f}Y) = \lambda g^{A\to X}.f\circ g$. Clearly, $\mathcal{G}(1_X) = \lambda g^{A\to X}.1_X\circ g = \lambda g^{A\to X}.g = 1_{g^{A\to X}} = 1_{\mathcal{G}(X)}$. For proving composition is preserved, let $f : X\to Y$ and $g : Y\to Z$. Then clearly,

$$\begin{aligned}
\mathcal{G}(g)\circ\mathcal{G}(f) &= \lambda h.\mathcal{G}(g)(\mathcal{G}(f)h)\\
&= \lambda h.\mathcal{G}(g)((\lambda h^{A\to X}.f\circ h)h)\\
&= \lambda h.\mathcal{G}(g)(f\circ h)\\
&= \lambda h.(\lambda h^{A\to Y}.g\circ h)(f\circ h)\\
&= \lambda h.g\circ(f\circ h)\\
&= \lambda h.(g\circ f)\circ h\\
&= \mathcal{G}(g\circ f).
\end{aligned}$$

Therefore, $\mathcal{G} = A\to(-)$ is a functor.

□

Exercise 2.17. *In any category* $\mathbf{C}$ *with products, define the graph of an arrow to be the monomorphism*

$$\Gamma(f) = \langle 1_A, f\rangle : A \rightarrowtail A\times B.$$

Show that for $\mathbf{C} = \mathbf{Sets}$ *this determines a functor* $\Gamma : \mathbf{Sets}\to\mathbf{Rel}$ *to the category* $\mathbf{Rel}$ *of relations.*

Proof. Define functor $\Gamma : \mathbf{C}\to\mathbf{Sets}$ so that $\Gamma(A) = A$ for any $A\in\mathrm{obj}(\mathbf{C})$ and $\Gamma(f) = \langle 1_A, f\rangle$ for any $f : A\to B$. This is the same functor we defined in exercise 1.1. □

Exercise 2.18. *Show that the forgetful functor* $U : \mathbf{Mon}\to\mathbf{Sets}$ *from monoids to sets is representable. Infer that* U *preserves all (small) products.*

Proof. The idea we are going to employ is to construct a natural isomorphism between functors $\mathrm{Hom}(\mathbb{N},-) : \mathbf{Mon}\to\mathbf{Sets}$ and $U : \mathbf{Mon}\to\mathbf{Sets}$ which would clealy lead to the desired result. Define natural transformation $\eta : \mathrm{Hom}(\mathbb{N},-)\to U$ as $\eta_A : \mathrm{Hom}(\mathbb{N},A)\to U(A)$ which maps $f\mapsto f(1)$ for any monoid A. First we show η_A is isomorphism. By easy induction, we have η_A to be injective. For surjection, for any

$a \in A$, consider the map $f : \mathbb{N} \to A$ such that $n \mapsto a^n$. We proceed to show η preserves the morphisms. Let A and B be monoids, and consider monoidal homomorphism $k : A \to B$. Clearly

$$\begin{aligned}
(U(k) \circ \eta_A)f &= U(k)(\eta_A(f)) \\
&= U(k)(f(1)) \\
&= k(f(1)) \\
&= (k \circ f)(1) \\
&= \eta_B(k \circ f) \\
&= \eta_B(\hom(\mathbb{N}, K)(f)) \\
&= (\eta_B \circ \hom(\mathbb{N}, k))f.
\end{aligned}$$

Thus, η makes

$$\begin{tikzcd}
\mathrm{Hom}(\mathbb{N}, A) \arrow[r, "\mathrm{Hom}(\mathbb{N},k)"] \arrow[d, "\eta_A"] & \mathrm{Hom}(\mathbb{N}, B) \arrow[d, "\eta_B"'] \\
U(A) \arrow[r, "U(k)"] & U(B)
\end{tikzcd}$$

commute. Consequently, η is a natural isomorphism. Therefore, by corollary 2.22, forgetful functor U preserves the products. $\square$