Fundamentals of Mathematical Logic

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Chapter 1

Propositional Logic and Other Fundamentals

Exercise 1.1. *(Section I, 12) Give a careful proof of Proposition 1.1.9, and show how theorem 1.1.7 is an application of it.*

Proposition 1.1.9: Sentence recursion theorem. *For any set Z, any function $F_0 : Sent_0 \to Z$ and any functions $G_\neg : Z \to Z$ and $G_\bullet : Z \times Z \to Z$, for $\bullet$ any of $\vee$, $\wedge$, $\to$, $\leftrightarrow$, there exists a unique function $F : Sent_L \to Z$ such that F extends F_0 and for all L-senteces ϕ and ψ,*

$$F(\neg\phi) = G_\neg(F(\phi)) \text{ and } F(\bullet\phi\psi) = G_\bullet(F(\phi), F(\psi)).$$

Proof. We shall define recursively a sequence of functions $F_n : \mathrm{Sent}_n \to Z$ for $n \in \omega$ such that for each n, F_{n+1} is extension of F_n and respect the desired relationship. Indeed, given such a F_n, we simply use the unqiue readability for propositional sentences to define values of F_{n+1} on the members of Sent_{n+1}. Thus, for $\phi \in \mathrm{Sent}_{n+1}$, set

$$F_{n+1}(\phi) = \begin{cases} F_n(\phi), & \text{if } \phi \in \mathrm{Sent}_n \\ G_\neg(F_{n+1}(\psi)), & \text{if } \phi = \neg\psi \text{ where } \psi \in \mathrm{Sent}_n \\ G_\bullet(F_{n+1}(\psi), F_{n+1}(\gamma)), & \text{if } \phi = \bullet\psi\gamma \text{ where } \psi, \gamma \in \mathrm{Sent}_n \end{cases}$$

Now we define $F : \mathrm{Sent}_L \to Z$ by $F(\theta) = F_n(\theta)$ for some n such that $\theta \in \mathrm{Sent}_n$. Clearly, F extends F_0 and satisfy the desired relationship because all F_n do. Furthermore, unique extension of atomic truth assignemnt is a special case where $Z = \{T, F\}$, and $G_\neg$ and $G_\bullet$ are truth assignemnt functions. ☐