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LECTURES ON HYPERREALS  
BOOK BY ROBERT GOLDBLATT

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## Chapter 1

# What are Hypereals?



## Chapter 2

### Large sets

**Exercise 2.1.** If  $\emptyset \neq A \subseteq I$ , there is an ultrafilter  $\mathcal{F}$  on  $I$  with  $A \in \mathcal{F}$ .

*Proof.* Define  $\mathcal{H} = \{B \in \mathcal{P}(I) : A \subseteq B\}$ . Since  $A$  is non empty, it follows that  $\mathcal{H}$  has fip. Clearly,  $A \in \mathcal{H}$ , and so  $A \in \mathcal{F}^{\mathcal{H}}$ . Therefore, by zorn lemma, we know that there exists ultrafilter,  $\mathcal{F}$ , such that  $A \in \mathcal{F}^{\mathcal{H}} \subseteq \mathcal{F}$ .  $\square$

**Exercise 2.2.** There exists a nonprincipal ultrafilter on  $\mathbb{N}$  containing the set of even numbers, and another containing the set of odd numbers.

*Proof.* Define  $I = \{x \in \mathbb{N} : x = 2n \text{ for some } n \in \mathbb{N}\}$ . Since  $I$  is infinite, we know that  $\mathcal{F}^{\text{co}}$  is proper and thus, have fip. Applying the zorn's lemma, we see that there exists ultrafilter  $\mathcal{F}$  such that  $\mathcal{F} \subseteq \mathcal{F}^{\text{co}}$ . However, for any  $i \in I$ , we have  $I - \{i\} \in \mathcal{F}^{\text{co}} \subseteq \mathcal{F}$  but, since  $\mathcal{F}$  is ultrafilter,  $\{i\} \in \mathcal{F}$ . However, clearly  $\{i\} \in \mathcal{F}^i$ . Therefore,  $\mathcal{F} \neq \mathcal{F}^i$  for any  $i \in I$ . Likewise we could argue about set of odd numbers.  $\square$

**Exercise 2.3.** An ultrafilter on finite set must be principal.

*Proof.* Let the set be  $A = \{a_1, a_2, \dots, a_n\}$  and  $\mathcal{F}$  be ultrafilter on it. Since  $\{a_1\} \cup \{a_2\} \cup \dots \cup \{a_n\} = A \in \mathcal{F}$ , it must be the case that  $a_i \in \mathcal{F}$  for some  $i$  such that  $1 \leq i \leq n$ . Since  $\mathcal{F}$  is proper, for any  $B \in \mathcal{F}$ , we have  $B \cap \{a_i\} \neq \emptyset$ , but that means  $\mathcal{F} = \mathcal{F}^{a_i}$ .  $\square$

**Exercise 2.4.** For  $\mathcal{H} \subseteq \mathcal{P}(I)$ .

(i) Show that  $\mathcal{F}^{\mathcal{H}}$  is a filter that includes  $\mathcal{H}$ , i.e.  $\mathcal{H} \subseteq \mathcal{F}^{\mathcal{H}}$ .

(ii) Show that  $\mathcal{F}^{\mathcal{H}}$  is included in any other filter that includes  $\mathcal{H}$ .

*Proof.*

(i) By the definition.

(ii) Let  $\mathcal{F}$  be a filter such that  $\mathcal{H} \subseteq \mathcal{F}$ . Let  $A \in \mathcal{F}^{\mathcal{H}}$ , then  $B_1 \cap B_2 \cap \dots \cap B_n \subseteq A$  for some  $B_1, \dots, B_n \in \mathcal{H}$ . Clearly,  $B_1 \cap \dots \cap B_n \in \mathcal{F}$  and so is  $A \in \mathcal{F}$ .  $\square$

**Exercise 2.5.** Let  $\mathcal{F}$  be a proper filter on  $I$ .

(I) Show that  $\mathcal{F} \cup \{A^c\}$  has fip iff  $A \notin \mathcal{F}$ .

(II) Use (I) to deduce that  $\mathcal{F}$  is an ultrafilter iff it is maximal proper filter on  $I$ .

*Proof.*

(I) First we prove forward implication. Notice that if  $A \in \mathcal{F}$  then since  $A \cap A^c = \emptyset$ , it follows  $\mathcal{F} \cup \{A^c\}$  does not have fip.

We prove the converse implication. Notice that since  $\mathcal{F}$  is a proper filter, ie it does not contain  $\emptyset$ , the only way for  $\mathcal{F} \cup \{A^c\}$  to not have fip would be  $A \in \mathcal{F}$ .

(II) We prove the forward implication. Suppose  $\mathcal{F}$  is an ultrafilter. Applying Zorn's lemma on proper filter  $\mathcal{F}$ , we see that there exists a maximal proper filter,  $K$ , such that  $\mathcal{F} \subseteq K$ . If  $K \neq \mathcal{F}$ , it would mean there is some  $A \in K$  such that  $A \notin \mathcal{F}$  but then that would mean  $A^c \in \mathcal{F}$ , and so  $A^c \in K$  which contradicts the

fact that  $K$  is proper since it fails to satisfy  $\text{fip}$ . Therefore, we must have  $F = K$ .

We prove the converse implication. Suppose  $\mathcal{F}$  is maximal proper filter. For the sake of contradiction assume  $\mathcal{F}$  is not an ultra filter i.e. there exists some  $A \in \mathcal{P}(I)$  such that both  $A, A^c \notin \mathcal{F}$ . Clearly, either  $\mathcal{F} \cup \{A\}$  has  $\text{fip}$  or  $\mathcal{F} \cup \{A^c\}$  does. Wlog assume  $\mathcal{F} \cup \{A\}$  has  $\text{fip}$ . Applying Zorn's lemma on  $\mathcal{F} \cup \{A\}$ , we see that there is maximal proper filter  $K$  such that  $\mathcal{F} \cup \{A\} \subseteq K$ . However this means that  $\mathcal{F}$  is not maximal proper filter since  $F \subsetneq F \cup \{A\} \subset P(I)$ .  $\square$

## Chapter 3

# Ultrapower Construction of the hyperreals

**Exercise 3.1.** (section 3.3, I).  $\equiv$  is an equivalence relation on  $\mathbb{R}^{\mathbb{N}}$ .

*Proof.* Let  $\langle a_n \rangle, \langle b_n \rangle, \langle c_n \rangle \in \mathbb{R}^{\mathbb{N}}$ . First we prove that reflexivity. Clearly,  $\{n \in \mathbb{N} : a_n = a_n\} = \mathbb{N} \in \mathcal{F}^{\text{co}} \subseteq \mathcal{F}$ . Thus,  $\langle a_n \rangle \equiv \langle a_n \rangle$ .

Symmetricity is trivial since  $\{n \in \mathbb{N} : a_n = b_n\} = \{n \in \mathbb{N} : b_n = a_n\}$ , if one of them is in  $\mathcal{F}$  then so is the other.

Finally, we prove transitivity. Suppose  $\langle a_n \rangle \equiv \langle b_n \rangle$  and  $\langle b_n \rangle \equiv \langle c_n \rangle$  i.e.  $\{n \in \mathbb{N} : a_n = b_n\}, \{n \in \mathbb{N} : b_n = c_n\} \in \mathcal{F}$ . Clearly,  $\{n \in \mathbb{N} : a_n = b_n = c_n\} = \{n \in \mathbb{N} : a_n = b_n\} \cap \{n \in \mathbb{N} : b_n = c_n\} \in \mathcal{F}$ . Therefore,  $\langle a_n \rangle \equiv \langle c_n \rangle$ .  $\square$

**Exercise 3.2.** (section 3.3, II)  $\equiv$  is a congruence on the ring  $\langle \mathbb{R}^{\mathbb{N}}, \oplus, \odot \rangle$ , which means that if  $r \equiv r'$  and  $s \equiv s'$ , then  $r \oplus s = r' \oplus s'$  and  $r \odot s = r' \odot s'$ .

*Proof.* Suppose  $r \equiv r'$  and  $s \equiv s'$  i.e.  $\{n \in \mathbb{N} : r_n = r'_n\}, \{n \in \mathbb{N} : s_n = s'_n\} \in \mathcal{F}$ . Thus, since  $\{n \in \mathbb{N} : r_n = r'_n, s_n = s'_n\} = \{n \in \mathbb{N} : r_n = r'_n\} \cap \{n \in \mathbb{N} : s_n = s'_n\} \in \mathcal{F}$ , we have  $\{n \in \mathbb{N} : r_n = r'_n, s_n = s'_n\} \subseteq \{n \in \mathbb{N} : r_n + s_n = r'_n + s'_n\} \in \mathcal{F}$ . Similarly,  $\{n \in \mathbb{N} : r_n s_n = r'_n s'_n\} \in \mathcal{F}$ . Therefore,  $r \oplus s \equiv r' \oplus s'$  and  $r \odot s \equiv r' \odot s'$ .  $\square$

**Exercise 3.3.** (section 3.3, III)  $\langle 1, \frac{1}{2}, \frac{1}{3}, \dots \rangle \not\equiv \langle 0, 0, 0, \dots \rangle$ .

*Proof.* Let  $\langle r \rangle = \langle 1, \frac{1}{2}, \frac{1}{3}, \dots \rangle$ . For the sake of simplicity, define  $[[r \neq 0]] = \{n \in \mathbb{N} : r_n \neq 0\}$ . Clearly,  $[[r \neq 0]] \subseteq \mathbb{N}$ . Notice that for any  $n \in \mathbb{N}$ , we have  $n \in [[r \neq 0]]$ . Thus,  $[[r \neq 0]] = \mathbb{N}$ . So,  $[[r \neq 0]] \in \mathcal{F}^{\text{co}} \subseteq \mathcal{F}$  but then it would mean  $[[r = 0]] = [[r \neq 0]]^c \notin \mathcal{F}$ . Therefore,  $\langle r \rangle \not\equiv \langle 0 \rangle$ .  $\square$

**Exercise 3.4.** (section 3.5, I)  $[[r = s]] \cap [[s = t]] \subseteq [[r = t]]$ .

*Proof.* Let  $n \in [[r = s]] \cap [[s = t]]$ . Then it follows that  $r_n = s_n$  and  $s_n = t_n$ , and so  $r_n = t_n$ . Therefore,  $n \in [[r = t]]$ .  $\square$

**Exercise 3.5.** (Section 3.5, II)  $[[r = r']] \cap [[s = s']] \subseteq [[r \oplus s = r' \oplus s']] \cap [[r \oplus s = r' \oplus s']]$

*Proof.* Let  $n \in [[r = r']] \cap [[s = s']]$ . Thus,  $r_n = r'_n$  and  $s_n = s'_n$ , and so  $r_n \oplus s_n = r'_n \oplus s'_n$  and  $r_n \odot s_n = r'_n \odot s'_n$ . Therefore,  $n \in [[r \oplus s = r' \oplus s']] \cap [[r \oplus s = r' \oplus s']]$ .  $\square$

**Exercise 3.6.** (Section 3.5, III)  $[[r = r']] \cap [[s = s']] \cap [[r < s]] \subseteq [[r' < s']]$ .

*Proof.* Let  $n \in [[r = r']] \cap [[s = s']] \cap [[r < s]]$ . Thus,  $r_n = r'_n, s_n = s'_n, r_n < s_n$  but then that means  $r'_n < s'_n$ , and so  $n \in [[r' < s']]$ .  $\square$

**Exercise 3.7.** (Section 3.5, IV) If  $r \equiv r'$  and  $s \equiv s'$ , then  $[[r < s]] \in \mathcal{F}$  iff  $[[r' < s']] \in \mathcal{F}$ .

*Proof.* Suppose  $r \equiv r'$  and  $s \equiv s'$ , that is to say,  $\{n \in \mathbb{N} : r_n = r'_n\}, \{n \in \mathbb{N} : s_n = s'_n\} \in \mathcal{F}$ . First assume  $[[r < s]] \in \mathcal{F}$ . Therefore,  $\{n \in \mathbb{N} : r_n = r'_n\} \cap \{n \in \mathbb{N} : s_n = s'_n\} \cap \{n \in \mathbb{N} : r_n < s_n\} \subseteq \{n \in \mathbb{N} : r' < s'\} \in mc\mathcal{F}$ . Next assume  $[[r' < s']] \in \mathcal{F}$ . It follows  $\{n \in \mathbb{N} : r_n = r'_n\} \cap \{n \in \mathbb{N} : s_n = s'_n\} \cap \{n \in \mathbb{N} : r'_n < s'_n\} \subseteq \{n \in \mathbb{N} : r < s\} \in \mathcal{F}$ .  $\square$

**Exercise 3.8.** (Section 3.8, I) Use only general properties of ordered fields to deduce from the fact that  $[\varepsilon]$  is a positive infinitesimal, the conclusion that  $[\varepsilon]^{-1}$  is greater than every real number.

*Proof.* Notice that for any  $r \in \mathbb{R}^+$ , we have  $*r = [\langle r, r, \dots, \rangle] \in *\mathbb{R}$ . Since  $\varepsilon < *r$ , by fields properties, we have  $0 < *r^{-1} < \varepsilon^{-1}$ . Since  $r$  is arbitrary positive real number, setting  $r = r^{-1}$ , we see that  $*r < \varepsilon^{-1}$  for any  $r \in \mathbb{R}^+$ .  $\square$

**Exercise 3.9.** (Section 3.10, I) If  $A$  is finite, show that  $*A = A$ , and hence  $A$  has non nonstandard members.

*Proof.* Let  $A = \{a_1, \dots, a_n\}$ . Consider some  $[r] \in *A$ . If we could prove that  $r \equiv t$  for some constant sequence  $t \in A$ , we would be done. So we set that as our goal. Notice that since  $[r] \in *A$ , we have  $\{n \in \mathbb{N} : r_n \in A\} = \{n \in \mathbb{N} : r_n = a_1 \vee r_n = a_2 \vee \dots \vee r_n = a_n\} = \{n \in \mathbb{N} : r_n = a_1\} \cup \dots \cup \{n \in \mathbb{N} : r_n = a_n\} \in \mathcal{F}$ . But since  $\mathcal{F}$  is ultrafilter, only one of such set is in  $\mathcal{F}$ . Therefore,  $\{n \in \mathbb{N} : r_n = a_k\} \in \mathcal{F}$  for some  $k$  such that  $1 \leq k \leq n$ . Hence,  $r \equiv a_k$ .  $\square$

**Exercise 3.10.** (Section 3.10, II) Prove:

- (a)  $A \subseteq B$  iff  $*A \subseteq *B$ .
- (b)  $A = B$  iff  $*A = *B$ .

*Proof.*

(a) First we prove the forward implication. Suppose  $A \subseteq B$ . Consider  $[r] \in *A$ . Therefore, we have  $[[r \in A]] \in \mathcal{F}$ , but since  $[[r \in A]] \subseteq [[r \in B]]$ , we have  $[[r \in B]] \in \mathcal{F}$  and so  $[r] \in *B$ . We prove the converse implication. Suppose  $*A \subseteq *B$ . Consider some  $r \in A$ . Then we have  $[r] \in *A$ , and so  $[r] \in *B$ . However, since  $[r] \in B$ , we have  $[[r \in B]] \in \mathcal{F}$  which means  $r \in B$ .

(b) Trivially follows from (a).  $\square$

**Exercise 3.11.** (Section 3.10, III)

*Prove:*

- (a)  $*(A \cup B) = *A \cup *B$ ,
- (b)  $*(A \cap B) = *A \cap *B$ ,
- (c)  $*(A - B) = *A - *B$ ,
- (d)  $*\emptyset = \emptyset$ .

*Proof.*

(a) First we prove the forward implication. Since  $A \subseteq A \cup B$ , applying (2), we have  $*A \subseteq *(A \cup B)$ . Similarly, we have  $*B \subseteq *(A \cup B)$ , and so  $*A \cup *B \subseteq *(A \cup B)$ .

We prove the converse implication. Let  $[r] \in *A \cup *B$ . Then we have following cases;

**case-1.**  $[r] \in *A$ . Thus,  $[[r \in A]] \in \mathcal{F}$ , and since  $[[r \in A]] \subseteq [[r \in A \cup B]]$ , we have  $[[r \in A \cup B]] \in \mathcal{F}$  but that means  $[r] \in *(A \cup B)$ .

**case-2.**  $[r] \in *B$ . The proof is analogous to **case-1**.

(b) First we prove the forward implication. Since  $A \cap B \subseteq A$ , applying (2), we have  $*(A \cap B) \subseteq *A$ . Similarly we have  $*(A \cap B) \subseteq *B$ , and so  $*(A \cap B) \subseteq *A \cap *B$ .

We prove the converse implication. Consider some  $[r] \in *A \cap *B$ . It means that  $[r] \in *A$  and  $[r] \in *B$ , thus  $[[r \in A]], [[r \in B]] \in \mathcal{F}$ . Therefore,  $[[r \in A \cap B]] = [[r \in A]] \cap [[r \in B]] \in \mathcal{F}$ , and so  $[r] \in *(A \cap B)$ .

(c) First we prove the forward implication. Let  $[r] \in *(A \setminus B)$ . Then we have  $[[r \in A]] \cap [[r \notin B]] = [[r \in A \setminus B]] \in \mathcal{F}$ . Thus,  $[[r \in A]], [[r \notin B]] \in \mathcal{F}$ . Hence,  $[r] \in *A$  and  $[r] \notin *B$ , and so  $[r] \in *A \setminus *B$ .

We prove the converse implication. Let  $[r] \in *A \setminus *B$  i.e.  $[r] \in *A$  and  $[r] \notin *B$ . Then we have  $[[r \in A]] \in \mathcal{F}$  and  $[[r \notin B]] \in \mathcal{F}$ . Thus,  $[[r \in A \setminus B]] = [[r \in A]] \cap [[r \notin B]] \in \mathcal{F}$ . Therefore,  $r \in *(A \setminus B)$ .

(d) Clearly,  $*\emptyset = \{[r] \in *\mathbb{R} : [[r \in \emptyset]] \in \mathcal{F}\} = \emptyset$ .  $\square$

**Exercise 3.12.** (Section 3.10, IV) Is it true that  $*(\bigcup_{n=1}^{\infty} A_n) = \bigcup_{n=1}^{\infty} *A_n$ ?



*Proof.* No. Define  $A_n = \{n\}$  for any  $n \in \mathbb{N}$ . Notice that since  $A_n$  is finite, using (1), we have  $*A_n = A_n = \{n\}$ . Since  $[[\omega \in \bigcup_{n=1}^{\infty} A_n]] = \mathbb{N} \in \mathcal{F}$ , we have  $\omega \in *(\bigcup_{n=1}^{\infty} A_n)$ . However,  $[[\omega \in A_n]] = \{n\} \notin \mathcal{F}$ , thus  $\omega \notin *A_n$  for any  $n \in \mathbb{N}$ . This mean  $\omega \notin \bigcup_{n=1}^{\infty} *A_n$ .  $\square$

**Exercise 3.13.** (Section 3.10, V) Show that if  $A \subseteq \mathbb{R}$ , then  $*A \cap \mathbb{R} = A$ .

*Proof.* Suppose  $A \subseteq \mathbb{R}$ . Let  $[r] \in *A \cap \mathbb{R}$ . Since  $\langle r \rangle$  is a constant sequence and  $[r] \in *A$ , it follows  $[r] = \langle r \rangle \in A$ . Thus,  $*A \cap \mathbb{R} \subseteq A$ . This means that  $*A \cap \mathbb{R}$  contains only the constant sequences. Let  $\langle r \rangle \in A$ . Then clearly  $[r] \in *A \cap \mathbb{R}$ .  $\square$

**Exercise 3.14.** For  $a, b \in \mathbb{R}$ , let  $[a, b]$  be the closed interval  $\{x \in \mathbb{R} : a \leq x \leq b\}$ . Prove that  $*[a, b] = \{x \in *\mathbb{R} : a \leq x \leq b\}$ .

*Proof.* We first prove the forward inclusion. Let  $[r] \in *[a, b]$ . Then we have  $\{n \in \mathbb{N} : a \leq r_n \leq b\} = [[r \in [a, b]]] \in \mathcal{F}$  but that means  $[r] \in \{x \in *\mathbb{R} : a \leq x \leq b\}$ . We prove the converse inclusion. Let  $[r] \in \{x \in *\mathbb{R} : a \leq x \leq b\}$ . It follows  $[[r \in [a, b]]] = \{n \in \mathbb{N} : a \leq r_n \leq b\} \in mc\mathcal{F}$ . Therefore,  $[r] \in \{x \in \mathbb{R} : a \leq x \leq b\}$ .  $\square$

**Exercise 3.15.** (Section 3.10, VII)  $*\mathbb{Z}$  is a subring of  $*\mathbb{R}$ .

*Proof.* Clearly,  $*\mathbb{Z} \neq \emptyset$ . Consider some  $[r], [s] \in *\mathbb{Z}$ . Then we have  $[[r \in \mathbb{Z}]], [[s \in \mathbb{Z}]] \in \mathcal{F}$ . Therefore, since  $[[r \in \mathbb{Z}]] \cap [[s \in \mathbb{Z}]] \subseteq [[r+s \in \mathbb{Z}]], [[rs \in \mathbb{Z}]]$ , we have  $[[r+s \in \mathbb{Z}]], [[rs \in \mathbb{Z}]] \in \mathcal{F}$ . Hence,  $[r+s], [rs] \in *\mathbb{Z}$ .  $\square$

**Exercise 3.16.** (Section 3.10, VIII) If  $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ , show that  $*(\mathbb{R}^+) = \{x \in *\mathbb{R}^+ : x > 0\}$ , i.e.,  $*(\mathbb{R}^+) = (*\mathbb{R})^+$ .

*Proof.* First we prove the forward inclusion. Let  $[r] \in *(\mathbb{R}^+)$ . It means that  $\{n \in \mathbb{N} : r_n > 0\} = [[r \in \mathbb{R}^+]] \in \mathcal{F}$  which implies  $[r] \in (*\mathbb{R})^+$ . Next we prove the converse inclusion. Let  $[r] \in (*\mathbb{R})^+$ . Then we have  $\{n \in \mathbb{N} : r_n > 0\} = [[r > 0]] = [[r \in \mathbb{R}^+]] \in \mathcal{F}$  from which it follows  $[r] \in *(\mathbb{R}^+)$ .  $\square$

**Exercise 3.17.** (Section 3.12, I) Show that  $*f$  agrees with  $f$  on  $\mathbb{R}$ : if  $r \in \mathbb{R}$ , then  $*f(r) = f(r)$ .

*Proof.* Let  $r \in \mathbb{R}$ . Notice that  $r$  can be indetified with the constant sequence  $\mathbf{r} \in *R$ . Then we have  $*f([\mathbf{r}]) = [f(\mathbf{r})]$  which is isomorphic to  $f(r)$ .  $\square$

**Exercise 3.18.** (Section 3.12, II) If  $f$  is injective, so is  $*f$ . What about surjectivity?

*Proof.* Suppose  $f$  is injective. Consider some  $[r], [s] \in *\mathbb{R}$  such that  $*f([r]) = *f([s])$ , that is to say  $[f \circ r] = [f \circ s]$ . Since  $f$  is injective, we have  $[[f \circ r = f \circ s]] \subseteq [[r = s]]$  but then, since  $[[f \circ r = f \circ s]] \in \mathcal{F}$ , it follows  $[[r = s]] \in \mathcal{F}$ . Therefore,  $[r] = [s]$ .

Yes, surjectivity is preserved. Suppose  $f$  is surjective. Consider some  $[s] \in *\mathbb{R}$ . Our goal is to prove that there is some  $[r] \in *\mathbb{R}$  such that  $*f([r]) = [s]$ . Define sequence  $(r_n)$  such that, for each  $s_n$ ,  $f(r_n) = s_n$ . We know such a  $r_n$  exists for each  $s_n$  since  $f$  is surjective. We claim that  $[r] \in *\mathbb{R}$  the element we looking for. To prove this claim, we need to show  $[f \circ r] = f([r]) = [s]$ , so we set that as our goal. Clearly,  $[[f \circ r = s]] = \{n \in \mathbb{N} : f(r_n) = s_n\} = \mathbb{N} \in \mathcal{F}$ . Therefore,  $[f \circ r] = f[r] = [s]$ .  $\square$

**Exercise 3.19.** (Section 3.12, III) For  $x \in *\mathbb{R}$ , let

$$|x| = \begin{cases} x, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$$

be the usual definition of absolute value function. Show that this extends the definition of  $|\cdot|$  on  $\mathbb{R}$ :  $||[r]|| = [|r_1|, |r_2|, \dots] = [r_n]$ .

*Proof.* We have the following cases;

**Case-1.** Suppose  $[r] > 0$ . Then  $[[r > 0]] \in \mathcal{F}$ . Since  $[[r > 0]] \subseteq [[[r] = [|r|]]] \in \mathcal{F}$ , we have  $[[r] = [r]] = [r] = [|r_n|]$ .

**Case-2.** Suppose  $[r] = 0$ . That is to say  $[[r = 0]] \in \mathcal{F}$ . Since  $[[r = 0]] \subseteq [[0 = |r|]] \in \mathcal{F}$ , we have  $[[r] = 0 = [|r|]]$ .

**Case-3.** Suppose  $[r] < 0$ . Then  $[[r]] = -[r] = [-r] = [|r|]$ .  $\square$

**Exercise 3.20.** (Section 3.12, IV) Let  $\chi_A$  be the characteristic function of a set  $A \subseteq \mathbb{R}$ . Show that  $^*(\chi_A) = \chi_{^*A}$ .

*Proof.* Let  $[r] \in ^*\mathbb{R}$ . Consider the expression  $^*(\chi_A)[r]$ , we have the following cases;

**Case-1.** Suppose  $[r] \in ^*A$ . Notice that  $^*(\chi_A)[r] = [\chi_A \circ r] = [\chi_A(r_n)]$  but then, since  $[[r \in A]] \in \mathcal{F}$  and  $[[r \in A]] \subseteq [[\chi_A \circ r = 1]]$ , we have  $[[\chi_A \circ r = 1]] \in \mathcal{F}$ . Therefore,  $[\chi_A \circ r] = \chi_A[r] = 1$ .

**Case-2.** Suppose  $[r] \notin ^*A$ . Then we have  $[[r \notin A]] = [[r \in A]]^c \in \mathcal{F}$ . Clearly,  $[[r \notin A]] \subseteq [[\chi_A \circ r = 0]]$ . Thus,  $[[\chi_A \circ r = 0]] \in \mathcal{F}$  and so  $[\chi_A \circ r] = \chi_A[r] = 0$ .  $\square$

**Exercise 3.21.** (Section 3.12, V) Show how to define  $^*f$  when  $f$  is a function of more than one argument.

*Proof.* Define  $^*f([r^1], \dots, [r^n]) = [f(r_1^1, \dots, r_1^n), f(r_2^1, \dots, r_2^n), \dots]$ . We prove that it's well defined. Clearly,  $[[r^1 = r^{1'}]] \cap \dots \cap [[r^n = r^{n'}]] \subseteq [[f \circ (r^1, \dots, r^n) = f \circ (r^{1'}, \dots, r^{n'})]]$ . Therefore,  $r^1 \equiv r^{1'}, \dots, r^n \equiv r^{n'}$  implies  $f \circ (r^1, \dots, r^n) \equiv f \circ (r^{1'}, \dots, r^{n'})$ .  $\square$

**Exercise 3.22.** (Section 3.15, I) If  $A_1, \dots, A_k$  are subsets of  $\mathbb{R}$ , put  $P = A_1 \times \dots \times A_k$  and apply the definition of  $^*P$  to show that

$$^*(A_1 \times \dots \times A_k) = ^*A_1 \times \dots \times ^*A_k.$$

*Proof.* For  $([r^1], [r^2], \dots, [r^k]) \in ^*P$ , we must have  $[[r^1, r^2, \dots, r^k] \in P] \in \mathcal{F}$ . Clearly,  $[[r^1, r^2, \dots, r^k] \in P] \subseteq [[r^n \in A_n]]$ , thus  $[[r^n \in A_n]]$  for any  $n$  such that  $1 \leq n \leq k$ . This means that  $[r^n] \in A_n$  for all such  $n$ . Hence,  $([r^1], \dots, [r^n]) \in ^*A_1 \times \dots \times ^*A_n$ . From this, it follows that  $^*(\mathbb{R}^k) = (^*\mathbb{R})^k$ .  $\square$

**Exercise 3.23.** (Section 3.15, II) Let  $\text{dom } P$  denote the domain of a binary relation. If  $P \subseteq \mathbb{R}^2$ , show that  $^*(\text{dom } P) = \text{dom } ^*P$ .

*Proof.* First we prove the forward inclusion. Let  $[r] \in ^*(\text{dom } P)$ . Then we have  $\{n \in \mathbb{N} : r_n \in \text{dom } P\} = [[r \in \text{dom } P]] \in \mathcal{F}$ .

$$\text{Define sequence } s_n = \begin{cases} s \text{ such that } (r_n, s) \in P, & \text{if } r_n \in \text{dom } P \\ 0, & \text{if } r_n \notin \text{dom } P \end{cases}$$

Then clearly,  $[[r, s] \in P] = [[r \in \text{dom } P]] \in \mathcal{F}$ , but that means  $([r], [s]) \in ^*P$ , and so  $[r] \in \text{dom } ^*P$ .

Now we prove the converse inclusion. Let  $[r] \in \text{dom } ^*P$ . It follows that there exists  $[s] \in ^*\mathbb{R}$  such that  $([r], [s]) \in ^*P$ , but that means  $[[r \in \text{dom } P]] = [[(r, s) \in P]] \in \mathcal{F}$ . Hence,  $[r] \in ^*(\text{dom } P)$ .  $\square$

## Chapter 4

# The Transfer Principle

**Exercise 4.1.** (Section 4.1, I) Verify the truth the  $*$ -transform given in 1 – 5 above, which are;

- (1) The Eudoxus-Archimedes Principle:  $\forall x \in {}^*\mathbb{R} : \exists m \in {}^*\mathbb{N}(x < m)$ .
- (2) Density of the rationals:  $\forall x, y \in {}^*\mathbb{R}(x < y \text{ implies } \exists q \in {}^*\mathbb{Q}(x < q < y))$ .
- (3) Finiteness. Let  $A = \{r_1, \dots, r_k\}$  be a finite subset of  $\mathbb{R}$ . Then  $\forall x \in {}^*A(x = {}^*r_1 \vee x = {}^*r_2 \vee \dots \vee x = {}^*r_k)$ .
- (4) Finitary set operations.  $\forall x \in {}^*\mathbb{R}(x \in {}^*(A \cup B))$  iff  $x \in {}^*A$  or  $x \in {}^*B$ .
- (5) Discreteness of  $\mathbb{N}$ .  $\forall x \in {}^*\mathbb{N}({}^*n \leq x \leq {}^*(n+1) \text{ implies } x = {}^*n \text{ or } x = {}^*(n+1))$ .

*Proof.*

- (1) Let  $x \in {}^*\mathbb{R}$ . Define sequence  $m$  such that  $x_i < m_i$  for every  $i$ . The existence of such  $m$  is guaranteed by Eudoxus-Archimedes Principle in reals. Clearly,  $m \in {}^*\mathbb{N}$  and  $[[x < m]] \in \mathcal{F}$ . Therefore,  $x < m$ .
- (2) Let  $x, y \in {}^*\mathbb{R}$ . Define sequence  $q$  such that  $x_i < q_i < y_i$  for each  $i$ , where  $q_i$ 's are rational. The existence of such  $q_i$ 's is guaranteed by density of rationals in reals. Clearly,  $q \in {}^*\mathbb{Q}$  and  $[[x < q < y]] = \mathbb{N} \in \mathcal{F}$ . Therefore,  $x < q < y$ .
- (3) Let  $x \in {}^*A$ . Then  $[[x = r_1]] \cup [[x = r_2]] \cup \dots \cup [[x = r_n]] = [[x \in A]] \in \mathcal{F}$ . Since finite union in  $\mathcal{F}$  means that exactly one of the set in  $\mathcal{F}$ , we have  $x = {}^*r_1$  or  $x = {}^*r_2$  or  $\dots$  or  $x = {}^*r_n$ .
- (4) Proved in section 3.10, II.
- (5) Let  $x \in {}^*\mathbb{N}$  and suppose  ${}^*n \leq x \leq {}^*(n+1)$ . Then we have  $[[n \leq x \leq n+1]] \in \mathcal{F}$  but due to discreteness of  $\mathbb{N}$ , it follows that  $[[x = n]] \cup [[x = n+1]] = [[x = n \vee x = n+1]] = [[n \leq x \leq n+1]] \in \mathcal{F}$ . Hence,  $x = {}^*n$  or  $x = {}^*(n+1)$ .  $\square$



## Chapter 5

# Hypereals Great and Small

**Exercise 5.1.** (Section 5.2, I) For any positive hyperreal  $a$ , explain why the function  $x \rightarrow a^x$  is defined for all  $x \in {}^*\mathbb{R}$ . Use transfer to explore its properties.

*Proof.* □

**Exercise 5.2.** (Section 5.5, I) Verify that  $\simeq$  and  $\sim$  are equivalence relationships.

*Proof.* We will first argue about  $\simeq$ . Let  $x, y, z \in {}^*\mathbb{R}$ . Clearly, it's reflexive since  $x - x = 0$ . If  $x \simeq y$  then we have  $x - y = \epsilon$ , thus  $y - x = -\epsilon$ , and so  $y \simeq x$ . If  $x \simeq y$  and  $y \simeq z$  then we have  $x - y = \epsilon$  and  $y - z = \delta$ , adding the two get us  $x \simeq z$ . The argument of  $\sim$  is similar, so we omit the details. □

**Exercise 5.3.** (Section 5.5, II) If  $b \simeq x \leq y \simeq c$  with  $b$  and  $c$  real, show that  $b \leq c$ . What if  $b$  and/or  $c$  are not real?

*Proof.* First we will prove an useful lemma, if  $y > 0$  and  $y \simeq k$  then  $k > 0$ . Let  $k = y + \epsilon$ . In the case where  $\epsilon = 0$  and  $\epsilon > 0$ , it follows trivially. So assume  $\epsilon < 0$ . Thus,  $y - \frac{1}{n} < y + \epsilon = k$  for all  $n \in \mathbb{N}$ . Since  $\frac{y}{2}$  a real number, we know that there exists  $m \in \mathbb{N}$  such that  $\frac{1}{m} < \frac{y}{2}$  which implies  $0 < \frac{y}{2} = y - \frac{y}{2} < y - \frac{1}{m} < k$ . This proves the lemma.

Now assume to the contrary that  $c < b$ . Thus,  $0 < c - b$  but it implies that  $0 < x - y$ , since  $c - b \simeq x - y$ , which is a contradiction.

The result would not be true if  $b$  is not real. Consider  $\epsilon \simeq 0 \leq 0 \simeq 0$  for some positive infinitesimal  $\epsilon$ , this would imply  $\epsilon \leq 0$  which is false. □

**Exercise 5.4.** (Section 5.5, III)  $\text{hal}(b) = \{b + \epsilon : \epsilon \in \text{hal}(0)\}$ .

*Proof.* The converse inclusion is trivial. For forward, notice that  $x \in \text{hal}(b)$  means that  $x - b = \epsilon$ , and so  $x = b + \epsilon$ . □

**Exercise 5.5.** (Section 5.5, IV)  $\text{gal}(b) = \{b + c : c \in \text{gal}(0)\}$ .

*Proof.* Again, the converse implication is trivial. For forward, we see that  $x \in \text{gal}(b)$  means  $x - b = c$  where  $c$  is a limited number, and so  $c \in \text{gal}(0)$ . Thus,  $x = b + c$  for some  $c \in \text{gal}(0)$ . □

**Exercise 5.6.** (Section 5.5, V) If  $x \simeq y$  and  $b$  is limited, prove that  $b \cdot x \simeq b \cdot y$ . Show that the result can fail for unlimited  $b$ .

*Proof.* Since  $x \simeq y$ , we have  $x - y = \epsilon$  for some  $\epsilon \in \text{hal}(0)$ . Multiplying by  $b$  on both sides get us,  $bx - by = b \cdot \epsilon$ . It follows that  $b \cdot \epsilon \in \text{hal}(0)$ , so  $bx \simeq by$ . Consider  $b = \frac{1}{\epsilon}$  then  $b \cdot \epsilon = 1$  which is not infinitesimal. □

**Exercise 5.7.** (Section 5.5, VI) Show that any galaxy contains members of  ${}^*\mathbb{Z}$ , of  ${}^*\mathbb{Q} - {}^*\mathbb{Z}$ , and of  ${}^*\mathbb{R} - {}^*\mathbb{Q}$ .

*Proof.* Let  $b \in {}^*\mathbb{R}$ . Clearly,  $b - a$  for  $a \in {}^*\mathbb{Z}$ , such that  $a$  is intger, is limited. □

**Exercise 5.8.** (Section 5.7, I) Prove theorem 5.6.2:

- (a)  $sh(b \pm c) = sh(b) \pm sh(c)$ .
- (b)  $sh(bc) = sh(b) \cdot sh(c)$ .
- (c)  $sh(b/c) = sh(b)/sh(c)$  if  $sh(c) \neq 0$  (ie  $c$  is appreciable).
- (d)  $sh(b^n) = sh(b)^n$ .
- (e)  $sh(|b|) = |sh(b)|$ .
- (f)  $sh(\sqrt[n]{b}) = \sqrt[n]{sh(b)}$ , if  $b \geq 0$ .
- (g) If  $b \leq c$  then  $sh(b) \leq sh(c)$ .

*Proof.* Consider  $b = sh(b) + \varepsilon$  and  $c = sh(c) + \delta$  for some infinitesimal  $\varepsilon$  and  $\delta$ .

(a) For  $sh(b + c)$ , notice that  $b + c - (sh(b) + sh(c)) = \varepsilon + \delta$ . Thus,  $sh(b + c) \simeq b + c \equiv sh(b) + sh(c)$ . From the uniqueness of shadow, it follows  $sh(b + c) = sh(b) + sh(c)$ . We get identical result in case of  $sh(b - c)$  by considering  $b - c - (sh(b) - sh(c)) = \varepsilon - \delta$ .

(b) Notice that  $bc = sh(b)sh(c) + sh(b)\delta + sh(c)\varepsilon + \varepsilon\delta$  where  $sh(b)\delta, sh(c)\varepsilon, \varepsilon\delta$  are infinitesimals. So,  $sh(bc) \simeq bc \simeq sh(b)sh(c)$ , and thus by uniqueness of shadow we have  $sh(bc) = sh(b)sh(c)$ .

(c) Notice that, applying (b), we have  $sh(b/c) = sh(b) \cdot sh(1/c)$ . So if we could prove  $sh(1/c) = 1/sh(c)$ , we would be done. Therefore, we set that as our goal. Clearly,  $\frac{1}{c} - \frac{1}{sh(c)} = \frac{1}{sh(c)+\delta} - \frac{1}{sh(c)} = \frac{\delta}{sh(c)^2+sh(c)\delta}$ . Since  $sh(c)^2 + sh(c)\delta$  is appreciable it follows that  $\frac{\delta}{sh(c)^2+sh(c)\delta}$  is infinitesimal, and thus  $sh(1/c) \simeq 1/c \simeq 1/sh(c)$ . Hence,  $sh(1/c) = 1/sh(c)$  by uniqueness of shadow.

(d) Trivially follows from induction on (b).

(e) If we could prove that  $|b| \simeq |sh(b)|$ , we would be done since from uniqueness of shadow,  $sh|b| = |sh(b)|$  would hold. Notice that  $|b| - |sh(b)| \leq |sh(b)| + |e| - |sh(b)| = |e|$ . If  $0 \leq |b| - |sh(b)|$  then clearly  $|b| - |sh(b)| \leq |e| < 1/n$  for all  $n \in \mathbb{N}$ . If  $|b| - |sh(b)| < 0$  then  $-1/n < -|e| \leq |b| - |sh(b)|$  for all  $n \in \mathbb{N}$ . Thus,  $||b| - |sh(b)|| < 1/n$  for every natural  $n$ . Hence,  $|b| - |sh(b)|$  is infinitesimal as desired.

(f) If  $sh(b^{\frac{1}{n}}) < (sh(b))^{\frac{1}{n}}$  then by squaring both side and applying part (d), we would get  $sh(b) = sh(b^{1/n \cdot n}) = (sh(b^{1/n}))^n < (sh(b))^{1/n \cdot n} = sh(b)$  which is a contradiction. We would get identical contradiction in case of  $(sh(b))^{\frac{1}{n}} < sh(b^{\frac{1}{n}})$ .

(g) It follows from 2nd exercise of section 5.5. □

**Exercise 5.9.** (Section 5.7, I) Show that if  $b, c$  are limited, and  $b \simeq b', c \simeq c'$ , then  $b \pm c \simeq b' \pm c', b \cdot c \simeq b' \cdot c'$ , and  $b/c \simeq b'/c'$  if  $c \not\simeq 0$ . Show that the last result can fail when  $c \simeq 0$ .

*Proof.* Let  $b = b' + \varepsilon$  and  $c = c' + \delta$  for some infinitesimal  $\varepsilon$  and  $\delta$ . Then  $b \pm c - (b' \pm c') = \varepsilon \pm \delta$  which is known to be infinitesimal. Thus,  $b \pm c \simeq b' \pm c'$ . For multiplication we have  $bc - b'c' = b'c' + b'\delta + c'\varepsilon + \varepsilon\delta - b'c' = b'\delta + c'\varepsilon + \varepsilon\delta$  which is infinitesimal. Finally for quotient, we have  $\frac{b}{c} - \frac{b'}{c'} = \frac{b'+\varepsilon}{c'+\delta} - \frac{b'}{c'} = \frac{b'c'+c'\varepsilon-b'c'-b'\delta}{(c')^2+c'\delta} = \frac{c'\varepsilon+b'\delta}{(c')^2+c'\delta}$ . Since  $(c')^2 + c'\delta$  is appreciable, we have  $\frac{b}{c} - \frac{b'}{c'}$  to be infinitesimal. Hence,  $b/c \simeq b'/c'$ . For proving the result does not hold for  $c \simeq 0$ . Consider  $b = 0$ . Then we have  $b/c - b'/c' = 0 - \frac{\varepsilon}{\varepsilon-\delta}$  which is not infinitesimal necessarily. □

**Exercise 5.10.** (Section 5.7, II)

If  $\varepsilon$  is infinitesimal, show that

- (a)  $\sin \varepsilon \simeq 0$ ,
- (b)  $\cos \varepsilon \simeq 1$ ,
- (c)  $\tan \varepsilon \simeq 0$ ,
- (d)  $\sin \varepsilon / \varepsilon \simeq 1$ ,
- (e)  $(\sin \varepsilon) / \varepsilon \simeq 1$ ,
- (f)  $(\cos \varepsilon - 1) / \varepsilon \simeq 0$ .

*Proof.*

(a) Since  $\forall x \in \mathbb{R} (|\sin x| \leq |x|)$  holds in  $\mathbb{R}$ , applying transfer on it, it must be the case that  $\forall x \in {}^*\mathbb{R} (|\sin x| \leq$

$|x|)$  holds in  ${}^*\mathbb{R}$ . Therefore,  $|\sin \varepsilon| \leq |\varepsilon|$ , and so  $\sin \varepsilon$  is infinitesimal. Hence,  $\sin \varepsilon \simeq 0$ .

(b) Since  $\cos \varepsilon = \sqrt{1 - \sin^2 \varepsilon}$ , we have  $\text{sh}(\cos \varepsilon) = \text{sh}\sqrt{1 - \sin^2 \varepsilon} = \sqrt{(\text{sh}(1) - \text{sh}(\sin^2 \varepsilon))} = \sqrt{1 - \text{sh}(\delta^2)} = \sqrt{1 - 0} = 1$ . Clearly,  $\cos \varepsilon \simeq \text{sh}(\cos \varepsilon) = 1$ .

(c)  $\text{sh}(\tan \varepsilon) = \text{sh}(\sin \varepsilon)/\text{sh}(\cos \varepsilon) = 0/1 = 0$ . Hence,  $\tan \varepsilon \simeq 0$ .

(d) Notice that  $|\sin \varepsilon| \leq |\varepsilon|$ , so  $|\sin(\varepsilon)/\varepsilon| \leq 1$ . Thus,  $\text{sh}(|\frac{\sin \varepsilon}{\varepsilon}|) \leq 1$ . We will now show that  $\text{sh}(|\frac{\sin \varepsilon}{\varepsilon}|) < 1$  leads to contradiction which would get us  $\text{sh}(|\frac{\sin \varepsilon}{\varepsilon}|) = 1$ . Clearly if  $\text{sh}(|\frac{\sin \varepsilon}{\varepsilon}|) < 1$  then  $0 = \text{sh}(|\sin \varepsilon|) - \text{sh}|\varepsilon| < 0$  which is false. Therefore,  $(\sin \varepsilon)/\varepsilon \simeq \text{sh}(|\frac{\sin \varepsilon}{\varepsilon}|) = 1$ .

(e) By the sine inequality,  $|\sin \varepsilon|/|\varepsilon| \leq 1$ . By the tangent inequality,  $|x| \leq |\tan(x)|$  for  $x \in [0, \pi/2)$ , we have  $|\cos x| \leq |\sin x|/|x|$ . Combining the two inequalities and taking shadow we see  $1 = \text{sh}(|\cos \varepsilon|) \leq \text{sh}(|\sin(\varepsilon)/\varepsilon|) \leq 1$ . Therefore,  $\text{sh}(|\sin(\varepsilon)/\varepsilon|) = 1$ . Hence,  $(\sin \varepsilon)/\varepsilon \simeq 1$ .

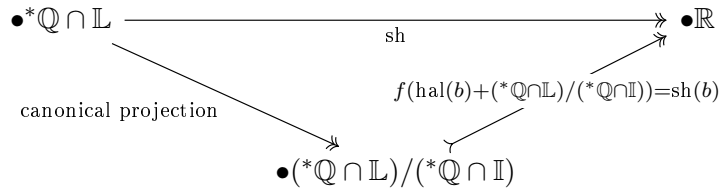
(f) By cosine inequality,  $1 - \frac{\varepsilon^2}{2} \leq \cos \varepsilon$ , we have  $-\varepsilon/2 \leq (\cos \varepsilon - 1)/\varepsilon$ . Since  $\cos \varepsilon \leq 1$ , we have  $(\cos \varepsilon - 1)/\varepsilon \leq 0/\varepsilon = 0$ . Hence  $|(\cos \varepsilon - 1)/\varepsilon| \leq \varepsilon/2$  and so  $(\cos \varepsilon - 1)/\varepsilon$  is infinitesimal, which makes it infinitely close to 0.  $\square$

**Exercise 5.11.** (Section 5.7, III) Show that every hyperreal is infinitely close to some hyperrational number.

*Proof.* Since  $\forall x, y \in \mathbb{R} (x < y \text{ implies } \exists q \in \mathbb{Q} (x < q < y))$  holds in reals (density of rationals), by transfer, we have  $\forall x, y \in {}^*\mathbb{R} (x < y \text{ implies } \exists q \in {}^*\mathbb{Q} (x < q < y))$ . In particular there is a hyperrational  $q$  such that  $r < q < r + \varepsilon$  for any hyperreal  $r$ . Clearly such a  $q$  is infinitely close to  $r$ .  $\square$

**Exercise 5.12.** (Section 5.7, IV) Show that  $\mathbb{R}$  is isomorphic to the ring of limited hyperrationals  ${}^*\mathbb{Q} \cap \mathbb{L}$  factored by its ideal  ${}^*\mathbb{Q} \cap \mathbb{I}$  of hyperreal infinitesimal.

*Proof.*



Consider shadow function restricted to  ${}^*\mathbb{Q} \cap \mathbb{L}$ , in what follows we would use  $\text{sh}$  notation for it instead of  $\text{sh}_{{}^*\mathbb{Q} \cap \mathbb{L}}$ . We will first justify it's surjective. Since  $\mathbb{R} \subseteq {}^*\mathbb{R}$ , applying previous exercise, we see that for any  $x \in \mathbb{R}$ , there is  $q \in {}^*\mathbb{Q}$  such that  $x \simeq q$  but then it would mean  $x \simeq \text{sh}(q)$ . Since both  $x$  and  $\text{sh}(q)$  are reals, it must be the case that  $x = \text{sh}(q)$ . Hence,  $\text{sh}(q) = x$  as desired. Notice that for any  $[r] \in ({}^*\mathbb{Q} \cap \mathbb{L})/({}^*\mathbb{Q} \cap \mathbb{I})$ , we have  $[r] = r + {}^*\mathbb{Q} \cap \mathbb{I}$  which is nothing but  $\text{hal}(r)$ . Define  $f : ({}^*\mathbb{Q} \cap \mathbb{L})/({}^*\mathbb{Q} \cap \mathbb{I}) \rightarrow \mathbb{R}$  as  $\text{hal}(b) \mapsto \text{sh}(b)$ . By the first isomorphism theorem of rings, it follows  $({}^*\mathbb{Q} \cap \mathbb{L})/({}^*\mathbb{Q} \cap \mathbb{I}) \simeq \mathbb{R}$ .  $\square$

**Exercise 5.13.** (Section 5.9, Exercise on Dedekind completeness) For theorem 5.8.1, instead of showing that cauchy sequences converge we can develop a direct proof that any subset  $A \subseteq \mathbb{R}$  with a real upper bound has a least real upper bound. First, for each  $n \in \mathbb{N}$ , let  $s_n$  be the least  $k \in \mathbb{Z}$  such that  $k/n$  is an upper bound of  $A$ . Then take an unlimited  $N \in {}^*\mathbb{N}_\infty$  and let  $L \in \mathbb{R}$  be infinitely close to  $s_N/N$ .

(a) Verify that  $s_n$  exists as defined for  $n \in \mathbb{N}$ .

(b) Show the  $s_N/N$  is limited, so that such a real  $L$  exists under hypothesis of theorem 5.8.1.

(c) Prove that  $L$  is a least upper bound of  $A$  in  $\mathbb{R}$ .

*Proof.* Let  $r$  be an upper bound of  $A \subseteq \mathbb{R}$ .

(a) Let  $n \in \mathbb{N}$ . Consider the set  $B = \{m \in \mathbb{N} : m/n \text{ is an upper bound of } A\}$ . Since  $r \in \mathbb{R}$ , by archimedean property, we know that there exists  $l \in \mathbb{N}$  such that  $r < l$ . Clearly  $l \cdot n \in B$ , and so  $B$  is non empty. By the well ordering principle, it follows that there exists a least element in  $B$ , or in other words there exists a least  $k \in \mathbb{N}$  such that  $k/n$  is an upperbound of  $A$  for each  $n$ .

(b) Let  $n \in \mathbb{N}$ . Since  $s_n$  is the least integer such that  $s_n/n$  is upperbound of  $A$ , clearly  $(s_n - 1)/n$  is not an upper bound. But that means there exists some  $a \in A$  such that  $(s_n - 1)/n \leq a \leq s_n/n$ . Applying transfer on the statement  $(\forall n \in \mathbb{N})(\exists a \in A)((s_n - 1)/n \leq a \leq s_n/n)$ , we get  $(\forall n \in {}^*\mathbb{N})(\exists a \in {}^*A)((s_n - 1)/n \leq a \leq s_n/n)$ . In particular, we have  $(s_N - 1)/N \leq a \leq s_N/N$ . However since  $s_N/N$  and  $(s_N - 1)/N$  are infinitely close, it must be the case that  $a$  is infinitely close to  $s_N/N$ . Applying transfer on the statement  $(\forall a \in A)(a \leq r)$ , where  $r$  is the real upper bound we are given, we get  $(\forall a \in {}^*A)(a \leq r)$ . Thus,  $s_N/N \leq r$ . Next we will prove that is not negative unlimited either. Since  $A \subseteq {}^*A$  and  $s_N/N$  is an upper bound of  ${}^*A$ , we have some  $x \in A$  such that  $x \leq s_N/N$ . Hence,  $x \leq s_N/N \leq r$  for some real  $r, s$ .

(c) For the sake of contradiction assume there is an upper bound,  $K$ , of  $A$  such that  $K < L$ . Notice that there does not exist any  $a \in A$  such that  $a \in [K, L]$  since otherwise  $K$  would not be an upper bound. Applying transfer on  $(\forall a \in A)(a \notin [K, L])$ , we get  $(\forall a \in {}^*A)(a \notin {}^*[K, L])$  which is false since in previous paragraph we demonstrated that there is an element of  ${}^*A$  which is infinitely close to  $L$ .  $\square$

**Exercise 5.14.** (Section 5.11, I) Provide an analogous description of the order structure of the hyperintegers  ${}^*\mathbb{Z}$ .

*Proof.* By transfer, we know that  ${}^*\mathbb{Z}$  is closed under addition, subtraction and multiplication. Observe that only limited hyperinteger are members of  $\mathbb{Z}$  since if  $k \in {}^*\mathbb{Z}$  is limited then  $|k| \leq n$  for some  $n \in \mathbb{N}$  but that means  $-n \leq k \leq n$ . Applying transfer on  $\forall x \in \mathbb{Z}(-n \leq x \leq n \rightarrow x = -n \vee x = -n+1 \vee \dots \vee x = 1 \vee \dots \vee x = n)$ , thus  $k \in \{-n, -n+1, \dots, n\}$ , and so  $k \in \mathbb{Z}$ . Therefore, all numbers of  ${}^*\mathbb{Z} - \mathbb{Z}$  are unlimited and thus either less than all members of  $\mathbb{Z}$  or greater than it. Let  $K \in {}^*\mathbb{Z} - \mathbb{Z}$ . Define  $\gamma(K) = \{K\} \cup \{K \pm n : n \in \mathbb{N}\}$ , clearly all members of  $\gamma(K)$  are unlimited and  $\gamma(K)$  is isomorphic to  $\mathbb{Z}$  under the usual ordering. Alternatively, we can define  $\gamma(K) = \{H \in {}^*\mathbb{Z} : K \sim H\} = \text{gal}(K) \cap {}^*\mathbb{Z}$ . Notice that  $\mathbb{Z}$  itself is a  ${}^*\mathbb{Z}$  galaxy,  $\mathbb{Z} = \text{gal}(0) \cap {}^*\mathbb{Z}$ .

We define order on  ${}^*\mathbb{Z}$  galaxy as  $\gamma(Y) < \gamma(H)$  iff  $Y \leq H$  whenever  $K \not\sim H$ . Clearly there is no greatest  ${}^*\mathbb{Z}$  galaxy since  $\gamma(K) < \gamma(2K)$  for any positive unlimited  $K$ . Similarly there is no smallest galaxy either. To see why notice that either  $K$  is even or  $K + 1$  is, by transfer. If  $K + 1$  is even, we have  $\gamma(K + 1/2) < \gamma(K + 1) = \gamma(K)$ . If  $K$  is even then we have  $\gamma(K) < \gamma(K)$ .

We can imagine ordering on  ${}^*\mathbb{Z}$  as ordering of  $\mathbb{Z}$  surrounded by ordered set of  ${}^*\mathbb{Z}$  galaxies with no greatest or least.  $\square$

**Exercise 5.15.** (Section 5.11, II) Show that for any  $M \in {}^*\mathbb{N}$  there is an  $N \in {}^*\mathbb{N}$  that is divisible by all members of  $\{1, 2, \dots, M\}$ . Hence show that there exists a hypernatural number  $N$  that is divisible by every standard positive integer.

*Proof.* We extend the factorial function  $! : \mathbb{N} \rightarrow \mathbb{N}$  to  $! : {}^*\mathbb{N} \rightarrow {}^*\mathbb{N}$  using extension of function. Applying transfer on the statement  $(\forall n \in \mathbb{N})(k \leq n \rightarrow k | n!)$ , we get  $(\forall n \in {}^*\mathbb{N})(k \leq n \rightarrow k | n!)$ . Thus, letting  $N = M!$  does the job.  $\square$

**Exercise 5.16.** (Section 5.11, III) Develop a theory of prime factors in  ${}^*\mathbb{N}$ : if  $\Pi$  is the set of standard prime numbers, with enlargement  ${}^*\Pi \subseteq {}^*\mathbb{N}$ , prove the following.

- (a)  ${}^*\Pi$  consists precisely of those hypernaturals  $> 1$  that have no nontrivial factors in  ${}^*\mathbb{N}$ .
- (b) Every hypernatural number  $> 1$  has a hyperprime factor, i.e., is divisible by some members of  ${}^*\Pi$ .
- (c) Two hypernaturals are equal if they have exactly the same factors of the form  $p^n$  with  $p \in {}^*\Pi$  and  $n \in {}^*\mathbb{N}$ .
- (d) A hypernatural number is divisible by every standard positive integer iff it is divisible by  $p^n$  for every standard prime  $p$  and every  $n \in \mathbb{N}$ .

*Proof.*

- (a) By transfer on  $(\forall x \in \Pi)(x > 1 \rightarrow x \text{ have no nontrivial factors in } \mathbb{N})$ .
- (b) By transfer on  $(\forall n \in \mathbb{N})(n > 1 \rightarrow \exists p \in \Pi)(p | n)$ .
- (c) By transfer on  $(\forall x \in \mathbb{N})(\forall y \in \mathbb{N})(x = y \leftrightarrow \text{they have exactly the same factors of the form } p^n \text{ with } p \in \Pi \text{ and } n \in \mathbb{N})$ .
- (d) Trivial.  $\square$



## Chapter 6

# Convergence of Sequences and Series

**Exercise 6.1.** (Section 6.3, I) Prove theorem 6.3.1. If  $\lim_{n \rightarrow \infty} s_n = L$  and  $\lim_{n \rightarrow \infty} t_n = M$  in  $\mathbb{R}$ . Then

- (a)  $\lim_{n \rightarrow \infty} (s_n + t_n) = L + M$ .
- (b)  $\lim_{n \rightarrow \infty} (cs_n) = cL$ , for any  $c \in \mathbb{R}$ ,
- (c)  $\lim_{n \rightarrow \infty} (s_n t_n) = LM$ ,
- (d)  $\lim_{n \rightarrow \infty} (s_n/t_n) = L/M$ , if  $M \neq 0$ .

*Proof.* For what follows consider  $N \in {}^*\mathbb{N}_\infty$ . It means that  $s_N \simeq L$  and  $t_N \simeq M$ .

(a) Since  $s_N \simeq L$  and  $t_N \simeq M$ , it follows  $s_N + t_N \simeq L + M$ . Since  $N$  was arbitrary, we have  $s_n + t_n \simeq L + M$  for all unlimited  $n$ . Hence,  $\lim_{n \rightarrow \infty} (s_n + t_n) = L + M$ .

(b) Since  $s_N \simeq L$ , we have  $cs_N \simeq cL$ . Since  $N$  was arbitrary,  $cs_n \simeq cL$  for all unlimited  $n$ . Hence,  $\lim_{n \rightarrow \infty} cs_n = cL$ .

(c) Since  $s_N \simeq L$  and  $t_N \simeq M$ , we have  $s_N t_N \simeq LM$  but since  $N$  was arbitrary, it follows  $s_n t_n \simeq LM$  for all unlimited  $n$ . Hence,  $\lim_{n \rightarrow \infty} (s_n t_n) = LM$ .

(d) Again, from  $s_N \simeq L$  and  $t_N \simeq M$ , we have  $s_N/t_N \simeq L/M$  but since  $N$  was arbitrary,  $\lim_{n \rightarrow \infty} s_n/t_n = L/M$ .  $\square$

**Exercise 6.2.** (Section 6.4, I) Prove theorem 6.4.2. A real valued sequence

- (a) diverges to infinity iff all of its extended terms are positive unlimited; and
- (b) diverges to negative infinity iff all of its extended terms are negative unlimited.

*Proof.* Let the sequence be  $(s_n)$ .

(a) First we prove forward direction. Let  $r \in \mathbb{R}$ . Since the sequence diverges, we know that there is some  $n \in \mathbb{N}$  such that  $(\forall m \in \mathbb{N})(m \geq n \rightarrow s_m > r)$  holds true. Applying transfer on it, we have  $(\forall m \in {}^*\mathbb{N})(m \geq n \rightarrow s_m > r)$  to be true. In particular,  $s_N > r$  for any  $N \in {}^*\mathbb{N}_\infty$  since  $N > n$ . It follows that  $s_N$  is positive unlimited for all  $N \in {}^*\mathbb{N}_\infty$ . We prove the converse direction. Let  $r \in \mathbb{R}$ . Notice that for any unlimited  $N \in {}^*\mathbb{N}_\infty$  if  $M \geq N$  then  $M \in {}^*\mathbb{N}_\infty$ . Therefore, it follows that  $(\exists n \in {}^*\mathbb{N})(\forall m \in {}^*\mathbb{N})(m \geq n \rightarrow s_m > r)$  since  $s_m$  is positive unlimited by assumption. Transferring the statement to  $\mathbb{R}$ , we have  $(\exists n \in \mathbb{N})(\forall m \in \mathbb{N})(m \geq n \rightarrow s_m > r)$  to be true in  $\mathbb{R}$ . Since  $r$  was arbitrary, we have  $(s_n)$  to be divergent to positive infinity.

(b) Follows from (a) by considering the sequence  $(-s_n)$ .  $\square$

**Exercise 6.3.** (Section 6.5, I) Prove theorem 6.5.1. A real values sequence  $\langle s_n \rangle$  is Cauchy in  $\mathbb{R}$  iff all it's extended terms are infinitely close to each other, i.e., iff  $s_m \simeq s_n$  for all  $m, n \in {}^*\mathbb{N}_\infty$ .

*Proof.* First we prove the forward implication. Suppose  $\langle s_n \rangle$  is cauchy. Let  $\varepsilon \in \mathbb{R}^+$  and  $j \in \mathbb{N}$  be such that  $(\forall m, n \in \mathbb{N})(m, n \geq j \rightarrow |s_m - s_n| < \varepsilon)$ . Applying transfer on the latter we know  $(\forall m, n \in {}^*\mathbb{N})(m, n \geq j \rightarrow |s_m - s_n| < \varepsilon)$  is true in  ${}^*\mathbb{R}$ . In particular, we have  $|s_M - s_N| < \varepsilon$  for arbitrary  $M, N \in {}^*\mathbb{N}_\infty$  since  $j$  is limited. Since  $\varepsilon$  was arbitrary positive real number, we have  $s_M \simeq s_N$ . We now prove the converse direction. Suppose  $s_m \simeq s_n$  for all  $m, n \in {}^*\mathbb{N}_\infty$ . Let  $\varepsilon \in \mathbb{R}^+$  be arbitrary. Notice that for any  $K \in {}^*\mathbb{N}_\infty$ , if  $L \geq K$  then  $L \in {}^*\mathbb{N}_\infty$ , and also notice for any  $L, K \in {}^*\mathbb{N}_\infty$ ,  $s_L - s_K = \delta$  where  $\delta$  is infinitesimal. Therefore, we have  $(\exists j \in {}^*\mathbb{N})(\forall m, n \in {}^*\mathbb{N})(m, n \geq j \rightarrow |s_m - s_n| \leq \varepsilon)$  to be true in  ${}^*\mathbb{R}$ . Applying transfer to it, we

have  $(\exists j \in \mathbb{N})(\forall m, n \in \mathbb{N})(m, n \geq j \rightarrow |s_m - s_n| < \varepsilon)$  to be true in  $\mathbb{R}$ . Since  $\varepsilon \in \mathbb{R}^+$  was arbitrary, we have  $(s_n)$  to be Cauchy.  $\square$

**Exercise 6.4.** (Section 6.5, II). *Cauchy's convergence Criterion.*

*Prove the forward direction of A real valued sequence converges in  $\mathbb{R}$  iff it is Cauchy.*

*Proof.* Let  $(s_n)$  be a real valued sequence and suppose it converges in  $\mathbb{R}$  to  $L$ . Therefore, by theorem 6.1.1, we have  $s_N \simeq L$  for all  $n \in {}^*\mathbb{N}_\infty$  but that means all extended terms are infinitely close to each other i.e. for any  $n, m \in {}^*\mathbb{N}_\infty$ ,  $s_n \simeq L \simeq s_m$ . Therefore, by theorem 6.5.1, we have  $(s_n)$  to be Cauchy.  $\square$

**Exercise 6.5.** (Section 6.7, I) *Let  $(s_n)$  and  $(t_n)$  be real valued sequences with limits,  $L, M$  respectively. Show that if  $s_n \leq t_n$  for  $n \in \mathbb{N}$ . Then  $L \leq M$ .*

*Proof.* Applying transfer on  $(\forall n \in \mathbb{N})(s_n \leq t_n)$ , we have  $(\forall n \in \mathbb{N})(s_n \leq t_n)$  to be true in  ${}^*\mathbb{R}$ . In particular for any  $N \in {}^*\mathbb{N}_\infty$ , we have  $s_N \leq t_N$ , however, since  $L \simeq s_N \leq t_N \simeq M$ , we have  $L \leq M$  by exercise 2 of section 5.5.  $\square$

**Exercise 6.6.** (Section 6.7, II) *If  $r_n \leq s_n \leq t_n$  in  $\mathbb{R}$  for all  $n \in \mathbb{N}$ , and  $\lim_{n \rightarrow \infty} r_n = \lim_{n \rightarrow \infty} t_n$ , show that  $(s_n)$  converges to this same limit.*

*Proof.* Applying transfer on  $(\forall n \in \mathbb{N})(r_n \leq s_n \leq t_n)$ , we have  $(\forall n \in {}^*\mathbb{N})(r_n \leq s_n \leq t_n)$  to be true in  ${}^*\mathbb{R}$ . In particular, for any  $N \in {}^*\mathbb{N}_\infty$ , we have  $r_N \leq s_N \leq t_N$ . However, since  $(r_N), (t_N)$  are limited,  $s_N$  is limited. By the shadow inequality, we have  $\text{sh}(r_N) \leq \text{sh}(s_N) \leq \text{sh}(t_N) = \text{sh}(r_N)$ . Hence,  $\text{sh}(s_N) = \text{sh}(r_N) = \lim_{n \rightarrow \infty} r_n$  but that means  $r_N \simeq \lim_{n \rightarrow \infty} r_n$  for all  $N \in {}^*\mathbb{N}_\infty$ . Hence,  $(s_n)$  converges to  $\lim_{n \rightarrow \infty} r_n$ .  $\square$

**Exercise 6.7.** (Section 6.7, III) *If a sequence converges in  $\mathbb{R}$ , show that it has exactly one cluster point.*

*Proof.* Suppose  $(s_n)$  converges to  $L$  and assume it has two cluster point  $A$  and  $B$ . By definition of convergence we have  $s_N \simeq L$  for all  $N \in {}^*\mathbb{N}_\infty$  but that would imply  $B \simeq L \simeq A$  by the definition of cluster point, since they are reals, it follows  $A = L = B$ .  $\square$

**Exercise 6.8.** (Section 6.7, IV) *Suppose that a real valued sequence has a single cluster point. If the sequence is bounded, must it be convergent? what if it is unbounded?*

*Proof.* Let  $(s_n)$  be a bounded sequence which has a unique cluster point i.e. for some  $M \in {}^*\mathbb{N}_\infty$ , we have  $s_M \simeq L$  for some real  $L$ . Suppose it is not convergent. Therefore, there must exist  $N \in {}^*\mathbb{N}_\infty$  such that  $s_N \not\simeq L$ . However since  $(s_n)$  is bounded, we have  $s_N$  to be limited and hence infinitely close to a real number ie its shadow. But if  $\text{sh}(s_N) \neq L$ , clearly  $\text{sh}(s_N)$  is a cluster point which contradicts the assumption that there was a unique cluster point.

Consider the sequence  $\langle s \rangle = (0, 1, 0, 2, 0, 3, \dots)$ . Clearly this sequence has only one cluster point and is unbounded but it does not converge to any real number.  $\square$

**Exercise 6.9.** (Section 6.8, I) *Prove, by nonstandard reasoning, that both the limit superior and limit inferior are cluster points of the sequence  $s$ .*

*Proof.* Let  $L = \limsup s$  and  $\varepsilon \in \mathbb{R}^+$ . Since  $L$  is supremum of  $C_s$ , we know there exists  $N \in {}^*\mathbb{N}_\infty$  such that  $L - \varepsilon < s_N \leq L$ . Consider some arbitrary  $n \in \mathbb{N}$ . We have  $(\exists m \in {}^*\mathbb{N})(m > n \wedge L - \varepsilon < s_m)$  to be true, and applying transfer on it, it follows  $(\exists m \in \mathbb{N})(m > n \wedge L - \varepsilon < s_n)$  is true in  $\mathbb{R}$ . Since  $\varepsilon \in \mathbb{R}^+$  and  $n \in \mathbb{N}$  was arbitrary,  $(\forall \varepsilon \in \mathbb{R}^+)(\forall n \in \mathbb{N})(\exists m \in \mathbb{N})(m > n \wedge L - \varepsilon < s_m)$  is true in  $\mathbb{R}$ . Applying transfer we see  $(\forall \varepsilon \in {}^*\mathbb{R}^+)(\forall n \in {}^*\mathbb{N})(\exists m \in {}^*\mathbb{N})(m > n \wedge L - \varepsilon < s_n)$  holds in  ${}^*\mathbb{R}$ . Picking  $\varepsilon$  as some infinitesimal and  $n$  as unlimited, we follow there is some  $m \in {}^*\mathbb{N}$  making  $m > n$  and  $L - e < s_m$  true, that is to say, there is unlimited  $m$  such that  $L < s_m + e$ . Therefore, it follows  $L \leq \text{sh}(s_m)$  for an unlimited  $m$ . However, since  $L$  is upper bound of  $C_s$ , we must have  $L = \text{sh}(s_m)$  making it a cluster point. We can argue about  $\liminf$  in a similar fashion.  $\square$

**Exercise 6.10.** (Section 6.9, I) *Formulate the definition of the limit inferior of an arbitrary real valued sequence.*

answer:

**Proposition** A real number  $L$  is equal to  $\liminf s$  if and only if

- (1)  $L < s_n$  or  $L \simeq s_n$  for all unlimited  $n$ ; and
- (2)  $s_n \simeq L$  for at least one unlimited  $n$ .

*Proof.* The condition (1) holds iff  $L \leq \text{sh}(s_n)$ . Thus, it implies  $L$  is lower bound of set of cluster points,  $C_s$ . The condition (2) asserts that  $L$  is a cluster point i.e.  $L \in C_s$ . Together they imply that  $L$  is infimum (since it's the smallest element) of the set. Hence, it is equal to  $\liminf s$ .  $\square$

**Exercise 6.11.** (Section 6.9, III) Formulate and prove theorems about the limit inferior of a bounded sequence that correspond to theorems 6.8.4 and 6.8.5.

answer:

**Proposition-1.** If  $s$  is a bounded real valued sequence with limit inferior,  $\liminf$ , then for any positive real  $\varepsilon$ :

- (1) Some standard tail of  $s$  has all its terms bigger than  $\liminf - \varepsilon$  i.e.  $\liminf - \varepsilon < s_n$  for all but finitely many  $n \in \mathbb{N}$ .
- (2)  $s_n < \liminf + \varepsilon$  for infinitely many  $n \in \mathbb{N}$ .

*Proof.* Let  $L = \liminf s$ . Consider  $\varepsilon \in \mathbb{R}^+$ . First we prove (1). For any unlimited  $N$ , we have  $L < s_N$  or  $L \simeq s_N$ . Thus,  $L - \varepsilon < s_N$  holds for any unlimited  $N$ . Transferring  $(\exists n \in {}^*\mathbb{N})(\forall m \in {}^*\mathbb{N})(m \geq n \wedge L - \varepsilon < s_m)$ , we have  $(\exists n \in \mathbb{N})(\forall m \in \mathbb{N})(m > n \wedge L - \varepsilon < s_m)$  to be true in  $\mathbb{R}$  but that simply means  $L - \varepsilon < s_m$  is true for all but finitely many  $n$ . We now prove (2). Let  $m \in \mathbb{N}$ . We know that  $L$  is infinitely close to  $s_N$  for some unlimited  $N$ . Thus,  $(\exists n \in {}^*\mathbb{N})(n > m \wedge s_n < L + \varepsilon)$  is true in  ${}^*\mathbb{R}$ . Transferring it, we have  $(\exists n \in \mathbb{N})(n > m \wedge s_n < L + \varepsilon)$  to be true in  $\mathbb{R}$ . However since  $m \in \mathbb{N}$  was arbitrary, it follows that for any  $m \in \mathbb{N}$  there exists  $n \in \mathbb{N}$  such that  $n > m$  and  $s_n < L + \varepsilon$  satisfying the proposition.  $\square$

**Proposition-2.** For any bounded real valued sequence  $s$ ,

$$\liminf_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} I_n = \lim_{n \rightarrow \infty} \left( \inf_{m \geq n} s_m \right)$$

where  $I_n = \inf_{m \geq n} s_m$ .

*Proof.* First we will prove that  $\liminf$  is upper bound of  $I$ . Take an extended term  $s_N$  infinitely close to the cluster point  $\liminf$ . For any  $m \in \mathbb{N}$ , we have  $I_m \leq s_n$  for all limited  $n \geq m$ , and hence it holds for all hypernatural  $n \geq m$  by transfer. In particular,  $I_m \leq s_N$ , as  $\liminf \simeq s_N$ , we have  $I_m \leq \liminf$ .

Let  $L = \lim I_n$ . Then clearly  $L$  is supremum of  $I$  since  $I$  is nondecreasing. However, as we showed  $\liminf$  is an upper bound, we must have  $L \leq \liminf$ . We will now show that  $L < \liminf$  is impossibility. Suppose  $L < \liminf$  and consider some  $\varepsilon \in \mathbb{R}^+$  such that  $L < \liminf - \varepsilon$ . By proposition-1, we know that there exists  $n \in \mathbb{N}$  such that standard tail beyond  $n$  is bounded below by  $\liminf - \varepsilon$ . But that means that  $\liminf - \varepsilon \leq I_n$ , and so  $L < \liminf - \varepsilon \leq I_n$  which contradicts the fact that  $L$  was upper bound of  $I$ . Hence,  $\liminf = L$ .  $\square$

**Exercise 6.12.** (Section 6.9, III) If  $s$  is a bounded sequence, show that for each  $\varepsilon \in \mathbb{R}^+$  there is some  $n \in \mathbb{N}$  such that the standard tail  $s_n, s_{n+1}, \dots$  is contained in the interval  $(\liminf - \varepsilon, \limsup + \varepsilon)$ .

*Proof.* By theorem 6.8.4 and previous exercise, we know that there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $s_n < \limsup + \varepsilon$  and there exists  $N' \in \mathbb{N}$  such that for all  $n \geq N'$ ,  $s_n < \limsup + \varepsilon$ . Defining  $m = \max\{N, N'\}$ , the tail following  $s_m$  is the desired one.  $\square$

**Exercise 6.13.** (Section 6.11, I) Give an example of a series that diverges but has  $a_n$  infinitesimal for all unlimited  $n$ .

*Proof.* Consider the sequence  $(s_n)$  defined as  $(s_n) = (1/n)$ . Clearly for any unlimited  $n$ ,  $a_n$  is infinitesimal. However, notice that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n} &\geq 1 + \sum_{n=1}^{\infty} \sum_{k=1}^{2^{n-1}} \frac{1}{n} \\ &\geq 1 + \sum_{n=1}^{\infty} \frac{1}{2}\end{aligned}$$

Since the last series is divergent, so is  $\sum_{n=1}^{\infty} \frac{1}{n}$ . □

**Exercise 6.14.** (Section 6.11, II) Give nonstandard proofs of the usual rules of arithmetically combining convergent series.

- (a)  $\sum_1^{\infty} a_i + \sum_1^{\infty} b_i = \sum_1^{\infty} (a_i + b_i)$ ,  
(b)  $\sum_1^{\infty} a_i - \sum_1^{\infty} b_i = \sum_1^{\infty} (a_i - b_i)$ ,

*Proof.* We will first prove (a), then (c), and then use (a) and (c) for proving (b). For what follow, Let  $L = \sum_1^{\infty} a_i$  and  $M = \sum_1^{\infty} b_i$ .

(a) Let  $N \in {}^*\mathbb{N}_{\infty}$ . Since  $\sum_1^n (a_i + b_i) = \sum_1^n a_i + \sum_1^n b_i$  holds in  $\mathbb{R}$  for all  $n \in \mathbb{N}$ , by transfer we know it also holds in  ${}^*\mathbb{R}$  for all  $n \in {}^*\mathbb{N}$ . Hence,

$$\sum_1^N (a_i + b_i) = \sum_1^N a_i + \sum_1^N b_i.$$

and thus,

$$\begin{aligned}\text{sh} \left( \sum_1^N (a_i + b_i) \right) &= \text{sh} \left( \sum_1^N a_i + \sum_1^N b_i \right) \\ &= \text{sh} \left( \sum_1^N a_i \right) + \text{sh} \left( \sum_1^N b_i \right) \\ &= L + M\end{aligned}$$

As desired.

(c) Let  $N \in {}^*\mathbb{N}_{\infty}$ . Then it follows

$$\begin{aligned}\text{sh} \left( \sum_1^N ca_i \right) &= \text{sh} \left( c \sum_1^N a_i \right) && \text{(by transfer)} \\ &= c \cdot \text{sh} \left( \sum_1^N a_i \right) \\ &= c \cdot L\end{aligned}$$

as desired.

(b) Let  $N \in {}^*\mathbb{N}_\infty$ . Then

$$\begin{aligned}
\text{sh} \left( \sum_1^N (a_i - b_i) \right) &= \text{sh} \left( \sum_1^N (a_i + (-b_i)) \right) \\
&= \text{sh} \left( \sum_1^N a_i \right) + \text{sh} \left( \sum_1^N (-b_i) \right) && \text{(by part a)} \\
&= \text{sh} \left( \sum_1^N a_i \right) + (-1) \cdot \text{sh} \left( \sum_1^N (-b_i) \right) && \text{(by part c)} \\
&= L + (-M) \\
&= L - M.
\end{aligned}$$

□

**Exercise 6.15.** (Section 6.11, III) Suppose that  $a_i \geq 0$  for all  $i \in \mathbb{N}$ . Prove that  $\sum_1^\infty a_i$  converges iff  $\sum_1^n a_i$  is limited for all unlimited  $n$ , and that this holds iff  $\sum_1^n a_i$  is limited for some unlimited  $n$ .

*Proof.* First we prove (1)  $\Rightarrow$  (2). Suppose  $\sum_1^\infty a_i$  converges to a real number  $L$ . Therefore, we must have  $\sum_1^N a_i \simeq L$  for all unlimited  $N$  but that means  $\sum_1^N a_i$  is limited. The direction (2)  $\Rightarrow$  (3) is easy to see.

We prove (3)  $\Rightarrow$  (2). Let  $n \in \mathbb{N}$ . Then we know that there exists  $m \in {}^*\mathbb{N}$  such that  $m > n$  and  $\sum_1^m a_i \leq L+1$  where  $L$  is the shadow of the limited extended term. Transferring it, we see that for any  $n \in \mathbb{N}$ , there exists  $m \in \mathbb{N}$  such that  $m > n$  and  $\sum_1^m a_i \leq L+1$ . Since  $a_i \geq 0$  for all  $i \in \mathbb{N}$ , we have  $|\sum_1^m a_i| = \sum_1^m |a_i| \leq L+1$ . Hence, the sequence of partial sum is bounded which implies all extended term are limited by theorem 6.4.1.

We prove (2)  $\Rightarrow$  (1). Since  $\sum_1^n a_i$  is limited for all unlimited  $n$ , by theorem 6.4.1, we know that sequence of partial sum is bounded. Also notice that since  $a_i \geq 0$ , for each  $n \in \mathbb{N}$ ,  $\sum_1^n a_i \leq \sum_1^{n+1} a_i$ . Therefore, the sequence of partial sums is non decreasing. Hence, by theorem 6.2.1, we know it is convergent. □

**Exercise 6.16.** (Section 6.11, IV) Comparison test.

Let  $\sum_1^\infty a_i$  and  $\sum_1^\infty b_i$  be two real series of non negative terms, with  $\sum_1^\infty b_i$  convergent. If  $a_i \leq b_i$  for all  $i \in \mathbb{N}$ , use result (III) to show that  $\sum_1^\infty a_i$  is convergent.

*Proof.* Since  $a_i \leq b_i$  for all  $i \in \mathbb{N}$ , by transfer, we know  $a_n \leq b_n$  for all unlimited  $n$ . By exercise (III), it follows  $\sum_1^n b_i$  is limited for all unlimited  $n$  but that means  $\sum_1^n a_i$  is also limited for all unlimited  $n$ , and thus by exercise (III), is convergent. □

**Exercise 6.17.** (Section 6.11, V) Show that the comparison test holds under the weaker assumption that  $a_n \leq b_n$  for all unlimited  $n$ . Show that this weaker assumption is equivalent to requiring that there be some limited  $k \in \mathbb{N}$  with  $a_n \leq b_n$  for all  $n \geq k$ .

*Proof.* For the first part, notice that since  $a_n \leq b_n$  for all unlimited  $n$ , we have

$$\sum_m^n a_i \leq \sum_m^n b_i \simeq \text{sh} \left( \sum_m^n b_i \right) = 0$$

for any unlimited  $m, n$  with  $m \leq n$ , because  $\sum_1^\infty b_i$  is convergent. However, since  $\sum_m^n a_i$  is finite, we have  $\text{sh}(\sum_m^n a_i) \leq 0$ . Since partial sums of  $a_i$  is non-decreasing sequence of non-negative terms, it follows  $0 \leq \sum_1^n a_i - \sum_1^{m-1} a_i = \sum_m^n a_i$  which gives us  $0 = \text{sh}(\sum_m^n a_i) \simeq \sum_m^n a_i$ . Hence, by cauchy convergence criterion, we know that  $\sum_1^\infty a_i$  converges.

For the second part, we know there exists  $k \in {}^*\mathbb{N}$  such that for any all  $n \geq k$ ,  $a_n \leq b_n$  i.e. when  $k$  is unlimited. Transferring it, we get the desired result. For converse, we know for any  $n \geq k$ ,  $a_n \leq b_n$ . Transferring it, we have for any  $n \in {}^*\mathbb{N}$  such that  $n \geq k$ ,  $a_n \leq b_n$  but then since  $k$  is limited, the inequality holds for all unlimited numbers. □

**Exercise 6.18.** (Section 6.11, VI) Let  $\sum_1^\infty a_i$  and  $\sum_1^\infty b_i$  be two series of positive terms such that the sequence  $\langle a_i/b_i : i \in \mathbb{N} \rangle$  is convergent in  $\mathbb{R}$ . Deduce that either both series converge, or both diverge.

*Proof.*

**Notation.** Define  $\langle s \rangle = \langle a_i/b_i : i \in \mathbb{N} \rangle$  and  $L = \lim_{n \rightarrow \infty} s$ .

We claim that the following result is true.

**Theorem 1.** If  $\sum_1^\infty a_i$  and  $\sum_1^\infty b_i$  be two series of positive terms such that the sequence  $\langle a_i/b_i : i \in \mathbb{N} \rangle$  is convergent in  $\mathbb{R}$  Then for any unlimited  $m$  and  $n$  it follows that

$$\sum_m^n a_i \text{ is infinitesimal if and only if } \sum_m^n b_i \text{ is infinitesimal.}$$

Assuming this claim for the moment, it follows from cauchy's convergence criterion that  $\sum_1^\infty a_i$  is convergent if and only if  $\sum_1^\infty b_i$  is convergent, which is the first part of the problem.

We prove our claim now. Suppose  $\sum_m^n a_i$  is infinitesimal for all unlimited  $m$  and  $n$  with  $n \geq m$ . Since the sequence  $\langle s \rangle$  is convergent, we have  $b_N \simeq 1/L \cdot a_N$  for any unlimited  $N$ . Thus,

$$\sum_m^n b_i \simeq \sum_{i=m}^n (1/L \cdot a_i) = 1/L \cdot \sum_{i=m}^n a_i$$

which is known to be infinitesimal. The other direction can be proved by identical argument considering  $a_N \simeq Lb_N$  for any unlimited  $N$ .

We can apply the same chain of thought for proving divergence. Recall that a sequence is divergent if all of its extended terms are unlimited. Since we have  $a_N \simeq Lb_N$ , it follows immediately that if one of them is divergent then so is the other.  $\square$

**Exercise 6.19.** (Section 6.11, VII) Let  $c \in \mathbb{R}$ . Recall the identity

$$1 + c + c^2 + \cdots + c^n = \frac{1 - c^{n+1}}{1 - c}. \quad (6.1)$$

- (a) Considering the case of unlimited  $n$ , show that the series  $\sum_1^\infty c^i$  converges if  $|c| < 1$ .  
(b) Show that  $\sum_m^n c^i$  is infinitesimal when  $m$  and  $n$  are unlimited, either by applying result (a) or by making further use of the above inequality.

*Proof.*

(a) If  $c = 0$  then the proposition is trivially true, so suppose  $c \neq 0$  for what follows. We claim the following result is true.

**Theorem 1.** For any  $c \in \mathbb{R}$  such that  $0 < |c| < 1$ , the series  $\sum_1^\infty |c|^i$  converges.

Before we get to the proof, we see that if the theorem 1 is true then the series  $\sum_1^\infty c^i$  is convergent because it is absolutely convergent which is what we desire.

We begin the proof by noticing that  $\langle |c|^i \rangle = \langle |c|^i \rangle$  for  $0 < c < 1$  is a non increasing sequence with a lower bound, and thus by monotone convergence theorem, is convergent. Let  $L = \lim |c|^n$  and consider some unlimited  $N$ . Then applying transfer on the identity 6.1, we have

$$\sum_1^N |c|^i = \frac{1 - |c|^{N+1}}{1 - |c|} - 1 \simeq \frac{1 - L}{1 - |c|} - 1$$

which is clearly limited. Therefore, it follows that  $\sum_1^\infty c^i$  is convergent when  $|c| < 1$ .

(b) We are going to give alternative proof to it instead of using part (a). Again we will prove that  $\sum_1^\infty c^i$  is absolutely convergent. We already know that the sequence  $\langle |c^i| \rangle = \langle |c|^i \rangle$  for  $0 < c < 1$  is convergent. Let  $L = \lim |c^i|$ . Then we have

$$\sum_m^n |c^i| = \sum_1^n |c^i| - \sum_1^m |c^i| \simeq \frac{1-L}{1-c} - \frac{1-L}{1-c} = 0.$$

□

**Exercise 6.20.** (Section 6.11, VIII) *Ratio test: Convergence.*  
Suppose that

$$\limsup_{i \rightarrow \infty} \frac{|a_{i+1}|}{|a_i|} < 1$$

in  $\mathbb{R}$ . Prove that the series  $\sum_1^\infty a_i$  is absolutely convergent.

*Proof.* We will prove the series  $\sum_1^\infty a_i$  is absolutely convergent by demonstrating that difference of partial sum of sequence is always less than or equal to difference of partial sum of a convergent sequence which will show that difference of partial sum of our sequence gets infinitely close eventually. We start off by proving the following theorem.

**Theorem 1.** Suppose that  $\limsup_{i \rightarrow \infty} \frac{|a_{i+1}|}{|a_i|} < 1$  in  $\mathbb{R}$ . Then there exists a positive real  $c < 1$  with  $|a_{n+1}| < c|a_n|$  for all unlimited  $n$ .

Before we discuss the proof of it, there is an interesting corollary which follows from this. If theorem-1 is true for all unlimited numbers then by easy transfer, we must have  $|a_{n+1}| < c|a_n|$  for all but finitely many  $n \in \mathbb{N}$ .

We will now prove theorem-1. If  $a_i = 0$  for any  $i \in \mathbb{N}$  then  $\frac{|a_{i+1}|}{|a_i|}$ , i.e. the  $i$ th term of the sequence, become undefined. Therefore, we must have  $a_i \neq 0$  for any  $i \in \mathbb{N}$ . Thus,  $0 < \frac{|a_{n+1}|}{|a_n|}$  for all  $n \in \mathbb{N}$ . Let  $N$  be arbitrary unlimited natural number. Then we have  $0 < \frac{|a_{N+1}|}{|a_N|} \leq \limsup_{i \rightarrow \infty} \frac{|a_{i+1}|}{|a_i|} < 1$ . Let  $L = \limsup_{i \rightarrow \infty} \frac{|a_{i+1}|}{|a_i|}$ . Define  $c = \frac{L+1}{2}$ . Then we have  $0 < \frac{|a_{N+1}|}{|a_N|} < c$ . Thus,  $|a_{N+1}| < c|a_N|$ . Since  $N$  was arbitrary, we have our result.

**Corollary 1.1.** There is some limited  $k \in \mathbb{N}$  such that  $|a_{n+1}| < c|a_n|$  for all  $n \geq k$ .

**Theorem 2.** Suppose that  $\limsup_{i \rightarrow \infty} \frac{|a_{i+1}|}{|a_i|} < 1$  in  $\mathbb{R}$ . Then there is some  $k$  such that for all  $n \geq k$ , we have  $|a_{k+n}| < c^n|a_k|$ , and hence  $\sum_{k+m}^{k+n} |a_i| \leq \sum_m^n c^i |a_k|$ .

The existence of such a  $k$  is guaranteed by corollary 1.1, furthermore, it also implies

$$|a_{k+n}| < c|a_{k+n-1}| < c^2|a_{k+n-2}| < \cdots < c^n|a_{k+n-n}| = c^n|a_k|.$$

Thus,  $|a_{k+n}| < c^n|a_k|$ . It is easy to see that the last inequality implies

$$0 < \sum_{k+m}^{k+n} |a_i| \leq \left( \sum_m^n c^i \right) |a_k|.$$

By an easy transfer we see that the last inequality also hold in hyperreals. However, by exercise VII, we know that  $\sum_m^n c^i$  is infinitesimal for any unlimited  $n$  and  $m$  such that  $n \geq m$  but that means  $\sum_{k+m}^{k+n} |a_i|$  is infinitesimal for any unlimited  $m$  and  $n$ . In particular, replacing  $m$  by  $m - k$  and  $n$  by  $n - k$ , we get the following result.

**Corollary 2.1.**  $\sum_1^\infty |a_i|$  is convergent, that is to say,  $\sum_1^\infty a_i$  is absolutely convergent.

□

**Exercise 6.21.** (Section 6.11, IX) *Ratio Test: divergence.*

Suppose that

$$\liminf_{i \rightarrow \infty} \frac{|a_{i+1}|}{|a_i|} > 1$$

in  $\mathbb{R}$ . Prove that the series  $\sum_1^\infty a_i$  diverges.

*Proof.* Before we get into the proof, notice that  $a_i \neq 0$  for any  $i \in \mathbb{N}$  since otherwise  $\frac{|a_{i+1}|}{|a_i|}$  becomes undefined. Consider some unlimited  $n$ . Then we have

$$1 < \liminf_{i \rightarrow \infty} \frac{|a_{i+1}|}{|a_i|} \leq \frac{|a_{N+1}|}{|a_N|},$$

which implies  $|a_{N+1}| > |a_N|$ . We summarize it into a theorem.

**Theorem 1.** Suppose that  $\liminf_{i \rightarrow \infty} \frac{|a_{i+1}|}{|a_i|} > 1$ . Then  $a_i \neq 0$  for any  $i \in \mathbb{N}$  and  $|a_{N+1}| > |a_N|$  for any unlimited  $N$ .

Next we prove the following result.

**Theorem 2.** Suppose that  $\liminf_{i \rightarrow \infty} \frac{|a_{i+1}|}{|a_i|} > 1$ . Then there exists  $k \in \mathbb{N}$  such that for all  $m > n$ ,  $|a_m| > |a_k|$ .

In natural numbers, we know that for any  $n, m \in \mathbb{N}$ , there is some  $p \in \mathbb{N}$  such that if  $m > n$  then  $m = n + p$ . Therefore, by transfer, it is also true in hypernaturals. In particular, it holds for unlimited numbers. Let  $m$  and  $n$  be unlimited numbers such that  $m > n$ . Then, by theorem-1, it follows

$$|a_m| = |a_{n+p}| > |a_{(n+p)-1}| > \cdots > |a_{n+1}| > |a_n|.$$

Transferring  $(\exists n \in {}^*\mathbb{N})(\forall m \in {}^*\mathbb{N})(m > n \rightarrow |a_m| > |a_n|)$  back to  $\mathbb{N}$ , we have our result.

By existential transfer, we have following corollary to be true.

**Corollary 2.1.** For all unlimited  $n$ ,  $|a_n| > |a_k|$ .

For the sake of contradiction, assume  $\sum_1^\infty a_i$  is convergent. Therefore, for any unlimited  $N$ , we must have  $\sum_1^{N+1} a_i - \sum_1^N a_i \simeq 0$  but then it implies

$$0 < |a_k| < |a_{N+1}| = \sum_1^{N+1} a_i - \sum_1^N a_i \simeq 0$$

which means  $0 < |a_k| \leq 0$ . A contradiction. □

**Exercise 6.22.** (Section 6.11, X) Show that for any real number  $x$ , the hyperreal  $x^n/n!$  is infinitesimal when  $n$  is unlimited.

*Proof.* Define  $a_i = (x^i/i!)$ . We will first prove that  $\sum_1^\infty a_i$  is convergent which would easily get us  $\sum_{n-1}^n a_i = \sum_{n-1}^n x^i/i! = x^n/n!$  to be infinitesimal. For any unlimited  $N$ , we have  $\frac{(x^{N+1}/(N+1)!)}{(x^N/N!)} = \frac{x}{N+1}$  which is clearly infinitesimal. It means that  $\limsup_{i \rightarrow \infty} \frac{|a_{i+1}|}{|a_i|} < 1$ . Therefore, by ratio test, it follows  $\sum_1^\infty a_i$  is convergent. Hence,  $\sum_m^n a_i$  is infinitesimal for any unlimited  $m$  and  $n$  such that  $n \geq m$ . In particular, setting  $m = n - 1$  gets us  $\sum_{n-1}^n a_i = \sum_{n-1}^n x^i/i! = x^n/n!$  to be infinitesimal. □

**Exercise 6.23.** (Section 6.11, XI) *Leibniz's alternating series test.*

Suppose  $\langle a_i : i \in \mathbb{N} \rangle$  is a real sequence that is nonincreasing (i.e.  $a_i \geq a_{i+1}$ ) and converges to 0. Prove that the alternating series

$$\sum_1^\infty (-1)^{i+1} a_i = a_1 - a_2 + a_3 - a_4 + \cdots$$

converges.



*Proof.* We first make the observation that all the terms in the sequence are positive since otherwise it would not be convergent to 0. We are going to first prove  $|\sum_m^n (-1)^{i+1} a_i| \leq |a_m|$  for  $m, n \in \mathbb{N}$  from which our result easily follow using the transfer principle and shadow inequality.

**Theorem 1.** Suppose  $\langle a_i : i \in \mathbb{N} \rangle$  is a real sequence that is nonincreasing (i.e.  $a_i \geq a_{i+1}$ ) and converges to 0. Then for all  $m, n \in \mathbb{N}$  such that  $n \geq m$ , we have

$$\left| \sum_m^n (-1)^{i+1} a_i \right| \leq |a_m|$$

to be true.

We divide the proof into cases.

First suppose  $m$  is even. Then we have

$$\sum_m^n (-1)^{i+1} a_i = \sum_{\substack{m \leq i \leq n \\ i \text{ is even}}} (-a_i + a_{i+1}) = \sum_{\substack{m \leq i \leq n-1 \\ i \text{ is even}}} (-a_i + a_{i+1}) < 0 < a_m$$

and

$$-a_m < (a_n - a_m) + \sum_{\substack{m+1 \leq i \leq n-1 \\ i \text{ is odd}}} (a_i - a_{i+1}) = \sum_m^n (-1)^{i+1} a_i$$

when  $n$  is odd because  $a_n < a_m$  and  $a_{i+1} < a_i$  for each  $i$ . In case of  $n$  being even, consider

$$\sum_{\substack{m \leq i \leq n-1 \\ i \text{ is even}}} (-a_i + a_{i+1}) - a_n < \sum_{\substack{m \leq i \leq n-1 \\ i \text{ is even}}} (-a_i + a_{i+1}) < 0 < a_m$$

and

$$-a_m < -a_m + \sum_{\substack{m+1 \leq i \leq n \\ i \text{ is odd}}} (a_i - a_{i+1}) = \sum_m^n (-1)^{i+1} a_i$$

Similarly, in case of  $m$  being odd, we have

$$-a_m < 0 < \sum_{\substack{m \leq i \leq n \\ i \text{ is odd}}} (a_i - a_{i+1}) = \sum_m^n (-1)^{i+1} a_i$$

and

$$\sum_m^n (-1)^{i+1} a_i = a_m - a_n + \sum_{\substack{m-1 \leq i \leq n-1 \\ i \text{ is even}}} (-a_i + a_{i+1}) < a_m$$

when  $n$  is even. In case of  $n$  being odd, consider

$$-a_m < 0 < \sum_{\substack{m \leq i \leq n-1 \\ i \text{ is odd}}} (a_i - a_{i+1}) < \sum_{\substack{m \leq i \leq n-1 \\ i \text{ is odd}}} (a_i - a_{i+1}) + a_n = \sum_m^n (-1)^{i+1} a_i$$

and

$$\sum_m^n (-1)^{i+1} a_i = a_m + \sum_{\substack{m-1 \leq i \leq n \\ i \text{ is even}}} (-a_i + a_{i+1}) < a_m.$$

This concludes the proof of theorem 1. By an easy transfer, we know that theorem 1 also holds in hypernaturals. Therefore,  $\sum_m^n (-1)^{i+1} a_i$  is infinitesimal in case of unlimited  $m$ . Hence, by cauchy convergence theorem, it follows  $\sum_1^\infty (-1)^{i+1} a_i$  converges.  $\square$