Lectures on Hyperreals
Book by Robert Goldblatt

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What are Hypereals?

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Large sets

Exercise 2.1. *If $\emptyset \neq A \subseteq I$, there is an ultrafilter $\mathcal{F}$ on I with $A \in \mathcal{F}$.*

Proof. Define $\mathcal{H} = \{B \in \mathscr{P}(I) : A \subseteq B\}$. Since A is non empty, it follows that $\mathcal{H}$ has fip. Clearly, $A \in \mathcal{H}$, and so $A \in \mathcal{F}^{\mathcal{H}}$. Therefore, by zorn lemma, we know that there exists ultrafilter, $\mathcal{F}$, such that $A \in \mathcal{F}^{\mathcal{H}} \subseteq \mathcal{F}$. □

Exercise 2.2. *There exists a nonprincipal ultrafilter on $\mathbb{N}$ containing the set of even numbers, and another containing the set of odd numbers.*

Proof. Define $I = \{x \in \mathbb{N} : x = 2n \text{ for some } n \in \mathbb{N}\}$. Since I is infinite, we know that $\mathcal{F}^{\text{co}}$ is proper and thus, have fip. Applying the zorn's lemma, we see that there exists ultrafilter $\mathcal{F}$ such that $\mathcal{F} \subseteq \mathcal{F}^{\text{co}}$. However, for any $i \in I$, we have $I - \{i\} \in \mathcal{F}^{\text{co}} \subseteq \mathcal{F}$ but, since $\mathcal{F}$ is ultrafilter, $\{i\} \in \mathcal{F}$. However, clearly $\{i\} \in \mathcal{F}^i$. Therefore, $F \neq F^i$ for any $i \in I$. Likewise we could argue about set of odd numbers. □

Exercise 2.3. *An ultrafilter on finite set must be principal.*

Proof. Let the set be $A = \{a_1, a_2, \cdots, a_n\}$ and $\mathcal{F}$ be ultrafilter on it. Since $\{a_1\} \cup \{a_2\} \cup \cdots \cup \{a_n\} = A \in \mathcal{F}$, it must be the case that $a_i \in \mathcal{F}$ for some i such that $1 \leq i \leq n$. Since $\mathcal{F}$ is proper, for any $B \in \mathcal{F}$, we have $B \cap \{a_i\} \neq \emptyset$, but that means $\mathcal{F} = \mathcal{F}^{a_i}$. □

Exercise 2.4. *For $\mathcal{H} \subseteq \mathscr{P}(I)$.*
(i) Show that $\mathcal{F}^{\mathcal{H}}$ is a filter that includes $\mathcal{H}$, i.e. $\mathcal{H} \subseteq \mathcal{F}^{\mathcal{H}}$.
(ii) Show that $\mathcal{F}^{\mathcal{H}}$ is included in any other filter that includes $\mathcal{H}$.

Proof.
(i) By the definition.
(ii) Let $\mathcal{F}$ be a filter such that $H \subseteq \mathcal{F}$. Let $A \in \mathcal{F}^{\mathcal{H}}$, then $B_1 \cap B_2 \cap \cdots \cap B_n \subseteq A$ for some $B_1, \cdots, B_n \in \mathcal{H}$. Clearly, $B_1 \cap \cdots \cap B_n \in F$ and so is $A \in F$. □

Exercise 2.5. *Let $\mathcal{F}$ be a proper filter on I.*
(I) Show that $\mathcal{F} \cup \{A^c\}$ has fip iff $A \notin \mathcal{F}$.
(II) Use (I) to deduce that $\mathcal{F}$ is an ultrafilter iff it is maximal proper filter on I.

Proof.
(I) First we prove forward implication. Notice that if $A \in \mathcal{F}$ then since $A \cap A^c = \emptyset$, it follows $\mathcal{F} \cup \{A^c\}$ does not have fip.
We prove the converse implication. Notice that since F is a proper filter, ie it does not contain $\emptyset$, the only way for $\mathcal{F} \cup \{A^c\}$ to not have fip would be $A \in \mathcal{F}$.

(II) We prove the foward implication. Suppose $\mathcal{F}$ is an ultrafilter. Applying Zorn's lemma on proper filter $\mathcal{F}$, we see that there exists a maximal proper filter, K, such that $\mathcal{F} \subseteq K$. If $K \neq F$, it would mean there is some $A \in K$ such that $A \notin \mathcal{F}$ but then that would mean $A^C \in \mathcal{F}$, and so $A^c \in K$ which contradicts the fact that K is proper since it fails to satisfy fip. Therefore, we must have $F = K$.

We prove the converse implication. Suppose $\mathcal{F}$ is maximal proper filter. For the sake of contradiction assume $\mathcal{F}$ is not an ultra filter i.e. there exists some $A \in \mathscr{P}(I)$ such that both $A, A^c \notin F$. Clearly, either $\mathcal{F} \cup \{A\}$ has fip or $\mathcal{F} \cup \{A^c\}$ does. Wlog assume $\mathcal{F} \cup \{A\}$ has fip. Applying Zorn's lemma on $\mathcal{F} \cup \{A\}$, we see that there is maximal proper filter K such that $\mathcal{F} \cup \{A\} \subseteq K$. However this means that $\mathcal{F}$ is not maximal proper filter since $F \subsetneqq F \cup \{A\} \subset P(I)$. $\square$

Chapter 3

Ultrapower Construction of the hyperreals

Exercise 3.1. *(section 3.3, I).* $\equiv$ *is an equivalence relation on* $\mathbb{R}^{\mathbb{N}}$.

Proof. Let $\langle a_n\rangle, \langle b_n\rangle, \langle c_n\rangle \in \mathbb{R}^{\mathbb{N}}$. First we prove that reflexivity. Clearly, $\{n \in \mathbb{N} : a_n = a_n\} = \mathbb{N} \in \mathcal{F}^{\text{co}} \subseteq \mathcal{F}$. Thus, $\langle a_n\rangle \equiv \langle a_n\rangle$.
Symmetricity is trivial since $\{n \in \mathbb{N} : a_n = b_n\} = \{n \in \mathbb{N} : b_n = a_n\}$, if one of them is in F then so is the other.
Finally, we prove transitivity. Suppose $\langle a_n\rangle \equiv \langle b_n\rangle$ and $\langle b_n\rangle \equiv \langle c_n\rangle$ i.e. $\{n \in \mathbb{N} : a_n = b_n\}, \{n \in \mathbb{N} : b_n = c_n\} \in \mathcal{F}$. Clearly, $\{n \in \mathbb{N} : a_n = b_n = c_n\} = \{n \in \mathbb{N} : a_n = b_n\} \cap \{n \in \mathbb{N} : b_n = c_n\} \in \mathcal{F}$. Therefore, $\langle a_n\rangle \equiv \langle c_n\rangle$. $\square$

Exercise 3.2. *(section 3.3, II)* $\equiv$ *is a congruence on the ring* $\langle \mathbb{R}^{\mathbb{N}}, \oplus, \odot\rangle$, *which means that if* $r \equiv r'$ *and* $s \equiv s'$, *then* $r \oplus s = r' \oplus s'$ *and* $r \odot s = r' \odot s'$.

Proof. Suppose $r \equiv r'$ and $s \equiv s'$ i.e. $\{n \in \mathbb{N} : r_n = r'_n\}, \{n \in \mathbb{N} : s_n = s'_n\} \in \mathcal{F}$. Thus, since $\{n \in \mathbb{N} : r_n = r'_n, s_n = s'_n\} = \{n \in \mathbb{N} : r_n = r'_n\} \cap \{n \in \mathbb{N} : s_n = s'_n\} \in \mathcal{F}$, we have $\{n \in \mathbb{N} : r_n = r'_n, s_n = s'_n\} \subseteq \{n \in \mathbb{N} : r_n + s_n = r'_n + s'_n\} \in \mathcal{F}$. Similarly, $\{n \in \mathbb{N} : r_n s_n = r'_n s'_n\} \in \mathcal{F}$. Therefore, $r \oplus s \equiv r' \oplus s'$ and $r \odot s \equiv r' \odot s'$. $\square$

Exercise 3.3. *(section 3.3, III)* $\langle 1, \frac{1}{2}, \frac{1}{3}, \cdots\rangle \not\equiv \langle 0, 0, 0, \cdots\rangle$.

Proof. Let $\langle r\rangle = \langle 1, \frac{1}{2}, \frac{1}{3}, \cdots\rangle$. For the sake of simplicity, define $[[r \neq 0]] = \{n \in \mathbb{N} : r_n \neq 0\}$. Clearly, $[[r \neq 0]] \subseteq \mathbb{N}$. Notice that for any $n \in \mathbb{N}$, we have $n \in [[r \neq 0]]$. Thus, $[[r \neq 0]] = \mathbb{N}$. So, $[[r \neq 0]] \in \mathcal{F}^{\text{co}} \subseteq \mathcal{F}$ but then it would mean $[[r = 0]] = [[r \neq 0]]^c \notin \mathcal{F}$. Therefore, $\langle r_n\rangle \not\equiv \langle 0\rangle$. $\square$

Exercise 3.4. *(section 3.5, I)* $[[r = s]] \cap [[s = t]] \subseteq [[r = t]]$.

Proof. Let $n \in [[r = s]] \cap [[s = t]]$. Then it follows that $r_n = s_n$ and $s_n = t_n$, and so $r_n = t_n$. Therefore, $n \in [[s = t]]$. $\square$

Exercise 3.5. *(Section 3.5, II)* $[[r = r']] \cap [[s = s']] \subseteq [[r \oplus s = r' \oplus s']] \cap [[r \oplus s = r' \oplus s']]$

Proof. Let $n \in [[r = r']] \cap [[s = s']]$. Thus, $r_n = r'_n$ and $s_n = s'_n$, and so $r_n \oplus s_n = r'_n \oplus s'_n$ and $r_n \odot s_n = r'_n \odot s'_n$. Therefore, $n \in [[r \oplus s = r' \oplus s']] \cap [[r \oplus s = r' \oplus s']]$. $\square$

Exercise 3.6. *(Section 3.5, III)* $[[r = r']] \cap [[s = s']] \cap [[r < s]] \subseteq [[r' < s']]$.

Proof. Let $n \in [[r = r']] \cap [[s = s']] \cap [[r < s]]$. Thus, $r_n = r'_n, s_n = s'_n, r_n < s_n$ but then that means $r'_n < s'_n$, and so $n \in [[r' < s']]$. $\square$

Exercise 3.7. *(Section 3.5, IV) If* $r \equiv r'$ *and* $s \equiv s'$, *then* $[[r < s]] \in \mathcal{F}$ *iff* $[[r' < s']] \in \mathcal{F}$.

Proof. Suppose $r \equiv r'$ and $s \equiv s'$, that is to say, $\{n \in \mathbb{N} : r_n = r'_n\}, \{n \in \mathbb{N} : s_n = s'_n\} \in \mathcal{F}$. First assume $[[r < s]] \in \mathcal{F}$. Therefore, $\{n \in \mathbb{N} : r_n = r'_n\} \cap \{n \in \mathbb{N} : s_n = s'_n\} \cap \{n \in \mathbb{N} : r_n < s_n\} \subseteq \{n \in \mathbb{N} : r' < s'\} \in mcF$. Next assume $[[r' < s^p]] \in \mathcal{F}$. It follows $\{n \in \mathbb{N} : r_n = r'_n\} \cap \{n \in \mathbb{N} : s_n = s'_n\} \cap \{n \in \mathbb{N} : r'_n < s'_n\} \subseteq \{n \in \mathbb{N} : r < s\} \in \mathcal{F}$ $\square$

Exercise 3.8. *(Section 3.8, I) Use only general properties of ordered fields to deduce from the fact that $[\varepsilon]$ is a positive infinitismal, the conclusion that $[\varepsilon]^{-1}$ is greater than every real number.*

Proof. Notice that for any $r \in \mathbb{R}^+$, we have ${}^*r = [\langle r, r, \cdots, \rangle] \in {}^*\mathbb{R}$. Since $\varepsilon < {}^*r$, by fields properties, we have $0 < {}^*r^{-1} < \varepsilon^{-1}$. Since r is arbitrary positive real number, setting $r = r^{-1}$,we see that ${}^*r < \varepsilon^{-1}$ for any $r \in \mathbb{R}^+$. □

Exercise 3.9. *(Section 3.10, I) If A is finite, show that ${}^*A = A$, and hence A has non nonstandard members.*

Proof. Let $A = \{a_1, \cdots, a_n\}$. Consider some $[r] \in {}^*A$. If we could prove that $r \equiv t$ for some constant sequence $t \in A$, we would be done. So we set that as our goal. Notice that since $[r] \in {}^*A$, we have $\{n \in \mathbb{N} : r_n \in A\} = \{n \in \mathbb{N} : r_n = a_1 \vee r_n = a_2 \vee \cdots r_n = a_n\} = \{n \in \mathbb{N} : r_n = a_1\} \cup \cdots \cup \{n \in \mathbb{N} : r_n = a_n\} \in \mathcal{F}$. But since $\mathcal{F}$ is ultrafilter, only one of such set is in $\mathcal{F}$. Therefore, $\{n \in \mathbb{N} : r_n = a_k\} \in \mathcal{F}$ for some k such that $1 \leq k \leq n$. Hence, $r \equiv a_k$. □

Exercise 3.10. *(Section 3.10, II) Prove:*
(a) $A \subseteq B$ iff ${}^*A \subseteq {}^*B$.
(b) $A = B$ iff ${}^*A = {}^*B$.

Proof.
(a) First we prove the forward implication. Suppose $A \subseteq B$. Consider $[r] \in {}^*A$. Therefore, we have $[[r \in A]] \in \mathcal{F}$, but since $[[r \in A]] \subseteq [[r \in B]]$, we have $[[r \in B]]$ and so $[r] \in {}^*B$.
We prove the converse implication. Suppose ${}^*A \subseteq {}^*B$. Consider some $r \in A$. Then we have $[\mathbf{r}] \in {}^*A$, and so $[\mathbf{r}] \in {}^*B$. However, since $[\mathbf{r}] \in B$, we have $[[r \in B]] \in \mathcal{F}$ which means $r \in B$.

(b) Trivially follows from (a). □

Exercise 3.11. *(Section 3.10, III)*
Prove:
(a) ${}^*(A \cup B) = {}^*A \cup {}^*B$,
(b) ${}^*(A \cap B) = {}^*A \cap {}^*B$,
(c) ${}^*(A - B) = {}^*A - {}^*B$,
(d) ${}^*\emptyset = \emptyset$.

Proof.
(a) First we prove the foward implication. Since $A \subset A \cup B$, applying (2), we have ${}^*A \subseteq {}^*(A \cup B)$. Similarly, we have ${}^*B \subseteq {}^*(A \cup B)$, and so ${}^*A \cup {}^*B \subseteq {}^*(A \cup B)$.
We prove the converse implication. Let $[r] \in {}^*A \cup {}^*B$. Then we have following cases;
case-1. $[r] \in {}^*A$. Thus, $[[r \in A]] \in mcF$, and since $[[r \in A]] \subseteq [[r \in A \cup B]]$, we have $[[r \in A \cup B]] \in \mathcal{F}$ but that means $[r] \in {}^*A \cup {}^*B$.
case-2. $[r] \in {}^*B$. The proof is analogous to **case-1**.

(b) First we prove the forward implication. Since $A \cap B \subseteq A$, applying (2), we have ${}^*(A \cap B) \subseteq {}^*A$. Similarly we have ${}^*(A \cap B) \subseteq {}^*B$, and so ${}^*(A \cap B) \subseteq {}^*A \cap {}^*B$.
We prove the converse implication. Consider some $[r] \in {}^*A \cap {}^*B$. It means that $[r] \in {}^*A$ and $[r] \in {}^*B$, thus $[[r \in A]], [[r \in B]] \in \mathcal{F}$. Therefore, $[[r \in A \cap B]] = [[r \in A]] \cap [[r \in B]] \in \mathcal{F}$, and so $[r] \in {}^*(A \cap B)$.

(c) First we prove the foward implication. Let $[r] \in {}^*(A \setminus B)$. Then we have $[[r \in A]] \cap [[r \notin B]] = [[r \in A \setminus B]] \in \mathcal{F}$. Thus, $[[r \in A]], [[r \notin B]] \in \mathcal{F}$. Hence, $[r] \in {}^*A$ and $[r] \notin {}^*B$, and so $[r] \in {}^*A \setminus {}^*B$.
We prove the converse implication. Let $[r] \in {}^*A \setminus {}^*B$ i.e. $[r] \in {}^*A$ and $[r] \notin {}^*B$. Then we have $[[r \in A]] \in \mathcal{F}$ and $[[r \notin B]] \in \mathcal{F}$. Thus, $[[r \in A \setminus B]] = [[r \in A]] \cap [[r \notin B]] \in \mathcal{F}$. Therefore, $r \in {}^*(A \setminus B)$.

(d) Clearly, ${}^*\emptyset = \{[r] \in {}^*\mathbb{R} : [[r \in \emptyset]] \in \mathcal{F}\} = \emptyset$. □

Exercise 3.12. *(Section 3.10, IV) Is it true that ${}^*(\bigcup_{n=1}^{\infty} A_n) = \bigcup_{n=1}^{\infty} {}^*A_n$?*

Proof. No. Define $A_n = \{n\}$ for any $n \in \mathbb{N}$. Notice that since A_n is finite, using (1), we have ${}^*A_n = A_n = \{n\}$. Since $[[\omega \in \bigcup_{n=1}^{\infty} A_n]] = \mathbb{N} \in \mathcal{F}$, we have $\omega \in {}^*(\bigcup_{n=1}^{\infty} A_n)$. However, $[[\omega \in A_n]] = \{n\} \notin \mathcal{F}$, thus $\omega \notin {}^*A_n$ for any $n \in \mathbb{N}$. This mean $\omega \notin \bigcup_{n=1}^{\infty} {}^*A_n$. □

Exercise 3.13. *(Section 3.10, V) Show that if $A \subseteq \mathbb{R}$, then ${}^*A \cap \mathbb{R} = A$.*

Proof. Suppose $A \subseteq \mathbb{R}$. Let $[r] \in {}^*A \cap \mathbb{R}$. Since $\langle r \rangle$ is a constant sequence and $[r] \in {}^*A$, it follows $[r] = \langle r \rangle \in A$. Thus, ${}^*A \cap \mathbb{R} \subseteq A$. This means that ${}^*A \cap \mathbb{R}$ contains only the constant sequences. Let $\langle r \rangle \in A$. Then clearly $[r] \in {}^*A \cap \mathbb{R}$. □

Exercise 3.14. *For $a, b \in \mathbb{R}$, let $[a, b]$ be the closed interval $\{x \in \mathbb{R} : a \leq x \leq n\}$. Prove that ${}^*[a, b] = \{x \in {}^*\mathbb{R} : a \leq x \leq b\}$.*

Proof. We first prove the forward inclusion. Let $[r] \in {}^*[a, b]$. Then we have $\{n \in \mathbb{N} : a \leq r_n \leq b\} = [[r \in [a, b]]] \in \mathcal{F}$ but that means $[r] \in \{x \in {}^*\mathbb{R} : a \leq x \leq b\}$. We prove the converse inclusion. Let $[r] \in \{x \in {}^*\mathbb{R} : a \leq x \leq b\}$. It follows $[[r \in [a, b]]] = \{n \in \mathbb{N} : a \leq r_n \leq b\} \in mcF$. Therefore, $[r] \in \{x \in \mathbb{R} : a \leq x \leq n\}$. □

Exercise 3.15. *(Section 3.10, VII) ${}^*\mathbb{Z}$ is a subring of ${}^*\mathbb{R}$.*

Proof. Clearly, ${}^*\mathbb{Z} \neq \emptyset$. Consider some $[r], [s] \in {}^*\mathbb{Z}$. Then we have $[[r \in \mathbb{Z}]], [[s \in \mathbb{Z}]] \in \mathcal{F}$. Therefore, since $[[r \in \mathbb{Z}]] \cap [[s \in \mathbb{Z}]] \subseteq [[r+s \in \mathbb{Z}]], [[rs \in \mathbb{Z}]]$, we have $[[r+s \in \mathbb{Z}]], [[rs \in \mathbb{Z}]] \in \mathcal{F}$. Hence, $[r+s], [rs] \in {}^*\mathbb{Z}$. □

Exercise 3.16. *(Section 3.10, VIII) If $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$, show that ${}^*(\mathbb{R}^+) = \{x \in {}^*\mathbb{R}^+ : x > 0\}$, i.e., ${}^*(R^+) = ({}^*\mathbb{R})^+$.*

Proof. First we prove the forward inclusion. Let $[r] \in {}^*(R^+)$. It means that $\{n \in \mathbb{N} : r_n > 0\} = [[r \in R^+]] \in \mathcal{F}$ which implies $[r] \in ({}^*\mathbb{R})^+$. Next we prove the converse inclusion. Let $[r] \in ({}^*\mathbb{R})^+$. Then we have $\{n \in \mathbb{N} : r_n > 0\} = [[r > 0]] = [[r \in \mathbb{R}^+]] \in \mathcal{F}$ from which it follows $[r] \in {}^*(\mathbb{R}^+)$. □

Exercise 3.17. *(Section 3.12, I) Show that ${}^*f$ agrees with f on $\mathbb{R}$: if $r \in \mathbb{R}$, then ${}^*f(r) = f(r)$.*

Proof. Let $r \in \mathbb{R}$. Notice that r can be indetified with the constant sequence $\mathbf{r} \in {}^*R$. Then we have ${}^*f([\mathbf{r}]) = [f(\mathbf{r})]$ which is isomorphic to $f(r)$. □

Exercise 3.18. *(Section 3.12, II) If f is injective, so is ${}^*f$. What about surjectivity?*

Proof. Suppose f is injective. Consider some $[r], [s] \in {}^*\mathbb{R}$ such that ${}^*f([r]) = {}^*f([s])$, that is to say $[f \circ r] = [f \circ s]$. Since f is injective, we have $[[f \circ r = f \circ s]] \subseteq [[r = s]]$ but then, since $[[f \circ r = f \circ s]] \in \mathcal{F}$, it follows $[[r = s]] \in \mathcal{F}$. Therefore, $[r] = [s]$.

Yes, surjecitivity is preserved. Suppose f is surjective. Consider some $[s] \in {}^*\mathbb{R}$. Our goal is to prove that there is some $[r] \in {}^*\mathbb{R}$ such that ${}^*f([r]) = [s]$. Define sequenece (r_n) such that, for each s_n, $f(r_n) = s_n$. We know such a r_n exists for each s_n since f is surjective. We claim that $[r] \in {}^*\mathbb{R}$ the element we looking for. To prove this claim, we need to show $[f \circ r] = f([r]) = [s]$, so we set that as our goal. Clearly, $[[f \circ r = s]] = \{n \in \mathbb{N} : f(r_n) = s_n\} = \mathbb{N} \in \mathcal{F}$. Therefore, $[f \circ r] = f[r] = [s]$. □

Exercise 3.19. *(Section 3.12, III) For $x \in {}^*\mathbb{R}$, let*

$$|x| = \begin{cases} x, & \textit{if } x > 0 \\ 0, & \textit{if } x = 0 \\ -x & \textit{if } x < 0 \end{cases}$$

be the usal definition of absolute value function. Show that this extends the definition of $|.|$ on $\mathbb{R}$: $|[r]| = [\langle |r_1|, |r_2|, \cdots \rangle] = [|r_n|]$.

Proof. We have the following cases;
Case-1. Suppose $[r] > 0$. Then $[[r > 0]] \in \mathcal{F}$. Since $[[r > 0]] \subseteq [[[r] = [|r|]]] \in \mathcal{F}$, we have $|[r]| = [r] = [|r_n|]$.
Case-2. Suppose $[r] = 0$. That is to say $[[r = 0]] \in \mathcal{F}$. Since $[[r = 0]] \subseteq [[0 = |r|]] \in \mathcal{F}$, we have $|[r]| = 0 = [|r|]$.
Case-3. Suppose $[r] < 0$. Then $|[r]| = -[r] = [-r] = [|r|]$. □

Exercise 3.20. *(Section 3.12, IV) Let χ_A be the characteristic function of a set $A \subseteq \mathbb{R}$. Show that ${}^*(\chi_A) = \chi_{{}^*A}$.*

Proof. Let $[r] \in {}^*\mathbb{R}$. Consider the expression ${}^*(\chi_A)[r]$, we have the following cases;
Case-1. Suppose $[r] \in {}^*A$. Notice that ${}^*(\chi_A)[r] = [\chi_A \circ r] = [\chi_A(r_n)]$ but then, since $[[r \in A]] \in \mathcal{F}$ and $[[r \in A]] \subseteq [[\chi_A \circ r = 1]]$, we have $[[\chi_A \circ r = 1]] \in \mathcal{F}$. Therefore, $[\chi_A \circ r] = \chi_A[r] = 1$.
Case-2. Suppose $[r] \notin {}^*A$. Then we have $[[r \notin A]] = [[r \in A]]^c \in \mathcal{F}$. Clearly, $[[r \notin A]] \subseteq [[\chi_A \circ r = 0]]$. Thus, $[[\chi_A \circ r = 0]] \in \mathcal{F}$ and so $[\chi_A \circ r] = \chi_A[r] = 0$. □

Exercise 3.21. *(Section 3.12, V) Show how to define ${}^*f$ when f is a function of more than one arguement.*

Proof. Define ${}^*f([r^1], \cdots, [r^n]) = [f(r^1_1, \cdots, r^n_1), f(r^1_2, \cdots, r^n_2), \cdots]$. We prove that it's well defined. Clearly, $[[r^1 = r^{1'}]] \cap \cdots [[r^n = r^{n'}]] \subseteq [[f \circ (r^1, \cdots, r^n) = f \circ (r^{1'}, \cdots, r^{n'})]]$. Therefore, $r^1 \equiv r^{1'}, \cdots, r^n \equiv r^{n'}$ implies $f \circ (r^1, \cdots, r^n) \equiv f \circ (r^{1'}, \cdots, r^{n'}$. □

Exercise 3.22. *(Section 3.15, I) If $A_1, \cdots, A_k$ are subsets of $\mathbb{R}$, put $P = A_1 \times \cdots \times A_k$ and apply the definition of ${}^*P$ to show that*

$$
{}^*(A_1 \times \cdots \times A_k) = {}^*A_1 \times \cdots \times {}^*A_k.
$$

Proof. For $([r^1], [r^2], \cdots, [r^k]) \in {}^*P$, we must have $[[(r^1, r^2, \cdots, r^k) \in P]] \in \mathcal{F}$. Clearly, $[[(r^1, r^2, \cdots, r^k) \in P]] \subseteq [[r^n \in A_n]]$, thus $[[r^n \in A_n]]$ for any n such that $1 \leq n \leq k$. This means that $[r^n] \in A^n$ for all such n. Hence, $([r^1], \cdots, [r^n]) \in {}^*A_1 \times \cdots {}^*A_n$. From this, it follows that ${}^*(\mathbb{R}^k) = ({}^*\mathbb{R})^k$. □

Exercise 3.23. *(Section 3.15, II) Let dom P denote the domain of a binary relation. If $P \subseteq \mathbb{R}^2$, show that ${}^*(dom\ P) = dom\ {}^*P$.*

Proof. First we prove the forward inclusion. Let $[r] \in {}^*(\text{dom } P)$. Then we have $\{n \in \mathbb{N} : r_n \in \text{dom } P\} = [[r \in \text{dom } P]] \in \mathcal{F}$.

$$
\text{Define sequence } s_n = \begin{cases} s \text{ such that } (r_n, s) \in P, \text{ if } r_n \in \text{dom } P \\ 0, \text{ if } r_n \notin \text{dom } P \end{cases}
$$

Then clearly, $[[(r, s) \in P]] = [[r \in \text{dom } P]] \in \mathcal{F}$, but that means $([r], [s]) \in {}^*P$, and so $[r] \in \text{dom } {}^*P$.
Now we prove the converse inclusion. Let $[r] \in \text{dom } {}^*P$. It follows that there exists $[s] \in {}^*\mathbb{R}$ such that $([r], [s]) \in {}^*P$, but that means $[[r \in \text{dom } P]] = [[(r, s) \in P]] \in \mathcal{F}$. Hence, $[r] \in {}^*(\text{dom } P)$. □

Chapter 4

The Transfer Principle

Exercise 4.1. *(Section 4.1, I) Verify the truth the* $*-$*transform given in* $1-5$ *above, which are;*
(1) The Eudoxus-Archimedes Principle: $\forall x \in {}^*\mathbb{R} : \exists m \in {}^*\mathbb{N}(x < m)$.
(2) Density of the rationals: $\forall x, y \in {}^*\mathbb{R}(x < y$ *implies* $\exists q \in {}^*\mathbb{Q}(x < q < y))$.
(3) Finiteness. Let $A = \{r_1, \cdots, r_k\}$ *be a finite subset of* $\mathbb{R}$*. Then* $\forall x \in {}^*A(x = {}^*r_1 \vee x = {}^*r_2 \vee \cdots x = {}^*r_k)$.
(4) Finitary set operations. $\forall x \in {}^*\mathbb{R}(x \in {}^*(A \cup B))$ *iff* $x \in {}^*A$ *or* $x \in {}^*B$.
(5) Discreteness of $\mathbb{N}$. $\forall x \in {}^*\mathbb{N}({}^*n \leq x \leq {}^*(n+1)$ *implies* $x = {}^*n$ *or* $x = {}^*(n+1)$.

Proof.

(1) Let $x \in {}^*\mathbb{R}$. Define sequence m such that $x_i < m_i$ for every i. The existence of such m is guranteed by Eudoxus-Archimedes Principle in reals. Clearly, $m \in {}^*\mathbb{N}$ and $[[x < m]] \in \mathcal{F}$. Therefore, $x < m$.
(2) Let $x, y \in {}^*\mathbb{R}$. Define sequence q such that $x_i < q_i < y_i$ for each i, where q_i's are rational. The existence of such q_i's is guranteed by density of rationals in reals. Clearly, $q \in {}^*\mathbb{Q}$ and $[[x < q < y]] = \mathbb{N} \in \mathcal{F}$. Therefore, $x < q < y$.
(3) Let $x \in {}^*A$. Then $[[x = r_1]] \cup [[x = r_2]] \cup \cdots [[x = r_n]] = [[x \in A]] \in \mathcal{F}$. Since finite union in $\mathcal{F}$ means that exactly one of the set in $\mathcal{F}$, we have $x = {}^*r_1$ or $x = {}^*r_2$ or $\cdots$ or $x = {}^*r_n$.
(4) Proved in section 3.10, II.
(5) Let $x \in {}^*\mathbb{N}$ and suppose ${}^*n \leq x \leq {}^*(n+1)$. Then we have $[[n \leq x \leq n+1]] \in \mathcal{F}$ but due to discreteness of $\mathbb{N}$, it folows that $[[x = n]] \cup [[x = n+1]] = [[x = n \vee x = n+1]] = [[n \leq x \leq n+1]] \in \mathcal{F}$. Hence, $x = {}^*n$ or $x = {}^*(n+1)$. $\square$

Chapter 5

Hypereals Great and Small

Exercise 5.1. *(Section 5.2, I) For any positive hyperreal a, explain why the function $x \to a^x$ is defined for all $x \in {}^*\mathbb{R}$. Use transfer to explore its properties.*

Proof. $\square$

Exercise 5.2. *(Section 5.5, I) Verify that $\simeq$ and $\sim$ are equivalence relationships.*

Proof. We will first argue about $\simeq$. Let $x, y, z \in {}^*\mathbb{R}$. Clearly, it's reflexive since $x - x = 0$. If $x \simeq y$ then we have $x - y = \epsilon$, thus $y - x = -\epsilon$, and so $y \simeq x$. If $x \simeq y$ and $y \simeq z$ then we have $x - y = \epsilon$ and $y - z = \delta$, adding the two get us $x \simeq z$. The arguement of $\sim$ is similar, so we omit the details. $\square$

Exercise 5.3. *(Section 5.5, II) If $b \simeq x \leq y \simeq c$ with b and c real, show that $b \leq c$. What if b and/or c are not real?*

Proof. First we will prove an useful lemma, if $y > 0$ and $y \simeq k$ then $k > 0$. Let $k = y + \varepsilon$. In the case where $\varepsilon = 0$ and $\varepsilon > 0$, it follows trivially. So assume $\varepsilon < 0$. Thus, $y - \frac{1}{n} < y + \varepsilon = k$ for all $n \in \mathbb{N}$. Since $\frac{y}{2}$ a real number, we know that there exists $m \in \mathbb{N}$ such that $\frac{1}{m} < \frac{y}{2}$ which implies $0 < \frac{y}{2} = y - \frac{y}{2} < y - \frac{1}{m} < k$. This proves the lemma.
Now assume to the contrary that $c < b$. Thus, $0 < c - b$ but it implies that $0 < x - y$, since $c - b \simeq x - y$, which is a contradiction.
The result would not be true if b is not real. Consider $\varepsilon \simeq 0 \leq 0 \simeq 0$ for some positive infintisimal ε, this would imply $\varepsilon \leq 0$ which is false. $\square$

Exercise 5.4. *(Section 5.5, III) $hal(b) = \{b + \varepsilon : e \in hal(0)\}$.*

Proof. The converse inclusion is trivial. For forward, notice that $x \in \text{hal}(b)$ means that $x - b = \varepsilon$, and so $x = b + \varepsilon$. $\square$

Exercise 5.5. *(Section 5.5, IV) $gal(b) = \{b + c : c \in gal(0)\}$.*

Proof. Again, the converse implication is trivial. For forward, we see that $x \in \text{gal}(b)$ means $x - b = c$ where c is a limited number, and so $c \in \text{gal}(0)$. Thus, $x = b + c$ for some $c \in \text{gal}(0)$. $\square$

Exercise 5.6. *(Section 5.5, V) If $x \simeq y$ and b is limited, prove that $b \cdot x \simeq b \cdot y$. Show that the result can fail for unlimited b.*

Proof. Since $x \simeq y$, we have $x - y = \varepsilon$ for some $\varepsilon \in \text{hal}(0)$. Multiplying by b on both sides get us, $bx - by = b \cdot \varepsilon$. It follows that $b \cdot \varepsilon \in \text{hal}(0)$, so $bx \simeq by$. Consider $b = \frac{1}{\varepsilon}$then $b \cdot \varepsilon = 1$ which is not infinitisimal. $\square$

Exercise 5.7. *(Section 5.5, VI) Show that any galaxy contains members of ${}^*\mathbb{Z}$, of ${}^*\mathbb{Q} - {}^*\mathbb{Z}$, and of ${}^*\mathbb{R} - {}^*\mathbb{Q}$.*

Proof. Let $b \in {}^*\mathbb{R}$. Clearly, $b - a$ for $a \in {}^*Z$, such that a is intger, is limited. $\square$

Exercise 5.8. *(Section 5.7, I) Prove theorem 5.6.2:*
(a) $sh(b \pm c) = sh(b) \pm sh(c)$.
(b) $sh(bc) = sh(b) \cdot sh(c)$.
(c) $sh(b/c) = sh(b)/sh(c)$ *if* $sh(c) \neq 0$ *(ie c is appreciable).*
(d) $sh(b^n) = sh(b)^n$.
(e) $sh(|b|) = |sh(b)|$.
(f) $sh(\sqrt[n]{b}) = \sqrt[n]{sh(b)}$, *if* $b \geq 0$.
(g) If $b \leq c$ *then* $sh(b) \leq sh(c)$.

Proof. Consider $b = \mathrm{sh}(b) + \varepsilon$ and $c = \mathrm{sh}(c) + \delta$ for some infinitisimal ε and δ.
(a) For $\mathrm{sh}(b+c)$, notice that $b + c - (\mathrm{sh}(b) + \mathrm{sh}(c)) = \varepsilon + \delta$. Thus, $\mathrm{sh}(b+c) \simeq b + c \equiv \mathrm{sh}(b) + sh(c)$. From the uniqueness of shadow, it follows $\mathrm{sh}(b+c) = \mathrm{sh}(b) + \mathrm{sh}(c)$. We get identical result in case of $\mathrm{sh}(b-c)$ by considering $b - c - (\mathrm{sh}(b) - \mathrm{sh}(c)) = \varepsilon - \delta$.

(b) Notice that $bc = \mathrm{sh}(b)\mathrm{sh}(c) + \mathrm{sh}(b)\delta + \mathrm{sh}(c)\varepsilon + \varepsilon\delta$ where $\mathrm{sh}(b)\delta, \mathrm{sh}(c)\varepsilon, \varepsilon\delta$ are infintisimals. So, $\mathrm{sh}(bc) \simeq bc \simeq \mathrm{sh}(b)\mathrm{sh}(c)$, and thus by uniqueness of shadow we have $\mathrm{sh}(bc) = \mathrm{sh}(b)\mathrm{sh}(c)$.

(c) Notice that, applying (b), we have $\mathrm{sh}(b/c) = \mathrm{sh}(b) \cdot \mathrm{sh}(1/c)$. So if we could prove $\mathrm{sh}(1/c) = 1/\mathrm{sh}(c)$, we would be done. Therefore, we set that as our goal. Clearly, $\frac{1}{c} - \frac{1}{\mathrm{sh}(c)} = \frac{1}{\mathrm{sh}(c)+\delta} - \frac{1}{\mathrm{sh}(c)} = \frac{\delta}{\mathrm{sh}(c)^2+\mathrm{sh}(c)\delta}$. Since $\mathrm{sh}(c)^2 + \mathrm{sh}(c)\delta$ is appreciable it follows that $\frac{\delta}{\mathrm{sh}(c)^2+\mathrm{sh}(c)\delta}$ is infinitismal, and thus $\mathrm{sh}(1/c) \simeq 1/c \simeq 1/\mathrm{sh}(c)$. Hence, $\mathrm{sh}(1/c) = 1/sh(c)$ by uniqueness of shadow.

(d) Trivially follows from induction on (b).

(e) If we could prove that $|b| \simeq |\mathrm{sh}(b)|$, we would be done since from uniqueness of shadow, $\mathrm{sh}|b| = |\mathrm{sh}(b)|$ would hold. Notice that $|b| - |\mathrm{sh}(b)| \leq |\mathrm{sh}(b)| + |e| - |\mathrm{sh}(b)| = |e|$. If $0 \leq |b| - |\mathrm{sh}(b)|$ then clearly $|b| - |\mathrm{sh} b| \leq |e| < 1/n$ for all $n \in \mathbb{N}$. If $|b| - |\mathrm{sh}(b)| < 0$ then $-1/n < -|e| \leq |b| - |\mathrm{sh}(b)|$ for all $n \in \mathbb{N}$. Thus, $||b| - |\mathrm{sh}(b)|| < 1/n$ for every natural n. Hence, $|b| - |\mathrm{sh}(b)$ is infinitisimal as desired.
(f) If $\mathrm{sh}(b^{\frac{1}{n}}) < (\mathrm{sh}(b))^{\frac{1}{n}}$ then by squaring both side and applying part (d), we would get $\mathrm{sh}(b) = \mathrm{sh}(b^{1/n \cdot n}) = (\mathrm{sh}(b^{1/n}))^n < (\mathrm{sh} b)^{1/n \cdot n} = \mathrm{sh}(b)$ which is a contradiction. We would get identical contradiction in case of $(\mathrm{sh}(b))^{\frac{1}{n}} < \mathrm{sh}(b^{\frac{1}{n}})$.
(g) It follows from 2nd exercise of section 5.5. □

Exercise 5.9. *(Section 5.7, I) Show that if* b, c *are limited, and* $b \simeq b', c \simeq c'$, *then* $b \pm c \simeq b' \pm c', b \cdot c \simeq b' \cdot c'$, *and* $b/c \simeq b'/c'$ *if* $c \not\simeq 0$. *Show that the last result can fail when* $c \simeq 0$.

Proof. Let $b = b' + \varepsilon$ and $c = c' + \delta$ for some infintisimal ε and δ. Then $b \pm c - (b' \pm c') = \varepsilon \pm \delta$ which is known to be infinitisimal. Thus, $b \pm c \simeq b' \pm c'$. For multiplication we have $bc - b'c' = b'c' + b'\delta + c'\varepsilon + \varepsilon\delta - b'c' = b'\delta + c'\varepsilon + \varepsilon\delta$ which is infintisimal. Finally for quotient, we have $\frac{b}{c} - \frac{b'}{c'} = \frac{b'+\varepsilon}{c'+\delta} - \frac{b'}{c'} = \frac{b'c'+c'\varepsilon-b'c'-b'\delta}{(c')^2+c'\delta} = \frac{c'\varepsilon+b'\delta}{(c')^2+c'\delta}$. Since $(c')^2 + c'\delta$ is appreciable, we have $\frac{b}{c} - \frac{b'}{c'}$ to be infinitsimal. Hence, $b/c \simeq b'/c'$. For proving the result does not hold for $c \simeq 0$. Consider $b = 0$. Then we have $b/c - b'/c^p = 0 - \frac{\varepsilon}{\epsilon - \delta}$ which is not infinitisimal necessarily. □

Exercise 5.10. *(Section 5.7, II)*
If ε *is infinitesimal, show that*
(a) $\sin \varepsilon \simeq 0$,
(b) $\cos \varepsilon \simeq 1$,
(c) $\tan \varepsilon \simeq 0$,
(d) $\sin \varepsilon/\varepsilon \simeq 1$,
(e) $(\sin \varepsilon)/\varepsilon \simeq 1$,
(f) $(\cos \varepsilon - 1)/\varepsilon \simeq 0$.

Proof.
(a) Since $\forall x \in \mathbb{R}(|\sin x| \leq |x|)$ holds in $\mathbb{R}$, applying transfer on it, it must be the case that $\forall x \in {}^*\mathbb{R}(|\sin x| \leq$

$|x|)$ holds in ${}^*\mathbb{R}$. Therefore, $|\sin\varepsilon| \leq |\varepsilon|$, and so $\sin\varepsilon$ is infintisimal. Hence, $\sin\varepsilon \simeq 0$.

(b) Since $\cos\varepsilon = \sqrt{(1-\sin^2\varepsilon)}$, we have $\mathrm{sh}(cos\varepsilon) = \mathrm{sh}\sqrt{(1-\sin^2\varepsilon)} = \sqrt{(\mathrm{sh}(1)-\mathrm{sh}(\sin^2\varepsilon))} = \sqrt{(1-\mathrm{sh}(\delta^2))} = \sqrt{(1-0)} = 1$. Clearly, $\cos\varepsilon \simeq \mathrm{sh}(\cos\varepsilon) = 1$.

(c) $\mathrm{sh}(\tan\varepsilon) = \mathrm{sh}(\sin\varepsilon)/\mathrm{sh}(\cos\varepsilon) = 0/1 = 0$. Hence, $\tan\varepsilon \simeq 0$.

(d) Notice that $|\sin\varepsilon| \leq |\varepsilon|$, so $|\sin(\varepsilon)/\varepsilon| \leq 1$. Thus, $\mathrm{sh}\left(\left|\frac{\sin\varepsilon}{\varepsilon}\right|\right) \leq 1$. We will now show that $\mathrm{sh}\left(\left|\frac{\sin\varepsilon}{\varepsilon}\right|\right) < 1$ leads to contradiction which would get us $\mathrm{sh}\left(\left|\frac{\sin\varepsilon}{\varepsilon}\right|\right) = 1$. Clearly if $\mathrm{sh}\left(\left|\frac{\sin\varepsilon}{\varepsilon}\right|\right) < 1$ then $0 = \mathrm{sh}(|\sin\varepsilon|) - \mathrm{sh}|\varepsilon| < 0$ which is false. Therefore, $(\sin\varepsilon)/\varepsilon \simeq \mathrm{sh}\left(\left|\frac{\sin\varepsilon}{\varepsilon}\right|\right) = 1$.

(e) By the sine inequality, $|\sin\varepsilon|/|\varepsilon| \leq 1$. By the tangent inequality, $|x| \leq |\tan(x)|$ for $x \in [0, \pi/2)$, we have $|\cos x| \leq |\sin x|/|x|$. Combining the two inequalities and taking shadow we see $1 = \mathrm{sh}(|\cos\varepsilon|) \leq \mathrm{sh}(|\sin(\varepsilon)/\varepsilon|) \leq 1$. Therefore, $\mathrm{sh}(|\sin(\varepsilon)/\varepsilon|) = 1$. Hence, $(\sin\varepsilon)/\varepsilon \simeq 1$.

(f) By cosine inequality, $1-\frac{\varepsilon^2}{2} \leq \cos\varepsilon$, we have $-\varepsilon/2 \leq (\cos\varepsilon - 1)/\varepsilon$. Since $cos\varepsilon \leq 1$, we have $(\cos\varepsilon-1)/\varepsilon \leq 0/\varepsilon = 0$. Hence $|(\cos\varepsilon-1)/\varepsilon| \leq \varepsilon/2$ and so $(\cos\varepsilon-1)/\varepsilon$ is infinitesimal, which makes it infinitely close to 0. $\square$

Exercise 5.11. *(Section 5.7, III) Show that every hyperreal is infinitely close to some hyperrational number.*

Proof. Since $\forall x, y \in \mathbb{R}(x < y$ implies $\exists q \in \mathbb{Q}(x < q < y))$ holds in reals (density of rationals), by transfer, we have $\forall x, y \in {}^*\mathbb{R}(x < y$ implies $\exists q \in {}^*\mathbb{Q}(x < q < y))$. In particular there is a hyperrational q such that $r < q < r + \varepsilon$ for any hyperreal r. Clearly such a q is infinitely close to r. $\square$

Exercise 5.12. *(Section 5.7, IV) Show that $\mathbb{R}$ is isomorphic to the ring of limited hyperrationals ${}^*\mathbb{Q} \cap \mathbb{L}$ factored by its ideal ${}^*\mathbb{Q} \cap \mathbb{I}$ of hyperreational infintisimal.*

Proof.

$$\begin{tikzcd}
\bullet {}^*\mathbb{Q} \cap \mathbb{L} \arrow[rr, two heads, "\mathrm{sh}"] \arrow[dr, two heads, "\text{canonical projection}"'] & & \bullet \mathbb{R} \\
& \bullet({}^*\mathbb{Q} \cap \mathbb{L})/({}^*\mathbb{Q} \cap \mathbb{I}) \arrow[ur, tail, "f(\mathrm{hal}(b)+({}^*\mathbb{Q}\cap\mathbb{L})/({}^*\mathbb{Q}\cap\mathbb{I}))=\mathrm{sh}(b)"'] &
\end{tikzcd}$$

Consider shadow function restricted to ${}^*\mathbb{Q} \cap \mathbb{L}$, in what follows we would use sh notation for it instead of $\mathrm{sh}_{{}^*\mathbb{Q}\cap\mathbb{L}}$. We will first justify it's surjective. Since $\mathbb{R} \subseteq {}^*\mathbb{R}$, applying previous exercise, we see that for any $x \in \mathbb{R}$, there is $q \in {}^*\mathbb{Q}$ such that $x \simeq q$ but then it would mean $x \simeq \mathrm{sh}(q)$. Since both x and $\mathrm{sh}(q)$ are reals, it must be the case that $x = \mathrm{sh}(q)$. Hence, $\mathrm{sh}(q) = x$ as desired. Notice that for any $[r] \in ({}^*\mathbb{Q} \cap \mathbb{L})/({}^*Q \cap \mathbb{I})$, we have $[r] = r + {}^*Q \cap \mathbb{I}$ which is nothing but $\mathrm{hal}(r)$. Define $f : ({}^*\mathbb{Q} \cap \mathbb{L})/({}^*Q \cap \mathbb{I}) \to \mathbb{R}$ as $\mathrm{hal}(b) \mapsto \mathrm{sh}(b)$. By the first isomorphism theorem of rings, it follows $({}^*\mathbb{Q} \cap \mathbb{L})/({}^*Q \cap \mathbb{I}) \simeq \mathbb{R}$. $\square$

Exercise 5.13. *(Section 5.9, Exercise on Dedekind completeness) For threoem 5.8.1, instead of showing that cauchy sequences converge we can develop a direct proof that any subset $A \subseteq \mathbb{R}$ with a real upper bound has a least real upper bound. First, for each $n \in \mathbb{N}$, let s_n be the least $k \in \mathbb{Z}$ such that k/n is an upper bound of A. Then take an unlimited $N \in {}^*\mathbb{N}_\infty$ and let $L \in \mathbb{R}$ be infinitely close to s_N/N.*
(a) Verify that s_n exists as defined for $n \in \mathbb{N}$.
(b) Show the s_N/N is limited, so that such a real L exists under hypothesis of theorem 5.8.1.
(c) Prove that L is a least upper bound of A in $\mathbb{R}$.

Proof. Let r be an upper bound of $A \subseteq \mathbb{R}$.
(a) Let $n \in \mathbb{N}$. Consider the set $B = \{m \in \mathbb{N} : m/n$ is an upper bound of $A\}$. Since $r \in \mathbb{R}$, by archimedean property, we know that there exists $l \in \mathbb{N}$ such that $r < l$. Clearly $l \cdot n \in B$, and so B is non empty. By the well ordering principle, it follows that there exists a least element in B, or in other words there exists a least $k \in \mathbb{N}$ such that k/n is an upperbound of A for each n.

(b) Let $n \in \mathbb{N}$. Since s_n is the least integer such that s_n/n is upperbound of A, clearly $(s_n - 1)/n$ is not an upper bound. But that means there exists some $a \in A$ such that $(s_n - 1)/n \leq a \leq s_n/n$. Applying transfer on the statement $(\forall n \in \mathbb{N})(\exists a \in A)((s_n - 1)/n \leq a \leq s_n/n)$, we get $(\forall n \in {}^*\mathbb{N})(\exists a \in {}^*A)((s_n - 1)/n \leq a \leq s_n/n)$. In particular, we have $(s_N - 1)/N \leq a \leq s_N/N$. However since s_N/N and $(s_N - 1)/N$ are infinitely close, it must be the case that a is infinitely close to s_N/N. Applying transfer on the statement $(\forall a \in A)(a \leq r)$, where r is the real upper bound we are given, we get $(\forall a \in {}^*A)(a \leq r)$. Thus, $s_N/N \leq r$. Next we will prove that is not negative unlimited either. Since $A \subseteq {}^*A$ and s_N/N is an upper bound of ${}^*A$, we have some $x \in A$ such that $x \leq s_N/N$. Hence, $x \leq s_N/N \leq r$ for some real r, s.

(c) For the sake of contradiction assume there is an upper bound, K, of A such that $K < L$. Notice that there does not exists any $a \in A$ such that $a \in [K, L]$ since otherwise K would not be an upper bound. Applying transfer on $(\forall a \in A)(a \notin [K, L])$, we get $(\forall a \in {}^*A)(a \notin {}^*[K, L])$ which is false since in previous paragraph we demonstrated that there is an element of ${}^*A$ which is infinitely close to L. $\square$

Exercise 5.14. *(Section 5.11, I) Provide an analogous description of the order structure of the hyperintgers ${}^*\mathbb{Z}$.*

Proof. By transfer, we know that ${}^*\mathbb{Z}$ is closed under addition, subtraction and multiplication. Observe that only limited hyperinteger are members of $\mathbb{Z}$ since if $k \in {}^*Z$ is limited then $|k| leqn$ for some $n \in \mathbb{N}$ but that means $-n \leq k \leq n$. Applying transfer on $\forall x \in \mathbb{Z}(-n \leq x \leq n \to x = -n \vee x = -n{+}1 \vee \cdots x = 1 \vee \cdots x = n)$, thus $k \in \{-n, -n{+}1, \cdots, n\}$, and so $k \in \mathbb{Z}$. Therefore, all numbers of ${}^*\mathbb{Z} - \mathbb{Z}$ are unlimited and thus either less than all members of $\mathbb{Z}$ or greater than it. Let $K \in {}^*\mathbb{Z} - \mathbb{Z}$. Define $\gamma(K) = \{K\} \cup \{K \pm n : n \in \mathbb{N}\}$, clearly all members of $\gamma(K)$ are unlimited and $\gamma(K)$ is isomorphic to $\mathbb{Z}$ under the usual ordering. Alternatively, we can defined $\gamma(K) = \{H \in {}^*\mathbb{Z} : Z \sim H\} = \mathrm{gal}(K) \cap {}^*\mathbb{Z}$. Notice that Z itself is a ${}^*\mathbb{Z}$ galaxy, $\mathbb{Z} = \mathrm{gal}(0) \cap {}^*\mathbb{Z}$.

We define order on ${}^*\mathbb{Z}$ galaxy as $\gamma(Y) < \gamma(H)$ iff $Y \leq H$ whenever $K \not\sim H$. Clearly there is no greatest ${}^*\mathbb{Z}$ galaxy since $\gamma(K) < \gamma(2K)$ for any positive unlimited K. Simillarly there is no smallest galaxy either. To see why notice that either K is even or $K + 1$ is, by transfer. If $K + 1$ is even, we have $\gamma(K + 1/2) < \gamma(K + 1) = \gamma(K)$. If K is even then we have $\gamma(K) < \gamma(K)$.

We can imagine ordering on ${}^*\mathbb{Z}$ as ordering of $\mathbb{Z}$ surrounded by ordered set of ${}^*\mathbb{Z}$ galaxies with no greatest or least. $\square$

Exercise 5.15. *(Section 5.11, II) Show that for any $M \in {}^*\mathbb{N}$ there is an $N \in {}^*\mathbb{N}$ that is divisble by all members of $\{1, 2, \cdots, M\}$. Hence show that there exists a hypernatural number N that is divisble by every standard positive integer.*

Proof. We extend the factorial function $! : \mathbb{N} \to \mathbb{N}$ to $! : {}^*\mathbb{N} \to {}^*\mathbb{N}$ using extension of function. Applying transfer on the statement $(\forall n \in \mathbb{N})(k \leq n \to k | n!)$, we get $(\forall n \in {}^*\mathbb{N})(k \leq n \to k | n!)$. Thus, letting $N = M!$ does the job. $\square$

Exercise 5.16. *(Section 5.11, III) Develop a theory of prime factors in ${}^*\mathbb{N}$: if Π is the set of standard prime numbers, with enlargement ${}^*\Pi \subseteq {}^*\mathbb{N}$, prove the following.*
(a) ${}^*\Pi$ consists precisely of those hypernaturals > 1 that have no nontrivial factors in ${}^*\mathbb{N}$.
(b) Every hypernatural number > 1 has a hyperprime factor, i.e., is divisble by some members of ${}^*\Pi$.
(c) Two hypernaturals are equal if they have exactly the same factors of the form p^n with $p \in {}^*\Pi$ and $n \in {}^*\mathbb{N}$.
(d) A hypernatural number is divisible by every standard positive integer iff it is divisible by p^n for every standard prime p and every $n \in \mathbb{N}$.

Proof.
(a) By transfer on $(\forall x \in \Pi)(x > 1 \to$ x have no nontrivial factors in $\mathbb{N})$.
(b) By transfer on $(\forall n \in \mathbb{N})(n > 1 \to \exists p \in \Pi)(p | n)$.
(c) By transfer on $(\forall x \in \mathbb{N})(\forall y \in \mathbb{N})(x = y \leftrightarrow$ they have exactly the same factors of the form p^n with $p \in \Pi$ and $n \in \mathbb{N})$.
(d) Trivial. $\square$

Chapter 6

Convergence of Sequences and Series

Exercise 6.1. *(Section 6.3, I) Prove theorem 6.3.1. If* $\lim_{n\to\infty} s_n = L$ *and* $\lim_{n\to\infty} t_n = M$ *in* $\mathbb{R}$. *Then*
(a) $\lim_{n\to\infty}(s_n + t_n) = L + M$.
(b) $\lim_{n\to\infty}(cs_n) = cL$, *for any* $c \in \mathbb{R}$,
(c) $\lim_{n\to\infty}(s_n t_n) = LM$,
(d) $\lim_{n\to\infty}(s_n/t_n) = L/M$, *if* $M \neq 0$.

Proof. For what follows consider $N \in {}^*\mathbb{N}_\infty$. It means that $s_N \simeq L$ and $t_N \simeq M$.
(a) Since $s_N \simeq L$ and $t_N \simeq M$, it follows $s_N + t_N \simeq L + M$. Since N was arbitrary, we have $s_n + t_n \simeq L + M$ for all unlimited n. Hence, $\lim_{n\to\infty}(s_n + t_n) = L + M$.
(b) Since $S_N \simeq L$, we have $cS_N \simeq cL$. Since N was arbitrary, $cs_n \simeq cL$ for all unlimited n. Hence, $\lim_{n\to\infty} cs_n = cL$.
(c) Since $s_N \simeq L$ and $t_N \simeq M$, we have $s_N t_N \simeq LM$ but since N was arbitrary, it follows $s_n t_n \simeq LM$ for all unlimited n. Hence, $\lim_{n\to\infty}(s_n t_n) = LM$.
(d) Again, from $s_N \simeq L$ and $t_N \simeq M$, we have $s_N/t_N \simeq L/M$ but since N was arbitrary, $\lim_{n\to\infty} s_n/t_n = L/M$. □

Exercise 6.2. *(Section 6.4, I) Prove theorem 6.4.2. A real valued sequence*
(a) diverges to infinity iff all of its extended terms are positive unlimited; and
(b) diverges to negative infinity iff all of its extended terms are negative unlimited.

Proof. Let the sequence be (s_n).
(a) First we prove forward direction. Let $r \in \mathbb{R}$. Since the sequence diverges, we know that there is some $n \in \mathbb{N}$ such that $(\forall m \in \mathbb{N})(m \geq n \to s_m > r)$ holds true. Applying transfer on it, we have $(\forall m \in {}^*\mathbb{N})(m \geq n \to s_m > r)$ to be true. In particular, $s_N > r$ for any $N \in {}^*\mathbb{N}_\infty$ since $N > n$. It follows that s_N is positive unlimited for all $N \in {}^*\mathbb{N}_\infty$. We prove the converse direction. Let $r \in \mathbb{R}$. Notice that for any unlimited $N \in {}^*\mathbb{N}_\infty$ if $M \geq N$ then $M \in {}^*\mathbb{N}_\infty$. Therefore, it follows that $(\exists n \in {}^*\mathbb{N})(\forall m \in {}^*\mathbb{N})(m \geq n \to s_m > r)$ since s_m is positive unlimited by assumption. Transferring the statement to $\mathbb{R}$, we have $(\exists n \in \mathbb{N})(\forall m \in \mathbb{N})(m \geq n \to s_m > r)$ to be true in $\mathbb{R}$. Since r was arbitrary, we have (s_n) to be divergent to positive infinity.

(b) Follows from (a) by considering the sequence $(-s_n)$. □

Exercise 6.3. *(Section 6.5, I) Prove theorem 6.5.1. A real values sequence $\langle s_n \rangle$ is Cauchy in $\mathbb{R}$ iff all it's extended terms are infinitely close to each other, i.e., iff $s_m \simeq s_n$ for all $m, n \in {}^*\mathbb{N}_\infty$.*

Proof. First we prove the forward implication. Suppose $\langle s_n \rangle$ is cauchy. Let $\varepsilon \in \mathbb{R}^+$ and $j \in \mathbb{N}$ be such that $(\forall m, n \in \mathbb{N})(m, n \geq j \to |s_m - s_n| < \varepsilon)$. Applying transfer on the latter we know $(\forall m, n \in {}^*\mathbb{N})(m, n \geq j \to |s_m - s_n| < \varepsilon)$ is true in ${}^*\mathbb{R}$. In particular, we have $|s_M - s_N| < \varepsilon$ for arbitrary $M, N \in {}^*\mathbb{N}_\infty$ since j is limited. Since ε was arbitrary positive real number, we have $s_M \simeq s_N$. We now prove the converse direction. Suppose $s_m \simeq s_n$ for all $m, n \in {}^*\mathbb{N}_\infty$. Let $\varepsilon \in \mathbb{R}^+$ be arbitrary. Notice that for any $K \in {}^*\mathbb{N}_\infty$, if $L \geq N$ then $L \in {}^*\mathbb{N}_\infty$, and also notice for any $L, K \in {}^*\mathbb{N}_\infty$, $s_L - s_K = \delta$ where δ is infintisimal. Therefore, we have $(\exists j \in {}^*\mathbb{N})(\forall m, n \in {}^*\mathbb{N})(m, n \geq j \to |s_m - s_n| \leq \varepsilon)$ to be true in ${}^*\mathbb{R}$. Applying transfer to it, we have $(\exists j \in \mathbb{N})(\forall m, n \in \mathbb{N})(m, n \geq j \to |s_m - s_n| < \varepsilon)$ to be true in $\mathbb{R}$. Since $\varepsilon \in \mathbb{R}^+$ was arbitrary, we have (s_n) to be cauchy. ☐

Exercise 6.4. *(Section 6.5, II). Cauchy's convergence Criterion.*
Prove the foward direction of A real valued sequence converges in $\mathbb{R}$ iff it is cauchy.

Proof. Let (s_n) be a real valued sequence and suppose it convergence in $\mathbb{R}$ to L. Therefore, by theorem 6.1.1, we have $s_N \simeq L$ for all $n \in {}^*\mathbb{N}_\infty$ but that means all extended terms are infinitely close to each other i.e. for any $n, m \in {}^*\mathbb{N}_\infty$, $s_n \simeq L \simeq s_m$. Therefore, by theorem 6.5.1, we have (s_n) to be cauchy. ☐

Exercise 6.5. *(Section 6.7, I) Let (s_n) and (t_n) be real valued sequences with limits, L, M respectively. Show that if $s_n \leq t_n$ for $n \in \mathbb{N}$. Then $L \leq M$.*

Proof. Applying transfer on $(\forall n \in \mathbb{N})(s_n \leq t_n)$, we have $(\forall n \in \mathbb{N})(s_n \leq t_n)$ to be true in ${}^*\mathbb{R}$. In particular for any $N \in {}^*\mathbb{N}_\infty$, we have $s_N \leq t_N$, however, since $L \simeq s_N \leq t_N \simeq M$, we have $L \leq M$ by exercise 2 of section 5.5. ☐

Exercise 6.6. *(Section 6.7, II) If $r_n \leq s_n \leq t_n$ in $\mathbb{R}$ for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} r_n = \lim_{n\to\infty} t_n$, show that (s_n) converges to this same limit.*

Proof. Applying transfer on $(\forall n \in \mathbb{N})(r_n \leq s_n \leq t_n)$, we have $(\forall n \in {}^*\mathbb{N})(r_n \leq s_n \leq t_n)$ to be true in ${}^*\mathbb{R}$. In particular, for any $N \in {}^*\mathbb{N}_\infty$, we have $r_N \leq s_N \leq t_N$. However, since $(r_N), (t_N)$ are limited, s_N is limited. By the shadow inequalitiy, we have $\mathrm{sh}(r_N) \leq \mathrm{sh}(s_N) \leq \mathrm{sh}(t_N) = \mathrm{sh}(r_N)$. Hence, $\mathrm{sh}(s_N) = \mathrm{sh}(r_N) = \lim_{n\to\infty} r_n$ but that means $r_N \simeq \lim_{n\to\infty} r_n$ for all $N \in {}^*\mathbb{N}_\infty$. Hence, (s_n) converges to $\lim_{n\to\infty} r_n$. ☐

Exercise 6.7. *(Section 6.7, III) If a sequence converges in $\mathbb{R}$, show that it has exactly one cluster point.*

Proof. Suppose (s_n) converges to L and assume it has two cluster point A and B. By definition of convergence we have $s_N \simeq L$ for all $N \in {}^*N_\infty$ but that would imply $B \simeq L \simeq A$ by the definition of cluster point, since they are reals, it follows $A = L = B$. ☐

Exercise 6.8. *(Section 6.7, IV) Suppose that a real valued sequence has a single cluster point. If the sequence is bounded, must it be convergent? what if it is unbounded?*

Proof. Let (s_n) be a bounded sequence which has an unique cluster point i.e. for some $M \in {}^*\mathbb{N}_\infty$, we have $s_M \simeq L$ for some real L. Suppose it is not convergent. Therefore, there must exists $N \in {}^*\mathbb{N}_\infty$ such that $s_N \not\simeq L$. However since (s_n) is bounded, we have s_N to be limited and hence infinitely close to a real number ie its shadow. But if $\mathrm{sh}(s_N) \neq L$, clearly $\mathrm{sh}(s_N)$ is a cluster point which contradicts the assumption that there was an unique cluster point.
Consider the sequence $\langle s \rangle = (0, 1, 0, 2, 0, 3, \cdots)$. Clearly this sequence has only one cluster point and is unbounded but it does not converges to any real number. ☐

Exercise 6.9. *(Section 6.8, I) Prove, by nonstandard reasoning, that both the limit superior and limit inferior are cluster points of the sequence s.*

Proof. Let $L = \limsup s$ and $\varepsilon \in \mathbb{R}^+$. Since L is supremum of C_s, we know there exists $N \in {}^*\mathbb{N}_\infty$ such that $L - \varepsilon < s_N \le L$. Consider some arbitrary $n \in \mathbb{N}$. We have $(\exists m \in {}^*\mathbb{N})(m > n \wedge L - \varepsilon < s_m)$ to be true, and applying transfer on it, it follows $(\exists m \in \mathbb{N})(m > n \wedge L - \varepsilon < s_n)$ is true in $\mathbb{R}$. Since $\varepsilon \in \mathbb{R}^+$ and $n \in \mathbb{N}$ was arbitrary, $(\forall \varepsilon \in \mathbb{R}^+)(\forall n \in \mathbb{N})(\exists m \in \mathbb{N})(m > n \wedge L - \varepsilon < s_m)$ is true in $\mathbb{R}$. Applying transfer we see $(\forall \varepsilon \in {}^*\mathbb{R}^+)(\forall n \in {}^*\mathbb{N})(\exists m \in {}^*\mathbb{N})(m > n \wedge L - \varepsilon < s_n)$ holds in ${}^*\mathbb{R}$. Picking ε as some infinitesimal and n as unlimted, we follow there is some $m \in {}^*\mathbb{N}$ making $m > n$ and $L - e < s_m$ true, that is to say, there is unlimited m such that $L < s_m + e$. Therefore, it follows $L \le \operatorname{sh}(s_m)$ for an unlimited m. However, since L is upper bound of C_s, we must have $L = \operatorname{sh}(s_m)$ making it a cluster point. We cab argue about lim inf in a similar fashion. ☐

Exercise 6.10. *(Section 6.9, I) Formulate the definition of the limit inferior of an arbitrary real valued sequence.*

answer:
Proposition A real number L is equal to $\liminf s$ if and only if
(1) $L < s_n$ or $L \simeq s_n$ for all unlimited n; and
(2) $s_n \simeq L$ fdor atleast one unlimited n.

Proof. The condition (1) holds iff $L \le \operatorname{sh}(s_n)$. Thus, it implies L is lower bound of set of cluster points, C_s. The condition (2) asserts that L is a cluster point i.e. $L \in C_s$. Together they imply that L is infimum (since it's the smallest element) of the set. Hence, it is equal to $\liminf s$. ☐

Exercise 6.11. *(Section 6.9, III) Formulate and prove theorems about the limit inferior of a bounded sequence that correspond to theorems 6.8.4 and 6.8.5.*

answer:
Proposition-1. If s is a bounded real valued sequence with limit inferior, lim inf, then for any positive real ε:
(1) Some standard tail of s has all its terms bigger than $\liminf -\varepsilon$ i.e. $\liminf -\varepsilon < s_n$ for all but finitely many $n \in \mathbb{N}$.
(2) $s_n < \liminf +\varepsilon$ for infinitely many $n \in \mathbb{N}$.

Proof. Let $L = \liminf s$. Consider $\varepsilon \in \mathbb{R}^+$. First we prove (1). For any unlimited N, we have $L < s_N$ or $L \simeq s_N$. Thus, $L - \varepsilon < s_N$ holds for any unlimited N. Trasnferring $(\exists n \in {}^*\mathbb{N})(\forall m \in {}^*\mathbb{N})(m \ge n \wedge L - \varepsilon < s_m)$, we have $(\exists n \in \mathbb{N})(\forall m \in \mathbb{N})(m > n \wedge L - \varepsilon < s_m)$ to be true in $\mathbb{R}$ but that simply means $L - \varepsilon < s_m$ is true for all but finitely many n. We now prove (2). Let $m \in \mathbb{N}$. We know that L is infinitely close to s_N for some unlimited N. Thus, $(\exists n \in {}^*\mathbb{N})(n > m \wedge s_n < L + \varepsilon)$ is true in ${}^*\mathbb{R}$. Transfering it, we have $(\exists n \in \mathbb{N})(n > m \wedge s_n < L + \varepsilon)$ to be true in $\mathbb{R}$. However since $m \in \mathbb{N}$ was arbitrary, it follows that for any $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $n > m$ and $s_n < L + e$ satisfying the proposition. ☐

Proposition-2. For any bounded real valued sequence s,

$$\liminf_{n\to\infty} s_n = \lim_{n\to\infty} I_n = \lim_{n\to\infty} \left(\inf_{m \ge n} s_m \right)$$

where $I_n = \inf_{m \ge n} s_m$.

Proof. First we will prove that lim inf is upper bound of I. Take an extended term s_N infinitely close to the cluster point lim inf. For any $m \in \mathbb{N}$, we have $I_m \le s_n$ for all limited $n \ge m$, and hence it holds for all hypernatural $n \ge m$ by transfer. In particular, $I_m \le s_N$, as $\liminf \simeq s_N$, we have $I_m \le \liminf$.
Let $L = \lim I_n$. Then clearly L is supremum of I since I is nondecreasing. However, as we showed lim inf is an upper bound, we must have $L \le \liminf$. We will now show that $L < \liminf$ is impossibility. Suppose $L < \liminf$ and consider some $\varepsilon \in \mathbb{R}^+$ such that $L < \liminf -\varepsilon$. By proposition-1, we know that there

exists $n \in \mathbb{N}$ such that standard tail beyond n is bounded below by $\liminf -\varepsilon$. But that means that $\liminf -\varepsilon \leq I_n$, and so $L < \liminf -\varepsilon \leq I_n$ which contradicts the fact that L was upper bound of I. Hence, $\liminf = L$. □

Exercise 6.12. *(Section 6.9, III) If s is a bounded sequence, show that for each $\varepsilon \in \mathbb{R}^+$ there is some $n \in \mathbb{N}$ such that the standard tail $s_n, s_{n+1}, \cdots$ is contained in the interval* $(\liminf -\varepsilon, \limsup +\varepsilon)$.

Proof. By theorem 6.8.4 and previous exericse, we know that there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $s_n < \limsup +\varepsilon$ and there exists $N' \in \mathbb{N}$ such that for all $n \geq N'$, $s_n < \limsup +\varepsilon$. Defining $m = \max\{N, N'\}$, the tail following s_m is the desired one. □

Exercise 6.13. *(Section 6.11, I) Give an example of a series that diverges but has a_n infinitesimal for all unlimted n.*

Proof. Consider the sequence (s_n) defined as $(s_n) = (1/n)$. Clearly for any unlimited n, a_n is infinitisimal. However, notice that

□

Exercise 6.14. *(Section 6.11, II) Give nonstandard proofs of the usual rules of arithemtically combining convergent series.*
(a) $\sum_1^\infty a_i + \sum_1^\infty b_i = \sum_1^\infty (a_i + b_i)$,
(b) $\sum_1^\infty a_i - \sum_1^\infty b_i = \sum_1^\infty (a_i - b_i)$,

Proof. We will first prove (a), then (c), and then use (a) and (c) for proving (b). For what follow, Let $L = \sum_1^\infty a_i$ and $M = \sum_1^\infty b_i$.
(a) Let $N \in {}^*\mathbb{N}_\infty$. Since $\sum_1^n (a_i + b_i) = \sum_1^n a_i + \sum_1^n b_i$ holds in $\mathbb{R}$ for all $n \in \mathbb{N}$, by transfer we know it also holds in ${}^*\mathbb{R}$ for all $n \in {}^*\mathbb{N}$. Hence,

$$\sum_1^N (a_i + b_i) = \sum_1^N a_i + \sum_1^N b_i.$$

and thus,

$$\begin{aligned} \mathrm{sh}\left(\sum_1^N (a_i + b_i)\right) &= \mathrm{sh}\left(\sum_1^N a_i + \sum_1^N b_i\right) \\ &= \mathrm{sh}\left(\sum_1^N a_i\right) + \mathrm{sh}\left(\sum_1^N b_i\right) \\ &= L + M \end{aligned}$$

As desired.

(c) Let $N \in {}^*\mathbb{N}_\infty$. Then it follows

$$\begin{aligned} \mathrm{sh}\left(\sum_1^N c a_i\right) &= \mathrm{sh}\left(c \sum_1^N a_i\right) && \text{(by transfer)} \\ &= c \cdot \mathrm{sh}\left(\sum_1^N a_i\right) \\ &= c \cdot L \end{aligned}$$

as desired.

(b) Let $N \in {}^*\mathbb{N}_\infty$. Then

$$\begin{aligned}
\operatorname{sh}\left(\sum_1^N (a_i - b_i)\right) &= \operatorname{sh}\left(\sum_1^N (a_i + (-b_i))\right) \\
&= \operatorname{sh}\left(\sum_1^N a_i\right) + \operatorname{sh}\left(\sum_1^N (-b_i)\right) && \text{(by part a)} \\
&= \operatorname{sh}\left(\sum_1^N a_i\right) + (-1) \cdot \operatorname{sh}\left(\sum_1^N (-b_i)\right) && \text{(by part c)} \\
&= L + (-M) \\
&= L - M.
\end{aligned}$$

□

Exercise 6.15. *(Section 6.11, III) Suppose that $a_i \geq 0$ for all $i \in \mathbb{N}$. Prove that $\sum_1^\infty$ converges iff $\sum_1^n a_i$ is limited for all unlimited n, and that this holds iff $\sum_1^n a_i$ is limited for some unlimted n.*

Proof. First we prove (1) ⇒ (2). Suppose $\sum_1^\infty a_i$ converges to a real number L. Therefore, we must have $\sum_1^N a_i \simeq L$ for all unlimited N but that means $\sum_1^N a_i$ is limited. The direction (2) ⇒ (3) is easy to see. □