Lectures on Hyperreals

Book by Robert Goldblatt

:

Contents

1 What are Hypereals? 3

2 Large sets 4

3 Ultrapower Construction of the hyperreals 6

4 The Transfer Principle 10

5 Hypereals Great and Small 11

6 Convergence of Sequences and Series 15

7 Continuous Functions 24

8 Differentiation 29

9 The Riemann Integral 34

10 Topology of the Reals 38

Chapter 1

What are Hypereals?

Chapter 2

Large sets

Exercise 2.1. *If $\emptyset \neq A \subseteq I$, there is an ultrafilter $\mathcal{F}$ on I with $A \in \mathcal{F}$.*

Proof. Define $\mathcal{H} = \{B \in \mathscr{P}(I) : A \subseteq B\}$. Since A is non empty, it follows that $\mathcal{H}$ has fip. Clearly, $A \in \mathcal{H}$, and so $A \in \mathcal{F}^{\mathcal{H}}$. Therefore, by zorn lemma, we know that there exists ultrafilter, $\mathcal{F}$, such that $A \in \mathcal{F}^{\mathcal{H}} \subseteq \mathcal{F}$. □

Exercise 2.2. *There exists a nonprincipal ultrafilter on $\mathbb{N}$ containing the set of even numbers, and another containing the set of odd numbers.*

Proof. Define $I = \{x \in \mathbb{N} : x = 2n \text{ for some } n \in \mathbb{N}\}$. Since I is infinite, we know that $\mathcal{F}^{\text{co}}$ is proper and thus, have fip. Applying the zorn's lemma, we see that there exists ultrafilter $\mathcal{F}$ such that $\mathcal{F} \subseteq \mathcal{F}^{\text{co}}$. However, for any $i \in I$, we have $I - \{i\} \in \mathcal{F}^{\text{co}} \subseteq \mathcal{F}$ but, since $\mathcal{F}$ is ultrafilter, $\{i\} \in \mathcal{F}$. However, clearly $\{i\} \in \mathcal{F}^i$. Therefore, $F \neq F^i$ for any $i \in I$. Likewise we could argue about set of odd numbers. □

Exercise 2.3. *An ultrafilter on finite set must be principal.*

Proof. Let the set be $A = \{a_1, a_2, \cdots, a_n\}$ and $\mathcal{F}$ be ultrafilter on it. Since $\{a_1\} \cup \{a_2\} \cup \cdots \cup \{a_n\} = A \in \mathcal{F}$, it must be the case that $a_i \in \mathcal{F}$ for some i such that $1 \leq i \leq n$. Since $\mathcal{F}$ is proper, for any $B \in \mathcal{F}$, we have $B \cap \{a_i\} \neq \emptyset$, but that means $\mathcal{F} = \mathcal{F}^{a_i}$. □

Exercise 2.4. *For $\mathcal{H} \subseteq \mathscr{P}(I)$.*
(i) Show that $\mathcal{F}^{\mathcal{H}}$ is a filter that includes $\mathcal{H}$, i.e. $\mathcal{H} \subseteq \mathcal{F}^{\mathcal{H}}$.
(ii) Show that $\mathcal{F}^{\mathcal{H}}$ is included in any other filter that includes $\mathcal{H}$.

Proof.
(i) By the definition.
(ii) Let $\mathcal{F}$ be a filter such that $H \subseteq \mathcal{F}$. Let $A \in \mathcal{F}^{\mathcal{H}}$, then $B_1 \cap B_2 \cap \cdots \cap B_n \subseteq A$ for some $B_1, \cdots, B_n \in \mathcal{H}$. Clearly, $B_1 \cap \cdots \cap B_n \in F$ and so is $A \in F$. □

Exercise 2.5. *Let $\mathcal{F}$ be a proper filter on I.*
(I) Show that $\mathcal{F} \cup \{A^c\}$ has fip iff $A \notin \mathcal{F}$.
(II) Use (I) to deduce that $\mathcal{F}$ is an ultrafilter iff it is maximal proper filter on I.

Proof.
(I) First we prove forward implication. Notice that if $A \in \mathcal{F}$ then since $A \cap A^c = \emptyset$, it follows $\mathcal{F} \cup \{A^c\}$ does not have fip.
We prove the converse implication. Notice that since F is a proper filter, ie it does not contain $\emptyset$, the only way for $\mathcal{F} \cup \{A^c\}$ to not have fip would be $A \in \mathcal{F}$.

(II) We prove the foward implication. Suppose $\mathcal{F}$ is an ultrafilter. Applying Zorn's lemma on proper filter $\mathcal{F}$, we see that there exists a maximal proper filter, K, such that $\mathcal{F} \subseteq K$. If $K \neq F$, it would mean there is some $A \in K$ such that $A \notin \mathcal{F}$ but then that would mean $A^C \in \mathcal{F}$, and so $A^c \in K$ which contradicts the fact that K is proper since it fails to satisfy fip. Therefore, we must have $F = K$.

We prove the converse implication. Suppose $\mathcal{F}$ is maximal proper filter. For the sake of contradiction assume $\mathcal{F}$ is not an ultra filter i.e. there exists some $A \in \mathscr{P}(I)$ such that both $A, A^c \notin F$. Clearly, either $\mathcal{F} \cup \{A\}$ has fip or $\mathcal{F} \cup \{A^c\}$ does. Wlog assume $\mathcal{F} \cup \{A\}$ has fip. Applying Zorn's lemma on $\mathcal{F} \cup \{A\}$, we see that there is maximal proper filter K such that $\mathcal{F} \cup \{A\} \subseteq K$. However this means that $\mathcal{F}$ is not maximal proper filter since $F \subsetneqq F \cup \{A\} \subset P(I)$. $\square$

Chapter 3

Ultrapower Construction of the hyperreals

Exercise 3.1. *(section 3.3, I).* $\equiv$ *is an equivalence relation on* $\mathbb{R}^{\mathbb{N}}$.

Proof. Let $\langle a_n\rangle, \langle b_n\rangle, \langle c_n\rangle \in \mathbb{R}^{\mathbb{N}}$. First we prove that reflexivity. Clearly, $\{n \in \mathbb{N} : a_n = a_n\} = \mathbb{N} \in \mathcal{F}^{\text{co}} \subseteq \mathcal{F}$. Thus, $\langle a_n\rangle \equiv \langle a_n\rangle$.
Symmetricity is trivial since $\{n \in \mathbb{N} : a_n = b_n\} = \{n \in \mathbb{N} : b_n = a_n\}$, if one of them is in F then so is the other.
Finally, we prove transitivity. Suppose $\langle a_n\rangle \equiv \langle b_n\rangle$ and $\langle b_n\rangle \equiv \langle c_n\rangle$ i.e. $\{n \in \mathbb{N} : a_n = b_n\}, \{n \in \mathbb{N} : b_n = c_n\} \in \mathcal{F}$. Clearly, $\{n \in \mathbb{N} : a_n = b_n = c_n\} = \{n \in \mathbb{N} : a_n = b_n\} \cap \{n \in \mathbb{N} : b_n = c_n\} \in \mathcal{F}$. Therefore, $\langle a_n\rangle \equiv \langle c_n\rangle$. □

Exercise 3.2. *(section 3.3, II)* $\equiv$ *is a congruence on the ring* $\langle \mathbb{R}^{\mathbb{N}}, \oplus, \odot\rangle$*, which means that if* $r \equiv r'$ *and* $s \equiv s'$*, then* $r \oplus s = r' \oplus s'$ *and* $r \odot s = r' \odot s'$.

Proof. Suppose $r \equiv r'$ and $s \equiv s'$ i.e. $\{n \in \mathbb{N} : r_n = r'_n\}, \{n \in \mathbb{N} : s_n = s'_n\} \in \mathcal{F}$. Thus, since $\{n \in \mathbb{N} : r_n = r'_n, s_n = s'_n\} = \{n \in \mathbb{N} : r_n = r'_n\} \cap \{n \in \mathbb{N} : s_n = s'_n\} \in \mathcal{F}$, we have $\{n \in \mathbb{N} : r_n = r'_n, s_n = s'_n\} \subseteq \{n \in \mathbb{N} : r_n + s_n = r'_n + s'_n\} \in \mathcal{F}$. Similarly, $\{n \in \mathbb{N} : r_n s_n = r'_n s'_n\} \in \mathcal{F}$. Therefore, $r \oplus s \equiv r' \oplus s'$ and $r \odot s \equiv r' \odot s'$. □

Exercise 3.3. *(section 3.3, III)* $\langle 1, \frac{1}{2}, \frac{1}{3}, \cdots\rangle \not\equiv \langle 0, 0, 0, \cdots\rangle$.

Proof. Let $\langle r\rangle = \langle 1, \frac{1}{2}, \frac{1}{3}, \cdots\rangle$. For the sake of simplicity, define $[[r \neq 0]] = \{n \in \mathbb{N} : r_n \neq 0\}$. Clearly, $[[r \neq 0]] \subseteq \mathbb{N}$. Notice that for any $n \in \mathbb{N}$, we have $n \in [[r \neq 0]]$. Thus, $[[r \neq 0]] = \mathbb{N}$. So, $[[r \neq 0]] \in \mathcal{F}^{\text{co}} \subseteq \mathcal{F}$ but then it would mean $[[r = 0]] = [[r \neq 0]]^c \notin \mathcal{F}$. Therefore, $\langle r_n\rangle \not\equiv \langle 0\rangle$. □

Exercise 3.4. *(section 3.5, I)* $[[r = s]] \cap [[s = t]] \subseteq [[r = t]]$.

Proof. Let $n \in [[r = s]] \cap [[s = t]]$. Then it follows that $r_n = s_n$ and $s_n = t_n$, and so $r_n = t_n$. Therefore, $n \in [[s = t]]$. □

Exercise 3.5. *(Section 3.5, II)* $[[r = r']] \cap [[s = s']] \subseteq [[r \oplus s = r' \oplus s']] \cap [[r \oplus s = r' \oplus s']]$

Proof. Let $n \in [[r = r']] \cap [[s = s']]$. Thus, $r_n = r'_n$ and $s_n = s'_n$, and so $r_n \oplus s_n = r'_n \oplus s'_n$ and $r_n \odot s_n = r'_n \odot s'_n$. Therefore, $n \in [[r \oplus s = r' \oplus s']] \cap [[r \oplus s = r' \oplus s']]$. □

Exercise 3.6. *(Section 3.5, III)* $[[r = r']] \cap [[s = s']] \cap [[r < s]] \subseteq [[r' < s']]$.

Proof. Let $n \in [[r = r']] \cap [[s = s']] \cap [[r < s]]$. Thus, $r_n = r'_n, s_n = s'_n, r_n < s_n$ but then that means $r'_n < s'_n$, and so $n \in [[r' < s']]$. □

Exercise 3.7. *(Section 3.5, IV) If* $r \equiv r'$ *and* $s \equiv s'$*, then* $[[r < s]] \in \mathcal{F}$ *iff* $[[r' < s']] \in \mathcal{F}$.

Proof. Suppose $r \equiv r'$ and $s \equiv s'$, that is to say, $\{n \in \mathbb{N} : r_n = r'_n\}, \{n \in \mathbb{N} : s_n = s'_n\} \in \mathcal{F}$. First assume $[[r < s]] \in \mathcal{F}$. Therefore, $\{n \in \mathbb{N} : r_n = r'_n\} \cap \{n \in \mathbb{N} : s_n = s'_n\} \cap \{n \in \mathbb{N} : r_n < s_n\} \subseteq \{n \in \mathbb{N} : r' < s'\} \in mcF$. Next assume $[[r' < s^p]] \in \mathcal{F}$. It follows $\{n \in \mathbb{N} : r_n = r'_n\} \cap \{n \in \mathbb{N} : s_n = s'_n\} \cap \{n \in \mathbb{N} : r'_n < s'_n\} \subseteq \{n \in \mathbb{N} : r < s\} \in \mathcal{F}$ □

Exercise 3.8. *(Section 3.8, I) Use only general properties of ordered fields to deduce from the fact that $[\varepsilon]$ is a positive infinitismal, the conclusion that $[\varepsilon]^{-1}$ is greater than every real number.*

Proof. Notice that for any $r \in \mathbb{R}^+$, we have ${}^*r = [\langle r, r, \cdots, \rangle] \in {}^*\mathbb{R}$. Since $\varepsilon < {}^*r$, by fields properties, we have $0 < {}^*r^{-1} < \varepsilon^{-1}$. Since r is arbitrary positive real number, setting $r = r^{-1}$,we see that ${}^*r < \varepsilon^{-1}$ for any $r \in \mathbb{R}^+$. $\square$

Exercise 3.9. *(Section 3.10, I) If A is finite, show that ${}^*A = A$, and hence A has non nonstandard members.*

Proof. Let $A = \{a_1, \cdots, a_n\}$. Consider some $[r] \in {}^*A$. If we could prove that $r \equiv t$ for some constant sequence $t \in A$, we would be done. So we set that as our goal. Notice that since $[r] \in {}^*A$, we have $\{n \in \mathbb{N} : r_n \in A\} = \{n \in \mathbb{N} : r_n = a_1 \vee r_n = a_2 \vee \cdots r_n = a_n\} = \{n \in \mathbb{N} : r_n = a_1\} \cup \cdots \cup \{n \in \mathbb{N} : r_n = a_n\} \in \mathcal{F}$. But since $\mathcal{F}$ is ultrafilter, only one of such set is in $\mathcal{F}$. Therefore, $\{n \in \mathbb{N} : r_n = a_k\} \in \mathcal{F}$ for some k such that $1 \leq k \leq n$. Hence, $r \equiv a_k$. $\square$

Exercise 3.10. *(Section 3.10, II) Prove:*
(a) $A \subseteq B$ iff ${}^*A \subseteq {}^*B$.
(b) $A = B$ iff ${}^*A = {}^*B$.

Proof.
(a) First we prove the forward implication. Suppose $A \subseteq B$. Consider $[r] \in {}^*A$. Therefore, we have $[[r \in A]] \in \mathcal{F}$, but since $[[r \in A]] \subseteq [[r \in B]]$, we have $[[r \in B]]$ and so $[r] \in {}^*B$.
We prove the converse implication. Suppose ${}^*A \subseteq {}^*B$. Consider some $r \in A$. Then we have $[\mathbf{r}] \in {}^*A$, and so $[\mathbf{r}] \in {}^*B$. However, since $[\mathbf{r}] \in B$, we have $[[r \in B]] \in \mathcal{F}$ which means $r \in B$.

(b) Trivially follows from (a). $\square$

Exercise 3.11. *(Section 3.10, III)*
Prove:
(a) ${}^*(A \cup B) = {}^*A \cup {}^*B$,
(b) ${}^*(A \cap B) = {}^*A \cap {}^*B$,
(c) ${}^*(A - B) = {}^*A - {}^*B$,
(d) ${}^*\emptyset = \emptyset$.

Proof.
(a) First we prove the foward implication. Since $A \subset A \cup B$, applying (2), we have ${}^*A \subseteq {}^*(A \cup B)$. Similarly, we have ${}^*B \subseteq {}^*(A \cup B)$, and so ${}^*A \cup {}^*B \subseteq {}^*(A \cup B)$.
We prove the converse implication. Let $[r] \in {}^*A \cup {}^*B$. Then we have following cases;
case-1. $[r] \in {}^*A$. Thus, $[[r \in A]] \in mcF$, and since $[[r \in A]] \subseteq [[r \in A \cup B]]$, we have $[[r \in A \cup B]] \in \mathcal{F}$ but that means $[r] \in {}^*A \cup {}^*B$.
case-2. $[r] \in {}^*B$. The proof is analogous to **case-1**.

(b) First we prove the forward implication. Since $A \cap B \subseteq A$, applying (2), we have ${}^*(A \cap B) \subseteq {}^*A$. Similarly we have ${}^*(A \cap B) \subseteq {}^*B$, and so ${}^*(A \cap B) \subseteq {}^*A \cap {}^*B$.
We prove the converse implication. Consider some $[r] \in {}^*A \cap {}^*B$. It means that $[r] \in {}^*A$ and $[r] \in {}^*B$, thus $[[r \in A]], [[r \in B]] \in \mathcal{F}$. Therefore, $[[r \in A \cap B]] = [[r \in A]] \cap [[r \in B]] \in \mathcal{F}$, and so $[r] \in {}^*(A \cap B)$.

(c) First we prove the foward implication. Let $[r] \in {}^*(A \setminus B)$. Then we have $[[r \in A]] \cap [[r \notin B]] = [[r \in A \setminus B]] \in \mathcal{F}$. Thus, $[[r \in A]], [[r \notin B]] \in \mathcal{F}$. Hence, $[r] \in {}^*A$ and $[r] \notin {}^*B$, and so $[r] \in {}^*A \setminus {}^*B$.
We prove the converse implication. Let $[r] \in {}^*A \setminus {}^*B$ i.e. $[r] \in {}^*A$ and $[r] \notin {}^*B$. Then we have $[[r \in A]] \in \mathcal{F}$ and $[[r \notin B]] \in \mathcal{F}$. Thus, $[[r \in A \setminus B]] = [[r \in A]] \cap [[r \notin B]] \in \mathcal{F}$. Therefore, $r \in {}^*(A \setminus B)$.

(d) Clearly, ${}^*\emptyset = \{[r] \in {}^*\mathbb{R} : [[r \in \emptyset]] \in \mathcal{F}\} = \emptyset$. $\square$

Exercise 3.12. *(Section 3.10, IV) Is it true that ${}^*(\bigcup_{n=1}^{\infty} A_n) = \bigcup_{n=1}^{\infty} {}^*A_n$?*

Proof. No. Define $A_n = \{n\}$ for any $n \in \mathbb{N}$. Notice that since A_n is finite, using (1), we have $^*A_n = A_n = \{n\}$. Since $[[\omega \in \bigcup_{n=1}^{\infty} A_n]] = \mathbb{N} \in \mathcal{F}$, we have $\omega \in {}^*(\bigcup_{n=1}^{\infty} A_n)$. However, $[[\omega \in A_n]] = \{n\} \notin \mathcal{F}$, thus $\omega \notin {}^*A_n$ for any $n \in \mathbb{N}$. This mean $\omega \notin \bigcup_{n=1}^{\infty} {}^*A_n$. $\square$

Exercise 3.13. *(Section 3.10, V) Show that if $A \subseteq \mathbb{R}$, then $^*A \cap \mathbb{R} = A$.*

Proof. Suppose $A \subseteq \mathbb{R}$. Let $[r] \in {}^*A \cap \mathbb{R}$. Since $\langle r \rangle$ is a constant sequence and $[r] \in {}^*A$, it follows $[r] = \langle r \rangle \in A$. Thus, $^*A \cap \mathbb{R} \subseteq A$. This means that $^*A \cap \mathbb{R}$ contains only the constant sequences. Let $\langle r \rangle \in A$. Then clearly $[r] \in {}^*A \cap \mathbb{R}$. $\square$

Exercise 3.14. *For $a, b \in \mathbb{R}$, let $[a, b]$ be the closed interval $\{x \in \mathbb{R} : a \leq x \leq n\}$. Prove that $^*[a, b] = \{x \in {}^*\mathbb{R} : a \leq x \leq b\}$.*

Proof. We first prove the forward inclusion. Let $[r] \in {}^*[a, b]$. Then we have $\{n \in \mathbb{N} : a \leq r_n \leq b\} = [[r \in [a, b]]] \in \mathcal{F}$ but that means $[r] \in \{x \in {}^*\mathbb{R} : a \leq x \leq b\}$. We prove the converse inclusion. Let $[r] \in \{x \in {}^*\mathbb{R} : a \leq x \leq b\}$. It follows $[[r \in [a, b]]] = \{n \in \mathbb{N} : a \leq r_n \leq b\} \in mcF$. Therefore, $[r] \in \{x \in \mathbb{R} : a \leq x \leq n\}$. $\square$

Exercise 3.15. *(Section 3.10, VII) $^*\mathbb{Z}$ is a subring of $^*\mathbb{R}$.*

Proof. Clearly, $^*\mathbb{Z} \neq \emptyset$. Consider some $[r], [s] \in {}^*\mathbb{Z}$. Then we have $[[r \in \mathbb{Z}]], [[s \in \mathbb{Z}]] \in \mathcal{F}$. Therefore, since $[[r \in \mathbb{Z}]] \cap [[s \in \mathbb{Z}]] \subseteq [[r+s \in \mathbb{Z}]], [[rs \in \mathbb{Z}]]$, we have $[[r+s \in \mathbb{Z}]], [[rs \in \mathbb{Z}]] \in \mathcal{F}$. Hence, $[r+s], [rs] \in {}^*\mathbb{Z}$. $\square$

Exercise 3.16. *(Section 3.10, VIII) If $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$, show that $^*(\mathbb{R}^+) = \{x \in {}^*\mathbb{R}^+ : x > 0\}$, i.e., $^*(R^+) = ({}^*\mathbb{R})^+$.*

Proof. First we prove the forward inclusion. Let $[r] \in {}^*(R^+)$. It means that $\{n \in \mathbb{N} : r_n > 0\} = [[r \in R^+]] \in \mathcal{F}$ which implies $[r] \in ({}^*\mathbb{R})^+$. Next we prove the converse inclusion. Let $[r] \in ({}^*\mathbb{R})^+$. Then we have $\{n \in \mathbb{N} : r_n > 0\} = [[r > 0]] = [[r \in \mathbb{R}^+]] \in \mathcal{F}$ from which it follows $[r] \in {}^*(\mathbb{R}^+)$. $\square$

Exercise 3.17. *(Section 3.12, I) Show that $^*f$ agrees with f on $\mathbb{R}$: if $r \in \mathbb{R}$, then $^*f(r) = f(r)$.*

Proof. Let $r \in \mathbb{R}$. Notice that r can be indetified with the constant sequence $\mathbf{r} \in {}^*R$. Then we have $^*f([\mathbf{r}]) = [f(\mathbf{r})]$ which is isomorphic to $f(r)$. $\square$

Exercise 3.18. *(Section 3.12, II) If f is injective, so is $^*f$. What about surjectivity?*

Proof. Suppose f is injective. Consider some $[r], [s] \in {}^*\mathbb{R}$ such that $^*f([r]) = {}^*f([s])$, that is to say $[f \circ r] = [f \circ s]$. Since f is injective, we have $[[f \circ r = f \circ s]] \subseteq [[r = s]]$ but then, since $[[f \circ r = f \circ s]] \in \mathcal{F}$, it follows $[[r = s]] \in \mathcal{F}$. Therefore, $[r] = [s]$.

Yes, surjecitivity is preserved. Suppose f is surjective. Consider some $[s] \in {}^*\mathbb{R}$. Our goal is to prove that there is some $[r] \in {}^*\mathbb{R}$ such that $^*f([r]) = [s]$. Define sequenece (r_n) such that, for each s_n, $f(r_n) = s_n$. We know such a r_n exists for each s_n since f is surjective. We claim that $[r] \in {}^*\mathbb{R}$ the element we looking for. To prove this claim, we need to show $[f \circ r] = f([r]) = [s]$, so we set that as our goal. Clearly, $[[f \circ r = s]] = \{n \in \mathbb{N} : f(r_n) = s_n\} = \mathbb{N} \in \mathcal{F}$. Therefore, $[f \circ r] = f[r] = [s]$. $\square$

Exercise 3.19. *(Section 3.12, III) For $x \in {}^*\mathbb{R}$, let*

$$|x| = \begin{cases} x, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$$

be the usal definition of absolute value function. Show that this extends the definition of $|.|$ on $\mathbb{R}$: $|[r]| = [\langle |r_1|, |r_2|, \cdots \rangle] = [|r_n|]$.

Proof. We have the following cases;
Case-1. Suppose $[r] > 0$. Then $[[r > 0]] \in \mathcal{F}$. Since $[[r > 0]] \subseteq [[[r] = [|r|]]] \in \mathcal{F}$, we have $|[r]| = [r] = [|r_n|]$.
Case-2. Suppose $[r] = 0$. That is to say $[[r = 0]] \in \mathcal{F}$. Since $[[r = 0]] \subseteq [[0 = |r|]] \in \mathcal{F}$, we have $|[r]| = 0 = [|r|]$.
Case-3. Suppose $[r] < 0$. Then $|[r]| = -[r] = [-r] = [|r|]$. □

Exercise 3.20. *(Section 3.12, IV) Let χ_A be the characteristic function of a set $A \subseteq \mathbb{R}$. Show that ${}^*(\chi_A) = \chi_{{}^*A}$.*

Proof. Let $[r] \in {}^*\mathbb{R}$. Consider the expression ${}^*(\chi_A)[r]$, we have the following cases;
Case-1. Suppose $[r] \in {}^*A$. Notice that ${}^*(\chi_A)[r] = [\chi_A \circ r] = [\chi_A(r_n)]$ but then, since $[[r \in A]] \in \mathcal{F}$ and $[[r \in A]] \subseteq [[\chi_A \circ r = 1]]$, we have $[[\chi_A \circ r = 1]] \in \mathcal{F}$. Therefore, $[\chi_A \circ r] = \chi_A[r] = 1$.
Case-2. Suppose $[r] \notin {}^*A$. Then we have $[[r \notin A]] = [[r \in A]]^c \in \mathcal{F}$. Clearly, $[[r \notin A]] \subseteq [[\chi_A \circ r = 0]]$. Thus, $[[\chi_A \circ r = 0]] \in \mathcal{F}$ and so $[\chi_A \circ r] = \chi_A[r] = 0$. □

Exercise 3.21. *(Section 3.12, V) Show how to define ${}^*f$ when f is a function of more than one arguement.*

Proof. Define ${}^*f([r^1], \cdots, [r^n]) = [f(r^1_1, \cdots, r^n_1), f(r^1_2, \cdots, r^n_2), \cdots]$. We prove that it's well defined. Clearly, $[[r^1 = r^{1'}]] \cap \cdots [[r^n = r^{n'}]] \subseteq [[f \circ (r^1, \cdots, r^n) = f \circ (r^{1'}, \cdots, r^{n'})]]$. Therefore, $r^1 \equiv r^{1'}, \cdots, r^n \equiv r^{n'}$ implies $f \circ (r^1, \cdots, r^n) \equiv f \circ (r^{1'}, \cdots, r^{n'}$. □

Exercise 3.22. *(Section 3.15, I) If $A_1, \cdots, A_k$ are subsets of $\mathbb{R}$, put $P = A_1 \times \cdots \times A_k$ and apply the definition of ${}^*P$ to show that*

$${}^*(A_1 \times \cdots \times A_k) = {}^*A_1 \times \cdots \times {}^*A_k.$$

Proof. For $([r^1], [r^2], \cdots, [r^k]) \in {}^*P$, we must have $[[(r^1, r^2, \cdots, r^k) \in P]] \in \mathcal{F}$. Clearly, $[[(r^1, r^2, \cdots, r^k) \in P]] \subseteq [[r^n \in A_n]]$, thus $[[r^n \in A_n]]$ for any n such that $1 \leq n \leq k$. This means that $[r^n] \in A^n$ for all such n. Hence, $([r^1], \cdots, [r^n]) \in {}^*A_1 \times \cdots {}^*A_n$. From this, it follows that ${}^*(\mathbb{R}^k) = ({}^*\mathbb{R})^k$. □

Exercise 3.23. *(Section 3.15, II) Let dom P denote the domain of a binary relation. If $P \subseteq \mathbb{R}^2$, show that ${}^*(dom\ P) = dom\ {}^*P$.*

Proof. First we prove the forward inclusion. Let $[r] \in {}^*(\text{dom } P)$. Then we have $\{n \in \mathbb{N} : r_n \in \text{dom } P\} = [[r \in \text{dom } P]] \in \mathcal{F}$.

$$\text{Define sequence } s_n = \begin{cases} s \text{ such that } (r_n, s) \in P, & \text{if } r_n \in \text{dom } P \\ 0, & \text{if } r_n \notin \text{dom } P \end{cases}$$

Then clearly, $[[(r, s) \in P]] = [[r \in \text{dom } P]] \in \mathcal{F}$, but that means $([r], [s]) \in {}^*P$, and so $[r] \in \text{dom } {}^*P$.
Now we prove the converse inclusion. Let $[r] \in \text{dom } {}^*P$. It follows that there exists $[s] \in {}^*\mathbb{R}$ such that $([r], [s]) \in {}^*P$, but that means $[[r \in \text{dom } P]] = [[(r, s) \in P]] \in \mathcal{F}$. Hence, $[r] \in {}^*(\text{dom } P)$. □

Chapter 4

The Transfer Principle

Exercise 4.1. *(Section 4.1, I) Verify the truth the* $*-$*transform given in* $1-5$ *above, which are;*
(1) The Eudoxus-Archimedes Principle: $\forall x \in {}^*\mathbb{R} : \exists m \in {}^*\mathbb{N}(x < m)$.
(2) Density of the rationals: $\forall x, y \in {}^*\mathbb{R}(x < y$ *implies* $\exists q \in {}^*\mathbb{Q}(x < q < y))$.
(3) Finiteness. Let $A = \{r_1, \cdots, r_k\}$ *be a finite subset of* $\mathbb{R}$. *Then* $\forall x \in {}^*A(x = {}^*r_1 \vee x = {}^*r_2 \vee \cdots x = {}^*r_k)$.
(4) Finitary set operations. $\forall x \in {}^*\mathbb{R}(x \in {}^*(A \cup B))$ *iff* $x \in {}^*A$ *or* $x \in {}^*B$.
(5) Discreteness of $\mathbb{N}$. $\forall x \in {}^*\mathbb{N}({}^*n \leq x \leq {}^*(n+1)$ *implies* $x = {}^*n$ *or* $x = {}^*(n+1)$.

Proof.

(1) Let $x \in {}^*\mathbb{R}$. Define sequence m such that $x_i < m_i$ for every i. The existence of such m is guranteed by Eudoxus-Archimedes Principle in reals. Clearly, $m \in {}^*\mathbb{N}$ and $[[x < m]] \in \mathcal{F}$. Therefore, $x < m$.
(2) Let $x, y \in {}^*\mathbb{R}$. Define sequence q such that $x_i < q_i < y_i$ for each i, where q_i's are rational. The existence of such q_i's is guranteed by density of rationals in reals. Clearly, $q \in {}^*\mathbb{Q}$ and $[[x < q < y]] = \mathbb{N} \in \mathcal{F}$. Therefore, $x < q < y$.
(3) Let $x \in {}^*A$. Then $[[x = r_1]] \cup [[x = r_2]] \cup \cdots [[x = r_n]] = [[x \in A]] \in \mathcal{F}$. Since finite union in $\mathcal{F}$ means that exactly one of the set in $\mathcal{F}$, we have $x = {}^*r_1$ or $x = {}^*r_2$ or $\cdots$ or $x = {}^*r_n$.
(4) Proved in section 3.10, II.
(5) Let $x \in {}^*\mathbb{N}$ and suppose ${}^*n \leq x \leq {}^*(n+1)$. Then we have $[[n \leq x \leq n+1]] \in \mathcal{F}$ but due to discreteness of $\mathbb{N}$, it folows that $[[x = n]] \cup [[x = n+1]] = [[x = n \vee x = n+1]] = [[n \leq x \leq n+1]] \in \mathcal{F}$. Hence, $x = {}^*n$ or $x = {}^*(n+1)$. □

Chapter 5

Hypereals Great and Small

Exercise 5.1. *(Section 5.2, I) For any positive hyperreal a, explain why the function $x \to a^x$ is defined for all $x \in {}^*\mathbb{R}$. Use transfer to explore its properties.*

Proof. ☐

Exercise 5.2. *(Section 5.5, I) Verify that $\simeq$ and $\sim$ are equivalence relationships.*

Proof. We will first argue about $\simeq$. Let $x, y, z \in {}^*\mathbb{R}$. Clearly, it's reflexive since $x - x = 0$. If $x \simeq y$ then we have $x - y = \epsilon$, thus $y - x = -\epsilon$, and so $y \simeq x$. If $x \simeq y$ and $y \simeq z$ then we have $x - y = \epsilon$ and $y - z = \delta$, adding the two get us $x \simeq z$. The arguement of $\sim$ is similar, so we omit the details. ☐

Exercise 5.3. *(Section 5.5, II) If $b \simeq x \leq y \simeq c$ with b and c real, show that $b \leq c$. What if b and/or c are not real?*

Proof. First we will prove an useful lemma, if $y > 0$ and $y \simeq k$ then $k > 0$. Let $k = y + \varepsilon$. In the case where $\varepsilon = 0$ and $\varepsilon > 0$, it follows trivially. So assume $\varepsilon < 0$. Thus, $y - \frac{1}{n} < y + \varepsilon = k$ for all $n \in \mathbb{N}$. Since $\frac{y}{2}$ a real number, we know that there exists $m \in \mathbb{N}$ such that $\frac{1}{m} < \frac{y}{2}$ which implies $0 < \frac{y}{2} = y - \frac{y}{2} < y - \frac{1}{m} < k$. This proves the lemma.
Now assume to the contrary that $c < b$. Thus, $0 < c - b$ but it implies that $0 < x - y$, since $c - b \simeq x - y$, which is a contradiction.
The result would not be true if b is not real. Consider $\varepsilon \simeq 0 \leq 0 \simeq 0$ for some positive infintisimal ε, this would imply $\varepsilon \leq 0$ which is false. ☐

Exercise 5.4. *(Section 5.5, III) $hal(b) = \{b + \varepsilon : e \in hal(0)\}$.*

Proof. The converse inclusion is trivial. For forward, notice that $x \in \text{hal}(b)$ means that $x - b = \varepsilon$, and so $x = b + \varepsilon$. ☐

Exercise 5.5. *(Section 5.5, IV) $gal(b) = \{b + c : c \in gal(0)\}$.*

Proof. Again, the converse implication is trivial. For forward, we see that $x \in \text{gal}(b)$ means $x - b = c$ where c is a limited number, and so $c \in \text{gal}(0)$. Thus, $x = b + c$ for some $c \in \text{gal}(0)$. ☐

Exercise 5.6. *(Section 5.5, V) If $x \simeq y$ and b is limited, prove that $b \cdot x \simeq b \cdot y$. Show that the result can fail for unlimited b.*

Proof. Since $x \simeq y$, we have $x - y = \varepsilon$ for some $\varepsilon \in \text{hal}(0)$. Multiplying by b on both sides get us, $bx - by = b \cdot \varepsilon$. It follows that $b \cdot \varepsilon \in \text{hal}(0)$, so $bx \simeq by$. Consider $b = \frac{1}{\varepsilon}$then $b \cdot \varepsilon = 1$ which is not infinitisimal. ☐

Exercise 5.7. *(Section 5.5, VI) Show that any galaxy contains members of ${}^*\mathbb{Z}$, of ${}^*\mathbb{Q} - {}^*\mathbb{Z}$, and of ${}^*\mathbb{R} - {}^*\mathbb{Q}$.*

Proof. Let $b \in {}^*\mathbb{R}$. Clearly, $b - a$ for $a \in {}^*Z$, such that a is intger, is limited. ☐

Exercise 5.8. *(Section 5.7, I) Prove theorem 5.6.2:*
(a) $sh(b \pm c) = sh(b) \pm sh(c)$.
(b) $sh(bc) = sh(b) \cdot sh(c)$.
(c) $sh(b/c) = sh(b)/sh(c)$ *if* $sh(c) \neq 0$ *(ie c is appreciable).*
(d) $sh(b^n) = sh(b)^n$.
(e) $sh(|b|) = |sh(b)|$.
(f) $sh(\sqrt[n]{b}) = \sqrt[n]{sh(b)}$, *if* $b \geq 0$.
(g) If $b \leq c$ *then* $sh(b) \leq sh(c)$.

Proof. Consider $b = \mathrm{sh}(b) + \varepsilon$ and $c = \mathrm{sh}(c) + \delta$ for some infinitisimal ε and δ.
(a) For $\mathrm{sh}(b+c)$, notice that $b + c - (\mathrm{sh}(b) + \mathrm{sh}(c)) = \varepsilon + \delta$. Thus, $\mathrm{sh}(b+c) \simeq b + c \equiv \mathrm{sh}(b) + sh(c)$. From the uniqueness of shadow, it follows $\mathrm{sh}(b+c) = \mathrm{sh}(b) + \mathrm{sh}(c)$. We get identical result in case of $\mathrm{sh}(b-c)$ by considering $b - c - (\mathrm{sh}(b) - \mathrm{sh}(c)) = \varepsilon - \delta$.

(b) Notice that $bc = \mathrm{sh}(b)\mathrm{sh}(c) + \mathrm{sh}(b)\delta + \mathrm{sh}(c)\varepsilon + \varepsilon\delta$ where $\mathrm{sh}(b)\delta, \mathrm{sh}(c)\varepsilon, \varepsilon\delta$ are infintisimals. So, $\mathrm{sh}(bc) \simeq bc \simeq \mathrm{sh}(b)\mathrm{sh}(c)$, and thus by uniqueness of shadow we have $\mathrm{sh}(bc) = \mathrm{sh}(b)\mathrm{sh}(c)$.

(c) Notice that, applying (b), we have $\mathrm{sh}(b/c) = \mathrm{sh}(b) \cdot \mathrm{sh}(1/c)$. So if we could prove $\mathrm{sh}(1/c) = 1/\mathrm{sh}(c)$, we would be done. Therefore, we set that as our goal. Clearly, $\frac{1}{c} - \frac{1}{\mathrm{sh}(c)} = \frac{1}{\mathrm{sh}(c)+\delta} - \frac{1}{\mathrm{sh}(c)} = \frac{\delta}{\mathrm{sh}(c)^2+\mathrm{sh}(c)\delta}$. Since $\mathrm{sh}(c)^2 + \mathrm{sh}(c)\delta$ is appreciable it follows that $\frac{\delta}{\mathrm{sh}(c)^2+\mathrm{sh}(c)\delta}$ is infinitismal, and thus $\mathrm{sh}(1/c) \simeq 1/c \simeq 1/\mathrm{sh}(c)$. Hence, $\mathrm{sh}(1/c) = 1/sh(c)$ by uniqueness of shadow.

(d) Trivially follows from induction on (b).

(e) If we could prove that $|b| \simeq |\mathrm{sh}(b)|$, we would be done since from uniqueness of shadow, $\mathrm{sh}|b| = |\mathrm{sh}(b)|$ would hold. Notice that $|b| - |\mathrm{sh}(b)| \leq |\mathrm{sh}(b)| + |e| - |\mathrm{sh}(b)| = |e|$. If $0 \leq |b| - |\mathrm{sh}(b)|$ then clearly $|b| - |\mathrm{sh} b| \leq |e| < 1/n$ for all $n \in \mathbb{N}$. If $|b| - |\mathrm{sh}(b)| < 0$ then $-1/n < -|e| \leq |b| - |\mathrm{sh}(b)|$ for all $n \in \mathbb{N}$. Thus, $||b| - |\mathrm{sh}(b)|| < 1/n$ for every natural n. Hence, $|b| - |\mathrm{sh}(b)$ is infinitisimal as desired.
(f) If $\mathrm{sh}(b^{\frac{1}{n}}) < (\mathrm{sh}(b))^{\frac{1}{n}}$ then by squaring both side and applying part (d), we would get $\mathrm{sh}(b) = \mathrm{sh}(b^{1/n \cdot n}) = (\mathrm{sh}(b^{1/n}))^n < (\mathrm{sh} b)^{1/n \cdot n} = \mathrm{sh}(b)$ which is a contradiction. We would get identical contradiction in case of $(\mathrm{sh}(b))^{\frac{1}{n}} < \mathrm{sh}(b^{\frac{1}{n}})$.
(g) It follows from 2nd exercise of section 5.5. □

Exercise 5.9. *(Section 5.7, I) Show that if* b, c *are limited, and* $b \simeq b', c \simeq c'$, *then* $b \pm c \simeq b' \pm c', b \cdot c \simeq b' \cdot c'$, *and* $b/c \simeq b'/c'$ *if* $c \not\simeq 0$. *Show that the last result can fail when* $c \simeq 0$.

Proof. Let $b = b' + \varepsilon$ and $c = c' + \delta$ for some infintisimal ε and δ. Then $b \pm c - (b' \pm c') = \varepsilon \pm \delta$ which is known to be infinitisimal. Thus, $b \pm c \simeq b' \pm c'$. For multiplication we have $bc - b'c' = b'c' + b'\delta + c'\varepsilon + \varepsilon\delta - b'c' = b'\delta + c'\varepsilon + \varepsilon\delta$ which is infintisimal. Finally for quotient, we have $\frac{b}{c} - \frac{b'}{c'} = \frac{b'+\varepsilon}{c'+\delta} - \frac{b'}{c'} = \frac{b'c'+c'\varepsilon - b'c' - b'\delta}{(c')^2+c'\delta} = \frac{c'\varepsilon + b'\delta}{(c')^2+c'\delta}$. Since $(c')^2 + c'\delta$ is appreciable, we have $\frac{b}{c} - \frac{b'}{c'}$ to be infinitsimal. Hence, $b/c \simeq b'/c'$. For proving the result does not hold for $c \simeq 0$. Consider $b = 0$. Then we have $b/c - b'/c^p = 0 - \frac{\varepsilon}{\epsilon - \delta}$ which is not infinitisimal necessarily. □

Exercise 5.10. *(Section 5.7, II)*
If ε *is infinitesimal, show that*
(a) $\sin\varepsilon \simeq 0$,
(b) $\cos\varepsilon \simeq 1$,
(c) $\tan\varepsilon \simeq 0$,
(d) $\sin\varepsilon/\varepsilon \simeq 1$,
(e) $(\sin\varepsilon)/\varepsilon \simeq 1$,
(f) $(\cos\varepsilon - 1)/\varepsilon \simeq 0$.

Proof.
(a) Since $\forall x \in \mathbb{R}(|\sin x| \leq |x|)$ holds in $\mathbb{R}$, applying transfer on it, it must be the case that $\forall x \in {}^*\mathbb{R}(|\sin x| \leq$

$|x|)$ holds in ${}^*\mathbb{R}$. Therefore, $|\sin\varepsilon| \leq |\varepsilon|$, and so $\sin\varepsilon$ is infintisimal. Hence, $\sin\varepsilon \simeq 0$.

(b) Since $\cos\varepsilon = \sqrt{(1-\sin^2\varepsilon)}$, we have $\mathrm{sh}(cos\varepsilon) = \mathrm{sh}\sqrt{(1-\sin^2\varepsilon)} = \sqrt{(\mathrm{sh}(1) - \mathrm{sh}(\sin^2\varepsilon))} = \sqrt{(1-\mathrm{sh}(\delta^2))} = \sqrt{(1-0)} = 1$. Clearly, $\cos\varepsilon \simeq \mathrm{sh}(\cos\varepsilon) = 1$.

(c) $\mathrm{sh}(\tan\varepsilon) = \mathrm{sh}(\sin\varepsilon)/\mathrm{sh}(\cos\varepsilon) = 0/1 = 0$. Hence, $\tan\varepsilon \simeq 0$.

(d) Notice that $|\sin\varepsilon| \leq |\varepsilon|$, so $|\sin(\varepsilon)/\varepsilon| \leq 1$. Thus, $\mathrm{sh}\left(\left|\frac{\sin\varepsilon}{\varepsilon}\right|\right) \leq 1$. We will now show that $\mathrm{sh}\left(\left|\frac{\sin\varepsilon}{\varepsilon}\right|\right) < 1$ leads to contradiction which would get us $\mathrm{sh}\left(\left|\frac{\sin\varepsilon}{\varepsilon}\right|\right) = 1$. Clearly if $\mathrm{sh}\left(\left|\frac{\sin\varepsilon}{\varepsilon}\right|\right) < 1$ then $0 = \mathrm{sh}(|\sin\varepsilon|) - \mathrm{sh}|\varepsilon| < 0$ which is false. Therefore, $(\sin\varepsilon)/\varepsilon \simeq \mathrm{sh}\left(\left|\frac{\sin\varepsilon}{\varepsilon}\right|\right) = 1$.

(e) By the sine inequality, $|\sin\varepsilon|/|\varepsilon| \leq 1$. By the tangent inequality, $|x| \leq |\tan(x)|$ for $x \in [0, \pi/2)$, we have $|\cos x| \leq |\sin x|/|x|$. Combining the two inequalities and taking shadow we see $1 = \mathrm{sh}(|\cos\varepsilon|) \leq \mathrm{sh}(|\sin(\varepsilon)/\varepsilon|) \leq 1$. Therefore, $\mathrm{sh}(|\sin(\varepsilon)/\varepsilon|) = 1$. Hence, $(\sin\varepsilon)/\varepsilon \simeq 1$.

(f) By cosine inequality, $1-\frac{\varepsilon^2}{2} \leq \cos\varepsilon$, we have $-\varepsilon/2 \leq (\cos\varepsilon - 1)/\varepsilon$. Since $cos\varepsilon \leq 1$, we have $(\cos\varepsilon - 1)/\varepsilon \leq 0/\varepsilon = 0$. Hence $|(\cos\varepsilon - 1)/\varepsilon| \leq \varepsilon/2$ and so $(\cos\varepsilon - 1)/\varepsilon$ is infinitesimal, which makes it infinitely close to 0. $\square$

Exercise 5.11. *(Section 5.7, III) Show that every hyperreal is infinitely close to some hyperrational number.*

Proof. Since $\forall x, y \in \mathbb{R}(x < y$ implies $\exists q \in \mathbb{Q}(x < q < y))$ holds in reals (density of rationals), by transfer, we have $\forall x, y \in {}^*\mathbb{R}(x < y$ implies $\exists q \in {}^*\mathbb{Q}(x < q < y))$. In particular there is a hyperrational q such that $r < q < r + \varepsilon$ for any hyperreal r. Clearly such a q is infinitely close to r. $\square$

Exercise 5.12. *(Section 5.7, IV) Show that $\mathbb{R}$ is isomorphic to the ring of limited hyperrationals ${}^*\mathbb{Q} \cap \mathbb{L}$ factored by its ideal ${}^*\mathbb{Q} \cap \mathbb{I}$ of hyperreational infintisimal.*

Proof.

$$\begin{tikzcd}
\bullet{}^*\mathbb{Q}\cap\mathbb{L} \arrow[rr, two heads, "\mathrm{sh}"] \arrow[dr, two heads, "\text{canonical projection}"'] & & \bullet\mathbb{R} \\
& \bullet({}^*\mathbb{Q}\cap\mathbb{L})/({}^*\mathbb{Q}\cap\mathbb{I}) \arrow[ur, tail, "f(\mathrm{hal}(b)+({}^*\mathbb{Q}\cap\mathbb{L})/({}^*\mathbb{Q}\cap\mathbb{I}))=\mathrm{sh}(b)"'] &
\end{tikzcd}$$

Consider shadow function restricted to ${}^*\mathbb{Q} \cap \mathbb{L}$, in what follows we would use sh notation for it instead of $\mathrm{sh}_{{}^*\mathbb{Q}\cap\mathbb{L}}$. We will first justify it's surjective. Since $\mathbb{R} \subseteq {}^*\mathbb{R}$, applying previous exercise, we see that for any $x \in \mathbb{R}$, there is $q \in {}^*\mathbb{Q}$ such that $x \simeq q$ but then it would mean $x \simeq \mathrm{sh}(q)$. Since both x and $\mathrm{sh}(q)$ are reals, it must be the case that $x = \mathrm{sh}(q)$. Hence, $\mathrm{sh}(q) = x$ as desired. Notice that for any $[r] \in ({}^*\mathbb{Q}\cap\mathbb{L})/({}^*Q \cap \mathbb{I})$, we have $[r] = r + {}^*Q \cap \mathbb{I}$ which is nothing but $\mathrm{hal}(r)$. Define $f : ({}^*\mathbb{Q}\cap\mathbb{L})/({}^*Q\cap\mathbb{I}) \to \mathbb{R}$ as $\mathrm{hal}(b) \mapsto \mathrm{sh}(b)$. By the first isomorphism theorem of rings, it follows $({}^*\mathbb{Q}\cap\mathbb{L})/({}^*Q\cap\mathbb{I}) \simeq \mathbb{R}$. $\square$

Exercise 5.13. *(Section 5.9, Exercise on Dedekind completeness) For threoem 5.8.1, instead of showing that cauchy sequences converge we can develop a direct proof that any subset $A \subseteq \mathbb{R}$ with a real upper bound has a least real upper bound. First, for each $n \in \mathbb{N}$, let s_n be the least $k \in \mathbb{Z}$ such that k/n is an upper bound of A. Then take an unlimited $N \in {}^*\mathbb{N}_\infty$ and let $L \in \mathbb{R}$ be infinitely close to s_N/N.*
(a) Verify that s_n exists as defined for $n \in \mathbb{N}$.
(b) Show the s_N/N is limited, so that such a real L exists under hypothesis of theorem 5.8.1.
(c) Prove that L is a least upper bound of A in $\mathbb{R}$.

Proof. Let r be an upper bound of $A \subseteq \mathbb{R}$.
(a) Let $n \in \mathbb{N}$. Consider the set $B = \{m \in \mathbb{N} : m/n$ is an upper bound of $A\}$. Since $r \in \mathbb{R}$, by archimedean property, we know that there exists $l \in \mathbb{N}$ such that $r < l$. Clearly $l \cdot n \in B$, and so B is non empty. By the well ordering principle, it follows that there exists a least element in B, or in other words there exists a least $k \in \mathbb{N}$ such that k/n is an upperbound of A for each n.

(b) Let $n \in \mathbb{N}$. Since s_n is the least integer such that s_n/n is upperbound of A, clearly $(s_n-1)/n$ is not an upper bound. But that means there exists some $a \in A$ such that $(s_n-1)/n \le a \le s_n/n$. Applying transfer on the statement $(\forall n \in \mathbb{N})(\exists a \in A)((s_n-1)/n \le a \le s_n/n)$, we get $(\forall n \in {}^*\mathbb{N})(\exists a \in {}^*A)((s_n-1)/n \le a \le s_n/n)$. In particular, we have $(s_N-1)/N \le a \le s_N/N$. However since s_N/N and $(s_N-1)/N$ are infinitely close, it must be the case that a is infinitely close to s_N/N. Applying transfer on the statement $(\forall a \in A)(a \le r)$, where r is the real upper bound we are given, we get $(\forall a \in {}^*A)(a \le r)$. Thus, $s_N/N \le r$. Next we will prove that is not negative unlimited either. Since $A \subseteq {}^*A$ and s_N/N is an upper bound of ${}^*A$, we have some $x \in A$ such that $x \le s_N/N$. Hence, $x \le s_N/N \le r$ for some real r, s.

(c) For the sake of contradiction assume there is an upper bound, K, of A such that $K < L$. Notice that there does not exists any $a \in A$ such that $a \in [K, L]$ since otherwise K would not be an upper bound. Applying transfer on $(\forall a \in A)(a \notin [K, L])$, we get $(\forall a \in {}^*A)(a \notin {}^*[K, L])$ which is false since in previous paragraph we demonstrated that there is an element of ${}^*A$ which is infinitely close to L. ☐

Exercise 5.14. *(Section 5.11, I) Provide an analogous description of the order structure of the hyperintgers ${}^*\mathbb{Z}$.*

Proof. By transfer, we know that ${}^*\mathbb{Z}$ is closed under addition, subtraction and multiplication. Observe that only limited hyperinteger are members of $\mathbb{Z}$ since if $k \in {}^*Z$ is limited then $|k|leqn$ for some $n \in \mathbb{N}$ but that means $-n \le k \le n$. Applying transfer on $\forall x \in \mathbb{Z}(-n \le x \le n \to x = -n \vee x = -n+1 \vee \cdots x = 1 \vee \cdots x = n)$, thus $k \in \{-n, -n+1, \cdots, n\}$, and so $k \in \mathbb{Z}$. Therefore, all numbers of ${}^*\mathbb{Z} - \mathbb{Z}$ are unlimited and thus either less than all members of $\mathbb{Z}$ or greater than it. Let $K \in {}^*\mathbb{Z} - \mathbb{Z}$. Define $\gamma(K) = \{K\} \cup \{K \pm n : n \in \mathbb{N}\}$, clearly all members of $\gamma(K)$ are unlimited and $\gamma(K)$ is isomorphic to $\mathbb{Z}$ under the usual ordering. Alternatively, we can defined $\gamma(K) = \{H \in {}^*\mathbb{Z} : Z \sim H\} = \text{gal}(K) \cap {}^*\mathbb{Z}$. Notice that Z itself is a ${}^*\mathbb{Z}$ galaxy, $\mathbb{Z} = \text{gal}(0) \cap {}^*\mathbb{Z}$.

We define order on ${}^*\mathbb{Z}$ galaxy as $\gamma(Y) < \gamma(H)$ iff $Y \le H$ whenever $K \not\sim H$. Clearly there is no greatest ${}^*\mathbb{Z}$ galaxy since $\gamma(K) < \gamma(2K)$ for any positive unlimited K. Simillarly there is no smallest galaxy either. To see why notice that either K is even or $K+1$ is, by transfer. If $K+1$ is even, we have $\gamma(K+1/2) < \gamma(K+1) = \gamma(K)$. If K is even then we have $\gamma(K) < \gamma(K)$.

We can imagine ordering on ${}^*\mathbb{Z}$ as ordering of $\mathbb{Z}$ surrounded by ordered set of ${}^*\mathbb{Z}$ galaxies with no greatest or least. ☐

Exercise 5.15. *(Section 5.11, II) Show that for any $M \in {}^*\mathbb{N}$ there is an $N \in {}^*\mathbb{N}$ that is divisble by all members of $\{1, 2, \cdots, M\}$. Hence show that there exists a hypernatural number N that is divisble by every standard positive integer.*

Proof. We extend the factorial function $! : \mathbb{N} \to \mathbb{N}$ to $! : {}^*\mathbb{N} \to {}^*\mathbb{N}$ using extension of function. Applying transfer on the statement $(\forall n \in \mathbb{N})(k \le n \to k|n!)$, we get $(\forall n \in {}^*\mathbb{N})(k \le n \to k|n!)$. Thus, letting $N = M!$ does the job. ☐

Exercise 5.16. *(Section 5.11, III) Develop a theory of prime factors in ${}^*\mathbb{N}$: if Π is the set of standard prime numbers, with enlargement ${}^*\Pi \subseteq {}^*\mathbb{N}$, prove the following.*
(a) ${}^*\Pi$ consists precisely of those hypernaturals > 1 that have no nontrivial factors in ${}^*\mathbb{N}$.
(b) Every hypernatural number > 1 has a hyperprime factor, i.e., is divisble by some members of ${}^*\Pi$.
(c) Two hypernaturals are equal if they have exactly the same factors of the form p^n with $p \in {}^*\Pi$ and $n \in {}^*\mathbb{N}$.
(d) A hypernatural number is divisible by every standard positive integer iff it is divisible by p^n for every standard prime p and every $n \in \mathbb{N}$.

Proof.
(a) By transfer on $(\forall x \in \Pi)(x > 1 \to$ x have no nontrivial factors in $\mathbb{N})$.
(b) By transfer on $(\forall n \in \mathbb{N})(n > 1 \to \exists p \in \Pi)(p|n)$.
(c) By transfer on $(\forall x \in \mathbb{N})(\forall y \in \mathbb{N})(x = y \leftrightarrow$ they have exactly the same factors of the form p^n with $p \in \Pi$ and $n \in \mathbb{N})$.
(d) Trivial. ☐

Chapter 6

Convergence of Sequences and Series

Exercise 6.1. *(Section 6.3, I) Prove theorem 6.3.1. If* $\lim_{n\to\infty} s_n = L$ *and* $\lim_{n\to\infty} t_n = M$ *in* $\mathbb{R}$*. Then*
(a) $\lim_{n\to\infty}(s_n + t_n) = L + M$*.*
(b) $\lim_{n\to\infty}(cs_n) = cL$*, for any* $c \in \mathbb{R}$*,*
(c) $\lim_{n\to\infty}(s_n t_n) = LM$*,*
(d) $\lim_{n\to\infty}(s_n/t_n) = L/M$*, if* $M \neq 0$*.*

Proof. For what follows consider $N \in {}^*\mathbb{N}_\infty$. It means that $s_N \simeq L$ and $t_N \simeq M$.
(a) Since $s_N \simeq L$ and $t_N \simeq M$, it follows $s_N + t_N \simeq L + M$. Since N was arbitrary, we have $s_n + t_n \simeq L + M$ for all unlimited n. Hence, $\lim_{n\to\infty}(s_n + t_n) = L + M$.
(b) Since $S_N \simeq L$, we have $cS_N \simeq cL$. Since N was arbitrary, $cs_n \simeq cL$ for all unlimited n. Hence, $\lim_{n\to\infty} cs_n = cL$.
(c) Since $s_N \simeq L$ and $t_N \simeq M$, we have $s_N t_N \simeq LM$ but since N was arbitrary, it follows $s_n t_n \simeq LM$ for all unlimited n. Hence, $\lim_{n\to\infty}(s_n t_n) = LM$.
(d) Again, from $s_N \simeq L$ and $t_N \simeq M$, we have $s_N/t_N \simeq L/M$ but since N was arbitrary, $\lim_{n\to\infty} s_n/t_n = L/M$. ☐

Exercise 6.2. *(Section 6.4, I) Prove theorem 6.4.2. A real valued sequence*
(a) diverges to infinity iff all of its extended terms are positive unlimited; and
(b) diverges to negative infinity iff all of its extended terms are negative unlimited.

Proof. Let the sequence be (s_n).
(a) First we prove forward direction. Let $r \in \mathbb{R}$. Since the sequence diverges, we know that there is some $n \in \mathbb{N}$ such that $(\forall m \in \mathbb{N})(m \geq n \to s_m > r)$ holds true. Applying transfer on it, we have $(\forall m \in {}^*\mathbb{N})(m \geq n \to s_m > r)$ to be true. In particular, $s_N > r$ for any $N \in {}^*\mathbb{N}_\infty$ since $N > n$. It follows that s_N is positive unlimited for all $N \in {}^*\mathbb{N}_\infty$. We prove the converse direction. Let $r \in \mathbb{R}$. Notice that for any unlimited $N \in {}^*\mathbb{N}_\infty$ if $M \geq N$ then $M \in {}^*\mathbb{N}_\infty$. Therefore, it follows that $(\exists n \in {}^*\mathbb{N})(\forall m \in {}^*\mathbb{N})(m \geq n \to s_m > r)$ since s_m is positive unlimited by assumption. Transferring the statement to $\mathbb{R}$, we have $(\exists n \in \mathbb{N})(\forall m \in \mathbb{N})(m \geq n \to s_m > r)$ to be true in $\mathbb{R}$. Since r was arbitrary, we have (s_n) to be divergent to positive infinity.

(b) Follows from (a) by considering the sequence $(-s_n)$. ☐

Exercise 6.3. *(Section 6.5, I) Prove theorem 6.5.1. A real values sequence* $\langle s_n \rangle$ *is Cauchy in* $\mathbb{R}$ *iff all it's extended terms are infinitely close to each other, i.e., iff* $s_m \simeq s_n$ *for all* $m, n \in {}^*\mathbb{N}_\infty$*.*

Proof. First we prove the forward implication. Suppose $\langle s_n \rangle$ is cauchy. Let $\varepsilon \in \mathbb{R}^+$ and $j \in \mathbb{N}$ be such that $(\forall m, n \in \mathbb{N})(m, n \geq j \to |s_m - s_n| < \varepsilon)$. Applying transfer on the latter we know $(\forall m, n \in {}^*\mathbb{N})(m, n \geq j \to |s_m - s_n| < \varepsilon)$ is true in ${}^*\mathbb{R}$. In particular, we have $|s_M - s_N| < \varepsilon$ for arbitrary $M, N \in {}^*\mathbb{N}_\infty$ since j is limited. Since ε was arbitrary positive real number, we have $s_M \simeq s_N$. We now prove the converse direction. Suppose $s_m \simeq s_n$ for all $m, n \in {}^*\mathbb{N}_\infty$. Let $\varepsilon \in \mathbb{R}^+$ be arbitrary. Notice that for any $K \in {}^*\mathbb{N}_\infty$, if $L \geq N$ then $L \in {}^*\mathbb{N}_\infty$, and also notice for any $L, K \in {}^*\mathbb{N}_\infty$, $s_L - s_K = \delta$ where δ is infintisimal. Therefore, we have $(\exists j \in {}^*\mathbb{N})(\forall m, n \in {}^*\mathbb{N})(m, n \geq j \to |s_m - s_n| \leq \varepsilon)$ to be true in ${}^*\mathbb{R}$. Applying transfer to it, we have $(\exists j \in \mathbb{N})(\forall m, n \in \mathbb{N})(m, n \geq j \to |s_m - s_n| < \varepsilon)$ to be true in $\mathbb{R}$. Since $\varepsilon \in \mathbb{R}^+$ was arbitrary, we have (s_n) to be cauchy. $\square$

Exercise 6.4. *(Section 6.5, II). Cauchy's convergence Criterion.*
Prove the foward direction of A real valued sequence converges in $\mathbb{R}$ iff it is cauchy.

Proof. Let (s_n) be a real valued sequence and suppose it convergence in $\mathbb{R}$ to L. Therefore, by theorem 6.1.1, we have $s_N \simeq L$ for all $n \in {}^*\mathbb{N}_\infty$ but that means all extended terms are infinitely close to each other i.e. for any $n, m \in {}^*\mathbb{N}_\infty$, $s_n \simeq L \simeq s_m$. Therefore, by theorem 6.5.1, we have (s_n) to be cauchy. $\square$

Exercise 6.5. *(Section 6.7, I) Let (s_n) and (t_n) be real valued sequences with limits, L, M respectively. Show that if $s_n \leq t_n$ for $n \in \mathbb{N}$. Then $L \leq M$.*

Proof. Applying transfer on $(\forall n \in \mathbb{N})(s_n \leq t_n)$, we have $(\forall n \in \mathbb{N})(s_n \leq t_n)$ to be true in ${}^*\mathbb{R}$. In particular for any $N \in {}^*\mathbb{N}_\infty$, we have $s_N \leq t_N$, however, since $L \simeq s_N \leq t_N \simeq M$, we have $L \leq M$ by exercise 2 of section 5.5. $\square$

Exercise 6.6. *(Section 6.7, II) If $r_n \leq s_n \leq t_n$ in $\mathbb{R}$ for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} r_n = \lim_{n\to\infty} t_n$, show that (s_n) converges to this same limit.*

Proof. Applying transfer on $(\forall n \in \mathbb{N})(r_n \leq s_n \leq t_n)$, we have $(\forall n \in {}^*\mathbb{N})(r_n \leq s_n \leq t_n)$ to be true in ${}^*\mathbb{R}$. In particular, for any $N \in {}^*\mathbb{N}_\infty$, we have $r_N \leq s_N \leq t_N$. However, since $(r_N), (t_N)$ are limited, s_N is limited. By the shadow inequalitiy, we have $\mathrm{sh}(r_N) \leq \mathrm{sh}(s_N) \leq \mathrm{sh}(t_N) = \mathrm{sh}(r_N)$. Hence, $\mathrm{sh}(s_N) = \mathrm{sh}(r_N) = \lim_{n\to\infty} r_n$ but that means $r_N \simeq \lim_{n\to\infty} r_n$ for all $N \in {}^*\mathbb{N}_\infty$. Hence, (s_n) converges to $\lim_{n\to\infty} r_n$. $\square$

Exercise 6.7. *(Section 6.7, III) If a sequence converges in $\mathbb{R}$, show that it has exactly one cluster point.*

Proof. Suppose (s_n) converges to L and assume it has two cluster point A and B. By definition of convergence we have $s_N \simeq L$ for all $N \in {}^*N_\infty$ but that would imply $B \simeq L \simeq A$ by the definition of cluster point, since they are reals, it follows $A = L = B$. $\square$

Exercise 6.8. *(Section 6.7, IV) Suppose that a real valued sequence has a single cluster point. If the sequence is bounded, must it be convergent? what if it is unbounded?*

Proof. Let (s_n) be a bounded sequence which has an unique cluster point i.e. for some $M \in {}^*\mathbb{N}_\infty$, we have $s_M \simeq L$ for some real L. Suppose it is not convergent. Therefore, there must exists $N \in {}^*\mathbb{N}_\infty$ such that $s_N \not\simeq L$. However since (s_n) is bounded, we have s_N to be limited and hence infinitely close to a real number ie its shadow. But if $\mathrm{sh}(s_N) \neq L$, clearly $\mathrm{sh}(s_N)$ is a cluster point which contradicts the assumption that there was an unique cluster point.
Consider the sequence $\langle s \rangle = (0, 1, 0, 2, 0, 3, \cdots)$. Clearly this sequence has only one cluster point and is unbounded but it does not converges to any real number. $\square$

Exercise 6.9. *(Section 6.8, I) Prove, by nonstandard reasoning, that both the limit superior and limit inferior are cluster points of the sequence s.*

Proof. Let $L = \limsup s$ and $\varepsilon \in \mathbb{R}^+$. Since L is supremum of C_s, we know there exists $N \in {}^*\mathbb{N}_\infty$ such that $L - \varepsilon < s_N \leq L$. Consider some arbitrary $n \in \mathbb{N}$. We have $(\exists m \in {}^*\mathbb{N})(m > n \wedge L - \varepsilon < s_m)$ to be true, and applying transfer on it, it follows $(\exists m \in \mathbb{N})(m > n \wedge L - \varepsilon < s_n)$ is true in $\mathbb{R}$. Since $\varepsilon \in \mathbb{R}^+$ and $n \in \mathbb{N}$ was arbitrary, $(\forall \varepsilon \in \mathbb{R}^+)(\forall n \in \mathbb{N})(\exists m \in \mathbb{N})(m > n \wedge L - \varepsilon < s_m)$ is true in $\mathbb{R}$. Applying transfer we see $(\forall \varepsilon \in {}^*\mathbb{R}^+)(\forall n \in {}^*\mathbb{N})(\exists m \in {}^*\mathbb{N})(m > n \wedge L - \varepsilon < s_n)$ holds in ${}^*\mathbb{R}$. Picking ε as some infinitesimal and n as unlimted, we follow there is some $m \in {}^*\mathbb{N}$ making $m > n$ and $L - e < s_m$ true, that is to say, there is unlimited m such that $L < s_m + e$. Therefore, it follows $L \leq \mathrm{sh}(s_m)$ for an unlimited m. However, since L is upper bound of C_s, we must have $L = \mathrm{sh}(s_m)$ making it a cluster point. We cab argue about $\liminf$ in a similar fashion. $\square$

Exercise 6.10. *(Section 6.9, I) Formulate the definition of the limit inferior of an arbitrary real valued sequence.*

answer:
Proposition A real number L is equal to $\liminf s$ if and only if
(1) $L < s_n$ or $L \simeq s_n$ for all unlimited n; and
(2) $s_n \simeq L$ fdor atleast one unlimited n.

Proof. The condition (1) holds iff $L \leq \mathrm{sh}(s_n)$. Thus, it implies L is lower bound of set of cluster points, C_s. The condition (2) asserts that L is a cluster point i.e. $L \in C_s$. Together they imply that L is infimum (since it's the smallest element) of the set. Hence, it is equal to $\liminf s$. □

Exercise 6.11. *(Section 6.9, III) Formulate and prove theorems about the limit inferior of a bounded sequence that correspond to theorems 6.8.4 and 6.8.5.*

answer:
Proposition-1. If s is a bounded real valued sequence with limit inferior, $\liminf$, then for any positive real ε:
(1) Some standard tail of s has all its terms bigger than $\liminf - \varepsilon$ i.e. $\liminf - \varepsilon < s_n$ for all but finitely many $n \in \mathbb{N}$.
(2) $s_n < \liminf + \varepsilon$ for infinitely many $n \in \mathbb{N}$.

Proof. Let $L = \liminf s$. Consider $\varepsilon \in \mathbb{R}^+$. First we prove (1). For any unlimited N, we have $L < s_N$ or $L \simeq s_N$. Thus, $L - \varepsilon < s_N$ holds for any unlimited N. Trasnferring $(\exists n \in {}^*\mathbb{N})(\forall m \in {}^*\mathbb{N})(m \geq n \wedge L - \varepsilon < s_m)$, we have $(\exists n \in \mathbb{N})(\forall m \in \mathbb{N})(m > n \wedge L - \varepsilon < s_m)$ to be true in $\mathbb{R}$ but that simply means $L - \varepsilon < s_m$ is true for all but finitely many n. We now prove (2). Let $m \in \mathbb{N}$. We know that L is infinitely close to s_N for some unlimited N. Thus, $(\exists n \in {}^*\mathbb{N})(n > m \wedge s_n < L + \varepsilon)$ is true in ${}^*\mathbb{R}$. Transfering it, we have $(\exists n \in \mathbb{N})(n > m \wedge s_n < L + \varepsilon)$ to be true in $\mathbb{R}$. However since $m \in \mathbb{N}$ was arbitrary, it follows that for any $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $n > m$ and $s_n < L + e$ satisfying the proposition. □

Proposition-2. For any bounded real valued sequence s,

$$\liminf_{n\to\infty} s_n = \lim_{n\to\infty} I_n = \lim_{n\to\infty} \left(\inf_{m \geq n} s_m \right)$$

where $I_n = \inf_{m \geq n} s_m$.

Proof. First we will prove that $\liminf$ is upper bound of I. Take an extended term s_N infinitely close to the cluster point $\liminf$. For any $m \in \mathbb{N}$, we have $I_m \leq s_n$ for all limited $n \geq m$, and hence it holds for all hypernatural $n \geq m$ by transfer. In particular, $I_m \leq s_N$, as $\liminf \simeq s_N$, we have $I_m \leq \liminf$.
Let $L = \lim I_n$. Then clearly L is supremum of I since I is nondecreasing. However, as we showed $\liminf$ is an upper bound, we must have $L \leq \liminf$. We will now show that $L < \liminf$ is impossibility. Suppose $L < \liminf$ and consider some $\varepsilon \in \mathbb{R}^+$ such that $L < \liminf - \varepsilon$. By proposition-1, we know that there exists $n \in \mathbb{N}$ such that standard tail beyond n is bounded below by $\liminf - \varepsilon$. But that means that $\liminf - \varepsilon \leq I_n$, and so $L < \liminf - \varepsilon \leq I_n$ which contradicts the fact that L was upper bound of I. Hence, $\liminf = L$. □

Exercise 6.12. *(Section 6.9, III) If s is a bounded sequence, show that for each $\varepsilon \in \mathbb{R}^+$ there is some $n \in \mathbb{N}$ such that the standard tail $s_n, s_{n+1}, \cdots$ is contained in the interval* $(\liminf - \varepsilon, \limsup + \varepsilon)$.

Proof. By theorem 6.8.4 and previous exericse, we know that there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $s_n < \limsup + \varepsilon$ and there exists $N' \in \mathbb{N}$ such that for all $n \geq N'$, $s_n < \limsup + \varepsilon$. Defining $m = \max\{N, N'\}$, the tail following s_m is the desired one. □

Exercise 6.13. *(Section 6.11, I) Give an example of a series that diverges but has a_n infinitesimal for all unlimted n.*

Proof. Consider the sequence (s_n) defined as $(s_n) = (1/n)$. Clearly for any unlimited n, a_n is infinitisimal. However, notice that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n} &\geq 1 + \sum_{n=1}^{\infty} \sum_{k=1}^{2^{n-1}} \frac{1}{n} \\ &\geq 1 + \sum_{n=1}^{\infty} \frac{1}{2}\end{aligned}$$

Since the last series is divergent, so is $\sum_{n=1}^{\infty} \frac{1}{n}$. □

Exercise 6.14. *(Section 6.11, II) Give nonstandard proofs of the usual rules of arithemtically combining convergent series.*
(a) $\sum_1^{\infty} a_i + \sum_1^{\infty} b_i = \sum_1^{\infty}(a_i + b_i)$,
(b) $\sum_1^{\infty} a_i - \sum_1^{\infty} b_i = \sum_1^{\infty}(a_i - b_i)$,

Proof. We will first prove (a), then (c), and then use (a) and (c) for proving (b). For what follow, Let $L = \sum_1^{\infty} a_i$ and $M = \sum_1^{\infty} b_i$.
(a) Let $N \in {}^*\mathbb{N}_{\infty}$. Since $\sum_1^n (a_i + b_i) = \sum_1^n a_i + \sum_1^n b_i$ holds in $\mathbb{R}$ for all $n \in \mathbb{N}$, by transfer we know it also holds in ${}^*\mathbb{R}$ for all $n \in {}^*\mathbb{N}$. Hence,

$$\sum_1^N (a_i + b_i) = \sum_1^N a_i + \sum_1^N b_i.$$

and thus,

$$\begin{aligned}\mathrm{sh}\left(\sum_1^N (a_i + b_i)\right) &= \mathrm{sh}\left(\sum_1^N a_i + \sum_1^N b_i\right) \\ &= \mathrm{sh}\left(\sum_1^N a_i\right) + \mathrm{sh}\left(\sum_1^N b_i\right) \\ &= L + M\end{aligned}$$

As desired.

(c) Let $N \in {}^*\mathbb{N}_{\infty}$. Then it follows

$$\begin{aligned}\mathrm{sh}\left(\sum_1^N c a_i\right) &= \mathrm{sh}\left(c \sum_1^N a_i\right) && \text{(by transfer)} \\ &= c \cdot \mathrm{sh}\left(\sum_1^N a_i\right) \\ &= c \cdot L\end{aligned}$$

as desired.

(b) Let $N \in {}^*\mathbb{N}_\infty$. Then

$$\begin{aligned}
\operatorname{sh}\left(\sum_1^N (a_i - b_i)\right) &= \operatorname{sh}\left(\sum_1^N (a_i + (-b_i))\right) \\
&= \operatorname{sh}\left(\sum_1^N a_i\right) + \operatorname{sh}\left(\sum_1^N (-b_i)\right) && \text{(by part a)} \\
&= \operatorname{sh}\left(\sum_1^N a_i\right) + (-1)\cdot\operatorname{sh}\left(\sum_1^N (-b_i)\right) && \text{(by part c)} \\
&= L + (-M) \\
&= L - M.
\end{aligned}$$

□

Exercise 6.15. *(Section 6.11, III) Suppose that $a_i \geq 0$ for all $i \in \mathbb{N}$. Prove that $\sum_1^\infty$ converges iff $\sum_1^n a_i$ is limited for all unlimited n, and that this holds iff $\sum_1^n a_i$ is limited for some unlimted n.*

Proof. First we prove (1) $\Rightarrow$ (2). Suppose $\sum_1^\infty a_i$ converges to a real number L. Therefore, we must have $\sum_1^N a_i \simeq L$ for all unlimited N but that means $\sum_1^N a_i$ is limited. The direction (2) $\Rightarrow$ (3) is easy to see.

We prove (3) $\Rightarrow$ (2). Let $n \in \mathbb{N}$. Then we know that there exists $m \in {}^*\mathbb{N}$ such that $m > n$ and $\sum_1^m a_i \leq L+1$ where L is the shadow of the limited extended term. Trasnfering it, we see that for any $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $m > n$ and $\sum_1^m a_i \leq L+1$. Since $a_i \geq 0$ for all $i \in \mathbb{N}$, we have $|\sum_1^m a_i| = \sum_1^m |a_i| \leq L+1$. Hence, the sequence of partial sum is bounded which implies all extended term are limited by theorem 6.4.1.

We prove (2) $\Rightarrow$ (1). Since $\sum_1^n a_i$ is limited for all unlimited n, by theorem 6.4.1, we know that sequence of partial sum is bounded. Also notice that since $a_i \geq 0$, for each $n \in \mathbb{N}$, $\sum_1^n a_i \leq \sum_1^{n+1} a_i$. Therefore, the sequence of partial sums is non decreasing. Hence, by theorem 6.2.1, we know it is convergent. □

Exercise 6.16. *(Section 6.11, IV) Comparison test.*
Let $\sum_1^\infty a_i$ and $\sum_1^\infty b_i$ be two real series of non negative terms, with $\sum_1^\infty b_i$ convergent. If $a_i \leq b_i$ for all $i \in \mathbb{N}$, use result (III) to show that $\sum_1^\infty a_i$ is convergent.

Proof. Since $a_i \leq b_i$ for all $i \in \mathbb{N}$, by transfer, we know $a_n \leq b_n$ for all unlimited n. By exercise (III), it follows $\sum_1^n b_i$ is limited for all unlimited n but that means $\sum_1^n a_i$ is also limited for all unlimited n, and thus by exercise (III), is convergent. □

Exercise 6.17. *(Section 6.11, V) Show that the comparison test holds under the weaker assumption that $a_n \leq b_n$ for all unlimited n. Show that this weaker assumption is equivalent to requiring that there be some limited $k \in \mathbb{N}$ with $a_n \leq b_n$ fora all $n \geq k$.*

Proof. For the first part, notice that since $a_n \leq b_n$ for all unlimited n, we have

$$\sum_m^n a_i \leq \sum_m^n b_i \simeq \operatorname{sh}\left(\sum_m^n b_i\right) = 0$$

for any unlimited m, n with $m \leq n$, because $\sum_1^\infty b_i$ is convergent. However, since $\sum_m^n a_i$ is finite, we have $\operatorname{sh}(\sum_m^n a_i) \leq 0$. Since partial sums of a_i is non-decreasing sequence of non-negative terms, it follows $0 \leq \sum_1^n a_i - \sum_1^{m-1} a_i = \sum_m^n a_i$ which gives us $0 = \operatorname{sh}(\sum_m^n a_i) \simeq \sum_m^n a_i$. Hence, by cauchy converegence criterion, we know that $\sum_1^\infty a_i$ converges.
For the second part, we know there exists $k \in {}^*\mathbb{N}$ such that for any all $n \geq k$, $a_n \leq b_n$ i.e. when k is unlimited. Transfering it, we get the desired result. For converse, we know for any $n \geq k$, $a_n \leq b_n$. Transfering it, we have for any $n \in {}^*\mathbb{N}$ such that $n \geq k$, $a_n \leq b_n$ but then since k is limited, the inequality holds for all unlimited numbers. □

Exercise 6.18. *(Section 6.11, VI) Let $\sum_1^\infty a_i$ and $\sum_1^\infty b_i$ be two series of positive terms such tha the sequence $\langle a_i/b_i : i \in \mathbb{N}\rangle$ is convergent in $\mathbb{R}$. Deduce that either both series converge, or both diverge.*

Proof.

Notation. *Define $\langle s\rangle = \langle a_i/b_i : i \in \mathbb{N}\rangle$ and $L = \lim_{n\to\infty} s$.*
We claim that the following result is true.

Theorem 1. *If $\sum_1^\infty a_i$ and $\sum_1^\infty b_i$ be two series of positive terms such that the sequence $\langle a_i/b_i : i \in \mathbb{N}\rangle$ is convergent in $\mathbb{R}$ Then for any unlimited m and n it follows that*

$$\sum_m^n a_i \text{ is infinitesimal if and only if } \sum_m^n b_i \text{ is infinitesimal.}$$

Assuming this claim for the moment, it follows from cauchy's convergence criterion that $\sum_1^\infty a_i$ is convergent if and only if $\sum_1^\infty b_i$ is convergent, which is the first part of the problem.

We prove our claim now. Suppose $\sum_m^n a_i$ is infintisimal for all unlimited m and n with $n \geq m$. Since the sequence $\langle s\rangle$ is convergent, we have $b_N \simeq 1/L \cdot a_N$ for any unlimited N. Thus,

$$\sum_m^n b_i \simeq \sum_{i=m}^n (1/L \cdot a_i) = 1/L \cdot \sum_{i=m}^n a_i$$

which is known to be infintisimal. The other direction can be proved by identical arguement considering $a_N \simeq Lb_N$ for any unlimited N.
We can apply the same chain of thought for proving divergence. Recall that a sequence is divergent if all of it's extended term are unlimted. Since we have $a_N \simeq Lb_N$, it follows immediately that if one of them is divergent then so is the other. □

Exercise 6.19. *(Section 6.11, VII) Let $c \in \mathbb{R}$. Recall the identity*

$$1 + c + c^2 + \cdots + c^n = \frac{1 - c^n + 1}{1 - c}. \tag{6.1}$$

(a) Considering the case of unlimited n, show that the series $\sum_1^\infty c^i$ converges if $|c| < 1$.
(b) Show that $\sum_m^n c^i$ is infintisimal when m and n are unlimited, either by applying result (a) or by making further use of the above inequality.

Proof.

(a) If $c = 0$ then the proposition is trivially true, so suppose $c \neq 0$ for what follows. We claim the following result is true.

Theorem 1. *For any $c \in \mathbb{R}$ such that $0 < |c| < 1$, the series $\sum_1^\infty |c^i|$ converges.*

Before we get to the proof, we see that if the theorem 1 is true then the series $\sum_1^\infty c^i$ is convergent because it is absolutely convergent which is what we desire.
We begin the proof by noticing that $\langle |c^i|\rangle = \langle |c|^i\rangle$ for $0 < c < 1$ is a non increasing sequence with a lower bound, and thus by montone convergence theorem, is convergent. Let $L = \lim |c|^n$ and consider some unlimited N. Then applying transfer on the identity 6.1, we have

$$\sum_1^N |c|^i = \frac{1 - |c|^{N+1}}{1 - |c|} - 1 \simeq \frac{1 - L}{1 - |c|} - 1$$

which is clearly limited. Therefore, it follows that $\sum_1^\infty c^i$ is convergent when $|c| < 1$.

(b) We are going to give alternative proof to it instead of using part (a). Again we will prove that $\sum_1^\infty c^i$ is absolutely convergent. We already know that the sequence $\langle |c^i| \rangle = \langle |c|^i \rangle$ for $0 < c < 1$ is convergent. Let $L = \lim |c^i|$. Then we have

$$\sum_m^n |c^i| = \sum_1^n |c^i| - \sum_1^m |c_i| \simeq \frac{1-L}{1-c} - \frac{1-L}{1-c} = 0.$$

□

Exercise 6.20. *(Section 6.11, VIII) Ratio test: Convergence.*
Suppose that

$$\limsup_{i\to\infty} \frac{|a_{i+1}|}{|a_i|} < 1$$

in $\mathbb{R}$. Prove that the series $\sum_1^\infty a_i$ is absoultely convergent.

Proof. We will prove the series $\sum_1^\infty a_i$ is absoultely convergent by demonstrating that difference of partial sum of sequence is always less than or equal to difference of partial sum of a convergent sequence which will show that difference of partial sum of our sequence gets infinitely close eventually. We start off by proving the following theorem.

Theorem 1. *Suppose that* $\limsup_{i\to\infty} \frac{|a_{i+1}|}{|a_i|} < 1$ *in $\mathbb{R}$. Then there exists a positive real $c < 1$ with $|a_{n+1}| < c|a_n|$ for all unlimited n.*

Before we discuss the proof of it, there is an interesting corollary which follows from this. If theorem-1 is true for all unlimited numbers then by easy trasnfer, we must have $|a_{n+1}| < c|a_n|$ for all but finitely many $n \in \mathbb{N}$.
We will now prove theorem-1. If $a_i = 0$ for any $i \in \mathbb{N}$ then $\frac{|a_{i+1}|}{|a_i|}$, i.e. the ith term of the sequence, become undefined. Therefore, we must have $a_i \neq 0$ for any $i \in \mathbb{N}$. Thus, $0 < \frac{|a_{n+1}|}{|a_n|}$ for all $n \in \mathbb{N}$. Let N be arbitrary unlimited natural number. Then we have $0 < \frac{|a_{N+1}|}{|a_N|} \leq \limsup_{i\to\infty} \frac{|a_{i+1}|}{|a_i|} < 1$. Let $L = \limsup_{i\to\infty} \frac{|a_{i+1}|}{|a_i|}$. Define $c = \frac{L+1}{2}$. Then we have $0 < \frac{|a_{N+1}|}{|a_N|} < c$. Thus, $|a_{N+1}| < c|a_N|$. Since N was arbitrary, we have our result.

Corollary 1.1. *There is some limited $k \in \mathbb{N}$ such that $|a_{n+1}| < c|a_n|$ for all $n \geq k$.*

Theorem 2. *Suppose that* $\limsup_{i\to\infty} \frac{|a_{i+1}|}{|a_i|} < 1$ *in $\mathbb{R}$. Then there is some k such that for all $n \geq k$, we have $|a_{k+n}| < c^n|a_k|$, and hence $\sum_{k+m}^{k+n} |a_i| \leq \sum_m^n c^i|a_k|$.*

The existence of such a k is guranteed by corollary 1.1, furthermore, it also implies

$$|a_{k+n}| < c|a_{k+n-1}| < c^2|a_{k+n-2} < \cdots < c^n|a_{k+n-n}| = c^n|a_k|.$$

Thus, $|a_{k+n}| < c^n|a_k|$. It is easy to see that the last inequalitiy implies

$$0 < \sum_{k+m}^{k+n} |a_i| \leq \left(\sum_m^n c^i \right) |a_k|.$$

By an easy transfer we see that the last inequality also hold in hyperreals. However, by exercise VII, we know that $\sum_m^n c^i$ is infinitesimal for any unlimited n and m such that $n \geq m$ but that means $\sum_{k+m}^{k+n} |a_i|$ is infinitesimal for any unlimited m and n. In particular, replacing m by $m-k$ and n by $n-k$, we get the following result.

Corollary 2.1. $\sum_1^\infty |a_i|$ *is convergent, that is to say,* $\sum_1^\infty a_i$ *is absolutely convergent.*

□

Exercise 6.21. *(Section 6.11, IX) Ratio Test: divergence.*
Suppose that

$$\liminf_{i\to\infty} \frac{|a_{i+1}|}{|a_i|} > 1$$

in $\mathbb{R}$. Prove that the series $\sum_1^\infty a_i$ diverges.

Proof. Before we get into the proof, notice that $a_i \neq 0$ for any $i \in \mathbb{N}$ since otherwise $\frac{|a_{i+1}|}{|a_i|}$ becomes undefined. Consider some unlimited n. Then we have

$$1 < \liminf_{i\to\infty} \frac{|a_{i+1}|}{|a_i|} \leq \frac{|a_{N+1}|}{|a_N|},$$

which implies $|a_{N+1}| > |a_N|$. We summarize it into a theorem.

Theorem 1. *Suppose that $\liminf_{i\to\infty} \frac{|a_{i+1}|}{|a_i|} > 1$. Then $a_i \neq 0$ for any $i \in \mathbb{N}$ and $|a_{N+1}| > |a_N|$ for any unlimited N.*

Next we prove the following result.

Theorem 2. *Suppose that $\liminf_{i\to\infty} \frac{|a_{i+1}|}{|a_i|} > 1$. Then there exists $k \in \mathbb{N}$ such that for all $m > n$, $|a_m| > |a_k$.*

In natural numbers, we know that for any $n, m \in \mathbb{N}$, there is some $p \in \mathbb{N}$ such that if $m > n$ then $m = n+p$. Therefore, by transfer, it is also true in hypernaturals. In particular, it holds for unlimited numbers. Let m and n be unlimited numbers such that $m > n$. Then, by theorem-1, it follows

$$|a_m| = |a_{n+p}| > |a_{(n+p)-1}| > \cdots > |a_{n+1}| > |a_n|.$$

Trasnfering $(\exists n \in {}^*\mathbb{N})(\forall m \in {}^*\mathbb{N})(m > n \to |a_m| > |a_n|)$ back to $\mathbb{N}$, we have our result.

By existential transfer, we have following corollary to be true.

Corollary 2.1. *For all unlimited n, $|a_n| > |a_k|$.*

For the sake of contradiction, assume $\sum_1^\infty a_i$ is convergent. Therefore, for any unlimited N, we must have $\sum_1^{N+1} a_i - \sum_1^N a_i \simeq 0$ but then it implies

$$0 < |a_k| < |a_{N+1}| = \sum_1^{N+1} a_i - \sum_1^N a_i \simeq 0$$

which means $0 < |a_k| \leq 0$. A contradiction. □

Exercise 6.22. *(Section 6.11, X) Show that for any real number x, the hyperreal $x^n/n!$ is infinitesimal when n is unlimited.*

Proof. Define $a_i = (x^i/i!)$. We will first prove that $\sum_1^\infty a_i$ is convergent which would easily get us $\sum_{n-1}^n a_i = \sum_{n-1}^n x^i/i! = x^n/n!$ to be infintisimal. For any unlimited N, we have $\frac{(x^{N+1}/(N+1)!)}{(x^N/N!)} = \frac{x}{N+1}$ which is clearly infinitesimal. It means that $\limsup_{i\to\infty} \frac{|a_{i+1}|}{|a_i|} < 1$. Threfore, by ratio test, it follows $\sum_1^\infty a_i$ is convergent. Hence, $\sum_m^n a_i$ is infinitesimal for any unlimited m and n such that $n \geq m$. In particular, setting $m = n-1$ gets us $\sum_{n-1}^n a_i = \sum_{n-1}^n x^i/i! = x^n/n!$ to be infinitesimal. □

Exercise 6.23. *(Section 6.11, XI) Leibniz's alternating series test.*
Suppose $\langle a_i : i \in \mathbb{N}\rangle$ is a real sequence that is nonincreasing (i.e. $a_i \geq a_{i+1}$) and converges to 0. Prove that the alternating series

$$\sum_1^\infty (-1)^{i+1} a_i = a_1 - a_2 + a_3 - a_4 + \cdots$$

converges.

Proof. We first make the observation that all the terms in the sequence are positive since otherwise it would not be convergent to 0. We are going to first prove $\left|\sum_m^n (-1)^{i+1} a_i\right| \leq |a_m|$ for $m, n \in \mathbb{N}$ from which our result easily follow using the transfer principle and shadow inequalitiy.

Theorem 1. *Suppose $\langle a_i : i \in \mathbb{N}\rangle$ is a real sequence that is nonincreasing (i.e. $a_i \geq a_{i+1}$) and converges to 0. Then for all $m, n \in \mathbb{N}$ such that $n \geq m$, we have*

$$\left|\sum_m^n (-1)^{i+1} a_i\right| \leq |a_m|$$

to be true.

We divide the proof into cases.
First suppose m is even. Then we have

$$\sum_m^n (-1)^{i+1} a_i = \sum_{\substack{m \leq i \leq n \\ i \text{ is even}}} (-a_i + a_{i+1}) = \sum_{\substack{m \leq i \leq n-1 \\ i \text{ is even}}} (-a_i + a_{i+1}) < 0 < a_m$$

and

$$-a_m < (a_n - a_m) + \sum_{\substack{m+1 \leq i \leq n-1 \\ i \text{ is odd}}} (a_i - a_{i+1}) = \sum_m^n (-1)^{i+1} a_i$$

when n is odd because $a_n < a_m$ and $a_{i+1} < a_i$ for each i. In case of n being even, consider

$$\sum_{\substack{m \leq i \leq n-1 \\ i \text{ is even}}} (-a_i + a_{i+1}) - a_n < \sum_{\substack{m \leq i \leq n-1 \\ i \text{ is even}}} (-a_i + a_{i+1}) < 0 < a_m$$

and

$$-a_m < -a_m + \sum_{\substack{m+1 \leq i \leq n \\ i \text{ is odd}}} (a_i - a_{i+1}) = \sum_m^n (-1)^{i+1} a_i$$

Similarly, in case of m being odd, we have

$$-a_m < 0 < \sum_{\substack{m \leq i \leq n \\ i \text{ is odd}}} (a_i - a_{i+1}) = \sum_m^n (-1)^{i+1} a_i$$

and

$$\sum_m^n (-1)^{i+1} a_i = a_m - a_n + \sum_{\substack{m-1 \leq i \leq n-1 \\ i \text{ is even}}} (-a_i + a_{i+1}) < a_m$$

when n is even. In case of n being odd, consider

$$-a_m < 0 < \sum_{\substack{m \leq i \leq n-1 \\ i \text{ is odd}}} (a_i - a_{i+1}) < \sum_{\substack{m \leq i \leq n-1 \\ i \text{ is odd}}} (a_i - a_{i+1}) + a_n = \sum_m^n (-1)^{i+1} a_i$$

and

$$\sum_m^n (-1)^{i+1} a_i = a_m + \sum_{\substack{m-1 \leq i \leq n \\ i \text{ is even}}} (-a_i + a_{i+1}) < a_m.$$

This concludes the proof of theorem 1. By an easy transfer, we know that theorem 1 also holds in hypernautrals. Therefore, $\sum_m^n (-1)^{i+1} a_i$ is infintisimal in case of unlimited m. Hence, by cauchy convergence theorem, it follows $\sum_1^\infty (-1)^{i+1} a_i$ converges. $\square$

Chapter 7

Continuous Functions

Exercise 7.1. *(Section 7.4, I) Derive the following characterisations from standard definition of limits.*
(a) $\lim_{x\to c^+} f(x) = L$ *iff* $f(x) \simeq L$ *for all* $x \in {}^*A$ *with* $x \simeq c$ *and* $x > c$.
(b) $\lim_{x\to c^-} f(x) = L$ *iff* $f(x) \simeq L$ *for all* $x \in {}^*A$ *with* $x < c$.
(c) $\lim_{x\to c} f(x) = L$ *iff* $f(x) \simeq L$ *for all* $x \in {}^*A$ *with* $x \simeq c$ *and* $x \neq 0$.
(d) $\lim_{x\to c} f(x) = +\infty$ *iff* $f(x) \in {}^*\mathbb{R}^+_\infty$ *for all* $x \in {}^*A$ *with* $x \simeq c$ *and* $x \neq c$.
(e) $\lim_{x\to c} f(x) = -\infty$ *iff* $f(x) \in {}^*\mathbb{R}^-_\infty$ *for all* $x \in {}^*A$ *with* $x \simeq c$ *and* $x \neq c$.
(f) $\lim_{x\to +\infty} f(x) = L$ *iff* $f(x) \simeq L$ *for all positive unlimited* $x \in {}^*A$ *(and such* x *exists).*
(g) $\lim_{x\to -\infty} f(x) = L$ *iff* $f(x) \simeq L$ *for all positive unlimited* $x \in {}^*A$ *(and such* x *exists).*

Proof.
(a) The definition of $\lim_{x\to c^+} f(x) = L$ is

$$(\forall \varepsilon \in \mathbb{R}^+)(\exists \delta \in \mathbb{R}^+)(\forall x \in A)(c < x < c + \delta \to |f(x) - L| < \varepsilon) \tag{1}$$

First we prove the forward implication. Suppose (1) is true. Let $\varepsilon \in \mathbb{R}^+$ be arbitrary. Then by (1) there is a positive δ such that

$$(\forall x \in A)(c < x < c + \delta \to |f(x) - L| < \varepsilon)$$

is true, and hence by universal transfer we have

$$(\forall x \in {}^*A)(c < x < c + \delta \to |f(x) - L| < \varepsilon).$$

Now if $x \simeq c$ with $x > c$ then clearly $c < x < c + \delta$ and so by last statement, $|f(x) - L| < \varepsilon$. Since this holds for arbitrary $\varepsilon \in \mathbb{R}^+$, it follows that $f(x) \simeq L$. Next we prove the converse implication. Suppose "$f(x) \simeq L$ for all $x \in {}^*A$ with $x \simeq c$ and $x > c$" is true. Let ε be a positive real. If $x \simeq c$ with $x > c$, there is a positive infinitismal δ such that $x = c + \delta$ but then it means $c < x < c + 2\delta$. Thus, by assumption, we have $f(x) \simeq L$ to be true. Replacing 2δ by existentially qunatified variable, this shows

$$(\exists \delta \in {}^*\mathbb{R}^+)(\forall x \in {}^*A)(c < x < c + \delta \to |f(x) - f(c)| < \varepsilon)$$

is true. By existential transfer, we infer

$$(\exists \delta \in \mathbb{R}^+)(\forall x \in A)(c < x < c + \delta \to |f(x) - f(c)| < \varepsilon)$$

which is enough to conclude (1).

(b) Essentially the same arguement as (a).

(c) The definition of $\lim_{x\to c} f(x) = L$ is

$$(\forall \varepsilon \in \mathbb{R}^+)(\exists \delta \in \mathbb{R}^+)(\forall x \in A)(|x - c| < \delta \to |f(x) - L| < \varepsilon) \tag{1}$$

First we prove the forward implication. Let $\varepsilon \in \mathbb{R}^+$ be arbitrary. Then by (1), there exists positive real δ such that

$$(\forall x \in A)(|x - c| < \delta \to |f(x) - L| < \varepsilon)$$

holds and thus, by universal transfer,

$$(\forall x \in {}^*A)(|x - c| < \delta \to |f(x) - L| < \varepsilon)$$

is also true. Now if $x \simeq c$ with $x \neq c$ then clearly $|x - c| < \delta$ and so $|f(x) - L| < \varepsilon$. Since this holds for arbitrary ε, we have $f(x) \simeq L$. Next we prove the converse implication. Suppose that "$f(x) \simeq L$ for all $x \in {}^*A$ with $x \simeq c$ and $x \neq c$" is true. Let $\varepsilon \in \mathbb{R}^+$ be arbitrary. If δ is an infinitesimal and $|x - c| < \delta$ then clearly $x \simeq c$ with $x \neq c$. By our assumption, it implies $f(x) \simeq L$, and consequently $|f(x) - L| < \varepsilon$. Replacing δ by existentially quantified variable, we have

$$(\exists \delta \in {}^*\mathbb{R}^+)(|x - c| < \delta \to |f(x) - L| < \delta)$$

to be true, and so by existential transfer,

$$(\exists \delta \in \mathbb{R}^+)(|x - c| < \delta \to |f(x) - L| < \varepsilon)$$

is also true, which is enough to conclude (1).
(d) Recall that the definition of $\lim_{x \to c} f(x) = +\infty$ is

$$(\forall M \in \mathbb{R}^+)(\exists \delta \in \mathbb{R}^+)(\forall x \in A)(0 < |x - c| < \delta \to f(x) > M) \tag{1}$$

First we prove the forward implication. Suppose (1) is true. Let $M \in \mathbb{R}^+$ be arbitrary. Then by (1), there is a positive δ such that

$$(\forall x \in A)(0 < |x - c| < \delta \to f(x) > M)$$

is true, and hence by universal transfer, we have

$$(\forall x \in {}^*A)(0 < |x - c| < \delta \to f(x) > M).$$

Now if $x \simeq c$ with $x \neq c$, then clearly $0 < |x - c| < \delta$. Therefore, we have $f(x) > M$. Since this holds for arbitrary x and M, we have the desired result. Next we prove the converse implication. Suppose "$f(x) \in \mathbb{R}^+_\infty$ for all $x \in {}^*A$ with $x \simeq c$ and $x \neq c$" is true. Let M be an arbitrary positive real. If δ is any positive infinitisimal, then for any $x \in {}^*A$, if $0 < |x - c| < \delta$, we have $x \simeq c$ and $x \neq c$. Hence, $f(x) \in {}^*\mathbb{R}^+_\infty$ by our assumption, and so $f(x) > M$. Replacing δ by existentially qunatified variable, this shows that the sentence

$$(\exists \delta \in {}^*\mathbb{R}^+)(\forall x \in {}^*A)(0 < |x - c| < \delta \to f(x) > M)$$

is true. By existential transfer, we then infer

$$(\exists \delta \in \mathbb{R}^+)(\forall x \in A)(0 < |x - c| < \delta \to f(x) > M)$$

and finally since M was arbitrary, we conclude (1).
(e) Essentially the same arguement as (d).
(f) Recall that the definition of $\lim_{x \to +\infty} f(x) = L$ is

$$(\forall \varepsilon \in \mathbb{R}^+)(\exists \delta \in \mathbb{R}^+)(\forall x \in A)(x > \delta \to |f(x) - L| < \varepsilon) \tag{1}$$

First we prove the forward implication. Suppose (1) is true. Let $\varepsilon \in \mathbb{R}^+$ be arbitrary. Then by (1), there is a positive real δ such that

$$(\forall x \in A)(x > \delta \to |f(x) - L| < \varepsilon)$$

is true, and hence by universal transfer, we have

$$(\forall x \in {}^*A)(x > \delta \to |f(x) - L| < \varepsilon)$$

to be true. Now if there exists a positive unlimited $x \in {}^*A$, clearly $x > \delta$ but then by our assumption we have $|f(x) - L| < \varepsilon$. Since this holds for arbitrary real ε, it must be the case that $f(x) \simeq L$. Next we prove the converse implication. Suppose $f(x) \simeq L$ for all positive unlimited $x \in {}^*A$ and such x exists. Let

$\varepsilon \in \mathbb{R}^+$ be arbitrary. If δ is any positive unlimited number, then for any $x \in {}^*A$, if $x > \delta$ then $f(x) \simeq L$ by our assumption. Replacing δ by existentially qunatified variable, this shows that the sentence

$$(\exists \delta \in {}^*\mathbb{R}^+)(\forall x \in {}^*A)(x > \delta \to |f(x) - L| < \varepsilon)$$

is true, and thus by existential transfer, so is

$$(\exists \delta \in \mathbb{R}^+)(\forall x \in A)(x > \delta \to |f(x) - L| < \varepsilon).$$

Since ε was arbitrary, we conclude (1).
(g) Essentially the same arguement as (f). □

Exercise 7.2. *(Section 7.4, II) Prove the following standard results.*
(a) $\lim_{x\to c} f(x) = L$ *iff* $\lim_{x\to c^+} f(x) = L$ *and* $\lim_{x\to c^-} f(x) = L$.
(b) f *is continuous at* c *iff* $\lim_{x\to c} f(x) = f(c)$.
(c) if $\lim_{x\to c} f(x)$ *and* $\lim_{x\to c} g(x)$ *exists, then*
(i) $\lim[f(x) + g(x)] = \lim f(x) + \lim g(x)$
(ii) $\lim[f(x)g(x)] = \lim f(x) \cdot \lim g(x)$
(iii) $\lim[f(x)/g(x)] = \lim f(x)/\lim g(x)$, *if* $\lim g(x) \neq 0$.

Proof.
(a) If $\lim_{x\to c} f(x) = L$ then we must have $f(x) \simeq L$ for all $x \in A$ such that $x \simeq c$ and $x \neq c$. By universal transfer, law of trichotomy holds in hypereals, and so we have $x < c$ and $x > c$. This observation implies the desired result. Conversely if we have $\lim_{x\to c^+} f(x) = L$ and $\lim_{x\to c^-} f(x) = L$ then $f(x) \simeq L$ for any $x \simeq c$ such that $x > c$ and $x < c$ which, again by law of trichotomy, means $x \neq c$. This is enough to conclude $\lim_{x\to c} f(x) = L$.
(b) Trivial by the definition.
(c) Suppose $x \simeq c$ with $x \neq c$. Then $f(x) \simeq \lim f(x)$ and $g(x) \simeq \lim g(x)$. Then
(i) $f(x) + g(x) \simeq \lim f(x) + \lim g(x)$.
(ii) $f(x)g(x) \simeq \lim f(x) \cdot \lim g(x)$.
(iii) Suppose further $\lim g(x) \neq 0$, then clearly $f(x)/g(x) = \lim f(x)/\lim g(x)$. □

Exercise 7.3. *(Section 7.4, III) Use infintisimal to discuss continuity of the following functions.*

(a) $f_1(x) = \begin{cases} \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

(b) $f_2(x) = \begin{cases} x\sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

(c) $f_3(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is rational} \end{cases}$.

(d) $f_4(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ -x & \text{if } x \text{ is rational} \end{cases}$.

(e) $f_5(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \in \mathbb{Q} \text{ in simplest form with } n \geq 1 \end{cases}$.

Proof.
(a) By uinversal transfer on

$$(\forall n \in \mathbb{N}) \left(\sin\left((2n+1) \cdot \frac{\pi}{2}\right) = 1 \text{ or } \sin\left((2n+1) \cdot \frac{\pi}{2}\right) = -1\right)$$

we have

$$(\forall n \in {}^*\mathbb{N}) \left(\sin\left((2n+1) \cdot \frac{\pi}{2}\right) = 1 \text{ or } \sin\left((2n+1) \cdot \frac{\pi}{2}\right) = -1\right)$$

to be true. However, since it means that there is an infintisimal, $x = \frac{1}{\frac{2}{(2n+1)\pi}}$, such that $f(x) \neq 0$, it follows f have discontinuity at 0.

(b) $\sin 1/x$ is continuous at all points except 0. However, in case of $x \sin 1/x$, if $x > 0$ and $x \simeq 0$, clearly $x \sin 1/x \simeq 0$. If $x < 0$ and $x \simeq 0$, again we get $x \sin 1/x \simeq 0$. Therefore, we have $\lim_{x \to 0} x \sin 1/x = 0 = f_2(0)$.

(c) It is discontinous everywhere. By universal transfer of density of rationals in $\mathbb{R}$, we know $(\forall x, y \in {}^*\mathbb{R})(x < y \to (\exists q \in {}^*Q)(x < q < y))$ holds. Thus, we know there exists a hyperrational q between r and $r + \varepsilon$, where ε is some infintisimal and r is an irrational. Thus, $q \simeq r$. However, $1 = f(q) \not\simeq f(r) = 0$. Similarly, since each open interval (a, b), where $a, b \in Q$, contain irrational. We have some irrational infinite close to each rational number. This implies that f is discontinous on rational as well as irrational, and so all of reals.

(d) For any non zero x, we clearly have the function to be discontinous at x following the line of arguement of part (c). However, the case $x = 0$ is interesting. For any irrational, ε, infinitely close to 0, we have $f(\varepsilon) = -\varepsilon \simeq 0 = f(0)$. Therefore, f is continuous at 0.

(e) The function is continous at irrational and discontinous at rationals. First we prove it is discontinous at rational points. Let $q \in \mathbb{Q}$. Assume further $q = \frac{m}{n}$ in its lowest form with $n \geq 1$. However since n is a finite number, $\frac{1}{n}$ can never be infinitely close to 0. Thus, $f(q) \not\simeq f(r)$ for any irrational r infinitely close to q. Now we prove that f is continous at irrational. Take any irrational r and consider some hyperrational q infinitely close to it. Assume further that $q = \frac{m}{n}$ in its lowest form and $n \geq 1$. Multiplying by n, we get $m \simeq rn$. It means $\mathrm{sh}(m - rn) \simeq 0$ but that would imply $m = rn$ since $m, n, r \in \mathbb{R}$. This is a contradiction. Thus, there is no q infinitely close to r which satisfy the given property and thus the function is vacously continous. $\square$

Exercise 7.4. *(Section 7.8, I) Explain why the argument just given fails for intervals* $(a, b), (a, b], (a, +\infty)$, $(-\infty, b)$, *etc that are not closed.*

Proof. In case of (semi)bounded open intervals, it fails because we can have a situation where $\mathrm{sh}(x) \notin (a, b]$ for some $x \in (a, b]$. Whereas in case of unbounded sequence, it fails because unlimited numbers do not have shadows. $\square$

Exercise 7.5. *(Section 7.8, II) Show that* $f(x) = \frac{1}{x}$ *is not uniformly continuous on* $(0, 1)$.

Proof. Consider two positive infinitisimals ε and ε^2. Clearly, $\varepsilon \simeq \mathrm{sh}(\varepsilon) = 0 = \mathrm{sh}(\varepsilon^2) \simeq \varepsilon^2$ but $f(\varepsilon) = \frac{1}{\varepsilon} \not\simeq \frac{1}{\varepsilon^2} = f(\varepsilon^2)$. $\square$

Exercise 7.6. *(Section 7.8, III) If f is uniformly continuous on $\mathbb{R}$ and $\langle s_n : n \in \mathbb{N} \rangle$ is a cauchy sequence, show that $\langle f(s_n) : n \in \mathbb{N} \rangle$ is a cauchy sequence.*

Proof. Consider some arbitrary unlimited hypernaturals M and N. Since $\langle s_n \rangle$ is cauchy, we have $s_N \simeq s_M$, however, uniform continuity of f further implies $f(s_N) \simeq f(s_M)$. This means $\langle f(s_n) \rangle$ is cauchy. $\square$

Exercise 7.7. *(Section 7.8, IV) Let the real function f be monotonic on $[a, b]$ and suppose that for all real r between $f(a)$ and $f(b)$ there exists a real $c \in [a, b]$ such that $f(c) = r$. Prove that f is continuous on $[a, b]$.*

Proof. Lets beging by assuming, without loss of generality, f is an increasing function. For the sake of contradiction suppose f is not continuous on $[a, b]$. This implies that there is some real x such that for some y infinitely close to it, we have $f(x) \not\simeq f(y)$. Assume further $x < y$ and consider some real c between $f(x)$ and $\mathrm{sh}(f(y))$. By the intermidiate value property, it follows there is some $d \in [a, b]$ such that $f(d) = c$. Since f is monotonic, we must have $x < d < y$. This implies $x \simeq d$. However, this is not possible since x and d are both unequal real numbers. $\square$

Exercise 7.8. *(Section 7.11, I) If f is a real function and $c \in \mathbb{R}$, verify in detail that $f(x)$ is defined for all $x \simeq c$ if and only if $f(x)$ is defined for all real x in some open interval $(c - \varepsilon, c + \varepsilon)$ with real radius $\varepsilon > 0$.*

Proof. We first prove the forward implication. Suppose $f(x)$ is defined for all $x \simeq c$. If ε is a positive infintisimal, then clearly $f(x)$ is defined for all $|x-c| < \varepsilon$. This means that

$$(\exists \varepsilon \in {}^*\mathbb{R}^+)(\forall x \in {}^*\mathbb{R})(|x-c| < \varepsilon \to f(x) \text{ is defined at } x)$$

holds true in ${}^*\mathbb{R}$ and so by existential transfer,

$$(\exists \varepsilon \in \mathbb{R}^+)(\forall x \in \mathbb{R})(|x-c| < \varepsilon \to f(x) \text{ is defined at } x)$$

is true in $\mathbb{R}$. We proceed with the converse implication. Suppose $f(x)$ is defined for all real x in some open interval $(c-\varepsilon, c+\varepsilon)$ with $\varepsilon > 0$. That means

$$(\exists \varepsilon \in \mathbb{R}^+)(\forall x \in \mathbb{R})(|x-c| < \varepsilon \to f(x) \text{ is defined at } x)$$

is true in $\mathbb{R}$ and so by universal transfer,

$$(\exists \varepsilon \in {}^*\mathbb{R}^+)(\forall x \in {}^*\mathbb{R})(|x-c| < \varepsilon \to f(x) \text{ is defined at } x)$$

holds in ${}^*\mathbb{R}$. In particular, this implies that $f(x)$ is defined for all x infintely close to c. □

Exercise 7.9. *(Section 7.11, II) Let f be a real function that is defined on some open neighbourhood of $c \in \mathbb{R}$. Show that if f is constant on hal(c), then it is constant on some interval $(c-\varepsilon, c+\varepsilon) \subseteq \mathbb{R}$.*

Proof. Suppose f is constant on $\mathrm{hal}(c)$. This means that for any infinitesimal ε, $|x-c| < \varepsilon$ implies $f(x)$ is constant. Thus,

$$(\exists \varepsilon \in {}^*\mathbb{R}^+)(\forall x \in {}^*\mathbb{R})(|x-c| < \varepsilon \to f(x) \text{ is constant})$$

holds in ${}^*\mathbb{R}$ and so by existential transfer,

$$(\exists \varepsilon \in \mathbb{R}^+)(\forall x \in \mathbb{R})(|x-c < \varepsilon \to f(x) \text{ is constant})$$

holds in $\mathbb{R}$. This means that there is some open neighbourhood of c in $\mathbb{R}$ where f is constant. □

Chapter 8

Differentiation

Exercise 8.1. *(Section 8.1, I) Prove that derivative of* $\sin x$ *is* $\cos x$ *at real* x.

Proof. Let ε be an infinitesimal. Then we have

$$\begin{aligned}\frac{\sin(x+\varepsilon)-\sin x}{\varepsilon} &= \frac{\sin x\cos\varepsilon+\cos x\sin\varepsilon-\sin x}{\varepsilon}\\ &= \frac{\sin x(\cos\varepsilon-1)+\cos x\sin\varepsilon}{\varepsilon}\\ &= \sin x\cdot\frac{\cos\varepsilon-1}{\varepsilon}+\cos x\cdot\frac{\sin\varepsilon}{\varepsilon}\\ &\simeq \sin x\cdot 0+\cos x\cdot 1\\ &= \cos x.\end{aligned}$$

□

Exercise 8.2. *(Section 8.2) If* f *and* g *are differentiable at* $x\in\mathbb{R}$*. Then*
(a) $f+g$ *is differentiable and* $(f+g)'(x)=f'(x)+g'(x)$.
(b) If $g\neq 0$ *then* f/g *is differentiable and* $(f/g)'=\frac{f'(x)g(x)-f(x)g'(x)}{g(x)^2}$.

Proof. (a) Let Δx be a non zero infinitesimal. Then we must have $f(x+\Delta x)$ and $g(x+\Delta x)$ to be defined because they are differentiable. So, $(f+g)(x+\Delta x)=f(x+\Delta)+g(x+\Delta x)$. Thus, increment of $f+g$ at x corresponding to Δx is

$$\begin{aligned}\Delta(f+g) &= (f+g)(x+\Delta xl)-(f+g)(x)\\ &= f(x+\Delta x)+g(x+\Delta x)-f(x)-g(x)\\ &= (f(x)+\Delta f)+(g(x)+\Delta g)-f(x)-g(x)\\ &= \Delta f+\Delta g.\end{aligned}$$

It follows that

$$\begin{aligned}\frac{\Delta(f+g)}{\Delta x} &= \frac{\Delta f}{\Delta x}+\frac{\Delta g}{\Delta x}\\ &\simeq f'(x)+g'(x).\end{aligned}$$

Hence, by theorem 8.11, $(f+g)'(x)=f'(x)+g'(x)$.

(b) Suppose $g\neq 0$. Consider some non zero infinitesimal Δx. Then we must have $f(x+\Delta x)$ and $g(x+\Delta x)$ to be defined because they are differentiable. Also, we have $(f/g)(x+\Delta x)=f(x+\Delta x)/g(x+\Delta x)$. Thus, increment of f/g at x corresponding to Δx is

$$\begin{aligned}
\Delta(f/g) &= (f/g)(x+\Delta x) - (f/g)(x) \\
&= \frac{f(x+\Delta x)}{g(x+\Delta x)} - \frac{f(x)}{g(x)} \\
&= \frac{f(x)+\Delta f}{g(x)+\Delta g} - \frac{f(x)}{g(x)} \\
&= \frac{(f(x)+\Delta f)g(x) - f(x)(g(x)+\Delta g)}{g(x)(g(x)+\Delta g)} \\
&= \frac{\Delta f g(x) - f(x)\Delta g}{g(x)^2 + g(x)\Delta g}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\Delta f/\Delta x &= \frac{1}{\Delta x} \cdot \frac{\Delta f g(x) - f(x)\Delta g}{g(x)^2 + g(x)\Delta g} \\
&= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2 + g(x)\Delta g} \\
&\simeq \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}
\end{aligned}$$

because the fraction is finite and Δg is infinitesimal. □

Exercise 8.3. *(Section 8.5, I). Prove Rolle theorem. That is Suppose f is continuous at $[a,b]$ and differentiable at (a,b). Then if $f(a) = f(b) = 0$, then $f'(x) = 0$ for some $x \in (a,b)$.*

Proof. Firstly, notice that if f acheives maxima at one of a and b and minima at the other, then clearly proposition is true because $f'(x) = 0$ for all $x \in (a,b)$. In any other case, we would either have minima or maxima for some $x \in (a,b)$ by extreme value theorem, and so by critical point theorem $f'(x) = 0$. □

Exercise 8.4. *(Section 8.5, II). Prove mean value theorem. That is for some $x \in (a,b)$,*

$$f'(x) = \frac{f(b)-f(a)}{b-a}.$$

Proof. Consider the function $h(x) = (f(b)-f(a))x - (b-a)f(x)$. Clearly, $h(b) = 0 = h(a)$. So by rolle theoerm, we know there is some $x \in (a,b)$ for which $(f(b)-f(a)) - (b-a)f'(x) = h'(x) = 0$. This implies $f'(x) = \frac{f(b)-f(a)}{b-a}$. □

Exercise 8.5. *(Section 8.5, III). If f' is zero/positive/negative on (a,b) then f is constant/increasing/decreasing on $[a,b]$.*

Proof. For f' being zero on (a,b). Consider some $c,d \in (a,b)$ such that $f(c) \neq f(d)$. Then by mean value theorem, we know there is some $x \in (c,d) \subseteq (a,b)$ with the property $f'(x) = \frac{f(d)-f(c)}{d-c} \neq 0$. Hence, $f(c) = f(d)$ for any $c,d \in (a,b)$. In case derivative is positive on (a,b), the function must be increasing clearly, otherwise we can argue in same sense as that of f' being 0 to argue that there is some x for which $f'(x)$ is negative. We can prove for f' being negative on (a,b) implies f is decreasing similarly. □

Exercise 8.6. *(Section 8.8, I) Show that if f is smooth at (a,b), then it is continuous at (a,b).*

Proof. Trivial by increment equation. □

Exercise 8.7. *(Section 8.8, II) Let f be smooth at (a,b). Given infinitesimal Δx, Δy show that the difference between Δf and df is itself infinitely smaller than the infinitesimal distance $\Delta l = \sqrt{\Delta x^2 + \Delta y^2}$ between (a,b) and $(a+\Delta x, b+\Delta y)$, in the sense that*

$$\frac{\Delta f - df}{\Delta l} \simeq 0.$$

Proof. Since f is smooth, we have $\Delta f - df = \varepsilon\Delta x + \delta\Delta y$ for some infinitesimal ε and δ. Thus,

$$\begin{aligned}\frac{\Delta f - df}{\Delta l} &= \frac{\varepsilon\Delta x + \delta\Delta y}{\sqrt{\Delta x^2 + \Delta y^2}} \\ &= \varepsilon \cdot \frac{\Delta x}{\sqrt{\Delta x^2 + \Delta y^2}} + \delta \cdot \frac{\Delta y}{\sqrt{\Delta x^2 + \Delta y^2}} \\ &= \varepsilon \cdot \frac{1}{\sqrt{1 + (\Delta y^2/\Delta x^2)}} + \delta \cdot \frac{1}{\sqrt{(\Delta x^2/\Delta y^2) + 1}}\end{aligned}$$

Which is clearly infinitesimal. ☐

Exercise 8.8. *(Section 8.9, I) Verify that the Maclaurin series for e^x converges to e^x at any $x \in \mathbb{R}$ by proving that the remainder $R_n(x)$ is infintisimal when n is unlimited.*

Proof. Let $x \in \mathbb{R}$. Applying transfer on lagrange form of the remainder, we see that

$$(\forall n \in {}^*\mathbb{N})\,(\exists c \in {}^*\mathbb{R})\,(R_n(x) = \frac{e^{n+1}(c)}{(n+1)!} \cdot x^{n+1})$$

holds in ${}^*\mathbb{R}$ where $0 < c < x$. Since $e^{n+1}(c) = e^c$, we have $e^{n+1}(c)$ to be finite while $\frac{x^{n+1}}{(n+1)!}$ to be infinitesimal for unlimited n. This implies $R_n(x)$ is infinitesimal. ☐

Exercise 8.9. *(Section 8.10, I) There are forms for the taylor remainder other than Lagrange's. One of these is*

$$R_n(x) = \frac{f^{(n)}(c) - f^{(n)}(a)}{(c-a)(n+1)!} \cdot (x-a)^{n+1}$$

for some c between a and x when $f^{(n+1)}$ exists between a and x. Prove the n-th order increment equation without using the hypothesis of continuity of $f^{(n)}$.

Proof. Applying transfer principle, we see there must exits some $c \in {}^*\mathbb{R}$ between a and x so that for any natural n,

$$R_n(x) = \frac{f^{(n)}(c) - f^{(n)}(a)}{(c-a)(n+1)!} \cdot (x-a)^{n+1} \tag{1}$$

This implies, for any infinitesimal Δx, there is some $c \in {}^*\mathbb{R}$ with $x < c < x + \Delta x$ so that (1) holds. Consequently, we have

$$\begin{aligned}f(x+\Delta x) &= p_{n-1}(x+\Delta x) + R_{n-1}(x+\Delta x) \\ &= \sum_{k=0}^{n-1} \frac{f^{(k)}(x)}{k!}(\Delta x)^k + \frac{f^{(n-1)}(c) - f^{(n-1)}(a)}{(c-x)n!} \cdot (\Delta x)^n \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!}(\Delta x)^k + \frac{(\Delta x)^k}{n!} \cdot \left(\frac{f^{(n-1)}(x) - f^{n-1}(c)}{x-c} - f^n(x)\right)\end{aligned}$$

where we know the latter term is infinitesimal because the derivative $f^{(n)}(x)$ is infinitely close to $\frac{f^{(n-1)}(x)-f^{n-1}(c)}{x-c}$ given $x \simeq c$. Hence, letting $\varepsilon = \frac{1}{n!} \cdot \left(\frac{f^{(n-1)}(x)-f^{n-1}(c)}{x-c} - f^n(x)\right)$, the desired result follows. ☐

Exercise 8.10. *(Section 8,12, I) Let f be differentiable and have f' continous on $(a,b) \subseteq \mathbb{R}$. Let x be well inside ${}^*(a,b)$ and $\Delta x \simeq 0$. Show that the incremental equation holds for any hyperreal well inside ${}^*(a,b)$, that is, $f(x+\Delta x) = f(x) + f'(x)\Delta x + \varepsilon\Delta x$.*

Proof. We claim the following result is true.

Theorem 1. *Let f be differentiable and f' be continuous on (a,b). Then for any hyperreal x well inside $^*(a,b)$ and $\Delta x \simeq 0$, we have*

$$f'(x) = \frac{f(x+\Delta x) - f(x)}{\Delta}.$$

Before we discuss the proof of it, we see that if the theorem-1 is true then the desired result easily follow because we would have $f'(x) + \varepsilon = \frac{f(x+\Delta x)-f(x)}{\Delta}$, multiplying which by Δx and rearranging proves the incremental equation.

We begin the proof by noticing that if f is differentiable on (a,b) then so it is on $(\mathrm{sh}(x), \mathrm{sh}(x)+\Delta x)$. Similarly, f' is also continuous on $(\mathrm{sh}(x), \mathrm{sh}(x)+\Delta x)$. Applying the transfered mean value theorem on f and the interval $(\mathrm{sh}(x), \mathrm{sh}(x)+\Delta x)$, we see that

$$f'(y) = \frac{f(\mathrm{sh}(x)+\Delta x) - f(\mathrm{sh}(x))}{\Delta x}$$

holds for some $y \in (\mathrm{sh}(x), \mathrm{sh}(x)+\Delta x)$. However, since f and f' are continuous on (a,b) along with $y \simeq x$ and $\mathrm{sh}(x) \simeq x$, we have

$$f'(x) \simeq f'(y) = \frac{f(\mathrm{sh}(x)+\Delta x) - f(\mathrm{sh}(x))}{\Delta x} \simeq \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

to be true. This concludes the proof of theorem-1. Hence, the following corollary holds.

Corollary 1.1. *If f is differentiable and f' be continuous on (a,b) then $f(x+\Delta x) = f(x) + f'(x)\Delta x + \varepsilon\Delta x$.*

□

Exercise 8.11. *(Section 8.12, II) Let*

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

(a) Prove that f' exists at 0 by is not continous there.
(b) Let $x = 1/(2\pi N)$ with N unlimited. Show that there is an infinitesimal Δx such that incremental equation fails for any $\varepsilon \simeq 0$ at $f(x+\Delta x)$.

Proof.
(a) We start by noticing that for any infinitesimal ε,

$$\begin{aligned} \frac{f(\varepsilon) - f(0)}{\varepsilon} &= \frac{\varepsilon^2 \sin(1/\varepsilon)}{\varepsilon} \\ &= \varepsilon \cdot sin(1/\varepsilon) \end{aligned}$$

Which is clearly infinitesimal because $\sin(1/\varepsilon)$ is limited. However, f' is not continous at $x = 0$. To see that consider the infinitesimal $\varepsilon = 1/(2\pi N)$ where N is unlimited natural. Since $f'(x) = 2x \cdot \sin 1/x - \cos 1/x$ when $x \neq 0$, we have

$$\begin{aligned} f'(\varepsilon) &= 2\varepsilon \cdot \sin\frac{1}{\varepsilon} - \cos\frac{1}{\varepsilon} \\ &= 2\frac{1}{2\pi N} \cdot \sin\frac{1}{1/(2\pi N)} - \cos\frac{1}{1/(2\pi N)} \\ &= 2\frac{1}{2\pi N} \cdot \sin 2\pi N - \cos 2\pi N \\ &= 0 - \cos 2\pi N \\ &= -\cos(2\pi N) \\ &= -1 \\ &\not\simeq 0 \\ &= f'(0) \end{aligned}$$

Therefore, we have $0 \simeq \varepsilon$ but $f'(0) \not\simeq f'(\varepsilon)$ proving that f' is not continuous at $x = 0$.

(b) Let N be unlimited natural and $x = 1/(2\pi N)$. Consider $\Delta x = -1/(2\pi N)$ and any infinitesimal ε. Then we have

$$\begin{aligned}
f(x) + f'(x)\Delta x + \varepsilon\Delta x &= \left(\frac{1}{2\pi N}\right)^2 \cdot \sin\frac{1}{1/(2\pi N)} - \cos(2\pi N)\Delta x + \varepsilon\Delta x \\
&= 0 + \Delta x(-\cos(2\pi N) + \varepsilon) \\
&= \Delta x(\varepsilon - 1) \\
&\neq 0 \\
&= f(x + \Delta x)
\end{aligned}$$

$\square$

Chapter 9

The Riemann Integral

Exercise 9.1. *(Section 9.1, I). Monotonic Functions.*
Prove that theorem 9.1.1 holds for monotonic function as well. That is if f is monotonic on $[a, b]$ then for any positive infinitesimal Δx, $L_a^b(f, \Delta x) \simeq U_a^b(f, \Delta x)$.

Proof. Without loss of generality assume f is non decreasing function. We begin our proof by observing that for any $\Delta x \in \mathbb{R}^+$, we have

$$
\begin{aligned}
U_a^b(f, \Delta x) - L_a^b(f, \Delta x) &= \sum_{i=1}^{n} M_i \Delta x_i - \sum_{i=1}^{n} m_i \Delta x_i \\
&= \sum_{i=1}^{n} (M_i - m_i)\Delta x_i \\
&= \sum_{i=1}^{n} (M_i - m_i)\Delta x && \text{(the partitions have equal width)} \\
&= (f(b) - f(a))\Delta x + \sum_{i=1}^{n} (M_i - m_{i+1})\Delta x && \text{(f is non-decreasing)} \\
&\leq (f(b) - f(a))\Delta x. && \text{(summation is non positive since $M_i \leq m_{i+1}$)}
\end{aligned}
$$

By transfer, we know it holds in $^*\mathbb{R}$, that is, for any $\Delta x \in {}^*\mathbb{R}$, $U_a^b - L_a^b \leq (f(b) - f(a))\Delta x$ must be true. In particular if Δx is any positive infinitesimal, $U_a^b \simeq L_a^b$. □

Exercise 9.2. *(Section 9.2, I) For each (standard) $n \in \mathbb{N}$, let $U_a^b(f, n), L_a^b(f, n), S_a^b(f, n)$ be the upper, lower, and ordinary reimann sums for the partition determined by the number $\Delta x = \frac{b-a}{n}$. Prove that if $n \in {}^*\mathbb{N}$ is unlimited, then*

$$L_a^b(f, n) \simeq S_a^b(f, n) \simeq U_a^b(f, n).$$

Show how the definition and proof of existence for the reiemann integral could be developed just using these functions of (hyper)natural numbers.

Proof. Since $\Delta x = \frac{b-a}{n}$ is infinitesimal for unlimited n, we can directly apply theorem 9.2.1 to conclude $L_a^b(f, n) \simeq U_a^b(f, n)$ and consequently, $L_a^b(f, n) \simeq S_a^b(f, n) \simeq U_a^b(f, n)$ by the transfer of the inequality $L_a^b(f, P) \leq S_a^b(f, P) \leq U_a^b(f, P)$. Furthermore, because Δx is infinitesimal, we can Essentially use the same arguement to define and justify existence of riemann integrals. □

Exercise 9.3. *(9.3, I) Derove proofs for the other properties of the integral listed above. That is*
(a) $\int_a^b cf(x)dx = c\int_a^b f(x)dx$.
(b) $\int_a^b f(x) + g(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$.
(c) $\int_a^b f(x)dx \leq \int_a^b g(x)dx$ *if* $f(x) \leq g(x)$ *on* $[a, b]$.
(d) $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$ *if* $m \leq f(x) \leq M$ *on* $[a, b]$.

Proof.
(a) For any $\Delta x \in \mathbb{R}$, we have

$$\begin{aligned} S_a^b(cf, \Delta x) &= \sum_{i=1}^{n} cf(x_{i-1}) \cdot \Delta x \\ &= c \cdot \sum_{i=1}^{n} f(x_{i-1}) \Delta x \\ &= c \cdot S_a^b(f, \Delta x). \end{aligned}$$

By transfer, it also holds in $^*R$. In particular, if Δx is any infinitesimal,

$$\begin{aligned} \int_a^b cf(x)dx &= \mathrm{sh}\left(S_a^b(cf, \Delta x)\right) \\ &\simeq S_a^b(cf, \Delta x) \\ &= c \cdot S_a^b(f, \Delta x) \\ &\simeq c \cdot \mathrm{sh}\left(S_a^b(f, \Delta x)\right) \\ &= c \cdot \int_a^b f(x)dx. \end{aligned}$$

Thus, $\int_a^b cf(x)dx \simeq c \cdot \int_a^b f(x)dx$. However, we must have $\int_a^b cf(x)dx = c \cdot \int_a^b f(x)dx$ because they both are real. This concludes the proof.

(b) For any $\Delta x \in \mathbb{R}$, the following holds.

$$\begin{aligned} S_a^b(f+g, \Delta x) &= \sum_{i=1}^{n} (f+g)(x_{i-1}) \cdot \Delta x \\ &= \sum_{i=1}^{n} f(x_{i-1}) \cdot \Delta x + \sum_{i=1}^{n} g(x_{i-1}) \cdot \Delta x \\ &= S_a^b(f, \Delta x) + S_a^b(g, \Delta x). \end{aligned}$$

Thus, by transfer, it also holds in $^*\mathbb{R}$. If Δx is any infinitesimal, we have

$$\begin{aligned} \int_a^b f(x) + g(x)dx &= \mathrm{sh}(S_a^b(f+g, \Delta x)) \\ &\simeq S_a^b(f+g, \Delta x) \\ &= S_a^b(f, \Delta x) + S_a^b(g, \Delta x) \\ &\simeq \mathrm{sh}(S_a^b(f, \Delta x) + S_a^b(g, \Delta x)) \\ &= \mathrm{sh}(S_a^b(f, \Delta x)) + \mathrm{sh}(S_a^b(g, \Delta x)) \\ &= \int_a^b f(x)dx + \int_a^b g(x)dx. \end{aligned}$$

Hence, $\int_a^b f(x) + g(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$ since they both are real numbers.

(c) Suppose $f(x) \leq g(x)$ for all $x \in [a, b]$. It is easy to see that

$$S_a^b(f, \Delta x) = \sum_{i=1}^{n} f(x_{i-1}) \Delta x \leq \sum_{i=1}^{n} g(x_{i-1}) \Delta x = S_a^b(g, \Delta x),$$

and so by transfer, it holds in ${}^*\mathbb{R}$. Let Δx be an infinitesimal. Notice that

$$\begin{aligned}\int_a^b f(x)dx &= \mathrm{sh}(S_a^b(f,\Delta x)) \\ &\simeq S_a^b(f,\Delta x) \\ &\leq S_a^b(g,\Delta x) \\ &\simeq \mathrm{sh}(S_a^b(g,\Delta x)) \\ &= \int_a^b g(x)dx.\end{aligned}$$

Hence, $\int_a^b f(x)dx \leq \int_a^b g(x)dx$.

(d) Suppose $m \leq f(x) \leq M$ on $[a,b]$. Then we must have

$$m(b-a) = m \cdot \sum_{i=1}^n \Delta x = \sum_{i=1}^n m\Delta x \leq \sum_{i=1}^n f(x_{i-1})\Delta x \leq \sum_{i=1}^n M\Delta x = M \cdot \sum_{i=1}^n \Delta x = M(b-a)$$

for any $\Delta x \in \mathbb{R}$. Therefore, the inequalitiy also holds in ${}^*\mathbb{R}$. In particular, letting Δx to be infinitesimal, we see that

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a).$$

□

Exercise 9.4. *(Section 9.5, I) Average Function values.*
Let f be continous on $[a,b] \subseteq \mathbb{R}$. Define the "sample average" function Av by putting, for each $n \in \mathbb{N}$,

$$Av(n) = \frac{f(x_0) + \cdots + f(x_{n-1})}{n},$$

where $x_i = a + \frac{i(b-a)}{n}$.
Prove that if $N \in {}^*\mathbb{N}$ is unlimited, then

$$Av(N) \simeq \frac{1}{b-a}\int_a^b f(x)dx$$

(ie, the average value of f on $[a,b]$ is given by the shadow of $Av(N)$).

Proof. Let $n \in \mathbb{N}$ be arbitrary. Then we see that

$$\begin{aligned}Av(n) &= \frac{\sum_{i=1}^n f(x_{i-1})}{n} \\ &= \frac{\sum_{i=1}^n f(x_{i-1})}{n} \cdot \frac{\Delta x}{\Delta x} \\ &= \frac{\sum_{i=1}^n f(x_{i-1})\Delta x}{n \cdot \Delta x} \\ &= \frac{\sum_{i=1}^n f(x_{i-1})\Delta x}{\sum_{i=1}^n \Delta x} \\ &= \frac{\sum_{i=1}^n f(x_{i-1})\Delta x}{\sum_{i=1}^n \Delta x_i - \Delta x_i} \\ &= \frac{\sum_{i=1}^n f(x_{i-1})\Delta x}{b-a} \\ &= \frac{1}{b-a} \cdot \sum_{i=1}^n f(x_{i-1})\Delta x \\ &= \frac{1}{b-a} \cdot S_a^b(f,n).\end{aligned}$$

Thus, by transfer, it follows that for any $n \in {}^*\mathbb{N}$, we must have $Av(N) = \frac{1}{b-a} \cdot S_a^b(f, n)$. In particular, if N is any unlimited hypernatural

$$\begin{aligned}
Av(N) &= \frac{1}{b-a} \cdot S_a^b(f, N) \\
&\simeq \mathrm{sh}\left(\frac{1}{b-a} \cdot S_a^b(f, N)\right) \\
&= \mathrm{sh}\left(\frac{1}{b-a}\right) \cdot \mathrm{sh}(S_a^b(f, N)) \\
&= \frac{1}{b-a} \cdot \int_a^b f(x)dx.
\end{aligned}$$

This proves the desired conclusion. $\square$

Chapter 10

Topology of the Reals

Exercise 10.1. *(Section 10.2, I) Prove that the collection of topologically closed sets is closed under finite union and arbitrary intersections.*

Proof. First we will demonstrate that it is closed under finite union. Let $A_1, \cdots, A_n$ be closed. We wish to show that $\bigcup_{i=1}^{n} A_i$ is closed. Consider some $x \in \mathbb{R}$ such that $x \simeq r \in {}^*(\bigcup_{i=1}^{n} A_i)$. Assume $x \notin \bigcup_{i=1}^{n} A_i$. This would mean $x \in (\bigcup_{i=1}^{n} A_i)^c = \bigcap_{i=1}^{n} A_i^c$. So $r \in {}^*(\bigcup_{i=1}^{n} A_i)^c$ because all of A_i^c's are open and $x \simeq r$. A contradiction. So we must have $x \in \bigcup_{i=1}^{n} A_i$. Thus proving $\bigcup_{i=1}^{n} A_i$ is closed.
Now let $\{A_i : i \in I\}$ be some collection of closed sets. Consider some $x \simeq r \in {}^*\left(\bigcap_{i \in I} A_i\right)$ and suppose $x \notin \bigcap_{i \in I} A_i$. Thus, $x \in \left(\bigcap_{i \in I} A_i\right)^c = \bigcup_{i \in I} A_i^c$. Since it is open being arbitrary union of open sets, we have $r \in {}^*\left(\bigcap_{i \in I} A_i\right)^c$ which is a contradiction. Therefore, $x \in \bigcap_{i \in I} A_i$ □

Exercise 10.2. *(Section 10.2, II) Show that the proof of 10.2.1 (2) does not work for infinite intersection by showing that*

$$^*\left(\bigcap_{n \in \mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right)\right) \neq \bigcap_{n \in \mathbb{N}} {}^*\left(-\frac{1}{n}, \frac{1}{n}\right).$$

Proof. Intutively, for any $n \in \mathbb{N}$, it can be easily seen that ${}^*\left(-\frac{1}{n}, \frac{1}{n}\right)$ contains infintisimal numbers. So we must have $\bigcap_{n \in \mathbb{N}} {}^*\left(-\frac{1}{n}, \frac{1}{n}\right)$ to be equal to set of infinitesimal. However, the only infinitesimal which is a real number is 0, so $\bigcap_{n \in N} \left(-\frac{1}{n}, \frac{1}{n}\right)$ must be $\{0\}$ which is enough to prove the desired result. We will now formalise our intutive arguments first by proving ${}^*\left(\bigcap_{n \in N} \left(-\frac{1}{n}, \frac{1}{n}\right)\right) = \{0\}$ and then proving $\bigcap_{n \in \mathbb{N}} {}^*\left(-\frac{1}{n}, \frac{1}{n}\right) = \mathbb{I}$ where $\mathbb{I}$ is set of infinitesimal (or in other words, $\mathbb{I} = \text{hal}(0)$).

Theorem 1. ${}^*\left(\bigcap_{n \in \mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right)\right) = \{0\}$

We will prove this claim by arguing that $\bigcap_{n \in \mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$ and since extension of finite set equals to the set, the result would follow. It is easy to see $0 \in \bigcap_{n \in \mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right)$. To prove it must only be 0, let r be some non-zero real number such that $r \in \bigcap_{n \in \mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right)$. That means the inequalitiy $-1/n < r < 1/n$ hold for any natural n. Furthermore, without loss of generality, assume r is positive. By archimedean property, we know there is some $n \in \mathbb{N}$ so that $1/r < n$. This implies that $r > 1/n$. A contradiction. We can derive similar contradiction in case r is negative real number. Thus, $\bigcap_{n \in \mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$. Therefore, ${}^*\left(\bigcap_{n \in N} \left(-\frac{1}{n}, \frac{1}{n}\right)\right) = \bigcap_{n \in \mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$ because $\bigcap_{n \in \mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right)$ is finite.

Theorem 2. $\bigcap_{n \in \mathbb{N}} {}^*\left(-\frac{1}{n}, \frac{1}{n}\right) = \mathbb{I}$*, where I is the set of infinitesimals.*

We are going to prove this theorem by standard inclusion arguement. Let $x \in \bigcap_{n \in \mathbb{N}} {}^*\left(-\frac{1}{n}, \frac{1}{n}\right)$. This means that for any $n \in \mathbb{N}$, we have x to be in ${}^*\left(-\frac{1}{n}, \frac{1}{n}\right)$, that is $x \in {}^*\mathbb{R}$ and $-1/n < x < 1/n$. We wish to show that $x \simeq 0$. Suppose not. Then it follows that $-1/n < \text{sh}(x) \simeq x < 1/n$ for all $n \in \mathbb{N}$. This contradicts Theorem-1. Hence, x must be infinitesimal i.e. $x \in \mathbb{I}$. Conversely, Let $x \in \mathbb{I}$. Then it is readily seen that $-1/n < x < 1/n$ for any $n \in \mathbb{N}$ because $-1/n$ and $1/n$ are real numbers. So $x \in {}^*\left(-\frac{1}{n}, \frac{1}{n}\right)$ for each natural n. This is sufficient to conclude theorem-2.

Finally, theorem-1 and theorem-2 together implies ${}^*\left(\bigcap_{n \in \mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right)\right) \neq \bigcap_{n \in \mathbb{N}} {}^*\left(-\frac{1}{n}, \frac{1}{n}\right)$. As desired. □

Exercise 10.3. *(Section 10.3, I) Prove that in $\mathbb{R}$ a closed subset of a compact set is compact.*

Proof. Let A be some real compact set. Consider some closed subset, B, of A. It is easy to see that B is bounded because we know A is from heine-borel. Therefore, applying heine borel again, we see that B must be compact. $\square$

Exercise 10.4. *(Section 10.5, I) Show that:*
(a) any S-open set is real open.
(b) each S-open set is union of halos, but a union of halos is not necessarily S-open.
(c) no real radius neighbourhood can be S-open.

Proof.
Consider some S-interval $((r-\varepsilon, r+\varepsilon))$. Let $s \in ((r-\varepsilon, r+\varepsilon))$ and define $d = \frac{\text{sh}\min\{|(r-\varepsilon)-s|,|(r+\varepsilon)-s|\}}{2}$. It is easy to see that the interval $(s-d, s+d)$ does not contain any point infinitely close $r-\varepsilon$ and $r+\varepsilon$ because $|r-\varepsilon-s|, |r+\varepsilon-s| > d$. Therefore, $S = \bigcup_{s\in S}(s-d, s+d)$ proving S-interval is R-open, and consequently S-open set is R-open.

(b) The former part follows from definition. For latter, consider hal(2)$\cup$hal(3). It is easy to see hal(2)$\cup$hal(3) can not be written as union of $S-$ open sets because only way to do that would be to show hal(2) and hal(3) are both S-open. This is clearly not the case because any subinterval of hal(2) and hal(3) contains point infinitely close to end points.

(c) Consider some real radius neighbourhood (a, b). To write it as union of S-open set, we would require hal(a) to be subset of one of the intervals. However, this is not possible because $a \in hal(a)$ but $a \notin (a, b)$. $\square$

Exercise 10.5. *(Section 10.5, II) Let A be an open subset of $\mathbb{R}$.*
(a) Show that ${}^*A$ is the interval-open in ${}^*\mathbb{R}$.
(b) Suppose A is the union of a sequence $\langle A_n : n \in \mathbb{N}\rangle$ of pairwsie disjoint open intervals in $\mathbb{R}$, with the length of A_n being less than $1/n$. Use transfer to show that some elements of ${}^*A$ is infinitely close to something not in ${}^*A$. Deduce that ${}^*A$ is not S-open.
(c) Show further that ${}^*A$ contains a point that does not belong to any real-radius neighbourhood that is included in ${}^*A$. Hence, deduce the stronger result that ${}^*A$ is non real-open.

Proof.

(a) Consider some open set $A = \bigcup_{i\in I}(a_i, b_i)$ of $\mathbb{R}$. Extending it we see $A^* = {}^*\left(\bigcup_{i\in I}(a_i, b_i)\right) = \bigcup_{i\in {}^*I} {}^*(a_i, b_i)$ which is union of intervals in ${}^*\mathbb{R}$ i.e. interval open.

(b) Let $A_n = (a_n, b_n)$ denote a term in the sequence, where $n \in \mathbb{N}$. Consider $x = \frac{bn+an}{2} \in A_n$. Clearly, $|x - b_n| = \left|\frac{b_n - a_n}{2}\right| < \frac{1}{2n}$. By transfer, this inequalitiy ${}^*\mathbb{R}$ as well. That is for any $A_n = (a_n, b_n)$, where $n \in {}^*\mathbb{N}$, there exists $x \in A_n$ such that $|x - b_n| < \frac{1}{2n}$. In particular, if N is unlimited natural, there is some $x \in A_N$ such that $|x - b_N| < \frac{1}{2n}$. That is $x \simeq b_N$, however, by definition $b_N \notin A_N = (a_N, b_N)$. Furthermore, b_n can not be in any other term of the sequence either because otherwise it would contradict the hypothesis that all terms are pairwsie disjoint open intervals. Consequently, ${}^*A$ is not S-open because there are elements $x \in {}^*A$ such that hal$(x) \not\subseteq {}^*A$.
(c) Let $A_n = (a_n, b_n)$ denote a term of the sequence, where n is an unlimited natural. Consider $x = \frac{b_n - a_n}{2}$. Clearly $x \in A_n \subseteq {}^*A$ but it is not contained in any real neighbourhood y of ${}^*A$. For if it is the neighbourhood would also contain hal(x), in particular b_n which is a contradiction. $\square$