
LECTURES ON HYPERREALS
BOOK BY ROBERT GOLDBLATT

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Contents

1	What are Hypereals?	3
2	Large sets	5
3	Ultrapower Construction of the hyperreals	7
4	The Transfer Principle	11
5	Hypereals Great and Small	13
6	Convergence of Sequences and Series	17

Chapter 1

What are Hypereals?

Chapter 2

Large sets

Exercise 2.1. If $\emptyset \neq A \subseteq I$, there is an ultrafilter \mathcal{F} on I with $A \in \mathcal{F}$.

Proof. Define $\mathcal{H} = \{B \in \mathcal{P}(I) : A \subseteq B\}$. Since A is non empty, it follows that \mathcal{H} has fip. Clearly, $A \in \mathcal{H}$, and so $A \in \mathcal{F}^{\mathcal{H}}$. Therefore, by zorn lemma, we know that there exists ultrafilter, \mathcal{F} , such that $A \in \mathcal{F}^{\mathcal{H}} \subseteq \mathcal{F}$. \square

Exercise 2.2. There exists a nonprincipal ultrafilter on \mathbb{N} containing the set of even numbers, and another containing the set of odd numbers.

Proof. Define $I = \{x \in \mathbb{N} : x = 2n \text{ for some } n \in \mathbb{N}\}$. Since I is infinite, we know that \mathcal{F}^{co} is proper and thus, have fip. Applying the zorn's lemma, we see that there exists ultrafilter \mathcal{F} such that $\mathcal{F} \subseteq \mathcal{F}^{\text{co}}$. However, for any $i \in I$, we have $I - \{i\} \in \mathcal{F}^{\text{co}} \subseteq \mathcal{F}$ but, since \mathcal{F} is ultrafilter, $\{i\} \in \mathcal{F}$. However, clearly $\{i\} \in \mathcal{F}^i$. Therefore, $\mathcal{F} \neq \mathcal{F}^i$ for any $i \in I$. Likewise we could argue about set of odd numbers. \square

Exercise 2.3. An ultrafilter on finite set must be principal.

Proof. Let the set be $A = \{a_1, a_2, \dots, a_n\}$ and \mathcal{F} be ultrafilter on it. Since $\{a_1\} \cup \{a_2\} \cup \dots \cup \{a_n\} = A \in \mathcal{F}$, it must be the case that $a_i \in \mathcal{F}$ for some i such that $1 \leq i \leq n$. Since \mathcal{F} is proper, for any $B \in \mathcal{F}$, we have $B \cap \{a_i\} \neq \emptyset$, but that means $\mathcal{F} = \mathcal{F}^{a_i}$. \square

Exercise 2.4. For $\mathcal{H} \subseteq \mathcal{P}(I)$.

(i) Show that $\mathcal{F}^{\mathcal{H}}$ is a filter that includes \mathcal{H} , i.e. $\mathcal{H} \subseteq \mathcal{F}^{\mathcal{H}}$.

(ii) Show that $\mathcal{F}^{\mathcal{H}}$ is included in any other filter that includes \mathcal{H} .

Proof.

(i) By the definition.

(ii) Let \mathcal{F} be a filter such that $\mathcal{H} \subseteq \mathcal{F}$. Let $A \in \mathcal{F}^{\mathcal{H}}$, then $B_1 \cap B_2 \cap \dots \cap B_n \subseteq A$ for some $B_1, \dots, B_n \in \mathcal{H}$. Clearly, $B_1 \cap \dots \cap B_n \in \mathcal{F}$ and so is $A \in \mathcal{F}$. \square

Exercise 2.5. Let \mathcal{F} be a proper filter on I .

(I) Show that $\mathcal{F} \cup \{A^c\}$ has fip iff $A \notin \mathcal{F}$.

(II) Use (I) to deduce that \mathcal{F} is an ultrafilter iff it is maximal proper filter on I .

Proof.

(I) First we prove forward implication. Notice that if $A \in \mathcal{F}$ then since $A \cap A^c = \emptyset$, it follows $\mathcal{F} \cup \{A^c\}$ does not have fip.

We prove the converse implication. Notice that since \mathcal{F} is a proper filter, ie it does not contain \emptyset , the only way for $\mathcal{F} \cup \{A^c\}$ to not have fip would be $A \in \mathcal{F}$.

(II) We prove the forward implication. Suppose \mathcal{F} is an ultrafilter. Applying Zorn's lemma on proper filter \mathcal{F} , we see that there exists a maximal proper filter, K , such that $\mathcal{F} \subseteq K$. If $K \neq \mathcal{F}$, it would mean there is some $A \in K$ such that $A \notin \mathcal{F}$ but then that would mean $A^c \in \mathcal{F}$, and so $A^c \in K$ which contradicts the

fact that K is proper since it fails to satisfy fip . Therefore, we must have $F = K$.

We prove the converse implication. Suppose \mathcal{F} is maximal proper filter. For the sake of contradiction assume \mathcal{F} is not an ultra filter i.e. there exists some $A \in \mathcal{P}(I)$ such that both $A, A^c \notin \mathcal{F}$. Clearly, either $\mathcal{F} \cup \{A\}$ has fip or $\mathcal{F} \cup \{A^c\}$ does. Wlog assume $\mathcal{F} \cup \{A\}$ has fip . Applying Zorn's lemma on $\mathcal{F} \cup \{A\}$, we see that there is maximal proper filter K such that $\mathcal{F} \cup \{A\} \subseteq K$. However this means that \mathcal{F} is not maximal proper filter since $F \subsetneq F \cup \{A\} \subset P(I)$. \square

Chapter 3

Ultrapower Construction of the hyperreals

Exercise 3.1. (section 3.3, I). \equiv is an equivalence relation on $\mathbb{R}^{\mathbb{N}}$.

Proof. Let $\langle a_n \rangle, \langle b_n \rangle, \langle c_n \rangle \in \mathbb{R}^{\mathbb{N}}$. First we prove that reflexivity. Clearly, $\{n \in \mathbb{N} : a_n = a_n\} = \mathbb{N} \in \mathcal{F}^{\text{co}} \subseteq \mathcal{F}$. Thus, $\langle a_n \rangle \equiv \langle a_n \rangle$.

Symmetricity is trivial since $\{n \in \mathbb{N} : a_n = b_n\} = \{n \in \mathbb{N} : b_n = a_n\}$, if one of them is in \mathcal{F} then so is the other.

Finally, we prove transitivity. Suppose $\langle a_n \rangle \equiv \langle b_n \rangle$ and $\langle b_n \rangle \equiv \langle c_n \rangle$ i.e. $\{n \in \mathbb{N} : a_n = b_n\}, \{n \in \mathbb{N} : b_n = c_n\} \in \mathcal{F}$. Clearly, $\{n \in \mathbb{N} : a_n = b_n = c_n\} = \{n \in \mathbb{N} : a_n = b_n\} \cap \{n \in \mathbb{N} : b_n = c_n\} \in \mathcal{F}$. Therefore, $\langle a_n \rangle \equiv \langle c_n \rangle$. \square

Exercise 3.2. (section 3.3, II) \equiv is a congruence on the ring $\langle \mathbb{R}^{\mathbb{N}}, \oplus, \odot \rangle$, which means that if $r \equiv r'$ and $s \equiv s'$, then $r \oplus s = r' \oplus s'$ and $r \odot s = r' \odot s'$.

Proof. Suppose $r \equiv r'$ and $s \equiv s'$ i.e. $\{n \in \mathbb{N} : r_n = r'_n\}, \{n \in \mathbb{N} : s_n = s'_n\} \in \mathcal{F}$. Thus, since $\{n \in \mathbb{N} : r_n = r'_n, s_n = s'_n\} = \{n \in \mathbb{N} : r_n = r'_n\} \cap \{n \in \mathbb{N} : s_n = s'_n\} \in \mathcal{F}$, we have $\{n \in \mathbb{N} : r_n = r'_n, s_n = s'_n\} \subseteq \{n \in \mathbb{N} : r_n + s_n = r'_n + s'_n\} \in \mathcal{F}$. Similarly, $\{n \in \mathbb{N} : r_n s_n = r'_n s'_n\} \in \mathcal{F}$. Therefore, $r \oplus s \equiv r' \oplus s'$ and $r \odot s \equiv r' \odot s'$. \square

Exercise 3.3. (section 3.3, III) $\langle 1, \frac{1}{2}, \frac{1}{3}, \dots \rangle \not\equiv \langle 0, 0, 0, \dots \rangle$.

Proof. Let $\langle r \rangle = \langle 1, \frac{1}{2}, \frac{1}{3}, \dots \rangle$. For the sake of simplicity, define $[[r \neq 0]] = \{n \in \mathbb{N} : r_n \neq 0\}$. Clearly, $[[r \neq 0]] \subseteq \mathbb{N}$. Notice that for any $n \in \mathbb{N}$, we have $n \in [[r \neq 0]]$. Thus, $[[r \neq 0]] = \mathbb{N}$. So, $[[r \neq 0]] \in \mathcal{F}^{\text{co}} \subseteq \mathcal{F}$ but then it would mean $[[r = 0]] = [[r \neq 0]]^c \notin \mathcal{F}$. Therefore, $\langle r \rangle \not\equiv \langle 0 \rangle$. \square

Exercise 3.4. (section 3.5, I) $[[r = s]] \cap [[s = t]] \subseteq [[r = t]]$.

Proof. Let $n \in [[r = s]] \cap [[s = t]]$. Then it follows that $r_n = s_n$ and $s_n = t_n$, and so $r_n = t_n$. Therefore, $n \in [[r = t]]$. \square

Exercise 3.5. (Section 3.5, II) $[[r = r']] \cap [[s = s']] \subseteq [[r \oplus s = r' \oplus s']] \cap [[r \oplus s = r' \oplus s']]$

Proof. Let $n \in [[r = r']] \cap [[s = s']]$. Thus, $r_n = r'_n$ and $s_n = s'_n$, and so $r_n \oplus s_n = r'_n \oplus s'_n$ and $r_n \odot s_n = r'_n \odot s'_n$. Therefore, $n \in [[r \oplus s = r' \oplus s']] \cap [[r \oplus s = r' \oplus s']]$. \square

Exercise 3.6. (Section 3.5, III) $[[r = r']] \cap [[s = s']] \cap [[r < s]] \subseteq [[r' < s']]$.

Proof. Let $n \in [[r = r']] \cap [[s = s']] \cap [[r < s]]$. Thus, $r_n = r'_n, s_n = s'_n, r_n < s_n$ but then that means $r'_n < s'_n$, and so $n \in [[r' < s']]$. \square

Exercise 3.7. (Section 3.5, IV) If $r \equiv r'$ and $s \equiv s'$, then $[[r < s]] \in \mathcal{F}$ iff $[[r' < s']] \in \mathcal{F}$.

Proof. Suppose $r \equiv r'$ and $s \equiv s'$, that is to say, $\{n \in \mathbb{N} : r_n = r'_n\}, \{n \in \mathbb{N} : s_n = s'_n\} \in \mathcal{F}$. First assume $[[r < s]] \in \mathcal{F}$. Therefore, $\{n \in \mathbb{N} : r_n = r'_n\} \cap \{n \in \mathbb{N} : s_n = s'_n\} \cap \{n \in \mathbb{N} : r_n < s_n\} \subseteq \{n \in \mathbb{N} : r'_n < s'_n\} \in mc\mathcal{F}$. Next assume $[[r' < s']] \in \mathcal{F}$. It follows $\{n \in \mathbb{N} : r_n = r'_n\} \cap \{n \in \mathbb{N} : s_n = s'_n\} \cap \{n \in \mathbb{N} : r'_n < s'_n\} \subseteq \{n \in \mathbb{N} : r_n < s_n\} \in \mathcal{F}$. \square

Exercise 3.8. (Section 3.8, I) Use only general properties of ordered fields to deduce from the fact that $[\varepsilon]$ is a positive infinitesimal, the conclusion that $[\varepsilon]^{-1}$ is greater than every real number.

Proof. Notice that for any $r \in \mathbb{R}^+$, we have $*r = [\langle r, r, \dots, \rangle] \in *\mathbb{R}$. Since $\varepsilon < *r$, by fields properties, we have $0 < *r^{-1} < \varepsilon^{-1}$. Since r is arbitrary positive real number, setting $r = r^{-1}$, we see that $*r < \varepsilon^{-1}$ for any $r \in \mathbb{R}^+$. \square

Exercise 3.9. (Section 3.10, I) If A is finite, show that $*A = A$, and hence A has non nonstandard members.

Proof. Let $A = \{a_1, \dots, a_n\}$. Consider some $[r] \in *A$. If we could prove that $r \equiv t$ for some constant sequence $t \in A$, we would be done. So we set that as our goal. Notice that since $[r] \in *A$, we have $\{n \in \mathbb{N} : r_n \in A\} = \{n \in \mathbb{N} : r_n = a_1 \vee r_n = a_2 \vee \dots \vee r_n = a_n\} = \{n \in \mathbb{N} : r_n = a_1\} \cup \dots \cup \{n \in \mathbb{N} : r_n = a_n\} \in \mathcal{F}$. But since \mathcal{F} is ultrafilter, only one of such set is in \mathcal{F} . Therefore, $\{n \in \mathbb{N} : r_n = a_k\} \in \mathcal{F}$ for some k such that $1 \leq k \leq n$. Hence, $r \equiv a_k$. \square

Exercise 3.10. (Section 3.10, II) Prove:

- (a) $A \subseteq B$ iff $*A \subseteq *B$.
- (b) $A = B$ iff $*A = *B$.

Proof.

(a) First we prove the forward implication. Suppose $A \subseteq B$. Consider $[r] \in *A$. Therefore, we have $[[r \in A]] \in \mathcal{F}$, but since $[[r \in A]] \subseteq [[r \in B]]$, we have $[[r \in B]] \in \mathcal{F}$ and so $[r] \in *B$. We prove the converse implication. Suppose $*A \subseteq *B$. Consider some $r \in A$. Then we have $[r] \in *A$, and so $[r] \in *B$. However, since $[r] \in B$, we have $[[r \in B]] \in \mathcal{F}$ which means $r \in B$.

(b) Trivially follows from (a). \square

Exercise 3.11. (Section 3.10, III)

Prove:

- (a) $*(A \cup B) = *A \cup *B$,
- (b) $*(A \cap B) = *A \cap *B$,
- (c) $*(A - B) = *A - *B$,
- (d) $*\emptyset = \emptyset$.

Proof.

(a) First we prove the forward implication. Since $A \subseteq A \cup B$, applying (2), we have $*A \subseteq *(A \cup B)$. Similarly, we have $*B \subseteq *(A \cup B)$, and so $*A \cup *B \subseteq *(A \cup B)$.

We prove the converse implication. Let $[r] \in *A \cup *B$. Then we have following cases;

case-1. $[r] \in *A$. Thus, $[[r \in A]] \in \mathcal{F}$, and since $[[r \in A]] \subseteq [[r \in A \cup B]]$, we have $[[r \in A \cup B]] \in \mathcal{F}$ but that means $[r] \in *(A \cup B)$.

case-2. $[r] \in *B$. The proof is analogous to **case-1**.

(b) First we prove the forward implication. Since $A \cap B \subseteq A$, applying (2), we have $*(A \cap B) \subseteq *A$. Similarly we have $*(A \cap B) \subseteq *B$, and so $*(A \cap B) \subseteq *A \cap *B$.

We prove the converse implication. Consider some $[r] \in *A \cap *B$. It means that $[r] \in *A$ and $[r] \in *B$, thus $[[r \in A]], [[r \in B]] \in \mathcal{F}$. Therefore, $[[r \in A \cap B]] = [[r \in A]] \cap [[r \in B]] \in \mathcal{F}$, and so $[r] \in *(A \cap B)$.

(c) First we prove the forward implication. Let $[r] \in *(A \setminus B)$. Then we have $[[r \in A]] \cap [[r \notin B]] = [[r \in A \setminus B]] \in \mathcal{F}$. Thus, $[[r \in A]], [[r \notin B]] \in \mathcal{F}$. Hence, $[r] \in *A$ and $[r] \notin *B$, and so $[r] \in *A \setminus *B$.

We prove the converse implication. Let $[r] \in *A \setminus *B$ i.e. $[r] \in *A$ and $[r] \notin *B$. Then we have $[[r \in A]] \in \mathcal{F}$ and $[[r \notin B]] \in \mathcal{F}$. Thus, $[[r \in A \setminus B]] = [[r \in A]] \cap [[r \notin B]] \in \mathcal{F}$. Therefore, $r \in *(A \setminus B)$.

(d) Clearly, $*\emptyset = \{[r] \in *\mathbb{R} : [[r \in \emptyset]] \in \mathcal{F}\} = \emptyset$. \square

Exercise 3.12. (Section 3.10, IV) Is it true that $*(\bigcup_{n=1}^{\infty} A_n) = \bigcup_{n=1}^{\infty} *A_n$?

Proof. No. Define $A_n = \{n\}$ for any $n \in \mathbb{N}$. Notice that since A_n is finite, using (1), we have $*A_n = A_n = \{n\}$. Since $[[\omega \in \bigcup_{n=1}^{\infty} A_n]] = \mathbb{N} \in \mathcal{F}$, we have $\omega \in *(\bigcup_{n=1}^{\infty} A_n)$. However, $[[\omega \in A_n]] = \{n\} \notin \mathcal{F}$, thus $\omega \notin *A_n$ for any $n \in \mathbb{N}$. This mean $\omega \notin \bigcup_{n=1}^{\infty} *A_n$. \square

Exercise 3.13. (Section 3.10, V) Show that if $A \subseteq \mathbb{R}$, then $*A \cap \mathbb{R} = A$.

Proof. Suppose $A \subseteq \mathbb{R}$. Let $[r] \in *A \cap \mathbb{R}$. Since $\langle r \rangle$ is a constant sequence and $[r] \in *A$, it follows $[r] = \langle r \rangle \in A$. Thus, $*A \cap \mathbb{R} \subseteq A$. This means that $*A \cap \mathbb{R}$ contains only the constant sequences. Let $\langle r \rangle \in A$. Then clearly $[r] \in *A \cap \mathbb{R}$. \square

Exercise 3.14. For $a, b \in \mathbb{R}$, let $[a, b]$ be the closed interval $\{x \in \mathbb{R} : a \leq x \leq b\}$. Prove that $*[a, b] = \{x \in *\mathbb{R} : a \leq x \leq b\}$.

Proof. We first prove the forward inclusion. Let $[r] \in *[a, b]$. Then we have $\{n \in \mathbb{N} : a \leq r_n \leq b\} = [[r \in [a, b]]] \in \mathcal{F}$ but that means $[r] \in \{x \in *\mathbb{R} : a \leq x \leq b\}$. We prove the converse inclusion. Let $[r] \in \{x \in *\mathbb{R} : a \leq x \leq b\}$. It follows $[[r \in [a, b]]] = \{n \in \mathbb{N} : a \leq r_n \leq b\} \in mc\mathcal{F}$. Therefore, $[r] \in \{x \in \mathbb{R} : a \leq x \leq b\}$. \square

Exercise 3.15. (Section 3.10, VII) $*\mathbb{Z}$ is a subring of $*\mathbb{R}$.

Proof. Clearly, $*\mathbb{Z} \neq \emptyset$. Consider some $[r], [s] \in *\mathbb{Z}$. Then we have $[[r \in \mathbb{Z}]], [[s \in \mathbb{Z}]] \in \mathcal{F}$. Therefore, since $[[r \in \mathbb{Z}]] \cap [[s \in \mathbb{Z}]] \subseteq [[r+s \in \mathbb{Z}]], [[rs \in \mathbb{Z}]]$, we have $[[r+s \in \mathbb{Z}]], [[rs \in \mathbb{Z}]] \in \mathcal{F}$. Hence, $[r+s], [rs] \in *\mathbb{Z}$. \square

Exercise 3.16. (Section 3.10, VIII) If $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$, show that $*(\mathbb{R}^+) = \{x \in *\mathbb{R}^+ : x > 0\}$, i.e., $*(\mathbb{R}^+) = (*\mathbb{R})^+$.

Proof. First we prove the forward inclusion. Let $[r] \in *(\mathbb{R}^+)$. It means that $\{n \in \mathbb{N} : r_n > 0\} = [[r \in \mathbb{R}^+]] \in \mathcal{F}$ which implies $[r] \in (*\mathbb{R})^+$. Next we prove the converse inclusion. Let $[r] \in (*\mathbb{R})^+$. Then we have $\{n \in \mathbb{N} : r_n > 0\} = [[r > 0]] = [[r \in \mathbb{R}^+]] \in \mathcal{F}$ from which it follows $[r] \in *(\mathbb{R}^+)$. \square

Exercise 3.17. (Section 3.12, I) Show that $*f$ agrees with f on \mathbb{R} : if $r \in \mathbb{R}$, then $*f(r) = f(r)$.

Proof. Let $r \in \mathbb{R}$. Notice that r can be indetified with the constant sequence $\mathbf{r} \in *R$. Then we have $*f([\mathbf{r}]) = [f(\mathbf{r})]$ which is isomorphic to $f(r)$. \square

Exercise 3.18. (Section 3.12, II) If f is injective, so is $*f$. What about surjectivity?

Proof. Suppose f is injective. Consider some $[r], [s] \in *\mathbb{R}$ such that $*f([r]) = *f([s])$, that is to say $[f \circ r] = [f \circ s]$. Since f is injective, we have $[[f \circ r = f \circ s]] \subseteq [[r = s]]$ but then, since $[[f \circ r = f \circ s]] \in \mathcal{F}$, it follows $[[r = s]] \in \mathcal{F}$. Therefore, $[r] = [s]$.

Yes, surjectivity is preserved. Suppose f is surjective. Consider some $[s] \in *\mathbb{R}$. Our goal is to prove that there is some $[r] \in *\mathbb{R}$ such that $*f([r]) = [s]$. Define sequence (r_n) such that, for each s_n , $f(r_n) = s_n$. We know such a r_n exists for each s_n since f is surjective. We claim that $[r] \in *\mathbb{R}$ the element we looking for. To prove this claim, we need to show $[f \circ r] = f([r]) = [s]$, so we set that as our goal. Clearly, $[[f \circ r = s]] = \{n \in \mathbb{N} : f(r_n) = s_n\} = \mathbb{N} \in \mathcal{F}$. Therefore, $[f \circ r] = f[r] = [s]$. \square

Exercise 3.19. (Section 3.12, III) For $x \in *\mathbb{R}$, let

$$|x| = \begin{cases} x, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$$

be the usual definition of absolute value function. Show that this extends the definition of $|\cdot|$ on \mathbb{R} : $||[r]| = [|r_1|, |r_2|, \dots]\rangle = [r_n]$.

Proof. We have the following cases;

Case-1. Suppose $[r] > 0$. Then $[[r > 0]] \in \mathcal{F}$. Since $[[r > 0]] \subseteq [[[r] = [|r|]]] \in \mathcal{F}$, we have $[[r] = [r]] = [r] = [|r_n|]$.

Case-2. Suppose $[r] = 0$. That is to say $[[r = 0]] \in \mathcal{F}$. Since $[[r = 0]] \subseteq [[0 = |r|]] \in \mathcal{F}$, we have $[[r] = 0 = [|r|]]$.

Case-3. Suppose $[r] < 0$. Then $[[r]] = -[r] = [-r] = [|r|]$. \square

Exercise 3.20. (Section 3.12, IV) Let χ_A be the characteristic function of a set $A \subseteq \mathbb{R}$. Show that $^*(\chi_A) = \chi_{^*A}$.

Proof. Let $[r] \in ^*\mathbb{R}$. Consider the expression $^*(\chi_A)[r]$, we have the following cases;

Case-1. Suppose $[r] \in ^*A$. Notice that $^*(\chi_A)[r] = [\chi_A \circ r] = [\chi_A(r_n)]$ but then, since $[[r \in A]] \in \mathcal{F}$ and $[[r \in A]] \subseteq [[\chi_A \circ r = 1]]$, we have $[[\chi_A \circ r = 1]] \in \mathcal{F}$. Therefore, $[\chi_A \circ r] = \chi_A[r] = 1$.

Case-2. Suppose $[r] \notin ^*A$. Then we have $[[r \notin A]] = [[r \in A]]^c \in \mathcal{F}$. Clearly, $[[r \notin A]] \subseteq [[\chi_A \circ r = 0]]$. Thus, $[[\chi_A \circ r = 0]] \in \mathcal{F}$ and so $[\chi_A \circ r] = \chi_A[r] = 0$. \square

Exercise 3.21. (Section 3.12, V) Show how to define *f when f is a function of more than one argument.

Proof. Define $^*f([r^1], \dots, [r^n]) = [f(r_1^1, \dots, r_1^n), f(r_2^1, \dots, r_2^n), \dots]$. We prove that it's well defined. Clearly, $[[r^1 = r^{1'}]] \cap \dots \cap [[r^n = r^{n'}]] \subseteq [[f \circ (r^1, \dots, r^n) = f \circ (r^{1'}, \dots, r^{n'})]]$. Therefore, $r^1 \equiv r^{1'}, \dots, r^n \equiv r^{n'}$ implies $f \circ (r^1, \dots, r^n) \equiv f \circ (r^{1'}, \dots, r^{n'})$. \square

Exercise 3.22. (Section 3.15, I) If A_1, \dots, A_k are subsets of \mathbb{R} , put $P = A_1 \times \dots \times A_k$ and apply the definition of *P to show that

$$^*(A_1 \times \dots \times A_k) = ^*A_1 \times \dots \times ^*A_k.$$

Proof. For $([r^1], [r^2], \dots, [r^k]) \in ^*P$, we must have $[[r^1, r^2, \dots, r^k] \in P] \in \mathcal{F}$. Clearly, $[[r^1, r^2, \dots, r^k] \in P] \subseteq [[r^n \in A_n]]$, thus $[[r^n \in A_n]]$ for any n such that $1 \leq n \leq k$. This means that $[r^n] \in A_n$ for all such n . Hence, $([r^1], \dots, [r^n]) \in ^*A_1 \times \dots \times ^*A_n$. From this, it follows that $^*(\mathbb{R}^k) = (^*\mathbb{R})^k$. \square

Exercise 3.23. (Section 3.15, II) Let $\text{dom } P$ denote the domain of a binary relation. If $P \subseteq \mathbb{R}^2$, show that $^*(\text{dom } P) = \text{dom } ^*P$.

Proof. First we prove the forward inclusion. Let $[r] \in ^*(\text{dom } P)$. Then we have $\{n \in \mathbb{N} : r_n \in \text{dom } P\} = [[r \in \text{dom } P]] \in \mathcal{F}$.

$$\text{Define sequence } s_n = \begin{cases} s \text{ such that } (r_n, s) \in P, & \text{if } r_n \in \text{dom } P \\ 0, & \text{if } r_n \notin \text{dom } P \end{cases}$$

Then clearly, $[[r, s] \in P] = [[r \in \text{dom } P]] \in \mathcal{F}$, but that means $([r], [s]) \in ^*P$, and so $[r] \in \text{dom } ^*P$.

Now we prove the converse inclusion. Let $[r] \in \text{dom } ^*P$. It follows that there exists $[s] \in ^*\mathbb{R}$ such that $([r], [s]) \in ^*P$, but that means $[[r \in \text{dom } P]] = [[(r, s) \in P]] \in \mathcal{F}$. Hence, $[r] \in ^*(\text{dom } P)$. \square

Chapter 4

The Transfer Principle

Exercise 4.1. (Section 4.1, I) Verify the truth the $*$ -transform given in 1 – 5 above, which are;

- (1) *The Eudoxus-Archimedes Principle:* $\forall x \in {}^*\mathbb{R} : \exists m \in {}^*\mathbb{N}(x < m)$.
- (2) *Density of the rationals:* $\forall x, y \in {}^*\mathbb{R}(x < y \text{ implies } \exists q \in {}^*\mathbb{Q}(x < q < y))$.
- (3) *Finiteness.* Let $A = \{r_1, \dots, r_k\}$ be a finite subset of \mathbb{R} . Then $\forall x \in {}^*A(x = {}^*r_1 \vee x = {}^*r_2 \vee \dots \vee x = {}^*r_k)$.
- (4) *Finitary set operations.* $\forall x \in {}^*\mathbb{R}(x \in {}^*(A \cup B))$ iff $x \in {}^*A$ or $x \in {}^*B$.
- (5) *Discreteness of \mathbb{N} .* $\forall x \in {}^*\mathbb{N}({}^*n \leq x \leq {}^*(n+1) \text{ implies } x = {}^*n \text{ or } x = {}^*(n+1))$.

Proof.

- (1) Let $x \in {}^*\mathbb{R}$. Define sequence m such that $x_i < m_i$ for every i . The existence of such m is guaranteed by Eudoxus-Archimedes Principle in reals. Clearly, $m \in {}^*\mathbb{N}$ and $[[x < m]] \in \mathcal{F}$. Therefore, $x < m$.
- (2) Let $x, y \in {}^*\mathbb{R}$. Define sequence q such that $x_i < q_i < y_i$ for each i , where q_i 's are rational. The existence of such q_i 's is guaranteed by density of rationals in reals. Clearly, $q \in {}^*\mathbb{Q}$ and $[[x < q < y]] = \mathbb{N} \in \mathcal{F}$. Therefore, $x < q < y$.
- (3) Let $x \in {}^*A$. Then $[[x = r_1]] \cup [[x = r_2]] \cup \dots \cup [[x = r_n]] = [[x \in A]] \in \mathcal{F}$. Since finite union in \mathcal{F} means that exactly one of the set in \mathcal{F} , we have $x = {}^*r_1$ or $x = {}^*r_2$ or \dots or $x = {}^*r_n$.
- (4) Proved in section 3.10, II.
- (5) Let $x \in {}^*\mathbb{N}$ and suppose ${}^*n \leq x \leq {}^*(n+1)$. Then we have $[[n \leq x \leq n+1]] \in \mathcal{F}$ but due to discreteness of \mathbb{N} , it follows that $[[x = n]] \cup [[x = n+1]] = [[x = n \vee x = n+1]] = [[n \leq x \leq n+1]] \in \mathcal{F}$. Hence, $x = {}^*n$ or $x = {}^*(n+1)$. \square

Chapter 5

Hypereals Great and Small

Exercise 5.1. (Section 5.2, I) For any positive hyperreal a , explain why the function $x \rightarrow a^x$ is defined for all $x \in {}^*\mathbb{R}$. Use transfer to explore its properties.

Proof. □

Exercise 5.2. (Section 5.5, I) Verify that \simeq and \sim are equivalence relationships.

Proof. We will first argue about \simeq . Let $x, y, z \in {}^*\mathbb{R}$. Clearly, it's reflexive since $x - x = 0$. If $x \simeq y$ then we have $x - y = \epsilon$, thus $y - x = -\epsilon$, and so $y \simeq x$. If $x \simeq y$ and $y \simeq z$ then we have $x - y = \epsilon$ and $y - z = \delta$, adding the two get us $x \simeq z$. The argument of \sim is similar, so we omit the details. □

Exercise 5.3. (Section 5.5, II) If $b \simeq x \leq y \simeq c$ with b and c real, show that $b \leq c$. What if b and/or c are not real?

Proof. First we will prove an useful lemma, if $y > 0$ and $y \simeq k$ then $k > 0$. Let $k = y + \epsilon$. In the case where $\epsilon = 0$ and $\epsilon > 0$, it follows trivially. So assume $\epsilon < 0$. Thus, $y - \frac{1}{n} < y + \epsilon = k$ for all $n \in \mathbb{N}$. Since $\frac{y}{2}$ a real number, we know that there exists $m \in \mathbb{N}$ such that $\frac{1}{m} < \frac{y}{2}$ which implies $0 < \frac{y}{2} = y - \frac{y}{2} < y - \frac{1}{m} < k$. This proves the lemma.

Now assume to the contrary that $c < b$. Thus, $0 < c - b$ but it implies that $0 < x - y$, since $c - b \simeq x - y$, which is a contradiction.

The result would not be true if b is not real. Consider $\epsilon \simeq 0 \leq 0 \simeq 0$ for some positive infinitesimal ϵ , this would imply $\epsilon \leq 0$ which is false. □

Exercise 5.4. (Section 5.5, III) $\text{hal}(b) = \{b + \epsilon : \epsilon \in \text{hal}(0)\}$.

Proof. The converse inclusion is trivial. For forward, notice that $x \in \text{hal}(b)$ means that $x - b = \epsilon$, and so $x = b + \epsilon$. □

Exercise 5.5. (Section 5.5, IV) $\text{gal}(b) = \{b + c : c \in \text{gal}(0)\}$.

Proof. Again, the converse implication is trivial. For forward, we see that $x \in \text{gal}(b)$ means $x - b = c$ where c is a limited number, and so $c \in \text{gal}(0)$. Thus, $x = b + c$ for some $c \in \text{gal}(0)$. □

Exercise 5.6. (Section 5.5, V) If $x \simeq y$ and b is limited, prove that $b \cdot x \simeq b \cdot y$. Show that the result can fail for unlimited b .

Proof. Since $x \simeq y$, we have $x - y = \epsilon$ for some $\epsilon \in \text{hal}(0)$. Multiplying by b on both sides get us, $bx - by = b \cdot \epsilon$. It follows that $b \cdot \epsilon \in \text{hal}(0)$, so $bx \simeq by$. Consider $b = \frac{1}{\epsilon}$ then $b \cdot \epsilon = 1$ which is not infinitesimal. □

Exercise 5.7. (Section 5.5, VI) Show that any galaxy contains members of ${}^*\mathbb{Z}$, of ${}^*\mathbb{Q} - {}^*\mathbb{Z}$, and of ${}^*\mathbb{R} - {}^*\mathbb{Q}$.

Proof. Let $b \in {}^*\mathbb{R}$. Clearly, $b - a$ for $a \in {}^*\mathbb{Z}$, such that a is intger, is limited. □

Exercise 5.8. (Section 5.7, I) Prove theorem 5.6.2:

- (a) $sh(b \pm c) = sh(b) \pm sh(c)$.
- (b) $sh(bc) = sh(b) \cdot sh(c)$.
- (c) $sh(b/c) = sh(b)/sh(c)$ if $sh(c) \neq 0$ (ie c is appreciable).
- (d) $sh(b^n) = sh(b)^n$.
- (e) $sh(|b|) = |sh(b)|$.
- (f) $sh(\sqrt[n]{b}) = \sqrt[n]{sh(b)}$, if $b \geq 0$.
- (g) If $b \leq c$ then $sh(b) \leq sh(c)$.

Proof. Consider $b = sh(b) + \varepsilon$ and $c = sh(c) + \delta$ for some infinitesimal ε and δ .

(a) For $sh(b + c)$, notice that $b + c - (sh(b) + sh(c)) = \varepsilon + \delta$. Thus, $sh(b + c) \simeq b + c \equiv sh(b) + sh(c)$. From the uniqueness of shadow, it follows $sh(b + c) = sh(b) + sh(c)$. We get identical result in case of $sh(b - c)$ by considering $b - c - (sh(b) - sh(c)) = \varepsilon - \delta$.

(b) Notice that $bc = sh(b)sh(c) + sh(b)\delta + sh(c)\varepsilon + \varepsilon\delta$ where $sh(b)\delta, sh(c)\varepsilon, \varepsilon\delta$ are infinitesimals. So, $sh(bc) \simeq bc \simeq sh(b)sh(c)$, and thus by uniqueness of shadow we have $sh(bc) = sh(b)sh(c)$.

(c) Notice that, applying (b), we have $sh(b/c) = sh(b) \cdot sh(1/c)$. So if we could prove $sh(1/c) = 1/sh(c)$, we would be done. Therefore, we set that as our goal. Clearly, $\frac{1}{c} - \frac{1}{sh(c)} = \frac{1}{sh(c)+\delta} - \frac{1}{sh(c)} = \frac{\delta}{sh(c)^2+sh(c)\delta}$. Since $sh(c)^2 + sh(c)\delta$ is appreciable it follows that $\frac{\delta}{sh(c)^2+sh(c)\delta}$ is infinitesimal, and thus $sh(1/c) \simeq 1/c \simeq 1/sh(c)$. Hence, $sh(1/c) = 1/sh(c)$ by uniqueness of shadow.

(d) Trivially follows from induction on (b).

(e) If we could prove that $|b| \simeq |sh(b)|$, we would be done since from uniqueness of shadow, $sh|b| = |sh(b)|$ would hold. Notice that $|b| - |sh(b)| \leq |sh(b)| + |e| - |sh(b)| = |e|$. If $0 \leq |b| - |sh(b)|$ then clearly $|b| - |sh(b)| \leq |e| < 1/n$ for all $n \in \mathbb{N}$. If $|b| - |sh(b)| < 0$ then $-1/n < -|e| \leq |b| - |sh(b)|$ for all $n \in \mathbb{N}$. Thus, $||b| - |sh(b)|| < 1/n$ for every natural n . Hence, $|b| - |sh(b)|$ is infinitesimal as desired.

(f) If $sh(b^{\frac{1}{n}}) < (sh(b))^{\frac{1}{n}}$ then by squaring both side and applying part (d), we would get $sh(b) = sh(b^{1/n \cdot n}) = (sh(b^{1/n}))^n < (sh(b))^{1/n \cdot n} = sh(b)$ which is a contradiction. We would get identical contradiction in case of $(sh(b))^{\frac{1}{n}} < sh(b^{\frac{1}{n}})$.

(g) It follows from 2nd exercise of section 5.5. □

Exercise 5.9. (Section 5.7, I) Show that if b, c are limited, and $b \simeq b', c \simeq c'$, then $b \pm c \simeq b' \pm c', b \cdot c \simeq b' \cdot c'$, and $b/c \simeq b'/c'$ if $c \not\simeq 0$. Show that the last result can fail when $c \simeq 0$.

Proof. Let $b = b' + \varepsilon$ and $c = c' + \delta$ for some infinitesimal ε and δ . Then $b \pm c - (b' \pm c') = \varepsilon \pm \delta$ which is known to be infinitesimal. Thus, $b \pm c \simeq b' \pm c'$. For multiplication we have $bc - b'c' = b'c' + b'\delta + c'\varepsilon + \varepsilon\delta - b'c' = b'\delta + c'\varepsilon + \varepsilon\delta$ which is infinitesimal. Finally for quotient, we have $\frac{b}{c} - \frac{b'}{c'} = \frac{b'+\varepsilon}{c'+\delta} - \frac{b'}{c'} = \frac{b'c'+c'\varepsilon-b'c'-b'\delta}{(c')^2+c'\delta} = \frac{c'\varepsilon+b'\delta}{(c')^2+c'\delta}$. Since $(c')^2 + c'\delta$ is appreciable, we have $\frac{b}{c} - \frac{b'}{c'}$ to be infinitesimal. Hence, $b/c \simeq b'/c'$. For proving the result does not hold for $c \simeq 0$. Consider $b = 0$. Then we have $b/c - b'/c' = 0 - \frac{\varepsilon}{\varepsilon - \delta}$ which is not infinitesimal necessarily. □

Exercise 5.10. (Section 5.7, II)

If ε is infinitesimal, show that

- (a) $\sin \varepsilon \simeq 0$,
- (b) $\cos \varepsilon \simeq 1$,
- (c) $\tan \varepsilon \simeq 0$,
- (d) $\sin \varepsilon / \varepsilon \simeq 1$,
- (e) $(\sin \varepsilon) / \varepsilon \simeq 1$,
- (f) $(\cos \varepsilon - 1) / \varepsilon \simeq 0$.

Proof.

(a) Since $\forall x \in \mathbb{R} (|\sin x| \leq |x|)$ holds in \mathbb{R} , applying transfer on it, it must be the case that $\forall x \in {}^*\mathbb{R} (|\sin x| \leq$

$|x|)$ holds in ${}^*\mathbb{R}$. Therefore, $|\sin \varepsilon| \leq |\varepsilon|$, and so $\sin \varepsilon$ is infinitesimal. Hence, $\sin \varepsilon \simeq 0$.

(b) Since $\cos \varepsilon = \sqrt{1 - \sin^2 \varepsilon}$, we have $\text{sh}(\cos \varepsilon) = \text{sh}\sqrt{1 - \sin^2 \varepsilon} = \sqrt{(\text{sh}(1) - \text{sh}(\sin^2 \varepsilon))} = \sqrt{1 - \text{sh}(\delta^2)} = \sqrt{1 - 0} = 1$. Clearly, $\cos \varepsilon \simeq \text{sh}(\cos \varepsilon) = 1$.

(c) $\text{sh}(\tan \varepsilon) = \text{sh}(\sin \varepsilon)/\text{sh}(\cos \varepsilon) = 0/1 = 0$. Hence, $\tan \varepsilon \simeq 0$.

(d) Notice that $|\sin \varepsilon| \leq |\varepsilon|$, so $|\sin(\varepsilon)/\varepsilon| \leq 1$. Thus, $\text{sh}(|\frac{\sin \varepsilon}{\varepsilon}|) \leq 1$. We will now show that $\text{sh}(|\frac{\sin \varepsilon}{\varepsilon}|) < 1$ leads to contradiction which would get us $\text{sh}(|\frac{\sin \varepsilon}{\varepsilon}|) = 1$. Clearly if $\text{sh}(|\frac{\sin \varepsilon}{\varepsilon}|) < 1$ then $0 = \text{sh}(|\sin \varepsilon|) - \text{sh}|\varepsilon| < 0$ which is false. Therefore, $(\sin \varepsilon)/\varepsilon \simeq \text{sh}(|\frac{\sin \varepsilon}{\varepsilon}|) = 1$.

(e) By the sine inequality, $|\sin \varepsilon|/|\varepsilon| \leq 1$. By the tangent inequality, $|x| \leq |\tan(x)|$ for $x \in [0, \pi/2)$, we have $|\cos x| \leq |\sin x|/|x|$. Combining the two inequalities and taking shadow we see $1 = \text{sh}(|\cos \varepsilon|) \leq \text{sh}(|\sin(\varepsilon)/\varepsilon|) \leq 1$. Therefore, $\text{sh}(|\sin(\varepsilon)/\varepsilon|) = 1$. Hence, $(\sin \varepsilon)/\varepsilon \simeq 1$.

(f) By cosine inequality, $1 - \frac{\varepsilon^2}{2} \leq \cos \varepsilon$, we have $-\varepsilon/2 \leq (\cos \varepsilon - 1)/\varepsilon$. Since $\cos \varepsilon \leq 1$, we have $(\cos \varepsilon - 1)/\varepsilon \leq 0/\varepsilon = 0$. Hence $|(\cos \varepsilon - 1)/\varepsilon| \leq \varepsilon/2$ and so $(\cos \varepsilon - 1)/\varepsilon$ is infinitesimal, which makes it infinitely close to 0. \square

Exercise 5.11. (Section 5.7, III) Show that every hyperreal is infinitely close to some hyperrational number.

Proof. Since $\forall x, y \in \mathbb{R} (x < y \text{ implies } \exists q \in \mathbb{Q} (x < q < y))$ holds in reals (density of rationals), by transfer, we have $\forall x, y \in {}^*\mathbb{R} (x < y \text{ implies } \exists q \in {}^*\mathbb{Q} (x < q < y))$. In particular there is a hyperrational q such that $r < q < r + \varepsilon$ for any hyperreal r . Clearly such a q is infinitely close to r . \square

Exercise 5.12. (Section 5.7, IV) Show that \mathbb{R} is isomorphic to the ring of limited hyperrationals ${}^*\mathbb{Q} \cap \mathbb{L}$ factored by its ideal ${}^*\mathbb{Q} \cap \mathbb{I}$ of hyperreal infinitesimal.

Proof.

$$\begin{array}{ccc}
 \bullet {}^*\mathbb{Q} \cap \mathbb{L} & \xrightarrow{\text{sh}} & \bullet \mathbb{R} \\
 \searrow \text{canonical projection} & \nearrow f(\text{hal}(b) + ({}^*\mathbb{Q} \cap \mathbb{L})/({}^*\mathbb{Q} \cap \mathbb{I})) = \text{sh}(b) & \\
 & \bullet ({}^*\mathbb{Q} \cap \mathbb{L})/({}^*\mathbb{Q} \cap \mathbb{I}) &
 \end{array}$$

Consider shadow function restricted to ${}^*\mathbb{Q} \cap \mathbb{L}$, in what follows we would use sh notation for it instead of $\text{sh}_{{}^*\mathbb{Q} \cap \mathbb{L}}$. We will first justify it's surjective. Since $\mathbb{R} \subseteq {}^*\mathbb{R}$, applying previous exercise, we see that for any $x \in \mathbb{R}$, there is $q \in {}^*\mathbb{Q}$ such that $x \simeq q$ but then it would mean $x \simeq \text{sh}(q)$. Since both x and $\text{sh}(q)$ are reals, it must be the case that $x = \text{sh}(q)$. Hence, $\text{sh}(q) = x$ as desired. Notice that for any $[r] \in ({}^*\mathbb{Q} \cap \mathbb{L})/({}^*\mathbb{Q} \cap \mathbb{I})$, we have $[r] = r + {}^*\mathbb{Q} \cap \mathbb{I}$ which is nothing but $\text{hal}(r)$. Define $f : ({}^*\mathbb{Q} \cap \mathbb{L})/({}^*\mathbb{Q} \cap \mathbb{I}) \rightarrow \mathbb{R}$ as $\text{hal}(b) \mapsto \text{sh}(b)$. By the first isomorphism theorem of rings, it follows $({}^*\mathbb{Q} \cap \mathbb{L})/({}^*\mathbb{Q} \cap \mathbb{I}) \simeq \mathbb{R}$. \square

Exercise 5.13. (Section 5.9, Exercise on Dedekind completeness) For theorem 5.8.1, instead of showing that cauchy sequences converge we can develop a direct proof that any subset $A \subseteq \mathbb{R}$ with a real upper bound has a least real upper bound. First, for each $n \in \mathbb{N}$, let s_n be the least $k \in \mathbb{Z}$ such that k/n is an upper bound of A . Then take an unlimited $N \in {}^*\mathbb{N}_\infty$ and let $L \in \mathbb{R}$ be infinitely close to s_N/N .

(a) Verify that s_n exists as defined for $n \in \mathbb{N}$.

(b) Show the s_N/N is limited, so that such a real L exists under hypothesis of theorem 5.8.1.

(c) Prove that L is a least upper bound of A in \mathbb{R} .

Proof. Let r be an upper bound of $A \subseteq \mathbb{R}$.

(a) Let $n \in \mathbb{N}$. Consider the set $B = \{m \in \mathbb{N} : m/n \text{ is an upper bound of } A\}$. Since $r \in \mathbb{R}$, by archimedean property, we know that there exists $l \in \mathbb{N}$ such that $r < l$. Clearly $l \cdot n \in B$, and so B is non empty. By the well ordering principle, it follows that there exists a least element in B , or in other words there exists a least $k \in \mathbb{N}$ such that k/n is an upperbound of A for each n .

(b) Let $n \in \mathbb{N}$. Since s_n is the least integer such that s_n/n is upperbound of A , clearly $(s_n - 1)/n$ is not an upper bound. But that means there exists some $a \in A$ such that $(s_n - 1)/n \leq a \leq s_n/n$. Applying transfer on the statement $(\forall n \in \mathbb{N})(\exists a \in A)((s_n - 1)/n \leq a \leq s_n/n)$, we get $(\forall n \in {}^*\mathbb{N})(\exists a \in {}^*A)((s_n - 1)/n \leq a \leq s_n/n)$. In particular, we have $(s_N - 1)/N \leq a \leq s_N/N$. However since s_N/N and $(s_N - 1)/N$ are infinitely close, it must be the case that a is infinitely close to s_N/N . Applying transfer on the statement $(\forall a \in A)(a \leq r)$, where r is the real upper bound we are given, we get $(\forall a \in {}^*A)(a \leq r)$. Thus, $s_N/N \leq r$. Next we will prove that is not negative unlimited either. Since $A \subseteq {}^*A$ and s_N/N is an upper bound of *A , we have some $x \in A$ such that $x \leq s_N/N$. Hence, $x \leq s_N/N \leq r$ for some real r, s .

(c) For the sake of contradiction assume there is an upper bound, K , of A such that $K < L$. Notice that there does not exist any $a \in A$ such that $a \in [K, L]$ since otherwise K would not be an upper bound. Applying transfer on $(\forall a \in A)(a \notin [K, L])$, we get $(\forall a \in {}^*A)(a \notin {}^*[K, L])$ which is false since in previous paragraph we demonstrated that there is an element of *A which is infinitely close to L . \square

Exercise 5.14. (Section 5.11, I) Provide an analogous description of the order structure of the hyperintegers ${}^*\mathbb{Z}$.

Proof. By transfer, we know that ${}^*\mathbb{Z}$ is closed under addition, subtraction and multiplication. Observe that only limited hyperinteger are members of \mathbb{Z} since if $k \in {}^*\mathbb{Z}$ is limited then $|k| \leq n$ for some $n \in \mathbb{N}$ but that means $-n \leq k \leq n$. Applying transfer on $\forall x \in \mathbb{Z}(-n \leq x \leq n \rightarrow x = -n \vee x = -n+1 \vee \dots \vee x = 1 \vee \dots \vee x = n)$, thus $k \in \{-n, -n+1, \dots, n\}$, and so $k \in \mathbb{Z}$. Therefore, all numbers of ${}^*\mathbb{Z} - \mathbb{Z}$ are unlimited and thus either less than all members of \mathbb{Z} or greater than it. Let $K \in {}^*\mathbb{Z} - \mathbb{Z}$. Define $\gamma(K) = \{K\} \cup \{K \pm n : n \in \mathbb{N}\}$, clearly all members of $\gamma(K)$ are unlimited and $\gamma(K)$ is isomorphic to \mathbb{Z} under the usual ordering. Alternatively, we can define $\gamma(K) = \{H \in {}^*\mathbb{Z} : K \sim H\} = \text{gal}(K) \cap {}^*\mathbb{Z}$. Notice that \mathbb{Z} itself is a ${}^*\mathbb{Z}$ galaxy, $\mathbb{Z} = \text{gal}(0) \cap {}^*\mathbb{Z}$.

We define order on ${}^*\mathbb{Z}$ galaxy as $\gamma(Y) < \gamma(H)$ iff $Y \leq H$ whenever $K \not\sim H$. Clearly there is no greatest ${}^*\mathbb{Z}$ galaxy since $\gamma(K) < \gamma(2K)$ for any positive unlimited K . Similarly there is no smallest galaxy either. To see why notice that either K is even or $K + 1$ is, by transfer. If $K + 1$ is even, we have $\gamma(K + 1/2) < \gamma(K + 1) = \gamma(K)$. If K is even then we have $\gamma(K) < \gamma(K)$.

We can imagine ordering on ${}^*\mathbb{Z}$ as ordering of \mathbb{Z} surrounded by ordered set of ${}^*\mathbb{Z}$ galaxies with no greatest or least. \square

Exercise 5.15. (Section 5.11, II) Show that for any $M \in {}^*\mathbb{N}$ there is an $N \in {}^*\mathbb{N}$ that is divisible by all members of $\{1, 2, \dots, M\}$. Hence show that there exists a hypernatural number N that is divisible by every standard positive integer.

Proof. We extend the factorial function $! : \mathbb{N} \rightarrow \mathbb{N}$ to $! : {}^*\mathbb{N} \rightarrow {}^*\mathbb{N}$ using extension of function. Applying transfer on the statement $(\forall n \in \mathbb{N})(k \leq n \rightarrow k | n!)$, we get $(\forall n \in {}^*\mathbb{N})(k \leq n \rightarrow k | n!)$. Thus, letting $N = M!$ does the job. \square

Exercise 5.16. (Section 5.11, III) Develop a theory of prime factors in ${}^*\mathbb{N}$: if Π is the set of standard prime numbers, with enlargement ${}^*\Pi \subseteq {}^*\mathbb{N}$, prove the following.

- (a) ${}^*\Pi$ consists precisely of those hypernaturals > 1 that have no nontrivial factors in ${}^*\mathbb{N}$.
- (b) Every hypernatural number > 1 has a hyperprime factor, i.e., is divisible by some members of ${}^*\Pi$.
- (c) Two hypernaturals are equal if they have exactly the same factors of the form p^n with $p \in {}^*\Pi$ and $n \in {}^*\mathbb{N}$.
- (d) A hypernatural number is divisible by every standard positive integer iff it is divisible by p^n for every standard prime p and every $n \in \mathbb{N}$.

Proof.

- (a) By transfer on $(\forall x \in \Pi)(x > 1 \rightarrow x \text{ have no nontrivial factors in } \mathbb{N})$.
- (b) By transfer on $(\forall n \in \mathbb{N})(n > 1 \rightarrow \exists p \in \Pi)(p | n)$.
- (c) By transfer on $(\forall x \in \mathbb{N})(\forall y \in \mathbb{N})(x = y \leftrightarrow \text{they have exactly the same factors of the form } p^n \text{ with } p \in \Pi \text{ and } n \in \mathbb{N})$.
- (d) Trivial. \square

Chapter 6

Convergence of Sequences and Series

Exercise 6.1. (Section 6.3, I) Prove theorem 6.3.1. If $\lim_{n \rightarrow \infty} s_n = L$ and $\lim_{n \rightarrow \infty} t_n = M$ in \mathbb{R} . Then

- (a) $\lim_{n \rightarrow \infty} (s_n + t_n) = L + M$.
- (b) $\lim_{n \rightarrow \infty} (cs_n) = cL$, for any $c \in \mathbb{R}$,
- (c) $\lim_{n \rightarrow \infty} (s_n t_n) = LM$,
- (d) $\lim_{n \rightarrow \infty} (s_n/t_n) = L/M$, if $M \neq 0$.

Proof. For what follows consider $N \in {}^*\mathbb{N}_\infty$. It means that $s_N \simeq L$ and $t_N \simeq M$.

(a) Since $s_N \simeq L$ and $t_N \simeq M$, it follows $s_N + t_N \simeq L + M$. Since N was arbitrary, we have $s_n + t_n \simeq L + M$ for all unlimited n . Hence, $\lim_{n \rightarrow \infty} (s_n + t_n) = L + M$.

(b) Since $s_N \simeq L$, we have $cs_N \simeq cL$. Since N was arbitrary, $cs_n \simeq cL$ for all unlimited n . Hence, $\lim_{n \rightarrow \infty} cs_n = cL$.

(c) Since $s_N \simeq L$ and $t_N \simeq M$, we have $s_N t_N \simeq LM$ but since N was arbitrary, it follows $s_n t_n \simeq LM$ for all unlimited n . Hence, $\lim_{n \rightarrow \infty} (s_n t_n) = LM$.

(d) Again, from $s_N \simeq L$ and $t_N \simeq M$, we have $s_N/t_N \simeq L/M$ but since N was arbitrary, $\lim_{n \rightarrow \infty} s_n/t_n = L/M$. \square

Exercise 6.2. (Section 6.4, I) Prove theorem 6.4.2. A real valued sequence

- (a) diverges to infinity iff all of its extended terms are positive unlimited; and
- (b) diverges to negative infinity iff all of its extended terms are negative unlimited.

Proof. Let the sequence be (s_n) .

(a) First we prove forward direction. Let $r \in \mathbb{R}$. Since the sequence diverges, we know that there is some $n \in \mathbb{N}$ such that $(\forall m \in \mathbb{N})(m \geq n \rightarrow s_m > r)$ holds true. Applying transfer on it, we have $(\forall m \in {}^*\mathbb{N})(m \geq n \rightarrow s_m > r)$ to be true. In particular, $s_N > r$ for any $N \in {}^*\mathbb{N}_\infty$ since $N > n$. It follows that s_N is positive unlimited for all $N \in {}^*\mathbb{N}_\infty$. We prove the converse direction. Let $r \in \mathbb{R}$. Notice that for any unlimited $N \in {}^*\mathbb{N}_\infty$ if $M \geq N$ then $M \in {}^*\mathbb{N}_\infty$. Therefore, it follows that $(\exists n \in {}^*\mathbb{N})(\forall m \in {}^*\mathbb{N})(m \geq n \rightarrow s_m > r)$ since s_m is positive unlimited by assumption. Transferring the statement to \mathbb{R} , we have $(\exists n \in \mathbb{N})(\forall m \in \mathbb{N})(m \geq n \rightarrow s_m > r)$ to be true in \mathbb{R} . Since r was arbitrary, we have (s_n) to be divergent to positive infinity.

(b) Follows from (a) by considering the sequence $(-s_n)$. \square

Exercise 6.3. (Section 6.5, I) Prove theorem 6.5.1. A real values sequence $\langle s_n \rangle$ is Cauchy in \mathbb{R} iff all it's extended terms are infinitely close to each other, i.e., iff $s_m \simeq s_n$ for all $m, n \in {}^*\mathbb{N}_\infty$.

Proof. First we prove the forward implication. Suppose $\langle s_n \rangle$ is cauchy. Let $\varepsilon \in \mathbb{R}^+$ and $j \in \mathbb{N}$ be such that $(\forall m, n \in \mathbb{N})(m, n \geq j \rightarrow |s_m - s_n| < \varepsilon)$. Applying transfer on the latter we know $(\forall m, n \in {}^*\mathbb{N})(m, n \geq j \rightarrow |s_m - s_n| < \varepsilon)$ is true in ${}^*\mathbb{R}$. In particular, we have $|s_M - s_N| < \varepsilon$ for arbitrary $M, N \in {}^*\mathbb{N}_\infty$ since j is limited. Since ε was arbitrary positive real number, we have $s_M \simeq s_N$. We now prove the converse direction. Suppose $s_m \simeq s_n$ for all $m, n \in {}^*\mathbb{N}_\infty$. Let $\varepsilon \in \mathbb{R}^+$ be arbitrary. Notice that for any $K \in {}^*\mathbb{N}_\infty$, if $L \geq N$ then $L \in {}^*\mathbb{N}_\infty$, and also notice for any $L, K \in {}^*\mathbb{N}_\infty$, $s_L - s_K = \delta$ where δ is infinitesimal. Therefore, we have $(\exists j \in {}^*\mathbb{N})(\forall m, n \in {}^*\mathbb{N})(m, n \geq j \rightarrow |s_m - s_n| \leq \varepsilon)$ to be true in ${}^*\mathbb{R}$. Applying transfer to it, we have $(\exists j \in \mathbb{N})(\forall m, n \in \mathbb{N})(m, n \geq j \rightarrow |s_m - s_n| < \varepsilon)$ to be true in \mathbb{R} . Since $\varepsilon \in \mathbb{R}^+$ was arbitrary, we have (s_n) to be cauchy. \square

Exercise 6.4. (Section 6.5, II). Cauchy's convergence Criterion.

Prove the foward direction of A real valued sequence converges in \mathbb{R} iff it is cauchy.

Proof. Let (s_n) be a real valued sequence and suppose it convergence in \mathbb{R} to L . Therefore, by theorem 6.1.1, we have $s_N \simeq L$ for all $n \in {}^*\mathbb{N}_\infty$ but that means all extended terms are infinitely close to each other i.e. for any $n, m \in {}^*\mathbb{N}_\infty$, $s_n \simeq L \simeq s_m$. Therefore, by theorem 6.5.1, we have (s_n) to be cauchy. \square

Exercise 6.5. (Section 6.7, I) Let (s_n) and (t_n) be real valued sequences with limits, L, M respectively. Show that if $s_n \leq t_n$ for $n \in \mathbb{N}$. Then $L \leq M$.

Proof. Applying transfer on $(\forall n \in \mathbb{N})(s_n \leq t_n)$, we have $(\forall n \in {}^*\mathbb{N})(s_n \leq t_n)$ to be true in ${}^*\mathbb{R}$. In particular for any $N \in {}^*\mathbb{N}_\infty$, we have $s_N \leq t_N$, however, since $L \simeq s_N \leq t_N \simeq M$, we have $L \leq M$ by exercise 2 of section 5.5. \square

Exercise 6.6. (Section 6.7, II) If $r_n \leq s_n \leq t_n$ in \mathbb{R} for all $n \in \mathbb{N}$, and $\lim_{n \rightarrow \infty} r_n = \lim_{n \rightarrow \infty} t_n$, show that (s_n) converges to this same limit.

Proof. Applying transfer on $(\forall n \in \mathbb{N})(r_n \leq s_n \leq t_n)$, we have $(\forall n \in {}^*\mathbb{N})(r_n \leq s_n \leq t_n)$ to be true in ${}^*\mathbb{R}$. In particular, for any $N \in {}^*\mathbb{N}_\infty$, we have $r_N \leq s_N \leq t_N$. However, since $(r_N), (t_N)$ are limited, s_N is limited. By the shadow inequality, we have $\text{sh}(r_N) \leq \text{sh}(s_N) \leq \text{sh}(t_N) = \text{sh}(r_N)$. Hence, $\text{sh}(s_N) = \text{sh}(r_N) = \lim_{n \rightarrow \infty} r_n$ but that means $r_N \simeq \lim_{n \rightarrow \infty} r_n$ for all $N \in {}^*\mathbb{N}_\infty$. Hence, (s_n) converges to $\lim_{n \rightarrow \infty} r_n$. \square

Exercise 6.7. (Section 6.7, III) If a sequence converges in \mathbb{R} , show that it has exactly one cluster point.

Proof. Suppose (s_n) converges to L and assume it has two cluster point A and B . By definition of convergence we have $s_N \simeq L$ for all $N \in {}^*\mathbb{N}_\infty$ but that would imply $B \simeq L \simeq A$ by the definition of cluster point, since they are reals, it follows $A = L = B$. \square

Exercise 6.8. (Section 6.7, IV) Suppose that a real valued sequence has a single cluster point. If the sequence is bounded, must it be convergent? what if it is unbounded?

Proof. Let (s_n) be a bounded sequence which has an unique cluster point i.e. for some $M \in {}^*\mathbb{N}_\infty$, we have $s_M \simeq L$ for some real L . Suppose it is not convergent. Therefore, there must exists $N \in {}^*\mathbb{N}_\infty$ such that $s_N \not\simeq L$. However since (s_n) is bounded, we have s_N to be limited and hence infinitely close to a real number ie its shadow. But if $\text{sh}(s_N) \neq L$, clearly $\text{sh}(s_N)$ is a cluster point which contradicts the assumption that there was an unique cluster point.

Consider the sequence $\langle s \rangle = (0, 1, 0, 2, 0, 3, \dots)$. Clearly this sequence has only one cluster point and is unbounded but it does not converges to any real number. \square

Exercise 6.9. (Section 6.8, I) Prove, by nonstandard reasoning, that both the limit superior and limit inferior are cluster points of the sequence s .

Proof. Let $L = \limsup s$ and $\varepsilon \in \mathbb{R}^+$. Since L is supremum of C_s , we know there exists $N \in {}^*\mathbb{N}_\infty$ such that $L - \varepsilon < s_N \leq L$. Consider some arbitrary $n \in \mathbb{N}$. We have $(\exists m \in {}^*\mathbb{N})(m > n \wedge L - \varepsilon < s_m)$ to be true, and applying transfer on it, it follows $(\exists m \in \mathbb{N})(m > n \wedge L - \varepsilon < s_m)$ is true in \mathbb{R} . Since $\varepsilon \in \mathbb{R}^+$ and $n \in \mathbb{N}$ was arbitrary, $(\forall \varepsilon \in \mathbb{R}^+)(\forall n \in \mathbb{N})(\exists m \in \mathbb{N})(m > n \wedge L - \varepsilon < s_m)$ is true in \mathbb{R} . Applying transfer we see $(\forall \varepsilon \in {}^*\mathbb{R}^+)(\forall n \in {}^*\mathbb{N})(\exists m \in {}^*\mathbb{N})(m > n \wedge L - \varepsilon < s_m)$ holds in ${}^*\mathbb{R}$. Picking ε as some infinitesimal and n as unlimited, we follow there is some $m \in {}^*\mathbb{N}$ making $m > n$ and $L - e < s_m$ true, that is to say, there is unlimited m such that $L < s_m + e$. Therefore, it follows $L \leq \text{sh}(s_m)$ for an unlimited m . However, since L is upper bound of C_s , we must have $L = \text{sh}(s_m)$ making it a cluster point. We can argue about \liminf in a similar fashion. \square

Exercise 6.10. (Section 6.9, I) Formulate the definition of the limit inferior of an arbitrary real valued sequence.

answer:

Proposition A real number L is equal to $\liminf s$ if and only if

- (1) $L < s_n$ or $L \simeq s_n$ for all unlimited n ; and
- (2) $s_n \simeq L$ for at least one unlimited n .

Proof. The condition (1) holds iff $L \leq \text{sh}(s_n)$. Thus, it implies L is lower bound of set of cluster points, C_s . The condition (2) asserts that L is a cluster point i.e. $L \in C_s$. Together they imply that L is infimum (since it's the smallest element) of the set. Hence, it is equal to $\liminf s$. \square

Exercise 6.11. (Section 6.9, III) Formulate and prove theorems about the limit inferior of a bounded sequence that correspond to theorems 6.8.4 and 6.8.5.

answer:

Proposition-1. If s is a bounded real valued sequence with limit inferior, \liminf , then for any positive real ε :

- (1) Some standard tail of s has all its terms bigger than $\liminf - \varepsilon$ i.e. $\liminf - \varepsilon < s_n$ for all but finitely many $n \in \mathbb{N}$.
- (2) $s_n < \liminf + \varepsilon$ for infinitely many $n \in \mathbb{N}$.

Proof. Let $L = \liminf s$. Consider $\varepsilon \in \mathbb{R}^+$. First we prove (1). For any unlimited N , we have $L < s_N$ or $L \simeq s_N$. Thus, $L - \varepsilon < s_N$ holds for any unlimited N . Transferring $(\exists n \in {}^*\mathbb{N})(\forall m \in {}^*\mathbb{N})(m \geq n \wedge L - \varepsilon < s_m)$, we have $(\exists n \in \mathbb{N})(\forall m \in \mathbb{N})(m > n \wedge L - \varepsilon < s_m)$ to be true in \mathbb{R} but that simply means $L - \varepsilon < s_m$ is true for all but finitely many n . We now prove (2). Let $m \in \mathbb{N}$. We know that L is infinitely close to s_N for some unlimited N . Thus, $(\exists n \in {}^*\mathbb{N})(n > m \wedge s_n < L + \varepsilon)$ is true in ${}^*\mathbb{R}$. Transferring it, we have $(\exists n \in \mathbb{N})(n > m \wedge s_n < L + \varepsilon)$ to be true in \mathbb{R} . However since $m \in \mathbb{N}$ was arbitrary, it follows that for any $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $n > m$ and $s_n < L + \varepsilon$ satisfying the proposition. \square

Proposition-2. For any bounded real valued sequence s ,

$$\liminf_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} I_n = \lim_{n \rightarrow \infty} \left(\inf_{m \geq n} s_m \right)$$

where $I_n = \inf_{m \geq n} s_m$.

Proof. First we will prove that \liminf is upper bound of I . Take an extended term s_N infinitely close to the cluster point \liminf . For any $m \in \mathbb{N}$, we have $I_m \leq s_n$ for all limited $n \geq m$, and hence it holds for all hypernatural $n \geq m$ by transfer. In particular, $I_m \leq s_N$, as $\liminf \simeq s_N$, we have $I_m \leq \liminf$.

Let $L = \lim I_n$. Then clearly L is supremum of I since I is nondecreasing. However, as we showed \liminf is an upper bound, we must have $L \leq \liminf$. We will now show that $L < \liminf$ is impossibility. Suppose $L < \liminf$ and consider some $\varepsilon \in \mathbb{R}^+$ such that $L < \liminf - \varepsilon$. By proposition-1, we know that there

exists $n \in \mathbb{N}$ such that standard tail beyond n is bounded below by $\liminf -\varepsilon$. But that means that $\liminf -\varepsilon \leq I_n$, and so $L < \liminf -\varepsilon \leq I_n$ which contradicts the fact that L was upper bound of I . Hence, $\liminf = L$. \square

Exercise 6.12. (Section 6.9, III) If s is a bounded sequence, show that for each $\varepsilon \in \mathbb{R}^+$ there is some $n \in \mathbb{N}$ such that the standard tail s_n, s_{n+1}, \dots is contained in the interval $(\liminf -\varepsilon, \limsup +\varepsilon)$.

Proof. By theorem 6.8.4 and previous exercise, we know that there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $s_n < \limsup +\varepsilon$ and there exists $N' \in \mathbb{N}$ such that for all $n \geq N'$, $s_n > \liminf -\varepsilon$. Defining $m = \max\{N, N'\}$, the tail following s_m is the desired one. \square

Exercise 6.13. (Section 6.11, I) Give an example of a series that diverges but has a_n infinitesimal for all unlimited n .

Proof. Consider the sequence (s_n) defined as $(s_n) = (1/n)$. Clearly for any unlimited n , a_n is infinitesimal. However, notice that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n} &\geq 1 + \sum_{n=1}^{\infty} \sum_{k=1}^{2^{n-1}} \frac{1}{n} \\ &\geq 1 + \sum_{n=1}^{\infty} \frac{1}{2} \end{aligned}$$

Since the last series is divergent, so is $\sum_{n=1}^{\infty} \frac{1}{n}$. \square

Exercise 6.14. (Section 6.11, II) Give nonstandard proofs of the usual rules of arithmetically combining convergent series.

- (a) $\sum_1^{\infty} a_i + \sum_1^{\infty} b_i = \sum_1^{\infty} (a_i + b_i)$,
(b) $\sum_1^{\infty} a_i - \sum_1^{\infty} b_i = \sum_1^{\infty} (a_i - b_i)$,

Proof. We will first prove (a), then (c), and then use (a) and (c) for proving (b). For what follow, Let $L = \sum_1^{\infty} a_i$ and $M = \sum_1^{\infty} b_i$.

(a) Let $N \in {}^*\mathbb{N}_{\infty}$. Since $\sum_1^n (a_i + b_i) = \sum_1^n a_i + \sum_1^n b_i$ holds in \mathbb{R} for all $n \in \mathbb{N}$, by transfer we know it also holds in ${}^*\mathbb{R}$ for all $n \in {}^*\mathbb{N}$. Hence,

$$\sum_1^N (a_i + b_i) = \sum_1^N a_i + \sum_1^N b_i.$$

and thus,

$$\begin{aligned} \text{sh} \left(\sum_1^N (a_i + b_i) \right) &= \text{sh} \left(\sum_1^N a_i + \sum_1^N b_i \right) \\ &= \text{sh} \left(\sum_1^N a_i \right) + \text{sh} \left(\sum_1^N b_i \right) \\ &= L + M \end{aligned}$$

As desired.

(c) Let $N \in {}^*\mathbb{N}_{\infty}$. Then it follows

$$\begin{aligned} \text{sh} \left(\sum_1^N c a_i \right) &= \text{sh} \left(c \sum_1^N a_i \right) && \text{(by transfer)} \\ &= c \cdot \text{sh} \left(\sum_1^N a_i \right) \\ &= c \cdot L \end{aligned}$$

as desired.

(b) Let $N \in {}^*\mathbb{N}_\infty$. Then

$$\begin{aligned}
\text{sh} \left(\sum_1^N (a_i - b_i) \right) &= \text{sh} \left(\sum_1^N (a_i + (-b_i)) \right) \\
&= \text{sh} \left(\sum_1^N a_i \right) + \text{sh} \left(\sum_1^N (-b_i) \right) && \text{(by part a)} \\
&= \text{sh} \left(\sum_1^N a_i \right) + (-1) \cdot \text{sh} \left(\sum_1^N (-b_i) \right) && \text{(by part c)} \\
&= L + (-M) \\
&= L - M.
\end{aligned}$$

□

Exercise 6.15. (Section 6.11, III) Suppose that $a_i \geq 0$ for all $i \in \mathbb{N}$. Prove that $\sum_1^\infty a_i$ converges iff $\sum_1^n a_i$ is limited for all unlimited n , and that this holds iff $\sum_1^n a_i$ is limited for some unlimited n .

Proof. First we prove (1) \Rightarrow (2). Suppose $\sum_1^\infty a_i$ converges to a real number L . Therefore, we must have $\sum_1^N a_i \simeq L$ for all unlimited N but that means $\sum_1^N a_i$ is limited. The direction (2) \Rightarrow (3) is easy to see.

We prove (3) \Rightarrow (2). Let $n \in \mathbb{N}$. Then we know that there exists $m \in {}^*\mathbb{N}$ such that $m > n$ and $\sum_1^m a_i \leq L+1$ where L is the shadow of the limited extended term. Trasnferring it, we see that for any $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $m > n$ and $\sum_1^m a_i \leq L+1$. Since $a_i \geq 0$ for all $i \in \mathbb{N}$, we have $|\sum_1^m a_i| = \sum_1^m |a_i| \leq L+1$. Hence, the sequence of partial sum is bounded which implies all extended term are limited by theorem 6.4.1.

We prove (2) \Rightarrow (1). Since $\sum_1^n a_i$ is limited for all unlimited n , by theorem 6.4.1, we know that sequence of partial sum is bounded. Also notice that since $a_i \geq 0$, for each $n \in \mathbb{N}$, $\sum_1^n a_i \leq \sum_1^{n+1} a_i$. Therefore, the sequence of partial sums is non decreasing. Hence, by theorem 6.2.1, we know it is convergent. □

Exercise 6.16. (Section 6.11, IV) Comparison test.

Let $\sum_1^\infty a_i$ and $\sum_1^\infty b_i$ be two real series of non negative terms, with $\sum_1^\infty b_i$ convergent. If $a_i \leq b_i$ for all $i \in \mathbb{N}$, use result (III) to show that $\sum_1^\infty a_i$ is convergent.

Proof. Since $a_i \leq b_i$ for all $i \in \mathbb{N}$, by transfer, we know $a_n \leq b_n$ for all unlimited n . By exercise (III), it follows $\sum_1^n b_i$ is limited for all unlimited n but that means $\sum_1^n a_i$ is also limited for all unlimited n , and thus by exercise (III), is convergent. □

Exercise 6.17. (Section 6.11, V) Show that the comparison test holds under the weaker assumption that $a_n \leq b_n$ for all unlimited n . Show that this weaker assumption is equivalent to requiring that there be some limited $k \in \mathbb{N}$ with $a_n \leq b_n$ for all $n \geq k$.

Proof. For the first part, notice that since $a_n \leq b_n$ for all unlimited n , we have

$$\sum_m^n a_i \leq \sum_m^n b_i \simeq \text{sh} \left(\sum_m^n b_i \right) = 0$$

for any unlimited m, n with $m \leq n$, because $\sum_1^\infty b_i$ is convergent. However, since $\sum_m^n a_i$ is finite, we have $\text{sh}(\sum_m^n a_i) \leq 0$. Since partial sums of a_i is non-decreasing sequence of non-negative terms, it follows $0 \leq \sum_1^n a_i - \sum_1^{m-1} a_i = \sum_m^n a_i$ which gives us $0 = \text{sh}(\sum_m^n a_i) \simeq \sum_m^n a_i$. Hence, by cauchy convergence criterion, we know that $\sum_1^\infty a_i$ converges.

For the second part, we know there exists $k \in {}^*\mathbb{N}$ such that for any all $n \geq k$, $a_n \leq b_n$ i.e. when k is unlimited. Transferring it, we get the desired result. For converse, we know for any $n \geq k$, $a_n \leq b_n$. Transferring it, we have for any $n \in {}^*\mathbb{N}$ such that $n \geq k$, $a_n \leq b_n$ but then since k is limited, the inequality holds for all unlimited numbers. \square

Exercise 6.18. (Section 6.11, VI) Let $\sum_1^\infty a_i$ and $\sum_1^\infty b_i$ be two series of positive terms such that the sequence $\langle a_i/b_i : i \in \mathbb{N} \rangle$ is convergent in \mathbb{R} . Deduce that either both series converge, or both diverge.

Proof.

Notation. Define $\langle s \rangle = \langle a_i/b_i : i \in \mathbb{N} \rangle$ and $L = \lim_{n \rightarrow \infty} s$.

We claim that the following result is true.

Theorem 1. If $\sum_1^\infty a_i$ and $\sum_1^\infty b_i$ be two series of positive terms such that the sequence $\langle a_i/b_i : i \in \mathbb{N} \rangle$ is convergent in \mathbb{R} Then for any unlimited m and n it follows that

$$\sum_m^n a_i \text{ is infinitesimal if and only if } \sum_m^n b_i \text{ is infinitesimal.}$$

Assuming this claim for the moment, it follows from cauchy's convergence criterion that $\sum_1^\infty a_i$ is convergent if and only if $\sum_1^\infty b_i$ is convergent, which is the first part of the problem.

We prove our claim now. Suppose $\sum_m^n a_i$ is infinitesimal for all unlimited m and n with $n \geq m$. Since the sequence $\langle s \rangle$ is convergent, we have $b_N \simeq 1/L \cdot a_N$ for any unlimited N . Thus,

$$\sum_m^n b_i \simeq \sum_{i=m}^n (1/L \cdot a_i) = 1/L \cdot \sum_{i=m}^n a_i$$

which is known to be infinitesimal. The other direction can be proved by identical argument considering $a_N \simeq L b_N$ for any unlimited N .

We can apply the same chain of thought for proving divergence. Recall that a sequence is divergent if all of its extended term are unlimited. Since we have $a_N \simeq L b_N$, it follows immediately that if one of them is divergent then so is the other. \square

Exercise 6.19. (Section 6.11, VII) Let $c \in \mathbb{R}$. Recall the identity

$$1 + c + c^2 + \cdots + c^n = \frac{1 - c^{n+1}}{1 - c}. \quad (6.1)$$

(a) Considering the case of unlimited n , show that the series $\sum_1^\infty c^i$ converges if $|c| < 1$.

(b) Show that $\sum_m^n c^i$ is infinitesimal when m and n are unlimited, either by applying result (a) or by making further use of the above inequality.

Proof.

(a) If $c = 0$ then the proposition is trivially true, so suppose $c \neq 0$ for what follows. We claim the following result is true.

Theorem 1. For any $c \in \mathbb{R}$ such that $0 < |c| < 1$, the series $\sum_1^\infty |c|^i$ converges.

Before we get to the proof, we see that if the theorem 1 is true then the series $\sum_1^\infty c^i$ is convergent because it is absolutely convergent which is what we desire.

We begin the proof by noticing that $\langle |c|^i \rangle = \langle |c|^i \rangle$ for $0 < c < 1$ is a non increasing sequence with a lower bound, and thus by monotone convergence theorem, is convergent. Let $L = \lim |c|^n$ and consider some unlimited N . Then applying transfer on the identity 6.1, we have

$$\sum_1^N |c|^i = \frac{1 - |c|^{N+1}}{1 - |c|} - 1 \simeq \frac{1 - L}{1 - |c|} - 1$$

which is clearly limited. Therefore, it follows that $\sum_1^\infty c^i$ is convergent when $|c| < 1$.

(b) We are going to give alternative proof to it instead of using part (a). Again we will prove that $\sum_1^\infty c^i$ is absolutely convergent. We already know that the sequence $\langle |c|^i \rangle = \langle |c|^i \rangle$ for $0 < c < 1$ is convergent. Let $L = \lim |c|^i$. Then we have

$$\sum_m^n |c|^i = \sum_1^n |c|^i - \sum_1^m |c|^i \simeq \frac{1 - L}{1 - c} - \frac{1 - L}{1 - c} = 0.$$

□

Exercise 6.20. (Section 6.11, VIII) Ratio test: Convergence.

Suppose that

$$\limsup_{i \rightarrow \infty} \frac{|a_{i+1}|}{|a_i|} < 1$$

in \mathbb{R} . Prove that the series $\sum_1^\infty a_i$ is absolutely convergent.

Proof. We will prove the series $\sum_1^\infty a_i$ is absolutely convergent by demonstrating that difference of partial sum of sequence is always less than or equal to difference of partial sum of a convergent sequence which will show that difference of partial sum of our sequence gets infinitely close eventually. We start off by proving the following theorem.

Theorem 1. Suppose that $\limsup_{i \rightarrow \infty} \frac{|a_{i+1}|}{|a_i|} < 1$ in \mathbb{R} . Then there exists a positive real $c < 1$ with $|a_{n+1}| < c|a_n|$ for all unlimited n .

Before we discuss the proof of it, there is an interesting corollary which follows from this. If theorem-1 is true for all unlimited numbers then by easy transfer, we must have $|a_{n+1}| < c|a_n|$ for all but finitely many $n \in \mathbb{N}$.

We will now prove theorem-1. If $a_i = 0$ for any $i \in \mathbb{N}$ then $\frac{|a_{i+1}|}{|a_i|}$, i.e. the i th term of the sequence, become undefined. Therefore, we must have $a_i \neq 0$ for any $i \in \mathbb{N}$. Thus, $0 < \frac{|a_{n+1}|}{|a_n|}$ for all $n \in \mathbb{N}$. Let N be arbitrary unlimited natural number. Then we have $0 < \frac{|a_{N+1}|}{|a_N|} \leq \limsup_{i \rightarrow \infty} \frac{|a_{i+1}|}{|a_i|} < 1$. Let $L = \limsup_{i \rightarrow \infty} \frac{|a_{i+1}|}{|a_i|}$. Define $c = \frac{L+1}{2}$. Then we have $0 < \frac{|a_{N+1}|}{|a_N|} < c$. Thus, $|a_{N+1}| < c|a_N|$. Since N was arbitrary, we have our result.

Corollary 1.1. *There is some limited $k \in \mathbb{N}$ such that $|a_{n+1}| < c|a_n|$ for all $n \geq k$.*

Theorem 2. *Suppose that $\limsup_{i \rightarrow \infty} \frac{|a_{i+1}|}{|a_i|} < 1$ in \mathbb{R} . Then there is some k such that for all $n \geq k$, we have $|a_{k+n}| < c^n|a_k|$, and hence $\sum_{k+m}^{k+n} |a_i| \leq \sum_m^n c^i |a_k|$.*

The existence of such a k is guaranteed by corollary 1.1, furthermore, it also implies

$$|a_{k+n}| < c|a_{k+n-1}| < c^2|a_{k+n-2}| < \cdots < c^n|a_{k+n-n}| = c^n|a_k|.$$

Thus, $|a_{k+n}| < c^n|a_k|$. It is easy to see that the last inequality implies

$$0 < \sum_{k+m}^{k+n} |a_i| \leq \left(\sum_m^n c^i \right) |a_k|.$$

By an easy transfer we see that the last inequality also hold in hyperreals. However, by exercise VII, we know that $\sum_m^n c^i$ is infinitesimal for any unlimited n and m such that $n \geq m$ but that means $\sum_{k+m}^{k+n} |a_i|$ is infinitesimal for any unlimited m and n . In particular, replacing m by $m - k$ and n by $n - k$, we get the following result.

Corollary 2.1. *$\sum_1^\infty |a_i|$ is convergent, that is to say, $\sum_1^\infty a_i$ is absolutely convergent.*

□

Exercise 6.21. (Section 6.11, IX) *Ratio Test: divergence.*

Suppose that

$$\liminf_{i \rightarrow \infty} \frac{|a_{i+1}|}{|a_i|} > 1$$

in \mathbb{R} . Prove that the series $\sum_1^\infty a_i$ diverges.

Proof. Before we get into the proof, notice that $a_i \neq 0$ for any $i \in \mathbb{N}$ since otherwise $\frac{|a_{i+1}|}{|a_i|}$ becomes undefined. Consider some unlimited n . Then we have

$$1 < \liminf_{i \rightarrow \infty} \frac{|a_{i+1}|}{|a_i|} \leq \frac{|a_{N+1}|}{|a_N|},$$

which implies $|a_{N+1}| > |a_N|$. We summarize it into a theorem.

Theorem 1. *Suppose that $\liminf_{i \rightarrow \infty} \frac{|a_{i+1}|}{|a_i|} > 1$. Then $a_i \neq 0$ for any $i \in \mathbb{N}$ and $|a_{N+1}| > |a_N|$ for any unlimited N .*

Next we prove the following result.

Theorem 2. *Suppose that $\liminf_{i \rightarrow \infty} \frac{|a_{i+1}|}{|a_i|} > 1$. Then there exists $k \in \mathbb{N}$ such that for all $m > n$, $|a_m| > |a_k|$.*

In natural numbers, we know that for any $n, m \in \mathbb{N}$, there is some $p \in \mathbb{N}$ such that if $m > n$ then $m = n + p$. Therefore, by transfer, it is also true in hypernaturals. In particular, it holds for unlimited numbers. Let m and n be unlimited numbers such that $m > n$. Then, by theorem-1, it follows

$$|a_m| = |a_{n+p}| > |a_{(n+p)-1}| > \cdots > |a_{n+1}| > |a_n|.$$

Trasnferring $(\exists n \in {}^*\mathbb{N})(\forall m \in {}^*\mathbb{N})(m > n \rightarrow |a_m| > |a_n|)$ back to \mathbb{N} , we have our result.

By existential transfer, we have following corollary to be true.

Corollary 2.1. *For all unlimited n , $|a_n| > |a_k|$.*

For the sake of contradiction, assume $\sum_1^\infty a_i$ is convergent. Therefore, for any unlimited N , we must have $\sum_1^{N+1} a_i - \sum_1^N a_i \simeq 0$ but then it implies

$$0 < |a_k| < |a_{N+1}| = \sum_1^{N+1} a_i - \sum_1^N a_i \simeq 0$$

which means $0 < |a_k| \leq 0$. A contradiction. \square

Exercise 6.22. (Section 6.11, X) Show that for any real number x , the hyperreal $x^n/n!$ is infinitesimal when n is unlimited.

Proof. Define $a_i = (x^i/i!)$. We will first prove that $\sum_1^\infty a_i$ is convergent which would easily get us $\sum_{n-1}^n a_i = \sum_{n-1}^n x^i/i! = x^n/n!$ to be infinitesimal. For any unlimited N , we have $\frac{(x^{N+1}/(N+1)!)}{(x^N/N!)} = \frac{x}{N+1}$ which is clearly infinitesimal. It means that $\limsup_{i \rightarrow \infty} \frac{|a_{i+1}|}{|a_i|} < 1$. Therefore, by ratio test, it follows $\sum_1^\infty a_i$ is convergent. Hence, $\sum_m^n a_i$ is infinitesimal for any unlimited m and n such that $n \geq m$. In particular, setting $m = n-1$ gets us $\sum_{n-1}^n a_i = \sum_{n-1}^n x^i/i! = x^n/n!$ to be infinitesimal. \square

Exercise 6.23. (Section 6.11, XI) *Leibniz's alternating series test.*

Suppose $\langle a_i : i \in \mathbb{N} \rangle$ is a real sequence that is nonincreasing (i.e. $a_i \geq a_{i+1}$) and converges to 0. Prove that the alternating series

$$\sum_1^\infty (-1)^{i+1} a_i = a_1 - a_2 + a_3 - a_4 + \cdots$$

converges.

Proof. We first make the observation that all the terms in the sequence are positive since otherwise it would not be convergent to 0. We are going to first prove $|\sum_m^n (-1)^{i+1} a_i| \leq |a_m|$ for $m, n \in \mathbb{N}$ from which our result easily follow using the transfer principle and shadow inequality.

Theorem 1. Suppose $\langle a_i : i \in \mathbb{N} \rangle$ is a real sequence that is nonincreasing (i.e. $a_i \geq a_{i+1}$) and converges to 0. Then for all $m, n \in \mathbb{N}$ such that $n \geq m$, we have

$$\left| \sum_m^n (-1)^{i+1} a_i \right| \leq |a_m|$$

to be true.

We divide the proof into cases.

First suppose m is even. Then we have

$$\sum_m^n (-1)^{i+1} a_i = \sum_{\substack{m \leq i \leq n \\ i \text{ is even}}} (-a_i + a_{i+1}) = \sum_{\substack{m \leq i \leq n-1 \\ i \text{ is even}}} (-a_i + a_{i+1}) < 0 < a_m$$

and

$$-a_m < (a_n - a_m) + \sum_{\substack{m+1 \leq i \leq n-1 \\ i \text{ is odd}}} (a_i - a_{i+1}) = \sum_m^n (-1)^{i+1} a_i$$

when n is odd because $a_n < a_m$ and $a_{i+1} < a_i$ for each i . In case of n being even, consider

$$\sum_{\substack{m \leq i \leq n-1 \\ i \text{ is even}}} (-a_i + a_{i+1}) - a_n < \sum_{\substack{m \leq i \leq n-1 \\ i \text{ is even}}} (-a_i + a_{i+1}) < 0 < a_m$$

and

$$-a_m < -a_m + \sum_{\substack{m+1 \leq i \leq n \\ i \text{ is odd}}} (a_i - a_{i+1}) = \sum_m^n (-1)^{i+1} a_i$$

Similarly, in case of m being odd, we have

$$-a_m < 0 < \sum_{\substack{m \leq i \leq n \\ i \text{ is odd}}} (a_i - a_{i+1}) = \sum_m^n (-1)^{i+1} a_i$$

and

$$\sum_m^n (-1)^{i+1} a_i = a_m - a_n + \sum_{\substack{m-1 \leq i \leq n-1 \\ i \text{ is even}}} (-a_i + a_{i+1}) < a_m$$

when n is even. In case of n being odd, consider

$$-a_m < 0 < \sum_{\substack{m \leq i \leq n-1 \\ i \text{ is odd}}} (a_i - a_{i+1}) < \sum_{\substack{m \leq i \leq n-1 \\ i \text{ is odd}}} (a_i - a_{i+1}) + a_n = \sum_m^n (-1)^{i+1} a_i$$

and

$$\sum_m^n (-1)^{i+1} a_i = a_m + \sum_{\substack{m-1 \leq i \leq n \\ i \text{ is even}}} (-a_i + a_{i+1}) < a_m.$$

This concludes the proof of theorem 1. By an easy transfer, we know that theorem 1 also holds in hyper-naturals. Therefore, $\sum_m^n (-1)^{i+1} a_i$ is infinitesimal in case of unlimited m . Hence, by cauchy convergence theorem, it follows $\sum_1^\infty (-1)^{i+1} a_i$ converges. \square