Lectures on Hyperreals

Book by Robert Goldblatt

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Chapter 1

What are Hypereals?

Chapter 2

Large sets

Exercise 2.1. *If $\emptyset \neq A \subseteq I$, there is an ultrafilter $\mathcal{F}$ on I with $A \in \mathcal{F}$.*

Proof. Define $\mathcal{H} = \{B \in \mathscr{P}(I) : A \subseteq B\}$. Since A is non empty, it follows that $\mathcal{H}$ has fip. Clearly, $A \in \mathcal{H}$, and so $A \in \mathcal{F}^{\mathcal{H}}$. Therefore, by zorn lemma, we know that there exists ultrafilter, $\mathcal{F}$, such that $A \in \mathcal{F}^{\mathcal{H}} \subseteq \mathcal{F}$. □

Exercise 2.2. *There exists a nonprincipal ultrafilter on $\mathbb{N}$ containing the set of even numbers, and another containing the set of odd numbers.*

Proof. Define $I = \{x \in \mathbb{N} : x = 2n \text{ for some } n \in \mathbb{N}\}$. Since I is infinite, we know that $\mathcal{F}^{\text{co}}$ is proper and thus, have fip. Applying the zorn's lemma, we see that there exists ultrafilter $\mathcal{F}$ such that $\mathcal{F} \subseteq \mathcal{F}^{\text{co}}$. However, for any $i \in I$, we have $I - \{i\} \in \mathcal{F}^{\text{co}} \subseteq \mathcal{F}$ but, since $\mathcal{F}$ is ultrafilter, $\{i\} \in \mathcal{F}$. However, clearly $\{i\} \in \mathcal{F}^i$. Therefore, $F \neq F^i$ for any $i \in I$. Likewise we could argue about set of odd numbers. □

Exercise 2.3. *An ultrafilter on finite set must be principal.*

Proof. Let the set be $A = \{a_1, a_2, \cdots, a_n\}$ and $\mathcal{F}$ be ultrafilter on it. Since $\{a_1\} \cup \{a_2\} \cup \cdots \cup \{a_n\} = A \in \mathcal{F}$, it must be the case that $a_i \in \mathcal{F}$ for some i such that $1 \leq i \leq n$. Since $\mathcal{F}$ is proper, for any $B \in \mathcal{F}$, we have $B \cap \{a_i\} \neq \emptyset$, but that means $\mathcal{F} = \mathcal{F}^{a_i}$. □

Exercise 2.4. *For $\mathcal{H} \subseteq \mathscr{P}(I)$.*
(i) Show that $\mathcal{F}^{\mathcal{H}}$ is a filter that includes $\mathcal{H}$, i.e. $\mathcal{H} \subseteq \mathcal{F}^{\mathcal{H}}$.
(ii) Show that $\mathcal{F}^{\mathcal{H}}$ is included in any other filter that includes $\mathcal{H}$.

Proof.
(i) By the definition.
(ii) Let $\mathcal{F}$ be a filter such that $H \subseteq \mathcal{F}$. Let $A \in \mathcal{F}^{\mathcal{H}}$, then $B_1 \cap B_2 \cap \cdots \cap B_n \subseteq A$ for some $B_1, \cdots, B_n \in \mathcal{H}$. Clearly, $B_1 \cap \cdots \cap B_n \in F$ and so is $A \in F$. □

Exercise 2.5. *Let $\mathcal{F}$ be a proper filter on I.*
(I) Show that $\mathcal{F} \cup \{A^c\}$ has fip iff $A \notin \mathcal{F}$.
(II) Use (I) to deduce that $\mathcal{F}$ is an ultrafilter iff it is maximal proper filter on I.

Proof.
(I) First we prove forward implication. Notice that if $A \in \mathcal{F}$ then since $A \cap A^c = \emptyset$, it follows $\mathcal{F} \cup \{A^c\}$ does not have fip.
We prove the converse implication. Notice that since F is a proper filter, ie it does not contain $\emptyset$, the only way for $\mathcal{F} \cup \{A^c\}$ to not have fip would be $A \in \mathcal{F}$.

(II) We prove the foward implication. Suppose $\mathcal{F}$ is an ultrafilter. Applying Zorn's lemma on proper filter $\mathcal{F}$, we see that there exists a maximal proper filter, K, such that $\mathcal{F} \subseteq K$. If $K \neq F$, it would mean there is some $A \in K$ such that $A \notin \mathcal{F}$ but then that would mean $A^C \in \mathcal{F}$, and so $A^c \in K$ which contradicts the fact that K is proper since it fails to satisfy fip. Therefore, we must have $F = K$.

We prove the converse implication. Suppose $\mathcal{F}$ is maximal proper filter. For the sake of contradiction assume $\mathcal{F}$ is not an ultra filter i.e. there exists some $A \in \mathscr{P}(I)$ such that both $A, A^c \notin F$. Clearly, either $\mathcal{F} \cup \{A\}$ has fip or $\mathcal{F} \cup \{A^c\}$ does. Wlog assume $\mathcal{F} \cup \{A\}$ has fip. Applying Zorn's lemma on $\mathcal{F} \cup \{A\}$, we see that there is maximal proper filter K such that $\mathcal{F} \cup \{A\} \subseteq K$. However this means that $\mathcal{F}$ is not maximal proper filter since $F \subsetneqq F \cup \{A\} \subset P(I)$. $\square$

Chapter 3

Ultrapower Construction of the hyperreals

Exercise 3.1. *(section 3.3, I).* $\equiv$ *is an equivalence relation on* $\mathbb{R}^{\mathbb{N}}$.

Proof. Let $\langle a_n\rangle, \langle b_n\rangle, \langle c_n\rangle \in \mathbb{R}^{\mathbb{N}}$. First we prove that reflexivity. Clearly, $\{n \in \mathbb{N} : a_n = a_n\} = \mathbb{N} \in \mathcal{F}^{\text{co}} \subseteq \mathcal{F}$. Thus, $\langle a_n\rangle \equiv \langle a_n\rangle$.
Symmetricity is trivial since $\{n \in \mathbb{N} : a_n = b_n\} = \{n \in \mathbb{N} : b_n = a_n\}$, if one of them is in F then so is the other.
Finally, we prove transitivity. Suppose $\langle a_n\rangle \equiv \langle b_n\rangle$ and $\langle b_n\rangle \equiv \langle c_n\rangle$ i.e. $\{n \in \mathbb{N} : a_n = b_n\}, \{n \in \mathbb{N} : b_n = c_n\} \in \mathcal{F}$. Clearly, $\{n \in \mathbb{N} : a_n = b_n = c_n\} = \{n \in \mathbb{N} : a_n = b_n\} \cap \{n \in \mathbb{N} : b_n = c_n\} \in \mathcal{F}$. Therefore, $\langle a_n\rangle \equiv \langle c_n\rangle$. □

Exercise 3.2. *(section 3.3, II)* $\equiv$ *is a congruence on the ring* $\langle \mathbb{R}^{\mathbb{N}}, \oplus, \odot\rangle$, *which means that if* $r \equiv r'$ *and* $s \equiv s'$, *then* $r \oplus s = r' \oplus s'$ *and* $r \odot s = r' \odot s'$.

Proof. Suppose $r \equiv r'$ and $s \equiv s'$ i.e. $\{n \in \mathbb{N} : r_n = r'_n\}, \{n \in \mathbb{N} : s_n = s'_n\} \in \mathcal{F}$. Thus, since $\{n \in \mathbb{N} : r_n = r'_n, s_n = s'_n\} = \{n \in \mathbb{N} : r_n = r'_n\} \cap \{n \in \mathbb{N} : s_n = s'_n\} \in \mathcal{F}$, we have $\{n \in \mathbb{N} : r_n = r'_n, s_n = s'_n\} \subseteq \{n \in \mathbb{N} : r_n + s_n = r'_n + s'_n\} \in \mathcal{F}$. Similarly, $\{n \in \mathbb{N} : r_n s_n = r'_n s'_n\} \in \mathcal{F}$. Therefore, $r \oplus s \equiv r' \oplus s'$ and $r \odot s \equiv r' \odot s'$. □

Exercise 3.3. *(section 3.3, III)* $\langle 1, \frac{1}{2}, \frac{1}{3}, \cdots\rangle \not\equiv \langle 0, 0, 0, \cdots\rangle$.

Proof. Let $\langle r\rangle = \langle 1, \frac{1}{2}, \frac{1}{3}, \cdots\rangle$. For the sake of simplicity, define $[[r \neq 0]] = \{n \in \mathbb{N} : r_n \neq 0\}$. Clearly, $[[r \neq 0]] \subseteq \mathbb{N}$. Notice that for any $n \in \mathbb{N}$, we have $n \in [[r \neq 0]]$. Thus, $[[r \neq 0]] = \mathbb{N}$. So, $[[r \neq 0]] \in \mathcal{F}^{\text{co}} \subseteq \mathcal{F}$ but then it would mean $[[r = 0]] = [[r \neq 0]]^c \notin \mathcal{F}$. Therefore, $\langle r_n\rangle \not\equiv \langle 0\rangle$. □

Exercise 3.4. *(section 3.5, I)* $[[r = s]] \cap [[s = t]] \subseteq [[r = t]]$.

Proof. Let $n \in [[r = s]] \cap [[s = t]]$. Then it follows that $r_n = s_n$ and $s_n = t_n$, and so $r_n = t_n$. Therefore, $n \in [[s = t]]$. □

Exercise 3.5. *(Section 3.5, II)* $[[r = r']] \cap [[s = s']] \subseteq [[r \oplus s = r' \oplus s']] \cap [[r \oplus s = r' \oplus s']]$

Proof. Let $n \in [[r = r']] \cap [[s = s']]$. Thus, $r_n = r'_n$ and $s_n = s'_n$, and so $r_n \oplus s_n = r'_n \oplus s'_n$ and $r_n \odot s_n = r'_n \odot s'_n$. Therefore, $n \in [[r \oplus s = r' \oplus s']] \cap [[r \oplus s = r' \oplus s']]$. □

Exercise 3.6. *(Section 3.5, III)* $[[r = r']] \cap [[s = s']] \cap [[r < s]] \subseteq [[r' < s']]$.

Proof. Let $n \in [[r = r']] \cap [[s = s']] \cap [[r < s]]$. Thus, $r_n = r'_n, s_n = s'_n, r_n < s_n$ but then that means $r'_n < s'_n$, and so $n \in [[r' < s']]$. □

Exercise 3.7. *(Section 3.5, IV) If* $r \equiv r'$ *and* $s \equiv s'$, *then* $[[r < s]] \in \mathcal{F}$ *iff* $[[r' < s']] \in \mathcal{F}$.

Proof. Suppose $r \equiv r'$ and $s \equiv s'$, that is to say, $\{n \in \mathbb{N} : r_n = r'_n\}, \{n \in \mathbb{N} : s_n = s'_n\} \in \mathcal{F}$. First assume $[[r < s]] \in \mathcal{F}$. Therefore, $\{n \in \mathbb{N} : r_n = r'_n\} \cap \{n \in \mathbb{N} : s_n = s'_n\} \cap \{n \in \mathbb{N} : r_n < s_n\} \subseteq \{n \in \mathbb{N} : r' < s'\} \in$ mcF. Next assume $[[r' < s^p]] \in \mathcal{F}$. It follows $\{n \in \mathbb{N} : r_n = r'_n\} \cap \{n \in \mathbb{N} : s_n = s'_n\} \cap \{n \in \mathbb{N} : r'_n < s'_n\} \subseteq$ $\{n \in \mathbb{N} : r < s\} \in \mathcal{F}$ □

Exercise 3.8. *(Section 3.8, I) Use only general properties of ordered fields to deduce from the fact that* $[\epsilon]$ *is a positive infinitismal, the conclusion that* $[\epsilon]^{-1}$ *is greater than every real number.*

Proof. Notice that for any $r \in \mathbb{R}^+$, we have $^*r = [\langle r, r, \cdots, \rangle] \in {}^*\mathbb{R}$. Since $\epsilon < {}^*r$, by fields properties, we have $0 < {}^*r^{-1} < \epsilon^{-1}$. Since r is arbitrary positive real number, setting $r = r^{-1}$,we see that $^*r < \epsilon^{-1}$ for any $r \in \mathbb{R}^+$. □

Exercise 3.9. *(Section 3.10, I) If* A *is finite, show that* $^*A = A$*, and hence* A *has non nonstandard members.*

Proof. Let $A = \{a_1, \cdots, a_n\}$. Consider some $[r] \in {}^*A$. If we could prove that $r \equiv t$ for some constant sequence $t \in A$, we would be done. So we set that as our goal. Notice that since $[r] \in {}^*A$, we have $\{n \in \mathbb{N} : r_n \in A\} = \{n \in \mathbb{N} : r_n = a_1 \vee r_n = a_2 \vee \cdots r_n = a_n\} = \{n \in \mathbb{N} : r_n = a_1\} \cup \cdots \cup \{n \in \mathbb{N} : r_n = a_n\} \in \mathcal{F}$. But since $\mathcal{F}$ is ultrafilter, only one of such set is in $\mathcal{F}$. Therefore, $\{n \in \mathbb{N} : r_n = a_k\} \in \mathcal{F}$ for some k such that $1 \leq k \leq n$. Hence, $r \equiv a_k$. □

Exercise 3.10. *(Section 3.10, II) Prove:*
(a) $A \subseteq B$ *iff* $^*A \subseteq {}^*B$.
(b) $A = B$ *iff* $^*A = {}^*B$.

Proof.
(a) First we prove the forward implication. Suppose $A \subseteq B$. Consider $[r] \in {}^*A$. Therefore, we have $[[r \in A]] \in \mathcal{F}$, but since $[[r \in A]] \subseteq [[r \in B]]$, we have $[[r \in B]]$ and so $[r] \in {}^*B$.
We prove the converse implication. Suppose $^*A \subseteq {}^*B$. Consider some $r \in A$. Then we have $[\mathbf{r}] \in {}^*A$, and so $[\mathbf{r}] \in {}^*B$. However, since $[\mathbf{r}] \in B$, we have $[[r \in B]] \in \mathcal{F}$ which means $r \in B$.

(b) Trivially follows from (a). □

Exercise 3.11. *(Section 3.10, III)*
Prove:
(a) $^*(A \cup B) = {}^*A \cup {}^*B$,
(b) $^*(A \cap B) = {}^*A \cap {}^*B$,
(c) $^*(A - B) = {}^*A - {}^*B$,
(d) $^*\emptyset = \emptyset$.

Proof.
(a) First we prove the foward implication. Since $A \subset A \cup B$, applying (2), we have $^*A \subseteq {}^*(A \cup B)$. Similarly, we have $^*B \subseteq {}^*(A \cup B)$, and so $^*A \cup {}^*B \subseteq {}^*(A \cup B)$.
We prove the converse implication. Let $[r] \in {}^*A \cup {}^*B$. Then we have following cases;
case-1. $[r] \in {}^*A$. Thus, $[[r \in A]] \in mcF$, and since $[[r \in A]] \subseteq [[r \in A \cup B]]$, we have $[[r \in A \cup B]] \in \mathcal{F}$ but that means $[r] \in {}^*A \cup {}^*B$.
case-2. $[r] \in {}^*B$. The proof is analogous to **case-1**.

(b) First we prove the forward implication. Since $A \cap B \subseteq A$, applying (2), we have $^*(A \cap B) \subseteq {}^*A$. Similarly we have $^*(A \cap B) \subseteq {}^*B$, and so $^*(A \cap B) \subseteq {}^*A \cap {}^*B$.
We prove the converse implication. Consider some $[r] \in {}^*A \cap {}^*B$. It means that $[r] \in {}^*A$ and $[r] \in {}^*B$, thus $[[r \in A]], [[r \in B]] \in \mathcal{F}$. Therefore, $[[r \in A \cap B]] = [[r \in A]] \cap [[r \in B]] \in \mathcal{F}$, and so $[r] \in {}^*(A \cap B)$.

(c) First we prove the foward implication. Let $[r] \in {}^*(A \setminus B)$. Then we have $[[r \in A]] \cap [[r \notin B]] = [[r \in A \setminus B]] \in \mathcal{F}$. Thus, $[[r \in A]], [[r \notin B]] \in \mathcal{F}$. Hence, $[r] \in {}^*A$ and $[r] \notin {}^*B$, and so $[r] \in {}^*A \setminus {}^*B$.
We prove the converse implication. Let $[r] \in {}^*A \setminus {}^*B$ i.e. $[r] \in {}^*A$ and $[r] \notin {}^*B$. Then we have $[[r \in A]] \in \mathcal{F}$ and $[[r \notin B]] \in \mathcal{F}$. Thus, $[[r \in A \setminus B]] = [[r \in A]] \cap [[r \notin B]] \in \mathcal{F}$. Therefore, $r \in {}^*(A \setminus B)$.

(d) Clearly, $^*\emptyset = \{[r] \in {}^*\mathbb{R} : [[r \in \emptyset]] \in \mathcal{F}\} = \emptyset$. □

Exercise 3.12. *(Section 3.10, IV) Is it true that* $^*(\bigcup_{n=1}^{\infty} A_n) = \bigcup_{n=1}^{\infty} {}^*A_n$*?*

Proof. No. Define $A_n = \{n\}$ for any $n \in \mathbb{N}$. Notice that since A_n is finite, using (1), we have ${}^*A_n = A_n = \{n\}$. Since $[[\omega \in \bigcup_{n=1}^{\infty} A_n]] = \mathbb{N} \in \mathcal{F}$, we have $\omega \in {}^*(\bigcup_{n=1}^{\infty} A_n)$. However, $[[\omega \in A_n]] = \{n\} \notin \mathcal{F}$, thus $\omega \notin {}^*A_n$ for any $n \in \mathbb{N}$. This mean $\omega \notin \bigcup_{n=1}^{\infty} {}^*A_n$. □

Exercise 3.13. *(Section 3.10, V) Show that if $A \subseteq \mathbb{R}$, then ${}^*A \cap \mathbb{R} = A$.*

Proof. Suppose $A \subseteq \mathbb{R}$. Let $[r] \in {}^*A \cap \mathbb{R}$. Since $\langle r \rangle$ is a constant sequence and $[r] \in {}^*A$, it follows $[r] = \langle r \rangle \in A$. Thus, ${}^*A \cap \mathbb{R} \subseteq A$. This means that ${}^*A \cap \mathbb{R}$ contains only the constant sequences. Let $\langle r \rangle \in A$. Then clearly $[r] \in {}^*A \cap \mathbb{R}$. □

Exercise 3.14. *For $a, b \in \mathbb{R}$, let $[a, b]$ be the closed interval $\{x \in \mathbb{R} : a \leq x \leq n\}$. Prove that ${}^*[a, b] = \{x \in {}^*\mathbb{R} : a \leq x \leq b\}$.*

Proof. We first prove the forward inclusion. Let $[r] \in {}^*[a, b]$. Then we have $\{n \in \mathbb{N} : a \leq r_n \leq b\} = [[r \in [a, b]]] \in \mathcal{F}$ but that means $[r] \in \{x \in {}^*\mathbb{R} : a \leq x \leq b\}$. We prove the converse inclusion. Let $[r] \in \{x \in {}^*\mathbb{R} : a \leq x \leq b\}$. It follows $[[r \in [a, b]]] = \{n \in \mathbb{N} : a \leq r_n \leq b\} \in mcF$. Therefore, $[r] \in \{x \in \mathbb{R} : a \leq x \leq n\}$. □

Exercise 3.15. *(Section 3.10, VII) ${}^*\mathbb{Z}$ is a subring of ${}^*\mathbb{R}$.*

Proof. Clearly, ${}^*\mathbb{Z} \neq \emptyset$. Consider some $[r], [s] \in {}^*\mathbb{Z}$. Then we have $[[r \in \mathbb{Z}]], [[s \in \mathbb{Z}]] \in \mathcal{F}$. Therefore, since $[[r \in \mathbb{Z}]] \cap [[s \in \mathbb{Z}]] \subseteq [[r+s \in \mathbb{Z}]], [[rs \in \mathbb{Z}]]$, we have $[[r+s \in \mathbb{Z}]], [[rs \in \mathbb{Z}]] \in \mathcal{F}$. Hence, $[r+s], [rs] \in {}^*\mathbb{Z}$. □

Exercise 3.16. *(Section 3.10, VIII) If $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$, show that ${}^*(\mathbb{R}^+) = \{x \in {}^*\mathbb{R}^+ : x > 0\}$, i.e., ${}^*(R^+) = ({}^*\mathbb{R})^+$.*

Proof. First we prove the forward inclusion. Let $[r] \in {}^*(R^+)$. It means that $\{n \in \mathbb{N} : r_n > 0\} = [[r \in R^+]] \in \mathcal{F}$ which implies $[r] \in ({}^*\mathbb{R})^+$. Next we prove the converse inclusion. Let $[r] \in ({}^*\mathbb{R})^+$. Then we have $\{n \in \mathbb{N} : r_n > 0\} = [[r > 0]] = [[r \in \mathbb{R}^+]] \in \mathcal{F}$ from which it follows $[r] \in {}^*(\mathbb{R}^+)$. □

Exercise 3.17. *(Section 3.12, I) Show that ${}^*f$ agrees with f on $\mathbb{R}$: if $r \in \mathbb{R}$, then ${}^*f(r) = f(r)$.*

Proof. Let $r \in \mathbb{R}$. Notice that r can be indetified with the constant sequence $\mathbf{r} \in {}^*R$. Then we have ${}^*f([\mathbf{r}]) = [f(\mathbf{r})]$ which is isomorphic to $f(r)$. □

Exercise 3.18. *(Section 3.12, II) If f is injective, so is ${}^*f$. What about surjectivity?*

Proof. Suppose f is injective. Consider some $[r], [s] \in {}^*\mathbb{R}$ such that ${}^*f([r]) = {}^*f([s])$, that is to say $[f \circ r] = [f \circ s]$. Since f is injective, we have $[[f \circ r = f \circ s]] \subseteq [[r = s]]$ but then, since $[[f \circ r = f \circ s]] \in \mathcal{F}$, it follows $[[r = s]] \in \mathcal{F}$. Therefore, $[r] = [s]$.

Yes, surjecitivity is preserved. Suppose f is surjective. Consider some $[s] \in {}^*\mathbb{R}$. Our goal is to prove that there is some $[r] \in {}^*\mathbb{R}$ such that ${}^*f([r]) = [s]$. Define sequenece (r_n) such that, for each s_n, $f(r_n) = s_n$. We know such a r_n exists for each s_n since f is surjective. We claim that $[r] \in {}^*\mathbb{R}$ the element we looking for. To prove this claim, we need to show $[f \circ r] = f([r]) = [s]$, so we set that as our goal. Clearly, $[[f \circ r = s]] = \{n \in \mathbb{N} : f(r_n) = s_n\} = \mathbb{N} \in \mathcal{F}$. Therefore, $[f \circ r] = f[r] = [s]$. □

Exercise 3.19. *(Section 3.12, III) For $x \in {}^*\mathbb{R}$, let*

$$|x| = \begin{cases} x, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$$

be the usal definition of absolute value function. Show that this extends the definition of $|.|$ on $\mathbb{R}$: $|[r]| = [\langle |r_1|, |r_2|, \cdots \rangle] = [|r_n|]$.

Proof. We have the following cases;
Case-1. Suppose $[r] > 0$. Then $[[r > 0]] \in \mathcal{F}$. Since $[[r > 0]] \subseteq [[[r] = [|r|]]] \in \mathcal{F}$, we have $|[r]| = [r] = [|r_n|]$.
Case-2. Suppose $[r] = 0$. That is to say $[[r = 0]] \in \mathcal{F}$. Since $[[r = 0]] \subseteq [[0 = |r|]] \in \mathcal{F}$, we have $|[r]| = 0 = [|r|]$.
Case-3. Suppose $[r] < 0$. Then $|[r]| = -[r] = [-r] = [|r|]$. ☐

Exercise 3.20. *(Section 3.12, IV) Let χ_A be the characteristic function of a set $A \subseteq \mathbb{R}$. Show that $^*(\chi_A) = \chi_{^*A}$.*

Proof. Let $[r] \in {}^*\mathbb{R}$. Consider the expression $^*(\chi_A)[r]$, we have the following cases;
Case-1. Suppose $[r] \in {}^*A$. Notice that $^*(\chi_A)[r] = [\chi_A \circ r] = [\chi_A(r_n)]$ but then, since $[[r \in A]] \in \mathcal{F}$ and $[[r \in A]] \subseteq [[\chi_A \circ r = 1]]$, we have $[[\chi_A \circ r = 1]] \in \mathcal{F}$. Therefore, $[\chi_A \circ r] = \chi_A[r] = 1$.
Case-2. Suppose $[r] \notin {}^*A$. Then we have $[[r \notin A]] = [[r \in A]]^c \in \mathcal{F}$. Clearly, $[[r \notin A]] \subseteq [[\chi_A \circ r = 0]]$. Thus, $[[\chi_A \circ r = 0]] \in \mathcal{F}$ and so $[\chi_A \circ r] = \chi_A[r] = 0$. ☐

Exercise 3.21. *(Section 3.12, V) Show how to define $^*f$ when f is a function of more than one arguement.*

Proof. Define $^*f([r^1], \cdots, [r^n]) = [f(r^1_1, \cdots, r^n_1), f(r^1_2, \cdots, r^n_2), \cdots]$. We prove that it's well defined. Clearly, $[[r^1 = r^{1'}]] \cap \cdots [[r^n = r^{n'}]] \subseteq [[f \circ (r^1, \cdots, r^n) = f \circ (r^{1'}, \cdots, r^{n'})]]$. Therefore, $r^1 \equiv r^{1'}, \cdots, r^n \equiv r^{n'}$ implies $f \circ (r^1, \cdots, r^n) \equiv f \circ (r^{1'}, \cdots, r^{n'}$. ☐

Exercise 3.22. *(Section 3.15, I) If $A_1, \cdots, A_k$ are subsets of $\mathbb{R}$, put $P = A_1 \times \cdots \times A_k$ and apply the definition of $^*P$ to show that*

$$^*(A_1 \times \cdots \times A_k) = {}^*A_1 \times \cdots \times {}^*A_k.$$

Proof. For $([r^1], [r^2], \cdots, [r^k]) \in {}^*P$, we must have $[[(r^1, r^2, \cdots, r^k) \in P]] \in \mathcal{F}$. Clearly, $[[(r^1, r^2, \cdots, r^k) \in P]] \subseteq [[r^n \in A_n]]$, thus $[[r^n \in A_n]]$ for any n such that $1 \leq n \leq k$. This means that $[r^n] \in A^n$ for all such n. Hence, $([r^1], \cdots, [r^n]) \in {}^*A_1 \times \cdots {}^*A_n$. From this, it follows that $^*(\mathbb{R}^k) = ({}^*\mathbb{R})^k$. ☐

Exercise 3.23. *(Section 3.15, II) Let dom P denote the domain of a binary relation. If $P \subseteq \mathbb{R}^2$, show that $^*(dom\ P) = dom\ {}^*P$.*

Proof. First we prove the forward inclusion. Let $[r] \in {}^*(\text{dom } P)$. Then we have $\{n \in \mathbb{N} : r_n \in \text{dom } P\} = [[r \in \text{dom } P]] \in \mathcal{F}$.

$$\text{Define sequence } s_n = \begin{cases} s \text{ such that } (r_n, s) \in P, \text{ if } r_n \in \text{dom } P \\ 0, \text{ if } r_n \notin \text{dom } P \end{cases}$$

Then clearly, $[[(r, s) \in P]] = [[r \in \text{dom } P]] \in \mathcal{F}$, but that means $([r], [s]) \in {}^*P$, and so $[r] \in \text{dom } {}^*P$.
Now we prove the converse inclusion. Let $[r] \in \text{dom } {}^*P$. It follows that there exists $[s] \in {}^*\mathbb{R}$ such that $([r], [s]) \in {}^*P$, but that means $[[r \in \text{dom } P]] = [[(r, s) \in P]] \in \mathcal{F}$. Hence, $[r] \in {}^*(\text{dom } P)$. ☐

Chapter 4

The Transfer Principle

Exercise 4.1.