This document contains a model description, descriptions of the blocks in my code, the derivation for the Euler equation used to solve the household problem and a description of PIGL preferences.

1 Model

I consider a small open economy heterogeneous-agent New-Keynesian (SOE-HANK), featuring idiosyncratic income risk and borrowing constraints, sticky wages and flexible prices and an exogenously given foreign economy. There is a domestic non-tradable and tradable domestic sector, in addition, there are imports of foreign tradable goods and energy. The model builds on the I-HANK model presented (cite GEModelTools version). I extend the model to include household non-homothetic preferences between tradable and non-tradable goods and the energy good, nested in the tradable consumption aggregate.

The model features discrete time, and the horizon is infinite. There is no aggregate uncertainty in the model meaning households have prefect foresight with regards to aggregate variables and prices, following the shock which occurs in period 0. The variable I shock in this analysis is the price of energy in a foreign currency P_E^*.

Domestic Households

The home economy is inhabited by a continuum of infinitely lived households indexed by $i \in [0,1]$. Households are *ex ante* heterogeneous in terms of there sector of employment s_i which is either the non-tradable goods sector (NT) of the domestic tradable goods sector (HT).

Households are *ex post* heterogeneous in their time-varying stochastic productivity, z_{it}, and their end-of-period nominal savings, a_{it-1}. The distribution of households over idiosyncratic states is denoted $\underline{\boldsymbol{D}}_t$ before shocks are realized and $\boldsymbol{D}_t$ after the shocks are realized.

Households in each sector supply labor, n_{is}, determined by a sector specific union. They receive sector specific wages on their productive labor supply W_{st} and pay income taxes τ_t. Households can save in nominal assets a_{it} on which they recive nominal interest rates i_t. Households are constrained in borrowing, in this case set to 0. Households choose their own expenditure level e_{it} to optimize their recursive consumption-savings problem in nominal terms given by:

$$
\begin{aligned}
v_t(s_i, z_t, a_{t-1}) &= \max_{e_{it}} u(e_{it}, P_{NT,t}, P_{T,t}) - \xi(n_{is,t}) + \beta \mathbb{E}_t \left[v_{t+1}(s_i, z_{t+1}, a_t) \right] \\
\text{s.t. } a_{it} + e_{it} &= (1 + i_{t-1}) a_{it-1} + (1 - \tau_t) w_{s_i,t} n_{s_i,t} z_{it} \\
\log z_{it+1} &= \rho_z \log z_{it} + \psi_{it+1}, \quad \psi_{it} \sim \mathcal{N}(\mu_\psi, \sigma_\psi), \quad \mathbb{E}[z_{it}] = 1 \\
a_{it} &\geq 0
\end{aligned}
$$

where $z_{i,t}$ captures the idiosyncratic part of household earnings, which follows and AR(1) process in logs, β_t is the discount factor and $u(e_{it}, P_{NT,t}, P_{T,t})$ is the indirect utility function, which takes the PIGL functional form (discussed in section 3).

The disutility of labor takes the form

$$
\xi(n_{si,t}) = \varphi_s \frac{n_{is,t}^{1+\nu}}{1+\nu}
$$

where φ_s scales the labor disutility, and is ν the inverse of the Frisch elasticity. Since labor supply is chosen by unions it does not enter as a control variable in the individuals consumption-savings problem.

Household consumption basket

Household consumption nests several layers. Households have non-homothetic preferences between tradable and non-tradable goods c_{NT}, c_T (I omit the household and time indices i, t). Tradable goods are a CES combination of energy consumption c_E and non energy tradable consumptionc_{THF}, which in itself nests domestically produced tradable goods c_{HT} and foreign produced tradable goods c_{TF} (see figure 1). The consumption following the node with c_T is identical to the form in (Auclert, Monnery, Rognlie, and Straub 2023).

Household utility from tradable and non-tradable goods takes the form:

$$
v(e_, P_T, P_{NT}) = \frac{1}{\epsilon} \left[\left(\frac{e}{P_{NT}} \right)^{\epsilon} - 1 \right] - \frac{\nu}{\gamma} \left[\left(\frac{P_T}{P_{NT}} \right)^{\gamma} - 1 \right] \tag{1}
$$

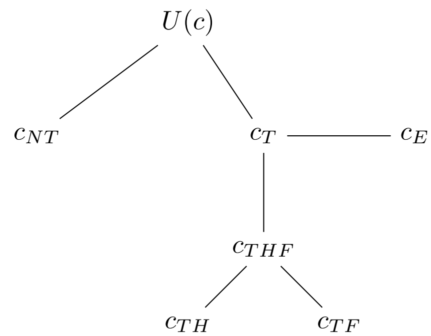

Figure 1: Composition of household consumption

Tradable goods are a CES nest of energy c_E and non-energy tradable consumption c_{THF}. Non-energy consumption consists of a CES nest of home tradable consumption and foreign tradable consumption. The bundles take the following form

$$c_T = [\alpha_E^{\frac{1}{\eta_E}} (c_E)^{\frac{\eta_E - 1}{\eta_E}} + (1 - \alpha_E)^{\frac{1}{\eta_E}} (c_{THF})^{\frac{\eta_E - 1}{\eta_E}}]^{\frac{\eta_E}{\eta_E - 1}}$$

$$c_{THF} = [\alpha_F^{\frac{1}{\eta_F}} (c_F)^{\frac{\eta_F - 1}{\eta_F}} + (1 - \alpha_F)^{\frac{1}{\eta_F}} (c_{THF})^{\frac{\eta_F - 1}{\eta_F}}]^{\frac{\eta_F}{\eta_F - 1}}$$

Where $\eta_E > 0$ is the elasticities of substitution between energy and non-energy consumption. $\alpha_E \in [0, 1]$ determines the share of energy in tradable consumption. $\eta_F > 0$ is the elasticities of substitution between foreign and domestic tradable goods and $(1 - \alpha_E) \in [0, 1]$ determines the home bias.

The price incidences for the intermediate bundles are:

$$P_{T,t} = \left[\alpha_E P_{E,t}^{1-\eta_E} + (1 - \alpha_E) P_{THF}^{1-\eta_E}\right]^{\frac{1}{1-\eta_E}}$$

$$P_{THF} = \left[\alpha_F P_F^{1-\eta_F} + (1 - \alpha_F) P_{TH}^{1-\eta_F}\right]^{\frac{1}{1-\eta_F}}$$

which is used to solve the consumption of tradable and non-tradable goods.

Using Roy's identity and the indirect utility form of PIGL preferences, I can derive the demand functions for tradable and non-tradable goods:

$$c_{T,it} = -\frac{\frac{\partial \upsilon}{\partial P_T}}{\frac{\partial \upsilon}{\partial e}} = \frac{e}{P_T} \left[\nu \left(\frac{P_{NT}}{e}\right)^{\epsilon} \left(\frac{P_T}{P_{NT}}\right)^{\gamma}\right]$$

$$c_{NT} = -\frac{\frac{\partial \upsilon}{\partial P_{NT}}}{\frac{\partial \upsilon}{\partial e}} = \frac{e}{P_N} \left[1 - \nu \left(\frac{P_{NT}}{e}\right)^{\epsilon} \left(\frac{P_T}{P_{NT}}\right)^{\gamma}\right]$$

Households differ in their consumption of tradable and non-tradable goods, but have the same consumption composition within tradable goods consumption. The CES form gives rise to the demand system:

$$c_E = \alpha_E \left(\frac{P_E}{P_T}\right)^{-\eta_E} c_T$$

$$c_{THF} = (1 - \alpha_E) \left(\frac{P_{THF}}{P_T}\right)^{-\eta_E} c_T$$

$$c_{TF} = \alpha_F \left(\frac{P_{TF}}{P_{THF}} \right)^{-\eta_E} c_{THF}$$

$$c_{TH} = (1 - \alpha_F) \left(\frac{P_{TH}}{P_{THF}} \right)^{-\eta_E} c_{THF}$$

Firms

A representative firm in each sector $s \in \{TH, NT\}$ hires labor $N_{s,t}$ to produce goods, with the production technology:

$$Y_{s,t} = Z_{s,t} N_{s,t}$$

where Z_t^s is the exogenous technology level, set to 1. Profits are:

$$\Pi_{s,t} = P_{s,t} Y_{s,t} - W_{s,t} N_{s,t}$$

Given $Z_{s,t} = 1$ first order condition for labor implies:

$$W_{s,t} = P_{s,t}$$

Unions

Labor supply is determined by unions. Households in each sector supply the same amount of labor

$$n_{s,t} = N_{s,t}^{hh}, \quad s \in \{T, NT\}$$

The sector specific New Keynesian wage Philips Curves are given by
Defining $\tilde{w}_{s,t} = (1 - \tau) \frac{W_{s,t}}{P_{NT,t}}$

$$\pi_{s,t}^w = \kappa \left(\underbrace{\varphi \left(N_{s,t} \right)^{\nu}}_{\text{mar. disutility of labor}} - \frac{1}{\mu} \underbrace{(1 - \tau_t) \tilde{w}_{s,t} U' \left(\tilde{E}_{s,t} \right)}_{\text{marg. utility of expenditure}} \right) + \beta \pi_{s,t+1}^w$$

where $1 + \pi_{s,t}^w = W_{s,t} / W_{s,t-1}$, κ is the slope parameter, and μ is a wage mark-up.

Central Bank

The domestic central bank controls the nominal interest rate i_t according to the Taylor rule:

$$i_t = i_{ss} + \phi_\pi \pi_{t+1} + \epsilon^i$$

where π_t is the inflation from period t to $t+1$. The coefficient on inflation $\phi_\pi = 1$ in my baseline analysis, resembling a *neutral* monetary policy (Auclert 2019).

Government

The government exogenously chooses real public consumption G_t and labor income τ_t and issues nominal bonds B_t. Public consumption is only in terms of non-tradable goods so the cost of government consumption is $G_t P_{NT,t}$.

$$B_t = (1 + i_{t-1}) B_{t-1} + P_{NT,t} G_t - \tau_t \sum_{s \in \{TH, NT\}} W_{s,t} N_{s,t}$$

The tax rule is:

$$\tau_t = \tau^{ss} + \omega_B \left(\frac{\frac{B_-}{P_{NT}} - \frac{B^{ss}}{P_{NT}}}{Y_{TH}^{ss} + Y_{NT}^{ss}} \right)$$

In steady state the budget is balanced, and the tax rule insures that the

The foreign economy

The rest of the world is taken exogenously. All foreign variables are sub-scripted (*).

Prices
The foreign price level in foreign currency is denoted $P_{F,t}^*$ and the nominal exchange rate is E. In home currency the price of foreign tradable is:

$$P_{F,t} = P_{F,t}^* E_t$$

The price of energy is converted with the same exchange rate:

$$P_{E,t} = P_{E,t}^* E_t$$

The price of domestically produced tradable, met by foreign households is

$$P_{TH}^* = \frac{P_{TH,t}}{E_t}$$

Households
The foreign demand for home tradable is given by and Armington relation:

$$C_{TH,t}^* = \alpha^* \left(\frac{P_{HT,t}^*}{P_{F,t}^*} \right) M^*$$

where the CES share is $\alpha^* \in [0, 1]$, the elasticity of forign demand is η^* and M^* is the size of the foreign markets.

Capital markets

Capital markets are free such that the uncovered interest rate parity must hold,

$$1 + i_t = (1 + i_t^f) \frac{E_{t+1}}{E_t}$$

Where i_t^f is the foreign nominal interest rate.

Lastly I have two clearing conditions on the domestic market:

$$Y_{T,t} = C_{TH,t} + C_{TH,t}^*$$
$$Y_{NT,t} = C_{NT,t} + G_t$$

2 Model Blocks - for code

Variables and Target Equations

Exogenous Variables (Shocks)

The exogenous variables/external shocks to the equation system are productivity levels, the size of the foreign economy, real foreign interest rates, foreign prices in foreign currency, and government consumption:

$$\{Z_{TH}, Z_{NT}, M_s^*, r^f, PF_s^*, G\}$$

Endogenous Variables

The endogenous variables are the nominal interest rate or the nominal exchange rate, depending on whether the exchange rate is fixed, labor supply in each sector, and wage inflation in each sector:

$$\{N_{NT}, N_{TH}, \pi_{W,TH}, \pi_{W,NT}, E/i\}$$

Target Equations

The target equations must hold for the system to be in equilibrium. These include the New Keynesian Wage Phillips Curves, the Uncovered Interest Parity (UIP), and market clearing conditions for the markets for non-tradable and domestic tradable goods:

$$\{\text{NKWC}_T, \text{NKWC}_{NT}, \text{clearing}_{YTH}, \text{clearing}_{YNT}, \text{UIP}\}$$

To simplify, I drop the sector subscript and use $s \in \{NT, HT\}$ to denote the domestic sectors: non-tradable goods (NT) and home tradable goods (HT), when there is no confusion.

The blocks are calculated for each $t \in T$. I drop the time subscripts for within-period variables and, when necessary, use subscript $+$ to indicate the next period and $-$ to indicate the preceding period.

1. **Firm Block**
   Inputs: $Z_s, N_s, \pi_{w,s}, W_s^{SE}$
   Outputs: Y_s, W_s, P_s

   This block calculates, for sectors $s \in \{NT, T\}$:

   - **Production:**
     Based on the guesses for labor supply (N_s) and the exogenous productivity (Z_s) for each sector s. Production is linear and given by:
     $$Y_s = Z_s N_s$$
   - **Wages:**
     Based on the guesses for wage inflation and the preceding nominal wage:
     $$W_s = \begin{cases} W_s^{SS}(1+\pi_{w,s}), & \text{if } t = 0, \\ W_{s-}(1+\pi_{w,s}), & \text{if } t > 0. \end{cases}$$
   - **Prices:**
     Given perfect competition, prices are equal to the marginal cost:
     $$P_s = \frac{W_s}{Z_s}$$

2. **Price Block**
   Inputs: $PF^*, E, PTH, PNT, WTH, WNT, \alpha_F, \eta_F, \alpha_T, \eta_T$
   Outputs: $PF, PTH_s, PT, \pi_{F^*}, \pi_F, \pi_{NT}, \pi_{TH}, \pi_T, \pi_{TH_s}$

   This block calculates:

- **Currency Conversion:**
The nominal exchange rate in home currency units per foreign currency is denoted E. Taking the foreign price level P_F^* as exogenously given, in home currency, the foreign price level is:

$$PF = PF^* E$$

The price of home tradable goods in the foreign currency is:

$$PTH^* = \frac{PTH}{E}$$

Where E is either an endogenous variable determined by a Taylor rule and the UIP or fixed $E = \bar{E}$, depending on the exchange rate regime.

- **Price Indies for intermediate Tradable Goods bundles:**

Home tradable and domestic tradable goods are combined in a CES aggregator with price index.

$$P_{THF} = \left(\alpha_F PF^{1-\eta_F} + (1-\alpha_F) PTH^{1-\eta_F}\right)^{\frac{1}{1-\eta_F}}$$

This composite is combined with energy to get the tradable goods price index:

$$P_{THF} = \left(\alpha_E P_E^{1-\eta_E} + (1-\alpha_E) P_{THF}^{1-\eta_E}\right)^{\frac{1}{1-\eta_E}}$$

- **Cost of Living price index**
In my baseline model I use a CES price index

$$P = \left(\bar{\omega}_T P_E^{1-\bar{\eta}_T} + (1-\bar{\omega}_T) P_{THF}^{1-\bar{\eta}_T}\right)^{\frac{1}{1-\bar{\eta}_T}}$$

Where $\bar{\omega}_T$ is the average expenditure on tradable in steady state and $\bar{\eta}_T$ is the elasticity of substitution of a household with the average expenditure.

- **Inflation:**
Inflation is calculated for all relevant variables so they can be used to inform monetary policy:

$$\pi_{F_s} = \frac{PF^*}{PF^*_-} - 1, \quad \pi_F = \frac{PF}{PF_-} - 1, \quad \pi_{NT} = \frac{PNT}{PNT_-} - 1$$

$$\pi_{TH} = \frac{PTH}{PTH_-} - 1, \quad \pi_T = \frac{PT}{PT_-} - 1 \quad \pi^{PIGL} = \frac{P^{PIGL}}{P^{PIGL}_-} - 1$$

3. **Monetary Policy Block**
Inputs: $\pi, \pi^{ss}, i^{ss}, i_{\text{shock}}, E/i$
Outputs: E/i

Monetary policy controls domestic nominal interest rates i. It is set either following a floating or to fix the exchange rate.

- **Floating Exchange Rate:**
i is set according to

$$i = i^{ss} + \phi^\pi \pi_+ + \epsilon^i$$

The inflation π_+ is the inflation type that the central bank targets. In the baseline scenario i set $\phi^\pi = 1$.

- **Fixed Exchange Rate:**
E is exogenous, and i is endogenously determined:

$$E = \bar{E}$$

4. **Government Budget Block**
   Inputs: $PNT, WTH, NTH, WNT, NNT, i_-, G, \tau^{ss}, B^{ss}, YTH^{ss}, YNT^{ss}$
   Outputs: B, τ

   The government budget constraint, in nominal terms, is:

$$B = (1 + i_-)B_- + P_{NT}G - \tau \sum_{s=1}^{2} W_s N_s$$

   Where B represents nominal government bonds, G represents the real government size.

   Taxes adjust dynamically:

$$\tau_t = \tau^{ss} + \omega \left(\frac{\frac{B_-}{P_{NT}} - \frac{B^{ss}}{P_{NT}}}{Y_{TH}^{ss} + Y_{NT}^{ss}} \right)$$

5. **Household Block**
   Inputs: $w_{\tau,TH}, w_{\tau,NT}, r_a$
   Outputs: $C, A, C_{NT}, C_T, C_{TH}, C_{TF}$

   - **Preparing**

     I calculate the inputs, deflated with the price of non-tradable goods:

$$\tilde{e} = \frac{e}{P_{NT}}, \quad \tilde{a} = \frac{a}{P_{NT}}, \quad p = \frac{P_T}{P_{NT}}, \quad \tilde{w}_{NT} = \frac{W_{NT}}{P_{NT}}, \quad \tilde{w}_{HT} = \frac{W_{HT}}{P_{NT}}, \quad (1 + \tilde{r}^a) = (1 + i_-)\frac{P_{NT,-}}{P_{NT}} \tag{2}$$

     where $(1 + r^a)$ is the return on beginning-of-period assets, deflated with prices $(\tilde{a}_-)$.

   - **Household problem**

     The household problem is solved using EGM. The Euler equation derivation can be seen in section 2.1.1

   - **Post calculations**

     The households problem outputs the aggregate household values for expenditure, assets measured in units of non-tradable goods, and consumption of: non-tradable goods, energy tradable goods, non-energy tradable goods, home tradable goods, and foreign tradable goods. Multiplied by P_{NT} to get nominal values for expenditure for assets:

$$E = P_{NT}E^{hh}, \quad A = P_{NT}A^{hh}$$

     The real aggregate consumption levels are given directly

$$\{C_{NT} = C_{NT}^{hh}, \quad C_T = C_T^{hh}, \quad C_{TH} = C_{TH}^{hh}, \quad C_{TF} = C_{TF}^{hh}, \quad C_E = C_E^{hh}, \quad C_{THF} = C_{THF}^{hh}\}$$

   - **Foreign Consumption Block**
     Inputs: P_{TH}^*, P_F^*, M^*
     Outputs: C_{HT}^*
     The foreign demand for home tradable goods is:

$$C_{HT}^* = \left(\frac{P_{TH}^*}{P_F^*} \right)^{-\eta^*} M^*$$

     Where M^* is the size of the foreign market, calibrated in steady state, and η^* is the elasticity of foreign demand.

6. **New Keynesian Wage Phillips Curve Block**
   Inputs: $\pi_{w,TH}, \pi_{w,NT}, N_{TH}, N_{NT}, w_{TH}, w_{NT}, \tau, E_{TH,hh}, E_{NT,hh}$
   Outputs: Phillips Curves Targets

   The wage inflation dynamics are as follows:

- **Tradable Sector:**

$$\pi_{W,TH} = \kappa \left[\varphi_{TH} \left(\frac{N_{TH}}{s_T} \right)^{-\nu} - \frac{1}{\mu_w}(1-\tau) W_{TH} U'(E^{hh}_{TH}) \right] + \beta \pi_{W,TH+} \quad \text{(Target)}$$

- **Non-Tradable Sector:**

$$\pi_{W,NT} = \kappa \left[\varphi_{NT} \left(\frac{N_{NT}}{1-s_T} \right)^{-\nu} - \frac{1}{\mu_w}(1-\tau) W_{NT} U'(E^{hh}_{NT}) \right] + \beta \pi_{W,NT+} \quad \text{(Target)}$$

Where $U'(E^{hh}_s)$ is the marginal utility of (average) nominal expenditure of household employed in sector s.

7. **Uncovered Interest Parity**
Inputs: i, i^f, E
Outputs: UIP Target

The nominal UIP is given by

$$1 + i = (1 + i^f) \frac{E_+}{E} \quad \text{(Target)}$$

A target which pins down the (endogenous) nominal exchange rate or interest rate, depending on the monetary regime. In my analysis i shock energy prices, whish are separate from the foreign price level, the foreign nominal interest rate is fixed.

8. **Market Clearing Block**
Inputs: $YTH, CTH, CTH_s, YNT, CNT, G$
Outputs: Targets and Clearing Conditions

- **Non-Tradable Goods:**

$$YNT = CNT + G \quad \text{(Target)}$$

- **Tradable Goods:**

$$YTH = CTH + CTH^* \quad \text{(Non-targeted clearing)}$$

9. **Accounting Block**
Inputs: $PTH, YTH, PNT, YNT, P, C_{hh}, G, A_{hh}, B, r_a$
Outputs: GDP, NX, CA, NFA, Walras

This block includes:

- **Nominal GDP:**

$$GDP = P_{TH} Y_{TH} + P_{NT} Y_{NT}$$

- **Net Exports (NX):**

Nominal production minus total expenditure (private EX and public $P_{NT}G$), is the nominal net exports

$$NX = GDP - EX - P_{NT} G$$

- **Current Account (CA):**

(Nominal)

$$CA = NX + i^f_- NFA_- \frac{E}{E_-}$$

- **Net Foreign Assets (NFA):**

(Nominal)

$$NFA = A - B$$

- **Walras' Law:**

$$\text{Walras} = (NFA - NFA_-) - CA$$

2.1 Household problem

2.1.1 Deriving the Euler equation

The dynamic household problem can be described by the bellman equation indirect utility function and the nominal budget constraint. i index dropped.

$$v_t(z_t, a_{t+1}) = \max_{e_t} \ u(e_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)] \tag{3}$$

$$e_t + a_t = (1 + i_{t-1})a_{t-1} + y(z_t) \tag{4}$$

$$y(z_t) = (1 - \tau)\ell W_t \tag{5}$$

$$e_t = P_{NT,t}C_{NT,t} + P_{T,t}C_{T,t} \tag{6}$$

$$\log z_t = \rho_z \log z_{t-1} + \psi_t^z \quad \psi_t^z \sim \mathcal{N}(0, \sigma_z^2) \tag{7}$$

$$u(e_t) = \frac{1}{\epsilon}\left[\left(\frac{e_t}{P_t^{NT}}\right)^{\epsilon} - 1\right] - \frac{\nu}{\gamma}\left[\left(\frac{P_t^T}{P_t^{NT}}\right)^{\gamma} - 1\right] \tag{8}$$

I define the following variables, deflated with the price of non-tradables

$$\tilde{e}_t = \frac{e_t}{P_{NT,t}} \quad \tilde{a}_t = \frac{a_t}{P_{NT,t}} \quad p_t = \frac{P_{T,t}}{P_{NT,t}} \quad \tilde{y}_t = \frac{y_t}{P_{NT}} \tag{9}$$

Dividing the budget constraint with $P_{NT,t}$

$$\tilde{e}_t + \tilde{a}_t = \frac{1}{P_{NT,t}}(1 + i_{t-1})a_{t-1} + \tilde{y}(z_t) \tag{10}$$

multiplying and dividing a_{t-1} by $P_{NT,t-1}$ to get ind real in terms of Non-tradables

$$\frac{a_{t-1}}{P_{NT,t-1}}\frac{P_{NT,t-1}}{P_{NT,t}} = \tilde{a}_{t-1}\frac{P_{NT,t-1}}{P_{NT,t}} \tag{11}$$

And defining $R_t = (1 + i_{t-1})\frac{P_{NT,t-1}}{P_{NT,t}}$ as the *ex post* "real" interest rate, the return on $\tilde{a}$. Gives the familiar budget constraint:

$$\tilde{e}_t + \tilde{a}_t = R_t\tilde{a}_{t-1} + \tilde{y}(z_t) \tag{12}$$

This can likewise be inserted in the utility function to get:

$$u(\tilde{e}_t) = \frac{1}{\epsilon}\left[(\tilde{e}_t)^{\epsilon} - 1\right] - \frac{\nu}{\gamma}\left[(p_t)^{\gamma} - 1\right] \tag{13}$$

The first order condition of w.r.t. the new choice variable $\tilde{e}_t$ from the bellman equation.

$$\begin{aligned} u'(\tilde{e}_t) + \beta\frac{\partial \tilde{a}}{\partial \tilde{e}_t}\mathbb{E}_t[v'_{t+1}(z_{t+1}, \tilde{a}_t)] &= 0 \\ \frac{\partial \tilde{a}}{\partial \tilde{e}_t} &= -1 \\ u'(\tilde{e}_t) &= \beta\mathbb{E}_t[v'_{t+1}(z_{t+1}, \tilde{a}_t)] \end{aligned} \tag{14}$$

Defining the beginning of period value function as $\underline{v}_{t+1} = \mathbb{E}_t[v_{t+1}(z_{t+1}, \tilde{a}_t)]$ with derivative $\underline{v}'_{t+1} = \mathbb{E}_t[v'_{t+1}(z_{t+1}, \tilde{a}_t)]$ (wrt. $\tilde{a}_{t-1}$) Inserting in the FOC:

$$u'(\tilde{e}_t) = \beta\underline{v}'_{t+1} \tag{15}$$

Defining the $\tilde{e}(\tilde{a}_{t-1})$ as the optimal expenditure, given $\tilde{a}_{t-1}$. I insert this in the value function and differentiate, using the chain rule and the envelope theorem:

$$\begin{aligned} v_t'(\tilde{a}_{t-1}, z_t) &= \frac{\partial}{\partial \tilde{a}_{t-1}}[u(\tilde{e}(\tilde{a}_{t-1})) + \beta \mathbb{E}[v_{t+1}(\tilde{a}_t, z_{t+1})]] \\ &= \beta \frac{\partial \tilde{a}_t}{\partial \tilde{a}_{t-1}} \mathbb{E}[v_{t+1}'(\tilde{a}_t, z_t)] \end{aligned}$$

From the budget constraint

$$\frac{\partial \tilde{a}_t}{\partial \tilde{a}_{t-1}} = R_t \tag{16}$$

And notation for beginning of period value function

$$v_t' = R_t \beta \underline{v}_{t+1}' \tag{17}$$

I insert from the FOC w.r.t. $\tilde{e}_t$

$$v_t' = R_t u'(\tilde{e}_t) \tag{18}$$

Iterating one period forwards

$$v_{t+1}' = R_{t+1} u'(\tilde{e}_{t+1}) \tag{19}$$

Inserting v_{t+1}' in the FOC:

$$u'(\tilde{e}_t) = \beta R_{t+1} u'(\tilde{e}_{t+1}) \tag{20}$$

Which is the familiar Euler equation

For the specific utility function the first order condition is

$$\begin{aligned} u'(\tilde{e}_t) &= \tilde{e}_t^{-(1-\epsilon)} = \beta \underline{v}_{t+1}' \\ \leftrightarrow \tilde{e}_t &= \left(\beta \underline{v}_{t+1}'(z_t, \tilde{a}_t)\right)^{\frac{1}{-(1-\epsilon)}} \end{aligned} \tag{21}$$

And the beginning of period marginal value function is:

$$\underline{v}_{t+1}'(z_t, \tilde{a}_t) = \mathbb{E}_t[R_{t+1} u(\tilde{e}_t)] = \mathbb{E}_t[R_{t+1} \tilde{e}_t^{-(1-\epsilon)}] \tag{22}$$

Households nominal expenditure e_t and nominal end of period assets a_t can of course be found by multiplying with the price of non-tradable goods.

3 Non-homothetic preferences

My paper examines the implications of differences in consumption bundles in the face of foreign price shocks, both at the household and aggregate levels. This calls for a non-homothetic preference structure, that is, a preference type which accommodates income-dependent consumption choices[1]. To address this, I follow Boehnert, de Ferra, Mitman, and Romei (2023) and adopt preferences introduced by Boppart (2014), which are a subclass of Price Independent Generalized Linearity (PIGL), as defined by Muellbauer, Muellbauer (1975, 1976). These PIGL preferences are non-homothetic. Following Boehnert et al. (2023), I classify tradable goods as necessities and non-tradable goods as luxuries[2]. Figure 2 provides empirical evidence supporting this relationship. This could be thought of as low-income households buying groceries, while high-income households dine out.

PIGL is a very geneal class of preferences which nests both non-homothetic CES preferences and Cobb-Douglas preferences. This flexibility enables me to compare the implications of incorporating non-homotheticity into the models, both at the individual level and for aggregate quantities. An additional feature of this preference class is

[1]Expenditure is monotonically increasing in income, so differences caused by income differences are the same as differences due to expenditure levels.

[2]This is in documented in much literature such as Carroll and Hur (2020) for the US, Fajgelbaum and Khandelwal, **?** (2016, **?**) on a country level, and as a robustnes also with microdata within the US, who also find evidence of non-constant elasticities of substitution. While some find evidence that point in the opposit direction, in the US (Borusyak and Jaravel 2021)

its tractable aggregation properties, which make it possible to analysis of how groups are affected by price changes, using a functional form equivalent to that of individual households(Hochmuth, Pettersson, and Weissert 2023).

The preference class has other desirable properties and was proposed by Boppart (2014) to satisfy both the Kaldor facts and sectoral changes both changes. It has since been used mostly in the structural change literature.[3]

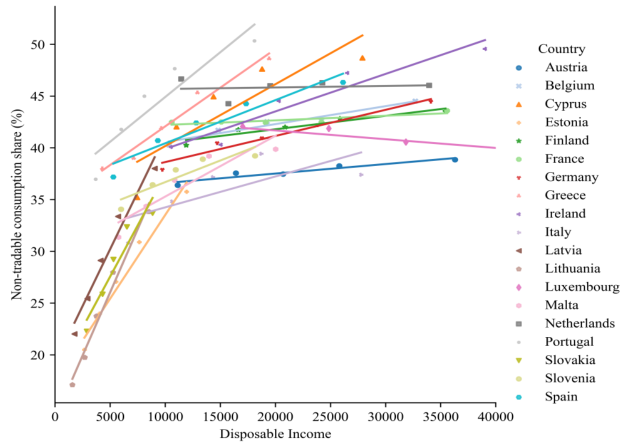

Figure 2: Non-tradable consumption share

3.0.1 PIGL - Functional form

It is not possible to have an explicit direct utility function. The indirect intratemporal utility function takes the form:

$$\upsilon(e_{it}, P_{T,t}, P_{NT,t}) = \frac{1}{\epsilon}\left[\left(\frac{e_{it}}{P_{NT,t}}\right)^{\epsilon} - 1\right] - \frac{\nu}{\gamma}\left[\left(\frac{P_{T,t}}{P_{NT,t}}\right)^{\gamma} - 1\right] \tag{23}$$

Where e_{it} is the total nominal expenditure of household i in period t. $P_{T,t}$ and $P_{NT,t}$ denote the prices of tradables and non-tradables. The expenditure satisfies $e_{it} = c_{T,it}P_{T,t} + c_{NT,it}P_{NT,t}$, where $c_{T,it}$ and $c_{NT,it}$ are households' consumption of tradable and non-tradable goods, respectively. In accordance with the empirical evidence presented by Boehnert et al. (2023), tradables are a necessary good, and non-tradables are a luxury good. The parameters have the restrictions $\epsilon, \gamma \in (0,1)$ and $\nu \geq 0$, as in Boppart (2014). .

Using Roy's identity, I can derive the demand functions for tradables and non-tradables:

$$\begin{aligned} c_{T,it} &= -\frac{\frac{\partial \upsilon}{\partial P_{T,t}}}{\frac{\partial \upsilon}{\partial e_{it}}} = \frac{e_{jt}}{P_T, t}\left[\nu\left(\frac{P_{NT,t}}{e_{it}}\right)^{\epsilon}\left(\frac{P_{T,t}}{P_{NT,t}}\right)^{\gamma}\right] \\ c_{NT,it} &= -\frac{\frac{\partial \upsilon}{\partial P_{NT,t}}}{\frac{\partial \upsilon}{\partial e_{it}}} = \frac{e_{jt}}{P_N, t}\left[1 - \nu\left(\frac{P_{NT,t}}{e_{it}}\right)^{\epsilon}\left(\frac{P_{T,t}}{P_{NT,t}}\right)^{\gamma}\right] \end{aligned} \tag{24}$$

With the corresponding expenditure shares:

$$\begin{aligned} \omega_{T,it} &\equiv \frac{P_{T,t}c_{T,t}}{e_{it}} = \nu\left(\frac{P_{N,t}}{e_{it}}\right)^{\epsilon}\left(\frac{P_{T,t}}{P_{NT,t}}\right)^{\gamma} \\ \omega_{NT,it} &= 1 - \omega_{T,it} = 1 - \nu\left(\frac{P_{N,t}}{e_{it}}\right)^{\epsilon}\left(\frac{P_{T,t}}{P_{NT,t}}\right)^{\gamma} \end{aligned} \tag{25}$$

Given that expenditure is above a certain threshold of expenditure e_{it}, such that the shares make sense.

[3]Such as quantifying the role of population aging in the structural transformation process (Cravino, Levchenko, and Rojas 2022). How structural transformation in most currently developing countries takes the form of a rapid rise in services but limited industrialization (Fan, Peters, and Zilibotti 2023)

3.0.2 PIGL - Properties and parameters

Three key properties of this utility form are worth emphasizing: (1) non-homothetic preferences with respect to income (2) non-constant elasticity of substitution and (3) relative risk aversion. These properties are governed by only two parameters.

1. **Non-homothetic consumption:**
The parameter ϵ controls the degree of non-homotheticity. The expenditure elasticity for the demand for tradables is given by:
$$1-\epsilon$$
Given $> \epsilon > 0$, the expenditure elasticity of tradable good consumption is positive and less than unity, making tradables a necessary good, meaning low-income households allocate a larger share of their total expenditure toward tradables compared to high-income households. The larger ϵ is, the more sensitive the expenditure share is to changes in total expenditure (Figure 3). In the limit where $\epsilon \to 0$, I obtain homothetic preferences as the expenditure elasticity for both tradables and non-tradables approaches unity. (Boppart 2014)

2. **Non-constant elasticity of substitution:**
The parameter γ controls the non-constant elasticity of substitution between tradable and non-tradable goods, given by:
$$\sigma(e_i) = 1-\gamma - \frac{\nu\left[\frac{P_T}{PNT}\right]^{\gamma}}{\left[\frac{e_i}{P_{PNT}}\right]^{\epsilon} - \nu\left[\frac{P_T}{P_{NT}}\right]^{\gamma}}(\gamma-\epsilon)$$
as follows from Lemma 3 of Boppart (2014). Adding the additional assumption $0 \leq \epsilon \leq \gamma < 1$ ensures that this elasticity is always less than unity, which will hold with my parameter choices. Under these restrictions, the elasticity of substitution is weakly increasing with the expenditure level (figure 4. This is in line with empirical evidence (**? ?**). An intuitive argument is that if you have very low income, you still need to buy groceries. On the other hand, if all your basic needs are met, you may not be as reliant on individual goods to make you happy. If $\gamma = \epsilon$, the last term drops out, resulting in a constant elasticity of substitution. So the function nests non-homothetic CES preferences, where consumption bundles still vary with income.
If $(\epsilon, \gamma) \to (0, 0)$, preferences become homothetic Cobb-Douglas.

3. **Relative risk aversion:**
The parameter ϵ also controls the relative risk aversion (RRA), given by $1-\epsilon$ (equal to the expenditure elasticity for the demand for tradables). This determines the inter-temporal elasticity of substitution (EIS), given by $\frac{1}{1-\epsilon}$. Figure 5 shows how ϵ affects the utility of expenditure. All very standard.

4. **Scaler**:
Lastly, ν is a scaling parameter that controls the demand for tradable goods. When $\nu = 0$, preferences become homothetic, and households consume only non-tradable goods.

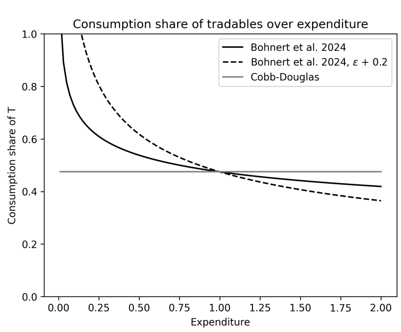

Figure 3: Non-homothetic

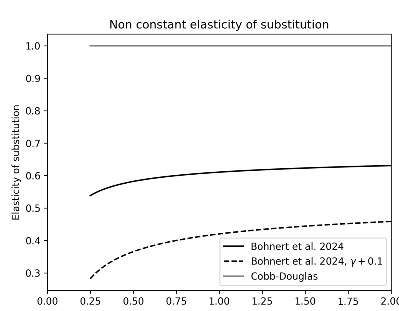

Figure 4: Substitutability

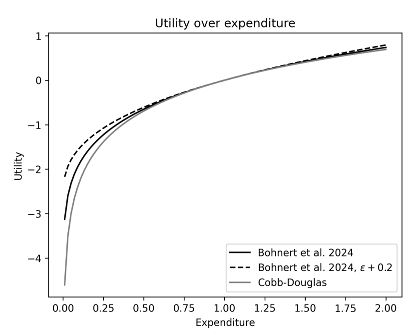

Figure 5: Utility

Note: Parameters compared to Boehnert et al. (2023)

3.0.3 Measuring Welfare

I conduct a comprehensive welfare analysis using two key measures to assess changes in well-being. The first is the PIGL cost of living index, which captures welfare changes resulting from price changes accounting for variations in consumption bundles. The second measure is consumption variation, which takes into account the general equilibrium affects.

I measure the welfare changes associated in consumption variation; the amount of additional initial assets a household would need to reach their ex ante initial life time utility (expected) after a shock occurs. In order to gain a full picture, I calculate it for each state in the state space. The ϵ which solves

$$v(s, z, a_{t+1} + \epsilon) = v'(s, z, a_{t+1}) \tag{26}$$

Where

I calculate welfare changes as the change in initial assets required to maintain a before shock utility level for each state. I calculate it as follows.

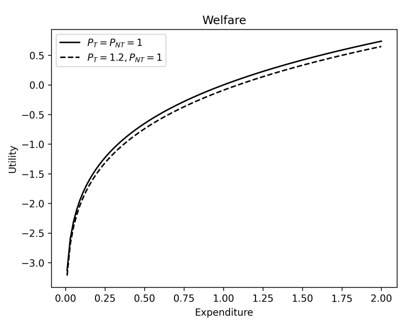

Figure 6: Welfare levels

TBD

Figure 7: Welfare changes

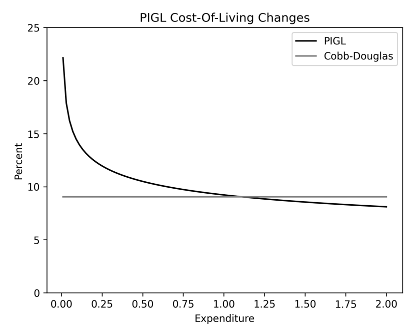

Figure 8: Price inequality

3.0.4 PIGL cost-of-living index

The non-homothetic preferences complicate the measurement of individual and aggregate inflation. I follow the recent contribution to the literature on price index theory by Hochmuth et al. (2023) and use their PIGL cost of living index. This index is derived from the Konüs (1939) definition of a "*true index of the cost of living*" as the percentage change in expenditure needed to maintain a fixed utility level following a price change. By holding utility constant, this measure captures only price changes and substitution effects.[4]

The cost-of-living index for an individual, relative to a base period s, in period t, is given by:

$$p_{it} = \tilde{p}_{it}^{\frac{\gamma}{\epsilon}} P_{NT,t}^{1-\frac{\gamma}{\epsilon}} \tag{27}$$

where

$$\tilde{p}_{it} = \left[\left(1 - \frac{\epsilon \omega_{T,is}}{\gamma} \right) P_{NT,t}^{\gamma} + \frac{\epsilon \omega_{T,is}}{\gamma} P_{T,t}^{\gamma} \right]^{\frac{1}{\gamma}}$$

This is Proposition 1 from Hochmuth et al. (2023). When poorer households spend a larger share of their expenditure on tradables (ω_T), the prices of tradables are weighted more heavily when determining the overall change in the cost of living.

The aggregate form of the cost-of-living index takes the same form as Equation **??**, with the average expenditure on tradable goods $\bar{\omega}_T$ serving as the tradable share. This yields the aggregate price index P^{RA}, which is consistent with the aggregation of individual households' price indices:

$$P_{RA,t} = \left[\int_0^N \mu_h (P_{ht})^{\epsilon} dh \right] \tag{28}$$

[4]See also Hochmuth et al. (2023) for a comparison of the income biases that can arise when using other superlative price indices.

Where μ_h reflects each individual's importance in the aggregated expenditure share on tradables. P^{RA} corresponds to a representative agent that embodies a social welfare function with distributional weights μ_h. One way these weights can be assigned is based on expenditure share, $\mu_h = e_h/E$, where E is the aggregate expenditure, also known as the *plutocratic* index (Prais 1959). An alternative is to assign equal weight to each household, $\mu_h = 1/N$, referred to as the *democratic* index (Prais 1959).

This cost-of-living index allows for capturing the effects of increased prices at both the individual and group levels, without relying on how distributions are affected, as the index requires only information about initial consumption shares and price changes

3.0.5 Other non-homothetic preferences

An alternative natural candidat for non-homothetic preferences are stone geary preferences, which have the functional form

$$U(C) = \sum_{i=1}^{I} \omega_i \frac{(C_i - \bar{C}_i)^{\frac{\sigma-1}{\sigma}}}{\frac{\sigma-1}{\sigma}} \tag{29}$$

Though this form has been used both in empirical work (**? ?**) and in in HANK literature see Auclert, Rognlie, Souchier, and Straub (2021) and Auclert et al. (2023) and –, it has several drawbacks. Firstly, preferences are asymptotically homothetic, which means the non-homotheticities are only relevant for poor households in poor countries secondly it does not retain the aggration properties, relevant for constructing a meaningful price level, for which to do monetary policy, which could be problem, when studying monetary policy. For more see Romero (2020).

15

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