UNIT - V

MULTIVARIATE ANALYSIS

Random Vectors and Matrices

A random vector is a vector whose elements are random variables.

A random matrix is a matrix whose elements are random variables.

The expected value of a random matrix is the matrix consisting of the expected values of each of its elements.

ie) Let $X = \{x_{ij}\}$ be an $n \times p$ random matrix. Then the expected value of X, denoted by $E(X)$, is the $n \times p$ matrix of numbers.

$$E(X) = \begin{bmatrix} E(X_{11}) & E(X_{12}) & \cdots & E(X_{1p}) \\ E(X_{21}) & E(X_{22}) & \cdots & E(X_{2p}) \\ \vdots & \vdots & \ddots & \vdots \\ E(X_{n1}) & E(X_{n2}) & \cdots & E(X_{np}) \end{bmatrix}$$

where, for each element of the matrix,

$$E(X_{ij}) = \begin{cases} \int_{-\infty}^{\infty} x_{ij} f_{ij}(x_{ij})\, dx_{ij} & \text{if } X_{ij} \text{ is a continuous random variable with density fn. } f_{ij}(x_{ij}) \\ \sum_{\text{all } x_{ij}} x_{ij} P_{ij}(x_{ij}) & \text{if } X_{ij} \text{ is a discrete random variable with probability fn. } P_{ij}(x_{ij}) \end{cases}$$

Example:

(computing expected values for discrete random variables)

Suppose $p=2$ and $n=1$, and consider the random vector $x' = [x_1, x_2]$. Let the discrete r.v X_1 have the following probability function:

x_1	-1	0	1
$P_1(x_1)$	$.3$	$.3$	$.4$

Then $E(X_1) = \sum_{\text{all } x_1} x_1 \, P_1(x_1) = (-1)(0.3) + (0)(0.3) + (0.4)\cdot 1$

$= .1$

x_2	0	1
$P_2(x_2)$	$.8$	$.2$

Then $E(X_2) = \sum_{\text{all } x_2} x_2 \, P_2(x_2) = (0)(.8) + 1(.2)$

$= .2$

Thus $E(X) = \begin{bmatrix} E(X_1) \\ E(X_2) \end{bmatrix} = \begin{bmatrix} .1 \\ .2 \end{bmatrix}$

Mean Vectors and covariance matrices

Suppose $x' = [x_1, x_2, \ldots, x_p]$ is a $p \times 1$ random vector. Then each element of x is a random variable with its own marginal probability distribution.

The marginal means μ_i and variances σ_i^2 are defined as $\mu_i = E(X_i)$ and $\sigma_i^2 = E(X_i - \mu_i)^2$, $i = 1, 2, \ldots, p$.

Specifically,

$$\mu_i = \begin{cases} \int_{-\infty}^{\infty} x_i f_i(x_i)\, dx_i & \text{if } X_i \text{ is a continuous r.v with probability density fn. } f_i(x_i) \\ \sum_{\text{all } x_i} x_i\, p_i(x_i) & \text{if } X_i \text{ is a discrete r.v with probability fn. } p_i(x_i) \end{cases}$$

$$\sigma_i^2 = \begin{cases} \int_{-\infty}^{\infty} (x_i-\mu_i)^2 f_i(x_i)\, dx_i & \text{if } X_i \text{ is a continuous r.v with probability density function } f_i(x_i) \\ \sum_{\text{all } x_i} (x_i-\mu_i)^2\, p_i(x_i) & \text{if } X_i \text{ is a discrete random variable with probability function } p_i(x_i) \end{cases}$$

For any pair of random variables, such as X_i and X_k, their covariance is defined as follows

$$\boxed{\sigma_{ik} = E(X_i-\mu_i)(X_k-\mu_k)}$$

$$= \begin{cases} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x_i-\mu_i)(x_k-\mu_k)\, f_{ik}(x_i, x_k)\, dx_i\, dx_k & \text{if } X_i, X_k \text{ are continuous random variables with joint density function } f_{ik}(x_i, x_k). \\ \sum_{\text{all } x_i}\sum_{\text{all } x_k} (x_i-\mu_i)(x_k-\mu_k)\, p_{ik}(x_i, x_k) & \text{if } X_i, X_k \text{ are discrete random variables with joint probability function } p_{ik}(x_i, x_k) \end{cases}$$

The means and covariances of the $p \times 1$ random vector X can be set out as matrices.

The expected value of each element is contained in the vector of means $\mu = E(X)$, and the p variances σ_{ii} and the $p(p-1)/2$ distinct covariances σ_{ik} $(i<k)$ are contained in the symmetric variance covariance matrix

$$\Sigma = E(X-\mu)(X-\mu)'.$$

Specifically,

$$E(X) = \begin{bmatrix} E(X_1) \\ E(X_2) \\ \vdots \\ E(X_p) \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_p \end{bmatrix} = \mu.$$

and

$$\Sigma = E(X-\mu)(X-\mu)'$$

$$= E\left(\begin{bmatrix} X_1-\mu_1 \\ X_2-\mu_2 \\ \vdots \\ X_p-\mu_p \end{bmatrix} \begin{bmatrix} X_1-\mu_1 & X_2-\mu_2 & \dots & X_p-\mu_p \end{bmatrix} \right)$$

$p \times 1$

$1 \times p$

$$= E \begin{bmatrix} (X_1-\mu_1)^2 & (X_1-\mu_1)(X_2-\mu_2) & \dots & (X_1-\mu_1)(X_p-\mu_p) \\ (X_2-\mu_2)(X_1-\mu_1) & (X_2-\mu_2)^2 & \dots & (X_2-\mu_2)(X_p-\mu_p) \\ \vdots & \vdots & \ddots & \vdots \\ (X_p-\mu_p)(X_1-\mu_1) & (X_p-\mu_p)(X_2-\mu_2) & \dots & (X_p-\mu_p)^2 \end{bmatrix}$$

2×3

3×2

$$
= \begin{bmatrix}
E(X_1-\mu_1)^2 & E(X_1-\mu_1)(X_2-\mu_2) & \cdots & E(X_1-\mu_1)(X_p-\mu_p) \\
E(X_2-\mu_2)(X_1-\mu_1) & E(X_2-\mu_2)^2 & \cdots & E(X_2-\mu_2)(X_p-\mu_p) \\
\vdots & \vdots & \ddots & \vdots \\
E(X_p-\mu_p)(X_1-\mu_1) & E(X_p-\mu_p)(X_2-\mu_2) & \cdots & E(X_p-\mu_p)^2
\end{bmatrix}
$$

(or)

$$
\Sigma = \operatorname{cov}(X) = \begin{bmatrix}
\sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p} \\
\sigma_{21} & \sigma_{22} & \cdots & \sigma_{2p} \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_{p1} & \sigma_{p2} & \cdots & \sigma_{pp}
\end{bmatrix}
$$

Example:. (computing the covariance matrix)

Find the covariance matrix for the two random variables X_1 and X_2 introduced in the above Example when their joint probability function $P_{12}(x_1, x_2)$ is represented by the entries in the body of the following table

x_1 \ x_2	0	1	$P_1(x_1)$
-1	.24	.06	.3
0	.16	.14	.3
1	.40	.00	.4
$P_2(x_2)$	.8	.2	1

We have already shown that

$$\mu_1 = E(X_1) = 0.1 , \quad \mu_2 = E(X_2) = 0.2 .$$

Now $\sigma_{11} = E\,(X_1 - \mu_1)\,(X_1 - \mu_1)$

$$= \sum_{\text{all } x_1} (x_1 - 0.1)\,(x_1 - 0.1)\; P_1(x_1)$$

$$= \sum_{\text{all } x_1} (x_1 - 0.1)^2 \; P_1(x_1)$$

$$= (-1 - 0.1)^2 (0.3) + (0 - 0.1)^2 (0.3) + (1 - 0.1)^2 (0.4)$$

$$= 0.69$$

$$\sigma_{12} = E\,(X_1 - \mu_1)\,(X_2 - \mu_2)$$

$$= \sum_{\text{all pairs}(x_1, x_2)} (x_1 - 0.1)\,(x_2 - 0.2)\; P_{12}(x_1, x_2)$$

$$= (-1-0.1)\,(0-0.2)\,(0.24) +$$
$$(-1-0.1)\,(1-0.2)\,(0.06) +$$
$$(0-0.1)\,(0-0.2)\,(0.16) +$$
$$(0-0.1)\,(1-0.2)\,(0.14) +$$
$$(1-0.1)\,(0-0.2)\,(0.40) +$$
$$(1-0.1)\,(1-0.2)\,(0.00)$$

$$= -0.08$$

$$\sigma_{21} = E\,(X_2 - \mu_2)\,(X_1 - \mu_1)$$

$$= E(X_1 - \mu_1)\,(X_2 - \mu_2) = \sigma_{12} = -0.08$$

$$\sigma_{22} = E\,(X_2 - \mu_2)^2$$

$$= \sum_{\text{all } x_2} (x_2 - 0.2)^2 \; P_2(x_2)$$

$$= (0 - 0.2)^2 (0.8) + (1 - 0.2)^2 (0.2)$$

$$\sigma_{22} = 0.16$$

consequently, with $x' = [x_1, x_2]$,

$$\mu = E(X) = \begin{bmatrix} E(X_1) \\ E(X_2) \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}'$$

and $\Sigma = E(x-\mu)(x-\mu)'$

$$= E\left(\begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix} [x_1-\mu_1 \quad x_2-\mu_2] \right)$$

$$= E \begin{bmatrix} (x_1-\mu_1)^2 & (x_1-\mu_1)(x_2-\mu_2) \\ (x_2-\mu_2)(x_1-\mu_1) & (x_2-\mu_2)^2 \end{bmatrix}$$

$$= \begin{bmatrix} E(x_1-\mu_1)^2 & E(x_1-\mu_1)(x_2-\mu_2) \\ E(x_2-\mu_2)(x_1-\mu_1) & E(x_2-\mu_2)^2 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 0.69 & -0.08 \\ -0.08 & 0.16 \end{bmatrix}$$

which is the required covariance matrix.

Correlation matrix from the covariance matrix

The correlation coefficient ρ_{ik} is defined as

$$\rho_{ik} = \frac{\sigma_{ik}}{\sqrt{\sigma_{ii}}\sqrt{\sigma_{kk}}}$$

Let the population correlation matrix be the $p \times p$ symmetric matrix

$$\rho = \begin{bmatrix} \frac{\sigma_{11}}{\sqrt{\sigma_{11}}\sqrt{\sigma_{11}}} & \frac{\sigma_{12}}{\sqrt{\sigma_{11}}\sqrt{\sigma_{22}}} & \cdots & \frac{\sigma_{1p}}{\sqrt{\sigma_{11}}\sqrt{\sigma_{pp}}} \\ \frac{\sigma_{12}}{\sqrt{\sigma_{11}}\sqrt{\sigma_{22}}} & \frac{\sigma_{22}}{\sqrt{\sigma_{22}}\sqrt{\sigma_{22}}} & \cdots & \frac{\sigma_{2p}}{\sqrt{\sigma_{22}}\sqrt{\sigma_{pp}}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\sigma_{1p}}{\sqrt{\sigma_{11}}\sqrt{\sigma_{pp}}} & \frac{\sigma_{2p}}{\sqrt{\sigma_{22}}\sqrt{\sigma_{pp}}} & \cdots & \frac{\sigma_{pp}}{\sqrt{\sigma_{pp}}\sqrt{\sigma_{pp}}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \rho_{12} & \cdots & \rho_{1p} \\ \rho_{12} & 1 & \cdots & \rho_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{1p} & \rho_{2p} & \cdots & 1 \end{bmatrix}$$

$\frac{\sqrt{\sigma_{11}}\ \sigma_{12}}{\sqrt{\sigma_{11}}\sqrt{\sigma_{22}}}$

and let the $p \times p$ standard deviation matrix be

$$V^{1/2} = \begin{bmatrix} \sqrt{\sigma_{11}} & 0 & \cdots & 0 \\ 0 & \sqrt{\sigma_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{\sigma_{nn}} \end{bmatrix}$$

Then it is easily verified that $V^{1/2} \rho V^{1/2} = \Sigma$.

$$\Rightarrow \rho = (V^{1/2})^{-1} \Sigma (V^{1/2})^{-1}.$$

Example:

[computing the correlation matrix from the covariance matrix.]

Suppose $\Sigma = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 9 & -3 \\ 2 & -3 & 25 \end{bmatrix}$

Obtain $V^{1/2}$ and ρ.

Solution:

Given $\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 9 & -3 \\ 2 & -3 & 25 \end{bmatrix}$

$$V^{1/2} = \begin{bmatrix} \sqrt{\sigma_{11}} & 0 & 0 \\ 0 & \sqrt{\sigma_{22}} & 0 \\ 0 & 0 & \sqrt{\sigma_{33}} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

We know that $\rho = (V^{1/2})^{-1} \Sigma (V^{1/2})^{-1}$.

Now $(V^{1/2})^{-1} = \dfrac{1}{|V^{1/2}|} \operatorname{adj}(V^{1/2})$

$$|V^{1/2}| = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{vmatrix} = 2(15) = 30.$$

$$\operatorname{adj}(V^{1/2}) = \begin{bmatrix} 15 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 6 \end{bmatrix}.$$

$$(V^{1/2})^{-1} = \frac{1}{30}\begin{bmatrix} 15 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/5 \end{bmatrix}$$

$$P = (V^{1/2})^{-1} \Sigma (V^{1/2})^{-1}$$

$$= \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/5 \end{bmatrix} \begin{bmatrix} 4 & 1 & 2 \\ 1 & 9 & -3 \\ 2 & -3 & 25 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1/2 & 1 \\ 1/3 & 3 & -1 \\ 2/5 & -3/5 & 5 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1/6 & 1/5 \\ 1/6 & 1 & -1/5 \\ 1/5 & -1/5 & 1 \end{bmatrix}$$

Partitioning the covariance matrix

We can partition the p characteristics contained in the $p \times 1$ random vector X into two groups of size q and $p-q$, respectively.

For example, we can write

$$X = \begin{bmatrix} X_1 \\ \vdots \\ X_q \\ \hdashline X_{q+1} \\ \vdots \\ X_p \end{bmatrix} \begin{matrix} \}q \\ \\ \}p-q \end{matrix} = \begin{bmatrix} X^{(1)} \\ \hline X^{(2)} \end{bmatrix} \text{ and}$$

$$\mu = E(X) = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_q \\ \hline \mu_{q+1} \\ \vdots \\ \mu_p \end{bmatrix} = \begin{bmatrix} \mu^{(1)} \\ \hline \mu^{(2)} \end{bmatrix}$$

Now,

$$(X^{(1)} - \mu^{(1)})(X^{(2)} - \mu^{(2)})'$$

$$= \begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \\ \vdots \\ x_q - \mu_q \end{bmatrix} \begin{bmatrix} x_{q+1} - \mu_{q+1} & x_{q+2} - \mu_{q+2} & \cdots & x_p - \mu_p \end{bmatrix}$$

$$= \begin{bmatrix} (x_1-\mu_1)(x_{q+1}-\mu_{q+1}) & (x_1-\mu_1)(x_{q+2}-\mu_{q+2}) & \cdots & (x_1-\mu_1)(x_p-\mu_p) \\ (x_2-\mu_2)(x_{q+1}-\mu_{q+1}) & (x_2-\mu_2)(x_{q+2}-\mu_{q+2}) & \cdots & (x_2-\mu_2)(x_p-\mu_p) \\ \vdots & \vdots & & \vdots \\ (x_q-\mu_q)(x_{q+1}-\mu_{q+1}) & (x_q-\mu_q)(x_{q+2}-\mu_{q+2}) & \cdots & (x_q-\mu_q)(x_p-\mu_p) \end{bmatrix}$$

Taking expectation, we get

$$E(X^{(1)} - \mu^{(1)})(X^{(2)} - \mu^{(2)})' = \begin{bmatrix} \sigma_{1,q+1} & \sigma_{1,q+2} & \cdots & \sigma_{1p} \\ \sigma_{2,q+1} & \sigma_{2,q+2} & \cdots & \sigma_{2p} \\ \vdots & \vdots & & \vdots \\ \sigma_{q,q+1} & \sigma_{q,q+2} & \cdots & \sigma_{qp} \end{bmatrix}$$

$$= \Sigma_{12}$$

Which gives all the covariance σ_{ij}, $i = 1, 2, \ldots q$ and $j = q+1, q+2, \ldots p$ between a component of $X^{(1)}$ and a component of $X^{(2)}$.

Making use of the partitioning, we can easily demonstrate that $(X-\mu)(X-\mu)'$

$$= \begin{bmatrix} \underset{(q\times 1)}{(X^{(1)}-\mu^{(1)})}\underset{(1\times q)}{(X^{(1)}-\mu^{(1)})'} & \underset{(q\times 1)}{(X^{(1)}-\mu^{(1)})}\underset{(1\times(p-q))}{(X^{(2)}-\mu^{(2)})'} \\ \underset{((p-q)\times 1)}{(X^{(2)}-\mu^{(2)})}\underset{(1\times q)}{(X^{(1)}-\mu^{(1)})'} & \underset{((p-q)\times 1)}{(X^{(2)}-\mu^{(2)})}\underset{(1\times(p-q))}{(X^{(2)}-\mu^{(2)})'} \end{bmatrix}$$

Now $\underset{(p\times p)}{\Sigma} = E(X-\mu)(X-\mu)'$

$$= \begin{array}{c} \\ q \\ p-q \end{array} \overset{\begin{array}{cc} q & p-q \end{array}}{\begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}}_{(p\times p)}$$

Similar to the above solved Σ_{12};

$$= \left[\begin{array}{ccc|ccc} \sigma_{11} & \cdots & \sigma_{1q} & \sigma_{1,q+1} & \cdots & \sigma_{1p} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{q1} & \cdots & \sigma_{qq} & \sigma_{q,q+1} & \cdots & \sigma_{qp} \\ \hline \sigma_{q+1,1} & \cdots & \sigma_{q+1,q} & \sigma_{q+1,q+1} & \cdots & \sigma_{q+1,p} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{p1} & \cdots & \sigma_{pq} & \sigma_{p,q+1} & \cdots & \sigma_{pp} \end{array}\right]$$

The mean vector and covariance matrix for linear combinations of Random variables

Let X_1 and X_2 be two random variables.

For the linear combination aX_1+bX_2,

$$E(aX_1+bX_2) = aE(X_1)+bE(X_2) = a\mu_1+b\mu_2. \quad (1)$$

$$\text{var}(aX_1+bX_2) = E\left[(aX_1+bX_2)-(a\mu_1+b\mu_2)\right]^2$$

$$= E\left[a(X_1-\mu_1)+b(X_2-\mu_2)\right]^2$$

$$= E\left[a^2(x_1-\mu_1)^2 + b^2(x_2-\mu_2)^2 + 2ab\,(x_1-\mu_1)(x_2-\mu_2)\right]$$

$$= a^2 E[x_1-\mu_1]^2 + b^2 E[x_2-\mu_2]^2 + 2ab\, E[x_1-\mu_1][x_2-\mu_2]$$

$$= a^2 \operatorname{var}(x_1) + b^2 \operatorname{var}(x_2) + 2ab \operatorname{var}(x_1, x_2)$$

$$= a^2\sigma_{11} + b^2\sigma_{22} + 2ab\,\sigma_{12} \longrightarrow (2)$$

Take $c' = [a \ \ b]$.

We have $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$, $\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}$

$$c'X = [a \ \ b]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = ax_1 + bx_2 \longrightarrow (a)$$

$$c'\mu = [a \ \ b]\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = a\mu_1 + b\mu_2 \longrightarrow (b)$$

$$c'\Sigma c = [a \ \ b]\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$$

$$= [a\sigma_{11} + b\sigma_{12} \quad a\sigma_{12} + b\sigma_{22}]\begin{bmatrix} a \\ b \end{bmatrix}$$

$$= a^2\sigma_{11} + ab\sigma_{12} + ab\sigma_{12} + b^2\sigma_{22}$$

$$= a^2\sigma_{11} + 2ab\sigma_{12} + b^2\sigma_{22} \longrightarrow (c)$$

By substituting (a), (b), (c) in (1) and (2)

So $E(c'X) = c'\mu$

$\operatorname{Var}(c'x) = c'\Sigma c$.

The preceding results can be extended to a linear combination of p random variables.

The linear combination $c'X = c_1x_1 + \dots + c_px_p$ has

mean $= E(c'X) = c'\mu$.

Variance $= \operatorname{Var}(c'X) = c'\Sigma c$

where $\mu = E(X)$ & $\Sigma = \operatorname{cov}(X)$.

In general, consider the q linear combinations of the p random variables $x_1, \ldots, x_p$

$$z_1 = c_{11}x_1 + c_{12}x_2 + \cdots + c_{1p}x_p$$

$$z_2 = c_{21}x_1 + c_{22}x_2 + \cdots + c_{2p}x_p$$

$$\vdots \qquad\qquad \vdots$$

$$z_q = c_{q1}x_1 + c_{q2}x_2 + \cdots + c_{qp}x_p$$

(or) $$Z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_q \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & \vdots & & \vdots \\ c_{q1} & c_{q2} & \cdots & c_{qp} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} = CX$$

The linear combinations $Z = CX$. have

$$\mu_Z = E(Z) = E(CX) = C\mu_X$$

$$\Sigma_Z = \operatorname{cov}(Z) = \operatorname{cov}(CX) = C\Sigma_X C'.$$

where μ_X and Σ_X are the mean vector and variance - covariance matrix of X, respectively.

<u>Example</u>: (Means and covariances of linear combinations).

Let $X' = [x_1 \; x_2]$ be a random vector with mean vector $\mu_X' = [\mu_1 \; \mu_2]$ and variance-covariance matrix

$$\Sigma_X = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}$$

Find the mean vector and covariance matrix for the linear combinations

$$z_1 = x_1 - x_2$$
$$z_2 = x_1 + x_2.$$

in terms of μ_X & Σ_X

(or) $$Z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = CX$$

Here
$$\mu_Z = E(Z) = C\mu_X = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} \mu_1 - \mu_2 \\ \mu_1 + \mu_2 \end{bmatrix}$$

and $\Sigma_Z = \text{cov}(Z) = C \Sigma_X C'$

$$= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{11} - \sigma_{12} & \sigma_{12} - \sigma_{22} \\ \sigma_{11} + \sigma_{12} & \sigma_{12} + \sigma_{22} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\Sigma_Z = \begin{bmatrix} \sigma_{11} - 2\sigma_{12} + \sigma_{22} & \sigma_{11} - \sigma_{22} \\ \sigma_{11} - \sigma_{22} & \sigma_{11} + 2\sigma_{12} + \sigma_{22} \end{bmatrix}$$

Exercise Problems

1. verify the relationships $V^{1/2} \rho V^{1/2} = \Sigma$ and $\rho = (V^{1/2})^{-1} \Sigma (V^{1/2})^{-1}$, where Σ is the $p \times p$ population variance matrix, ρ is the $p \times p$ population correlation matrix, and $V^{1/2}$ is the population standard deviation matrix.

2. Let X have covariance matrix $\Sigma = \begin{bmatrix} 25 & -2 & 4 \\ -2 & 4 & 1 \\ 4 & 1 & 9 \end{bmatrix}$

a) Determine ρ and $V^{1/2}$.

b) check $V^{1/2} \rho V^{1/2} = \Sigma$.

Soln:

$$V^{1/2} = \begin{bmatrix} \sqrt{\sigma_{11}} & 0 & 0 \\ 0 & \sqrt{\sigma_{22}} & 0 \\ 0 & 0 & \sqrt{\sigma_{33}} \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$(V^{1/2})^{-1} = \begin{bmatrix} 1/5 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$$

$$\rho = (V^{1/2})^{-1} \Sigma (V^{1/2})^{-1}$$

$$= \begin{bmatrix} 1/5 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} 25 & -2 & 4 \\ -2 & 4 & 1 \\ 4 & 1 & 9 \end{bmatrix} \begin{bmatrix} 1/5 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -2/5 & 4/5 \\ -1 & 2 & 1/2 \\ 4/3 & 1/3 & 3 \end{bmatrix} \begin{bmatrix} 1/5 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$$

$$\rho = \begin{bmatrix} 1 & -1/5 & 4/15 \\ -1/5 & 1 & 1/6 \\ 4/15 & 1/6 & 1 \end{bmatrix}$$

Now, $V^{1/2} \rho V^{1/2} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1/5 & 4/15 \\ -1/5 & 1 & 1/6 \\ 4/15 & 1/6 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 5 & -1 & 4/3 \\ -2/5 & 2 & 1/3 \\ 12/15 & 1/2 & 3 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 25 & -2 & 4 \\ -2 & 4 & 1 \\ 4 & 1 & 9 \end{bmatrix} = \Sigma.$$

Hence proved.

(3) For $\Sigma = \begin{bmatrix} 25 & -2 & 4 \\ -2 & 4 & 1 \\ 4 & 1 & 9 \end{bmatrix}$

a) Find ρ_{13}

b) Find the correlation between ~~to check~~ X_1 and $\frac{X_2}{2} + \frac{X_3}{3}$.

Solution:

We have $\rho_{ik} = \dfrac{\sigma_{ik}}{\sqrt{\sigma_{ii}}\sqrt{\sigma_{kk}}}$.

$$\therefore \rho_{13} = \frac{\sigma_{13}}{\sqrt{\sigma_{11}}\sqrt{\sigma_{33}}} = \frac{4}{\sqrt{25}\sqrt{9}} = \frac{4}{5\times 3}$$

$$= \frac{4}{15} = 0.27$$

b) correlation between x_1 and $\frac{1}{2}x_2+\frac{1}{2}x_3$ is

$$\frac{cov\left(x_1, \frac{1}{2}x_2+\frac{1}{2}x_3\right)}{\sqrt{var(x_1)}\sqrt{var\left(\frac{1}{2}x_2+\frac{1}{2}x_3\right)}}$$

Now, $cov\left(x_1, \frac{1}{2}x_2+\frac{1}{2}x_3\right)$

$$= E\left[x_1-\mu_1\right]\left[\left(\frac{1}{2}x_2+\frac{1}{2}x_3\right)-\left(\frac{1}{2}\mu_2+\frac{1}{2}\mu_3\right)\right]$$

$$= E\left[x_1-\mu_1\right]\left[\frac{1}{2}(x_2-\mu_2)+\frac{1}{2}(x_3-\mu_3)\right]$$

$$= \frac{1}{2}E(x_1-\mu_1).\left((x_2-\mu_2)+(x_3-\mu_3)\right)$$

$$= \frac{1}{2}\left[E(x_1-\mu_1)(x_2-\mu_2)+E(x_1-\mu_1)(x_3-\mu_3)\right]$$

$$= \frac{1}{2}\left[\sigma_{12}+\sigma_{13}\right]$$

$$= \frac{1}{2}\left[-2+4\right] = 1$$

Now $var\left(\frac{1}{2}x_2+\frac{1}{2}x_3\right) = \left(\frac{1}{2}\right)^2\sigma_{22}+\left(\frac{1}{2}\right)^2\sigma_{33}+2\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\sigma_{23}$

$\left(\because var(ax_1+bx_2) = a^2\sigma_{11}+b^2\sigma_{22}+2ab\,\sigma_{12}\right)$

$$= \left(\frac{1}{2}\right)^2(4)+\left(\frac{1}{2}\right)^2(9)+\frac{1}{2}$$

$$= \frac{4}{4}+\frac{9}{4}+\frac{2}{4} = \frac{15}{4}$$

$$k = \frac{1}{\sqrt{25}\,\sqrt{15/4}}$$

$$= \frac{2}{5\times\sqrt{15}} = 0.1033.$$

(4) Derive expressions for the mean and variances of the following linear combinations interms of the mean and covariances of the random variables x_1, x_2 and x_3.

a) $x_1 - 2x_2$ b) $-x_1 + 3x_2$ c) $x_1 + x_2 + x_3$ d) $x_1 + 2x_2 - x_3$

e) $3x_1 - 4x_2$ if x_1 and x_2 are independent random variables.

Soln:

We know that mean $= E(c'x) = c'\mu$

variance $= \text{var}(c'x) = c'\Sigma c$.

where $c'x = c_1 x_1 + \dots + c_p x_p$.

$$c' = [c_1 \; c_2 \; \dots \; c_p] \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}$$

a)

$$E(x_1 - 2x_2) = [1 \;\; -2]\begin{bmatrix}\mu_1 \\ \mu_2\end{bmatrix} = \mu_1 - 2\mu_2$$

$$\text{var}(x_1 - 2x_2) = [1 \;\; -2]\begin{bmatrix}\sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22}\end{bmatrix}\begin{bmatrix}1 \\ -2\end{bmatrix}$$

$ax_1 + bx_2$

$a^2\sigma_{11} + b^2\sigma_{22} + 2ab\sigma_{12}$

$$= [\sigma_{11} - 2\sigma_{21} \quad \sigma_{12} - 2\sigma_{22}]\begin{bmatrix}1 \\ -2\end{bmatrix}$$

$$= \sigma_{11} - 2\sigma_{21} - 2\sigma_{12} + 4\sigma_{22}$$

$$\cancel{= \sigma_{11} + 4\sigma_{22} - 2\sigma_{21}}$$

$$= \sigma_{11} + 4\sigma_{22} - 4\sigma_{12}$$

b) mean $= -\mu_1 + 3\mu_2$

variance $= (-1)^2 \sigma_{11} + 3^2 \sigma_{22} + 2(-1)(3)\sigma_{12}$

$= \sigma_{11} + 9\sigma_{22} - 6\sigma_{12}$

c) $E(x_1 + x_2 + x_3) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cancel{[\mu}$

$= \mu_1 + \mu_2 + \mu_3$

$$\operatorname{var}(x_1 + x_2 + x_3) = [1 \ 1 \ 1] \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= [\sigma_{11} + \sigma_{12} + \sigma_{13} \quad \sigma_{12} + \sigma_{22} + \sigma_{23} \quad \sigma_{13} + \sigma_{23} + \sigma_{33}] \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \sigma_{11} + \sigma_{12} + \sigma_{13} + \sigma_{12} + \sigma_{22} + \sigma_{23} + \sigma_{13} + \sigma_{23} + \sigma_{33}$$

$$= \sigma_{11} + \sigma_{22} + \sigma_{33} + 2(\sigma_{12} + \sigma_{13} + \sigma_{23})$$

W.H.

d) mean $(x_1 + 2x_2 - x_3) = \mu_1 + 2\mu_2 - \mu_3$

$$\operatorname{var}(x_1 + 2x_2 - x_3) = [1 \ 2 \ -1] \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$= [\sigma_{11} + 2\sigma_{21} - \sigma_{31} \quad \sigma_{12} + 2\sigma_{22} - \sigma_{23} \quad \sigma_{13} + 2\sigma_{23} - \sigma_{33}] \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$= \sigma_{11} + 2\sigma_{21} - \sigma_{31} + 2\sigma_{12} + 4\sigma_{22} - 2\sigma_{23} - \sigma_{13} - 2\sigma_{23} + \sigma_{33}$$

$$= \sigma_{11} + 4\sigma_{22} + \sigma_{33} - 2\sigma_{13} - 4\sigma_{23} + 4\sigma_{12}$$

e) mean = $3\mu_1 - 4\mu_2$

variance = $9\sigma_{11} + 16\sigma_{22}$

[Since x_1 and x_2 are independent variables, $\sigma_{12} = 0$.]

5) Consider the arbitrary random vector $x' = [x_1\ x_2\ x_3\ x_4\ x_5]$ with mean vector $\mu' = [\mu_1\ \mu_2\ \mu_3\ \mu_4\ \mu_5]$.

Partition x into $x = \begin{bmatrix} x^{(1)} \\ \cdots \\ x^{(2)} \end{bmatrix}$

where $x^{(1)} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $x^{(2)} = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix}$

Let Σ be the covariance matrix of x with general element σ_{ik}. Partition Σ into the covariance matrices of $x^{(1)}$ and $x^{(2)}$ and the covariance matrix of an element of $x^{(1)}$ and an element of $x^{(2)}$.

H.W. 6) You are given the random vector $x' = [x_1\ x_2\ x_3\ x_4]$ with mean vector $\mu_x' = [4\ 3\ 2\ 1]$ and variance-covariance matrix $\Sigma_x = \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 1 & 1 & 0 \\ 2 & 1 & 9 & -2 \\ 2 & 0 & -2 & 4 \end{bmatrix}$

Partition x as $x = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x^{(1)} \\ \cdots \\ x^{(2)} \end{bmatrix}$

Let $A = [1\ \ 2]$ and $B = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}$

and consider the linear combinations $Ax^{(1)}$ and $Bx^{(2)}$.

Find (a) $E(x^{(1)})$ b) $E(Ax^{(1)})$ c) $\mathrm{cov}(x^{(1)})$ 6

d) $\mathrm{cov}(Ax^{(1)})$ e) $E(x^{(2)})$ f) $E(Bx^{(2)})$

g) $\mathrm{cov}(x^{(2)})$ h) $\mathrm{cov}(Bx^{(2)})$ (i) $\mathrm{cov}(x^{(1)}, x^{(2)})$

(j) $\mathrm{cov}(Ax^{(1)}, Bx^{(2)})$.

(7) Repeat the Pbm (6) for $A = [1 \ \ -1]$ $B = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$.

(8) You are given the random vector $x' = [x_1 \ x_2 \ldots x_5]$ with mean vector $\mu_x' = [2 \ \ 4 \ \ -1 \ \ 3 \ \ 0]$ and variance - covariance matrix

$$\Sigma_x = \begin{bmatrix} 4 & -1 & 1/2 & -1/2 & 0 \\ -1 & 3 & 1 & -1 & 0 \\ 1/2 & 1 & 6 & 1 & -1 \\ -1/2 & -1 & 1 & 4 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{bmatrix}$$

Partition x as

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x^{(1)} \\ \cdots \\ x^{(2)} \end{bmatrix}$$

Let $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$

and consider the linear combinations $Ax^{(1)}$ and $Bx^{(2)}$.

Find a) $E(x^{(1)})$ b) $E(Ax^{(1)})$ c) $\mathrm{cov}(x^{(1)})$

d) $\mathrm{cov}(Ax^{(1)})$ e) $E(x^{(2)})$ f) $E(Bx^{(2)})$

g) $\mathrm{cov}(x^{(2)})$ h) $\mathrm{cov}(Bx^{(2)})$ (i) $\mathrm{cov}(x^{(1)}, x^{(2)})$

(j) $\mathrm{cov}(Ax^{(1)}, Bx^{(2)})$.

a)

$$Ax^{(1)} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \end{bmatrix}$$

$$Bx^{(2)} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_3 + x_4 + x_5 \\ x_3 + x_4 - 2x_5 \end{bmatrix}$$

a) $E\left[X^{(1)}\right] = E\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

b) $E\left[A X^{(1)}\right] = E\begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \end{bmatrix}$

$$= \begin{bmatrix} \mu_1 - \mu_2 \\ \mu_1 + \mu_2 \end{bmatrix} = \begin{bmatrix} 2-4 \\ 2+4 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$$

c) $\operatorname{cov}(X^{(1)}) = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix}$

d) $\operatorname{cov}\left(\begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \end{bmatrix}\right) = E\left[\begin{bmatrix} (x_1 - x_2) - (\mu_1 - \mu_2) \\ (x_1 + x_2) - (\mu_1 + \mu_2) \end{bmatrix} \left[(x_1 - x_2) - (\mu_1 - \mu_2) \quad (x_1 + x_2) - (\mu_1 + \mu_2)\right]\right]$

$$= E\left[\begin{bmatrix} (x_1 - \mu_1) - (x_2 - \mu_2) \\ (x_1 - \mu_1) + (x_2 - \mu_2) \end{bmatrix} \left[(x_1 - \mu_1) - (x_2 - \mu_2) \quad (x_1 - \mu_1) + (x_2 - \mu_2)\right]\right]$$

$$= E\begin{bmatrix} (x_1 - \mu_1)^2 + (x_2 - \mu_2)^2 - 2(x_1 - \mu_1)(x_2 - \mu_2) & (x_1 - \mu_1)^2 - (x_2 - \mu_2)^2 \\ (x_1 - \mu_1)^2 - (x_2 - \mu_2)^2 & (x_1 - \mu_1)^2 + (x_2 - \mu_2)^2 + 2(x_1 - \mu_1)(x_2 - \mu_2) \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{11} + \sigma_{22} - 2\sigma_{12} & \sigma_{11} - \sigma_{22} \\ \sigma_{11} - \sigma_{22} & \sigma_{11} + \sigma_{22} + 2\sigma_{12} \end{bmatrix}$$

$$= \begin{bmatrix} 4+3-2(-1) & 4-3 \\ 4-3 & 4+3+2(-1) \end{bmatrix}$$

$$\mathrm{cov}(AX^{(1)}) = \begin{bmatrix} 9 & 1 \\ 1 & 5 \end{bmatrix}$$

e) $E[X^{(2)}] = E\begin{bmatrix} X_3 \\ X_4 \\ X_5 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$

f) $E[BX^{(2)}] = E\begin{bmatrix} X_3+X_4+X_5 \\ X_3+X_4-2X_5 \end{bmatrix}$

$$= \begin{bmatrix} \mu_3+\mu_4+\mu_5 \\ \mu_3+\mu_4-2\mu_5 \end{bmatrix}$$

$$= \begin{bmatrix} -1+3+0 \\ -1+3-0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

g) $\mathrm{cov}(X^{(2)}) = E\begin{bmatrix} X_3-\mu_3 \\ X_4-\mu_4 \\ X_5-\mu_5 \end{bmatrix} [X_3-\mu_3 \quad X_4-\mu_4 \quad X_5-\mu_5]$

$$= \begin{bmatrix} \sigma_{33} & \sigma_{34} & \sigma_{35} \\ \sigma_{34} & \sigma_{44} & \sigma_{45} \\ \sigma_{35} & \sigma_{54} & \sigma_{55} \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 1 & -1 \\ 1 & 4 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

h) $\operatorname{cov}(BX^{(2)}) = E\left[\begin{bmatrix}(X_3+X_4+X_5)-(\mu_3+\mu_4+\mu_5)\\(X_3+X_4-2X_5)-(\mu_3+\mu_4-2\mu_5)\end{bmatrix}\begin{bmatrix}(X_3+X_4+X_5)-(\mu_3+\mu_4+\mu_5) & (X_3+X_4-2X_5)-(\mu_3+\mu_4-2\mu_5)\end{bmatrix}\right]$

$$= E\left[\begin{bmatrix}(X_3-\mu_3)+(X_4-\mu_4)+(X_5-\mu_5)\\(X_3-\mu_3)+(X_4-\mu_4)-2(X_5-\mu_5)\end{bmatrix}\begin{bmatrix}(X_3-\mu_3)+(X_4-\mu_4)+(X_5-\mu_5) & (X_3-\mu_3)+(X_4-\mu_4)-2(X_5-\mu_5)\end{bmatrix}\right]$$

$$= E\begin{bmatrix}(X_3-\mu_3)^2+(X_4-\mu_4)^2+(X_5-\mu_5)^2+2(X_3-\mu_3)(X_4-\mu_4)+2(X_3-\mu_3)(X_5-\mu_5)+2(X_4-\mu_4)(X_5-\mu_5) & (X_3-\mu_3)^2+(X_4-\mu_4)^2-2(X_5-\mu_5)^2+2(X_3-\mu_3)(X_4-\mu_4)-(X_3-\mu_3)(X_5-\mu_5)-(X_4-\mu_4)(X_5-\mu_5)\\(X_3-\mu_3)^2+(X_4-\mu_4)^2-2(X_5-\mu_5)^2+2(X_3-\mu_3)(X_4-\mu_4)-(X_3-\mu_3)(X_5-\mu_5)-(X_4-\mu_4)(X_5-\mu_5) & (X_3-\mu_3)^2+(X_4-\mu_4)^2+4(X_5-\mu_5)^2+2(X_3-\mu_3)(X_4-\mu_4)-4(X_3-\mu_3)(X_5-\mu_5)-4(X_4-\mu_4)(X_5-\mu_5)\end{bmatrix}$$

$$= \begin{bmatrix}\sigma_{33}+\sigma_{44}+\sigma_{55}+2\sigma_{34}+2\sigma_{35}+2\sigma_{45} & \sigma_{33}+\sigma_{44}-2\sigma_{55}+2\sigma_{34}-\sigma_{35}-\sigma_{45}\\\sigma_{33}+\sigma_{44}-2\sigma_{55}+2\sigma_{34}-\sigma_{35}-\sigma_{45} & \sigma_{33}+\sigma_{44}+4\sigma_{55}+2\sigma_{34}-4\sigma_{35}-4\sigma_{45}\end{bmatrix}$$

$$= \begin{bmatrix}6+4+2+2(1)+2(-1)+2(0) & 6+4-2(2)+2(1)+1+0\\6+4-2(2)+2(1)+1+0 & 6+4+4(2)+2(1)-4(-1)-4(0)\end{bmatrix}$$

22 $\quad = \begin{bmatrix}12 & 9\\9 & 24\end{bmatrix}$

$$\operatorname{cov}\left(X^{(1)}, X^{(2)}\right) = E\begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix}\begin{bmatrix} x_3-\mu_3 & x_4-\mu_4 & x_5-\mu_5 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{13} & \sigma_{14} & \sigma_{15} \\ \sigma_{23} & \sigma_{24} & \sigma_{25} \end{bmatrix}$$

$$= \begin{bmatrix} \tfrac{1}{2} & -\tfrac{1}{2} & 0 \end{bmatrix}$$

$$\operatorname{cov}\left(AX^{(1)}, BX^{(2)}\right) = E\begin{bmatrix} (x_1-x_2)-(\mu_1-\mu_2) \\ (x_1+x_2)-(\mu_1+\mu_2) \end{bmatrix}\begin{bmatrix} (x_3+x_4+x_5)-(\mu_3+\mu_4+\mu_5) & (x_3+x_4-2x_5)-(\mu_3+\mu_4-2\mu_5) \end{bmatrix}$$

$$= E\begin{bmatrix} (x_1-\mu_1)-(x_2-\mu_2) \\ (x_1-\mu_1)+(x_2-\mu_2) \end{bmatrix}\begin{bmatrix} (x_3-\mu_3)+(x_4-\mu_4)+(x_5-\mu_5), & (x_3-\mu_3)+(x_4-\mu_4)-2(x_5-\mu_5) \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{13}+\sigma_{14}+\sigma_{15}-\sigma_{23}-\sigma_{24}-\sigma_{25} & \sigma_{13}+\sigma_{14}-2\sigma_{15}-\sigma_{23}-\sigma_{24}+2\sigma_{25} \\ \sigma_{13}+\sigma_{14}+\sigma_{15}+\sigma_{23}+\sigma_{24}+\sigma_{25} & \sigma_{13}+\sigma_{14}-2\sigma_{15}+\sigma_{23}+\sigma_{24}-2\sigma_{25} \end{bmatrix}$$

$$= \begin{bmatrix} \tfrac{1}{2}-\tfrac{1}{2}+0-1+0+1 & \tfrac{1}{2}-\tfrac{1}{2}-0-1+1+0 \\ \tfrac{1}{2}-\tfrac{1}{2}+0+1-1+0 & \tfrac{1}{2}-\tfrac{1}{2}-0+1-1+0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(9) Repeat the same pbm, but with x partitioned as

$$x_t = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \cdots \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x^{(1)} \\ \cdots \\ x^{(2)} \end{bmatrix}$$ and with A and B

replaced by $A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$

(10) You are given the random vector $x' = [x_1 \; x_2 \; x_3 \; x_4]$ with mean vector $\mu_x' = [3 \; 2 \; -2 \; 0]$ and variance-covariance matrix

$$\Sigma_x = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Let $A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & -2 & 0 \\ 1 & 1 & 1 & -3 \end{bmatrix}$

a) Find $E(AX)$, the mean of AX.

b) Find $\text{cov}(AX)$, the variances and covariances of AX.

c) Which pairs of linear combinations have zero covariances?

soln:: $E(AX) = A\,E(X) = A\mu_x = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & -2 & 0 \\ 1 & 1 & 1 & -3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -2 \\ 0 \end{bmatrix}$

$$= \begin{bmatrix} 1 \\ 9 \\ 3 \end{bmatrix}$$

$$\text{cov}(AX) = A \Sigma_x A'$$

$$= \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & -2 & 0 \\ 1 & 1 & 1 & -3 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -3 & 0 & 0 \\ 3 & 3 & -6 & 0 \\ 3 & 3 & 3 & -9 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix}$$

$$\mathrm{cov}(AX) = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 36 \end{bmatrix}$$

$$Z = AX \Rightarrow \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & -2 & 0 \\ 1 & 1 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \end{bmatrix}$$

$$\Rightarrow Z_1 = x_1 - x_2$$

$$Z_2 = x_1 + x_2 - 2x_3$$

$$Z_3 = x_1 + x_2 + x_3 - 3x_4$$

$$\mathrm{cov}(Z_1, Z_2) = \mathrm{cov}\left([x_1 - x_2], [x_1 + x_2 - 2x_3]\right)$$

$x_1 + x_2 - \mu$

$$= E\left[(x_1-\mu_1)-(x_2-\mu_2)\right]\left[(x_1-\mu_1)+(x_2-\mu_2) - 2(x_3-\mu_3)\right]$$

$$= \sigma_{11} + \sigma_{12} - 2\sigma_{13} - \sigma_{21} - \sigma_{22} + 2\sigma_{23}$$

$$= 3 + 0 - 0 - 0 - 3 + 0 = 0$$

$$\mathrm{cov}(Z_1, Z_3) = \mathrm{cov}\left([x_1 - x_2], [x_1 + x_2 + x_3 - 3x_4]\right)$$

$$= E\left[(x_1-\mu_1)-(x_2-\mu_2)\right]\left[(x_1-\mu_1)+(x_2-\mu_2)+(x_3-\mu_3) - 3(x_4-\mu_4)\right]$$

$$= \sigma_{11} + \sigma_{12} + \sigma_{13} - 3\sigma_{14} - \sigma_{12} - \sigma_{22} - \sigma_{23} + 3\sigma_{24}$$

$$= 3 + 0 + 0 - 0 - 0 - 3 - 0 + 0 = 0$$

$$\mathrm{cov}(Z_2, Z_3) = \mathrm{cov}\left(x_1 + x_2 - 2x_3,\ x_1 + x_2 + x_3 - 3x_4\right)$$

$$= E\left[(x_1-\mu_1)+(x_2-\mu_2)-2(x_3-\mu_3)\right]\left[(x_1-\mu_1)+(x_2-\mu_2)+(x_3-\mu_3)-3(x_4-\mu_4)\right]$$

$$= \sigma_{11} + \sigma_{12} + \sigma_{13} - 3\sigma_{14} + \sigma_{12} + \sigma_{22} + \sigma_{23} - 3\sigma_{24} - 2\sigma_{13} - 2\sigma_{23} - 2\sigma_{33} + 6\sigma_{34}$$

$$= 3 + 3 - 6 = 0$$

(11) Repeat the same Problem for $\Sigma_x = \begin{bmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix}$

$67 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\begin{bmatrix} 67 \\ 67 \end{bmatrix}$ $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

$\begin{bmatrix} 67 & 66 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix}$

adj A =

The Multivariate Normal Density and its Properties.

The multivariate normal density is a generalization of the univariate normal density to $P \geq 2$ dimensions.

The univariate normal distribution, with mean μ and variance σ^2 has the probability density function

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\left[\frac{x-\mu}{\sigma}\right]^2/2} \quad -\infty < x < \infty$$

A P-dimensional normal density for the random vector $x' = [x_1 \ x_2 \ \ldots \ x_p]$ has the form

$$f(x) = \frac{1}{(2\pi)^{P/2} |\Sigma|^{1/2}} e^{-(x-\mu)'\Sigma^{-1}(x-\mu)/2} \longrightarrow (1)$$

where $-\infty < x_i < \infty$, $i = 1, 2, \ldots, P$.

We denote the P-dimensional normal density is $N_p(\mu, \Sigma)$.

Bivariate Normal Density

Let us evaluate the $P = 2$ variate normal density in terms of the individual parameters $\mu_1 = E(x_1)$, $\mu_2 = E(x_2)$

$\sigma_{11} = var(x_1)$, $\sigma_{22} = var(x_2)$ and $\rho_{12} = \frac{\sigma_{12}}{\sqrt{\sigma_{11}}\sqrt{\sigma_{22}}} = corr(x_1, x_2)$

The covariance matrix is $\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}$

$$\Rightarrow \Sigma^{-1} = \frac{1}{\sigma_{11}\sigma_{22} - \sigma_{12}^2} \begin{bmatrix} \sigma_{22} & -\sigma_{12} \\ -\sigma_{12} & \sigma_{11} \end{bmatrix} \qquad A^{-1} = \frac{1}{|A|} adj A$$

and we have $\sigma_{12} = \rho_{12}\sqrt{\sigma_{11}}\sqrt{\sigma_{22}}$

Now

$$(x-\mu)'\Sigma^{-1}(x-\mu) = [(x_1-\mu_1) \ (x_2-\mu_2)] \frac{1}{(\sigma_{11}\sigma_{22}-\sigma_{12}^2)} \begin{bmatrix} \sigma_{22} & -\sigma_{12} \\ -\sigma_{12} & \sigma_{11} \end{bmatrix} \begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix}$$

(1)

$$= [(x_1-\mu_1) \quad (x_2-\mu_2)] \frac{1}{\sigma_{11}\sigma_{22}(1-\rho_{12}^2)} \begin{bmatrix} \sigma_{22} & -\rho_{12}\sqrt{\sigma_{11}}\sqrt{\sigma_{22}} \\ -\rho_{12}\sqrt{\sigma_{11}}\sqrt{\sigma_{22}} & \sigma_{11} \end{bmatrix} \begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix}$$

$$= \frac{1}{\sigma_{11}\sigma_{22}(1-\rho_{12}^2)} \left[(x_1-\mu_1)\sigma_{22} - \rho_{12}\sqrt{\sigma_{11}}\sqrt{\sigma_{22}}(x_2-\mu_2) \quad -(x_1-\mu_1)\rho_{12}\sqrt{\sigma_{11}}\sqrt{\sigma_{22}} + (x_2-\mu_2)\sigma_{11} \right] \begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix}$$

$$= \frac{1}{\sigma_{11}\sigma_{22}(1-\rho_{12}^2)} \left\{ (x_1-\mu_1)^2\sigma_{22} - 2\rho_{12}\sqrt{\sigma_{11}}\sqrt{\sigma_{22}}(x_1-\mu_1)(x_2-\mu_2) + (x_2-\mu_2)^2\sigma_{11} \right\}$$

$$= \frac{1}{(1-\rho_{12}^2)} \left[\frac{(x_1-\mu_1)^2}{\sigma_{11} \cancel{1-\rho_{12}}} + \frac{(x_2-\mu_2)^2}{\sigma_{22}} - 2\rho_{12}\frac{(x_1-\mu_1)}{\sqrt{\sigma_{11}}}\frac{(x_2-\mu_2)}{\sqrt{\sigma_{22}}} \right]$$

Now, $|\Sigma| = \sigma_{11}\sigma_{22} - \sigma_{12}^2 = \sigma_{11}\sigma_{22}\left(1 - \frac{\sigma_{12}^2}{\sigma_{11}\sigma_{22}}\right)$

$$= \sigma_{11}\sigma_{22}(1-\rho_{12}^2)$$

Now

From ①, For 2-dimensional,

$$f(x) = \frac{1}{2\pi \, |\Sigma|^{1/2}} e^{-(x-\mu)'\Sigma^{-1}(x-\mu)/2}$$

$$f(x_1, x_2) = \frac{1}{2\pi\sqrt{\sigma_{11}\sigma_{22}(1-\rho_{12}^2)}} \exp\left\{ \frac{-1}{2(1-\rho_{12}^2)} \left[\left(\frac{x_1-\mu_1}{\sqrt{\sigma_{11}}}\right)^2 + \left(\frac{x_2-\mu_2}{\sqrt{\sigma_{22}}}\right)^2 - 2\rho_{12}\left(\frac{x_1-\mu_1}{\sqrt{\sigma_{11}}}\right)\left(\frac{x_2-\mu_2}{\sqrt{\sigma_{22}}}\right) \right] \right\}$$

Result: If Σ is positive definite, so that Σ^{-1} exists, then $\Sigma e = \lambda e$ implies $\Sigma^{-1} e = \left(\frac{1}{\lambda}\right) e$.

So (λ, e) is an eigen value - eigenvector pair for Σ corresponding to the pair $(1/\lambda, e)$ for Σ^{-1}. Also Σ^{-1} is positive definite.

Proof:

Since Σ is positive definite and $e \neq 0$ an eigenvector,

We have $0 < e'\Sigma e = e'(\Sigma e) = e'(\lambda e) = \lambda e'e = \lambda$.

$\Rightarrow \lambda > 0$.

Now, $e = \Sigma^{-1}(\Sigma e) = \Sigma^{-1}(\lambda e) = \lambda \Sigma^{-1} e$.

$\Rightarrow e = \lambda \Sigma^{-1} e \longrightarrow (1)$

Since $\lambda > 0$, division (1) by λ we get

$$\left(\frac{1}{\lambda}\right) e = \Sigma e.$$

Thus $(1/\lambda, e)$ is an eigen value - eigenvector pair for Σ^{-1}.

To prove Σ^{-1} is positive definite.

For any $P \times 1$ vector x, we have

$$x'\Sigma x = x'\left(\sum_{i=1}^{p}\left(\frac{1}{\lambda_i}\right) e_i e_i'\right) x$$

$$= \sum_{i=1}^{p}\left(\frac{1}{\lambda_i}\right)(x'e_i)^2 \geq 0 \quad \left(\because \lambda_i^{-1}(x'e_i)^2 > 0\right)$$

In addition

$x'e_i = 0$ for all i iff $x = 0$.

So $x \neq 0$ implies that $\sum_{i=1}^{p}(1/\lambda_i)(x'e_i)^2 > 0$.

So Σ^{-1} is positive definite.

contours of the bivariate normal density

We shall obtain the axes of constant Probability density contours for a bivariate normal distribution when $\sigma_{11} = \sigma_{22}$.

Now $|\Sigma - \lambda I| = 0$

$$\Rightarrow 0 = \begin{vmatrix} \sigma_{11} - \lambda & \sigma_{12} \\ \sigma_{12} & \sigma_{22} - \lambda \end{vmatrix}$$

$$= (\sigma_{11} - \lambda)(\sigma_{22} - \lambda) - \sigma_{12}^2$$

Since $\sigma_{11} = \sigma_{22}$, $(\sigma_{11} - \lambda)^2 - \sigma_{12}^2 = 0$.

$$\Rightarrow (\sigma_{11} - \lambda + \sigma_{12})(\sigma_{11} - \lambda - \sigma_{12}) = 0.$$

$$\Rightarrow (\lambda - \sigma_{11} - \sigma_{12})(\lambda - \sigma_{11} + \sigma_{12}) = 0.$$

$\therefore$ The eigenvalues are $\lambda_1 = \sigma_{11} + \sigma_{12}$ and $\lambda_2 = \sigma_{11} - \sigma_{12}$.

The eigenvector e_1 is determined from

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{11} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = (\sigma_{11} + \sigma_{12}) \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

$$\sigma_{11} e_1 + \sigma_{12} e_2 = (\sigma_{11} + \sigma_{12}) e_1$$

$$\sigma_{12} e_1 + \sigma_{11} e_2 = (\sigma_{11} + \sigma_{12}) e_2$$

From this $e_1 = e_2$.

Put $e_1 = 1$, $e_2 = 1$.

The Normalized eigen vector is $e_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

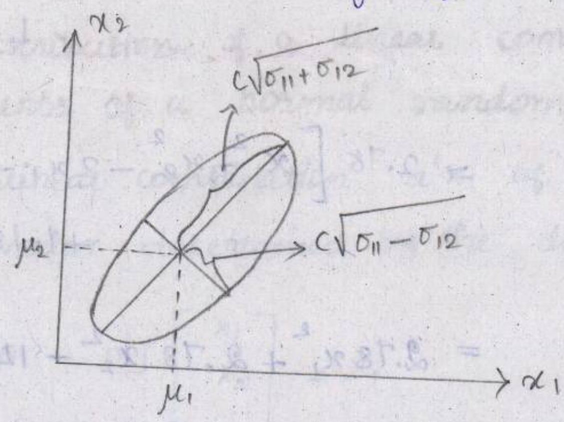
Similarly for $\lambda_2 = e_{11} - e_{12}$, The Normalized eigen vector is

$$e_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

When the covariance σ_{12} is positive, $\lambda_1 = \sigma_{11} + \sigma_{12}$ is the largest eigen value, and its associated eigenvector $e_1' = [1/\sqrt{2} \;\; 1/\sqrt{2}]$ lies along the 45° line through the point $\mu' = [\mu_1, \mu_2]$. This is true for any positive value of the covariance.

Since the axes of the constant-density ellipses are given by $\pm c\sqrt{\lambda_1}\, e_1$ and $\pm c\sqrt{\lambda_2}\, e_2$. and the eigenvectors each have length unity, the major axis will be associated with the largest eigen value.

For positively correlated normal random variables, then, the major axis of the constant-density ellipses will be along the 45° line through μ.



When the covariance is negative $\lambda_2 = \sigma_{11} - \sigma_{12}$ will be the largest eigen value, and the major axes of the constant-density ellipses will lie along a line at right angles to the 45° line through μ.

The axes of the ellipses of constant density for a bivariate Normal distribution with $\sigma_{11} = \sigma_{22}$ are determined by

$$\pm c\sqrt{\sigma_{11}+\sigma_{12}} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \text{ and } \pm c\sqrt{\sigma_{11}-\sigma_{12}} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

Qns. consider a bivariate Normal distribution with $\mu_1 = 1$, $\mu_2 = 3$, $\sigma_{11} = 2$, $\sigma_{22} = 1$ and $\rho_{12} = -0.8$.

a) Write out the bivariate normal density.

b) Write out the squared statistical distance expression $(x-\mu)' \Sigma^{-1} (x-\mu)$ as a quadratic function of x_1 and x_2.

Soln:

$$(X-\mu)' \Sigma^{-1} (X-\mu) = \frac{1}{1-\rho_{12}^2}\left[\left(\frac{x_1-\mu_1}{\sqrt{\sigma_{11}}}\right)^2 + \left(\frac{x_2-\mu_2}{\sqrt{\sigma_{22}}}\right)^2 - 2\rho_{12}\left(\frac{x_1-\mu_1}{\sqrt{\sigma_{11}}}\right)\left(\frac{x_2-\mu_2}{\sqrt{\sigma_{22}}}\right)\right]$$

$$= 2.78\left[\frac{(x_1-1)^2}{2} + \frac{(x_2-3)^2}{1} + 1.6\,\frac{(x_1-1)}{\sqrt{2}}\cdot\frac{(x_2-3)}{1}\right]$$

$$= 2.78\left[x_1^2+1-2x_1+x_2^2+9-6x_2+1.1314(x_1x_2-x_2-3x_1+3)\right]$$

$$= 2.78\left[x_1^2+x_2^2-2x_1-3.394x_1-6x_2-1.1314x_2+1.1314x_1x_2+13.3941\right]$$

$$= 2.78x_1^2+2.78x_2^2-14.9953x_1-19.8253x_2+3.1453x_1x_2+37.2356$$

$$K = (x-\mu)'\Sigma^{-1}(x-\mu)/2 = 1.39x_1^2+1.39x_2^2-7.4977x_1-9.9127x_2 \; 1.5727x_1x_2+18.6178$$

$$f(x_1,x_2) = \frac{1}{2\pi\sqrt{\sigma_{11}\sigma_{22}(1-\rho_{12}^2)}}\, e^{-K}$$

$$= \frac{1}{2\pi\sqrt{2(1-0.8^2)}}\exp\left[-1.39x_1^2-1.39x_2^2+7.4977x_1+9.9127x_2-1.5727x_1x_2-18.6178\right].$$

$$= 0.3128\exp\left[-1.39x_1^2-1.39x_2^2+7.4977x_1+9.9127x_2-1.5727x_1x_2-18.6178\right].$$

2) Consider a bivariate normal population with $\mu_1 = 0$, $\mu_2 = 2$, $\sigma_{11} = 2$, $\sigma_{22} = 1$ and $\rho_{12} = 0.5$.

a) Write out the bivariate normal density.

b) Write out the squared generalized distance expression $(x-\mu)'\Sigma^{-1}(x-\mu)$ as a function of x_1 and x_2.

c) Determine the constant-density contour that contains 50% of the probability.

Additional properties of the multivariate Normal Distribution

Result: 1

If X is distributed as $N_p(\mu, \Sigma)$, then any linear combination of variables $a'X = a_1x_1 + a_2x_2 + \cdots + a_px_p$ is distributed as $N(a'\mu, a'\Sigma a)$. Also, if $a'x$ is distributed as $N(a'\mu, a'\Sigma a)$ for every a, then ~~Σ~~ X must be $N_p(\mu, \Sigma)$.

Example: (The distribution of a linear combination of the components of a normal random vector).

Consider the linear combination $a'x$ of a multivariate normal random vector determined by the choice $a' = [1, 0, \ldots, 0]$

Since
$$a'x = [1 \quad 0 \cdots 0]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} = x_1$$

$$a'\mu = [1 \quad 0 \quad 0 \cdots 0]\begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_p \end{bmatrix} = \mu_1$$

$$a'\Sigma a = [1 \quad 0 \cdots 0]\begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{12} & \sigma_{22} & \cdots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1p} & \sigma_{2p} & \cdots & \sigma_{pp} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \sigma_{11}$$

It follows from the above result ~~+~~ that X_1 is distributed as $N(\mu_1, \sigma_{11})$.

Result: 2

If x is distributed as $N_p(\mu, \Sigma)$, the q-linear combinations

$$\underset{(q\times p)}{A}\ \underset{(p\times 1)}{x} = \begin{bmatrix} a_{11}x_1 + \cdots + a_{1p}x_p \\ a_{21}x_1 + \cdots + a_{2p}x_p \\ \vdots \\ a_{q1}x_1 + \cdots + a_{qp}x_p \end{bmatrix}$$

are distributed as $N_q(A\mu, A\Sigma A')$.

Also, $\underset{(p\times 1)}{x} + \underset{(p\times 1)}{d}$, where d is a vector of constants, is distributed as $N_p(\mu + d, \Sigma)$.

Example: (The distribution of two linear combinations of the components of a normal random vector)

For x distributed as $N_3(\mu, \Sigma)$, find the distribution of

$$\begin{bmatrix} x_1 - x_2 \\ x_2 - x_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = Ax$$

By above result, the distribution of Ax is multivariate normal with mean $A\mu = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} = \begin{bmatrix} \mu_1 - \mu_2 \\ \mu_2 - \mu_3 \end{bmatrix}$

and covariance matrix

$$A\Sigma A' = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{11} - \sigma_{12} & \sigma_{12} - \sigma_{22} & \sigma_{13} - \sigma_{23} \\ \sigma_{12} - \sigma_{13} & \sigma_{22} - \sigma_{23} & \sigma_{23} - \sigma_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{11} - 2\sigma_{12} + \sigma_{22} & \sigma_{12} + \sigma_{23} - \sigma_{22} - \sigma_{13} \\ \sigma_{12} + \sigma_{23} - \sigma_{22} - \sigma_{13} & \sigma_{22} - 2\sigma_{23} + \sigma_{33} \end{bmatrix}$$

Result: 3

All subsets of X are normally distributed. If we respectively partition X, its mean vector μ, and its covariance matrix Σ as

$$\underset{(p\times 1)}{X} = \begin{bmatrix} \underset{(q\times 1)}{x_1} \\ \cdots\cdots \\ \underset{(p-q)\times 1}{x_2} \end{bmatrix} \qquad \underset{(p\times 1)}{\mu} = \begin{bmatrix} \underset{(q\times 1)}{\mu_1} \\ \cdots\cdots \\ \underset{(p-q)\times 1}{\mu_2} \end{bmatrix}$$

and

$$\underset{p\times p}{\Sigma} = \left[\begin{array}{c:c} \underset{q\times q}{\Sigma_{11}} & \underset{q\times(p-q)}{\Sigma_{12}} \\ \hdashline \underset{(p-q)\times q}{\Sigma_{21}} & \underset{(p-q)\times(p-q)}{\Sigma_{22}} \end{array}\right]$$

then x_1 is distributed as $N_q(\mu_1, \Sigma_{11})$.

Example: (The distribution of a subset of a normal random vector)

If x is distributed as $N_5(\mu, \Sigma)$, find the distribution of $\begin{bmatrix} x_2 \\ x_4 \end{bmatrix}$.

We set $x_1 = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix}$, $\mu_1 = \begin{bmatrix} \mu_2 \\ \mu_4 \end{bmatrix}$, $\Sigma_{11} = \begin{bmatrix} \sigma_{22} & \sigma_{24} \\ \sigma_{24} & \sigma_{44} \end{bmatrix}$

and Note that with this assignment, X, μ and Σ can respectively be arranged and partitioned as

$$X = \begin{bmatrix} x_2 \\ x_4 \\ \cdots \\ x_1 \\ x_3 \\ x_5 \end{bmatrix}, \quad \mu = \begin{bmatrix} \mu_2 \\ \mu_4 \\ \cdots \\ \mu_1 \\ \mu_3 \\ \mu_5 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sigma_{22} & \sigma_{24} & \sigma_{12} & \sigma_{23} & \sigma_{25} \\ \sigma_{24} & \sigma_{44} & \sigma_{14} & \sigma_{43} & \sigma_{45} \\ \sigma_{12} & \sigma_{14} & \sigma_{11} & \sigma_{13} & \sigma_{15} \\ \sigma_{23} & \sigma_{34} & \sigma_{13} & \sigma_{33} & \sigma_{35} \\ \sigma_{25} & \sigma_{25} & \sigma_{15} & \sigma_{35} & \sigma_{55} \end{bmatrix}$$

(or) $X = \begin{bmatrix} X_1 \\ (2\times1) \\ \cdots \\ X_2 \\ (3\times1) \end{bmatrix}$ $\mu = \begin{bmatrix} \mu_1 \\ (2\times1) \\ \cdots \\ \mu_2 \\ (3\times1) \end{bmatrix}$, $\Sigma = \begin{bmatrix} \Sigma_{11} & \vdots & \Sigma_{12} \\ (2\times2) & \vdots & (2\times3) \\ \cdots & & \cdots \\ \Sigma_{21} & \vdots & \Sigma_{22} \\ (3\times2) & \vdots & (3\times3) \end{bmatrix}$

Thus by the above result,

for $X_1 = \begin{pmatrix} x_2 \\ x_4 \end{pmatrix}$

we have the distribution $N_2(\mu_1, \Sigma_{11})$

$$= N_2\left(\begin{bmatrix} \mu_2 \\ \mu_4 \end{bmatrix}, \begin{bmatrix} \sigma_{22} & \sigma_{24} \\ \sigma_{24} & \sigma_{44} \end{bmatrix}\right)$$

Result: 4.

a) If $\underset{(q_1\times1)}{X_1}$ and $\underset{(q_2\times1)}{X_2}$ are independent,

then $\operatorname{cov}(X_1, X_2) = 0$, a q_1+q_2 matrix of zeros.

b) If $\begin{bmatrix} X_1 \\ \cdots \\ X_2 \end{bmatrix}$ is $N_{q_1+q_2}\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \Sigma_{11} & \vdots & \Sigma_{12} \\ \cdots & & \cdots \\ \Sigma_{21} & \vdots & \Sigma_{22} \end{bmatrix}\right)$,

then X_1 and X_2 are independent iff $\Sigma_{12} = 0$.

c) If X_1 and X_2 are independent and are distributed as $N_{q_1}(\mu_1, \Sigma_{11})$ and $N_{q_2}(\mu_2, \Sigma_{22})$ respectively, then $\begin{bmatrix} X_1 \\ \cdots \\ X_2 \end{bmatrix}$ has the multivariate Normal distribution

$$N_{q_1+q_2}\left(\begin{bmatrix} \mu_1 \\ \cdots \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \Sigma_{11} & \vdots & 0 \\ \cdots & & \cdots \\ 0' & \vdots & \Sigma_{22} \end{bmatrix}\right)$$

Example:

(The equivalence of zero covariance and independence for normal variables).

Let $\underset{(3\times1)}{X}$ be $N_3(\mu, \Sigma)$ with $\Sigma = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Are X_1 and X_2 independent? What about (X_1, X_2) and X_3?

since x_1 and x_2 have covariance $\sigma_{12} = 1$, they are not independent.

Partitioning X and Σ as

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \cdots \\ X_3 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 4 & 1 & \vdots & 0 \\ 1 & 3 & \vdots & 0 \\ \cdots & \cdots & & \cdots \\ 0 & 0 & \vdots & 2 \end{bmatrix} = \begin{bmatrix} \underset{(2\times 2)}{\Sigma_{11}} & \vdots & \underset{(2\times 1)}{\Sigma_{12}} \\ \cdots & & \cdots \\ \underset{(1\times 2)}{\Sigma_{21}} & \vdots & \underset{(1\times 1)}{\Sigma_{22}} \end{bmatrix}$$

$X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and x_3 have covariance matrix

$$\Sigma_{12} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$\therefore \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ and X_3 are independent by above result.

$\Rightarrow X_3$ is independent of X_1 and also of X_2.

Result: 5

Let $X = \begin{bmatrix} X_1 \\ \cdots \\ X_2 \end{bmatrix}$ be distributed as $N_p(\mu, \Sigma)$ with

$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$, $\Sigma = \begin{bmatrix} \Sigma_{11} & \vdots & \Sigma_{12} \\ \cdots & & \cdots \\ \Sigma_{21} & \vdots & \Sigma_{22} \end{bmatrix}$, and $|\Sigma_{22}| > 0$.

Then the conditional distribution of X_1, given that $X_2 = x_2$, is normal and has

$$\text{mean} = \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2) \text{ and}$$

$$\text{covariance} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}.$$

Note that the covariance does not depend on the value x_2 of the conditioning variable.

Example: (The conditional density of a bivariate normal distribution)

The conditional density of X_1, given that $X_2 = x_2$ for any bivariate distribution, is defined by