

Unit - IIIEstimation theoryDefn: (Estimator)

Any function of the random sample x_1, x_2, \dots, x_n that are being observed, say $T_n(x_1, x_2, \dots, x_n)$ is called a statistic. Clearly, a statistic is a random variable. If it is used to estimate an unknown parameter θ of the distribution, it is called an estimator.

Defn: (Unbiased estimator)

An estimator $T_n(x_1, x_2, \dots, x_n)$ is said to be an unbiased estimator of the corresponding population parameter θ , if $E(T_n) = \theta$.

⇒ The mean of the sampling distribution of the statistic is equal to the parameter of the population.

Example:

1. The sample mean \bar{x} is an unbiased estimator of population parameter μ if $E(\bar{x}) = \mu$.

Similarly, $E(p) = p$, where $P = \text{Population proportion}$
 $p = \text{sample proportion}$.

2. Let s^2 denote the sample variance, then s^2 is defined as $s^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2$

But s^2 is not an unbiased estimator of the population variance σ^2 .

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But on the other hand $S^2 = \frac{1}{n-1} \sum (x_i - \bar{x})^2$

provides an unbiased estimator of the population variance σ^2 .

Thus $E(S^2) = \sigma^2$ but $E(\hat{s}^2) \neq \sigma^2$.

It is important to note that

$$1. nS^2 = (n-1)\hat{s}^2 \Rightarrow S^2 = \left(\frac{n}{n-1}\right) \hat{s}^2$$

when the sample size n is large, $(n-1)$ can be approximated by n and in that case

$$\text{a bias of } (nS^2 - \sigma^2) = \left(\frac{n}{n-1}\right) S^2 - \hat{s}^2.$$

So, the sample variance S^2 gives an estimate of the population variance σ^2 .

i.e. $E(S^2) = \sigma^2$ for large samples.

Consistency:

A statistics $t_n = t(x_1, x_2, \dots, x_n)$ based on the sample size n is said to be consistent estimator of the parameter θ if $t_n \rightarrow \theta$ as $n \rightarrow \infty$.

$$\text{In other words } \lim_{n \rightarrow \infty} P(|t_n - \theta| < \epsilon) = 1$$

Efficiency:

If t_1 and t_2 are two consistent estimators of a parameter θ such that $\text{var}(t_1) < \text{var}(t_2)$ for all sample size n , then t_1 is said to be more efficient than t_2 .

In other words, an estimator with lesser variability is said to be more efficient and consequently more reliable than the other.

Sufficiency:

A statistic $t = t(x_1, x_2, \dots, x_n)$ is said to be a sufficient estimator of parameter θ if it contains all the information in the sample regarding the parameter.

Random variable. If it is used to estimate an unknown parameter θ of the distribution, it is called an estimator.

Defn: (Unbiased estimator)

An estimator $T_n(x_1, x_2, \dots, x_n)$ is said to be an unbiased estimator of the corresponding population parameter θ , if $E(T_n) = \theta$.

i. The mean of the sampling distribution of the statistic is equal to the parameter of the population.

Example:

i. The sample mean \bar{x} is an unbiased estimator of population parameter if $E(\bar{x}) = \mu$.

Similarly, $E(p) = P$, where P = Population proportion
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2. Let s^2 denote the sample variance,

then s^2 is defined as $s^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2$

But s^2 is not an unbiased estimator of population variance σ^2 .

- ① Let (x_1, x_2, \dots, x_n) be a random sample of x having unknown mean μ . Show that the estimator of μ defined by $M = \frac{1}{n} \sum_{i=1}^n x_i = \bar{x}$ is an unbiased estimator of μ .

Solution

Given $M = \frac{1}{n} \sum_{i=1}^n x_i$ and $\lambda_2 = \frac{1}{2}(x_1 + x_2)$

$$E(M) = E\left(\frac{1}{n} \sum_{i=1}^n x_i\right)$$

$$= \frac{1}{n} \sum_{i=1}^n E(x_i)$$

$$= \frac{1}{n} \sum_{i=1}^n \mu = \frac{1}{n} n \mu = \mu$$

$$= \frac{1}{n} \cdot n \mu = \mu$$

$$= E(M) = \mu$$

Thus, M is an unbiased estimator of μ .

- ② Let (x_1, \dots, x_n) be a random sample of x having unknown mean μ and variance σ^2 . Show that the estimator of σ^2 defined by $s^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2$

where \bar{x} is the sample mean, is a biased estimator of σ^2 .

Soln:

$$\text{By defn, we have } \sigma^2 = E[(x_i - \mu)^2]$$

Now,

$$\text{Given } s^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2$$

$$E(s^2) = E\left[\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2\right]$$

$$= \frac{1}{n} \sum_{i=1}^n E(x_i - \bar{x})^2$$

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b) $\text{Var}(\bar{x}) = E\left\{\frac{1}{n} \sum_{i=1}^n [x_i - \mu + \mu - \bar{x}]^2\right\}$

x fo elgant $E\left\{\frac{1}{n} \sum_{i=1}^n [(x_i - \mu) - (\bar{x} - \mu)]^2\right\}$ \rightarrow given \rightarrow (x_1, \dots, x_n) tot. \rightarrow $\text{Var}(\bar{x})$ \rightarrow $\text{Var}(\bar{x})$ \rightarrow $\text{Var}(\bar{x})$ \rightarrow $\text{Var}(\bar{x})$ \rightarrow $\text{Var}(\bar{x})$

mo $\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$ \rightarrow $\bar{x} = M$ \rightarrow $\text{Var}(\bar{x})$ \rightarrow $\text{Var}(\bar{x})$ \rightarrow $\text{Var}(\bar{x})$

$$= E\left\{\frac{1}{n} \sum_{i=1}^n \left[(x_i - \mu)^2 - 2(x_i - \mu)(\bar{x} - \mu) + (\bar{x} - \mu)^2 \right] \right\}$$

$$= E\left\{\frac{1}{n} \left[\sum_{i=1}^n (x_i - \mu)^2 - 2 \sum_{i=1}^n (x_i - \mu)(\bar{x} - \mu) + n(\bar{x} - \mu)^2 \right] \right\}$$

$$\text{var}(M_2) = \text{var}\left(\frac{1}{n} \sum_{i=1}^n x_i \right) = \frac{1}{n} \sum_{i=1}^n (\bar{x} - \mu)^2$$

$$= E\left\{\frac{1}{n} \left[\sum_{i=1}^n (x_i - \mu)^2 - 0 + n(\bar{x} - \mu)^2 \right] \right\}$$

$$= \frac{1}{n} \sum_{i=1}^n E(x_i - \mu)^2 + E(\bar{x} - \mu)^2$$

$$= \frac{1}{n} \cdot n \cdot E(x_i - \mu)^2 + E(\bar{x} - \mu)^2$$

\cdot x fo $\text{Var}(\bar{x})$ \rightarrow $\text{Var}(\bar{x})$ \rightarrow $\text{Var}(\bar{x})$ \rightarrow $\text{Var}(\bar{x})$ \rightarrow $\text{Var}(\bar{x})$ \rightarrow $\text{Var}(\bar{x})$

$= E(x_i - \mu)^2 + E(\bar{x} - \mu)^2$

given x fo elgant mabnor \rightarrow (x_1, \dots, x_n) tot. \rightarrow $\text{Var}(\bar{x}) = \sigma^2 + \frac{\sigma^2}{n}$ \rightarrow $\text{Var}(\bar{x})$ \rightarrow $\text{Var}(\bar{x})$

$\therefore (\bar{x} - \mu)^2$ \rightarrow $\text{Var}(\bar{x})$ \rightarrow $\text{Var}(\bar{x})$ \rightarrow $\text{Var}(\bar{x})$ \rightarrow $\text{Var}(\bar{x})$ \rightarrow $\text{Var}(\bar{x})$

We have $\text{Var}\left(\sum_{i=1}^n a_i x_i\right) = \sum_{i=1}^n a_i^2 \text{Var}(x_i)$

Let $\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$ \rightarrow $\text{Var}(\bar{x})$ \rightarrow $\text{Var}(\bar{x})$ \rightarrow $\text{Var}(\bar{x})$ \rightarrow $\text{Var}(\bar{x})$

mean μ and variance σ^2 \rightarrow $\text{Var}(\bar{x})$ \rightarrow $\text{Var}(\bar{x})$ \rightarrow $\text{Var}(\bar{x})$ \rightarrow $\text{Var}(\bar{x})$

So, $\sigma_{\bar{x}}^2 = \text{Var}(\bar{x}) = \text{Var}\left(\frac{1}{n} \sum_{i=1}^n x_i\right)$

is defined to be \sqrt{n} \rightarrow $\text{Var}(\bar{x})$ \rightarrow $\text{Var}(\bar{x})$ \rightarrow $\text{Var}(\bar{x})$

$\sqrt{n}(\bar{x} - \mu)$ \rightarrow $\text{Var}(\bar{x})$ \rightarrow $\text{Var}(\bar{x})$ \rightarrow $\text{Var}(\bar{x})$

$\therefore (\bar{x}_1, \dots, \bar{x}_n)$, shows that the linear estimator defined

by $M = \frac{1}{n} \sum_{i=1}^n x_i = \bar{x}$ \rightarrow $\text{Var}(M) = \frac{1}{n^2} \sum_{i=1}^n \text{Var}(x_i)$

linear unbiased estimator of all \rightarrow $\text{Var}(\bar{x})$ \rightarrow $\text{Var}(\bar{x})$ \rightarrow $\text{Var}(\bar{x})$ \rightarrow $\text{Var}(\bar{x})$

$$\begin{aligned} \text{Var}(\bar{x}) &= \frac{1}{n^2} \sum_{i=1}^n \sigma^2 \\ &= \frac{1}{n^2} n \sigma^2 = \frac{1}{n} \sigma^2 \end{aligned}$$

$$\text{Thus } E(S^2) = \sigma^2 + \frac{1}{n} \sigma^2 = \left(\frac{n+1}{n}\right) \sigma^2 \quad (\text{d})$$

which shows that S^2 is a biased estimator of σ^2 .

- ③ Let (x_1, x_2, \dots, x_n) be a random sample of a Poisson r.v. X with unknown parameter λ .

a) Show that $\lambda_1 = \frac{1}{n} \sum_{i=1}^n x_i$ and $\lambda_2 = \frac{1}{2}(x_1+x_2)$

are both unbiased estimators of λ .

b) Which estimator is more efficient?

Solution:

$$\text{Given } \lambda_1 = \frac{1}{n} \sum_{i=1}^n x_i$$

$$E(\lambda_1) = E\left(\frac{1}{n} \sum_{i=1}^n x_i\right)$$

$$= \frac{1}{n} \cdot n = \frac{1}{n} \sum_{i=1}^n E(x_i)$$

$$= \frac{1}{n} \sum_{i=1}^n \lambda$$

thus, λ_1 is an unbiased estimator of λ .

- ④ Let (x_1, x_2, \dots, x_n) be a random sample of X having unknown mean μ and variance σ^2 . Show that the estimator λ_2 defined by

$$E(\lambda_2) = E\left(\frac{1}{2}(x_1+x_2)\right)$$

where x_1 is the first sample mean, is an unbiased estimator of μ .

If we denote the first n observations by x_1, x_2, \dots, x_n and the last n observations by $x_{n+1}, x_{n+2}, \dots, x_{2n}$, then $\lambda_2 = \frac{1}{2} E(x_1) + \frac{1}{2} E(x_2)$ is a unbiased estimator of μ .

Since $E(x_1) = \frac{1}{n} \sum_{i=1}^n E(x_i) = \frac{1}{n} \sum_{i=1}^n \mu = \mu$

$$\text{Given } E(x_2) = \frac{1}{n} \sum_{i=n+1}^{2n} E(x_i) = \frac{1}{n} \sum_{i=n+1}^{2n} \mu = \mu$$

$\therefore E(\lambda_2) = \frac{1}{2} \mu + \frac{1}{2} \mu = \mu$

$\therefore \lambda_2$ and λ_1 are unbiased estimators of μ .

$$E(S^2) = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2$$

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$$b) \text{ var}(\lambda_1) = \text{var}\left(\frac{1}{n} \sum_{i=1}^n x_i\right)$$

$$= \sum_{i=1}^n \frac{1}{n^2} \text{var}(x_i)$$

\rightarrow if elements of random sample (x_1, x_2, \dots, x_n) are iid

λ estimated by $\frac{1}{n} \sum_{i=1}^n x_i$ is var λ

$$(ex + x) \stackrel{!}{=} \lambda \text{ bms} ; x \sum_{i=1}^n \frac{1}{n} = \lambda \text{ resultants}$$

$$\lambda \text{ go to } \frac{1}{n^2} \sum_{i=1}^n n \lambda = \frac{1}{n} \cdot \lambda$$

$$\text{admissible exam & notarise} \quad \text{var}(\lambda_2) = \text{var}\left(\frac{1}{2} [x_1 + x_2]\right)$$

$$= E\left\{\left(x_1 + x_2\right)^2\right\} - \text{mean}^2$$

$$= \sum_{i=1}^2 \frac{1}{2^2} \text{var}(x_i) + \frac{1}{2^2} \text{var}(x_2)$$

$$\left(\sum_{i=1}^2 \frac{1}{2^2}\right) = (1A) \exists$$

$$= \frac{1}{4} \lambda + \frac{1}{4} \lambda = \frac{2}{4} \lambda = \lambda/2$$

[Here mean = variance = λ , since x is a poisson variable]

$$\text{If } n > 2, \lambda_1 < \lambda_2$$

thus if $n > 2$, λ_1 is a more efficient estimator of λ than λ_2 .

- (4) Let (x_1, \dots, x_n) be a random sample of x with mean μ and variance σ^2 . A linear estimator of μ is defined to be a linear function of x_1, \dots, x_n , $\lambda(x_1, \dots, x_n)$. Show that the linear estimator defined by $M = \frac{1}{n} \sum_{i=1}^n x_i = \bar{x}$ is the most efficient linear unbiased estimator of μ .

$$= \frac{1}{n^2} n \sigma^2 = \frac{1}{n} \sigma^2$$



Assume that $M_1 = \sum_{i=1}^n a_i x_i$

is a linear unbiased estimator of μ
with lower variance than M .

Since M_1 is unbiased, we must have

$$E(M_1) = \mu \Rightarrow \sum_{i=1}^n a_i E(x_i) = \mu$$

$$\Rightarrow \sum_{i=1}^n a_i \mu = \mu \text{ (unbiased)}$$

$$\Rightarrow \mu \sum_{i=1}^n a_i = \mu M \quad (\because E(M_1) = \mu)$$

$$\Rightarrow \sum_{i=1}^n a_i = 1.$$

$$0 = (\sigma^2) \text{Var} \quad \text{and} \quad 0 = (\sigma^2) \sum_{i=1}^n a_i^2 \quad \text{and}$$

$$\text{Var}(M_1) = \frac{1}{n} \sigma^2 \text{ and}$$

$$\text{Var}(M_1) = \sigma^2 \sum_{i=1}^n a_i^2$$

By assumption

$$\sigma^2 \sum_{i=1}^n a_i^2 < \frac{1}{n} \sigma^2$$

$$\text{As variance is diff. positive} \Rightarrow \sum_{i=1}^n a_i^2 < \frac{1}{n} \text{ variance}$$

$$\text{Also } \sum_{i=1}^n a_i^2 - \frac{1}{n} < 0$$

$$\text{Now consider } 0 \leq \sum_{i=1}^n (a_i - \frac{1}{n})$$

$$[\theta - (\sigma^2) \sum_{i=1}^n a_i + (\sigma^2) \sum_{i=1}^n - \sigma^2] = \sum_{i=1}^n (a_i^2 - 2a_i \frac{1}{n} + \frac{1}{n^2})$$

$$+ [\theta - (\sigma^2) \sum_{i=1}^n + [(\sigma^2) \sum_{i=1}^n - \sigma^2] = \sum_{i=1}^n a_i^2 - \frac{2}{n} \sum_{i=1}^n a_i + \sum_{i=1}^n \frac{1}{n^2}$$

$$\{[\theta - (\sigma^2) \sum_{i=1}^n] [(\sigma^2) \sum_{i=1}^n - \sigma^2]\}$$

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$$\sum_{i=1}^n x_i = \sum_{i=1}^n a_i - \frac{\theta}{n} \cdot 11 + \frac{1}{n^2} \cdot n$$

to estimate θ we will \rightarrow

$$0 \leq \sum_{i=1}^n a_i - \frac{1}{n} = \sum_{i=1}^n a_i - \frac{1}{n}$$

and $\sum_{i=1}^n a_i - \frac{1}{n}$ is unbiased for θ .

$$\Rightarrow \sum_{i=1}^n a_i - \frac{1}{n} \neq 0$$

$$M = \sum_{i=1}^n a_i - \frac{1}{n} = (M)$$

This is contradiction to our assumption,

$$\text{since } \left(\sum_{i=1}^n a_i - \frac{1}{n} < 0 \right).$$

($M = (M)$) So M is the most efficient linear unbiased estimator of μ .

⑤ Show that if $\lim_{n \rightarrow \infty} E(T_n) = \theta$ and $\lim_{n \rightarrow \infty} \text{var}(T_n) = 0$.

then the estimator T_n is consistent.

Solution:

[for any $a > 0$, show that

$$P(|T_n - \theta| > a) \leq \frac{\sigma^2}{a^2}$$

where μ_x and σ_x^2 are the mean and variance of x respectively. This is known as Chebyshov inequality]

Using Chebyshov's inequality, we can write

$$P(|T_n - \theta| > \epsilon) \leq \frac{E[(T_n - \theta)^2]}{\epsilon^2}$$

$$\left(\frac{1}{n} + \frac{1}{n^2} - \frac{1}{n^3} \right) = \frac{1}{\epsilon^2} E \left[T_n - E(T_n) + E(T_n) - \theta \right]^2$$

$$\frac{1}{n} + \frac{1}{n^2} - \frac{1}{n^3} = \frac{1}{\epsilon^2} E \left\{ [T_n - E(T_n)]^2 + [E(T_n) - \theta]^2 + 2[T_n - E(T_n)][E(T_n) - \theta] \right\}$$

$$\text{Assume } = \frac{1}{n} \sum_{i=1}^n [\text{var}(T_i) + E(E(T_i) - \theta)^2 + \epsilon_i]$$

is a linear unbiased estimator of $E\{[T_n - E(T_n)][E(T_n) - \theta]\}$

Given $\lim_{n \rightarrow \infty} \text{var}(T_n) = 0$ then M_1 is unbiased, we must have

$$\lim_{n \rightarrow \infty} \text{var}(M_1) = 0 \quad \Rightarrow \quad \sum_{i=1}^n E(X_i) = \mu$$

taking $\lim_{n \rightarrow \infty}$ in ①,

$$\lim_{n \rightarrow \infty} P(|T_n - \theta| > \epsilon) = \frac{1}{\sqrt{n}} \left(\lim_{n \rightarrow \infty} \text{var}(T_n) + \sum_{i=1}^n E(X_i) - \mu \right) = 0 \quad (\because E(M_1) = \mu)$$

$$\Rightarrow \lim_{n \rightarrow \infty} E(E(T_n) - \theta)^2 +$$

$$2 \lim_{n \rightarrow \infty} E\{[T_n - E(T_n)][E(T_n) - \theta]\}$$

$$\text{var}(M_1) = \sigma^2 = \sum_{i=1}^n \sigma_i^2$$

By assumption T_n is consistent.

$$\sigma^2 \sum_{i=1}^n a_i^2 \leq \frac{1}{n} \sigma^2$$

$$\Rightarrow \sum_{i=1}^n a_i^2 \leq \frac{1}{n}$$

$$\Rightarrow \sum_{i=1}^n a_i^2 + \frac{1}{n} \leq 0$$

Now consider

$$0 \leq \sum_{i=1}^n (a_i - b_i)$$

$$0 = \sum_{i=1}^n (a_i^2 - 2a_i b_i + b_i^2)$$

$$\sum_{i=1}^n a_i^2 - \frac{2}{n} \sum_{i=1}^n a_i + \frac{1}{n^2}$$

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Maximum-Likelihood Estimation

Let $f(x; \theta) = f(x_1, x_2, \dots, x_n; \theta)$ denote the joint pmf of the r.v's x_1, \dots, x_n when they are discrete and let it be their joint pdf when they are continuous.

$$\text{Let } L(\theta) = f(x; \theta) = f(x_1, \dots, x_n; \theta)$$

Note $L(\theta)$ represents the likelihood that the values x_1, x_2, \dots, x_n will be observed when θ is the true value of the parameter. Thus $L(\theta)$ is often referred to as the likelihood function of the random sample.

Let $L(\theta_M) = S(x_1, x_2, \dots, x_n)$ be the maximizing value of $L(\theta)$.

i.e., $L(\theta_M) = \max_{\theta} L(\theta)$
maximum-likelihood estimator of θ .

Then the maximum-likelihood estimator of θ is θ_M which is $S(x_1, \dots, x_n)$ and

θ_M is the maximum-likelihood estimate of θ .

Ques: ① Let (x_1, \dots, x_n) be a random sample of a binomial r.v X with parameters (m, p) , where m is assumed to be known and p unknown. Determine the maximum-likelihood estimator of p .

Solution:

The likelihood function is given by

$$L(p) = f(x_1, x_2, \dots, x_n; p) = \binom{m}{x_1} p^{x_1} (1-p)^{m-x_1}$$

$$\text{where } \binom{m}{x_1} p^{x_1} (1-p)^{m-x_1}$$

$$\frac{d}{dp} \ln L(p) = \sum_{i=1}^n x_i - n$$

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Setting $\text{soft stored} = \left(\frac{m}{x_1}\right) \dots \left(\frac{m}{x_n}\right) p^{\sum_{i=1}^n x_i} (1-p)^{(mn - \sum_{i=1}^n x_i)}$

Taking Natural logarithm of the above expression,
we get

$$\ln L(p) = \ln \left[\left(\frac{m}{x_1} \right) \dots \left(\frac{m}{x_n} \right) p^{\sum_{i=1}^n x_i} (1-p)^{(mn - \sum_{i=1}^n x_i)} \right]$$

Hence the $= \ln \left[\left(\frac{m}{x_1} \right) \dots \left(\frac{m}{x_n} \right) \right] + \ln p^{\sum_{i=1}^n x_i} + \ln (1-p)^{(mn - \sum_{i=1}^n x_i)}$

$$= \ln C + \sum_{i=1}^n x_i (\ln p) + (mn - \sum_{i=1}^n x_i) \ln (1-p)$$

Determine the maximum likelihood estimate \hat{p}

$$(0) \text{ max } = \text{where } C = \prod_{i=1}^n \frac{m}{x_i}$$

The likelihood function is given by

$$\text{Now } \frac{d}{dp} \ln(L(p)) = \sum_{i=1}^n x_i \left(\frac{1}{p} \right) + \left(mn - \sum_{i=1}^n x_i \right) \frac{1}{(1-p)}$$

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Thus $\frac{d}{dp} \ln(L(p)) = \frac{1}{p} \sum_{i=1}^n x_i - \left(mn - \sum_{i=1}^n x_i \right) \frac{1}{(1-p)}$

Set $\frac{d}{dp} \ln(L(p)) = 0$ to find \hat{p}

Setting $\frac{d}{dp} \ln(L(p)) / dp = 0$, the maximum

likelihood estimate \hat{p}_{ML} of p is obtained as

\hat{p}_{ML} for maximum likelihood estimate

$$0 = \frac{1}{\hat{p}_{ML}} \sum_{i=1}^n x_i - \left(mn - \sum_{i=1}^n x_i \right) \frac{1}{(1-\hat{p}_{ML})}$$

Setting

$$\Rightarrow \frac{1}{\hat{p}_{ML}} \sum_{i=1}^n x_i = \left(mn - \sum_{i=1}^n x_i \right) \frac{1}{(1-\hat{p}_{ML})}$$

$$\Rightarrow \frac{(1-\hat{p}_{ML})}{\hat{p}_{ML}} \sum_{i=1}^n x_i = mn - \sum_{i=1}^n x_i$$

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Maximum Likelihood Estimation

$$\Rightarrow \frac{1}{\hat{P}_{ML}} \sum_{i=1}^n x_i - \sum_{i=1}^n \ln x_i = mn - \sum_{i=1}^n \ln x_i$$

Let $L(\theta) = \prod_{i=1}^n f(x_i; \theta)$

$$\Rightarrow \frac{1}{\hat{P}_{ML}} \sum_{i=1}^n x_i = mn$$

$$\Rightarrow \frac{1}{mn} \sum_{i=1}^n x_i = \hat{P}_{ML}$$

Hence, the maximum likelihood estimator of P is given by

$$\hat{P}_{ML} = \frac{1}{mn} \sum_{i=1}^n x_i = \bar{x}$$

2) Let (x_1, \dots, x_n) be a random sample of a Poisson r.r with unknown parameter λ . Determine the maximum likelihood estimator of λ .

Solution:

$$L(\lambda) = f(x_1, \dots, x_n; \lambda) = \prod_{i=1}^n \frac{e^{-\lambda} \lambda^{x_i}}{x_i!}$$

$$\begin{aligned} L(\lambda) &= e^{-n\lambda} \lambda^{\sum_{i=1}^n x_i} \\ &\quad \times \left[\frac{e^{-n\lambda} \lambda^{\sum_{i=1}^n x_i}}{x_1! x_2! \dots x_n!} \right] \end{aligned}$$

Solution: The likelihood function is given by

$$L(\lambda) = e^{-n\lambda} + \sum_{i=1}^n x_i \ln \lambda - \sum_{i=1}^n \ln x_i$$

$$\begin{aligned} \frac{d}{d\lambda} \ln L(\lambda) &= -n + \sum_{i=1}^n x_i - \sum_{i=1}^n \frac{x_i}{\lambda} \\ &\quad \text{where } C = x_1! \dots x_n! \end{aligned}$$

$$\frac{d}{d\lambda} \ln L(\lambda) = -n + \sum_{i=1}^n x_i - \frac{\sum_{i=1}^n x_i}{\lambda}$$

Setting $\frac{d}{d\lambda} \ln L(\lambda) = 0$, we get

Taking $0 = -n + \frac{1}{\hat{\lambda}_{ML}} \sum_{i=1}^n x_i$ of the above expression,

we get

$$\Rightarrow \frac{1}{\hat{\lambda}_{ML}} \sum_{i=1}^n x_i = n$$

$$\Rightarrow \hat{\lambda}_{ML} = \frac{1}{n} \sum_{i=1}^n x_i = \bar{x}$$

Hence, the maximum-likelihood estimator of λ is given by

$$\lambda_{ML} = \frac{1}{n} \sum_{i=1}^n x_i = \bar{x}$$

- ③ Let (x_1, \dots, x_n) be a random sample of an exponential r.r. X with unknown parameter λ . Determine the maximum-likelihood estimator of λ .

solution
The likelihood function is given by

$$L(\lambda) = f(x_1, \dots, x_n; \lambda) = \prod_{i=1}^n \lambda e^{-\lambda x_i}$$

$$\text{thus } \ln L(\lambda) = \ln \left(\lambda^n e^{-\lambda \sum_{i=1}^n x_i} \right)$$

$$\text{setting } \ln L(\lambda) = \ln \lambda^n + \ln e^{-\lambda \sum_{i=1}^n x_i}$$

$$\text{likelihood } \ln L(\lambda) = n \ln \lambda + \lambda \sum_{i=1}^n x_i$$

$$\Rightarrow \frac{d}{d\lambda} \ln(L(\lambda)) = \frac{n}{\lambda} - \sum_{i=1}^n x_i$$

Setting $\frac{d}{d\lambda} \ln L(\lambda) = 0$, we get

$$0 = \frac{n}{\hat{\lambda}_{ML}} - \sum_{i=1}^n x_i$$

$$\Rightarrow \hat{\lambda}_{ML} = \frac{n}{\sum_{i=1}^n x_i} = \frac{n}{mn - (\lambda) \sum_{i=1}^n x_i} = \frac{n}{mn - \bar{x}n} = \frac{n}{n(\bar{x} - 1)}$$



$\Rightarrow \hat{\lambda}_{ML} = \frac{n}{\sum_{i=1}^n x_i}$

(H) $\Rightarrow \hat{\lambda}_{ML} = \frac{1}{\bar{x}}$

Hence, the maximum-likelihood estimator of λ is given by $\hat{\lambda}_{ML} = \frac{n}{\sum_{i=1}^n x_i} = \frac{1}{\bar{x}}$.

- A) Let (x_1, \dots, x_n) be a random sample of a normal random variable x with unknown mean μ and unknown variance σ^2 . Determine the maximum-likelihood estimators of μ and σ^2 .

solution

The likelihood function is given by

$$L(\mu, \sigma) = f(x_1, x_2, \dots, x_n; \mu, \sigma)$$

$$= \prod_{i=1}^n \frac{1}{\sqrt{2\pi}\sigma} \exp \left[-\frac{1}{2\sigma^2} (x_i - \mu)^2 \right]$$

$$= \left(\frac{1}{2\pi} \right)^{n/2} \frac{1}{\sigma^n} \exp \left[-\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2 \right]$$

Thus

$$\ln L(\mu, \sigma) = \ln \left[\left(\frac{1}{2\pi} \right)^{n/2} \frac{1}{\sigma^n} \exp \left[-\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2 \right] \right]$$

$$= \ln \left(\frac{1}{2\pi} \right)^{n/2} + \ln \frac{1}{\sigma^n} + \left(-\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2 \right)$$

$$= \frac{n}{2} \ln \frac{1}{2\pi} - n \ln \sigma - \frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2$$

To maximize the log-likelihood function w.r.t μ , we differentiate w.r.t μ and equate to zero.

$$\frac{\partial \ln L(\mu, \sigma)}{\partial \mu} = -\frac{n}{2\sigma^2} \sum_{i=1}^n (x_i - \mu) = 0$$

$$\Rightarrow \sum_{i=1}^n x_i = n\bar{x}$$

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In order to find the values of μ and σ^2 maximizing the above, we compute

$$\Rightarrow \frac{\partial}{\partial \mu} \ln L(\mu, \sigma) = -\frac{1}{2\sigma^2} \sum_{i=1}^n 2(x_i - \mu) \quad (\rightarrow)$$

$$= \frac{1}{\sigma^2} \sum_{i=1}^n (x_i - \mu) \rightarrow 0$$

$$\Rightarrow \frac{\partial}{\partial \sigma} \ln L(\mu, \sigma) = -\frac{n}{\sigma} - \frac{1}{2} \cdot \left(-\frac{2}{\sigma^3} \right) \sum_{i=1}^n (x_i - \mu)^2$$

Intention is to obtain solution so that (μ, \dots, σ) is the maximum likelihood estimator. Then we have $\frac{\partial \ln L}{\partial \mu} + \frac{1}{\sigma^3} \sum_{i=1}^n (x_i - \mu)^2 = 0$

Maximum likelihood estimator is obtained by equating these equations to zero, we get

$$\textcircled{1} \Rightarrow \frac{1}{\sigma^2} \sum_{i=1}^n (x_i - \hat{\mu}_{ML}) = 0$$

$$\Leftrightarrow \sum_{i=1}^n (x_i - \hat{\mu}_{ML}) = 0 \quad \text{and}$$

$$\left[\frac{(x_i - \bar{x})}{\hat{\sigma}_{ML}} \right] \frac{1}{\hat{\sigma}_{ML}} + \frac{1}{\hat{\sigma}_{ML}^3} \sum_{i=1}^n (x_i - \hat{\mu}_{ML})^2 = 0$$

$$\textcircled{2} \Rightarrow \left[\frac{(x_i - \bar{x})}{\hat{\sigma}_{ML}} \right] \frac{1}{\hat{\sigma}_{ML}^3} + \frac{1}{\hat{\sigma}_{ML}^3} \sum_{i=1}^n (x_i - \hat{\mu}_{ML})^2 = \frac{n}{\hat{\sigma}_{ML}^2}$$

$$\left[\frac{(x_i - \bar{x})}{\hat{\sigma}_{ML}} \right] \frac{1}{\hat{\sigma}_{ML}^3} + \frac{1}{\hat{\sigma}_{ML}^3} \sum_{i=1}^n (x_i - \hat{\mu}_{ML})^2 = \frac{n}{\hat{\sigma}_{ML}^2}$$

$$\Rightarrow \frac{1}{\hat{\sigma}_{ML}^2} \sum_{i=1}^n (x_i - \hat{\mu}_{ML})^2 = \frac{n}{\hat{\sigma}_{ML}^2}$$

$$\Rightarrow \frac{1}{n} \sum_{i=1}^n (x_i - \hat{\mu}_{ML})^2 = \frac{\hat{\sigma}_{ML}^2}{n}$$

Hence, the maximum likelihood estimators of μ and σ^2 are given respectively by

$$\Rightarrow M_{ML} = \frac{1}{n} \sum_{i=1}^n x_i = \bar{x}$$

$$S_{ML}^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2$$

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5) Find the MLE for θ of the Bin. $(1, \theta)$ distribution

$$f(x; \theta) = \theta^x (1-\theta)^{1-x}, \quad x=0,1; \quad 0 \leq \theta \leq 1.$$

Solution The likelihood function is given by

$$L(\theta) = f(x_1, x_2, \dots, x_n, \theta)$$

$$= \prod_{i=1}^n \theta^{x_i} (1-\theta)^{1-x_i}$$

$$= \frac{\theta^{\sum_{i=1}^n x_i}}{(1-\theta)^{n - \sum_{i=1}^n x_i}}$$

$$\text{Thus } \ln L(\theta) = \ln \left[\theta^{\sum_{i=1}^n x_i} (1-\theta)^{n - \sum_{i=1}^n x_i} \right]$$

$$= \frac{n}{\theta} \ln \theta + (n - \sum_{i=1}^n x_i) \ln (1-\theta)$$

$$= \sum_{i=1}^n x_i \ln \theta + (n - \sum_{i=1}^n x_i) \ln (1-\theta)$$

Setting Now $\frac{d}{d\theta} \ln L(\theta) = \frac{1}{\theta} \sum_{i=1}^n x_i - \frac{1}{1-\theta} (n - \sum_{i=1}^n x_i)$

Setting $\frac{d}{d\theta} \ln L(\theta) = 0$, we get

$$0 = \frac{1}{\hat{\theta}_{ML}} \sum_{i=1}^n x_i - \frac{1}{1-\hat{\theta}_{ML}} (n - \sum_{i=1}^n x_i)$$

$$\Rightarrow \frac{1}{\hat{\theta}_{ML}} \sum_{i=1}^n x_i = \frac{1}{(1-\hat{\theta}_{ML})} (n - \sum_{i=1}^n x_i)$$

$$\Rightarrow \frac{(1-\hat{\theta}_{ML})}{\hat{\theta}_{ML}} \sum_{i=1}^n x_i = n - \sum_{i=1}^n x_i$$

(19)

In order to find the values of μ and σ^2

$$\Rightarrow \left(\frac{1}{\hat{\theta}_{ML}} - 1 \right) \sum_{i=1}^n x_i = n + \sum_{i=1}^n x_i$$

$$\Rightarrow \frac{1}{\hat{\theta}_{ML}} \leq x_i - \sum_{i=1}^n x_i = n - \sum_{i=1}^n x_i \quad (1)$$

which implies $(\theta, 1)$ is not MLE est. for σ^2 .

$$\Rightarrow 1 \geq \theta \geq 10 \sum_{i=1}^n x_i = \frac{n}{10} (\theta - 1) \Rightarrow \theta = \frac{n}{10} + \sum_{i=1}^n x_i$$

$$\Rightarrow \hat{\theta}_{ML} = \frac{1}{n} \sum_{i=1}^n x_i$$

Hence, the maximum likelihood estimator of θ

$$\text{is given by } \hat{\theta}_{ML} = \frac{1}{n} \sum_{i=1}^n x_i = \bar{x}.$$

Equating the equations,

$$\left[\begin{array}{l} \frac{1}{\hat{\theta}_{ML}} - 1 = 0 \\ \sum_{i=1}^n (x_i - \hat{\theta}_{ML}) = 0 \end{array} \right] \text{ and } (2)$$

$$\Rightarrow \sum_{i=1}^n (x_i - \hat{\theta}_{ML}) = 0 \text{ and}$$

$$(2-1) \Rightarrow \frac{1}{\hat{\theta}_{ML}} - 1 = 0 \Rightarrow \hat{\theta}_{ML} = \frac{n}{n-1}$$

$$(2-1) \Rightarrow \sum_{i=1}^n (x_i - \hat{\theta}_{ML})^2 = 0$$

$$\left(\sum_{i=1}^n (x_i - \bar{x})^2 \right) \frac{1}{\hat{\theta}_{ML}} - \frac{1}{\hat{\theta}_{ML}} \sum_{i=1}^n (x_i - \bar{x})^2 = 0 \Rightarrow \hat{\theta}_{ML} \text{ won't pass}$$

$$\Rightarrow \hat{\theta}_{ML} = \bar{x} = \frac{1}{n} \sum_{i=1}^n x_i = \frac{1}{n} \sum_{i=1}^n (x_i - \hat{\theta}_{ML})^2 = \sigma^2$$

$$\left(\sum_{i=1}^n (x_i - \bar{x})^2 \right) \frac{1}{\hat{\theta}_{ML}} - \frac{1}{\hat{\theta}_{ML}} = \sum_{i=1}^n \frac{1}{\hat{\theta}_{ML}} = \sigma^2$$

Hence, the maximum likelihood estimators of μ and σ^2 are given respectively by

$$\hat{\mu} = \bar{x} = \frac{1}{n} \sum_{i=1}^n x_i \quad (2)$$

(1)

(2)

Method of moments:

The method of moments consists of equating the first few moments of a population to the corresponding moment of a sample, thus getting as many equations as are needed to solve for the unknown parameters of the population.

Definition:

The r^{th} sample moment of a set of observations x_1, x_2, \dots, x_n is the mean of their powers and it is denoted by m_r' .

$$\text{i.e. } m_r' = \frac{1}{n} \sum_{i=1}^n x_i^r \text{ and } \bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$$

\therefore if a population has r parameters, the method of moments consists of solving the system of equations $m_r' = \mu_r'$, $r=1, 2, \dots, k$. for the k parameters.

Example:

Let x_1, x_2, \dots, x_n be a random sample of size n from a poisson population with parameter. Obtain the estimator of λ by the method of moments.

~~Soln:~~

The poisson population has the probability function $P(x, \lambda) = \frac{e^{-\lambda} \lambda^x}{x!}$ $x=0, 1, 2, \dots$

$$E(x) = \lambda$$

$$E(m_1') = \frac{1}{n} \sum_{i=1}^n x_i = \bar{x}$$

(2)

Solving the equations $m'_1 = \mu$, we get $\bar{x} = \lambda$.

The method of moment estimator of λ is \bar{x} .

Example: 2 Let x_1, x_2, \dots, x_n be a random sample of size n from the normal population $N(\mu, \sigma^2)$. Obtain the estimators of μ and σ^2 by the method of moments.

The P.D.F. of the Normal distribution

$$f(x; \mu, \sigma^2) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}, -\infty < x < \infty$$

$$\text{Mean } M'_1 = E(x) = \mu$$

$$\text{Var } \mu'_2 = E(x^2) = \mu^2 + \sigma^2 \text{ or } (\mu^2 + \sigma^2 = E(x^2) - E(x)^2)$$

$$\text{But } m'_1 = \frac{1}{n} \sum x_i = \bar{x} \quad (i)$$

$$m'_2 = \frac{1}{n} \sum x_i^2$$

Solving the equations $\mu'_1 = m'_1$,

$$\mu'_2 = m'_2$$

$$\text{we get } \mu'_1 = \bar{x} \quad \mu'_2 = \frac{1}{n} \sum x_i^2$$

$$\mu^2 + \sigma^2 = \frac{1}{n} \sum x_i^2$$

$$\sigma^2 = \frac{1}{n} \sum x_i^2 - \bar{x}^2$$

$$\text{Variance } = \frac{1}{n} \sum (x_i - \bar{x})^2$$

$$\begin{aligned} \left[\frac{1}{n} \sum (x_i^2 + \bar{x}^2 - 2x_i \bar{x}) \right] \\ = \frac{1}{n} \left[\sum x_i^2 + n\bar{x}^2 - 2\bar{x} \sum x_i \right] \end{aligned}$$

$$= \frac{1}{n} \left[\sum x_i^2 + n\bar{x}^2 - 2\bar{x}(n\bar{x}) \right]$$

$$= \frac{1}{n} \left[\sum x_i^2 + n\bar{x}^2 - 2n\bar{x}^2 \right]$$

$$\bar{x} = \frac{1}{n} \left[\sum x_i^2 - n\bar{x}^2 \right] = \frac{1}{n} \sum x_i^2 - \bar{x}^2$$

\therefore The estimator of the parameter μ is \bar{x} .

The estimator of the parameter σ^2 is $\sum (x_i - \bar{x})^2$.

Example : 3

Obtain the estimator of the parameter P of a Binomial population using the method of moments.

Soln:

$$P(x, n, P) = n C_x P^x q^{n-x}, \quad x=0, 1, 2, \dots, n.$$

Definition:

$$m'_i = E(x^i) = np^i$$

Let x_1, x_2, \dots, x_n be a random sample from the Binomial population then $m'_i = \frac{\sum x_i}{n} = \bar{x}$.

\therefore By the method of moments $m'_i = \mu_i$

$$\bar{x} = np \Rightarrow \frac{\bar{x}}{n} = p$$

$\therefore \frac{\bar{x}}{n}$ is the estimator of p .

Example : 4

The number of hours an electron tube will work is assumed to be an exponential variable with parameter λ . Given a sample of n life tubes of this sort, compute the method of moments estimator of λ .

Solution:

The pdf of the exponential distribution

$$f(x, \lambda) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$M = \int_0^\infty x \lambda e^{-\lambda x} dx \quad [\because f \text{ is continuous}]$$

$$u = x \quad dv = \lambda e^{-\lambda x} dx$$

$$du = dx \quad v = \frac{e^{-\lambda x}}{-\lambda}$$

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$$X \sim \text{Exp}(\lambda) \text{ determined with respect to } f(x)$$

Solving the equation $m_1 = \int_{-\infty}^{\infty} x e^{-\lambda x} dx$

$\Rightarrow \lambda = \left[\int_0^{\infty} x e^{-\lambda x} dx + \frac{1}{\lambda} \int_0^{\infty} e^{-\lambda x} dx \right]$

\Rightarrow Example 2

\Rightarrow To find $\hat{\lambda}$ determine $m_1 = \int_{-\infty}^{\infty} x e^{-\lambda x} dx$

Method \Rightarrow by partial integration $= \left[x e^{-\lambda x} \right]_{-\infty}^{\infty} + \left[e^{-\lambda x} \right]_{-\infty}^{\infty}$

$= 0 + \frac{1}{\lambda} [0 - 1] = \frac{1}{\lambda}$

i.e. $\mu_1' = \frac{1}{\lambda}$

Method of moments: Let x_1, x_2, \dots, x_n be a random sample from the population.

Let $\bar{x} = \frac{x_1 + x_2 + \dots + x_n}{n}$ be the sample mean.

$$\bar{x} = \frac{x_1 + x_2 + \dots + x_n}{n}$$

To estimate μ_1' as \bar{x} .

By the method of moments,

This will make $\mu_1' = m_1'$ equal to sample \bar{x} .
 Method of moments used between \bar{x} & μ_1' is called a linear estimator of μ_1' .
 $\Rightarrow \lambda = \frac{1}{\bar{x}}$ will estimate λ with respect to \bar{x} .

Method of moments will be for $f(x)$.

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

$$[\text{and estimate of } f(x)] = \frac{1}{n} \sum_{i=1}^n \lambda e^{-\lambda x_i} = \lambda$$

$$= \frac{1}{n} [2x_1^2 + 10x_1 - 3x_1(n+1)]$$

$$= \frac{1}{n} [2x_1^2 + 10x_1 - 3nx_1 - 3nx]$$

$$= \frac{1}{n} [2x_1^2 + 10x_1 - 3nx]$$

$$= \frac{1}{n} [2x_1^2 + 10x_1 - 3nx]$$