Unit-III

Estimation theory

Defn: (Estimator).

Any function of the random sample $x_1, x_2, \ldots, x_n$ that are being observed, say $T_n(x_1, x_2, \ldots, x_n)$ is called a statistic. clearly, a statistic is a random variable. If it is used to estimate an unknown parameter θ of the distribution, it is called an estimator.

Defn: (Unbiased estimator)

An estimator $T_n(x_1, x_2, \ldots x_n)$ is said to be an unbiased estimator of the corresponding population parameter θ, if $E(T_n) = \theta$.

$\Rightarrow$ The mean of the sampling distribution of the statistic is equal to the parameter of the populat

Example:

1. The sample mean $\bar{x}$ is an unbiased estimat of population parameter if $E(\bar{x}) = \mu$.

Similarly, $E(p) = P$, where P = Population proportic
p = sample proportion.

2. Let s^2 denote the sample variance, then s^2 is defined as $s^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2$

But s^2 is not an unbiased estimator of population variance σ^2.

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Estimation method

But on the otherhand $S^2 = \frac{1}{n-1} \sum (x_i - \bar{x})^2$ provides an unbiased estimator of the population variance σ^2.

Thus $E(S^2) = \sigma^2$ but $E(s^2) \neq \sigma^2$.

It is important to note that

1. $ns^2 = (n-1) S^2 \Rightarrow S^2 = \left(\frac{n}{n-1}\right) s^2$

when the sample size is large, $(n-1)$ can be approximated by n and in that case

$$S^2 = \left(\frac{n}{n-1}\right) s^2 \simeq s^2.$$

So, the sample variance s^2 gives an estimate of the population variance σ^2.

i.e) $E(s^2) = \sigma^2$ for large samples.

Consistency:-

A statistics $t_n = t(x_1, x_2, \ldots, x_n)$ based on the sample size n is said to be consistent estimator of the parameter θ if $t_n \to \theta$ as $n \to \infty$.

In otherwords $\lim_{n\to\infty} P(|T_n - \theta| < \varepsilon) = 1$

$\lim_{n\to\infty} P(|T_n - \theta| \geq \varepsilon) = 0$

Efficiency:-

If t_1 and t_2 are ~~more~~ two consistent estimators of a parameter θ such that $var(t_1) < var(t_2)$ for all sample size n, then t_1 is said to be more efficient than t_2.

In otherwords, an estimator with lesser variability is said to be more efficient and consequently more reliable than the other.

Sufficiency:

A statistic $t = t(x_1, x_2, \ldots, x_n)$ is said to be a sufficient estimator of parameter θ if it contains all the information in the sample regarding the parameter.

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Defn: (Unbiased estimator)

An estimator $T_n(x_1, x_2, \ldots x_n)$ is said to be an unbiased estimator of the corresponding population parameter θ, if $E(T_n) = \theta$.

(ie) The mean of the sampling distribution of the statistic is equal to the parameter of the population.

Example.

1. The sample mean $\bar{x}$ is an unbiased estimator of population parameter if $E(\bar{x}) = \mu$

Similarly, $E(p) = P$, where P = Population proportion, p = sample proportion.

2. Let s^2 denote the sample variance, then s^2 is defined as $s^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2$

But s^2 is not an unbiased estimator of population variance σ^2.

① Let $(x_1, x_2, \dots, x_n)$ be a random sample of X having unknown mean μ. Show that the estimator of μ defined by $M = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x}$ is an unbiased estimator of μ.

Solution

Given $M = \frac{1}{n}\sum_{i=1}^{n} x_i$

$$E(M) = E\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n} E(x_i)$$

$$= \frac{1}{n}\sum_{i=1}^{n} \mu$$

$$= \frac{1}{n}\cdot n\mu = \mu.$$

$$E(M) = \mu.$$

Thus, M is an unbiased estimator of μ.

② Let $(x_1, \dots, x_n)$ be a random sample of x having unknown mean μ and variance σ^2. Show that the estimator of σ^2 defined by $s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$ where $\bar{x}$ is the sample mean, is a biased estimator of σ^2.

Soln:

By defn. we have $\sigma^2 = E\left[(x_i - \mu)^2\right]$

Now,

Given $s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$

$$E(S^2) = E\left[\frac{1}{n}\sum_{i=n}^{n}(x_i - \bar{x})^2\right]$$

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b) $E\left\{\frac{1}{n}\sum_{i=1}^{n}\left[x_i-\mu+\mu-\bar{x}\right]^2\right\}$

$$= E\left\{\frac{1}{n}\sum_{i=1}^{n}\left[(x_i-\mu)-(\bar{x}-\mu)\right]^2\right\}$$

$$= E\left\{\frac{1}{n}\sum_{i=1}^{n}\left[(x_i-\mu)^2-2(x_i-\mu)(\bar{x}-\mu)+(\bar{x}-\mu)^2\right]\right\}$$

$$= E\left\{\frac{1}{n}\left[\sum_{i=1}^{n}(x_i-\mu)^2-2\sum_{i=1}^{n}(x_i-\mu)(\bar{x}-\mu)+\sum_{i=1}^{n}(\bar{x}-\mu)^2\right]\right\}$$

$$= E\left\{\frac{1}{n}\left[\sum_{i=1}^{n}(x_i-\mu)^2-0+n(\bar{x}-\mu)^2\right]\right\}$$

$$= \frac{1}{n}\left[\sum_{i=1}^{n}E(x_i-\mu)^2+E(\bar{x}-\mu)^2\right]$$

$$= \frac{1}{n}\cdot nE(x_i-\mu)^2+E(\bar{x}-\mu)^2$$

$$= E(x_i-\mu)^2+E(\bar{x}-\mu)^2$$

$$= \sigma^2+\sigma_{\bar{x}}^2$$

We have $\operatorname{var}\left(\sum_{i=1}^{n}a_i x_i\right)=\sum_{i=1}^{n}a_i^2\operatorname{var}(x_i)$

So, $\sigma_{\bar{x}}^2=\operatorname{var}(\bar{x})=\operatorname{var}\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)$

$$= \sum_{i=1}^{n}\frac{1}{n^2}\operatorname{var}(x_i)$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}\sigma^2$$

$$= \frac{1}{n^2}n\sigma^2=\frac{1}{n}\sigma^2$$

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Thus $E(S^2) = \sigma^2 + \frac{1}{n}\sigma^2 = \left(\frac{n+1}{n}\right)\sigma^2$.

which shows that S^2 is a biased estimator of σ^2.

(5) Let $(x_1, x_2, \ldots, x_n)$ be a random sample of a poisson r.v X with unknown parameter λ.

a) show that $\Lambda_1 = \frac{1}{n}\sum_{i=1}^{n} x_i$ and $\Lambda_2 = \frac{1}{2}(x_1+x_2)$ are both unbiased estimators of λ.

b) which estimator is more efficient?

Solution:

Given $\Lambda_1 = \frac{1}{n}\sum_{i=1}^{n} x_i$

$$E(\Lambda_1) = E\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n} E(x_i)$$

mean = variance

$$= \frac{1}{n}\sum_{i=1}^{n} \lambda$$

$$= \frac{1}{n}\cdot n\lambda = \lambda$$

$$E(\Lambda_1) = \lambda.$$

$$E(\Lambda_2) = E\left(\frac{1}{2}(x_1+x_2)\right)$$

$$= \frac{1}{2}E(x_1) + \frac{1}{2}E(x_2)$$

$$= \frac{1}{2}\lambda + \frac{1}{2}\lambda$$

$$E(\Lambda_2) = \lambda$$

$\therefore$ Λ_1 and Λ_2 are unbiased estimators of λ.

(7)

b) $\text{var}(\Lambda_1) = \text{var}\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)$

$$= \sum_{i=1}^{n} \frac{1}{n^2}\,\text{var}(x_i)$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}\lambda$$

$$= \frac{1}{n^2}\, n\lambda = \frac{1}{n}\cdot\lambda$$

$$\text{var}(\Lambda_2) = \text{var}\left(\frac{1}{2}[x_1 + x_2]\right)$$

$$= \frac{1}{2^2}\,\text{var}(x_1) + \frac{1}{2^2}\,\text{var}(x_2)$$

$$= \frac{1}{4}\lambda + \frac{1}{4}\lambda = \frac{2}{4}\lambda = \lambda/2.$$

[Here mean = variance $= \lambda$, since x is a poisson variable]

If $n > 2$, $\lambda/n < \lambda/2$;

Thus if $n > 2$, Λ_1 is a more efficient estimator of λ than Λ_2.

(4) Let $(x_1, \dots, x_n)$ be a random sample of x with mean μ and variance σ^2. A linear estimator of μ is defined to be a linear function of $x_1, \dots, x_n$, $l(x_1, \dots, x_n)$. Show that the linear estimator defined by $M = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x}$ is the most efficient linear unbiased estimator of μ.

Assume that $M_1 = l(x_1, \ldots, x_n) = \sum_{i=1}^{n} a_i x_i$

is a linear unbiased estimator of μ

with lower variance than M.

Since M_1 is unbiased, we must have

$$E(M_1) = \mu \Rightarrow \sum_{i=1}^{n} a_i E(x_i) = \mu$$

$$\Rightarrow \sum_{i=1}^{n} a_i \mu = \mu$$

$$\Rightarrow \mu \sum_{i=1}^{n} a_i = \mu \quad (\because E(M_1) = \mu)$$

$$\Rightarrow \sum_{i=1}^{n} a_i = 1.$$

$Var(M) = \frac{1}{n}\sigma^2$ and

$$Var(M_1) = \sigma^2 \sum_{i=1}^{n} a_i^2$$

By assumption $\sigma^2 \sum_{i=1}^{n} a_i^2 < \frac{1}{n}\sigma^2$

$$\Rightarrow \sum_{i=1}^{n} a_i^2 < \frac{1}{n}.$$

$$\Rightarrow \sum_{i=1}^{n} a_i^2 - \frac{1}{n} < 0$$

Now consider $0 \le \sum_{i=1}^{n} (a_i - 1/n)$

$$= \sum_{i=1}^{n} \left(a_i^2 - 2a_i \tfrac{1}{n} + \frac{1}{n^2}\right)$$

$$= \sum_{i=1}^{n} a_i^2 - 2/n \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} \frac{1}{n^2}$$

$var\left(\sum_{i=1}^{n} \frac{x_i}{n}\right) = \frac{1}{n^2}\sum var(x_i) = \frac{1}{n^2}\sum_{i=1}^{n} \sigma^2 = \frac{1}{n^2} n\sigma^2 = \sigma^2/n$

$var\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} a_i^2 var(x_i) = \sum_{i=1}^{n} a_i^2 \sigma^2 = \sigma^2\left(\sum_{i=1}^{n} a_i^2\right)$

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$$= \sum_{i=1}^{n} a_i^2 - \frac{2}{n}.(1) + \frac{1}{n^2}.n$$

$$= \sum_{i=1}^{n} a_i^2 - \frac{2}{n} + \frac{1}{n} = \sum_{i=1}^{n} a_i^2 - \frac{1}{n}$$

$$0 \le \sum_{i=1}^{n} a_i^2 - \frac{1}{n}$$

$$\Rightarrow \cancel{\sum_{i=1}^{n} a_i^2 - \frac{1}{n} < 0}$$

This is contraction to our assumption,

since $\left(\sum_{i=1}^{n} a_i^2 - \frac{1}{n} < 0\right)$.

So) M is the most efficient linear unbiased estimator of μ.

(5) Show that if $\lim_{n\to\infty} E(T_n) = \theta$ and $\lim_{n\to\infty} var(T_n) = 0$.

then the estimator T_n is consistent.

Solution:

[For any $a>0$, show that

$$P(|X-\mu_x| \geq a) \le \frac{\sigma_x^2}{a^2}$$

where μ_x and σ_x^2 are the mean and variance of x & respectively. This is known as chebyshev inequality].

Using chebyshev's inequality, we can write

$$P(|T_n - \theta| \geq \epsilon) \le \frac{E[(T_n-\theta)^2]}{\epsilon^2}$$

$$= \frac{1}{\epsilon^2} E\left[T_n - E(T_n) + E(T_n) - \theta\right]^2$$

$$= \frac{1}{\epsilon^2} E\left\{[T_n - E(T_n)]^2 + [E(T_n)-\theta]^2 + 2[T_n - E(T_n)][E(T_n)-\theta]\right\}$$

(10)

$$= \frac{1}{\varepsilon^2}\left[var(T_n) + E\left(E(T_n)-\theta\right)^2 + 2\, E\left\{[T_n - E(T_n)]\,[E(T_n)-\theta]\right\}\right] \longrightarrow ①$$

Given $\lim\limits_{n\to\infty} E(T_n) = \theta$

$$\lim_{n\to\infty} var(T_n) = 0.$$

taking $\lim\limits_{n\to\infty}$ in ①,

$$\lim_{n\to\infty} P(|T_n-\theta| \geq \varepsilon) = \frac{1}{\varepsilon^2}\left[\lim_{n\to\infty} var(T_n) + \lim_{n\to\infty} E\left(E(T_n)-\theta\right)^2 + 2 \lim_{n\to\infty} E\left\{[T_n-E(T_n)]\cdot[E(T_n)-\theta]\right\}\right]$$

$\therefore T_n$ is consistent.

Maximum-Likelihood Estimation

Let $f(x;\theta) = f(x_1, x_2, \ldots, x_n; \theta)$ denote the joint pmf of the r.v's $X_1, \ldots, X_n$ when they are discrete and let it be their joint pdf when they are continuous.

Let $L(\theta) = f(x;\theta) = f(x_1, \ldots, x_n; \theta)$

Now $L(\theta)$ represents the likelihood that the values $x_1, x_2, \ldots, x_n$ will be observed when θ is the true value of the parameter. Thus $L(\theta)$ is often reffered to as the likelihood function of the random sample.

Let $L(\theta_{ML}) = s(x_1, x_2, \ldots, x_n)$ be the maximizing value of $L(\theta)$.

ie) $L(\theta_{ML}) = \max_{\theta} L(\theta)$

Then the maximum-likelihood estimator of θ is $\hat{\theta}_{ML}$ is $s(X_1, \ldots, X_n)$ and

$\hat{\theta}_{ML}$ is the maximum-likelihood estimate of θ.

Pbms

① Let $(X_1, \ldots, X_n)$ be a random sample of a binomial r.v X with parameters (m, p), where m is assumed to be known and p unknown. Determine the maximum-likelihood estimator of p.

Solution:

The likelihood function is given by

$$L(p) = f(x_1, x_2, \ldots, x_n; p) = \binom{m}{x_1} p^{x_1} (1-p)^{m-x_1} \cdots \cdots \binom{m}{x_n} p^{x_n} (1-p)^{m-x_n}.$$

$$= \binom{m}{x_1}\cdots\cdots\binom{m}{x_n} p^{\sum_{i=1}^{n} x_i} (1-p)^{\left(mn - \sum_{i=1}^{n} x_i\right)}$$

Taking Natural logarithm of the above expression, we get

$$\ln L(p) = \ln\left[\binom{m}{x_1}\cdots\binom{m}{x_n} p^{\sum_{i=1}^{n} x_i} (1-p)^{\left(mn - \sum_{i=1}^{n} x_i\right)}\right]$$

$$= \ln\left[\binom{m}{x_1}\cdots\cdots\binom{m}{x_n}\right] + \ln p^{\sum_{i=1}^{n} x_i} + \ln (1-p)^{\left(mn - \sum_{i=1}^{n} x_i\right)}$$

$$= \ln C + \sum_{i=1}^{n} x_i (\ln p) + \left(mn - \sum_{i=1}^{n} x_i\right) \ln (1-p)$$

where $C = \prod_{i=1}^{n} \binom{m}{x_i}$

Now $$\frac{d}{dp} \ln (L(p)) = \sum_{i=1}^{n} x_i \left(\frac{1}{p}\right) + \left(mn - \sum_{i=1}^{n} x_i\right) \frac{1}{(1-p)} (-1)$$

$$= \frac{1}{p} \sum_{i=1}^{n} x_i - \left(mn - \sum_{i=1}^{n} x_i\right) \frac{1}{(1-p)}$$

Setting $d[\ln L(p)]/dp = 0$, the maximum likelihood estimate $\hat{p}_{ML}$ of p is obtained as

$$0 = \frac{1}{\hat{p}_{ML}} \sum_{i=1}^{n} x_i - \left(mn - \sum_{i=1}^{n} x_i\right) \frac{1}{(1-\hat{p}_{ML})}$$

$$\Rightarrow \frac{1}{\hat{p}_{ML}} \sum_{i=1}^{n} x_i = \left(mn - \sum_{i=1}^{n} x_i\right) \frac{1}{(1-\hat{p}_{ML})}$$

$$\Rightarrow \frac{(1-\hat{p}_{ML})}{\hat{p}_{ML}} \sum_{i=1}^{n} x_i = mn - \sum_{i=1}^{n} x_i$$

$$\Rightarrow \frac{1}{\hat{p}_{ML}} \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i = mn - \sum_{i=1}^{n} x_i$$

$$\Rightarrow \frac{1}{\hat{p}_{ML}} \sum_{i=1}^{n} x_i = mn$$

$$\Rightarrow \frac{1}{mn} \sum_{i=1}^{n} x_i = \hat{p}_{ML}$$

Hence, the maximum-likelihood estimator of p is given by

$$p_{ML} = \frac{1}{mn} \sum_{i=1}^{n} x_i = \frac{1}{m} \overline{x}.$$

2) Let $(x_1, \ldots, x_n)$ be a random sample of a poisson r.v with unknown parameter λ. Determine the maximum-likelihood estimator of λ.

Solution:

$$L(\lambda) = f(x_1, \ldots, x_n; \lambda) = \prod_{i=1}^{n} \frac{e^{-\lambda} \lambda^{x_i}}{x_i!}$$

$$= \frac{e^{-n\lambda} \lambda^{\sum_{i=1}^{n} x_i}}{x_1! \ldots x_n!}$$

$$\ln (L(\lambda)) = \ln \left[\frac{e^{-n\lambda} \lambda^{\sum_{i=1}^{n} x_i}}{x_1! x_2! \ldots x_n!} \right]$$

$$= \ln e^{-n\lambda} + \ln \lambda^{\sum_{i=1}^{n} x_i} - \ln (x_1! x_2! \ldots x_n!)$$

$$= -n\lambda + \sum_{i=1}^{n} x_i \ln \lambda - \ln c.$$

where $c = x_1! \ldots x_n!$

$$\frac{d}{d\lambda} \ln L(\lambda) = -n + \sum_{i=1}^{n} x_i \cdot \frac{1}{\lambda}$$

Setting $\frac{d}{d\lambda} \ln L(\lambda) = 0$, we get

$$0 = -n + \frac{1}{\hat{\lambda}_{ML}} \sum_{i=1}^{n} x_i$$

$$\Rightarrow \frac{1}{\hat{\lambda}_{ML}} \sum_{i=1}^{n} x_i = n$$

$$\Rightarrow \hat{\lambda}_{ML} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

Hence, the maximum-likelihood estimator of λ is given by $\lambda_{ML} = \frac{1}{n} \sum_{i=1}^{n} x_i = \bar{X}$.

(3) Let $(x_1, \dots, x_n)$ be a random sample of an exponential r.v X with unknown parameter λ. Determine the maximum-likelihood estimator of λ.

Solution

The likelihood function is given by

$$L(\lambda) = f(x_1, \dots, x_n ; \lambda) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i}$$

$$= \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i}$$

Thus $\ln L(\lambda) = \ln (\lambda^n e^{-\lambda \sum_{i=1}^{n} x_i})$

$$= \ln \lambda^n + \ln e^{-\lambda \sum_{i=1}^{n} x_i}$$

$$\ln L(\lambda) = n \ln \lambda - \lambda \sum_{i=1}^{n} x_i$$

$$\Rightarrow \frac{d}{d\lambda} \ln (L(\lambda)) = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i$$

Setting $\frac{d}{d\lambda} \ln L(\lambda) = 0$, we get

$$0 = \frac{n}{\hat{\lambda}_{ML}} - \sum_{i=1}^{n} x_i$$

$$\Rightarrow \sum_{i=1}^{n} x_i = \frac{n}{\hat{\lambda}_{ML}}$$

$$\Rightarrow \hat{\lambda}_{ML} = \frac{n}{\sum_{i=1}^{n} x_i}$$

$$\cancel{\Rightarrow \hat{\lambda}_{ML} = \frac{1}{\bar{x}}}.$$

Hence, the maximum-likelihood estimator of λ is given by $\lambda_{ML} = \frac{n}{\sum_{i=1}^{n} x_i} = \frac{1}{\bar{x}}$.

4) Let $(X_1, \ldots, X_n)$ be a random sample of a normal random variable X with unknown mean μ and unknown variance σ^2. Determine the maximum-likelihood estimators of μ and σ^2.

Solution: The likelihood function is given by

$$L(\mu, \sigma) = f(x_1, x_2, \ldots, x_n; \mu, \sigma)$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[\frac{-1}{2\sigma^2}(x_i - \mu)^2\right]$$

$$= \left(\frac{1}{2\pi}\right)^{n/2} \frac{1}{\sigma^n} \exp\left[\frac{-1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2\right]$$

Thus $\ln L(\mu, \sigma) = \ln\left[\left(\frac{1}{2\pi}\right)^{n/2} \frac{1}{\sigma^n} \exp\left[\frac{-1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2\right]\right]$

$$= \ln (2\pi)^{-n/2} + \ln \frac{1}{\sigma^n} + \left(\frac{-1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2\right)$$

$$= -\frac{n}{2} \ln 2\pi - \ln \sigma^n - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2$$

$$= -\frac{n}{2} \ln 2\pi - n \ln \sigma - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2$$

Inorder to find the values of μ and σ maximizing the above, we compute

$$\frac{\partial}{\partial \mu} \ln L(\mu, \sigma) = -\frac{1}{2\sigma^2} \sum_{i=1}^{n} 2(x_i - \mu)(-1)$$

$$= \frac{1}{\sigma^2} \sum_{i=1}^{n} (x_i - \mu) \rightarrow (1)$$

$$\frac{\partial}{\partial \sigma} \ln L(\mu, \sigma) = \frac{-n}{\sigma} - \frac{1}{2} \cdot \left(\frac{-2}{\sigma^3}\right) \sum_{i=1}^{n} (x_i - \mu)^2$$

$$= \frac{-n}{\sigma} + \frac{1}{\sigma^3} \sum_{i=1}^{n} (x_i - \mu)^2 \rightarrow (2)$$

Equating these equations to zero, we get

$$(1) \Rightarrow \frac{1}{\sigma^2} \sum_{i=1}^{n} (x_i - \hat{\mu}_{ML}) = 0$$

$$\Rightarrow \sum_{i=1}^{n} (x_i - \hat{\mu}_{ML}) = 0 \text{ and}$$

$$(2) \Rightarrow \frac{-n}{\hat{\sigma}_{ML}} + \frac{1}{\hat{\sigma}^3_{ML}} \sum_{i=1}^{n} (x_i - \hat{\mu}_{ML})^2 = 0$$

$$\Rightarrow \frac{1}{\hat{\sigma}^3_{ML}} \sum_{i=1}^{n} (x_i - \hat{\mu}_{ML})^2 = \frac{n}{\hat{\sigma}_{ML}}$$

$$\Rightarrow \frac{1}{\hat{\sigma}^2_{ML}} \sum_{i=1}^{n} (x_i - \hat{\mu}_{ML})^2 = n$$

$$\Rightarrow \frac{1}{n} \sum_{i=1}^{n} (x_i - \hat{\mu}_{ML})^2 = \hat{\sigma}^2_{ML}$$

$x_1 - \hat{\mu}_{ML} + x_2 - \hat{\mu}_{ML} + \dots + x_n - \hat{\mu}_{ML} = 0$

$\sum_{i=1}^{n} x_i - n\hat{\mu}_{ML} = 0$

$\sum_{i=1}^{n} x_i = n\hat{\mu}_{ML}$

$\frac{1}{n}\sum_{i=1}^{n} x_i = \hat{\mu}_{ML}$

Hence, the maximum-likelihood estimators of μ and σ^2 are given, respectively by

$$\hat{M}_{ML} = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x}$$

$$S_{ML}^2 = \frac{1}{n}\sum_{i=1}^{n} (x_i - \bar{x})^2$$

5) Find the MLE for θ of the Bin. $(1,\theta)$ distribution

$$f(x;\theta) = \theta^{x}(1-\theta)^{1-x}, \quad x = 0,1; \quad 0 \le \theta \le 1.$$

Solution The likelihood function is given by

$$L(\theta) = f(x_1, x_2, \ldots, x_n, \theta)$$

$$= \prod_{i=1}^{n} \theta^{x_i}(1-\theta)^{1-x_i}$$

$$= \theta^{\sum_{i=1}^{n} x_i}(1-\theta)^{n-\sum_{i=1}^{n} x_i}$$

Thus $\ln L(\theta) = \ln\left[\theta^{\sum_{i=1}^{n} x_i}(1-\theta)^{n-\sum_{i=1}^{n} x_i}\right]$

$$= \ln \theta^{\sum_{i=1}^{n} x_i} + \ln (1-\theta)^{n-\sum_{i=1}^{n} x_i}$$

$$= \sum_{i=1}^{n} x_i \ln\theta + \left(n - \sum_{i=1}^{n} x_i\right)\ln(1-\theta)$$

Now $\frac{d}{d\theta}\ln(L(\theta)) = \frac{1}{\theta}\sum_{i=1}^{n} x_i - \frac{1}{1-\theta}\left(n - \sum_{i=1}^{n} x_i\right)$

Setting $\frac{d}{d\theta}\ln L(\theta) = 0$, we get

$$0 = \frac{1}{\hat{\theta}_{ML}}\sum_{i=1}^{n} x_i - \frac{1}{1-\hat{\theta}_{ML}}\left(n - \sum_{i=1}^{n} x_i\right)$$

$$\Rightarrow \frac{1}{\hat{\theta}_{ML}}\sum_{i=1}^{n} x_i = \frac{1}{(1-\hat{\theta}_{ML})}\left(n - \sum_{i=1}^{n} x_i\right)$$

$$\Rightarrow \left(\frac{1-\hat{\theta}_{ML}}{\hat{\theta}_{ML}}\right)\sum_{i=1}^{n} x_i = n - \sum_{i=1}^{n} x_i$$

$$\Rightarrow \left(\frac{1}{\hat{\theta}_{ML}} - 1\right) \sum_{i=1}^{n} x_i = n - \sum_{i=1}^{n} x_i$$

$$\Rightarrow \frac{1}{\hat{\theta}_{ML}} \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i = n - \sum_{i=1}^{n} x_i$$

$$\Rightarrow \frac{1}{\hat{\theta}_{ML}} \sum_{i=1}^{n} x_i = n$$

$$\Rightarrow \hat{\theta}_{ML} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

Hence, the maximum likelihood estimator of θ is given by $\theta_{ML} = \frac{1}{n} \sum_{i=1}^{n} x_i = \overline{x}$.

Method of moments:

The method of moments consists of equating the first few moments of a population to the corresponding moment of a sample, thus getting as many equations as are needed to solve for the unknown parameters of the population.

Definition:

The r^{th} sample moment of a set of observations $x_1, x_2, \ldots, x_n$ is the mean of their powers and it is denoted by m_r'.

ie) $$m_r' = \frac{\sum_{i=1}^{n} x_i^r}{n}$$

$\therefore$ if a population has r parameters, the method of moments consists of solving the system of equations $m_r' = \mu_r'$, $r = 1, 2, \ldots, k$. for the k parameters.

Example :1.

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from a poisson population with parameter. Obtain the estimator of λ by the method of moments.

Soln:

The poisson population has the probability function $p(x, \lambda) = \dfrac{e^{-\lambda}\lambda^x}{x!}$ $x = 0, 1, 2, \ldots$

$$\mu_1' = E(x) = \lambda.$$

$$m_1' = \frac{\sum_{i=1}^{n} x_i}{n} = \bar{x}$$

Solving the equations $m_1' = \mu_1'$ we get $\bar{x} = \lambda$.

The method of moment estimator of λ is $\bar{x}$.

Example: 2 Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from the normal population $N(\mu, \sigma^2)$, Obtain the estimators of μ and σ^2 by the method of moments.

Soln:

The pdf of the Normal distribution

$$f(x,\mu,\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{\frac{-1}{2\sigma^2}(x-\mu)^2}, \quad -\infty < x < \infty.$$

$$\mu_1' = E(x) = \mu$$

$$\mu_2' = E(x^2) = \mu^2+\sigma^2 \quad \left[\because \sigma^2 = E(x^2) - E(x)^2\right]$$

But $m_1' = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x}$

$$m_2' = \frac{1}{n}\sum^{n} x_i^2$$

Solving the equations $\mu_1' = m_1'$

$\mu_2' = m_2'$

we get $\mu_1' = \bar{x}$, $\mu_2' = \frac{1}{n}\sum x_i^2$

$$\mu^2+\sigma^2 = \frac{1}{n}\sum x_i^2$$

$$\Rightarrow \sigma^2 = \frac{1}{n}\sum x_i^2 - \mu^2$$

$$= \frac{1}{n}\sum x_i^2 - \bar{x}^2$$

$$= \frac{1}{n}\sum (x_i - \bar{x})^2$$

$$\left[\frac{1}{n}\sum (x_i^2 + \bar{x}^2 - 2x_i\bar{x})\right.$$

$$= \frac{1}{n}\left[\sum x_i^2 + n\bar{x}^2 - 2\bar{x}\sum x_i\right]$$

$$= \frac{1}{n}\left[\sum x_i^2 + n\bar{x}^2 - 2\bar{x}(n\bar{x})\right]$$

$$= \frac{1}{n}\left[\sum x_i^2 + n\bar{x}^2 - 2n\bar{x}^2\right]$$

$$\left. = \frac{1}{n}\left[\sum x_i^2 - n\bar{x}^2\right] = \frac{1}{n}\sum x_i^2 - \bar{x}^2\right]$$

$\therefore$ The estimator of the parameter μ is $\bar{x}$.

The estimator of the parameter σ^2 is $\frac{1}{n}\sum(x_i-\bar{x})^2$.

Example: 3

Obtain the estimator of the parameter p of a Binomial population using the method of moments.

Soln:

$$p(x,n,p) = nC_x\, p^x q^{n-x}, \quad x = 0,1,2,\dots n$$

$$\mu_1' = E(x) = np$$

Let $x_1, x_2, \dots, x_n$ be a random sample from the Binomial population then $m_1' = \frac{\sum x_i}{n} = \bar{x}$.

$\therefore$ By the method of moments $m_1' = \mu_1'$

$$\bar{x} = np \Rightarrow \bar{x}/n = p$$

$\therefore \bar{x}/n$ is the estimator of p.

Example: 4

The number of hours an electron tube will work is assumed to be an exponential variable with parameter λ. Given a sample of n life tubes of this sort, compute the method of moments estimator of λ.

Solution:

The pdf of the exponential distribution

$$f(x,\lambda) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\mu = \int_0^\infty x\lambda e^{-\lambda x}\,dx \quad [\because f \text{ is continuous}]$$

$u = x \qquad dv = \lambda e^{-\lambda x}\,dx$

$du = dx \qquad v = \frac{e^{-\lambda x}}{-\lambda}$

$$= \lambda \left[\left[\frac{x e^{-\lambda x}}{-\lambda} \right]_0^\infty + \frac{1}{\lambda}\int_0^\infty e^{-\lambda x}\, dx \right]$$

$$= \left[-x e^{-\lambda x} \right]_0^\infty + \left[\frac{e^{-\lambda x}}{-\lambda} \right]_0^\infty$$

$$= 0 + \frac{1}{-\lambda}[0-1] = \frac{1}{\lambda}.$$

ie) $\mu_1' = \frac{1}{\lambda}$

Let $x_1, x_2, \ldots, x_n$ be a random sample from the population.

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

By the method of moments,

$$\mu_1' = m_1'$$

$$\frac{1}{\lambda} = \bar{x}$$

$$\Rightarrow \boxed{\lambda = \frac{1}{\bar{x}}}$$