Topology optimisation of Stokes flow

with FEniCS

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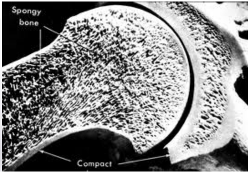
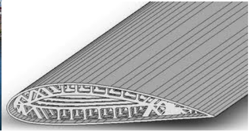
Topology optimisation

What is topology optimisation?

Find the optimal shape of a physical system. The response is captured by the solution of a partial differential equation that depends on the shape.

Applications

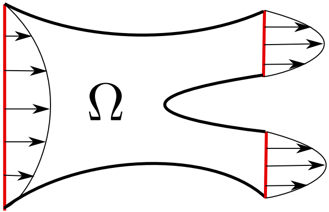



Topology optimisation of Stokes flow

Goal of this talk

Compute the shape and topology of the domain Ω that minimises the dissipation into heat of a 2D Stokes flow for given boundary conditions (in < 100 lines of code).

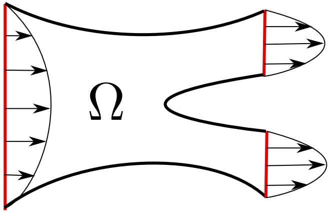


Model problem

$$\min_{u,p,\Omega} \frac{1}{2}\int_\Omega \nabla u \cdot \nabla u$$

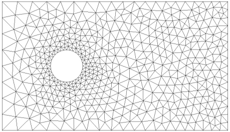
subject to

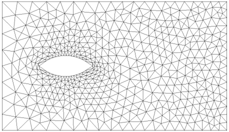
$$\begin{aligned} \Delta u - \nabla p &= 0 \qquad \text{in } \Omega, \\ \nabla \cdot u &= 0 \qquad \text{in } \Omega, \\ \int_\Omega 1 &\leqslant V_0. \end{aligned}$$



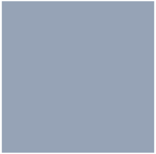
Solution approaches

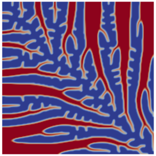
- The domain topology is known a priori.
  Methods: Shape optimisation with mesh movement.
  Example: Optimal shape of an airplane wing.





- The domain topology is unknown a priori.
  Methods: SIMP, level-set methods or
  shape optimisation with mesh movement and topology derivatives.
  Example: Heat conduction optimisation.

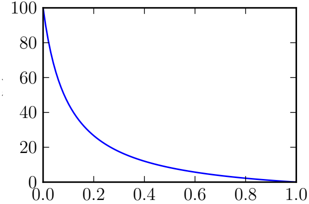




SIMP

Solid **I**sotropic **M**icrostructures with **P**enalization for intermediate densities.

- Mesh a sufficiently large domain $\bar{\Omega}$.
- Parametrise the geometry via a density function $\rho : \bar{\Omega} \to [0, 1]$ ($\rho = 0 \to$ material, $\rho = 1 \to$ no material).
- To obtain "binary" solutions, we augment the problem with $\alpha : [0, 1] \to [0, \bar{\alpha}]$ with $\bar{\alpha} \gg 0$.



SIMP formulation of the model problem

Let $\bar{\Omega}$ be a sufficiently large domain.

$$\min_{u,p,\rho} \int_{\bar{\Omega}} \left(\frac{1}{2} \nabla u \cdot \nabla u + \frac{\alpha(\rho)}{2} u \cdot u \right)$$

$$\text{subject to}$$

$$\Delta u + \alpha(\rho) u - \nabla p = 0 \qquad \text{in } \bar{\Omega},$$

$$\nabla \cdot u = 0 \qquad \text{in } \bar{\Omega},$$

$$\int_{\bar{\Omega}} \rho \leqslant V_0,$$

$$0 \leqslant \rho \leqslant 1.$$

Reduced SIMP problem

Consider the u as a function of ρ, i.e. we can write $u(\rho)$.

$$
\begin{aligned}
\min_{\rho} \int_{\bar{\Omega}} \left(\frac{1}{2} \nabla u(\rho) \cdot \nabla u(\rho) + \frac{\alpha(\rho)}{2} u(\rho) \cdot u(\rho) \right) \\
\text{subject to} \\
\int_{\bar{\Omega}} \rho \leqslant V_0, \\
0 \leqslant \rho \leqslant 1.
\end{aligned}
$$

This can be solved with FEniCS + dolfin-adjoint.optimize + SQP.

(Shortened) FEniCS implementation

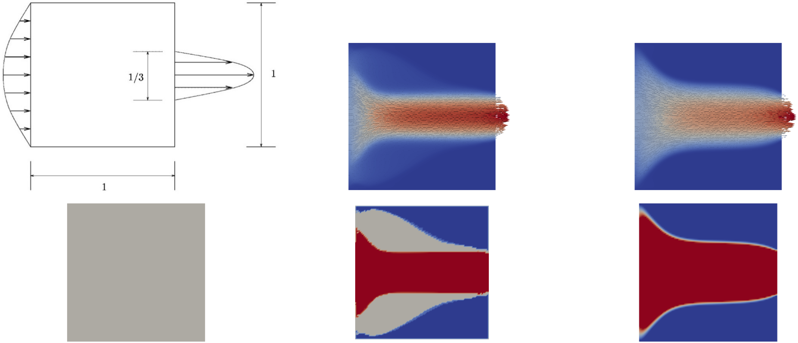
```python
from dolfin import *
from dolfin_adjoint import *
# Solve the forward problem
F = (mu * inner(grad(u), grad(v)) +
    alpha(rho) * inner(u, v) -
    inner(grad(p), v)) * dx
F += div(u) * q * dx
solve(F == 0, w, bcs=bc)

# Define the functional
J = Functional(0.5*inner(alpha(rho)*u, u)*dx +
               0.5*inner(grad(u), grad(u))*dx)
m = SteadyParameter(rho)
Jhat = ReducedFunctional(J, m)

# Solve the optimisation problem
minimize(Jhat, bounds=..., constraints=...)
```

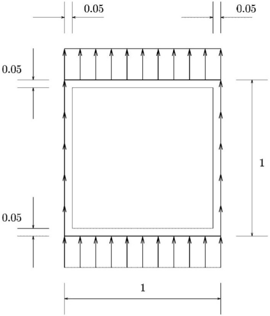
Diffuser problem

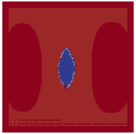
$V_0 = 0.5$

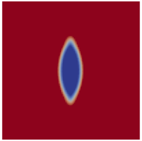


Rugby problem

$V_0 = 0.95$

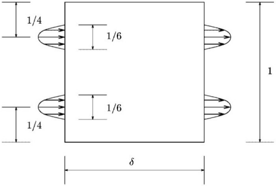


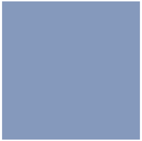


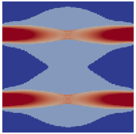


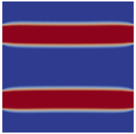
Double pipe problem

$V_0 = 0.33$









Summary and discussion

Summary and discussion

- Solved flow topology optimisation in the reduced formulation in $<$ 100 lines of code.
- Extension to time-dependent Navier-Stokes straight-forward.
- Performance could be improved using a one-shoot approach.
- Potential links to FSI?

References & literature

- fenicsproject.org
- dolfin-adjoint.org (includes the shown examples)
- Example problems: Borrvall & Petersson, 2003
- Introduction to topology optimisation: Bendsøe, Sigmund, 2013
- Topology optimisation with level-set methods: Dijk et al, 2013