A continuous approach for the optimisation of tidal turbine farms

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***Abstract*—The optimal placement of individual devices within a tidal turbine farm is a challenging problem that may benefit greatly from automated optimisation methods. A previously published gradient-based approach for the optimal placement/arrangement of turbines significantly reduces the number of required iterations over traditional optimisation methods. This allows for a higher level of computational cost, and hence realism, in the hydrodynamic model used in every optimisation iteration.**

Here, we introduce a closely related approach that optimises for a turbine density field (referred to here as the "continuous" approach) instead of the positions of individual turbines (the "discrete" approach). Its advantages are: (1) it requires less mesh resolution than the discrete approach and hence has lower computational costs; (2) the number of turbines does not need to be chosen in advance – this allows for the inclusion of per-turbine-costs to be included in the optimisation, and as a by-product returns an estimate for the optimal number of turbines on a site; (3) it allows for the inclusion of more complex site design constraints.

We present a number of cases to demonstrate the validity of the method. The optimal number of turbines predicted by the continuous approach is shown to agree well with the results of running several discrete optimisations with different numbers of turbines, giving confidence to the validity of the new approach. In a realistic case we show how non-convex domain sites can be optimised. Furthermore, this approach naturally supports complex constraints such as maximum bathymetry gradients above which turbines cannot be installed, and the simultaneous optimisation of multiple, potentially interacting farms.

***Index Terms*—Tidal stream arrays/farms, farm layout optimisation, PDE-constrained optimisation, energy resource assessment**

I. Introduction

Due to the complex interactions between turbines and their wakes within a farm, the optimal placement of turbines is a challenging problem that may benefit greatly from automated optimisation methods. In many approaches the power output for a given configuration is estimated using a simplified model, which is run for (potentially a large number of) different configurations. Different strategies exist for the choice of configurations, including the use of designer expertise as well as through computational methods that evolve the configuration using genetic algorithms [1], simulated annealing or Powell's conjugate direction method to give just some examples. The advantage of such approaches is that the model can be treated as a black-box and hence implementation is relatively straightforward. However, the number of evaluations required of the model tends to be very large, and rapidly increases with the complexity of the problem, e.g. increasing the number of turbines or the extent of the farm area(s).

In [2] it was shown that using a model for which gradient information is available (i.e. for which one can compute the sensitivity of the power output with respect to the turbine positions), the number of iterations required can be greatly reduced. Consequently this facilitates the use of greater computational costs per optimisation iteration, and hence higher levels of realism in the hydrodynamic model used. The approach introduced in [2] is being developed within the open source OpenTidalFarm framework (opentidalfarm.org). Currently it solves the 2D nonlinear shallow water equations for the tidal hydrodynamics, with the effect of each individual turbine parameterised via a turbine-sized region of increased drag. The efficient evaluation of the gradient information is based on an application of the adjoint method.

Despite the greatly reduced number of iterations, the fact that each turbine needs to be represented ("resolved" in the loosest sense) individually in the model whilst modelling the much larger scale tidal flow environment, means that the cost of this approach can still be significant even when utilising multi-scale computational meshes [3]. In this paper we introduce a closely related approach that instead of optimising for the position of each turbine individually, tries to find the optimal turbine density, i.e. the number of turbines per unit area represented as a continuous field. This turbine density can be incorporated in the model as an enhanced bottom drag which is proportional to the density function. This means that we no longer need to resolve individual turbines, as the effect of multiple turbines can be "smeared" out over a larger area. Thus, the spatial mesh resolution, and therefore the computational costs can be significantly reduced. The overall optimisation strategy is then still very similar to that outlined in [2], in that the gradient of the power output with respect to, in this case, the turbine density function is evaluated via the adjoint method, and used to accelerate the optimisation via gradient-based optimisation algorithms.

In addition to a reduced computational cost, an advantage over the discrete approach in which turbines are modelled individually, is that in the continuous approach presented here the number of turbines does not need to be chosen in advance. In [4] it was shown how the discrete approach can be combined with economic models, to include additional costs that are dependent on the turbine positioning, such as cabling, or areas in which installation is more costly. Assuming a certain revenue model that is a function of the produced power, it then becomes possible to optimise for profit rather than power output only. However, the most important factor in determining the cost of a farm is the number of turbines deployed. Thus, to obtain the optimal number of turbines that achieves the most profit, the discrete optimisation approach has to be run for a different number of turbines, with an outer optimisation iteration honing in on the optimal number of turbines [5]. In the continuous approach however, we consider all configurations for all possible number of turbines, limited by a user-defined maximum local turbine density, at once. The total number of turbines for a given turbine density function, can then be determined by integrating it over the domain. The *per turbine cost* can therefore be included in the profit functional, and the optimal number of turbines can be determined from the optimal turbine density function that was found.

The efficiency of the approach presented here opens up the possibility to study farm optimisation for which the discrete approach is computationally too expensive. For example, it potentially allows for optimisation of an entire region made up of multiple farms which may or may not be interacting. Comparing the optimal farm configurations and power outputs when the farms are designed in isolation, or when considered in-tandem, will yield insight into whether the industry needs to be concerned with competition for resource and the potential benefits of teamwork in multi-farm development areas.

In the remainder of this paper we introduce the underlying methodology for the continuous farm optimisation approach, including the use of a turbine density field and the inclusion of a simple economic model. We then summarise its implementation within the OpenTidalFarm framework and conduct some preliminary validation via comparisons between the discrete and continuous optimisation approaches. We also take this opportunity to demonstrate how the resulting configuration from a continuous optimisation might be used to provide a good initial guess with which to accelerate the more costly discrete optimisation approach. We finish with some early examples of the application of the method to multi-farm optimisation and a realistic case motivated by the Inner Sound of the Pentland Firth.

II. Method

In the continuous optimisation approach introduced here each farm layout is characterised by a turbine density function $d : \Omega \to \mathbb{R}$ that specifies the number of turbines per square metre in each part of the domain Ω. Typically the turbines are restricted to be inside some smaller domain $\Omega_{\text{farm}} \subset \Omega$, so that $d|_{\Omega\setminus\Omega_{\text{farm}}} = 0$. In addition, inside the designated farm area, turbines cannot be placed arbitrarily close to one another, hence we restrict the density function by some maximum density $\overline{d}$: $0 \leq d \leq \overline{d}$.

For each turbine density function, d, we can calculate a financial profit as the difference between revenue and costs:

$$\text{Profit}(d) = \text{Revenue}(d) - \text{Cost}(d).$$

The revenue and cost functions will be considered in more detail in the following section. However here we note that the revenue is a function of a farm design's power production P, which itself is a function of the turbine density *and* the velocity $\mathbf{u}$ in the farm area. Because of the effect of the turbines on the hydrodynamics, the velocity will also depend on the turbine density function. In this work we use a nonlinear shallow water model to compute the velocity throughout the domain for each given turbine density function. The overall profit function can therefore still be formulated as a function of d only:

$$\text{Profit}(d) = \text{Revenue}(P(d, \mathbf{u}(d))) - \text{Cost}(d),$$

where a solve of the hydrodynamics is required to evaluate this. This allows us to formulate the optimal farm design problem as a PDE-constrained optimisation:

$$\max_{d} \ \text{Profit}(d),$$

subject to the following set of constraints

$$\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} - \nu\nabla^2\mathbf{u} + g\nabla\eta + \frac{c_b + c_t(d)}{H}\|\mathbf{u}\|\mathbf{u} &= 0, \\
\frac{\partial \eta}{\partial t} + \nabla\cdot(H\mathbf{u}) &= 0, \\
0 \leq d \leq \overline{d} \qquad &\text{in } \Omega_{\text{farm}}, \\
d = 0 \qquad &\text{in } \Omega\setminus\Omega_{\text{farm}}.
\end{aligned}$$

Here η is the free surface elevation, H is the total water depth, ν is a kinematic viscosity used to parameterise subgrid-scale turbulence, and g is the gravitational acceleration. The bottom drag/friction is parameterised via a quadratic term with dimensionless coefficient c_b. The effect of the turbines on the flow is parameterised through the inclusion of an additional friction field $c_t(d)$ which is a function of the turbine density.

We assume that the thrust of a single turbine is given by

$$\mathbf{F}_{\text{turbine}}(\mathbf{u}) = \tfrac{1}{2}\rho C_T A_T \|\mathbf{u}\|\mathbf{u}, \tag{1}$$

where C_T and A_T are respectively the thrust coefficient and cross-sectional area of the turbine. If we choose

$$c_t(d) = \tfrac{1}{2} C_T A_T d, \tag{2}$$

then we can derive that the total force on the flow exerted by the additional drag over the entire domain, is given by:

$$\mathbf{F}_{\text{farm}}(d) = \int_\Omega \rho c_t(d)\|\mathbf{u}\|\mathbf{u} = \int_\Omega \mathbf{F}_{\text{turbine}}(\mathbf{u}) d. \tag{3}$$

In other words, this choice for $c_t(d)$ results in the thrust of individual turbines being smeared out over the farm area and scaled by d.

III. Economic model

For a given turbine configuration, the power taken out of the flow can be calculated as

$$P(d, \mathbf{u}) = \rho \int_{\Omega} c_t(d) \|\mathbf{u}\|^3.$$

Note that this power includes both energy usefully extracted at the turbines and energy losses due to mixing.

The revenue is now calculated as

$$\text{Revenue}(d) = Ik \int_{t=0}^{L} P(d, \mathbf{u}),$$

where L is the lifetime of the turbine farm. The coefficient I represents the income per energy unit, and the efficiency coefficient k denotes the percentage of the power P which is available to be sold on the market. Since P is an overestimate of the energy usefully extracted at the turbines, k will include mixing losses in the farm area in addition to losses incurred in transporting the usefully extracted energy to the grid on shore.

As mentioned in the introduction, the number of turbines for a given turbine density can be calculated by integrating the turbine density function. A simple cost model is therefore given by

$$\text{Cost}(d) = C \int_{\Omega} d, \tag{4}$$

where C is the cost per turbine. The inclusion of location specific costs is discussed in [4].

The profit model thus contains a number of coefficients, C, I, k and L which may be hard to determine in practice, due to uncertainty or their commercially sensitive nature. We can simplify the optimisation however by rescaling the functional and optimising instead for the profit divided by IkL:

$$\max_{d} \left[\frac{\int_{t=0}^{L} P(d, \mathbf{u})}{L} - \frac{C}{IkL} \int_{\Omega} d \right]. \tag{5}$$

The first term represents the average power production over the farm lifetime, which as a first approximation in this work is simply taken as the average over a tidal cycle modelled by the shallow water model. A rough estimate of the factor $C/(IkL)$ can be obtained by considering an idealised single ($\int_{\Omega} d = 1$) turbine case. For a turbine with given peak power production of

$$P_{\text{peak}} = \tfrac{1}{2} \rho C_T A_T u_{\text{peak}}^3, \tag{6}$$

experiencing a simple sinusoidal tide, due to the cubic dependency of P on u, the time-averaged power production can be estimated as $0.42 P_{\text{peak}}$. Now if we assume, for this highly idealised case a certain profit margin m, i.e.:

$$\text{Profit} = m\text{Revenue} \implies \text{Cost} = (1 - m)\text{Revenue}, \tag{7}$$

then we may derive:

$$\frac{C}{IkL} = \frac{0.42}{2} \rho C_T A_T (1 - m) u_{\text{peak}}^3. \tag{8}$$

Although this will only give a very crude estimation of what the factor would be in reality, it gives us enough information

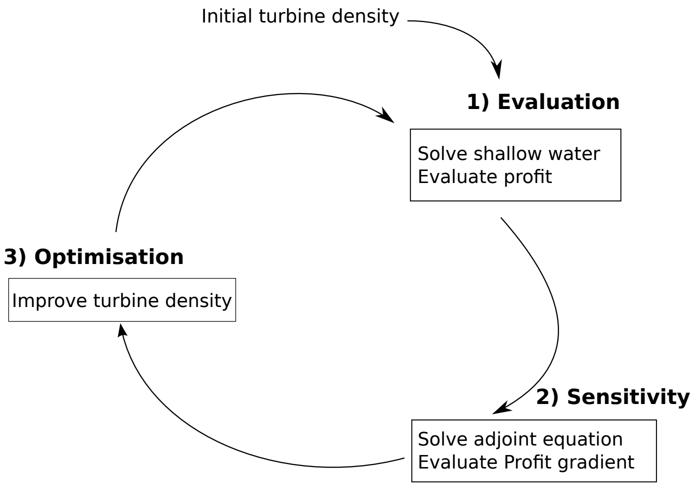


Fig. 1. Schematic of the iterative optimisation loop implemented within OpenTidalFarm.

to start evaluating the method proposed here. This can always be refined when more detailed information becomes available, or this tool is used in-house by industry. The units of the factor are the same as those for power production (W). Considering a generic farm of $N = \int d$ turbines and an average power production P_{avg} per turbine, it is easy to see that the factor may be interpreted as the average production P_{avg} required to break even, since in this case:

$$\frac{\text{Profit}}{IkL} = N P_{\text{avg}} - \frac{C}{IkL} N = 0. \tag{9}$$

IV. Implementation

The nonlinear shallow water equations are discretised using the finite element method employing a Taylor-Hood velocity-pressure basis function pair. This is implemented within the FEniCS [6] finite element framework. The time-integration, for the unsteady case in section VII, uses the implicit backward Euler scheme. The adjoint of the shallow water model is automatically derived using dolfin-adjoint [7]. This allows for the efficient evaluation of the gradient of the scaled profit functional with respect to the turbine density. All the functionality required for the continuous approach has been implemented within OpenTidalFarm (opentidalfarm.org), an open source software package for tidal resource assessment and tidal farm optimisation, as an extension of the discrete approach. More details can be found in [2].

The steps taken in one iteration of the optimisation loop in OpenTidalFarm are shown in figure 1 and can be described as:

1) Solve the shallow water equations, including the effect of a given turbine friction field on the flow, over the specified simulation period with specified boundary conditions, for the current turbine density function d. From the model results, use the computed velocity **u** to evaluate the Profit function.

TABLE I
PARAMETERS USED FOR THE CONTINUOUS VS. DISCRETE OPTIMISATION COMPARISON

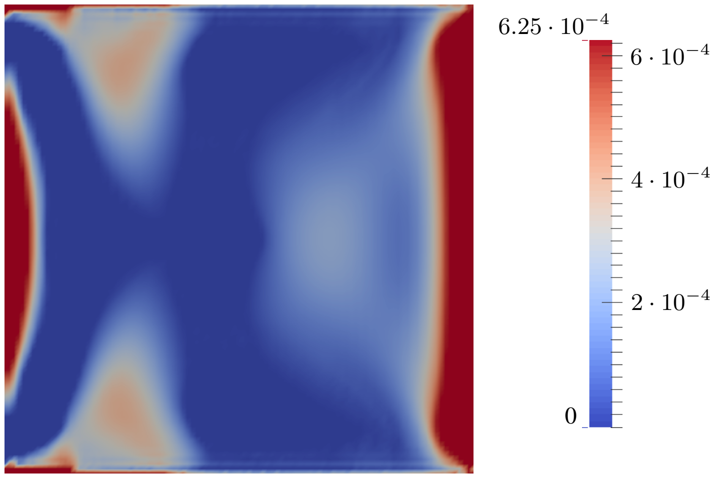
Description	Symbol	Value	Units
Water density	ρ	1000	$\mathrm{kg\,m^{-3}}$
Viscosity	ν	0.5	$\mathrm{m^2\,s^{-1}}$
Water depth	h	50	m
Gravity	g	9.81	$\mathrm{m\,s^{-2}}$
Natural bottom friction	c_b	0.0025	–
Turbine diameter	D	20	m
Minimum turbine distance	dist	40	m
Maximum turbine density	$\bar{d}$	$6.25 \cdot 10^{-4}$	$\mathrm{m^{-2}}$
Thrust coefficient	C_T	0.6	–
Turbine cross section	A_T	314.15	$\mathrm{m^2}$
Profit margin	m	40	%
Cost coefficient	$\frac{C}{LIk}$	452.39	kW

2) Solve the adjoint shallow water equations to obtain the sensitivity (gradient) of the Profit with respect to the turbine density function (see [2] for more details).
3) The outcome of the evaluated Profit function and its gradient are fed into a gradient-based optimisation algorithm. In the work presented here we use the limited memory BFGS method with bound constraints (L-BFGS-B) [8], [9]. Based on this information the algorithm updates the turbine density function and a convergence criterion is applied to determine whether we have reached a (local) maximum. If convergence has not yet been achieved the steps above are repeated.

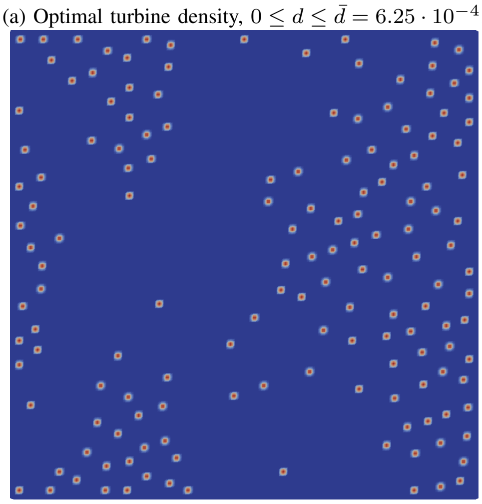
V. DISCRETE VS. CONTINUOUS APPROACH

In order to evaluate the continuous method and to investigate consistency with the pre-existing discrete approach, a simple test case was set up consisting of a square farm area of 1 km×1 km inside a larger flow domain of 4 km×4 km. A 2 $\mathrm{ms^{-1}}$ velocity inflow condition was applied at the Western boundary, and a free surface elevation of $\eta = 0$ at the Eastern boundary, with free slip velocity boundary conditions at the North and South boundaries. To simplify matters, only a steady state solution of the shallow water equations was solved for. Other parameters used in this test case are given in table I. The drag parameterisation was based on an idealised turbine with cross-sectional area $A_T = \pi \cdot 100\ \mathrm{m^2}$ and thrust coefficient $C_T = 0.6$, that is kept constant (i.e. no cut-in speed or rating were applied in this work). The cost of the turbines were based on formula (8) with $u_{\text{peak}} = 2\ \mathrm{ms^{-1}}$ but, since we consider a steady state, leaving out the factor of 0.42.

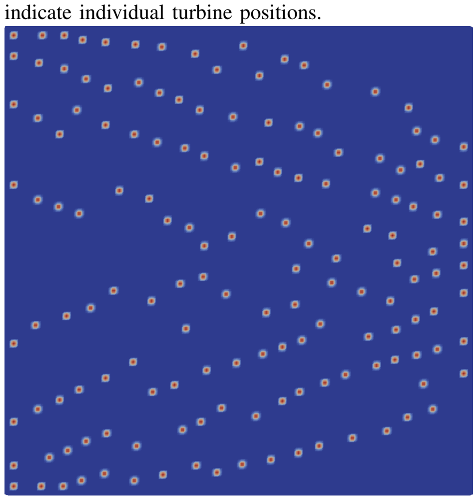
For the continuous method we started the optimisation with an initial turbine density function that is zero everywhere. The turbine density was bounded above by 6.25×10^{-4} turbines per square metre, based on a minimum distance constraint of 40 m between turbines. The optimal turbine density that was found



(a) Optimal turbine density, $0 \leq d \leq \bar{d} = 6.25 \cdot 10^{-4}$.



(b) Optimal discrete turbine configuration. Red dots indicate individual turbine positions.



(c) Optimal discrete turbine configuration started from regular layout. Red dots indicate individual turbines.

Fig. 2. Optimal turbine configuration found by the continuous (top) and discrete approaches (middle and bottom). The middle solution was found after starting from an initial guess based on the optimal continuous solution. The bottom solution was found starting from a regular layout. The figures show only the farm area within the larger domain.

is displayed in figure 2a with the applied convergence criterion reached after 277 iterations (a fairly good approximation was however reached after only 90 iterations). The optimal solution shows a wide band of maximum turbine density at the downstream end of the farm (indicated in red) and a smaller band of maximum density at the upstream end, suggesting a barrage-like zone that maximally blocks the flow. In addition there are areas of intermediate turbine density on either side behind the upstream "barrage", and in the middle in front of the downstream one. Other areas are left completely empty (blue). The optimal solution was found to achieve a power production of $P = 89.21$ MW, whereas the cost, scaled by the factor IKL, was 68.81 MW leading to a scaled revenue of 20.39 MW. The integral of the optimal turbine density function evaluates to 152.11, suggesting an optimal number of $\sim$152 turbines.

Using exactly the same setup we applied the discrete optimisation approach, as described in [2], for a farm of 152 turbines. Starting with a regular layout, the optimal solution that was found (figure 2c) is quite different to that suggested by the turbine density function, with a quite specific, staggered, north-south symmetric configuration. However, starting from different initial configurations quite different solutions were found that all produced roughly the same amount of power, $P \approx 84.5$ MW. This suggests that for this optimisation problem there are multiple local maxima, that are found when the turbines have been reconfigured such that the optimal balance is found between the opposing objectives of extracting the maximum amount of energy and avoiding excessive redirection of flow around the farm. The fact that the amount of power found to be extracted in the optimal discrete solutions is slightly lower than the extracted power in the continuous solution, is likely due to the fact that more of the mixing losses in the farm are actually resolved in the discrete approach and thus not included in the power extraction by the drag. Thus the amount of usefully extracted energy as a percentage of the power extracted by the drag is higher in the discrete case.

It was found however that the continuous optimal solution could be used to find an initial guess at the turbine locations for the discrete approach that was much closer to a local maximum, thus reducing the number of iterations required in the discrete approach. This initial guess was generated using a probabilistic approach, where the likelihood of a turbine being placed in any area was proportional to the optimal turbine density function. This creates an initial and optimal turbine configuration that roughly adheres to the turbine density prescribed by the optimal solution of the continuous approach, see figure 2b. As can be seen in figure 3, this initial guess already gives a power extraction of 80 MW, and reaches 82 MW after 7 iterations, after which it only increases very slowly to 84 MW in 89 iterations. Starting from the regular layout however, we start with a power extraction of only 41 MW; 80 MW is reached after 12 iterations, 82 MW after 37 iterations. Thus initialising the discrete approach using the solution from the continuous approach may be used in cases where the forward model is very expensive and only a few iterations can be afforded. When the optimisation is run to a tight convergence criterion (typically $>$100 iterations), the saving becomes negligible as the optimisation started from the regular layout quickly catches up with the one started from the optimised layout.

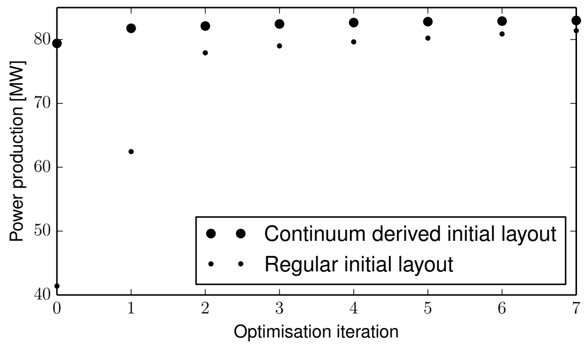


Fig. 3. Power extraction vs. iteration in two discrete optimisations for the same problem but with different initial turbine positions. The small dots indicate the optimisation started from a regular layout, where the turbine are placed in 8 rows of 19 turbines. The larger dots start from an initial layout that is derived from the optimal turbine density found in the continuous optimisation.

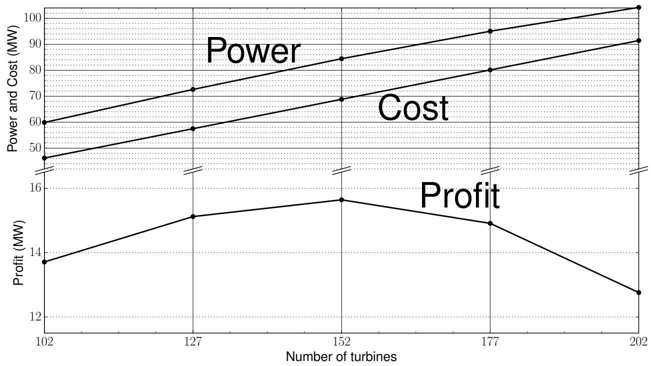
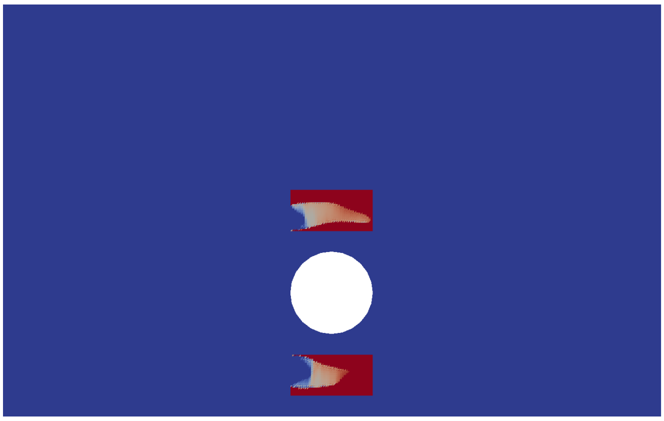
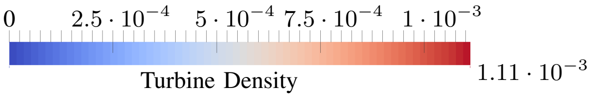


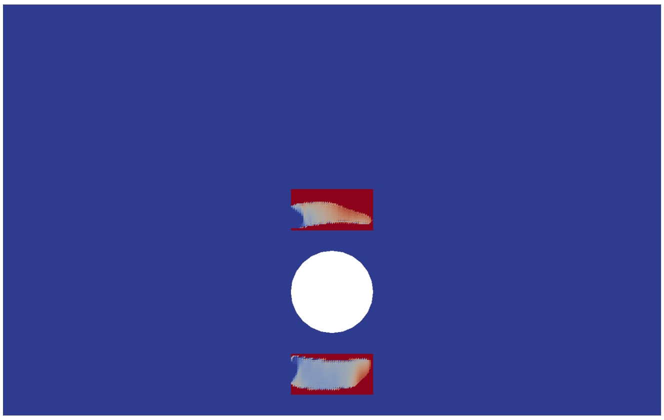
Fig. 4. The continuous approach predicts an optimal number of 152 turbines. This is verified by running discrete optimisations with $N = 102, 127, 152, 177,$ and 202 turbines. The Power, Cost and resulting Profit for which are displayed here.

Next, we compared whether the optimal number of turbines predicted by the continuous approach ($\sim$152 turbines) agrees with the discrete approach. To do this, the discrete approach was run for a different number of turbines, centred around 152. Note that the discrete optimisations are actually optimised for maximum power extraction, since the number of turbines, and hence the cost, is fixed in each of them. The cost can be computed easily afterwards as Cost $= CN$ and after division by IKL subtracted from the power extraction to obtained the scaled Revenue. The results for $N =$ 102, 127, 152, 177 and 202 are displayed in figure 4. As can be seen, the diminishing returns with increasing turbines in the extracted power, lead to a maximum scaled Profit after subtracting the scaled Cost at $\sim$152 turbines, which agrees with the continuous result.



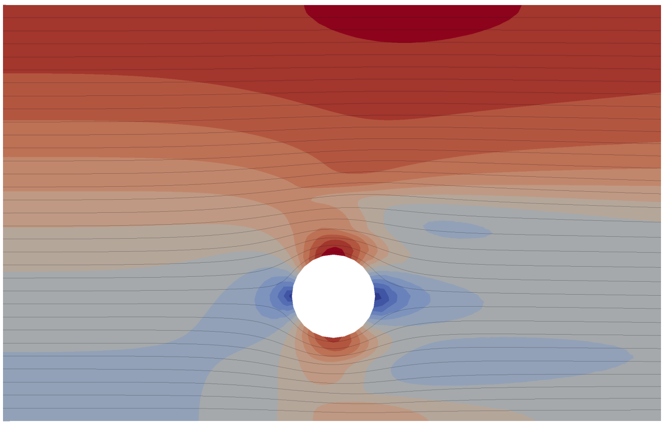
(a) Separate "greedy" optimisation



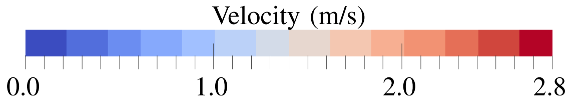


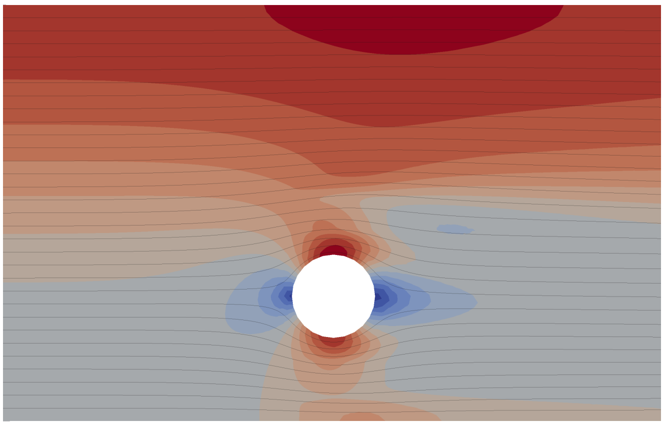
(b) Simultaneous "team" optimisation

Fig. 5. Comparison between the optimal turbine density for cases where each farm is optimised separately without taking the other farm into account (a), and where both farms are optimised simultaneously (b). The density is constrained to be within the range $0 \leq d \leq \bar{d} = 1.11 \cdot 10^{-3}$.



(a) Velocity field "greedy" optimisation





(b) Velocity field "team" optimisation

Fig. 6. Comparison of the velocity fields in the presence of the optimised farms configurations, shown in figure 5, produced by the "greedy" and "team" approaches.

We thus have some initial confidence in the implementation of the continuous farm optimisation approach, as well as its potential utility in the meaningful design of turbine farms and the estimation of optimal device numbers.

VI. Two farm case

It is clear that in many geometries suitable for the exploitation of tidal stream based power generation, the inclusion of many devices could significantly alter the hydrodynamics. For example, the tidal currents that make up the resource may be diverted through an alternative route in a channelised system [10], [11]. Thus in a region with multiple sub-channels where different developers may be designing different farms, some degree of collaboration may be mutually beneficial. One can imagine related scenarios where farms are down- or up-stream of one another during flood and ebb tide, where similar issues should be considered. This is of importance both for the overall resource assessment of a region, as well as the optimal design of farms.

To demonstrate the utility of the continuous optimisation approach to study such scenarios, a highly idealised case of two interacting farms is considered. The simulation domain is 2560 m $\times$ 1600 m large with a circular island in the southern half of 160 m radius, its centre at a distance of 480 m from the southern boundary. The domain contains two farms, one north and one south of the island. Both farm areas are 320 m $\times$ 160 m in size, and use a minimum distance constraint of 30 m, so that maximally 56 turbines can be packed in each farm. The flow is driven by a 0.1 m head difference between west and east boundaries with a free-slip velocity boundary condition on the island as well as on the northern and southern boundaries. The viscosity was set to a relatively high value of 10 m^2s^{-1}, and as in the previous example only the steady state solution was considered. The bottom friction c_b was set to 0.0025 and

TABLE II
BREAKDOWN OF THE POWER PRODUCTION

	Individual	Greedy	Team
Total power (N turbines):		39.3 MW (88)	39.6 MW (79)
North farm:	31.3 MW (45)	24.3 MW (45)	25.2 MW (45)
South farm:	19.3 MW (43)	15.0 MW (43)	14.3 MW (34)

Breakdown of the power production: (1) when only a single farm is considered and optimised without the presence of the other ("Individual"); (2) when the two individual designs are considered operating together ("Greedy"); and (3) when both farms are considered together and optimised simultaneously ("Team"). Note that the individual approach over-predicts the power production for both farms significantly. The numbers in brackets are the number of turbines in the optimised farm layouts.

an unperturbed water depth h of 50 m was used.

First, we studied the case where the farms are optimised separately, i.e. the influence of the other farm is neglected by removing the other farm during the optimisation of each of the farms individually. Here for simplicity the power alone is optimised for, with the cost term neglected. The optimal farm layouts from these two individual optimisations are then added back together to give the combined layout of the "greedy" approach shown in figure 5a. The individual optimisation, without the presence of the second farm, predicts a power production of 31.3 MW for the north farm using 45 turbines, and 19.3 MW for the south farm using 43 turbines. However, if both of these farm designs are combined in a computation of the hydrodynamics and power, the actual production reduces to 24.3 MW and 15.0 MW for the north and south farms, respectively. The power estimates in this "greedy" case are thus significantly over-predicted, see table II. Hence, one would expect the farm designs to also be of questionable value.

Next, we optimised the turbine density of both farms simultaneously within the same calculation in order to maximise the total power production. The resulting turbine density distribution is shown in figure 5b. In this case the northern farm produced 25.2 MW and the southern farm 14.3 MW. Comparing the optimal layouts produced by the "greedy" and "team" optimisations, it can be seen that a slightly more efficient configuration is achieved with the "team" approach by reducing the number of turbines in the southern farm (from 43 to 34 turbines), the northern farm remaining very similar (still 45 turbines). This can be explained by looking at the velocity solutions for both configurations, displayed in figure 6. The "greedy" approach has overpredicted the optimal amount of friction, i.e. the total number of turbines, that would achieve the maximum power output. The "team" approach shows that slightly more power can be produced by reducing the amount of friction (number of turbines), where it is more efficient to reduce it in the southern farm as the northern farm is better located to take advantage of the faster flow in the northern half of the channel. As can be seen in figure 6b, the reduction of the number of the turbines in the southern farm has indeed increased the speed in the entire channel. It must be acknowledged that the result may be dependent on the amount of viscosity that is used and would need to be further investigated.

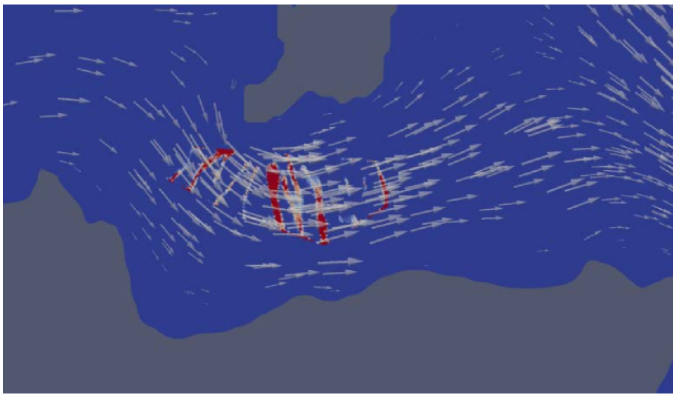


Fig. 7. Example of an optimised turbine density field in a semi-realistic case motivated by the Inner Sound of the Pentland Firth. Red indicates high turbine density regions, and blue indicates the presence of no turbines.

VII. ORKNEY CASE

Finally, we present a very preliminary application of the continuous approach to a more realistic test case. As in [2], we consider the Inner Sound of the Pentland Firth. Here however we incorporate realistic bathymetry and the farm site is non-convex. Note that the minimum mesh resolution used here is approximately 100 m, leading to a mesh of approximately 100k elements. This should be compared with the turbine "resolving" mesh used with the discrete approach in [2] which comprised over 1M elements. Further details of this physical set-up as well as its validation against tide gauge and ADCP data will be the subject of a future publication.

An optimised turbine density field is presented in figure 7. Although very preliminary, this design displays the interesting feature of barrage-like structures aligned perpendicular to the flood flow which is also plotted.

Ongoing work is extending this test case to incorporate other proposed farms in the Pentland Firth region and the inclusion of more complex installation constraints such as a maximum allowed bathymetry gradient.

VIII. CONCLUSION

Building on a gradient-based turbine placement optimisation methodology recently developed [2], here we have presented a new, related approach which seeks to maximise the power or profit of a farm through the optimisation of a turbine density field. We refer to this as the continuous as opposed to the existing discrete optimisation approach. Its advantages stem from far lower mesh resolution requirements, and hence computational costs, as well as the fact that the optimal number of turbines on a site is established automatically as part of the optimisation. This is valuable in its own right as well as facilitating the inclusion of per-turbine costs within the optimisation of a profit, rather than simply power, based functional. In addition, although this was not demonstrated here, the new approach simplifies the inclusion of complex site design constraints.

To demonstrate the applicability and correctness of the continuous approach we presented a very simple test case which was shown to compare favourably with the existing discrete optimisation approach. We also used this test to demonstrate the potential use of the continuous approach to provide a good starting guess for, and hence accelerate, the more costly discrete approach. As such the continuous approach should perhaps be best seen as playing a role within a hierarchy of design methodologies [12].

Future work will further examine the issue of competition and teamwork in the optimal design of multiple turbine farms in a region. We also intend to incorporate a variety of site-specific design constraints and the inclusion of environmental impact measures within the optimisation functional.

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