

FEniCS Course

Lecture 4: Time-dependent PDEs

Contributors

Hans Petter Langtangen

Anders Logg

Marie E. Rognes



The heat equation

We will solve the simplest extension of the Poisson problem into the time domain, the heat equation:

$$\frac{\partial u}{\partial t} - \Delta u = f \quad \text{in } \Omega \text{ for } t > 0$$

$$u = g \quad \text{on } \partial\Omega \text{ for } t > 0$$

$$u = u^0 \quad \text{in } \Omega \text{ at } t = 0$$

The solution $u = u(x, t)$, the right-hand side $f = f(x, t)$ and the boundary value $g = g(x, t)$ may vary in space ($x = (x_0, x_1, \dots)$) and time (t). The initial value u^0 is a function of space only.

Time-discretization of the heat equation

We discretize in time using the implicit Euler (dG(0)) method:

$$\frac{\partial u}{\partial t}(t^n) \approx \frac{u^n - u^{n-1}}{\Delta t}, \quad u(t^n) \approx u^n, \quad f^n = f(t^n)$$

Semi-discretization of the heat equation:

$$\frac{u^n - u^{n-1}}{\Delta t} - \Delta u^n = f^n$$

Algorithm

- 1 Start with u^0 and choose a timestep $\Delta t > 0$.
- 2 For $n = 1, 2, \dots$, solve for u^n :

$$u^n - \Delta t \Delta u^n = u^{n-1} + \Delta t f^n$$

Variational problem for the heat equation

Find $u^n \in V^n$ such that

$$a(u^n, v) = L^n(v)$$

for all $v \in \hat{V}$ where

$$\begin{aligned} a(u, v) &= \int_{\Omega} uv + \Delta t \nabla u \cdot \nabla v \, dx \\ L^n(v) &= \int_{\Omega} u^{n-1} v + \Delta t f^n v \, dx \end{aligned}$$

Note that the bilinear form $a(u, v)$ is constant while the linear form L^n depends on n

Detailed time-stepping algorithm for the heat equation

Define the boundary condition

Compute u^0 as the projection of the given initial value

Define the forms a and L

Assemble the matrix A from the bilinear form a

$t \leftarrow \Delta t$

while $t \leq T$ **do**

Assemble the vector b from the linear form L

Apply the boundary condition

Solve the linear system $AU = b$ for U and store in u^1

$t \leftarrow t + \Delta t$

$u^0 \leftarrow u^1$ (get ready for next step)

end while

Test problem

We construct a test problem for which we can easily check the answer. We first define the exact solution by

$$u = 1 + x^2 + \alpha y^2 + \beta t$$

We insert this into the heat equation:

$$f = u_t - \Delta u = \beta - 2 - 2\alpha$$

The initial condition is

$$u^0 = 1 + x^2 + \alpha y^2$$

This technique is called the *method of manufactured solutions*

Handling time-dependent expressions

We define a time-dependent expression for the boundary value:

```
alpha = 3; beta = 1.2
t = 0.0
g = Expression("1 + x[0]*x[0] + \
               alpha*x[1]*x[1] + beta*t",
               alpha=alpha, beta=beta, t=t,
               degree=2)
```

Then, we must explicitly update t and g:

```
t = 1.0
g.t = t
```

An alternative (robust) approach is to define t as a Constant:

```
t = Constant(0.0)
g = Expression("...", ..., t=t, ...)
t.assign(1.0)
# No need to update g itself
```

Projection and interpolation

We need to project the initial value into V_h :

```
u0 = project(g, V)
```

We can also interpolate the initial value into V_h :

```
u0 = interpolate(g, V)
```


A closer look at solve

For linear problems, this code

```
solve(a == L, u, bcs)
```

is equivalent to this

```
# Assembling a bilinear form yields a matrix
A = assemble(a)
# Assembling a linear form yields a vector
b = assemble(L)

# Applying boundary condition info to system
for bc in bcs:
    bc.apply(A, b)

# Solve Ax = b
solve(A, u.vector(), b)
```

Implementing the variational problem

```
# Decide on a time step
dt = 0.3

# Create Functions for previous and current sol.s
u0 = project(g, V)
u1 = Function(V)

# Define the variational formulation
u = TrialFunction(V)
v = TestFunction(V)
f = Constant(beta - 2 - 2*alpha)
a = u*v*dx + dt*inner(grad(u), grad(v))*dx
L = u0*v*dx + dt*f*v*dx

# Define the boundary condition
bc = DirichletBC(V, g, "on_boundary")

# Assemble only once, before time-stepping
A = assemble(a)
```

Implementing the time-stepping loop

```
T = 2          # Set end time
t.assign(dt)   # Solve on [0, dt] first

while t <= T:
    b = assemble(L)  # Assemble the rhs vector
    bc.apply(A, b)   # Apply boundary conditions

    # Solve linear system
    solve(A, u1.vector(), b)

    # Update time and previous solution
    t1 = float(t + dt)
    t.assign(t1)      # t := t1 + dt
    u0.assign(u1)     # u0 := u1
```

FEniCS programming exercise: heat equation

Consider the heat equation problem:

$$\frac{\partial u}{\partial t} - \Delta u = f \quad \text{in } \Omega = [0, 1]^2 \text{ for } t > 0$$

$$u(x, t) = g(x, t) \quad \text{for } x \in \partial\Omega \text{ for } t > 0$$

$$u(x, 0) = g(x, 0) \quad \text{for } x \in \Omega$$

with

$$f = \beta - 2 - 2\alpha$$

$$g(x, t) = 1 + x_0^2 + \alpha x_1^2 + \beta t \quad (x = (x_0, x_1))$$

Ex. 1 Compute an approximate solution at $T = 1.8$

Ex. 2 Compare the approximate solution to the exact solution at $T = 1.8$. How large is the error (in the eyenorm and in the $L^2(\Omega)$ norm)?

Ex. 3 Compute an approximate solution with the same set-up but on $\Omega = [0, 1]^3 \subset \mathbb{R}^3$.