

FEniCS Course

Lecture 16: Optimal control of the Navier-Stokes equations

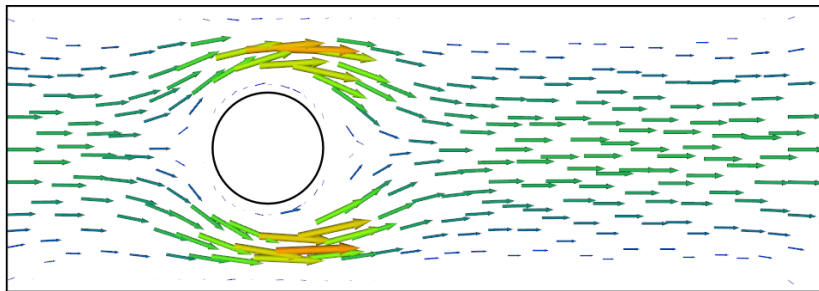
Contributors

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The FEniCS challenge!

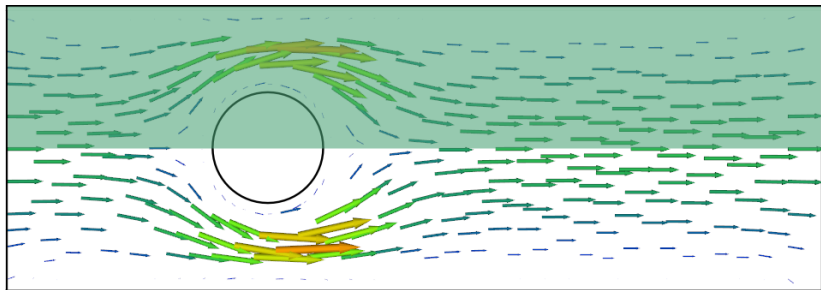
Consider steady flow around a cylinder driven by a pressure difference at the left and right boundaries:



- Imagine you can place sponges in the top half (light green) area of the domain.
- How would you place the sponges in order to minimise dissipation of the flow into heat?

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$$\min_{u,f} \int_{\Omega} \langle \nabla u, \nabla u \rangle \, dx + \alpha \int_{\Omega} \langle f, f \rangle \, dx$$

subject to:

$$\begin{aligned} -\nu \Delta u + \nabla u \cdot u - \nabla p &= -fu && \text{in } \Omega, \\ \operatorname{div}(u) &= 0 && \text{in } \Omega, \end{aligned}$$

with:

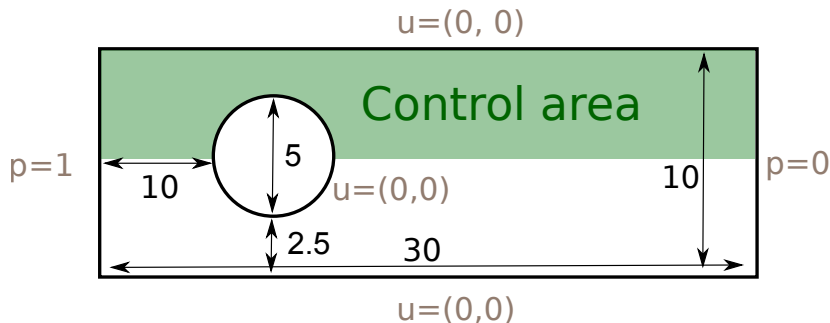
- u the velocity,
- p the pressure,
- f the control function.
- $\nu = 1$ the viscosity,
- α the regularisation parameter,

The boundary conditions are:

- $p = 1$ on the left,
- $p = 0$ on the right,
- $u = (0, 0)$ on the top and bottom and circle.

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Domain



(Or rather, put the circle with center (10.0, 5.0) and radius 2.5)

- 1 Write a new program and generate a mesh with the above domain.

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Navier-Stokes solver

The variational formulation of the Navier-Stokes equations is:
Find $u, p \in V \times Q$ such that:

$$\int_{\Omega} \nu \nabla u \cdot \nabla v + (\nabla u \cdot u + fu) \cdot v + p \operatorname{div} v \, dx = - \int_{\partial\Omega} p_0 v \cdot n \, ds,$$
$$\int_{\Omega} \operatorname{div} u \, q \, dx = 0$$

for all $v \in V$ and all $q \in Q$.

- ② Create a `MixedFunctionSpace` consisting of continuous piecewise quadratic vector fields for the velocity and continuous piecewise linears for the pressure.
- ③ Solve the Navier-Stokes equation for $f = 0$ and the given boundary conditions.

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Optimise

- 3 Import `dolfin-adjoint` and define the control parameter and functional.
- 4 Minimise the functional and plot the optimal control.
- 5 What is the minimised functional?

Note

Multiply the $\int_{\Omega} f u \cdot v \, dx$ term in the variational formulation with the indicator function:

```
x = SpatialCoordinate(mesh)
chi = conditional(x[1] >= 5, 1, 0)
```

to ensure that the control is only active in the top half of the domain.