

# FEniCS Course

## Lecture 14: From sensitivities to optimisation

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*Contributors*

Simon Funke



# What is PDE-constrained optimisation?

*Optimisation problems where at least one constrained is a partial differential equation*

## Applications

- Data assimilation.  
*Example:* Weather modelling.
- Shape and topology optimisation.  
*Example:* Optimal shape of an aerfoil.
- Parameter estimation.
- Optimal control.
- ...

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# *Hello World* of PDE-constrained optimisation!

We will solve the optimal control of the Poisson equation:

$$\min_{u,m} \frac{1}{2} \int_{\Omega} \|u - u_d\|^2 dx + \frac{\alpha}{2} \int_{\Omega} \|m\|^2 dx$$

subject to

$$\begin{aligned} -\Delta u &= m && \text{in } \Omega \\ u &= u_0 && \text{on } \partial\Omega \end{aligned}$$

- This problem can be physically interpreted as: Find the heating/cooling term  $m$  for which  $u$  best approximates the desired heat distribution  $u_d$ .
- The second term in the objective functional, known as Tikhonov regularisation, ensures existence and uniqueness for  $\alpha > 0$ .

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# The canonical abstract form

$$\min_{u,m} J(u, m)$$

subject to:

$$F(u, m) = 0,$$

with

- the objective functional  $J$ .
- the parameter  $m$ .
- the PDE operator  $F$  with solution  $u$ , parametrised by  $m$ .

# The reduced problem

$$\min_m \tilde{J}(m) = J(u(m), m)$$

with

- the reduced functional  $\tilde{J}$ .
- the parameter  $m$ .

How do we solve this problem?

- Gradient descent.
- Newton method.
- Quasi-Newton methods.

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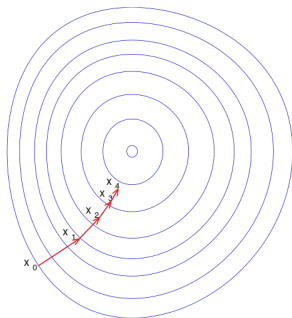
# Gradient descent

## Algorithm

- 1 Choose initial parameter value  $m^0$  and  $\gamma > 0$ .
- 2 For  $i = 0, 1, \dots$ :
  - $m^{i+1} = m^i - \gamma \nabla J(m^i)$

## Features

- + Easy to implement.
- Slow convergence.



# Newton method

Optimisation problem:  $\min_m \tilde{J}(m)$ .

Optimality condition:

$$\nabla \tilde{J}(m) = 0. \quad (1)$$

Newton method applied to (1):

- 1 Choose initial parameter value  $m^0$ .
- 2 For  $i = 0, 1, \dots$ :
  - $H(J)\delta m = -\nabla J(m^i)$ , where  $H$  denotes the Hessian.
  - $m^{i+1} = m^i + \delta m$

## Features

- + Fast (locally quadratic) convergence.
- Requires iteratively solving a linear system with the Hessian, which might require many Hessian action computations.
- Hessian might not be positive definite, resulting in an update  $\delta m$  which is not a descent direction.

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# Quasi-Newton methods

Like Newton method, but use approximate, low-rank Hessian approximation using gradient information only. A common approximation method is *BFGS*.

## Features

- + Robust: Hessian approximation is always positive definite.
- + Cheap: No Hessian computation required, only gradient computations.
- Only superlinear convergence rate.

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# Solving the optimal Poisson problem

```
from fenics import *
from dolfin_adjoint import *

# Solve Poisson problem
# ...

J = Functional(inner(s, s)*dx)
m = SteadyParameter(f)

rf = ReducedFunctional(J, m)
m_opt = minimize(rf, method="L-BFGS-B", tol=1e-2)
```

## Tipps

- You can call `print_optimization_methods()` to list all available methods.
- Use `maximize` if you want to solve a maximisation problem.



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## Bound constraints

Sometimes it is usefull to specify lower and upper bounds for parameters:

$$l_b \leq m \leq u_b. \quad (2)$$

Example:

```
lb = interpolate(0, V)
ub = interpolate(Expression("x[0]", degree=1), V)
m_opt = minimize(rf, method="L-BFGS-B",
                 bounds=[lb, ub])
```

*Note:* Not all optimisation algorithms support bound constraints.

# Inequality constraints

Sometimes it is usefull to specify (in-)equality constraints on the parameters:

$$g(m) \leq 0. \tag{3}$$

You can do that by overloading the **InequalityConstraint** class.

For more information visit the *Example* section on `dolfin-adjoint.org`.

## *The FEniCS challenge!*

- 1 Solve the "Hello world" PDE-constrained optimisation problem on the unit square with  $u_d(x, y) = \sin(\pi x) \sin(\pi y)$ , homogenous boundary conditions and  $\alpha = 10^{-6}$ .
- 2 Compute the difference between optimised heat profile and  $u_d$  before and after the optimisation.
- 3 Use the optimisation algorithms SLSQP, Newton-CG and L-BFGS-B and compare them.
- 4 What happens if you increase  $\alpha$ ?