FEniCS Course

Lecture 12: Computing sensitivities

Contributors
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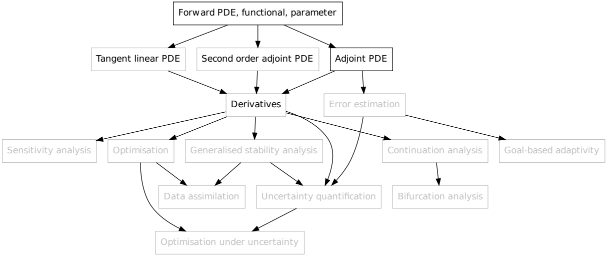

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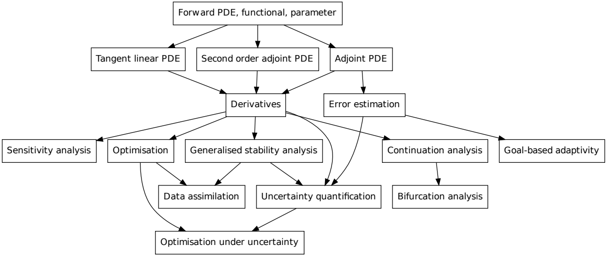

Example

Consider the Poisson's equation

$$\begin{aligned} -\nu \Delta u &= m \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

together with the *objective functional*

$$J(u) = \frac{1}{2} \int_\Omega \|u - u_d\|^2 \, \mathrm{d}x,$$

where u_d is a known function.

Goal

Compute the sensitivity of J with respect to the *parameter* m: $\mathrm{d}J/\mathrm{d}m$.

Comput. deriv. (i) General formulation

Given

- Parameter m,
- PDE $F(u, m) = 0$ with solution u.
- Objective functional $J(u, m) \to \mathbb{R}$,

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Compute $\mathrm{d}J/\mathrm{d}m$.

Reduced functional

Consider u as an implicit function of m by solving the PDE.
With that we define the *reduced functional* R:

$$R(m) = J(u(m), m)$$

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Comput. deriv. (ii) Reduced functional

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$$R(m) \equiv J(u(m), m).$$

Taking the derivative of with respect to m yields:

$$\frac{\mathrm{d}R}{\mathrm{d}m} = \frac{\mathrm{d}J}{\mathrm{d}m} = \frac{\partial J}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}m} + \frac{\partial J}{\partial m}.$$

Computing $\frac{\partial J}{\partial u}$ and $\frac{\partial J}{\partial m}$ is straight-forward, but how handle $\frac{\mathrm{d}u}{\mathrm{d}m}$?

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Comput. deriv. (iii) Computing $\frac{\mathrm{d}u}{\mathrm{d}m}$

Taking the derivative of $F(u, m) = 0$ with respect to m yields:

$$\frac{\mathrm{d}F}{\mathrm{d}m} = \frac{\partial F}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}m} + \frac{\partial F}{\partial m} = 0$$

Hence:

$$\frac{\mathrm{d}u}{\mathrm{d}m} = -\left(\frac{\partial F}{\partial u}\right)^{-1}\frac{\partial F}{\partial m}$$

Final formula for functional derivative

$$\frac{\mathrm{d}J}{\mathrm{d}m} = -\overbrace{\frac{\partial J}{\partial u}\left(\frac{\partial F}{\partial u}\right)^{-1}}^{\text{adjoint PDE}}\underbrace{\left(\frac{\partial F}{\partial u}\right)^{-1}\frac{\partial F}{\partial m}}_{\text{tangent linear PDE}} + \frac{\partial J}{\partial m},$$

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Dimensions of a finite dimensional example

$$\frac{\mathrm{d}J}{\mathrm{d}m} = \overbrace{\boxed{-\frac{\partial J}{\partial u}} \times \mathrlap{\underbrace{\phantom{\boxed{\left(\frac{\partial F}{\partial u}\right)^{-1}} \times \boxed{\frac{\partial F}{\partial m}}}}_{\text{discretised tangent linear PDE}}}\boxed{\left(\frac{\partial F}{\partial u}\right)^{-1}}}^{\text{discretised adjoint PDE}} \times \boxed{\frac{\partial F}{\partial m}} + \boxed{\frac{\partial J}{\partial m}}$$

The tangent linear solution is a matrix of dimension $|u| \times |m|$ and requires the solution of m linear systems. The adjoint solution is a vector of dimension $|u|$ and requires the solution of one linear systems.

Adjoint approach

1. Solve the adjoint equation for λ

$$\frac{\partial F}{\partial u}^* \lambda = -\frac{\partial J^*}{\partial u}.$$

2. Compute

$$\frac{\mathrm{d}J}{\mathrm{d}m} = \lambda^* \frac{\partial F}{\partial m} + \frac{\partial J}{\partial m}.$$

The computational expensive part is (1). It requires solving the (linear) adjoint PDE, and its cost is independent of the choice of parameter m.