

# FEniCS Course

## Lecture 12: Computing sensitivities

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*Contributors*

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But often we are also interested the sensitivity with respect to certain parameters, for example

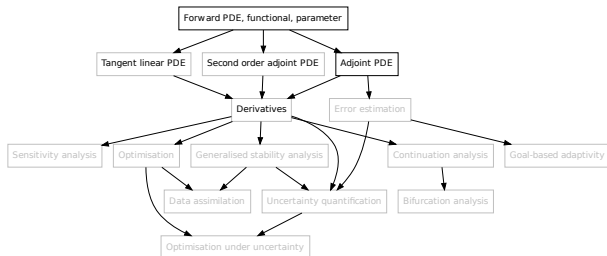
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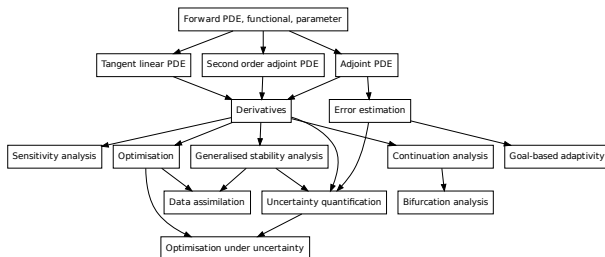


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## Example

Consider the Poisson's equation

$$\begin{aligned} -\nu \Delta u &= m && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

together with the *objective functional*

$$J(u) = \frac{1}{2} \int_{\Omega} \|u - u_d\|^2 \, dx,$$

where  $u_d$  is a known function.

### Goal

Compute the sensitivity of  $J$  with respect to the *parameter*  $m$ :  $dJ/dm$ .

# Comput. deriv. (i) General formulation

## Given

- Parameter  $m$ ,
- PDE  $F(u, m) = 0$  with solution  $u$ .
- Objective functional  $J(u, m) \rightarrow \mathbb{R}$ ,

## Goal

Compute  $dJ/dm$ .

## Reduced functional

Consider  $u$  as an implicit function of  $m$  by solving the PDE.  
With that we define the *reduced functional*  $R$ :

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## Comput. deriv. (ii) Reduced functional

Reduced functional:

$$R(m) \equiv J(u(m), m).$$

Taking the derivative of with respect to  $m$  yields:

$$\frac{dR}{dm} = \frac{dJ}{dm} = \frac{\partial J}{\partial u} \frac{du}{dm} + \frac{\partial J}{\partial m}.$$

Computing  $\frac{\partial J}{\partial u}$  and  $\frac{\partial J}{\partial m}$  is straight-forward, but how handle  $\frac{du}{dm}$ ?

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Taking the derivative of  $F(u, m) = 0$  with respect to  $m$  yields:

$$\frac{dF}{dm} = \frac{\partial F}{\partial u} \frac{du}{dm} + \frac{\partial F}{\partial m} = 0$$

Hence:

$$\frac{du}{dm} = - \left( \frac{\partial F}{\partial u} \right)^{-1} \frac{\partial F}{\partial m}$$

## Final formula for functional derivative

$$\frac{dJ}{dm} = - \overbrace{\frac{\partial J}{\partial u} \left( \frac{\partial F}{\partial u} \right)^{-1} \frac{\partial F}{\partial m}}^{\text{adjoint PDE}} + \frac{\partial J}{\partial m},$$

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# Dimensions of a finite dimensional example

$$\frac{dJ}{dm} = \boxed{-\frac{\partial J}{\partial u}} \times \underbrace{\boxed{\left(\frac{\partial F}{\partial u}\right)^{-1}} \times \boxed{\frac{\partial F}{\partial m}}}_{\text{discretised tangent linear PDE}} + \boxed{\frac{\partial J}{\partial m}}$$

discretised adjoint PDE

The tangent linear solution is a matrix of dimension  $|u| \times |m|$  and requires the solution of  $m$  linear systems. The adjoint solution is a vector of dimension  $|u|$  and requires the solution of one linear systems.

# Adjoint approach

- 1 Solve the adjoint equation for  $\lambda$

$$\frac{\partial F^*}{\partial u} \lambda = -\frac{\partial J^*}{\partial u}.$$

- 2 Compute

$$\frac{dJ}{dm} = \lambda^* \frac{\partial F}{\partial m} + \frac{\partial J}{\partial m}.$$

The computational expensive part is (1). It requires solving the (linear) adjoint PDE, and its cost is independent of the choice of parameter  $m$ .