Truth Learning in Social and Adversarial Settings

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July 17, 2024

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Introduction

Definition (Social learning network)

A social learning network is $\mathcal{N} := (G, q, p)$*, where*

1. $G = (V, E)$ *is a directed graph with agents as vertices,*
2. $q \in (0, 1)$ *is the prior probability of* $\theta = 1$*,*
3. $p \in \left(\frac{1}{2}, 1\right)$ *is the accuracy of agents' private signals* $s_v \in \{0, 1\}$*.*

Introduction

Agents choose actions in some sequence $\sigma \in \Sigma_n$.

When making decisions, agent $v \in V$ has access to

- the private signal s_v,
- actions of neighbors, who chose before v:

$$N_v = \{u \in V \mid uv \in E \wedge \sigma(v) > \sigma(u)\}.$$

When making decisions, agents use an *aggregation rule* μ.

Aggregation Rules

- Bayesian model

$$\mu^B(s_v, N_v) = \begin{cases} 1 & \text{if } \Pr[\theta = 1 \mid s_v, N_v] > \frac{1}{2}, \\ 0 & \text{if } \Pr[\theta = 0 \mid s_v, N_v] > \frac{1}{2}, \\ s_v & \text{otherwise.} \end{cases}$$

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- Simple majority vote

$$\mu^M(s_v, N_v) = \begin{cases} 1 & \text{if } s_v + \sum_{u \in N_v} a_u > \frac{1}{2}(|N_v| + 1), \\ 0 & \text{if } s_v + \sum_{u \in N_v} a_u < \frac{1}{2}(|N_v| + 1), \\ s_v & \text{otherwise.} \end{cases}$$

Learning Rate

Definition

The expected learning rate of a network $\mathcal{N}$ under the ordering σ and aggregation rule μ is

$$\mathcal{L}(\mathcal{N}, \sigma, \mu) := \frac{1}{n} \sum_{v \in V} \Pr_{\theta, s}[a_v = \theta].$$

NP-hardness

Definition (Network Learning)

Suppose some given aggregation rule μ. NETWORK LEARNING *problem is to decide for a network* $\mathcal{N}$, *and a constant* $\varepsilon \in (0, 1)$ *whether*

$$(\exists \sigma \in \Sigma_n) \quad \mathcal{L}(\mathcal{N}, \sigma, \mu) \geq 1 - \varepsilon.$$

We will focus on the Majority Dynamics setting, meaning $\mu = \mu^M$.

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- Reduce 3-SAT to Network Learning.
- *Goal:* Given a formula φ, construct a network $\mathcal{N}$ and choose ε such that the maximal $\mathcal{L}$ exceeds $1 - \varepsilon$ iff φ is satisfiable.

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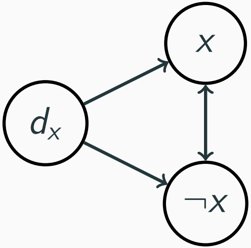
- Ordering over *variable gadgets (cells)* encodes boolean variable assignments $x_i = \{T, F\}$.
- *Clause gadgets* aggregate variables so learning rate is much higher if satisfied.

NP-hardness: Proof Construction

- *Variable cell* for each variable



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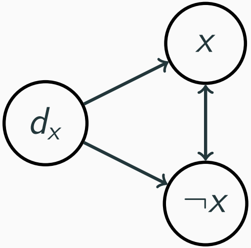

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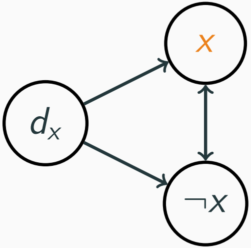

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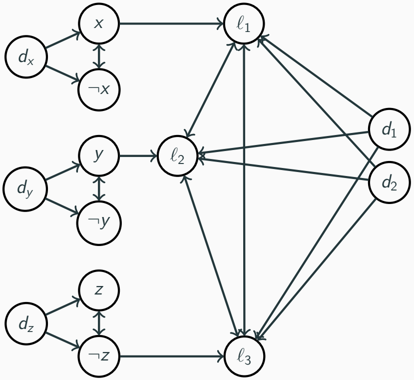

Example for $\varphi = x \vee y \vee \neg z$.

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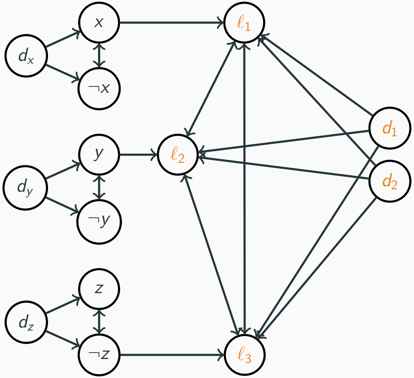

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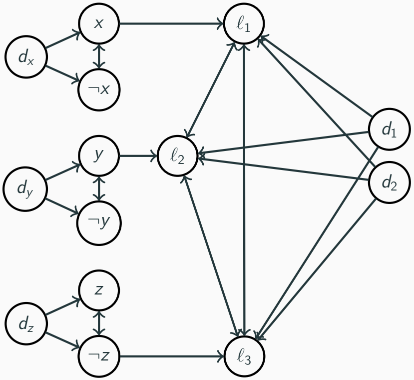

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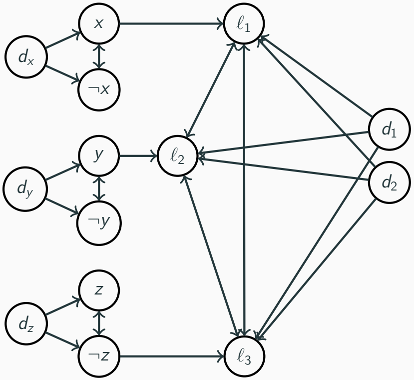
- Gap between SAT and non-SAT



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For suitably chosen p, q, ε, it holds that

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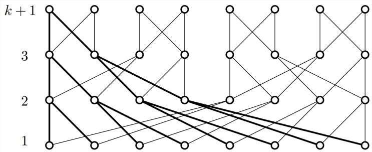
formula φ is satisfiable $\iff$ $\textsc{Network Learning}$ answers yes.

$\therefore$ $\textsc{Network Learning}$ with μ^M is NP-hard.

Adversarial Agents

- Some agents might deliberately mislead the group by reporting the opposite of θ.



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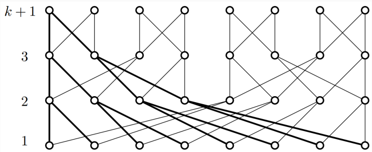

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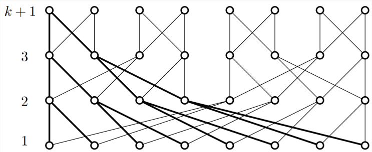

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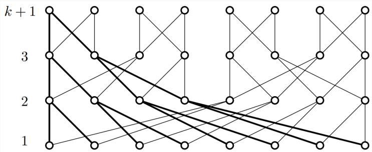
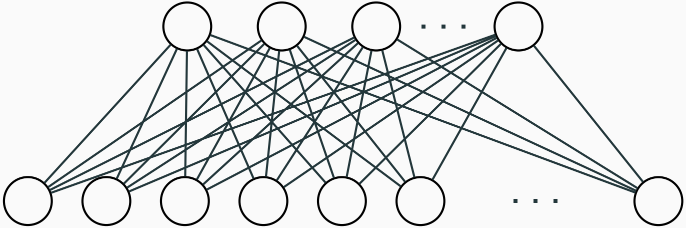
- We studied the Butterfly Network and the Celebrity Network.



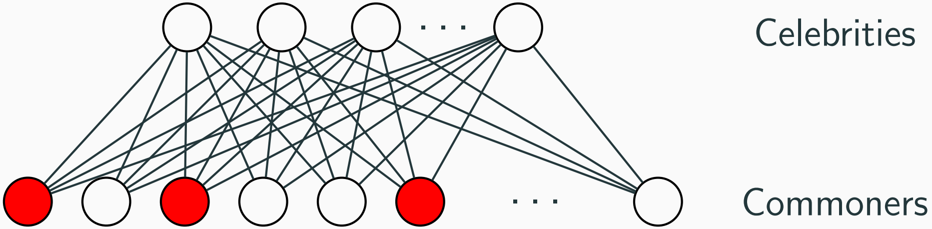
The Celebrity Network

- Complete Bipartite graph with M "celebrities" and N "commoners", with $N >> M$. (Bahar et al., 2020)



The Celebrity Network

- Complete Bipartite graph with M "celebrities" and N "commoners", with $N >> M$. (Bahar et al., 2020)
- We show that it is robust against $\mathcal{O}(N)$ adversarial commoners under a uniformly random decision ordering.



Robustness of Celebrity Network

- Suppose the non-adversarial agents' private signals have probability $0.5 + \delta$ of being correct, for $\delta \in (0, 1/2)$.

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- Suppose the decision order is uniformly random.
- Given any $\epsilon > 0$, the expected learning rate for the network is at least $1-\alpha-\epsilon$ for sufficiently large networks.

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- A majority of these commoners will correctly predict the ground truth θ WHP (this can be shown using Chebyshev's inequality).
- The first celebrity will mimic this majority.
- All non-adversaries after this first celebrity will mimic the action of the first celebrity.

References i

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Acknowledgments

This work was made possible by the Rutgers DIMACS REU program. Thank you as well to Professor Jie Gao and Ph.D. student Kevin Lu for their help and leadership.

Thank you to the National Science Foundation for funding this project through the grant CNS-2150186 and the REU supplement to NSF 2208663 -Collaborative Research: AF: Small: Promoting Social Learning Amid Interference in the Age of Social Media.

Filip Úradník and Júlia Križanová were supported by CoSP, a project funded by European Union's Horizon 2020 research and innovation programme, grant agreement No. 823748.