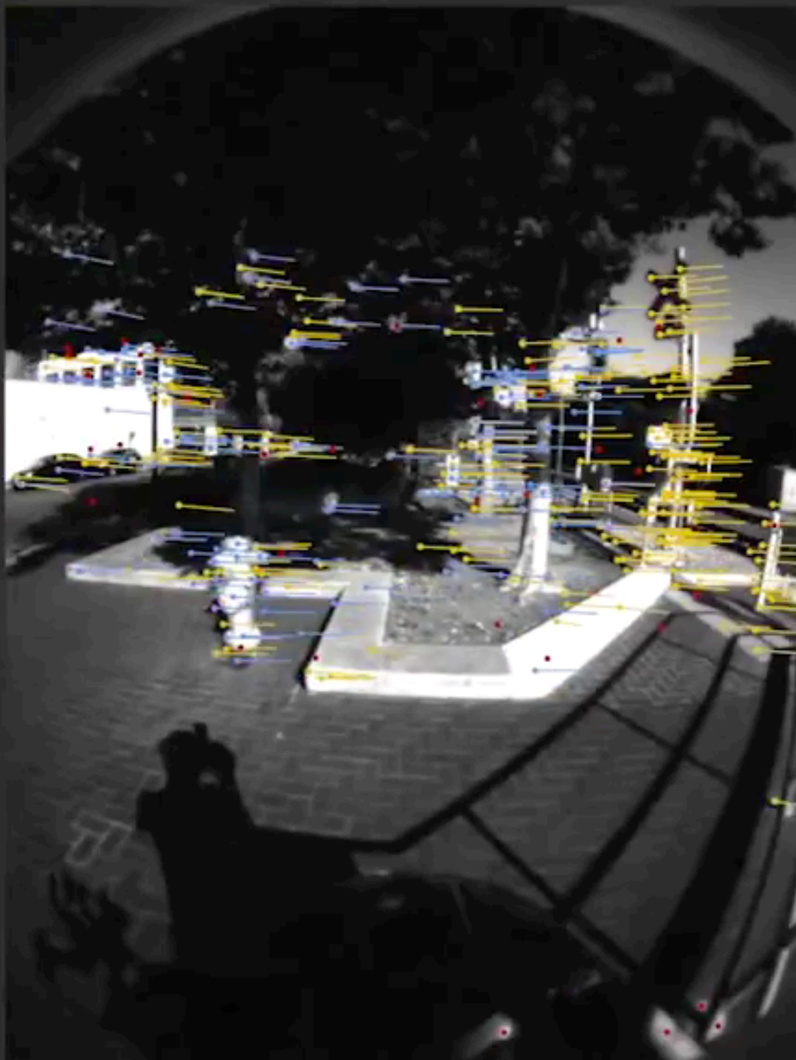
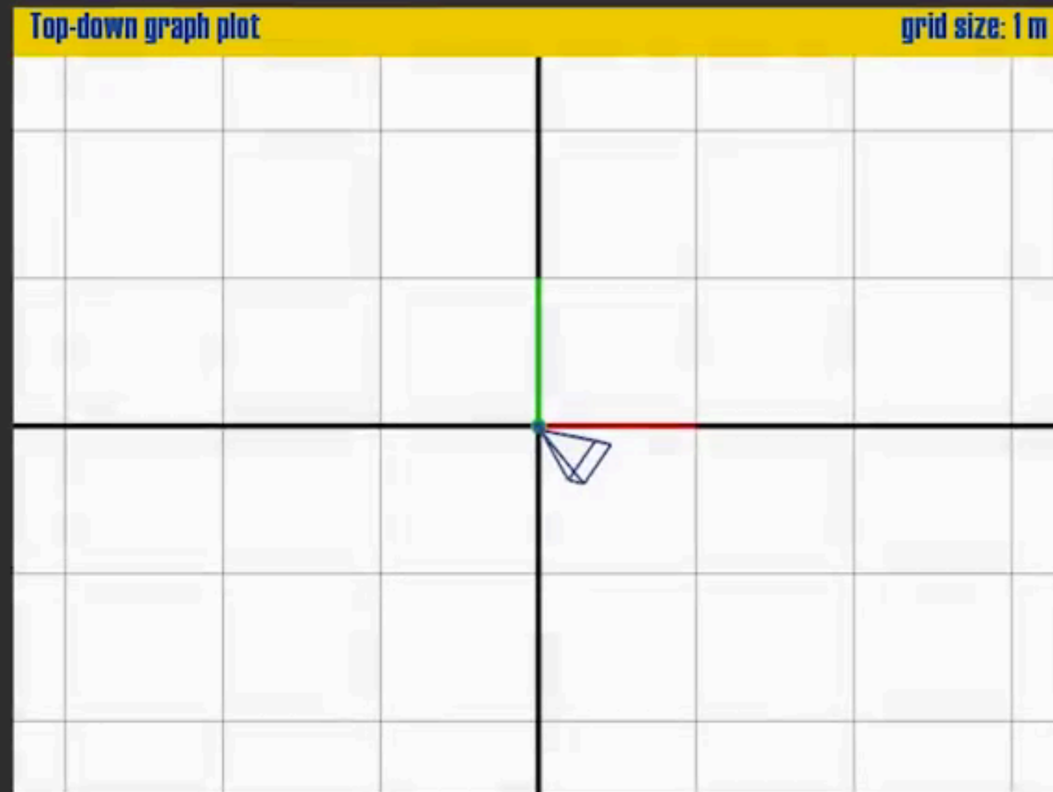
ETH zürich

Representing Robot Pose

The First-Order View

Paul Furgale, Hannes Sommer





Top-down graph plot
grid size: 1 m

A Brief Sketch of Nonlinear Least Squares

Nonlinear Least Squares

$$\text{Goal: estimate } \mathbf{x}^\star = \underset{\mathbf{x}}{\operatorname{argmin}} \left(J\left(\mathbf{x}\right)\right)$$

Sum of Squared Errors:

$$J(\mathbf{x}) := \frac{1}{2} \sum_{n=1}^{N} \mathbf{e}_n(\mathbf{x})^T \mathbf{R}_n^{-1} \mathbf{e}_n(\mathbf{x})$$

Nonlinear Least Squares

$$\text{Goal: estimate } \mathbf{x}^\star = \underset{\mathbf{x}}{\operatorname{argmin}} \left(J\left(\mathbf{x}\right)\right)$$

$$\mathbf{e}(\mathbf{x}) := \begin{bmatrix} \mathbf{e}_1(\mathbf{x}) \\ \vdots \\ \mathbf{e}_N(\mathbf{x}) \end{bmatrix}, \quad \mathbf{R} := \operatorname{diag}\left\{\mathbf{R}_1, \ldots, \mathbf{R}_N\right\},$$

Block form:

$$J(\mathbf{x}) = \frac{1}{2}\mathbf{e}(\mathbf{x})^T \mathbf{R}^{-1} \mathbf{e}(\mathbf{x}).$$

Nonlinear Least Squares

$$\text{Goal: estimate } \mathbf{x}^{\star} = \underset{\mathbf{x}}{\operatorname{argmin}} \left(J\left(\mathbf{x}\right)\right)$$

Current Guess

Perturbation

$$J(\bar{\mathbf{x}} + \boldsymbol{\epsilon}_x) = \frac{1}{2}\mathbf{e}(\bar{\mathbf{x}} + \boldsymbol{\epsilon}_x)^T\mathbf{R}^{-1}\mathbf{e}(\bar{\mathbf{x}} + \boldsymbol{\epsilon}_x)$$

Linearized:

Nonlinear Least Squares

$$\text{Goal: estimate } \mathbf{x}^\star = \underset{\mathbf{x}}{\operatorname{argmin}} \left(J \left(\mathbf{x} \right) \right)$$

Current Guess Perturbation

$$J(\bar{\mathbf{x}} + \boldsymbol{\epsilon}_x) = \frac{1}{2} \mathbf{e}(\bar{\mathbf{x}} + \boldsymbol{\epsilon}_x)^T \mathbf{R}^{-1} \mathbf{e}(\bar{\mathbf{x}} + \boldsymbol{\epsilon}_x)$$

$$\approx \frac{1}{2} \left(\mathbf{e}(\bar{\mathbf{x}}) + \left. \frac{\partial \mathbf{e}(\mathbf{x})}{\partial \mathbf{x}} \right|_{\bar{\mathbf{x}}} \boldsymbol{\epsilon}_x \right)^T \mathbf{R}^{-1} \left(\mathbf{e}(\bar{\mathbf{x}}) + \left. \frac{\partial \mathbf{e}(\mathbf{x})}{\partial \mathbf{x}} \right|_{\bar{\mathbf{x}}} \boldsymbol{\epsilon}_x \right)$$

Nonlinear Least Squares

Goal: estimate $\mathbf{x}^\star = \underset{\mathbf{x}}{\operatorname{argmin}} \left(J \left(\mathbf{x} \right) \right)$

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$$J(\bar{\mathbf{x}} + \boldsymbol{\epsilon}_x) = \frac{1}{2}\mathbf{e}(\bar{\mathbf{x}} + \boldsymbol{\epsilon}_x)^T \mathbf{R}^{-1} \mathbf{e}(\bar{\mathbf{x}} + \boldsymbol{\epsilon}_x)$$
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$$J(\boldsymbol{\epsilon}_x) = \frac{1}{2}\mathbf{e}(\boldsymbol{\epsilon}_x)^T \mathbf{R}^{-1}\mathbf{e}(\boldsymbol{\epsilon}_x),$$

Nonlinear Least Squares

$$\text{Goal: estimate } \mathbf{x}^\star = \underset{\mathbf{x}}{\operatorname{argmin}} \left(J \left(\mathbf{x} \right) \right)$$

An intermediate goal: $\boldsymbol{\epsilon}_x^\star = \underset{\boldsymbol{\epsilon}_x}{\operatorname{argmin}} \left(J \left(\boldsymbol{\epsilon}_x \right) \right)$

$$\frac{\partial J(\boldsymbol{\epsilon}_x)}{\partial \boldsymbol{\epsilon}_x}^T = \mathbf{E}^T \mathbf{R}^{-1} \left(\bar{\mathbf{e}} + \mathbf{E} \boldsymbol{\epsilon}_x^\star \right) = 0$$

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$$\mathbf{E}^T\mathbf{R}^{-1}\mathbf{E}\boldsymbol{\epsilon}_x^\star = -\mathbf{E}^T\mathbf{R}^{-1}\bar{\mathbf{e}}$$

Solve the linear system:

$$\boldsymbol{\epsilon}_x^\star = -\left(\mathbf{E}^T\mathbf{R}^{-1}\mathbf{E}\right)^{-1}\mathbf{E}^T\mathbf{R}^{-1}\bar{\mathbf{e}}$$

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Solve the linear system:

$$\boldsymbol{\epsilon}_x^\star = - \left(\mathbf{E}^T \mathbf{R}^{-1} \mathbf{E} \right)^{-1} \mathbf{E}^T \mathbf{R}^{-1} \bar{\mathbf{e}}$$

Update the current guess:

$$\bar{\mathbf{x}} \leftarrow \bar{\mathbf{x}} + \boldsymbol{\epsilon}_x^\star$$

Nonlinear Least Squares / Gauss Newton

$$\text{Goal: estimate } \mathbf{x}^\star = \underset{\mathbf{x}}{\operatorname{argmin}} \, J(\mathbf{x}) = \frac{1}{2}\mathbf{e}(\mathbf{x})^T\mathbf{R}^{-1}\mathbf{e}(\mathbf{x}).$$

- Start with an initial guess for $\mathbf{x}$
- Iterate
  - Linearize

$$\bar{\mathbf{e}} := \mathbf{e}(\bar{\mathbf{x}}), \quad \mathbf{E} := \left.\frac{\partial \mathbf{e}(\mathbf{x})}{\partial \mathbf{x}}\right|_{\bar{\mathbf{x}}}, \quad \mathbf{e}(\boldsymbol{\epsilon}_x) := \bar{\mathbf{e}} + \mathbf{E}\boldsymbol{\epsilon}_x.$$

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Recover the covariance:

$$\mathbf{P} := \left(\mathbf{E}^T\mathbf{R}^{-1}\mathbf{E}\right)^{-1},$$

$$\mathbf{x} = \mathbf{x}^\star + \boldsymbol{\epsilon}_x, \quad \boldsymbol{\epsilon}_x \sim \mathcal{N}(\mathbf{0}, \mathbf{P})$$

Calculus

$$\left.\frac{\partial \mathbf{e}(\mathbf{x})}{\partial \mathbf{x}}\right|_{\overline{\mathbf{x}}}$$

Linear Algebra

$$\boldsymbol{\epsilon}_x^{\star} = -\left(\mathbf{E}^T\mathbf{R}^{-1}\mathbf{E}\right)^{-1}\mathbf{E}^T\mathbf{R}^{-1}\overline{\mathbf{e}}$$

Probabilities

$$\mathbf{P} := \left(\mathbf{E}^T \mathbf{R}^{-1} \mathbf{E}\right)^{-1},$$

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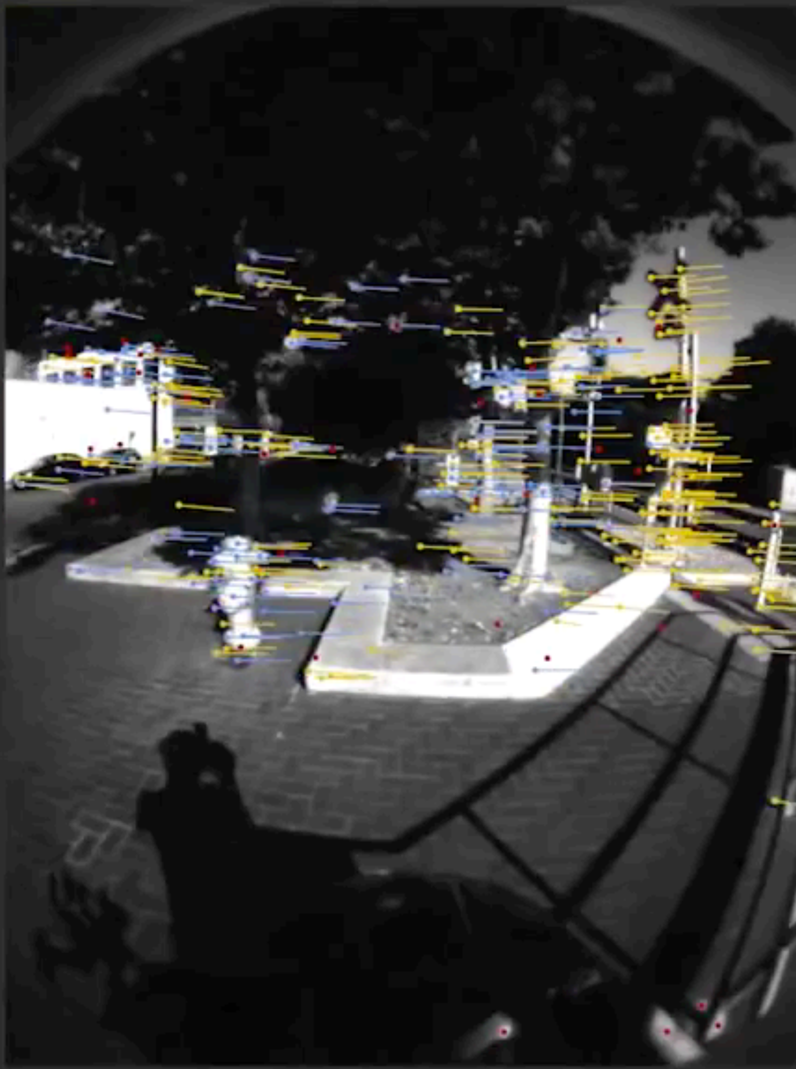
Basic assumption

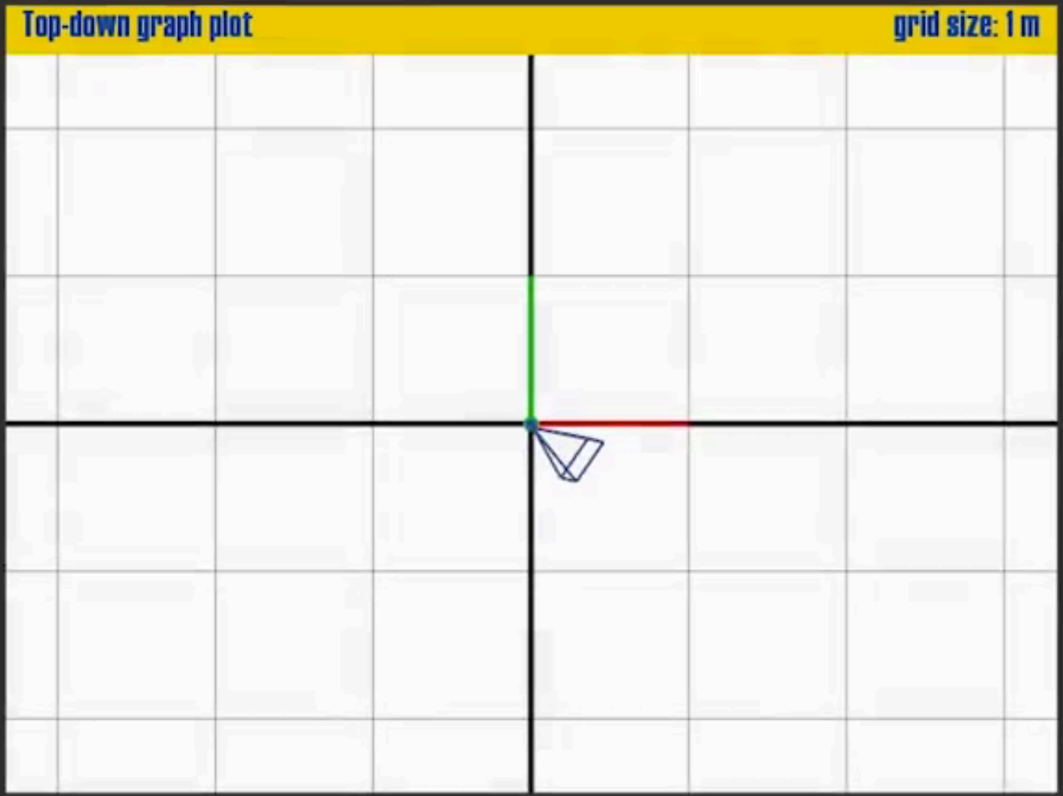
$$\mathbf{x} \in \mathbb{R}^M$$

Basic assumption

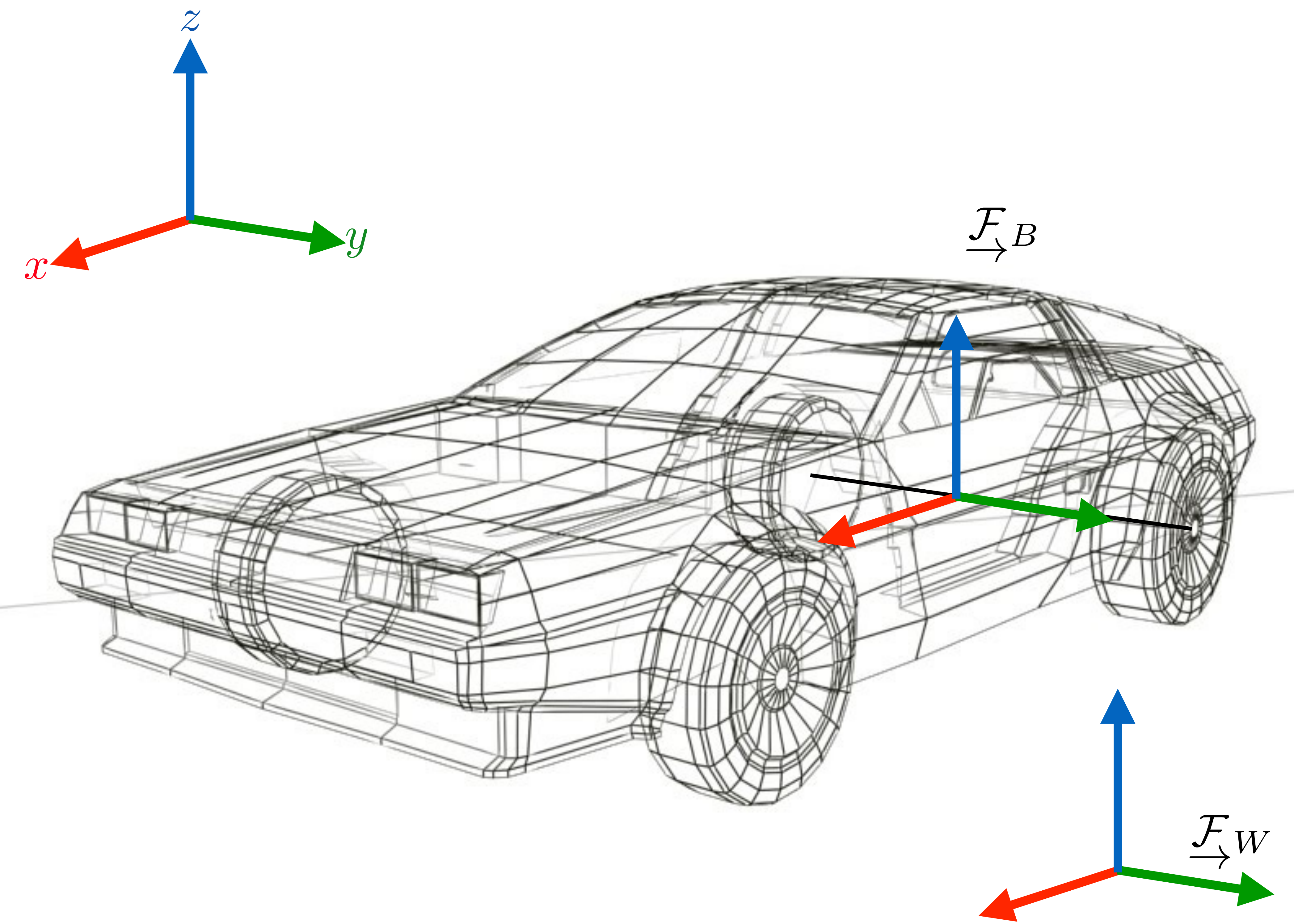
$$\mathbf{x} \in \mathbb{R}^M$$

Not always true



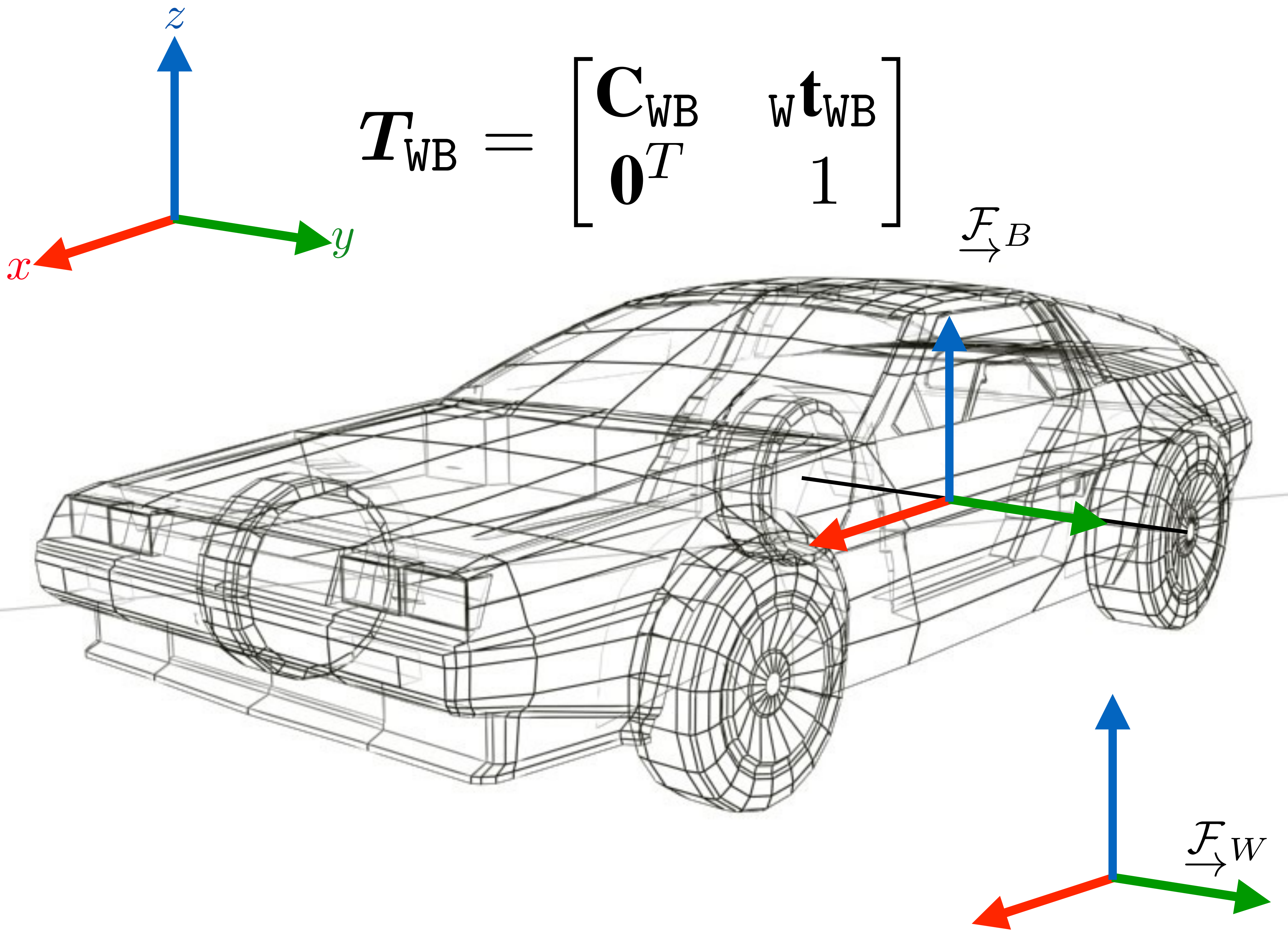


Top-down graph plot
grid size: 1 m



Super sweet wireframe Delorian from http://nanahra.blogspot.com

$$\boldsymbol{T}_{\mathtt{WB}} = \begin{bmatrix} \mathbf{C}_{\mathtt{WB}} & {}_{\mathtt{W}}\mathbf{t}_{\mathtt{WB}} \\ \mathbf{0}^T & 1 \end{bmatrix}$$



Super sweet wireframe Delorian from http://nanahra.blogspot.com

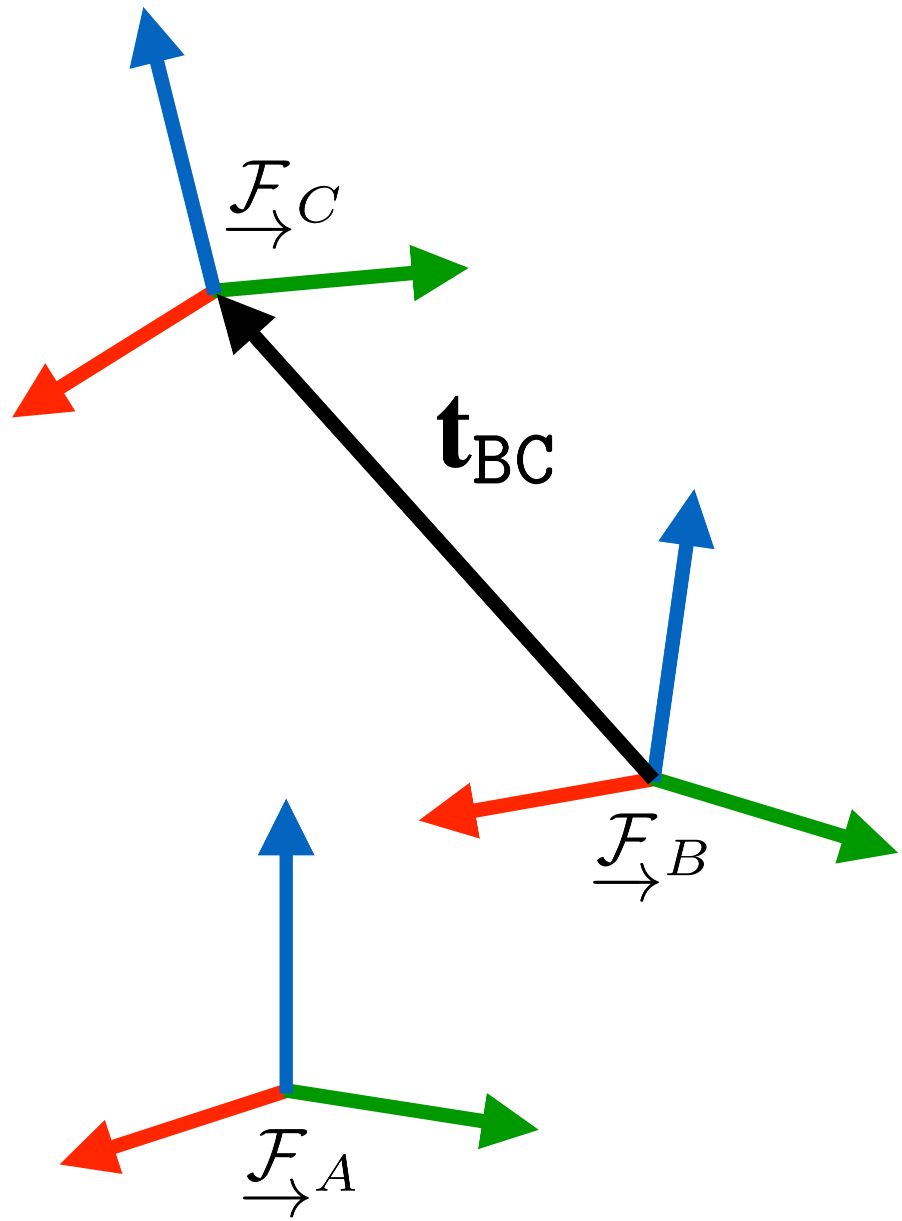
Physical quantity

$$ {}_{\mathrm{A}}\mathbf{t}_{\mathrm{BC}} $$

Expressed in

With respect to

of



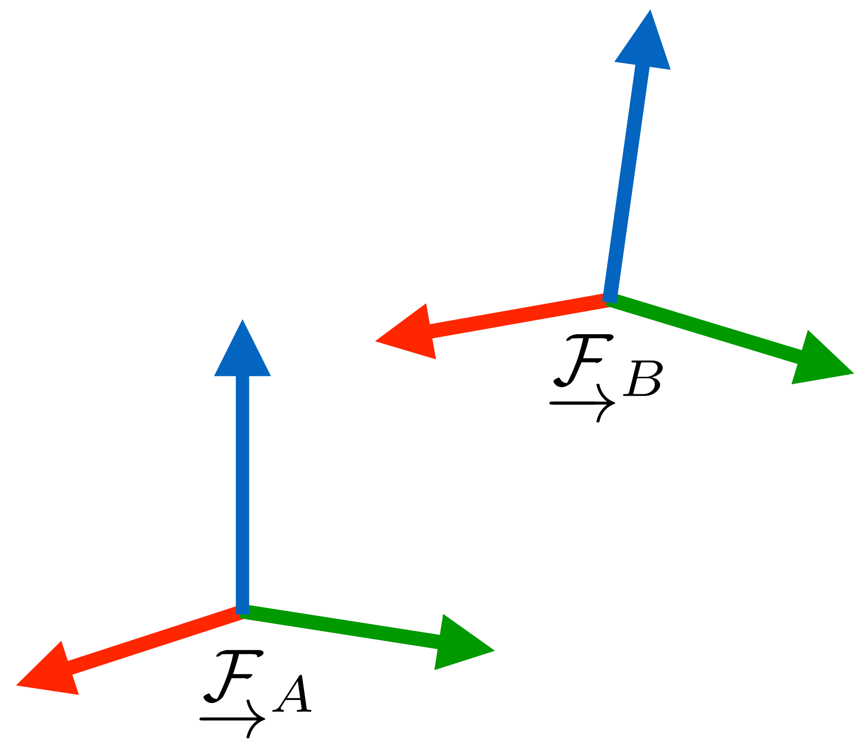
“The vector from B to C expressed in A”

$$\mathbf{C}_{AB}$$

into

rotates vectors from

$$ {}_{A}\mathbf{v}_{BC} = \mathbf{C}_{AB}\,{}_{B}\mathbf{v}_{BC} $$



“The orientation/attitude of B with respect to A”

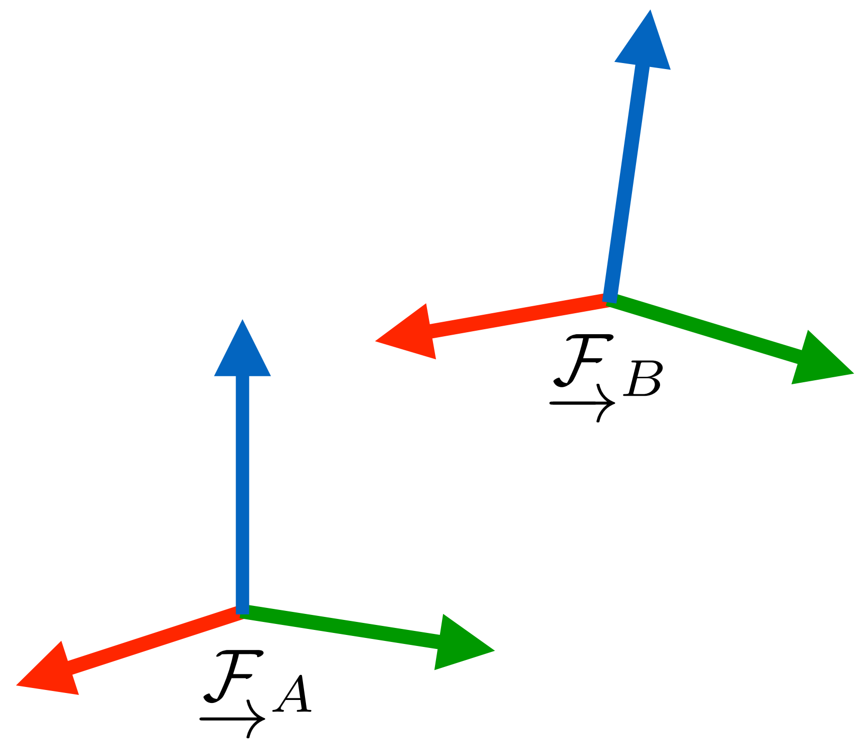
“Rotates vectors from frame B into frame A”

$$\mathbf{C}_{\mathtt{AB}}$$

With respect to

of

$$_{\mathtt{A}}\mathbf{v}_{\mathtt{BC}} = \mathbf{C}_{\mathtt{AB}\,\mathtt{B}}\mathbf{v}_{\mathtt{BC}}$$



"The pose of B with respect to A"

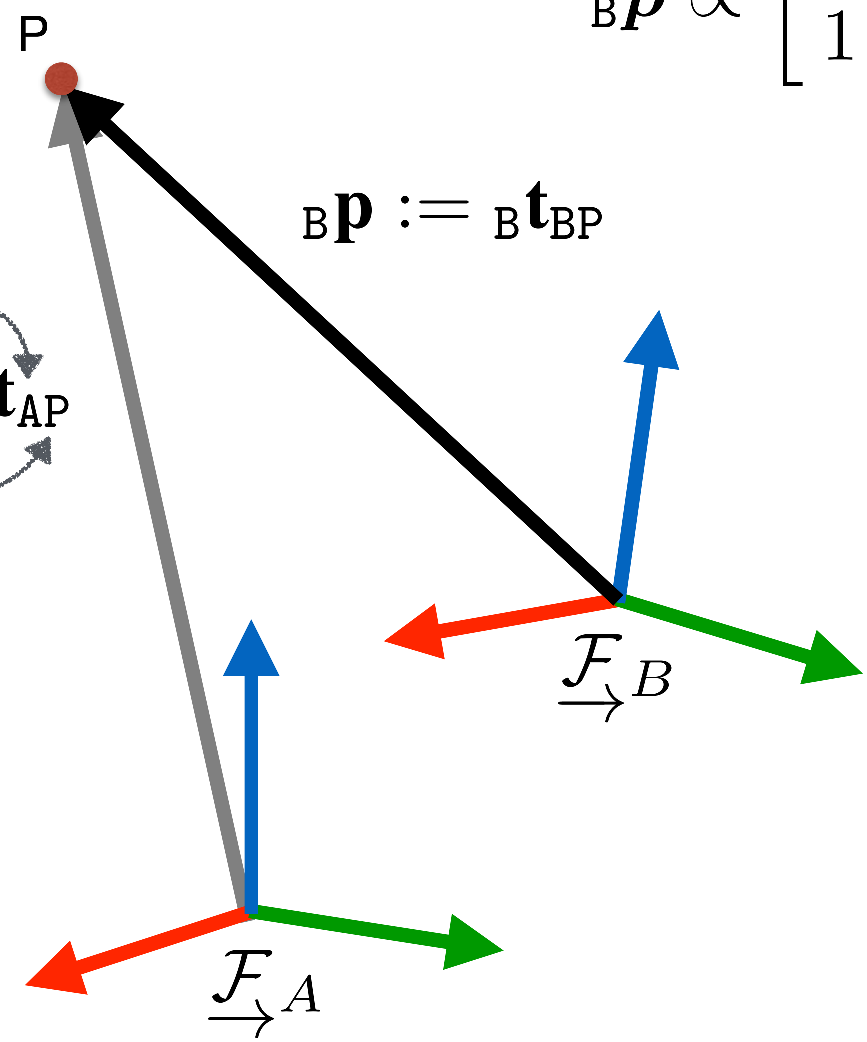
"Transforms point from frame B into frame A"

$${}_{\mathrm{B}}\boldsymbol{p} \propto \begin{bmatrix} {}_{\mathrm{B}}\mathbf{p} \\ 1 \end{bmatrix}$$

$$\boldsymbol{T}_{\mathrm{AB}}$$

into — transforms points from

$${}_{\mathrm{A}}\mathbf{p} := {}_{\mathrm{A}}\mathbf{t}_{\mathrm{AP}}$$



$${}_{\mathrm{A}}\boldsymbol{p} = \boldsymbol{T}_{\mathrm{AB}}\,{}_{\mathrm{B}}\boldsymbol{p}$$

“The pose of B with respect to A”

“Transforms point from frame B into frame A”

$${}_{\mathrm{B}}\boldsymbol{p} \propto \begin{bmatrix} {}_{\mathrm{B}}\mathbf{p} \\ 1 \end{bmatrix}$$

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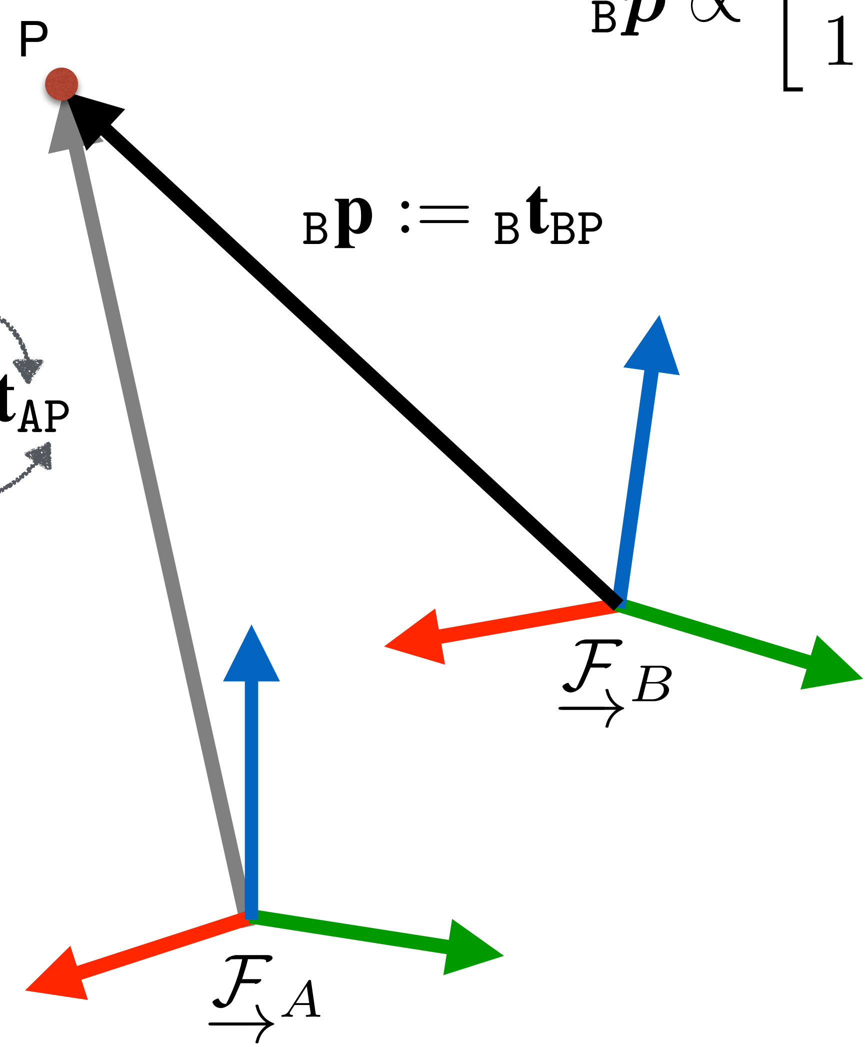
With respect to

of

$${}_{\mathrm{A}}\mathbf{p} := {}_{\mathrm{A}}\mathbf{t}_{\mathrm{AP}}$$



$${}_{\mathrm{A}}\boldsymbol{p} = \boldsymbol{T}_{\mathrm{AB}}\,{}_{\mathrm{B}}\boldsymbol{p}$$

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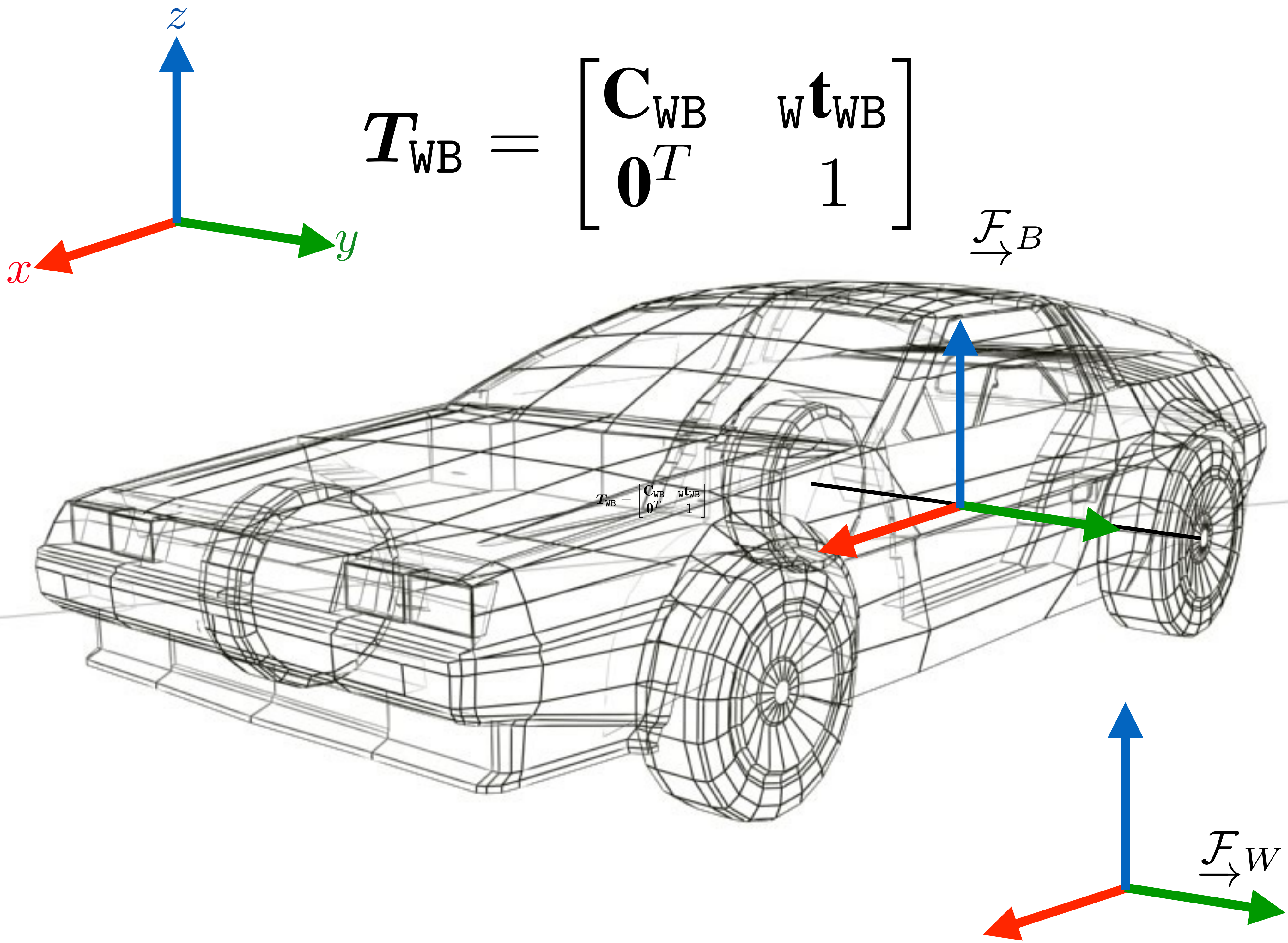

Super sweet wireframe Delorian from http://nanahra.blogspot.com

Basic assumption

$$\mathbf{x} \in \mathbb{R}^M$$

Does not hold for…

$$\mathbf{C}_{\mathtt{AB}}, \boldsymbol{T}_{\mathtt{AB}}$$

These elements exist on manifolds

$$\mathbf{C}_{\mathtt{AB}}, \boldsymbol{T}_{\mathtt{AB}}$$

SO(3)

The Special Orthogonal Group

SE(3)

The Special Euclidean Group

Nonlinear Least Squares / Gauss Newton

$$\text{Goal: estimate } \mathbf{x}^\star = \underset{\mathbf{x}}{\operatorname{argmin}}\ J(\mathbf{x}) = \frac{1}{2}\mathbf{e}(\mathbf{x})^T\mathbf{R}^{-1}\mathbf{e}(\mathbf{x}).$$

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Linear Algebra

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Probabilities

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These elements exist on manifolds

$$\mathbf{C}_{\mathtt{AB}}, \boldsymbol{T}_{\mathtt{AB}}$$

SO(3)

The Special Orthogonal Group

SE(3)

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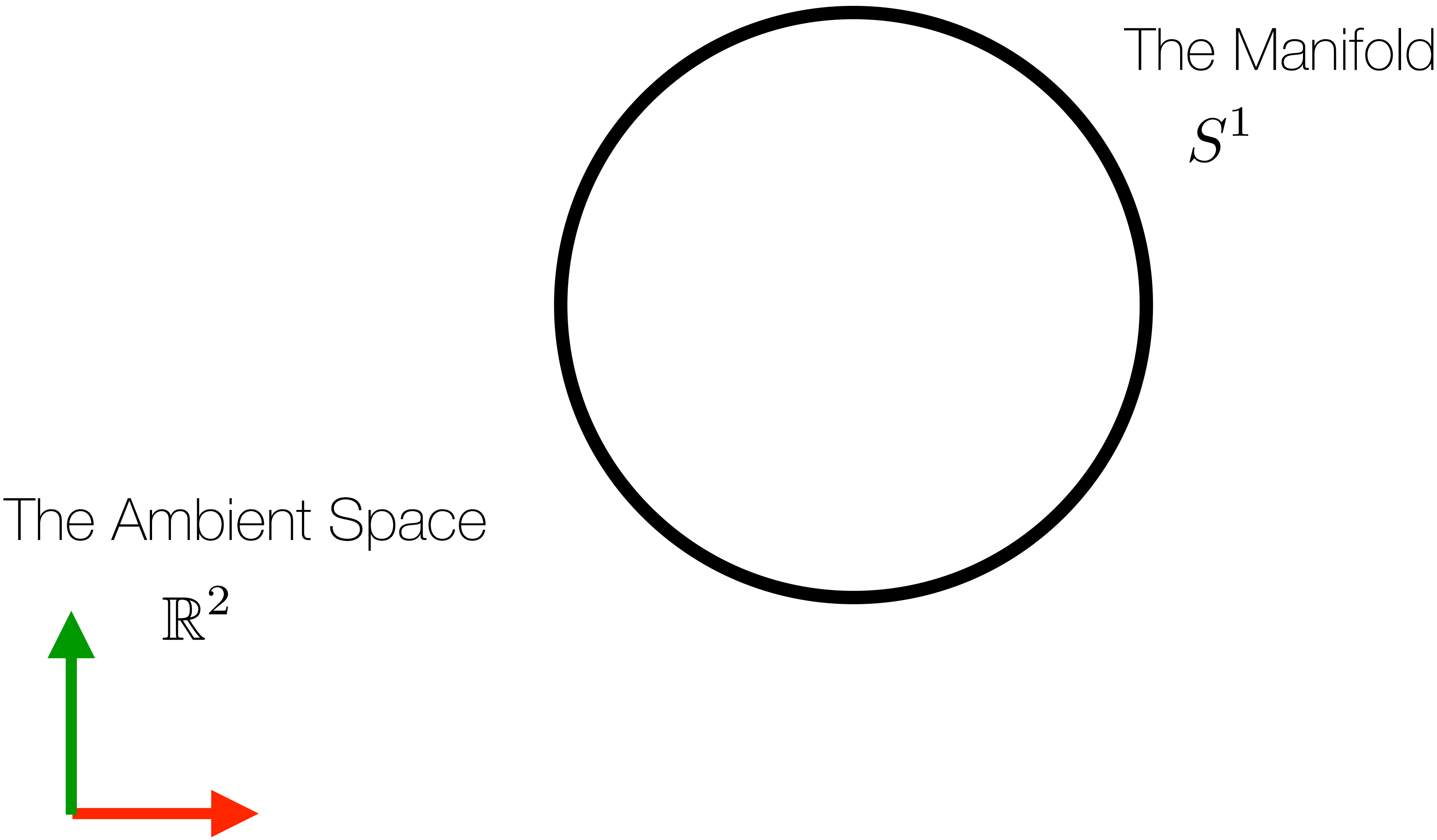
A Brief Introduction to Manifolds


The Manifold
S^1
The Ambient Space
$\mathbb{R}^2$

A Brief Introduction to Manifolds

- Topological Spaces
  - A set of points with the notion of a neighbourhood
  - This is enough to define continuous functions and curves

A Brief Introduction to Manifolds

- Topological Spaces
- N-Dimensional Topological Manifolds
  - A topological space that locally "looks like" $\mathbb{R}^N$

A Brief Introduction to Manifolds

- Topological Spaces
- N-Dimensional Topological Manifolds
- N-Dimensional Differentiable Manifolds
  - Curves have well-defined tangent vectors
  - Real-valued functions have well-defined directional derivatives

A Brief Introduction to Manifolds

- Topological Spaces
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We can start to do calculus

A Brief Introduction to Manifolds

- Topological Spaces
- N-Dimensional Topological Manifolds
- N-Dimensional Differentiable Manifolds (DM)

These are all restrictions

A Brief Introduction to Manifolds

- N-Dimensional Differentiable Manifolds (DM)

extension

restriction

- Riemannian Manifolds
  - Introduce the notion of distance and angles
  - We …. shortest connecting curves between points
- Lie Groups
  - Adds the structure of a group: differentiable multiplication between points

Both Differentiable Manifolds.
Both Lie Groups.

$$\mathbf{C}_{\mathtt{AB}}, \boldsymbol{T}_{\mathtt{AB}}$$

SO(3)

The Special Orthogonal Group

SE(3)

The Special Euclidean Group

Choosing an Atlas

- Both SO(3) and SE(3) are Differentiable Manifolds
- This implies that differentiation is possible. How?
- By choosing an *Atlas*

Choosing an Atlas

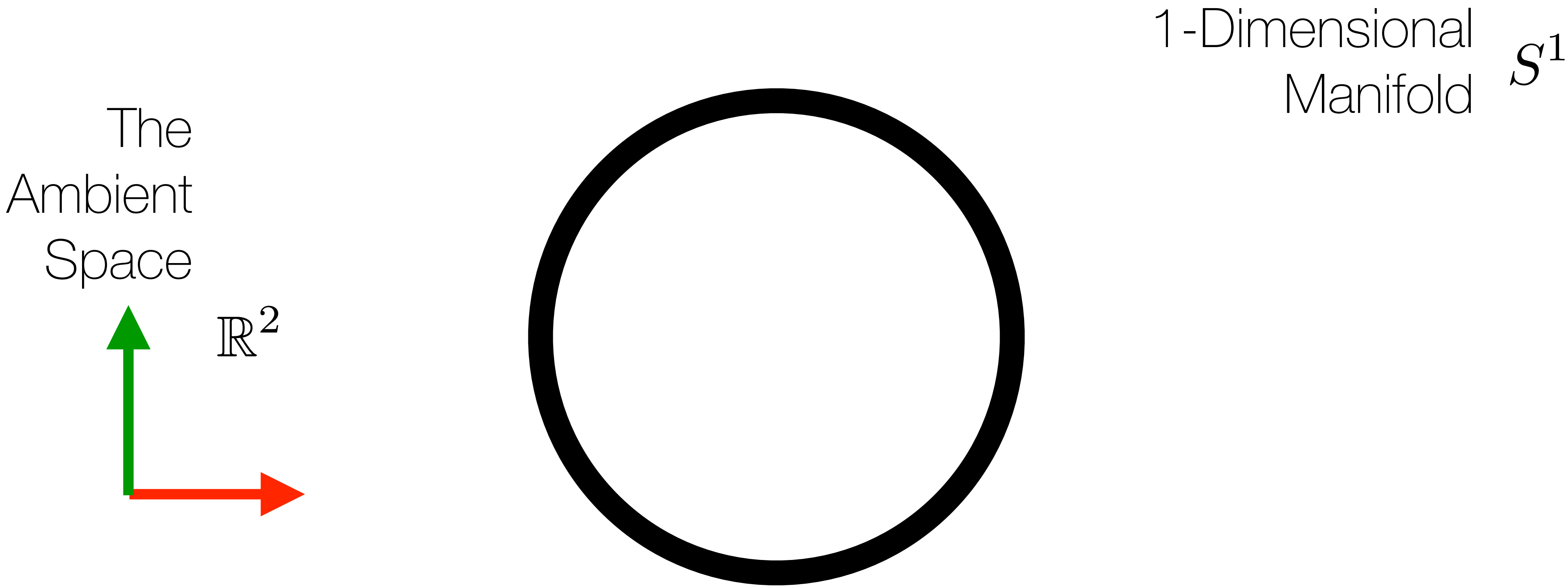
- Charts:
  - For an M-Dimensional DM, $\mathcal{M}$, a *chart* is

Invertible Mapping

The Manifold

$$\varphi : U_\varphi \to \mathcal{M}$$

Open subset of $\mathbb{R}^M$



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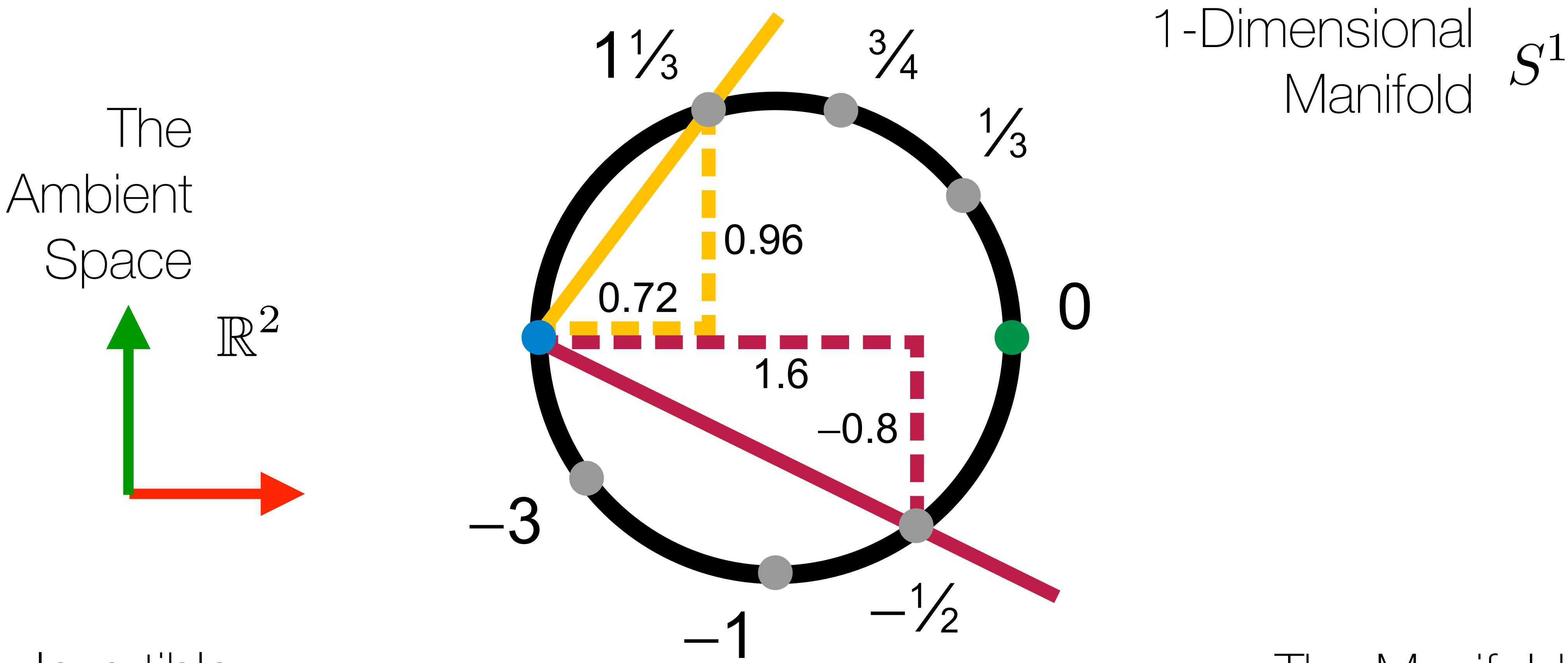

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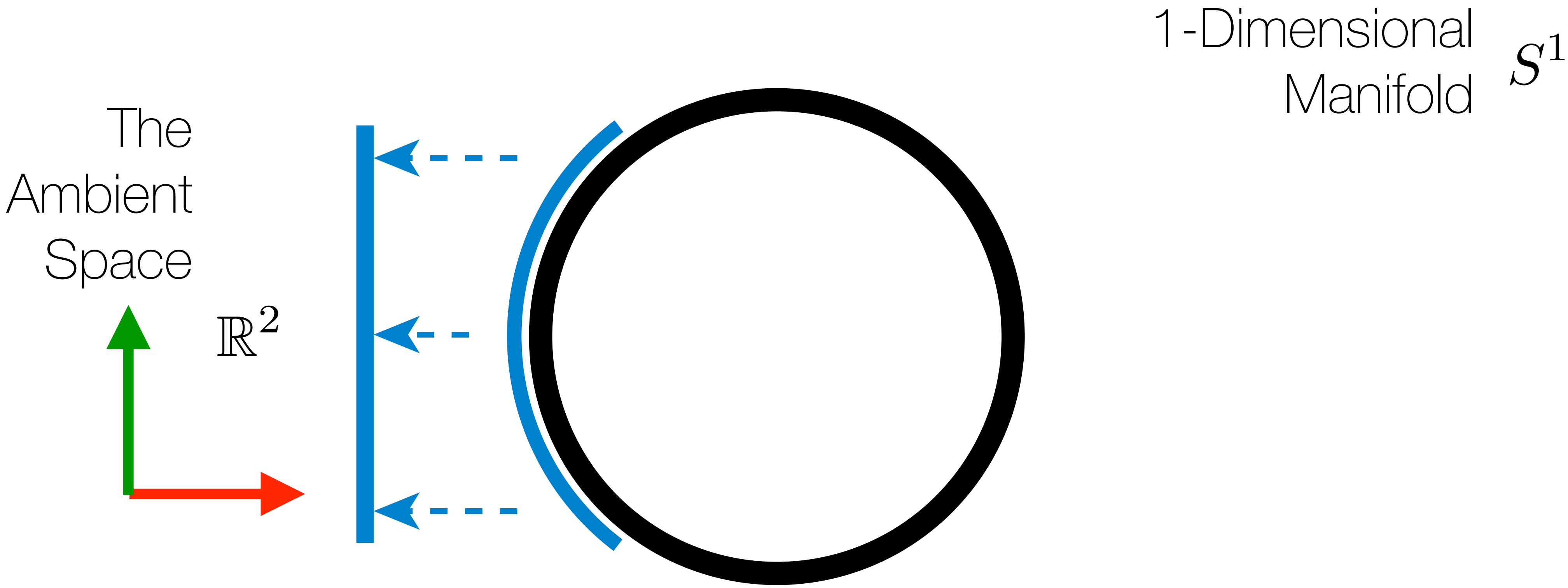

Invertible Mapping

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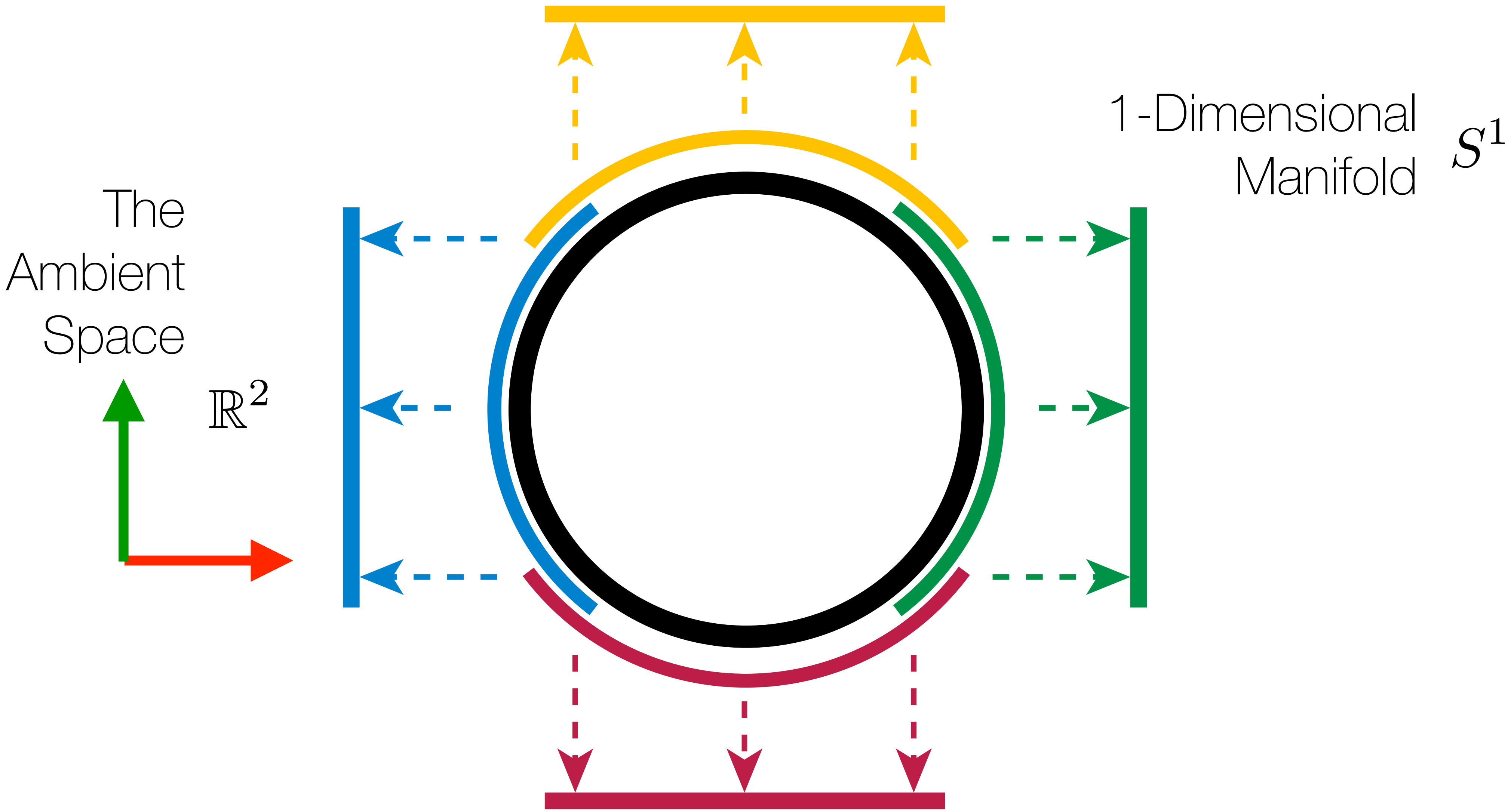
Open subset of $\mathbb{R}^M$



- An *Atlas* is a collection of charts such that cover the manifold

The Atlas Unlocks Calculus

M-dimensional

N-dimensional

$$m \in \mathcal{M}, \quad n \in \mathcal{N}$$

function

$$f : \mathcal{M} \to \mathcal{N}$$

The Atlas Unlocks Calculus

$\mathcal{M}$-dimensional

N-dimensional

$$m \in \mathcal{M}, \quad n \in \mathcal{N}$$

function

$$n = f(m)$$

The Atlas Unlocks Calculus

M-dimensional

N-dimensional

$$m \in \mathcal{M}, \quad n \in \mathcal{N}$$

function

$$f : \mathcal{M} \rightarrow \mathcal{N}$$

The Atlas Unlocks Calculus

$$f : \mathcal{M} \rightarrow \mathcal{N}$$

- The Jacobian encodes the first-order relationship that says "How do small changes in m become small changes in $f(m)$?"
- What should the "Jacobian" look like?

$$N \times M$$

The Atlas Unlocks Calculus

$$f : \mathcal{M} \rightarrow \mathcal{N}$$

- Choose charts for m and $f(m)$

$$\varphi_m : U_{\varphi_m} \rightarrow \mathcal{M}$$

$$\varphi_{f(m)} : U_{\varphi_{f(m)}} \rightarrow \mathcal{N}$$

- Define a mapping from $\mathbb{R}^M$ to $\mathbb{R}^N$ using the charts

$$\mathbf{f} := \varphi_{f(m)}^{-1} \circ f \circ \varphi_m$$

The Atlas Unlocks Calculus

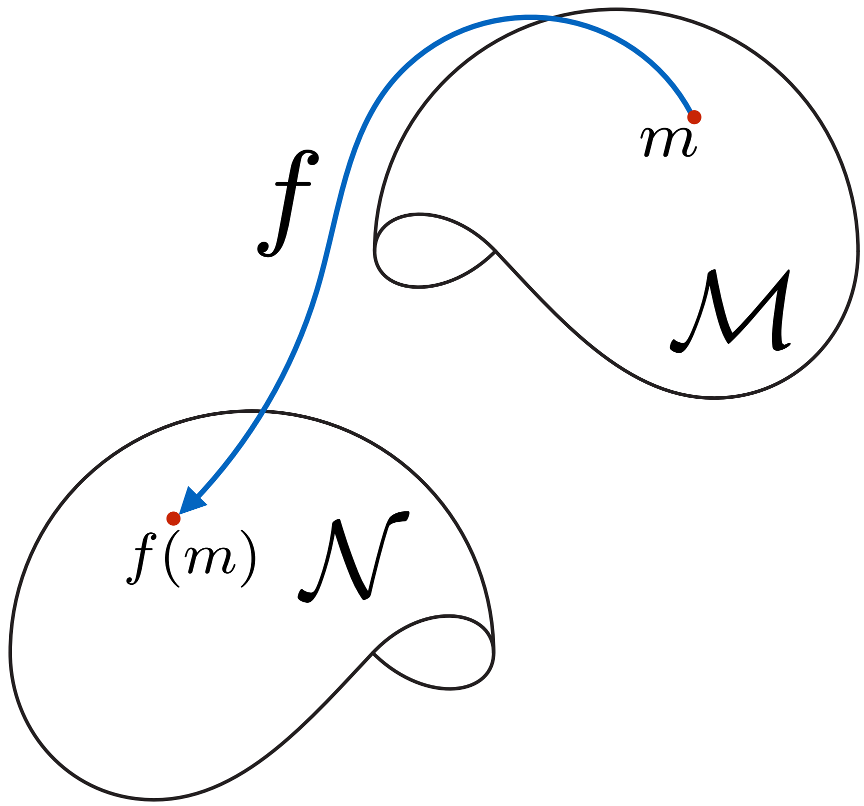
$$f : \mathcal{M} \rightarrow \mathcal{N}$$

- Define a mapping from $\mathbb{R}^M$ to $\mathbb{R}^N$ using the charts

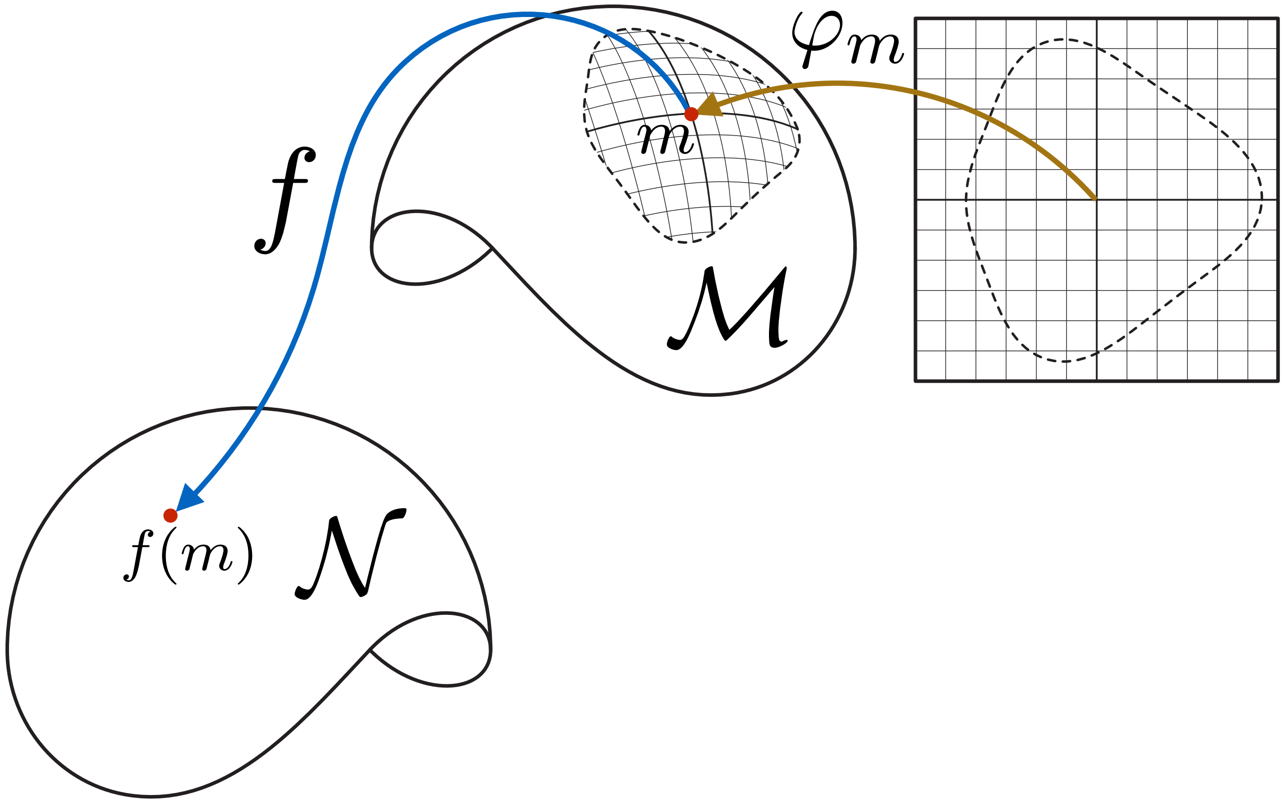
$$\mathbf{f} := \varphi_{f(m)}^{-1} \circ f \circ \varphi_m$$

- Define the "Representing Matrix" of the differential of f at m, which depends on our choice of charts

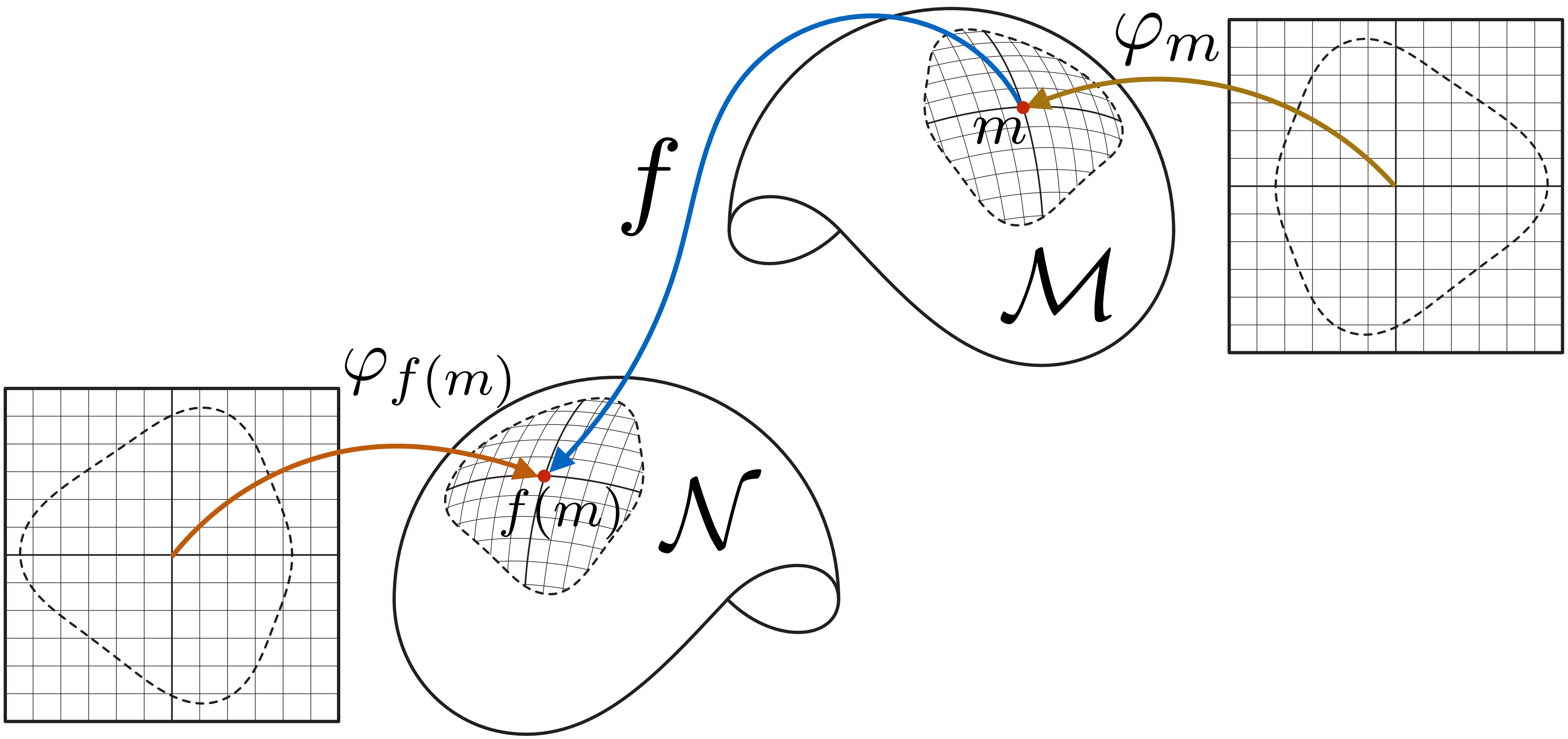
$$\mathbf{J}_m f := \left. \frac{\partial \mathbf{f}}{\partial \mathbf{m}} \right|_{\varphi_m^{-1}(m)} \in \mathbb{R}^{N \times M}$$



f
m
$\mathcal{M}$
$f(m)$
$\mathcal{N}$

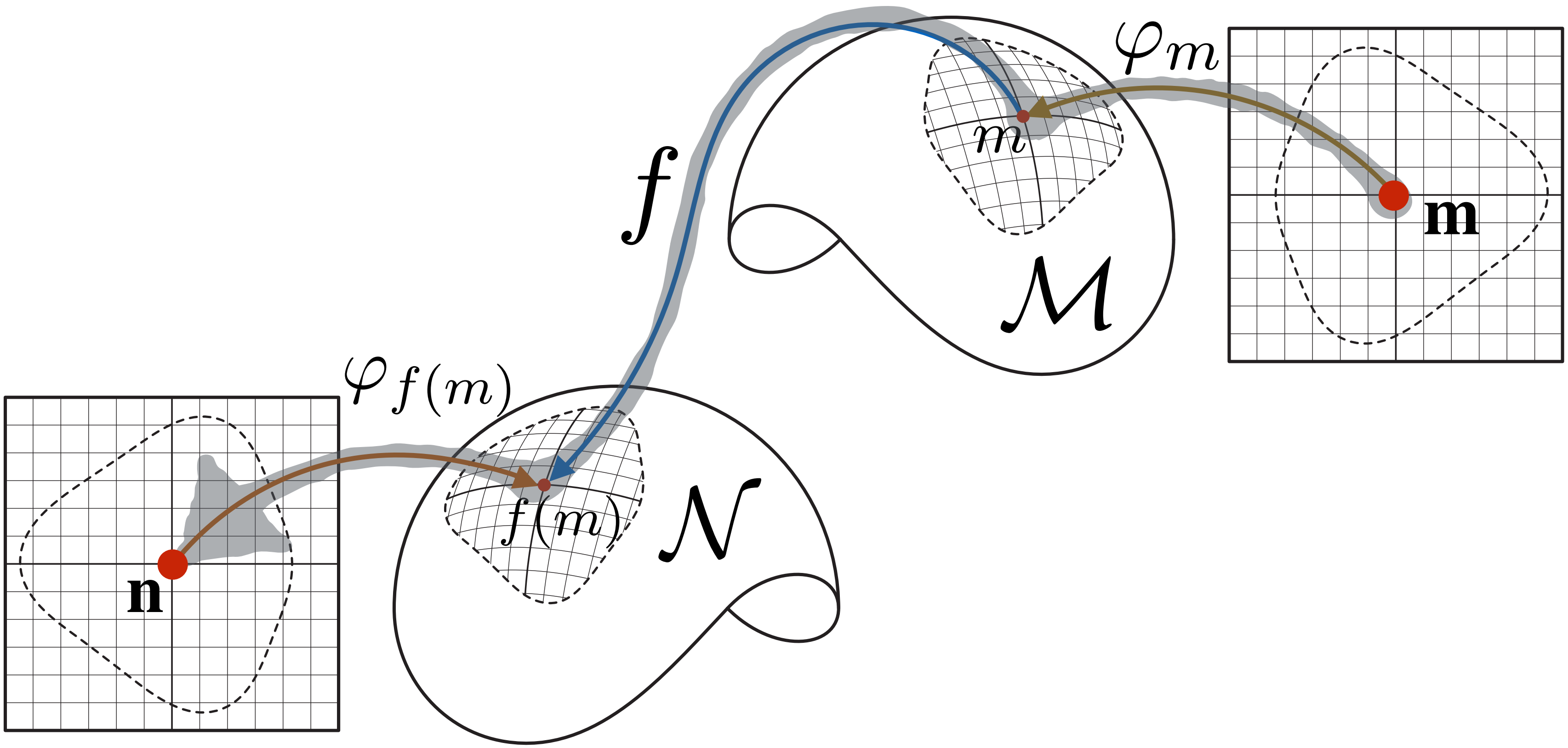


φ_m
m
f
$\mathcal{M}$
$f(m)$
$\mathcal{N}$

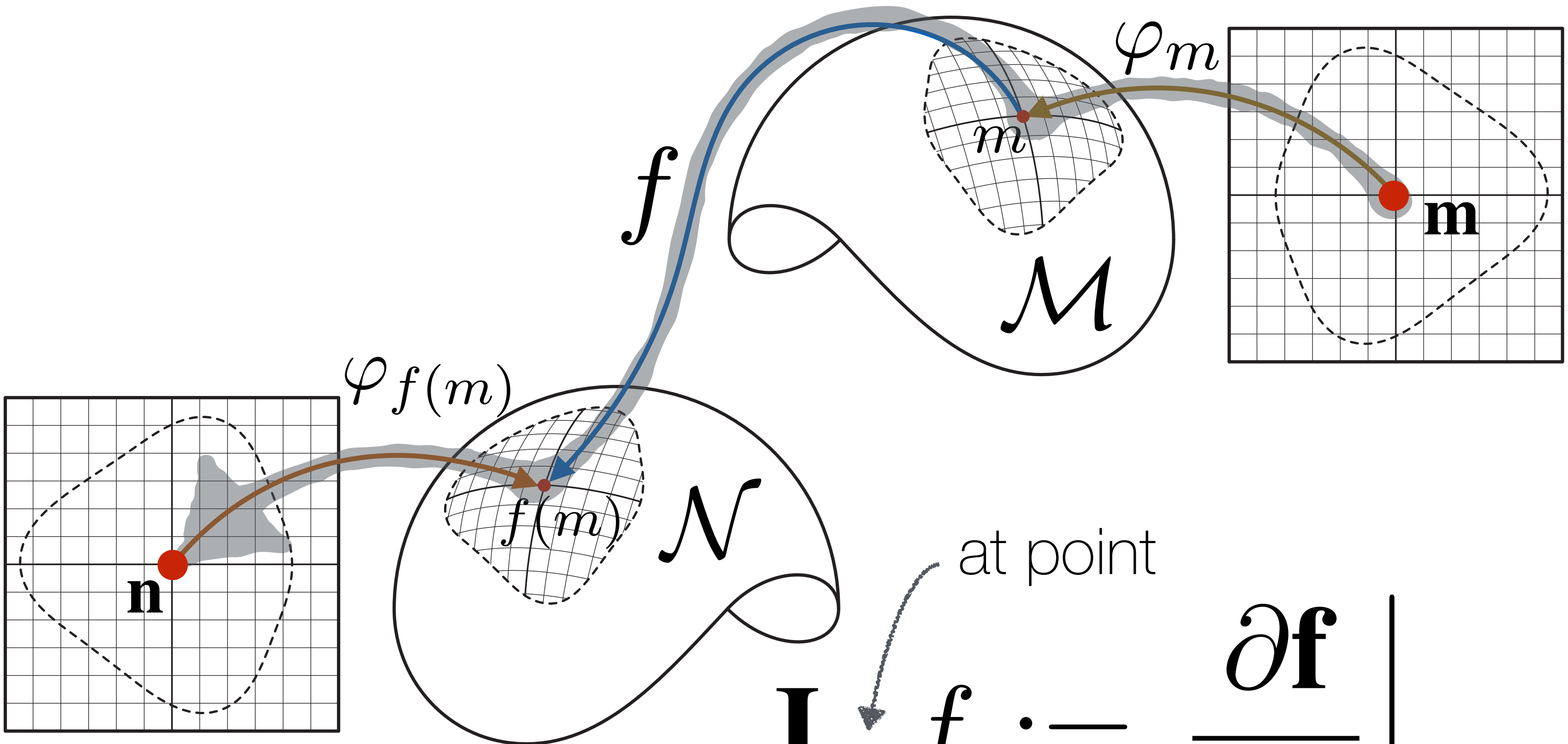


φ_m
m
f
$\mathcal{M}$
$\varphi_{f(m)}$
$f(m)$
$\mathcal{N}$

$$\mathbf{f} := \varphi_{f(m)}^{-1} \circ f \circ \varphi_m$$



$$\mathbf{f} := \varphi_{f(m)}^{-1} \circ f \circ \varphi_m$$



at point

$$\mathbf{J}_m f := \left.\frac{\partial \mathbf{f}}{\partial \mathbf{m}}\right|_{\varphi_m^{-1}(m)}$$

Representing matrix
of the differential

of mapping

Calculus

$$\left.\frac{\partial \mathbf{e}(\mathbf{x})}{\partial \mathbf{x}}\right|_{\overline{\mathbf{x}}}$$

Nonlinear Least Squares / Gauss Newton

$$\text{Goal: estimate } \mathbf{x}^\star = \underset{\mathbf{x}}{\text{argmin}}\ J(\mathbf{x}) = \frac{1}{2}\mathbf{e}(\mathbf{x})^T\mathbf{R}^{-1}\mathbf{e}(\mathbf{x}).$$

- Start with an initial guess for $\mathbf{x}$
- Iterate
  - Linearize
  - Solve the linear system
  - Update the parameters
  - Repeat until convergence

$$\bar{\mathbf{e}} := \mathbf{e}(\bar{\mathbf{x}}), \quad \mathbf{E} := \left.\frac{\partial \mathbf{e}(\mathbf{x})}{\partial \mathbf{x}}\right|_{\bar{\mathbf{x}}}, \quad \mathbf{e}(\boldsymbol{\epsilon}_x) := \bar{\mathbf{e}} + \mathbf{E}\boldsymbol{\epsilon}_x$$

$$\mathbf{E}^T\mathbf{R}^{-1}\mathbf{E}\boldsymbol{\epsilon}_x^\star = -\mathbf{E}^T\mathbf{R}^{-1}\bar{\mathbf{e}}$$

$$\bar{\mathbf{x}} \leftarrow \bar{\mathbf{x}} + \boldsymbol{\epsilon}_x^\star$$

Nonlinear Least Squares / Gauss Newton

Goal: estimate $\mathbf{x}^\star = \underset{\mathbf{x}}{\operatorname{argmin}}\ J(\mathbf{x}) = \frac{1}{2}\mathbf{e}(\mathbf{x})^T\mathbf{R}^{-1}\mathbf{e}(\mathbf{x}).$

Use the representing matrix and the chain rule

- Start with an initial guess for $\mathbf{x}$
- Iterate
  - Linearize
  - Solve the linear system
  - Update the parameters
  - Repeat until convergence

$$\bar{\mathbf{e}} := \mathbf{e}(\bar{\mathbf{x}}), \quad \boxed{\mathbf{E} := \left.\frac{\partial \mathbf{e}(\mathbf{x})}{\partial \mathbf{x}}\right|_{\bar{\mathbf{x}}}} \quad \mathbf{e}(\boldsymbol{\epsilon}_x) := \bar{\mathbf{e}} + \mathbf{E}\boldsymbol{\epsilon}_x$$

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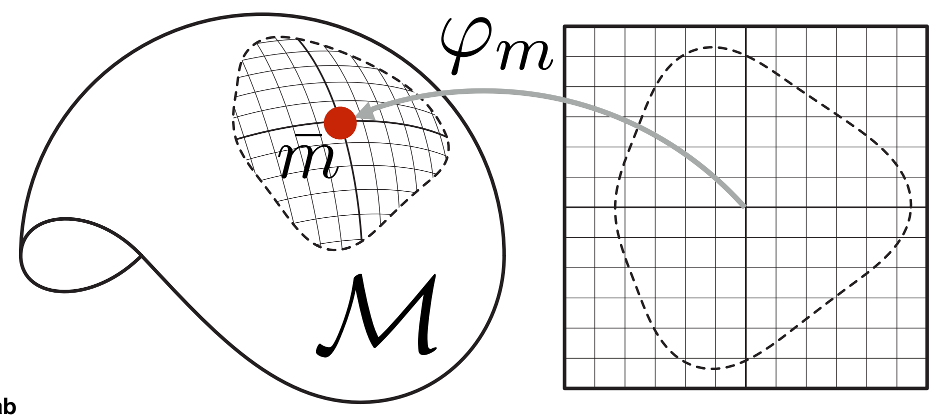

Nonlinear Least Squares / Gauss Newton

$$\text{Goal: estimate } \mathbf{x}^{\star} = \underset{\mathbf{x}}{\operatorname{argmin}}\, J(\mathbf{x}) = \frac{1}{2}\mathbf{e}(\mathbf{x})^T\mathbf{R}^{-1}\mathbf{e}(\mathbf{x}).$$

- Start with an initial guess for $\mathbf{x}$
- Iterate
  - Linearize
  - Solve the linear system
  - Update the parameters
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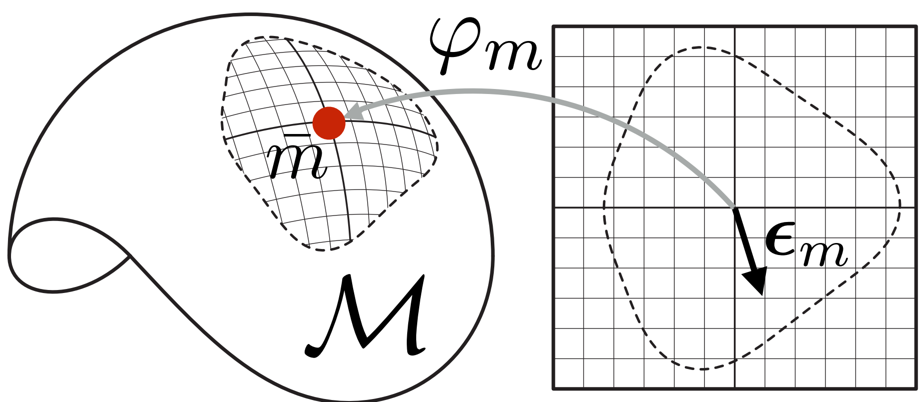
$$\bar{\mathbf{e}} := \mathbf{e}(\bar{\mathbf{x}}), \quad \mathbf{E} := \left.\frac{\partial \mathbf{e}(\mathbf{x})}{\partial \mathbf{x}}\right|_{\bar{\mathbf{x}}}, \quad \mathbf{e}(\boldsymbol{\epsilon}_x) := \bar{\mathbf{e}} + \mathbf{E}\boldsymbol{\epsilon}_x$$

$$\boxed{\mathbf{E}^T\mathbf{R}^{-1}\mathbf{E}\boldsymbol{\epsilon}_x^{\star} = -\mathbf{E}^T\mathbf{R}^{-1}\bar{\mathbf{e}}}$$

$$\bar{\mathbf{x}} \leftarrow \bar{\mathbf{x}} + \boldsymbol{\epsilon}_x^{\star}$$



Solve for the update

Nonlinear Least Squares / Gauss Newton

$$\text{Goal: estimate } \mathbf{x}^\star = \underset{\mathbf{x}}{\operatorname{argmin}}\ J(\mathbf{x}) = \frac{1}{2}\mathbf{e}(\mathbf{x})^T\mathbf{R}^{-1}\mathbf{e}(\mathbf{x}).$$

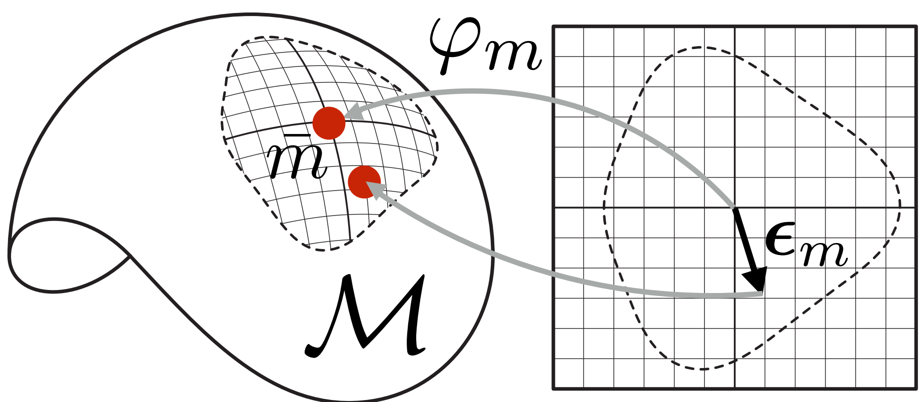
- Start with an initial guess for $\mathbf{x}$
- Iterate
  - Linearize
  - Solve the linear system
  - Update the parameters
  - Repeat until convergence

$$\bar{\mathbf{e}} := \mathbf{e}(\bar{\mathbf{x}}), \quad \mathbf{E} := \left.\frac{\partial \mathbf{e}(\mathbf{x})}{\partial \mathbf{x}}\right|_{\bar{\mathbf{x}}}, \quad \mathbf{e}(\boldsymbol{\epsilon}_x) := \bar{\mathbf{e}} + \mathbf{E}\boldsymbol{\epsilon}_x$$

$$\mathbf{E}^T\mathbf{R}^{-1}\mathbf{E}\boldsymbol{\epsilon}_x^\star = -\mathbf{E}^T\mathbf{R}^{-1}\bar{\mathbf{e}}$$

$$\boxed{\bar{\mathbf{x}} \leftarrow \bar{\mathbf{x}} + \boldsymbol{\epsilon}_x^\star}$$

$$\bar{m} \leftarrow \varphi_m\left(\varphi_m^{-1}(\bar{m}) + \boldsymbol{\epsilon}_m\right)$$



Apply the update

Linear Algebra

$$\boldsymbol{\epsilon}_x^\star = -\left(\mathbf{E}^T\mathbf{R}^{-1}\mathbf{E}\right)^{-1}\mathbf{E}^T\mathbf{R}^{-1}\overline{\mathbf{e}}$$

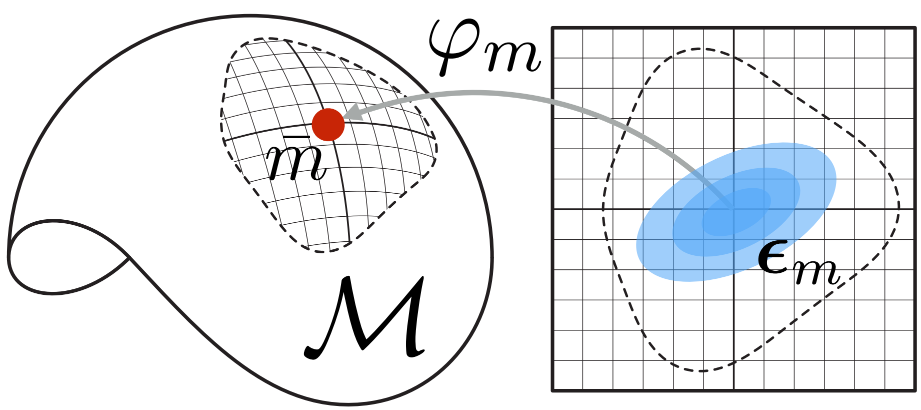
Nonlinear Least Squares / Gauss Newton

Recover the covariance:

$$\mathbf{P} := \left(\mathbf{E}^T \mathbf{R}^{-1} \mathbf{E}\right)^{-1},$$

$$\mathbf{x} = \mathbf{x}^{\star} + \boldsymbol{\epsilon}_x, \quad \boldsymbol{\epsilon}_x \sim \mathcal{N}\left(\mathbf{0}, \mathbf{P}\right).$$

$$m = \varphi_m\left(\varphi_m^{-1}(\bar{m}) + \boldsymbol{\epsilon}_m\right)$$



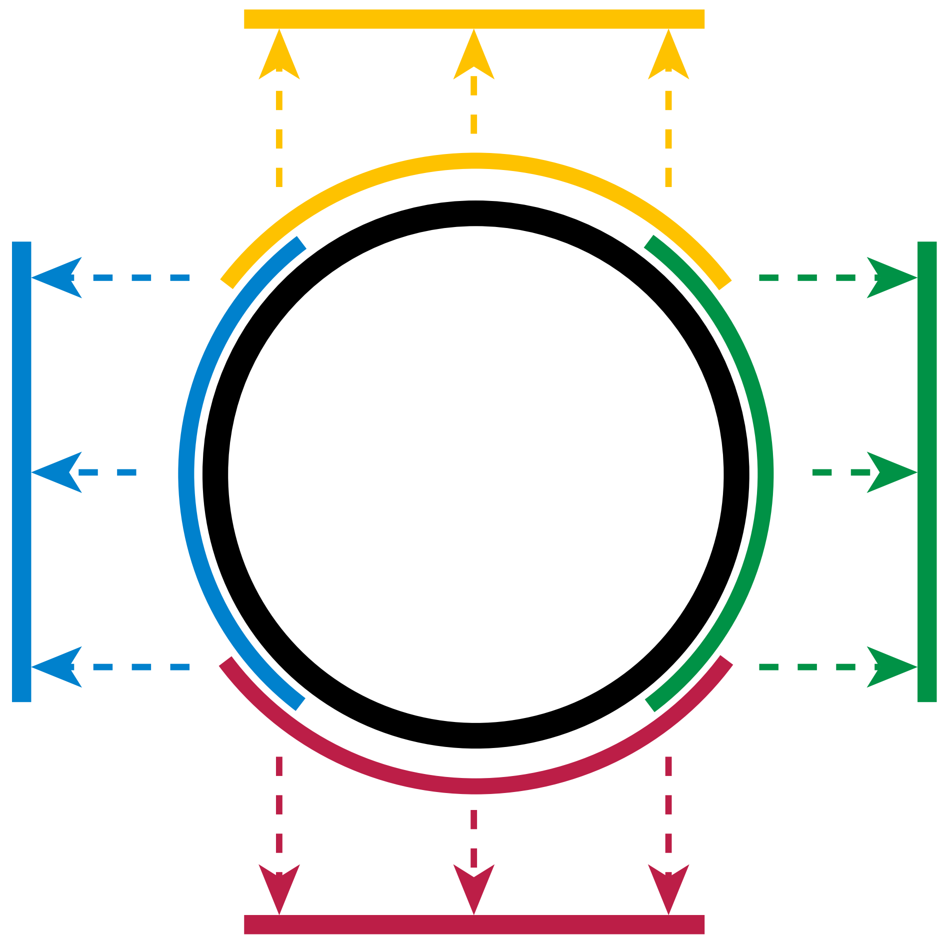
Probabilities

$$\mathbf{P} := \left(\mathbf{E}^T\mathbf{R}^{-1}\mathbf{E}\right)^{-1},$$

$$\mathbf{x} = \mathbf{x}^{\star} + \boldsymbol{\epsilon}_x, \quad \boldsymbol{\epsilon}_x \sim \mathcal{N}\left(\mathbf{0}, \mathbf{P}\right).$$

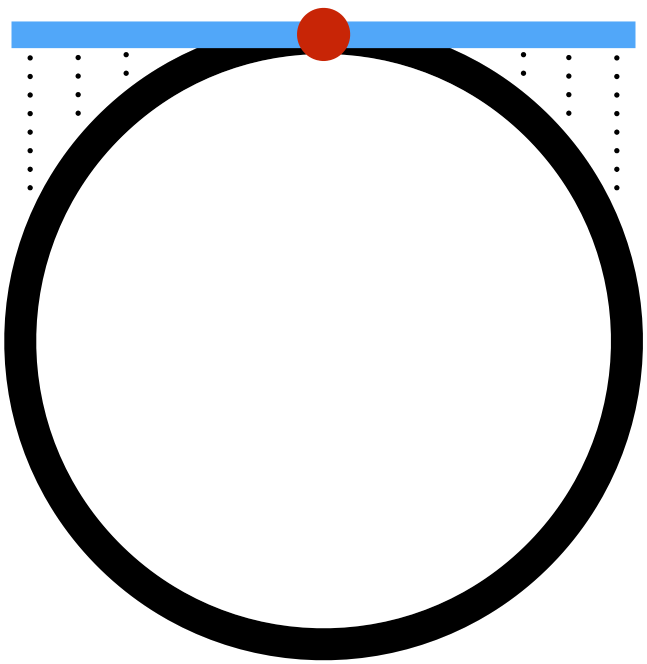
Choosing an Atlas

- You can choose the Atlas you want
- Different choices will have benefits and drawbacks



Point-Wise Charts

- It is possible to define an atlas by specifying point-wise charts



Point-Wise Charts

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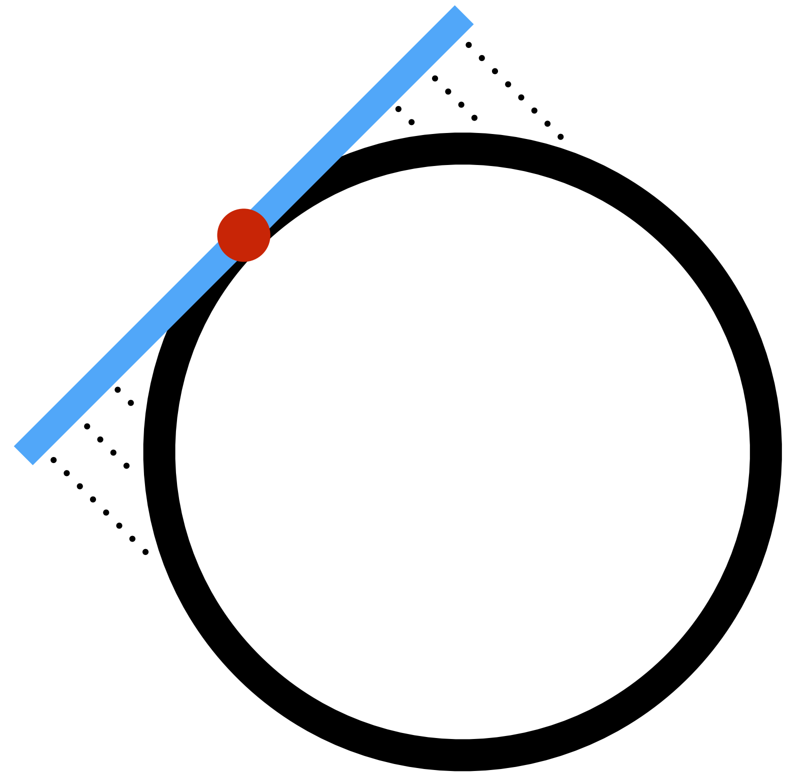
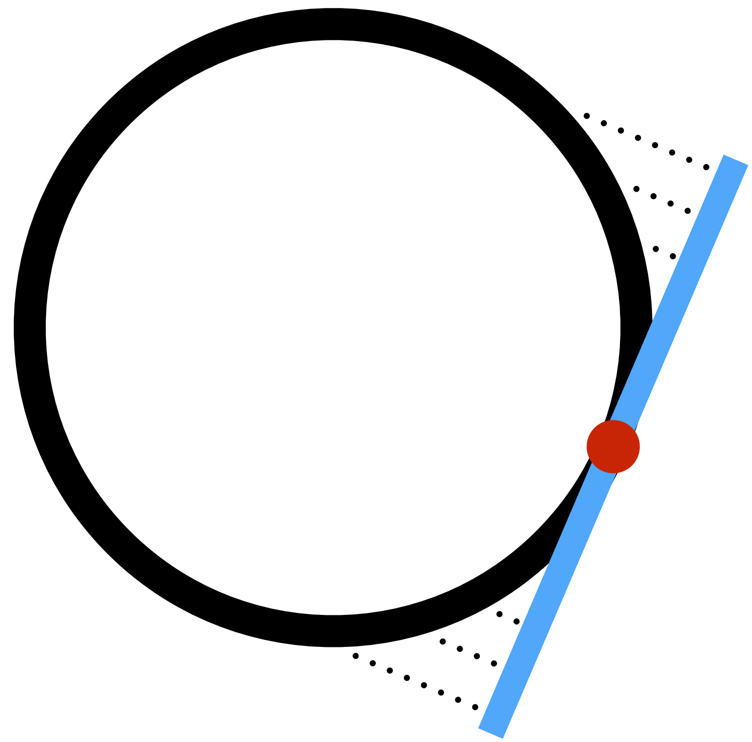


Point-Wise Charts

- It is possible to define an atlas by specifying point-wise charts
- Benefits:
  - Locally the best approximation to the manifold
  - Simplifies the computation of representing matrices



A Brief Introduction to Manifolds: Summary

Choosing an Atlas

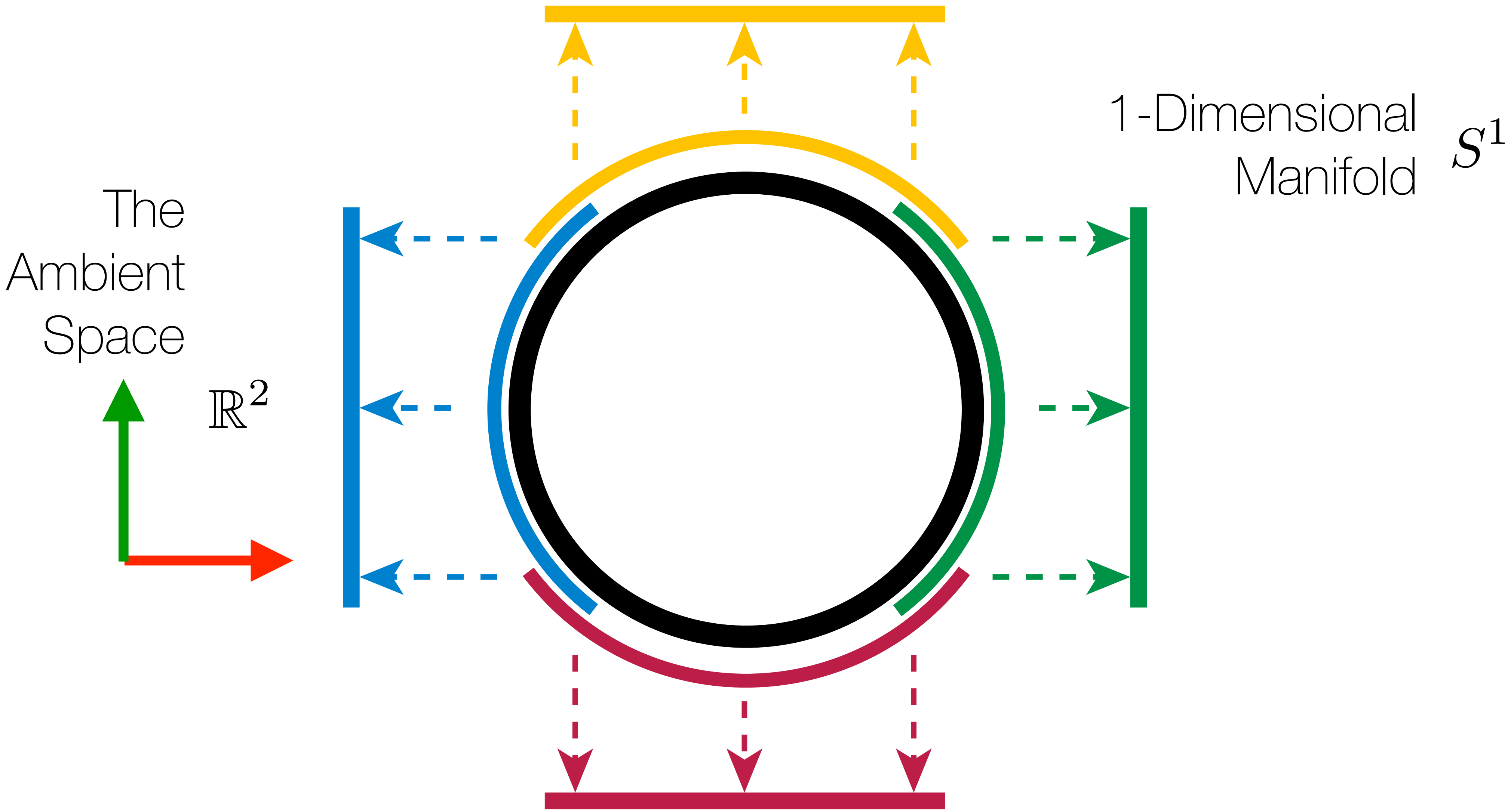
- Charts:
  - For an M-Dimensional DM, $\mathcal{M}$, a *chart* is

Invertible Mapping

The Manifold

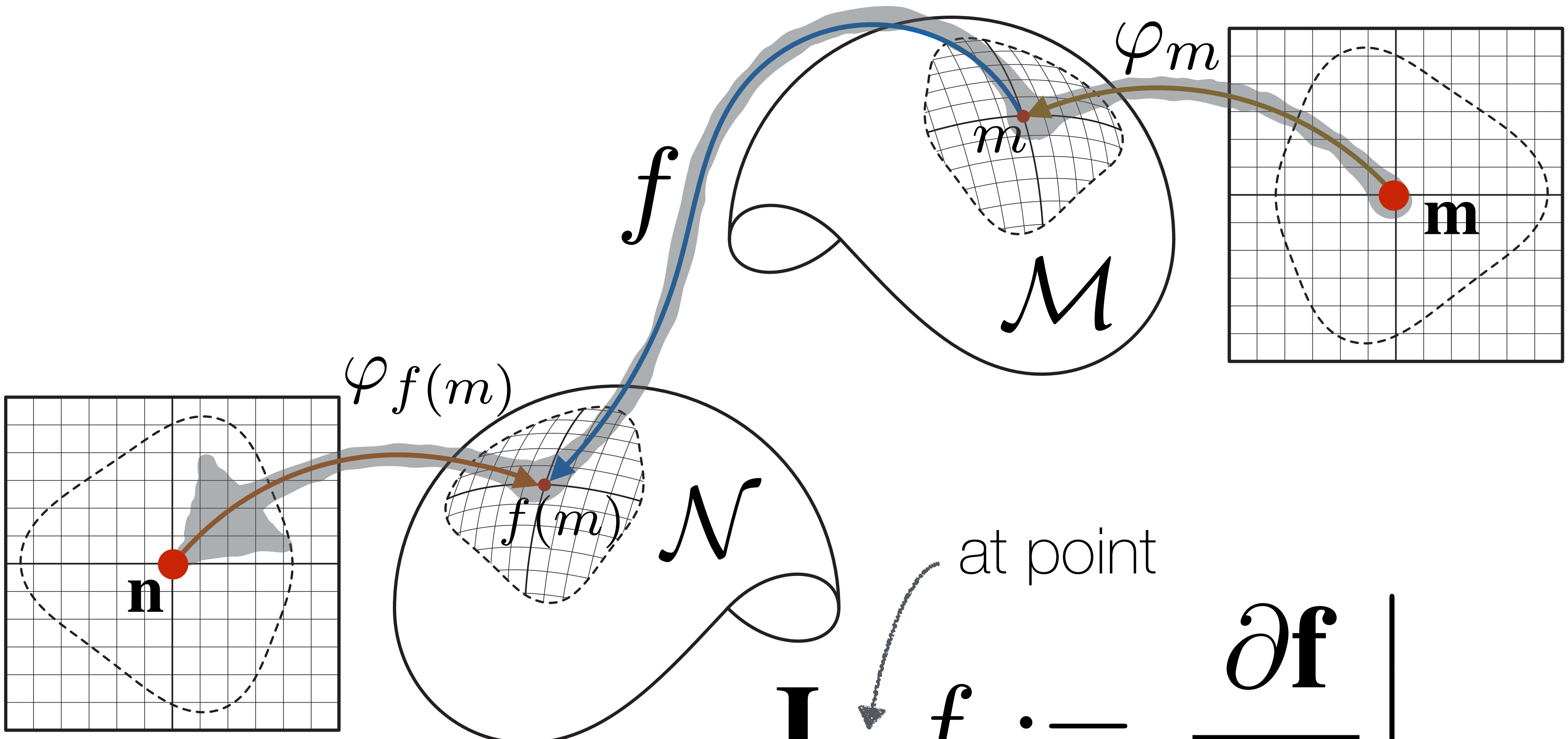
$$\varphi : U_\varphi \to \mathcal{M}$$

Open subset of $\mathbb{R}^M$



- An *Atlas* is a collection of charts such that cover the manifold

$$\mathbf{f} := \varphi_{f(m)}^{-1} \circ f \circ \varphi_m$$



at point

$$\mathbf{J}_m f := \left.\frac{\partial \mathbf{f}}{\partial \mathbf{m}}\right|_{\varphi_m^{-1}(m)}$$

Representing matrix
of the differential

of mapping

$$C_{\mathtt{AB}}$$

SO(3)

The Special Orthogonal Group

The Special Orthogonal Group

Ambient Space

The Identity Matrix

$$SO(3) := \left\{ \mathbf{C} \in \mathbb{R}^{3\times 3} \,\middle|\, \mathbf{C}^T\mathbf{C} = \mathbf{1}, \det \mathbf{C} = 1 \right\}$$

$$\mathbf{C}^T \equiv \mathbf{C}^{-1}$$

- SO(3) is a group, closed under standard matrix multiplication

Atlases Covering SO(3)

- There are *many* atlases that cover SO(3)
  - Compositions of fundamental rotation matrices (Euler angles, Tait-Bryan angles)
  - Cayley rotational parameters
  - Axis/angle
  - Euler/Rodriguez parameters
- Each of these atlases has a singularity

Atlases Covering SO(3)

- A singlarity-free point-wise atlas:

$$\varphi_{\mathrm{C}}(\boldsymbol{\phi}) := \exp(\boldsymbol{\phi}^\wedge)\mathbf{C}$$

Skew-symmetric matrix

$$\begin{bmatrix}\phi_1\\\phi_2\\\phi_3\end{bmatrix}^\wedge = \begin{bmatrix}0 & -\phi_3 & \phi_2\\\phi_3 & 0 & -\phi_1\\-\phi_2 & \phi_1 & 0\end{bmatrix}$$

The Exponential Map of SO(3)

$$\exp(\boldsymbol{\phi}^{\wedge})$$

The Exponential Map of SO(3)

$$\exp(\boldsymbol{\phi}^\wedge)$$

$$\exp(\boldsymbol{\phi}^\wedge) \in SO(3)$$

The Exponential Map of SO(3)

$$\exp(\boldsymbol{\phi}^\wedge)$$

$$\exp(\boldsymbol{\phi}^\wedge)\mathbf{C} \in SO(3)$$

The Exponential Map of SO(3)

$$\exp(\boldsymbol{\phi}^\wedge)$$

$$\exp(\boldsymbol{\phi}^\wedge) = \sum_{n=0}^{\infty} \frac{1}{n!} (\boldsymbol{\phi}^\wedge)^n$$

The Exponential Map of SO(3)

$$\exp(\boldsymbol{\phi}^\wedge)$$

$$\phi = \|\boldsymbol{\phi}\|, \quad \mathbf{a} = \boldsymbol{\phi}/\phi$$

$$\exp(\boldsymbol{\phi}^\wedge) = \cos\phi\mathbf{1} + (1-\cos\phi)\mathbf{a}\mathbf{a}^T + \sin\phi\mathbf{a}^\wedge$$

The Exponential Map of SO(3)

$$\exp(\boldsymbol{\phi}^\wedge)$$

$$\exp(\boldsymbol{\phi}^\wedge) = \sum_{n=0}^{\infty} \frac{1}{n!} (\boldsymbol{\phi}^\wedge)^n \approx \mathbf{1} + \boldsymbol{\phi}^\wedge$$

A first-order approximation

The Exponential Map of SO(3)

$$\log(\mathbf{C})$$

$$\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}^\wedge = \begin{bmatrix} 0 & -\phi_3 & \phi_2 \\ \phi_3 & 0 & -\phi_1 \\ -\phi_2 & \phi_1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -\phi_3 & \phi_2 \\ \phi_3 & 0 & -\phi_1 \\ -\phi_2 & \phi_1 & 0 \end{bmatrix}^\vee = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}$$

The Exponential Map of SO(3)

$$\log(\mathbf{C})$$

$$\theta = \arccos\left(\frac{\operatorname{trace}(\mathbf{C}) - 1}{2}\right)$$

$$\log(\mathbf{C}) = \left(\frac{\theta}{2\sin\theta}\left(\mathbf{C} - \mathbf{C}^T\right)\right)^{\vee}$$

The Left-Exponential-Map Atlas for SO(3)

- A point-wise atlas:

$$\varphi_{\mathrm{C}}(\boldsymbol{\phi}) := \exp(\boldsymbol{\phi}^{\wedge})\mathbf{C}$$

The Left-Exponential-Map Atlas for SO(3)

- A point-wise atlas:

$$\varphi_{\mathrm{C}}(\boldsymbol{\phi}) := \exp(\boldsymbol{\phi}^{\wedge})\mathbf{C}$$

$$\varphi_{\mathrm{C}}^{-1}(\mathbf{R}) := \log(\mathbf{R}\mathbf{C}^{T})$$

The Left-Exponential-Map Atlas for SO(3)

- A point-wise atlas:

$$\varphi_{\mathrm{C}}(\boldsymbol{\phi}) := \exp(\boldsymbol{\phi}^{\wedge})\mathbf{C}$$

$$\varphi_{\mathrm{C}}^{-1}(\mathbf{R}) := \log(\mathbf{R}\mathbf{C}^{T})$$

$$\varphi_{\mathrm{C}}^{-1}(\mathbf{C}) = \mathbf{0}$$

The Left-Exponential-Map Atlas for SO(3)

- A point-wise atlas:

$$\begin{aligned}\varphi_{\mathbf{C}}(\boldsymbol{\phi}) &:= \exp(\boldsymbol{\phi}^{\wedge})\mathbf{C}\\ &\approx (\mathbf{1} + \boldsymbol{\phi}^{\wedge})\mathbf{C}\end{aligned}$$

The Left-Exponential-Map Atlas for SO(3)

$$\varphi_{\mathbf{C}}(\boldsymbol{\phi}) := \exp(\boldsymbol{\phi}^{\wedge})\mathbf{C}$$
$$\approx (\mathbf{1} + \boldsymbol{\phi}^{\wedge})\mathbf{C}$$

- Benefits of the left-exponential-map atlas
  - Extremely simple linearized expression
  - Identities for manipulating expressions of this form
  - Known first order relationship with other rotation parameterizations

Fundamental Identities (Cheat Sheet)

$$\mathbf{a}^{\wedge}\mathbf{b} = -\mathbf{b}^{\wedge}\mathbf{a}$$

$$\mathbf{C}\mathbf{a}^{\wedge} = (\mathbf{C}\mathbf{a})^{\wedge}\mathbf{C}$$

$$\mathbf{a}^{\wedge}\mathbf{C} = \mathbf{C}(\mathbf{C}^{T}\mathbf{a})^{\wedge}$$

$$(\mathbf{a}^{\wedge})^{T} = -\mathbf{a}^{\wedge}$$

Linearizing Expressions

$$
\begin{aligned}
\mathbf{a}^\wedge \mathbf{b} &= -\mathbf{b}^\wedge \mathbf{a} \\
\mathbf{C}\mathbf{a}^\wedge &= (\mathbf{C}\mathbf{a})^\wedge \mathbf{C} \\
\mathbf{a}^\wedge \mathbf{C} &= \mathbf{C}(\mathbf{C}^T \mathbf{a})^\wedge \\
(\mathbf{a}^\wedge)^T &= -\mathbf{a}^\wedge
\end{aligned}
$$

- When someone gives you a rotation expression, you can linearize by (1) substituting in the linearized form (based on your chosen atlas)

$$\mathbf{C}_i = (\mathbf{1} + \boldsymbol{\phi}_i^\wedge)\overline{\mathbf{C}}_i$$

The perturbation   The “large” rotation

Linearizing Expressions

$$\mathbf{a}^\wedge \mathbf{b} = -\mathbf{b}^\wedge \mathbf{a}$$
$$\mathbf{C}\mathbf{a}^\wedge = (\mathbf{C}\mathbf{a})^\wedge \mathbf{C}$$
$$\mathbf{a}^\wedge \mathbf{C} = \mathbf{C}(\mathbf{C}^T \mathbf{a})^\wedge$$
$$(\mathbf{a}^\wedge)^T = -\mathbf{a}^\wedge$$

- Then you can rearrange the expression to bring it back into the form below. Use these two tools:
  1. Apply the fundamental identities
  2. Drop the products of “small” perturbations (this is consistent with the first order approximation).
- The result implicitly recovers the representing matrix

$$\mathbf{C}_i = (\mathbf{1} + \phi_i^\wedge)\overline{\mathbf{C}}_i$$

Example: Transpose

$$\mathbf{a}^\wedge \mathbf{b} = -\mathbf{b}^\wedge \mathbf{a}$$
$$\mathbf{C}\mathbf{a}^\wedge = (\mathbf{C}\mathbf{a})^\wedge \mathbf{C}$$
$$\mathbf{a}^\wedge \mathbf{C} = \mathbf{C}(\mathbf{C}^T \mathbf{a})^\wedge$$
$$(\mathbf{a}^\wedge)^T = -\mathbf{a}^\wedge$$

$$\mathbf{C}^T \approx \left((\mathbf{1} + \phi^\wedge)\overline{\mathbf{C}}\right)^T$$

Example: Transpose

$$\mathbf{a}^\wedge \mathbf{b} = -\mathbf{b}^\wedge \mathbf{a}$$

$$\mathbf{C}\mathbf{a}^\wedge = (\mathbf{C}\mathbf{a})^\wedge \mathbf{C}$$

$$\mathbf{a}^\wedge \mathbf{C} = \mathbf{C}(\mathbf{C}^T \mathbf{a})^\wedge$$

$$(\mathbf{a}^\wedge)^T = -\mathbf{a}^\wedge$$

$$\begin{aligned} \mathbf{C}^T &\approx \left((\mathbf{1} + \phi^\wedge)\overline{\mathbf{C}}\right)^T \\ &= \overline{\mathbf{C}}^T (\mathbf{1} + \phi^\wedge)^T \end{aligned}$$

Example: Transpose

$$\mathbf{a}^\wedge \mathbf{b} = -\mathbf{b}^\wedge \mathbf{a}$$
$$\mathbf{C}\mathbf{a}^\wedge = (\mathbf{C}\mathbf{a})^\wedge \mathbf{C}$$
$$\mathbf{a}^\wedge \mathbf{C} = \mathbf{C}(\mathbf{C}^T \mathbf{a})^\wedge$$
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$$\begin{aligned}
\mathbf{C}^T &\approx \left((\mathbf{1} + \phi^\wedge)\overline{\mathbf{C}}\right)^T \\
&= \overline{\mathbf{C}}^T (\mathbf{1} + \phi^\wedge)^T \\
&= \overline{\mathbf{C}}^T (\mathbf{1}^T + (\phi^\wedge)^T)
\end{aligned}$$

Example: Transpose

$$\mathbf{a}^\wedge \mathbf{b} = -\mathbf{b}^\wedge \mathbf{a}$$
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$$\begin{aligned}
\mathbf{C}^T &\approx \left((\mathbf{1} + \phi^\wedge)\overline{\mathbf{C}}\right)^T \\
&= \overline{\mathbf{C}}^T (\mathbf{1} + \phi^\wedge)^T \\
&= \overline{\mathbf{C}}^T (\mathbf{1}^T + (\phi^\wedge)^T) \\
&= \overline{\mathbf{C}}^T (\mathbf{1} - \phi^\wedge)
\end{aligned}$$

Example: Transpose

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\mathbf{C}^T &\approx \left((\mathbf{1} + \phi^\wedge)\overline{\mathbf{C}}\right)^T \\
&= \overline{\mathbf{C}}^T (\mathbf{1} + \phi^\wedge)^T \\
&= \overline{\mathbf{C}}^T \left(\mathbf{1}^T + (\phi^\wedge)^T\right) \\
&= \boxed{\overline{\mathbf{C}}^T (\mathbf{1} - \phi^\wedge)}
\end{aligned}$$

Example: Transpose

$$\mathbf{a}^\wedge \mathbf{b} = -\mathbf{b}^\wedge \mathbf{a}$$
$$\mathbf{C}\mathbf{a}^\wedge = (\mathbf{C}\mathbf{a})^\wedge \mathbf{C}$$
$$\mathbf{a}^\wedge \mathbf{C} = \mathbf{C}(\mathbf{C}^T \mathbf{a})^\wedge$$
$$(\mathbf{a}^\wedge)^T = -\mathbf{a}^\wedge$$

$$\begin{aligned}
\mathbf{C}^T &\approx \left((\mathbf{1} + \boldsymbol{\phi}^\wedge)\overline{\mathbf{C}}\right)^T \\
&= \overline{\mathbf{C}}^T (\mathbf{1} + \boldsymbol{\phi}^\wedge)^T \\
&= \overline{\mathbf{C}}^T (\mathbf{1}^T + (\boldsymbol{\phi}^\wedge)^T) \\
&= \overline{\mathbf{C}}^T (\mathbf{1} - \boldsymbol{\phi}^\wedge) \\
&= \overline{\mathbf{C}}^T - \overline{\mathbf{C}}^T \boldsymbol{\phi}^\wedge
\end{aligned}$$

Example: Transpose

$$\mathbf{a}^\wedge\mathbf{b} = -\mathbf{b}^\wedge\mathbf{a}$$
$$\mathbf{C}\mathbf{a}^\wedge = (\mathbf{C}\mathbf{a})^\wedge\mathbf{C}$$
$$\mathbf{a}^\wedge\mathbf{C} = \mathbf{C}(\mathbf{C}^T\mathbf{a})^\wedge$$
$$(\mathbf{a}^\wedge)^T = -\mathbf{a}^\wedge$$

$$\begin{aligned}
\mathbf{C}^T &\approx \left((\mathbf{1}+\phi^\wedge)\overline{\mathbf{C}}\right)^T \\
&= \overline{\mathbf{C}}^T(\mathbf{1}+\phi^\wedge)^T \\
&= \overline{\mathbf{C}}^T(\mathbf{1}^T+(\phi^\wedge)^T) \\
&= \overline{\mathbf{C}}^T(\mathbf{1}-\phi^\wedge) \\
&= \overline{\mathbf{C}}^T-\overline{\mathbf{C}}^T\phi^\wedge \\
&= \overline{\mathbf{C}}^T-(\overline{\mathbf{C}}^T\phi)^\wedge\overline{\mathbf{C}}^T
\end{aligned}$$

Example: Transpose

$$\mathbf{a}^\wedge \mathbf{b} = -\mathbf{b}^\wedge \mathbf{a}$$
$$\mathbf{C}\mathbf{a}^\wedge = (\mathbf{C}\mathbf{a})^\wedge \mathbf{C}$$
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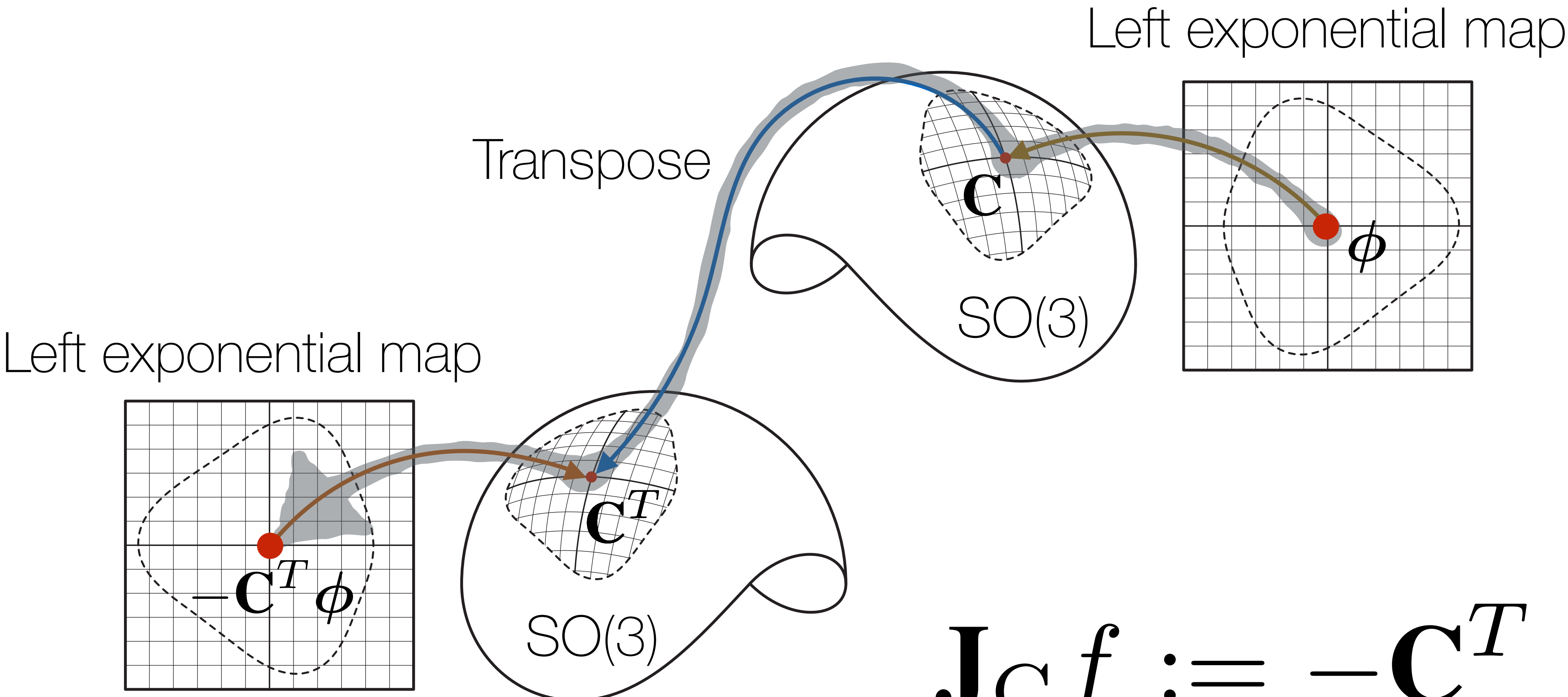
\mathbf{C}^T &\approx \left((\mathbf{1} + \phi^\wedge)\overline{\mathbf{C}}\right)^T \\
&= \overline{\mathbf{C}}^T (\mathbf{1} + \phi^\wedge)^T \\
&= \overline{\mathbf{C}}^T (\mathbf{1}^T + (\phi^\wedge)^T) \\
&= \overline{\mathbf{C}}^T (\mathbf{1} - \phi^\wedge) \\
&= \overline{\mathbf{C}}^T - \overline{\mathbf{C}}^T \phi^\wedge \\
&= \overline{\mathbf{C}}^T - (\overline{\mathbf{C}}^T \phi)^\wedge \overline{\mathbf{C}}^T \\
&= (\mathbf{1} + (-\overline{\mathbf{C}}^T \phi)^\wedge)\overline{\mathbf{C}}^T
\end{aligned}$$



$$\mathbf{J}_{\mathrm{C}} f := -\mathbf{C}^T$$

Example: Multiplication

$$\mathbf{a}^\wedge\mathbf{b} = -\mathbf{b}^\wedge\mathbf{a}$$
$$\mathbf{C}\mathbf{a}^\wedge = (\mathbf{C}\mathbf{a})^\wedge\mathbf{C}$$
$$\mathbf{a}^\wedge\mathbf{C} = \mathbf{C}(\mathbf{C}^T\mathbf{a})^\wedge$$
$$(\mathbf{a}^\wedge)^T = -\mathbf{a}^\wedge$$

$$\mathbf{C}_1\mathbf{C}_2 \approx (\mathbf{1} + \boldsymbol{\phi}_1^\wedge)\overline{\mathbf{C}}_1(\mathbf{1} + \boldsymbol{\phi}_2^\wedge)\overline{\mathbf{C}}_2$$

Example: Multiplication

$$\mathbf{a}^\wedge \mathbf{b} = -\mathbf{b}^\wedge \mathbf{a}$$
$$\mathbf{C}\mathbf{a}^\wedge = (\mathbf{C}\mathbf{a})^\wedge \mathbf{C}$$
$$\mathbf{a}^\wedge \mathbf{C} = \mathbf{C}(\mathbf{C}^T \mathbf{a})^\wedge$$
$$(\mathbf{a}^\wedge)^T = -\mathbf{a}^\wedge$$

$$\begin{aligned}\mathbf{C}_1\mathbf{C}_2 &\approx (\mathbf{1} + \phi_1^\wedge)\overline{\mathbf{C}}_1(\mathbf{1} + \phi_2^\wedge)\overline{\mathbf{C}}_2 \\ &= \overline{\mathbf{C}}_1\overline{\mathbf{C}}_2 + \phi_1^\wedge\overline{\mathbf{C}}_1\overline{\mathbf{C}}_2 + \phi_1^\wedge\overline{\mathbf{C}}_1\phi_2^\wedge\overline{\mathbf{C}}_2 + \overline{\mathbf{C}}_1\phi_2^\wedge\overline{\mathbf{C}}_2\end{aligned}$$

Example: Multiplication

$$\mathbf{a}^\wedge \mathbf{b} = -\mathbf{b}^\wedge \mathbf{a}$$
$$\mathbf{C}\mathbf{a}^\wedge = (\mathbf{C}\mathbf{a})^\wedge \mathbf{C}$$
$$\mathbf{a}^\wedge \mathbf{C} = \mathbf{C}(\mathbf{C}^T \mathbf{a})^\wedge$$
$$(\mathbf{a}^\wedge)^T = -\mathbf{a}^\wedge$$

$$\begin{aligned}
\mathbf{C}_1\mathbf{C}_2 &\approx (\mathbf{1} + \phi_1^\wedge)\overline{\mathbf{C}}_1(\mathbf{1} + \phi_2^\wedge)\overline{\mathbf{C}}_2 \\
&= \overline{\mathbf{C}}_1\overline{\mathbf{C}}_2 + \phi_1^\wedge\overline{\mathbf{C}}_1\overline{\mathbf{C}}_2 + \phi_1^\wedge\overline{\mathbf{C}}_1\phi_2^\wedge\overline{\mathbf{C}}_2 + \overline{\mathbf{C}}_1\phi_2^\wedge\overline{\mathbf{C}}_2 \\
&\approx \overline{\mathbf{C}}_1\overline{\mathbf{C}}_2 + \phi_1^\wedge\overline{\mathbf{C}}_1\overline{\mathbf{C}}_2 + \overline{\mathbf{C}}_1\phi_2^\wedge\overline{\mathbf{C}}_2
\end{aligned}$$

Example: Multiplication

$$\begin{aligned}
\mathbf{a}^\wedge\mathbf{b} &= -\mathbf{b}^\wedge\mathbf{a} \\
\mathbf{C}\mathbf{a}^\wedge &= (\mathbf{C}\mathbf{a})^\wedge\mathbf{C} \\
\mathbf{a}^\wedge\mathbf{C} &= \mathbf{C}(\mathbf{C}^T\mathbf{a})^\wedge \\
(\mathbf{a}^\wedge)^T &= -\mathbf{a}^\wedge
\end{aligned}$$

$$\begin{aligned}
\mathbf{C}_1\mathbf{C}_2 &\approx (\mathbf{1} + \boldsymbol{\phi}_1^\wedge)\overline{\mathbf{C}}_1(\mathbf{1} + \boldsymbol{\phi}_2^\wedge)\overline{\mathbf{C}}_2 \\
&= \overline{\mathbf{C}}_1\overline{\mathbf{C}}_2 + \boldsymbol{\phi}_1^\wedge\overline{\mathbf{C}}_1\overline{\mathbf{C}}_2 + \boldsymbol{\phi}_1^\wedge\overline{\mathbf{C}}_1\boldsymbol{\phi}_2^\wedge\overline{\mathbf{C}}_2 + \overline{\mathbf{C}}_1\boldsymbol{\phi}_2^\wedge\overline{\mathbf{C}}_2 \\
&\approx \overline{\mathbf{C}}_1\overline{\mathbf{C}}_2 + \boldsymbol{\phi}_1^\wedge\overline{\mathbf{C}}_1\overline{\mathbf{C}}_2 + \overline{\mathbf{C}}_1\boldsymbol{\phi}_2^\wedge\overline{\mathbf{C}}_2 \\
&= \overline{\mathbf{C}}_1\overline{\mathbf{C}}_2 + \boldsymbol{\phi}_1^\wedge\overline{\mathbf{C}}_1\overline{\mathbf{C}}_2 + (\overline{\mathbf{C}}_1\boldsymbol{\phi}_2)^\wedge\overline{\mathbf{C}}_1\overline{\mathbf{C}}_2
\end{aligned}$$

Example: Multiplication

$$
\begin{aligned}
\mathbf{a}^\wedge \mathbf{b} &= -\mathbf{b}^\wedge \mathbf{a} \\
\mathbf{C}\mathbf{a}^\wedge &= (\mathbf{C}\mathbf{a})^\wedge \mathbf{C} \\
\mathbf{a}^\wedge \mathbf{C} &= \mathbf{C}(\mathbf{C}^T \mathbf{a})^\wedge \\
(\mathbf{a}^\wedge)^T &= -\mathbf{a}^\wedge
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{C}_1\mathbf{C}_2 &\approx (\mathbf{1} + \boldsymbol{\phi}_1^\wedge)\overline{\mathbf{C}}_1(\mathbf{1} + \boldsymbol{\phi}_2^\wedge)\overline{\mathbf{C}}_2 \\
&= \overline{\mathbf{C}}_1\overline{\mathbf{C}}_2 + \boldsymbol{\phi}_1^\wedge\overline{\mathbf{C}}_1\overline{\mathbf{C}}_2 + \boldsymbol{\phi}_1^\wedge\overline{\mathbf{C}}_1\boldsymbol{\phi}_2^\wedge\overline{\mathbf{C}}_2 + \overline{\mathbf{C}}_1\boldsymbol{\phi}_2^\wedge\overline{\mathbf{C}}_2 \\
&\approx \overline{\mathbf{C}}_1\overline{\mathbf{C}}_2 + \boldsymbol{\phi}_1^\wedge\overline{\mathbf{C}}_1\overline{\mathbf{C}}_2 + \overline{\mathbf{C}}_1\boldsymbol{\phi}_2^\wedge\overline{\mathbf{C}}_2 \\
&= \overline{\mathbf{C}}_1\overline{\mathbf{C}}_2 + \boldsymbol{\phi}_1^\wedge\overline{\mathbf{C}}_1\overline{\mathbf{C}}_2 + (\overline{\mathbf{C}}_1\boldsymbol{\phi}_2)^\wedge\overline{\mathbf{C}}_1\overline{\mathbf{C}}_2 \\
&= (\mathbf{1} + (\boldsymbol{\phi}_1 + \overline{\mathbf{C}}_1\boldsymbol{\phi}_2)^\wedge)\overline{\mathbf{C}}_1\overline{\mathbf{C}}_2
\end{aligned}
$$

Example: Multiplication with a Vector

$$\mathbf{a}^\wedge \mathbf{b} = -\mathbf{b}^\wedge \mathbf{a}$$

$$\mathbf{C}\mathbf{a}^\wedge = (\mathbf{C}\mathbf{a})^\wedge \mathbf{C}$$

$$\mathbf{a}^\wedge \mathbf{C} = \mathbf{C}(\mathbf{C}^T \mathbf{a})^\wedge$$

$$(\mathbf{a}^\wedge)^T = -\mathbf{a}^\wedge$$

$$\mathbf{C}\mathbf{v} \approx (\mathbf{1} + \boldsymbol{\phi}^\wedge)\overline{\mathbf{C}}(\overline{\mathbf{v}} + \boldsymbol{\epsilon}_v)$$

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$$(\mathbf{a}^\wedge)^T = -\mathbf{a}^\wedge$$

$$\begin{aligned}\mathbf{C}\mathbf{v} &\approx (\mathbf{1} + \boldsymbol{\phi}^\wedge)\overline{\mathbf{C}}(\overline{\mathbf{v}} + \boldsymbol{\epsilon}_v) \\ &= \overline{\mathbf{C}}\overline{\mathbf{v}} + \boldsymbol{\phi}^\wedge \overline{\mathbf{C}}\overline{\mathbf{v}} + \boldsymbol{\phi}^\wedge \overline{\mathbf{C}}\boldsymbol{\epsilon}_v + \overline{\mathbf{C}}\boldsymbol{\epsilon}_v\end{aligned}$$

Example: Multiplication with a Vector

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$$\begin{aligned}\mathbf{C}\mathbf{v} &\approx (\mathbf{1} + \boldsymbol{\phi}^\wedge)\overline{\mathbf{C}}(\overline{\mathbf{v}} + \boldsymbol{\epsilon}_v) \\ &= \overline{\mathbf{C}}\overline{\mathbf{v}} + \boldsymbol{\phi}^\wedge \overline{\mathbf{C}}\overline{\mathbf{v}} + \boldsymbol{\phi}^\wedge \overline{\mathbf{C}}\boldsymbol{\epsilon}_v + \overline{\mathbf{C}}\boldsymbol{\epsilon}_v \\ &\approx \overline{\mathbf{C}}\overline{\mathbf{v}} + \boldsymbol{\phi}^\wedge \overline{\mathbf{C}}\overline{\mathbf{v}} + \overline{\mathbf{C}}\boldsymbol{\epsilon}_v\end{aligned}$$

Example: Multiplication with a Vector

$$\mathbf{a}^\wedge\mathbf{b} = -\mathbf{b}^\wedge\mathbf{a}$$
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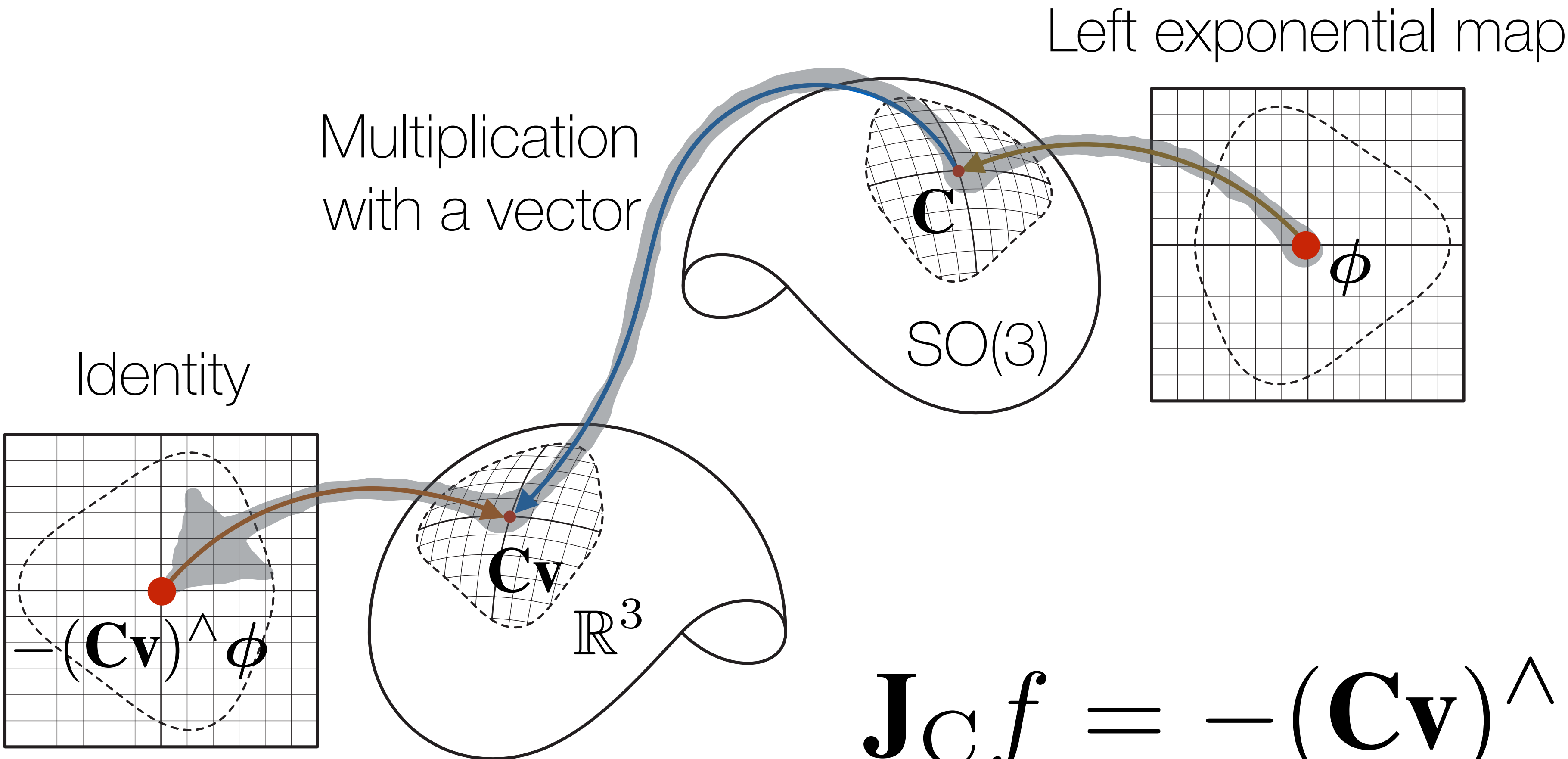
$$\begin{aligned}
\mathbf{C}\mathbf{v} &\approx (\mathbf{1} + \boldsymbol{\phi}^\wedge)\overline{\mathbf{C}}(\overline{\mathbf{v}} + \boldsymbol{\epsilon}_v) \\
&= \overline{\mathbf{C}}\overline{\mathbf{v}} + \boldsymbol{\phi}^\wedge\overline{\mathbf{C}}\overline{\mathbf{v}} + \boldsymbol{\phi}^\wedge\overline{\mathbf{C}}\boldsymbol{\epsilon}_v + \overline{\mathbf{C}}\boldsymbol{\epsilon}_v \\
&\approx \overline{\mathbf{C}}\overline{\mathbf{v}} + \boldsymbol{\phi}^\wedge\overline{\mathbf{C}}\overline{\mathbf{v}} + \overline{\mathbf{C}}\boldsymbol{\epsilon}_v \\
&= \overline{\mathbf{C}}\overline{\mathbf{v}} + (-\overline{\mathbf{C}}\overline{\mathbf{v}})^\wedge\boldsymbol{\phi} + \overline{\mathbf{C}}\boldsymbol{\epsilon}_v
\end{aligned}$$



$$\mathbf{J}_{\mathrm{C}} f = -(\mathbf{Cv})^\wedge$$

Checking your answers

- When developing code for optimization packages, always always always check your answers using finite differences
- General strategy:
  - Write down the exact charts that you are using
  - Implement the charts and the nonlinear function
  - Evaluate the Jacobian with respect to the charts
  - Estimate derivatives with:
    - http://www.mathworks.com/matlabcentral/fileexchange/13490-adaptive-robust-numerical-differentiation
  - Or, write unit tests for your functions

Checking your answers: Example

jacobianest

```
gradest: estimate of the Jacobian matrix of a vector valued function of n variables
usage: [jac,err] = jacobianest(fun,x0)


arguments: (input)
 fun - (vector valued) analytical function to differentiate.
       fun must be a function of the vector or array x0.

 x0  - vector location at which to differentiate fun
       If x0 is an nxm array, then fun is assumed to be
       a function of n*m variables.


arguments: (output)
 jac - array of first partial derivatives of fun.
       Assuming that x0 is a vector of length p
       and fun returns a vector of length n, then
       jac will be an array of size (n,p)

 err - vector of error estimates corresponding to
       each partial derivative in jac.
```

Checking your answers: Example

```
>> jacobianest( @(phi) so3log( (so3exp(phi)*C)'*C ), zeros(3,1) )
```

Checking your answers: Example

```
>> jacobianest( @(phi) so3log( (so3exp(phi)*C)'*C ), zeros(3,1) )
```

Matlab anonymous function of phi

Checking your answers: Example

$$\varphi_{\mathrm{C}}(\phi) := \exp(\phi^{\wedge})\mathbf{C}$$

```
>> jacobianest( @(phi) so3log( (so3exp(phi)*C)'*C ), zeros(3,1) )
```

Chart on the input

Checking your answers: Example

$$\varphi_{\mathrm{C}}^{-1}(\mathbf{C}) = \mathbf{0}$$

```
>> jacobianest( @(phi) so3log( (so3exp(phi)*C)'*C ), zeros(3,1) )
```

The linearization point of the chart on the input

Checking your answers: Example

$$\mathbf{C}^T$$

```
>> jacobianest( @(phi) so3log( (so3exp(phi)*C)'*C ), zeros(3,1) )
```

The function (transpose on the result)

Checking your answers: Example

$$\varphi_{\mathrm{C}}^{-1}(\mathbf{R}) := \log(\mathbf{R}\mathbf{C}^T)$$

```
>> jacobianest( @(phi) so3log( (so3exp(phi)*C)'*C ), zeros(3,1) )
```

The inverse chart of the result

Checking your answers: Example

```
>> jacobianest( @(phi) so3log( (so3exp(phi)*C)'*C ), zeros(3,1) )

ans =

   -0.6315   -0.4226    0.6501
   -0.2276   -0.7005   -0.6764
   -0.7413    0.5751   -0.3462

>> C'

ans =

    -0.6315  -0.4226    0.6501
    -0.2276  -0.7005   -0.6764
    -0.7413   0.5751   -0.3462
```

$$\mathbf{C}_{\mathtt{AB}}$$

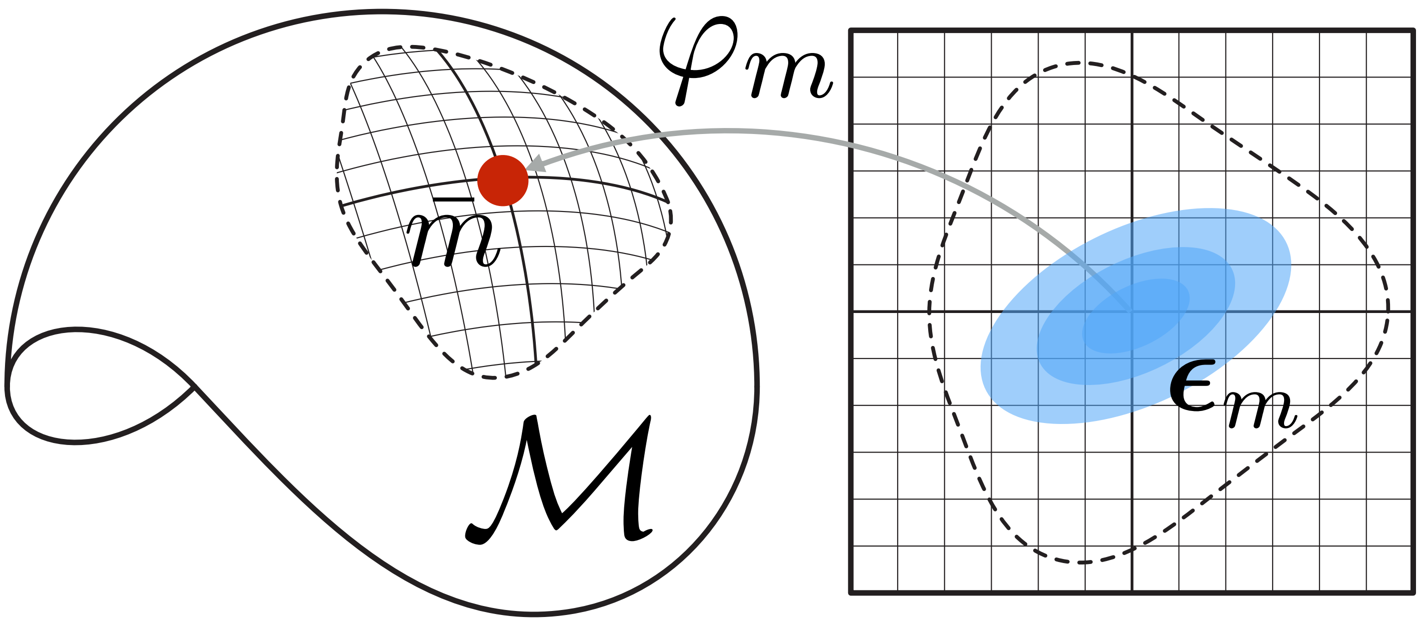
SO(3)

Uncertainties on The Special Orthogonal Group

SO(3) — Uncertainty

- Uncertainty is often represented in the chart
- Whatever chart you use for optimization, that is where the uncertainty lives

$$\epsilon_m \sim \mathcal{N}(\mathbf{0}, \mathbf{P})$$



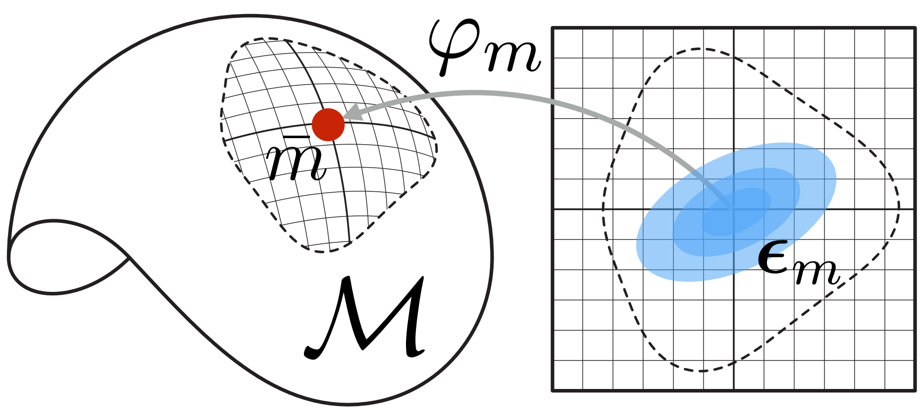
SO(3) — Uncertainty

Recover the covariance:

$$\mathbf{P} := \left(\mathbf{E}^T\mathbf{R}^{-1}\mathbf{E}\right)^{-1},$$

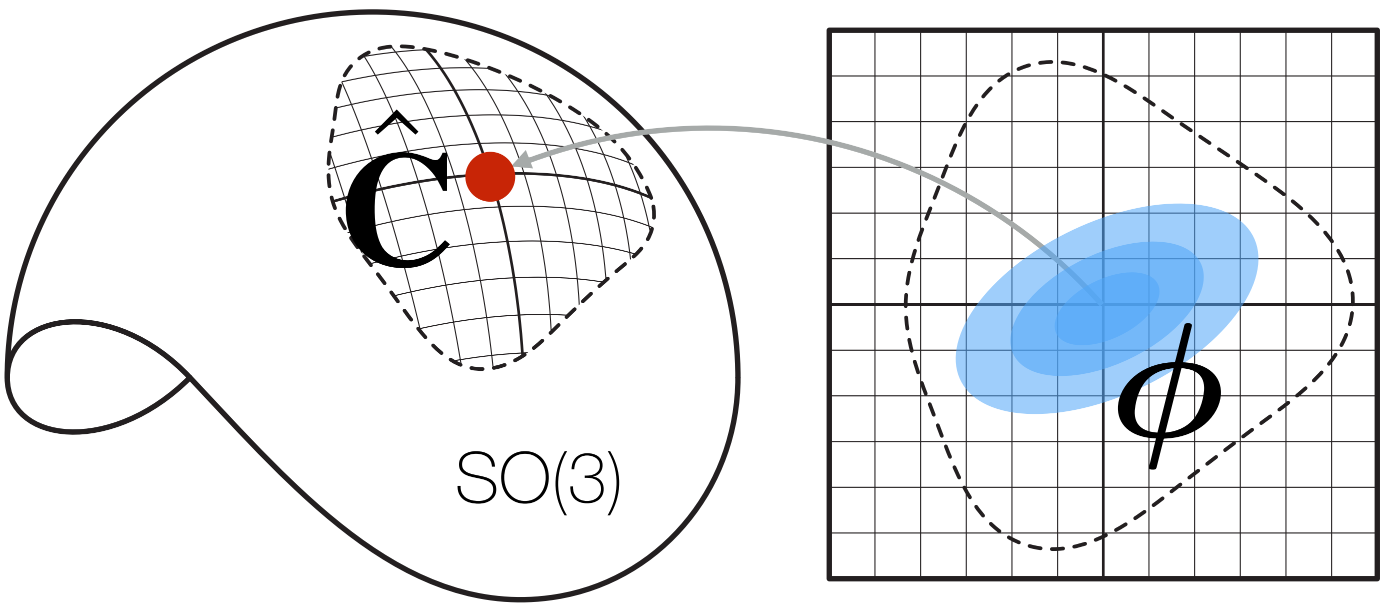
$$\mathbf{x} = \mathbf{x}^\star + \boldsymbol{\epsilon}_x, \quad \boldsymbol{\epsilon}_x \sim \mathcal{N}\left(\mathbf{0}, \mathbf{P}\right).$$

$$m = \varphi_m\left(\varphi_m^{-1}(\bar{m}) + \boldsymbol{\epsilon}_m\right) \qquad \boldsymbol{\epsilon}_m \sim \mathcal{N}(\mathbf{0}, \mathbf{P})$$



SO(3) — Uncertainty

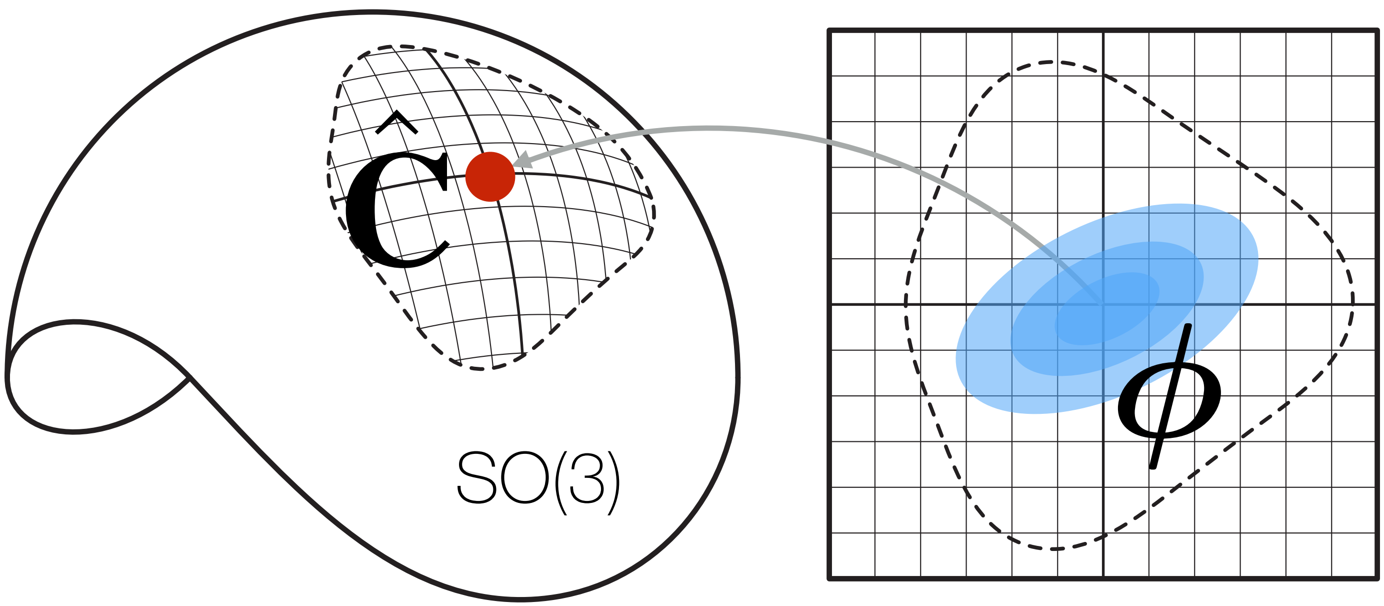
$$\mathbf{C} = \exp(\boldsymbol{\phi})\hat{\mathbf{C}}, \quad \boldsymbol{\phi} \sim \mathcal{N}(\mathbf{0}, \mathbf{P})$$



This is called an *injection* of noise onto SO(3)

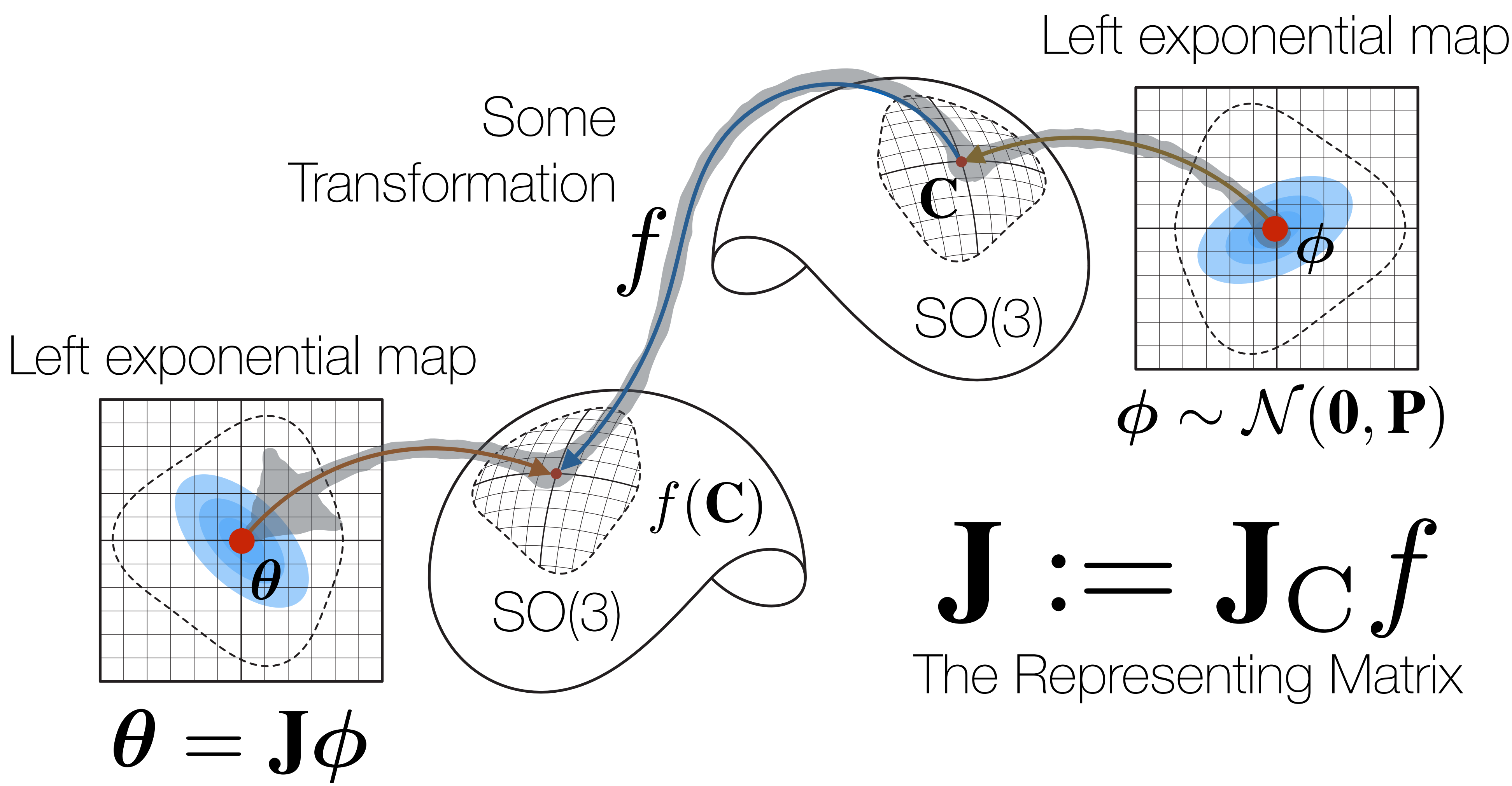
SO(3) — Uncertainty

$$\mathbf{C} \sim \mathcal{N}(\hat{\mathbf{C}}, \mathbf{P})$$



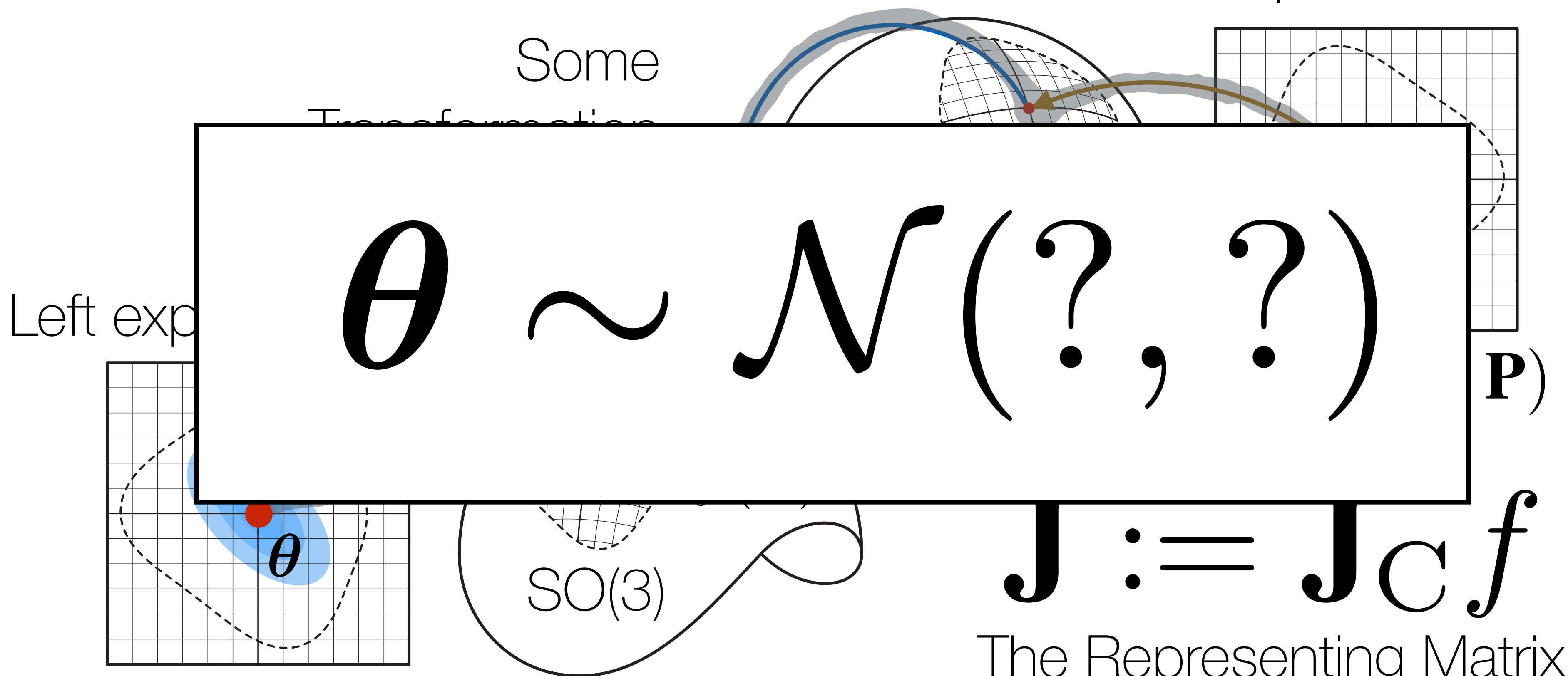
This is called an *injection* of noise onto SO(3)

SO(3) — Transforming Uncertainties (to first order)



SO(3) — Transforming Uncertainties (to first order)

Left exponential map



$$\boldsymbol{\theta} \sim \mathcal{N}(?, ?)$$

$$\mathbf{J} := \mathbf{J}_{\mathrm{C}} f$$

The Representing Matrix

$$\boldsymbol{\theta} = \mathbf{J}\boldsymbol{\phi}$$

SO(3) – Transforming Uncertainties (to first order)

$$\boldsymbol{\theta} = \mathbf{J}\boldsymbol{\phi}, \quad \boldsymbol{\phi} \sim \mathcal{N}(\mathbf{0}, \mathbf{P})$$

$$E\left[\boldsymbol{\theta}\right]$$

SO(3) – Transforming Uncertainties (to first order)

$$\boldsymbol{\theta} = \mathbf{J}\boldsymbol{\phi}, \quad \boldsymbol{\phi} \sim \mathcal{N}(\mathbf{0}, \mathbf{P})$$

$$\begin{aligned} & E\left[\boldsymbol{\theta}\right] \\ = & E\left[\mathbf{J}\boldsymbol{\phi}\right] \end{aligned}$$

SO(3) – Transforming Uncertainties (to first order)

$$\boldsymbol{\theta} = \mathbf{J}\boldsymbol{\phi}, \quad \boldsymbol{\phi} \sim \mathcal{N}(\mathbf{0}, \mathbf{P})$$

$$\begin{aligned} &E\left[\boldsymbol{\theta}\right] \\ =&E\left[\mathbf{J}\boldsymbol{\phi}\right] \\ =&\mathbf{J}E\left[\boldsymbol{\phi}\right] \end{aligned}$$

SO(3) – Transforming Uncertainties (to first order)

$$\boldsymbol{\theta} = \mathbf{J}\boldsymbol{\phi}, \quad \boldsymbol{\phi} \sim \mathcal{N}(\mathbf{0}, \mathbf{P})$$

$$\begin{aligned} & E\left[\boldsymbol{\theta}\right] \\ = & E\left[\mathbf{J}\boldsymbol{\phi}\right] \\ = & \mathbf{J}E\left[\boldsymbol{\phi}\right] \\ = & \mathbf{0} \end{aligned}$$

SO(3) – Transforming Uncertainties (to first order)

$$\boldsymbol{\theta} = \mathbf{J}\boldsymbol{\phi}, \quad \boldsymbol{\phi} \sim \mathcal{N}(\mathbf{0}, \mathbf{P})$$

$$E\left[\boldsymbol{\theta}\right]$$

$$\boxed{\boldsymbol{\theta} \sim \mathcal{N}(?, ?)}$$

$$= \mathbf{0}$$

SO(3) – Transforming Uncertainties (to first order)

$$\boldsymbol{\theta} = \mathbf{J}\boldsymbol{\phi}, \quad \boldsymbol{\phi} \sim \mathcal{N}(\mathbf{0}, \mathbf{P})$$

$$\underbrace{E\left[\boldsymbol{\theta}\right]}_{=\mathbf{0}}$$

$$\boxed{\boldsymbol{\theta} \sim \mathcal{N}(\mathbf{0}, ?)}$$

SO(3) – Transforming Uncertainties (to first order)

$$\boldsymbol{\theta} = \mathbf{J}\boldsymbol{\phi}, \quad \boldsymbol{\phi} \sim \mathcal{N}(\mathbf{0}, \mathbf{P})$$

$$E\left[\boldsymbol{\theta}\boldsymbol{\theta}^T\right]$$

SO(3) – Transforming Uncertainties (to first order)

$$\boldsymbol{\theta} = \mathbf{J}\boldsymbol{\phi}, \quad \boldsymbol{\phi} \sim \mathcal{N}(\mathbf{0}, \mathbf{P})$$

$$\begin{aligned} & E\left[\boldsymbol{\theta}\boldsymbol{\theta}^T\right] \\ = & E\left[\mathbf{J}\boldsymbol{\phi}\boldsymbol{\phi}^T\mathbf{J}^T\right] \end{aligned}$$

SO(3) – Transforming Uncertainties (to first order)

$$\boldsymbol{\theta} = \mathbf{J}\boldsymbol{\phi}, \quad \boldsymbol{\phi} \sim \mathcal{N}(\mathbf{0}, \mathbf{P})$$

$$\begin{aligned} & E\left[\boldsymbol{\theta}\boldsymbol{\theta}^T\right] \\ =& E\left[\mathbf{J}\boldsymbol{\phi}\boldsymbol{\phi}^T\mathbf{J}^T\right] \\ =& \mathbf{J}E\left[\boldsymbol{\phi}\boldsymbol{\phi}^T\right]\mathbf{J}^T \end{aligned}$$

SO(3) – Transforming Uncertainties (to first order)

$$\boldsymbol{\theta} = \mathbf{J}\boldsymbol{\phi}, \quad \boldsymbol{\phi} \sim \mathcal{N}(\mathbf{0}, \mathbf{P})$$

$$\begin{aligned} & E\left[\boldsymbol{\theta}\boldsymbol{\theta}^T\right] \\ = & E\left[\mathbf{J}\boldsymbol{\phi}\boldsymbol{\phi}^T\mathbf{J}^T\right] \\ = & \mathbf{J}E\left[\boldsymbol{\phi}\boldsymbol{\phi}^T\right]\mathbf{J}^T \\ = & \mathbf{J}\mathbf{P}\mathbf{J}^T \end{aligned}$$

SO(3) – Transforming Uncertainties (to first order)

$$\boldsymbol{\theta} = \mathbf{J}\boldsymbol{\phi}, \quad \boldsymbol{\phi} \sim \mathcal{N}(\mathbf{0}, \mathbf{P})$$

$$E\left[\boldsymbol{\theta}\boldsymbol{\theta}^T\right]$$

$$\boxed{\boldsymbol{\theta} \sim \mathcal{N}(\mathbf{0}, ?)}$$

$$= \mathbf{J}\mathbf{P}\mathbf{J}^T$$

SO(3) – Transforming Uncertainties (to first order)

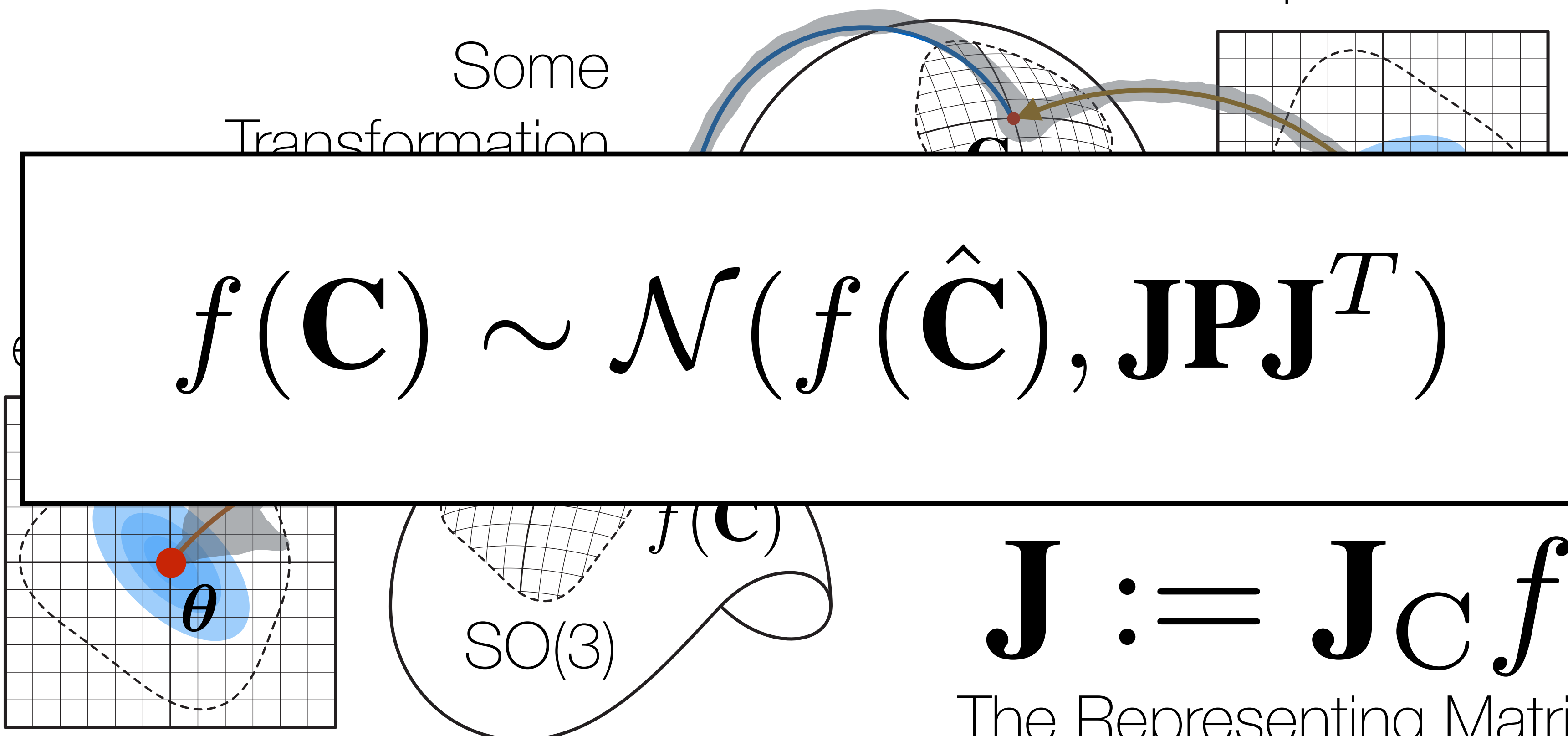
$$\boldsymbol{\theta} = \mathbf{J}\boldsymbol{\phi}, \quad \boldsymbol{\phi} \sim \mathcal{N}(\mathbf{0}, \mathbf{P})$$

$$E\left[\boldsymbol{\theta}\boldsymbol{\theta}^T\right]$$

$$\boxed{\boldsymbol{\theta} \sim \mathcal{N}(\mathbf{0}, \mathbf{J}\mathbf{P}\mathbf{J}^T)}$$

$$= \mathbf{J}\mathbf{P}\mathbf{J}^T$$

SO(3) — Transforming Uncertainties (to first order)



$$f(\mathbf{C}) \sim \mathcal{N}\left(f(\hat{\mathbf{C}}), \mathbf{J}\mathbf{P}\mathbf{J}^T\right)$$

$$\mathbf{J} := \mathbf{J}_{\mathbf{C}} f$$

The Representing Matrix

$$\boldsymbol{\theta} = \mathbf{J}\boldsymbol{\phi}$$

Relating Different Parameterizations

- We know the mappings between the left-exponential chart and other rotation parameterizations
- Amazingly, this is the same relationship between angular velocity and parameter rates
- The formulas are available in standard textbooks on spacecraft attitude dynamics

Relating Different Parameterizations

- We know the mappings between the left-exponential chart and other rotation parameterizations
- Amazingly, this is the same relationship between angular velocity and parameter rates
- The formulas are available in standard textbooks on spacecraft attitude dynamics

The “S” matrix

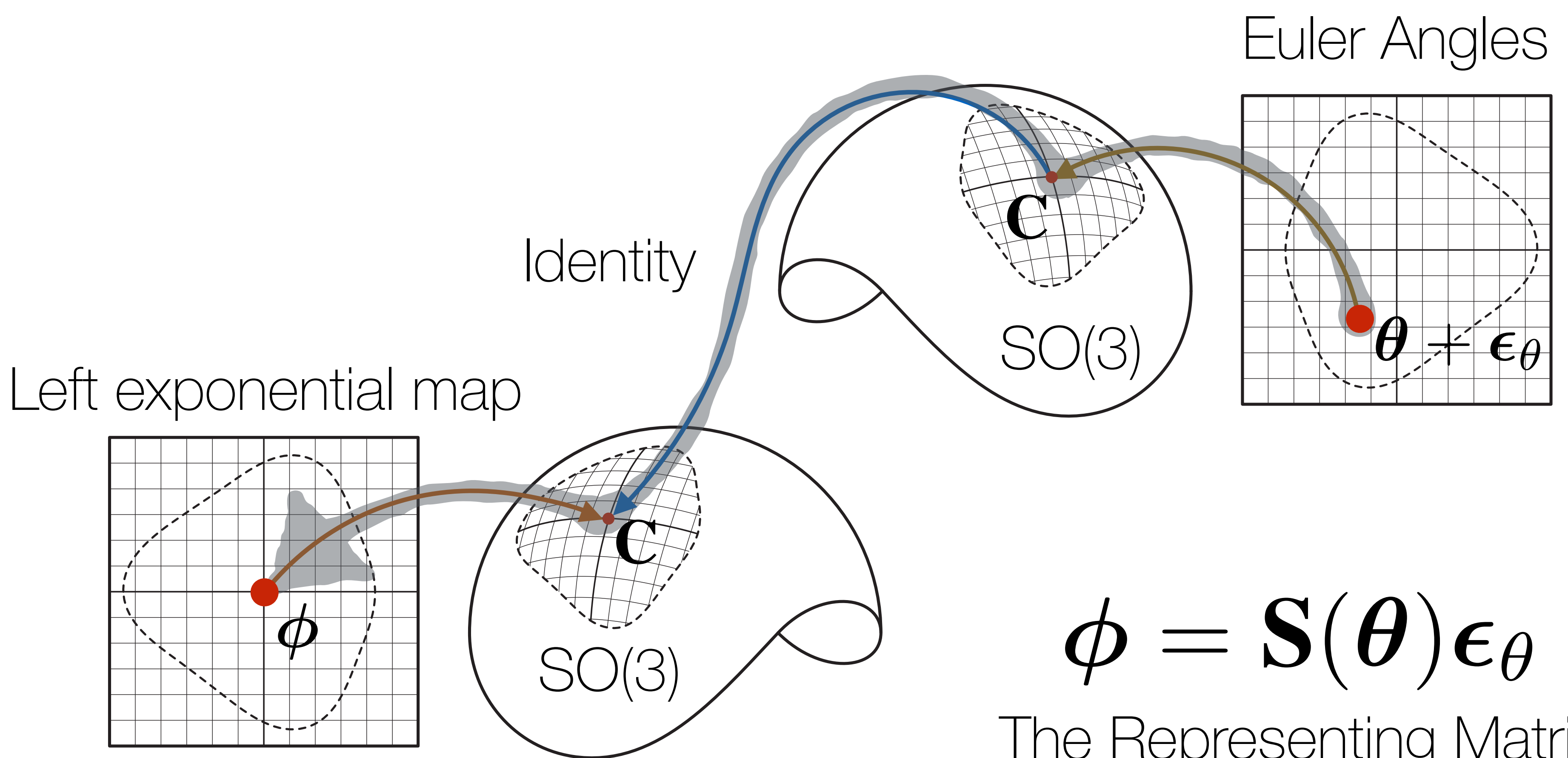
Rotation parameters

$$\boldsymbol{\phi} = \mathbf{S}(\boldsymbol{\theta})\boldsymbol{\epsilon}_\theta$$

Small changes in the left-exponential chart

Small changes in rotation parameters

Relating Different Parameterizations



$$\phi = \mathbf{S}(\boldsymbol{\theta})\boldsymbol{\epsilon}_\theta$$

The Representing Matrix

TABLE 2.3
Comparison of Parameter Alternatives

Name	n	Parameters, v	Definition	$\mathbf{C}(\mathbf{v})$
Direction cosines	9	c_{ij}	(2.1, 6)	$\mathbf{C} = \{c_{ij}\}$
Direction cosines	6	c_{ij}	(2.1, 6)	$\mathbf{C} = \{c_{ij}\}$
Axis / angle variables	4	a_1, a_2, a_3, ϕ	Euler's theorem	$\mathbf{C} = \mathbf{1}\cos\phi + (1 - \cos\phi)\mathbf{a}\mathbf{a}^T - \sin\phi\,\mathbf{a}^\times$
Axis / angle variables	3	ϕ_1, ϕ_2, ϕ_3	$\boldsymbol{\phi} = \phi\mathbf{a}$	$\mathbf{C} = \mathbf{1} + b_1(\phi)\boldsymbol{\phi}^\times + b_2(\phi)\boldsymbol{\phi}^\times\boldsymbol{\phi}^\times$; $b_1 \triangleq -\phi^{-1}\sin\phi$; $b_2 \triangleq 2\phi^{-2}\sin^2\frac{1}{2}\phi$
Euler – Rodriguez (see Probs. 2.12, 19)	3	p_1, p_2, p_3	$\mathbf{p} = \mathbf{a}\tan\frac{1}{2}\phi$	$\mathbf{C} = \mathbf{1} + \dfrac{2}{1 + \mathbf{p}^T\mathbf{p}}(\mathbf{p}^\times\mathbf{p}^\times - \mathbf{p}^\times)$
Euler parameters	4	$\varepsilon_1, \varepsilon_2, \varepsilon_3, \eta$	$\boldsymbol{\varepsilon} = \mathbf{a}\sin\frac{1}{2}\phi$; $\eta = \cos\frac{1}{2}\phi$	$\mathbf{C} = (\eta^2 - \boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon})\mathbf{1} + 2\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^T - 2\eta\boldsymbol{\varepsilon}^\times$
Euler parameters	3	$\varepsilon_1, \varepsilon_2, \varepsilon_3$	$\boldsymbol{\varepsilon} = \mathbf{a}\sin\frac{1}{2}\phi$	$\mathbf{C} = \mathbf{1} + 2\boldsymbol{\varepsilon}^\times\boldsymbol{\varepsilon}^\times - 2(1 - \boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon})^{1/2}\boldsymbol{\varepsilon}^\times$
Euler angles	3	$\theta_1, \theta_2, \theta_3$	Fig. 2.2	$\mathbf{C} = \mathbf{C}_i(\theta_1)\mathbf{C}_j(\theta_2)\mathbf{C}_k(\theta_3)$; $i \neq j$; $j \neq k$ (see Table 2.1)

Name	$\boldsymbol{\omega} = \mathbf{S}(\mathbf{v})\dot{\mathbf{v}}$	$\dot{\mathbf{v}} = \mathbf{S}^{-1}(\mathbf{v})\boldsymbol{\omega}$
Direction cosines	$\boldsymbol{\omega}^{\times} = -\dot{\mathbf{C}}\mathbf{C}^T \equiv \mathbf{C}\dot{\mathbf{C}}^T$	$[\mathbf{c}_1 \quad \mathbf{c}_2 \quad \mathbf{c}_3] \triangleq \mathbf{C}$ $\dot{\mathbf{c}}_1 = \mathbf{c}_1^{\times}\boldsymbol{\omega};\ \dot{\mathbf{c}}_2 = \mathbf{c}_2^{\times}\boldsymbol{\omega};\ \dot{\mathbf{c}}_3 = \mathbf{c}_3^{\times}\boldsymbol{\omega}$
Direction cosines	—	$\dot{\mathbf{c}}_1 = \mathbf{c}_1^{\times}\boldsymbol{\omega};\ \dot{\mathbf{c}}_2 = \mathbf{c}_2^{\times}\boldsymbol{\omega};\ \mathbf{c}_3 = \mathbf{c}_1^{\times}\mathbf{c}_2$
Axis / angle variables	$\boldsymbol{\omega} = \dot{\phi}\mathbf{a} - (1 - \cos\phi)\mathbf{a}^{\times}\dot{\mathbf{a}} + \dot{\mathbf{a}}\sin\phi$	$\dot{\mathbf{a}} = \frac{1}{2}(\mathbf{a}^{\times} - \cot\frac{1}{2}\phi\, \mathbf{a}^{\times}\mathbf{a}^{\times})\boldsymbol{\omega}$ $\dot{\phi} = \mathbf{a}^T\boldsymbol{\omega}$
Axis / angle variables	$\boldsymbol{\omega} = (\mathbf{1} + c_1(\phi)\boldsymbol{\phi}^{\times} + c_2(\phi)\boldsymbol{\phi}^{\times}\boldsymbol{\phi}^{\times})\dot{\boldsymbol{\phi}}$ $c_1 \triangleq -2\phi^{-2}\sin^2\frac{1}{2}\phi;$ $c_2 \triangleq \phi^{-3}(\phi - \sin\phi)$	$\dot{\boldsymbol{\phi}} = (\mathbf{1} + \frac{1}{2}\boldsymbol{\phi}^{\times} + a_1(\phi)\boldsymbol{\phi}^{\times}\boldsymbol{\phi}^{\times})\boldsymbol{\omega}$ $\phi \triangleq (\boldsymbol{\phi}^T\boldsymbol{\phi})^{1/2};\ a_1 \triangleq \phi^{-2}(1 - \frac{1}{2}\phi\cot\frac{1}{2}\phi)$
Euler – Rodriguez (see Probs. 2.12, 19)	$\boldsymbol{\omega} = \dfrac{2}{1 + \mathbf{p}^T\mathbf{p}}(\mathbf{1} - \mathbf{p}^{\times})\dot{\mathbf{p}}$	$\dot{\mathbf{p}} = \frac{1}{2}(\mathbf{p}\mathbf{p}^T + \mathbf{1} + \mathbf{p}^{\times})\boldsymbol{\omega}$
Euler parameters	$\boldsymbol{\omega} = 2(\eta\dot{\boldsymbol{\varepsilon}} - \dot{\eta}\boldsymbol{\varepsilon}) - 2\boldsymbol{\varepsilon}^{\times}\dot{\boldsymbol{\varepsilon}}$	$\dot{\boldsymbol{\varepsilon}} = \frac{1}{2}(\boldsymbol{\varepsilon}^{\times} + \eta\mathbf{1})\boldsymbol{\omega}$ $\dot{\eta} = -\frac{1}{2}\boldsymbol{\varepsilon}^T\boldsymbol{\omega}$
Euler parameters	$\boldsymbol{\omega} = 2\left[\dfrac{\mathbf{1} + \boldsymbol{\varepsilon}^{\times}\boldsymbol{\varepsilon}^{\times}}{(1 - \boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon})^{1/2}} - \boldsymbol{\varepsilon}^{\times}\right]\dot{\boldsymbol{\varepsilon}}$	$\dot{\boldsymbol{\varepsilon}} = \frac{1}{2}\{\boldsymbol{\varepsilon}^{\times} + (1 - \boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon})^{1/2}\mathbf{1}\}\boldsymbol{\omega}$
Euler angles	$\boldsymbol{\omega} = [\mathbf{1}_i \quad \mathbf{C}_i\mathbf{1}_j \quad \mathbf{C}_i\mathbf{C}_j\mathbf{1}_k]\dot{\boldsymbol{\theta}}$	$\dot{\boldsymbol{\theta}} = \mathbf{S}^{-1}(\theta_1, \theta_2)\boldsymbol{\omega}$ (see Table 2.2)

Relating Different Parameterizations

- We know the mappings between the left-exponential chart and other rotation parameterizations
- Once you know the S matrix for each parameterization, you can transform uncertainties between parameterizations

The "S" matrix

Rotation parameters

$$\boldsymbol{\phi} = \mathbf{S}(\boldsymbol{\theta})\boldsymbol{\epsilon}_{\theta}$$

Small changes in the left-exponential chart

Small changes in rotation parameters

Relating Different Rotation Parameterizations

$$\boldsymbol{\phi} = \mathbf{S}(\boldsymbol{\theta})\boldsymbol{\epsilon}_{\theta}$$

Hughes, P. C. (1986). *Spacecraft Attitude Dynamics.* John Wiley & Sons, New York.

$$T_{\mathtt{AB}}$$

SE(3)

The Special Euclidean Group

SE(3) - The Special Euclidean Group

- Things work exactly the same
  - Tim Barfoot and Paul Furgale (2014). Associating Uncertainty with Three-Dimensional Poses for use in Estimation Problems. Robotics, IEEE Transactions on, 30(3), pp. 679-693.

Summary

Nonlinear Least Squares / Gauss Newton

$$\text{Goal: estimate } \mathbf{x}^\star = \underset{\mathbf{x}}{\operatorname{argmin}}\, J(\mathbf{x}) = \frac{1}{2}\mathbf{e}(\mathbf{x})^T\mathbf{R}^{-1}\mathbf{e}(\mathbf{x}).$$

- Start with an initial guess for $\mathbf{x}$
- Iterate
  - Linearize
  - Solve the linear system
  - Update the parameters
  - Repeat until convergence

$$\bar{\mathbf{e}} := \mathbf{e}(\bar{\mathbf{x}}), \quad \mathbf{E} := \left.\frac{\partial \mathbf{e}(\mathbf{x})}{\partial \mathbf{x}}\right|_{\bar{\mathbf{x}}}, \quad \mathbf{e}(\boldsymbol{\epsilon}_x) := \bar{\mathbf{e}} + \mathbf{E}\boldsymbol{\epsilon}_x$$

$$\mathbf{E}^T\mathbf{R}^{-1}\mathbf{E}\boldsymbol{\epsilon}_x^\star = -\mathbf{E}^T\mathbf{R}^{-1}\bar{\mathbf{e}}$$

$$\bar{\mathbf{x}} \leftarrow \bar{\mathbf{x}} + \boldsymbol{\epsilon}_x^\star$$

Recover the covariance:

$$\mathbf{P} := \left(\mathbf{E}^T\mathbf{R}^{-1}\mathbf{E}\right)^{-1},$$

$$\mathbf{x} = \mathbf{x}^\star + \boldsymbol{\epsilon}_x, \quad \boldsymbol{\epsilon}_x \sim \mathcal{N}(\mathbf{0}, \mathbf{P})$$

Calculus

$$\left.\frac{\partial \mathbf{e}(\mathbf{x})}{\partial \mathbf{x}}\right|_{\overline{\mathbf{x}}}$$

Linear Algebra

$$\boldsymbol{\epsilon}_x^{\star} = -\left(\mathbf{E}^T\mathbf{R}^{-1}\mathbf{E}\right)^{-1}\mathbf{E}^T\mathbf{R}^{-1}\overline{\mathbf{e}}$$

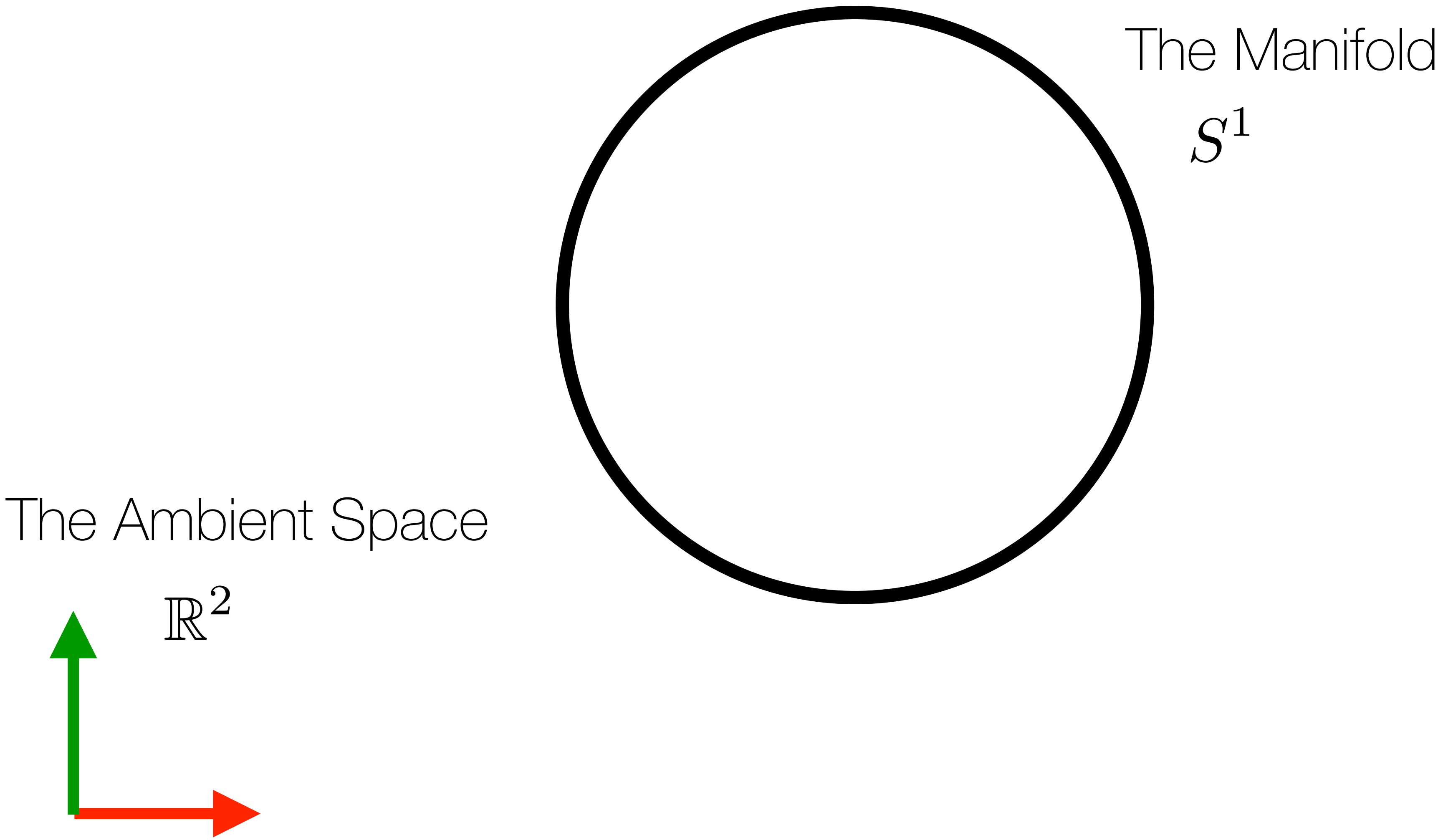
Probabilities

$$\mathbf{P} := \left(\mathbf{E}^T \mathbf{R}^{-1} \mathbf{E}\right)^{-1},$$

$$\mathbf{x} = \mathbf{x}^{\star} + \boldsymbol{\epsilon}_x, \quad \boldsymbol{\epsilon}_x \sim \mathcal{N}\left(\mathbf{0}, \mathbf{P}\right)$$

Basic assumption

$$\mathbf{x} \in \mathbb{R}^M$$


The Manifold
S^1
The Ambient Space
$\mathbb{R}^2$

Choosing an Atlas

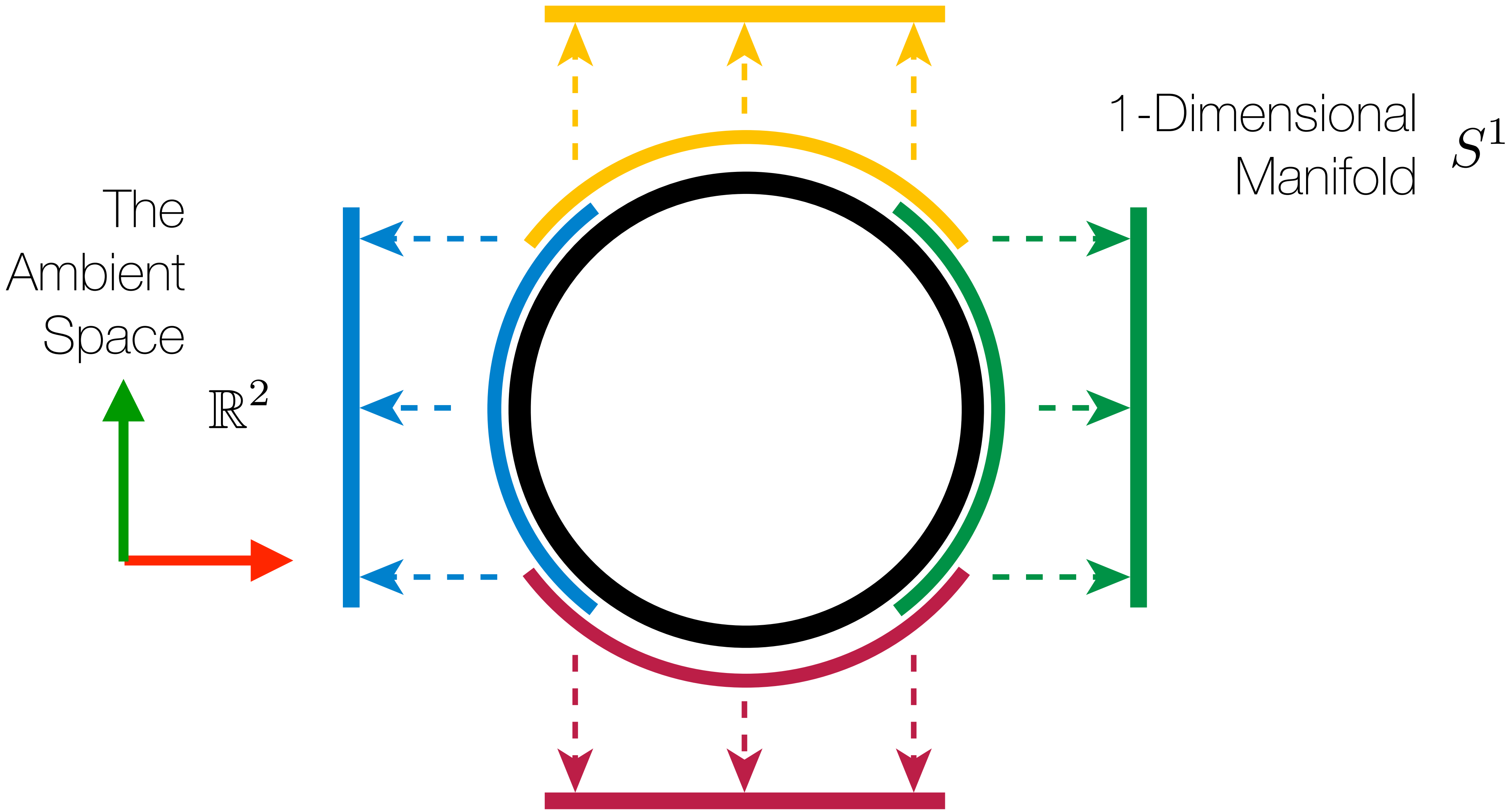
- Charts:
  - For an M-Dimensional DM, $\mathcal{M}$, a *chart* is

Invertible Mapping

The Manifold

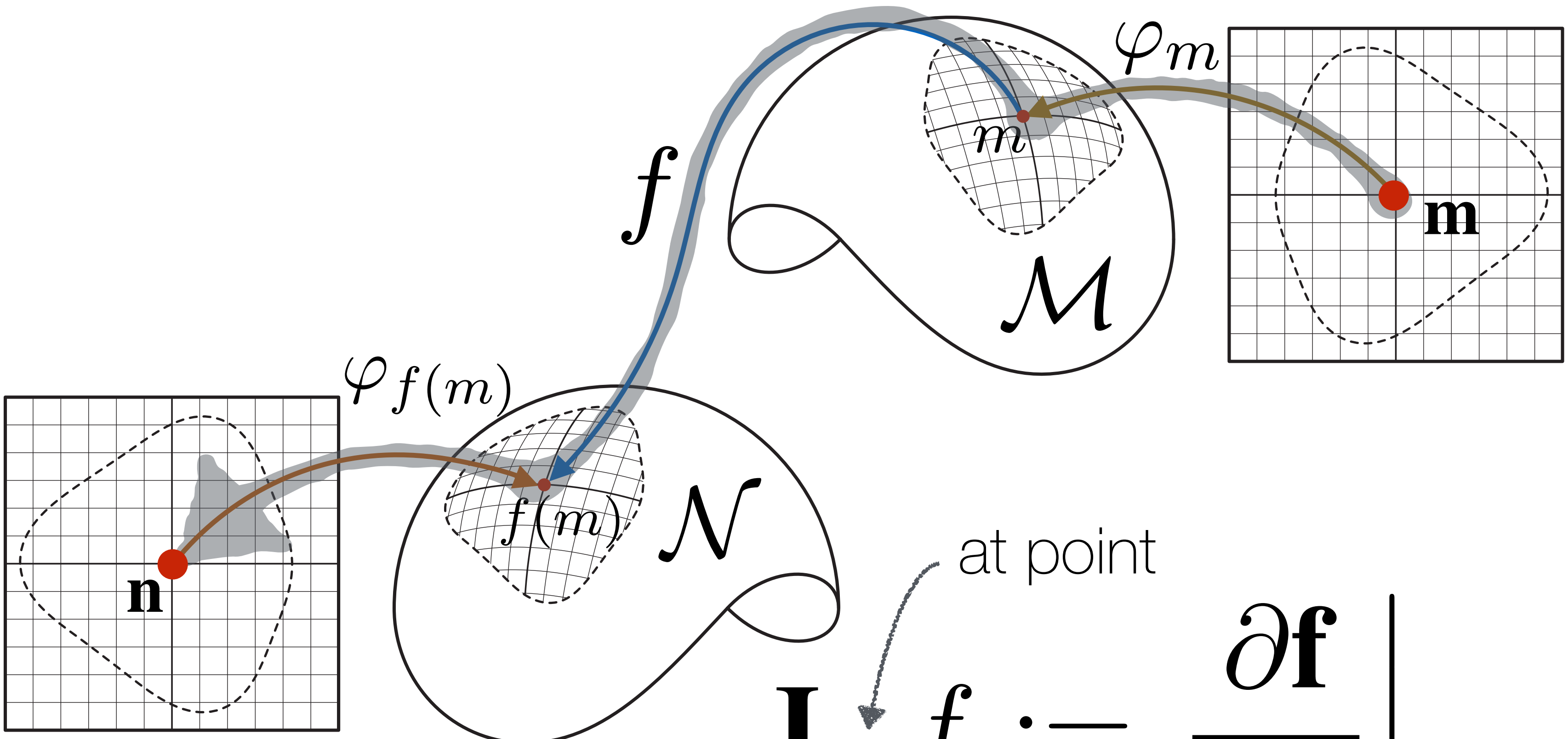
$$\varphi : U_\varphi \rightarrow \mathcal{M}$$

Open subset of $\mathbb{R}^M$



- An *Atlas* is a collection of charts such that cover the manifold

$$\mathbf{f} := \varphi^{-1}_{f(m)} \circ f \circ \varphi_m$$



at point

Representing matrix
of the differential

$$\mathbf{J}_m f := \left. \frac{\partial \mathbf{f}}{\partial \mathbf{m}} \right|_{\varphi_m^{-1}(m)}$$

of mapping

$$C_{\mathtt{AB}}$$

SO(3)

The Special Orthogonal Group

Fundamental Identities (Cheat Sheet)

$$\mathbf{a}^\wedge \mathbf{b} = -\mathbf{b}^\wedge \mathbf{a}$$

$$\mathbf{C}\mathbf{a}^\wedge = (\mathbf{C}\mathbf{a})^\wedge \mathbf{C}$$

$$\mathbf{a}^\wedge \mathbf{C} = \mathbf{C}(\mathbf{C}^T \mathbf{a})^\wedge$$

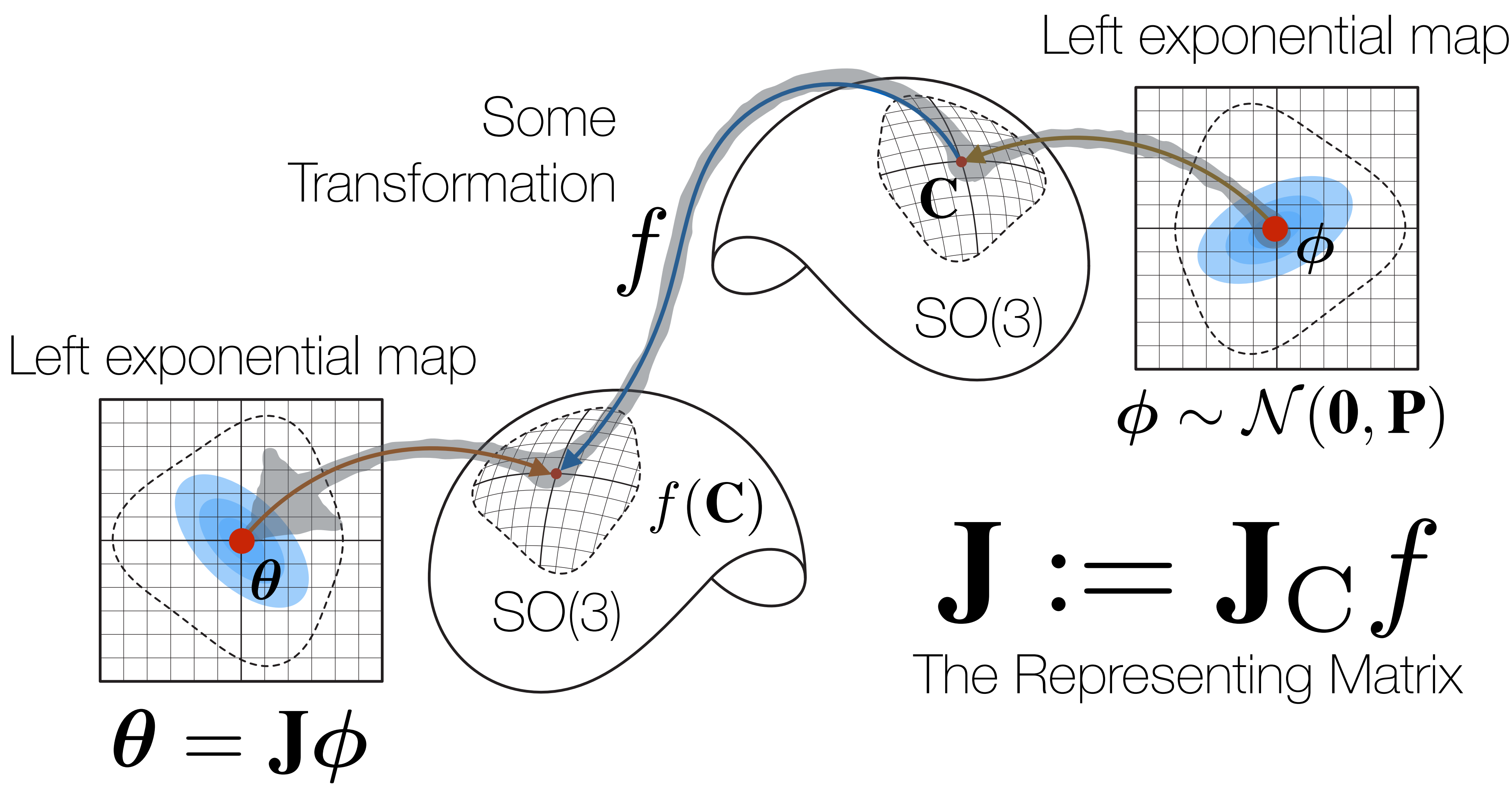
$$(\mathbf{a}^\wedge)^T = -\mathbf{a}^\wedge$$

$$\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}^\wedge = \begin{bmatrix} 0 & -\phi_3 & \phi_2 \\ \phi_3 & 0 & -\phi_1 \\ -\phi_2 & \phi_1 & 0 \end{bmatrix}$$

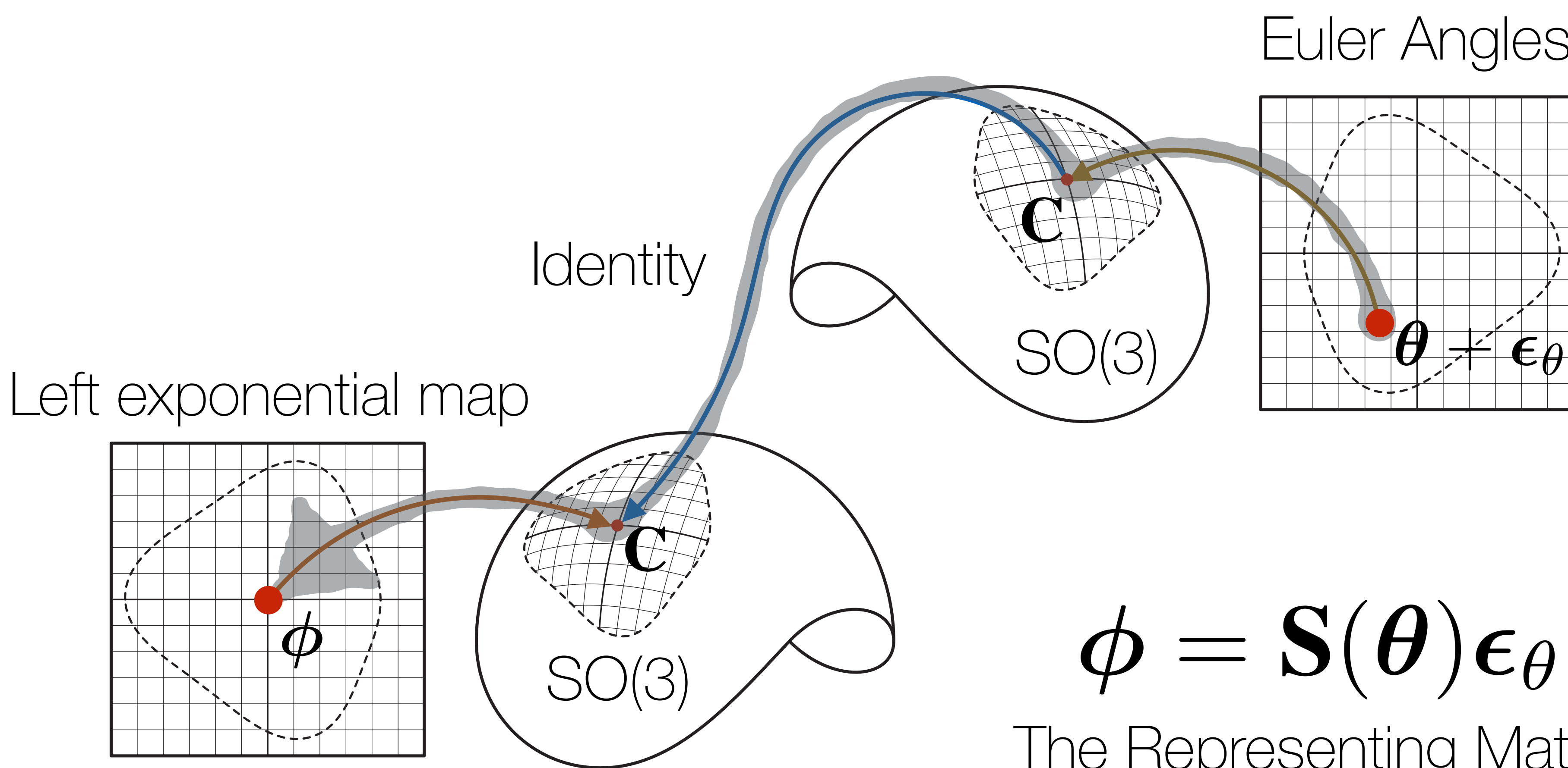
Checking your answers

```
>> jacobianest( @(phi) so3log( (so3exp(phi)*C)'*C ), zeros(3,1) )
```

SO(3) — Transforming Uncertainties (to first order)



Relating Different Parameterizations



$$\boldsymbol{\phi} = \mathbf{S}(\boldsymbol{\theta})\boldsymbol{\epsilon}_\theta$$

The Representing Matrix

Reading List

- Steven Lavalle, Planning Algorithms, Cambridge University Press 2006 (http://planning.cs.uiuc.edu/)
- Murray, Richard M., et al. A mathematical introduction to robotic manipulation. CRC press, 1994. (http://nyx-www.informatik.uni-bremen.de/444/1/li_book_06.pdf)
- Hughes, Peter C. Spacecraft attitude dynamics. Courier Dover Publications, 2012.