Synthesizing Robust Distributed Controllers: When Is Information Enough?

Luca Furieri

IfA Coffee Talk, February 8th, 2018

Motivations

Critical emerging large-scale systems

- Autonomous decision making agents with local information
- Physical limitations and safety: constraints on states and inputs
- Model uncertainties and external disturbances

Motivations

Critical emerging large-scale systems

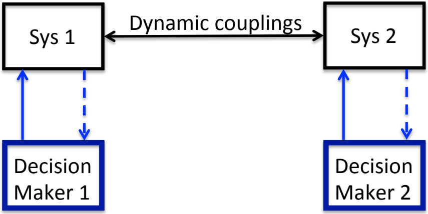
- Autonomous decision making agents with local information
- Physical limitations and safety: constraints on states and inputs
- Model uncertainties and external disturbances

Motivations

Critical emerging large-scale systems

- Autonomous decision making agents with local information
- Physical limitations and safety: constraints on states and inputs
- Model uncertainties and external disturbances

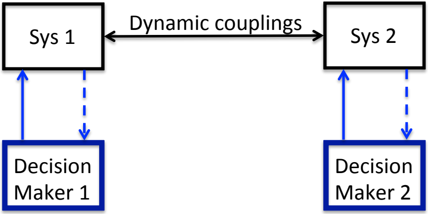
Example: Asymmetric Information and Complexity



- Need of inferring the decisions of others
- Observations of others must be reconstructed...
- ...by **computing** how decisions would propagate through dynamical couplings over time

$\Longrightarrow$ **Recipe for intractability**

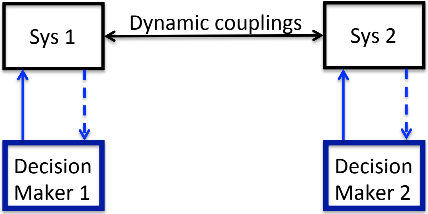
Example: Asymmetric Information and Complexity



- Need of inferring the decisions of others
- Observations of others must be reconstructed...
- ...by **computing** how decisions would propagate through dynamical couplings over time

$\Longrightarrow$ **Recipe for intractability**

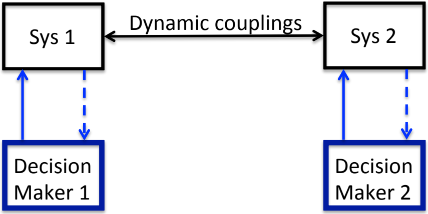
Example: Asymmetric Information and Complexity



- Need of inferring the decisions of others
- Observations of others must be reconstructed...
- ...by **computing** how decisions would propagate through dynamical couplings over time

$\Longrightarrow$ **Recipe for intractability**

Example: Asymmetric Information and Complexity



- Need of inferring the decisions of others
- Observations of others must be reconstructed...
- ...by **computing** how decisions would propagate through dynamical couplings over time

$\Longrightarrow$ **Recipe for intractability**

A Brief History (1)

LQG problem with asymmetric information

- Nonlinear strategies may outperform linear ones (Witsenhausen's counterexample) [Witsenhausen, 1968]
- Witsenhausen's control problem is NP-complete

  [Papadimitriou et al., 1986]

Restriction to linear policies

- Intractable problem in general! Why?
  $\Rightarrow$ Need of computing missing information by inverting system dynamics [Rotkowitz, 2008]

A Brief History (1)

LQG problem with asymmetric information

- Nonlinear strategies may outperform linear ones (Witsenhausen's counterexample) [Witsenhausen, 1968]
- Witsenhausen's control problem is NP-complete [Papadimitriou et al., 1986]

Restriction to linear policies

- Intractable problem in general! Why?
$\Rightarrow$ Need of computing missing information by inverting system dynamics [Rotkowitz, 2008]

A Brief History (2) and Open Questions

Tractable decentralization schemes (no need of dynamic inversion):

- Partially Nested schemes [Ho et al., 1972]
- Quadratically Invariant (QI) schemes [Rotkowitz, 2006]
  $\Rightarrow \mathcal{H}_2$ norm minimization, no states/inputs constraints...

Open questions:

1. What about state-space frameworks in finite horizon?
2. Can we include robust satisfaction of state/input constraints? Is tractability affected?
3. How can communication be exploited for tractability in the presence of challenging systems?

A Brief History (2) and Open Questions

Tractable decentralization schemes (no need of dynamic inversion):

- Partially Nested schemes [Ho et al., 1972]
- Quadratically Invariant (QI) schemes [Rotkowitz, 2006]
  $\Rightarrow \mathcal{H}_2$ norm minimization, no states/inputs constraints...

Open questions:

1. What about state-space frameworks in finite horizon?
2. Can we include robust satisfaction of state/input constraints? Is tractability affected?
3. How can communication be exploited for tractability in the presence of challenging systems?

Table of Content

Problem Definition

Robust Distributed Control Problem

$$\min_{\pi_k(\cdot)} \quad J(x_0, \dots, x_N, u_0, \dots, u_{N\text{-}1})$$

$$\begin{aligned} \text{s.t.} \quad & x_{k+1} = Ax_k + Bu_k + w_k\,, \\ & x_k \in \mathbb{X}_k,\ u_k \in \mathbb{U}_k\,, \\ & \forall w_k \in \mathbb{W} \subseteq \mathbb{R}^n\,, \forall k \in \mathbb{Z}_{[0,N\text{-}1]}\,, \\ & u_k = \pi_k(x_0, \dots, x_k),\ \pi_k(\cdot) \in \text{Information Structure}\,. \end{aligned}$$

- Linear dynamics with disturbances
- States and inputs constraints. Robust satisfaction
- State feedback controller
- Compliance with an information structure. Different controllers measure different states

Problem Definition

Robust Distributed Control Problem

$$\begin{aligned}
\min_{\pi_k(\cdot)} \quad & J(x_0, \dots, x_N, u_0, \dots, u_{N\text{-}1}) \\
\text{s.t.} \quad & x_{k+1} = Ax_k + Bu_k + w_k\,, \\
& x_k \in \mathbb{X}_k,\ u_k \in \mathbb{U}_k\,, \\
& \forall w_k \in \mathbb{W} \subseteq \mathbb{R}^n\,, \forall k \in \mathbb{Z}_{[0,N\text{-}1]}\,, \\
& u_k = \pi_k(x_0, \dots, x_k),\ \pi_k(\cdot) \in \text{Information Structure}\,.
\end{aligned}$$

- Linear dynamics with disturbances
- States and inputs constraints. Robust satisfaction
- State feedback controller
- Compliance with an information structure. Different controllers measure different states

Problem Definition

Robust Distributed Control Problem

$$\begin{aligned}
\min_{\pi_k(\cdot)} \quad & J(x_0, \ldots, x_N, u_0, \ldots, u_{N\text{-}1}) \\
\text{s.t.} \quad & x_{k+1} = Ax_k + Bu_k + w_k\,, \\
& x_k \in \mathbb{X}_k,\ u_k \in \mathbb{U}_k\,, \\
& \forall w_k \in \mathbb{W} \subseteq \mathbb{R}^n\,, \forall k \in \mathbb{Z}_{[0,N\text{-}1]}\,, \\
& u_k = \pi_k(x_0, \ldots, x_k),\ \pi_k(\cdot) \in \text{Information Structure}\,.
\end{aligned}$$

- Linear dynamics with disturbances
- States and inputs constraints. Robust satisfaction
- State feedback controller
- Compliance with an information structure. Different controllers measure different states

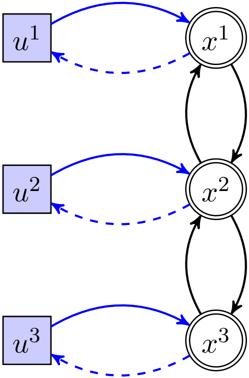
Information Structures

Past state measurements known to controllers at each time

$$\mathcal{I}_k^i = \left\{ x_{\mathbf{l}}^{\mathbf{r}} \text{ s.t. } \begin{array}{l} x_{\mathbf{l}}^{\mathbf{r}} \text{ is known to controller } i \text{ at time } k\,, \\ \mathbf{r} \in \mathbb{Z}_{[1,n]},\ \mathbf{l} \in \mathbb{Z}_{[0,k]}\,. \end{array} \right\}.$$

- Commonly: **direct sensing** and **communication** between controllers
- Time varying network topologies

Sensor-Information Structures



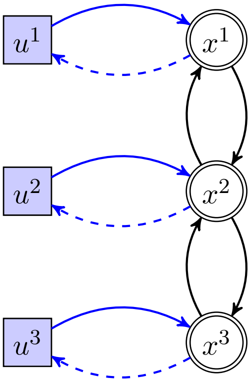
$S = I_3$

$$\begin{bmatrix} x^1_{k+1} \\ x^2_{k+1} \\ x^3_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & \star & 0 \\ \star & 0 & \star \\ 0 & \star & 0 \end{bmatrix} \begin{bmatrix} x^1_k \\ x^2_k \\ x^3_k \end{bmatrix} + \begin{bmatrix} u^1_k \\ u^2_k \\ u^3_k \end{bmatrix},$$

- Fixed sensing topology S

$$S(i,j) = 1 \iff u^i \text{ knows } x^j$$

Sensor-Information Structures



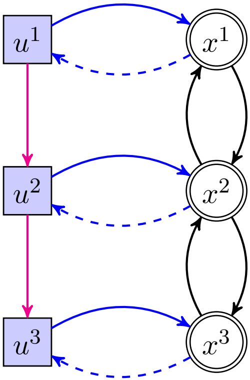
$S = I_3$

$$\begin{bmatrix} x^1_{k+1} \\ x^2_{k+1} \\ x^3_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & \star & 0 \\ \star & 0 & \star \\ 0 & \star & 0 \end{bmatrix} \begin{bmatrix} x^1_k \\ x^2_k \\ x^3_k \end{bmatrix} + \begin{bmatrix} u^1_k \\ u^2_k \\ u^3_k \end{bmatrix},$$

- Fixed sensing topology S

$$S(i,j) = 1 \iff u^i \text{ knows } x^j$$

Sensing and Communication Information Structures



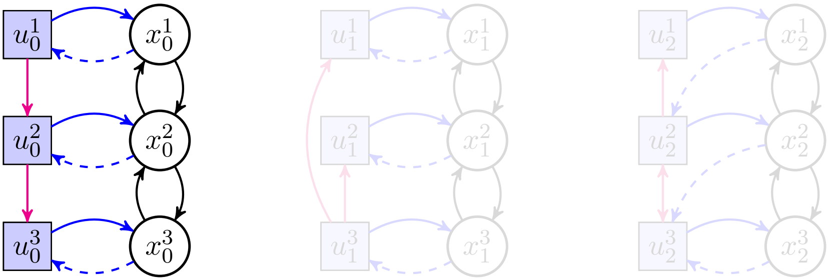
$$S = I_3,\ Z = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x^1_{k+1} \\ x^2_{k+1} \\ x^3_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & \star & 0 \\ \star & 0 & \star \\ 0 & \star & 0 \end{bmatrix} \begin{bmatrix} x^1_k \\ x^2_k \\ x^3_k \end{bmatrix} + \begin{bmatrix} u^1_k \\ u^2_k \\ u^3_k \end{bmatrix},$$

- Sensing topology S and communication topology Z
- Sensor measurements are propagated with one time-step delay
  $\Rightarrow u^3_k$ knows $x^3_{[0,k]}$, $x^2_{[0,k\text{-}1]}$, $x^1_{[0,k\text{-}2]}$

General Information Structures

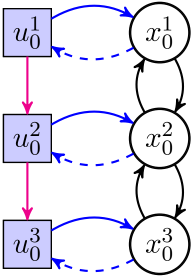
time $k = 0$, time $k = 1$, time $k = 2$, ...



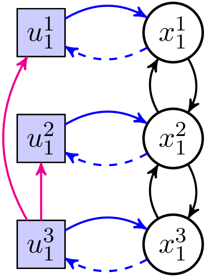
How do we encode compliance
with an information structure?

General Information Structures

time $k = 0$,



time $k = 1$,

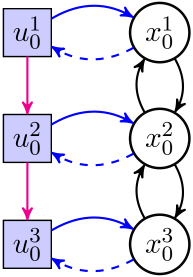


time $k = 2$, ...

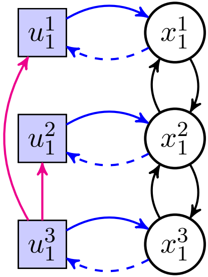
How do we encode compliance
with an information structure?

General Information Structures

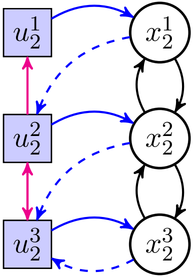
time $k = 0$,



time $k = 1$,



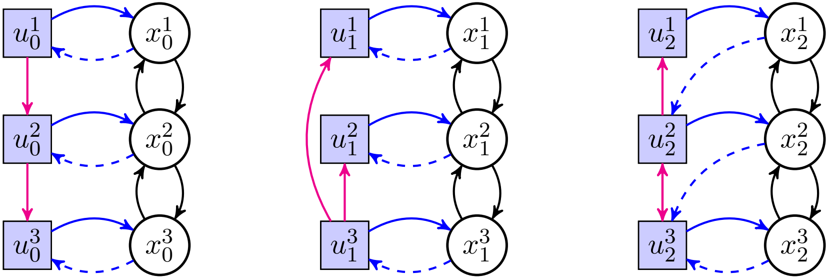
time $k = 2$, ...



How do we encode compliance
with an information structure?

General Information Structures

time $k = 0$, time $k = 1$, time $k = 2$, ...



How do we encode compliance
with an information structure?

Sparsity constraints

- Causal, affine state feedback

$$u_k = \sum_{j=0}^{k} L_{k,j} x_j + g_k$$

- The $L_{k,j}$' must lie in subspaces $\mathbb{S}_{k,j}$'s, defining their zero-patterns

- Subspaces $\mathbb{S}_{k,j}$ generated by binary matrices $S_{k,j}$ $\Rightarrow$ we say $\mathbb{S}_{k,j} = \text{Sparse}(S_{k,j})$

Example: sensor-information structures

$L_{k,j} \in \mathbb{S} = \text{Sparse}(S),\ \forall k, j$

$\implies$ time invariant sparsity pattern

Sparsity constraints

- Causal, affine state feedback

$$u_k = \sum_{j=0}^{k} L_{k,j} x_j + g_k$$

- The $L_{k,j}$' must lie in subspaces $\mathbb{S}_{k,j}$'s, defining their zero-patterns

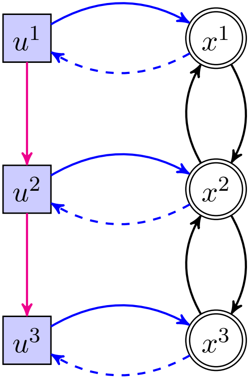
- Subspaces $\mathbb{S}_{k,j}$ generated by binary matrices $S_{k,j}$
  $\Rightarrow$ we say $\mathbb{S}_{k,j} = \text{Sparse}(S_{k,j})$

Example: sensor-information structures

$L_{k,j} \in \mathbb{S} = \text{Sparse}(S), \; \forall k, j$

$\implies$ time invariant sparsity pattern

Sparsity Constraints: Sensing and Communication Information Structures



$S = I_3$, $Z = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

- u_k^i knows $x_{k\text{-}r}^j$ iff there is a magenta path of length r or less from u^j to u^i, or equivalently iff $Z^r(i,j) \neq 0$
  $\Rightarrow Z^r S(i,j) \neq 0$ in general

$$L_{k,k-r} \in \mathbb{S}_{k,k-r} = \text{Sparse}(Z^r S)$$

$\Rightarrow$ time varying sparsity pattern

Table of Content

Inherent Intractability of State Feedback Decisions

State Feedback Problem

$$\min_{L_{k,j},g_k} \quad J(L_{k,j},\dots,g_k,\dots,x_0)$$

$$\text{s.t.} \quad x_k \in \mathbb{X}_k,\; u_k \in \mathbb{U}_k\,,$$

$$\forall w_k \in \mathbb{W} \subseteq \mathbb{R}^n\,,\; \forall k \in \mathbb{Z}_{[0,N\text{-}1]}\,,$$

$$u_k = \sum_{j=0}^{k} L_{k,j}x_j + g_k\,,\; L_{k,j} \in \mathbb{S}_{k,j}\,.$$

- Decisions are nonlinearly propagated through dynamics

$$u_0 = L_0 x_0,\; x_1 = Ax_0 + Bu_0\,,$$

$$u_1 = L_1 x_1 = L_1(A + BL_0)x_0\,.$$

- Sparsity constraints are linear in the $L_{k,j}$'s

The inherent non-convexity of state feedback decisions hides the complexity of asymmetric information...

Inherent Intractability of State Feedback Decisions

State Feedback Problem

$$\min_{L_{k,j}, g_k} \quad J(L_{k,j}, \dots, g_k, \dots, x_0)$$

$$\text{s.t.} \quad x_k \in \mathbb{X}_k, \; u_k \in \mathbb{U}_k \,, \quad \forall w_k \in \mathbb{W} \subseteq \mathbb{R}^n \,, \; \forall k \in \mathbb{Z}_{[0,N\text{-}1]} \,,$$

$$u_k = \sum_{j=0}^{k} L_{k,j} x_j + g_k \,, \; L_{k,j} \in \mathbb{S}_{k,j} \,.$$

- Decisions are nonlinearly propagated through dynamics

$$u_0 = L_0 x_0, \; x_1 = A x_0 + B u_0 \,,$$
$$u_1 = L_1 x_1 = L_1 (A + B L_0) x_0 \,.$$

- Sparsity constraints are linear in the $L_{k,j}$'s

The inherent non-convexity of state feedback decisions hides the complexity of asymmetric information...

Inherent Intractability of State Feedback Decisions

State Feedback Problem

$$\min_{L_{k,j},g_k} \quad J(L_{k,j},\dots,g_k,\dots,x_0)$$

$$\text{s.t.} \quad x_k \in \mathbb{X}_k,\ u_k \in \mathbb{U}_k\,,$$

$$\forall w_k \in \mathbb{W} \subseteq \mathbb{R}^n\,,\ \forall k \in \mathbb{Z}_{[0,N\text{-}1]}\,,$$

$$u_k = \sum_{j=0}^{k} L_{k,j}x_j + g_k\,,\ L_{k,j} \in \mathbb{S}_{k,j}\,.$$

- Decisions are nonlinearly propagated through dynamics

$$u_0 = L_0x_0,\ x_1 = Ax_0 + Bu_0\,,$$

$$u_1 = L_1x_1 = L_1(A + BL_0)x_0\,.$$

- Sparsity constraints are linear in the $L_{k,j}$'s

The inherent non-convexity of state feedback decisions hides the complexity of asymmetric information...

Solution Approach: Disturbance Feedback [Goulart, P. J. et al., 2006]

Disturbance Feedback Problem

$$\min_{\mathbf{Q},\mathbf{v}} \quad J(x_0, \mathbf{Q}, \mathbf{v})\,,$$

$$\text{s.t.} \quad \max_{\mathbf{w}\in\mathbb{W}^{N+1}} (F\mathbf{Q}\mathbf{E}+G)\mathbf{w} \leq c_{\mathbf{v}}\,,$$

$$(I + \mathbf{Q}\mathbf{B})^{-1}\mathbf{Q} \in \mathbb{S}\,.$$

- Stacked dynamics $\mathbf{x} = \mathbf{A}x_0 + \mathbf{B}\mathbf{u} + \mathbf{E}\mathbf{w}$
- Decisions over past disturbances
- $\mathbf{u} = \mathbf{Q}\mathbf{E}\mathbf{w}+\mathbf{v}$
- $\mathbb{S}$ stacks the $\mathbb{S}_{k,j}$'s

- Disturbance feedback removes nonlinear propagation of decisions
- Equivalent to the original problem. $\mathbf{L} = (I+\mathbf{Q}\mathbf{B})^{-1}\mathbf{Q}$
- Non-convexity isolated in the sparsity constraints

Main question

Under which conditions are the sparsity constraints convex?

Solution Approach: Disturbance Feedback [Goulart, P. J. et al., 2006]

Disturbance Feedback Problem

$$\min_{\mathbf{Q},\mathbf{v}} \quad J(x_0, \mathbf{Q}, \mathbf{v}) \,,$$

$$\text{s.t.} \quad \max_{\mathbf{w}\in\mathbb{W}^{N+1}} (F\mathbf{QE}+G)\mathbf{w} \leq c_{\mathbf{v}} \,,$$

$$(I + \mathbf{QB})^{-1}\mathbf{Q} \in \mathbb{S} \,.$$

- Stacked dynamics $\mathbf{x} = \mathbf{A}x_0 + \mathbf{Bu} + \mathbf{Ew}$
- Decisions over past disturbances
- $\mathbf{u} = \mathbf{QEw}+\mathbf{v}$
- $\mathbb{S}$ stacks the $\mathbb{S}_{k,j}$'s

- Disturbance feedback removes nonlinear propagation of decisions
- Equivalent to the original problem. $\mathbf{L} = (I+\mathbf{QB})^{-1}\mathbf{Q}$
- Non-convexity isolated in the sparsity constraints

Main question

Under which conditions are the sparsity constraints convex?

Solution Approach: Disturbance Feedback [Goulart, P. J. et al., 2006]

Disturbance Feedback Problem

$$\min_{\mathbf{Q},\mathbf{v}} \quad J(x_0, \mathbf{Q}, \mathbf{v})\,,$$

$$\text{s.t.} \quad \max_{\mathbf{w}\in\mathbb{W}^{N+1}} (F\mathbf{Q}\mathbf{E}+G)\mathbf{w} \le c_{\mathbf{v}}\,,$$

$$(I+\mathbf{Q}\mathbf{B})^{-1}\mathbf{Q} \in \mathbb{S}\,.$$

- Stacked dynamics $\mathbf{x} = \mathbf{A}x_0 + \mathbf{B}\mathbf{u} + \mathbf{E}\mathbf{w}$
- Decisions over past disturbances
- $\mathbf{u} = \mathbf{Q}\mathbf{E}\mathbf{w}+\mathbf{v}$
- $\mathbb{S}$ stacks the $\mathbb{S}_{k,j}$'s

- Disturbance feedback removes nonlinear propagation of decisions
- Equivalent to the original problem. $\mathbf{L} = (I+\mathbf{Q}\mathbf{B})^{-1}\mathbf{Q}$
- Non-convexity isolated in the sparsity constraints

Main question

Under which conditions are the sparsity constraints convex?

Solution Approach: Disturbance Feedback [Goulart, P. J. et al., 2006]

Disturbance Feedback Problem

$$\min_{\mathbf{Q},\mathbf{v}} \quad J(x_0,\mathbf{Q},\mathbf{v})\,,$$
$$\text{s.t.} \quad \max_{\mathbf{w}\in\mathbb{W}^{N+1}} (F\mathbf{QE}+G)\mathbf{w} \leq c_{\mathbf{v}}\,,$$
$$(I+\mathbf{QB})^{-1}\mathbf{Q} \in \mathbb{S}\,.$$

- Stacked dynamics $\mathbf{x} = \mathbf{A}x_0 + \mathbf{Bu} + \mathbf{Ew}$
- Decisions over past disturbances
- $\mathbf{u} = \mathbf{QEw}+\mathbf{v}$
- $\mathbb{S}$ stacks the $\mathbb{S}_{k,j}$'s

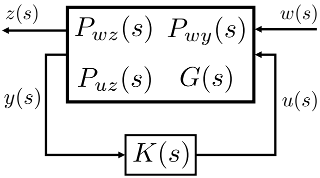
- Disturbance feedback removes nonlinear propagation of decisions
- Equivalent to the original problem. $\mathbf{L} = (I+\mathbf{QB})^{-1}\mathbf{Q}$
- Non-convexity isolated in the sparsity constraints

Main question

Under which conditions are the sparsity constraints convex?

Sparse $\mathcal{H}_2$ Norm Minimization and Youla Parametrizations

[Youla D. et al., 1976]



- Choose $K(s)$ to stabilize the loop
- Minimize the $\mathcal{H}_2$ norm of the closed-loop map $w(s) \rightarrow z(s)$

$\Longrightarrow$ Both are nonlinear in $K(s)$

Define the Youla decision variable $Q(s) = K(s)(I - G(s)K(s))^{-1}$

Sparse $\mathcal{H}_2$ norm minimization (Youla)

$$\min_{Q(s)} \quad ||P_{wz}(s)+P_{uz}(s)Q(s)P_{wy}(s)||_2^2$$

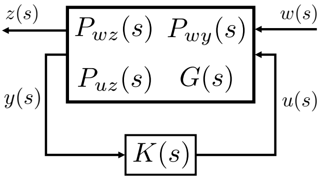
$$\text{s.t.} \quad Q(s) \text{ is stable},$$

$$(I+Q(s)G(s))^{-1}Q(s) \in \mathbb{S}.$$

- Non-convexity isolated in the sparsity constraints

Sparse $\mathcal{H}_2$ Norm Minimization and Youla Parametrizations

[Youla D. et al., 1976]



- Choose $K(s)$ to stabilize the loop
- Minimize the $\mathcal{H}_2$ norm of the closed-loop map $w(s) \rightarrow z(s)$

$\Longrightarrow$ Both are nonlinear in $K(s)$

Define the Youla decision variable $Q(s) = K(s)(I - G(s)K(s))^{-1}$

Sparse $\mathcal{H}_2$ norm minimization (Youla)

$$\min_{Q(s)} \quad ||P_{wz}(s)+P_{uz}(s)Q(s)P_{wy}(s)||_2^2$$

$$\text{s.t.} \quad Q(s) \text{ is stable}\,,$$

$$(I+Q(s)G(s))^{-1}Q(s) \in \mathbb{S}\,.$$

- Non-convexity isolated in the sparsity constraints

Establishing a Connection

Disturbance Feedback Problem

$$\min_{\mathbf{Q},\mathbf{v}} \quad J(x_0,\mathbf{Q},\mathbf{v})\,,$$

$$\text{s.t.} \quad \max_{\mathbf{w}\in\mathbb{W}^{N+1}} (F\mathbf{Q}\mathbf{E}+G)\mathbf{w} \leq c_{\mathbf{v}}$$

$$(I+\mathbf{Q}\mathbf{B})^{-1}\mathbf{Q} \in \mathbb{S}\,.$$

Sparse $\mathcal{H}_2$ minimization (Youla)

$$\min_{Q(s)} ||P_{wz}(s)+P_{uz}(s)Q(s)P_{wy}(s)||_2^2$$

s.t. $Q(s)$ is stable ,

$$(I+Q(s)G(s))^{-1}Q(s) \in \mathbb{S}\,.$$

Disturbance feedback for finite-horizon in state space

$$\equiv$$

Youla parametrization for infinite-horizon in transfer functions

Quadratic Invariance [Rotkowitz M. et al., 2006]

The following sparsity constraints are equivalent

$$(I+Q(s)G(s))^{-1}Q(s) \in \mathbb{S}\,.$$

$$Q(s) \in \mathbb{S}\,.$$

if and only if $\mathbb{S}$ is Quadratically Invariant (QI) w.r.t $G(s)$, that is

$$Q(s)G(s)Q(s) \in \mathbb{S}\,, \forall Q(s) \in \mathbb{S}\,. \tag{QI}$$

- Insight: $(I+QG)^{-1}Q = Q+QGQ+QG(QGQ)\ldots$ lies in $\mathbb{S}$
- The sparse $\mathcal{H}_2$ norm minimization problem is convex if and only if $\mathbb{S}$ is QI with respect to $G(s)$ [Lessard L. et al., 2011]

Quadratic Invariance [Rotkowitz M. et al., 2006]

The following sparsity constraints are equivalent

$$(I+Q(s)G(s))^{-1}Q(s) \in \mathbb{S}\,.$$

$$Q(s) \in \mathbb{S}\,.$$

if and only if $\mathbb{S}$ is Quadratically Invariant (QI) w.r.t $G(s)$, that is

$$Q(s)G(s)Q(s) \in \mathbb{S}\,, \forall Q(s) \in \mathbb{S}\,. \qquad \text{(QI)}$$

- Insight: $(I+QG)^{-1}Q = Q+QGQ+QG(QGQ)\ldots$ lies in $\mathbb{S}$
- The sparse $\mathcal{H}_2$ norm minimization problem is convex if and only if $\mathbb{S}$ is QI with respect to $G(s)$ [Lessard L. et al., 2011]

Convexity of the Robust Distributed Control Problem

The following robust LP is equivalent to the problem under study

Disturbance Feedback Problem

$$\begin{aligned} \min_{\mathbf{Q},\mathbf{v}} \quad & J(x_0, \mathbf{Q}, \mathbf{v})\,, \\ \text{s.t.} \quad & \max_{\mathbf{w}\in\mathbb{W}^{N+1}} (F\mathbf{Q}\mathbf{E}+G)\mathbf{w} \leq c_{\mathbf{v}}\,, \\ & \mathbf{Q} \in \mathbb{S}\,. \end{aligned}$$

if and only if $\mathbb{S}$ is QI with respect to the system.

- General information structure $\mathbb{S}$ stacking the $\mathbb{S}_{k,j}$'s
- Checking QI by verifying $\mathbf{QBQ} \in \mathbb{S}$ for all $\mathbf{Q} \in \mathbb{S}$ is demanding: grows with N

Derive *few* conditions for classes of information structures

Table of Content

Convexity for Sensor-Information Structures

[Furieri L., Kamgarpour M., 2017]

Theorem

Let $\Delta_r = \text{bin}(A^r B)$. The following statements are equivalent.

1. *The Robust Distributed Control Problem is convex.*
2. *Sparse(S) is QI with respect to $A^r B$, $\forall r \in \mathbb{Z}_{[0,n\text{-}1]}$.*
3. *bin$(S\Delta_r S) \leq S$, $\forall r \in \mathbb{Z}_{[0,n\text{-}1]}$.*

- Reachability matrix and convexity are related
- Such relationship allows graph theoretic insights

Convexity for Sensor-Information Structures

[Furieri L., Kamgarpour M., 2017]

Theorem

Let $\Delta_r = \mathit{bin}(A^r B)$. The following statements are equivalent.

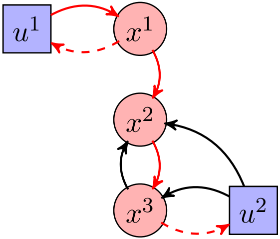
1. *The Robust Distributed Control Problem is convex.*
2. *$\mathit{Sparse}(S)$ is QI with respect to $A^r B$, $\forall r \in \mathbb{Z}_{[0,n\text{-}1]}$.*
3. *$\mathit{bin}(S\Delta_r S) \leq S$, $\forall r \in \mathbb{Z}_{[0,n\text{-}1]}$.*

- Reachability matrix and convexity are related
- Such relationship allows graph theoretic insights

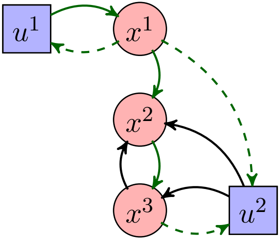
Graph Theoretic Convexity for Sensor-Information Structures

Every time there is a walk $u^i \to \ldots \to x^j \to \ldots \to u^k$, then for every edge $x^l \to u^i$ there must also be an edge $x^l \to u^k$.

non-convex

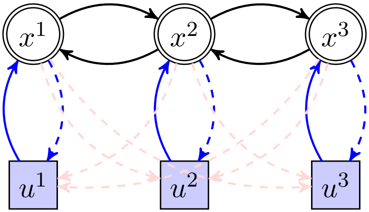


convex



Sensor-Information Structures and Strongly Connected Systems

Every time there is a walk $u^i \to \ldots \to x^j \to \ldots \to u^k$, then for every edge $x^l \to u^i$ there must also be an edge $x^l \to u^k$.

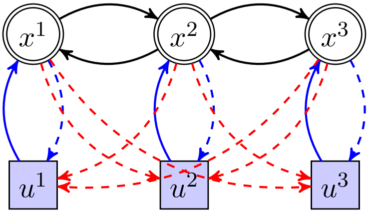


- Full information is required whenever the graph of states is strongly connected

Can we exploit communication to restore convexity?

Sensor-Information Structures and Strongly Connected Systems

Every time there is a walk $u^i \to \ldots \to x^j \to \ldots \to u^k$, then for every edge $x^l \to u^i$ there must also be an edge $x^l \to u^k$.



- Full information is required whenever the graph of states is strongly connected

Can we exploit communication to restore convexity?

Convexity for Sensing and Communication Information Structures [Furieri L., Kamgarpour M., 2018]

Theorem

Let $\Delta_k = \textit{bin}(A^k B)$. Let $\mathcal{D}(Z)$ be the diameter of the communication graph. The following statements are equivalent.

1. *The Robust Distributed Control Problem is convex*
2. $\textit{bin}(S\Delta_k Z^r S) \leq \textit{bin}\left(Z^{k+r+1}S\right)$, $\forall k \in \mathbb{Z}_{[0,n\text{-}1]}$, $\forall r \in \mathbb{Z}_{[0,\mathcal{D}(Z)]}$ *s.t.* $k+r \leq N\text{-}2$

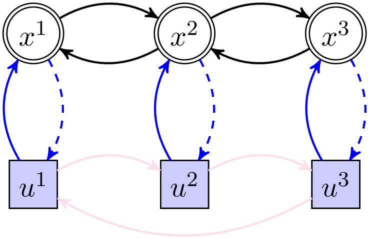
- Few conditions, their number is determined by n and $\mathbb{D}(Z)$
- A^k may grow to be full if system strongly connected
  $\Rightarrow$ The term $Z^{k+r+1}S$ accommodates the growth

Graph Theoretic Convexity for Sensing and Communication Information Structures

- Communication paths: $u^i \rightarrow \dots \rightarrow u^f$
- Dynamic paths: $\underbrace{u^{i_1} \rightarrow \dots \rightarrow u^{i_r}}_{Z^r} \underbrace{\rightarrow}_{B} \underbrace{x^{l_1} \rightarrow \dots \rightarrow x^{l_k}}_{A^k} \rightarrow u^f$

Recall $\text{bin}(S\Delta_k Z^r S) \leq \text{bin}\left(Z^{k+r+1}S\right)$, then

> For every dynamic path from u^{i_1} to u^f, there is a communication path from u^{i_1} to u^f which is **at most as long**



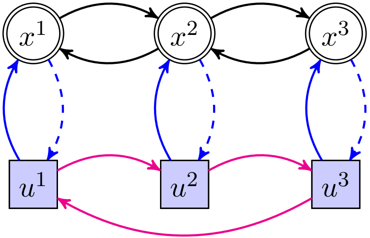
- Convexity for strongly connected systems

Graph Theoretic Convexity for Sensing and Communication Information Structures

- Communication paths: $u^i \to \dots \to u^f$
- Dynamic paths: $\underbrace{u^{i_1} \to \dots \to u^{i_r}}_{Z^r} \underbrace{\to}_{B} \underbrace{x^{l_1} \to \dots \to x^{l_k}}_{A^k} \to u^f$

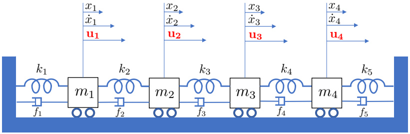
Recall $\text{bin}(S\Delta_k Z^r S) \leq \text{bin}\left(Z^{k+r+1}S\right)$, then

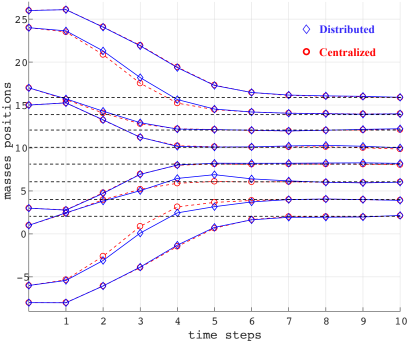
> For every dynamic path from u^{i_1} to u^f, there is a communication path from u^{i_1} to u^f which is **at most as long**



- Convexity for strongly connected systems

Example: masses, springs and dampers





- Minimize total mechanical energy. Max velocity and input are constrained
- Neighbouring controllers share information

- Convexity restored through communication
- Performance of distributed controller is close to full information

Table of Content

Conclusions

1. QI arguments can be ported to the state-space framework in finite horizon
   - Main insight: disturbance feedback parametrizations
   - More general information structures
2. Robust satisfaction of state/input constraints can be included quite naturally
   - Tractability is not affected
3. Communication can be exploited to restore convexity for strongly connected systems

Future work

- Application to receding horizon control is worth investigation. Distributed set invariance [Sadraddini S. et al., 2017]
- The stringent conditions for QI should be overcome. Convex parametrization of solution subspaces for non-QI systems
- Sharing control inputs applied in the past might benefit convexity [Wang et al., 2014, 2017]
- Application to dynamic games where agents have asymmetric information [Colombino et al., 2017]

Thank you for your attention!