An Input-Output Parametrization of Stabilizing Controllers: amidst Youla and System Level Synthesis

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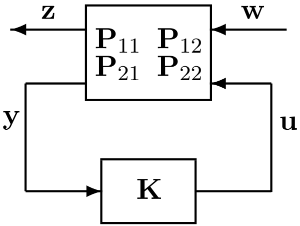
A Fundamental Problem in Control

Given an LTI system (discrete- or continuous-time):

$$\mathbf{z} = \mathbf{P}_{11}\mathbf{w} + \mathbf{P}_{12}\mathbf{u}$$
$$\mathbf{y} = \mathbf{P}_{21}\mathbf{w} + \mathbf{P}_{22}\mathbf{u}$$
$$\mathbf{u} = \mathbf{K}\mathbf{y}$$



Designing optimal stabilizing controller: intractable program

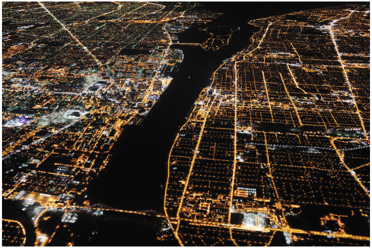
$$\min_{\mathbf{K}} \quad \|f_{\mathbf{w}\to\mathbf{z}}(\mathbf{K})\|_{(\mathcal{H}_2/\mathcal{H}_\infty)} = \|\mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{K}(I - \mathbf{P}_{22}\mathbf{K})^{-1}\mathbf{P}_{21}\|$$

subject to $\quad \mathbf{K} \in \mathcal{C}_{\text{stab}}\,, \quad \mathbf{K} \in \mathcal{K}$

$\Longrightarrow$ A *tractable parametrization* is needed

Beyond Standard Optimal Control

Parametrizing controllers is critical in emerging applications



- Distributed control for large-scale systems
  - Structural requirements on controllers
  - *Tractable parametrization* of structured controllers

    [Furieri et al., 2019]
- Model-based learning control [Dean et al., 2017-2019]
  - Estimate model from noisy data
  - *Tractable parametrization* of robustly stabilizing controllers

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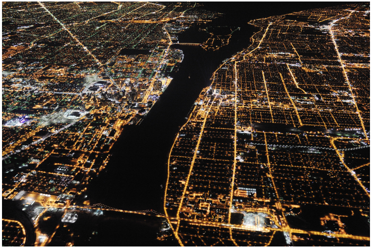

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Classical Approach: Youla Parametrization

[Youla et al., 1986]

1. Find 8 stable transfer matrices $\mathbf{U}_{l,r}, \mathbf{V}_{l,r}, \mathbf{N}_{l,r}, \mathbf{M}_{l,r}$ such that:

$$\mathbf{P}_{22} = \mathbf{N}_r \mathbf{M}_r^{-1} = \mathbf{M}_l^{-1} \mathbf{N}_l \,, \qquad \begin{bmatrix} \mathbf{U}_l & -\mathbf{V}_l \\ -\mathbf{N}_l & \mathbf{M}_l \end{bmatrix} \begin{bmatrix} \mathbf{M}_r & \mathbf{V}_r \\ \mathbf{N}_r & \mathbf{U}_r \end{bmatrix} = I \,.$$

   The above is a *doubly-coprime* factorization of $\mathbf{P}_{22}$

2. Solve a convex program in $\mathbf{Q}$ (equivalent to the original):

$$\min_{\mathbf{Q}} \quad \|\mathbf{T}_{11} + \mathbf{T}_{12}\mathbf{Q}\mathbf{T}_{21}\|$$

$$\text{subject to} \quad \mathbf{Q} \text{ is stable.}$$

   where $\mathbf{T}_{11}$, $\mathbf{T}_{12}$, $\mathbf{T}_{21}$ functions of the *doubly-coprime* factors

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Beyond Doubly-Coprime Factorizations

Youla requires a *doubly-coprime* factorization of $\mathbf{P}_{22}$

- Factorization does not exist in some domains [Anantharam et al., 1985]
- Might be challenging to compute, e.g. delayed plants [Foias et al., 1996], [Laakkonen, 2016]

Existing approaches to avoid factorization of $\mathbf{P}_{22}$:

1. *Coordinate-free* parametrization [Mori,2004]
   - Requires knowing stabilizing controller $\mathbf{K}_0 \in \mathcal{C}_{\text{stab}}$ a-priori
2. *System-level* parametrization [Wang et al, 2017]
   - Based on state-space representation
   - Not directly applicable beyond finite-dimensional LTIs

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Question

Can we bypass plant factorization/$\mathbf{K}_0$ in the frequency domain?

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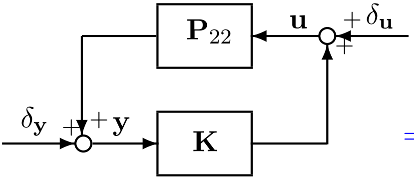
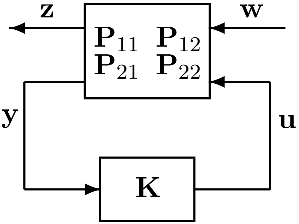
The Input-Output Parametrization (IOP)

We exploit classical internal stability concept



$\Longrightarrow$



Internally stable

Overall stable

The Input-Output Parametrization (IOP)

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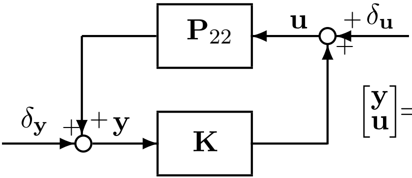

$$\begin{bmatrix}\mathbf{y}\\ \mathbf{u}\end{bmatrix}=\begin{bmatrix}(I\text{-}\mathbf{P}_{22}\mathbf{K})^{-1} & (I\text{-}\mathbf{P}_{22}\mathbf{K})^{-1}\mathbf{P}_{22}\\ \mathbf{K}(I\text{-}\mathbf{P}_{22}\mathbf{K})^{-1} & (I\text{-}\mathbf{K}\mathbf{P}_{22})^{-1}\end{bmatrix}\begin{bmatrix}\delta_{\mathbf{y}}\\ \delta_{\mathbf{u}}\end{bmatrix}$$

- $\mathbf{K} \in \mathcal{C}_{\text{stab}}$ iff the 4 transfer functions above are all stable.

Simple idea: proceed as follows.

1. Denote these functions as $\begin{bmatrix}\mathbf{y}\\ \mathbf{u}\end{bmatrix} = \begin{bmatrix}\mathbf{Y} & \mathbf{W}\\ \mathbf{U} & \mathbf{Z}\end{bmatrix}\begin{bmatrix}\delta_{\mathbf{y}}\\ \delta_{\mathbf{u}}\end{bmatrix}$
2. Derive *mutual relationship* between $(\mathbf{Y}, \mathbf{U}, \mathbf{W}, \mathbf{Z})$
3. Enforce relationship and require $(\mathbf{Y}, \mathbf{U}, \mathbf{W}, \mathbf{Z})$ stable!

The Input-Output Parametrization (IOP)

Theorem 1

The optimal control problem admits the following reformulation:

$$\min_{\mathbf{Y},\mathbf{U},\mathbf{W},\mathbf{Z}} \quad \|\mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{U}\mathbf{P}_{21}\|$$

$$\text{subject to} \quad \begin{bmatrix} I & -\mathbf{P}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{Y} & \mathbf{W} \\ \mathbf{U} & \mathbf{Z} \end{bmatrix} = \begin{bmatrix} I & 0 \end{bmatrix}, \qquad \text{(aff1)}$$

$$\begin{bmatrix} \mathbf{Y} & \mathbf{W} \\ \mathbf{U} & \mathbf{Z} \end{bmatrix} \begin{bmatrix} -\mathbf{P}_{22} \\ I \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix}, \qquad \text{(aff2)}$$

$$\mathbf{Y}, \mathbf{U}, \mathbf{W}, \mathbf{Z} \text{ are stable.} \qquad \text{(stab)}$$

- Convex in $\mathbf{Y}, \mathbf{U}, \mathbf{W}, \mathbf{Z}$! Recover $\mathbf{K} = \mathbf{U}\mathbf{Y}^{-1}$
- Solve directly; no preliminary steps
- Theorem 2: exact mappings Youla $\iff$ IOP (backup slide)
  - They are affine! Convexity independent of parametrization...

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Distributed Control with Youla

Consider the optimal *distributed* control task

$$\begin{aligned} \min_{\mathbf{K}} \quad & \|\mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{K}(I - \mathbf{P}_{22}\mathbf{K})^{-1}\mathbf{P}_{21}\| \\ \text{subject to} \quad & \mathbf{K} \in \mathcal{C}_{\text{stab}}\,, \quad \mathbf{K} \in \mathcal{K} \quad \left(\text{e.g. } \mathbf{K} = \begin{bmatrix} \star & 0 & \star \\ \star & \star & 0 \end{bmatrix} \right) \end{aligned}$$

- Iff Quadratic Invariance (QI) holds, ($\mathbf{K}\mathbf{P}_{22}\mathbf{K} \in \mathcal{K}$, $\forall \mathbf{K} \in \mathcal{K}$):

$$\begin{aligned} \min_{\mathbf{Q}} \quad & \|\mathbf{T}_{11} + \mathbf{T}_{12}\mathbf{Q}\mathbf{T}_{21}\| \\ \text{subject to} \quad & \mathbf{Q} \text{ is stable}, \quad \mathbf{Q} \in \mathcal{K} \end{aligned}$$

  is an equivalent formulation [Rotkowitz et al., 2006]
- Can we solve this problem with IOP?

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Distributed Control with IOP

Theorem 3

Let $\mathcal{K}$ be a subspace such that QI holds. Then, the optimal distributed controller in $\mathcal{K}$ is found by solving

$$\min_{\mathbf{Y},\mathbf{U},\mathbf{W},\mathbf{Z}} \quad \|\mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{U}\mathbf{P}_{21}\|$$

subject to

$$\begin{bmatrix} I & -\mathbf{P}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{Y} & \mathbf{W} \\ \mathbf{U} & \mathbf{Z} \end{bmatrix} = \begin{bmatrix} I & 0 \end{bmatrix}, \qquad \text{(aff1)}$$

$$\begin{bmatrix} \mathbf{Y} & \mathbf{W} \\ \mathbf{U} & \mathbf{Z} \end{bmatrix} \begin{bmatrix} -\mathbf{P}_{22} \\ I \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix}, \qquad \text{(aff2)}$$

$\mathbf{Y}, \mathbf{U}, \mathbf{W}, \mathbf{Z}$ are stable (stab)

$\mathbf{U} \in \mathcal{K}$. (sparsity)

Approximating stable Transfer Functions (TFs)

Both in Youla and IOP, space of stable TFs is *infinite-dimensional*

- In practice, *finite-dimensional* approximation of stable TFs:

$$\mathbf{X} = \sum_{i=0}^{N} X[i] z^{-i}\,, \quad N \in \mathbb{N}\,,\ X[i] \text{ real matrix}\,.$$

- If $N \to \infty$, we encode TFs having any norm.
- For continuous-time, similar approximation

Then, in IOP...

- Constraints (aff1)-(aff2) are affine in $Y[i], U[i], W[i], Z[i]$!
- Sparsity constraints translate to $U[i] \in \mathcal{K}$.

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Application of IOP: Discrete-Time Example

Let $v(z) = \frac{0.1}{z-0.5}$ and $u(z) = \frac{1}{z-2}$, and

$$\mathbf{P}_{22} = \begin{bmatrix} v(z) & 0 & 0 & 0 & 0 \\ v(z) & u(z) & 0 & 0 & 0 \\ v(z) & u(z) & v(z) & 0 & 0 \\ v(z) & u(z) & v(z) & v(z) & 0 \\ v(z) & u(z) & v(z) & v(z) & u(z) \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

with $\mathbf{P}_{11} = \begin{bmatrix} \mathbf{G} & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbf{P}_{12} = \begin{bmatrix} \mathbf{G} \\ I \end{bmatrix}$, $\mathbf{P}_{21} = [\mathbf{G} \quad I]$.

Goal: Compute $\mathcal{H}_2$-optimal controller $\mathbf{K}$ with sparsity of $\mathbf{S}$

(Clearly, $\mathbf{K}\mathbf{P}_{22}\mathbf{K} \in \mathcal{S}$ for $\mathbf{K} \in \mathcal{S}$, hence QI holds)

Application of IOP: Discrete-Time Example

- Cost function $\sum_{i=0}^{N} \text{Trace}(J[i]^\mathsf{T} J[i])$, where

$$J[i] = \begin{bmatrix} W[i] & Y[i] \\ Z[i] & U[i] \end{bmatrix} .$$

- Solve a Quadratic Program (QP)!
  - Efficiently solvable (≤ 1 second)
- For $N \geq 10$, negligible improvement on minimal cost
- $J^\star = 5.67$ without sparsity constraints on $\mathbf{U}$ (centralized)
- $J^\star = 6.73$ with sparsity constraints on $\mathbf{U}$ (distributed)

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Conclusions

- New input-output perspective on controller synthesis
  - Makes output-feedback controller synthesis intuitive
  - Potential application beyond finite-dimensional systems, e.g. *delayed systems*
- Simple implementation for complex control tasks
  - Any convex problem in Youla is convex in IOP and vice-versa
- Open directions include
  - Thorough study of numerical properties in comparison with Youla and SLP
  - Effect of chosen parametrization on sample-complexity bounds for model-based learning

More on Controller Parametrizations

- We have recently shown that

$$\text{Youla} \equiv \text{SLP} \equiv \text{IOP}$$

On the Equivalence of Youla, System-level and Input-output Parameterizations

Yang Zheng, Luca Furieri, Antonis Papachristodoulou, Na Li, and Maryam Kamgarpour

(**Conditionally accepted to TAC**)

- Any convex System-Level-Synthesis (SLS) problem is a convex problem both in the Youla and the IOP parametrizations
- New insights on the SLS approach; check the paper! https://arxiv.org/abs/1907.06256

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Thank you for your attention.

Equivalence With Youla

Q: is Youla $\equiv$ IOP? **A: YES**... when Youla exists!

Theorem 1

Assume that $\mathbf{P}_{22}$ has a doubly-coprime factorization. Then

1. *For any stable Youla parameter $\mathbf{Q}$, the matrices*

$$\mathbf{Y} = (\mathbf{U}_r - \mathbf{N}_r\mathbf{Q})\mathbf{M}_l\,, \quad \mathbf{U} = (\mathbf{V}_r - \mathbf{M}_r\mathbf{Q})\mathbf{M}_l\,,$$
$$\mathbf{W} = (\mathbf{U}_r - \mathbf{N}_r\mathbf{Q})\mathbf{N}_l\,, \quad \mathbf{Z} = I + (\mathbf{V}_r - \mathbf{M}_r\mathbf{Q})\mathbf{N}_l\,,$$

   belong to (aff1)-(aff2) and are stable.

2. *For any stable $\mathbf{Y}, \mathbf{U}, \mathbf{W}, \mathbf{Z}$ in (aff1)-(aff2), the matrix*

$$\mathbf{Q} = \mathbf{V}_l\mathbf{Y}\mathbf{U}_r - \mathbf{U}_l\mathbf{U}\mathbf{U}_r - \mathbf{V}_l\mathbf{W}\mathbf{V}_r + \mathbf{U}_l\mathbf{Z}\mathbf{V}_r - \mathbf{V}_l\mathbf{U}_r\,,$$

   is the Youla parameter giving same closed-loop responses.

The mappings are affine! Convexity not dependent on parametrization...

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Application of IOP: Continuous-Time Example

Consider the same example with $u(s) = \frac{1}{s-1}$ and $v(s) = \frac{1}{s+1}$.

1. Finite-dimensional approximation with $N = 2$ and $a = 3$.
2. Solve (aff1)-(aff2) with no cost. We obtain initial controller

$$\mathbf{K}_0 = \frac{8}{s+7}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 \\ 0 & 2\frac{(s+5)(s+3)}{(s+1)(s+7)} & 0 & 0 & -2 \end{bmatrix}$$

3. Implement SDP similar to [Alavian et al., 2013]

Results

- $J^\star = 6.38$ without sparsity constraints on $\mathbf{U}$ (centralized)
- $J^\star = 7.36$ with sparsity constraints on $\mathbf{U}$ (distributed)
- 0.24 seconds for LP, 7 seconds for SDP