UC Irvine Math 147 Fall 2023

Complex Analysis

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Lecture Note Series #**2**

Introduction

These notes come from both the lecture and the discussion. Sections are numbered chronologically using the following scheme by taking the section number modulo 10:

Date	Lecture	Discussion
Monday	0	1
Tuesday	2	3
Wednesday	4	5
Thursday	6	7
Friday	8	9

Additionally, the first digit (first two if the section number is three digits long) denotes the week that the lecture/discussion occurred in. It should be noted that not every lecture is recorded in these notes: some lectures were skipped, but despite this the notes should be comprehensible.

The text used was *Complex Variables and Applications*, 9e, by Brown and Churchill. Numbers in [brackets] refer to sections in this text.

8 Complex Numbers

[This is the Week 0 Friday lecture.]

We define the notion of a *complex number*, the objects of study in this course.

Definition 8.1. A *complex number* is an element of the field $\mathbb{R}[x]/\left\langle x^2+1\right\rangle$.

To an algebraist, this definition makes sense, but it is not very geometric. We can spell this out some more:

Definition 8.2. A *complex number* $z := (a, b)$ is a point in $\mathbb{R}^2$, where addition and multiplication are defined as follows:

$$(a_1, b_1) + (a_2, b_2) := (a_1 + a_2, b_1 + b_2);$$

$$(a_1, b_1) \cdot (a_2, b_2) := (a_1 a_2 - b_1 b_2, a_2 b_1 + a_1 b_2).$$

We will write these points as $z = (a, b) = a + bi$, where we $i := (0, 1)$, and we define the *real* resp. *imaginary* parts of z by $\mathrm{Re}(z) := a$ and $\mathrm{Im}(z) := b$.

The *set* of complex numbers is denoted by $\mathbb{C}$.

Readers with experience in Math 120B can show the equivalence of these two definitions. Immediately, we get the following consequence, which the reader can verify.

Proposition 8.3. *Let* $z, w \in \mathbb{C}$*. Then* $\mathrm{Re}(z+w) = \mathrm{Re}(z) + \mathrm{Re}(w)$ *and* $\mathrm{Im}(z+w) = \mathrm{Im}(z) + \mathrm{Im}(w)$.

Similarly, the reader, using Definition 8.2, can verify the following:

Theorem 8.4. *The complex numbers* $\mathbb{C}$*, as defined in Definition 8.2, form a field.*

With that being said, division over the complex numbers deserves some special treatment. Take some $z \in \mathbb{C} \setminus \{0\}$. Then since $\mathbb{C}$ is a field, there exists a unique $w \in \mathbb{C}$ such that $zw = 1$. Letting $z = a + bi$, we write

$$w = \frac{1}{z} = \frac{1}{a+bi} = \frac{a-bi}{a^2+b^2} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2} i,$$

and we leave it for the reader to verify that

$$zw = (a+bi)\left(\frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i\right) = 1.$$

We can simplify our formulas from above by introducing two fundamental concepts.

Definition 8.5. Let $z := a+bi$. The *complex conjugate* $\bar{z}$ is the complex number $\bar{z} := a - bi$.

Definition 8.6. Let $z := a+bi$. The *modulus* $|z|$ is the real number $|z| := \sqrt{a^2+b^2}$.

Intuitively, the complex conjugate is the reflection of z about the x-axis, and the modulus is the length of z regarded as a vector in $\mathbb{R}^2$. We get the following proposition:

Proposition 8.7. *Let $z \in \mathbb{C}$ be nonzero. Then $z^{-1} = \bar{z}/|z|^2$.*

Of course, a useful rearrangement of this is as follows:

Corollary 8.8. *Let $z \in \mathbb{C}$ be nonzero. Then $z\bar{z} = |z|^2$.*

10 Complex Conjugation and Polar Form

[6] Complex Conjugation

We introduce these basic properties of complex conjugation, which the reader can verify.

Proposition 10.1. *Let $z, w \in \mathbb{C}$. Then $\bar{\bar{z}} = z$, $|\bar{z}| = |z|$, $\overline{z+w} = \bar{z} + \bar{w}$, and $\overline{zw} = \bar{z}\cdot\bar{w}$.*

We can also take the conjugate of the inverse:

Proposition 10.2. *Let $z \neq 0$. Then $\overline{z^{-1}} = (\bar{z})^{-1}$.*

Proof. Recall that $z^{-1} = \bar{z}/|z|^2$. Then

$$\overline{z^{-1}} = \overline{\left(\frac{\bar{z}}{|z|^2}\right)} = \frac{\bar{\bar{z}}}{|z|^2} = \frac{z}{|z|^2},$$

as $|z|^2$ is real, and

$$(\bar{z})^{-1} = \left(\frac{|\bar{z}|^2}{\bar{\bar{z}}}\right)^{-1} = \frac{z}{|\bar{z}|^2} = \frac{z}{|z|^2} = \overline{z^{-1}}.$$

$\square$

Also, we note the following relationships between a complex number and its conjugate:

Proposition 10.3. *For any $z \in \mathbb{C}$, we have $\mathrm{Re}(z) = \dfrac{z+\bar{z}}{2}$ and $\mathrm{Im}(z) = \dfrac{z-\bar{z}}{2i}$.*

Finally, the following is true regarding the moduli of complex numbers:

Proposition 10.4. *For any $z \in \mathbb{C}$, we have $|zw| = |z||w|$ and $|z^n| = |z|^n$ for any $n \in \mathbb{N}$.*

Proof. We have $|zw|^2 = zw\overline{zw} = zw\bar{z}\bar{w} = z\bar{z}\cdot w\bar{w} = |z|^2|w|^2$, so taking the square root gives us the first of these. Now induct on $n \in \mathbb{N}$. $\square$

We leave it for the reader to show that $|z/w| = |z|/|w|$ when w is nonzero.

[7] Exponential Form

Let $z := x + yi$ be nonzero. We define $r := |z| > 0$, and we define θ to be the angle between the positive x-axis and the vector determined by z, measured counterclockwise. By drawing this out and using trigonometry, we establish that $z = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta) =: r\operatorname{cis}\theta$. This allows us to express a complex number in terms of length and angle, but the angle θ is not unique. We thus define the *argument* of $z := r\operatorname{cis}\theta_0$ to be the set of all θ such that $r\operatorname{cis}\theta = z$.

Example 10.5. Let $z = 1 + i = \sqrt{2}\operatorname{cis}(\pi/4)$. Since both sine and cosine are periodic with period 2π, we see that $\operatorname{cis}(\pi/4) = \operatorname{cis}(\pi/4 + 2k\pi)$ for all $k \in \mathbb{Z}$. Hence

$$\arg(1+i) = \boxed{\left\{\frac{\pi}{4} + 2k\pi : k \in \mathbb{Z}\right\}},$$

or in algebraic coset notation, $\arg(1+i) = \pi/4 + 2\pi\mathbb{Z}$.

From this infinite set of arguments, we often find it helpful to distinguish the "preferred" value of the argument.

Definition 10.6. The *principal argument* of z, denoted $\operatorname{Arg} z$, is the unique element of $\arg z \cap (-\pi, \pi]$.

Of course, this uniqueness comes from the fact that sine and cosine have period 2π.

Now, recall from Math 2B that we have the following Taylor series expansion for cosine and sine, and we can write the following:

$$\cos\theta = \sum_{n=0}^{\infty}(-1)^n\frac{\theta^{2n}}{(2n)!} = \sum_{n=0}^{\infty} i^{2n}\frac{\theta^{2n}}{(2n)!}, \text{ and}$$

$$i\sin\theta = i\sum_{n=0}^{\infty}(-1)^n\frac{\theta^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{i^{2n+1}\theta^{2n+1}}{(2n+1)!},$$

so that

$$\operatorname{cis}\theta = \sum_{n=0}^{\infty}\frac{(i\theta)^n}{n!}.$$

By using the Taylor series for e^x, we see that the equation above can be formally written as $\operatorname{cis}\theta = e^{i\theta}$, which we will do so.

Definition 10.7. We write $e^{i\theta} = \exp(i\theta) := \operatorname{cis}\theta$.

This gives us the *exponential form* of a complex number, $z = re^{i\theta}$. We can verify that this "new" exponential satisfies the properties we are used to:

Proposition 10.8. *For all $\theta, \varphi \in \mathbb{R}$, we have $e^{i\theta}e^{i\varphi} = e^{i(\theta+\varphi)}$.*

Proof. We use the trigonometric definition to see that

$$e^{i\theta}e^{i\varphi} = (\cos\theta + i\sin\theta)(\cos\varphi + i\sin\varphi = (\cos\theta\cos\varphi - \sin\theta\sin\varphi) + i(\cos\theta\sin\varphi + \sin\theta\cos\varphi)$$

$$= \cos(\theta+\varphi) + i\sin(\theta+\varphi) = e^{i(\theta+\varphi)},$$

as desired. □

13 The Triangle Inequality

Often, we will be describing shapes in the complex plane through equations involving moduli of complex numbers. Let us view an example.

Example 13.1. The equation $|z+3-4i| = 5$ represents a circle of radius 5 centered at $-3+4i$, as $|z+3i-4i| = |z-(-3+4i)|$.

The next proposition shows up often in proofs, and is similar to its real analysis analogue.

Proposition 13.2 (Triangle Inequality)**.** *For any $z, w \in \mathbb{C}$, we have $|z+w| \leq |z| + |w|$.*

Proof. Let $z, w \in \mathbb{C}$. then $|z+w|^2 = (z+w)\overline{(z+w)} = (z+w)(\bar{z}+\bar{w}) = z\bar{z} + w\bar{w} + w\bar{z} + z\bar{w}$. We notice that $\overline{w\bar{z}} = z\bar{w}$, so we have that

$$z\bar{z} + w\bar{w} + w\bar{z} + z\bar{w} = z\bar{z} + w\bar{w} + 2\operatorname{Re}(w\bar{z}) = |z|^2 + |w|^2 + 2\operatorname{Re}(w\bar{z}).$$

Certainly, we have $\operatorname{Re}(w\bar{z}) \leq |w\bar{z}|$, so it follows that

$$|z+w|^2 = |z|^2 + |w|^2 + 2\operatorname{Re}(w\bar{z}) \leq |z|^2 + |w|^2 + 2|w\bar{z}| = |z|^2 + |w|^2 + 2|w||z|$$
$$\implies |z+w|^2 \leq (|z| + |w|)^2,$$

so taking square roots finishes the proof. □

The following corollary is extremely useful as well.

Corollary 13.3 (Reverse Triangle Inequality)**.** *For all $z, w \in \mathbb{C}$, we have $|z+w| \geq ||z| - |w||$.*

By induction, we have this:

Corollary 13.4. *Let $\{z_1, \ldots, z_n\} \subseteq \mathbb{C}$. Then $|z_1 + \cdots + z_n| \leq |z_1| + \cdots + |z_n|$.*

Example 13.5. Suppose $|z| = 2$. then $|3 + z + z^2| \leq |3| + |z| + |z^2| = 3 + 2 + 4 = 9$.

The following theorem will be useful when we prove the Fundamental Theorem of Algebra.

Theorem 13.6. *Let $p(z) \in \mathbb{C}[z]$, with $\deg p := n \geq 1$; say $p(z) = a_0 + a_1 z + \cdots + a_n z^n$. Then there exists some $R > 0$ such that whenever $|z| > R$,*

$$\left|\frac{1}{p(z)}\right| < \frac{2}{|a_n|R^n}.$$

Intuitively, this tells us that $1/p(z)$ is somehow "bounded" for a large enough $R > 0$.

Proof. Define the rational function $w(z) := p(z)/z^n - a_n$ for $z \neq 0$, so that $p(z) = (a_n + w(z))z^n$. Now, by the Triangle Inequality,

$$|w(z)| \leq \frac{|a_0|}{|z^n|} + \frac{|a_1|}{|z^{n-1}|} + \cdots + \frac{|a_{n-1}|}{|z|}.$$

The moduli in the numerators of each of the fractions above are fixed real numbers, so there exists a sufficiently large $R > 0$ such that $|a_j|/|z^{n-j}| < |a_n|/2$ for $j \leq n-1$ and any z with $|z| > R$. Hence $|w(z)| < n|a_n|/(2n) = |a_n|/2$. Now, applying the Reverse Triangle Inequality gives

$$|a_n + w(z)| \geq ||a_n| - |w|| > \frac{|a_n|}{2}$$

whenever $|z| > R$, so noting that $p(z) = (a_n + w(z))z^n$, we have $|p(z)| > \dfrac{|a_n|R^n}{2}$. Inverting everything finishes the proof. □

14 Products, Powers, Arguments, and Roots

[8] Products and Powers in Exponential Form

Using our work from Lecture 10, we see that $z_1 z_2 = r_1 r_2 \exp(i(\theta_1 + \theta_2))$, where $z_j = r_j \exp(i\theta)$. In the case that $z := z_1 = z_2$, we have the following.

Proposition 14.1. *For any $z \in \mathbb{C}$, $z^n = r^n \exp(in\theta)$ for all $n \geq 1$.*

Example 14.2. We have $(1-i)^8 = \left(\sqrt{2}\right)^8 \exp(i \cdot 8(-\pi/4)) = 16\exp(-2\pi i) = \boxed{16}$.

The following corollary is from precalculus.

Corollary 14.3 (De Moivre's Formula)**.** *For any $\theta \in \mathbb{R}$, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i \sin n\theta$, for any $n \geq 1$.*

[9] Arguments of Products and Quotients

Let $z = r\exp i\theta$ and $w = s \exp i\varphi$. Then $zw = rs\exp(i(\theta+\varphi))$. Certainly, we have the equality of argument sets $\arg(zw) = \arg(z) + \arg(w)$. Namely, we can view the argument as defining a group homomomorphism $\psi : \mathbb{C}^\times \to \mathbb{R}/2\pi\mathbb{Z}$ by $z \mapsto \arg(z)$. However, the same is not true for the principal argument.

Example 14.4. Let $z = -1$ and $w = i$. Then $zw = -i$, $\operatorname{Arg}(z) = \pi$, $\operatorname{Arg}(w) = \pi/2$, but $\operatorname{Arg}(-i) = -\pi/2 \neq 3\pi/2 = \operatorname{Arg}(z) + \operatorname{Arg}(w)$.

[10] Roots of Complex Numbers

Here, we again see why having infinitely many arguments causes complex analysis to be interesting. Take some $z_0 \in \mathbb{C}$, and fix $n \geq 2$. What is the set of nth roots of z_0; i.e., which $z \in \mathbb{C}$ satisfy $z^n \neq z_0$? The case $z_0 = 0$ is a degenerate case, so we assume $z_0 \neq 0$ and write $z_0 = r_0 \exp(i\theta_0)$. If z is an nth root of z_0, write $z = r\exp(i\theta)$ so that

$$z^n = r^n \exp(in\theta) = r_0 \exp(i\theta_0) = z_0,$$

so we immediately see $r^n = r_0 \implies r = \sqrt[n]{r_0}$. However, we need to use the multivalued nature of the argument and write $in\theta = i\theta_0 \implies n\theta - \theta_0 \in 2\pi\mathbb{Z}$, thus

$$\theta = \frac{\theta_0}{n} + \frac{2k\pi}{n},$$

for any $k \in \mathbb{Z}$. Putting it all together and dealing with periodicity as necessary, we have the following.

Proposition 14.5. *Let $z_0 \neq 0$. Then the nth roots of z_0 are c_k, for $0 \leq k \leq n-1$, given by*

$$c_k := \sqrt[n]{r_0} \exp\left[i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)\right].$$

18 Roots of Complex Numbers, Regions of $\mathbb{C}$

We once again have *many* roots of a complex number z_0. Hence, we fix the following notation.

Definition 18.1. For any $z \in \mathbb{C}^\times$, we define the *set of nth roots* by

$$z^{1/n} := \left\{ \sqrt[n]{r} \exp\left[i \left(\frac{\theta_0}{n} + \frac{2k\pi}{n} \right) \right] \right\},$$

where $r := |z|$.

Again, we would like to select one "preferred" root from our list.

Definition 18.2. For any $z \in \mathbb{C}^\times$, we define the *principal nth root of z* by

$$c_0 := \sqrt[n]{r_0} \exp\left(\frac{i \operatorname{Arg}(z_0)}{n} \right).$$

The nth roots of 1 are of particular interest. If we let $\zeta_n := \exp(2\pi i/n)$, then via Definition 18.2, the set in Definition 18.1 can be easily written as $z^{1/n} = \{c_0 \zeta_n^k\}_{k=0}^{n-1}$, so that $c_j = c_0 \zeta_n^j$.

Example 18.3. Let $z_0 = i$ and $n = 2$. Then $z_0 = 1 \exp(\pi i/2)$, so $r_0 = 1$ and $\operatorname{Arg}(z_0) = \pi/2$. Now we compute

$$c_0 = \sqrt{r_0} \exp\left(\frac{i \operatorname{Arg}(z_0)}{n} \right) = \sqrt{1} \exp\left(\frac{\pi i}{4} \right) = \frac{1}{\sqrt{2}}(1 + i).$$

Seeing that $\zeta_2 = -1$, we have $c_1 = c_0 \zeta_2 = -(1+i)/\sqrt{2}$. Hence $i^{1/2} = \boxed{\{\pm(1+i)/\sqrt{2}\}}$.

Example 18.4. Let us find $(16i)^{1/4}$. We have $|z_0| = 16$ and $\operatorname{Arg}(z_0) = \pi/2$. Now

$$c_0 = \sqrt[4]{16} \exp\left(\frac{i\pi}{2 \cdot 4} \right) = 2 \exp\left(\frac{\pi i}{8} \right).$$

We see $\zeta_4 = i$, so $c_j = c_0 i^j$, hence $c_1 = 2\exp(5\pi i/8)$, $c_2 = -c_0$, and $c_3 = -c_1$.

[12] Regions in $\mathbb{C}$

We now shift to discussing the topology of $\mathbb{C}$. We start with these basic definitions.

Definition 18.5. Let $z_0 \in \mathbb{C}$ and $\varepsilon > 0$. We define the *ε-neighborhood of z_0* by

$$B(z_0, \varepsilon) := \{z \in \mathbb{C} : |z - z_0| < \varepsilon\}.$$

Similarly, we define the *deleted ε-neighborhood of z_0* by

$$D(z_0, \varepsilon) := B(z_0, \varepsilon) \setminus \{z_0\}.$$

These definitions allow us to rigorously say what it means to be "inside" or "outside" a region.

Definition 18.6. Fix some $S \subseteq \mathbb{C}$ and $z_0 \in \mathbb{C}$. We say that:

1. $z_0 \in S$ is an *interior point* of S if there exists some $\varepsilon > 0$ such that $B(z_0, \varepsilon) \subseteq S$.
2. $z_0 \in S$ is an *exterior point* of S if z_0 is an interior point of S^c.
3. z_0 is a boundary point of S if it is neither interior nor exterior. We write $z_0 \in \partial S$.

20 Regions and Functions

We continue discussing the basic topology of $\mathbb{C}$.

Definition 20.1. Fix some $S \subseteq \mathbb{C}$. We say that S is *open* if all of its points are interior. Similarly, we say that S is *closed* if S^c is open.

It follows that if S is closed, then $\partial S \subseteq S$. Notice that a set can be neither open nor closed:

Example 20.2. Take $S := \{|z| \geq 1\} \setminus \{i\}$. Notice that $\partial S \setminus \{i\} \subseteq S$, but $i \in S$, so S contains *most* of its boundary points except for i. Hence, S is neither open nor closed.

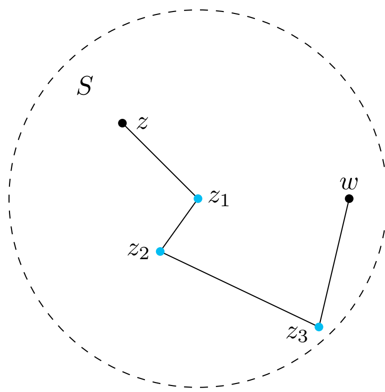
When a set is not closed, we sometimes find it helpful to close it.

Definition 20.3. Let $S \subseteq \mathbb{C}$. We say that the *closure* of S is the set $\overline{S} = S \cup \partial S$.

We also define what it means for an open set to be "all in one piece."

Definition 20.4. Let $S \subseteq \mathbb{C}$ be open. Then S is *connected* if for any $z, w \in S$, there exists a finite subset $\{z_i\}_{i=0}^n \subseteq S$ such that $z_0 = z$, $z_n = w$, and the line segment from w_j to w_{j+1} is contained in S for all $j \leq n-1$.

Example 20.5. Let S be the open disk below and consider the two black points z and w.



Notice that the polygonal line going through z_1, z_2, and z_3 is completely within S. Of course, this line is not unique, but the point stands that S is connected.

These definitions will be important once we begin our discussion of functions.

Definition 20.6. Let $S \subseteq \mathbb{C}$. Then S is a *domain* if S is nonempty, open, and connected.

Definition 20.7. Let S be a domain, and $T \subseteq \partial S$. Then a *region* is any set of the form $S \cup T$.

Example 20.8. Consider the *annulus* $S := \{1 < |z| < 2\}$. Then S is a domain, as it is open and connected. Now, the set $\{|z| < 1\} \cup \{|z| > 2\}$, the "pseudo-complement" of S, is not a domain as it is not connected.

Definition 20.9. Let $S \subseteq \mathbb{C}$. Then S is *bounded* if there exists some $M > 0$ such that $S \subseteq B(0, M)$.

That is, S is bounded if it is contained inside a large enough circle centered at the origin.

Definition 20.10. Let $S \subseteq \mathbb{C}$. Then $z_0 \in \mathbb{C}$ is an *accumulation point of* S if for every $\varepsilon > 0$, we have that $D(z_0, \varepsilon) \cap S$ is nonempty.

The reader can verify that a set is closed if and only if it contains all of its accumulation points.

[13] Functions and Mappings

Now, we are ready to discuss functions which take in a complex variable.

Definition 20.11. Take $S \subseteq \mathbb{C}$. A *function* f is a rule $f : S \to \mathbb{C}$. We write $w = f(z)$ to specify a function.

Usually, we will not specify the domain set S, but it will be inferred from context.

Example 20.12. Let $f(z) := 1/z$. Then, the function's understood domain is $\operatorname{dom} f = \mathbb{C}^\times$.

Example 20.13. Let $p, q \in \mathbb{C}[z]$. Then the rational function $f(z) := p(z)/q(z)$ has domain $\operatorname{dom} f = \mathbb{C} \setminus \ker q$.

Now, we know that a complex number can be decomposed into its real and imaginary parts, say $z = x + yi$. But the same is true for functions: if f is a function, we can certainly write $w = f(z) = f(x + yi) = u(x, y) + v(x, y)i$. This tells us that f behaves a lot like a pair of functions of real numbers.

Example 20.14. Let $f(z) = 1/z = 1/(x + yi)$. Then

$$f(z) = \frac{x}{x^2 + y^2} - \frac{y}{x^2 + y^2} i,$$

so we have $u(x, y) = x/(x^2 + y^2)$ and $v(x, y) = -y/(x^2 + y^2)$.

Similarly, we can decompose a complex function into polar coordinates. Instead of $z = x+yi$, write $z = re^{i\theta}$ and thus $w = f(z) = u(r, \theta) + iv(r, \theta)$.

Example 20.15. Let $f(z) = 1/z$. Taking $z = re^{i\theta}$, we see that

$$f(z) = \frac{1}{r} e^{-i\theta} = \frac{1}{r} \cos\theta - \frac{1}{r} i \sin\theta,$$

so $u(r, \theta) = \frac{1}{r} \cos\theta$ and $v(r, \theta) = -\frac{1}{r} \sin\theta$.

Of course, the multivalued nature of the argument can cause issues. Hence, we define the following.

Definition 20.16. Let $S \subseteq C$. Then a *multivalued function* is a rule $f : S \to \mathcal{P}(\mathbb{C})$.

Example 20.17. Let $f(z) = z^{1/2}$. Then $f(z) = \pm\sqrt{r} \exp((i \operatorname{Arg} z)/2)$, which sends a complex number z to a set with 2 elements (or 1 if $z = 0$). This can be split into two single-valued functions by either choosing the positive or negative square root.

23 The Mapping $w = z^2$

This is [14] in the text.

24 Limits and Derivatives

[15] Limits

Many of the limit/derivative rules from real analysis carry over to complex analysis. However, we should be responsible and define the terms below.

Definition 24.1. Let $S \subseteq \mathbb{C}$ and $f : S \to \mathbb{C}$, and suppose $D(z_0, r) \subseteq S$ for a fixed $z_0 \in \mathbb{C}$. Then we say that $\lim_{z \to z_0} f(z) = w_0$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for each $z \in D(z_0, r)$, we have $|z - z_0| < \delta$ implies $|f(z) - w_0| < \varepsilon$.

Theorem 24.2. *If $\lim_{z \to z_0} f(z)$ exists, then it is unique.*

Example 24.3. Let $f : \{z : |z| < 1\} \to \mathbb{C}$ by $z \mapsto \bar{z}i/2$. Notice that 1 is not in the domain of f, but we claim that $\boxed{\lim_{z \to 1} f(z) = \frac{i}{2}}$.

Proof. Fix $\varepsilon > 0$, and let $\delta := 2\varepsilon$. Now, if $|z| < 1$ and $|z - 1| < \delta = 2\varepsilon$, we see that

$$\varepsilon > \frac{|z-1|}{2} = \frac{|\bar{z}-1|}{2} = \left|\frac{i}{2}\right| |\bar{z} - 1| = \left|f(z) - \frac{i}{2}\right|,$$

so our claim is proved. $\square$

The next example illustrates how Theorem 24.2 is useful for causing a contradiction.

Example 24.4. We show that $\lim_{z \to 0} \bar{z}/z$ does not exist. If $z \in \mathbb{R}^\times$, then $\bar{z}/z = 1$, so the limit approaching along the real line is 1. However, if $z \in i\mathbb{R}^\times$, then $\bar{z}/z = -1$. Hence, the limit along the imaginary axis is -1. If the limit exists, we must have $1 = -1$, which is impossible.

[16] Limit Theorems

All of these should be familiar to us from real analysis, but the first of these theorems allows us to split a complex limit into two real limits:

Theorem 24.5. *Let $z = x + yi$, and write $f(z) = f(x + yi) = u(x, y) + iv(x, y)$, where u, v are real-valued. Let $z_0 := x_0 + y_0 i$ and $w_0 = u_0 + v_0 i$. Then $\lim_{z \to z_0} f(z) = w_0$ if and only if*

$$\lim_{(x,y) \to (x_0,y_0)} u(x, y) = u_0 \text{ and } \lim_{(x,y) \to (x_0,y_0)} v(x, y) = v_0.$$

Theorem 24.6 (Limit Laws)**.** *Suppose $\lim_{z \to z_0} f(z) = L_1$, and $\lim_{z \to z_0} F(z) = L_2$. Then*

1. $\lim_{z \to z_0} (f(z) \pm F(z)) = L_1 \pm L_2$;
2. $\lim_{z \to z_0} f(z)F(z) = L_1 L_2$,
3. *If $L_2 \neq 0$, then* $\lim_{z \to z_0} f(z)/F(z) = L_1/L_2$.

These basic limit calculations are the same as in real analysis.

Example 24.7. Let c be a constant, and p, q be polynomials. Then for any $z_0 \in \mathbb{C}$:

1. $\lim_{z \to z_0} c = c$.
2. $\lim_{z \to z_0} z = z_0$.
3. $\lim_{z \to z_0} p(z) = p(z_0)$.
4. If $q(z_0) \neq 0$, then $\lim_{z \to z_0} p(z)/q(z) = p(z_0)/q(z_0)$.

All of the functions above are examples of *continuous* functions.

[18] Continuity

Definition 24.8. We say that f is *continuous* at z_0 if $\lim_{z \to z_0} f(z) = f(z_0)$. Similarly, f is *continuous on a region* $R \subseteq \mathbb{C}$ if f is continuous at every point $z_0 \in R$.

Again, all of the usual facts carry over.

Theorem 24.9. *If f and g are continuous at z_0, then so are $f \pm g$, and fg, and if $g(z_0) \neq 0$, then f/g is also continuous.*

Theorem 24.10. *If f is continuous at z_0 and g is continuous at $f(z_0)$, then $g \circ f$ is continuous at z_0.*

Theorem 24.11. *If f is continuous at z_0 and $f(z_0) \neq 0$, then there exists some $\delta > 0$ such that $f(z) \neq 0$ for all $z \in B(z_0, \delta)$.*

Again, we can consider splitting a function into real and imaginary parts:

Theorem 24.12. *Let $z = x + yi$, and $f(z) = u(x, y) + iv(x, y)$, and let $z_0 := x_0 + y_0 i$. Then f is continuous at z_0 if and only if both u and v are continuous at (x_0, y_0).*

Recall the Extreme Value Theorem from real analysis, which tells us that a function on a closed interval is bounded. The following is a similar analogue.

Theorem 24.13. *Suppose $R \subseteq \mathbb{C}$ is a closed, bounded region and f is continuous on R. Then f is bounded on R, i.e., there exists some $M > 0$ such that $|f(z)| \leq M$ for all $z \in R$.*

[19] Derivatives

Finally, we begin our discussion of derivatives. Notice that the definition below seems innocuous, but we will quickly see how much more powerful having a complex derivative is.

Definition 24.14. Let f be defined on a neighborhood of z_0. Then f is *differentiable* at z_0 if the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} =: f'(z_0)$$

exists. We call the number $f'(z_0)$ the *derivative of f at z_0* whenever it exists.

28 Derivatives and the Cauchy-Riemann Equations

The familiar derivatives still hold in complex analysis.

Proposition 28.1. *Let c be a constant. Then $c' = 0$, $z' = 1$, and $(cf(z))' = cf'(z)$.*

Proposition 28.2 (Derivative Laws). *Let f, g be differentiable. Then*

1. $(f + g)' = f' + g'$;
2. $(fg)' = f'g + g'f$;
3. $(f \circ g)' = (f' \circ g)(g')$.

Now, we view some examples.

Example 28.3. Let $f(z) = \bar{z}$. Does $f'(0)$ exist? If $w = f(z)$, then

$$\frac{\Delta w}{\Delta z} = \frac{\overline{0 + \Delta z} - \bar{0}}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z}.$$

From Example 24.4, we see that the limit $\lim\limits_{\Delta z \to 0} \overline{\Delta z}/\Delta z$ cannot exist, so $f'(0)$ does not exist.

Example 28.4. Let $f(z) = |z|^2$. Now $\Delta w = |z + \Delta z|^2 - |z|^2 = z\overline{\Delta z} + \bar{z}\Delta z + \Delta z\overline{\Delta z}$, so that

$$\frac{\Delta w}{\Delta z} = z\frac{\overline{\Delta z}}{\Delta z} + \bar{z} + \overline{\Delta z}.$$

Now, as $\Delta z \to 0$ along the real axis, $\Delta w/\Delta z \to z + \bar{z}$. Along the imaginary axis, we have $\Delta w/\Delta z \to -z + \bar{z}$. If the derivative exists, it can only exist when $z = 0$, and indeed, we can check that at 0, we have $f'(0) = 0$.

The above example demonstrated a pathology. Writing $f(x + yi) = x^2 + y^2$ as per above, we see that $u(x, y) = x^2 + y^2$ and $v(x, y) = 0$, which are both nicely differentiable for any point in $\mathbb{R}^2$. However, f behaves strangely over $\mathbb{C}$, so what conditions do we need so that $f'(z)$ is "nice" over $\mathbb{C}$?

[21] Cauchy-Riemann Equations

Suppose $f'(z_0)$ exists, and we write $f(z) = u(x, y) + iv(x, y)$. Then we see that

$$\Delta w = [u(x_0 + \Delta x, y_0 + \Delta y) + iv(x_0 + \Delta x, y_0 + \Delta y)] - [u(x_0, y_0) - iv(x_0, y_0)],$$

so by rearranging, we write

$$\Delta w = [u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)] + i[v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)].$$

If we fix $\Delta y = 0$ and let Δx approach 0, we can check that $f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$, where u_x, v_x are partial derivatives with respect to x. Similarly, we check that $f'(z_0) = -i(u_y(x_0, y_0) + iv_y(x_0, y_0))$. This gives us a system of equations to solve, and thus the following theorem.

Theorem 28.5 (Cauchy-Riemann Equations). *Let f be differentiable at $z_0 := x_0 + y_0 i$. Then if $f(z) = u(x, y) + iv(x, y)$, the partials of u and v must satisfy*

$$u_x(x_0, y_0) = v_y(x_0, y_0) \text{ and } v_x(x_0, y_0) = -u_y(x_0, y_0).$$

Let us view some examples.

Example 28.6. Let $f(z) = z^2$. Then $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$. Hence $u_x = v_y = 2x$ and $u_y = -2y = -v_x$, so the Cauchy-Riemann Equations (CR) both hold. Hence, f *might be* differentiable everywhere, as the converse to Theorem 28.5 does not necessarily hold.

Example 28.7. Referring to Example 28.4, let $f(z) = |z|^2$. We saw that $u(x, y) = x^2 + y^2$ and $v(x, y) = 0$, so $v_x = v_y = 0$, $u_x = 2x$, and $u_y = 2y$. Hence, CR holds only at $x = y = 0$, which makes sense.

Example 28.8. Let $f(z)$ be the piecewise function defined below:

$$f(z) := \begin{cases} \bar{z}^2/z & z \neq 0 \\ 0 & z = 0. \end{cases}$$

Then u, v are given piecewise as well:

$$u(x, y) = \begin{cases} \frac{x^3 - 3xy^2}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = 0 \end{cases} \text{ and } v(x, y) = \begin{cases} \frac{y^3 - 3x^2 y}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = 0. \end{cases}$$

Now we can check that CR holds at $z = 0$, but that $f'(0)$ does not exist. We stress again that CR is necessary but not sufficient.

30 CR Equations and Analytic Functions

Recall that stating that $f : \mathbb{C} \to \mathbb{C}$ being differentiable is much stronger than saying that the corresponding $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable. For one, f needs to satisfy the CR equations, but even this is not sufficient. However, this is very close to a sufficient condition.

Theorem 30.1. *Let $f : \mathbb{C} \to \mathbb{C}$, and write $f(z) = f(x + yi) = u(x, y) + iv(x, y)$. Assume f is defined in some ε-neighborhood of some $z_0 := x_0 + y_0 i \in \mathbb{C}$, and also assume that the first order partial derivatives of u, v exist and are continuous at z_0, and that these partials satisfy the Cauchy-Riemann equations at z_0. Then $f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$.*

Example 30.2. Let $f(z) = |z|^2 = x^2 + y^2$. Though we can compute $f'(0)$ using the limit definition of the derivative, we can also use the previous theorem.

Proof. We have $u(x, y) = x^2 + y^2$ and $v(x, y) = 0$. Clearly, u and v has partial derivatives that exist at 0 and are continuous at 0, and we have seen before that the CR equations are satisfied at $z = 0$. Hence, $f'(0)$ exists. □

Again, in view of Example 28.7, $f(z) = |z|^2$ is differentiable at exactly 1 point. This contrasts with the map $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$ by $(x, y) \mapsto (x^2 + y^2, 0)$, which is differentiable everywhere.

[25] Analytic Functions

Normally, we concern ourselves with functions that are differentiable at least on a certain region. Hence, we make the following definition.

Definition 30.3. Let $f : \mathbb{C} \to \mathbb{C}$. We say that f is *analytic in an open set $S \subseteq \mathbb{C}$* if f is differentiable at every $s \in S$. Similarly, we say that f is *analytic at $z_0 \in \mathbb{C}$* if there exists a neighborhood of z_0 such that f is analytic in that neighborhood.

Analytic functions are all we really care about in complex analysis!

Example 30.4. From Example 30.2, $f(z) = |z|^2$ is not analytic anywhere: it is differentiable at the single point $z_0 = 0$ but not in every neighborhood around z_0.

Example 30.5. In contrast, $f(z) = 1/z$ is analytic everywhere on its domain, due to the derivative laws.

Analytic functions satisfy many nice properties. One of them is below, which should be intuitive.

Theorem 30.6. *Assume $f'(z) = 0$ for all z in a domain $D \subseteq \mathbb{C}$. Then $f(z)$ is constant on D.*

[27] Harmonic Functions

Now, we combine analyticity with the Cauchy-Riemann equations to give us information about real-valued functions.

Definition 30.7. Let $H : \mathbb{R}^2 \to \mathbb{R}$. We say that H is *harmonic* if $H_{xx} + H_{yy} = 0$.

The equation that defines a harmonic function is known as the *Laplace equation*. It turns out that analytic functions are very much related to harmonic ones:

Theorem 30.8. *If $f = u + vi$ is analytic on a domain D, then the components $u, v : \mathbb{R}^2 \to \mathbb{R}$ are both harmonic functions.*

Proof. Because f is analytic, we know $u_x = v_y$ and $u_y = -v_x$. We claim that we can just continue to differentiate these partials, and we will see why later in these notes, but then we observe $u_{xx} = v_{yx}$ and $u_{yy} = -v_{xy}$. By equality of mixed partials, $v_{xy} = v_{yx}$, yielding $u_{xx} + u_{yy} = 0$, so u is harmonic. Using a similar argument, v is harmonic. $\square$

Proposition 30.9. *Assume that $f(z)$ and $\overline{f(z)}$ are both analytic in a neighborhood. Then $f(z)$ is constant. Furthermore, if $|f(z)|$ is constant and f is analytic, then f is constant.*

Proof. We prove the second part. Write $f \cdot \bar{f} = |f|$, which is constant by assumption, so $\bar{f} = |f|/f$. Then, apply the first claim. $\square$

34 Exponentiation

We have already seen the notation $e^{i\theta}$ before. But what about e^z, where z is any complex number?

Definition 34.1. If $z = x + yi \in \mathbb{C}$, then we define $e^{x+yi} := e^x \cdot e^{yi} = e^x(\cos y + i \sin y)$.

It immediately follows that $|e^{x+yi}| = e^x$ and $\arg(e^{x+yi}) = y + 2\pi\mathbb{Z}$. Hence, $|e^z| \neq 0 \implies e^z \neq 0$ for all $z \in \mathbb{C}$, but we should note that it is possible that $e^z < 0$: take $z = \pi i$.

Proposition 34.2. *For every $z, w \in \mathbb{C}$, we have $e^z \cdot e^w = e^{z+w}$.*

Hence, the exponential still behaves the way we are used to it behaving.

Corollary 34.3. *For every $z, w \in \mathbb{C}$, we have $e^{z-w} = e^z/e^w$.*

Proof. Write $e^z = e^{z-w+w} = e^{z-w}e^w$. Solving for e^{z-w} gives the desired result. $\square$

Theorem 34.4. *The function $f(z) = e^z$ is analytic on $\mathbb{C}$, with derivative $(e^z)' = e^z$.*

Proof. Write $z = x + yi$, so $f(z) = f(x + yi) = e^x \cos y + ie^x \sin y$, so $u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$. We have $u_x = e^x \cos y = v_y$ and $u_y = -e^x \sin y = -v_x$, so the CR equations are satisfied. Furthermore, these partials are continuous on $\mathbb{R}^2$, so f is differentiable on $\mathbb{C}$. By computing, we see $f'(z) = u_x + v_x i = e^x(\cos y + i \sin y) = f(z)$. $\square$

37 CR Equations

Definition 37.1. We define the following operator: $\dfrac{\partial}{\partial \bar{z}} := \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y}\right)$.

Using this operator, we will find an alternative form of the Cauchy-Riemann equations. Suppose $f = u + vi$ satisfies CR. Then, we can check that

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} + i\frac{\partial v}{\partial y}\right)$$

$$= \frac{1}{2}\left[u_x - v_y + i(u_y + v_x)\right] = \frac{1}{2}(0 + 0) = 0,$$

where the simplification was done by assuming that f satisfied CR. Thus, we have derived a different form of the Cauchy-Riemann equations.

Proposition 37.2. *If f is differentiable at z_0, then we have $\partial f/\partial \bar{z} = 0$ at z_0.*

Intuitively, this tells us that if a function is analytic, then there is no "dependence" on varying $\bar{z}$. Of course, that $\partial/\partial \bar{z}$ is defined in a way that makes sense needs to be checked. Recall that

$$x = \frac{z + \bar{z}}{2} \text{ and } y = \frac{z - \bar{z}}{2i}.$$

By the multivariable chain rule, we see

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right),$$

which of course motivates our definition.

[28] Uniqueness

In this section, we prove a very strong condition involving analytic functions being constant.

Theorem 37.3. *If f is analytic on a domain D, and if $f(z) = 0$ on some open ball $B(z_0, \delta) \subseteq D$ or $f(z) = 0$ on some line segment $L \subseteq D$, then $f = 0$ on all of D.*

This statement is obviously false for real-valued functions, yet it works in the complex realm. To prove this, we first state a lemma to be proven later.

Lemma 37.4. *If f is analytic on a domain D containing z_0, and $f = 0$ on some neighborhood $N_0 \subseteq D$ containing z_0 or on some line segment $L \subseteq D$ through z_0, then $f = 0$ on D.*

With this, we are ready to prove the main theorem.

Proof of Thm. 37.3. Let f be analytic on a domain D.

First, suppose that there exists some open ball $B(z_0, \delta) \subseteq D$ such that $f(z) = 0$ for all $z \in B(z_0, \delta)$. Pick any point $p \in D$. Since D is connected, then there is some piecewise curve γ connecting z_0 and p. Let d be the minimum of δ and the distance from γ to ∂D, and construct a finite sequence of points on γ, namely $z = z_0, z_1, \ldots, z_{n-1}, z_n = p$ such that $|z_j - z_{j-1}| < d$ for every $2 \leq j \leq n$. Construct the open neighborhoods $N_j := B(z_j, d)$. Our choice of d forces $N_j \cap N_{j+1} \neq \varnothing$ and $N_j \subseteq D$. By Lemma 37.4, we have $f = 0$ on N_0, and $z_1 \in N_0 \cap N_1$. This implies that $f = 0$ on N_1 by Lemma 37.4, and keep applying the lemma to see that $f(p) = 0$. Hence, we are done.

The case of the line segment is similar. ☐

Corollary 37.5 (Uniqueness Theorem)**.** *Suppose f, g are analytic on a domain D, and $f(z) = g(z)$ on some subdomain $E \subseteq D$ or on some line segment $L \subset D$. Then $f = g$ on D.*

Proof. Take $h := f - g$ and apply Theorem 37.3. ☐

38 Logarithms and Branch Cuts

Let $z \neq 0$, and let $w = \log z$, whatever that means. We expect $z = e^w$. Over the real numbers, there is no problem inverting the exponential, but over $\mathbb{C}$, the exponential function is not injective: if $w = u + vi$, then $z = e^w \implies re^{i\theta} = e^u e^{vi}$, so $v \in \theta + 2\pi\mathbb{Z}$. Hence, the logarithm is multivalued.

Definition 38.1. If $z := re^{i\theta}$, then $\log z := \ln r + i(\theta + 2\pi\mathbb{Z})$, where $\ln$ denotes the natural logarithm over the positive real numbers.

Again, we would like to find the "principal" value of the logarithm. It would be nice if we had $\log e^x = x$, so by writing $z = x + yi$, we see

$$\log e^z = \ln e^x + i(y + 2\pi\mathbb{Z}) = x + yi + 2\pi i\mathbb{Z} = z + 2\pi i\mathbb{Z}.$$

Taking $\log e^z = z + 2\pi n$ for some n, we want $n = 0$ so that $\log e^z = z$.

Definition 38.2. Let $z = re^{i\theta} \neq 0$. We define the *principal logarithm* by $\operatorname{Log}(z) := \ln r + i \operatorname{Arg} z$.

We immediately have that $\log z = \operatorname{Log} z + 2\pi i\mathbb{Z}$. We also see the following.

Proposition 38.3. *Let $z \neq 0$. Then $e^{\operatorname{Log} z} = z$ and $\operatorname{Log}(e^z) = z$.*

Let us view some examples.

Example 38.4. Let $z = 1 - i = \sqrt{2} \exp\left(-i\frac{\pi}{4}\right)$. Then $\operatorname{Log} z = \ln \sqrt{2} + i\left(-\frac{\pi}{4}\right) = \frac{1}{2}\ln 2 - \frac{\pi i}{4}$, and $\log z = \ln \sqrt{2} + i\left(-\frac{\pi}{4} + 2\pi\mathbb{Z}\right)$.

Example 38.5. Let $z = -1 = 1\exp(\pi i)$. Then $\operatorname{Log}(-1) = \ln 1 + \pi i = \pi i$, and $\log(-1) = i(\pi + 2\pi\mathbb{Z})$.

Example 38.6. Notice that $\operatorname{Log}[(-1+i)^2] = \operatorname{Log}(-2i) = \ln 2 - \frac{\pi i}{2}$, but

$$2\operatorname{Log}(-1+i) = 2\left[\ln\sqrt{2} + i\left(\frac{3\pi}{4}\right)\right] = \ln 2 + \frac{3\pi i}{2} \neq \operatorname{Log}[(-1+i)]^2.$$

Hence, not all properties hold over for the complex principal logarithm, so this is not a problem for the multivalued $\log$, at least here.

Example 38.7. From Example 38.5, $\log(i^2) = \log(-1) = i(\pi + 2\pi\mathbb{Z})$. But $2\log i = i(\pi + 4\pi\mathbb{Z})$, which are not equal as sets.

[33] Branch Cuts

We have seen that $\operatorname{Log} z = \ln r + i\theta$, where $\theta = \operatorname{Arg} z$. However, other angle restrictions are possible: given any $\alpha \in \mathbb{R}$, choose $\theta \in (\alpha, \alpha + 2\pi)$, and we write $\underset{\alpha}{\log} z := \ln r + i\theta$. This allows us to consider a single-valued *branch* of the logarithm multivalued functions. We exclude the endpoints $\alpha, \alpha + 2\pi$ so that we can take the derivative — in fact, this exclusion causes $\underset{\alpha}{\log}$ to be analytic.

We first state the polar form of the CR equations.

Proposition 38.8. *Let $z = re^{i\theta}$, and let $f(z) = u(r,\theta) + iv(r,\theta)$. If $f'(z)$ exists , then $ru_r = v_\theta$ and $u_\theta = -rv_r$.*

For $f(z) = \underset{\alpha}{\log} z$, we see that $u(r,\theta) = \ln r$ and $v(r,\theta) = \theta$. We have $u_r = 1/r$, $u_\theta = v_R = 0$, and $v_\theta = 1$, so CR is satisfied. Furthermore, the derivatives are continuous, so $\underset{\alpha}{\log}$ is differentiable, and we get the expected formula:

Proposition 38.9. *Let $\alpha \in \mathbb{R}$. Then the branch cut $\underset{\alpha}{\log}$ is analytic on its domain, and*

$$\frac{d}{dz}\left[\underset{\alpha}{\log} z\right] = \frac{1}{z}.$$

In particular, choosing $\alpha = -\pi$ gives us the principal logarithm, so $(\operatorname{Log} z)' = 1/z$.

Example 38.10. For the multivalued logarithm, we saw that $\log(i^2) \neq 2\log i$. But when we choose the branch cut $\underset{\pi/4}{\log}$, we see

$$\underset{\pi/4}{\log}(-1) = \underset{\pi/4}{\log}(i^2) = \pi i,$$

and

$$2\underset{\pi/4}{\log} i = 2 \cdot \frac{\pi i}{2} = \pi i = \underset{\pi/4}{\log}(i^2).$$

We end with the formal definition of a branch.

Definition 38.11. If f is a multivalued function, a *branch* of f is any single-valued function F, analytic in some domain such that at each point in the domain, we have $F(z) \in f(z)$.

44 Complex Trigonometry

Recall that $e^{ix} = \cos x + i\sin x$, where x is real. But this implies $e^{-ix} = \cos x - i\sin x$, so that we see that the (real) sine and cosine are given by

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \text{ and } \sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$

We generalize this definition to arrive at the complex sine and cosine.

Definition 44.1. The *sine* and *cosine* functions of a complex variable z are defined in terms of the exponential:

$$\cos z := \frac{e^{iz} + e^{-iz}}{2} \text{ and } \sin z := \frac{e^{iz} - e^{-iz}}{2i}.$$

These two functions still have their expected derivatives.

Proposition 44.2. *The functions $\sin z$ and $\cos z$ are entire, and we have $(\sin z)' = \cos z$ and $(\cos z)' = -\sin z$.*

Proof. By the definition and the fact that the exponential is entire, we observe that

$$\frac{d}{dz} \sin z = \frac{1}{2i}(ie^{iz} - (-i)e^{-iz}) = \frac{1}{2}(e^{iz} + e^{-iz}) = \cos z.$$

The other derivative follows similarly. □

Additionally, sine and cosine still satisfy their usual trigonometric properties.

Proposition 44.3. *Let $z, w \in \mathbb{C}$. Then*

1. $\sin(-z) = -\sin z$, $\cos(-z) = \cos z$;
2. $e^{iz} = \cos z + i \sin z$;
3. $\sin(z + w) = \sin z \cos w + \sin w \cos z$;
4. $\cos(z + w) = \cos z \cos w - \sin z \sin w$;
5. $\sin^2 z + \cos^2 z = 1$;
6. $\sin z = \sin(z + 2\pi)$, $\cos z = \cos(z + 2\pi)$.

In the complex realm, the (trigonometric) sine and cosine are highly related to their hyperbolic versions. Recall that if $y \in \mathbb{R}$, we define

$$\sinh y := \frac{e^y - e^{-y}}{2} \text{ and } \cosh y := \frac{e^y + e^{-y}}{2}.$$

But then this implies

$$\sin yi = \frac{e^{i(iy)} - e^{-i(iy)}}{2i} = \frac{e^{-y} - e^y}{2i} = \frac{i}{2}(e^y - e^{-y}) = i \sinh y, \text{ and}$$

$$\cos yi = \frac{e^{i(iy)} + e^{-i(iy)}}{2} = \frac{e^{-y} + e^y}{2} = \cosh y.$$

Hence, if $z = x + yi \in \mathbb{C}$, then we arrive at the following component decomposition of complex sine and cosine.

Proposition 44.4. *Let $z = x + yi \in \mathbb{C}$. Then*

1. $\sin(x + yi) = \sin x \cosh y + i \cos x \sinh y$;
2. $\cos(x + yi) = \cos x \cosh y - i \sin x \sinh y$.

These can be used to prove the following proposition.

Proposition 44.5. *Let $z = x + yi \in \mathbb{C}$. Then $|\sin z|^2 = \sin^2 x + \sinh^2 y$ and $|\cos z|^2 = \cos^2 x + \sinh^2 y$. This means that sine and cosine are unbounded on $\mathbb{C}$.*

Proof. Write $|\sin z|^2 = \sin z \cdot \overline{\sin z}$ and expand, and similarly for $|\cos z|^2$. $\square$

With that being said, we might worry whether sine and cosine have any "new" zeros outside of just the usual ones on the real axis. Luckily, the following theorem tells us that the extension into $\mathbb{C}$ does not have any new zeros.

Theorem 44.6. *The zeros and $\sin z$ and $\cos z$ are precisely the real zeros.*

Proof. Write $z = x + yi$, and suppose $\sin z = 0$, so this implies $|\sin z|^2 = \sin^2 x + \sinh^2 y = 0$. By non-negativity, we see that $\sin^2 x = \sinh^2 y = 0$, and we can check that this implies $y = 0$. Hence, all of the zeros of $\sin z$ are real, and are thus precisely the normal real zeros. The proof for $\cos z$ is similar. $\square$

Of course, we define the other four trigonometric functions in the same way we do on the real line.

Definition 44.7. We define the *tangent, cotangent, secant, and cosecant* functions as follows:

$$\tan z := \frac{\sin z}{\cos z}; \ \cot z := \frac{\cos z}{\sin z}; \ \sec z := \frac{1}{\cos z}; \ \csc z := \frac{1}{\sin z},$$

provided that the denominators of these are nonzero.

It follows that these functions are analytic on their domains, with their usual derivatives.

47 Powers and Exponentials

We have seen polynomials before, and we have seen that there are multiple nth roots. Here, we define the general power function.

Definition 47.1. For any $z \in \mathbb{C}^\times$ and a fixed $c \in C$, we define the *power function* $z^c := e^{c \log z}$.

Of course, the case where $c = 1/n$ for $n \in \mathbb{N}$ gives us our familiar nth roots of a complex number z. Hence, it is easy to see that z^c is multivalued, as $\log z$ is multivalued. We thus define the *principal value* of z^c to be PV $z^c := \exp(c \operatorname{Log} z)$. The power function has its usual derivative.

Proposition 47.2 (Power Rule)**.** *Let $f(z) = z^c$ be a branch of a power function. Then $f'(z) = cz^{c-1}$.*

Similarly, we can define exponentials in other bases besides e:

Definition 47.3. Fix some $c \in \mathbb{C}$. We define the *exponential function with base c*: $c^z := e^{z \log c}$.

Again, the principal branch is PV $c^z := \exp(z \operatorname{Log} c)$, and it has its usual derivative:

Proposition 47.4. *Let $f(z) = c^z$ be a branch of an exponential function. Then $f'(z) = c^z \log c$.*

We view some examples.

Example 47.5. Show that $(-1+\sqrt{3}i)^{3/2} = \pm 2\sqrt{2}$.

Solution. We compute by definition:

$$(-1+\sqrt{3}i)^{3/2} = \exp\left(\frac{3}{2}\log(-1+\sqrt{3}i)\right) = \exp\left(\frac{3}{2}\log(2e^{2\pi i/3+2\pi ki})\right)$$

$$= \exp\left(\frac{3}{2}(\ln 2 + \frac{2\pi i}{3} + 2\pi ki)\right) = \exp\left(\frac{3}{2}\ln 2 + \pi i + 3\pi ki\right) = \exp\left(\frac{3}{2}\ln 2\right)\cdot \exp(\pi i + 3\pi ki)$$

$$= 2\sqrt{2}\cdot(\pm 1) = \pm 2\sqrt{2},$$

as expected. •

Example 47.6. What is the principal value of $(1-i)^{4i}$?

Solution. Following the definition, we write

$$(1-i)^{4i} := \exp(4i\,\mathrm{Log}(1-i)) = \exp(4i\,\mathrm{Log}(\sqrt{2}e^{-\pi i/4})) = \exp(4i(\ln\sqrt{2} - \pi i/4))$$

$$= \exp(2i\ln 2 + \pi) = e^{\pi}e^{2i\ln 2},$$

which we can write as $(1-i)^{4i} = \boxed{e^{\pi}\cos(2\ln 2) + e^{\pi}\sin(2\ln 2)i}$. •

50 Contours

We now move into the integration theory part of the course. Over the real line, there was only one way to integrate: along some interval, but complex numbers are inherently two dimensional. Hence, we will often be integrating along a *curve* instead. We start with some basic terminology.

Definition 50.1. An *arc* is a function $z(t) = x(t) + y(t)i$, where x and y are continuous real-valued functions $x, y : [a,b] \to \mathbb{R}$.

We should note that an arc is **not** the same as the *image* of the arc, which is the shape it draws out!

Definition 50.2. An arc is *differentiable* if $x'(t)$ and $y'(t)$ both exist, and we define the arc *derivative* as $z'(t) := x'(t) + y'(t)i$ whenever it exists.

Definition 50.3. If an arc $z(t)$ is differentiable and $z'(t)$ is continuous, the *length* of the arc is

$$s := \int_a^b \sqrt{(x')^2 + (y')^2}\,dt = \int_a^b |z'(t)|\,dt.$$

Often, we want our arcs to be nice and *smooth*.

Definition 50.4. An arc $z : [a,b] \to \mathbb{C}$ is *smooth* if $z'(t)$ is nonzero and continuous on (a,b).

Hence, we arrive at the definition of a *contour*, which is an arc which is smooth, except at possibly finitely many points.

Definition 50.5. A *contour* $z : [a,b] \to \mathbb{C}$ is a piecewise smooth arc. We say that a contour is *simple* if is an injective function. We say that a contour is *closed* if $z(a) = z(b)$. Finally, a contour is *simple closed* if it is closed, and the restriction $z|_{[a,b)}$ is injective.

Theorem 50.6 (Jordan Curve Theorem)**.** *If C is a simple closed contour, then there is a bounded "inside" and an unbounded "outside" of the contour.*

Proof. Very hard. □

Intuitively, the Jordan Curve Theorem allows us to refer to the inside and outside of a contour, which is often needed for integration. Using this, we are ready to define the contour integral.

Definition 50.7. Let $C : z(t)$ be a contour $z : [a,b] \to \mathbb{C}$, and let $f : \operatorname{im} C \to \mathbb{C}$ such that $f \circ C : [a,b] \to \mathbb{C}$ is piecewise continuous. The *contour integral* of f over C is given by

$$\int_C f(z)\,dz := \int_a^b f(z(t))z'(t)\,dt.$$

We note that $\int_C f(z)\,dz$ is invariant under reparameterization, unless the parametrization reverses the endpoints. Contour integration satisfies many of the same properties we are used to over the real line.

Proposition 50.8. *Let $z_0 \in \mathbb{C}$, and let f, g be functions over a contour $C = z(t)$, $z : [a,b] \to \mathbb{C}$. Then*

1. $\displaystyle\int_C z_0 f(z)\,dz = z_0 \int_C f(z)\,dz$;
2. $\displaystyle\int_C (f+g)\,dz = \int_C f\,dz + \int_C g\,dz$;
3. $\displaystyle\int_{-C} f\,dz = -\int_C f\,dz$,

where the contour $-C : [-b,-a] \to \mathbb{C}$ is defined by $t \mapsto z(-t)$. Additionally, if $C_1 : [a,b] \to \mathbb{C}$ and $C_2 : [b,c] \to \mathbb{C}$ are contours satisfying $z_1(b) = z_2(b)$, then

4. $\displaystyle\int_{C_1+C_2} f\,dz = \int_{C_1} f\,dz + \int_{C_2} f\,dz$,

where the contour $C_1 + C_2 : [a,c] \to \mathbb{C}$ is defined by $t \mapsto z_1(t)$ if $t \in [a,b]$, and $t \mapsto z_2(t)$ if $t \in [b,c]$.

We now view some examples.

Example 50.9. Let $C_1 : z = e^{i\theta}$, $\theta \in [0,\pi]$. What is $\int_{C_1} dz/z$? From definition, we can calculate

$$\int_{C_1} \frac{dz}{z} = \int_0^\pi \frac{1}{e^{i\theta}} ie^{i\theta}\,d\theta = \int_0^\pi i\,d\theta = \pi i.$$

Now, consider $C_2 : z = e^{-i\theta}$, $0 \le \theta \le \pi$. Then

$$\int_{C_2} \frac{dz}{z} = \int_0^\pi \frac{1}{e^{-i\theta}}(-i)e^{-i\theta} = -\pi i.$$

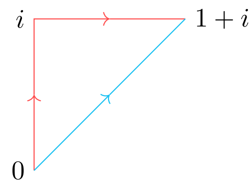
Finally, if $C : z = e^{i\theta}$, $0 \le \theta \le 2\pi$, we observe $\int_C dz/z = 2\pi i$.

Example 50.10. Let C be a smooth arc $C : z = z(t)$, with $t \in [a,b]$. Then by the Chain Rule,

$$\int_C z\,dz = \int_a^b z(t)z'(t)\,dt = \frac{1}{2}\int_a^b \frac{d}{dt}[z(t)]^2\,dt = \frac{1}{2}z^2(t)\Big|_a^b = \boxed{\frac{1}{2}(z^2(b) - z^2(a))}.$$

This is an example of a *path-independent* integral: the answer is only dependent on the end-points. Thus, if C is closed, we see $\displaystyle\oint_C z\,dz = z^2(a) - z^2(a) = 0$.

Example 50.11. Let C_1 (in red) and C_2 (in blue) be both contours from 0 to $1+i$:



Define $f(z) = f(x+yi) := y - x + 3x^2 i$. Notice that we can decompose C_1 into its vertical piece C_{11} and its horizontal piece C_{12}. We parameterize $C_{11} : z(t) = ti$, $t \in [0,1]$, so that

$$\int_{C_{11}} f\,dz = \int_0^1 (t-0)i\,dt = i\int_0^1 t\,dt = \frac{i}{2}.$$

Similarly, we parameterize $C_{12} : z(t) = t+i$, $t \in [0,1]$, so that

$$\int_{C_{12}} f\,dz = \int_0^1 (1-t+3t^2)\cdot 1\,dt = \frac{1}{2} - i.$$

Hence $\displaystyle\int_{C_1} f\,dz = \frac{i}{2} + \left(\frac{1}{2} - i\right) = \boxed{\frac{1-i}{2}}$.

Finally, we parameterize $C_2 : z(t) = t + ti$, $t \in [0,1]$. Then

$$\int_{C_2} f\,dz = \int_0^1 (t - t + 3t^2 i)(1+i)\,dt = \boxed{1-i}.$$

Hence, this integral is *not* path-independent.

54 Integrals at Branch Cuts, Bounds

[46] Branch Cuts

Example 54.1. Let $C : z = 3e^{i\theta}$, $\theta \in [0,\pi]$, and let $f(z) := z^{1/2} = \exp\left(\frac{1}{2}\log z\right)$, for $\theta \in (0, 2\pi)$. Notice that our contour hits points where the function f is not defined; namely 3 lies on the branch cut and thus $f(3)$ is not defined. However, this it not an issue: we can still compute

$$f(z(\theta)) = \exp\left(\frac{1}{2}\log 3e^{i\theta}\right) = \sqrt{3}\exp\left(\frac{i\theta}{2}\right), \text{ for } \theta \in (0,\pi].$$

Now $z'(\theta) = 3ie^{i\theta}$, so

$$\int_C z^{1/2}\,dz = \int_0^\pi \sqrt{3}e^{i\theta/2}\cdot 3ie^{i\theta}\,d\theta = 3\sqrt{3}i\int_0^\pi e^{3i\theta/2}\,d\theta = \boxed{-2\sqrt{3}(1+i)}.$$

In general, this process of disregarding the branch cut is legal for $g(\theta)$, $\theta \in (a, \beta]$, if $\lim_{\theta\to\alpha^+} g(\theta)$ exists. What we are doing is really hiding the improper integral computation.

Example 54.2. Let $C : z = e^{i\theta}$, $\theta \in [-\pi, \pi]$, and let $f(z) := z^{-1+i} = \exp((-1+i)\log z)$. We choose the principal branch PV $z^{-1+i} = \exp((-1+i)\operatorname{Log} z)$. Here, neither the beginning nor the end of C lie in the domain of f, but we can still compute directly:

$$f(z(\theta))z'(\theta) = \exp((-1+i)\cdot i\theta)\cdot ie^{i\theta} = ie^{-\theta}, \text{ for } \theta \in (-\pi, \pi).$$

The one-sided limits at the endpoints both exist, so we can just integrate:

$$\int_C z^{-1+i}\, dz = \int_{-\pi}^{\pi} ie^{-\theta}\, d\theta = \boxed{2i\sinh\pi}.$$

[47] The Estimation Lemma

Often, it is useful to find a bound of an integral instead of calculating it directly. Our next theorem gives us a very common way to do this.

Theorem 54.3 (Estimation Lemma)**.** *Let C be a contour of length L, and let $f(z)$ be piecewise continuous on C with $|f(z)| \le M$ for every $z \in \operatorname{im} C$. Then*

$$\left|\int_C f(z)\, dz\right| \le M \cdot L.$$

This allows us to quickly see the sizes of very complicated integrals.

Example 54.4. Let C be a quarter-circle of radius 2 in the first quadrant. Then we claim

$$\left|\int_C \frac{z-2}{z^4+1}\, dz\right| \le \frac{4\pi}{15}.$$

To see this, we observe that the length of C is $2\pi(2)/4 = \pi$. Now, for all $z \in C$, we have by the triangle inequality $|z-2| \le 4$, and $|z^4+1| \ge \big||z|^4 - 1\big| = |16-1| = 15$. Hence

$$\left|\frac{z-2}{z^4+1}\right| \le \frac{4}{15},$$

so the Estimation Lemma proves the claim.

Example 54.5. Fix $R > 3$. Define $C_R : z = Re^{i\theta}$, for $\theta \in [0, \pi]$. We claim that

$$\lim_{R\to\infty} \int_C \frac{z+1}{(z^2+4)(z^2+9)}\, dz = 0.$$

Proof. Take z on C_R. Then we have $|z+1| \le |z| + |1| = R+1$, $|z^2+4| \ge R^2 - 4$, and $|z^2+9| \ge R^2 - 9$. Hence

$$\left|\frac{z+1}{(z^2+4)(z^2+9)}\right| \le \frac{R+1}{(R^2-4)(R^2-9)} =: M.$$

The length of C_R is $\pi R =: L$. By the Estimation Lemma, we see

$$\left|\int_C \frac{z+1}{(z^2+4)(z^2+9)}\, dz\right| \le M \cdot L = \frac{\pi R(R+1)}{(R^2-4)(R^2-9)} \to 0 \text{ as } R \to \infty,$$

so we are done. □

To prove the Estimation Lemma, we use the following statement:

Lemma 54.6. *Let $w : [a,b] \to \mathbb{C}$ be piecewise continuous. Then* $\left|\int_a^b w(t)\,dt\right| \le \int_a^b |w(t)|\,dt.$

With this, we prove Theorem 54.3.

Proof of Thm. 54.3. By definition, we write

$$\left|\int_C f(z)\,dz\right| = \left|\int_a^b f(z(t))z'(t)\,dt\right|.$$

Using the lemma, we see

$$\left|\int_a^b f(z(t))z'(t)\,dt\right| \le \int_a^b |f(z(t))| \cdot |z'(t)|\,dt \le \int_a^b M|z'(t)|\,dt = ML,$$

where the last equality follows from the definition of the length of the contour. □

60 Antiderivatives and the Cauchy-Goursat Theorem

[48] Antiderivatives

The Fundamental Theorem of Calculus (FTC) tells us that a real integral is related to the integrand's antiderivative. A similar thing happens for complex functions.

Definition 60.1. Let f be continuous on a domain $D \subseteq \mathbb{C}$. An *antiderivative* of f on D is a function F such that $F'(z) = f(z)$ for all $z \in D$.

As in real analysis, the antiderivative F is unique up to a constant term. The following is the complex version of the FTC.

Theorem 60.2 (FTC for Complex Analysis)**.** *Let f be continuous n a domain $D \subseteq \mathbb{C}$. Then the following are equivalent:*

1. *The function f has an antiderivative on D.*
2. *All integrals of f on contours in D are path-independent.*
3. *If C is a contour and with endpoints z_1 and z_2, then $\int_C f\,dz = \int_{z_1}^{z_2} f\,dz = F(z_2) - F(z_1)$, where F is an antiderivative of f.*

Proof. Notice that $(2) \implies (3)$ is trivial.

$(1) \implies (2)$: Suppose F is an antiderivative of f on $D \subseteq \mathbb{C}$. Let $C : z(t)$, $t \in [a,b]$ be a contour on D, and we calculate $\int_C f(z)\,dz$ from definition:

$$\int_C f(z)\,dz = \int_a^b f(z(t))z'(t)\,dt = \int_a^b F'(z(t))z'(t)\,dt = \int_a^b \frac{d}{dt}[F(z(t))]\,dt,$$

where the last step follows from the chain rule. Hence $\int_C f(z)\,dt = F(z(b)) - F(z(a))$ from the regular FTC, so the integral is path-independent.

$(2) \implies (1)$: Since any integral is path independent, we claim that $F(z) = \displaystyle\int_{z_0}^{z} f(s)\,ds$ for some choice of $z_0 \in D$. Using the fact that f is continuous, it is readily shown that $F'(z) = f(z)$.

$(3) \implies (2)$: Take the contours C_1, C_2 that start and end at the same points. It follows that $C_1 - C_2$ is a closed loop ending at the same point z_0, so

$$\int_{C_1 - C_2} f\,dz = F(z_0) - F(z_0) = 0 \implies \int_{C_1} f\,dz = \int_{C_2} f\,dz$$

by integral properties. $\square$

We now turn to some examples.

Example 60.3. Let $f(z) = e^{\pi z}$ on the whole complex plane. We can verify that an antiderivative is $F(z) = \frac{1}{\pi}e^{\pi z}$, so by FTC, every integral is path independent. We can thus calculate

$$\int_{i}^{i/2} e^{\pi z}\,dz = \frac{1}{\pi}e^{\pi z}\Big|_{i}^{i/2} = \boxed{\frac{1+i}{\pi}}.$$

Example 60.4. Let $D := \mathbb{C}^\times$, and let $f(z) := 1/z^2$. Then f is continuous on D, and $F(z) = -1/z$ is an antiderivative. Hence, if C is any contour lying in D, then $\displaystyle\oint_C \frac{dz}{z^2} = 0$.

Notice that in the example above, if we had $f(z) = 1/z$, then $F(z) = \log z$ (or some branch of it), but if C is a circle centered at 0, the circle crosses a branch cut and thus the theorem does not apply. However, we can choose branches wisely, as seen in the next example.

Example 60.5. Let $C_1 : z = e^{i\theta}$, $\theta \in [-\pi/2, \pi/2]$, and let $f(z) := 1/z$. Then $F(z) = \operatorname{Log} z$ is an antiderivative on some domain containing C_1. Notice that $F(z)$ does not cross a branch cut on $[-\pi/2, \pi/2]$, so FTC applies and

$$\int_{C_1} \frac{dz}{z} = \operatorname{Log} z\Big|_{-i}^{i} = \frac{\pi i}{2} - \left(-\frac{\pi i}{2}\right) = \pi i.$$

Now, let $C_2 : z = e^{i\theta}$, $\theta \in [\pi/2, 3\pi/2]$. We choose the branch $\underset{0}{\log} z$ to be an antiderivative in some domain containing C_2. FTC applies and

$$\int_{C_2} \frac{dz}{z} = \underset{0}{\log} z\Big|_{i}^{-i} = \frac{3\pi i}{2} - \frac{\pi i}{2} = \pi i.$$

Hence, the integral along the whole unit circle $C : z = e^{i\theta}$, $\theta \in [-\pi/2, 3\pi/2]$ is given by

$$\oint_C \frac{dz}{z} = \int_{C_1} \frac{dz}{z} + \int_{C_2} \frac{dz}{z} = \pi i + \pi i = \boxed{2\pi i}.$$

[50] Cauchy-Goursat

The FTC tells us that $\oint_C f(z)\,dz = 0$ if and only if f has an antiderivative on a domain containing C. However, the theorem is restrictive in that *having* an antiderivative in the first place is a very strong assumption. Our next theorem gives us a weaker condition in which an integral around a closed contour equals 0.

Theorem 60.6 (Cauchy-Goursat Theorem)**.** *If f is analytic at all points on and inside a simple closed contour C, then $\oint_C f\,dz = 0$.*

We will prove this theorem in the special case that f' is continuous on the region R inside C. To do so, we recall the following theorem from Math 2E.

Theorem 60.7 (Green's Theorem). *Let $P(x,y), Q(x,y) : \mathbb{R}^2 \to \mathbb{R}$ have continuous first-order partial derivatives son a closed region $R \subseteq \mathbb{R}^2$ with $\partial R \subseteq R$. Then*

$$\int_C (P\,dx + Q\,dy) = \iint_C (Q_x - P_y)\,dA.$$

We now proceed with the proof of a case of the Cauchy-Goursat Theorem.

Proof of Thm. 60.6, assuming f' is continuous in R. Write $f = u + vi$, and let $C : z(t) = x(t) + y(t)i$ be positively-oriented. Let R denote the interior of C, and assume that f' is continuous on R. Then

$$\begin{aligned}\int_C f(z)\,dz = \int_a^b (u+vi)\cdot(x'+y'i)\,dt &= \int_a^b (ux' - vy')dt + i\int_a^b (vx' + uy')\,dt \\ &= \int_C (u\,dx - v\,dy) + i\int (v\,dx + u\,dy).\end{aligned}$$

Applying Green's Theorem, we have

$$\int_C f\,dz = \iint_R (-v_x - u_y)\,dA + i\iint_R (u_x - v_y)\,dA = 0 + 0 = 0,$$

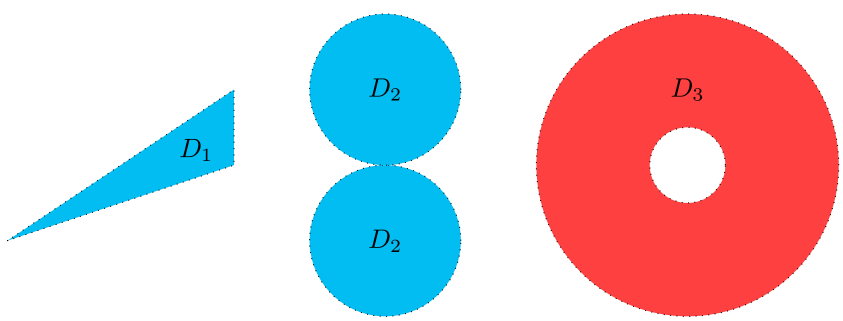
where the rightmost equalities come from Cauchy-Riemann. □

64 Connected Domains, Cauchy's Integral Formula

[52]/[53] Simply- and Multiply-Connected Domains

Definition 64.1. Let $D \subseteq \mathbb{C}$ be a domain. Then D is *simply connected* if for any simple closed contour inside D, the inside of C is also contained in D.

Intuitively, a domain is simply connected if it has no holes. The blue regions below are simply connected, while the red one is not.



From here, we get a theorem similar to Cauchy-Goursat.

Theorem 64.2. *If D is simply connected and C is any closed contour in D, and if f is analytic on D, then $\oint_C f\,dz = 0$.*

The proof of this can be done by visualizing a domain such as D_2, then splitting a contour into its "simple closed" bits, then applying Cauchy-Goursat.

Example 64.3. For any closed contour C inside $D : |z| < 2$, we have

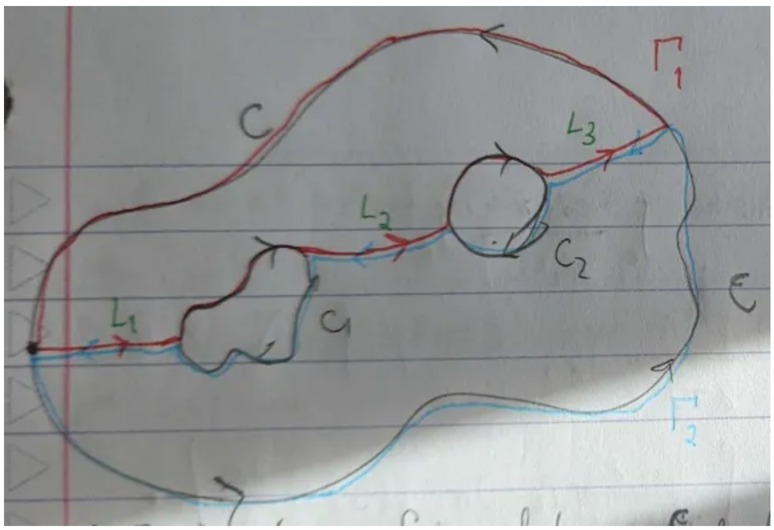
$$\int_C \frac{\sin z}{(z^2+9)^5}\,dz = 0.$$

The next corollary is particularly powerful.

Corollary 64.4. *If f is analytic on a simply connected domain D, then f has an antiderivative on D. In particular, this implies that entire functions have antiderivatives on all of $\mathbb{C}$.*

Proof. Apply Theorem 64.2, so that FTC applies. $\square$

What if we now allow holes in our domains? Consider the *multiply-connected* domain below:



Notice that we can split our contour C into many pieces, as long as we choose good orientations.

Theorem 64.5. *Let C, $\{C_k\}_{k=1}^n$ be contours, and assume f is analytic on C, all of the C_k, and all of the space in between the C_k and C. Assume also that the C_k have opposite orientation to C. Then*

$$\int_C f\,dz + \sum_{k=1}^n \left(\int_{C_k} f\,dz\right) = 0.$$

Proof Sketch for the case above. Consider the figure above. Now, Γ_1 and Γ_2 are obviously simple closed curves, and by construction, f is analytic inside Γ_1 and Γ_2, so Cauchy-Goursat applies and

$$\oint_{\Gamma_1} f\,dz + \oint_{\Gamma_2} f\,dz = 0 + 0 = 0.$$

But staring at the figure for long enough gives that this is the same as saying $\int_C f\,dz + \int_{C_1} f\,dz + \int_{C_2} f\,dz = 0.$ $\square$

This gives us the following useful corollary.

Corollary 64.6 (Deformation of Paths). *Let C_1, C_2 be positively oriented and simple closed contours, and say C_1 is interior to C_2. If f is analytic on and in between C_1 and C_2, then*

$$\oint_{C_1} f\,dz = \oint_{C_2} f\,dz.$$

Example 64.7. Let C be any arbitrary contour exterior to the unit circle. Then $\int_C dz/z = 2\pi i$ by Deformation of Paths.

[54] The Cauchy Integral Formula

Now, we give an extremely important theorem in complex analysis — arguably, the rest of this course lies upon this theorem.

Theorem 64.8 (Cauchy Integral Formula). *Let C be a simple closed positively-oriented contour, and take f to be analytic on and inside C. For any z_0 on the interior of C, we have*

$$\int_C \frac{f(z)}{z - z_0}\,dz = 2\pi i \cdot f(z_0).$$

Proof Sketch. Pick z_0 inside C, and draw a circle C_ρ of radius ρ around z_0 and inside C. By Deformation of Contours,

$$\int_C \frac{f(z)}{z - z_0}\,dz = \int_{C_\rho} \frac{f(z)}{z - z_0}\,dz.$$

Now, as $z \to z_0$, we can check that

$$\int_{C\rho} \frac{f(z)}{z - z_0}\,dz \to \int_{C\rho} \frac{f(z_0)}{z - z_0}\,dz = f(z_0)\int_{C\rho} \frac{dz}{z - z_0} = 2\pi i f(z_0)$$

by virtue of previous examples. □

This is a dramatic simplification in how certain contour integrals are computed: the type of integral in the Cauchy Integral Formula (CIF) depends only on the function's value at z_0.

Example 64.9. Let $C : |z| = 1$, positively oriented. Evaluate $\displaystyle\int_C \frac{\cos z}{z(z^2+9)}\,dz$.

Solution. Write $f(z) := \cos z/(z^2 + 9)$, so

$$\int_C \frac{\cos z}{z(z^2+9)} = \int_C \frac{f(z)}{z - 0}\,dz.$$

Notice that f is analytic on and inside C, so the CIF applies and

$$\int_C \frac{\cos z}{z(z^2+9)}\,dz = 2\pi i \cdot f(0) = \boxed{\frac{2\pi i}{9}}.$$

•

70 The Generalized CIF and Its Consequences (I)

The next theorem, as its name suggests, generalizes the CIF. We will see a short proof sketch for the version where $n = 1$.

Theorem 70.1 (Generalized CIF)**.** *Let C be a simple closed positively-oriented contour, and let f be analytic on and inside C. Then for any z_0 inside C, we have*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z - z_0)^{n+1}}\,dz.$$

Notationally, if z is arbitrary within C, we can instead write $\boxed{f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(s)\,ds}{(s-z)^{n+1}}}$.

Proof Sketch when $n = 1$. From the CIF, we know $f(z) = \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{s - z}$. Now

$$\begin{aligned}\frac{f(z+\Delta z) - f(z)}{\Delta z} &= \frac{1}{2\pi i}\int_C \left(\frac{1}{s - z - \Delta z} - \frac{1}{s-z}\right)\frac{f(s)}{\Delta s}\,ds \\ &= \frac{1}{2\pi i}\int_C \left[\frac{1}{(s-z)^2} + \frac{\Delta z}{(s - z - \Delta z)(s-z)^2}\right] f(s)\,ds.\end{aligned}$$

Rearranging, we see that

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} - \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{(s-z)^2} = \frac{1}{2\pi i}\int_C \frac{f(s)\Delta z\,ds}{(s-z-\Delta z)(s-z)^2}.$$

Now, $f(s)$ is bounded as it is analytic on and inside C, and the denominator $(s-z-\Delta z)(s-z)^2$ approaches $(s-z)^3$ as $\Delta z \to 0$. After some verification of some technicalities, taking the limit as $\Delta z \to 0$ finishes the proof. □

Example 70.2. Let $f(z) := 1$, and pick some $z_0 \in \mathbb{C}$. Pick any simple closed positively-oriented contour C around z_0. Then

$$\int_C \frac{dz}{z - z_0} = 2\pi i \text{ and } \int_C \frac{dz}{(z-z_0)^{n+1}} = 0$$

when $n \geq 1$, as $f'(z) = 0$.

Example 70.3. Let $f(z) := e^{2z}$, and let $C : |z| = 1$, positively oriented. Evaluate $\int_C \frac{e^{2z}}{z^3}\,dz$.

Solution. Using the notation of the Generalized CIF (GCIF), pick $z_0 = 0$ and $n = 3 - 1 = 2$. Then

$$\int_C \frac{e^{2z}\,dz}{z^3} = \int_C \frac{f(z)}{(z - z0)^{2+1}}\,dz,$$

and certainly f is entire and z_0 is inside C. Hence

$$\int_C \frac{e^{2z}\,dz}{z^3} = f''(0)\cdot\frac{2\pi i}{2!} = 4\cdot\pi i = \boxed{4\pi i}.$$

•

The GCIF, in addition to being a powerful computational tool, has many interesting theoretical consequences. We list some of them below.

Theorem 70.4. *If f is analytic at $z_0 \in \mathbb{C}$, then so is f'. Hence, f has infinitely many derivatives at z_0.*

Proof. Saying that f is analytic at a point z_0 is the same as saying that f is differentiable in a ball $B(z_0, \varepsilon)$, for appropriate $\varepsilon > 0$. Pick some simple closed contour C lying in $B(z_0, \varepsilon)$, say $C : |z - z_0| < \varepsilon/3$, positively-oriented. Then, the GCIF provides a formula for the derivative, so it exists. By induction, $f^{(n)}(z_0)$ exists for all $n \in \mathbb{N}$. $\square$

We have the following corollary, which provides the justification for our proof of Theorem 30.8 earlier in the course.

Corollary 70.5. *If $f = u + vi$ is analytic at $z_0 = x_0 + y_0 i \in \mathbb{C}$, then all partial derivatives of u and v exist and are continuous at $(x_0, y_0) \in \mathbb{R}^2$.*

Our next theorem is slightly strange — it gives a condition for *differentiability* based on integrals.

Theorem 70.6 (Morera's Theorem)**.** *Suppose f is continuous on a domain $D \subseteq \mathbb{C}$, and that $\int_C f\, dz = 0$ for all closed contours C inside D. Then f is analytic on D.*

Proof. By FTC, f has an antiderivative F on D. But then F is analytic on D, as $F' = f$. Applying Theorem 70.4 finishes the proof, as $F'' = f'$ exists. $\square$

Using integration theory, we can in fact go back and estimate derivatives.

Theorem 70.7 (Cauchy's Inequality)**.** *Let the circle $C_R : |z - z_0| = R$ be positively oriented, and let f be analytic on and inside C_R. If $M_R := \max\{|f(z)| : z \in C_R\}$, then*

$$\left|f^{(n)}(z_0)\right| = \frac{n! M_R}{R^n}.$$

Proof. By the GCIF, we know that $f^{(n)}(z_0) = \dfrac{n!}{2\pi i} \displaystyle\int_{C_R} \frac{f(z)\, dz}{(z - z_0)^{n+1}}$. Using this and the Estimation Lemma, we have

$$\left|f^{(n)}(z_0)\right| = \frac{n!}{2\pi} \left| \int_{C_R} \frac{f(z)\, dz}{(z - z_0)^{n+1}} \right| \leq \frac{n!}{2\pi} \cdot \frac{M_R}{R^{n+1}} \cdot 2\pi R = \frac{n! M_R}{R^n},$$

which completes the proof. $\square$

74 The Generalized CIF and Its Consequences (II)

The Generalized CIF also gives us the following powerful result, which is obviously false in real analysis.

Theorem 74.1 (Liouville's Theorem)**.** *If f is entire and bounded, then f is constant.*

Proof. Take some $R > 0$ and $z_0 \in \mathbb{C}$. Say that f is entire and bounded by $M > 0$; i.e., $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Then, Cauchy's Inequality applies and for a circle C_R and M_R as in that theorem,

$$0 \leq |f'(z_0)| = \frac{M_R}{R} \leq \frac{M}{R} \to 0$$

as $R \to \infty$. Hence $f'(z_0) = 0$, so f is constant. $\square$

This allows us an easy proof of the algebraic closure of $\mathbb{C}$. Note that this statement is neither fundamental, nor algebraic.

Theorem 74.2 (Fundamental Theorem of Algebra)**.** *Let* $p(z) = a_0 + a_1 z + \cdots + a_n z^n \in \mathbb{C}[z]$*, with* $a_n \neq 0$*,* $n \geq 1$*. Then* p *has a root in* $\mathbb{C}$*. Equivalently,* $\mathbb{C}$ *is algebraically closed.*

Proof. Assume for contradiction that p does not have a root in $\mathbb{C}$. Then $f(z) := 1/p(z)$ is entire, and Theorem 13.6 provides a bound as follows: there exists an $R > 0$ such that for all $|z| > R$, we have

$$|f(z)| = \left|\frac{1}{p(z)}\right| < \frac{2}{|a_n|R^n}.$$

This shows that f is bounded outside of the circle $|z| = R$. But the interior of the circle is closed, so f is bounded there, say by some $M_0 > 0$. Take M to be the maximum of the two bounds, so f is bounded everywhere and entire, so f is constant. Hence p is constant, but we assumed otherwise — a contradiction. ☐

[59] The Maximum Modulus Principle

The (regular) CIF has this important consequence, which saves us work if we ever have to do an optimization problem.

Theorem 74.3 (Maximum Modulus Principle)**.** *Let* $D \subseteq \mathbb{C}$ *be a domain (which is open), and let* f *be a nonconstant analytic function on* D*. Then* $|f(z)|$ *has no maximum on* D*.*

We first state and prove a simpler version of the Maximum Modulus Principle (MMP) for circles.

Lemma 74.4 (MMP for Disks)**.** *Let* f *be analytic on* $B(z_0, \varepsilon)$*, for some* $z_0 \in \mathbb{C}$ *and* $\varepsilon > 0$*. Suppose* $|f(z)| \leq |f(z_0)|$ *for every* $z \in B(z_0, \varepsilon)$*. Then* f *is constant on* $B(z_0, \varepsilon)$*.*

Proof. Pick an arbitrary $z_1 \in B(z_0, \varepsilon)$. We show that $|f(z_0)| = |f(z_1)|$, so that Proposition 30.9 finishes the proof. Take $\rho := |z_1 - z_0|$, and define $C_\rho : |z - z_0| = \rho$, positively-oriented. Then C_ρ is a circle centered at z_0, passing through z_1, lying entirely in $B(z_0, \varepsilon)$. By the CIF, we see that after parameterizing C_ρ in the obvious way,

$$f(z_0) = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(z)\,dz}{z - z_0} \implies f(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + \rho e^{i\theta})}{\rho e^{i\theta}} \cdot \rho i e^{i\theta}\,d\theta,$$

which simplifies further to $f(z_0) = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} f(z_0 + \rho e^{i\theta})\,d\theta$. By our assumption, we have

$$|f(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})|\,d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0)|\,d\theta,$$

but we know that $\dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} |f(z_0)|\,d\theta = |f(z_0)|$, hence the inequalities above are actually equalities and we write

$$|f(z_0)| = \frac{1}{2\pi} \int_0^{2\pi} |f(z_0)|\,d\theta = \frac{1}{2\pi} \int_0^{2\pi} \left|f(z_0 + \rho e^{i\theta})\right|\,d\theta$$

$$\implies \int_0^{2\pi} \left[|f(z_0)| - \left|f(z_0 + \rho e^{i\theta})\right|\right]\,d\theta = 0.$$

The integrand of the above is continuous and nonnegative, so $|f(z_0)| = \left|f(z_0 = \rho e^{i\theta})\right|$. Since z_1 lies on C_ρ, we have $|f(z_0)| = |f(z_1)|$. ☐

78 The MMP, Sequences and Series

[59] The Maximum Modulus Principle

Continuing on, we now prove Theorem 74.3.

Proof of Thm. 74.3. We prove by contrapositive. Take some $z_0 \in D$ such that $|f(z)| \leq |f(z_0)|$. Since D is connected, draw a piecewise linear curve γ between z_0 and any point $p \in D$. Picking appropriate points z_j along γ with $p = z_n$, draw balls of radius δ within D, such that $B(z_j, \delta) \cap B(z_{j+1}, \delta) \neq \varnothing$ and each $B(z_j, \delta)$ is contained in D. Apply Lemma 74.4 along each point to see that $f(z_0) = f(z_1) = \cdots = f(z_n) = f(p)$, so f is constant. □

From here, we get the following corollary.

Corollary 78.1. *Let $R \subseteq \mathbb{C}$ be a closed bounded region, and let f be non-constant and analytic on the interior of R as well as continuous on ∂R. Then f attains its maximum on ∂R.*

Additionally, if $f = u + vi$, then u has a maximum on ∂R, and never on the interior of R. To see this, let $g := \exp f$, and apply the corollary. We should notice the asymmetry between maximums and minimums here: we know that $f(z) = z^2$ attains its minimum *within* the region $R = \{|z| \leq 1\}$, and not on the boundary.

[60]/[61] Sequences and Series

We now take a short break to discuss sequences and series, which will return powerfully once we begin discussing integrals again. Most of this section is review or at least intuitive.

Definition 78.2. A sequence (z_n) of complex numbers *converges to* $z \in \mathbb{C}$ if for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n > N$, we have $|z_n - z| < \varepsilon$. If such a z exists, then we say that z is the *limit* of the sequence and write $\lim z_n = z$.

Of course, the limit of a sequence is *unique* when it exists. If no such limit exists, we say that the sequence *diverges*.

Theorem 78.3. *Let $z_n = x_n + y_n i$, and $z = x + yi \in \mathbb{C}$. Then $(z_n) \to z$ if and only if we have both $(x_n) \to x$ and $(y_n) \to y$.*

This is not too surprising: a complex sequence can be considered a pair of real sequences.

Example 78.4. Let $z_n = 5 + \dfrac{(-1)^n i}{n}$. Then $x_n = 5 \to 5$ and $y_n = \dfrac{(-1)^n}{n} \to 0$, so $z_n \to 5 + 0i = 5$.

Example 78.5. Let $z_n = -1 - \dfrac{i}{n^2}$. Then $z_n \to -1$, but if we write

$$r_n = |z_n| = \sqrt{1 + n^{-4}} \to 1 = r = |x_n|, \text{ and}$$

$$\operatorname{Arg}(z_n) \to -\pi,$$

we see that the polar analogue of Theorem 78.3 does not work.

We move on to discussing series.

Definition 78.6. Let (z_n) be a sequence in $\mathbb{C}$. We define the *infinite series* by

$$\sum_{n=0}^{\infty} z_n := \lim_{n\to\infty} \sum_{j=0}^{n} z_j,$$

whenever this limit exists.

Of course, complex series can be split into a pair of real series in the obvious way.

Theorem 78.7. *Let (z_n) be a sequence, $z_n = x_n + y_n i$, and $z = x + yi \in C$. Then $\sum z_n = z$ if and only if we have both $\sum x_n = x$ and $\sum y_n = y$.*

The nth term test still holds from real analysis.

Corollary 78.8 (nth Term Test)**.** *If $\sum z_n$ converges, then $\lim z_n = 0$.*

We also have the notion of *absolute convergence*:

Definition 78.9. We say that $\sum z_n$ *converges absolutely* if $\sum |z_n|$ converges.

Of course, it goes without saying that if $\sum |z_n|$ converges, so does $\sum z_n$.

Complex power series are defined in the same way.

Definition 78.10. A *power series* is a function

$$f(z) := \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

for fixed $a_n, z_0 \in \mathbb{C}$, whenever the sum converges.

80 Taylor and Laurent Series (I)

Let us view an example of a familiar power series.

Example 80.1. Let $f(z) = \sum z^n$. Then $f(z) = 1/(1-z)$ whenever $|z| < 1$, as

$$\sum_{n=0}^{N} z^n = \frac{1 - z^{N+1}}{1 - z}; \text{ hence}$$

$$\frac{1}{1-z} - \sum_{n=0}^{N} z^n = \frac{z^{N+1}}{1-z} \to 0$$

if $|z| < 1$. If $|z| \geq 1$, the sum diverges as usual.

The following theorem should be familiar to us.

Theorem 80.2 (Taylor's Theorem)**.** *Let f be analytic on some $B(z_0, R_0)$. Then $f(z) = \sum a_n (z - z_0)^n$ for all $z \in B(z_0, R_0)$, where $a_n = f^{(n)}(z_0)/n!$.*

Since Taylor series are differentiable, we see that a function is analytic in some ball if and only if it can be expanded as a Taylor series in that ball. In particular, if f is entire, then it has a Taylor series expansion everywhere.

For a proof of Taylor's Theorem, see [63] in the text. We view an example.

Example 80.3. Let $f(z) = 1/(1-z)$, and $z_0 = i$. Note that f is analytic on $B(i, \sqrt{2})$, so Taylor's Theorem guarantees a power series representation on that ball. Write

$$\frac{1}{1-z} = \frac{1}{1-i-(z-i)} = \frac{1}{1-(z-i)/(1-i)} \cdot \frac{1}{1-i}.$$

Now, whenever $z \in B(i, \sqrt{2})$, we observe that this implies $|(z-i)/(1-i)| < 1$, so we can write

$$\frac{1}{1-(z-i)/(1-i)} \cdot \frac{1}{1-i}. = \frac{1}{1-i} \sum_{n=0}^{\infty} \left(\frac{z-i}{1-i} \right)^n$$

on that ball. We can simplify this as we wish: $\dfrac{1}{1-z} = \boxed{\displaystyle\sum_{n=0}^{\infty} \frac{1}{(1-i)^{n+1}} (z-i)^n}$ for all $z \in B(i, \sqrt{2})$.

Now, we turn our attention to a phenomenon that occurs rather exclusively to complex functions: we can have negative powers in our "power series" representations.

Example 80.4. Let $f(z) = e^{-z}/z^2$. Then f is certainly not analytic at $z = 0$, but

$$f(z) = \frac{1}{z^2} \sum_{n=0}^{\infty} \frac{(-z)^n}{n!} = \frac{1}{z^2} - \frac{1}{z} + \frac{1}{2} - \frac{z}{6} + \frac{z^2}{24} - \cdots, \ (z \neq 0),$$

which is a "power series" save for the negative powers -2 and -1 appearing.

Example 80.5. Let $f(z) = \dfrac{1+2z^2}{z^3+z^5}$. Then f is not analytic at $z = 0, \pm i$. But for $0 < |z| < 1$, we can safely write

$$f(z) = \frac{1+2z^2}{z^3(1+z^2)} = \frac{1}{z^3} \left(2 - \frac{1}{1+z^2} \right) = \frac{1}{z^3} \left(2 - \sum_{n=0}^{\infty} (-1)^n z^{2n} \right)$$

$$\implies f(z) = \frac{1}{z^3} + \frac{1}{z} - z + z^3 - \cdots, \ (0 < |z| < 1),$$

where the negative powers -3 and -1 appear.

This is the idea of a *Laurent series*, which we introduce below.

Theorem 80.6 (Laurent's Theorem)**.** *Let f be analytic on some annulus $R_1 < |z - z_0| < R_2$, and let C be a positively oriented, simple closed contour on the annulus. If*

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}}, \ n \geq 0 \text{ and}$$

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z-z_0)^{-n+1}}, \ n \geq 1,$$

then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

on the annulus. Alternatively, we can write $c_n = \dfrac{1}{2\pi i} \displaystyle\int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}}$ for all $n \in \mathbb{Z}$, so that

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n.$$

83 Taylor and Laurent Series (II)

In this section, we view examples of Taylor and Laurent series.

Example 83.1. Let $f(z) := \sin z$, and $z_0 = \pi/2$. Find the Taylor series of f centered at z_0.

Solution. Notice that $\sin z = \cos(z - \pi/2)$. Then we can just use the Taylor series for cosine:

$$\sin z = \cos(z-\pi/2) = 1 - \frac{(z-\pi/2)^2}{2!} + \frac{(z-\pi/2)^4}{4!} - \cdots = \boxed{\sum_{n=0}^{\infty}(-1)^n \frac{(z-\pi/2)^{2n}}{(2n)!}}.$$

This example illustrates an important point: apply shifts before resorting to computation of derivatives. •

Example 83.2. Let $f(z) = z/(z^4+4)$. Find the *Maclaurin series* of f.

Solution. Certainly, f is analytic at 0, so the Maclaurin series exists. Using some algebra, we write

$$f(z) = z \cdot \frac{1}{z^4+4} = z\left(\frac{1}{1+\frac{z^4}{4}}\right)\cdot\frac{1}{4} = \frac{z}{4}\left(\frac{1}{1+\frac{z^4}{4}}\right).$$

Now

$$\frac{1}{1+\frac{z^4}{4}} = 1 + \left(-\frac{z^4}{4}\right) + \left(-\frac{z^4}{4}\right)^2 + \cdots = 1 - \frac{z^4}{4} + \frac{z^8}{16} - \frac{z^{12}}{64} + \cdots,$$

whenever $|z^4/4| < 1 \implies |z| < \sqrt{2}$. Hence we have the following series for f, valid whenever $|z| < \sqrt{2}$:

$$f(z) = \frac{1}{4}\left(z - \frac{z^5}{4} + \frac{z^9}{16} - \frac{z^{13}}{64} + \cdots\right) = \boxed{\sum_{n=0}^{\infty}(-1)^n \frac{z^{4n+1}}{4^{n+1}}}.$$

•

Example 83.3. Find the Laurent series for $f(z) = e^z/z^3$, centered at $z_0 = 0$.

Solution. Notice that f is not analytic at $z_0 = 0$, but definitely whenever $0 < |z| < \infty$, so a Laurent series exists. Now

$$f(z) = \frac{1}{z^3}e^z = z^{-3}\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right) = \boxed{\sum_{n=-3}^{\infty} \frac{z^n}{(n+3)!}},$$

provided that $z \neq 0$. •

Example 83.4. Find the Laurent series for $f(z) := z/(z+1)^3$, centered at $z_0 = -1$.

Solution. We compute from definition here. The coefficients c_n for the Laurent series are given by

$$c_n = \frac{1}{2\pi i}\int_C \frac{z/(z+1)^3}{(z+1)^{n+1}}\,dz = \frac{1}{2\pi i}\int_C \frac{z\,dz}{(z+1)^{n+4}},$$

where C is any contour with $z=-1$ in its interior. By the GCIF, if we let $g(z)=z$, we see that $g^{(n+3)}(-1)/n! = c_n$, so that $c_{-2}=1$, $c_{-3}=-1$, and $c_n=0$ otherwise. Thus, our Laurent series is in fact finite:

$$f(z) = \boxed{\frac{1}{(z+1)^2} - \frac{1}{(z+1)^3}},$$

provided that $z \neq -1$. Notice that this is just the partial fraction decomposition of f, which is not surprising: just as polynomials have finite Taylor expansions everywhere, rational functions should have finite Laurent expansions on their domains. •

We finally comment that if f is analytic at z_0, the Laurent series expansion coincides with the Taylor series, and no negative terms are generated.

90 Laurent Series, Singularities, and Residues

We view two more applications of Laurent series.

Example 90.1. Let $f(z) := (z+1)/(z-1)$, and let $z_0 := 0$. Consider the open disk $B(z_0,1) = B(0,1)$. Then f is analytic there, and we have the Taylor series

$$\frac{z+1}{z-1} = (z+1)\left[-\sum_{n=0}^{\infty} z^n\right] = -1 - 2\sum_{n=1}^{\infty} z^n.$$

[We could have done this by writing $f(z) = 1 + 2/(z-1)$ as well.] Additionally, f is analytic whenever $|z| > 1 \iff |1/z| < 1$. Hence, we get a Laurent series on the annulus $1 < |z| < \infty$:

$$f(z) = \frac{z+1}{z-1} = \frac{1+1/z}{1-1/z} = \left(1+\frac{1}{z}\right)\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n, \ |z|>1.$$

Expanding this gives

$$f(z) = \sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n + \sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^{n+1} = 1 + 2\sum_{n=1}^{\infty}\left(\frac{1}{z}\right)^n.$$

Example 90.2. We claim that $\int_C e^{1/z}\,dz = 2\pi i$ for any positively oriented closed contour C around 0. To see this, we write the Laurent expansion for $e^{1/z}$ for $z \neq 0$: $e^{1/z} = \sum_{n=0}^{\infty} z^{-n}/n!$. Notice that if $f(z) = e^{1/z}$, then from definition of the coefficients

$$c_{-1} = \frac{1}{1} = 1 = \frac{1}{2\pi i}\int_C f(z)\,dz = \frac{1}{2\pi i}\int_C e^{1/z}\,dz,$$

so solving for the integral proves our claim.

[74] Singularities

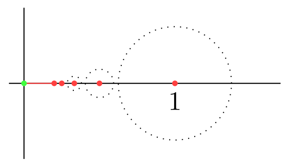
The above example is particularly useful: it gave us the answer to an integral because we knew a function's Laurent series. Recall that Laurent series only have negative powered terms around some annulus, centered at a non-analytic point. This is the topic of study in this section.

Definition 90.3. Let f be a function. We say that $z_0 \in \mathbb{C}$ is a *singularity of f* if f is not analytic at z_0, but analytic at points arbitrarily close to z_0. A singular point of f is an *isolated singularity* if f is analytic on $D(z_0, \varepsilon)$ for some $\varepsilon > 0$.

Example 90.4. Let $f(z) = \dfrac{z-1}{z^5(z^2+9)}$. Then f has exactly 3 singularities: $z = 0, \pm 3i$, and all of these are isolated, as f is analytic on the punctured balls $D(0,1)$, $D(3i,1)$, and $D(-3i,1)$.

Example 90.5. Let $f(z) = \operatorname{Log} z$. Then f has singularities along the entire negative real axis due to a branch cut. These are *not* isolated.

Example 90.6. Let $f(z) = \csc\left(\frac{\pi}{2}\right)$. Then f has singularities at all $z = 0, 1/n$ for $n \in \mathbb{N}$. The singularities $z = 1/n$ are isolated, but the singularity $z = 0$ is not:



We remark that if C is a simple closed contour, and f is analytic inside C except for finitely many points z_j, then all of the z_j are isolated. Now, consider one isolated singularity z_0 of f. This means we can take some $R > 0$ such that f is analytic on $0 < |z - z_0| < R$, so we have a Laurent series expansion in that region. We make the following definition.

Definition 90.7. If $z_0 \in \mathbb{C}$ is an isolated singularity of f, we define the *residue* of f at z_0 by

$$\operatorname*{Res}_{z=z_0} f(z) := c_{-1} = \frac{1}{2\pi i}\int_C f(z)\,dz,$$

where c_{-1} is the coefficient of the $1/(z - z_0)$ term in the Laurent series expansion that occurs centered at z_0.

This gives us a fast way to compute integrals, like we have seen before: just find the residue.

93 More Examples, Calculus with Power Series

[This discussion section just had more helpful examples.]

Example 93.1. Find the Laurent series for $f(z) := z^2 \sin(1/z^2)$ on the region $0 < |z| < \infty$.

Solution. When $z \neq 0$, we may write

$$\sin\left(\frac{1}{z^2}\right) - \frac{1}{z^2} - \frac{(1/z^2)^3}{3!} + \frac{(1/z^2)^5}{5!} - \cdots = z^{-2} - \frac{z^{-6}}{3!} + \frac{z^{-10}}{5!} - \cdots,$$

so that

$$f(z) = z^0 - \frac{z^{-4}}{3!} + \frac{z^{-8}}{5!} - \cdots = \boxed{\sum_{k=0}^{\infty} \frac{1}{z^{4k}} \frac{(-1)^k}{(2k+1)!}}.$$

•

Example 93.2. Find the Laurent series for $g(z) := a/(z - a)$, on $|a| < |z| < \infty$.

Solution. Notice that when $|z| > |a|$, we have $|1/z| < |1/a|$ as we are comparing positive real numbers. Write

$$g(z) = \frac{a}{z-1} = \frac{a/z}{1 - a/z},$$

which is analytic whenever $z \neq a$. Now when $|1/z| < |1/a|$, we have

$$g(z) = \frac{a}{z}\left(\frac{1}{1-a/z}\right) = \frac{a}{z}\left(1 + \frac{a}{z} + \frac{a^2}{z^2} + \cdots\right) = \boxed{\sum_{k=0}^{\infty}\left(\frac{a}{z}\right)^{k+1}}.$$

•

As in real analysis, complex power series may be differentiated and integrated term by term.

Theorem 93.3. *A power series $S(z) := \sum a_n(z-z_0)^n$ defines an analytic function in its circle of convergence. In particular, S can be differentiated term by term:*

$$\frac{d}{dz}\sum_{n=0}^{\infty} a_n(z-z_0)^n = \sum_{n=0}^{\infty}\frac{d}{dz}[a_n(z-z_0)^n],$$

and if C is any contour lying within the circle of convergence of S, then the contour integral can be done term by term:

$$\int_C\left(\sum_{n=0}^{\infty} a_n(z-z_0)^n\right)dz = \sum_{n=0}^{\infty}\left(\int_C a_n(z-z_0)^n\,dz\right).$$

94 Residues and Poles

We view an example of calculating a residue to solve an integral.

Example 94.1. Let $C : |z| = 1$, positively oriented. Find $\displaystyle\int_C \frac{e^z - 1}{z^4}\,dz$.

Solution. Let $f(z) := (e^z - 1)/z^4$. Then f is analytic on $0 < |z| < \infty$, so it has a Laurent series expansion there:

$$\frac{e^z-1}{z^4} = \frac{1}{z^4}\sum_{n=1}^{\infty}\frac{z^n}{n!} = \sum_{n=1}^{\infty}\frac{z^{n-4}}{n!}.$$

The residue of $f(z)$ is the z^{-1} coefficient: $z^{3-4}/3! = z/6$. Hence

$$\int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0} f(z) = 2\pi i \cdot \frac{1}{6} = \boxed{\frac{\pi i}{3}}.$$

•

This example is in fact an indication of a more general statement.

Theorem 94.2 (Cauchy's Residue Theorem)**.** *Let C be a positively-oriented simple closed contour, and let f be analytic on and inside C, except for finitely many $\{z_j\}_{j=1}^n$, which are isolated singularities. Then*

$$\int_C f(z)\,dz = 2\pi i\sum_{j=1}^{n}\operatorname*{Res}_{z=z_j} f(z).$$

Proof. Inside C, draw non-overlapping balls $B(z_j, \delta_j)$ around each z_j, and let the C_j be the boundary of each ball, positively oriented. By Theorem 64.5 (and taking care of the signs), we see that

$$\int_C f\,dz - \sum_{j=1}^{n}\int_{C_j} f\,dz = 0.$$

But by definition $\displaystyle\int_{C_j} f\,dz = 2\pi i \operatorname*{Res}_{z=z_j} f(z)$, so this completes the proof. $\square$

Example 94.3. Find $\displaystyle\int_C \frac{4z-5}{z(z-1)}\,dz$, where $C : |z| = 2$, positively oriented.

Solution. Letting $f(z) = 4z - 5/(z(z-1))$, we have two isolated singularities inside C, namely $z = 0$ and $z = 1$. At $z = 0$, we can write

$$f(z) = -\frac{4z-5}{z} \cdot \frac{1}{1-z} = -\left(4 - \frac{5}{z}\right)\sum_{n=0}^{\infty} z^n,\ 0 < |z| < 1.$$

By inspection, we read off the residue: $\operatorname*{Res}_{z=0} f(z) = 5$. Similarly, when $z = 1$, note that

$$\frac{1}{z} = \frac{1}{1+(z-1)} = \sum_{n=0}^{\infty}(-1)^n(z-1)^n,\ |z-1| < 1.$$

Hence

$$f(z) = \frac{1}{z-1}(4z-5)\sum_{n=0}^{\infty}(-1)^n(z-1)^n = \left(4 - \frac{1}{z-1}\right)\sum_{n=0}^{\infty}(-1)^n(z-1)^n,\ 0 < |z-1| < 1,$$

so we read off the residue again: $\operatorname*{Res}_{z=1} f(z) = -1$. By Cauchy's Residue Theorem, we have $\displaystyle\int_C f(z)\,dz = 2\pi i(5-1) = \boxed{8\pi i}$. •

We should note that not all singularities have meaningful residues. In this course, we will distinguish three types of singularities, but will focus only on one.

Definition 94.4. Let $z_0 \in \mathbb{C}$ be a singularity of f. If the Laurent coefficients c_k satisfy $c_k = 0$ for all $k < 0$, then z_0 is a *removable singularity* of f.

Example 94.5. Let $f(z) := (1 - \cos z)/z^2$. Then f has an isolated singularity at $z_0 = 0$ but when $z \neq 0$, f has the Laurent series expansion

$$f(z) = \frac{1}{2} - \frac{z^2}{4!} + \frac{z^4}{6!} - \cdots,$$

which has no negative terms. We can "patch" the singularity by extending $f(0) = 1/2$, and then f would in fact be an entire function.

We can consider the opposite extreme, though we will not see examples.

Definition 94.6. Let $z_0 \in \mathbb{C}$ be a singularity of f. If infinitely many of the c_k are nonzero when $k < 0$, then z_0 is an *essential singularity* of f.

Finally, we consider the "usual" case.

Definition 94.7. Let $z_0 \in \mathbb{C}$ be a singularity of f. If only finitely many of the c_k are nonzero when $k < 0$, then z_0 is a *pole*. The *order* of the pole is the "last" k where c_k is nonzero: i.e., it is the integer $m > 0$ such that $c_{-m} \neq 0$ but $c_{-n} - 0$ for all $n > m$. In the case that $m = 1$, we say that z_0 is a *simple pole*.

98 Residues at Poles

When we know that a singular point is a pole, we can calculate its residue easily.

Theorem 98.1. *Suppose that f has an isolated singularity at $z_0 \in \mathbb{C}$. Then z_0 is a pole of order m if and only if $f(z) = \varphi(z)/(z - z_0)^m$, where φ is analytic and nonzero at z_0. If this is the case, then*

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{\varphi^{(m-1)}(z_0)}{(m-1)!}.$$

Example 98.2. Let $f(z) = \dfrac{z+4}{z^2+1}$. Then f has singularities at $\pm i$. At $z = i$, we write

$$f(z) = \frac{1}{z-i} \cdot \frac{z+4}{z+i} =: \frac{1}{z-1}\varphi(z).$$

We check that φ is analytic and nonzero at i, so $\operatorname*{Res}_{z=i} f(z) = \varphi(i) = \boxed{\dfrac{i+4}{2i}}$.

We also define the dual notion: a *zero* of order m.

Definition 98.3. Suppose f is analytic at z_0. We say that f has a *zero of order* m at z_0 if $f(z_0) = f'(z_0) = \cdots = f^{(m-1)}(z_0) = 0$, but $f^{(m)}(z_0) \neq 0$.

Theorem 98.4 (Factorization Theorem)**.** *Suppose f is analytic at z_0. Then z_0 is a zero of order m if and only if $f(z) = (z - z_0)^m g(z)$, where g is analytic and nonzero at z_0.*

The next theorem states that zeros become poles when placed in the denominator.

Theorem 98.5. *Suppose p, q are analytic at z_0, $p(z_0) \neq 0$, and q has a zero of order m at z_0. Then p/q has a pole of order m at z_0.*

Proof. Applying Theorem 98.4, write $p/q = p/((z - z_0)^m g)$, where g is analytic and nonzero at z_0. By Theorem 98.1, write $\varphi := p/g$, so $p/q = \varphi/(z - z_0)^m$. Now $\varphi(z_0) \neq 0$ and φ is analytic at z_0 since p and g are analytic at z_0. Hence z_0 is a pole of order m. $\square$

Example 98.6. Let $f(z) := 1/(1 - \cos z)$. Then $p(z) = 1$ and $q(z) = 1 - \cos z$. Fix $z_0 = 0$. Then $q(0) = q'(0) = 0$, but we verify $q''(0) = \cos 0 = 1 \neq 0$. Hence, z_0 is a zero of order 2 of q, so it is a pole of order 2 of f.

The next theorem simplifies our calculations when a pole is simple.

Theorem 98.7. *Let p, q be analytic at z_0, $p(z_0) \neq 0$, and z_0 be a simple zero of q: i.e., $q(z_0) = 0$ and $q'(z_0) \neq 0$. Then p/q has a simple pole at z_0, and the residue is given by*

$$\operatorname*{Res}_{z=z_0} \frac{p}{q} = \frac{p(z_0)}{q'(z_0)}.$$

Proof. Write $p/q = p/[(z - z_0)g] = \varphi/(z - z_0)$ as before. By Theorem 98.1, we have $\operatorname*{Res}_{z=z_0} p/q = \varphi(z_0)$, so that $\varphi(z_0) = p(z_0)/g(z_0)$. The product rule shows that $q' = (z - z_0)g' + g$, so $q'(z_0) = g(z_0)$. $\square$

We now view some examples.

Example 98.8. Let $f(z) := \cot z = \cos z / \sin z$. Notice that $\sin z$ has simple zeros at $n\pi$, $n \in \mathbb{Z}$, so Theorem 98.7 tells us that $\cot z$ has simple poles at $n\pi$, and

$$\operatorname*{Res}_{z=n\pi} \cot z = \frac{\cos(n\pi)}{\cos(n\pi)} = \boxed{1}.$$

Example 98.9. Let $f(z) = z/(z^4+4)$. Then z^4+4 has a simple zero at $z_0 = 1+i$, so z_0 is a simple pole of f. Hence

$$\operatorname*{Res}_{z=1+i} \frac{z}{z^4+4} = \frac{z_0}{4z_0^3} = \boxed{-\frac{i}{8}}.$$

100 Improper Real Integrals

Recall that if $f : [0,\infty) \to \mathbb{R}$ is continuous, we define $\displaystyle\int_0^\infty f(x)\,dx := \lim_{t\to\infty} \int_0^t f(x)\,dx$, whenever the limit exists. Similarly, if $f : \mathbb{R} \to \mathbb{R}$ is continuous, we define by "splitting" the integral

$$\int_{-\infty}^{\infty} f(x)\,dx := \lim_{s\to\infty} \int_{-s}^{0} f(x)\,dx + \lim_{t\to\infty} \int_0^t f(x)\,dx.$$

But this is not the only way to define an integral across the real line.

Definition 100.1. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. We define the *Cauchy principal value* by

$$\mathsf{PV} \int_{\mathbb{R}} f(x)\,dx := \lim_{t\to\infty} \int_{-t}^{t} f(x)\,dx.$$

Lemma 100.2. *If $\int_{\mathbb{R}} f(x)\,dx$ exists, then* PV $\int_{\mathbb{R}} f(x)\,dx = \int_{\mathbb{R}} f(x)\,dx$.

Hence, the principal value agrees with the standard integral whenever the standard integral exists. However, the converse is not always true.

Example 100.3. Clearly, $\int_{-t}^{t} x\,dx = 0$, so PV $\int_{\mathbb{R}} x\,dx = 0$. But $\int_{\mathbb{R}} x\,dx$ diverges.

However, the converse holds if f is even:

Proposition 100.4. *Let f be even. Then $\int_{\mathbb{R}} f(x)\,dx =$ PV $\int_{\mathbb{R}} f(x)\,dx$, provided that the standard integral exists.*

Proof. Since f is even, we have $\int_0^t f(x)\,dx = \frac{1}{2}\int_{-t}^{t} f(x)\,dx$, so as $t \to \infty$, we observe that

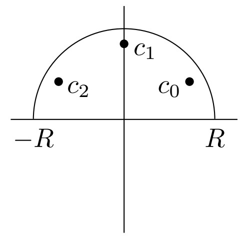
$$\int_0^t f(x)\,dx = \frac{1}{2}\,\mathsf{PV} \int_{\mathbb{R}} f(x)\,dx.$$

Doubling finishes the proof. □

We now show how to calculate PV $\int_{\mathbb{R}} f(x)\,dx$ in a special case.

Example 100.5. Find $\displaystyle\int_{\mathbb{R}} \frac{dx}{x^6+1}$.

Solution. Let $f(x) = 1/(x^6+1)$. Notice that x^6+1 only has complex roots, and we consider the ones with positive imaginary part:



We also draw a semicircle $C_R : |z| = R$, with $R > 1$, oriented positively. Notice that by Cauchy's Residue Theorem, we have that

$$\int_{-R}^{R} f(x)\,dx + \int_{C_R} f(x)\,dx = 2\pi i \sum_{k=0}^{2} \operatorname*{Res}_{z=c_k} f(z).$$

We have that

$$\operatorname*{Res}_{z=c_k} f(z) = \frac{1}{6c_k^5} = -\frac{c_k}{6}, \text{ so that}$$

$$\sum_{k=0}^{2} \operatorname*{Res}_{z=c_k} f(z) = -\frac{1}{6}(2i) = -\frac{i}{3}.$$

Now, we claim that $\int_{C_R} f(z)\,dz \to 0$ as $R \to \infty$. Indeed, if z lies on C_R, we have $|z^6 + 1| \geq \big||z^6| - 1\big| = R^6 - 1$, so that

$$|f(z)| \leq \left|\frac{1}{R^6 - 1}\right| = \frac{1}{R^6 - 1}$$

on C_R, so by the Estimation Lemma,

$$\left|\int_{C_R} f(z)\,dz\right| \leq \frac{1}{R^6 - 1} \cdot \pi R \to 0 \text{ as } R \to \infty.$$

Taking the limit as $R \to \infty$, we have

$$\text{PV} \int_{\mathbb{R}} f(x)\,dx = 0 + 2\pi i \left(-\frac{i}{3}\right) = \frac{2\pi}{3}.$$

Since f is even, we have $\int_{\mathbb{R}} f(x)\,dx = \boxed{2\pi/3}$. •

This motivates our generalized case.

Theorem 100.6. *Let $f(x) = p(x)/q(x)$, where $p, q \in \mathbb{R}[x]$, $\gcd(p, q) = 1$, $\deg q \geq \deg p + 2$, and q has no real roots but complex roots $\{z_k\}_{k=1}^{n}$ with $\operatorname{Im} z_k > 0$ for all k. Then*

$$\text{PV} \int_{\mathbb{R}} f(x)\,dx = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z).$$

104 Trigonometric Improper Integrals

Consider the integral $\int_{\mathbb{R}} f(x) \sin ax\,dx$, where $f(x) \in \mathbb{R}(x)$. Here, the semicircle technique would not work, as $|\sin az|$ is unbounded. Instead, we note that

$$\int_{-R}^{R} f(x)\cos ax\,dx + i\int_{-R}^{R} f(x)\sin ax\,dx = \int_{-R}^{R} f(x)e^{iax}\,dx,$$

so now $|e^{iaz}| = |e^{iax}| \cdot |e^{-ay}| = |e^{-ay}| \leq 1$, which is nicely bounded.

Example 104.1. Evaluate $\displaystyle\int_0^\infty \frac{\cos 2x}{(x^2+4)^2}\,dx.$

Solution. We define $f(x) = 1/(x^2+4)^2$, so f only has $2i$ as a zero *above* the real axis. Denote $C_R : |z| = R$, $\operatorname{Im} z \geq 0$ and $R > 2$. Now, if $B := \operatorname*{Res}_{z=2i} f(z)e^{2zi}$, by the Cauchy Residue Theorem

$$\int_{-R}^{R} \frac{e^{2xi}}{(x^2+4)^2}\,dx = 2\pi iB - \int_{C_R} f(z)e^{2iz}\,dz. \tag{1}$$

Write $f(z)e^{2iz} = \varphi(z)/(z-2i)^2$, where $\varphi(z) := e^{2iz}/(z+2i)^2$, which is analytic and nonzero at $z = 2i$. Hence $z = 2i$ is a zero of order 2 of $f(z)e^{2iz}$, and

$$B = \operatorname*{Res}_{z=2i} f(z)e^{2iz} = -\frac{5i}{32e^4}.$$

Hence, equation 1 becomes

$$\int_{-R}^{R} \frac{e^{2ix}}{(x^2+4)^2}\,dx = \frac{5\pi}{16e^4} - \int_{C_R} f(z)e^{2iz}\,dz.$$

Taking the real part gives

$$\int_{-R}^{R} \frac{\cos 2x}{(x^2+4)^2}\,dx = \frac{5\pi}{16e^4} - \operatorname{Re}\int_{C_R} f(z)e^{2\pi i}\,dz.$$

Now, all is left is an Estimation Lemma argument. On the contour C_R, we check that $|f(z)| \leq 1/(R^2-4)$, and $|e^{2iz}| \leq 1$. Now

$$\left|\operatorname{Re}\int_{C_R} f(z)e^{2iz}\,dz\right| \leq \left|\int_{C_R} f(z)e^{2iz}\,dz\right| \leq \frac{\pi R}{(R^2-4)^2} \to 0$$

as $R \to 0$, so PV $\displaystyle\int_{\mathbb{R}} \frac{\cos 2x\,dx}{(x^2+4)^2} = \frac{5\pi}{16e^4}$. Halving due to evenness gives our desired integral: $\displaystyle\int_0^\infty \frac{\cos 2x\,dx}{(x^2+4)^2} = \boxed{\frac{5\pi}{32e^4}}$. •

107 Index of Integration Techniques

Review these: Definition 50.7 (by definition), Theorem 60.2 (FTC/Antiderivatives), Theorem 60.6 (Cauchy-Goursat), Corollary 64.6 (Deformation of Paths), Theorem 70.1 (GCIF), Theorem 94.2 (Cauchy Residue Theorem), and Sections 100 and 104 for techniques for real improper integrals.