UC Irvine Math 141 Spring 2024
Introduction to Topology

Professor: Dong (Doris) Yan
Teaching Assistant: Devansh Saluja
Notes: Timothy Cho

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Lecture Note Series #10

Introduction

These notes come from both the lecture and the discussion, and are roughly sorted by content. Sections are numbered chronologically using the following scheme by taking the section number modulo 10. Note that we have occasionally merged two sections for continuity reasons.

Day	Lecture	Discussion
Monday	0	1
Tuesday	2	3
Wednesday	4	5
Thursday	6	7
Friday	8	9

Additionally, the first digit (first two if the section number is three digits long) denotes the week that the lecture/discussion occurred in. It should be noted that not every lecture is recorded in these notes: some lectures were skipped, but despite this the notes should be comprehensible.

The text used was *Introduction to Topology*, 2e, by Theodore W. Gamelin and Robert E. Greene. Numbers in [brackets] refer to sections in the text.

10 Overview

In this course, we will study *point-set topology*, which comprises the basic language of topology. This forms the framework of *algebraic topology*, which will be a natural next step. In general, topology deals with geometric properties of a set like length or distance, which, intuitively, is how "far" the points are in some set S. In Euclidean space $\mathbb{R}^n$, we can compute the distance between two points by taking the magnitude of their difference as vectors. We will generalize this to metric spaces, and then later to topological spaces. We thus begin with our study of metric spaces below.

[1.1] Open and Closed Sets

Definition 10.1. A *metric* on a set X is a function $d : X \times X \to \mathbb{R}$ satisfying four properties:

1. Non-negativity: for all $x, y \in X$, we have $d(x, y) \geq 0$.
2. We have $d(x, y) = 0$ if and only if $x = y$.
3. Symmetry: we have $d(x, y) = d(y, x)$ for all $x, y \in X$.
4. Triangle Inequality: for all $x, y, z \in X$, we have $d(x, z) \leq d(x, y) + d(y, z)$.

From this, we get the natural structure a metric imposes on a set.

Definition 10.2. A *metric space* (X, d) is a set X, equipped with a metric $d : X \times X \to \mathbb{R}$.

Example 10.3. Familiar Euclidean space $\mathbb{R}^n$ is a metric space for all $n \in \mathbb{Z}^+$, with the metric

$$d(x, y) := \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2},$$

the standard distance function in n dimensions.

Example 10.4. Let S be any set, and let $B(S)$ denote the set of bounded functions $f : S \to \mathbb{R}$. Define a metric $d : B(S) \times B(S) \to \mathbb{R}$ by $d(f, g) = \sup\{|f(x) - g(x)| : x \in S\}$. Then d makes $(B(S), d)$ into a metric space. For example, we check that the triangle inequality holds for d. If $f, g, h \in B(S)$, then we know that

$$|f(x) - h(x)| = |f(x) - g(x) + g(x) - h(x)| \leq |f(x) - g(x)| + |g(x) - h(x)|.$$

Taking supremums of both sides of the inequality, we see

$$\sup_{x \in S} |f(x)-h(x)| \leq \sup_{x \in S} (|f(x) - g(x)| + |g(x) - h(x)|) \leq \sup_{x \in S} |f(x)-g(x)|+\sup_{x \in S} |g(x)-h(x)|,$$

from which it follows $d(f, h) \leq d(f, g) + d(g, h)$.

For make things more concrete, let $S = [0, 1]$. Then $d(\sin x, \cos x) = \sup_{x \in [0,1]} |\sin x - \cos x| = 1$.

13 Metric Spaces

We can often impose many metrics on a set X, and we may be interested in combining a couple of metrics. However, we must be careful with how we do this, as a combination of metrics with other functions may or may not give a metric. See the examples below.

Example 13.1. Let S be any set, and d be any metric on S.

(a) $4 + d$ is **not** a metric on S. This is because $(4 + d)(x, x) = 4 + d(x, x) = 4 + 0 = 4 \neq 0$, so $4 + d$ does not identify whether two points are identical.

(b) $e^d - 1$ is **not** a metric on S. For example, the triangle inequality fails: let $S = \mathbb{R}$ and d be the standard metric. Then $d(0, 1) = d(1, 2) = 1$, and $d(0, 2) = 2$, so if $e^d - 1$ were a metric, we must have

$$(e^d - 1)(0, 1) + (e^d - 1)(1, 2) \geq (e^d - 1)(0, 2),$$

which occurs if and only if $e^2 \leq 2e - 1 \leq 2e$, a contradiction as $e > 2$.

(c) $d - |d|$ is **not** a metric on S: notice that since $d \geq 0$, we must have $d - |d| = 0$, so that whenever $x \neq y$, $d(x, y) = 0$, which is not allowed.

(d) d^2 is **not** a metric on S. Again, we check the triangle inequality. Take $S = \mathbb{R}$, so that under the standard metric, $d(0, 1) = 1$, and $d(0, 1/2) = d(1/2, 1) = 1/2$. But now

$$d^2\left(\frac{1}{2}, 1\right) + d^2\left(0, \frac{1}{2}\right) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \not\geq 1 = d^2(0, 1),$$

so the triangle inequality fails.

However, we compare the following:

Proposition 13.2. *Let S be any set, and d be a metric on S. Then $\sqrt{d}$ is a metric on S.*

Proof. Notice that $d \geq 0$, so the function $\sqrt{d}$ is actually well-defined. Now, properties (1) through (3) of a metric obviously hold for $\sqrt{d}$, so we check the triangle inequality. Set $d(x, z) =: \alpha$, $d(x, y) =: \beta$, and $d(y, z) =: \gamma$. Since d itself is a metric, we have $\alpha \leq \beta + \gamma$, but this implies

$$\alpha \leq \beta + \gamma + 2\sqrt{\beta\gamma} = \left(\sqrt{\beta} + \sqrt{\gamma}\right)^2.$$

Taking square roots shows $\sqrt{\alpha} \leq \sqrt{\beta} + \sqrt{\gamma}$, so back-substituting shows $\sqrt{d}$ is a metric. $\square$

Proposition 13.3. *Let S be any set, and d be a metric on S. Then $k \cdot d$ is also a metric on S, for any $k > 0$. That is, positive scalar multiples of a metric are still metrics.*

Proof. For brevity, we verify only the triangle inequality; the rest is trivial. Let $x, y, z \in S$, so that $d(x,z) \leq d(x,y) + d(y,z)$. Now, since $k > 0$, the result follows immediately upon multiplying both sides of the inequality by k. $\square$

Proposition 13.4. *Let d_1, d_2 be metrics on a set S. Then $d_1 + d_2$ is also a metric.*

Proof. Notice $d_1 + d_2 \geq d_1, d_2$ so the basic properties hold. We verify the triangle inequality; let $x, y, z \in X$ so that

$$d_1(x,z) \leq d_1(x,y) + d_1(y,z) \text{ and}$$
$$d_2(x,z) \leq d_2(x,y) + d_2(y,z).$$

Adding the inequalities gives

$$d_1(x,z) + d_2(x,z) \leq (d_1(x,y) + d_2(x,y) + (d_1(y,z) + d_2(y,z))$$
$$\iff (d_1 + d_2)(x,z) \leq (d_1 + d_2)(x,y) + (d_1 + d_2)(y,z).$$

Thus, the triangle inequality holds for $d_1 + d_2$, so it is a metric. $\square$

The two propositions above (with induction on Prop. 13.4) combine to give us the following statement: *any positive linear combination of metrics is a metric.*

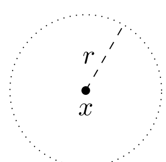
14 Open and Closed Sets

In this section, let (X, d) be a metric space.

Definition 14.1. Let $x \in X$ and $r \in \mathbb{R}^+$. We define the *open ball centered at x with radius r* by

$$B(x,r) := \{y \in X : d(x,y) < r\}.$$

Geometrically, say in $X = \mathbb{R}^2$ with the standard metric, we do indeed get an open two-dimensional ball (i.e., disk):



It is not too difficult to see the following properties about the open ball, so we state them without proof.

Proposition 14.2. *Let $x \in X$. Then*

1. *If $0 \leq r_1 \leq r_2$, then $B(x, r_1) \subseteq B(x, r_2)$;*
2. *We have $\bigcup_{r>0} B(x,r) = X$, and*
3. *We have $\bigcap_{r>0} B(x,r) = \{x\}$.*

Balls are useful, in that they allow us to describe whether a point is "inside" a set.

Definition 14.3. Let $Y \subseteq X$. We say that $x \in X$ is an *interior point* of Y if there exists some $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq Y$. The set of all interior points of Y is called the *interior of* Y, and is denoted $\text{int}(Y)$.

From here, we define open sets on a metric space X.

Definition 14.4. Let $Y \subseteq X$. Then Y is an *open set* if $\text{int}(Y) = Y$.

Unpacking this definition some more, we say Y is open if for every $y \in Y$, there is some (possibly) small $\varepsilon > 0$ such that the (possibly) small ball $B(y, \varepsilon)$ is contained within Y.

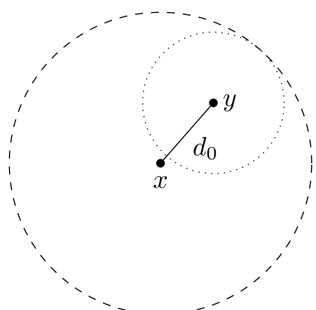
Proposition 14.5. *Let $x \in X$ and $r > 0$. Then $B(x, r)$, the open ball, is an open set.*

Proof. Let $y \in B(x, r)$, so that $d(x, y) =: d_0 < r$. Take $\varepsilon := r - d_0 > 0$, and consider the ball $B(y, r - d_0)$. We claim that $B(y, r - d_0) \subseteq B(x, r)$; to see this, pick $z \in B(y, r - d_0)$. Then $d(y, z) < r - d_0$ by definition. Hence

$$d(x, z) \leq d(x, y) + d(y, z) < d_0 + (r - d_0) = r,$$

so $z \in B(x, r)$, which completes the proof. □

The situation in the proof of Proposition 14.5 can be illustrated below.



Proposition 14.6. *Let X be a metric space, A be an indexing sets with $U_\alpha \subseteq X$ an open set for all $\alpha \in A$. Then $\bigcup_{\alpha \in A} U_\alpha$ is an open set. That is, the arbitrary union of open sets is also open.*

Proof. Take $x \in \bigcup_{\alpha \in A} U_\alpha$, so $x \in U_\alpha$ for some $\alpha \in A$. Since U_α is open by assumption, there exists $\varepsilon > 0$ such that $B(x, e) \subseteq U_\alpha \subseteq \bigcup_{\alpha \in A} U_\alpha$, so the indexed union is open. □

Proposition 14.7. *Let X be a metric space. Then $U \subseteq X$ is open if and only if U is a union of open balls.*

Proof. ($\Longleftarrow$): This follows directly from Proposition 14.6.

($\Longrightarrow$): Let $U \subseteq X$ be open. Then, for all $x \in U$, there exists some $r_x > 0$ such that $B(x, r_x) \subseteq U$. Then clearly $U = \bigcup_{x \in U} B(x, r_x)$, so now again appeal to the previous result. □

Proposition 14.8. *Let X be a metric space. Then if $U_i \in X$ is open for all $i = 1, 2, \ldots, n$, then $\bigcap_{i=1}^{n} U_i$ is open. That is, a **finite** intersection of open sets is open.*

Proof. Take $x \in \bigcap_{i=1}^{n} U_i$, so $x \in U_i$ for all $i \leq n$. Take radii $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ such that $B(x, \varepsilon_i) \subseteq U_i$ by the openness of each U_i, and set $r := \min\{\varepsilon_1, \ldots, \varepsilon_n\}$. Then $B(x, r) \subseteq B(x, \varepsilon_i) \subseteq U_i$ for all $i \leq n$, which completes the proof. $\square$

However, infinite intersections of open sets are not necessarily open.

Example 14.9. Over $\mathbb{R}$, consider the real intervals $\left(-\frac{1}{n}, \frac{1}{n}\right)$ for all $n \in \mathbb{Z}^+$. Then we know that $\bigcup_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$, which is not an open set: $B(0, \varepsilon) \nsubseteq \{0\}$ for all $\varepsilon > 0$.

We now turn to *closed* sets, after defining some terminology. Again, let X be a metric space.

Definition 14.10. Let $Y \subseteq X$. A point $x \in X$ is *adherent to* Y if for all $r > 0$, the set $B(x, r) \cap Y$ is nonempty.

Of course, if $x \in Y$, then x is automatically adherent to Y (the intersection $B(x, r) \cap Y$ contains $x \in Y$), but these are not all the adherent points of Y: if $Y := (-1, 1) \subseteq \mathbb{R}$, then $x = 1$ is adherent to Y without being a member of Y.

Definition 14.11. The *closure* of a subset $Y \subseteq X$, denoted $\overline{Y}$, is the set of all points in X that are adherent to Y.

From our remark above, it follows that $Y \subseteq \overline{Y}$.

Definition 14.12. A subset $Y \subseteq X$ is *closed* if $Y = \overline{Y}$.

In particular, $\varnothing$ and X are both vacuously closed (in fact, they are *also* vacuously open). We should also prove that the closure is in fact, indeed, closed:

Proposition 14.13. *Let $Y \subseteq X$. Then the closure $\overline{Y}$ is closed.*

Proof. We must show that $\overline{Y} = \overline{\overline{Y}}$. One inclusion is trivial, so suppose $x \in \overline{\overline{Y}}$. Let $r > 0$, so that it suffices to show $B(x, r) \cap Y$ is nonempty, from which it would follow that x is adherent to Y, i.e., $x \in \overline{Y}$. Now, since x is at least adherent to $\overline{Y}$, there exists a point $z \in B(x, r/2) \cap \overline{Y}$. But then $z \in \overline{Y}$, i.e., z is adherent to Y, so there exists some point $y \in B(z, r/2) \cap Y$. Then we see $d(x, y) \leq d(x, z) + d(z, y) < r/2 + r/2 = r$, so $y \in B(x, r) \cap Y$, so $B(x, r) \cap Y$ is nonempty. $\square$

17 Metric Spaces: Worked Examples

Example 17.1. We prove that $d_\infty(x, y) := \max\{|x_1 - y_1|, |x_2 - y_2|\}$ is a metric on $\mathbb{R}^2$.

Proof. Clearly, $d_\infty \geq 0$ a. That $d_\infty(x, y) = 0$ if and only if $x = y$ is obvious, and so is $d_\infty(x, y) = d_\infty(y, x)$ for all $x, y \in \mathbb{R}^2$. It remains to verify the triangle inequality. Write $x = (x_1, x_2)$, $y = (y_1, y_2)$, and $z = (z_1, z_2)$, all in $\mathbb{R}^2$. Then $d_\infty(x, z) = \max\{|x_1 - z_1|, |x_2 - z_2|\}$ but now note

$$|x_1 - z_1| = |x_1 - y_1 + y_1 - z_1| \leq |x_1 - y_1| + |y_1 - z_1| \text{ and}$$
$$|x_2 - z_2| = |x_2 - y_2 + y_2 - z_2| \leq |x_2 - y_2| + |y_2 - z_2|.$$

Thus,

$$|x_1 - z_1|, |x_2 - z_2| \leq \max\{|x_1 - y_1|, |x_2 - y_2|\} + \max\{|y_1 - z_1|, |y_2 - z_2|\},$$

so that $d_\infty(x, z) = \max\{|x_1 - z_1|, |x_2 - z_2|\} \leq d_\infty(x, y) + d_\infty(y, z)$. This completes the proof. $\square$

The above example can be generalized meaningfully.

Proposition 17.2. *If $\{d_i\}_{i=1}^n$ is a list of metrics on a set S, then $d_\infty := \max_{1\leq i\leq n} d_i$ is a metric on S.*

Proof. The basic properties of a metric hold trivially, so we again verify the triangle inequality. Let $x, y, z \in S$. Then $d_\infty(x,z) = \max_{1\leq i\leq n} d_i(x,z)$, but since the d_i are metrics, we have

$$d_i(x,z) \leq d_i(x,y) + d_i(y,z),$$

for all $1 \leq i \leq n$. Thus,

$$d_1(x,z), d_2(x,z), \ldots, d_n(x,z) \leq \max_{i\leq n} d_i(x,y) + \max_{i\leq n}(y,z) = d_\infty(x,y) + d_\infty(y,z).$$

Since $d_\infty(x,z) = \max_{i\leq n} d_i(x,z) = d_j(x,z)$ for some $j \leq n$, the triangle inequality follows. $\square$

Definition 17.3. Let (X,d) be a metric space, and $Y \subseteq X$. Then $x \in X$ is a *limit point of* Y if for all $\varepsilon > 0$, there exists some $y \in Y$, with $y \neq x$, such that $q \in B(x,\varepsilon)$.

Note the very subtle difference between limit points and adherent points. The limit point definition is asserting that the set $(B(x,\varepsilon) \cap Y) \setminus \{x\}$ is nonempty for every $\varepsilon > 0$, while the definition of an adherent point (Definition 14.10) only requires $B(x,\varepsilon) \cap Y$ to be nonempty for every $\varepsilon > 0$. Thus, every limit point is an adherent point, but not the other way around.

Example 17.4. Let X be a metric space, $Y \subseteq X$, and p a limit point of Y. We claim that for each $r > 0$, the subset $B(p,r) \cap Y$ is infinite:

Proof. Suppose for contradiction otherwise, so that there exists some $r > 0$ such that $B(p,r) \cap Y$ is finite, so we may enumerate in particular

$$(B(p,r) \cap Y) \setminus \{x\} = \{q_1, q_2, \ldots, q_n\},$$

for q_i distinct. Find q_i such that $d(p,q_i) =: r_0$ is minimized, and take $\varepsilon = \frac{1}{2}r_0$. Now $B(p,\varepsilon) \cap Y \subseteq B(p,r) \cap Y$, but none of the q_i are in $B(p,\varepsilon)$ by construction, so that $(B(p,\varepsilon) \cap Y) \setminus \{x\}$ is empty. This contradicts the fact that p is a limit point. $\square$

18 Open and Closed Sets

We continue with some more basic results about open and closed sets.

Proposition 18.1. *A subset Y of a metric space X is closed if and only if $X \setminus Y$ is open.*

Proof. ($\implies$): Suppose $Y \subseteq X$ is closed, so that $Y = \overline{Y}$, i.e., any $y \in X$ that is adherent to Y is a element of Y. Now, fix $x \in X \setminus Y$. Since $x \notin Y$, x cannot be adherent to Y, i.e., there exists some $r > 0$ such that $B(x,r) \cap Y$ is empty, i.e., $B(x,r) \subseteq X \setminus Y$. This shows x is an interior point of $X \setminus Y$, so $X \setminus Y$ is open.

($\impliedby$): Suppose $X \setminus Y$ is open, so that for any $x \in X \setminus Y$, there exists $r_x > 0$ such that $B(x,r_x) \subseteq X \setminus Y$. It thus follows that $B(x,r_x) \cap Y = \varnothing$, so x cannot be adherent to Y. Hence if $y \in X$ is any adherent point to Y, then $y \notin X \setminus Y \iff y \in Y$. Hence $\overline{Y} \subseteq Y$, implying $\overline{Y} = Y$, so Y is closed. $\square$

Again, we remark that it follows that $\varnothing$ and X are both open and closed. A set that is both open and closed is called *clopen*.[1]

As with open sets, we can examine unions and intersections of closed sets. Let X be a metric space.

Proposition 18.2. *Let A be an indexing set. If for all $\alpha \in A$, the subset $E_\alpha \subseteq X$ is closed, then $\bigcap_{\alpha \in A} E_\alpha$ is closed. If A is* ***finite****, then $\bigcup_{\alpha \in A} E_\alpha$ is closed. That is, an arbitrary intersection of closed sets is closed, but only finite unions of closed sets are closed.*

Proof. This follows from the set identities $X \setminus \bigcap_{\alpha \in A} E_\alpha = \bigcup_{\alpha \in A}(X \setminus E_\alpha)$ and $X \setminus \bigcup_{\alpha \in A} E_\alpha = \bigcap_{\alpha \in A}(X \setminus E_\alpha)$, the result about open sets in Proposition 14.8, and the previous Proposition 18.1, which states that a set is open if and only if its complement is closed. $\square$

Inside metric spaces, we can impose concepts from analysis.

Definition 18.3. Let X be a metric space. We say that a sequence $(x_n) \subseteq X$ *converges to* $x \in X$ if

$$\lim_{n \to \infty} d(x_n, x) = 0.$$

The element x is called the *limit* of (x_n), and we write $(x_n) \to x$ or $\lim x_n = x$.

It is not too difficult to see that $(x_n) \to x$ if and only if for all $\varepsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that $n > N$ implies $d(x_n, x) < \varepsilon$.

Proposition 18.4. *The limit of a convergent sequence in a metric space X is unique.*

Proof. Let $(x_n) \subseteq X$ be a convergent sequence, and x, y be limits of (x_n). Now there exist N_1, N_2 such that whenever $n > N_1$, we have $d(x_n, x) < \varepsilon$, and when $n > N_2$, we have $d(x_n, y) < \varepsilon$. Take $n > \max\{N_1, N_2\}$, so that

$$d(x, y) \leq d(x, x_n) + d(x_n, y) < \varepsilon + \varepsilon = 2\varepsilon,$$

which can be made arbitrarily small. Hence $d(x, y) = 0$, so $x = y$. $\square$

Theorem 18.5. *Let X be a metric space and $Y \subseteq X$. Then $x \in X$ is adherent to Y if and only if there exists a sequence in Y that converges to X.*

Proof. ($\Longleftarrow$): Let $(y_n) \subseteq Y$ converge to $x \in X$, so that for all $\varepsilon > 0$, there exists some $N > 0$ such that whenever $n > N$, we have $d(y_n, x) < \varepsilon$. This implies that there exists some sequence member $y_n \in Y$ in each ball $B(x, \varepsilon)$, for all $\varepsilon > 0$; i.e., $B(x, \varepsilon) \cap Y$ is nonempty for all $\varepsilon > 0$. Thus, x is adherent to Y.

($\Longleftarrow$): Suppose $x \in X$ is adherent to Y, so that $B(x, r) \cap Y$ is nonempty for all $r > 0$. For each $n \in \mathbb{Z}^+$, pick $y_n \in Y$ such that $d(x, y_n) < 1/n$, i.e., $y_n \in B(x, 1/n) \cap Y$. We claim $(y_n) \to x$, and this could be seen as for any $\varepsilon > 0$, by the Archimedean property of $\mathbb{R}$, there exists $N \in \mathbb{Z}+$ such that $1/N \leq \varepsilon$, so that whenever $n > N$,

$$d(x, y_n) < \frac{1}{n} < \frac{1}{N} \leq \varepsilon.$$

Thus $(y_n) \to x$, and we are done. $\square$

[1] And yes, this is an actual word.

[1.2] Completeness

The definition of convergence of a sequence $(x_n) \to x$, in a metric space, involves the limit x *existing*. However, this is a problem if x is unknown, or x does not exist (sequences in $\mathbb{Q}$, for example, do this). To fix this, let us weaken our requirements slightly.

Definition 18.6. Let X be a metric space. A sequence (x_n) is *Cauchy* if $\lim\limits_{m,n\to\infty} d(x_n, x_m) = 0$, i.e., for all $\varepsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that $d(x_n, x_m) < \varepsilon$ whenever $n, m > N$.

We check that this is indeed a weakening of our notion of convergence.

Proposition 18.7. *Any convergent sequence is a Cauchy sequence.*

Proof. Suppose that $(x_n) \to x$ in a metric space X. Fix $\varepsilon > 0$, and take $N \in \mathbb{Z}^+$ such that for all $n > N$, we have $d(x_n, x) < \varepsilon$. Take $n, m > N$, so that

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \varepsilon + \varepsilon = 2\varepsilon,$$

which can be made arbitrarily small. $\square$

Proposition 18.8. *If (x_n) is a Cauchy sequence in a metric space X, which contains a subsequence (x_{n_k}) converging to some $x \in X$, then $(x_n) \to X$.*

Proof. Since (x_n) is Cauchy, fix $\varepsilon > 0$ and pick N such that whenever $m, n > N$, we have $d(x_n, x_m) < \varepsilon$. Since $(x_{n_k}) \to x$, there exists N' such that whenever $k > N'$, we have $d(x_{n_k}, x) < \varepsilon$. Now, take $n_k, n > \max N, N'$, so that

$$d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x) < \varepsilon + \varepsilon = 2\varepsilon,$$

which can be made arbitrarily small. $\square$

20 Completeness (I)

With the vocabulary from the previous section, we make the following definition.

Definition 20.1. A metric space X is *complete* if every Cauchy sequence in X converges.

Example 20.2. Let $X = (0, 1)$, the real interval from 0 to 1, endowed with the standard Euclidean metric d. Then (X, d) is a metric space, but it is not complete: consider the sequence $(x_n) = \frac{1}{n}$. However, its closure $\overline{X} = [0, 1]$ is a complete metric space.

In general, every metric space X has a completion $\overline{X}$. This is not at all obvious to show, however. Let us view some more examples.

Example 20.3. Let $X = \mathbb{Q}$, under the standard Euclidean metric d. We see that $\mathbb{Q}$ is not complete; consider the sequence (x_n) given by the digits of π: $x_1 = 3.1$, $x_2 = 3.14$, $x_3 = 3.141, \ldots$, so that $x_n \to \pi \notin \mathbb{Q}$.

Next, we prove the following fact about complete metric spaces. We say that $Y \subseteq X$ is a *subspace* of X when we regard Y with the restricted metric $d|_Y$.

Proposition 20.4. *A closed subspace of a complete metric space is complete.*

Proof. Let X be a complete metric space, and $Y \subseteq X$ be a closed subspace. Let $(y_n) \subseteq Y$ be a Cauchy sequence. Since X is complete, $(y_n) \to X$ for some $x \in X$. Hence, it suffices to show that $x \in \overline{Y}$, as $Y = \overline{Y}$. Since $(y_n) \to x$, for all $\varepsilon > 0$, there exists an $n \in \mathbb{Z}^+$ such that $d(y_n, x) < \varepsilon$, i.e., $B(x, \varepsilon)$ contains an element of Y for all $\varepsilon > 0$. This shows $x \in \overline{Y} = Y$, so we are done. □

The converse to the previous proposition also holds.

Proposition 20.5. *A complete subspace of a metric space must be closed.*

Proof. Let X be a metric space and $Y \subseteq X$ be a subspace. Let $x \in X$ be adherent to Y, so that Theorem 18.5 tells us that there exists a sequence $(y_n) \subseteq Y$ with $(y_n) \to x$. By completeness of Y, we see $x \in Y$, so that $Y = \overline{Y}$ and Y is closed. □

We introduce a new concept: the idea of denseness.

Definition 20.6. Let X be a metric space. A subset $Y \subseteq X$ is *dense* in X if $\overline{Y} = X$.

That is, a set Y is dense if and only if every point in the whole space is adherent to Y. It immediately follows that X is dense in X, and to show that a set is dense, it suffices to show $X \subseteq \overline{Y}$, i.e., for any $x \in X$, there exists a sequence $(y_n) \subseteq Y$ with $(y_n) \to x$.

Example 20.7. Take $X = \mathbb{R}$ and $Y = \mathbb{Q}$. We know from analysis that $\mathbb{Q}$ is dense in $\mathbb{R}$, but we can also consider each real number x as an expansion $x = k + \sum_{i=1}^{\infty} 10^{-i} d_i$, where $k \in \mathbb{Z}$ and $0 \le d_i \le 9$. Then the sequence $(y_n) := k + \sum_{i=1}^{n} 10^{-i} d_i$ converges to x.

Theorem 20.8 (Baire Category Theorem)**.** *Let $U_1, U_2, \ldots$ be a family of dense, open subsets of a complete metric space X. Then $U := \bigcap_{n=1}^{\infty} U_n$ is also dense in X.*

The proof of the Baire Category Theorem is the first major result we will establish, so we first sketch the idea of the proof. We clearly know $\overline{U} \subseteq X$. Now, if $x \in X$, it suffices to show that for all $\varepsilon > 0$, we have $B(x, \varepsilon) \cap U \neq \varnothing$. We now see the proof.

Proof. That $\overline{U} \subseteq X$ is obvious. ow, fix some $x \in X$, and fix $\varepsilon > 0$. We must find an element $y \in B(x, \varepsilon) \cap U$. The rest of the proof now proceeds by repeated, chained definition chasing. Since U_1 is dense, there exists some $y_1 \in B(x, \varepsilon) \cap U_1$, as x is an adherent point of U_1. Since U_1 is open and $B(x, \varepsilon)$ is open, the intersection $B(x, \varepsilon) \cap U_1$ is open as well, i.e., there exists some $r_1 < 1$ (make the ball smaller if needed) such that

$$B(y_1, r_1) \subseteq B(x, \varepsilon) \cap U_1.$$

Now, we continue this chain inductively: find $y_2 \in U_2$ with $y_2 \in B(y_1, r_1) \cap U_2$ by density of U_2 (so $y_1 \in U_1 \subseteq X$ is adherent to U_2), and note that there exists $r_2 < 1/2$ (again, shrink the radius if necessary) with $\overline{B(y_2, r_2)} \subseteq B(y_1, r_1) \cap U_2$, and continue this to get the sequences $(y_k) \subseteq X$ and $(r_n) \subseteq \mathbb{R}$ (with $r_n < 1/n$ for all n), such that

$$\overline{B(y_k, r_k)} \subseteq B(y_{k-1}, r_{k-1}) \cap U_k \text{ for all } k \ge 2.$$

Now, it follows that $\overline{B(y_k, r_k)} \subseteq B(x, \varepsilon)$, and (y_n) is a Cauchy sequence: if $m > n$, note $d(y_n, y_m) < r_n \le 1/n \to 0$. Since X is complete, $(y_n) \to y$. But now $y \in B(x, \varepsilon) \cap U$, so we are done. □

24 Completeness (II)

Let X be a metric space, and $Y \subseteq X$. Recall that the following are equivalent:

1. Y is dense in X;
2. $\overline{Y} \supseteq X$;
3. For all $x \in X$, for any $\varepsilon > 0$, we have $Y \cap B(x, \varepsilon) \neq \varnothing$.
4. For all $U \subseteq X$ which is open and nonempty, then $Y \cap U \neq \varnothing$.
5. For all $x \in X$, there exists a sequence $(y_n) \subseteq Y$ such that $(y_n) \to x$.

Example 24.1. The rational numbers $\mathbb{Q}$ are dense in $\mathbb{R}$, as $\overline{\mathbb{Q}} = \mathbb{R}$. Taking any $x \in \mathbb{R}$, notice that for any $\varepsilon > 0$, there exists some rational number $r \in \mathbb{Q}$ with $r \in B(x, \varepsilon)$, by simply truncating x to enough decimal places.

Now, we examine the "opposite" of denseness.

Definition 24.2. Let X be a metric space. A subset $Y \subseteq X$ is *nowhere dense* if $\overline{Y}$ has no interior points: $\operatorname{int}(Y) = \varnothing$.

We remark that if a subset $Y \subseteq X$ is *any* subset, then the following holds:

1. $\operatorname{int}(Y) = X \setminus \overline{X \setminus Y}$;
2. $\overline{Y} = X \setminus \operatorname{int}(X \setminus Y)$,

which can be proven as an exercise. Then this follows immediately:

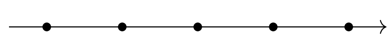
Proposition 24.3. *Let X be a metric space. Then $Y \subseteq X$ if nowhere dense if and only if $X \setminus \overline{Y}$ is dense in X.*

Proof. The following are logically equivalent:

$$\begin{aligned} Y \text{ is nowhere dense in } X &\iff \operatorname{int}(\overline{Y}) = \varnothing \\ &\iff X \setminus \overline{X \setminus \overline{Y}} = \varnothing \\ &\iff \overline{X \setminus \overline{Y}} = X, \end{aligned}$$

from which it follows that $X \setminus \overline{Y}$ is dense in X. $\square$

Example 24.4. The integers $\mathbb{Z}$ are nowhere dense in $\mathbb{R}$: the closure of $\mathbb{Z}$ is $\mathbb{Z}$ itself, which has no interior points.



Example 24.5. Let $S := \left\{ \frac{1}{n} : n \in \mathbb{Z}^+ \right\}$. Then $\overline{S} = S \cup \{0\}$, but $\overline{S}$ has no interior points, so S is nowhere dense in $\mathbb{R}$.

From here, we get the following restatement of the Baire Category Theorem.

Corollary 24.6. *Let $E_1, E_2, \ldots$ be a sequence of nowhere dense subsets of a complete metric space X. Then $\bigcup_{n=1}^{\infty} E_n$ has empty interior in X.*

It follows that nowhere dense sets are "small," in that the union of many of them still forms a set with empty interior. In contrast, the main statement of the Baire Category Theorem states that dense sets are "large" for a dual reason. This suggests two "categories" of sets, as suggested in the name of the theorem.

Definition 24.7. Let X be a metric space and $Y \subseteq X$.

1. We say Y is *of the first category* if it is the countable union of nowhere dense subsets.
2. We say Y is *of the second category* if it is not in the first category.

Thus, the statement of Cor. 24.6 can be rewritten: *The complement of a set of the first category is dense in* X. This gives an alternative way to establish the uncountability of $\mathbb{R}$.

Theorem 24.8. *The real numbers* $\mathbb{R}$ *are uncountable.*

Proof. Assume for contradiction that $\mathbb{R}$ were countable, so we enumerate $\mathbb{R} = \{x_1, x_2, x_3, \ldots\}$. For any $n \in \mathbb{Z}^+$, define $U_n := \{x_n\}$. Clearly, $U_n = \overline{U_n}$ is nowhere dense, as it is a singleton set. Now $\mathbb{R} = \cup_{i=1}^{\infty} U_i$ is of the first category, so Cor. 24.6 shows that $\mathbb{R}$ is nowhere dense. This is contradictory as $\overline{\mathbb{R}} = \mathbb{R}$, where all points are interior. $\square$

Example 24.9. To contrast the previous theorem, let $X = \mathbb{Q}$, which we can enumerate: $\mathbb{Q} = \{r_1, r_2, r_3, \ldots\}$. Take $U_n := \mathbb{Q} \setminus \{r_n\}$ for all $n \in \mathbb{Z}^+$, so that each U_n is open in $\mathbb{Q}$, and U_n is dense in $\mathbb{Q}$. It follows that $\overline{U_n} \cap \mathbb{Q} = \mathbb{Q}$, but $\bigcap_{n=1}^{\infty} U_n = \varnothing$. Note that this is not a contradiction with the Baire Category Theorem, as $\mathbb{Q}$ is *not* complete.

27 Closures and Interiors: Worked Examples

In this section, let X be a metric space.

Example 27.1. Let $A \subseteq X$. Show that $\overline{A}$ is the smallest closed set containing A.

Proof. We already know that $\overline{A}$ is closed. Take a closed subset $B \subseteq X$ containing A; we must show $\overline{A} \subseteq B$. If $x \in \overline{A}$, then for all $r > 0$, we have $B(x, r) \cap A \neq \varnothing$. But since $A \subseteq B$, we have $B(x, r) \cap B \neq \varnothing$, so $x \in \overline{B}$. But B is closed, so $x \in \overline{B} = B$ and we are done. $\square$

The above allows us to state $\overline{A}$ *is the intersection of all the closed subsets containing* A.

Example 27.2. It is not true that for any $A_1, A_2 \subseteq X$, we have $\overline{A_1 \cap A_2} \stackrel{?}{=} \overline{A_1} \cap \overline{A_2}$. To see this, take $X = \mathbb{R}$, $A_1 = (0, 1)$ and $A_2 = (1, 2)$, so that $\overline{A_1 \cap A_2} = \overline{\varnothing} = \varnothing$ and $\overline{A_1} \cap \overline{A_2} = [0, 1] \cap [1, 2] = \{1\}$.

Example 27.3. We prove $\overline{A_1} \cup \overline{A_2} = \overline{A_1 \cup A_2}$.

Proof. ($\subseteq$): Suppose $x \in \overline{A_1} \cup \overline{A_2}$. If $x \in \overline{A_1}$ (this proof is identical if $x \in \overline{A_2}$, the only other case), we see $B(x, r) \cap A_1 \neq \varnothing$, for all $r > 0$. Now

$$\varnothing \neq B(x, r) \cap A_1 \subseteq B(x, r) \cap (A_1 \cup A_2),$$

which shows $x \in \overline{A_1 \cup A_2}$.

($\supseteq$): Suppose $x \in \overline{A_1 \cup A_2}$. Then for all $r > 0$, $B(x, r) \cap (A_1 \cup A_2)$ is nonempty. By set algebra,

$$\varnothing \neq B(x, r) \cap (A_1 \cup A_2) = (B(x, r) \cap A_1) \cup (B(x, r) \cap A_2),$$

so at least one of the $B(x, r) \cap A_i$ must be nonempty. Hence $x \in \overline{A_1}$ or $x \in \overline{A_2}$, i.e., $x \in \overline{A_1} \cup \overline{A_2}$. $\square$

Example 27.4. However, it is *not* true that for any indexing set I and family of sets A_α for each $\alpha \in I$, that

$$\overline{\bigcup_{\alpha \in I} A_\alpha} \stackrel{?}{=} \bigcup_{\alpha \in I} \overline{A_\alpha}.$$

To see this, take $I = \mathbb{Z}^+$, and for each $n \in \mathbb{Z}^+$ take $A_n = \left(\frac{1}{n}, 2 - \frac{1}{n}\right)$ so that $\bigcup_{n=1}^\infty A_n = (0, 2)$. Thus $\overline{\bigcup_{n=1}^\infty A_n} = [0, 2]$, but $\bigcup_{n=1}^\infty \overline{A_n} = \bigcup_{n=1}^\infty \left[\frac{1}{n}, 2 - \frac{1}{n}\right] = (0, 2)$.

Example 27.5. We show that for any metric space X, the singleton $\{x\} \subseteq X$ is closed.

Proof. Take $y \in \overline{\{x\}}$. Then for all $r > 0$, $B(y, r) \cap \{x\}$ is nonempty, so $B(y, r) \supseteq \{x\}$, for all $r > 0$. If $y \neq x$, there exists some $\varepsilon > 0$ such that $\varepsilon < d(x, y)$, so $B(y, \varepsilon) \cap \{x\}$ must be empty, a contradiction. Thus $y = x$ and thus $\{x\}$ is closed. $\square$

28 Completeness of the Real Numbers

In this section, we begin by proving that $\mathbb{R}$ is complete, a fact that we have been using for a while. We start with the following definition, which should be familiar from analysis.

Definition 28.1. Let $S \subseteq \mathbb{R}$. We say S is *bounded above* if there exists some $M \in \mathbb{R}$ such that $s \leq M$ for all $s \in S$. Such an M is called an *upper bound*, the least of these upper bounds (if it exists) is called the *supremum* of the set S, denoted $\sup S$.

The following axiom also characterizes the real numbers.

Axiom 28.2 (Least Upper Bound Property)**.** If $S \subseteq \mathbb{R}$ is nonempty and bounded above, then $\sup S$ exists.

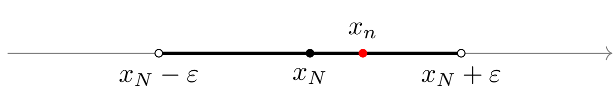
Again, this statement is used in the definition of what $\mathbb{R}$ is, and is enough to show that $\mathbb{R}$ is a complete metric space.

Theorem 28.3. *The real numbers $\mathbb{R}$, with the standard metric, form a complete metric space.*

Proof. Let $(x_n) \subseteq \mathbb{R}$ be a Cauchy sequence, and define

$$S := \{y \in \mathbb{R} : y > x_n \text{ for finitely many } n\}.$$

If $y \in S$, then any $z \in \mathbb{R}$ with $z < y$ satisfies $z \in S$, so that $(-\infty, y] \subseteq S$. That is, S is *closed below.* Now, since (x_n) is Cauchy, for every $\varepsilon > 0$ there exists some $N \in \mathbb{Z}^+$ such that $|x_n - x_m| < \varepsilon$ whenever $m, n \geq N$. Without loss of generality, we may write $|x_n - x_N| < \varnothing$, for any $n \geq N$:



Alternatively speaking, the interval $(x_N - \varepsilon, x_N + \varepsilon)$ contains all but finitely many terms in (x_n). This implies $x_N - \varepsilon \in S$, so S is, at the very least, nonempty. But S is bounded above by $x_N + \varepsilon$, so by Axiom 28.2, $\sup S$ must exist. Now, $\sup S \leq x_N + \varepsilon$ and $x_N - \varepsilon \leq \sup S$, so $|x_N - \sup S| \leq \varepsilon$. Now, the triangle inequality gives

$$|x_n - \sup S| \leq |x_n - x_N| + |x_N - \sup S| < \varepsilon + \varepsilon = 2\varepsilon$$

for all $n \geq N$, which can be made arbitrarily small, so it follows $(x_n) \to \sup S \in \mathbb{R}$. Thus, $\mathbb{R}$ is complete. $\square$

[1.4] Products of Metric Spaces

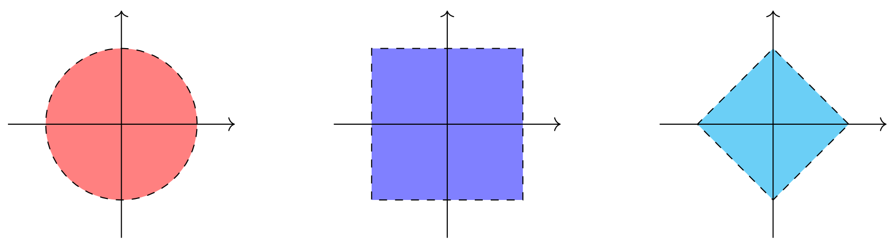
As with other mathematical structures, we are often concerned with putting them together to form larger structures. Thus, we begin our study of product spaces.

Definition 28.4. Let $X_1, \dots, X_n$ be metric spaces with respective metrics $d_1, \dots, d_n$. We define a *product space* $X_1 \times X_2 \times \cdots \times X_n$ to have the underlying Cartesian product as its underlying set, with any metric $d : (X_1 \times \cdots \times X_n)^2 \to \mathbb{R}$.

Example 28.5. We know $\mathbb{R}^n = \overbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}^{n \text{ times}}$. Define a metric d on $\mathbb{R}^n$ by

$$d(x, y) := \sqrt{\sum_{k=1}^{n} d_k^2(x_k, y_k)},$$

for any n metrics on $\mathbb{R}$ $d_1, d_2, \dots, d_n$. It is easy to show that d is a metric on $\mathbb{R}^n$, so that $(\mathbb{R}^n, d)$ is a metric space. Alternative metrics include $d'(x, y) = \max_{1 \le k \le n} d_k(x_k, y_k)$ or $d''(x, y) = \sum_{k=1}^{n} d_k(x_k, y_k)$. We know from previous work that all of these are metrics, but they induce different metric *spaces* on $\mathbb{R}^n$. For example, if we take all of the d_k to be the standard metric on $\mathbb{R}$, and take $n = 2$, the unit ball $B((0, 0), 1)$ looks like:



(a) In $(\mathbb{R}^2, d)$ (b) In $(\mathbb{R}^2, d')$ (c) In $(\mathbb{R}^2, d'')$

However, it turns out that most choices of metric on a space are inconsequential.

Theorem 28.6. *Let d be a metric on the product space $X_1 \times X_2 \times \cdots \times X_n =: X$ that satisfies the following property:*

($\star$) *A sequence $(x^j) \subseteq X$, where $x^j = (x_1^j, \dots, x_n^j)$ converges to $x = (x_1, \dots, x_n)$ if and only if for each $k \le n$, the sequence of components (x_k^j) converges to $x_k \in X_k$.*

Then the open sets in (X, d) are the unions of product sets of the form $U_1 \times \cdots \times U_n$, where the U_i are open in the X_i for all $i \le n$.

We remark that in dealing with many problems, the choices of the metric functions are not important. The theorem above shows that all metrics satisfying ($\star$) determine the same family of open sets, which we actually care about. We now turn to a proof of this theorem.

Proof. We first prove that $U_1 \times U_2 \times \cdots \times U_n \subseteq X$ is actually an open set, for all $U_i \subseteq X_i$ open. First, suppose generally that $E_k \subseteq X_k$ is closed for all $k \le n$; then for any $x = (x_1, x_2, \dots, x_n) \in E_1 \times \cdots \times E_n \subseteq X$, by closure of the E_i we have sequences $(x_i^j) \subseteq E_i$ that converge to x_i. By ($\star$), the sequence $(x^j) := (x_1^j, \dots, x_n^j)$ converges to $x = (x_1, \dots, x_n)$. But this means $E_1 \times \cdots \times E_n$ contains its limit points, so $E_1 \times \cdots \times E_n$ is closed. In particular, we now set $E_i = X_i \setminus U_i$ for one $i \in \{1, 2, \ldots, n\}$, and $E_j = X_j$ for all $i \neq j$, $j \leq n$. Then

$$X_1 \times \cdots \times X_{i-1} \times (X_i \setminus U_i) \times X_{i+1} \times \cdots \times X_n$$

is closed, so its complement

$$V_k := X_1 \times \cdots \times X_{i-1} \times U_i \times X_{i+1} \times \cdots \times X_n$$

is open. It follows that $U_1 \times \cdots \times U_n = \bigcap_{i=1}^n V_i$ is open. Moreover, the arbitrary union of open sets is open, so the unions of all such products of this form are open.

Now, we prove that any open subset of X is the union of products of the form $U_1 \times \cdots \times U_n$, where the $U_i \subseteq X_i$ are open. Let $U \subseteq X$ be open, so it suffices to show that if $x \in U$, there exist open subsets $U_i \subseteq X_i$, $i = 1, 2, \ldots, n$, such that $x \in U_1 \times \cdots \times U_n$. Suppose for contradiction otherwise; i.e., $x \notin U_1 \times \cdots \times U_n$ for all open $U_i \subseteq X_i$. In particular, for each $m \in \mathbb{Z}^+$, this product of open balls (writing $x = (x_1, x_2, \ldots, x_n)$ as usual) is not contained in U:

$$B\left(x_1, \frac{1}{m}\right) \times B\left(x_2, \frac{1}{m}\right) \times \cdots \times B\left(x_n, \frac{1}{m}\right) =: B_m.$$

Hence, there exists $x^m \in B_m \subseteq X \setminus U$ such that $d_i(x_i^m, x_i) < \frac{1}{m}$, where d_i is the metric on X_i. Now $(x_i^m) \to x_i$ for all $i \leq n$, so by $(\star)$, we have $(x^m) \to x$. But x is arbitrary and $X \setminus U$ is closed, so $x \in X \setminus U$, which is a contradiction as we assumed $x \in U$. $\square$

34 Compactness

Definition 34.1. Let I be an indexing set. A family $\{U_\alpha\}_{\alpha \in I}$ of sets in a metric space X *covers* a set $S \subseteq X$ if $S \subseteq \bigcup_{\alpha \in I} U_\alpha$. In addition, if all of the U_α are open sets, we say the family is an *open cover of* X if $X \subseteq \bigcup_{\alpha \in I} U_\alpha$.

From here, we define compactness, a crucial notion later in topology.

Definition 34.2. A metric space X is *compact* if every open cover of X has a finite subcover; that is, for every open cover $\{U_\alpha\}_{\alpha \in I}$, there exist $U_1, U_2, \ldots, U_k \in \{U_\alpha\}$ such that

$$X \subseteq \bigcup_{i=1}^{k} U_i.$$

Similarly, we may say that a subset is *compact*, and the definition is analogous.

Example 34.3. The real numbers $\mathbb{R}$ are not compact: take the open cover $\{(n, n+2) : n \in \mathbb{Z}\}$. Then no finite subcover can cover the whole real line, as the finite cover is bounded.

Example 34.4. The set $(0, 1) \subseteq \mathbb{R}$ is not compact: note $(0, 1) \subseteq \bigcup_{n=1}^{\infty} \left(\frac{1}{n}, 1\right)$. But, again, this cover has no finite subcover, because of boundedness once more.

Example 34.5. The subset $[0, 1] \cap \mathbb{Q} \subseteq \mathbb{R}$ is not compact: take any irrational number $x \in [0, 1]$, and consider the set

$$(-1, x) \cup \bigcup_{n=1}^{\infty} \left(x + \frac{1}{n}, 2\right).$$

Since $x \in \mathbb{Q}$, we see that

$$(-1, x) \cup \bigcup_{n=1}^{\infty} \left(x + \frac{1}{n}, 2 \right) = (-1, 2) \setminus \{x\} \supseteq [0, 1] \cap \mathbb{Q},$$

but again, there are no finite subcovers as $\mathbb{Q}$ is dense in $[0, 1]$.

We give another important definition.

Definition 34.6. A metric space X is *totally bounded* if for every $\varepsilon > 0$, there exists a finite number of open balls of radius ε that cover X.

This, in turn, allows us to state one of the fundamental theorems regarding metric spaces.

Theorem 34.7 (Bolzano-Weierstrass Theorem — BW)**.** *The following three statements are equivalent for a metric space X:*

1. *X is compact.*
2. *Every sequence $(x_n) \to x$ has a convergent subsequence.*
3. *X is totally bounded and complete.*

Proof. We prove (1) $\implies$ (2) and (2) $\implies$ (3) first, and leave (3) $\implies$ (1) for later.

(1) $\implies$ (2): Suppose X is compact. For any $x \in X$ and $\varepsilon > 0$, $\bigcup_{x \in X} B(x, \varepsilon)$ is a natural open cover for X. By compactness, there exist $\{x_1, \ldots, x_n\} \subseteq X$ such that $X \subseteq \bigcup_{k=1}^{n} B(x_k, \varepsilon)$. Now, let $(y_n) \subseteq X$ be any sequence, so $(y_n) \subseteq \bigcup_{k=1}^{n} B(x_k, \varepsilon)$. There must exist some $x^* \in X$ such that for any $\varepsilon > 0$, the ball $B(x^*, \varepsilon)$ contains infinitely many elements of (y_n). Now, take $\varepsilon = \frac{1}{k}$ for $k \in \mathbb{Z}^+$, and pick some y_{n_k} such that $d(y_{n_k}, x^*) < \frac{1}{k}$. It follows that $(y_{n_k}) \to x^* \in X$, which proves (2).

(2) $\implies$ (3): Suppose (2) holds. We first show that X is complete, so let (x_n) be a Cauchy sequence. Then (x_n) has a convergent subsequence, say $(x_{n_k}) \to x$. Now, Proposition 18.8 gives $(x_n) \to x$, so X is complete.

Now, we show that X is totally bounded, so let $\varepsilon > 0$. We must cover X with a finite number of open balls with radius ε. Let $y_1 \in X$. If $B(y_1, \varepsilon) \not\supseteq X$, choose $y_2 \in X \setminus B(y_1, \varepsilon)$. Repeat this process to find a list $y_1, y_2, \ldots, y_n, \ldots$, which either terminates, giving us an open cover $\bigcup_{i=1}^{n} B(y_i, \varepsilon) \supseteq X$, or never terminates, so that there is a sequence (y_n) such that $d(y_i, y_j) \geq \varepsilon$, i.e., (y_n) is not convergent. However, this sequence lacks a convergent subsequence, which contradicts (2), so this process must terminate and give a finite cover of open balls. □

38 The Bolzano-Weierstrass Theorem

The main purpose of this section is to prove (3) $\implies$ (1) in BW as stated in the previous section. To do this, we need more definitions.

Definition 38.1. A metric space X is *separable* if there is a sequence $(x_i) \subseteq X$ that is dense in X.

For example, $\mathbb{R}$ is separable after an enumeration of the rationals $\mathbb{Q}$. We are now on our way to finish the proof of BW, with this first step.

Proposition 38.2. *A totally bounded metric space is separable.*

Proof. Let $n \in \mathbb{Z}^+$, so that there exists a finite set $\{x_{n_1}, x_{n_2}, \dots x_{n_{m_n}}\}$ such that

$$X \subseteq \bigcup_{i=1}^{n} B\left(x_{n_i}, \frac{1}{n}\right).$$

Then the family $E := \{x_{n_i} : 1 \leq i \leq m_n, 1 \leq n < \infty\}$ is a countable subset of X. Fix $x \in X$ and any $n \in \mathbb{Z}^+$. Then there exists $x_{n_j} \in E$ by the fact that $d(x_{n_j}, x) < 1/n$, so E is dense in X as $1/n \to 0$. Hence X is separable. $\square$

Definition 38.3. A *base of open sets* for a metric space X is a family $\mathcal{B}$ of open subsets of X such that every open subset of X is a union of sets in $\mathcal{B}$.

Example 38.4. Let X be any metric space, and take $\mathcal{B} := \{B(x, \varepsilon) : x \in X, \varepsilon > 0\}$. Since every open set is the union of open balls, $\mathcal{B}$ is a base of open sets. This, in particular, shows that a base of open sets always exists.

Example 38.5. Let $X = \mathbb{R}$, and define $\mathcal{B} := \{B(x, r) : x \in \mathbb{Q}, r \in \mathbb{Q}^+\}$. Since $\mathbb{Q}$ is dense, we see that $\mathcal{B}$ is a base (taking arbitrary unions if necessary). In particular, the set $\mathcal{B}$ is *countable*, which will show up in an important way later.

Example 38.6. Let $X = X_1 \times \cdots \times X_n$, the product space of the metric spaces X_i. Define

$$\mathcal{B} := \{U_1 \times \cdots \times U_n : U_i \subseteq X_i \text{ open}\}.$$

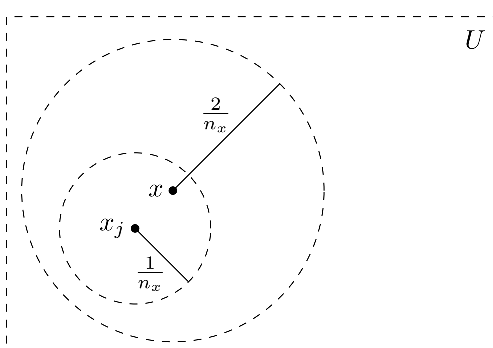
Then $\mathcal{B}$ is a base of open sets.

Definition 38.7. A metric space is *second-countable* if there exists a base of open sets of at most countable cardinality.

Second-countable metric spaces are important, as it is a necessary and sufficient condition for separability.

Theorem 38.8. *A metric space is second-countable if and only if it is separable.*

Proof. ($\Longleftarrow$): Suppose X is separable. Let $\{x_j\}_{j \in \mathbb{Z}^+}$ be a countable, dense subset. Consider a family of open sets $\mathcal{B} := \left\{B\left(x_j, \frac{1}{n}\right) : j, n \in \mathbb{Z}^+\right\}$. Clearly, $\mathcal{B}$ is countable. Now, let $U \subseteq X$ be an open subset. For each $x \in U$, pick some $n_x \in \mathbb{Z}^+$ such that $B\left(x, \frac{2}{n_x}\right) \subseteq U$. By density of the x_j, there exists $j_x \in \mathbb{Z}^+$ such that $d(x, x_{j_x}) < \frac{1}{n_x}$. Our situation is the following:



We observe that by construction, $x \in B\left(x_{j_x}, \frac{1}{n_x}\right) \subseteq B\left(x, \frac{2}{n_x}\right) \subseteq U$, so that it follows

$$U = \bigcup_{x \in U} B\left(x_{j_x}, \frac{1}{n_x}\right),$$

but each of those balls are found in $\mathcal{B}$; hence we have shown that $\mathcal{B}$ is a base of open sets that is countable, so X is second-countable.

($\Longrightarrow$): Let X be second-countable, so let $\mathcal{B} = \{U_1, U_2, \ldots\}$ be a countable base of open sets. Pick $x_n \in U_n$ for all $n \in \mathbb{Z}^+$. Thus every nonempty open subset of X contains a point of the sequence (x_n) because $\mathcal{B}$ is a base. But this is exactly the same as saying (x_n) is dense in X, so X is separable. $\square$

Definition 38.9 (Lindelöf's Theorem)**.** Let X be second-countable. Then every open cover of X has a countable subcover.

Proof. Let X be second-countable and $\{U_\alpha\}_{\alpha \in I}$ be an open cover for X. Let $\mathcal{B}$ be a countable base of open sets. Take $\mathcal{A} \subseteq \mathcal{B}$ by

$$\mathcal{A} := \{V \in \mathcal{B} : V \subseteq U_\alpha \text{ for some } \alpha \in I\}.$$

Clearly, $\mathcal{A}$ is (at most) countable; we claim that $\mathcal{A}$ covers X. Take any $x \in X$, so there exists some $\alpha \in I$ with $x \in U_\alpha$. But $\mathcal{B}$ is a base, so U_α is a union of sets in $\mathcal{B}$. Let $V \in \mathcal{B}$ be any member of this union that contains x, so that $x \in V \subseteq U_\alpha$. That is, $V \in \mathcal{A}$, so that $\mathcal{A}$ is a cover for X. $\square$

Using these theorems, we finish the proof of BW.

Proposition 38.10. *Let X be a totally bounded and complete metric space. Then any countable open cover of X has a finite subcover.*

Proof. Let $\{U_i : i \in \mathbb{Z}^+\}$ be a countable open cover for X. Suppose for contradiction that $\{U_i\}$ does not have a finite subcover, i.e., $X \neq \bigcup_{n=1}^m U_n$ for all $m \in \mathbb{Z}^+$. For each m, there exist some $x_n \in X \setminus \bigcup_{n=1}^m U_n$. Now, since X is totally bounded, there is an open ball B_1 with radius 1 that contains infinitely many elements in (x_m). Set $(x_{1,j}) := (x_m) \cap B_1$, which is an infinite sequence. Continuing this by total boundedness, for each $k \geq 2$, inductively pick some $B_k \subseteq B_{k-1}$, a ball of radius $1/k$, such that B_k contains infinitely many of the $x_{k-1,j}$, and set $(x_{k,j}) := B_k \cap (x_{k-1,j})$. Now, take $(y_k) := (x_{k,k}) \subseteq (x_m)$; note that whenever $i, j \geq k$, we see $d(y_i, y_j) < \frac{2}{k} \to 0$, so (y_k) is Cauchy. But X is complete, so $(y_) \to y \in X$, so we have $(x_m) \to y$. Now, note that for any $j \geq m$, we have

$$x_j \in X \setminus \bigcup_{n=1}^{\infty} U_n,$$

but now note $X \setminus \bigcup_{n=1}^{\infty} U_n$ is closed, yet $(x_m) \to y$. Because a closed set contains its limit points, we have $y \in X \setminus \bigcup_{n=1}^m U_n$ for all $m \in \mathbb{Z}^+$. But the U_i, $i \in \mathbb{Z}^+$ form a cover for X, which implies $y \in X \setminus X = \varnothing$, which is contradictory. $\square$

We remark that the proof of Proposition 38.10 completes the proof of the Bolzano-Weierstrass Theorem, following the chain of ideas from Definition 38.1.

40 Continuity (I)

Definition 40.1. Let (X, d) and (Y, ρ) be two metric spaces. A function $f : X \to Y$ is *continuous at* $x \in X$ if whenever $(x_n) \subseteq X$ is a sequence converging to x, then $f(x_n) \to f(x) \in Y$. Similarly, we say $f : X \to Y$ is *continuous* (on X) if it is continuous at all $x \in X$.

Theorem 40.2. *The following are equivalent for a function $f : X \to Y$, where (X, d) and (Y, ρ) are metric spaces:*

1. *The function f is continuous.*
2. *For every $x \in X$ and every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $z \in X$, if $d(x, z) < \delta$, then $\rho(f(x), f(z)) < \varepsilon$.*
3. *If $V \subseteq Y$ is open, its preimage $f^{-1}(V) \subseteq X$ is open.*

Proof. (1) $\implies$ (2): We proceed by contraposition: suppose there exists an $x \in X$ and some (fixed) $\varepsilon > 0$ such that for *every* $\delta > 0$, there exists some $z \in X$ where $d(x, z) < \delta$ but $\rho(f(x), f(z)) \geq \varepsilon$. For each $k \in \mathbb{Z}^+$, take $\delta_k := 1/k > 0$. Thus, there exists $z_k \in X$ such that $d(x, z_k) < \delta_k = 1/k$ and $\rho(f(x), f(z_k)) \geq \varepsilon$. But now $(z_k) \to x$, yet $f(z_k) \not\to f(x)$, so that f is not continuous at $x \in X$, so that f is not continuous on X.

(2) $\implies$ (3): Let $V \subseteq Y$ be open, and fix $x \in f^{-1}(V) \subseteq X$, so that $f(x) \in V$. Since V is open, pick some $\varepsilon > 0$ such that $B(f(x), \varepsilon) \subseteq V$. By (2), there exists $\delta > 0$ (corresponding to ε) such that for all $z \in X$, if $d(x, z) < \delta$ then $\rho(f(x), f(z)) < \varepsilon$, i.e., if $z \in B(x, \delta)$, then $f(z) \in B(f(x), \varepsilon) \subseteq V$, i.e., $f(B(x, \delta)) \subseteq B(f(x), \varepsilon) \subseteq V$, so that it follows

$$B(x, \delta) \subseteq f^{-1}(V).$$

This means $f^{-1}(V)$ is open, as claimed.

(3) $\implies$ (1): Suppose (3) holds. Let $(x_n) \to x$ be a convergent sequence in X. Now, note that for any $x \in X$, $B(f(x), \varepsilon) \subseteq Y$ is open in Y, for all $\varepsilon > 0$. By (3), $f^{-1}\left(B(f(x), \varepsilon)\right) \subseteq X$ is open. Since $f(x) \in B(f(x), \varepsilon)$, we have $x \in f^{-1}\left(B(f(x), \varepsilon)\right)$, but this set is open, so x is interior, so there exists $\delta > 0$ such that $B(x, \delta) \subseteq f^{-1}\big(B(f(x), \varepsilon)\big)$. Since $(x_n) \to x$, then for the $\delta > 0$ chosen above, there exists $N \in \mathbb{Z}^+$ such that $\{x_n\}_{n \in \mathbb{N}} \subseteq B(x, \delta) \subseteq f^{-1}\big(B(f(x), \varepsilon)\big)$, so that

$$\big\{f(x_n) : n \in \mathbb{Z}^+\big\} \subseteq B\big(f(x), \varepsilon\big).$$

Hence $f(x_n) \to f(x)$, so f is continuous. $\square$

43 Compactness: Worked Examples

Example 43.1. Prove that compact sets are closed.

Proof. Let $S \subseteq X$ be compact, where X is the underlying metric space. We show that $X \setminus S$ is open. Fix some $x \in X \setminus S$. Since $x \notin S$, there exist[2] balls $B(s, \varepsilon_s)$ and $B(x, \varepsilon_s)$ such that $B(s, \varepsilon_s) \cap B(x, \varepsilon_s) = \varnothing$. Then $S \subseteq \bigcup_{s \in S} B(s, \varepsilon_s)$, so these balls form an open cover for S. Since S is compact, there exists a finite subcover, say $\{B(s_1, \varepsilon_{s_1}), \ldots, B(s_n, \varepsilon_{s_n})\}$ for some $n \in \mathbb{Z}^+$. Now, take $r = \min(\varepsilon_{s_1}, \ldots, \varepsilon_{s_n})$, so now $B(x, r) \subseteq X \setminus S$, so $X \setminus S$ is open. Hence S is closed. $\square$

[2] Later on, we will see that this statement is really saying that metric spaces satisfy the T_1 separation axiom.

Example 43.2. Show that if K is compact, then every infinite subset $S \subseteq K$ has a limit point in K.

Proof. Suppose otherwise; let $S \subseteq K$ be an infinite subset which has no limit points in K. Then for each $x \in K$, there exists some open ball B_x containing finitely many points in S, centered at x, so define the cover $\bigcup_{x \in K} B_x$. Evidently, this is an infinite open cover, as S is infinite, but a finite subset of this cover can only contain finitely many points from S, contradicting the fact that K is compact and $S \subseteq K$ is infinite. $\square$

We remark that this shows that K is *complete*: consider a Cauchy sequence $(x_n) \subseteq K$. If the set $\{x_n\}$ is infinite, then the result above gives the proof. Now, if $\{x_n\}$ is finite, then x_n is eventually constant, so it obviously converges.

44 Continuity (II)

Recall this definition from analysis.

Definition 44.1. Let (X, d) and (Y, ρ) be two metric spaces. We say that a function $f : X \to Y$ is *uniformly continuous* if for any $\varepsilon > 0$, there exists some $\delta > 0$ such that if $x, z \in X$ satisfy $d(x, z) < \delta$, then $\rho\big(f(x), f(z)\big) < \varepsilon$.

We observe that every uniformly continuous function is indeed continuous, but the converse is not true:

Example 44.2. Take $f : \mathbb{R}^+ \to \mathbb{R}$ by $f(x) = 1/x$. Then f is continuous on $\mathbb{R}^+$, but it is not uniformly continuous: take $\varepsilon = 1$. Then given any x, we see that δ_x varies with x: for example $\delta_1 < 1/2$ yet $\delta_{1/10} < 1/110$.

However, in the case that the domain of f is compact, the converse we mentioned above holds.

Theorem 44.3. *Let X, Y be metric spaces, and suppose X is compact. Then every continuous function is uniformly continuous.*

Proof. Let $f : X \to Y$ be continuous, and suppose for contradiction that f is not uniformly continuous. Then there exists some $\varepsilon > 0$ such that for all $\delta = \frac{1}{n}$ $(n \in \mathbb{Z}^+)$, there exist $x_n, z_n \in X$ such that $d(x_n, z_n) < \frac{1}{n}$ but $\rho(f(x_n), f(z_n)) \geq \varepsilon$. Since X is compact, Bolzano-Weierstrass applies, and the sequence (x_n) has a convergent subsequence $(x_{n_k}) \to x \in X$. Since $d(x_n, z_n) < \frac{1}{k}$, pick a subsequence (z_{n_k}) such that $d(x_{n_k}, z_{n_k}) \to 0$, so $z_{n_k} \to x$. Since f is continuous, we see $f(x_{n_k}) \to f(x)$ and $f(z_{n_k}) \to f(x)$. Then by the triangle inequality,

$$\rho\big(f(x_{n_k}), f(z_{n_k})\big) \leq \rho\big(f(x_{n_k}), f(x)\big) + \rho\big(f(x), f(x_{n_k})\big) \to 0,$$

but this is contradictory as we assumed $\rho\big(f(x_n), f(z_n)\big) \geq \varepsilon$. Hence, f is forced to be uniformly continuous. $\square$

Proposition 44.4. *If $f : X \to Y$ is continuous and X is compact then* $\operatorname{im} f \subseteq Y$ *is compact.*

Proof. Let $(x_n) \subseteq X$ be a sequence. Since X is compact, (x_n) has a convergent subsequence (x_{n_k}). Now, f is continuous, so $f(x_{n_k}) \to f(x)$, i.e., every sequence $f(x_n) \subseteq Y$ has a convergent subsequence $f(x_{n_k})$, so $\operatorname{im} f$ is compact. $\square$

Definition 44.5. A function $f : X \to Y$ between two metric spaces is called a *homeomorphism* if it is a continuous bijection with a continuous inverse $f^{-1} : Y \to X$.

Intuitively, homeomorphisms are isomorphisms of metric spaces: if $f : X \to Y$ is a homeomorphism, then if X has some property definable by open sets (e.g., compactness, etc.), then Y has that same property, and vice versa (as f^{-1} is also a homeomorphism).

Example 44.6. We remark that the continuity of the inverse in the definition of a homeomorphism is important. Let $f : [0, 2\pi) \to S^1$ be[3] given by $\theta \mapsto e^{i\theta}$. Then f itself is a bijective continuous function, but $f^{-1} : S^1 \to [0, 2\pi)$ by $e^{i\theta} \mapsto \theta$ (where $0 \leq \theta < 2\pi$) is not continuous at $e^0 = 1$.

48 Normed Vector Spaces

Now, we look at metric spaces with extra structure. Recall the following definition from linear algebra:

Definition 48.1. A *vector space* V over a field F is an abelian group $(V, +)$, with a field action, called *scalar multiplication*, satisfying the following for all $a, b \in F$ and $v, w \in V$:

1. $1v = v$.
2. $(a + b)v = av + bv$.
3. $a(v + w) = av + aw$.
4. $a(bv) = (ab)v$.

Alternatively, we say that a vector space is an F-module, where F is a field.

The relevant thing about vector spaces for us is the following notion.

Definition 48.2. Let V be a vector space over $F = \mathbb{R}$ or $\mathbb{C}$. A *norm* on V is a function $\|\cdot\| : V \to \mathbb{R}$ such that

1. $\|x\| \geq 0$, with equality holding if and only if $x = 0$.
2. $\|cx\| = |c| \|x\|$, for all $c \in F$ and $x \in V$.
3. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in V$.

Example 48.3. Euclidean space $\mathbb{R}^n$ has the standard norm $\|x\|_2 := \left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{1/2}$. However, this is not the only possible norm on $\mathbb{R}^n$: we define the *L^p-norm* (for all $p \geq 1$) by

$$\|x\|_p := \left(|x_1|^p + |x_2|^p + \cdots + |x_n|^p\right)^{1/p}.$$

We can also define the *L^∞-norm* by

$$\|x\|_\infty := \max\{|x_1|, |x_2|, \ldots, |x_n|\}.$$

[3]Here, $S^1 \subseteq \mathbb{C}$ are the complex numbers with magnitude 1, i.e., S^1 is the unit circle.

Example 48.4. Let S be a nonempty set, and let $B(S)$ denote the set of all functions $f : S \to \mathbb{R}$. Define the norm

$$\|f\|_S := \sup_{s\in S} |f(s)| .$$

This is well-defined as $f \in B(S)$ is bounded. The norm $\|\cdot\|_S$ is known as the *norm of uniform convergence* on S.

Our definition for a norm looks very much like our definition for a metric. In fact, norms always induce metrics:

Proposition 48.5. *If V is a normed vector space, then the norm $\|\cdot\|$ induces a metric on V by $d(x, y) := \|x - y\|$.*

Proof. We verify the triangle inequality for d. Let $x, y, z \in V$. Then by the triangle inequality for the norm, we see

$$d(x, z) = \|x - z\| = \|x - y + y - z\| \leq \|x - y\| + \|y - z\| = d(x, y) + d(y, z),$$

which completes the proof. □

50 Transformations on Normed Spaces

Recall the following definition.

Definition 50.1. Let V, W be vector spaces over a field F. A *linear transformation* is a map $\varphi : V \to W$ such that $\varphi(x + y) = \varphi(x) + \varphi(y)$ and $\varphi(ax) = a\varphi(x)$ for all $x, y \in V$ and $a \in F$. In the case that $W = F$, the linear map $\varphi : V \to F$ is called a *linear functional.*

Definition 50.2. Let V, W be normed vector spaces over $\mathbb{R}$ or $\mathbb{C}$. If $\varphi : V \to W$ is a linear transformation satisfying

$$\sup \left\{ \|\varphi(x)\| : x \in X, \|x\| \leq 1 \right\} < \infty,$$

we say that φ is a *bounded linear transformation*, and we define $\|\varphi\|$ by the supremum above.

We remark that in the definition above, the symbols $\|\cdot\|$ mean three different things depending on the context — to be more clear, we may write

$$\|\varphi\|_{\mathsf{Hom}(V,W)} := \sup \left\{ \|\varphi(x)\|_W : x \in X, \|x\|_V \leq 1 \right\} < \infty.$$

From this, we remark that if φ is a bounded linear transformation, then we may write, for any $c \in F$, where $F = \mathbb{R}$ or $\mathbb{C}$, that $\|\varphi(cx)\| = |c| \|\varphi(x)\|$, so in order to "bound" φ, we restrict in the definition above to vectors $x \in V$ with less than unit norm.

Lemma 50.3. *If $\varphi : V \to W$ is a bounded linear operator, then for all $x \in V$, we have $\|\varphi(x)\| \leq \|\varphi\| \cdot \|x\|$.*

Proof. Clearly, the lemma holds when $x = 0$. If $x \neq 0$, then we know

$$\|\varphi\| = \sup_{\|x\|\leq 1} \|\varphi(x)\| = \sup_{\|x\|\leq 1} \left\| \varphi\left(\frac{x}{\|x\|}\right) \right\| \geq \left\| \varphi\left(\frac{x}{\|x\|}\right) \right\| = \frac{\|\varphi(x)\|}{\|x\|},$$

which completes the proof. □

Lemma 50.4. *Let V, W be two normed vector spaces. Then the space of bounded linear transformations $V \to W$ is itself a normed space, with the norm as defined above.*

Proof. It suffices to check that the norm we defined is actually a norm. Clearly $\|\varphi\| \geq 0$ for all bounded linear transformations $\varphi : V \to W$, with equality holding exactly when $\varphi = 0$. Now, let $c \in F$. Then $\|(c\varphi)(x)\| = \|c \cdot \varphi(x)\| = |c| \, \|\varphi(x)\|$ for all $x \in X$, so taking the supremum across $\|x\| \leq 1$ gives $\|c\varphi\| = |c| \, \|\varphi\|$. Finally, let $\varphi, \psi : V \to W$ be bounded linear transformations. Then

$$\begin{aligned}
\|(\varphi + \psi)(x)\| = \|\varphi(x) + \psi(x)\| &\leq \|\varphi(x)\| + \|\psi(x)\| \quad \text{(triangle inequality)} \\
&\leq \|\varphi\| \, \|x\| + \|\psi\| \, \|x\| \quad \text{(Lemma 50.3)} \\
&= \left(\|\varphi\| + \|\psi\| \right) \|x\| .
\end{aligned}$$

Taking supremums finishes the proof. $\square$

Definition 50.5. Let V, W be normed vector spaces. The space of all bounded linear transformations $V \to W$ is denoted $\mathsf{BL}(V, W)$. In the case $V = W$, we write $\mathsf{BL}(V)$ instead, and in the case $W = F$, the space of all bounded linear transformations $V \to F$ is denoted $V' := \mathsf{BL}(V, F)$, and is called the *dual space* of V.

[Note that this differs from the usual definition in abstract algebra — the dual space of an arbitrary vector space V is $V^* := \mathsf{Hom}(V, F)$, regardless of boundedness.]

Theorem 50.6. *Let V, W be normed spaces and $\varphi : V \to W$ be linear. Then φ is continuous if and only if φ is bounded.*

Proof. ($\implies$): Suppose φ is continuous. Then in particular, φ is continuous at $x = 0$, so for specifically $\varepsilon = 1$, there exists $\delta > 0$ such that whenever $\|x\| < \delta$, we have $\|\varphi(x)\| \leq 1$. Now, if $x \in X$ and $\|x\| \leq 1$, we see $\|\delta x\| \leq \delta$ and $\|\varphi(\delta x)\| \leq 1 \implies \|\varphi(x)\| \leq \frac{1}{\delta}$. Taking supremums gives $\|\varphi\| \leq \frac{1}{\delta}$, so φ is bounded.

($\impliedby$): Suppose φ is bounded. Pick $x_0 \in V$, so that $\|\varphi(x) - \varphi(x_0)\| = \|\varphi(x - x_0)\| \leq \|\varphi\| \, \|x - x_0\|$, for all $x \in X$. Now, $\|\varphi\| < \infty$ is constant, so let $x \to x_0$, so that $\varphi(x) \to \varphi(x_0)$, completing the proof. $\square$

Definition 50.7. A complete normed vector space (with respect to the induced metric) is called a *Banach space*.

Example 50.8. $\mathbb{R}^n, \mathbb{C}^n$, and L^p-space (where $p \geq 1$) are all Banach spaces. In contrast, $C([0,1])$ is not a Banach space, taking the norm $\|f\|_p := \left(\int_0^1 |f(x)|^p \, dx \right)^{1/p}$ for all $p \geq 1$.

60 Topological Spaces (I)

At this point, we are ready for a shift into more abstraction. In metric spaces, the metric d was a notion of distance, which we can do computation with. This is nice; however, we now seek to generalize the notion of "distance" and "place" regardless if a metric exists. We do this by identifying certain subsets of a larger set X which have desirable properties.

Definition 60.1. Let X be a set. A family τ of subsets of X is a *topology* if τ satisfies the following properties:

1. We have $X, \varnothing \in \tau$.
2. Any arbitrary union of sets in τ is also in τ.
3. Finite intersection of sets in τ belong to τ.

The pair (X, τ) is called a *topological space*, and the elements of τ are called *open sets*.

Notice that these properties of a topology are generalized from how open sets behave in metric spaces. Indeed, we will see that every metric space induces a topology. However, one large difference is that in a metric space, the metric determines what subsets are open; in a topology, we simply declare sets as open assuming they stay consistent with the three properties of a topology.

Example 60.2. Let X be a set. The *trivial topology* if the pair $\tau = \{X, \varnothing\}$, which is clearly a topology. However, it is not very interesting. In contrast, if we let $\tau = 2^X$, the entire power set of X, then every subset is open. This is called the *discrete topology* on X, and (X, τ) is called the *discrete topological space* on X. This is equivalent to the discrete metric space (X, d), where $d(x, y) = 1$ if $x \neq y$ and 0 otherwise, as every subset of (X, d) is open as well.

Definition 60.3. If (X, d) is a metric space, then the open subsets of X form a topology on X, called the *metric topology* on X.

Of course, this is immediate from how the a topology is defined. However, not every topological space can be made into a metric space.

Example 60.4. Let X be a set with at least 2 elements. If τ is the trivial topology $\tau = \{X, \varnothing\}$, then τ cannot be determined by any metric — if it were, take $x \in X$ and note $X \setminus \{x\} \neq \varnothing$. If X were a metric space, then $\{x\}$ must be closed, so that $X \setminus \{x\}$ must be open in the metric, and thus in the metric topology, a contradiction.

Nonetheless, topological spaces which can be made into metric spaces are important because we know how metric spaces work. Hence, it is occasionally important to know *when* a topological space could be made into a metric space.

Definition 60.5. A topological space (X, τ) is *metrizable* if the topology τ on X is the metric topology associated with some metric d on X.

Again, we stress that metrizable spaces are important because they "de-abstract" away from topologies and lets us work with actual distances. Obviously, any metric topology is metrizable.

Now, we generalize more definitions from our work in metric spaces.

Definition 60.6. Let X be[4] a topological space. A subset $S \subseteq X$ is *closed* if $X \setminus S$ is open.

Note that this is a *definition* in topological spaces, whereas it was a proven *proposition* for metric spaces. Again, as before X and $\varnothing$ are both open and closed. The following thus arises from definition:

Proposition 60.7. *Let X be a topological space. Then an arbitrary intersection of closed sets are closed, and finite unions of closed sets are closed.*

Proof. Trivial. □

Definition 60.8. Let X be a topological space. A subset $S \subseteq X$ is a *neighborhood* of a point $x \in X$ if there exists an open set $U \subseteq S$ such that $x \in U$. If S itself is open, we say that S is an *open neighborhood* of x.

Definition 60.9. Let $S \subseteq X$. We say $x \in X$ is an *interior point* of S if S is a neighborhood of X. The set of all interior points of S is called the *interior* of S, denoted $\text{int}(S)$.

[4]We will often suppress them mention of τ if it is unnecessary.

Evidently, $\text{int}(S) \subseteq S$. We may also ask about the other inclusion: when does $\text{int}(S) \supseteq S$?

Proposition 60.10. *Let $S \subseteq X$. Then S s open if and only if $S = \text{int}(S)$; i.e., S is a neighborhood for any point $s \in S$.*

Proof. ($\implies$): If S is open, pick $s \in S$. Then $s \in S \subseteq S$ and S is open, so $s \in \text{int}(S)$. Hence $S = \text{int}(S)$.

($\impliedby$): Suppose $S = \text{int}(S)$. Taking any $s \in S$, there exists an open set $U_s \subseteq S$ such that $s \in U_s$. Consider the union $\bigcup_{s \in S} U_s \subseteq S$, which is an open set. But now by construction, $S \subseteq \bigcup_{s \in S} U_s$, which establishes that S is open. ☐

63 Sequences in Topological Spaces

Definition 63.1. Let X be a topological space. A sequence $(x_n) \subseteq X$ *converges to* x if for all open neighborhoods U of x, there exists some $N \in \mathbb{Z}^+$ such that if $n > N$, then $x_n \in U$.

Of course, we note that open neighborhoods now take the role of open balls in metric spaces. We will revisit this next definition later on.

Definition 63.2. We say that a topological space X is *Hausdorff* if whenever $x \neq y$, there exist open neighborhoods N_x, N_y such that $x \in N_x$, $y \in N_y$, and $N_x \cap N_y = \varnothing$.

Evidently, metric spaces are Hausdorff, as given distinct points, we can always select disjoint open balls that contain the two points. However, not every topological space is Hausdorff.

Example 63.3. Let X have at least 2 elements, and endow it with the trivial topology. Clearly, X is not Hausdorff, and convergent sequences are very pathological: if (x_n) is *any* sequence, then x_n converges — in fact, it converges to *any* element $x \in X$, as x only has the neighborhood X — which is the whole set. Hence, the uniqueness of limits is not immediately granted in an arbitrary topological space.

However, in a Hausdorff space, sequences are more well-behaved.

Proposition 63.4. *Let X be a Hausdorff space and (x_n) be a convergent sequence. Then x_n has a unique limit.*

Proof. Say for contradiction that $(x_n) \to x, y$, where $x \neq y$. Because X is Hausdorff, pick *disjoint* open neighborhoods U_x, U_y of x resp. y. But by convergence, there exists a sufficiently large N such that if $n > N$, then $x_n \in U_x$ and $x_n \in U_y$, which contradicts disjointness. Hence, the limit of (x_n) must be unique. ☐

64 Topological Spaces (II)

We continue generalizing notions from metric spaces to topological spaces.

Definition 64.1. Let X be a topological space and $S \subseteq X$. A point $x \in X$ is *adherent* to S if for all neighborhoods U of x, $S \cap U$ is nonempty.

Definition 64.2. Let X be a topological space and $S \subseteq X$. The *closure* of S is the set of all adherent points to S, and is denoted $\overline{S}$.

Certainly, $S \subseteq \overline{S}$, as any neighborhood of $s \in S$ must contain s. But what about the inclusion $\overline{S} \subseteq S$?

Proposition 64.3. *Let X be a topological space and $S \subseteq X$. Then S is closed if and only if $S = \overline{S}$.*

Proof. ($\Longrightarrow$): Suppose S is closed; it suffices to show $\overline{S} \subseteq S$. Let $x \in \overline{S}$. Then $N \cap S$ is nonempty for all neighborhoods N of x. Since $X \setminus S$ is open, if for contradiction $x \in X \setminus S$, then x must be interior to $X \setminus S$, so that $X \setminus S$ is a neighborhood for x. But now $(X \setminus S) \cap S = \varnothing$, a contradiction, so $x \in S$.

($\Longleftarrow$): Suppose $S = \overline{S}$. We will show that $X \setminus S$ is open. Take any point $x \in X \setminus S$; it suffices to show that x is interior to $X \setminus S$. Notice $S = \overline{S}$ implies $x \notin \overline{S}$, i.e., x is not adherent to S. That is, there exists a neighborhood N of x such that $N \cap S = \varnothing$, i.e.., $N \subseteq X \setminus S$. This implies $x \in \operatorname{int}(X \setminus S)$, so $X \setminus S$ is open and S is closed. $\square$

We remark that the above is taken for definition in a metric space, while it is a proven proposition in topological spaces.

Definition 64.4. Let $S \subseteq X$. A point $x \in X$ is a *limit point* of S if every neighborhood of x intersects $S \setminus \{x\}$. The set of all limit points of S is denoted by S', and is called the *derived set* of S.

Definition 64.5. Let $S \subseteq X$. A point $x \in S$ is an *isolated point* of S if there exists a neighborhood N of x such that $N \cap S = \{x\}$.

Proposition 64.6. *Let $S \subseteq X$. Then $\overline{S}$ is the disjoint union of S' and the set of all isolated points of S.*

Proof. Exercise — prove that $S' \cap T = \varnothing$, where T is the set of isolated points, and show $\overline{S} = S' \cup T$. $\square$

Our next result relates sequences and adherent points.

Proposition 64.7. *If $S \subseteq X$ and $(x_n) \subseteq S$ converges to $x \in X$, then $x \in \overline{S}$.*

This is trivially true in metric spaces (in fact for its converse as well), but the converse of this proposition fails in a general topological space.

Proof. Suppose $(x_n) \to x$. Then for any open neighborhood U of x, there exists $N \in \mathbb{Z}^+$ such that whenever $n > N$, we have $x_n \in U$. Hence, $U \cap S$ is nonempty for all choices of U; i.e., $x \in \overline{S}$. $\square$

Definition 64.8. Let $S \subseteq X$. A point $x \in X$ is a *boundary point* of S is $x \in \overline{S}$ and $x \in \overline{X \setminus S}$. The set of all boundary points of S is called the *boundary* of S, and is denoted $\partial S := \overline{S} \cap \overline{X \setminus S}$.

Evidently, $\partial S = \partial(X \setminus S)$. It can be checked that $\overline{S}$ and $\overline{X \setminus S}$ are both closed themselves, so ∂S is closed.

Proposition 64.9. *Let $S \subseteq X$. Then $\overline{S} = \operatorname{int}(S) \cup \partial S$, and this union is disjoint.*

Proof. Since $\overline{S}$ contains all points adherent to S, for all $x \in \overline{S}$, we have for all neighborhoods U of x, we have $U \cap S = \varnothing$. If $U \subseteq S$, evidently $x \in \operatorname{int}(S)$. Otherwise, U intersects both S and $X \setminus S$. Since U was arbitrary, we see $x \in \overline{S}$ and $x \in \overline{X \setminus S}$, so $x \in \partial S$. Hence, $x \in \operatorname{int}(S) \cup \partial S$. Clearly, $\operatorname{int}(S) \subseteq \overline{S}$ and $\partial S \subseteq \overline{S}$, so the reverse inclusion holds. Finally, the above argument also shows us that $\operatorname{int}(S) \cap \partial S = \varnothing$. $\square$

68 Subspaces

Recall that if X is a metric space, then any subset $Y \subseteq X$ can be viewed as a subspace of X if we restrict the metric on X to Y, giving us an induced metric. We now generalize this to topological spaces.

Definition 68.1. Let (X, τ) be a topological space and $S \subseteq X$. The family of subsets $\tau_S := \{U \cap S : U \in \tau\}$ is called the *relative topology* on S, *inherited from* (X, τ). The elements of τ_S are called the *relatively open* subsets of S, and the sets $S \setminus V$, for some $V \in \tau_S$, are called the *relatively closed* subsets of S.

It can be checked that τ_S is indeed a well-defined topology on S — this is a straightforward verification of the axioms. Of course, if X is a metric space with metric subspace $Y \subseteq X$, then Y under the metric topology coincides with the relative topology (which is also metric) on X. Hence, this is a true generalization of what we have done before.

Example 68.2. A relatively open subset of some $S \subseteq X$ need not be open in the larger space X. For example, take $X = \mathbb{R}$ under the standard metric topology, and $S = [0, 1) \subseteq \mathbb{R}$. The set $A = \left(-\frac{1}{2}, \frac{1}{2}\right) \cap S = \left[0, \frac{1}{2}\right)$ is relatively open in S, but is clearly not open in $\mathbb{R}$.

Proposition 68.3. *Let S be a subspace of a topological space X. A subset $E \subseteq S$ is relatively closed if and only if $E = S \cap F$, where $F \subseteq X$ is some closed subset.*

Proof. ($\Longrightarrow$): Let $E \subseteq S$ be relatively closed. Then $S \setminus E$ is relatively open, i.e., $S \setminus E = U \cap S$ for some open $U \subseteq X$. Certainly, $X \setminus U$ is closed, but we claim that $S \cap (X \setminus U) = E$. To see this, pick some $x \in E$. Then certainly $x \in S$ and $x \in X$, and we need to show $x \notin U$. If $x \in U$, then the fact that $x \in S$ implies $x \in S \setminus E$, a contradiction. Hence, $x \in S \cap (X \setminus U)$. Conversely, pick $y \in S \cap (X \setminus U)$. Then $y \in S$ and $y \notin U$, so that $y \notin U \cap S = S \setminus E$, so that $y \in E$. Hence, we let $F = X \setminus U$.

($\Longleftarrow$): Let F be a closed subset of X, and $E = S \cap F$. We must show that E is relatively closed. Certainly, $X \setminus F$ is open, but now we claim $S \setminus E = S \cap (X \setminus F)$, which, if true, completes the proof. To see the set equality, take $x \in S \setminus E$. Then $x \in S$ and $x \notin E$, so $x \notin S \cap F$. This implies $x \notin F$, but clearly $x \in X$, so $x \in S \cap (X \setminus F)$. Conversely, take $y \in S \cap (X \setminus F)$. Then $y \in S$ and $y \notin F$, so that $y \notin S \cap F$. This implies $y \in E$ by definition of E. $\square$

Proposition 68.4. *Let S be a subspace of the topological space X, and let $E \subseteq S$. Then the relative closure of E in S is $\overline{E} \cap S$, where $\overline{E}$ is just the regular closure of E in X.*

Proof. Let $x \in S$. Then x is in the relative closure of E if and only if every relatively open neighborhood of x in S intersects E. That is, if N is an open neighborhood of x in X, then $(N \cap S) \cap E \neq \varnothing$. By associativity of the intersection, $\varnothing = (N \cap S) \cap E = N \cap (S \cap E) = N \cap E$. That is, $x \in \overline{E}$, which implies $x \in S \cap \overline{E}$. $\square$

70 Continuity in Topological Spaces

Definition 70.1. Let X and Y be topological spaces. A function $f : X \to Y$ is *continuous* at a point $x \in X$ if for every open set $V \subseteq Y$ such that $f(x) \in V$, there exists an open set $U \subseteq X$ such that $x \in U$ and $f(U) \subseteq V$. Similarly, we say that f is *continuous everywhere* (or simply, *continuous*) if $f^{-1}(V)$ is open in X whenever $V \subseteq Y$ is open.

Notice that this is an equivalent condition to continuity in metric spaces, while it is taken for definition in topological spaces.

Proposition 70.2. *Let X, Y be topological spaces and $f : X \to Y$ be a function. Then f is continuous everywhere if and only if f is continuous at every $x \in X$.*

Proof. ($\implies$): Suppose f is continuous everywhere. Fix some $x \in X$, and choose an open set V containing $f(x)$. By continuity, $f^{-1}(V)$ is open and certainly $x \in f^{-1}(V)$. Finally, $f(f^{-1}(V)) \subseteq V$ so f is continuous at x.

($\impliedby$): Suppose f is continuous at every $x \in X$. Fix an open set $V \subseteq Y$; we must show that $f^{-1}(V) \subseteq X$ is open. Take an arbitrary $x \in f^{-1}(V)$, so that f is continuous at x and $f(x) \in V$. Now, by continuity at x, there exists an open set $U_x \subseteq X$ such that $x \in U_x$ and $f(U_x) \subseteq V$, i.e., $U_x \subseteq f^{-1}(V)$. We claim that $f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x$, which essentially just holds by construction. Since the arbitrary union of open sets is open, $f^{-1}(V)$ is open, and so f is continuous everywhere. $\square$

Proposition 70.3. *A composition of two continuous functions is continuous.*

Proof. Let X, Y, Z be topological spaces, and take $f : X \to Y$ and $g : Y \to Z$ to be continuous functions. Let $V \subseteq Z$ be open in Z, so by continuity of g, we see $g^{-1}(V) \subseteq Y$ is open in Y. Similarly, by continuity of f, $f^{-1}(g^{-1}(V)) \subseteq X$ is open in X. We claim that $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$; to see this, pick $x \in f^{-1}(g^{-1}(V))$. This holds if and only if $f(x) \in g^{-1}(V)$, which holds if and only if $g(f(x)) = (g \circ f)(x) \in V$, if and only if $x \in (g \circ f)^{-1}(V)$. Thus, this completes the proof. $\square$

Definition 70.4. Let X, Y be topological spaces. A function $f : X \to Y$ is a *homeomorphism* if f is a bijective continuous function with a continuous inverse.

Alternatively, $f : X \to Y$ is a homeomorphism if f is bijective, and that $U \subseteq X$ is open if and only if $f(U) \subseteq V$ is open. That is, X and Y have "equivalent" open sets if there exists a homeomorphism between them; in that case, we say X and Y are *homeomorphic*. By Proposition 70.3, the property of being homeomorphic is an equivalence relation on the class of topological spaces.

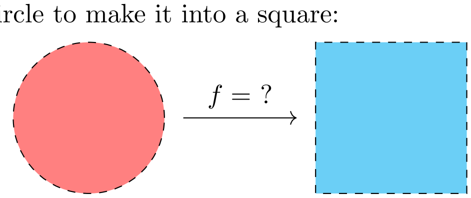
Our next definition is metamathematical in nature.

Definition 70.5. A property of a topological space is a *topological property* if it is preserved under homeomorphisms.

For example, the cardinality of a topological space is a topological property, and so are discreteness and metrizability. Knowing when a property is topological is occasionally important, as it allows us to say that two spaces are "same" up to some property.

Example 70.6. We claim that $(0, 1)$ is homeomorphic to $(0, \infty)$, both on the standard metric topology, and there are many such homeomorphisms: $f(x) = \frac{x}{1-x}$ is one example, and $g(x) = \tan\left(\frac{2}{\pi}x\right)$ is another, where $f, g : (0, 1) \to (0, \infty)$.

Example 70.7. As viewed in $\mathbb{R}^2$, the open unit ball $X := \left\{(x, y) : x^2 + y^2 < 1\right\}$ and the open unit square $Y := \{(x, y) : |x|, |y| < 1\}$ are homeomorphic. This is intuitive, if we think about "stretching" the circle to make it into a square:



We leave it for the reader to find the explicit homeomorphism $f : X \to Y$.

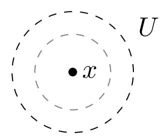
74 Base of a Topology

In a metric space, the open balls $B(x, r)$ have the property that the open sets are precisely the unions of open balls. However, this does not hold for an arbitrary topology — does such a list of "basic" sets exist? Hence, we make the following definition. [Let X be a topological space throughout.]

Definition 74.1. A family of open sets $\mathcal{B}$, of X, is a *base* for the topology on X if every open subset of X is a union of sets in $\mathcal{B}$.

Proposition 74.2. *A family $\mathcal{B}$ of open subsets if a base for the topology on X if and only if for each $x \in X$ and neighborhood U of x, there exists an open neighborhood $x \in V \subseteq U$.*

That is, a family $\mathcal{B}$ is a base if and only if given any point x and a neighborhood U of it, there is a neighborhood V, chosen from $\mathcal{B}$ that "interposes" between x and U, as follows:



Proof. ($\Longrightarrow$): If $\mathcal{B}$ is a base, then any open neighborhood U of x is a union of sets in $\mathcal{B}$; simply pick a V in that union that contains x.

($\Longleftarrow$): Let $U \subseteq X$ be open. For each $x \in U$, there exists an open neighborhood $V_x \in \mathcal{B}$ such that $x \in V_x \subseteq U$. Now, we simply have $U = \bigcup_{x \in U} V_x$, so U is a union of subsets in $\mathcal{B}$. Hence, $\mathcal{B}$ is a base. $\square$

We also have the following characterization of a base.

Proposition 74.3. *A family $\mathcal{B}$ if open subsets is a base if and only if $\mathcal{B}$ satisfies the following two properties:*

1. *For each $x \in X$, there exists $V \in \mathcal{B}$ with $x \in V$.*
2. *If $U, V \in \mathcal{B}$ and $x \in U \cap V$, there exists some $w \in \mathcal{B}$ with $x \in W \subseteq U \cap V$.*

Proof. ($\Longrightarrow$): This is obvious.

($\Longleftarrow$): Let $\mathcal{B}$ satisfy properties (1) and (2) above. We show that $\mathcal{B}$ is a base by constructing explicitly the topology that it comes from. Let τ be the family of all subsets of X that are unions of sets in $\mathcal{B}$. We claim that τ is a well-defined topology, so that $\mathcal{B}$ is automatically a base for τ. First, $\varnothing$ is the empty union[5] of sets in $\mathcal{B}$, so $\varnothing \in \tau$. Now, using (1), pick $V_x \in \mathcal{B}$ such that $x \in V_x \subseteq X$, so that $X = \bigcup_{x \in X} V_x$. Hence $X \in \tau$.

Now, let $\{U_\alpha\}_{\alpha \in A}$ be a collection of sets in τ, so that each U_α is a union of sets in $\mathcal{B}$. Clearly, $\bigcup_{\alpha \in A} U_\alpha$ is also a union of sets in $\mathcal{B}$, so $\bigcup_{\alpha \in A} \in \tau$. Finally, let $U_1, U_2, \ldots, U_n \in \tau$. We must show $\bigcap_{i=1}^n U_i \in \tau$. Pick $x \in \bigcap_{i=1}^n U_i =: U$. Since the U_i are unions of sets in $\mathcal{B}$, choose $V_{i,x} \subseteq U_i$ with $V_{i,x} \in \mathcal{B}$ and $x \in V_{i,x}$. Hence $x \in \bigcup_{i=1}^n V_{i,x}$. Applying (2) inductively, there exists $W_x \in \mathcal{B}$ with $x \in W_x \subseteq \bigcap_{i=1}^n V_{i,x}$. Now, take

$$\bigcap_{i=1}^{n} U_i = \bigcup_{x \in U} W_x,$$

so $\bigcap_{i=1}^n U_i \in \tau$. Hence, τ is a well-defined topology with base $\mathcal{B}$. $\square$

[5] Yes, this is real.

Recall this definition, which we now generalize to topological spaces.

Definition 74.4. A topological space X is *second-countable* if it has a countable base.

We also get Lindelöf's Theorem, whose proof is exactly the same as that for metric spaces after changing the necessary vocabulary.

Theorem 74.5 (Lindelöf's Theorem)**.** *Every open cover of a second-countable topological space has a countable subcover.*

Here, *open cover* means exactly what we think it means.

We can also consider denseness and separability, as before.

Definition 74.6. A subset $S \subseteq X$ is *dense* in X if $\overline{S} = X$.

Definition 74.7. A topological space X is *separable* if there exists a countable dense subset $S \subseteq X$.

Theorem 74.8. *If X is second-countable, then X is separable.*

The proof of this theorem is exactly the same as that for its metric space counterpart. However, the converse of this theorem is not true, unlike the version for metric spaces.

78 Separation Axioms

We saw how topological spaces are more general, and sometimes less well-behaved, than metric spaces. However, some spaces are "nicer" than others, in that distinct points are "easier to distinguish" — for example, in the discrete topology, every point can be discriminated against another, because the singletons are open; in contrast, $\mathbb{R}$ with the indiscrete topology is nearly useless due to how few open sets it contains. In this section, we rigorously define these ideas using *separation axioms.* We start with the most basic separation axiom below.

Definition 78.1. A topological space X is a T_1 *space* if for each pair of distinct points $x, y \in X$, there exists an open set U such that $y \in U$ and $x \notin U$.

We remark that being T_1 is relatively weak — many spaces have this property.

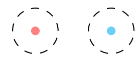
Proposition 78.2. *A topological space X is a T_1 space if and only if the singletons $\{x\}$ are closed in X.*

Proof. ($\implies$): Let X be a T_1 space, and let $x \in X$. Then for all $y \neq x$, there exists an open set U_y containing y but not x. Hence $X \setminus \{x\} = \bigcup_{y \neq x} U_y$ is open, so $\{x\}$ is closed.

($\impliedby$): Suppose $\{x\}$ is closed in X. Then $X \setminus x$ is open and contains all points $y \neq x$. In particular, this means that there is an open set containing y but not x, for an arbitrary $y \neq x$, so that X is T_1. $\square$

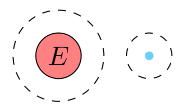
Now, let us "level up" our topological space. We have already introduced T_2 spaces when discussing convergence:

Definition 78.3. The space X is a T_2 *space*, or a *Hausdorff space*, if for each pair of distinct points $x, y \in X$, there exist disjoint open neighborhoods U, V such that $x \in U$ and $y \in V$.



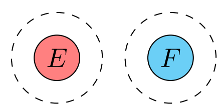
Clearly, every T_2 space is T_1. We may continue:

Definition 78.4. A space X is *regular* if for each closed subset $E \subseteq X$ and $x \in X \setminus E$, there exists disjoint open sets U, V such that $E \subseteq U$ and $x \in V$. A T_3 *space* is a regular T_1 space.



We remark that T_3 spaces are T_2 spaces.

Definition 78.5. A space X is *normal* if for each pair E, F of disjoint closed subsets of X, there exist disjoint open sets U, V such that $E \subseteq U$ and $F \subseteq V$. A T_4 *space* is a normal T_1 space.



Finally, T_4 implies T_3, so we have the chain $T_4 \implies T_3 \implies T_2 \implies T_1$.

Let us view some examples.

Example 78.6. Let $X = \{0, 1\}$ and take $\tau = \{X, \varnothing\}$ to be the trivial topology. Clearly, (X, τ) is not T_1. However, (X, τ) is normal: the only possible pair of disjoint closed subsets if X and $\varnothing$, and $\varnothing \subseteq \varnothing$ and $X \subseteq X$. Now, X and $\varnothing$ are also disjoint *open* sets, so X. Certainly, normality implies regularity, so X is also regular.

Theorem 78.7. *Every metric space is a T_4 space.*

Proof. Let X be a metric space with metric d, and pick $x \in X$. Let $y \neq x$, so that $d(x, y) =: r > 0$. Then $B(y, \frac{r}{2})$ contains y but not x, so X is T_1.

Now, we show X is normal. Let $E, F \subseteq X$ be disjoint closed subsets of X. For each $x \in E$, there exists some $r_x > 0$ such that $B(x, r_x) \cap F = \varnothing$, by disjointness of E and F. Likewise, if $y \in F$, there exists some $s_y > 0$ such that $B(y, s_y) \cap E = \varnothing$. Define the open sets $U := \bigcup_{x \in E} B(x, \frac{r_x}{2})$ and $V := \bigcup_{y \in F} B(y, \frac{s_y}{2})$. Certainly, $E \subseteq U$ and $F \subseteq V$, so it remains to show that $U \cap V = \varnothing$. Suppose for contradiction otherwise, so pick $z \in U \cap V$, so there exists $x_0 \in E$ with $d(x_0, z) < \frac{1}{2} r_{x_0}$; similarly, there exists $y_0 \in F$ with $d(y_0, z) < \frac{1}{2} s_{y_0}$. But now

$$d(x_0, y_0) \leq d(x_0, z) + d(z, y_0) < \frac{1}{2}(r_{x_0} + s_{y_0}) \leq \max\{r_{x_0}, s_{y_0}\},$$

and this forces $B(x_0, r_{x_0}) \cap B(y_0, s_{y_0})$ to be nonempty, a contradiction. Hence, $U \cap V$ is empty and X is normal, as claimed. $\square$

Lemma 78.8. *A topological space X is normal if and only if for each closed subset $E \subseteq X$ and each open set W with $E \subseteq W \subseteq X$, there is an open set U such that $E \subseteq U \subseteq \overline{U} \subseteq W$.*

Proof. ($\implies$): Suppose X is normal, E is closed, and $W \supseteq E$ is open. Then E and $X \setminus W$ are disjoint closed subsets in X, so there exists disjoint open subsets U, V with $E \subseteq U$ and $X \setminus W \subseteq V$. Taking complements, we observe $W \supseteq X \setminus V$, but by disjointness $U \subseteq X \setminus V$. Now, $X \setminus V$ is closed, so $\overline{U} \subseteq \overline{X \setminus V} = X \setminus V \subseteq W$.

($\impliedby$): Let E, F be disjoint closed subsets of X. Let $W = X \setminus F$ (which is open), so $E \subseteq W$, so that by assumption, there exists an open set $U \subseteq X$ with $E \subseteq U \subseteq \overline{U} \subseteq W = X \setminus F$. Now, notice $F = X \setminus W \subseteq X \setminus \overline{U}$ by taking complements, but $X \setminus \overline{U}$ is open as $\overline{U}$ is closed. But now, U and $X \setminus \overline{U}$ are disjoint open subsets of X such that $E \subseteq U$ and $F \subseteq X \setminus \overline{U}$, so X is normal. $\square$

80 Urysohn's Lemma

In this section, we establish an important result about continuous functions on normal spaces.

Theorem 80.1 (Urysohn's Lemma)**.** *Let E and F be disjoint closed subsets of a normal topological space X. Then there exists a continuous function $f : X \to [0,1]$, under the standard topology of $[0,1]$, such that $f = 0$ on E and $f = 1$ on F.*

Proof. Let $V := X \setminus F$, which is open, and $E \subseteq V$. Applying Lemma 78.8, take some open set $U_{1/2}$ with $E \subseteq U_{1/2} \subseteq \overline{U_{1/2}} \subseteq V$. But now, apply Lemma 78.8 to the pairs $(E, U_{1/2})$ and $(\overline{U_{1/2}}, V)$ to get open sets $U_{1/4}$ and $U_{3/4}$, respectively, to get the longer chain

$$E \subseteq U_{1/4} \subseteq \overline{U_{1/4}} \subseteq U_{1/2} \subseteq \overline{U_{1/2}} \subseteq U_{3/4} \subseteq \overline{U_{3/4}} \subseteq V.$$

Continue applying the lemma inductively to get a family of open sets $\{U_\alpha\}$, where $\alpha \in \mathbb{Z}[\frac{1}{2}] \cap (0,1) =: A$ [that is, $\alpha \in (0,1)$ is a rational number whose denominator is a power of 2], such that $E \subseteq U_\alpha \subseteq V$ for all $\alpha \in A$. Additionally, if $\alpha, \beta \in A$ satisfy $\alpha < \beta$, then $\overline{U_\alpha} \subseteq U_\beta$.

Now, define $f : X \to [0,1]$ so that the sets ∂U_r are the level sets of f (i.e., $f(x) = r$ whenever $x \in \partial U_r$). To do so, if $x \in U_r$ for all $r \in A$, then set $f(x) := 0$. Otherwise, there exists some U_r such that $x \notin U_r$, so take $f(x) := \sup\{r \in A : x \notin U_r\}$. Since $r \in A$, f is well-defined. Now, $E \subseteq U_r$ for all r, so $f(x) = 0$ on E. Similarly, since $F = X \setminus V$, we see that if $x \in F$, then $x \notin V$, so $x \notin U_r$ for all r, so $f(x) = 1$ on F.

Finally, we check that f is continuous. Since $f : X \to [0,1]$, applying the metric topology on $[0,1]$ shows that it suffices to prove that for all $\varepsilon > 0$, there exists an open neighborhood U of x such that when $y \in U_x$, we have $|f(x) - f(y)| < \varepsilon$. Fix $x \in X$. If $f(x) = 0$ or $f(x) = 1$, this is trivial, so suppose $f(x) \in (0,1)$. Take $\varepsilon > 0$, and by density of $\mathbb{Z}[\frac{1}{2}]$ in $\mathbb{R}$, there exists $r, s \in A$ such that $f(x) - \varepsilon < s < f(x) < r < f(x) + \varepsilon$. Then $x \notin U_t$ for all $t < f(x)$, so $x \notin \overline{U_s}$, but certainly $x \in U_r$. Hence $W := U_r \setminus \overline{U_s}$ is an open neighborhood of x. Now, if $y \in W$, then by definition of $f(y)$ we have $s \leq f(y) \leq r$, so $|f(x) - f(y)| < \varepsilon$. $\square$

84 Compactness (I)

Definition 84.1. Let X be a topological space. A subset $S \subseteq X$ is *compact* if every open cover has a finite subcover.

Let us see some properties of compactness.

Proposition 84.2. *Let $f : X \to Y$ be a continuous function, where X is compact. Then $f(X)$ is a compact subset of Y, i.e., continuous functions map compact sets to compact sets.*

Proof. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover for $f(X)$. Because f is continuous, $\{f^{-1}(U_\alpha)\}_{\alpha \in A}$ is a family of open sets of X, and it covers X as a function is always surjective on its range. Since X is compact, there exist finitely many $f^{-1}(U_1), f^{-1}(U_2), \ldots, f^{-1}(U_n)$ such that $X = \bigcup_{i=1}^n f^{-1}(U_i)$. Then $f(X) \subseteq \bigcup_{i=1}^n U_i$, so that $f(X)$ is compact in Y. $\square$

This means compactness is a topological property, as it is preserved under homeomorphisms, which are by definition continuous.

Proposition 84.3. *Any finite union of compact sets is compact.*

Proof. Let X be a topological space, and take $S_1, S_2, \ldots, S_n \subseteq X$ to be compact subsets. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover for $S := \bigcup_{i=1}^n S_i$. By compactness of each S_i, there exist sets $U_{i,1}, U_{i,2}, \ldots, U_{i,j_i} \in \{U_\alpha\}$ which cover S_i. Then

$$\bigcup_{i=1}^{n} \bigcup_{k=1}^{j_i} U_{i,k} \supseteq S$$

is a finite subcover of S, so S is compact. $\square$

Proposition 84.4. *A closed subset of a compact topological space is compact.*

Proof. Let $S \subseteq X$ be one such subset, and take an open cover $\{U_\alpha\}_{\alpha \in A}$. Because S is closed, $X \setminus S$ is open, so that $\{U_\alpha\} \cup \{X \setminus S\}$ is an open cover for X. By compactness of X, there exist finitely many $U_1, U_2, \ldots, U_n$ such that $X = \bigcup_{i=1}^n U_i \cup (X \setminus S)$. The corresponding open finite subcover will be a subcover for S, completing the proof. $\square$

Lemma 84.5. *Let S be a compact subset of a Hausdorff space X. For each $x \in X \setminus S$, there exist disjoint open neighborhoods U of x and V of S.*

Proof. For each $y \in S$, there exist disjoint open neighborhoods U_y of x and V_y of y (the double-subscripting of y here is intentional). Now, we see $S \subseteq \bigcup_{y \in S} V_y$, so the V_y are an open cover for S, so by compactness of S, there exist finitely many $y_1, y_2, \ldots, y_n$ in S such that $S \subseteq \bigcup_{i=1}^n V_{y_i} =: V$. In parallel, define $U := \bigcap_{i=1}^n U_{y_i}$, which contains x. Now, U and V are disjoint open neighborhoods of x and S, respectively, so we are done. $\square$

The upshot to this lemma is the following, which tells us that Hausdorff spaces are very nice.

Corollary 84.6. *A compact subset of a Hausdorff space is closed.*

Proof. Let $S \subseteq X$ be compact, where X is Hausdorff. Take any $x \in X \setminus S$, so that by Lemma 84.5, there exist disjoint U_x, V_x open neighborhoods of x resp. S. Certainly, $U_x \cap S = \varnothing$, and thus $U_x \subseteq X \setminus S$. Let $U := \bigcup_{x \in X \setminus X} U_x$, so that $U = X \setminus S$. But now U is open, so S is closed. $\square$

Theorem 84.7. *A compact Hausdorff space is normal.*

Proof. Let S, T be disjoint closed subsets of a compact Hausdorff space X. By Proposition 84.4, both S and T are compact. For any $x \in T$, by Lemma 84.5, there exist disjoint open neighborhoods U_x of x and V_x of S, so that the open sets $\{U_x\}_{x \in T}$ cover T. By compactness, there exist finitely many $x_1, x_2, \ldots, x_n \in T$ such that

$$T \subseteq \bigcup_{i=1}^{n} U_{x_i} =: U,$$

and similarly $S \subseteq \bigcap_{i=1}^{n} V_{x_i} =: V$. Then U, V are disjoint open neighborhoods of T and S, respectively. $\square$

88 Compactness (II)

In this section, we illustrate a few more properties of compactness, and give some examples.

Theorem 88.1. *Let $f : X \to Y$ be a continuous function from a compact space X to a Hausdorff space Y. If f is injective, then f induces a homeomorphism from X to $f(X)$.*

Proof. Clearly, f is surjective onto its range $f(X)$. We must show $f^{-1} : f(X) \to X$ is continuous. Let $U \subseteq X$ be open; we must show that $f(U)$ is open in $f(X)$. Now, because U is open, $X \setminus U$ is closed and thus $X \setminus U$ is compact by Proposition 84.4. Now, f is continuous, so it maps compact sets to compact sets (Proposition 84.2), so $f(X \setminus U)$ is compact By by Corollary 84.6, $f(X \setminus U)$ is closed, but now we claim $f(X \setminus U) = f(X) \setminus f(U)$. If this holds, then $f(U)$ is open, so we are reduced to showing this set equality. Take $y \in f(X) \setminus f(U)$. Then $y \notin f(U)$, so $y \neq f(x)$ for any $x \in U$. But since $y \in f(X)$, we must have $y \in f(X \setminus U)$.

Conversely, take $y \in f(X) \setminus f(U)$. Then there exists $x \in X \setminus U$ such that $f(x) = y$; by injectivity of f, this x is unique in $X \supseteq X \setminus U$. In particular, if $x' \in U$, then $x' \neq x$ so $f(x') \neq f(x) = y$, so $y \in f(X) \setminus f(U)$, which proves the inclusion and concludes the proof. $\square$

Example 88.2. Let X be any set, and τ be a finite topology (i.e., $|\tau| < \aleph_0$ as a set). Then (X, τ) is compact — any open cover of X is automatically finite as τ is finite. It immediately follows that the trivial topology is compact, and if X is finite, the power set 2^X is finite and X is automatically compact, given any topology.

Example 88.3. A compact topological space may not be T_1. If $X = \{0, 1\}$ and $\tau = \{X, \varnothing\}$, then by the previous example, X is compact. But we have already seen that X is not T_1 — no open neighborhood contains 0 but not 1 (and vice versa).

Example 88.4. We know that in a compact topological space, E is closed implies E is compact. The converse of this is not necessarily true. Again, take $X = \{0, 1\}$, and consider the topology $\tau = \{X, \{0\}, \varnothing\}$. Then (X, τ) is compact by Example 88.2, and clearly $\{0\}$ is compact. But $X \setminus \{0\} = \{1\}$, which is not open, so $\{0\}$ is not closed.

In fact, one can show the space above is T_1 but not T_2. This generalizes:

Proposition 88.5. *Any non-Hausdorff topological space contains compact subsets that are not closed.*

That is, being Hausdorff exactly means that closed sets are compact.

[2.7] Local Compactness

Sometimes, a space might not be compact, but many subsets of it can behave compactly. Hence, we make the following definition.

Definition 88.6. A topological space X is *locally compact* if for each $x \in X$, there exists an open set U such that $x \in U$ and $\overline{U}$ is compact.

Clearly, compactness implies local compactness. We also note that any nonempty subset of $\mathbb{R}^n$ is locally compact — if $E \subseteq \mathbb{R}^n$ is bounded, then $\overline{E}$ is closed and bounded, so it is compact in $\mathbb{R}^n$. Otherwise, just take a bounded subset $F \subseteq E$.

Example 88.7. A discrete space is locally compact: if $x \in X$, then $\{x\}$ is open and closed, and $\{x\}$ is finite so it is compact.

Proposition 88.8. *Local compactness is a topological property.*

Proof. Let $f : X \to Y$ be a homeomorphism, where X is locally compact. We must show that Y is locally compact. Take $y \in Y$, so that there exists $x \in X$ with $y = f(x)$. Since X is locally compact, there exists an open $U \subseteq X$ such that $x \in U$ and $\overline{U}$ is compact. Let $V := f(U)$. Since f^{-1} is continuous, the openness of U implies the openness of V. Clearly, $y \in V$, so it remains to show that $\overline{V}$ is compact. We claim that $\overline{V} = \overline{f(U)} = f(\overline{U})$; to see this, note that since f is continuous, we have $f(\overline{U}) \subseteq \overline{f(U)}$; applying f^{-1} gives the other inclusion. $\square$

The next proposition should be compared to Theorem 84.7.

Proposition 88.9. *A locally compact Hausdorff space is regular.*

Proof. Let X be a locally compact Hausdorff space, and let $C \subseteq X$ be closed. Pick $p \in X \setminus C$. Since X is locally compact, there exists an open neighborhood W of p such that $\overline{W}$ is compact. Consider the disjoint union

$$C = (\overline{W} \cap C) \cup (C \setminus \overline{W}).$$

We cover both parts of this union using an open set. Since $\overline{W} \cap C$ is closed inside the compact subspace $\overline{W}$, we see $\overline{W} \cap C$ is compact. Then, since X is Hausdorff, then for the compact set $\overline{W} \cap C$ disjoint from $p \in X \setminus C$, there exist disjoint open neighborhoods U of p and V of $\overline{W} \cap C$. Now, clearly $X \setminus \overline{W}$ is an open cover for $C \setminus \overline{W}$, so $V \cup (X \setminus \overline{W})$ is an open cover for C. Also, $p \in U \cap W$, but this is disjoint from the cover we found. $\square$

94 Compactification

Locally compact spaces are nice, but truly compact spaces have stronger properties than merely locally compact spaces. Hence, we define what it means to "make" a locally compact space, truly compact.

Definition 94.1. A compact space Y is a *compactification* of X if X is homeomorphic to a dense subspace of Y.

This definition is fairly straightforward, so we view an example.

Example 94.2. In $\mathbb{R}$, $[0, 1]$ is a compactification of $(0, 1), [0, 1), (0, 1]$, and $[0, 1] \cap \mathbb{Q}$.

Notice that in the above, some compactifications took more "effort" than others: for $[0, 1)$, we only "filled in" the missing point at 1, but for $[0, 1] \cap \mathbb{Q}$, we had to fill in an uncountable number of irrational numbers. What is the "lowest effort" compactification we can get? For certain nice spaces, we can compactify by just adding one point:

Theorem 94.3 (One-Point Compactification)**.** *Let X be a locally compact Hausdorff space, and let Y be a set consisting of X and one other point. Then there exists a unique topology for Y, called the* <u>*one-point compactification of X*</u>*, such that Y is a compact Hausdorff space and the relative topology for X inherited from Y coincides with the original topology on X.*

Proof. Write $Y = X \cup \{\infty\}$, where[6] $\infty \notin X$. Consider the family τ of subsets $U \subseteq Y$ such that either

1. $U \subseteq X$ is open in X, or
2. $\infty \in U$ and $X \setminus U$ is compact in X.

Since compact subsets of a Hausdorff space are closed, observe that type (2) sets are relatively open in X (that is, if U is type (2), then $U \cap X$ is relatively open). Hence, the relative topology for X inherited from τ coincides with the original topology on X... if indeed τ is actually a topology, which we check. Clearly, $\varnothing \in \tau$, and now $X \setminus Y = \varnothing$ is compact in X, so $Y \in \tau$ by (2). Take $\{U_\alpha\}_{\alpha \in A}$ to be a family of subsets in τ and write $U := \bigcup_{\alpha \in A} U_\alpha$. If every U_α is type (1), then clearly $U \in \tau$. Otherwise, take U_0 of type (2) in this family, so that $\infty \in U_0$ and $X \setminus U_0$ is compact in X. Now

$$X \setminus U = X \setminus \bigcup_{\alpha \in A} U_\alpha = \bigcap_{\alpha \in A} (X \setminus U_\alpha) = (X \setminus U_0) \cap \bigcap_{\alpha \in A} (X \setminus U_\alpha)$$

is closed in $X \setminus U_0$, so $X \setminus U$ is compact in X. Hence, U is type (2) in τ.

Finally, suppose $U_1, U_2, \ldots, U_n \in \tau$, and set $U := U_1 \cap \cdots \cap U_n$. If one U_i is of type 1), then $\infty \notin U$, so that $U \subseteq X$ and is of type (1). Otherwise, $\infty \in U$, and consider $X \setminus U = (X \setminus U_1) \cup \cdots \cup (X \setminus U_n)$. Since each U_i is of type (2), each $X \setminus U_i$ is compact in X. The finite union of compact subsets if compact, so $X \setminus U$ is compact in X. Hence, U is of type (2) in τ, which proves that τ is a well-defined topology.

Next, we show that (Y, τ) is compact. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover for Y. Certainly, there exists some U_0 such that $\infty \in U_0$, i.e., U_0 is of type (2). This forces $X \setminus U_0$ to be compact. Now, consider the open sets $\{U_\alpha \cap X\}_{\alpha \in A}$, which are relatively open in X. Certainly, $\{U_\alpha \cap X\}$ is an open cover of X, but in particular, it is also an open cover of $X \setminus U_0 = Y \setminus U_0$ (this is valid as U_0 contains ∞). By compactness of $X \setminus U_0$, there exist $U_1, U_2, \ldots, U_n$ such that $X \setminus U_0 = Y \setminus U_0 \subseteq \bigcup_{i=1}^n U_i$, so $Y \subseteq \bigcup_{i=0}^n U_i$, which shows Y is compact.

Finally, we show that (Y, τ) is Hausdorff. Since X is already Hausdorff, we only need to take $x \in X$ and $\infty \in Y \setminus X$, and find disjoint open neighborhoods in Y containing x and ∞. Since X is locally compact, there exists some open neighborhood $U \subseteq X$ of x such that $\overline{U} \subseteq X$ is compact. Take $V := Y \setminus \overline{U}$, which is open. Since $\overline{U} \subseteq X$, we see $\infty \in V$, but $U \cap V = \varnothing$, i.e., we have shown that Y is indeed Hausdorff. □

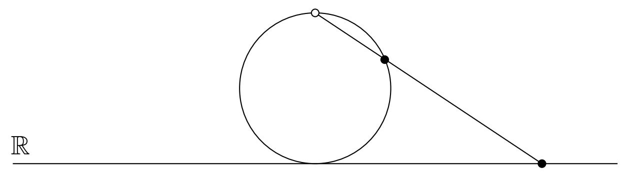
Though we have proven all claims in the theorem statement above, we note that the name "one-point *compactification*" does not quite make sense yet — we have not actually shown that Y is a real compactification! This is left as an exercise — one needs to show that ∞ is adherent to X.

[6] The reason why we choose ∞ to represent this point will be made evident later.

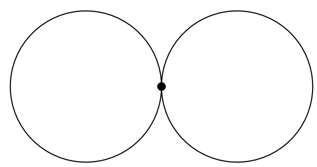
Example 94.4. The one-point compactification of $\mathbb{R}^n$ is homeomorphic to the *n-sphere* S^n, which is defined by

$$S^n := \left\{(y_1, y_2, \ldots, y_n) \in \mathbb{R}^{n+1} : y_1^2 + y_2^2 + \cdots + y_{n+1}^2 = 1\right\}.$$

For $n = 1$, we see $\mathbb{R} \cap \{\infty\} \cong S^1$ [here, "$\cong$" means "is homeomorphic to"]. A helpful picture is here.



Example 94.5. Let $X = (0,1) \cup (2,3)$. Notice that $(0,1) \cong \mathbb{R} \cong (2,3)$, but we know that the one-point compactification of $\mathbb{R}$ is homeomorphic to S^1. Hence, the one-point compactification of X is homeomorphic to two tangent circles as such:



98 Connectedness (I)

Definition 98.1. A topological space is *connected* if X is not the disjoint union of two nonempty clopen[7] sets. Alternatively, if $U \subseteq X$ is clopen, then either $U = \varnothing$ or $U = X$; i.e., the only clopen sets of X are X and $\varnothing$.

Let us see what connectedness is *not*, in order to get a feel for it.

Example 98.2. Let $X = (0,1) \cup (2,3)$, with its metric topology. Then $(0,1)$ and $(2,3)$ are certainly open in X, but now $X \setminus (0,1) = (2,3)$ and $X \setminus (2,3) = (0,1)$ are closed. Now, $(0,1)$ and $(2,3)$ are nonempty proper clopen subsets of X, so X is not connected.

As a union of two real intervals, X looks like a union of two disjoint open intervals — which, if we were to plot it, looks quite *disconnected*.

Hence, we conclude that intuitively, connectedness tells us that our space is in "one piece" — again, unlike $X = (0,1) \cup (2,3)$.

Proposition 98.3. *Let $f : X \to Y$ be continuous, where X is connected. Then $f(X)$ is connected. In particular, connectedness is a topological property.*

Proof. Let $U \subseteq f(X)$ be clopen. Since f is continuous, $f^{-1}(U)$ is also clopen, so by connectedness of X, we have $f^{-1}(U) = \varnothing$ or $f^{-1}(U) = X$. Now, this immediately implies $U = \varnothing$ or $U = f(X)$, so that $f(X)$ is connected. $\square$

While entire spaces may fail to be connected, subsets can still be connected.

[7]That is, both closed and open.

Definition 98.4. A subset Y of a topological space X is a *connected subset* if it is connected as a subspace.

Theorem 98.5. *Let $\{E_\alpha\}_{\alpha \in A}$ be a family of connected subsets of a topological space X, such that $E_\alpha \cap E_\beta$ is nonempty for all $\alpha, \beta \in A$. Then $\bigcup_{\alpha \in A} E_\alpha$ is connected.*

Intuitively, this should make sense — if several connected subsets all overlap with each other, then their union must be connected as well.

Proof. Write $E := \bigcup_{\alpha \in A} E_\alpha$, and pick $F \subseteq E$ to be clopen. Suppose $F \neq \varnothing$; we show $F = E$. By assumption, $F \subseteq E$, so it suffices to show $E \subseteq F$. Let $x \in F$. Then there exists some E_0 in the family $\{E_\alpha\}$ such that $x \in E_0$, so $x \in F \cap E_0$. But now $F \cap E_0$ is relatively clopen and nonempty in E_0; since E_0 is connected, $F \cap E_0 = E_0$. This forces $E_0 \subseteq F$. Moreover, if $\beta \in A$ is any index, then $F \cap E_\beta$ is relatively open. We claim that $F \cap E_\beta$ is nonempty; note that by assumption $\varnothing \neq E_0 \cap E_\beta \subseteq F \cap E_\beta$, so that by connectedness of E_β, we again have $E_\beta \subseteq F$. Since β was arbitrary, we observe $E \subseteq F$, so $E = F$. $\square$

100 Connectedness (II)

In general, spaces may fail to be connected. However, we can ask a related question: what is the "largest" connected subset that contains a point?

Definition 100.1. Let X be a topological space, and let $x \in X$. The *connected component* of x in X, denoted by $C(x)$, is the union of all connected subsets of X that contain x.

We remark that $C(x)$ is indeed the largest connected subset that contains x, ordered by inclusion. [It is connected by Theorem 98.5]. The following result explains why we care about connected components:

Theorem 100.2. *A topological space X is partitioned by its connected components into maximal connected subsets.*

Proof. Clearly, X is the union of its connected components. Now, let $x \neq y \in X$, and consider the connected components $C(x), C(y)$. Suppose $C(x) \cap C(y) \neq \varnothing$; we show $C(x) = C(y)$. To see this, note that by Theorem 98.5 we have that $C(x) \cup C(y)$ is connected, but by maximality of $C(x)$ and $C(y)$, we have $C(x) = C(x) \cup C(y) = C(y)$, completing the proof. That the sets are maximal comes from the definition of the connected components. $\square$

Hence, we can always view a space as a union of connected subspaces. We also have the following corollary:

Corollary 100.3. *A topological space X is connected if and only if X only has one connected component, namely X itself.*

Example 100.4. Let X be a set, endowed with the discrete topology. Now, this implies $\{x\}$ is clopen for any $x \in X$, so $C(x) = \{x\}$. Hence, the connected components of X are precisely the singletons $\{x\}$.

Proposition 100.5. *Each connected component of a topological space X is closed.*

We remark that the connected component may or may not be open.

Proposition 100.6. *Any interval in $\mathbb{R}$ is connected.*

Proof. There are three types of intervals in $\mathbb{R}$: closed, open, and half-open, and thus every interval in $\mathbb{R}$ is homeomorphic to $[0,1]$, $(0,1)$, or $[0,1)$. Since connectedness is a topological property, it suffices to show that these three intervals are connected. Let $X \subseteq [0,1]$ be a relatively clopen subset. Then $X = U \cap [0,1]$, where U is clopen in $\mathbb{R}$. But $\mathbb{R}$ is connected, so we have $U = \varnothing$ or $U = \mathbb{R}$, so it follows that $X = \varnothing$ or $X = [0,1]$. Hence, X is connected, and thus every bounded closed interval $[a,b]$ is connected.

From here, observe $(0,1) = \bigcup_{n=2}^{\infty} \left[\frac{1}{n}, 1 - \frac{1}{n}\right]$ and $[0,1) = \bigcup_{n=2}^{\infty} \left[0, 1 - \frac{1}{n}\right]$. By Theorem 98.5, these sets are connected, so we are done. □

[2.9] Path Connectedness

Definition 100.7. Let X be a topological space and $x_0, x_1 \in X$. A *path* in X from x_0 to x_1 is a continuous function $\gamma : [0,1] \to X$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$.

Certainly, this path is directional, as we force the "start" point to be x_0 and the "end" point to be x_1.

Definition 100.8. The space X is *path-connected* if, for every pair of points $x_0, x_1 \in X$, there exists a path γ from x_0 to x_1.

This definition is exactly what it sounds like: the space is path connected if every pair of points is connected by a path. We prove some basic properties about path connectedness.

Proposition 100.9. *Let X be a topological space, and define the relation $\sim$ on X by $x \sim y$ if there is a path from x to y. Then $\sim$ is an equivalence relation on X.*

Proof. Clearly, $\sim$ is reflexive as we have the constant-function path $\gamma(t) := x$, going from x to $x \in X$. Now, suppose $x \sim y$, so that there exists a path β from x to y. Now, we know $\beta(0) = x$ and $\beta(1) = y$. Define the path $\gamma(t) := \beta(1-t)$; this map is continuous because it is a composition of continuous functions $[0,1] \xrightarrow{1-t} [0,1] \xrightarrow{\beta} X$, and $\gamma(0) = y$ and $\gamma(1) = x$. Hence, $y \sim x$ and $\sim$ is symmetric.

Finally, suppose $x \sim y$ and $y \sim z$, so let α be a path from x to y and β be a path from y to z. We know $\alpha(0) = x$, $\alpha(1) = y$, $\beta(0) = y$, and $\beta(1) = z$. Define the map

$$\gamma(t) := \begin{cases} \alpha(2t) & 0 \le t \le \frac{1}{2} \\ \beta(2t-1) & \frac{1}{2} < t \le 1. \end{cases}$$

The only thing we need to check is continuity at $t = \frac{1}{2}$, but this is clear because $\alpha(1) = \beta(0)$. Hence, γ is a path from x to z, so $x \sim z$ and $\sim$ is transitive. □

Definition 100.10. The equivalence classes of the equivalence relation $\sim$ defined above on X are called *path components*.

It immediately follows that a topological space X is partitioned by its path components, and it is also true that X is path-connected if and only if there is one path component in X, namely X itself. Now, we discuss the relationship between path-connectedness and connectedness.

Proposition 100.11. *A path-connected space is connected.*

Proof. Fix $x_0 \in X$. By path-connectedness of X, there exists a path $\gamma_x : [0,1] \to X$ from x_0 to any $x \in X$. Since γ_x is continuous and $[0,1]$ is connected, we see $\gamma_x([0,1])$ is connected. But now clearly $X = \bigcup_{x \in X} \gamma_x([0,1])$, and we know $x_0 \in \gamma_x([0,1])$ for all $x \in X$, so by Theorem 98.5, X is connected. □

Example 100.12. However, it is not the case that every path connected space is path-connected. Define the *topologist's sine curve* as follows: take $E := \{0\} \times [-1,1] \subseteq \mathbb{R}^2$ and $F := \left\{(x, \sin \frac{1}{x}) : 0 < x \leq 1\right\}$. The topologist's sine curve is the space $X := E \cup F$, which is connected, but not path-connected. An image of the curve may be found here (clickable link to Wikipedia).

104 Course Review (I)

In this course, we learned about properties of two types of spaces: metric spaces and the more general topological spaces. Recall that a *metric* is a function $d : X \times X \to \mathbb{R}$ such that for all $x, y \in X$, we have $d(x,y) \geq 0$ with equality holding if and only if $x = y$; $d(x,y) = d(y,x)$, and the *triangle inequality* $d(x,z) \leq d(x,y) + d(y,z)$. Refer to Sections 10 and 13 for examples.

Example 104.1. Any metric function is a continuous function; see this link here.[8] Hence, any linear combination of metric functions gives a continuous function $X \times X \to \mathbb{R}$.

In contrast, a *topology* is defined by immediately declaring its open sets, subject to the conditions that $X, \varnothing \in \tau$, any union of sets in τ belongs to τ, and any finite intersection of sets in τ belongs to τ. Metric spaces are all topological spaces, where the metric does the job of declaring sets as open.

Example 104.2. A common example (see the homework) of a topological space is the *cofinite topology*: given a set X, let τ be the family of complements of finite sets, together with $\varnothing$:

$$\tau := \{S \subseteq X : X \setminus S \text{ is finite}\} \cup \{\varnothing\}.$$

In Homework 4 (Section 204), we checked that the cofinite topology is non-metrizable if X is infinite.

Two important notions occur in both metric and topological spaces in general. Recall that a set is *compact* if every open cover has a finite subcover. In a metric space, X is compact if and only if X is totally bounded and complete — this was the Bolzano-Weierstrass Theorem (Theorem 34.7). In specifically $\mathbb{R}^n$, X is compact if and only if it is closed and bounded. In topological spaces, however, these equivalences do not hold. We know that closed subsets of compact topological spaces are compact (Proposition 84.4), but a compact subset of a compact topological space may not be closed: see Example 88.4. Another simpler example is the following:

Example 104.3. Take $X = \{0,1\}$, and $\tau = \{X, \varnothing\}$ to be the trivial topology. Then $\{a\}$ and $\{b\}$ are both compact, as they are finite sets, but they are not closed.

In topological spaces, we also discussed the four separation axioms (Section 78):

[8] https://math.stackexchange.com/questions/287285/how-to-prove-the-continuity-of-the-metric-function.

1. A space is T_1 if for each pair of distinct points $x \neq y$, there exists an open set U such that $y \in U$ yet $x \notin U$.
2. A space is T_2 (or Hausdorff) if for each pair of distinct points $x \neq y$, there exist two disjoint open sets U, V with $x \in U$ and $y \in V$.
3. A space is T_3 if it is T_1, and for each closed subset $E \subseteq X$ and a point $x \in X \setminus E$, there exist disjoint open neighborhoods U, V such that $E \subseteq U$ and $x \in V$. That is, a T_3 space is a *regular* T_1 space.
4. A space is T_4 if it is T_1, and for each pair of disjoint closed subsets $E, F \subseteq X$, there exist disjoint open neighborhoods U, V such that $E \subseteq U$ and $F \subseteq V$. That is, a T_4 space is a *normal* T_1 space.

We recall that normal (or regular) spaces are not necessarily T_1 — see Example 78.6, and we have $T_4 \implies T_3 \implies T_2 \implies T_1$. Also, every metric space is a T_4 space (see Theorem 78.7), which is why we did not discuss separation axioms for metric spaces.

The other important notion in metric and topological spaces is *continuity*, which, if we recall, is characterized by preimages of open sets: a function between two topological spaces $f : X \to Y$ is *continuous* if whenever $V \subseteq Y$ is open in Y, we have that $f^{-1}(V) \subseteq X$ is open in X.

Example 104.4. If f is continuous, it is not the case that U is open implies $f(U)$ is open. Take $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \sin x$. Then $U := (-10, 10)$ is certainly open in $\mathbb{R}$, yet $f(U) = [-1, 1]$ is not open.

However, if f is a *homeomorphism*, then the implication above holds.

107 Compactness: Worked Examples

Lemma 107.1. *Let X be Hausdorff and $K \subseteq X$ be compact. Then for all $x \in X \setminus K$, there exist disjoint open sets U and V such that $x \in U$ and $K \subseteq V$.*

Proof. Because X is Hausdorff, for every $k \in K$, there exist disjoint open neighborhoods U_k, V_k containing x resp. k. Now, $\bigcup_{k \in K} V_k$ is an open cover of K, so there exists a finite subcover $V_1, V_2, \ldots, V_n$. Now $V := \bigcup_{i=1}^{n} V_i$ contains K, and $U := \bigcap_{i=1}^{n} U_i$ is open and disjoint from V, so we are done. $\square$

Example 107.2. Let X be a topological space. We prove that if every point of X has an open neighborhood U such that $\overline{U}$ is a compact Hausdorff space, then X is Hausdorff.

Proof. Suppose $x \neq y$ are distinct points in X. Pick open neighborhoods U, V containing x resp. y, such that $\overline{U}$ and $\overline{V}$ are compact and Hausdorff. Now, $\overline{U} \cap \overline{V}$ is closed in a compact space $\overline{U}$, so $\overline{U} \cap \overline{V}$ is compact. Suppose $x \in \overline{U} \cap \overline{V}$. Then $x, y \in \overline{V}$, so there exist open sets O_x, O_y such that $O_x \cap \overline{V} \ni x$, $O_y \cap \overline{V} \ni y$, and the two relatively open sets are disjoint. But this implies $O_x \cap O_y \cap \overline{V} = \varnothing$, but now $O_x \cap O_y \cap V \subseteq O_x \cap O_y \cap \overline{V} = \varnothing$. Hence O_x and $O_y \cap V$ are disjoint open sets containing x and y respectively.

If $y \in \overline{U} \cap \overline{V}$, the proof is identical, so consider the case $x, y \notin \overline{U} \cap \overline{V}$. If $\overline{U} \cap \overline{V} = \varnothing$, then we are done. Otherwise, by the lemma above, find open sets O_x, O such that $x \in O_x \cap \overline{U}$, $\overline{U} \cap \overline{V} \subseteq O \cap \overline{U}$, and the two relatively open sets are disjoint. Again, by similar argumentation to the first case, $O_x \cap U \ni x$ and $O \supseteq \overline{U} \cap \overline{V}$ are still disjoint. Now, repeat the same argument for y, so there exists $O_y \ni y$ and $O' \supseteq \overline{U} \cap \overline{V}$, again relatively disjoint. Now, consider the sets $O_x \cap U$ and $O_y \cap Y$, both open in the original topology. By construction, these two sets are disjoint, so we are done. $\square$

108 Course Review (II)

Recall that *topological properties* are properties that are preserved under homeomorphisms. Some examples of topological properties include:

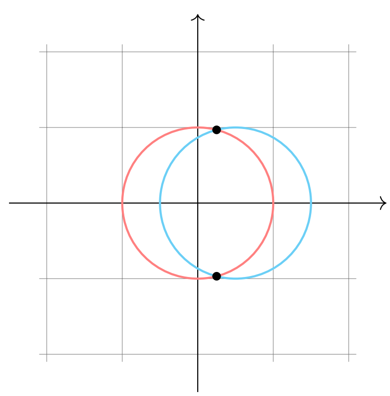
1. Compactness
2. Local Compactness
3. Connectedness
4. Path Connectedness
5. Separability
6. Separation axioms T_1 through T_4.

Note that the separation axioms are indeed topological properties, but they need not be preserved under a general continuous function:

Example 108.1. Let $X = \{0, 1\}$, $\tau_1 = 2^X$ (the discrete topology on X), and $\tau_2 = \{X, \varnothing\}$ (the indiscrete topology on X). Consider the map $f : (X, \tau_1) \to (X, \tau_2)$ from $f(x) = x$, the identity. This is easily checked to be continuous as the two preimages $f^{-1}(\varnothing) = \varnothing$ and $f^{-1}(X) = X$ are open in (X, τ_1). However, (X, τ_1) is metrizable, and hence T_4, but (X, τ_2) is not even T_1.

Now, we review connectedness. Recall that a topological space X is *connected* if the only clopen sets of X are X and $\varnothing$. Also, Theorem 98.5 tells us when unions of connected sets are connected. What about intersections?

Example 108.2. If we have $U, V \subseteq X$ to be connected with nonempty intersection, then this *does not* imply $U \cap V$ is connected. Take $X = \mathbb{R}^2$, $U = \left\{x^2 + y^2 = 1\right\}$ and $V = \left\{(x - \frac{1}{2})^2 + y^2 = 1\right\}$. These circles are clearly connected. Now, some algebra demonstrates that $U \cap V = \left\{\left(\frac{1}{2}, \pm\frac{\sqrt{15}}{4}\right)\right\}$, but this finite set is disconnected.



Recall that *path-connectedness* is a stronger condition than regular connectedness, and strictly so: the *topologist's sine curve* demonstrates that connected spaces need not be path-connected. However, in $\mathbb{R}$, connectedness does imply path connectedness, as every connected set in $\mathbb{R}$ is an interval.

Topological Set Identities

Much of our work in topology deals with definitions of sets, and showing inclusions and equality. Also, many supposed equalities are disproved by counterexamples as well.

Example 108.3. Is it true that $\operatorname{int}(A \cup B) \stackrel{?}{=} \operatorname{int}(A) \cup \operatorname{int}(B)$?

Solution. No — take $A = \mathbb{Q}$ and $B = \mathbb{R} \setminus \mathbb{Q}$. Then both A and B have empty interior, yet $A \cup B = \mathbb{R}$, and so $\operatorname{int}(A \cup B) = \mathbb{R} \neq \varnothing$. •

Example 108.4. Does the closure commute with the interior $\overline{\operatorname{int}(A)} \stackrel{?}{=} \operatorname{int}(\overline{A})$?

Solution. No — take $A = \mathbb{Q}$. Then $\overline{A} = \mathbb{R}$, so $\operatorname{int}(\overline{A}) = \mathbb{R}$, yet A has empty interior, so the left-hand side is the empty set. •

As one can see, trying $\mathbb{Q} \subsetneq \mathbb{R}$ as a counterexample is very helpful. There are some set equalities that are true, like $\overline{A} \cup \overline{B} = \overline{A \cup B}$: see Example 27.3. The proofs of these are *generally* Math 13-level subset inclusion proofs.

Homework Exercises

What follows are my attempted problem set solutions this quarter. These were graded on completion, so there is no guarentee of accuracy here. Also, I have omitted some exercises out of taste, and have revised some for clarity.

201 Homework 1

Exercise 1. This exercise concerns the standard norm in Euclidean space $\mathbb{R}^n$.

(a) Show that if $a, b, c \in \mathbb{R}$ are such that for all $\lambda \in \mathbb{R}$, $a\lambda^2 + b\lambda + c \geq 0$, then $b^2 - 4ac \leq 0$.

Proof. Consider the polynomial $f(x) := ax^2 + bx + c \geq 0$, so that our condition is equivalent to saying $f(\lambda) \geq 0$ for all $\lambda \in \mathbb{R}$. Notice that this implies that if $a = 0$, then $b = 0$ and $c \geq 0$, from which the result follows immediately, so assume that $a \neq 0$. If $a < 0$, then calculus tells us that $f(x)$ can be arbitrarily negative, so we must have $a > 0$.

The roots of f are

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

In the case $f(\lambda) > 0$ for all $\lambda \in \mathbb{R}$, the roots $x_{1,2}$ are non-real, so $b^2 - 4ac < 0$ follows immediately, so suppose f has a real root. We claim that f has one root; to see this, suppose $x_1 \neq x_2$ are roots of f, so that

$$\frac{x_1 + x_2}{2} = -\frac{b}{2a}.$$

From here, it is not too hard to show $f(-b/2a) < 0$, a contradiction, so that f has one root and thus $b^2 - 4ac = 0$. $\square$

(b) Show that for any $(x_1, \dots, x_n), (y_1, \dots, y_n) \in \mathbb{R}^n$, the *Cauchy-Schwarz inequality* holds:

$$\sum_{i=1}^n x_i y_i \leq \sqrt{\sum_{i=1}^n x_i^2}\sqrt{\sum_{i=1}^n y_i^2}.$$

Proof. Consider the polynomial $f(\lambda) = \sum_{i=1}^n (x_i - \lambda y_i)^2 = \sum_{i=1}^n (x_i^2 - 2\lambda x_i y_i + \lambda^2 y_i^2)$. By construction, $f(\lambda) \geq 0$ for all $\lambda \in \mathbb{R}$, so we write

$$f(\lambda) = \left(\sum_{i-1}^n y_i^2\right)\lambda^2 - \left(2\sum_{i=1}^n x_i y_i\right)\lambda + \left(\sum_{i=1}^n x_i^2\right) =: ax^2 + bx + c.$$

From part (a), it follows that $b^2 \leq 4ac$, i.e.,

$$4\left(\sum_{i=1}^n x_i y_i\right)^2 \leq 4\left(\sum_{i-1}^n y_i^2\right)\left(\sum_{i=1}^n x_i^2\right),$$

so dividing by 4 and extracting square roots finishes the proof of the inequality. $\square$

(c) Using Cauchy-Schwarz, show that the function $d : \mathbb{R}^n \to \mathbb{R}$ defined by

$$d(x, y) := \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

satisfies the triangle inequality.

Proof. Let $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n)$, and $z = (z_1, \dots, z_n)$. We must prove that $d(x, z) \leq d(x, y) + d(y, z)$. To do this, define the *norm* of a vector $v = (v_1, \dots, v_n) \in \mathbb{R}^n$ by

$$\|v\| := \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

Thus, it is equivalent to verify $\|x - z\| \leq \|x - y\| + \|y - z\|$. Letting $a := x - y$ and $b := y - z$, this is equivalent to showing $\|a + b\| \leq \|a\| + \|b\|$. Now letting $a_i := x_i - y_i$ and $b_i := y_i - z_i$ for all $i \leq n$, we have

$$\begin{aligned}
\|a + b\|^2 &= (a_1 + b_1)^2 + \cdots + (a_n + b_n)^2 \\
&= (a_1^2 + 2a_1 b_1 + b_1^2) + \cdots + (a_n^2 + 2a_n b_n + b_n^2) \\
&= \sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} b_i^2 + 2 \sum_{i=1}^{n} a_i b_i = \|a\|^2 + \|b\|^2 + 2 \sum_{i=1}^{n} a_i b_i \\
&\leq \|a\|^2 + \|b\|^2 + 2 \left(\sum_{i=1}^{n} a_i^2 \right)^{1/2} \left(\sum_{i=1}^{n} b_i^2 \right)^{1/2} \quad \text{(by Cauchy-Schwarz)} \\
&= \|a\|^2 + \|b\|^2 + 2 \|a\| \|b\| = (\|a\| + \|b\|)^2 .
\end{aligned}$$

Extracting square roots gives $\|a + b\| \leq \|a\| + \|b\|$, as desired. ☐

Exercise 2. Any set X can be made into a "discrete" metric space by associating with X the metric d defined by

$$d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y. \end{cases}$$

Verify that the above function is indeed a metric on X, and that every subset of the resulting metric space is both open and closed.

Proof. That $d(x, y) = 0$ if and only if $x = y$, $d(x, y) = d(y, x)$, and that d is non-negative everywhere is obvious. Now, take $x, y, z \in X$. Then if $x = z$, we have $d(x, z) = 0$, so that $d(x, z) \leq d(x, y) + d(y, z)$ follows immediately. If $x \neq z$, then $d(x, z) = 1$. Now, in the sum $d(x, y) + d(y, z)$, since $x \neq z$, at least one of $x \neq y$ or $y \neq z$ must hold, so the sum equals 1 and the triangle inequality holds.

Now, let $Y \subseteq X$. Take any point $y \in Y$, and consider the ball $B(y, 1/2)$, which by definition of d, is a singleton set: $B(y, 1/2) = \{y\} \subseteq Y$. Thus, Y is an open set. This implies that every subset of X is open, so the complement of every set is also open; i.e., every subset is both open and closed. ☐

Exercise 3. Suppose that $d : X \times X \to \mathbb{R}$ is a metric function.

(a) Is $\rho : X \times X \to \mathbb{R}$ by $(x, y) \mapsto d^2(x, y)$ also a metric function?

Solution. No, this is not the case. Let $d(x, y)$ be the standard metric on $\mathbb{R}$, given by $d(x, y) = |x - y|$. Now

$$d^2(0, 2) = |0 - 2|^2 = 4 \not\leq d^2(0, 1) + d^2(1, 2) = 1 + 1 = 2,$$

so the triangle inequality fails. •

(b) Is $\rho : X \times X \to \mathbb{R}$ by $(x, y) \mapsto \sqrt{d(x, y)}$ also a metric function?

Solution. Yes. The basic properties obviously hold, so for brevity, we check the triangle inequality; i.e., we check that $\sqrt{d(x, z)} \leq \sqrt{d(x, y)} + \sqrt{d(y, z)}$ for all $x, y, z \in X$. Let $\alpha := d(x, z), \beta := d(x, y)$, and $\gamma := d(y, z)$. By the triangle inequality applied to $d(x, z)$, we see $\alpha \leq \beta + \gamma$. We note the identity

$$\left(\sqrt{\beta} + \sqrt{\gamma}\right)^2 = \beta + \gamma + 2\sqrt{\beta\gamma},$$

so that $\alpha \leq \beta + \gamma \leq \beta + \gamma + 2\sqrt{\beta\gamma} = \left(\sqrt{\beta} + \sqrt{\gamma}\right)^2$ as $2\sqrt{\beta\gamma} \geq 0$. Extracting square roots proves the triangle inequality for $\sqrt{d}$. •

(c) Is $\rho : X \times X \to \mathbb{R}$ by $(x, y) \mapsto 3d(x, y)$ also a metric function? **Yes**, by Proposition 13.3.

(d) Is $\rho : X \times X \to \mathbb{R}$ by $(x, y) \mapsto \sqrt{d(x, y)} + 2d(x, y)$ a metric function? **Yes**, by applying part (b) and Proposition 13.4.

Exercise 4. Let U, V, W be subsets of some larger set.

(a) Prove that $(U \cup V) \setminus W = (U \setminus W) \cup (V \setminus W)$.

Proof. This is a straightforward double subset inclusion exercise. Let $x \in (U \cup V) \setminus W$. Then $x \in U \cup V$ and $x \notin W$. Now, either $x \in U$ or $x \in V$. In either case, we know $x \in W$, so that $x \in U \setminus W$ or $x \in V \setminus W$, so that it follows $x \in (U \setminus W) \cup (V \setminus W)$.

Conversely, suppose $x \in (U \setminus W) \cup (V \setminus W)$, i.e., $x \in U \setminus W$ or $x \in V \setminus W$. In the first case, we notice that $U \setminus W \subseteq (U \cup V) \setminus W$, so that $x \in (U \cup V) \setminus W$. The second case is similar, which completes the proof. □

(b) Prove that $(U \cap V) \setminus W = (U \setminus W) \cap (V \setminus W)$.

Proof. Let $x \in (U \cap V) \setminus W$. Then $x \in U$, $x \in V$, and $x \notin W$. This means $x \in U \setminus W$ and $x \in V \setminus W$, so the inclusion $(U \cap V) \setminus W \subseteq (U \setminus W) \cap (V \setminus W)$ follows.

Conversely, suppose $x \in (U \setminus W) \cap (V \setminus W)$. Then $x \in U, x \notin W, x \in V$, and $x \notin W$. The inclusion is obvious. □

(c) Is it true that $U \setminus (V \setminus W) = (U \setminus V) \setminus W$?

Solution. No. Take $U = \{1, 2\}$, $V = \{1\}$, and $W = \{2\}$. Then $V \setminus W = \{1\}$, so $U \setminus (V \setminus W) = \{2\}$, yet $U \setminus V = \{2\} = W$, so $(U \setminus V) \setminus W = W \setminus W = \varnothing \neq \{2\}$. •

(d) Let $\{E_\alpha\}_{\alpha\in A}$ be a family of subsets of a set X. Prove that

$$X \setminus \bigcap_{\alpha\in A} E_\alpha = \bigcup_{\alpha\in A} (X \setminus E_\alpha) \text{ and } X \setminus \bigcup_{\alpha\in A} E_\alpha = \bigcap_{\alpha\in A} (X \setminus E_\alpha).$$

Proof. For the first statement, let $x \in X \setminus \bigcap_{\alpha\in A} E_\alpha$. Then $x \in X$, and $x \notin \bigcap_{\alpha\in A} E_\alpha$. By definition of the arbitrary intersection, this means that there exists some $\alpha_0 \in A$ such that $X \notin E_{\alpha_0}$, so that $x \in X \setminus E_{\alpha_0}$. Now, note

$$X \setminus E_{\alpha_0} \subseteq \bigcup_{\alpha\in A} (X \setminus E_\alpha),$$

so this direction of the equality is established. Now, let $y \in \bigcup_{\alpha\in A}(X \setminus E_\alpha)$. Then there exists $\alpha_0 \in A$ such that $y \in X \setminus E_{\alpha_0}$, i.e., $y \in X$ but $y \notin E_{\alpha_0}$. By definition of the arbitrary intersection, this implies $y \notin \bigcap_{\alpha\in A} E_\alpha$, so that $y \in X \setminus \bigcap_{\alpha\in A} E_\alpha$.

The second statement is very similar, so we omit the proof for brevity. Note that we do not need the assumption that X is a metric space, just a set. □

Exercise 5. Let (X, d) be a metric space and $E \subseteq X$ be a subset. The boundary ∂E of a set E is defined to be $\partial E := \overline{E} \cap \overline{X \setminus E}$.

(a) Show that for any set E, the boundary ∂E is a closed set.

Proof. The sets $\overline{E}$ and $\overline{X \setminus E}$ are closed, as they are closures of certain sets. Now, the intersection of two closed sets is closed, so that $\partial E = \overline{E} \cap \overline{X \setminus E}$ is closed. □

(b) Show that E is open if and only if $E \cap \partial E = \varnothing$.

Proof. Suppose E is open, so that $X \setminus E$ is closed. Hence, $X \setminus E = \overline{X \setminus E}$, but now $\partial E \subseteq X \setminus E$, which implies $E \cap \partial E = \varnothing$.

Conversely, suppose $E \cap \partial E = \varnothing$. Unpacking this, we have $E \cap (\overline{E} \cap \overline{X \setminus E}) = \varnothing$, so noting $E \subseteq \overline{E}$,

$$E \cap \left(\overline{E} \cap \overline{X \setminus E}\right) = E \cap \overline{X \setminus E} = \varnothing.$$

Hence, no point in $E = X \setminus (X \setminus E)$ is adherent to any point in $X \setminus E$, so all of the adherent points must be inside $X \setminus E$; i.e., $X \setminus E \supseteq \overline{X \setminus E}$, so $X \setminus E$ is closed. This implies E is open. □

(c) Show that E is closed if and only if $\partial E \subseteq E$.

Proof. Suppose E is closed. It suffices to show $\partial E \cap E = \partial E$. We know that $\partial E \cap E = E \cap \overline{X \setminus E}$. Since E is closed, $E = \overline{E}$ so $\partial E \cap E = E \cap \overline{X \setminus E} = \overline{E} \cap \overline{X \setminus E} = \partial E$.

Conversely, suppose $\partial E \subseteq E$. Notice

$$\partial E = \overline{E} \cap \overline{X \setminus E} = \overline{X \setminus (X \setminus E)} \cap \overline{X \setminus E} = \partial(X \setminus E).$$

It follows that $\partial(X \setminus E) \subseteq E$, so that $\partial(X \setminus E) \cap (X \setminus E) = \varnothing$; i.e., $X \setminus E$ is open by part (b). Hence E is closed. □

202 Homework 2

Exercise 1. Prove that the set of irrational numbers is dense in $\mathbb{R}$.

Proof. Take $x \in \mathbb{R}$. We find a sequence of irrational numbers that converges to x. If x is irrational, then take the sequence $y_n := x + \frac{1}{n}$. Now, by closure of $\mathbb{Q}$ under addition, $y_n \in \mathbb{R} \setminus \mathbb{Q}$ for all $n \in \mathbb{Z}^+$, and clearly $(y_n) \to x$. If x is rational, then take the sequence $y'_n := x + \frac{\pi}{n}$. Again, by closure of $\mathbb{Q}$ under addition, $y'_n \in \mathbb{R} \setminus \mathbb{Q}$ for all $n \in \mathbb{Z}^+$, and once again by construction $(y_n) \to x$. Since for each real number x, we found a sequence of irrational numbers converging to x, we see that $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$. ☐

Exercise 2. Let (X, d) be a metric space, and let $Y \subseteq X$ be a dense subset such that every Cauchy sequence in Y converges in X. Prove that X is complete.

Proof. We must show that every Cauchy sequence in X converges in X, so let $(x_n) \subseteq X$ be a Cauchy sequence. Now, for each $n \in \mathbb{Z}^+$, pick some $y_n \in Y \cap B(x_n, 1/n)$; such a y_n exists because Y is dense in X. We show that (y_n) is a Cauchy sequence; to see this, fix $\varepsilon > 0$, so that because (x_n) is Cauchy, we have $d(x_m, x_n) < \varepsilon$ for all $m \geq n \geq N$, for some N, so pick N such that $N \geq 1/\varepsilon$. From this choice, we force $d(y_n, x_n) < 1/n < 1/N < \varepsilon$ and similarly for $d(y_m, x_m)$, so by the triangle inequality, we have

$$d(y_n, y_m) = d(y_n, x_n) + d(x_n, x_m) + d(x_m, y_m) < \varepsilon + \varepsilon + \varepsilon = 3\varepsilon,$$

which could be made arbitrarily small. Hence, Y is a Cauchy sequence, which by hypothesis, converges to some $x_0 \in X$. We now claim $(x_n) \to x_0$, but now

$$d(x_n, x_0) < d(x_n, y_n) + d(y_n, x_0) \to 0$$

by the triangle inequality. Hence, $(x_n) \to x_0$, so it follows that X is complete. ☐

Exercise 3. Let Y be a subset of a metric space X. Is it true that $\overline{\operatorname{int}(Y)} = \operatorname{int}(\overline{Y})$?

Solution. Let $X = \mathbb{R}$, and take $Y = [0,1] \cap \mathbb{Q}$. Then Y has no interior as $\mathbb{Q}$ has no interior, so $\overline{\operatorname{int}(Y)} = \overline{\varnothing} = \varnothing$, but $\overline{Y} = [0,1]$, so $\operatorname{int}(\overline{Y}) = (0,1) \neq \varnothing$, so $\overline{\operatorname{int}(Y)} \neq \operatorname{int}(\overline{Y})$ in general. •

Exercise 4. Let Y be a subset of a metric space X.

(a) Prove that $\operatorname{int}(Y) = X \setminus \overline{X \setminus Y}$.

Proof. Suppose that $y \in \operatorname{int}(Y)$. Then there exists some $r > 0$ such that $B(y,r) \subseteq Y$. We must show that y is not adherent to $X \setminus Y$, but this is clear as the open ball $B(y,r)$ does not contain any points from $X \setminus Y$ by construction, so that $y \in X \setminus \overline{X \setminus Y}$.

Conversely, suppose that $x \in X \setminus \overline{X \setminus Y}$. Then $x \notin \overline{X \setminus Y}$, so that x is not adherent to $X \setminus Y$. This implies that there exists an open ball $B(x,r) \subseteq X$ such that $B(x,r) \cap (X \setminus Y) = \varnothing$, i.e., $B(x,r) \subseteq Y$. It follows that x is interior to Y, completing the proof. ☐

(b) Prove that $\overline{Y} = X \setminus \operatorname{int}(X \setminus Y)$.

Proof. Suppose that $y \in \overline{Y}$, so that y is adherent to Y. This implies that for every $r > 0$, we have $B(y,r) \cap Y \neq \varnothing$, so that $B(y,r) \not\subseteq X \setminus Y$ for all $r > 0$. It follows that y cannot be interior to $X \setminus Y$, so that $y \in X \setminus \operatorname{int}(X \setminus Y)$.

Conversely, suppose $x \in X \setminus \operatorname{int}(X \setminus Y)$, so that x is not interior to Y. This means that for every $r > 0$, we have $B(x,r) \not\subseteq X \setminus Y$, so that there exists some $y_r \in B(x,r) \cap Y$ for all $r > 0$. It follows that x is adherent to Y, so $x \in \overline{Y}$. ☐

Exercise 5. A point $x \in S$ is an *isolated point* of S if there exists $r > 0$ such that $B(x, r) \cap S = \{x\}$.

(a) Show that the set of isolated points of a countable complete metric space X forms a dense subset of X.

Proof. Let X be a countable complete metric space, and let S be its set of isolated points. If X consists of one element, then we are done, so suppose otherwise. Take any $x \in X$ that is not an isolated point, and consider the set $S_x = X \setminus \{x\}$. Notice that because $\{x\}$ is closed, it follows that S_x is open, and we notice that S_x is dense: let $y \in X$; if $y \in S_x$, this is obvious, so let $y = x$. Because x is not an isolated point, notice that $B(x, 1/n) \cap S_x \neq \{x\}$ for all $n \in \mathbb{Z}^+$. Since S_x contains all elements of X except x, take $x_n \in B(x, 1/n) \cap S_x$ for every $n \in \mathbb{Z}^+$. Clearly, $(x_n) \to x$, so S_x is open and dense. Now, notice that the countable intersection

$$I := \bigcap_{x \in X}^{x \text{ not isolated}} S_x$$

is exactly the set of isolated points in X, but since X is complete, the Baire Category Theorem applies to X, and thus I is dense. This completes the proof. $\square$

(b) Regard the rational numbers $\mathbb{Q}$ as a subspace of $\mathbb{R}$. Does the metric space $\mathbb{Q}$ contain any isolated points? Why does this not contradict part (a)?

Solution. $\mathbb{Q}$ has no isolated points, but this is not a contradiction as while $\mathbb{Q}$ is countable, it is not complete. •

Exercise 6. Determine the interior, the closure, the limit points, and the isolated points of each of the following subsets of $\mathbb{R}$.

(a) $Y_1 := [0, 1)$.

Solution. We have $\text{int}(Y_1) = \boxed{(0, 1)}$, as every point except 0 is interior: 0 is not interior as the open ball $B(0, r)$ for any $r > 0$ contains a negative number, which is not included in $[0, 1)$. The closure is $\overline{Y_1} = \boxed{[0, 1]}$, which is also its set of limit points. Finally, as this is a real interval, there are no isolated points. •

(b) $Y_2 := \mathbb{Q}$.

Solution. Because the irrational numbers are dense in $\mathbb{R}$, we have $\text{int}(\mathbb{Q}) = \boxed{\varnothing}$, as every open ball $B(q, r)$ for any $q \in \mathbb{Q}$ and $r > 0$ contains an irrational number. We know that $\overline{\mathbb{Q}} = \boxed{\mathbb{R}}$, which is also its set of limit points. Finally, as $\mathbb{Q}$ is dense in $\mathbb{R}$, there are no isolated points. •

(c) $Y_3 := \{m + n\pi : m, n \in \mathbb{Z}^+\}$.

Solution. Notice that Y_3 has countably many elements, so that $\text{int}(Y_3) = \boxed{\varnothing}$. Now, we claim $\overline{Y_3} = \boxed{Y_3}$; this is because for any $y \in Y_3$, we have that $[0, y] \cap Y_3$ is finite, so it follows that Y_3 has no limit points and the claim follows. Also, because of the same reason, every point of Y_3 is isolated. •

(d) $Y_4 := \left\{\frac{1}{m} + \frac{1}{n} : m, n \in \mathbb{Z}^+\right\}$.

Solution. We observe $Y_4 \subseteq \mathbb{Q}$, so $\operatorname{int}(Y_4) = \boxed{\varnothing}$. The set of limit points of Y_4 can be found by considering an arbitrary Cauchy sequence $1/(x_k) + 1/(y_k) \subseteq Y_4$, which gives 0 if $(x_k), (y_k) \to \infty$, and the points $1/n$ whenever exactly one of $(x_k), (y_k)$ go to infinity. In the other case, we get the points $1/m + 1/n$, but these are not limit points, and are in fact isolated. Thus, the closure $\overline{Y_4}$ consists of the points $0, 1/n$, and everything in Y_4, and $\{0\} \cup \{1/n : n \in \mathbb{Z}^+\}$ is its set of limit points. •

Exercise 7. Determine the interior, the closure, the limit points, and the isolated points of each of the following subsets of $\mathbb{R}^2$.

(a) $A_1 := \left\{(x, y) : 0 < x^2 + y^2 < 1\right\}$.

Solution. Notice that A_1 is an open set, so $\operatorname{int}(A_1) = \boxed{A_1}$. The adherent points of A_1 are A_1 itself, 0, and the points on the unit circle $x^2 + y^2 = 1$, so we have $\overline{A_1} = \left\{(x, y) : x^2 + y^2 \leq 1\right\} = \boxed{\overline{B((0,0),1)}}$, the closed unit disk. This is also the set of limit points of A_1. Finally, A_1 has no isolated points. •

(b) $A_2 := \{(x, 0) : x \in \mathbb{R}\}$.

Solution. Note that for any $(x, 0) \in A_2$ and for all $\varepsilon > 0$, the point $(x, \varepsilon/2)$ lies in the ball $B(x, \varepsilon)$, so $\operatorname{int}(A_2) = \boxed{\varnothing}$. It is easy to observe that $\mathbb{R}^2 \setminus A_2$ is open, so that A_2 is closed and $\overline{A_2} = \boxed{A_2}$, which is also the set of its limit points. Finally, A_2 has no isolated points. •

(c) $A_3 := \{(m, n) : m, n \in \mathbb{Z}^+\}$.

Solution. We have $\operatorname{int}(A_3) = \boxed{\varnothing}$ as given any $(m, n) \in A_3$ and for all $r > 0$, the point $(m + 1/2, n) \notin A_3$ lies in $B((m, n), r)$ for all $r > 1/2$, and the point $(m + r, n) \notin A_3$ lies in $B((m, n), r)$ for all $r < 1/2$. Now, we observe $\overline{A_3} = \boxed{A_3}$; however, this is not the set of its limit points, as the set of limit points of A_3 is empty. Every point of A_3 is isolated. •

(d) $A_4 := \left\{\left(\frac{1}{m}, \frac{1}{n}\right) : m, n \in \mathbb{Z}^+\right\}$.

Solution. We have $\operatorname{int}(A_4) = \boxed{\varnothing}$, as $(\mathbb{R} \setminus \mathbb{Q})^2$ is dense in $\mathbb{R}^2$. Now, the set of limit points of A_4 consists of A_4 itself, the points of the form $(0, \frac{1}{n})$, of the form $(\frac{1}{m}, 0)$, and $(0, 0)$. Thus, $\overline{A_4}$ consists of all of these limit points. Now, every point of the form $p = (\frac{1}{m}, \frac{1}{n})$ is isolated, which can be seen by considering the ball $B(p, \frac{1}{m+1})$. •

203 Homework 3

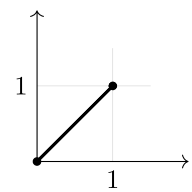
Exercise 1. Let $a = (0, 0)$ and $b = (1, 1)$. For each of the following metrics in $\mathbb{R}^2$, draw the set

$$S := \left\{x \in \mathbb{R}^2 : d(x, a) + d(x, b) = d(a, b)\right\}.$$

We remark that this problem has a nice geometric interpretation: when $d(x, a) + d(x, b) = d(a, b)$ on the standard metric in $\mathbb{R}^2$, we say that x is on the straight line between a and b. Hence, this problem is really asking us to find all analogues to "straight lines" in $(\mathbb{R}^2, d)$.

(a) $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$.

Solution. By the triangle inequality, we know that $d(x, a) + d(x, b) = d(a, x) + d(x, b) \geq d(a, b)$ in the usual case, so this problem is asking for places where equality holds. This choice of d is the standard metric on $\mathbb{R}^2$, and we know equality holds precisely when a, x, b are co-linear, and x is "in between" x and b. Hence, the set described is the line segment $x_2 = x_1$, where $x_1 \in [0, 1]$:



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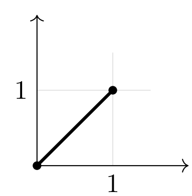
(b) $d(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$.

Solution. We have $d(a, b) = 1$, and clearly $d(x, a) = \max\{|x_1|, |x_2|\}$. If $\vec{x} = (x, y) \in S$, it follows that $x, y \in [0, 1]$, which can be easily verified. Furthermore, it is easy to show $x = y$: if $x < y$, then $d(a, x) = \max\{x, y\} = y$, so that $d(b, x) = 1 - x$, implying $d(a, x) + d(x, b) = y + 1 - x > 1$. A symmetric argument shows $x \not> y$, so $x = y$. Hence, the "straight line" S is just the usual straight line:



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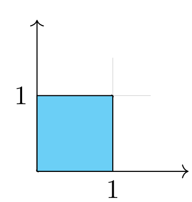
(c) $d(x, y) = |x_1 - y_1| + |x_2 - y_2|$.

Solution. Here, we have $d(a, b) = 2$, and again, it is easy to see that if $\vec{x} = (x, y) \in S$ we must have $(x, y) \in [0, 1]^2$ (otherwise, we "overshoot" the given total distance of 2). In this case, we have $d(\vec{x}, a) = |x| + |y| = x + y$ and $d(\vec{x}, b) = (1 - x) + (1 - y)$. Now for $\vec{x} \in [0, 1]^2$, observe

$$d(\vec{x}, a) + d(\vec{x}, b) = x + y + 1 - x + 1 - y = 2,$$

so $S = [0, 1]^2$, the entire unit square, i.e., the "straight line" is a square:



This makes sense — splitting up the path of travel from $(0,0)$ to $(1,1)$ via a point in the square does not change the distance traveled with respect to d. $\bullet$

Exercise 2. Show that $d((x_1,x_2,x_3),(y_1,y_2,y_3)) := |x_1-y_1| + \max\{|x_2-y_2|, |x_3-y_3|\}$ is a metric in $\mathbb{R}^3$.

Proof. Clearly, d is nonnegative and symmetric. Now, if $d((x_1,x_2,x_3),(y_1,y_2,y_3)) = 0$, we must have $x_1 = y_1$ and $\max\{|x_2-y_2|, |x_3-y_3|\} = 0$, i.e., $x_2 = y_2$ and $x_3 = y_3$, so the two points are the same. Now, let $(x_1,x_2,x_3),(y_1,y_2,y_3),(z_1,z_2,z_3) \in \mathbb{R}^3$. Then

$$\begin{aligned} d(x,z) &= |x_1-z_1| + \max\{|x_2-z_2|, |x_3-z_3|\} \\ &= |x_1-y_1+y_1-z_1| + \max\{|x_2-y_2+y_2-z_2|, |x_3-y_3+y_3-z_3|\} \\ &\le |x_1-y_1| + |y_1-z_1| + \max\{|x_2-y_2|+|y_2-z_2|, |x_3-y_3|+|y_3-z_3|\} \\ &\le |x_1-y_1| + |y_1-z_1| + \max\{|x_2-y_2|, |x_3-y_3|\} + \max\{|y_2-z_2|, |y_3-z_3|\} \\ &= d(x,y) + d(y,z), \end{aligned}$$

which is exactly what we needed to show. Hence, d is a metric on $\mathbb{R}^3$. $\square$

Exercise 3. Which of the following subsets in $\mathbb{R}$ are compact?

(a) $A = \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{Z}^+\right\}$.

Solution. We claim that A is compact. To prove this, let $\mathcal{U} = \{U_\alpha\}$ be an open cover for A, for $\alpha \in I$, an indexing set. Let $U_1 \in \mathcal{U}$ contain the element 0. Because U_1 is open, find a ball $B(0,r) \subseteq U_1$. However, this ball $B(0,r)$ contains all but finitely many points in the set $\left\{\frac{1}{n} : n \in \mathbb{Z}^+\right\}$; more specifically, it contains all such points $1/n$ where $n > 1/r$. Hence, to the finitely many points $a_2, a_3, \dots, a_k$ in $A \setminus B(0,r)$, take some $U_i \in \mathcal{U}$ such that $a_i \in U_i$. Then $\{U_1, U_2, \dots, U_k\}$ is a finite subcover of $\mathcal{U}$, so A is compact. $\bullet$

(b) $B = \bigcup_{n \in \mathbb{Z}^+} [2n, 2n+1]$.

Solution. Define the family of sets

$$\mathcal{U} := \left\{ \left(2n - \frac{1}{2}, 2n + \frac{3}{2}\right) : n \in \mathbb{Z}^+ \right\}.$$

By construction, $\mathcal{U}$ is an open cover for B. However, $\mathcal{U}$ has no finite subcover, as any finite subset of $\mathcal{U}$ is bounded above, while B is not bounded above. Hence, B is **not** compact. $\bullet$

(c) $C = \bigcup_{n \in \mathbb{Z}^+} \left[\frac{1}{2n+1}, \frac{1}{2n}\right]$.

Solution. Define the sequence (x_n) by $x_n := 1/(n+1)$ for all $n \in \mathbb{Z}^+$. Clearly, $x_n \in C$ for all $n \in \mathbb{Z}^+$, but (x_n) cannot have a convergent subsequence (in C) as $(x_n) \to 0 \in \mathbb{R} \setminus C$. [It is a fact in analysis that if a sequence converges to a limit x, then every subsequence converges to the same limit x.] Hence, C is **not** compact. $\bullet$

Exercise 4. Recall that the discrete metric spaces is any nonempty set X associated with the metric d which is 0 whenever $x = y$ and 1 otherwise. Prove that a discrete metric space is compact if and only if it is finite.

Proof. ($\Longleftarrow$): Suppose X is finite. First, we know that in a discrete metric, every set is open and closed (which we proved in Homework 1). Now, enumerate the elements of X: $X = \{x_1, \ldots, x_n\}$. Take any cover $\mathcal{U}$ of X, and for each x_i, take some $U_i \in \mathcal{U}$ containing x_i. Then $\{U_1, \ldots, U_n\}$ is evidently a finite open cover for X, so X is complete.

($\Longrightarrow$): We argue by contraposition; suppose X is infinite. Since X is discrete, every set is open and closed, so the singleton $\{x\}$ is open. Take the open cover $\mathcal{U}$ to consist of all the singleton sets in X. This is an infinite cover, but since X is an infinite set, every finite subset of $\mathcal{U}$ fails to cover some elements of X, so X is not compact. □

Exercise 5. Give an example of a totally bounded metric space which is not compact.

Solution. Take the space $\boxed{X = [0,1] \cap \mathbb{Q}}$, regarded as a subspace of $\mathbb{R}$. For each $\varepsilon > 0$, X can be covered by at most $2/\varepsilon + 1$ open balls of radius ε, which is finite. However, X is not compact: take any sequence (x_n) of rationals in X converging to $\sqrt{2}/2 \notin X$. Now, (x_n) does not have any convergent subsequences in X; therefore, X cannot be compact. •

Exercise 6. Give an example of a complete metric space which is not compact.

Solution. Take $\boxed{X = \mathbb{R}}$. We have proven already that $\mathbb{R}$ is complete, yet it is clearly not compact: take the open cover $\mathcal{U}$ to consist of the intervals of the form $(n, n+2)$ for some $n \in \mathbb{Z}$. A union of elements of a finite subset of $\mathcal{U}$ is bounded, and thus $\mathcal{U}$ has no finite subcovers and thus $\mathbb{R}$ is not compact. •

Exercise 7. Show directly that a compact metric space is totally bounded.

Proof. Let X be compact, and take $\varepsilon > 0$. Define the cover $\mathcal{U} := \{B(x, \varepsilon) : x \in X\}$. Clearly, $\mathcal{U}$ is a cover for X. But because X is compact, $\mathcal{U}$ has a finite subcover. But this just means that X can be covered by finitely many balls $B(x_i, \varepsilon)$, which means that X is totally bounded. □

204 Homework 4

Exercise 1. Let $X = \{a, b\}$ be a set with two elements. List all topologies of X.

Solution. The power set of X is $2^X = \{\varnothing, \{a\}, \{b\}, X\}$, consisting of four elements. Certainly, $\boxed{2^X}$ is the *discrete topology* on X, and $\{\varnothing, X\}$ is the *trivial topology* on X. Also, the three-element subsets $\boxed{\{\varnothing, \{a\}, X\}}$ and $\boxed{\{\varnothing, \{b\}, X\}}$ are topologies as well — a quick verification shows that they are closed under arbitrary unions and finite intersections. No other subset of 2^X is a topology, as those contain less than 2 elements, the minimum required to be a topology. •

Exercise 2. Let X be a set and $\tau := \{U \subseteq X : X \setminus U \text{ is finite}\} \cup \{\varnothing\}$. Show that τ is a topology, called the *cofinite topology* on X.

Proof. Clearly, $X, \varnothing \in \tau$. Hence, we just need to demonstrate that X is closed under arbitrary unions and finite intersections. Take $\{U_\alpha\}_{\alpha \in A}$ to be a family of subsets in X. then $X \setminus U_\alpha$ is finite for each $\alpha \in A$, or U_α is the empty set. If each U_α is the empty set, then clearly $\bigcup_{\alpha \in A} U_\alpha = \varnothing \in \tau$. Otherwise, note the set identity $X \setminus \bigcup_{\alpha \in A} U_\alpha = \bigcap_{\alpha \in A}(X \setminus U_\alpha)$, and since at least one of the U_α is nonempty, $X \setminus U_\alpha$ is finite, so that the indexed intersection is finite. This implies $\bigcup_{\alpha \in A} U_\alpha \in \tau$, so τ is closed under arbitrary unions.

Now, take $U_1, \dots, U_n \in \tau$. Then each U_i is either empty, or $X \setminus U_i$ is finite, for all $i \leq n$. If at least one of the U_i is empty, we just have $U_1 \cap U_2 \cap \dots \cap U_n = \varnothing \in \tau$, so there is nothing to do, so we suppose that all of the U_i are nonempty. In this case, $X \setminus U_i$ is finite for all $i \leq n$. Now, note the set identity $X \setminus \bigcap_{i=1}^n U_i = \bigcup_{i=1}^n (X \setminus U_i)$, and the union on the right is a union of finite sets, so it is finite. Hence $\bigcap_{i=1}^n U_i \in \tau$, so τ is closed under finite intersections. This proves that τ is a well-defined topology on X. $\square$

Exercise 3. Let τ be the cofinite topology on $\mathbb{Z}$. Show that the sequence $(1, 2, 3, \dots)$ converges in $(\mathbb{Z}, \tau)$ to each point in $\mathbb{Z}$. Describe the convergent sequences in $(\mathbb{Z}, \tau)$.

Proof. Let $x_n = n$, as given above, and pick $a \in \mathbb{Z}$ to be arbitrary. We must show that for all open neighborhoods of a, there exists some $N \in \mathbb{Z}^+$ such that whenever $n > N$, we have $x_n \in U_n$. Hence, take an open neighborhood U_a of a. Certainly, U_a is nonempty as $a \in U_a$, so by definition of τ, the set $\mathbb{Z} \setminus U_a$ is finite. In particular, it contains finitely many positive integers. If $\mathbb{Z} \setminus U_a$ contains no positive integers, then the entire sequence (x_n) sits in U_a, so pick $N = 0$. Otherwise, set $N := \max(\mathbb{Z} \setminus U_a) > 0$, so that whenever $n > N$, we have $x_n \in U_a$. Since the neighborhood U_a was arbitrary, and $a \in \mathbb{Z}$ was arbitrary, we have shown that (x_n) converges to every integer.

Now, for a general sequence (x_n), we see that x_n converges in $\mathbb{Z}$ (say to some $x \in \mathbb{Z}$) if and only if for every open neighborhood U_x of x, there exists $N \in \mathbb{Z}^+$ such that whenever $n > N$, we have $x_n \in U_x$. In particular, we may take the neighborhoods $U_x = \mathbb{Z} \setminus \{b\}$ for all $b \neq x$. The sequence x_n must sit inside the set U_x eventually, for all choices of b other than x; hence, every entry of the sequence that is not x must appear a finite number of times.

Conversely, if x_n is a sequence such that there is at most one element that occurs infinitely often, then consider two cases. First, if any entry of x_n only occurs finitely often, then x_n converges to any integer $x \in \mathbb{Z}$. Otherwise, x_n converges to the element that occurs infinitely often. Hence, the convergent sequences in $(\mathbb{Z}, \tau)$ are exactly the ones in which at most one entry occurs infinitely often. $\square$

Exercise 4. This exercise shows that the cofinite topology is not metrizable.

(a) Show that if X is a metrizable topological space and if $p \neq q$ are distinct points in X, then there exist open sets U, V such that $p \in U$, $q \in V$, and $U \cap V = \varnothing$.

Proof. Take the metric d associated with the topology on X, and set $r := d(p, q) > 0$. Now, take $U := B(p, \frac{r}{2})$ and $V := B(q, \frac{r}{2})$; these are open balls, so they are open in X, and certainly $p \in U$ and $q \in V$. Also, $U \cap V = \varnothing$; if for contradiction we had some $x \in U \cap V$, then $d(x, p) < \frac{r}{2}$ and $d(x, q) < \frac{r}{2}$, but now $d(p, q) \leq d(p, x) + d(x, q) < \frac{r}{2} + \frac{r}{2} = r$, a contradiction. Hence, $U \cap V = \varnothing$ as claimed. $\square$

(b) Let X be an infinite set and τ be the cofinite topology on X. Prove that the property described in part (a) does not hold for open sets in X, and hence that (X, τ) is not metrizable.

Proof. Take distinct points $p, q \in X$, and suppose for contradiction that there exist open sets U, V such that $p \in U$, $q \in V$, and $U \cap V = \varnothing$. Clearly, U, V are nonempty, so that $X \setminus U$ and $X \setminus V$ are finite. Furthermore, $X \setminus (U \cap V) = X$, which is infinite, but now we note the set identity $X = X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V)$, which implies that X is finite, a contradiction. Hence, the property of (a) does not hold, and (X, τ) is not metrizable. $\square$

Exercise 5. Let S be a subset of a topological space X. Show that the closure $\overline{S}$ is the disjoint union of the set of limit points of S and the isolated points of S.

Proof. It suffices to show that if $x \in \overline{S}$ is not a limit point, then s is isolated, and that no limit point is an isolated point. First, suppose $x \in \overline{S}$ is not a limit point. Then there exists a neighborhood N of x such that $N \cap (S \setminus \{x\})$ is empty. Certainly, $x \in N$, so this implies $N \cap S = \{x\}$, so that we see $x \in S$ and x is an isolated point of S. Hence, every point in $\overline{S}$ is either a limit point, or it is isolated, so that S is the union of the set of limit points of S and the isolated point of S.

Now, we show that if $x \in \overline{S}$ is a limit point of S, then it is not isolated, thus proving that the union is indeed disjoint. By definition, if N is any neighborhood of x, the intersection $N \cap (S \setminus \{x\})$ is nonempty. Hence, there exists some $y \in N \cap S$ such that $y \neq x$, so that we see $\{x, y\} \subseteq N \cap S$ for any neighborhood N of x. In particular, it is never true that $N \cap S = \{x\}$, so x is not isolated and we are done. □

Exercise 6. Let X be a set and τ be a family of subsets U of X such that $X \setminus U$ is at most *countable*, together with the empty set $\varnothing$.

(a) Prove that τ is a topology for X.

Proof. The proof is basically the same as that for the cofinite topology. First, clearly $X, \varnothing \in \tau$. We must show that τ is closed under arbitrary unions and finite intersections.

First, let $\{U_\alpha\}_{\alpha \in A}$ be a family of subsets in τ. Then either U_α is empty, or $X \setminus U_\alpha$ is at most countable for all $\alpha \in A$. If each U_α is empty, then there is nothing to do, and otherwise, we note the set identity $X \setminus \bigcup_{\alpha \in A} U_\alpha = \bigcap_{\alpha \in A}(X \setminus U_\alpha)$, and the intersection on the right is an intersection of arbitrarily many sets of at most countable cardinality, and is thus at most countable. This shows $\bigcup_{\alpha \in A} U_\alpha \in \tau$.

Now, let $U_1, U_2, \ldots, U_n \in \tau$. Then either U_i is empty, or $X \setminus U_i$ is at most countable for all $i \leq n$. If any of the U_i are empty, then $U_1 \cap U_2 \cap \cdots \cap U_n = \varnothing \in \tau$, so suppose otherwise. The set identity $X \setminus \bigcap_{i=1}^n U_i = \bigcup_{i=1}^n (X \setminus U_i)$ shows that $X \setminus \bigcap_{i=1}^n U_i$ is a finite union of at most countable sets, so it is at most countable. Thus, it follows that $\bigcap_{i=1}^n U_i \in \tau$, and we are done. □

(b) Describe the convergent sequences with respect to this topology.

Solution. A sequence (x_n) converges to some $x \in X$ if for all open neighborhoods U of x, there exists an $N \in \mathbb{Z}^+$ such that whenever $n > N$, we have $x_n \in U$. We claim that a sequence converges in this topology if and only if **it is eventually constant**. To see this, let (x_n) be a sequence that is eventually constant; say that it is eventually x_0 for some $x_0 \in X$. Then $x_n \to x_0$, as obviously every neighborhood of x_0 contains x_0, so it contains the "tail" of this sequence (take N to be the index at which $x_n = x_0$ for all $n > N$).

Conversely, we show that all other sorts of sequences are not convergent, and we do this by cases, depending on the number of elements that occur infinitely often:

Case I: Every entry of (x_n) occurs finitely many times. Since (x_n) is an infinite sequence, call the entries of (x_n) by $\{a_1, a_2, \ldots\}$, with each a_i having finite occurrence in (x_n). Take $a \in X$, and consider the neighborhood $U := X \setminus \{a_1, a_2, \ldots\}$. [If a is among one of the a_i, then without loss of generality take $a = a_1$ and consider $U' := X \setminus \{a_2, a_3 \ldots\}$.] Clearly, $X \setminus U$ is countable, but now none of the sequence lies in the neighborhood U, so (x_n) cannot converge.

Case II: At least two distinct entries of (x_n) occur infinitely often. Call two of these entries $a \neq b$, and take $c \in X$. If $c \neq a, b$, consider the neighborhood $U := X \setminus \{x_n\}$. Otherwise, without loss of generality say $c = a$. Then take the neighborhood $U := X \setminus A$, where A consists of all entries in (x_n) that are not a. Then clearly $b \notin U$, but (x_n) contains infinitely many b's, so $(x_n) \not\to a$.

Case III: Only one entry of (x_n) occurs infinitely often, but the sequence is not eventually constant. Call the entry that occurs infinitely often, $x \in X$. We see in this case that there must be infinitely many $a_1, a_2, a_3, \ldots$ that occur in (x_n), all with finite number of occurrences. Now, take $c \in X$. Again, if $c \notin (x_n)$, then take $U := X \setminus \{x_n\}$. If $c = a_i$, then take $U = X \setminus \{x_n : n > N\}$, where N is the last occurrence of a_i. Finally, if $c = x$, take $U = X \setminus \{a_1, a_2, a_3, \ldots\}$. Then none of the a_i sit in U, so the sequence cannot sit in U. Hence, (x_n) cannot converge. •

(c) Prove that if X is uncountable, then there is a subset $S \subseteq X$ whose closure contains points that are not limits of convergent sequences in S.

Proof. Let $x \in X$, and consider the set $S := X \setminus \{x\}$. Clearly, S is uncountable, and S is not closed, as the complement $\{x\}$ is clearly not open. Hence, $S \subsetneq \overline{S}$, so we are forced to have $\overline{S} = X$. But this is a problem, because the convergent sequences in S are the sequences which are eventually constant, so it suffices to show that such sequences cannot converge to $x \notin S$. To do this, take any sequence $(s_n) \subseteq S$, and consider the neighborhood of x given by $U := X \setminus \{s_n\}$. Clearly, because $x \notin S$, the entire sequence does not lie in S, so $(s_n) \not\to x$. Hence, x is not a limit of convergent sequences in S, completing the proof. □

Exercise 7. Let S be a subset of a topological space X. Show that a sequence $(x_i) \subseteq S$ converges to $x_0 \in S$ in the relative topology if and only if, considered as a sequence in X, the sequence (x_i) still converges to x_0.

Proof. ($\Longleftarrow$): Suppose $(x_i) \to x_0$, when viewed as a sequence in X. Then for every open neighborhood U of x_0, there exists an $N \in \mathbb{Z}^+$ such that whenever $n > N$, we have $x_n \in U$. By assumption $(x_i) \in S$, so clearly for such a U with its associated N, we have $x_n \in U \cap S$, which are exactly the relatively open neighborhoods of x_0 in S. This implies that $(x_i) \to x_0$ in the relative topology of S.

($\Longrightarrow$): Suppose $(x_i) \to x_0$ in the relative topology of S. We must show that $(x_i) \to x_0$ in X, so let U be an arbitrary *open* (not relatively open) neighborhood of x_0. Then $U \cap S$ is a relatively open neighborhood of x_0, so by assumption, take some $N \in \mathbb{Z}^+$ such that $x_n \in U \cap S \subseteq U$ for all $n > N$. But now U is arbitrary, so $(x_i) \to x_0$ in X. □

205 Homework 5

Exercise 1. Show that a function $f : X \to Y$ is continuous if and only if $f^{-1}(E)$ is a closed subset of X for every closed subset E of Y.

Proof. Let $E \subseteq Y$ be closed. Then $Y \setminus E$ is open, so because $f : X \to Y$ is continuous, $f^{-1}(Y \setminus E)$ is open. But then $f^{-1}(Y \setminus E) = X \setminus f^{-1}(E)$, which we can verify: $a \in f^{-1}(Y \setminus E)$ if and only if $f(a) \in Y \setminus E$, if and only if $f(a) \notin E$, if and only if $a \notin f^{-1}(E)$, if and only if $a \in X \setminus f^{-1}(E)$. This implies that $X \setminus f^{-1}(E)$ is open, so $f^{-1}(E)$ is closed. □

Exercise 2. Prove that all open intervals in $\mathbb{R}$ are homeomorphic.

Proof. It suffices to show that every open interval is homeomorphic to the unit open interval $(0,1)$, so that composing and inverting homeomorphisms (which gives homeomorphisms) completes the proof. Throwing out the degenerate open interval $\varnothing$ (from which the proposition is trivial), we consider the following cases:

Case I: Let (a,b) be an open interval, where $-\infty < a < b < \infty$. Then the linear function $f : (0,1) \to (a,b)$ by $x \mapsto \frac{1}{b-a}x + a$ is obviously a continuous bijection with continuous (linear) inverse.

Case II: Let (a,∞) be an open interval, where $a \in \mathbb{R}$. Then the map $f : (0,1) \to (a,\infty)$ by $x \mapsto a + \tan\left(\frac{\pi}{2}x\right)$ with continuous inverse $f^{-1}(x) = \frac{2}{\pi}\arctan(x-a)$.

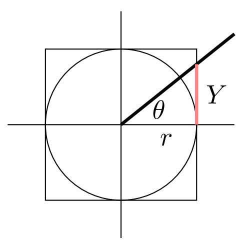
Case III: Let $(-\infty,b)$ be an open interval, where $b \in \mathbb{R}$. Similarly to the above, the map $f : (0,1) \to (-\infty,b)$ by $x \mapsto b - \tan\left(\frac{\pi}{2}x\right)$ is a homeomorphism.

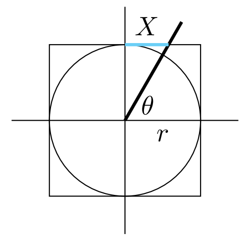
Case IV: $(-\infty,\infty) = \mathbb{R}$ is an open interval. The function $f(x) = \arctan(x)$ gives a homeomorphism $f : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$, from which composing with the inverse in Case I gives a homeomorphism $\mathbb{R} \to (0,1)$. $\square$

Exercise 3. Show that the following pairs of sets in $\mathbb{R}^2$ are homeomorphic.

(a) The unit ball $A = \left\{(x,y) : x^2 + y^2 < 1\right\}$ and the open square $B = \{(x,y) : |x|, |y| < 1\}$.

Proof. Write each point in A in the form $(r\cos\theta, r\sin\theta)$ for $0 \le r < 1$ and $\theta \in [0, 2\pi)$. The idea is that we take each "slice" of the ball $x^2 + y^2 < r^2$, where $0 \le r < 1$, and map it onto the square $\max\{|x|,|y|\} < r$. To do this, consider the figures below:





In the first figure on the previous page, we have $0 \le \theta \le \pi/4$, and in the second, we have $\pi/4 \le \theta \le \pi/2$ — we think the bolded line in each as specifying the map, sending the point of intersection with the circle to the point of intersection with the square. Right-triangle trigonometry immediately gives $Y = r\tan\theta$ and $X = r\cot\theta$.

Modifying this idea (i.e., taking appropriate negative signs) for other angles if necessary, we have the homeomorphism $f : A \to B$ by, where we fix $0 \le r < 1$ and $\theta \in [0, 2\pi)$:

$$f(r\cos\theta, r\sin\theta) = \begin{cases} (r, r\tan\theta) & \theta \in [0, \frac{\pi}{4}] \cup [\frac{7\pi}{4}, 2\pi) \\ (r\cot\theta, r) & \theta \in (\frac{\pi}{4}, \frac{3\pi}{4}], \theta \neq \frac{\pi}{2} \\ (-r, r\tan\theta) & \theta \in (\frac{3\pi}{4}, \frac{5\pi}{4}] \\ (r\cot\theta, -r) & \theta \in (\frac{5\pi}{4}, \frac{7\pi}{4}], \theta \neq \frac{3\pi}{2} \\ (r\cos\theta, r\sin\theta) & \theta = \frac{\pi}{2}, \frac{3\pi}{2}. \end{cases}$$

It is tedious (though doable) to check that this is a homeomorphism. $\square$

(b) The punctured plane $A = \mathbb{R}^2 \setminus \{(0,0)\}$ and the exterior of the closed unit ball $B = \mathbb{R}^2 \setminus \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$.

Proof. To make our lives easier, we consider A and B as subsets of $\mathbb{C}$. Then $A = \mathbb{C}^\times$ and $B = \{z \in \mathbb{C} : |z| > 1\}$. Then, the map $f : A \to B$ by $re^{i\theta} \mapsto (r+1)e^{i\theta}$, for $r > 0$ and $\theta \in [0, 2\pi)$ is well-defined, as $r + 1 > 1$ for all $r \in \mathbb{C}^\times$, and it is obviously continuous. Furthermore, its inverse $f^{-1} : re^{i\theta} \mapsto (r-1)e^{i\theta}$ is also continuous, so f is a homeomorphism. $\square$

(c) The open half plane $A = \{(x, y) : x > 0\}$ and the unit ball $B = \{(x, y) : x^2 + y^2 < 1\}$.

Proof. Again, to make our lives easier, regard A and B as subsets of $\mathbb{C}$. Consider the map $f : A \to H$, where $H := \{z : \mathrm{Im}(z) > 0\}$ is the open upper half plane, by $re^{i\theta} \mapsto re^{i(\theta+\pi/2)}$. This is evidently a homeomorphism from the right half plane to the upper half plane.

Now, we use the *Cayley transform* $g : H \to B$ by $z \mapsto \frac{i-z}{i+z}$. We claim that this is a homeomorphism. First, this function is well-defined, as we observe that if $z = x + yi$ with $y > 0$, we see $i - z = i - (x + yi) = -x + (1 - y)i$ and $i + z = x + (1 + y)i$. Since y is positive, we have $|1 - y| < |1 + y|$, so that $|i - z| < |i + z|$, and thus $|g(z)| < 1$. This means that g actually does map into the unit ball B. Now, g is continuous, as rational functions are continuous on their domains.

Finally, we compute the inverse of g as follows:

$$\begin{aligned} w = \frac{i-z}{i+z} \implies w(i+z) = (i-z) &\implies wi + wz = i - z \\ &\implies wi - i = -z - wz \\ &\implies i(w-1) = -z(1+w), \end{aligned}$$

from which it follows that g^{-1} is given by $w \mapsto i\frac{1-w}{1+w}$. Since this is a rational function, g^{-1} is continuous, which proves that g is also a homeomorphism. Finally, $g \circ f : A \to B$ is a homeomorphism, so we are done. $\square$

Exercise 4. Let X be a topological space with the discrete topology. Find a base $\mathcal{B}$ of open sets for X such that $\mathcal{B}$ is included in any other base of open sets for X.

Solution. In a discrete topological space, every subset of X is open. In particular, this implies the singletons are open. A base of a topology is a collection of open sets such that every open set in X is a union of sets in the base. In the discrete topology, this means we must write every singleton as a union of sets in $\mathcal{B}$. Hence, take $\mathcal{B}$ to be the set of all singletons in X — clearly, this forms a base, and by our discussion above, every other base must contain $\mathcal{B}$. •

Exercise 5. Show that a sequence in a Hausdorff space cannot converge to more than one point.

Proof. Let X be a Hausdorff space and $(x_n) \subseteq X$ be a convergent sequence. Suppose for contradiction that $(x_n) \to x$ and $(x_n) \to y$, where $x \neq y$. Since x and y are distinct, there must contain disjoint open neighborhoods U of x and V of y, but by convergence of (x_n), there must exist $N_1 \in \mathbb{Z}^+$ such that whenever $n > N_1$, we have $x_n \in U$, and there must exist $N_2 \in \mathbb{Z}^+$ such that whenever $n > N_2$, we have $x_n \in V$. Take $N := \max\{N_1, N_2\}$, so that whenever $n > N$, we have $x_n \in U \cap V = \varnothing$, a contradiction. $\square$

Exercise 6. Recall that the cofinite topology τ of a set X is the family of subsets $U \subseteq X$ such that $X \setminus U$ is finite, together with the empty set $\varnothing$. Let (X, τ) be a topological space with the cofinite topology.

(a) Show that (X, τ) is separable.

Proof. When X is at most countable, this is obvious, so take X to be an uncountable set. We claim that every countable subset $Y := \{y_1, y_2, y_3, \ldots\} \subseteq X$ is dense. Pick $x \in X$. If (for contradiction) x is not adherent to Y, then there exists an open neighborhood U of x such that $U \cap Y = \varnothing$. However, since U is open and nonempty (as $x \in U$), we see that $X \setminus U$ is finite, so $(X \setminus U) \cap Y$ is finite. But this implies $U \cap Y$ is infinite, a contradiction as the empty set is not infinite. Hence, x is adherent to Y, so that $\overline{Y} = X$ and thus Y is dense, so X is separable. □

(b) When is (X, τ) second-countable?

Solution. We claim that (X, τ) is second-countable exactly when X is at most countable. When X is finite, this is obvious, so assume X is countable. Then the set $\mathcal{S}$ of finite subsets of X is countable, but by definition of τ, the set $\mathcal{B} := \{X \setminus S : S \in \mathcal{S}\}$ contains precisely the open sets of X, excluding the empty set. There is a bijection $\mathcal{S} \to \mathcal{B}$ by sending each finite set in $\mathcal{S}$ to its infinite complement, so $\mathcal{B}$ is a countable base.

Now, suppose X is uncountable, and suppose for contradiction that X is second-countable, so that there exists a countable base $\mathcal{B} = \{U_1, U_2, U_3, \ldots\}$. Then $X \setminus U_i$ is finite for all $i \in \mathbb{Z}^+$. Consider the union $A := \bigcup_{i=1}^{\infty}(X \setminus U_i)$, which is an at most countable set; i.e., $A \subsetneq B$. Now, take $x \in X \setminus A$ and consider $U := X \setminus \{x\}$. Then U is open, as it has finite complement, but arbitrary unions of the U_i can only have complements that are subsets of A, which does not contain A. This is a contradiction, so the countable base $\mathcal{B}$ cannot exist. •

(c) When is (X, τ) a T_1 space?

Solution. We claim that (X, τ) is always a T_1 space. To see this, take any point $x \in X$. Then the set $X \setminus \{x\}$ is open, because its complement $\{x\}$ is obviously finite. This implies that the singletons in (X, τ) are closed, which is equivalent to being a T_1 space. •

(d) When is (X, τ) a Hausdorff space?

Solution. We claim that (X, τ) is Hausdorff exactly when X is finite. To see this, notice that when X is finite, then (X, τ) coincides with the discrete topology on X, which is metrizable and hence Hausdorff. Now, suppose that X is infinite. Then there exists an infinite sequence (x_n), where each x_i is distinct from each x_j, where $i \neq j$. Arguing by Exercise 5, we show that (x_n) does not converge to a unique point. First, we show that $(x_n) \to x_1$ by noting that every neighborhood U of x_1 has finite complement, which implies that because (x_n) is an infinite sequence, there exists sufficiently large N such that whenever $n > N$, we have $x_n \in U$. However, this argument does not use anything particular about x_1, so in fact $(x_n) \to x_i$ for all $i \in \mathbb{Z}^+$. This belies the uniqueness of limits, so that X is not Hausdorff. •

206 Homework 6

Exercise 1. Let X, Y be topological spaces and $f : X \to Y$ be continuous. Prove the following.

(a) The preimage of any closed set of Y is closed in X.

Proof. Let $V \subseteq Y$ be closed. Then $Y \setminus V$ is open, so that by continuity, $f^{-1}(Y \setminus V)$ is open. We claim that $f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$; to see this, pick $x \in f^{-1}(Y \setminus V)$, so that $f(x) \in Y \setminus V$, which holds if and only if $f(x) \notin V$, i.e., $x \notin f^{-1}(V)$, i.e., $x \in X \setminus f^{-1}(V)$. Hence $X \setminus f^{-1}(V)$ is open, so $f^{-1}(V)$ is closed. $\square$

(b) For each $x \in X$ and every neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subseteq V$.

Proof. Take a neighborhood V of $f(x)$. Then by continuity, $f^{-1}(V)$ is open, so take $U := f^{-1}(V)$. Certainly, $x \in U$. Now we show the inclusion $f(f^{-1}(V)) \subseteq V$; pick $b \in f(f^{-1}(V))$. Then there exists $a \in f^{-1}(V)$ such that $f(a) = b$, but because $a \in f^{-1}(V)$, this forces $b \in V$ so the inclusion holds and the proof is complete. $\square$

(c) For each subset $A \subseteq X$, we have $f(\overline{A}) \subseteq \overline{f(A)}$.

Proof. First, notice that $\overline{f(A)} \subseteq Y$ is a closed set, and since f is continuous, its preimage $f^{-1}(\overline{f(A)})$ is closed. We claim that $A \subseteq f^{-1}(\overline{f(A)})$; to see this, note $A \subseteq f^{-1}(f(A))$ as if $a \in A$, we see $f(a) \in f(A)$ so that $a \in f^{-1}(f(A))$, and now $f^{-1}(f(A)) \subseteq f^{-1}(\overline{f(A)})$. But $f^{-1}(\overline{f(A)})$ is closed, so it must contain the closure $\overline{A}$, and taking images yields $f(\overline{A}) = \overline{f(A)}$ as claimed. $\square$

(b) For each subset $B \subseteq Y$, we have $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$.

Proof. First, notice that $\overline{B}$ is closed, so $f^{-1}(\overline{B})$ is closed. Notice $f^{-1}(B) \subseteq f^{-1}(\overline{B})$, but the latter set is closed, so it must contain the closure $\overline{f^{-1}(B)}$, proving the claim. $\square$

Exercise 2. Show that any space with the cofinite topology is compact.

Proof. Let X be endowed with the cofinite topology, and take an open cover $\{U_\alpha\}_{\alpha \in A}$. It is safe to assume that each U_α is nonempty, so that $X \setminus U_\alpha$ is finite. Now, take U_1 as any member of this open cover; notice that $X \setminus U_1$ is finite. If $X \setminus U_1 = \varnothing$, then $U_1 = X$ and we are done. If not, because $\{U_\alpha\}$ is a cover, there must exist some U_2 (another member of the open cover) such that $X \setminus (U_1 \cup U_2) \subsetneq X \setminus U_1$. If $X \setminus (U_1 \cup U_2) = \varnothing$, we are done, and if not, continue inductively. This gives us a strictly decreasing chain of sets

$$X \setminus U_1 \supsetneq X \setminus (U_1 \cup U_2) \supsetneq X \setminus (U_1 \cup U_2 \cup U_3) \supsetneq X \setminus (U_1 \cup U_2 \cup U_3 \cup U_4) \supsetneq \cdots,$$

which must terminate as $X \setminus U_1$ is a finite set. Hence, at the end of this process, we have constructed a finite subcover $\{U_1, U_2, \ldots, U_n\}$ of X, which completes the proof. $\square$

Exercise 3. Prove that if every point of X has an open neighborhood U such that $\overline{U}$ is a compact Hausdorff space, then X is Hausdorff.

Proof. This is Example 107.2. $\square$

Exercise 4. Let X be a locally compact Hausdorff space that is not compact. Prove that X is dense in the one-point compactification $Y := X \cup \{\infty\}$.

Proof. Clearly, X is dense in X, so we must show that ∞ is a limit point to X. Let U be an open set containing ∞. Then by definition of the one-point compactification, $X \setminus U$ is compact. In particular, $X \setminus U \neq \varnothing$ because $\infty \notin \varnothing$, and so $\varnothing$ is a set of type (1) in the definition of the one-point compactification. Since $X \setminus U \subseteq X$ and nonempty, it contains a point in x. In particular, $x \neq \infty$, so ∞ is a limit point to X and hence X is dense in Y. $\square$

Exercise 5. Prove that the one-point compactification of any open ball in $\mathbb{R}^n$ is homeomorphic to the n-sphere S^n.

Proof. Every open ball is homeomorphic to $\mathbb{R}^n$, by a similar argument to what we did in Homework 5. It remains to show that the one-point compactification is homeomorphic to S^n, to do so, we use the *stereographic projection*, where the details may be found here: `https://en.wikipedia.org/wiki/Stereographic_projection`. Now, by construction, every point on S^n is identified with a point of $\mathbb{R}^n$, except for one point, so compactifying, we see that the given homeomorphism holds. $\square$

Exercise 6. Let X be a topological space, and $A \subseteq X$ be a connected set. Prove that $\overline{A}$ is also connected.

Proof. Let S be a clopen set in $\overline{A}$, and assume S is nonempty. We claim $S = \overline{A}$. Since A is connected, $S \cap A = A$ as S is a nonempty relatively clopen set in A. Now, a closed set must contain its closure, so $S = \overline{A}$. $\square$

Exercise 7. Suppose that X is a topological space such that A, B are connected subsets of X such that $A \cap \overline{B}$ is nonempty. Show that $A \cup B$ is connected.

Proof. We proceed by contraposition: suppose A, B are connected subsets of X such that $A \cup B$ is not connected. Then there exist disjoint clopen sets, both nonempty such that $X \cup Y = A \cup B$. Now, it follows that either $X \cap A$ is nonempty or $X \cap B$ is nonempty, so suppose without loss of generality that $X \cap A$ is nonempty. Then $X \cap A$ is relatively clopen in A, so by connectedness of A, we have $X \cap A = A$, so $A \subseteq X$. Now, by a similar argument, we know hat $Y \cap A$ is nonempty or $Y \cap B$ is nonempty; if $Y \cap A$ were nonempty, we would have $Y \cap A = A$, which contradicts the disjointness of X and Y. Hence, $Y \cap B$ is nonempty, so by connectedness of B, we have $B \subseteq Y$. But now disjointness forces $A = X$ and $B = Y$, so now by clopenness $A = \overline{A}$ and $B = \overline{B}$, so $A \cap \overline{B} = X \cap Y = \varnothing$ and we are done. $\square$