Topos

Def: $\mathcal{X}$ set , $\mathcal{T} \subseteq \mathcal{P}(\mathcal{X})$

$\mathcal{T}$; Topology on $\mathcal{X}$;

$\emptyset, \mathcal{X} \in \mathcal{T}$

$\{\mathcal{U}_i\}_{i \in J \subseteq \mathbb{N}} \subseteq \mathcal{T} \Longrightarrow \bigcup_{i \in J} \mathcal{U}_i \in \mathcal{T}$

$\{\mathcal{U}_i\}_{i \in \dot{J} \subset \mathbb{N}} \subseteq \mathcal{T} \Longrightarrow \bigcap_{i \in \dot{J}} \mathcal{U}_i \in \mathcal{T}$

Def: $\mathcal{Y}$; $\mathcal{T}$-open $:= \mathcal{Y} \in \mathcal{T}$

Def: C; $\mathcal{T}$-closed $:= \mathcal{X} \setminus C \in \mathcal{T}$

Def: J; $\mathcal{T}$-clopen $:= J$; $\mathcal{T}$-open $\wedge$ $\mathcal{T}$-closed

Ex: $(X, \mathcal{T}_{ind})$; $\mathcal{T}_{ind} = \{\emptyset, X\}$

$\emptyset, X \in \mathcal{T}_{ind}$

$\emptyset \cup X = X \in \mathcal{T}_{ind}$

$\emptyset \cap X = \emptyset \in \mathcal{T}_{ind}$

hence $\mathcal{T}_{ind}$ is Topology on X

Ex; $(X, \mathcal{T}_{dis})$; $\mathcal{T}_{dis} = \mathcal{P}(X)$

$\emptyset, X \in \mathcal{P}(X)$

$\bigcup_J U_i \in \mathcal{P}(X)$

$\bigcap_J U_i \in \mathcal{P}(X)$

hence $\mathcal{T}_{dis}$ is Topology on X

Ex; $(X, \mathcal{T}_{x_0})$; $\mathcal{T}_{x_0} = \{\emptyset, X, \{x_0 \in X\}\}$

if $|X| = 2$, $\mathcal{T}_{x_0}$ is Sierpinski Topology

Generally $\mathcal{T}_{x_0}$: particular point Topology

Ex: $(X, \mathcal{T})$: $\mathcal{T} = \{ \mathcal{U}_{\subseteq X} \mid |X \setminus \mathcal{U}| \in \mathbb{N} \} \cup \{ \emptyset \}$

$\emptyset \in \mathcal{T}$, since $|X \setminus X| = 0$, $X \in \mathcal{T}$

$\{\mathcal{U}_i\} \in \mathcal{T} \Rightarrow X \setminus \mathcal{U}_i$ fin $\Rightarrow \bigcup_i X \setminus \mathcal{U}_i$, $\bigcap_i X \setminus \mathcal{U}_i$ fin

$\Rightarrow X \setminus \bigcup_i \mathcal{U}_i$, $X \setminus \bigcap_i \mathcal{U}_i$ fin $\Rightarrow \bigcup_i \mathcal{U}_i, \bigcap_i \mathcal{U}_i \in \mathcal{T}$

$\mathcal{T}$: Zariski Topology

Ex:

$$X = \{x_0, x_1, x_2, x_3\}$$

$$\mathcal{T} = \{\emptyset, X, \{x_0\}, \{x_1\}, \{x_0, x_1\}, \{x_1, x_2, x_3\}\}$$

$$\mathcal{T}_c = \{X, \emptyset, \{x_1, x_2, x_3\}, \{x_0, x_2, x_3\}, \{x_2, x_3\}, \{x_0\}\}$$

$\{x_1\}, \{x_0, x_1\}$: op, $\neg$cl

$\{x_2, x_3\}$: cl, $\neg$op

$\{x_0\}, \{x_1, x_2, x_3\}$: op, cl ; clopen

$\{x_2\}, \{x_3\}, \{x_0, x_3\}$: $\neg$op, $\neg$cl

$\mathcal{T}^* = \{\emptyset, X, \{x_2\}, \{x_2, x_3\}\}$, $\mathcal{T}_c^* = \{X, \emptyset, \{x_0, x_1, x_3\}, \{x_0, x_1\}\}$

$\{x_0, x_1\}$: cl, $\neg$op ; $\{x_2\}, \{x_2, x_3\}$: op, $\neg$cl

$\mathcal{T}^{\varphi} = \{\emptyset, X, \{x_0\}, \{x_1\}\}$, since $\{x_0\} \cup \{x_1\} = \{x_0, x_1\} \notin \mathcal{T}^{\varphi}$

$\neg\mathcal{T}_0$:

$\mathcal{T}^{\sigma} = \{\emptyset, X, \{x_0, x_1\}, \{x_0, x_2\}\}$; $\{x_0, x_1\} \cap \{x_0, x_2\} = \{x_0\} \notin \mathcal{T}^{\sigma}$

Theorem; $\mathcal{T}_j$ $(j \in J)$ Topology on X

$\Rightarrow$ $\mathcal{T}_\sigma$ $(:= \bigcap_j \mathcal{T}_j)$ Topology on X.

Proof;

$$\emptyset, X \in \mathcal{T}_\sigma \Leftrightarrow \emptyset, X \in \mathcal{T}_j \ \forall j_{\in J}$$

$$\Leftrightarrow \mathcal{T}_j \text{ Top on } X \ \forall j_{\in J}$$

$$\{U_i\}_{i \in I} \subseteq \mathcal{T}_\sigma \Leftrightarrow \{U_i\}_{i \in I} \subseteq \mathcal{T}_j \ \forall j_{\in J}$$

$$\Rightarrow \bigcap_I U_i, \bigcup_I U_i \in \mathcal{T}_j \ \forall j_{\in J}$$

$$\Rightarrow \bigcap_I U_i, \bigcup_I U_i \in \mathcal{T}_\sigma .$$

- Ex: $\mathcal{T} = \{\emptyset, X, \{x_0\}, \{x_0, x_1\}\}$
  $\mathcal{T}_* = \{\emptyset, X, \{x_0\}, \{x_0, x_4\}\}$ Tops on X

  $\mathcal{T} \cap \mathcal{T}_* = \{\emptyset, X, \{x_0\}\}$ is also Topology on X

  Ex: $\mathcal{T}_i = \{\emptyset, X, \{x_i\}, \{x_i, x_{i+j}, x_{i-j}\}, \{x_i, x_{i-j}\}, \{x_i, x_{i+j}\}\}$

- $\mathcal{T}_n \cap \mathcal{T}_{n+j} = \{\emptyset, X, \{x_n, x_{n+j}\}\}$; Top on X

  Cou-ex: $\mathcal{T} = \{\emptyset, X, \{x_0\}\}$
  $\mathcal{T}_* = \{\emptyset, X, \{x_1\}\}$

  $\mathcal{T} \cup \mathcal{T}_* = \{\emptyset, X, \{x_0\}, \{x_1\}\}$, since ;

- $\{x_0\}_{\in \mathcal{T} \cup \mathcal{T}_*} \cup \{x_1\}_{\in \mathcal{T} \cup \mathcal{T}_*} = \{x_0, x_1\} \notin \mathcal{T} \cup \mathcal{T}_*$,

  $\mathcal{T} \cup \mathcal{T}_*$: $\neg$Top on X

  Cou-ex: $\mathcal{T}_i = \{\emptyset, X, \{x_i\}\}$,

  $\bigcup_i \mathcal{T}_i = \{\emptyset, X, \{x_0\}, \dots, \{x_n\}\}$, $\bigcup_i \mathcal{T}_i$: $\neg$Top on X ;

  $\{x_n\}_{\in \bigcup \mathcal{T}_i} \cup \{x_\sigma\}_{\in \bigcup \mathcal{T}_i} = \{x_n, x_\sigma\} \notin \bigcup_i \mathcal{T}_i$

- Ex: $\mathcal{T}_1 = \{\emptyset, X, \{x_0\}, \{x_0, x_2\}\}$, $\mathcal{T}_2 = \{\emptyset, X, \{x_2\}, \{x_0, x_2\}\}$, $\bigcup_{j=1,2} \mathcal{T}_j$: Top

Theorem: $\mathcal{T}$: Top on $X \Rightarrow$

$\emptyset, X \in \mathcal{T}_c$, $\bigcup_{J \subset \mathbb{N}} U_j , \bigcap_{J \subseteq \mathbb{N}} U_j \in \mathcal{T}_c$: $\{U_j\} \subseteq \mathcal{T}_c$

Proof: $\emptyset, X \in \mathcal{T}_c \Leftrightarrow X, \emptyset \in \mathcal{T}$,

$\{U_j\}_{j \in J \subset \mathbb{N}} \subseteq \mathcal{T}_c \Leftrightarrow \{X \setminus U_j\}_{j \in J \subset \mathbb{N}} \subseteq \mathcal{T}$

$\Rightarrow \bigcap_{J \subset \mathbb{N}} \{X \setminus U_j\}_{(= X \setminus \bigcup U_j)} \in \mathcal{T}$

$\Rightarrow X \setminus \bigcup_{J \subset \mathbb{N}} U_j \in \mathcal{T} \Leftrightarrow \bigcup_{J \subset \mathbb{N}} U_j \in \mathcal{T}_c$

$\{U_j\}_{j \in J \subseteq \mathbb{N}} \subseteq \mathcal{T}_c \Leftrightarrow \{X \setminus U_j\}_{j \in J \subseteq \mathbb{N}} \subseteq \mathcal{T}$,

$\Rightarrow \bigcup_{J \subseteq \mathbb{N}} \{X \setminus U_j\}_{(= X \setminus \bigcap U_j)} \in \mathcal{T}$

$\Rightarrow X \setminus \bigcap_{J \subseteq \mathbb{N}} U_j \in \mathcal{T} \Leftrightarrow \bigcup_{J \subseteq \mathbb{N}} U_j \in \mathcal{T}_c$.

Def: $\mathcal{T} \subseteq \mathcal{T}_*$: $\mathcal{T}$ weaker $\mathcal{T}_*$: $\mathcal{T}_*$ stronger $\mathcal{T}$

Lemma: $\mathcal{T}_{dis}$ strongest , $\mathcal{T}_{ind}$ weakest

Proof: $\mathcal{T}_\alpha$: Top ,

$\mathcal{T}_{ind} = \{\emptyset, X\} \subseteq \mathcal{T}_\alpha \subseteq \mathcal{P}(X) = \mathcal{T}_{dis}$.

Def: X_τ Topological Space

$\underset{\subseteq X}{Y} : G_\delta := \exists \{U_i \in \tau\}_{J \subseteq \mathbb{N}} ; \bigcap_J U_i = Y$

$K : F_\sigma := \exists \{U_i \in \tau_c\}_{J \subseteq \mathbb{N}} ; \bigcup_J U_i = K$

Ex: $\tau = \{\emptyset, X, \{x_i\}, \{x_j\}, \{x_i, x_j\}\}$

$\tau_c = \{X, \emptyset, X \setminus \{x_i\}, X \setminus \{x_j\}, X \setminus \{x_i, x_j\}\}$

$\{x_i\} : G_\delta \; ; \; \{x_i\} = \{x_i, x_j\}_{\in \tau} \cap \{x_i\}_{\in \tau}$

$X \setminus \{x_i, x_j\} : F_\sigma \; ; \; X \setminus \{x_i, x_j\} = X \setminus \{x_i\} \cup X \setminus \{x_j\}$

Ex: $X = \{x_i, x_j, x_v\}$

$\tau = \{\emptyset, X, \{x_i\}, \{x_j\}, \{x_v\}, \{x_i, x_j\}, \{x_i, x_v\}, \{x_j, x_v\}\}$

$\tau_c = \{X, \emptyset, \{x_j, x_v\}, \{x_i, x_v\}, \{x_i, x_j\}, \{x_v\}, \{x_j\}, \{x_i\}\}$

$\forall J_{\in \tau}, J : \tau$-clopen

$\{x_i\} : G_\delta \; ; \; \{x_i\} = \{x_i, x_j\} \cap \{x_i, x_v\}$

$\{x_i, x_j\} : F_\sigma \; ; \; \{x_i, x_j\} = \{x_i\} \cup \{x_j\}$

Def: $\mathcal{Y} \subseteq \mathcal{X}$, $\mathcal{T}$ Top on $\mathcal{X}$

$\mathcal{T}_{\mathcal{Y}}$ Sub-Top of $\mathcal{T}$:= $\mathcal{T}_{\mathcal{Y}} = \{\mathcal{Y} \cap \mathcal{U} \mid \mathcal{U} \in \mathcal{T}\}$

Theorem; $\mathcal{T}_{\mathcal{Y}}$; Top on $\mathcal{Y}$.

Proof;

$$\emptyset \in \mathcal{T}_{\mathcal{Y}} ; \ \emptyset = \mathcal{Y} \cap \emptyset_{\in \mathcal{T}}$$

$$\mathcal{Y} \in \mathcal{T}_{\mathcal{Y}} ; \ \mathcal{Y} = \mathcal{Y} \cap \mathcal{X}_{\in \mathcal{T}}$$

$$\{\mathcal{Y} \cap \mathcal{U}_i \mid \mathcal{U}_i \in \mathcal{T}\}_i \subseteq \mathcal{T}_{\mathcal{Y}}$$

$$\Longrightarrow \bigcup_i \{\mathcal{Y} \cap \mathcal{U}_i\}_{(= \mathcal{Y} \cap (\bigcup_i \mathcal{U}_i)_{\in \mathcal{T}} \in \mathcal{T}_{\mathcal{Y}})} \in \mathcal{T}_{\mathcal{Y}}$$

$$\bigcap_i \{\mathcal{Y} \cap \mathcal{U}_i\}_{(= \mathcal{Y} \cap (\bigcap_i \mathcal{U}_i)_{\in \mathcal{T}} \in \mathcal{T}_{\mathcal{Y}})} \in \mathcal{T}_{\mathcal{Y}} .$$

Theorem: $\mathcal{U}$; $\mathcal{T}_{\mathcal{Y}}$-open $\wedge$ $\mathcal{Y}$; $\mathcal{T}_{\mathcal{X}}$-open $\Longrightarrow$ $\mathcal{U}$: $\mathcal{T}_{\mathcal{X}}$-open

Proof: $\mathcal{U} \in \mathcal{T}_{\mathcal{Y}} \Longrightarrow \exists \mathcal{V}_{\in \mathcal{T}_{\mathcal{X}}} \ \mathcal{U} = \mathcal{V} \cap \mathcal{Y}_{\in \mathcal{T}_{\mathcal{X}}}$

$$\Longrightarrow \mathcal{U} \in \mathcal{T}_{\mathcal{X}}$$

Lemma: $\{\mathcal{T}_j\}_j$: Topologies on $\mathcal{X}$, $\bigcap_j \mathcal{T}_{j\,c} = (\bigcap_j \mathcal{T}_j)_c$

Proof: $\mathcal{V} \in \bigcap_j \mathcal{T}_{j\,c} \Longleftrightarrow \forall_j \ \mathcal{V} \in \mathcal{T}_{j\,c} \Longleftrightarrow \forall_j \ \mathcal{V}^c \in \mathcal{T}_j$

$$\Longleftrightarrow \mathcal{V}^c \in \bigcap_j \mathcal{T}_j \Longleftrightarrow \underset{(=\{\mathcal{V}^{c}\}^{c})}{\mathcal{V}} \in (\bigcap_j \mathcal{T}_j)_c$$

Theorem: $\mathcal{Y} \subseteq \mathcal{X}_{\mathcal{T}}$

$$G \in \mathcal{T}_{\mathcal{Y}c} \iff \exists K_{\in \mathcal{T}_c}\ G = \mathcal{Y} \cap K .$$

Proof:

$(\Rightarrow)$: $G \in \mathcal{T}_{\mathcal{Y}c} \Rightarrow \mathcal{Y} \setminus G \in \mathcal{T}_{\mathcal{Y}}$

$\Rightarrow \exists V_{\in \mathcal{T}} : \mathcal{Y} \setminus G = \mathcal{Y} \cap V$,

$G = \mathcal{Y} \setminus (\mathcal{Y} \setminus G) = \mathcal{Y} \setminus (\mathcal{Y} \cap V) = \mathcal{Y} \cap (\mathcal{X} \setminus V)$,

$V \in \mathcal{T} \Rightarrow \mathcal{X} \setminus V \in \mathcal{T}_c$, $K := (\mathcal{X} \setminus V)_{\in \mathcal{T}_c}$,

$\Rightarrow \exists K_{\in \mathcal{T}_c}\ G = \mathcal{Y} \cap K$

$(\Leftarrow)$: $\exists K_{\in \mathcal{T}_c}\ G = \mathcal{Y} \cap K$,

$K \in \mathcal{T}_c \Rightarrow \mathcal{X} \setminus K \in \mathcal{T} \Rightarrow \mathcal{Y} \cap (\mathcal{X} \setminus K)_{(= \mathcal{Y} \setminus (\mathcal{Y} \cap K))} \in \mathcal{T}_{\mathcal{Y}}$

$\Rightarrow \mathcal{Y} \setminus (\mathcal{Y} \cap K)_{(\mathcal{Y} \setminus G)} \in \mathcal{T}_{\mathcal{Y}} \Rightarrow G \in \mathcal{T}_{\mathcal{Y}c}$.

Defs: $\mathcal{F} \subseteq \mathcal{X}_\sigma$

$$\mathcal{F}^\circ = \bigcup_{\mathcal{J} \subseteq \mathcal{F},\ \mathcal{J} \in \mathcal{T}} \mathcal{J} \quad , \quad \overline{\mathcal{F}} = \bigcap_{C \supseteq \mathcal{F},\ C \in \mathcal{T}_c} C$$

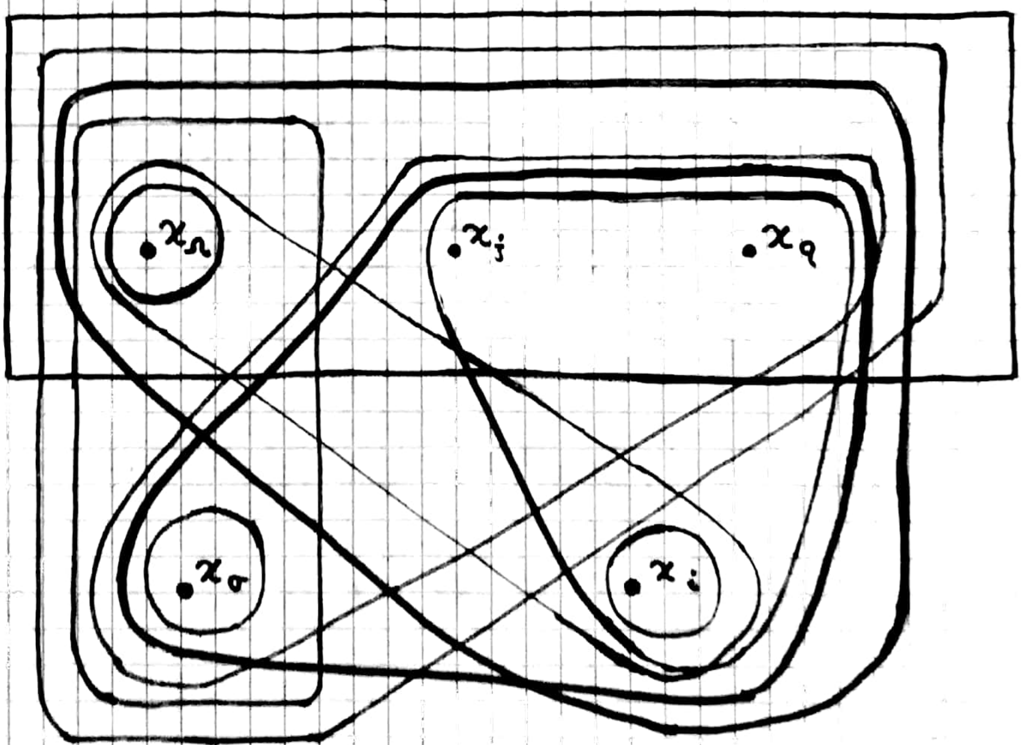
$$\partial \mathcal{F} = \overline{\mathcal{F}} \setminus \mathcal{F}^\circ \quad , \quad \mathcal{F}^e = (\mathcal{X} \setminus \mathcal{F})^\circ$$

Ex: $\mathcal{X} = \{x_i, x_j, x_r, x_\sigma, x_q\}$

$$\mathcal{T} = \{\emptyset, \mathcal{X}, \{x_r\}, \{x_\sigma\}, \{x_r, x_\sigma\}, \{x_j, x_\sigma, x_q\}, \{x_j, x_r, x_\sigma, x_q\}\}$$

$$\mathcal{T}_c = \{\mathcal{X}, \emptyset, \{x_i, x_j, x_\sigma, x_q\}, \{x_i, x_j, x_r, x_q\}, \{x_i, x_j, x_q\}, \{x_i, x_r\}, \{x_i\}\}$$

$$\mathcal{F} = \{x_j, x_r, x_q\}$$



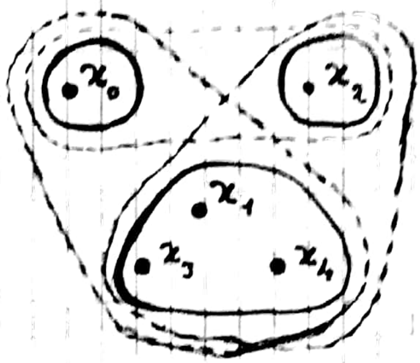
$$\mathcal{F}^\circ = \{x_r\} \ , \ \overline{\mathcal{F}} = \{x_i, x_j, x_r, x_q\} \ , \ \mathcal{F}^e = \{x_\sigma\}$$

$$\partial \mathcal{F} = \{x_i, x_j, x_q\}$$

Ex: $\mathcal{X}=\{x_0,x_1,x_2,x_3,x_4\}$

$$\mathcal{T}=\{\emptyset,\mathcal{X},\{x_0\},\{x_2\},\{x_1,x_3,x_4\}\}$$

$$\mathcal{T}_c=\{\mathcal{X},\emptyset,\{x_1,x_2,x_3,x_4\},\{x_0,x_1,x_3,x_4\},\{x_0,x_2\}\}$$



$\mathcal{F}=\{x_0,x_1,x_3\}$; $\mathcal{F}^\circ=\{x_0\}$, $\overline{\mathcal{F}}=\{x_0,x_1,x_3,x_4\}$

$\partial\mathcal{F}=\{x_1,x_3,x_4\}$, $\mathcal{F}^e=\{x_2\}$

$G=\{x_2,x_3,x_4\}$; $G^\circ=\{x_2\}$, $\overline{G}=\{x_1,x_2,x_3,x_4\}$

$\partial G=\{x_1,x_3,x_4\}$, $G^e=\{x_0\}$

$K=\{x_2,x_4\}$; $K^\circ=\{x_2\}$, $\overline{K}=\{x_1,x_2,x_3,x_4\}$

$\partial K=\{x_1,x_3,x_4\}$, $K^e=\{x_0\}$

$S=\{x_3,x_4\}$; $S^\circ=\emptyset$, $S^e=\{x_0,x_2\}$, $\partial S=\overline{S}\setminus\emptyset=\overline{S}$

$\overline{S}=\{x_0,x_1,x_3,x_4\}\cap\{x_1,x_2,x_3,x_4\}=\{x_1,x_3,x_4\}$

$\Omega=\{x_0,x_2\}$; $\Omega^\circ=\{x_0,x_2\}$, $\overline{\Omega}=\{x_0,x_2\}$, $\partial\Omega=\emptyset$

$\Omega^e=\{x_1,x_3,x_4\}$

Ex:

$$\mathcal{X} = \{x_0, x_1, x_2, x_3\}$$

$$\mathcal{F} = \{x_0, x_1\}$$

$$\mathcal{T}_i = \{\emptyset, \mathcal{X}, \{x_0\}, \{x_2\}, \{x_0, x_2\}\}$$

$$\mathcal{T}_{ic} = \{\mathcal{X}, \emptyset, \{x_1, x_2, x_3\}, \{x_0, x_1, x_3\}, \{x_1, x_3\}\}$$

$$\mathcal{F}^{\circ}_{(\mathcal{T}_i)} = \{x_0\}, \quad \overline{\mathcal{F}}_{(\mathcal{T}_i)} = \{x_0, x_1, x_3\},$$

$$\partial\mathcal{F}_{(\mathcal{T}_i)} = \{x_1, x_3\}, \quad \mathcal{F}^{e}_{(\mathcal{T}_i)} = \{x_2\}$$

$$\mathcal{T}_\sigma = \{\emptyset, \mathcal{X}, \{x_1\}, \{x_3\}, \{x_1, x_3\}\}$$

$$\mathcal{T}_{\sigma c} = \{\mathcal{X}, \emptyset, \{x_0, x_2, x_3\}, \{x_0, x_1, x_2\}, \{x_0, x_2\}\}$$

$$\mathcal{F}^{\circ}_{(\mathcal{T}_\sigma)} = \{x_1\}, \quad \overline{\mathcal{F}}_{(\mathcal{T}_\sigma)} = \{x_0, x_1, x_2\}$$

$$\partial\mathcal{F}_{(\mathcal{T}_\sigma)} = \{x_0, x_2\}, \quad \mathcal{F}^{e}_{(\mathcal{T}_\sigma)} = \{x_3\}$$

Defn: $\mathcal{F}$ regular open $:= \mathcal{F} = (\overline{\mathcal{F}})^{\circ}$

$\mathcal{F}$ regular closed $:= \mathcal{F} = (\overline{\mathcal{F}^{\circ}})$

Theorem; $\partial_{(\mathcal{Y})} \mathcal{F}_{\subseteq \mathcal{X}_{\mathcal{Y}}} = \emptyset \iff \mathcal{F}$: $\mathcal{Y}$-clop

Proof;

$$(\Rightarrow);\ \partial \mathcal{F} = \emptyset \Rightarrow \overline{\mathcal{F}}_{\supseteq \mathcal{F}} \setminus \mathcal{F}^\circ = \emptyset$$

$$\Rightarrow \overline{\mathcal{F}} = \mathcal{F}^\circ,$$

$$\overline{\mathcal{F}} \supseteq \mathcal{F} \supseteq \mathcal{F}^\circ ;$$

$$\overline{\mathcal{F}}_{\in \mathcal{Y}_c} = \mathcal{F} = \mathcal{F}^\circ_{\in \mathcal{Y}} \Rightarrow \mathcal{F} ;\ \mathcal{Y}\text{-clop}$$

$$(\Leftarrow);\ \mathcal{F} : \text{clop} \Rightarrow \mathcal{F} : \text{op}, \text{cl} ,$$

$$\mathcal{F}\ \text{op} \iff \mathcal{F} = \mathcal{F}^\circ$$

$$(\Rightarrow);\ \mathcal{F}^\circ = \bigcup_{\mathcal{J}_{\subseteq \mathcal{F}}^{\in \mathcal{T}}} \mathcal{J} \wedge \mathcal{F}\ \text{op} \Rightarrow \mathcal{F}^\circ = \bigcup_{\mathcal{J}_{\subseteq \mathcal{F}_{\in \mathcal{T}}}^{\in \mathcal{T}}} \mathcal{J} = \mathcal{F}$$

$$(\Leftarrow);\ \mathcal{F} = \mathcal{F}^\circ_{\in \mathcal{T}} \Rightarrow \mathcal{F} \in \mathcal{T},$$

$$\mathcal{F}\ \text{cl} \iff \mathcal{F} = \overline{\mathcal{F}}$$

$$(\Rightarrow);\ \overline{\mathcal{F}} = \bigcap_{C_{\supseteq \mathcal{F}}^{\in \mathcal{T}_c}} C \wedge \mathcal{F}\ \text{cl} \Rightarrow \overline{\mathcal{F}} = \bigcap_{C_{\supseteq \mathcal{F}_{\in \mathcal{T}_c}}^{\in \mathcal{T}_c}} C = \mathcal{F}$$

$$(\Leftarrow);\ \mathcal{F} = \overline{\mathcal{F}}_{\in \mathcal{T}_c} \Rightarrow \mathcal{F} \in \mathcal{T}_c ,$$

$$\mathcal{F} ;\ \text{cl}, \text{op} \Rightarrow \mathcal{F} = \overline{\mathcal{F}} \wedge \mathcal{F} = \mathcal{F}^\circ$$

$$\Rightarrow \partial \mathcal{F} = \overline{\mathcal{F}}_{(=\mathcal{F})} \setminus \mathcal{F}^\circ_{(=\mathcal{F})} = \emptyset .$$

Theorem: $\mathcal{F}, \mathcal{G} \subseteq \mathcal{X}_\tau$,

i. $\mathcal{F} \subseteq \mathcal{G} \Rightarrow \mathcal{F}^\circ \subseteq \mathcal{G}^\circ$

ii. $(\mathcal{F} \cap \mathcal{G})^\circ = \mathcal{F}^\circ \cap \mathcal{G}^\circ$

iii. $\mathcal{F}^\circ \cup \mathcal{G}^\circ \subseteq (\mathcal{F} \cup \mathcal{G})^\circ$.

Proof:

i. $x \in \mathcal{F}^\circ \Rightarrow \exists \mathcal{T}_{\subseteq \mathcal{F}_{\subseteq \mathcal{G}}} \in \tau : x \in \mathcal{T} \subseteq \mathcal{F}$

$\Rightarrow \exists \mathcal{T}_{\in \tau}, \mathcal{T}_{(\ni x)} \subseteq \mathcal{G} \Rightarrow x \in \mathcal{G}^\circ$

ii. $x \in (\mathcal{F} \cap \mathcal{G})^\circ \Rightarrow \exists \mathcal{T}_{\in \tau}\ x \in \mathcal{T} \subseteq \mathcal{F} \cap \mathcal{G}$

$\Rightarrow \exists \mathcal{T}_{\in \tau} : \mathcal{T}_{(\ni x)} \subseteq \mathcal{F} \wedge \mathcal{T}_{(\ni x)} \subseteq \mathcal{G}$

$\Rightarrow x \in \mathcal{F}^\circ \wedge x \in \mathcal{G}^\circ \Rightarrow x \in \mathcal{F}^\circ \cap \mathcal{G}^\circ$,

$x \in \mathcal{F}^\circ \cap \mathcal{G}^\circ \Rightarrow \exists \mathcal{T}_{\mathcal{F}}, \mathcal{T}_{\mathcal{G}} \in \mathcal{T}$

$x \in \mathcal{T}_{\mathcal{F}} \subseteq \mathcal{F} \wedge x \in \mathcal{T}_{\mathcal{G}} \subseteq \mathcal{G} \Rightarrow \exists \mathcal{Y}_{(=\mathcal{T}_{\mathcal{F}} \cap \mathcal{T}_{\mathcal{G}})} \in \mathcal{T}$

$x \in \mathcal{Y} \subseteq \mathcal{F} \cap \mathcal{G} \Rightarrow x \in (\mathcal{F} \cap \mathcal{G})^\circ$

iii. $x \in \mathcal{F}^\circ \cup \mathcal{G}^\circ \Rightarrow \exists \mathcal{T}_{\in \tau}\ x_{\in \mathcal{F}^\circ} \in \mathcal{T} \subseteq \mathcal{F}$

$\Rightarrow \exists \mathcal{T}_{\in \tau}\ x \in \mathcal{T} \subseteq \mathcal{F} \cup \mathcal{G} \Rightarrow x \in (\mathcal{F} \cup \mathcal{G})^\circ$.

Corollary: $\mathcal{X}_{\mathcal{Y}}$ Top $\Rightarrow \partial_{(\mathcal{Y})} \mathcal{X} = \partial \emptyset = \emptyset$.

Proof: $\mathcal{X}_{\mathcal{Y}}$ Top $\Rightarrow \mathcal{X}, \emptyset \in \mathcal{Y} \Rightarrow \emptyset, \mathcal{X} \in \mathcal{Y}_c$,
$\mathcal{X}, \emptyset$; $\mathcal{Y}$-clop $\Rightarrow \partial_{(\mathcal{Y})} \mathcal{X} = \partial \emptyset = \emptyset$.

Theorem; $\mathcal{J}^{\circ} = \mathcal{J}^{\circ \cdots \circ}$, $\overline{\overline{\mathcal{J}}^{\cdots}} = \overline{\mathcal{J}}$.

Proof; $(\mathcal{J}^{\circ})^{\circ} = \bigcup\limits_{\substack{\nu \in \mathcal{T} \\ \subseteq \mathcal{J}^{\circ}_{\in \mathcal{T}}}} \nu = \mathcal{J}^{\circ}$, $(\overline{\overline{\mathcal{J}}}) = \bigcap\limits_{\substack{y \in \mathcal{T}_c \\ \supseteq \overline{\mathcal{J}}_{\in \mathcal{T}_c}}} y = \overline{\mathcal{J}}$.

Theorem;

i. $\mathcal{F} \subseteq G \Rightarrow \overline{\mathcal{F}} \subseteq \overline{G}$

ii. $\overline{(\mathcal{F} \cup G)} = \overline{\mathcal{F}} \cup \overline{G}$

iii. $\overline{(\mathcal{F} \cap G)} \subseteq \overline{\mathcal{F}} \cap \overline{G}$

Proof;

i. $\mathcal{F} \subseteq G \Rightarrow \bigcap\limits_{\substack{j \in \mathcal{T}_c \\ \supseteq \mathcal{F}}} j \subseteq \overline{G}_{(\supseteq G \supseteq \mathcal{F}) \in \mathcal{T}_c}$

$\Rightarrow \overline{\mathcal{F}} \subseteq \overline{G}$

ii. $\overline{G}, \overline{\mathcal{F}} \subseteq \overline{(\mathcal{F} \cup G)}_{(\supseteq (\mathcal{F} \cup G))} \Rightarrow \overline{\mathcal{F}} \cup \overline{G} \subseteq \overline{(\mathcal{F} \cup G)}$

$\overline{(\mathcal{F} \cup G)} = \bigcap\limits_{\substack{J \in \mathcal{T}_c \\ \supseteq (\mathcal{F} \cup G)_{(\subseteq \overline{\mathcal{F}} \cup \overline{G})}}} J \subseteq \overline{\mathcal{F}} \cup \overline{G}$

iii. $\overline{(\mathcal{F} \cap G)} = \bigcap\limits_{\substack{J \in \mathcal{T}_c \\ \supseteq (\mathcal{F} \cap G)_{(\subseteq \overline{\mathcal{F}} \cap \overline{G})}}} J \subseteq \overline{\mathcal{F}} \cap \overline{G}$.

Lemma; $G_{\in \mathcal{T}_c} \supseteq \mathcal{F} \Rightarrow G \supseteq \overline{\mathcal{F}}, \mathcal{F}^{\circ}$.

Proof; $\overline{\mathcal{F}} = \bigcap\limits_{\substack{J \in \mathcal{T}_c \\ \supseteq \mathcal{F} (\subseteq G_{\in \mathcal{T}_c})}} J \subseteq G$, $\mathcal{F}^{\circ}_{(\subseteq \overline{\mathcal{F}})} \subseteq G$.

Lemma; $\mathcal{F}_{\in \mathcal{T}} \subseteq G \Rightarrow \mathcal{F} \subseteq G^{\circ}$

Proof; $G^{\circ} = \bigcup\limits_{\substack{J \in \mathcal{T} \\ \subseteq G_{(\supseteq \mathcal{F}_{\in \mathcal{T}})}}} \supseteq \mathcal{F}$.

Theorem: $G_{\subseteq \mathcal{X}_{\mathcal{T}}}$,

$$\overline{G} = \{x \mid \mathcal{U}^{\in \mathcal{T}}_{(\ni x)} \cap G \neq \emptyset\}$$

Proof:

$$(\Rightarrow);\ x_{\in \overline{G}} \in \mathcal{U}_{\in \mathcal{T}} \wedge \mathcal{U} \cap G = \emptyset$$

$$\Rightarrow G \subseteq (\mathcal{X} \setminus \mathcal{U})^{\in \mathcal{T}_c}_{(\not\ni x)} \Rightarrow \overline{G} \subseteq (\mathcal{X} \setminus \mathcal{U})_{(\not\ni x)}$$

$$\Rightarrow x \notin \overline{G} \quad ; \qquad x \in \overline{G} \Rightarrow \mathcal{U}_{(\ni x) \in \mathcal{T}} \cap G \neq \emptyset$$

$$(\Leftarrow);\ x_{\in \mathcal{U}_{\in \mathcal{T}}} \notin \overline{G} \wedge \mathcal{U} \cap G \neq \emptyset$$

$$\Rightarrow (\mathcal{X} \setminus \overline{G})^{\in \mathcal{T}}_{(\ni x)} \cap \underset{(\neq \emptyset)}{G} \neq \emptyset = (\mathcal{X} \setminus \overline{G}) \cap G_{(\subseteq \overline{G})}$$

$$;\ \mathcal{U}_{(\ni x) \in \mathcal{T}} \cap G \neq \emptyset \Rightarrow x \in \overline{G}.$$

Theorem: $\mathcal{F}_{\subseteq \mathcal{X}_{\mathcal{T}}}$

$$\partial \mathcal{F} := \overline{\mathcal{F}} \setminus \mathcal{F}^\circ = \overline{\mathcal{F}} \cap (\overline{\mathcal{F}^c}) = \{x \mid \mathcal{U}^{\in \mathcal{T}}_{(\ni x)} \cap^{\mathcal{F}}_{\mathcal{F}^c} \neq \emptyset\}$$

Proof:

$$\overline{\mathcal{F}} \setminus \mathcal{F}^\circ = \overline{\mathcal{F}} \cap (\mathcal{F}^\circ)^c, \left(x \in G \setminus Y \Leftrightarrow \begin{matrix} x \in G \\ x \notin Y \end{matrix} \Leftrightarrow \begin{matrix} x \in G \\ x \in Y^c \end{matrix} \Leftrightarrow x \in G \cap Y^c\right)$$

$$(\mathcal{F}^\circ)^c = (\overline{\mathcal{F}^c}) \ ; \quad \begin{matrix} (\mathcal{F}^{\circ}_{\substack{\in \mathcal{T} \\ \subseteq \mathcal{F}}})^c_{(\supseteq \mathcal{F}^c)} \in \mathcal{T}_c \Rightarrow (\mathcal{F}^\circ)^c \supseteq (\overline{\mathcal{F}^c}), \\ ((\overline{\mathcal{F}^c})_{\substack{\in \mathcal{T}_c \\ \supseteq \mathcal{F}^c}})^c_{(\subseteq (\mathcal{F}^c)^c_{(=\mathcal{F})})} \in \mathcal{T} \Rightarrow ((\overline{\mathcal{F}^c}))^c \subseteq \mathcal{F}^\circ \\ (\overline{\mathcal{F}^c})_{(=((\overline{\mathcal{F}^c})^c)^c)} \supseteq (\mathcal{F}^\circ)^c \end{matrix}$$

$$\Rightarrow \overline{\mathcal{F}} \setminus \mathcal{F}^\circ = \overline{\mathcal{F}} \cap (\overline{\mathcal{F}^c}),$$

$$\overline{\mathcal{F}} \cap (\overline{\mathcal{F}^c}) = \{x \mid \mathcal{U}^{\in \mathcal{T}}_{(\ni x)} \cap \mathcal{F} \neq \emptyset\} \cap \{x \mid \mathcal{U}^{\in \mathcal{T}}_{(\ni x)} \cap \mathcal{F}^c \neq \emptyset\}$$

$$= \{x \mid \mathcal{U}^{\in \mathcal{T}}_{(\ni x)} \cap^{\mathcal{F}}_{\mathcal{F}^c} \neq \emptyset\}.$$

Theorem: G ; cl $\Leftrightarrow \partial G \subseteq G$.

Proof: $(\Rightarrow)$: G ; cl $\Leftrightarrow G = \overline{G}$,

$$\overline{G} = \partial G \cup G^\circ \Rightarrow \underset{(=G)}{\overline{G}} \supseteq \partial G$$

$$\Rightarrow G \supseteq \partial G,$$

$(\Leftarrow)$: $\partial G \subseteq G$

$$\Rightarrow \overline{G} = G^\circ \cup \partial G \subseteq G \cup \underset{(\subseteq G)}{G^\circ} = G$$

$$\Rightarrow \underset{\in \tau_+}{\overline{G}} \subseteq \underset{(\subseteq \overline{G})}{G}$$

$$\Rightarrow \underset{(=\overline{G} \in \tau_c)}{G} \text{ ; cl}.$$

Theorem; G ; op $\Leftrightarrow \partial G \subseteq G^c$

Proof: $(\Rightarrow)$: G op $\Leftrightarrow G = G^\circ$,

$$\partial G \cap G = (\overline{G} \setminus G^\circ) \cap G = (\overline{G} \setminus G) \cap G = \emptyset,$$

$$\partial G \cap G = \emptyset \Rightarrow \partial G \subseteq G^c,$$

$(\Leftarrow)$: $\partial G \subseteq G^c$

$$\Rightarrow \overline{G} \setminus G^\circ \subseteq G^c$$

$$\Rightarrow \overline{G} \underset{(\supseteq G)}{=} (\overline{G} \setminus G^\circ) \cup G^\circ \subseteq G^c \cup G^\circ$$

$$\Rightarrow G \subseteq G^c \cup G^\circ \Rightarrow \underset{(\supseteq G^\circ)}{G} \subseteq G^\circ \Rightarrow G = G^\circ$$

$$\Rightarrow \underset{(=G^\circ)}{G} \text{ ; op}.$$

Theorem; $G_{\subseteq X_{\mathcal{T}}}$,

i. $\partial G \in \mathcal{T}_c$, ii. $\partial(G^\circ), \partial(\overline{G}) \subseteq \partial G$

iii. $\partial G = \partial(G^c)$, iv. $G \in \mathcal{T}_c \Rightarrow (\partial G)^\circ = \emptyset$

v. $\partial G \supseteq \partial(\partial G)$, vi. $\partial G = \partial(\partial G) \Leftrightarrow (\partial G)^\circ = \emptyset$

Proof;

i. $\partial G = \overline{G}_{\in \mathcal{T}_c} \cap (\overline{G^c})_{\in \mathcal{T}_c} \in \mathcal{T}_c$

ii. $\partial(G^\circ) = (\overline{G^\circ})_{\subseteq \overline{G}_{(=\overline{\overline{G}})}} \setminus G^\circ_{(=(G^\circ)^\circ)} \subseteq \overline{G} \setminus G^\circ = \partial G$

$\partial(\overline{G}) = (\overline{\overline{G}})_{(=\overline{G})} \setminus (\overline{G})^\circ_{(\supseteq (G^\circ)^\circ_{(=G^\circ)})} \subseteq \overline{G} \setminus G^\circ = \partial G$

iii. $\partial G = \overline{G} \cap (\overline{G^c}) = \overline{((G^c)^c)} \cap (\overline{G^c}) = \partial(G^c)$

iv. $G \in \mathcal{T}_c \Longrightarrow \partial G = \overline{G}_{(=G)} \setminus G^\circ$

$$\Rightarrow (\partial G)^\circ = (G \setminus G^\circ)^\circ = \bigcup_{J_{\in \mathcal{T}} \subseteq G \setminus G^\circ} J = \bigcup_{\substack{J_{\in \mathcal{T}} \subseteq G \\ \not\subseteq \bigcup_{\sigma_{\in \mathcal{T}} \subseteq G} \sigma}} J$$

$$= \bigcup_{J^{\subseteq G}_{\in \mathcal{T}} \not\subseteq \bigcup_{\sigma^{\subseteq G}_{\in \mathcal{T}}} \sigma} J = \emptyset$$

v. $\partial(\partial G) = \overline{((\partial G)_{\in \mathcal{T}_c})} \setminus (\partial G)^\circ = \partial G \setminus (\partial G)^\circ_{(\subseteq \partial G)} \subseteq \partial G$

vi. $\partial G = \partial(\partial G) \Leftrightarrow \partial G = \overline{((\partial G)_{\in \mathcal{T}_c})} \setminus (\partial G)^\circ$

$\Leftrightarrow \partial G = \partial G \setminus (\partial G)^\circ_{(\subseteq \partial G)} \Leftrightarrow (\partial G)^\circ = \emptyset$.

Def: $D_{\subseteq \mathcal{X}_\tau}$ dense $:= \overline{D} = \mathcal{X}$

Def: $G_{\subseteq \mathcal{X}_\tau}$, $x_{\in \mathcal{X}_\tau}\,\mathrm{acc}_G := (\mathcal{U}^{\in \tau}_{(\ni x)} \setminus \{x\}) \cap G \neq \emptyset$

$$\tilde{G} := \{x_{\in \mathcal{X}_\tau} \mid x;\ \mathrm{acc}_G\}$$

Ex:

$$\mathcal{X} = \{x_0, x_1, x_2, x_3, x_4\}$$

$$\mathcal{T} = \{\emptyset, \mathcal{X}, \{x_0\}, \{x_2, x_3\}, \{x_0, x_2, x_3\}, \{x_1, x_2, x_3, x_4\}\}$$

$$G = \{x_1, x_2, x_3\}$$

$\tilde{G} = \{x_1, x_2, x_3, x_4\}$, $\quad x_0 \notin \tilde{G}$; $\exists\, \mathcal{U}^{\in \tau}_{(\ni x_0)} (= \{x_0\})$;
$(\mathcal{U} \setminus \{x_0\}) \cap G = \emptyset$

Theorem: $\mathcal{F}, G \subseteq \mathcal{X}_\tau$

i. $\mathcal{F} \subseteq G \Rightarrow \tilde{\mathcal{F}} \subseteq \tilde{G}$, $\quad$ ii. $(\mathcal{F} \cup G)^\sim = \tilde{\mathcal{F}} \cup \tilde{G}$

iii. $(\mathcal{F} \cap G)^\sim \subseteq \tilde{\mathcal{F}} \cap \tilde{G}$

Proof:

i. $x \in \tilde{\mathcal{F}} \Rightarrow (\mathcal{U}^{\in \tau}_{(\ni x)} \setminus \{x\}) \cap \mathcal{F}_{\subseteq G} \neq \emptyset$
$\Rightarrow (\mathcal{U}^{\in \tau}_{(\ni x)} \setminus \{x\}) \cap G \neq \emptyset \Rightarrow x \in \tilde{G}$

ii. $(\Rightarrow)$: $\mathcal{F}, G_{\subseteq \mathcal{F} \cup G}$; $\tilde{\mathcal{F}}, \tilde{G} \subseteq (\mathcal{F} \cup G)^\sim \Rightarrow \tilde{\mathcal{F}} \cup \tilde{G} \subseteq (\mathcal{F} \cup G)^\sim$
$(\Leftarrow)$: $x_{\in (\mathcal{F} \cup G)^\sim}$: $(\mathcal{U}^{\in \tau}_{(\ni x)} \setminus \{x\}) \cap (\mathcal{F} \cup G) \neq \emptyset$
$\Rightarrow \big((\mathcal{U} \setminus \{x\}) \cap \mathcal{F}\big) \cup \big((\mathcal{U} \setminus \{x\}) \cap G\big) \neq \emptyset \Leftrightarrow \bigvee \begin{array}{l} (\mathcal{U} \setminus \{x\}) \cap \mathcal{F} \neq \emptyset \\ (\mathcal{U} \setminus \{x\}) \cap G \neq \emptyset \end{array}$
$\Leftrightarrow x \in \tilde{\mathcal{F}} \lor x \in \tilde{G} \Leftrightarrow x \in \tilde{\mathcal{F}} \cup \tilde{G}$

iii. $(\mathcal{F} \cap G)_{\subseteq \mathcal{F}, G} \Rightarrow (\mathcal{F} \cap G)^\sim \subseteq \tilde{\mathcal{F}}, \tilde{G} \Rightarrow (\mathcal{F} \cap G)^\sim \subseteq \tilde{\mathcal{F}} \cap \tilde{G}$.

Theorem; $G_{\subseteq X_\tau} \in \mathcal{T}_c \Leftrightarrow \tilde{G} \subseteq G$.

Proof; $(\Rightarrow)$; $x \notin G_{\in \mathcal{T}_c} \Rightarrow x \in G^c_{\in \tau}$

$\Rightarrow (G^c \setminus \{x\}) \cap G = \emptyset \Rightarrow x \notin \tilde{G}$

; $x \in \tilde{G} \Rightarrow x \in G$; $\tilde{G} \subseteq G$

$(\Leftarrow)$; $\tilde{G} \subseteq G \wedge x \in G^c$

$\Rightarrow x \notin \tilde{G} \Rightarrow \exists \mathcal{U}^{\in \tau}_{(\ni x)} \quad \mathcal{U} \cap G = \emptyset$

$\Rightarrow \forall x_{\in G^c} \exists \mathcal{U}_x{}^{\in \tau}_{(\ni x)} \quad \mathcal{U} \subseteq G^c$

$\Rightarrow G^c = \bigcup_x \mathcal{U}_x = \bigcup_{J_{\in\tau} \subseteq G^c} J = (G^c)^\circ$

$\Rightarrow G^c \in \mathcal{T} \Rightarrow G \in \mathcal{T}_c$.

Theorem; $(G \cup \tilde{G})_{\in \mathcal{T}_c} = \bar{G}$.

Proof; $x \in (G \cup \tilde{G})^c \Rightarrow x \notin \tilde{G}, G$

$\Rightarrow \exists \mathcal{U}^{\in \tau}_{(\ni x_{\notin G})} \; (\mathcal{U} \setminus \{x\}) \cap G = \emptyset \Rightarrow \mathcal{U} \cap G = \emptyset$,

$r \in \mathcal{U}_{|\mathcal{U} \cap G = \emptyset} \Rightarrow \exists y^{\in \tau}_{(\ni r)(=\mathcal{U})} [(y \setminus \{r\}) \cap G]_{(\subseteq y \cap G_{(=\emptyset)})} = \emptyset$

$\Rightarrow r \notin \tilde{G}$; $\mathcal{U} \cap \tilde{G} = \emptyset$,

$\mathcal{U} \cap (G \cup \tilde{G}) = (\mathcal{U} \cap G)_{(=\emptyset)} \cup (\mathcal{U} \cap \tilde{G})_{(=\emptyset)} = \emptyset$

$\Rightarrow \forall x_{\in (G \cup \tilde{G})^c} \exists \mathcal{U}_x{}^{\in \tau}_{(\ni x)} \quad \mathcal{U} \subseteq (G \cup \tilde{G})^c$

$\Rightarrow (G \cup \tilde{G})^c = \bigcup_x \mathcal{U}_x = \bigcup_{J_{\in\tau} \subseteq (G \cup \tilde{G})^c} J = ((G \cup \tilde{G})^c)^\circ \Rightarrow (G \cup \tilde{G})^c \in \mathcal{T}$

$\Rightarrow (G \cup \tilde{G}) \in \mathcal{T}_c$

Proof: $G = \emptyset \Rightarrow G \cup \tilde{G} = \emptyset = \overline{G}$

$$G \neq \emptyset ,$$

$$(\Rightarrow);\ x \in \overline{G} \Rightarrow \mathcal{U}^{\in \mathcal{T}}_{(\ni x)} \cap G \neq \emptyset$$

$$\Rightarrow \veebar \begin{cases} \mathcal{U} \cap G \ni x \Rightarrow x \in G \\ (\mathcal{U} \cap G)_{(\neq \emptyset)} \not\ni x \Rightarrow (\mathcal{U} \setminus \{x\}) \cap G \neq \emptyset \Rightarrow x \in \tilde{G} \end{cases}$$

$$\Rightarrow x \in G \cup \tilde{G} \quad ; \quad \overline{G} \subseteq G \cup \tilde{G} ,$$

$$(\Leftarrow);\quad x \in G \cup \tilde{G} \Rightarrow x \in G \vee x \in \tilde{G}$$

$$x \in G_{(\subseteq \overline{G})} \Rightarrow x \in \overline{G} ,$$

$$x_{\in G \cup \tilde{G}} \notin G \Rightarrow x \in \tilde{G} \Rightarrow (\mathcal{U}^{\in \mathcal{T}}_{(\ni x)} \setminus \{x_{\notin G}\}) \cap G \neq \emptyset$$

$$\Rightarrow \mathcal{U}^{\in \mathcal{T}}_{(\ni x)} \cap G \neq \emptyset \Rightarrow x \in \overline{G} \ ; \ G \cup \tilde{G} \subseteq \overline{G} ,$$

$$\Rightarrow G \cup \tilde{G} = \overline{G} \ .$$

Def: $\mathcal{N}_{\subseteq \mathcal{X}_\tau}$ $\text{neig}_{\mathfrak{r} \in \mathcal{X}_\tau} := \exists\, \mathcal{U}^{\in \mathcal{T}}_{(\ni \mathfrak{r})} ;\ \mathcal{U} \subseteq \mathcal{N}$

$$\mathcal{N}(\mathfrak{r}_{\in \mathcal{X}_\tau})_{\subseteq \mathcal{P}(\mathcal{X}_\tau)} := \{\mathcal{N} \mid \mathcal{N} ;\ \text{neig}_{\mathfrak{r}}\}$$

Ex: $\mathcal{X} = \{\mathfrak{r}, \sigma, \rho, x\}$

$$\mathcal{T} = \{\emptyset, \mathcal{X}, \{\mathfrak{r}\}, \{\mathfrak{r}, \sigma\}, \{\mathfrak{r}, \sigma, x\}\}$$

$$\mathcal{Y} = \{\emptyset, \mathcal{X}, \{\rho\}, \{\rho, \sigma\}, \{\rho, \sigma, \mathfrak{r}\}\}$$

$$\mathcal{N}(\sigma)_{|\mathcal{T}} = \{\mathcal{X}, \{\mathfrak{r}, \sigma\}, \{\mathfrak{r}, \sigma, \rho\}, \{\mathfrak{r}, \sigma, x\}\}$$

$$\mathcal{N}(\sigma)_{|\mathcal{Y}} = \{\mathcal{X}, \{\rho, \sigma\}, \{\rho, \sigma, x\}, \{\rho, \sigma, \mathfrak{r}\}\}$$

Theorem: i. $\forall \mathfrak{r}_{\in \mathcal{X}_\tau}\ \mathcal{N}(\mathfrak{r}) \neq \emptyset$

ii. $\mathcal{N}_{\subseteq \mathcal{M}} \in \mathcal{N}(\mathfrak{r}) \Rightarrow \mathcal{M} \in \mathcal{N}(\mathfrak{r})$

iii. $\{\mathcal{N}_i\}_{\mathcal{J}} \subseteq \mathcal{N}(\mathfrak{r}) \Rightarrow \bigcup_{\mathcal{J}} \mathcal{N}_i, \bigcap_{\mathcal{J}} \mathcal{N}_i \in \mathcal{N}(\mathfrak{r})$

iv. $\forall \mathcal{N}_{\in \mathcal{N}(\mathfrak{r})}\ \exists \mathcal{J}_{\in \mathcal{N}(\mathfrak{r})}\ \mathcal{J} \subseteq \mathcal{N}$

v. $\mathcal{N}_{\subseteq \mathcal{X}_\tau} \in \mathcal{N}(\mathfrak{r}) \Rightarrow \exists \mathcal{J}_{\in \mathcal{N}(\mathfrak{r})}\ \forall y_{\in \mathcal{J}}\ \mathcal{N} \in \mathcal{N}(y)$

Proof: i. $\mathfrak{r}_{\in \mathcal{X}_\tau \in \mathcal{T}} ;\ \mathcal{X}_\tau \in \mathcal{N}(\mathfrak{r}) ;\ \mathcal{N}(\mathfrak{r}) \neq \emptyset$

ii. $\mathcal{N}_{\subseteq \mathcal{M}} \in \mathcal{N}(\mathfrak{r}) \Rightarrow \exists\, \mathcal{U}^{\in \mathcal{T}}_{(\ni \mathfrak{r})}\ \mathcal{U} \subseteq \mathcal{N}_{\subseteq \mathcal{M}}$

$\Rightarrow \exists\, \mathcal{U}^{\in \mathcal{T}}_{(\ni \mathfrak{r})}\ \mathcal{U} \subseteq \mathcal{M} \Rightarrow \mathcal{M} \in \mathcal{N}(\mathfrak{r})$

Proof:

iii. $\{N_i\}_J \subseteq \mathcal{N}(\Omega) \Rightarrow \forall N_i \; \exists U_{i\,(\ni\Omega)}^{\in\mathcal{T}} \; U_i \subseteq N_i$

$\Rightarrow (\bigcup_J U_i)^{\in\mathcal{T}}_{(\ni\Omega)} \subseteq \bigcup_J N_i \; , \; (\bigcap_J U_i)^{\in\mathcal{T}}_{(\ni\Omega)} \subseteq \bigcap_J N_i$

$\Rightarrow \bigcup_J N_i \, , \, \bigcap_J N_i \in \mathcal{N}(\Omega)$

iv. $N \in \mathcal{N}(\Omega) \Rightarrow \exists U^{\in\mathcal{T}}_{(\ni\Omega)} \; U \subseteq N$

$\Rightarrow \exists J_{\subseteq X_\mathcal{T}} (= U \in \mathcal{N}(\Omega)) \; J \subseteq N$

v. $N_{\subseteq X_\mathcal{T}} \in \mathcal{N}(\Omega) \Rightarrow \exists U^{\in\mathcal{T}}_{(\ni\Omega)} \; U \subseteq N$,

$(\forall Y_{\in\mathcal{T}} \; Y \in \mathcal{N}(\sigma_{\in Y}))$

$\Rightarrow \exists J_{\in\mathcal{N}(\Omega)} (= U^{\in\mathcal{T}}_{(\ni\Omega)}) \; J_{\subseteq N} \in \mathcal{N}(\sigma_{\in J})$

$\Rightarrow \exists J_{\in\mathcal{N}(\Omega)} \; \forall y_{\in J} \; N \in \mathcal{N}(y)$.

Theorem: $N_{Y_\Omega \subseteq X_\mathcal{T}}(\sigma) = \{Y \cap N \mid N \in N_{X_\mathcal{T}}(\sigma)\}$

Proof: $J \in N_{Y_\Omega}(\sigma) \Leftrightarrow \exists U_J{}^{\in\Omega}_{(\ni\sigma)} \; U_J \subseteq J$

$\Leftrightarrow \exists G_J{}^{\in\mathcal{T}}_{(\ni\sigma)} \; G_J \cap Y \subseteq J$

$\Leftrightarrow \exists K^{(=G_J)}_{\in N_{X_\mathcal{T}}(\sigma)} \; K \cap Y \subseteq J$

$\Leftrightarrow \exists F^{\in N_{X_\mathcal{T}}(\sigma)}_{\supseteq K \in N_{X_\mathcal{T}}(\sigma)} \; F \cap Y = J \Leftrightarrow J \in \{Y \cap N \mid N \in N_{X_\mathcal{T}}(\sigma)\}$

Theorem: $\mathcal{X}_{\neq\emptyset}$, $\forall x_{\in\mathcal{X}}$ $\mathcal{B}(x) := \{\mathcal{N}_{\subseteq\mathcal{X}} : \text{Satisfies } i \vdots v\}$

$\Rightarrow \exists! \mathcal{T}_{\mathcal{X}}$ $\forall x_{\in\mathcal{X}}$ $\mathcal{N}_{\mathcal{T}_{\mathcal{X}}}(x) = \mathcal{B}(x)$.

Proof: define $\mathcal{T} = \{\mathcal{U}_{\subseteq\mathcal{X}} \mid \forall x_{\in\mathcal{U}}\ \mathcal{U} \in \mathcal{B}(x)\}$

1. Since $i.[\forall \mathcal{N}_{\in\mathcal{B}(x)}\ x \in \mathcal{N}]$, $\mathcal{B}(x) \neq \emptyset$

Then since $ii.[\mathcal{M} \in \mathcal{B}(x) \wedge \mathcal{N} \supseteq \mathcal{M} \Rightarrow \mathcal{N} \in \mathcal{B}(x)]$, $\mathcal{X} \in \mathcal{B}(x)$

hence $\mathcal{X}, \emptyset \in \mathcal{T}$

2. $\{\mathcal{U}_j^{\neq\emptyset}\} \subseteq \mathcal{T} \Rightarrow \bigcap_j \mathcal{U}_j \supset \mathcal{F}_{\in\mathcal{B}(x)}$, if $\exists_i\ \mathcal{U}_i = \emptyset$; $\bigcap_i \mathcal{U}_i = \emptyset \in \mathcal{T}$.

by $ii.[\mathcal{N}_{\subseteq\mathcal{M}}^{\in} \mathcal{B}(x) \Rightarrow \mathcal{M} \in \mathcal{B}(x)]$, $\bigcap_j \mathcal{U}_j^{\subseteq\mathcal{X}} \in \mathcal{B}(x) \Rightarrow \bigcap_j \mathcal{U}_j \in \mathcal{T}$

3. $\{\mathcal{U}_j\} \subseteq \mathcal{T} \Rightarrow \bigcup_j \mathcal{U}_j \supset \mathcal{K}_{\in\mathcal{B}(x)} \Rightarrow \bigcup_j \mathcal{U}_j^{\subseteq\mathcal{X}} \in \mathcal{B}(x)$

$\Rightarrow \bigcup_j \mathcal{U}_j \in \mathcal{T}$

Hence, $\mathcal{T}$; Topology on $\mathcal{X}$

Previously shown; $\forall \mathcal{N}_{\in\mathcal{N}_{\mathcal{T}}(x)}$ Satisfies $i \vdots v.$ So $\mathcal{N}_{\mathcal{T}}(x) \subseteq \mathcal{B}(x)$

$J \in \mathcal{B}(x)$, $V := \{y_{\in\mathcal{X}} \mid J_{\ni x, \ni y}^{\in} \mathcal{B}(y)\}$, note $V_{\ni x} \subseteq J$

$J \in \mathcal{B}(y) \Rightarrow \exists W_{\in\mathcal{B}(y)}\ \forall z_{\in W}\ J \in \mathcal{B}(z)$

by def of V, $W \subseteq V$, since $W \in \mathcal{B}(y)$, by $ii.$, $V \in \mathcal{B}(y)$

So $\forall y_{\in V} \exists G_{\subseteq V}^{(=V)}\ G \in \mathcal{B}(y) : V_{\subseteq J}^{\ni x} \in \mathcal{T}$; $J \in \mathcal{N}_{\mathcal{T}}(x)$; $\mathcal{B}(x) \subseteq \mathcal{N}_{\mathcal{T}}(x)$

Proof cons: Hence $\exists \mathcal{T} : \mathcal{B}(x) = \mathcal{N}_{\mathcal{T}}(x) \ \forall x_{\in \mathcal{X}}$

to show $\exists ! \mathcal{T} : \mathcal{B}(x) = \mathcal{N}_{\mathcal{T}}(x) \ \forall x_{\in \mathcal{X}}$,

i.e. $\mathcal{P} : \mathcal{N}_{\mathcal{P}}(x) = \mathcal{B}(x) \ \forall x_{\in \mathcal{X}} \implies \mathcal{P} = \mathcal{T}$

$\equiv \ \mathcal{P} \neq \mathcal{T} \implies \exists \omega_{\in \mathcal{X}} \ \mathcal{N}_{\mathcal{P}}(\omega) \neq \mathcal{B}(x)$

$\equiv \ \exists \mathcal{U}_{\in \mathcal{P}} \neg (\forall x_{\in \mathcal{U}} \mathcal{U} \in \mathcal{B}(x)) \implies \exists \omega_{\in \mathcal{X}} \ \mathcal{N}_{\mathcal{P}}(\omega) = \mathcal{B}(\omega)$

$\equiv \ \exists x_{\in \mathcal{U}^{\in \mathcal{P}}_{\subseteq \mathcal{X}}} \ \mathcal{U} \notin \mathcal{B}(x) \implies \exists \omega_{\in \mathcal{X}} \ \mathcal{N}_{\mathcal{P}}(\omega) \neq \mathcal{B}(\omega)$

Since $\mathcal{N}_{\mathcal{P}}(j) := \{ \mathcal{N}_{\subseteq \mathcal{X}} \mid \exists \mathcal{F}^{\in \mathcal{P}}_{\ni j} \ \mathcal{F} \subseteq \mathcal{N} \}$,

$\mathcal{U}^{\ni x}_{\subseteq \mathcal{X}} \ \exists \mathcal{F}^{\in \mathcal{P}, \ni x}_{(= \mathcal{U})} \ \mathcal{F} \subseteq \mathcal{U}$ so $\mathcal{U} \in \mathcal{N}_{\mathcal{P}}(x)$

So $\exists x_{\in \mathcal{U}^{\in \mathcal{P}}_{\subseteq \mathcal{X}}} \ \mathcal{U} \notin \mathcal{B}(x) \implies \exists \omega^{(=x)}_{\in \mathcal{X}} \ \mathcal{U}_{\notin \mathcal{B}(\omega)} \in \mathcal{N}_{\mathcal{P}}(\omega)$

hence, $\exists \omega^{(=x)}_{\in \mathcal{X}} \ \mathcal{N}_{\mathcal{P}}(\omega) \neq \mathcal{B}(\omega)$

So $\mathcal{P} = \mathcal{T}$,

Hence $\exists ! \mathcal{T} : \forall x_{\in \mathcal{X}} \ \mathcal{N}_{\mathcal{T}}(x) = \mathcal{B}(x)$.

Def: $\mathcal{B}_{\subseteq\mathcal{T}}$ basis of $(\mathcal{X},\mathcal{T}) := \forall \mathcal{U}_{\in\mathcal{T}}\ \exists\{\mathcal{B}_i\}_{\subseteq\mathcal{B}}$

$$\mathcal{U} = \bigcup_i \mathcal{B}_i$$

Lemma: $\mathcal{B}$ covers $\mathcal{X}$.

Proof: $\mathcal{X}\in\mathcal{T} \Rightarrow \exists\{\mathcal{B}_i\}_{\subseteq\mathcal{B}}\ \mathcal{X} = \bigcup_i \mathcal{B}_i \subseteq \bigcup_{\mathcal{B}\in\mathcal{B}} \mathcal{B}$

$$\Rightarrow \mathcal{X} \subseteq \bigcup_{\mathcal{B}\in\mathcal{B}} \mathcal{B} \qquad \blacksquare$$

Lemma: $x \in \mathcal{B}_i^{\in\mathcal{B}} \cap \mathcal{B}_j^{\in\mathcal{B}} \Rightarrow \exists \mathcal{B}_\sigma^{\in\mathcal{B}}\ x\in\mathcal{B}_\sigma \subseteq \mathcal{B}_i\cap\mathcal{B}_j$

Proof: $\mathcal{B}_i, \mathcal{B}_j \in \mathcal{B} \subseteq \mathcal{T} \Rightarrow \mathcal{B}_i, \mathcal{B}_j \in \mathcal{T}$

$\Rightarrow \mathcal{B}_i\cap\mathcal{B}_j \in \mathcal{T} \Rightarrow \exists\{\mathcal{B}_\lambda\}_{\subseteq\mathcal{B}}\ \mathcal{B}_i\cap\mathcal{B}_j = \bigcup_\lambda \mathcal{B}_\lambda$

$\Rightarrow$ since $x \in \bigcup_\lambda \mathcal{B}_\lambda\ (=\mathcal{B}_i\cap\mathcal{B}_j)\ \exists\lambda_0\ x\in\mathcal{B}_{\lambda_0} \subseteq \bigcup_\lambda \mathcal{B}_\lambda$

$\Rightarrow \exists \mathcal{B}_{\sigma\,(=\mathcal{B}_{\lambda_0})}^{\in\mathcal{B}}\ x\in\mathcal{B}_\sigma \subseteq \mathcal{B}_i\cap\mathcal{B}_j \qquad \blacksquare$

Def: Topology generated by $\mathcal{B}$:

$$\mathcal{T}_\mathcal{B} = \{\mathcal{U}_{\subseteq\mathcal{X}} \mid \forall x_{\in\mathcal{U}}\ \exists \mathcal{B}_{\in\mathcal{B}}\ x\in\mathcal{B}\subseteq\mathcal{U}\}$$

Lemma: $\mathcal{T}_\mathcal{B}$ is topology.

Proof: $\emptyset \in \mathcal{T}_\mathcal{B}$ trivially; since $\mathcal{B}$ covers $\mathcal{X}$, $\mathcal{X}\in\mathcal{T}_\mathcal{B}$

$\{\mathcal{U}_i\}\subseteq\mathcal{T}_\mathcal{B}$, $\forall x_{\in\bigcup\mathcal{U}_i}\ \exists\mathcal{U}_{i_0}\ x\in\mathcal{U}_{i_0}$, since $\mathcal{U}_{i_0}\in\mathcal{T}_\mathcal{B}$

$\exists\mathcal{B}_{\in\mathcal{B}}\ x\in\mathcal{B}\subseteq\mathcal{U}_{i_0}\subseteq\bigcup_i\mathcal{U}_i$ hence $\bigcup_i \mathcal{U}_i \in \mathcal{T}_\mathcal{B}$

Proof cont: $\{\mathcal{U}_i\} \subseteq \mathcal{T}_{\mathcal{B}}$, by induction

$n=1$: $\bigcap_{i\in\{1\}} \mathcal{U}_i = \mathcal{U}_1 \in \mathcal{T}_{\mathcal{B}}$

Suppose $\bigcap_{i=1}^{n-1} \mathcal{U}_i \in \mathcal{T}_{\mathcal{B}}$, since $\bigcap_{i=1}^{n} \mathcal{U}_i = \mathcal{U}_n \cap (\bigcap_{i=1}^{n-1} \mathcal{U}_i)$

it is sufficent to show if $\mathcal{U}_i, \mathcal{U}_j \in \mathcal{T}_{\mathcal{B}}$ then $\mathcal{U}_i \cap \mathcal{U}_j \in \mathcal{T}_{\mathcal{B}}$;

$x \in \mathcal{U}_i \cap \mathcal{U}_j \Rightarrow$ since $\mathcal{U}_i, \mathcal{U}_j \in \mathcal{T}_{\mathcal{B}}$, $\exists \mathcal{B}_i, \mathcal{B}_j \in \mathcal{B}$

$x \in \mathcal{B}_i \subseteq \mathcal{U}_i$, $x \in \mathcal{B}_j \subseteq \mathcal{U}_j \Rightarrow \exists \mathcal{B}_\sigma^{\in \mathcal{B}}$ such that

$x \in \mathcal{B}_\sigma \subseteq \mathcal{B}_i \cap \mathcal{B}_j$, since $\mathcal{B}_i \cap \mathcal{B}_j \subseteq \mathcal{B}_i, \mathcal{B}_j \subseteq \mathcal{U}_i, \mathcal{U}_j$,

$\mathcal{B}_i \cap \mathcal{B}_j \subseteq \mathcal{U}_i \cap \mathcal{U}_j$ hence $x \in \mathcal{B}_\sigma \subseteq \mathcal{U}_i \cap \mathcal{U}_j$

therefore, $\mathcal{U}_i \cap \mathcal{U}_j \in \mathcal{T}_{\mathcal{B}}$ $\blacksquare$

Lemma: $\mathcal{B} \subseteq \mathcal{T}$ then, $\mathcal{B}$: basis $\wedge\ \mathcal{T} = \mathcal{T}_{\mathcal{B}} \Leftrightarrow \mathcal{T} = \{\bigcup_{B \in \mathcal{B}'} B \mid \mathcal{B}' \subseteq \mathcal{B}\}$

Proof: $(\Rightarrow)$: $\mathcal{U} \in \mathcal{T} \Rightarrow \forall x_{\in \mathcal{U}} \exists B_x^{\in \mathcal{B}}\ x \in B_x \subseteq \mathcal{U}$

$\Rightarrow \mathcal{U} = \bigcup_{x \in \mathcal{U}} B_x \Rightarrow \mathcal{U} \in \{\bigcup_{B \in \mathcal{B}'} B \mid \mathcal{B}' \subseteq \mathcal{B}\}$,

On the other hand, since $\forall B_{\in \mathcal{B}}\ B \in \mathcal{T}$, $\bigcup B \in \mathcal{T}$

So $\mathcal{U} \in \{\bigcup_{B \in \mathcal{B}'} B \mid \mathcal{B}' \subseteq \mathcal{B}\} \Rightarrow \mathcal{U} \in \mathcal{T}$, hence $\mathcal{T} = \{\bigcup_{B \in \mathcal{B}'} B \mid \mathcal{B}' \subseteq \mathcal{B}\}$

$(\Leftarrow)$: Since $X \in \mathcal{T}$, $X = \bigcup B_i$ so $\mathcal{B}$ covers X,

$B_i, B_j \in \mathcal{B}_{\subseteq \sigma} \Rightarrow B_i \cap B_j \in \mathcal{T} \Rightarrow B_i \cap B_j = \bigcup_\lambda B_\lambda$

$\Rightarrow \forall x_{\in B_i \cap B_j} \exists B_{\lambda_x}^{\in \mathcal{B}}\ x \in B_{\lambda_x} \subseteq B_i \cap B_{j\,(=\bigcup_\lambda B_\lambda)}$, hence $\mathcal{B}$ is basis, Proof cons: since if $\mathcal{U} \in \mathcal{T}_{\mathcal{B}}$ so $\forall x_{\in \mathcal{U}} \exists B_x^{\in \mathcal{B}}\ x \in B_x \subseteq \mathcal{U}$

then $\mathcal{U} = \bigcup_{x \in \mathcal{U}} B_x \subseteq \bigcup_{B \in \mathcal{B}_{\subseteq \mathcal{T}}} B \in \mathcal{T}$ hence $\mathcal{T}_{\mathcal{B}} \subseteq \mathcal{T}$

On the other hand, if $\mathcal{U} \in \mathcal{T}$ so $\mathcal{U} = \bigcup_{\lambda} B_\lambda$

then $\forall x_{\in \mathcal{U}} \exists B_{\lambda_x}^{\in \mathcal{B}}\ x \in B_{\lambda_x} \subseteq \mathcal{U}$, hence $\mathcal{U} \in \mathcal{T}_{\mathcal{B}}$

so $\mathcal{T} \subseteq \mathcal{T}_{\mathcal{B}}$, therefore $\mathcal{T} = \mathcal{T}_{\mathcal{B}}$ ■

Lemma; $\mathcal{B} \subseteq \mathcal{T}$ then, $\mathcal{B}$: basis $\wedge$ $\mathcal{T} = \mathcal{T}_{\mathcal{B}}$

$\Longleftrightarrow \forall \mathcal{U}_{\in \mathcal{T}}$ satisfies: $\forall x_{\in \mathcal{U}} \exists B_x^{\in \mathcal{B}}\ x \in B_x \subseteq \mathcal{U}$.

Proof: $(\Rightarrow)$; trivial by the definition of $\mathcal{T}_{\mathcal{B}}$,

$(\Leftarrow)$; since $\mathcal{X} \in \mathcal{T}$, $\forall x_{\in \mathcal{X}} \exists B_x^{\in \mathcal{B}}\ x \in B_x \subseteq \mathcal{X}$

so $\mathcal{B}$ covers $\mathcal{X}$, $B_i, B_j \in \mathcal{B}_{\subseteq \mathcal{T}}$ then $B_i \cap B_j \in \mathcal{T}$

so $\forall x_{\in B_i \cap B_j} \exists B_\sigma^{\in \mathcal{B}}\ x \in B_\sigma \subseteq B_i \cap B_j$ so $\mathcal{B}$ is basis,

also $\mathcal{T} = \mathcal{T}_{\mathcal{B}}$ as shown above ■

Lemma: $\mathcal{B}, \mathcal{B}'$ basis of $\mathcal{T}, \mathcal{T}'$ on $\mathcal{X}$, then

$\mathcal{T} \subseteq \mathcal{T}' \Longleftrightarrow \forall B_{\in \mathcal{B}} \forall x_{\in B} \exists B'_{\in \mathcal{B}'}\ x \in B' \subseteq B$

Proof: $(\Rightarrow)$: $B \in \mathcal{B}_{\subseteq \mathcal{T}} \Rightarrow B \in \mathcal{T}' \Rightarrow B = \bigcup_{\lambda} B'_\lambda$,

$x \in B \Rightarrow x \in B'_{\lambda_x} \in \mathcal{B}' \Rightarrow x \in B'_{\lambda_x} \subseteq B$,

$(\Leftarrow)$: $x \in \mathcal{U} \in \mathcal{T} \Rightarrow \mathcal{U} = \bigcup_\lambda B_\lambda$ so $x \in B_{\lambda_x} \subseteq \mathcal{U}$

by assumption, $\exists B'^{\in \mathcal{B}'}_{\lambda_x}\ x \in B'_{\lambda_x} \subseteq B_{\lambda_x}$, $\mathcal{U} \subseteq \bigcup_{x \in \mathcal{U}} B'_{\lambda_x} \subseteq \bigcup_{x \in \mathcal{U}} B_{\lambda_x} = \mathcal{U}$

so $\mathcal{U} = \bigcup_{x \in \mathcal{U}} B'_{\lambda_x}$ hence $\mathcal{U} \in \mathcal{T}'$, therefore $\mathcal{T} \subseteq \mathcal{T}'$ ■

Lemma. $\mathcal{B} = \{ \mathcal{U} \times \mathcal{V}_{\subseteq \mathcal{X} \times \mathcal{Y}} \mid \mathcal{U} \in \mathcal{T}_{\mathcal{X}}, \mathcal{V} \in \mathcal{T}_{\mathcal{Y}} \}$

is a basis of $\mathcal{X} \times \mathcal{Y}$.

Proof: since $\mathcal{X} \times \mathcal{Y} \in \mathcal{B}$, $\forall (x,y)_{\in \mathcal{X} \times \mathcal{Y}} \exists \mathcal{B}^{\in \mathcal{B}}_{(= \mathcal{X} \times \mathcal{Y})}$ $(x,y) \in \mathcal{B}$

$\mathcal{U}_i \times \mathcal{V}_i, \mathcal{U}_j \times \mathcal{V}_j \in \mathcal{B} \Rightarrow (\mathcal{U}_i \times \mathcal{V}_i) \cap (\mathcal{U}_j \times \mathcal{V}_j)$

$= \{ (x,y) \mid x \in \mathcal{U}_i \cap \mathcal{U}_j, y \in \mathcal{V}_i \cap \mathcal{V}_j \} = (\mathcal{U}_i \cap \mathcal{U}_j)_{\in \mathcal{T}_{\mathcal{X}}} \times (\mathcal{V}_i \cap \mathcal{V}_j)_{\in \mathcal{T}_{\mathcal{Y}}}$ ∎

Def: $\mathcal{T}_{\mathcal{B}}$ product topology on $\mathcal{X} \times \mathcal{Y}$.

Theorem: $\mathcal{B}_{\mathcal{X}}$ basis of $(\mathcal{X}, \mathcal{T}_{\mathcal{X}}) \wedge \mathcal{B}_{\mathcal{Y}}$ basis of $(\mathcal{Y}, \mathcal{T}_{\mathcal{Y}})$

$\Rightarrow \mathcal{B}_{\mathcal{X} \times \mathcal{Y}} := \{ B \times C \mid B \in \mathcal{B}_{\mathcal{X}} \wedge C \in \mathcal{B}_{\mathcal{Y}} \}$ basis of $(\mathcal{X} \times \mathcal{Y}, \mathcal{T}_{\mathcal{B}}^{\text{product topology}})$.

Proof: To check $\mathcal{B}_{\mathcal{X} \times \mathcal{Y}}$, note the previous Lemma that

$\mathcal{B}$ basis for $\mathcal{T}$ iff $\forall \mathcal{U}_{\in \mathcal{T}}$ satisfies $\forall x_{\in \mathcal{U}} \exists B_{\in \mathcal{B}}$ $x \in B \subseteq \mathcal{U}$,

Let $(x,y) \in \mathcal{W} \in \mathcal{T}_{\mathcal{B}}$, by the definition of product topology $\mathcal{T}_{\mathcal{B}}$,

$\exists \mathcal{U}_{\in \mathcal{T}_{\mathcal{X}}}, \mathcal{V}_{\in \mathcal{T}_{\mathcal{Y}}}$ such that $(x,y) \in \mathcal{U} \times \mathcal{V} \subseteq \mathcal{W}$,

since $\mathcal{B}_{\mathcal{X}}, \mathcal{B}_{\mathcal{Y}}$ are basis, $\exists B_{\in \mathcal{B}_{\mathcal{X}}}, C_{\in \mathcal{B}_{\mathcal{Y}}}$ such that

$x \in B \subseteq \mathcal{U}$, $y \in C \subseteq \mathcal{V}$, hence $B \times C \subseteq \mathcal{U} \times \mathcal{V}$

So, $\exists J^{\in \mathcal{B}_{\mathcal{X} \times \mathcal{Y}}}_{(= B \times C)}$ $(x,y) \in J \subseteq \mathcal{W}$, hence, for $\mathcal{T}_{\mathcal{B}}$ product top.,

$\mathcal{B}_{\mathcal{X} \times \mathcal{Y}}$ is a basis ∎

Lemma: $\mathcal{B}$ basis for $\mathcal{T}_\mathcal{X}$, $\mathcal{Y} \subseteq \mathcal{X}$,

$\mathcal{B}_\mathcal{Y} := \{ \mathcal{Y} \cap B \mid B \in \mathcal{B} \}$ is a basis of the $\mathcal{T}_\mathcal{Y}$.

Proof: note the previous Lemma that $\mathcal{B}$ basis for $\mathcal{T}$ iff $\forall \mathcal{U}_{\in \mathcal{T}}$ satisfies $\forall x_{\in \mathcal{U}} \exists B^{\in \mathcal{B}}\ x \in B \subseteq \mathcal{U}$,

Let $V \in \mathcal{T}_\mathcal{Y}$ i.e. $\exists \mathcal{U}_{\in \mathcal{T}}\ V = \mathcal{Y} \cap \mathcal{U}$,

$x \in V \Rightarrow x \in \mathcal{U}_{\in \mathcal{T}} \Rightarrow \exists B^{\in \mathcal{B}}\ x \in B \subseteq \mathcal{U}$

$\Rightarrow \exists B_{\mathcal{Y}\,(= \mathcal{Y} \cap B)}^{\in \mathcal{B}_\mathcal{Y}}\ x \in B_\mathcal{Y} \subseteq V$,

hence $\mathcal{B}_\mathcal{Y}$ is a basis of the $\mathcal{T}_\mathcal{Y}$ ■

Def: $\mathcal{B}_x \subseteq \mathcal{T}$: local basis of $x_{\in \mathcal{X}}$:=

i. $\forall B_{\in \mathcal{B}_x}\ x \in B$ ii. $\forall \mathcal{U}_{\in \mathcal{T}}^{\ni x}\ \exists B_{\in \mathcal{B}_x}\ x \in B \subseteq \mathcal{U}$

Def: $(\mathcal{X}, \mathcal{T})$: firs-countable := $\forall x_{\in \mathcal{X}} \exists \mathcal{B}_x^{\text{local basis}}\ \mathcal{B}_x$: countable

Def: $(\mathcal{X}, \mathcal{T})$: second-countable := $\exists \mathcal{B}_{\subseteq \mathcal{T}}^{\text{basis}}\ \mathcal{B}$: countable

Def: $(\mathcal{X}, \mathcal{T})$ seperable := $\exists \mathcal{J}_{\subseteq \mathcal{X}}\ \mathcal{J}$: dense$^{\mathcal{T}}$ $\wedge$ countable

Lemma: $\mathcal{J}$ dence in $\mathcal{T}$ i.e. $\overline{\mathcal{J}} = \mathcal{X} \Leftrightarrow \forall \mathcal{U}_{\in \mathcal{T}}^{\neq \emptyset}\ \mathcal{J} \cap \mathcal{U} \neq \emptyset$

Proof: one of the initial Theorems states that

$\overline{\mathcal{J}} = \{ x_{\in \mathcal{X}} \mid \mathcal{U}_{(\ni x)}^{\in \mathcal{T}} \cap \mathcal{J} \neq \emptyset \}$, $\overline{\mathcal{J}} = \mathcal{X} : \forall x_{\in \mathcal{X}} \forall \mathcal{U}_{(\ni x)}^{\in \mathcal{T}} : \forall \mathcal{U}_{\in \mathcal{T}}^{\neq \emptyset}$ ■

Lemma: $\mathcal{B}$ basis of $\mathcal{T}$ $\Rightarrow$ $\forall x_{\in X}$ $\mathcal{B}_x = \{B^{\in \mathcal{B}}_{\ni x}\}$: local basis of x.

Proof: let $x \in X$, to prove the lemma we show that

$\forall U^{\in \mathcal{T}}_{\ni x}$ $\exists B_{\in \mathcal{B}_x}$ $x \in B \subseteq U$, let $U_{\ni x} \in \mathcal{T}$,

since $\mathcal{B}$ is a basis, $U = \bigcup_{B \in \mathcal{B}^*} B$ for some $\mathcal{B}^* \subseteq \mathcal{B}$,

since $x \in \bigcup_{B \in \mathcal{B}^*} B$, $\exists B_x^{\in \mathcal{B}^*}$ $x \in B_x$ so $B_x \in \mathcal{B}_x$ and

$x \in B_x \subseteq \bigcup_{B \in \mathcal{B}^*} B = U$ ■

Theorem: $(X, \mathcal{T})$: sec-cou $\Longrightarrow$ $(X, \mathcal{T})$: first cou.

Proof: $(X, \mathcal{T})$: sec-cou $\Longrightarrow$ $\exists \mathcal{B}^{\text{cou}}_{\subseteq \mathcal{T}}$ $\mathcal{B}$ basis of $\mathcal{T}$

$\Longrightarrow$ $\forall x_{\in X}$ $\mathcal{B}_x = \{B^{\in \mathcal{B}}_{\ni x}\}$: local basis of x $\wedge$ $\mathcal{B}_{x \subseteq \mathcal{B}}$: cou

$\Longrightarrow$ $\forall x_{\in X}$ $\exists \mathcal{J}_x^{\text{local basis}}{}_{(=\mathcal{B}_x)}$ $\mathcal{J}_x$: cou $\Longrightarrow$ $(X, \mathcal{T})$: first-cou ■

Theorem: $(X, \mathcal{T})$: sec-cou $\Longrightarrow$ $(X, \mathcal{T})$: seperable.

Proof: $(X, \mathcal{T})$: sec-cou $\Longrightarrow$ $\exists \mathcal{B}^{\text{basis}}_{\subseteq \mathcal{T}}$ $\mathcal{B}$: cou,

pick $x_i \in B_{i \in \mathcal{B}}$ from each $B_i^{\neq \emptyset}{}_{\in \mathcal{B}}$ define $J_{\subseteq X} = \{x_i\}_{i \in I \subset \mathbb{N}}$,

let $U_{\neq \emptyset} \in \mathcal{T}$, then $U = \bigcup_\lambda B_\lambda$ so $\exists B_i^{\neq \emptyset}$ $B_i \subseteq U$

so $x_i \in U$, hence $J \cap U \neq \emptyset$, by previous lemma,

$\overline{J} = X$, also by definition $J_{\subseteq X}$: cou, hence $(X, \mathcal{T})$: sep ■

Definition: Seperation Axioms:

T_0: Kolmogorov: $\forall x, y_{\in \mathcal{X}}^{(x \neq y)}\ \exists \mathcal{U}^{\ni x}, \mathcal{V}^{\ni y}_{\in \mathcal{T}}$

such that $x \notin \mathcal{V} \vee y \notin \mathcal{U}$.

T_1: Fréchet: $\forall x, y_{\in \mathcal{X}}^{(x \neq y)}\ \exists \mathcal{U}^{\ni x}, \mathcal{V}^{\ni y}_{\in \mathcal{T}}$

such that $x \notin \mathcal{V} \wedge y \notin \mathcal{U}$.

T_2: Hausdorff: $\forall x, y_{\in \mathcal{X}}^{(x \neq y)}\ \exists \mathcal{U}^{\ni x}, \mathcal{V}^{\ni y}_{\in \mathcal{T}}$

such that $\mathcal{U} \cap \mathcal{V} = \emptyset$.

T_3: Regular Hausdorff: $T_1 \wedge$ Regular.

T_4: Normal Hausdorff: $T_1 \wedge$ Normal.

Def: $\mathcal{U}^{\in \mathcal{T}}_{\supseteq \Omega}, \mathcal{V}^{\in \mathcal{T}}_{\supseteq \mathcal{H}}$ separate $\Omega, \mathcal{H}_{(\Omega \cap \mathcal{H} = \emptyset)} := \mathcal{U} \cap \mathcal{V} = \emptyset$.

Def: $(\mathcal{X}, \mathcal{T})$: Regular $:= \forall x_{\in \mathcal{X}}, \mathcal{F}^{\not\ni x}_{\in \mathcal{T}_c}\ \exists \mathcal{U}, \mathcal{V}_{\in \mathcal{T}}$

such that $\mathcal{U}, \mathcal{V}$ seperate $\mathcal{F}, \{x\}$.

Def: $(\mathcal{X}, \mathcal{T})$: Normal $:= \forall \mathcal{E}, \mathcal{F}^{(\mathcal{E} \cap \mathcal{F} = \emptyset)}_{\in \mathcal{T}_c}\ \exists \mathcal{U}, \mathcal{V}_{\in \mathcal{T}}$

such that $\mathcal{U} \subseteq \mathcal{E} \wedge \mathcal{V} \subseteq \mathcal{F}$.

Theorem: $T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$.

Proof:

$T_4 \Rightarrow T_3$: $\mathcal{X}: T_4 \Rightarrow \mathcal{X}: T_1 \wedge$ Normal,

Let $x \in \mathcal{X}$, $F_{\not\ni x} \in T_c$, since $\mathcal{X}: T_1$, by following Lemma,

$\{x\} \in T_c$, since $\mathcal{X}$: Normal, $\exists\, \mathcal{U}, \mathcal{V}_{\in \mathcal{T}}$ $(\mathcal{U} \supseteq \{x\}, \mathcal{V} \supseteq \mathcal{F})$

such that $\mathcal{U} \cap \mathcal{V} = \emptyset$ so $\mathcal{X}$: Reg, since also T_1,

$\mathcal{X}: T_1 \wedge$ Reg, hence $\mathcal{X}: T_3$, since $\mathcal{F}^{\not\ni x}_{\in T_c} \cap \{x\}_{\in T_c} = \emptyset$

$T_3 \Rightarrow T_2$: $\mathcal{X}: T_3 \Rightarrow \mathcal{X}: T_1 \wedge$ Reg, Let $x_{\in \mathcal{X}} \neq y_{\in \mathcal{X}}$

since $\mathcal{X}: T_1$, $\{x\}, \{y\} \in T_c$, since $\mathcal{X}$: Reg, $x \notin \{y\}_{\in T_c}$

$\exists\, \mathcal{U}_{\supseteq \{x\}}, \mathcal{V}_{\supseteq \{y\}} \in \mathcal{T}$ such that $\mathcal{U} \cap \mathcal{V} = \emptyset$; $\mathcal{X}: T_2$

$T_2 \Rightarrow T_1$: $\mathcal{X}: T_2 \Rightarrow \forall x, y_{\in \mathcal{X}}^{(x \neq y)}\ \exists\, \mathcal{U}^{\ni x}, \mathcal{V}^{\ni y} \in \mathcal{T}$

such that $\mathcal{U} \cap \mathcal{V} = \emptyset$: $x \notin \mathcal{V} \wedge y \notin \mathcal{U}$ so $\mathcal{X}: T_1$

$T_1 \Rightarrow T_0$: since $x \notin \mathcal{V} \wedge y \notin \mathcal{U} \Rightarrow x \notin \mathcal{V} \vee y \notin \mathcal{U}$

Hence, $T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$ ∎

Lemma: $\mathcal{X}:T_1 \Leftrightarrow \forall x_{\in\mathcal{X}}\ \{x\}\in\mathcal{T}_c$.

Proof: $(\Rightarrow)$: Let $\mathcal{X}:T_1$, $x\in\mathcal{X}$ then pick $y\in\{x\}^c$, if $\{x\}^c=\emptyset$, $\mathcal{X}=\{x\}$ so already $\{x\}_{(=\mathcal{X})}\in\mathcal{T}_c$. since $\mathcal{X}:T_1$ and $x\neq y_{\in\{x\}^c}$ $\exists V^{\ni y}_{\in\mathcal{T}}$ $x\notin V$ so $V_{\in\mathcal{T}}\subseteq\{x\}^c$

hence $V\subseteq(\{x\})^\circ$, since $\forall y_{\in\{x\}^c}$ $y\in(\{x\}^c)^\circ$,

$\{x\}^c_{\supseteq(\{x\}^c)^\circ}\subseteq(\{x\}^c)^\circ$ so $\{x\}^c=(\{x\}^c)^\circ$ so $\{x\}^c\in\mathcal{T}$

hence $\{x\}\in\mathcal{T}_c$,

$(\Leftarrow)$: Let $\forall x_{\in\mathcal{X}}\ \{x\}\in\mathcal{T}_c$, $x,y\underset{(x\neq y)}{\in}\mathcal{X}$ then

define $U_{\in\mathcal{T}}=\{y\}^c$, $V_{\in\mathcal{T}}=\{x\}^c$ hence

$x\notin V^{\ni y}_{\in\mathcal{T}}\wedge y\notin U^{\ni x}_{\in\mathcal{T}}$ so $\mathcal{X}:T_1$ ∎

Theorem: $\mathcal{X}:T_2$, $\Omega_{\subseteq\mathcal{X}}:|\Omega|\in\mathbb{N}\Rightarrow\Omega\in\mathcal{T}_c$.

Proof: $\mathcal{X}:T_2$ so $\mathcal{X}:T_1$ so $\forall x_{\in\mathcal{X}}\ \{x\}\in\mathcal{T}_c$

since $|\Omega|\in\mathbb{N}$, $\Omega=\bigcup\limits_{\substack{i\in I_{\in\mathbb{N}}\\ x_i\in\Omega}}\{x_i\}\in\mathcal{T}_c$ ∎

Def: $(x_n)_{n\,(=f(\mathbb{N});\ \mathbb{N}\to\mathcal{X})}$ converges to $\alpha_{\in\mathcal{X}}$:=

$\forall U^{\ni\alpha}_{\in\mathcal{T}}\ \exists N_{u\in\mathbb{N}}: x_n\in U,\ n\geq N_u$.

Theorem: $\mathcal{X}:T_2$, $x_n\to\alpha_{\in\mathcal{X}}\Rightarrow\exists!\sigma_{\in\mathcal{X}}\ x_n\to\sigma$.

Proof: Let $x_n\to\sigma_{1\in U_{\in\mathcal{T}}}$, $x_n\to\sigma_{2\in V_{\in\mathcal{T}}}:\sigma_1\neq\sigma_2$, $N:=\max\{N_u,N_v\}$ so $\forall x_{n\,(n\geq N)}\ x_n\in U,V$ but that is contradiction, since $\mathcal{X}:T_2$; $U\cap V=\emptyset$ ∎

Def: $f_{(\mathcal{X},\mathcal{T}) \to (\mathcal{Y},\mathcal{N})}$ continuous at $x_{\in \mathcal{X}}$ $:=$

$\exists \mathcal{B}^{loc}_{x}, \mathcal{B}^{loc}_{f(x)} : \forall B_{\in \mathcal{B}_{f(x)}} \exists B^*_{\in \mathcal{B}_x} \; f(B^*) \subseteq B$.

Def: $f_{(\mathcal{X},\mathcal{T}) \to (\mathcal{Y},\mathcal{N})}$ continuous on $\mathcal{U}_{\subseteq \mathcal{X}}$ $:=$

$\forall x_{\in \mathcal{U}}$ f continuous at x .

Theorem: $f_{\mathcal{X} \to \mathcal{Y}}$ $con_x \Leftrightarrow \forall \mathcal{V}^{\ni f(x)}_{\in \mathcal{N}} \exists \mathcal{U}^{\ni x}_{\in \mathcal{T}} \; f(\mathcal{U}) \subseteq \mathcal{V}$.

Proof: $(\Rightarrow)$: $\mathcal{V}^{\ni f(x)}_{\in \mathcal{N}}$, by definition of $\mathcal{B}^{loc}_{f(x)}$,

$$\exists B_{\in \mathcal{B}_{f(x)}} : f(x) \in B \subseteq \mathcal{V}$$

since $B \in \mathcal{B}_{f(x)}$, by con of f, $\exists B^*_{\in \mathcal{B}_x} : f(B^*) \subseteq B$

define $\mathcal{U} = B^{*\,\ni x}_{\in \mathcal{T}}$, hence $\forall \mathcal{V}^{\ni f(x)}_{\in \mathcal{N}} \exists \mathcal{U}^{\ni x}_{\in \mathcal{T}}$

$$f(\mathcal{U}) \subseteq B \subseteq \mathcal{V}$$

$(\Leftarrow)$: assume $\forall \mathcal{V}^{\ni f(x)}_{\in \mathcal{N}} \exists \mathcal{U}^{\ni x}_{\in \mathcal{T}} \; f(\mathcal{U}) \subseteq \mathcal{V}$

define $\mathcal{B}_x = \{ \mathcal{U}^{\ni x}_{\in \mathcal{T}} \}$, $\mathcal{B}_{f(x)} = \{ \mathcal{V}^{\ni f(x)}_{\in \mathcal{N}} \}$ local bases

$\forall \mathcal{U}_{\in \mathcal{B}_x} \; x \in \mathcal{U}$

$\forall \mathcal{U}^{\ni x}_{\in \mathcal{T}} \exists B^{(=\mathcal{U})}_{\in \mathcal{B}_x} \; x \in B \subseteq \mathcal{U}$

$\forall \mathcal{V}_{\in \mathcal{B}_{f(x)}} \; f(x) \in \mathcal{V}$

$\forall \mathcal{V}^{\ni x}_{\in \mathcal{N}} \exists B^{(=\mathcal{V})}_{\in \mathcal{B}_{f(x)}} \; f(x) \in B \subseteq \mathcal{V}$

so $\forall \mathcal{V}_{\in \mathcal{B}_{f(x)}} \exists \mathcal{U}_{\in \mathcal{B}_x} \; f(\mathcal{U}) \subseteq \mathcal{V}$

hence f con_x $\blacksquare$

Theorem: $f_{\mathcal{X}\to\mathcal{Y}}$, following are equivalent

i. f con on $\mathcal{X}$ ii. $\forall \mathcal{V}^{\in\mathcal{N}}_{\subseteq\mathcal{Y}}\ f^{-1}(\mathcal{V})\in\mathcal{T}$

iii. $\forall \mathcal{B}_{\mathcal{Y}}^{\text{basis}}\ \forall B_{\in\mathcal{B}_{\mathcal{Y}}}\ f^{-1}(B)\in\mathcal{T}$

Proof: $(i\Rightarrow ii)$: Let $\mathcal{V}_{\ni f(x)}\in\mathcal{N}$ so $\mathcal{V}=\mathcal{V}^{\circ}$

hence $\exists \mathcal{V}_{f(x)}{}^{\ni f(x)}_{\in\mathcal{N}}$: $f(x)\in\mathcal{V}_{f(x)}\subseteq\mathcal{V}$,

by previous Theorem, $\forall \mathcal{V}_{f(x)}{}^{\ni f(x)}_{\in\mathcal{N}}\ \exists \mathcal{U}_{x}{}^{\ni x}_{\in\mathcal{T}}\ f(\mathcal{U}_x)\subseteq\mathcal{V}_{f(x)}$

So $\mathcal{U}_x\subseteq f^{-1}(\mathcal{V})$ hence, since $f^{-1}(\mathcal{V})=\bigcup_{x\in f^{-1}(\mathcal{V})}\mathcal{U}_x$

$f^{-1}(\mathcal{V})\in\mathcal{T}$ since $\bigcup\mathcal{U}_x\in\mathcal{T}$ as $\mathcal{U}_x\in\mathcal{T}$

$(ii\Rightarrow iii)$: $\mathcal{B}_{\mathcal{Y}\subseteq\mathcal{N}}$: a basis of $\mathcal{Y}$, since $\mathcal{B}_{\mathcal{Y}}\subseteq\mathcal{N}$

$\forall B_{\in\mathcal{B}_{\mathcal{Y}}}$ $B\in\mathcal{N}$ so by assumption $f^{-1}(B)\in\mathcal{T}$

$(iii\Rightarrow i)$: define the local basis of $f(x)$: $\mathcal{B}_{f(x)}=\{B^{\ni f(x)}_{\in\mathcal{B}_{\mathcal{Y}}}\}$

Let $\mathcal{V}^{\ni f(x)}_{\in\mathcal{N}}$, so $\exists B_{\in\mathcal{B}_{f(x)}}$ $f(x)\in B\subseteq\mathcal{V}$

so $x\in f^{-1}(B)$ also since $B_{\in\mathcal{B}_{f(x)}}\in\mathcal{B}_{\mathcal{Y}}$ by assumption

$f^{-1}(B)\in\mathcal{T}$ so by previous Theorem, defining $\mathcal{U}=f^{-1}(B)$,

$f(\mathcal{U})=f(f^{-1}(B))=B\subseteq\mathcal{V}$ so f con $\blacksquare$

Theorem: $f^{\text{con}}_{\mathcal{X}\to\mathcal{Y}}$, $x_n\to x\Rightarrow f(x_n)\to f(x)$.

Proof: $\mathcal{V}^{\ni f(x)}_{\in\mathcal{N}}$, by con of f. $f^{-1}(\mathcal{V})_{\ni x}\in\mathcal{T}$, since $x_n\to x$

$\exists N_{f^{-1}(\mathcal{V})}: x_n\in f^{-1}(\mathcal{V}), n\geq N_{f^{-1}(\mathcal{V})}: f(x_n)\in\mathcal{V}$ so $f(x_n)\to f(x)$ $\blacksquare$

Theorem: $f_{(\mathcal{X},\mathcal{T})\to(\mathcal{Y},\mathcal{N})}, g_{(\mathcal{Y},\mathcal{N})\to(\Omega,\mathcal{M})}$: con

$\Longrightarrow g\circ f_{(\mathcal{X},\mathcal{T})\to(\Omega,\mathcal{M})}$: con .

Proof: note that $\forall V_{\in\mathcal{N}}\ f^{-1}(V)\in\mathcal{T}$
and $\forall J_{\in\mathcal{M}}\ g^{-1}(J)\in\mathcal{N}$

Let $W_{\subseteq\Omega}\in\mathcal{M}$, $(g\circ f)^{-1}(W)=f^{-1}(g^{-1}(W))$

since $W\in\mathcal{M}$ $g^{-1}(W)\in\mathcal{N}$ so $f^{-1}(g^{-1}(W))\in\mathcal{T}$

hence $\forall W_{\in\mathcal{M}}\ (g\circ f)^{-1}(W)\in\mathcal{T}$

so, $g\circ f$: con ■

Lemma: $\mathcal{X}\xrightarrow{f}\mathcal{Y}\Longrightarrow\forall J_{\subseteq\mathcal{Y}}\ f^{-1}(J^c)=(f^{-1}(J))^c$.

Proof: $x\in f^{-1}(J^c)\Rightarrow f(x)\in J^c\Rightarrow f(x)\notin J$

$\Rightarrow x\notin f^{-1}(J)\Rightarrow x\in(f^{-1}(J))^c$: $f^{-1}(J^c)\subseteq(f^{-1}(J))^c$

$x\in(f^{-1}(J))^c\Rightarrow x\notin f^{-1}(J)\Rightarrow f(x)\notin J$

$\Rightarrow f(x)\in J^c\Rightarrow x\in f^{-1}(J^c)$: $(f^{-1}(J))^c\subseteq f^{-1}(J^c)$

$f^{-1}(J^c)=(f^{-1}(J))^c$ ■

Theorem: $f_{\mathcal{X}\to\mathcal{Y}}$ con $\Leftrightarrow \forall V_{\in\mathcal{N}_c}\ f^{-1}(V)\in\mathcal{T}_c$.

Proof: $(\Rightarrow)$: $V\in\mathcal{N}_c \Rightarrow V^c\in\mathcal{N} \Rightarrow f^{-1}(V^c)\in\mathcal{T}$

$\Rightarrow \underbrace{(f^{-1}(V))^c}_{(=f^{-1}(V^c))}\in\mathcal{T} \Rightarrow f^{-1}(V)\in\mathcal{T}_c$

$(\Leftarrow)$: $V\in\mathcal{N} \Rightarrow V^c\in\mathcal{N}_c \Rightarrow f^{-1}(V^c)\in\mathcal{T}_c$

$\Rightarrow \underbrace{(f^{-1}(V))^c}_{(=f^{-1}(V^c))}\in\mathcal{T}_c \Rightarrow f^{-1}(V)\in\mathcal{T}$

$\Rightarrow f$: con ■

Theorem: $f_{\mathcal{X}\to\mathcal{Y}}$ con $\Leftrightarrow \forall J_{\subseteq\mathcal{X}}\ f(\overline{J})\subseteq\overline{f(J)}$.

Proof: $(\Rightarrow)$: $J = f^{-1}(f(J))\subseteq f^{-1}(\overline{f(J)})$

since $\overline{f(J)}\in\mathcal{N}_c$, $f^{-1}(\overline{f(J)})\in\mathcal{T}_c$

so $f^{-1}(\overline{f(J)}) \underset{\supseteq J}{\overset{\in\mathcal{T}_c}{}} \supseteq \overline{J}$ hence $f(\overline{J})\subseteq\overline{f(J)}$

$(\Leftarrow)$: Suppose f: $\neg$con, so $\exists C_{\in\mathcal{N}_c}$ such that

$f^{-1}(C)\notin\mathcal{T}_c$, so $f^{-1}(C)\neq\overline{f^{-1}(C)}$ so

let $\sigma\in\overline{f^{-1}(C)}\setminus f^{-1}(C)$, note: $C_{\in\mathcal{N}_c}=\overline{C}=\overline{f(f^{-1}(C))}$

since $f(\sigma)\in f(\overline{f^{-1}(C)}) \wedge f(\sigma)\notin\overline{f(f^{-1}(C))}$,

$\exists J^{\subseteq\mathcal{X}}_{(=f^{-1}(C))}\ f(\overline{J})\nsubseteq\overline{f(J)}$, hence

$\forall J_{\subseteq\mathcal{X}}\ f(\overline{J})\subseteq\overline{f(J)} \Rightarrow f$: con ■

Def: $f_{\mathcal{X}\to\mathcal{Y}}$ homeomorphism $:= f:\text{bij} \wedge f:\text{con} \wedge f^{-1}:\text{con}$

Def: $\mathcal{X} \cong \mathcal{Y} := \exists f_{\mathcal{X}\to\mathcal{Y}};\ f:\text{homeomorphism}$

Def: $\sim_{\subseteq \mathcal{X}\times\mathcal{X}}$ equivalence relation $:= \forall x,y,z_{\in\mathcal{X}}$

i. $x\sim x$ ii. $x\sim y \Leftrightarrow y\sim x$ iii. $x\sim y \wedge y\sim z \Rightarrow x\sim z$

Theorem: $\cong$: equivalence relation.

Proof: i. $\exists f^{(=id)}_{\mathcal{X}\to\mathcal{X}}\ f:\text{homeo} \Rightarrow \mathcal{X}\cong\mathcal{X}$

ii. $\mathcal{X}\cong\mathcal{Y} \Rightarrow \exists f_{\mathcal{X}\to\mathcal{Y}}\ f:\text{homeo}$

$\Rightarrow \exists g^{(=f^{-1})}_{\mathcal{Y}\to\mathcal{X}}\ g:\text{homeo} \Rightarrow \mathcal{Y}\cong\mathcal{X}$

iii. $\mathcal{X}\cong\mathcal{Y} \wedge \mathcal{Y}\cong\mathcal{Z} \Rightarrow \exists f_{\mathcal{X}\to\mathcal{Y}}, g_{\mathcal{Y}\to\mathcal{Z}}\ f,g:\text{homeo}$

$\Rightarrow \exists h^{(=g\circ f)}_{\mathcal{X}\to\mathcal{Z}}\ h:\text{homeo} \Rightarrow \mathcal{X}\cong\mathcal{Z}$ ∎

Lemma: $id_{\mathcal{X}\to\mathcal{X}}$: homeo.

Proof: i. $\forall x,y^{(x\neq y)}_{\in\mathcal{X}}\ id(x)_{(=x)} \neq id(y)_{(=y)}$; $id:\text{inj}$

$\forall y_{\in\mathcal{X}}\ \exists x^{(=y)}_{\in\mathcal{X}}\ id(x)=y$; $id:\text{surj}$; $id:\text{bij}$

ii. $\forall V_{\in\mathcal{T}}\ id^{-1}(V)_{(=V)} \in\mathcal{T}$; $id:\text{con}$

iii. $\forall V_{\in\mathcal{T}}\ (id^{-1})^{-1}(V)_{(=id(V)=V)} \in\mathcal{T}$; $id^{-1}:\text{con}$ ∎

Lemma: f: homeo $\Rightarrow f^{-1}$: homeo.

Proof: i. f: homeo $\Rightarrow f$: bij $\Rightarrow f^{-1}$: bij

ii. f homeo $\Rightarrow f^{-1}$ con

iii. f homeo $\Rightarrow \underbrace{(f^{-1})^{-1}}_{(=f)}$: con ■

Lemma: $f_{x\to y}, g_{y\to z}$: homeo $\Rightarrow g\circ f_{x\to z}$: homeo.

Proof: i. f,g: homeo $\Rightarrow f,g$: bij $\Rightarrow g\circ f$: bij

ii. f,g: homeo $\Rightarrow f,g$: con $\Rightarrow g\circ f$: con

iii. f,g: homeo $\Rightarrow f^{-1}, g^{-1}$: con $\Rightarrow f^{-1}\circ g^{-1}$: con

$\Rightarrow (g\circ f)^{-1}$: con ■

Def: $f_{x\to y}$: open $:= \forall U_{\in \mathcal{T}}\ f(U)\in \mathcal{N}$

$f_{x\to y}$: closed $:= \forall V_{\in \mathcal{T}_c}\ f(V)\in \mathcal{N}_c$

Lemma: $f_{x\to y}$ open $\Leftrightarrow \forall J_{\subseteq x}\ f(J^\circ)\subseteq (f(J))^\circ$

Proof: $(\Rightarrow)$: Let $J_{\subseteq x}$, since $J^\circ_{\subseteq J}\in \mathcal{T}$,

$f(J^\circ) = (f(J^\circ))^\circ \subseteq (f(J))^\circ$

$(\Leftarrow)$: Let $J_{\subseteq x}\in\mathcal{T}$ so $J=J^\circ$, by assumption

$f(J)=f(J^\circ)\subseteq (f(J))^\circ \subseteq f(J)$ so $f(J)=(f(J))^\circ$

so $f(J)\in\mathcal{N}$, hence f: open ■

Lemma: $f_{\mathcal{X}\to\mathcal{Y}}$ closed $\iff \forall J_{\subseteq \mathcal{X}}\ f(\overline{J}) \supseteq \overline{f(J)}$

Proof: $(\Rightarrow)$: Let $J_{\subseteq \mathcal{X}}$, since $\overline{J}_{\supseteq J} \in \mathcal{T}_c$

$f(\overline{J}) = \overline{f(\overline{J})} \supseteq \overline{f(J)}$

$(\Leftarrow)$: Let $J_{\subseteq \mathcal{X}} \in \mathcal{T}_c$ so $\overline{J} = J$, by assumption

$f(J) = f(\overline{J}) \supseteq \overline{f(J)} \supseteq f(J)$

So $f(J) = \overline{f(J)}$ hence $f(J) \in \mathcal{N}_c$

So f: closed ■

Theorem: $f_{\mathcal{X}\to\mathcal{Y}}$: bij,

i. f: open $\Rightarrow f^{-1}$: con ii. f: closed $\Rightarrow f^{-1}$: con

Proof: i. Since f: bij, $f^{-1}_{\mathcal{Y}\to\mathcal{X}}$ is a function,

by openness of f, $\forall V_{\in \mathcal{T}}\ (f^{-1})^{-1}(V)_{(=f(V))} \in \mathcal{N}$

So f^{-1}: con

ii. Since f: bij, $f^{-1}_{\mathcal{Y}\to\mathcal{X}}$ is a function,

by closeness of f, $\forall J_{\in \mathcal{T}_c}\ (f^{-1})^{-1}(J)_{(=f(J))} \in \mathcal{N}_c$

So f^{-1}: con ■

Theorem: $f_{X\to Y}$ homeo $\Rightarrow$ f: open $\wedge$ closed.

Proof: f: homeo $\Rightarrow$ f^{-1}: con.,

i. $\forall \mathcal{U}_{\in\mathcal{T}}\ (f^{-1})^{-1}(\mathcal{U})_{(=f(\mathcal{U}))} \in \mathcal{N} \Leftrightarrow f$: open

ii. $\forall V_{\in\mathcal{T}_c}\ (f^{-1})^{-1}(V)_{(=f(V))} \in \mathcal{N}_c \Leftrightarrow f$: closed $\blacksquare$

Theorem: $f_{X\to Y}$: homeo $\Rightarrow \forall J_{\subseteq X}\ f(J^\circ) = (f(J))^\circ$

Proof: $x \in f(J^\circ) \Rightarrow f^{-1}(x) \in J^\circ$

$\Rightarrow \exists \mathcal{U}_{\in\mathcal{T}}\ f^{-1}(x) \in \mathcal{U} \subseteq J$

$\Rightarrow x \in f(\mathcal{U})_{\in\mathcal{N}} \subseteq f(J) \Rightarrow x \in (f(J))^\circ$

$\Rightarrow f(J^\circ) \subseteq (f(J))^\circ$,

$x \in (f(J))^\circ \Rightarrow \exists V_{\in\mathcal{N}}\ x \in V \subseteq f(J)$

$\Rightarrow f^{-1}(x) \in f^{-1}(V)_{\in\mathcal{T}} \subseteq J \Rightarrow f^{-1}(x) \in J^\circ$

$\Rightarrow x \in f(J^\circ) \Rightarrow (f(J))^\circ \subseteq f(J^\circ)$,

hence $f(J^\circ) = (f(J))^\circ$ $\blacksquare$

Theorem: $f_{\mathcal{X}\to\mathcal{Y}}$: homeo $\Rightarrow \forall C_{\subseteq \mathcal{X}}\ f(\overline{C}) = \overline{f(C)}$.

Proof: Let $x \in f(\overline{C})$, so $f^{-1}(x) \in \overline{C}$

so $f^{-1}(x) \in C \cup \widetilde{C}$, if $f^{-1}(x) \in C$, then

$x \in f(C) \subseteq \overline{f(C)}$, so $f(\overline{C}) \subseteq \overline{f(C)}$,

if $f^{-1}(x) \in \widetilde{C}$, then $\forall \mathcal{U}^{\in \mathcal{T}}_{\ni f^{-1}(x)}\ (\mathcal{U} \setminus \{f^{-1}(x)\}) \cap C \neq \emptyset$

hence $f(C \cap \mathcal{U} \setminus \{f^{-1}(x)\}) = f(C) \cap f(\mathcal{U}) \setminus \{x\} \neq \emptyset$.

Since f: homeo, $\forall V_{\in \mathcal{N}}\ \exists \mathcal{U}_{\in \mathcal{T}}\ V = f(\mathcal{U})$,

so $\forall V^{\in \mathcal{N}}_{\ni x}\ f(C) \cap V \setminus \{x\} \neq \emptyset$ hence,

$x \in \widetilde{f(C)} \subseteq f(C) \cup \widetilde{f(C)} = \overline{f(C)}$

hence $f(\overline{C}) \subseteq \overline{f(C)}$,

Let $x \in \overline{f(C)} = f(C) \cup \widetilde{f(C)}$, if $x \in f(C)$,

$f^{-1}(x) \in C \subseteq \overline{C}$ so $x \in f(\overline{C})$ so $\overline{f(C)} \subseteq f(\overline{C})$,

if $x \in \widetilde{f(C)}$, $\forall V^{\in \mathcal{N}}_{\ni x}\ f(C) \cap V \setminus \{x\} \neq \emptyset$

so $\forall V^{\in \mathcal{N}}_{\ni x}\ C \cap f^{-1}(V) \setminus \{f^{-1}(x)\} \neq \emptyset$

Since f^{-1}: homeo, $\forall \mathcal{U}_{\in \mathcal{T}}\ \exists V_{\in \mathcal{N}}\ \mathcal{U} = f^{-1}(V)$

so $\forall \mathcal{U}^{\in \mathcal{T}}_{\ni f^{-1}(x)}\ C \cap \mathcal{U} \setminus \{f^{-1}(x)\} \neq \emptyset$ so $f^{-1}(x) \in \widetilde{C}$

so $x \in f(\widetilde{C}) \subseteq f(\overline{C})$ so $\overline{f(C)} \subseteq f(\overline{C})$ $\blacksquare$

Theorem: $\underset{\mathcal{X}\to\mathcal{Y}}{f}$: homeo $\Rightarrow \forall C_{\subseteq \mathcal{X}}\; f(\tilde{C}) = \widetilde{f(C)}$

Proof: $x \in f(\tilde{C}) \Rightarrow f^{-1}(x) \in \tilde{C}$

$\Rightarrow \forall \mathcal{U}^{\in \mathcal{T}}_{\ni f^{-1}(x)}\; C \cap \mathcal{U} \setminus \{f^{-1}(x)\} \neq \emptyset$

$\Rightarrow \forall \mathcal{U}^{\in \mathcal{T}}_{\ni f^{-1}(x)}\; f(C) \cap f(\mathcal{U}) \setminus \{x\} \neq \emptyset$

Since f: homeo, $\forall V_{\in \mathcal{N}}\; \exists \mathcal{U}_{\in \mathcal{T}}\; V = f(\mathcal{U})$

$\Rightarrow \forall V^{\in \mathcal{N}}_{\ni x}\; f(C) \cap V \setminus \{x\} \neq \emptyset \Rightarrow x \in \widetilde{f(C)}$

$\Rightarrow f(\tilde{C}) \subseteq \widetilde{f(C)}$,

$x \in \widetilde{f(C)} \Rightarrow \forall V^{\in \mathcal{N}}_{\ni x}\; f(C) \cap V \setminus \{x\} \neq \emptyset$

$\Rightarrow \forall V^{\in \mathcal{N}}_{\ni x}\; C \cap f^{-1}(V) \setminus \{f^{-1}(x)\} \neq \emptyset$

Since f: homeo, $\forall \mathcal{U}_{\in \mathcal{T}}\; \exists V_{\in \mathcal{N}}\; \mathcal{U} = f^{-1}(V)$

$\Rightarrow \forall \mathcal{U}^{\in \mathcal{T}}_{\ni f^{-1}(x)}\; C \cap \mathcal{U} \setminus \{f^{-1}(x)\} \neq \emptyset$

$\Rightarrow f^{-1}(x) \in \tilde{C} \Rightarrow x \in f(\tilde{C}) \Rightarrow \widetilde{f(C)} \subseteq f(\tilde{C})$,

hence $f(\tilde{C}) = \widetilde{f(C)}$ ■

Theorem: $\underset{\mathcal{X}\to\mathcal{Y}}{f}$: homeo $\Rightarrow \forall J_{\subseteq \mathcal{X}}\; f(\partial J) = \partial(f(J))$

Proof: $x \in f(\partial J) \Leftrightarrow f^{-1}(x) \in \partial J = \bar{J} \setminus J^{\circ}$

$\Leftrightarrow x \in f(\bar{J}) \setminus f(J^{\circ}) \underset{\text{by Previous Theorems}}{=} \overline{f(J)} \setminus (f(J))^{\circ} = \partial(f(J))$, so

$\forall J_{\subseteq \mathcal{X}}\; f(\partial J) = \partial(f(J))$ ■

Theorem: $\mathcal{X} \cong \mathcal{Y} \wedge \mathcal{X}$: first-cou $\Longrightarrow \mathcal{Y}$: first-cou .

Proof: $\mathcal{X}$: first-cou so $\forall x \, \exists \mathcal{B}_x \overset{\text{cou}}{\subseteq} \mathcal{T}$ $\mathcal{B}_x$: local basis of x,

Let $x \in \mathcal{X}$, $B \in \mathcal{B}_x$ so $f(x) \in f(B) \in \mathcal{N}$ as f: open ,

also since f: bij , $\forall y_{\in \mathcal{Y}} \exists x_{\in \mathcal{X}}$ $y = f(x)$, let $y \in \mathcal{Y}$

define $\mathcal{B}_y = \{ f(B) : B \in \mathcal{B}_{f^{-1}(y)} \}$; $\mathcal{B}_{f^{-1}(y)} \overset{\text{cou}}{}$: local basis of $f^{-1}(y)$

note that since $\mathcal{B}_{f^{-1}(y)}$: cou , $\mathcal{B}_y$: cou ,

Suppose $\mathcal{B}_y$ is not a local basis of y ; so

$\neg(\forall V^{\ni y}_{\in \mathcal{N}} \exists B_{\in \mathcal{B}_y} B \subseteq V) \equiv \exists V^{\ni y}_{\in \mathcal{N}} \forall B_{\in \mathcal{B}_y} B \nsubseteq V$

so $\forall B_{\in \mathcal{B}_{f^{-1}(y)}}$ $f^{-1}(y) \in B \nsubseteq f^{-1}(V) \in \mathcal{T}$

hence $\exists \mathcal{U}^{\in \mathcal{T}, \ni f^{-1}(y)}_{(= f^{-1}(V))} \forall B_{\in \mathcal{B}_{f^{-1}(y)}} B \nsubseteq \mathcal{U}$

so $\mathcal{B}_{f^{-1}(y)}$ is not a local basis of $f^{-1}(y)$, which is contradiction

so $\mathcal{B}_y$: local basis of y hence $\mathcal{Y}$: first-cou ■

Theorem: $\mathcal{X} \cong \mathcal{Y} \wedge \mathcal{X}$: sec-cou $\Longrightarrow \mathcal{Y}$: sec-cou.

Proof: $\mathcal{X}$: sec-cou so $\exists \mathcal{B}_{\subseteq \mathcal{T}}\ \forall \mathcal{U}_{\in \mathcal{T}}\ \exists \mathcal{B}^*_{\subseteq \mathcal{B}}$

$\mathcal{U} = \bigcup_{B \in \mathcal{B}^*} B$ so $f(\mathcal{U}) = f(\bigcup_{B \in \mathcal{B}^*} B) = \bigcup_{B \in \mathcal{B}^*} f(B)$

note that $\forall \mathcal{V}_{\in \mathcal{N}}\ \exists \mathcal{U}_{\in \mathcal{T}}\ \mathcal{V} = f(\mathcal{U})$

define $\tilde{\mathcal{B}} = \{ f(B) : B \in \mathcal{B} \}$, note that $\tilde{\mathcal{B}}$: cou

since $\mathcal{B}$: cou, suppose $\tilde{\mathcal{B}}$ is not a basis of $\mathcal{Y}$,

so $\exists \mathcal{V}_{\in \mathcal{N}}\ \forall \mathcal{B}^*_{\subseteq \mathcal{B}}\ \mathcal{V} \neq \bigcup_{B \in \mathcal{B}^*} f(B)$

so $\exists \mathcal{U}^{\in \mathcal{T}}_{(=f^{-1}(\mathcal{V}))}\ \forall \mathcal{B}^*_{\subseteq \mathcal{B}}\ \mathcal{U} \neq \bigcup_{B \in \mathcal{B}^*} B$

so $\mathcal{B}$ is not a basis of $\mathcal{X}$, which is contradiction

hence $\tilde{\mathcal{B}}$: basis of $\mathcal{Y}$ so, $\mathcal{Y}$: sec-cou $\blacksquare$

Theorem: $\mathcal{X} \cong \mathcal{Y} \wedge \mathcal{X} : T_2 \Longrightarrow \mathcal{Y} : T_2$.

Proof: $x, y \in \mathcal{Y}_{(x \neq y)} \Longrightarrow f^{-1}(x) \neq f^{-1}(y)$

$\Longrightarrow \exists \mathcal{U}^{\in \mathcal{T}}_{\ni f^{-1}(x)}, \mathcal{V}^{\in \mathcal{T}}_{\ni f^{-1}(y)}\ \mathcal{U} \cap \mathcal{V} = \emptyset$

$\Longrightarrow$ by bij of f, $f(\mathcal{U}) \cap f(\mathcal{V}) = \emptyset$

$\Longrightarrow \forall x, y^{(x \neq y)}_{\in \mathcal{Y}}\ \exists \mathcal{H}^{\ni x}_{(=f(\mathcal{U}))}, \Omega^{\ni y}_{(=f(\mathcal{V}))}\ \mathcal{H} \cap \Omega = \emptyset$

$\Longrightarrow \mathcal{Y} : T_2$ $\blacksquare$

Theorem: $\mathcal{X}$ sep $\wedge$ $\mathcal{X} \cong \mathcal{Y} \Longrightarrow \mathcal{Y}$: sep.

Proof: $\mathcal{X}$ sep so $\exists \Omega^{\text{cou}}_{\subseteq \mathcal{X}}$ $\overline{\Omega} = \mathcal{X}$

i.e. $\forall \mathcal{U}^{\neq \emptyset}_{\in \mathcal{T}}$ $\Omega \cap \mathcal{U} \neq \emptyset$,

note that since f: bij, $f(\Omega)$: cou

Let $V_{\neq \emptyset} \in \mathcal{N}$, since f: con, $f^{-1}(V) \in \mathcal{T}$

so $\Omega \cap f^{-1}(V) \neq \emptyset$ hence $f(\Omega) \cap V \neq \emptyset$

hence $\forall V^{\neq \emptyset}_{\in \mathcal{N}}$ $f(\Omega) \cap V \neq \emptyset$ i.e.

$\exists \mathfrak{J}^{\text{cou}}_{\substack{\subseteq \mathcal{Y} \\ (= f(\Omega))}}$ $\overline{\mathfrak{J}} = \mathcal{Y}$ so $\mathcal{Y}$: sep $\blacksquare$

Def: $\mathcal{T}_{\subseteq \mathcal{P}(\mathcal{X})}$: Initial Topology induced by $\{\underset{\mathcal{Y}_i \to \mathcal{X}}{f_i}\}_i$ on $\mathcal{X}$:=
$\mathcal{T} \subseteq \mathcal{T}^*$: $\forall i$ f_i: con on $\mathcal{T}^*$.

Def: $S_{\subseteq \mathcal{T}}$: subbasis := $\mathcal{B}_S = \{ \bigcap_{J \subset \mathbb{N}} \mathcal{U}_j : \forall_{j \in J} \mathcal{U}_j \in S \}$: basis

Def: $\mathcal{T}_{\subseteq \mathcal{P}(\mathcal{X})}$: Final Topology induced by $\{\underset{\mathcal{Y}_i \to \mathcal{X}}{f_i}\}_i$ on $\mathcal{X}$:=
$\mathcal{T} \supseteq \mathcal{T}^*$: $\forall i$ f_i: con on $\mathcal{T}^*$ -

Theorem: $\mathcal{T}$: final topology induced by $\{\underset{\mathcal{Y}_i \to \mathcal{X}}{f_i}\}$ on $\mathcal{X}$,

$$\mathcal{T} = \{\mathcal{U}_{\subseteq \mathcal{X}} : \forall i \; f_i^{-1}(\mathcal{U}) \in \mathcal{T}_{\mathcal{Y}_i}\} .$$

Proof: define $\vartheta_i = \{\mathcal{U}_{\subseteq \mathcal{X}} : f_i^{-1}(\mathcal{U}) \in \mathcal{T}_{\mathcal{Y}_i}\}$

Since $\forall \mathcal{Y}_{i \subseteq \vartheta_i}$ $f_i^{-1}(\bigcup_{u \in \mathcal{Y}_i} u) = \bigcup_{u \in \mathcal{Y}_i} f_i^{-1}(u) \in \mathcal{T}_{\mathcal{Y}_i}$

also $f_i^{-1}(\bigcap_{u \in \mathcal{Y}_i} u) = \bigcap_{u \in \mathcal{Y}_i} f_i^{-1}(u) \in \mathcal{T}_{\mathcal{Y}_i}$

so $\bigcup_{u \in \mathcal{Y}_i} u, \bigcap_{u \in \mathcal{Y}_i} u \in \vartheta_i$

and $f_i^{-1}(\mathcal{X}) = \mathcal{Y}_i \in \mathcal{T}_{\mathcal{Y}_i}$, $f_i^{-1}(\emptyset) = \emptyset \in \mathcal{T}_{\mathcal{Y}_i}$

so $\mathcal{X}, \emptyset \in \vartheta_i$

hence ϑ_i is a topology on $\mathcal{X}$,

$\mathcal{T} = \bigcap_i \vartheta_i$ is also a topology on $\mathcal{X}$,

also since if $\forall i \; f_i$: con $\mathcal{T}^*$ then $\mathcal{T}^* \subseteq \mathcal{T}$

$\mathcal{T}$: final topology induced by $\{\underset{\mathcal{Y}_i \to \mathcal{X}}{f_i}\}_i$ on $\mathcal{X}$ $\blacksquare$

Def: $[x] = \{y \mid y \sim x\}$

$$\mathcal{X}/\sim \; = \{[x] \mid x \in \mathcal{X}\}$$

Def: $q_{\mathcal{X}\to\mathcal{X}/\sim}$: quotient map $:=$ $\forall x_{\in\mathcal{X}}\ q(x)=[x]$.

Def: $(\mathcal{X}/\sim, \mathcal{T}_{\mathcal{X}/\sim})$: Topological Quotient $:=$

$$\mathcal{T}_{\mathcal{X}/\sim}=\{\mathcal{U}_{\subseteq\mathcal{X}/\sim} : q^{-1}(\mathcal{U})\in\mathcal{T}_{\mathcal{X}}\}.$$

Lemma: $\mathcal{U}_{\subseteq\mathcal{X}/\sim}\in\mathcal{T}_{\mathcal{X}/\sim} \iff \bigcup_{[x]\in\mathcal{U}}[x]\in\mathcal{T}_{\mathcal{X}}$.

Proof: $(\Rightarrow)$: $\mathcal{U}\in\mathcal{T}_{\mathcal{X}/\sim} \Rightarrow q^{-1}(\mathcal{U})_{(=\bigcup_{[x]\in\mathcal{U}}[x])}\in\mathcal{T}_{\mathcal{X}}$

$$\Rightarrow \bigcup_{[x]\in\mathcal{U}}[x]\in\mathcal{T}_{\mathcal{X}}$$

$(\Leftarrow)$: $\bigcup_{[x]\in\mathcal{U}}[x]\in\mathcal{T}_{\mathcal{X}} \Rightarrow q(\bigcup_{[x]\in\mathcal{U}}[x])=\mathcal{U} : q^{-1}(\mathcal{U})\in\mathcal{T}_{\mathcal{X}}$

$$\Rightarrow \mathcal{U}\in\mathcal{T}_{\mathcal{X}/\sim} \qquad \blacksquare$$

Lemma: $C_{\subseteq\mathcal{X}/\sim}\in\mathcal{T}_{\mathcal{X}/\sim}^{c} \iff \bigcup_{[x]\in C}[x]\in\mathcal{T}_{\mathcal{X}}^{c}$.

Proof: $(\Rightarrow)$: $C\in\mathcal{T}_{\mathcal{X}/\sim}^{c} \Rightarrow (\mathcal{X}/\sim)\setminus C\in\mathcal{T}_{\mathcal{X}/\sim}$

$\Rightarrow q^{-1}((\mathcal{X}/\sim)\setminus C)_{\in\mathcal{T}_{\mathcal{X}}} = \bigcup_{[x]\in(\mathcal{X}/\sim)\setminus C}[x]$

$\Rightarrow \mathcal{X}\setminus\bigcup_{[x]\in(\mathcal{X}/\sim)\setminus C}[x] = \bigcup_{[x]\in\mathcal{X}/\sim}[x]\setminus\bigcup_{[x]\in(\mathcal{X}/\sim)\setminus C}[x] = \bigcup_{[x]\in C}[x]\in\mathcal{T}_{\mathcal{X}}^{c}$

Since $\mathcal{X}=\bigcup_{[x]\in\mathcal{X}/\sim}[x]$ because of $\forall x_{\in\mathcal{X}}\ x\in[x]$ with $x\sim x$,

$(\Leftarrow)$: $\bigcup_{[x]\in C}[x]\in\mathcal{T}_{\mathcal{X}}^{c} \Rightarrow \mathcal{X}\setminus\bigcup_{[x]\in C}[x] = \bigcup_{[x]\in(\mathcal{X}/\sim)\setminus C}[x]\in\mathcal{T}_{\mathcal{X}}$

$\Rightarrow \mathcal{X}\setminus C\in\mathcal{T}_{\mathcal{X}/\sim} \Rightarrow C\in\mathcal{T}_{\mathcal{X}/\sim}^{c} \qquad \blacksquare$

by
Procrus Theorem

Theorem: $f^{\text{con}}_{\mathcal{X}\to\mathcal{Y}}$, $x_0 \sim_f x_1 := f(x_0) = f(x_1)$,

$g_{\mathcal{X}/\sim_f \to \mathcal{Y}}$, $g([x]) := f(x)$,

i. g : inj $\wedge$ con

ii. f : open $\vee$ closed $\implies \mathcal{X}/\sim_f \cong f(\mathcal{X})$ with g

Proof: i. $g([x_0]) = g([x_1]) \implies f(x_0) = f(x_1)$

$\implies x_0 \sim_f x_1 \implies [x_0] = [x_1]$; g : inj ,

Let $q_{\mathcal{X}\to\mathcal{X}/\sim_f}$: quotient map,

Since $(g\circ q)(x) = g(q(x)) = g([x]) = f(x)$, $f = g\circ q$

Let $\mathcal{U} \in \mathcal{T}_{\mathcal{Y}}$, since f : con, $f^{-1}(\mathcal{U}) \in \mathcal{T}_{\mathcal{X}}$ i.e.

$f^{-1}(\mathcal{U}) = (g\circ q)^{-1}(\mathcal{U}) = q^{-1}(g^{-1}(\mathcal{U})) = \bigcup_{[x]\in g^{-1}(\mathcal{U})} [x] \in \mathcal{T}_{\mathcal{X}}$

So, by previous Theorem, $g^{-1}(\mathcal{U}) \in \mathcal{T}_{\mathcal{X}/\sim_f}$

hence g : con

ii. f : open , since $f = g\circ q$, $g = f\circ q^{-1}$

So $\forall \mathcal{U}_{\in \mathcal{T}_{\mathcal{X}/\sim_f}}$ $q^{-1}(\mathcal{U}) \in \mathcal{T}_{\mathcal{X}}$ so $f(q^{-1}(\mathcal{U})) \in \mathcal{T}_{\mathcal{Y}}$

hence $g(\mathcal{U}) \in \mathcal{T}_{\mathcal{Y}}$ so g : open so g^{-1} : con ,

Since $\mathcal{X}/\sim_f \xrightarrow{g} f(\mathcal{X})$: surj , since $\forall y_{\in f(\mathcal{X})} \exists [x]^{\in \mathcal{X}/\sim_f}_{(=[f^{-1}(y)])}$

$g^{-1}(y) = [x]$, so g : bij , hence $\mathcal{X}/\sim_f \cong f(\mathcal{X})$ with g ∎

Lemma: $f_{\Omega_0 \to Y}, g_{\Omega_1 \to Y}$: con, $f|_{\Omega_0 \cap \Omega_1} = g|_{\Omega_0 \cap \Omega_1}$

$\Omega = \Omega_0^{\in \mathcal{T}_{\Omega c}} \cup \Omega_1^{\in \mathcal{T}_{\Omega c}}$, $(\Omega, \mathcal{T}_\Omega) \xrightarrow{h} (Y, \mathcal{T}_Y)$: $h := \begin{cases} f(x), & x \in \Omega_0 \\ g(x), & x \in \Omega_1 \end{cases}$

$\Rightarrow h$: con.

Proof: $V \in \mathcal{T}_{Yc} \Rightarrow f^{-1}(V) \in \mathcal{T}_{\Omega_0 c}$

$\Rightarrow \exists K_{0 \in \mathcal{T}_{\Omega c}}$ $f^{-1}(V) = \Omega_0 \cap K_0$ by a previous Theorem

$\Rightarrow f^{-1}(V) \in \mathcal{T}_{\Omega c}$, since $\Omega_0 \in \mathcal{T}_{\Omega c}$

$V \in \mathcal{T}_{Yc} \Rightarrow g^{-1}(V) \in \mathcal{T}_{\Omega_1 c}$

$\Rightarrow \exists K_{1 \in \mathcal{T}_{\Omega c}}$ $g^{-1}(V) = \Omega_1 \cap K_1$ by a previous Theorem

$\Rightarrow g^{-1}(V) \in \mathcal{T}_{\Omega c}$, since $\Omega_1 \in \mathcal{T}_{\Omega c}$

Hence, $\forall V_{\in \mathcal{T}_{Yc}}$ $h^{-1}(V)_{(= f^{-1}(V) \cup g^{-1}(V))} \in \mathcal{T}_{\Omega c}$

So, h : con ∎

Def: $\bigoplus_{i\in I} X_i := (\bigcup_i X_i, \mathcal{T}_{\mathcal{B}_\oplus})$: Topological Sum :=

$$\mathcal{B}_\oplus = \{ \mathcal{U}_{\subseteq \bigcup_i X_i} : \exists i \ \mathcal{U} \in \mathcal{T}_i \}, \quad X_{j_0} \cap X_{j_1} = \emptyset$$

Theorem: $V \in \mathcal{T}_{\mathcal{B}_\oplus} \iff \forall i \ V \cap X_i \in \mathcal{T}_i$

Proof: $(\Rightarrow)$: $V \in \mathcal{T}_{\mathcal{B}_\oplus} \Rightarrow \exists \mathcal{B}^*_{\subseteq \mathcal{B}_\oplus} \ V = \bigcup_{B\in\mathcal{B}^*} B$

since $\forall B_{\in \mathcal{B}^*} \ \exists i \ B \in \mathcal{T}_i$, for given i,

$$V \cap X_i = X_i \cap \bigcup_{B\in\mathcal{B}^*} B = \bigcup_{B\in\mathcal{B}^*} (X_i \cap B)$$

if $\exists B_{\in\mathcal{B}^*} \ B \subseteq X_i$, $V \cap X_i \in \mathcal{T}_i$

if $\forall B_{\in\mathcal{B}^*} \ B \cap X_i = \emptyset$, $V \cap X_i = \emptyset \in \mathcal{T}_i$

$(\Leftarrow)$: $\forall i \ V \cap X_i \in \mathcal{T}_i$, $\forall i \ V \cap X_i \in \mathcal{B}_\oplus$

$$V = V \cap \bigcup_i X_i = \bigcup_i (V \cap X_i) \in \mathcal{T}_{\mathcal{B}_\oplus} \quad \blacksquare$$

Def: Ω : disconnected $:= \exists \Omega_0, \Omega_1 \overset{\in \mathcal{T}_\Omega\ (\neq \emptyset)}{\underset{(\Omega_0 \cap \Omega_1 = \emptyset)}{}}$

$$\Omega = \Omega_0 \cup \Omega_1$$

Ω : connected $:= \Omega : \neg$ disconnected, Ω_0, Ω_1 : sep

Theorem: Ω : disconn $\Longleftrightarrow \Omega \cong \Omega_0 \oplus \Omega_1$.

Proof: $(\Longrightarrow)$: $\mathcal{U} \in \mathcal{T}_\Omega$,

if $\mathcal{U} \subseteq \Omega_0$, $\mathcal{U} = \mathcal{U} \cap \Omega_0 \in \mathcal{T}_{\Omega_0}$

and, $\emptyset = \mathcal{U} \cap \Omega_1 \in \mathcal{T}_{\Omega_1}$ so $\mathcal{U} \in \mathcal{T}_\oplus$

if $\mathcal{U} \subseteq \Omega_1$, $\mathcal{U} = \mathcal{U} \cap \Omega_1 \in \mathcal{T}_{\Omega_1}$

and, $\emptyset = \mathcal{U} \cap \Omega_0 \in \mathcal{T}_{\Omega_0}$ so $\mathcal{U} \in \mathcal{T}_\oplus$

if $\mathcal{U} \nsubseteq \Omega_0 \wedge \mathcal{U} \nsubseteq \Omega_1$, $\mathcal{U} \cap \Omega_0 \neq \emptyset \wedge \mathcal{U} \cap \Omega_1 \neq \emptyset$

so $\mathcal{U} = (\mathcal{U} \cap \Omega_0)_{\in \mathcal{T}_\oplus} \cup (\mathcal{U} \cap \Omega_1)_{\in \mathcal{T}_\oplus} \in \mathcal{T}_\oplus$

if $\mathcal{U} \subseteq \Omega_0 \wedge \mathcal{U} \subseteq \Omega_1$, $\mathcal{U} \subseteq \Omega_0 \cap \Omega_1 = \emptyset \in \mathcal{T}_\oplus$,

$\mathcal{U} \in \mathcal{T}_\oplus$, then $\mathcal{U} = \bigcup_\lambda \mathcal{B}_\lambda$: $\mathcal{B}_\lambda \in \mathcal{T}_{\Omega_0} \vee \mathcal{T}_{\Omega_1}$ so $\mathcal{B}_\lambda \in \mathcal{T}_\Omega$,

$\mathcal{U} = (\mathcal{U} \cap \Omega_0) \cup (\mathcal{U} \cap \Omega_1) \in \mathcal{T}_\Omega$

So $\mathcal{T}_\Omega = \mathcal{T}_\oplus$ and so $\Omega_{\in \mathcal{T}_\Omega} \in \mathcal{T}_\oplus \wedge \Omega_0 \oplus \Omega_1{}_{\in \mathcal{T}_\oplus} \in \mathcal{T}_\Omega$

So $\Omega \subseteq \Omega_0 \oplus \Omega_1 \wedge \Omega_0 \oplus \Omega_1 \subseteq \Omega$

So $\Omega = \Omega_0 \oplus \Omega_1$ hence $\Omega \cong \Omega_0 \oplus \Omega_1$

Proof cons: ($\Leftarrow$): Suppose $\Omega \cong \Omega_0 \oplus \Omega_1$

with $f_{\Omega \to \Omega_0 \oplus \Omega_1}$, since $\Omega_0 \in \mathcal{T}_{\Omega_0}$, $\Omega_1 \in \mathcal{T}_{\Omega_1}$

and $\emptyset = \Omega_0 \cap \Omega_1 \in \mathcal{T}_{\Omega_0}, \mathcal{T}_{\Omega_1}$; $\Omega_0, \Omega_1 \in \mathcal{T}_{\oplus}$,

$f^{-1}(\Omega_0), f^{-1}(\Omega_1) \in \mathcal{T}_\Omega$ also since $\Omega_0 \cap \Omega_1 = \emptyset$

$f^{-1}(\Omega_0) \cap f^{-1}(\Omega_1) = f^{-1}(\Omega_0 \cap \Omega_1) = f^{-1}(\emptyset) = \emptyset$

and also, since $f(\Omega) = \Omega_0 \cup \Omega_1$,

$f^{-1}(\Omega_0) \cup f^{-1}(\Omega_1) = f^{-1}(\Omega_0 \cup \Omega_1) = \Omega$

Hence, Ω: discon ■

Theorem: Ω: discon $\Leftrightarrow \exists f^{\text{con} \wedge \text{surj}}_{(\Omega, \mathcal{T}_\Omega) \to (\{0,1\}, \mathcal{T}_{dis})}$

Proof: ($\Rightarrow$): define f as

$$f(x) = \begin{cases} 0, & x \in \Omega_0 \\ 1, & x \in \Omega_1 \end{cases}$$, as $f(\Omega) = \{0,1\}$, f: surj

$f^{-1}(\emptyset) = \emptyset^{\in \mathcal{T}_\Omega}$, $f^{-1}(\{0\}) = \Omega_0^{\in \mathcal{T}_\Omega}$, $f^{-1}(\{1\}) = \Omega_1^{\in \mathcal{T}_\Omega}$

$f^{-1}(\{0,1\}) = \Omega_0 \cup \Omega_1 = \Omega^{\in \mathcal{T}_\Omega}$ so f: con

($\Leftarrow$): Since f: con; $f^{-1}(\{0\}), f^{-1}(\{1\}) \in \mathcal{T}_\Omega$

as f: surj, $f^{-1}(\{0\}), f^{-1}(\{1\}) \neq \emptyset$,

also $\Omega = f^{-1}(\{0,1\}) = f^{-1}(\{0\} \cup \{1\}) = f^{-1}(\{0\}) \cup f^{-1}(\{1\})$

and $f^{-1}(\{0\}) \cap f^{-1}(\{1\}) = \emptyset$, so Ω: discon ■
$(= f^{-1}(\{0\} \cap \{1\}) = f^{-1}(\emptyset) = \emptyset)$

Theorem: Ω : disconn $\iff \exists J^{\in \mathcal{T}_\Omega}_{\in \mathcal{T}_{\Omega^c}} \; J_{\subset\Omega} \neq \emptyset$

Proof: $(\Rightarrow)$: $\Omega_0{}^{\neq\emptyset}_{\subset\Omega} \in \mathcal{T}_\Omega$, also since $\Omega_1^{\in\mathcal{T}_\Omega} = \Omega_0^c$

$\Omega_0^{\in\mathcal{T}_\Omega} \in \mathcal{T}_{\Omega^c}$ ::

$(\Leftarrow)$: $J^{\in\mathcal{T}_\Omega}_{\in\mathcal{T}_{\Omega^c}}$ so $J^{c}{}^{\in\mathcal{T}_{\Omega^c}}_{\in\mathcal{T}_\Omega}$ also $J \cap J^c = \emptyset$

and since $J_{\subset\Omega} \neq \Omega, \emptyset$; $J^c_{\subset\Omega} \neq \emptyset, \Omega$

also $\Omega = J \cup J^c$, so $\{J, J^c\}$: sep of Ω

So, Ω : disconn

Def: $C_{\subseteq\Omega}$ conn $:= (C, \mathcal{T}_{\Omega|C})$: conn

Theorem: $J_{\subseteq\Omega}$: conn $\Rightarrow \overline{J}_{\subseteq\Omega}$: conn.

Proof: Let $J_{\subseteq\Omega}$: conn, assume $\overline{J}_{\subseteq\Omega}$: disconn,

then $\exists C_0, C_1{}^{\in\mathcal{T}_{\Omega|\overline{J}}}_{\in\mathcal{T}_{\Omega|\overline{J}^c}}$ $\overline{J} = C_0 \cup C_1$: $C_0{}^{\neq\emptyset}_{\subset\overline{J}} \cap C_1{}^{\neq\emptyset}_{\subset\overline{J}} = \emptyset$

note $C_0, C_1 \in \mathcal{T}_{\Omega|\overline{J}^c}$, $C_0 = \overline{C_0} \wedge C_1 = \overline{C_1}$,

since $J \subseteq \overline{J}$, $J = (J \cap C_0) \cup (J \cap C_1)$

Since $C \in \mathcal{T}_{\Omega|\overline{J}} \Rightarrow \exists \mathcal{U}_{\in\mathcal{T}_\Omega} \; C = \overline{J} \cap \mathcal{U}$

$\Rightarrow J \cap C = J \cap \overline{J}_{\supseteq J} \cap \mathcal{U}_{\in\mathcal{T}} = J \cap \mathcal{U}_{\in\mathcal{T}} \in \mathcal{T}_{\Omega|J}$,

$J \cap C_0 , J \cap C_1 \in \mathcal{T}_{\Omega|J}$, Proof cons: So, since J : conn,

$J \cap C_0 \vee J \cap C_1 = \emptyset$

if $J \cap C_0 = \emptyset$, then $J = J \cap C_1$

so $J \subseteq C_{1\, \in \mathcal{T}_{\Omega|\bar{J}c}}$ so $\bar{J} \subseteq \bar{C_1} = C_{1 \subset \bar{J}}$: contradiction

if $J \cap C_1 = \emptyset$, then $J = J \cap C_0$

so $J \subseteq C_{0\, \in \mathcal{T}_{\Omega|\bar{J}c}}$ so $\bar{J} \subseteq \bar{C_0} = C_{0 \subset \bar{J}}$: contradiction

hence $\bar{J}$: conn ■

Theorem: $\{\Omega_i^{\in \Omega}\}_I$: conn $\wedge \bigcap_I \Omega_i \neq \emptyset$

$\Longrightarrow \bigcup_I \Omega_i$: conn.

Proof: Suppose $\bigcup_I \Omega_i$: disconn,

so $\exists C^{\in \mathcal{T}_{\Omega|\bigcup_I \Omega_i}}_{\in \mathcal{T}_{\Omega|\bigcup_I \Omega_i}c}$ $C_{\neq \emptyset} \subset \bigcup_I \Omega_i$

Let $C \cap \Omega_j \neq \emptyset$, then suppose $\Omega_j \not\subseteq C$

since $C \in \mathcal{T}_{\Omega|\bigcup_I \Omega_i}$, $\exists \mathcal{U}_{\in \mathcal{T}_\Omega}$ $C = \bigcup_I \Omega_i \cap \mathcal{U}$, so

$\Omega_j \cap C = \Omega_j \cap \bigcup_I \Omega_i \cap \mathcal{U}_{\in \mathcal{T}_\Omega} = \Omega_j \cap \mathcal{U}_{\in \mathcal{T}_\Omega} \in \mathcal{T}_{\Omega|\Omega_j}$

also, since $C \in \mathcal{T}_{\Omega|\bigcup_I \Omega_i c}$, $C^c \in \mathcal{T}_{\Omega|\bigcup_I \Omega_i}$

$\exists V_{\in \mathcal{T}_\Omega}$ $C^c = \bigcup_I \Omega_i \cap V$, note since $C \cap \Omega_j \neq \emptyset$

$\wedge\ \Omega_j \not\subseteq C$, $(C^c \cap \Omega_j)_{\subset \Omega} \neq \emptyset$

Proof cons: also,

$$\Omega_j \cap C^c = \Omega_j \cap \bigcup_I \Omega_i \cap V_{\in \mathcal{T}_\Omega} = \Omega_j \cap V_{\in \mathcal{T}_\Omega} \in \mathcal{T}_{\Omega | \Omega_j}$$

since $(\Omega_j \cap C) \cap (\Omega_j \cap C^c) = \emptyset$

$\wedge\ (\Omega_j \cap C) \cup (\Omega_j \cap C^c) = \Omega_j$

$\{\Omega_j \cap C, \Omega_j \cap C^c\}$: sep of Ω_j

so Ω_j: disconn ; contradiction

hence, if $\Omega_j \cap C \neq \emptyset$ then $\Omega_j \subseteq C$

So $\exists J_{\subset I}\ C_{\subset \bigcup_I \Omega_i} = \bigcup_J \Omega_j$

Hence $\exists \sigma_{\in I \setminus J}\ \Omega_\sigma \not\subseteq C$ so $\Omega_\sigma \cap C = \emptyset$

Also since $\bigcap_I \Omega_i \neq \emptyset$, $\Omega_\sigma \cap C \neq \emptyset$

which is a contradiction, so $\bigcup_I \Omega_i$: conn ■

Theorem: $\{\Omega_i^{\in \Omega}\}_I$: conn $\wedge$ $\exists \Omega_{s \subseteq \Omega}\ \forall i_{\in I}\ \Omega_i \cap \Omega_s \neq \emptyset$

$$\Longrightarrow \Omega_s \cup \bigcup_I \Omega_i : \text{conn}.$$

Proof: Supposing $\Omega_s \cup \bigcup_I \Omega_i$, in a similar way

of previous proof, $\exists C^{\in \mathcal{T}_{\Omega | \Omega_s \cup \bigcup_I \Omega_i}}_{\in \mathcal{T}_{\Omega} | \Omega_s \cup \bigcup_I \Omega_i\ c}\ C_{\neq \emptyset} \subset \Omega_s \cup \bigcup_I \Omega_i$

and $\exists J_{\subset I \cup \{s\}}\ C = \bigcup_J \Omega_j$

Proof cons :

if $\Omega_s \subseteq C$ then $\forall i_{\in I}\ C \cap \Omega_i \neq \emptyset$

since $\forall i_{\in I}\ \Omega_i \cap \Omega_s \neq \emptyset$, so since if $\Omega_i \cap C \neq \emptyset$

then $\Omega_i \subseteq C$, $C_{\subset \Omega_s \cup \bigcup_I \Omega_i} = \Omega_s \cup \bigcup_I \Omega_i$

which is a contradiction,

if $\Omega_s \nsubseteq C$ then $\Omega_s \cap C = \emptyset$

so $\exists j_{\in I} : \Omega_s \cap \Omega_j = \emptyset$: contradiction

Hence, $\Omega_s \cup \bigcup_I \Omega_i$: conn $\blacksquare$

Theorem: Ω : conn $\wedge$ $f_{\Omega \to Y}$: con $\Longrightarrow$ $f(\Omega)$: conn .

Proof: Suppose $f(\Omega)$: discann ,

so $\exists V_0, V_1 {}^{\in \mathcal{T}_{Y|f(\Omega)}\ (\neq \emptyset)}_{(V_0 \cap V_1 = \emptyset)}$ $f(\Omega) = V_0 \cup V_1$

Since f : con ; $f^{-1}(V_0), f^{-1}(V_1) \in \mathcal{T}_\Omega$,

also $f^{-1}(V_0^{\neq\emptyset}), f^{-1}(V_1^{\neq\emptyset}) \neq \emptyset$,

$\Omega = f^{-1}(f(\Omega)) = f^{-1}(V_0 \cup V_1) = f^{-1}(V_0) \cup f^{-1}(V_1)$

$f^{-1}(V_0) \cap f^{-1}(V_1) = f^{-1}(V_0 \cap V_1) = f^{-1}(\emptyset) = \emptyset$

So, $\{f^{-1}(V_0), f^{-1}(V_1)\}$: Sep of Ω

So, Ω : discann : contradiction

Hence, $f(\Omega)$: conn $\blacksquare$

Theorem: $\forall o_{\in\Omega}\ \Omega\setminus\{o\}$: conn

$\wedge\ \exists y_{\in\mathcal{Y}}\ \mathcal{Y}\setminus\{y\}$: disconn $\Longrightarrow\ \Omega\not\cong\mathcal{Y}$.

Proof: Suppose $\Omega\cong\mathcal{Y}$ with $f_{\Omega\to\mathcal{Y}}$

Let $\{N_0, N_1\}$: sep of $\mathcal{Y}\setminus\{y\}$

following the steps in the previous proof,

$\{f^{-1}(N_0), f^{-1}(N_1)\}$: sep of $\Omega\setminus\{f^{-1}(y)\}$

which is a contradiction.

So, $\Omega\not\cong\mathcal{Y}$ ∎

Def: Ω: path conn $:=\ \forall x,y^{\in\Omega}_{(x\neq y)}\ \exists r^{\text{con}}_{I\to\Omega}\ r(0)=x,\ r(1)=y$

Theorem: Ω: path conn $\Longrightarrow$ Ω: conn.

Proof: Suppose Ω: disconn,

Let $\{\Omega_0, \Omega_1\}$: sep of Ω, since r: con,

following the steps in the previous proof,

$\{r^{-1}(\Omega_0), r^{-1}(\Omega_1)\}$: sep of I

So I: disconn : contradiction

So, Ω: conn ∎

Theorem: $\{\Omega_i^{\subseteq \Omega}\}_I$: path conn

$\wedge\ \forall i_{\in I}\ \Omega_i \cap \Omega_{i+1} \neq \emptyset \implies \bigcup_I \Omega_i$: path conn

Proof: note $\forall \sigma_i \in \Omega_i \cap \Omega_{i+1}\ \exists r_i^{\text{con}}\ r_i(0) = \sigma_i \quad r_i(1) = \sigma_{i+1}$

Let $x, y \in \bigcup_I \Omega_i$ with $x \neq y$

if $x, y \in \Omega_s$ then $\exists r_{\Omega_s}^{\text{con}}\ r_{\Omega_s}(0) = x \quad r_{\Omega_s}(1) = y$

if $x \in \Omega_{j_0}$, $y \in \Omega_{j_1}$ with $j_0 < j_1$,

$\exists \varphi_0, \varphi_1^{\text{con}}\ \varphi_0(0) = x \quad \varphi_0(1) = \sigma_{j_0} \quad \varphi_1(0) = y \quad \varphi_1(1) = \sigma_{j_1}$

define $*$: $S_0 * S_1 = \begin{cases} S_0(2x), & 0 \leq x \leq \frac{1}{2} \\ S_1(2x-1), & \frac{1}{2} \leq x \leq 1 \end{cases}$

define $\psi := \varphi_0 * r_{j_0} * \ldots * r_{j_1 - 1} * \varphi_1$

So $\exists r_{(=\psi)}^{\text{con}}\ r(0) = x \quad r(1) = y$

So, $\bigcup_I \Omega_i$: path conn ■

Theorem: Ω : path conn $\wedge$ $f_{\Omega \to Y}$: con

$\implies f(\Omega)$: path conn

Proof: Let $x, y \in f(\Omega)$, $\exists r_{I \to \Omega}^{\text{con}}$

$r(0) = f^{-1}(x)$, $r(1) = f^{-1}(y)$

define $\theta = f \circ r$ so $\theta(0) = f(r(0)) = x$

and $\theta(1) = f(r(1)) = y$, also θ : con as f, r : con

So, $f(\Omega)$: path conn ■

Theorem: $\mathcal{X}\times\mathcal{Y}\cong\mathcal{Y}\times\mathcal{X}$

Proof: define $f_{\mathcal{X}\times\mathcal{Y}\to\mathcal{Y}\times\mathcal{X}}$ as $f(x,y)=(y,x)$

$f(x,y)=f(z,w)$ then $(y,x)=(w,z)$

so $(x,y)=(z,w)$ hence f: inj

$(\Omega,\sigma)\in\mathcal{Y}$ then $f(\sigma,\Omega)=(\Omega,\sigma)$

so f: surj hence f: bij

Let $\bigcup_J(\mathcal{V}\times\mathcal{U})_j\in\mathcal{T}_{y\times x}$,

$f^{-1}(\bigcup_J(\mathcal{V}\times\mathcal{U})_j)=\bigcup_J f^{-1}((\mathcal{V}\times\mathcal{U}))$

$=\bigcup_J(\mathcal{U}\times\mathcal{V})_j\in\mathcal{T}_{x\times y}$ so f: con

Let $\bigcup_I(\mathcal{U}\times\mathcal{V})_i\in\mathcal{T}_{x\times y}$,

$(f^{-1})^{-1}(\bigcup_I(\mathcal{U}\times\mathcal{V})_i)=f(\bigcup_I(\mathcal{U}\times\mathcal{V})_i)$

$=\bigcup_I f((\mathcal{U}\times\mathcal{V})_i)=\bigcup_I(\mathcal{V}\times\mathcal{U})_i\in\mathcal{T}_{y\times x}$ so f^{-1}: con

Hence $\mathcal{X}\times\mathcal{Y}\cong\mathcal{Y}\times\mathcal{X}$ ∎

Def: P_j $\prod_i X_i \to X_j$ projection map :=

$$\forall x_{\in \prod_i X_i} \quad P_j(x) = x_j \; .$$

Theorem: P_j : surj, open, con.

Proof: i. $\forall \Omega_{\in X_j} \; \exists \sigma_{\in \prod_i X_i} \quad \sigma = (x_1, \dots, x_{j-1}, \Omega, \dots)$

so $P_j(\sigma) = \Omega$ so P_j : surj

ii. $\bigcup_\lambda (\prod_i U_i)_\lambda \in \mathcal{T}_{\prod X_i}$, $P_j(\bigcup_\lambda (\prod_i U_i)_\lambda) = \bigcup_\lambda P_j((\prod_i U_i)_\lambda)$

$= \bigcup_\lambda (U_j)_\lambda \in \mathcal{T}_{X_j}$ so P_j : open

iii. $U \in \mathcal{T}_{X_j}$, $P_j^{-1}(U) = X_1 \times \dots \times X_{j-1} \times U \times \dots \times X_n$

since also $\forall i \; X_i \in \mathcal{T}_{X_i}$, $P_j^{-1}(U) \in \mathcal{B}_{\prod_i X_i} \subseteq \mathcal{T}_{\prod_i X_i}$

so P_j : con since $\mathcal{T}_{\prod_i X_i} = \{ \bigcup_{B \in \mathcal{B}^*} B \mid \mathcal{B}^* \subseteq \mathcal{B}_{\prod_i X_i} \}$ ■

Theorem: $\forall \mathcal{T}^*$: $P_{j \in J}$ con on $\mathcal{T}^*$, $\mathcal{T}_{\prod_i X_i} \subseteq \mathcal{T}^*$.

Proof: Let $\prod_i U_i \in \mathcal{B}_{\prod_i X_i}$, since $\forall j_{\in J}$

P_j con on $\mathcal{T}^*$, $P_j^{-1}(U_{\in \mathcal{T}_{X_j}}) = X_1 \times \dots \times X_{j-1} \times U \times \dots \in \mathcal{T}^*$

So $\prod_i U_i = \bigcap_J P_j^{-1}(U_j) \in \mathcal{T}^*$ as $U_j \subseteq X_j$

So, since $\forall G_{\in \mathcal{T}_{\prod_i X_i}} \; \exists \mathcal{B}_0 \subseteq \mathcal{B}_{\prod_i X_i} \subseteq \mathcal{T}^*$

$G = \bigcup_{B \in \mathcal{B}_0} B \in \mathcal{T}^*$ so $\mathcal{T}_{\prod_i X_i} \subseteq \mathcal{T}^*$ ■

Theorem: $\forall i \; C_i \in \mathcal{T}_{X_i c} \implies \prod_i C_i \in \mathcal{T}_{\prod_i X_i c}$.

Proof: since $\forall i \; C_i \in \mathcal{T}_{X_i c}$ i.e. $X_i \setminus C_i \in \mathcal{T}_{X_i}$,

$$\big(\prod_i X_i\big)_{\in \mathcal{T}_{\prod_i X_i}} \setminus \big(\prod_i C_i\big) = \Big(\bigcup_i (X_1 \times \dots \times (X_i \setminus C_i) \times \dots)\Big)_{\in \mathcal{T}_{\prod_i X_i}}$$

so $\prod_i C_i \in \mathcal{T}_{\prod_i X_i c}$ $\blacksquare$

Theorem: $\forall i \; Y_i \subseteq X_i$, $\big(\prod_i Y_i\big)^\circ = \prod_i (Y_i)^\circ$.

Proof: $x \in \big(\prod_i Y_i\big)^\circ_{\in \mathcal{T}_{\prod_i X_i}} \iff \exists \big(\prod_i U_i\big)_{\in \mathcal{B}_{\prod_i X_i} \subseteq \mathcal{T}_{\prod_i X_i}}$

$x \in \prod_i U_i \subseteq \prod_i Y_i \iff \forall i \; x_i \in U_i^{\in \mathcal{T}_{X_i}} \subseteq Y_i$

$\iff \forall i \; x_i \in_{U_i \underset{\subseteq Y_i}{\in \mathcal{T}_{\pi X_i}}} (Y_i)^\circ \iff x \in \prod_i (Y_i)^\circ$ $\blacksquare$

Theorem: $\forall i \; Y_i \subseteq X_i$, $\overline{\prod_i Y_i} = \prod_i \overline{Y_i}$.

Proof: $x \in \overline{\prod_i Y_i} \iff \forall \big(\bigcup_j \prod_i U_i\big)^{\ni x}_{\in \mathcal{T}_{\prod_i X_i}}$

$\prod_i Y_i \cap \bigcup_j \prod_i U_i = \bigcup_j \big(\prod_i U_i \cap \prod_i Y_i\big) \neq \emptyset$

$\iff \forall j_0 \; \big(\prod_i U_i\big)_{j_0} \cap \prod_i Y_i \neq \emptyset$ since $\forall j_0 \big(\prod_i U_i\big)_j \in \mathcal{B}_{\pi} \subseteq \mathcal{T}_{\pi}$

$\iff \forall i_j \; U_{i_j} \cap Y_i \neq \emptyset \iff \forall i \; x_i \in \overline{Y_i}$

$\iff x \in \prod_i \overline{Y_i}$ $\blacksquare$

Theorem: $\partial(\prod_i Y_i) = \bigcup_i \overline{Y}_1 \times \dots \times \partial Y_i \times \dots \times \overline{Y}_n$.

Proof: $\partial(\prod_i Y_i) = \overline{\prod_i Y_i} \setminus (\prod_i Y_i)^\circ$

$= \prod_i \overline{Y}_i \setminus \prod_i Y_i^\circ = \bigcup_i \overline{Y}_1 \times \dots \times (\overline{Y}_i \setminus Y_i^\circ) \times \dots \times \overline{Y}_n$

$= \bigcup_i \overline{Y}_1 \times \dots \times \partial Y_i \times \dots \times \overline{Y}_n$ ∎

Theorem: $\forall i \; Y_i \subseteq X_i$, $\prod_i \tilde{Y}_i \subseteq (\widetilde{\prod_i Y_i})$.

Proof: Let $x \in \prod_i \tilde{Y}_i$ and $\bigcup_J (\prod_i U_i)_j \ni x \in \mathcal{T}_{\prod_i X_i}$

$(\prod_i Y_i) \cap (\bigcup_J \prod_i U_i) \setminus \{x\}$

$= \bigcup_J \prod_i Y_i \cap U_i \setminus \prod_i \{x_i\} = \bigcup_{j,i} Y_1 \cap U_1 \times \dots \times Y_i \cap U_i \setminus \{x_i\} \times \dots$

$\neq \emptyset$, since $U_{i \subseteq Y_i} \cap Y_i \neq \emptyset$ and

as $x \in \prod_i \tilde{Y}_i$, $Y_i \cap U_i \setminus \{x_i\} \neq \emptyset$,

hence $x \in (\widetilde{\prod_i Y_i})$ ∎

CamScanner

Theorem: f: homeo $\wedge$ g: homeo $\Longrightarrow$ $f\times g$: homeo

Proof: i. f: bij $\wedge$ g: bij $\Longrightarrow$ $f\times g$: bij

i.i. f: inj $\wedge$ g: inj $\Longrightarrow$ $f\times g$: inj

$(u_1,u_2)\neq(y_1,y_2) \Longrightarrow u_1\neq y_1 \wedge u_2\neq y_2$

$\Longrightarrow f(u_1)\neq f(y_1) \wedge g(u_2)\neq g(y_2)$

$\Longrightarrow (f\times g)(u_1,u_2) = (f(u_1),g(u_2))$

$\neq (f(y_1),g(y_2)) = (f\times g)(y_1,y_2)$

i.ii. f: surj $\wedge$ g: surj $\Longrightarrow$ $f\times g$: surj

$(v_1,v_2)\in V_1\times V_2 \Longrightarrow v_1\in V_1 \wedge v_2\in V_2$

$\Longrightarrow \exists u_1,u_2 : f(u_1)=v_1 \wedge g(u_2)=v_2$

$\Longrightarrow \exists (u_1,u_2) : (f\times g)(u_1,u_2) = (v_1,v_2)$

ii: f: con $\wedge$ g: con $\Longrightarrow$ $f\times g$: con

$\bigcup_J (\mathcal{V}_1\times\mathcal{V}_2)_j \in \mathcal{T}_{V_1\times V_2} \Longrightarrow (f\times g)^{-1}(\bigcup_J(\mathcal{V}_1\times\mathcal{V}_2)_j)$

$= (f^{-1}\times g^{-1})(\bigcup_J(\mathcal{V}_1\times\mathcal{V}_2))$

$= \bigcup_J (f^{-1}(\mathcal{V}_1^j), g^{-1}(\mathcal{V}_2^j)) \in \mathcal{T}_{U_1\times U_2}$

iii. f^{-1}: con $\wedge$ g^{-1}: con $\Longrightarrow$ $(f\times g)^{-1}$: con

by ii. since $(f\times g)^{-1} = f^{-1}\times g^{-1}$ $\blacksquare$

Theorem: $\forall i \ \mathcal{Y}_i : \mathcal{T}_{\mathcal{X}_i}$- dense

$\iff \prod_i \mathcal{Y}_i : \mathcal{T}_{\prod_i \mathcal{X}_i}$- dense.

Proof: $(\Rightarrow)$: $\forall i \ \mathcal{Y}_i : \mathcal{T}_{\mathcal{X}_i}$- dense $\wedge \bigcup_J \prod_i \mathcal{U}_i \in \mathcal{T}_{\prod_i \mathcal{X}_i}$

$\Rightarrow \forall \mathcal{U}_{i \in \mathcal{T}_{\mathcal{X}_i}} \ \mathcal{Y}_i \cap \mathcal{U}_i \neq \emptyset \Rightarrow \exists x_{i \in \mathcal{X}_i} \ x_i \in \mathcal{Y}_i \cap \mathcal{U}_i$

$\Rightarrow \exists x^{\in \prod \mathcal{X}_i}_{(=(x_1, \dots, x_i, \dots, x_n))} \ x \in (\prod_i \mathcal{Y}_i) \cap (\bigcup_J \prod_i \mathcal{U}_i)$

$\Rightarrow (\prod_i \mathcal{Y}_i) \cap (\bigcup_J \prod_i \mathcal{U}_i) \neq \emptyset$

$\Rightarrow \prod_i \mathcal{Y}_i : \mathcal{T}_{\prod_i \mathcal{X}_i}$- dense

$(\Leftarrow)$: $\mathcal{U}_i \in \mathcal{T}_{\mathcal{X}_i}$,

define $\mathcal{U} = \mathcal{X}_1 \times \dots \times \mathcal{X}_{i-1} \times \mathcal{U}_i \times \mathcal{X}_{i+1} \times \dots \times \mathcal{X}_n$

so $\mathcal{U} \in \mathcal{T}_{\prod_i \mathcal{X}_i}$ hence by assumption

$$(\prod_i \mathcal{Y}_i) \cap \mathcal{U} \neq \emptyset$$

so $(\mathcal{Y}_{1 \subseteq \mathcal{X}_1} \cap \mathcal{X}_1) \times \dots \times (\mathcal{Y}_i \cap \mathcal{U}_i) \times \dots \neq \emptyset$

so $\mathcal{Y}_1 \times \dots \times (\mathcal{Y}_i \cap \mathcal{U}_i) \times \dots \times \mathcal{Y}_n \neq \emptyset$

so $\mathcal{Y}_i \cap \mathcal{U}_i \neq \emptyset$

hence $\mathcal{Y}_i : \mathcal{T}_{\mathcal{X}_i}$- dense ■

Theorem: $\forall i\ X_i$: first-cou $\Longrightarrow \prod_i X_i$: first-cou

Proof: Let $x \in \prod X_i$ and $\forall i\ \mathcal{B}_i^{cou}$: loc basis of x_i

define $\mathcal{B}_x = \{\prod_i B_i : B_i \in \mathcal{B}_i\}$.

so, since $\forall i\ \mathcal{B}_i$: cou, $\mathcal{B}_x$: cou

Let $\bigcup_J \prod_i U_i \underset{\ni x}{\in} \mathcal{T}_{\pi x_i}$, since $\forall U^{j \in \mathcal{T}_{x_i}}_{i\, \ni x_i}\ \exists B_{i \in \mathcal{B}_i}$

such that $x_i \in U_i^j \subseteq B_i$,

$\exists Y^{\in \mathcal{B}_x}_{(=\prod_i B_i)}\quad x \in Y \subseteq \prod_i U_i^j \subseteq \bigcup_J \prod_i U_i$

so $\mathcal{B}_x$: loc basis of x

hence $\prod_i X_i$: first-cou ∎

Theorem: $\forall i\ X_i$: sec-cou $\Longrightarrow \prod_i X_i$: sec-cou

Proof: Let $\forall i\ \mathcal{B}_i^{cou}$: basis of X_i

define $\mathcal{B} = \prod_i \mathcal{B}_i$, let $x \in \bigcup_J \prod_i U_i \in \mathcal{T}_{\pi x_i}$

so $\exists_{j \in J}\ x \in \prod_i U_i^j$ so since $\forall U_i^j\ \forall x_{i \in U_i^j}\ \exists B_{i \in \mathcal{B}_i}$

$x_i \in B_i \subseteq U_i^j$, $x \in \prod_i B_{i\, \in \mathcal{B}} \subseteq \prod_i U_i^j \subseteq \bigcup_J \prod_i U_i$

so $\mathcal{B}$: cou basis of $\prod_i X_i$ since $\forall i\ \mathcal{B}_i$: cou

so $\prod_i X_i$: sec-cou ∎

Theorem: $\forall i\ X_i : \text{sep} \Longrightarrow \prod_i X_i : \text{sep}$.

Proof: $\forall i\ X_i : \text{sep} \Longrightarrow \forall i\ \exists G_i^{\text{cou}} : \overline{G}_i = X_i$

define $G = \prod_i G_i$ which is also cou

since $\forall i\ G_i : \text{cou}$,

Let $\bigcup_j \prod_i U_i \in \mathcal{T}_{\prod X_i}$, since $\forall i\ \overline{G}_i = X_i$

$\forall U_i^j \in \mathcal{T}_{X_i}\ G_i \cap U_i^j \neq \emptyset$

so $G \cap \bigcup_j \prod_i U_i \neq \emptyset$ so $\overline{G} = \prod_i X_i$

hence $\prod_i X_i$: sep ∎

Theorem: $\forall i\ X_i : T_2 \Longrightarrow \prod_i X_i : T_2$.

Proof: Let $x, y \in \prod_i X_i$ such that $x \neq y$

so $\exists j\ x_j \neq y_j$ hence by assumption

$\exists U_j^{\ni x_j}, V_j^{\ni y_j} \in \mathcal{T}_{X_j}\ U_j \cap V_j = \emptyset$

define $U = X_1 \times \ldots \times U_j \times \ldots \times X_n$

$V = X_1 \times \ldots \times V_j \times \ldots \times X_n$

if $U \cap V \neq \emptyset$ then $\exists z\ z \in U \cap V$

so $z_j \in (U_j \cap V_j)_{(=\emptyset)}$, hence $U \cap V = \emptyset$

so $\prod_i X_i : T_2$ ∎

Theorem: $X_{\mathcal{T}}$: first-cou

$\Longrightarrow$ $Y_{\mathcal{T}|_Y}$: first-cou

Proof: Let $\sigma \in Y$,

so $\exists \mathcal{B}_\sigma$; $\mathcal{B}_\sigma$: cou loc basis of $\sigma \in Y_{\subseteq X}$

define $\mathcal{B}_{\sigma|Y} = \{ Y \cap B : B \in \mathcal{B}_\sigma \}$

note that since $\mathcal{B}_\sigma$: cou , $\mathcal{B}_{\sigma|Y}$: cou

also $\forall (Y \cap B)_{\in \mathcal{B}_{\sigma|Y}}$ $\sigma \in Y_{(\ni\sigma)} \cap B^{\in \mathcal{B}_\sigma}_{(\ni\sigma)}$

Let $U^{\ni\sigma}_{\in \mathcal{T}|_Y}$ so $\exists V_{\in \mathcal{T}}$ $U = Y \cap V$

since $\sigma \in V_{\supseteq U}$, $\exists B_{\in \mathcal{B}_\sigma}$ $\sigma \in B \subseteq V$

so $\sigma \in (Y \cap B)_{\in \mathcal{B}_{\sigma|Y}} \subseteq Y \cap V = U$

so $\mathcal{B}_{\sigma|Y}$: cou loc basis of $\sigma \in Y_{\mathcal{T}|_Y}$

hence $Y_{\mathcal{T}|_Y}$: first-cou ∎

Theorem: $X_{\mathcal{T}}$: sec - cou

$\Longrightarrow Y_{\mathcal{T}|Y}$: sec - cou.

Proof: Let $\mathcal{B}$: cou basis of X

define $\mathcal{B}_Y = \{ Y \cap B : B \in \mathcal{B} \}$

note that since $\mathcal{B}$: cou, $\mathcal{B}_Y$: cou

Let $U \in \mathcal{T}_{|Y}$, so $\exists V_{\in \mathcal{T}}$ $U = Y \cap V$

since $V \in \mathcal{T}$, $\exists \mathcal{B}^*_{\subseteq \mathcal{B}}$ $V = \bigcup\limits_{B \in \mathcal{B}^*} B$

define $\mathcal{B}^*_Y = \{ Y \cap B : B \in \mathcal{B}^* \}$

so $U = Y \cap V = Y \cap \left(\bigcup\limits_{B \in \mathcal{B}^*} B \right)$

$= \bigcup\limits_{B \in \mathcal{B}^*} (Y \cap B) = \bigcup\limits_{G \in \mathcal{B}^*_Y \subseteq \mathcal{B}_Y} G$

so $\mathcal{B}_Y$: cou basis of $Y_{\mathcal{T}|Y}$

so $Y_{\mathcal{T}|Y}$: sec - cou $\blacksquare$

Theorem: $X_{\mathcal{T}} : T_2 \Longrightarrow Y_{\mathcal{T}|Y} : T_2$

Proof: Let $x, y \in Y_{\subseteq X}$ such that $x \neq y$

so $\exists U^{\ni x}, V^{\ni y}{}_{\in \mathcal{T}}$ $U \cap V = \emptyset$

so $(Y \cap U)^{\ni x}_{\in \mathcal{T}|Y} \cap (Y \cap V)^{\ni y}_{\in \mathcal{T}|Y} = Y \cap (U \cap V)$

$= Y \cap \emptyset = \emptyset$

so $Y_{\mathcal{T}|Y} : T_2$ $\blacksquare$

Def: $\mathcal{X}$: com :=

$\forall \{\mathcal{U}_i^{\in \mathcal{T}}\}_{I\,(\mathcal{X} \subseteq \bigcup_I \mathcal{U}_i)}\ \exists \{\mathcal{U}_j^{\in \mathcal{T}}\}_{J \subseteq I,\ \in \mathbb{N}}\ \mathcal{X} \subseteq \bigcup_J \mathcal{U}_j$.

Theorem: $\mathcal{Y}_{\subseteq \mathcal{X}}$ finite $\Longrightarrow$ $\mathcal{Y}_{\subseteq \mathcal{X}}$: com

Proof: Let $\mathcal{Y} = \{x_j\}_{J_{\in \mathbb{N}}}$ and $\{\mathcal{U}_i\}_I$: op cover of $\mathcal{Y}$

so $\forall x_{j \in \mathcal{Y}}\ \exists \mathcal{U}_{x_j \in \{\mathcal{U}_i\}_I}\ x_j \in \mathcal{U}_{x_j}$

so $\{\mathcal{U}_{x_j}\}_{\subseteq \{\mathcal{U}_i\}_I}$ covers $\mathcal{Y}$ and $|\{\mathcal{U}_{x_j}\}| \leqslant |J|_{\in \mathbb{N}}$

hence, $\mathcal{Y}$: com $\blacksquare$

Theorem: $\mathcal{X}$: com

$\Longleftrightarrow \forall \{\mathcal{F}_i^{\in \mathcal{T}_c}\}_I \left(\left[\forall J_{\subseteq I,\ \in \mathbb{N}}\ \bigcap_J \mathcal{F}_j \neq \emptyset \right] \Longrightarrow \bigcap_I \mathcal{F}_i \neq \emptyset \right)$

Proof: ($\Longrightarrow$): Let $\mathcal{X}$: com and pick $\{\mathcal{F}_i^{\in \mathcal{T}_c}\}_I$,

suppose $\bigcap_I \mathcal{F}_i = \emptyset$ so

$\bigcup_I \mathcal{F}_i^c = \left(\bigcap_I \mathcal{F}_i\right)^c = \emptyset^c = \mathcal{X}$

and $\forall i\ \mathcal{F}_i^c \in \mathcal{T}$ since $\forall i\ \mathcal{F}_i \in \mathcal{T}_c$

so $\{\mathcal{F}_i^c\}_I$: op cover of $\mathcal{X}$, by com of $\mathcal{X}$

$\exists J_{\subseteq I,\ \in \mathbb{N}}\ \bigcup_J \mathcal{F}_j^c = \mathcal{X}$, so $\exists J_{\subseteq I,\ \in \mathbb{N}}\ \bigcap_J \mathcal{F}_j = \emptyset$

which is a contradiction, so $\bigcap_I \mathcal{F}_i \neq \emptyset$

Proof cons: ($\Leftarrow$): suppose X: non-com

Let $\{U_i\}_I$: op cover of X

so $\forall J \underset{\subset \mathbb{N}}{\subseteq} I$ $\bigcup_J U_j \neq X$ therefore

$\bigcap_J U_j^c = (\bigcup_J U_j)^c \neq X^c = \emptyset$

so since for $\{U_j^c\}_J \subset \{U_i^c \in \mathcal{T}_c\}_I$, $\bigcap_J U_j^c \neq \emptyset$

by assumption, $\bigcap_I U_i^c \neq \emptyset$

so $\bigcup_I U_i \neq X$ which is a contradiction

so X: com $\blacksquare$

Theorem: X com $\wedge$ $Y \in \mathcal{T}_c \implies Y$: com

Proof: Let $\{U_i^{\circ}\}_I$ covers Y, i.e. $Y \subseteq \bigcup_I U_i$

since $Y \in \mathcal{T}_c$, $Y^c \in \mathcal{T}$ so

$\{U_i\}_I \cup \{Y^c\}$ covers X, so since X: com

$\exists J \underset{\subset \mathbb{N}}{\subseteq} I$ $X_{\supseteq Y} \subseteq (\bigcup_J U_j) \cup (Y^c_{\not\supseteq Y})$

so $Y \subseteq \bigcup_J U_j$ so Y: com $\blacksquare$

Theorem: $\mathcal{Y}_{\subseteq X}$: com $\wedge$ $f_{X \to Y}$: con

$\Rightarrow f(\mathcal{Y})$: com

Proof: Let $f(\mathcal{Y}) \subseteq \bigcup_I \mathcal{U}_i$, $\forall i \; \mathcal{U}_i \in \mathcal{T}_Y$

so $\mathcal{Y} \subseteq f^{-1}(\bigcup_I \mathcal{U}_i) = \bigcup_I f^{-1}(\mathcal{U}_i)$

since f : con , $\forall i \; f^{-1}(\mathcal{U}_i) \in \mathcal{T}_X$

so by com of $\mathcal{X}$,

$\exists J_{\subseteq I, \in \mathbb{N}} \quad \mathcal{Y} \subseteq \bigcup_J f^{-1}(\mathcal{U}_j)$

so, $f(\mathcal{Y}) \subseteq f(\bigcup_J f^{-1}(\mathcal{U}_j))$

$= \bigcup_J f(f^{-1}(\mathcal{U}_j)) = \bigcup_J \mathcal{U}_j$

so $f(\mathcal{Y})$: com

Theorem: $\mathcal{X}:T_2 \wedge \mathcal{Y}_{\subseteq \mathcal{X}}:\text{com} \implies \mathcal{Y} \in \mathcal{T}_c$

Proof: Let $\sigma \in \mathcal{Y}^c$, since $\mathcal{X}:T_2$

$\forall x_{\in \mathcal{X}}\ \exists\, \mathcal{U}_x^{\in\mathcal{T}}, V_{(\sigma,x)}^{\in \mathcal{T}} \quad \mathcal{U}_x^{\ni x}\cap V_{(\sigma,x)}^{\ni\sigma}=\emptyset$

note that $\{\mathcal{U}_x\}_{x\in\mathcal{Y}}$ covers $\mathcal{Y}$

so since $\mathcal{Y}$:com, $\exists\, J^{\le|\mathcal{Y}|}_{\subset\mathbb{N}} \quad \mathcal{Y}\subseteq \bigcup_J \mathcal{U}_{x_j}$

define $V := \bigcap_J V_{(\sigma,x_j)}$

note that $V_{\ni\sigma}\in\mathcal{T}$ since $\forall j_{\in J}\ V^{\ni\sigma}_{(\sigma,x_j)}\in\mathcal{T}$

also as $\forall j_{\in J}, K_{\subseteq V(=\bigcap_J V_{(\sigma,x_j)})} \quad K\cap \mathcal{U}_{x_j}=\emptyset$

$\mathcal{Y}_{\subseteq\bigcup_J \mathcal{U}_{x_j}}\cap V=\emptyset$ so $V^{\ni\sigma}_{\in\mathcal{T}}\subseteq \mathcal{Y}^c$

so $\sigma_{\in\mathcal{Y}^c}\in(\mathcal{Y}^c)^\circ$ so $\mathcal{Y}^c_{\supseteq(\mathcal{Y}^c)^\circ}\subseteq(\mathcal{Y}^c)^\circ$

so $\mathcal{Y}^c=(\mathcal{Y}^c)^\circ_{\in\mathcal{T}}$ so $\mathcal{Y}^c\in\mathcal{T}$

so $\mathcal{Y}_{(=(\mathcal{Y}^c)^c)}\in\mathcal{T}_c$

Def: $f_{\Omega_{\subseteq\mathcal{X}}\to\mathcal{Y}}$:con, $x_i\sim_f x_j := f(x_i)=x_j$

$$\mathcal{X}\cup_f\mathcal{Y} := (\mathcal{X}\oplus\mathcal{Y})/\sim_f$$

Def: $\mathcal{X}\xhookrightarrow{f}\mathcal{Y} := (\mathcal{X},\mathcal{T}_{\mathcal{X}})\underset{f}{\cong}(f(\mathcal{X}),\mathcal{T}_{\mathcal{Y}|f(\mathcal{X})})$

Theorem: $\mathcal{X}$: cam $\wedge$ $\mathcal{Y}$: T_2 $\wedge$ $f_{\mathcal{X}\to\mathcal{Y}}$: con bij

$\Longrightarrow$ f : homeo

Proof: Let $\mathcal{U} \in \mathcal{T}_{\mathcal{X}}$ so $\mathcal{U}^c \in \mathcal{T}_{\mathcal{X}^c}$

so as $\mathcal{X}$: cam, $\mathcal{U}^c_{\in \mathcal{T}_{\mathcal{X}^c}}$: cam

so as f : con, $f(\mathcal{U}^c)$: cam

so as $\mathcal{Y}$: T_2, $f(\mathcal{U}^c) \in \mathcal{T}_{\mathcal{Y}^c}$

so as $f(\mathcal{U}^c) = (f(\mathcal{U}))^c$

$(f(\mathcal{U}))^c \in \mathcal{T}_{\mathcal{Y}^c}$ so $f(\mathcal{U}) \in \mathcal{T}_{\mathcal{Y}}$

so f : op, hence as f : bij, f^{-1} : con

hence f : homeo ■

Theorem: $\mathcal{X}$: cam $\wedge$ $\mathcal{Y}$: T_2 $\wedge$ $f_{\mathcal{X}\to\mathcal{Y}}$: con surj

$\wedge$ $x_0 \sim_f x_1 := f(x_0) = f(x_1)$ $\Longrightarrow$ $\mathcal{X}/\sim_f \cong \mathcal{Y}$

Proof: by the proof of the previous theorem

f : op so by another previous theorem

$\mathcal{X}/\sim_f \cong f(\mathcal{X})$, since f : surj, $f(\mathcal{X}) = \mathcal{Y}$

so $\mathcal{X}/\sim_f \cong \mathcal{Y}$ ■

Theorem: $K_{\subseteq X}$: com $\wedge$ $Z_{\subseteq Y}$: com

$\wedge$ $K \times Z \subseteq N_{\in \mathcal{T}_{X \times Y}}$

$\Longrightarrow \exists\, \mathcal{U}_{\in \mathcal{T}_X}, \mathcal{V}_{\in \mathcal{T}_Y} \quad K \times Z \subseteq \mathcal{U} \times \mathcal{V} \subseteq N$

Proof: $N \in \mathcal{T}_{X \times Y}$

so $\exists J \quad N = \bigcup_J (\mathcal{U}_{\in \mathcal{T}_X} \times \mathcal{V}_{\in \mathcal{T}_Y})_j$

Let $(\sigma, \Omega) \in K \times Z \subseteq N$

so $\exists\, \mathcal{U}_{(\sigma,\Omega)}^{\in \mathcal{T}_X \ni \sigma}, \mathcal{V}_{(\sigma,\Omega)}^{\in \mathcal{T}_Y \ni \Omega}$ such that

$(\sigma, \Omega) \in \mathcal{U}_{(\sigma,\Omega)} \times \mathcal{V}_{(\sigma,\Omega)} \subseteq N$,

since Z: com and $\bigcup_{\Omega \in Z} \mathcal{V}_{(\sigma,\Omega)} \supseteq Z$,

$\exists\, Z_{* \subseteq Z}^{|Z_*| \in \mathbb{N}} \quad \bigcup_{\Omega \in Z_*} \mathcal{V}_{(\sigma,\Omega)} \supseteq Z$

since K: com and $\bigcup_{\sigma \in K} \bigcap_{\Omega \in Z_*} \mathcal{U}_{(\sigma,\Omega)} \supseteq K$

$\exists\, K_{* \subseteq K}^{|K_*| \in \mathbb{N}} \quad \bigcup_{\sigma \in K_*} \bigcap_{\Omega \in Z_*} \mathcal{U}_{(\sigma,\Omega)} \supseteq K$

define $\mathcal{U} := \bigcup_{\sigma \in K_*} \bigcap_{\Omega \in Z_*} \mathcal{U}_{(\sigma,\Omega)}$

$\mathcal{V} := \bigcap_{\sigma \in K_*} \bigcup_{\Omega \in Z_*} \mathcal{V}_{(\sigma,\Omega)}$

note that $\mathcal{U} \in \mathcal{T}_X$ as $|Z_*| \in \mathbb{N}$

and $\mathcal{V} \in \mathcal{T}_Y$ as $|K_*| \in \mathbb{N}$

Proof cons :

since $\mathcal{U} \supseteq K$ by def

and $\mathcal{V} \supseteq Z$ as $\forall \sigma_{\in K} \bigcup_{\Omega \in Z_*} \mathcal{V}_{(\sigma,\Omega)} \supseteq Z$

that is $Z \subseteq \bigcap_{\sigma \in K_* \subseteq K} \bigcup_{\Omega \in Z_*} \mathcal{V}_{(\sigma,\Omega)}$,

$K \times Z \subseteq \mathcal{U} \times \mathcal{V}$

Let $(u,v) \in \mathcal{U} \times \mathcal{V}$,

so $\exists (\sigma_0, \Omega_0)_{\in K_* \times Z_*}$ $u \in \mathcal{U}_{(\sigma_0,\Omega_0)}$

and $\exists (\sigma_1, \Omega_1)_{\in K_* \times Z_*}$ $v \in \mathcal{V}_{(\sigma_1,\Omega_1)}$

so also since $\forall (\sigma,\Omega)_{\in K \times Z \supseteq K_* \times Z_*}$

$$\mathcal{U}_{(\sigma,\Omega)} \times \mathcal{V}_{(\sigma,\Omega)} \subseteq \mathcal{N}$$

$(u,v) \in \mathcal{U}_{(\sigma_0,\Omega_0)} \times \mathcal{V}_{(\sigma_1,\Omega_1)} \subseteq \mathcal{N}$

so $\mathcal{U} \times \mathcal{V} \subseteq \mathcal{N}$

hence $\exists \mathcal{U}_{\in \mathcal{T}_x}, \mathcal{V}_{\in \mathcal{T}_y}$

$K \times Z \subseteq \mathcal{U} \times \mathcal{V} \subseteq \mathcal{N}$ ■

Theorem: $\mathcal{X}$: com $\wedge$ $\mathcal{Y}$: com

$\Longrightarrow$ $\mathcal{X} \times \mathcal{Y}$: com

Proof: Let $\bigcup_I \mathcal{U}_i^{\in \mathcal{J}_{\mathcal{X}\times\mathcal{Y}}} \supseteq \mathcal{X} \times \mathcal{Y}$

so $\forall x_{\in \mathcal{X}}$ $\bigcup_I \mathcal{U}_i \supseteq \{x\} \times \mathcal{Y}$

so since $\{x\} \times \mathcal{Y} \cong \mathcal{Y}$ and $\mathcal{Y}$: com, $\forall x_{\in \mathcal{X}}$

$\exists J_{x \subseteq I}^{|J| \in \mathbb{N}}$ $\bigcup_{J_x} \mathcal{U}_{j_x} \supseteq \{x\} \times \mathcal{Y}$

define $N_x = \bigcup_{J_x} \mathcal{U}_{j_x}$

so by previous theorem

since $N_x \in \mathcal{J}_{\mathcal{X}\times\mathcal{Y}}$ and $\{x\}$: com as $|\{x\}| \in \mathbb{N}$

$\exists W_x^{\in \mathcal{J}_\mathcal{X}}$ $\{x\} \times \mathcal{Y} \subseteq W_x \times \mathcal{Y} \subseteq N_x$

so $\bigcup_{x \in \mathcal{X}} W_x \supseteq \mathcal{X}$, so since $\mathcal{X}$: com

$\exists K_{\subseteq \mathbb{N}}^{|K| \in \mathbb{N}}$ $\bigcup_{\kappa} W_{x_\kappa} \supseteq \mathcal{X}$

hence $\mathcal{X} \times \mathcal{Y} \subseteq (\bigcup_{\kappa} W_{x_\kappa}) \times \mathcal{Y}$

$= \bigcup_{\kappa} (W_{x_\kappa} \times \mathcal{Y}) \subseteq \bigcup_{\kappa} N_{x_\kappa} \subseteq \bigcup_{\kappa} \bigcup_{J_{x_\kappa}} \mathcal{U}_{j_{x_\kappa}}$

define $J_* = \{ j \mid j \in J_{x_\kappa} \wedge \kappa \in K \}$

so $\exists S_{\subseteq I}^{\overset{(=J_*)}{|S| \in \mathbb{N}}}$ $\bigcup_S \mathcal{U}_s \supseteq \mathcal{X} \times \mathcal{Y}$

so $\mathcal{X} \times \mathcal{Y}$: com $\blacksquare$

Def: $\mathcal{M}$: n-manifold $:=$

$\mathcal{M} : T_2 \quad \wedge \quad \mathcal{M}$: sec cou

$\wedge\ \mathcal{M}$: loc euc i.e. $\forall x_{\in \mathcal{M}}\ \exists\, \mathcal{U}^{\in \mathcal{T}}_{\ni x}, \varphi_{\mathcal{U}} \quad \mathcal{U} \underset{\varphi_{\mathcal{U}}}{\cong} \mathbb{R}^n$

Ex:   

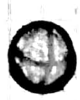
$\mathbb{S}^1 \qquad \mathbb{S}^2 \qquad \mathbb{S}^1 \times \mathbb{S}^1$

Def: $\mathcal{M}$: n-manifold with boundary $:=$

$\mathcal{M} : T_2 \quad \wedge \quad \mathcal{M}$: sec cou

$\wedge\ \forall x_{\in \mathcal{M}}\ \exists\, \mathcal{U}^{\in \mathcal{T}}_{\ni x}, \varphi_{\mathcal{U}} \quad \mathcal{U} \underset{\varphi_{\mathcal{U}}}{\cong} \mathbb{R}^n \ \veebar\ \mathcal{U} \underset{\varphi_{\mathcal{U}}}{\cong} \mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$

$\partial \mathcal{M} := \{ x_{\in \mathcal{M}} \mid \forall\, \mathcal{U}^{\in \mathcal{T}}_{\ni x} \quad \mathcal{U} \cong \mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1} \}$

Ex:   

$\mathbb{D}^1$

$\partial \mathbb{D}^1 = \mathbb{S}^0$

$\mathbb{D}^3$

$\partial \mathbb{D}^3 = \mathbb{S}^2$

$\mathbb{D}^2 \times \mathbb{S}^1$

$\partial(\mathbb{D}^2 \times \mathbb{S}^1) = \mathbb{S}^1 \times \mathbb{S}^1$