EEE 321
Signals and Systems
Lab Assignment 5
Spring 2024–2025

due: 16 April 2025, Wednesday by 23:55 on Moodle

Part 1: Fourier Series and its Convergence

Any periodic signal $x(t)$, with fundamental period T_o, has a Fourier series representation (assuming that Dirichlet conditions hold) that can be expressed as

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t}.$$

Here, $\omega_o = \frac{2\pi}{T_o}$, and the Fourier series coefficients $\{a_k\}_{k=-\infty}^{\infty}$ are given by

$$a_k = \int_{T_o} x(t) e^{-jk\omega_o t}\, dt,$$

where the integral can be taken over any period T_o of the signal.

For any $N \in \mathbb{Z}$, let $x_N(t)$ be defined as

$$x_N(t) \triangleq \sum_{k=-N}^{N} a_k e^{jk\omega_o t}.$$

Hence, if $x(t)$ has a Fourier series expansion, we would expect that

$$\lim_{N\to\infty} x_N(t) = x(t) \quad \forall t \in \mathbb{R}.$$

According to Dirichlet conditions, for any finite interval of time, the number of discontinuities of $x(t)$ must be finite. That is, even though $x(t)$ may have some discontinuities, it can be expressed as a weighted sum of harmonics, which are definitely continuous with respect to t.

- First, consider $x(t) = \cos(2t+\pi/4)$. Find its Fourier series expansion. For $t \in [-\pi/2, \pi/2]$, plot $x_1(t)$ and $x(t)$ on the same plot. Comment on the results.

- Second, consider $x(t) = \begin{cases} 1, & \text{for } t \in [2n, 2n+1] \text{ and } n \in \mathbb{Z} \\ 0, & \text{otherwise.} \end{cases}$

  Compute its Fourier series expansion. For $t \in [0, 2]$, plot $x_{10}(t)$, $x_{100}(t)$, and $x(t)$ on the same plot. Then, compare the values:

  $$\max_{t\in[0,2]} |x(t) - x_{10}(t)| \text{ and } \max_{t\in[0,2]} |x(t) - x_{100}(t)|.$$

  What do you observe?

Part 2: Identifying the Frequencies

Let $x(t)$ be given by

$$x(t) = \sin(2\pi f_1 t) + \sin(2\pi f_2 t),$$

where $f_1 \in \{2, 4, 6\}$ Hz and $f_2 \in \{60, 90, 120\}$ Hz.

Our aim is to identify f_1 and f_2 based on $x(t)$ and the prior knowledge about f_1 and f_2. Show that when $f_i \neq f_j$,

$$\int_{T_o} \sin(2\pi f_i t) \sin(2\pi f_j t)\, dt = 0.$$

where the integral is taken over a period T_o such that $\sin(2\pi f_i T_o) = \sin(2\pi f_j T_o) = 0$. Hence, a receiver structure can be defined to estimate the values of f_1 and f_2, denoted by $\hat{f}_1$ and $\hat{f}_2$, as follows:

$$\hat{f}_1 = \underset{f \in \{2,4,6\}}{\arg\max} \left| \int_{T_o} x(t) \sin(2\pi f t)\, dt \right| \quad \text{and} \quad \hat{f}_2 = \underset{f \in \{60,90,120\}}{\arg\max} \left| \int_{T_o} x(t) \sin(2\pi f t)\, dt \right| \tag{1}$$

- Let $f_1 = 2$ Hz and $f_2 = 120$ Hz. Choose the sampling rate as $f_s = 360$ Hz. Plot $x(t)$ for $t \in [0, 2]$. Then, try to identify f_1 and f_2 based on the procedure outlined above.

- Second, assume that an error occurred in the indicated receiver structure in Eqn.(1), yielding a corrupted receiver such that the integral is taken over $T_1 = \frac{19T_o}{20}$ instead of T_o. What would you expect? If you expect an error, show your work mathematically and repeat the previous part using such a corrupted receiver. Compare your results.

- Third, assume that the signal $x(t)$ is corrupted by some random noise signal $n(t)$. To implement this, use the following code in MATLAB:

  $$\mathtt{x} = \mathtt{x} + \mathtt{sigma} * \mathtt{randn}(1, \mathtt{length}(\mathtt{x}));$$

  For the values of $\sigma = 1$ and $\sigma = 10$, identify f_1 and f_2. Comment on the results.

- Now, suppose that σ can take values in $\{1, 10, 50, 100\}$. For each σ value, perform 10^5 different trials, and in each trial assume that both f_1 and f_2 are taken uniformly from $\{2, 4, 6\}$ and $\{60, 90, 120\}$, respectively. (You may use `randi`$(\cdot)$ command of MATLAB.) Then, for each σ value, find all the errorenous estimates of f_1 and f_2, count their number, and normalize the count with 2×10^5. Store this count value for each σ value and plot the count values with respect to σ. Comment on the results.

  Do you see any analogy between our methodology and Fourier series representation? If so, explain it.

  Note: In case of an equality in Eqn.(1), you may pick the estimate arbitrarily. For example, in the following situation:

  $$\left| \int_{T_o} x(t) \sin(2\pi 2t)\, dt \right| = \left| \int_{T_o} x(t) \sin(2\pi 4t)\, dt \right| > \left| \int_{T_o} x(t) \sin(2\pi 6t)\, dt \right|$$

  you may choose $\hat{f}_1$ as either 2 or 4 with equal probability.

Part 3: Decomposing Music into Notes

In this part of the assignment, we have composed a song based on given musical notes. First, we will decompose the given song into musical notes.

Let $\{\phi_k(t)\}_{k=1}^{12}$ denote the waveforms corresponding to 12 different musical notes. Let $T = 0.25$ sec be the duration of each $\phi_k(t)$. We know that:

$$\phi_k(t) = \sin(2\pi f_k t),$$

where $f_k = 440 \times 2^{(k-1)/12}$ Hz, for all $k = 1, 2, \ldots, 12$.

Let $x(t)$ denote the song signal and assume that there are N different musical notes in $x(t)$. Then, $x((m-1)T : mT)$ is one of the signals in $\{\phi_k(t)\}_{k=1}^{12}$ for any $1 \leq m \leq N$. In other words, we need to estimate the frequency content of $x(t)$ for each interval $t \in [(m-1)T, mT]$.

We know that spectrum plot of $\phi_k(t)$ comprises two impulses located at $\omega_1 = -2\pi f_k$ and $\omega_2 = 2\pi f_k$. Therefore, what we need to do is simply identify the location of peaks in the frequency spectrum of $x((m-1)T : mT)$, and find the closest location among the frequencies $\{f_k\}_{k=1}^{12}$. Noting that the given song (`songnote`) is of size 200000×1 and sampled at a rate of 4000 Hz, listen to `songnote` using the `sound(·)` command. Then, by simply mapping each interval of length T to the corresponding musical note ($\phi_k(t)$), rebuild the song into another variable named `qsong`. Again, using the `sound(·)` command, listen to `qsong` and compare it with `songnote`.

Comment on the similarities and differences between the problems discussed in Part II and Part III. Could we use the approach that we use in Part III for Part II?

Furthermore, assume that the actual signal is corrupted as in Part II. You may use the following MATLAB code to implement this:

$$\texttt{songnote} = \texttt{songnote} + \texttt{sigma} * \texttt{randn}(\texttt{length}(\texttt{songnote}), 1);$$

For various values of σ, plot the actual song and the corrupted song and listen to both of them. Try to regenerate the song as above. Assume that σ can take values $\{1, 2, 3, \ldots, 10\}$. Comment on the results.

Part 4: Filtering the Sound Signal

In the last part of the assignment, utilize the song data given in the previous section and corrupt them with noise using the following MATLAB command:

$$\texttt{Y} = \texttt{songdata} + 0.05 * \texttt{randn}(\texttt{length}(\texttt{songdata}), 1);$$

Then, plot Y and the song data on the same plot. We will use a fifth-order finite impulse response (FIR) filter to denoise Y. If the filtered output is denoted by Z, then $\{Z[n]\}_{n=0}^{N}$ is given by

$$Z[n] = \begin{cases} \frac{1}{5}\sum_{k=0}^{4} Y[n-k], & \text{if} \quad n \geq 5, \\ Y[n], & \text{otherwise.} \end{cases}$$

Give the expression for the impulse response of the FIR filter. Find its frequency response and plot it.

Then, plot the filtered signal, actual signal, and the corrupted one on the same plot. Listen to the filtered signal using the `sound`(·) command of MATLAB. Comment on the results.

Remarks:

Submit the results of your own work in the form of a well-documented lab report on Moodle. Borrowing full or partial code or other material from your peers or elsewhere is **not** allowed and will be penalized.

Throughout this assignment, you are **not** allowed to use symbolic operations in MATLAB. The axes of all plots should be scaled and labeled. To modify the styles of the plots, add labels, and scale the plots, use only MATLAB commands; do **not** use the GUI of the figure windows. When your program is executed, the figures must appear exactly the same as you provide in your solution. You need to write your MATLAB codes not only correctly but efficiently as well.

Please include all evidence (plots, screenshots, MATLAB codes, MATLAB command window print-outs, etc.) as needed in your report. Append your MATLAB code at the end of your assignment as text, not as an image, and do **not** upload it separately. You can use the "Publish" menu of MATLAB to generate a PDF file from your codes and their outputs and append it to the end of your report. If you do this, please also indicate the part that the code corresponds to with a label. Typing your report instead of handwriting some parts will be better. If you decide to write some parts by hand, please use plain white paper. Please do not upload any photos/images of your report or parts of it. Your complete report should be uploaded on Moodle as a **single** good-quality pdf file on plain white background by the given deadline. Please try to upload several hours before the deadline to avoid last-minute problems that may cause you to miss the deadline. Please do **not** submit files by e-mail or on memory stick/CD or as hard copies.

Please name the PDF file you submit on Moodle as follows, using only lower-case English characters for your first name, middle name (if any), and last name. Please use your full name as it appears on the Bilkent system.

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