ROBOT TASARIMI VE UYGULAMALARI

Robot Tasarımı ve Uygulamaları Dersi | 2025-2026 Güz Dönemi

Ders Sorumlusu: Furkan HANİLÇİ

Bölüm: Bilgisayar Mühendisliği

İletişim Kaynakları

LinkedIn: furkanhanilci

Autonomous Driving Software Engineer

GitHub: furkanhanilci

github.com/furkanhanilci

E-posta

furkan.hanilci@ext.karsan.com.tr

Agenda

5.1 Introduction

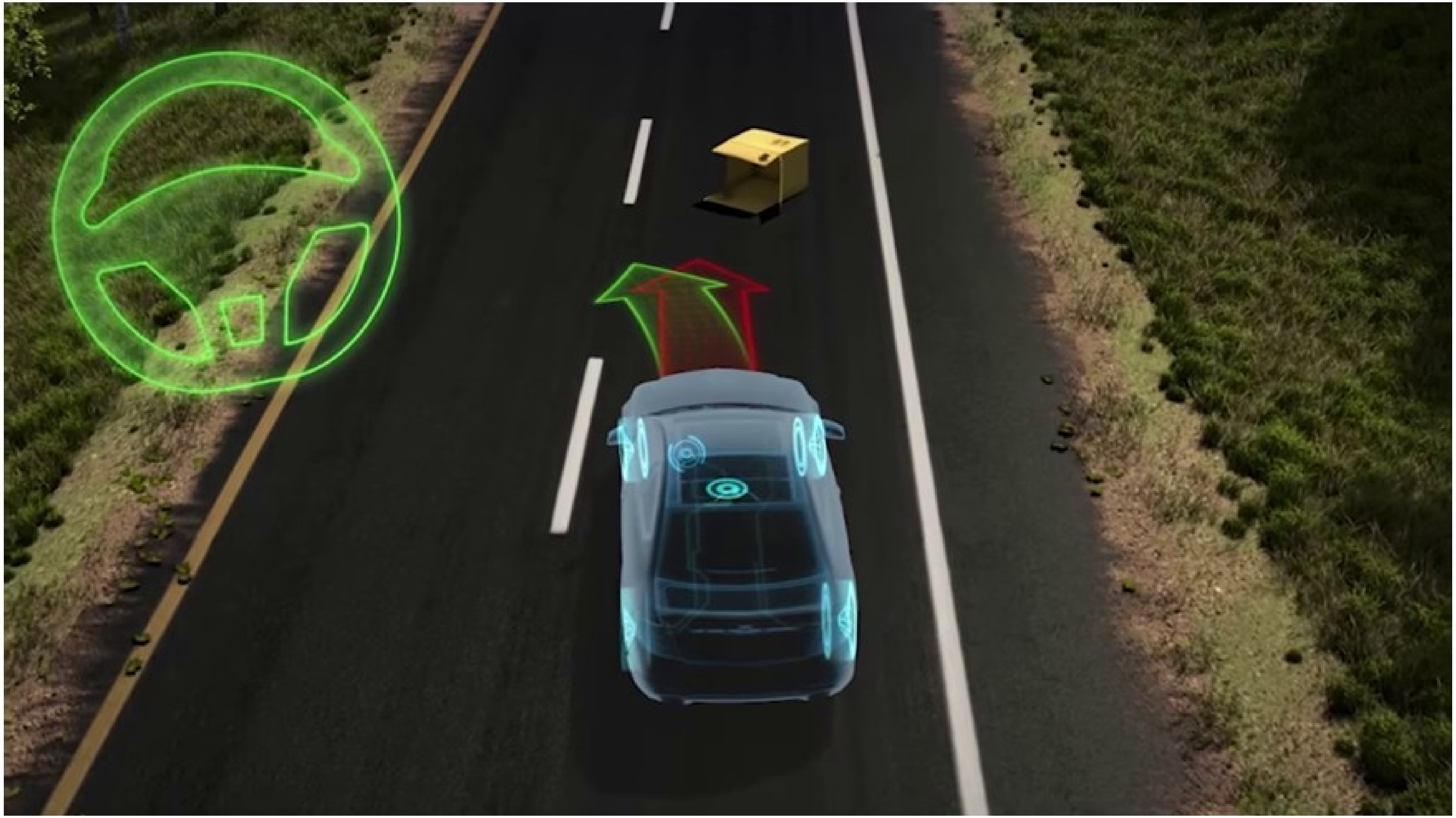
5.2 Kinematic Bicycle Model

5.3 Tire Models

5.4 Dynamic Bicycle Model

5.1
Introduction

Electronic Stability Program

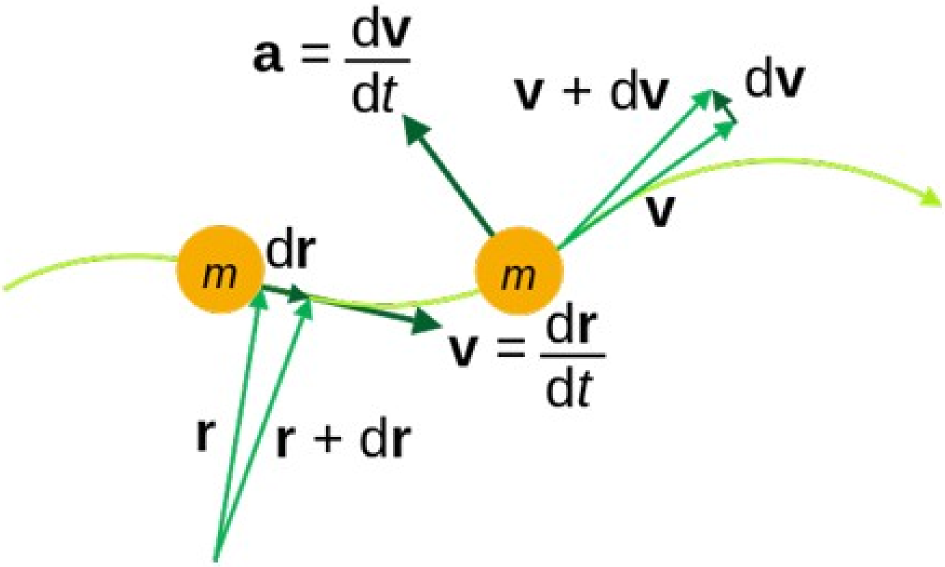


Knowledge of **vehicle dynamics** enables accurate **vehicle control**

Kinematics vs. Kinetics

Kinematics:

- Greek origin: “motion”, “moving”
- Describes motion of points and bodies
- Considers position, velocity, acceleration, ..
- Examples: Celestial bodies, particle systems, robotic arm, human skeleton



Kinetics:

- Describes causes of motion
- Effects of forces/moments
- Newton’s laws, e.g., $F = ma$

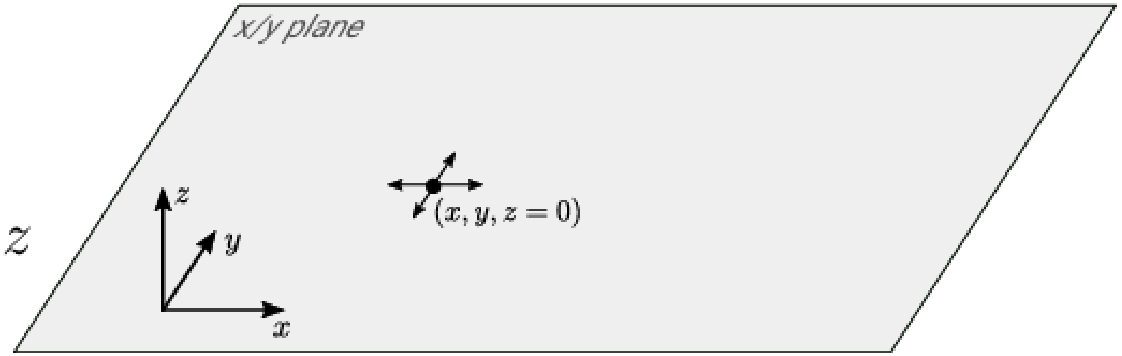
Holonomic Constraints

Holonomic constraints are constraints on the **configuration:**

- Assume a particle in three dimensions $(x, y, z) \in \mathbb{R}^3$
- We can constrain the particle to the x/y plane via:

$$z = 0$$

$$\Leftrightarrow \quad f(x, y, z) = 0 \quad \text{with} \quad f(x, y, z) = z$$



- Constraints of the form $f(x, y, z) = 0$ are called holonomic constraints
- They constrain the configuration space
- But the system can move freely in that space
- Controllable degrees of freedom equal total degrees of freedom (2)

Non-Holonomic Constraints

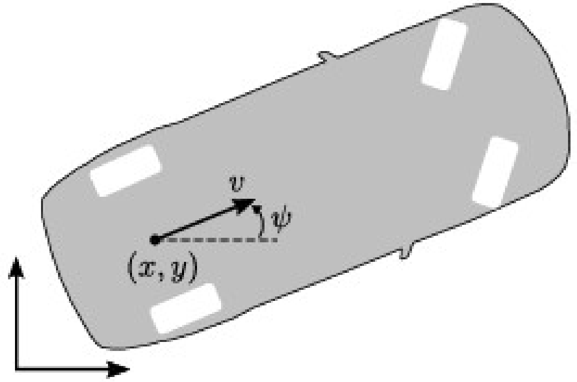
Non-Holonomic constraints are constraints on the **velocity:**

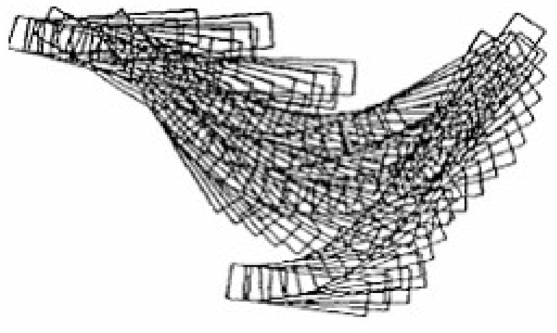
- Assume a vehicle that is parameterized by $(x, y, \psi) \in \mathbb{R}^2 \times [0, 2\pi]$
- The 2D vehicle velocity is given by:

$$\dot{x} = v\cos(\psi)$$

$$\dot{y} = v\sin(\psi)$$

$$\Rightarrow \quad \dot{x}\sin(\psi) - \dot{y}\cos(\psi) = 0$$





- This non-holonomic constraint cannot be expressed in the form $f(x, y, \psi) = 0$
- The car cannot freely move in any direction (e.g., sideways)
- It constrains the velocity space, but not the configuration space
- Controllable degrees of freedom less than total degrees of freedom (2 vs. 3)

Holonomic vs. Non-Holonomic Systems

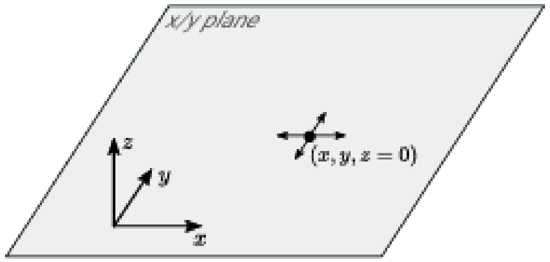
Holonomic Systems

- Constrain configuration space
- Can freely move in any direction
- Controllable degrees of freedom equal to total degrees of freedom
- Constraints **can** be described by $f(x_1, \ldots, x_N) = 0$

Example:

3D Particle

$z = 0$



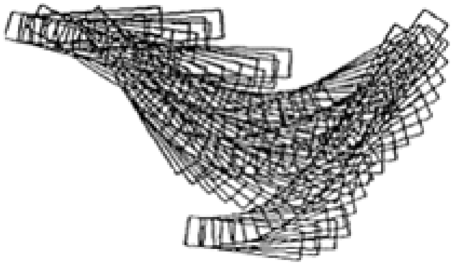
Nonholonomic Systems

- Constrain velocity space
- Cannot freely move in any direction
- Controllable degrees of freedom less than total degrees of freedom
- Constraints **cannot** be described by $f(x_1, \ldots, x_N) = 0$

Example:

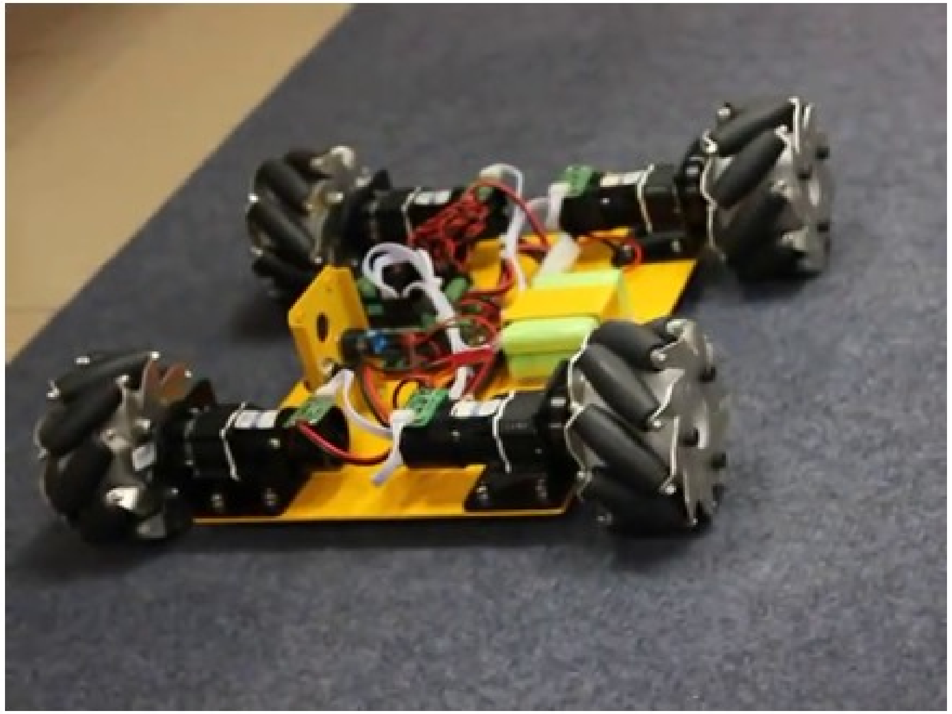
Car

$\dot{x}\sin(\psi) - \dot{y}\cos(\psi) = 0$

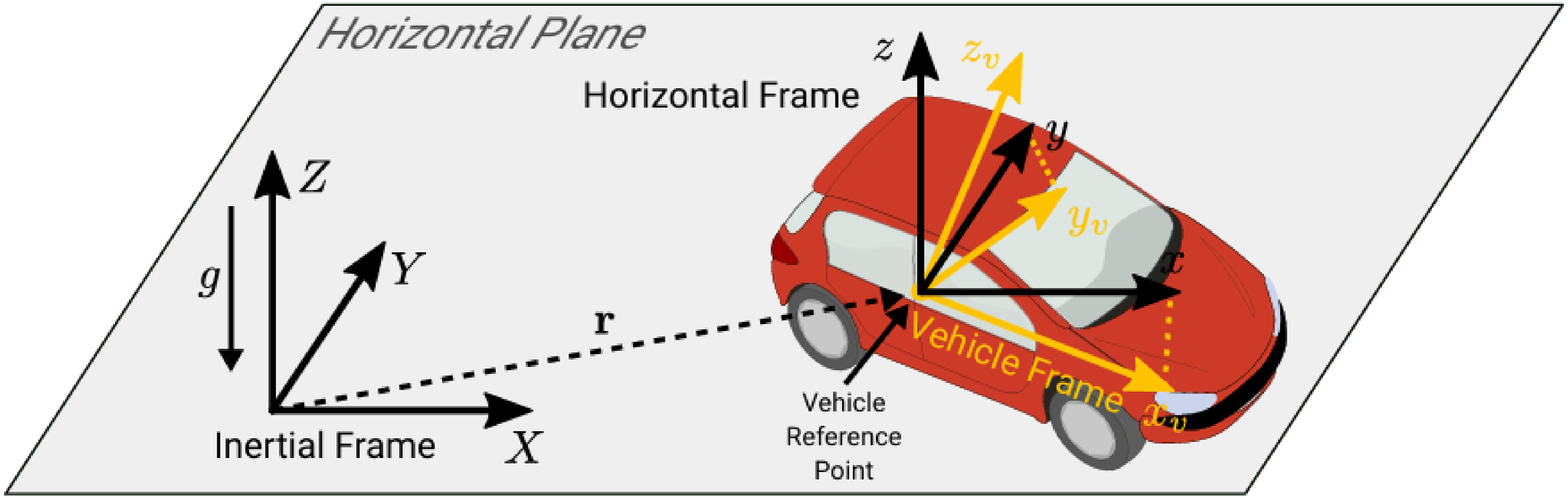


Holonomic vs. Non-Holonomic Systems

- A robot can be subject to both holonomic and non-holonomic constraints
- A car (rigid body in 3D) is kept on the ground by 3 holonomic constraints
- One additional non-holonomic constraint prevents sideways sliding



Coordinate Systems



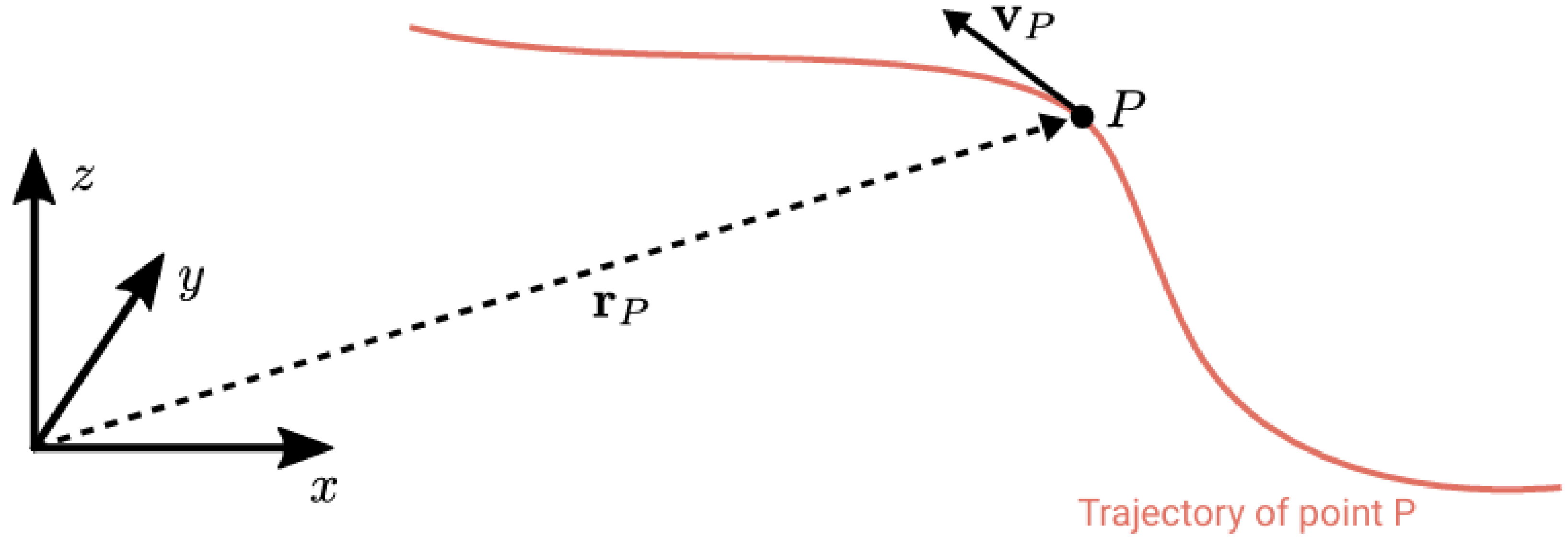
- **Inertial Frame:** Fixed to earth with vertical Z-axis and X/Y horizontal plane
- **Vehicle Frame:** Attached to vehicle at fixed reference point; x_v points towards the front, y_v to the side and z_v to the top of the vehicle (ISO 8855)
- **Horizontal Frame:** Origin at vehicle reference point (like vehicle frame) but x- and y-axes are projections of x_v- and y_v-axes onto the X/Y horizontal plane

Kinematics of a Point

The **position** $\mathbf{r}_P(t) \in \mathbb{R}^3$ of point P at time $t \in \mathbb{R}$ is given by 3 coordinates.

Velocity and **acceleration** are the first and second derivatives of the position $\mathbf{r}_P(t)$.

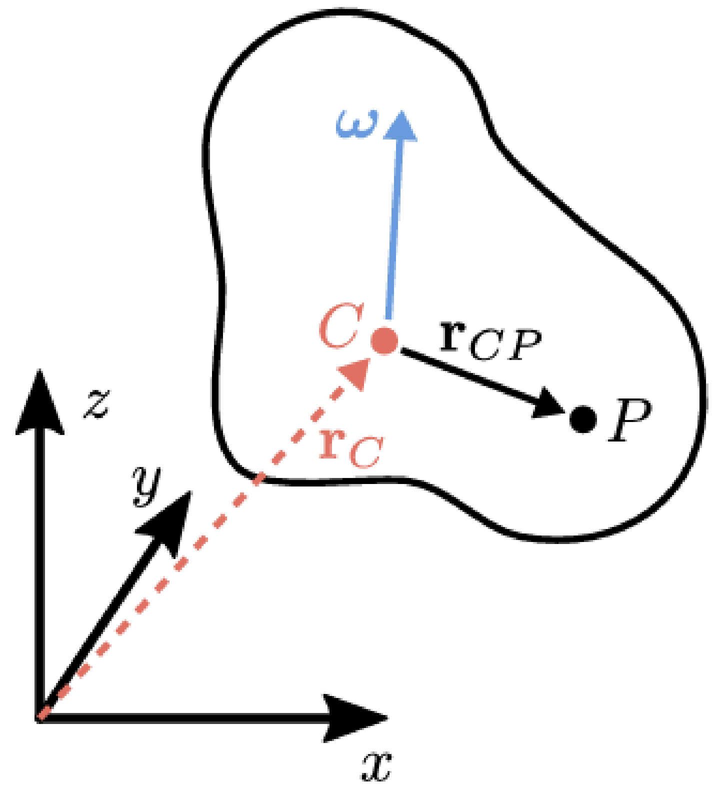
$$\mathbf{r}_P(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \qquad \mathbf{v}_P(t) = \dot{\mathbf{r}}_P(t) = \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} \qquad \mathbf{a}_P(t) = \ddot{\mathbf{r}}_P(t) = \begin{pmatrix} \ddot{x}(t) \\ \ddot{y}(t) \\ \ddot{z}(t) \end{pmatrix}$$



Kinematics of a Rigid Body

A **rigid body** refers to a collection of infinitely many infinitesimally small mass points which are rigidly connected, i.e., their relative position remains unchanged over time. It's **motion** can be compactly described by the motion of an (arbitrary) reference point C of the body plus the relative motion of all other points P with respect to C.

- C: Reference point fixed to rigid body
- P: Arbitrary point on rigid body
- $\boldsymbol{\omega}$: Angular velocity of rigid body
- Position: $\mathbf{r}_P = \mathbf{r}_C + \mathbf{r}_{CP}$
- Velocity: $\mathbf{v}_P = \mathbf{v}_C + \boldsymbol{\omega} \times \mathbf{r}_{CP}$
- Due to rigidity, points P can only rotate wrt. C
- Thus a rigid body has 6 DoF (3 pos., 3 rot.)



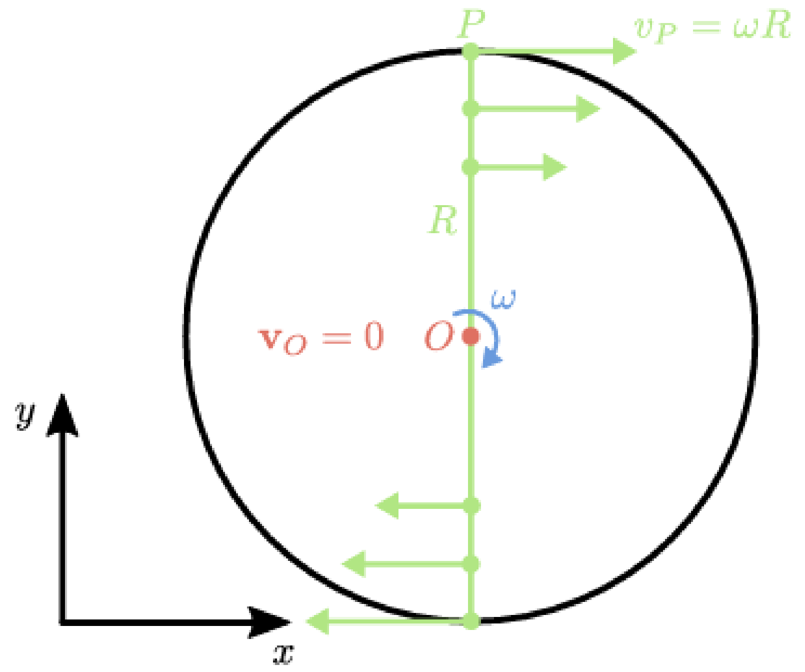
Instantaneous Center of Rotation

At each time instance $t \in \mathbb{R}$, there exists a particular reference point O (called the **instantaneous center of rotation**) for which $\mathbf{v}_O(t) = 0$. Each point P of the rigid body performs a pure rotation about O:

$$\mathbf{v}_P = \mathbf{v}_O + \boldsymbol{\omega} \times \mathbf{r}_{OP} = \boldsymbol{\omega} \times \mathbf{r}_{OP}$$

Example 1: Turning Wheel

- Wheel is completely lifted off the ground
- Wheel does not move in x or y direction
- Ang. vel. vector $\boldsymbol{\omega}$ points into x/y plane
- Velocity of point P: $v_P = \omega R$ with radius R



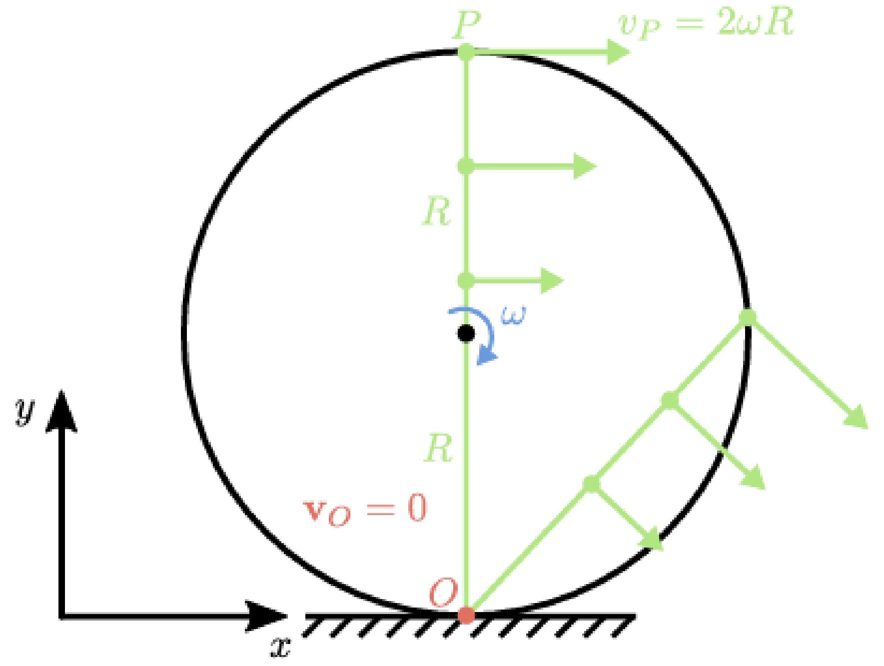
Instantaneous Center of Rotation

At each time instance $t \in \mathbb{R}$, there exists a particular reference point O (called the **instantaneous center of rotation**) for which $\mathbf{v}_O(t) = 0$. Each point P of the rigid body performs a pure rotation about O:

$$\mathbf{v}_P = \mathbf{v}_O + \boldsymbol{\omega} \times \mathbf{r}_{OP} = \boldsymbol{\omega} \times \mathbf{r}_{OP}$$

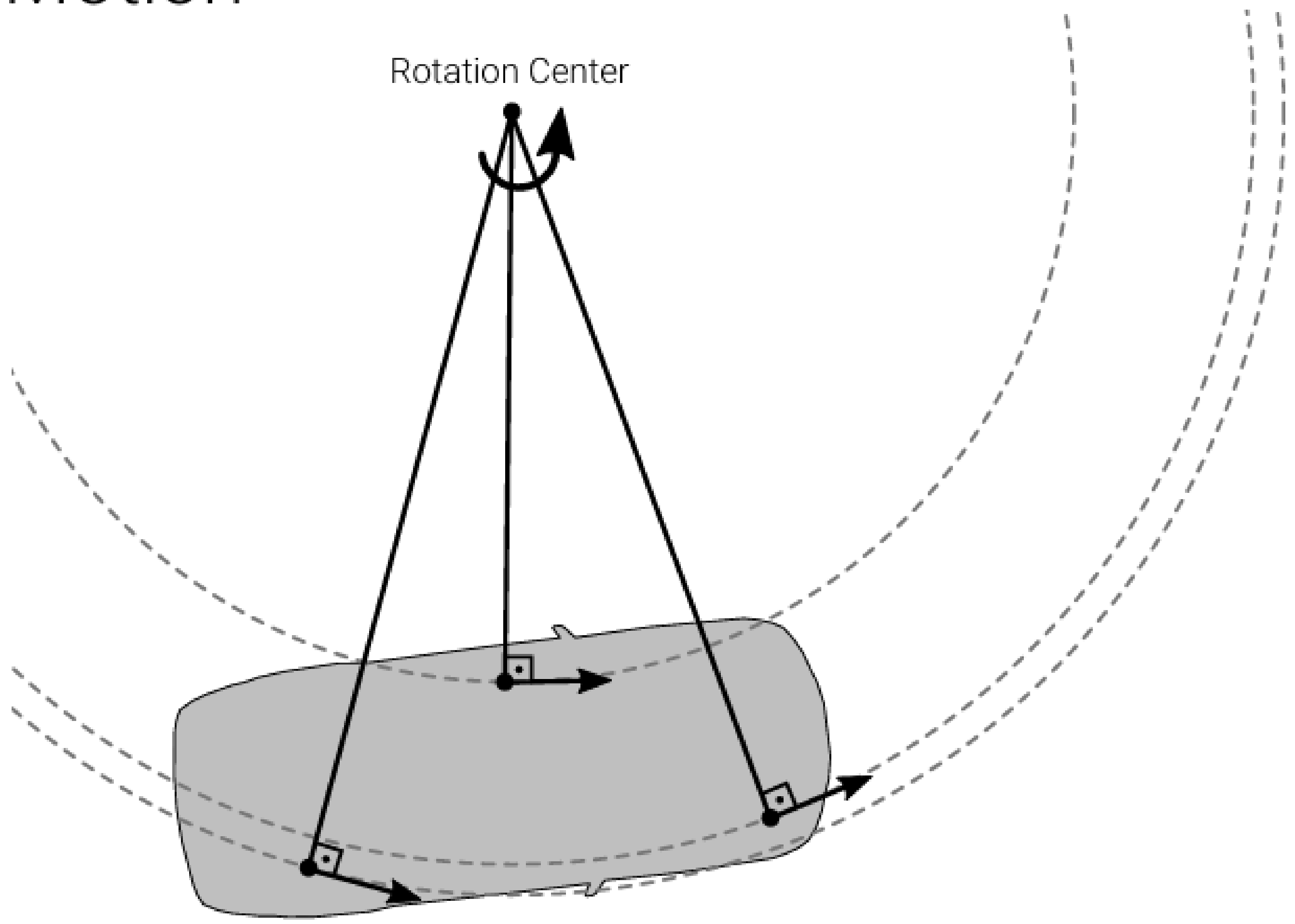
Example 2: Rolling Wheel

- Wheel is rolling on the ground without slip
- Ground is fixed in x/y plane
- Ang. vel. vector $\boldsymbol{\omega}$ points into x/y plane
- Velocity of point P: $v_P = 2\omega R$ with radius R



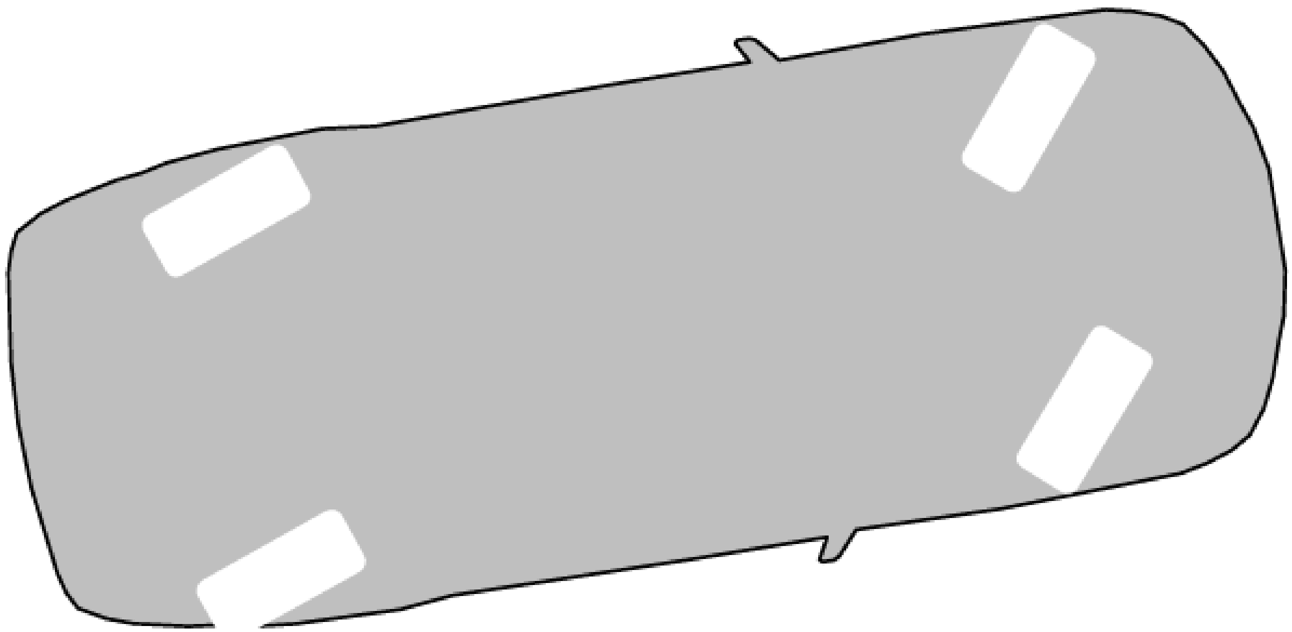
5.2
Kinematic Bicycle Model

Rigid Body Motion



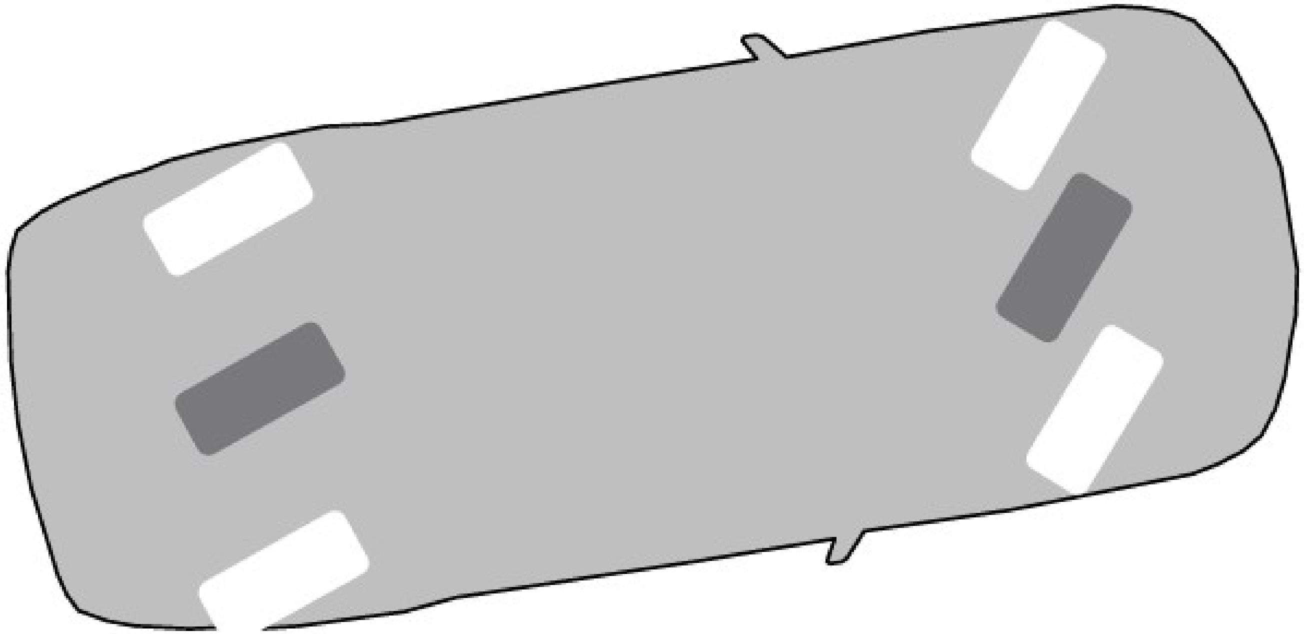
- Different points on the rigid body move along different circular trajectories

Kinematic Bicycle Model



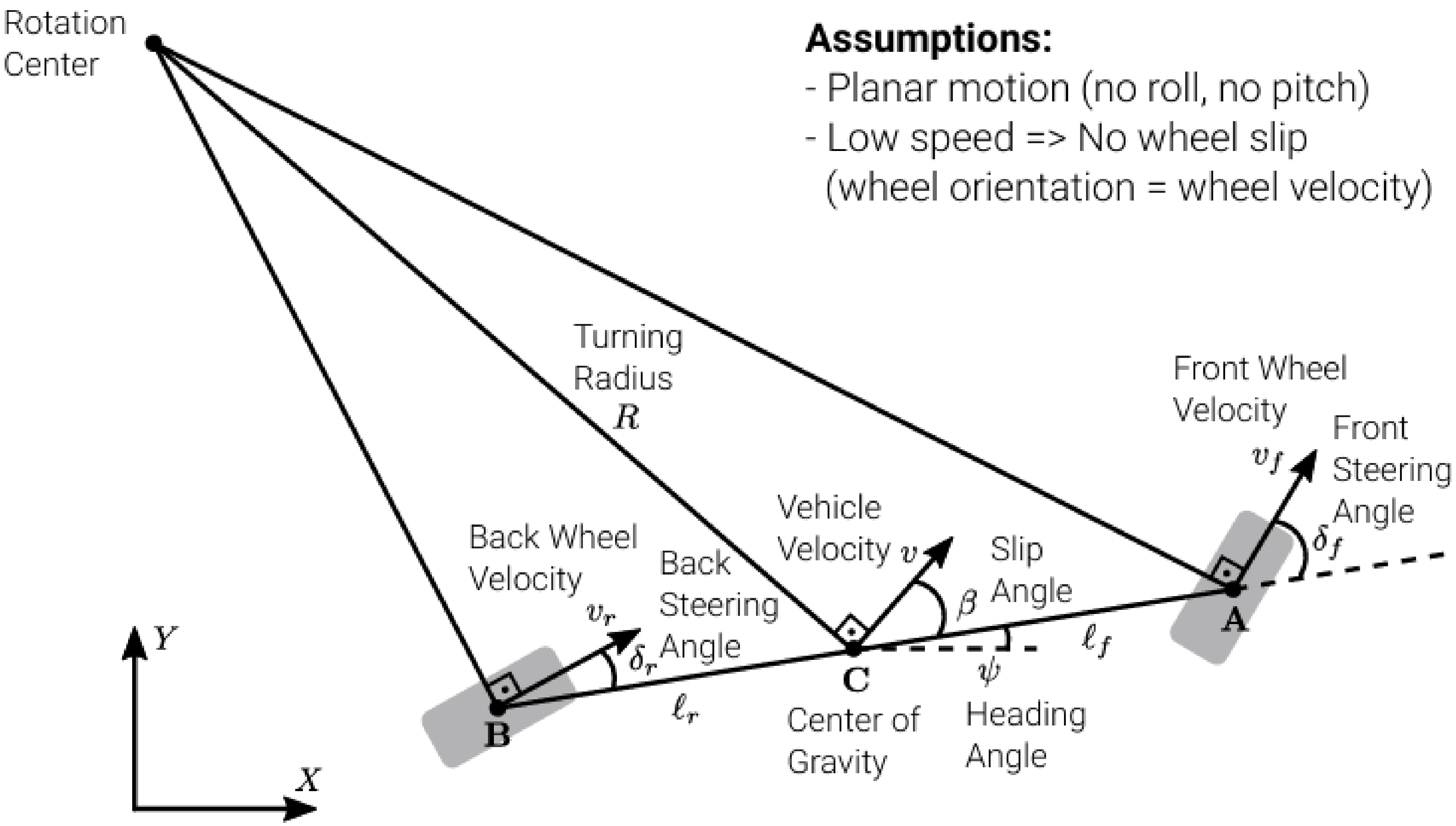
- The **kinematic bicycle model** approximates the 4 wheels with 2 imaginary wheels

Kinematic Bicycle Model



- The **kinematic bicycle model** approximates the 4 wheels with 2 imaginary wheels

Kinematic Bicycle Model



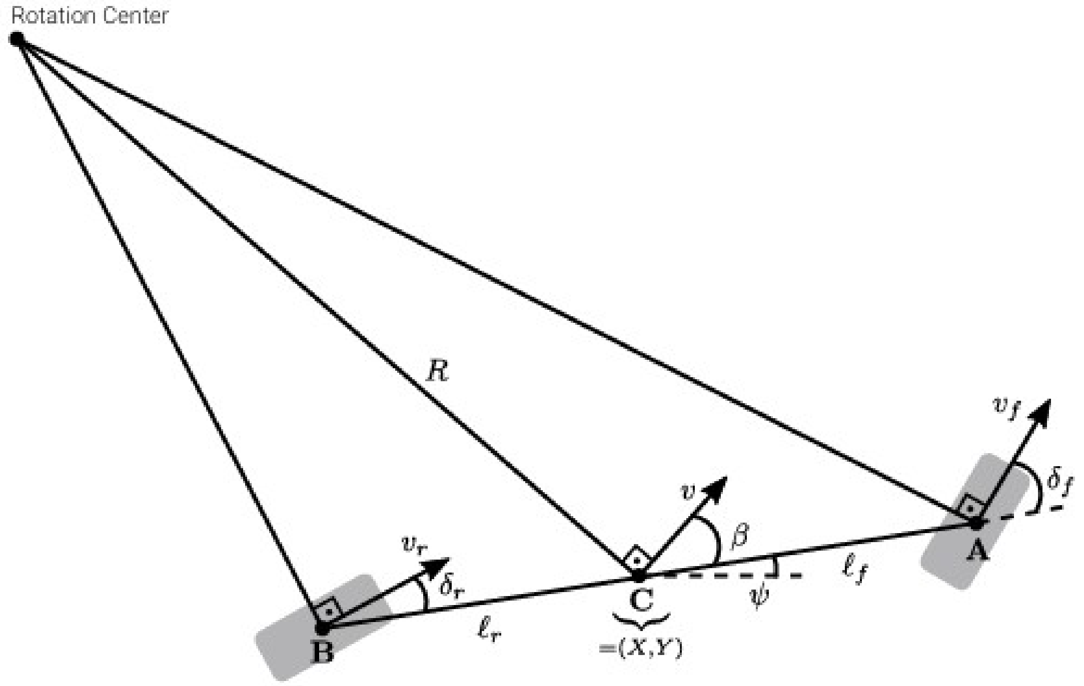
Assumptions:
- Planar motion (no roll, no pitch)
- Low speed => No wheel slip (wheel orientation = wheel velocity)

Wheelbase $L = \ell_r + \ell_f$ Course Angle $\psi + \beta$

► The **kinematic bicycle model** approximates the 4 wheels with 2 imaginary wheels

Kinematic Bicycle Model

Model



Motion Equations

$$\dot{X} = v\cos(\psi + \beta)$$

$$\dot{Y} = v\sin(\psi + \beta)$$

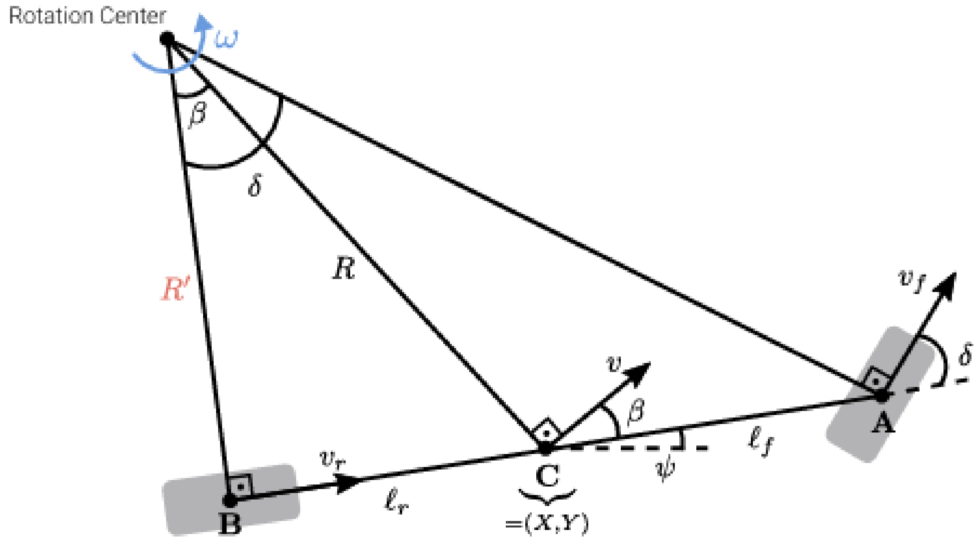
$$\dot{\psi} = \frac{v\cos(\beta)}{\ell_f + \ell_r}\left(\tan(\delta_f) - \tan(\delta_r)\right)$$

$$\beta = \tan^{-1}\left(\frac{\ell_f\tan(\delta_r) + \ell_r\tan(\delta_f)}{\ell_f + \ell_r}\right)$$

(proof as exercise)

Kinematic Bicycle Model

Model



Motion Equations

$$\dot{X} = v\cos(\psi + \beta)$$

$$\dot{Y} = v\sin(\psi + \beta)$$

$$\dot{\psi} = \frac{v\cos(\beta)}{\ell_f + \ell_r}\tan(\delta)$$

$$\beta = \tan^{-1}\left(\frac{\ell_r \tan(\delta)}{\ell_f + \ell_r}\right)$$

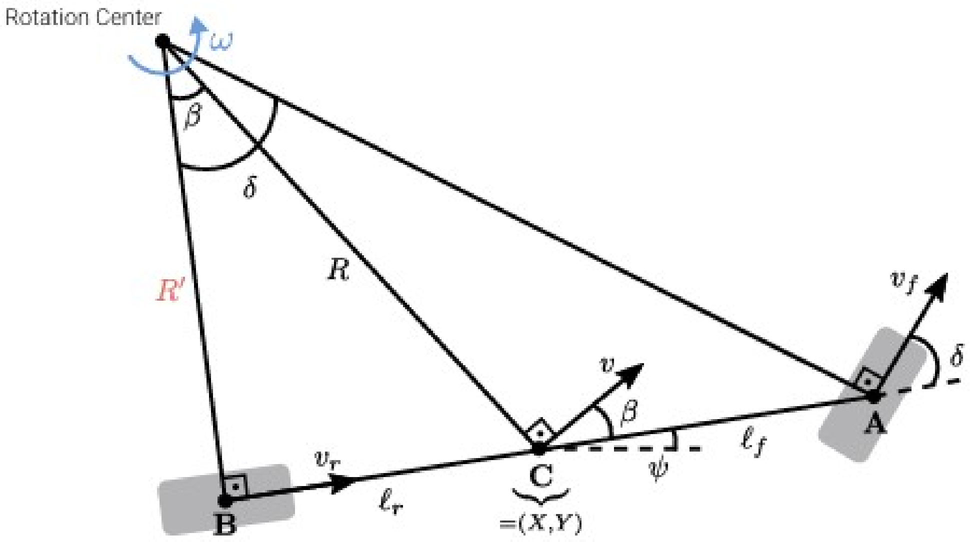
(only front steering)

$$\tan\delta = \frac{l_f + l_r}{R'} \quad \Rightarrow \quad \frac{1}{R'} = \frac{\tan\delta}{l_f + l_r} \quad \Rightarrow \quad \tan\beta = \frac{l_r}{R'} = \frac{l_r \tan\delta}{l_f + l_r}$$

$$\cos\beta = \frac{R'}{R} \quad \Rightarrow \quad \frac{1}{R} = \frac{\cos\beta}{R'} \quad \Rightarrow \quad \dot{\psi} = \omega = \frac{v}{R} = \frac{v\cos(\beta)}{R'} = \frac{v\cos(\beta)}{l_f + l_r}\tan(\delta)$$

Kinematic Bicycle Model

Model



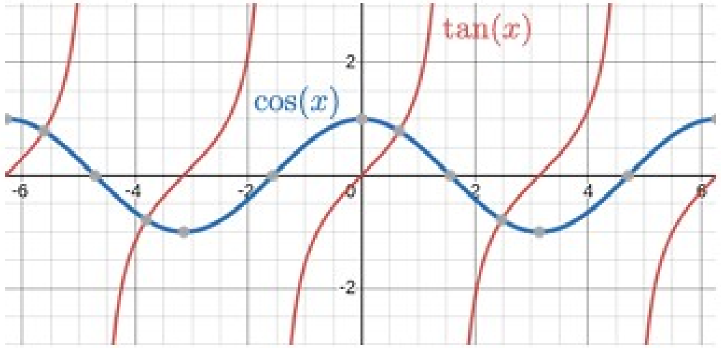
Motion Equations

$$\dot{X} = v\cos(\psi)$$

$$\dot{Y} = v\sin(\psi)$$

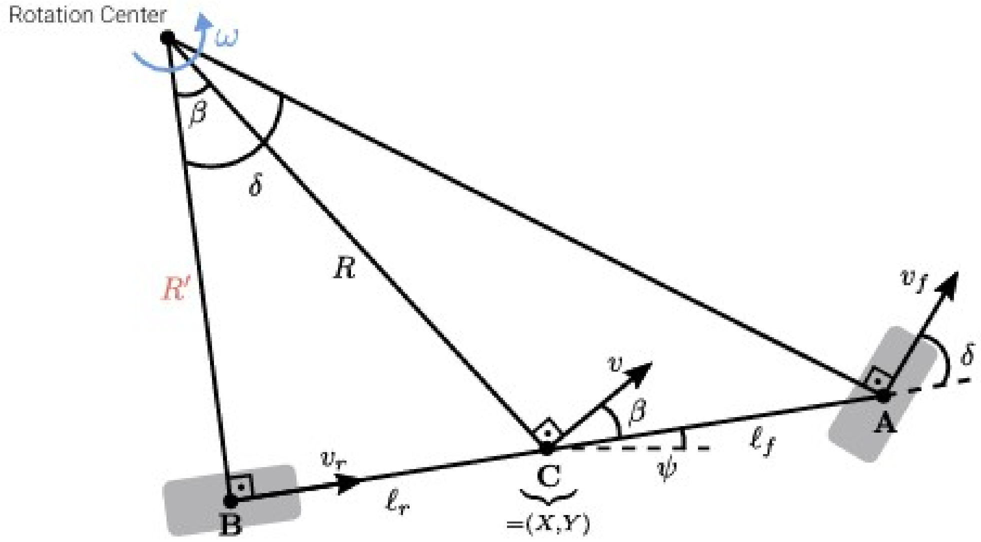
$$\dot{\psi} = \frac{v\delta}{\ell_f + \ell_r}$$

(assuming β and δ are very small)



Kinematic Bicycle Model

Model



Motion Equations

$$X_{t+1} = X_t + v\cos(\psi_t)\,\Delta t$$

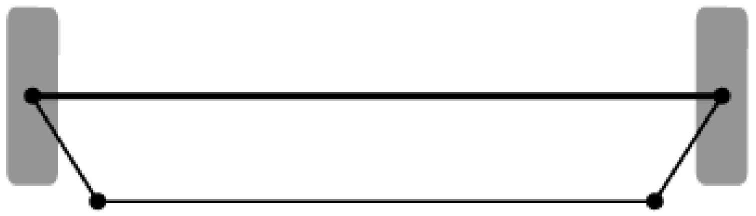
$$Y_{t+1} = Y_t + v\sin(\psi_t)\,\Delta t$$

$$\psi_{t+1} = \psi_t + \frac{v\delta}{\ell_f + \ell_r}\,\Delta t$$

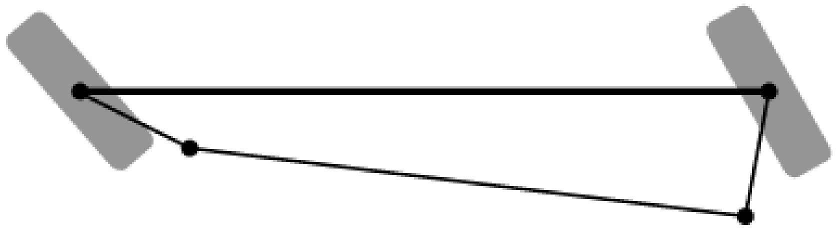
(time discretized model)

Ackermann Steering Geometry

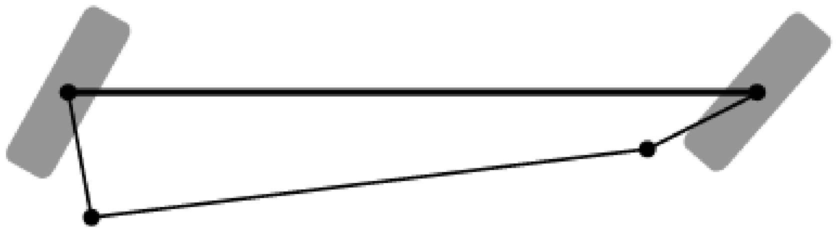
Trapezoidal Geometry



Left Turn



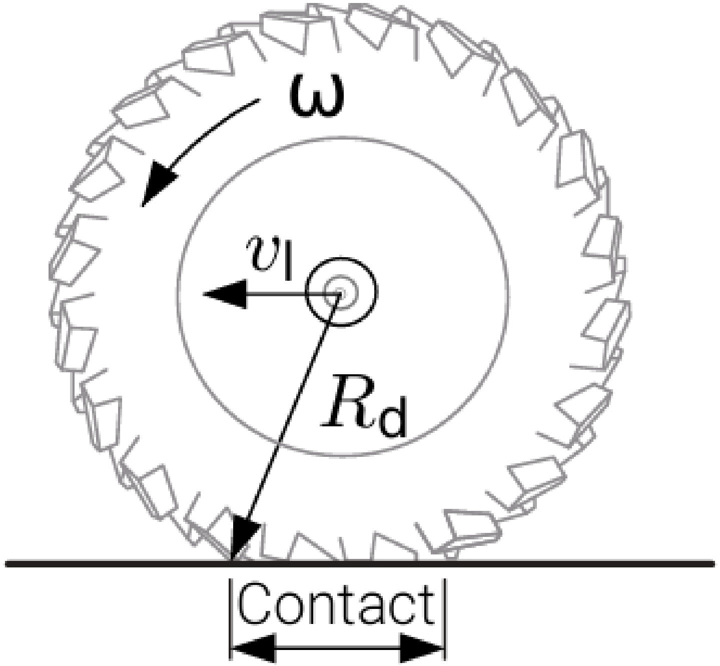
Right Turn



- In practice, this setup can be realized using a trapezoidal tie rod arrangement

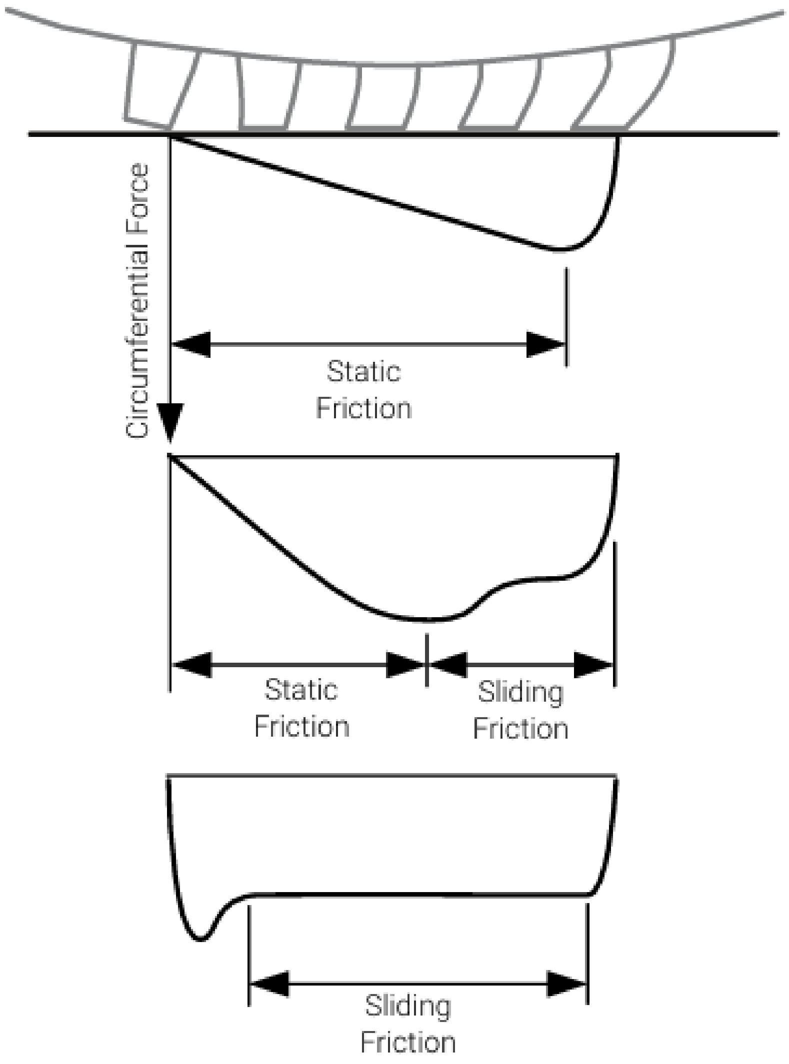
5.3
Tire Models

Tire Models



- Tire models describe the lateral and longitudinal forces at the tires
- There exist many different tire models at various levels of complexity
- For a simple qualitative description we consider the **tread block model**
- **Question:** Why do tires “slip”?

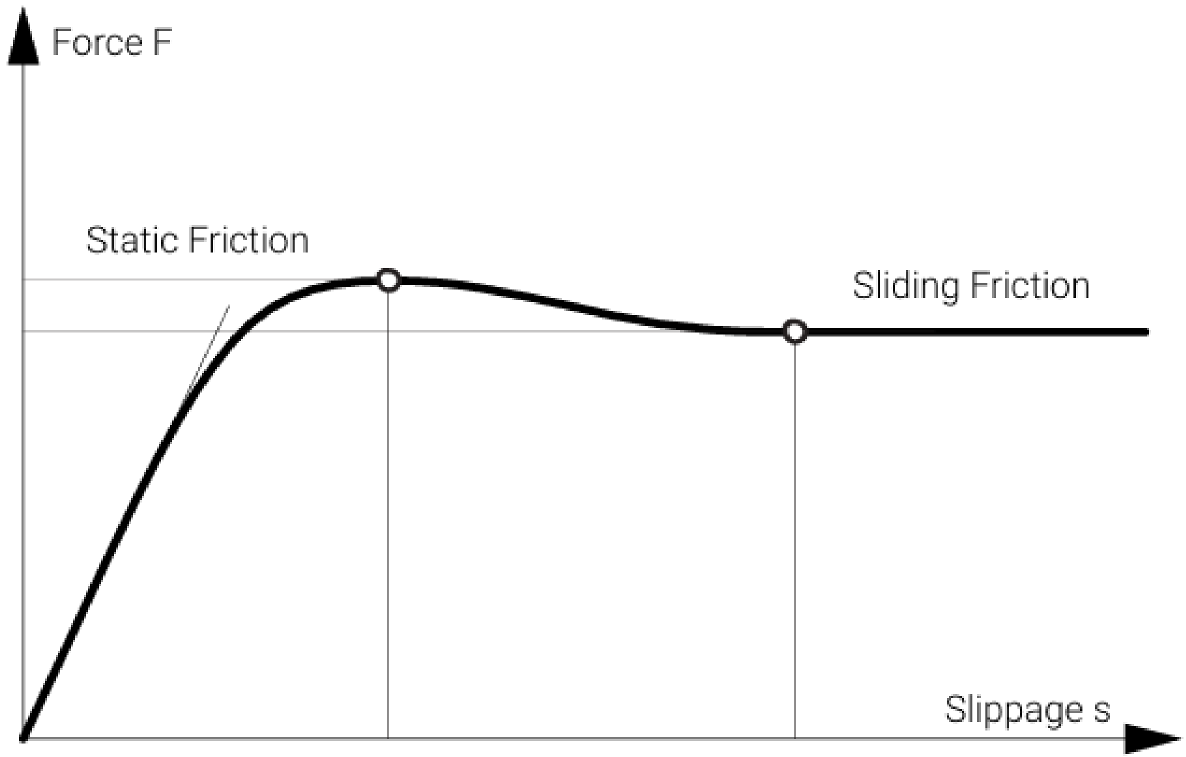
Tread Block Model



Longitudinal Force:

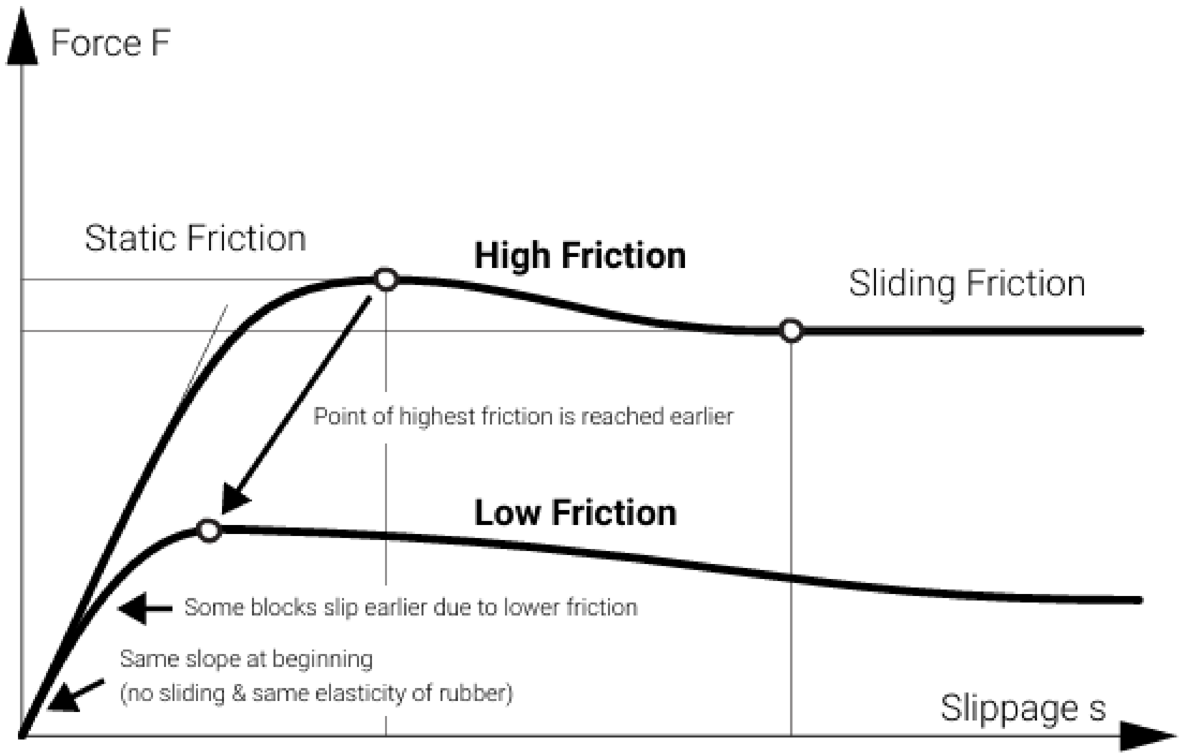
- As soon as the wheel is driven externally, the **tire tread blocks** start deforming and slipping
- The tire tread blocks adhere to the ground, **deform** and **slip** when loosing contact
- When the driving force increases and static friction is exceeded the **blocks slip earlier**
- As **sliding friction** is smaller than **static friction,** this decreases the transmitted driving force
- If the tire tread blocks start sliding at the beginning, only **sliding friction** can be applied

Tread Block Model



- **Slippage:** Difference between surface speed of the wheel and vehicle speed
- The force F grows **linearly** with the slippage s in the beginning (linear deform.)
- Large slippage s leads to a **reduction** of F (sliding friction $<$ static friction)

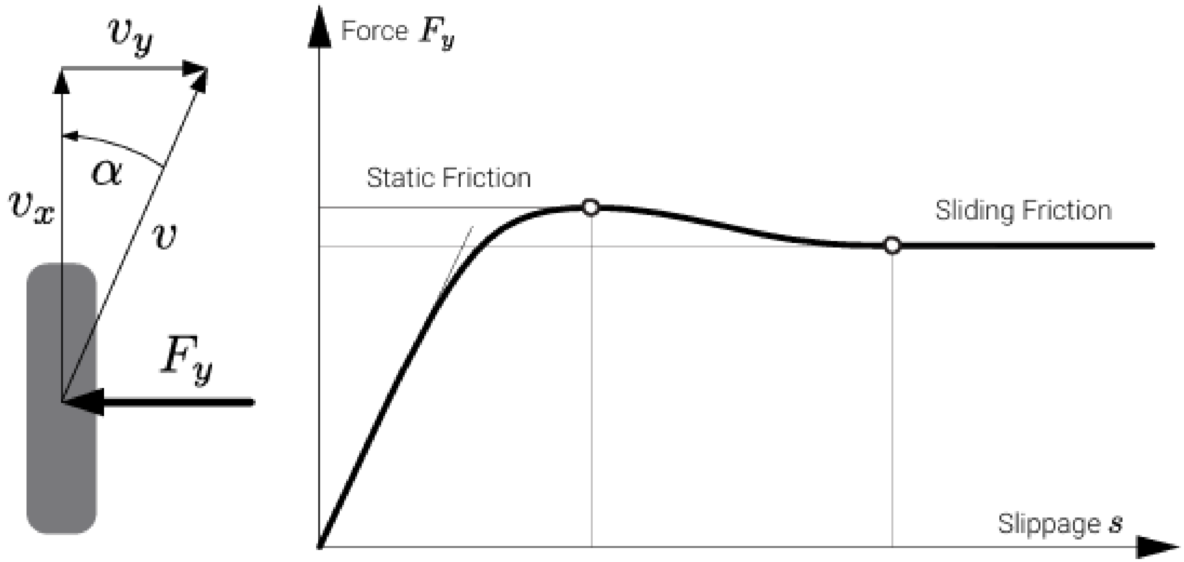
Tread Block Model



How does the force curve $F(s)$ change for **slippery terrain** (low friction)?

- Start of the curve doesn't change as the elasticity of the blocks doesn't change
- However, the **maximum reduces** due to the decreased static friction, i.e., the tread blocks start sliding earlier due to a decrease in friction

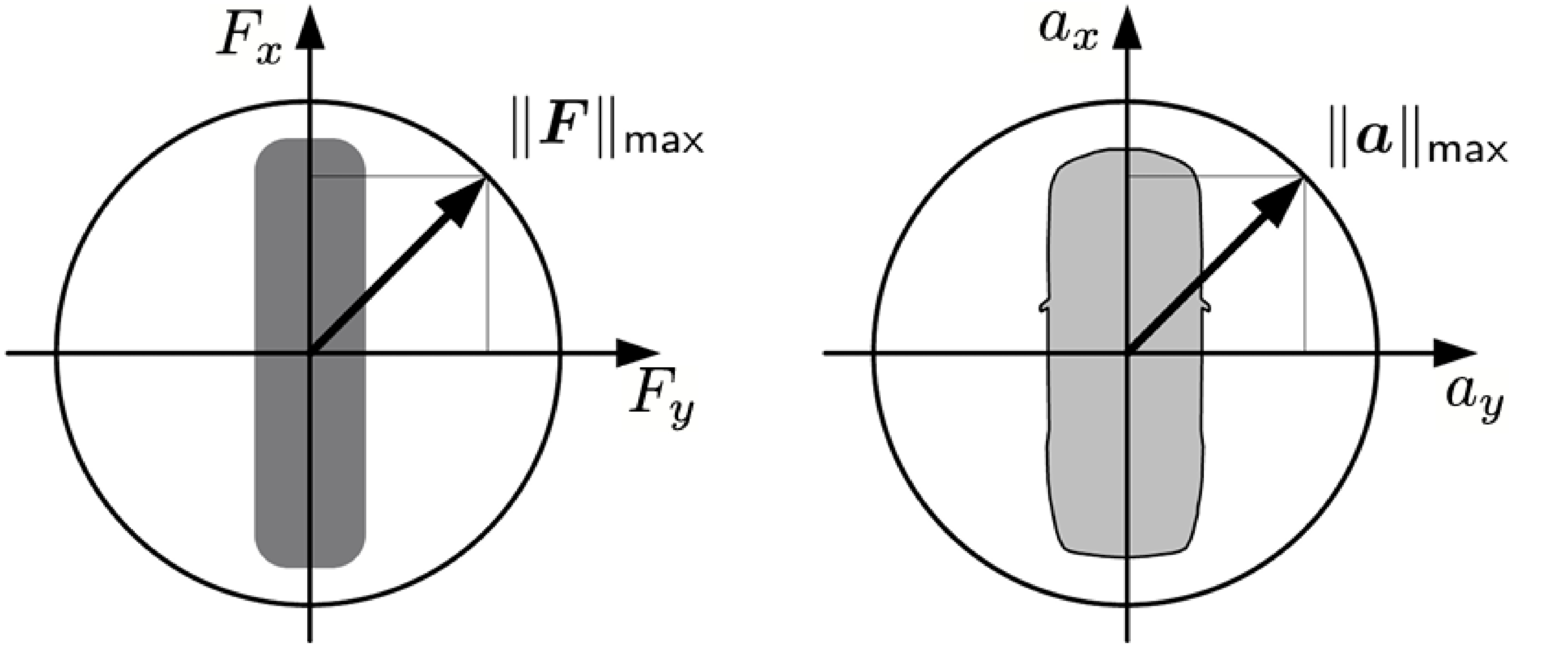
Tread Block Model



Lateral Force: (the figure above shows a top view onto a wheel)

- Lateral force F_y analogous to longitudinal force but blocks move laterally now
- Lateral force for small s and α given by: $F_y = c\,s = c\,\tan(\alpha) \approx c\,\alpha$
- v = wheel velocity, v_x = longitudinal vel., v_y = lateral vel., c = cornering stiffness

Circle of Forces



Circle of Forces:

- Lateral F_y and longitudinal F_x force cannot exceed max. friction force $\|F\|_{max}$
- More long. force implies less lat. force; max. acceleration only for straight driving
- Allows to make statements about maximal possible vehicle accelerations

5.4
Dynamic Bicycle Model

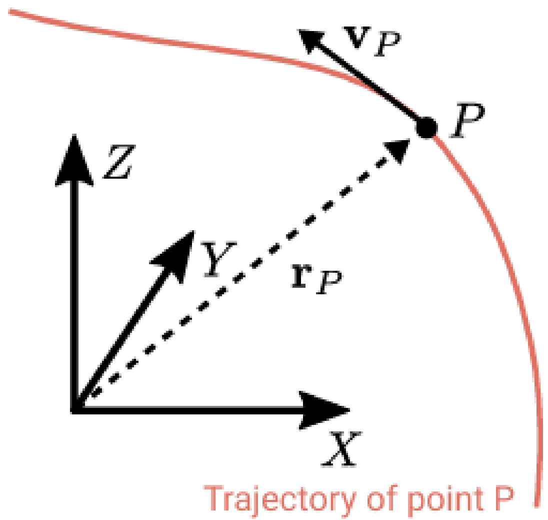
Dynamics of a Rigid Body

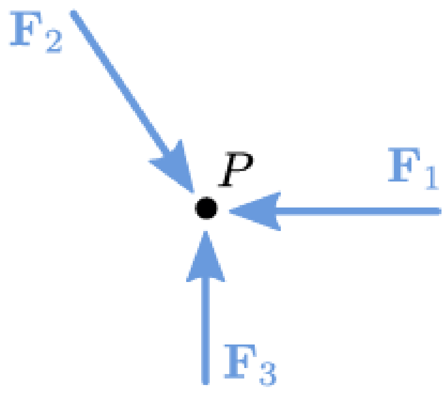
Translatory Motion of a Point:

- Consider **point** P with mass m in $\mathbb{R}^3$
- Let $\mathbf{r}_P(t) \in \mathbb{R}^3$ be its **position** in an inertial reference frame
- Let $\mathbf{v}_P(t)$ denote its **velocity** and $\mathbf{a}_P(t)$ its **acceleration**
- The **linear momentum** of P is defined as $\mathbf{p}_P(t) = m\mathbf{v}_P(t)$
- By **Newton's second law** we have

$$\frac{d}{dt}\mathbf{p}_P(t) = m\,\mathbf{a}_P(t) = \mathbf{F}_{net}(t) = \sum_i \mathbf{F}_i(t)$$

where $\mathbf{F}_i(t)$ represent all forces acting on the point mass P





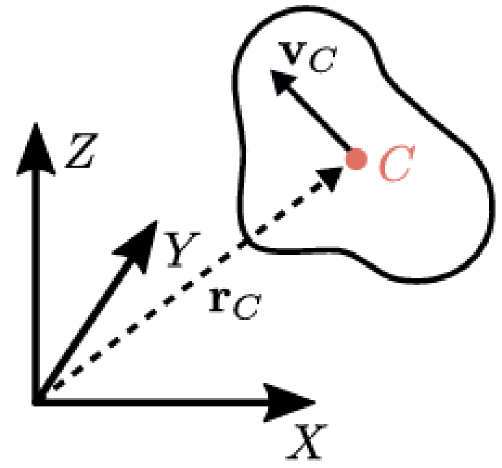
Dynamics of a Rigid Body

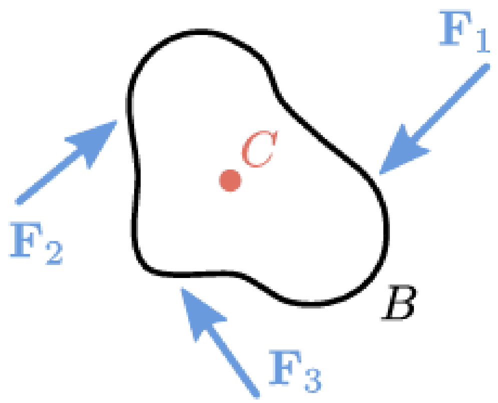
Translatory Motion of a Rigid Body:

- Consider a **rigid body** B with mass m in $\mathbb{R}^3$
- Let $\mathbf{r}_C(t) \in \mathbb{R}^3$ be the **position** of its **center of gravity C**
- Let $\mathbf{v}_C(t)$ denote its **velocity** and $\mathbf{a}_C(t)$ its **acceleration**
- The **linear momentum** of B is defined as $\mathbf{p}_B(t) = m\mathbf{v}_C(t)$
- The **center of gravity** of a rigid body **behaves like a point mass** with mass m and as if all forces act on that point

$$\frac{d}{dt}\mathbf{p}_B(t) = m\,\mathbf{a}_C(t) = \mathbf{F}_{net}(t) = \sum_i \mathbf{F}_i(t)$$

where $\mathbf{F}_i(t)$ represent all forces acting on the rigid body B



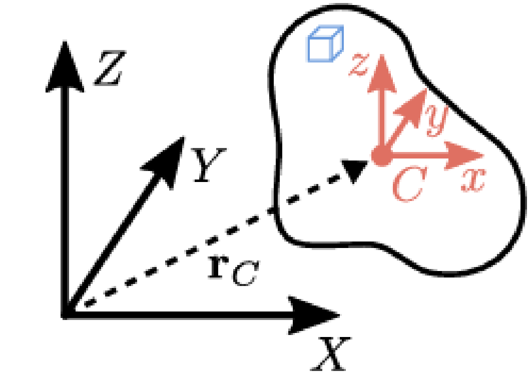


Dynamics of a Rigid Body

Rotatory Motion of a Rigid Body:

- For the **rotatory motion,** also the geometric shape of B and the spatial distribution of its mass is important
- Let $\rho(x, y, z)$ be the **body's density function:**

$$dm = \rho(x, y, z)\, dx\, dy\, dz$$



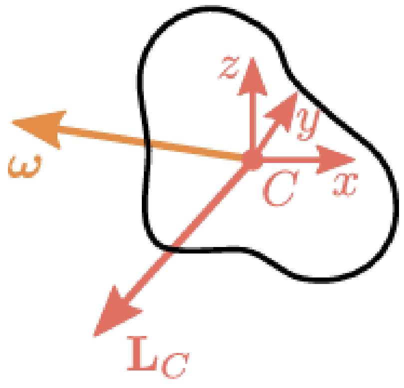
$$m = \int_B \rho(x, y, z)\, dx\, dy\, dz = \int_B dm$$

- The **inertia tensor** of B is defined as

$$\mathbf{\Theta} = \begin{bmatrix} I_x & I_{xy} & I_{xz} \\ I_{yx} & I_y & I_{yz} \\ I_{zx} & I_{zy} & I_z \end{bmatrix} \qquad \underbrace{\begin{aligned} I_x &= \textstyle\int_B (y^2 + z^2)\, dm \\ I_y &= \textstyle\int_B (x^2 + z^2)\, dm \\ I_z &= \textstyle\int_B (x^2 + y^2)\, dm \end{aligned}}_{\text{moments of inertia}} \qquad \underbrace{\begin{aligned} I_{xy} &= I_{yx} = -\textstyle\int_B xy\, dm \\ I_{xz} &= I_{zx} = -\textstyle\int_B xz\, dm \\ I_{yz} &= I_{zy} = -\textstyle\int_B yz\, dm \end{aligned}}_{\text{moments of deviation}}$$

Dynamics of a Rigid Body

Rotatory Motion of a Rigid Body:



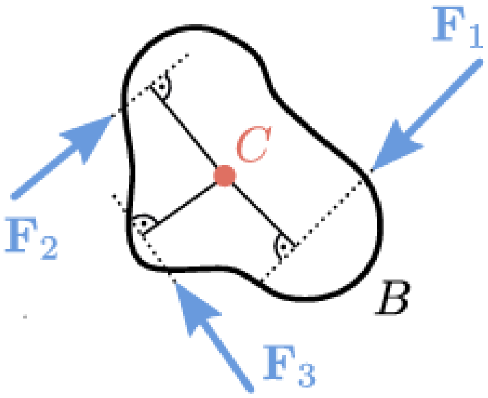
- Let $\boldsymbol{\omega}$ be the vector of **angular velocities:**

$$\boldsymbol{\omega} = (\omega_x \; \omega_y \; \omega_z)^\top$$

- The **angular momentum** $\mathbf{L}_C$ of the rigid body B is given by

$$\mathbf{L}_C = \boldsymbol{\Theta} \, \boldsymbol{\omega}$$

- By the **angular momentum principle**



$$\frac{d}{dt}\mathbf{L}_C(t) = \boldsymbol{\Theta} \, \dot{\boldsymbol{\omega}} = \mathbf{M}_{net}(t) = \sum_i \mathbf{M}_i(t)$$

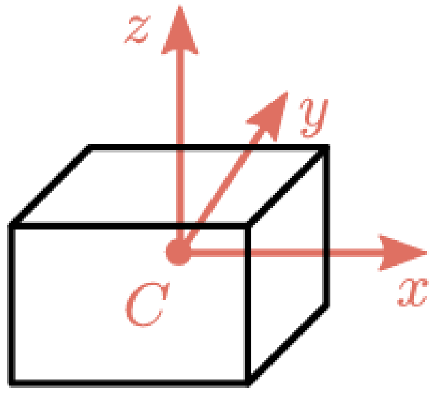
where $\mathbf{M}_i(t)$ are the moments of all forces acting on B with respect to the center of gravity C.

Dynamics of a Rigid Body

Rotatory Motion of a Rigid Body with Canonical Coordinates:

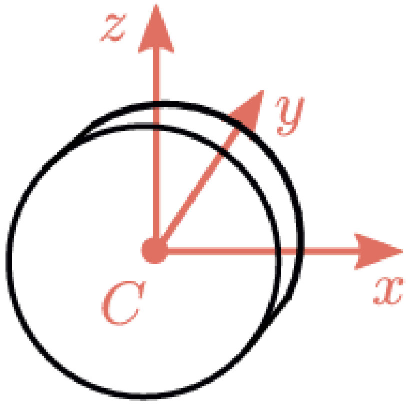
- If the body frame is chosen as a principal axis system for the rigid body (symmetry axes), the inertia tensor is diagonal:

$$\boldsymbol{\Theta} = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix}$$



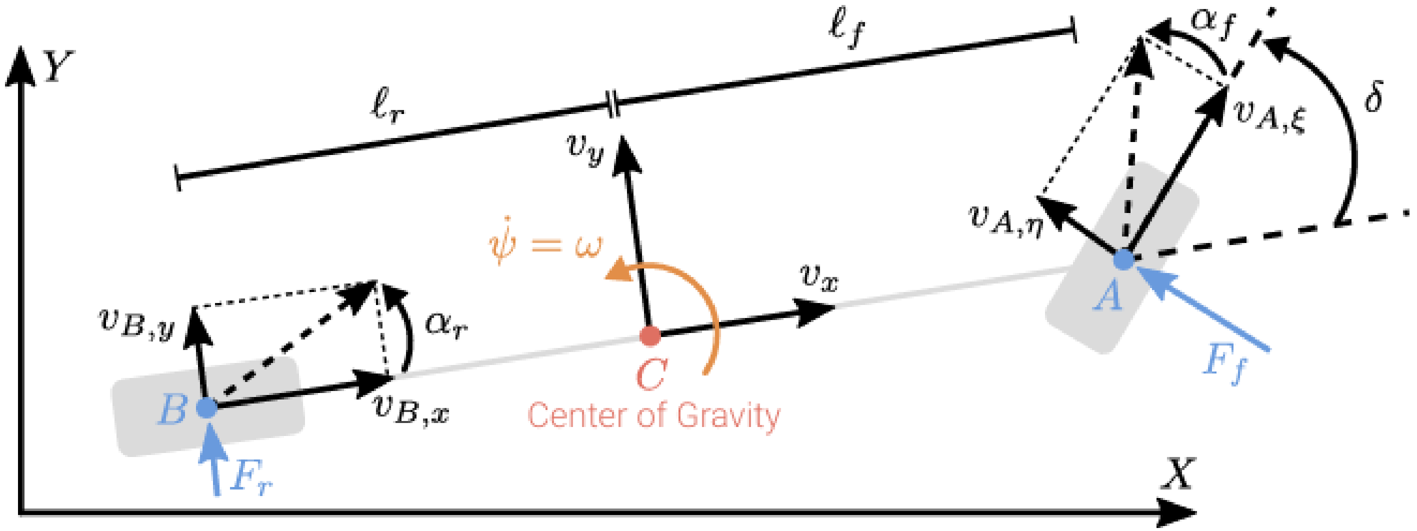
- For the planar motion of a rigid body in the x/y-plane:

$$\omega_x = \omega_y = 0 \quad \text{and} \quad M_x = M_y = 0$$



- Hence the angular momentum becomes $L_z = I_z\,\omega_z(t)$ and the angular momentum principle yields $I_z\,\dot{\omega}_z = \sum_i M_i$

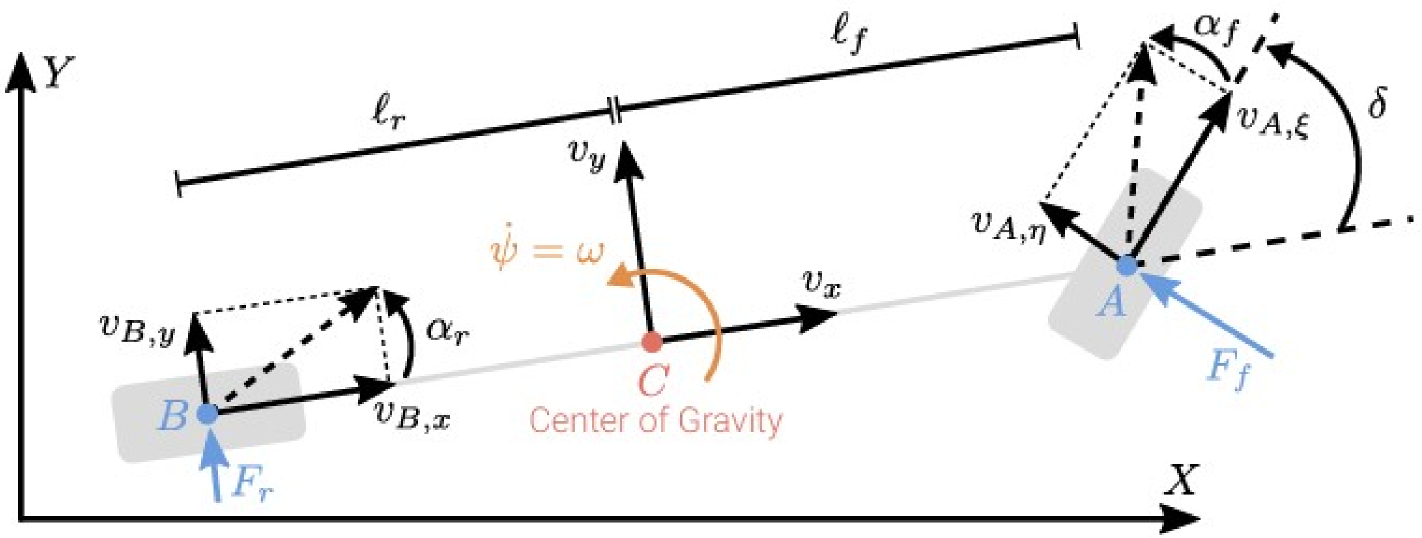
Dynamic Bicycle Model



Assumptions:

- The vehicle's motion is restricted to the X/Y plane
- The vehicle is considered as a rigid body
- Only lateral tire forces, generated by a linear tire model
- Small steering angle δ: $\sin\delta \approx \delta \quad \tan\delta \approx \delta \quad \cos\delta \approx 1$
- Constant longitudinal velocity v_x

Dynamic Bicycle Model



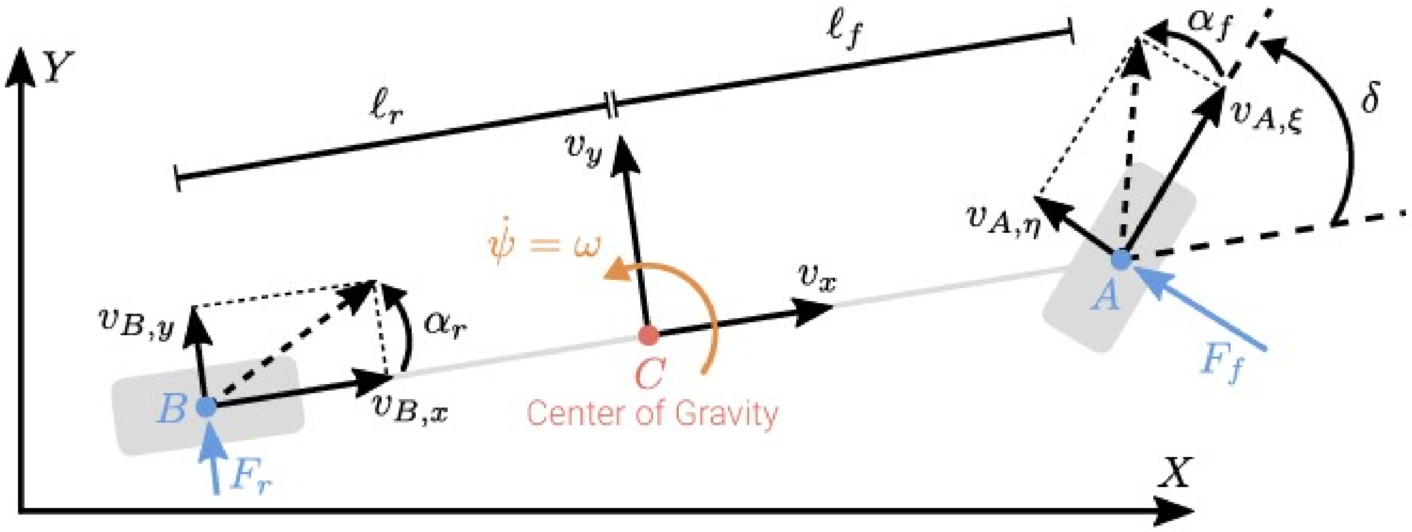
Lateral Dynamics:

$$m a_y = \sum_i F_{y,i} = F_r + F_f \cos\delta \approx F_r + F_f$$

$$a_y = \dot{v}_y + \omega\, v_x \qquad (\omega\, v_x = \text{centripetal acc.})$$

$$\Rightarrow \quad m(\dot{v}_y + \omega v_x) = F_r + F_f$$

Dynamic Bicycle Model

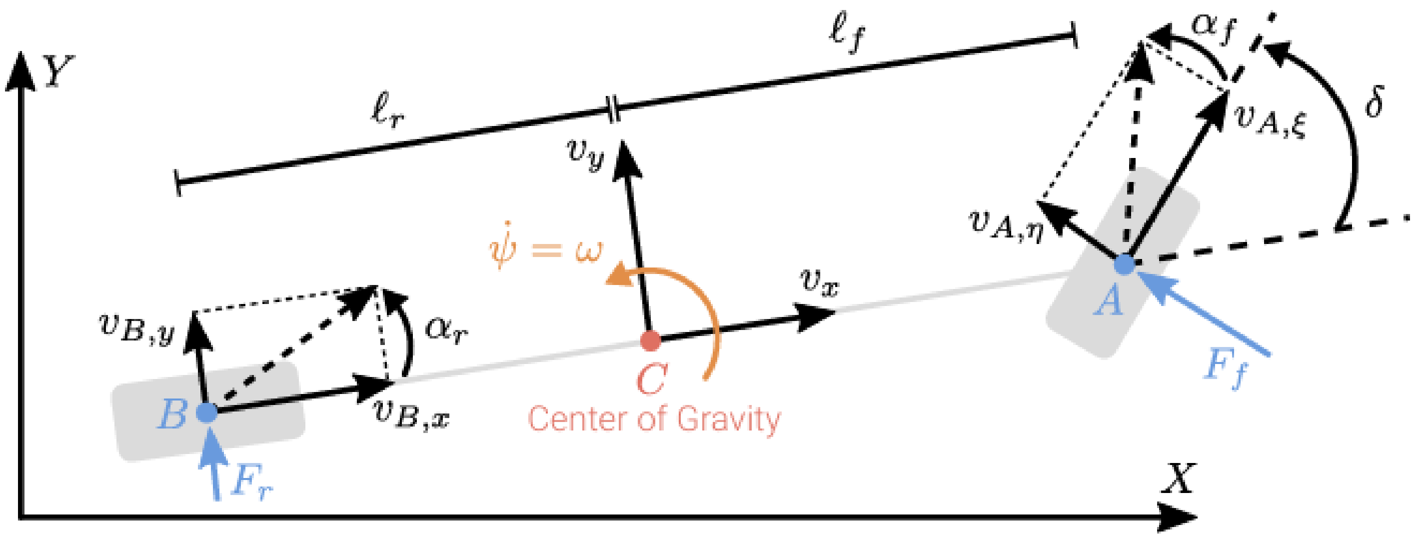


Yaw Dynamics:

$$I_z \dot{\omega} = \sum_i M_i = -l_r F_r + l_f F_f \underbrace{\cos\delta}_{\approx 1}$$

$$\Rightarrow \quad I_z \dot{\omega} = -l_r F_r + l_f F_f$$

Dynamic Bicycle Model



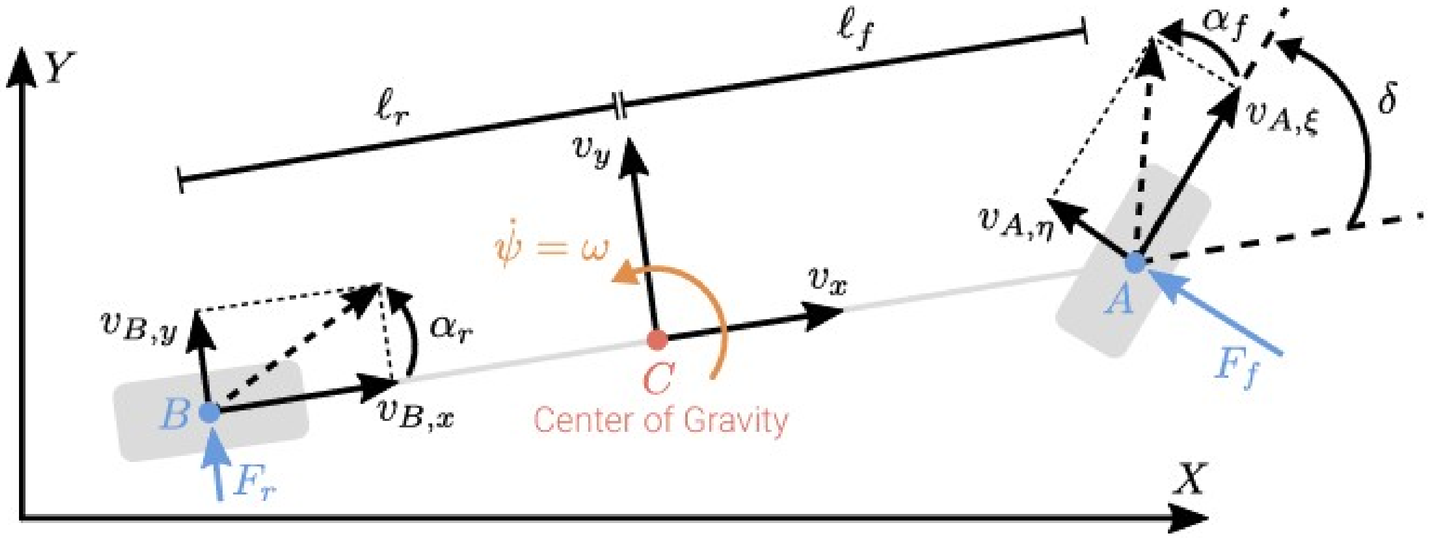
Tire Forces:

$$F_r = -c_r \alpha_r \approx -c_r \tan(\alpha_r) = -c_r \frac{v_{B,y}}{v_{B,x}} \qquad F_f = -c_f \alpha_f \approx -c_f \tan(\alpha_f) = -c_f \frac{v_{A,\eta}}{v_{A,\xi}}$$

$$v_{B,x} = v_x \quad v_{B,y} = v_y - \omega l_r \qquad v_{A,x} = v_x \quad v_{A,y} = v_y + \omega l_f$$

$$v_{A,\xi} = v_{A,x} \underbrace{\cos(\delta)}_{\approx 1} + v_{A,y} \underbrace{\sin(\delta)}_{\approx \delta} \qquad v_{A,\eta} = -v_{A,x} \underbrace{\sin(\delta)}_{\approx \delta} + v_{A,y} \underbrace{\cos(\delta)}_{\approx 1}$$

Dynamic Bicycle Model



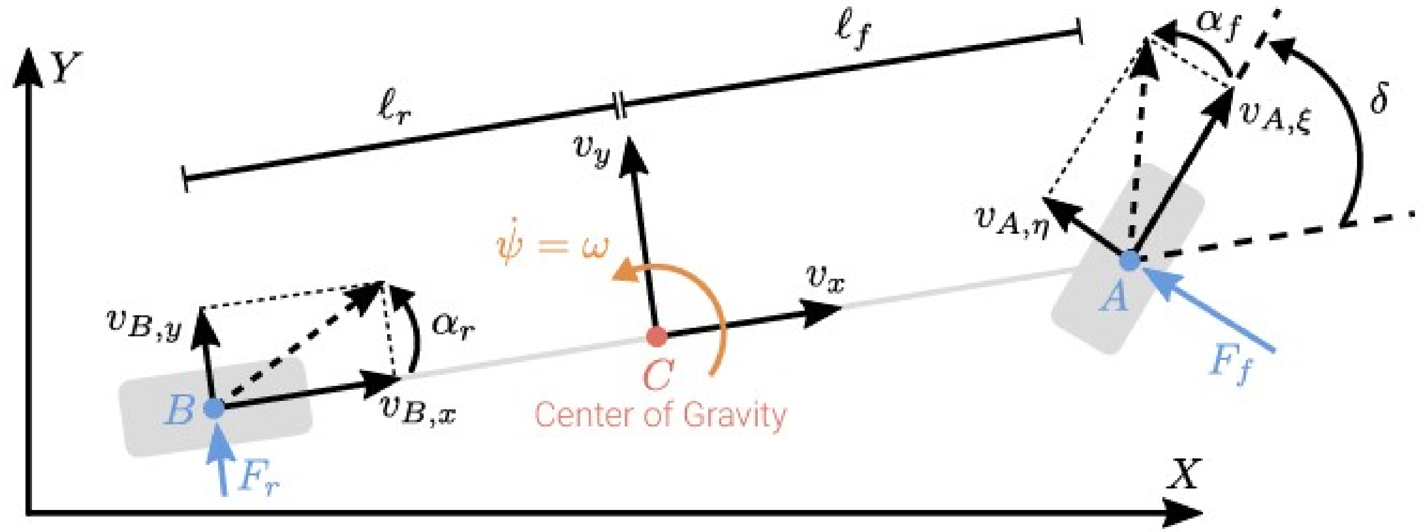
Tire Forces:

$$F_r = -c_r \frac{v_{B,y}}{v_{B,x}} = -c_r \frac{v_y - \omega l_r}{v_x}$$

$$F_f = -c_f \frac{v_{A,\eta}}{v_{A,\xi}} = -c_f \frac{-v_x \delta + v_y + \omega l_f}{v_x + (v_y + \omega l_f)\delta} \approx c_f \delta - c_f \frac{v_y + \omega l_f}{v_x}$$

Last approximation due to: $v_x \gg (v_y + \omega l_f)\delta$

Dynamic Bicycle Model

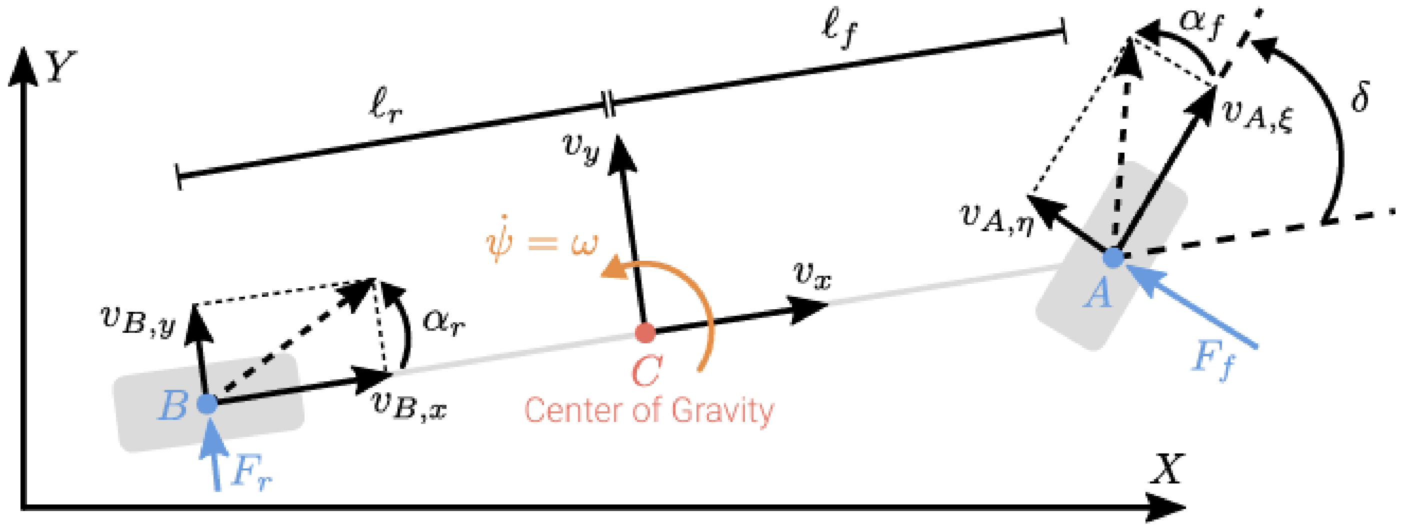


State Space Representation:

$$m(\dot{v}_y + \omega v_x) = \underbrace{-c_r \frac{v_y - \omega l_r}{v_x}}_{=F_r} + \underbrace{c_f \delta - c_f \frac{v_y + \omega l_f}{v_x}}_{=F_f}$$

$$I_z \dot{\omega} = -l_r \underbrace{\left(-c_r \frac{v_y - \omega l_r}{v_x}\right)}_{F_r} + l_f \underbrace{\left(c_f \delta - c_f \frac{v_y + \omega l_f}{v_x}\right)}_{=F_f}$$

Dynamic Bicycle Model



State Space Representation:

$$\begin{bmatrix} \dot v_y \\ \dot\psi \\ \dot\omega \end{bmatrix} = \begin{bmatrix} -\frac{c_r+c_f}{m v_x} & 0 & \frac{c_r l_r - c_f l_f}{m v_x} - v_x \\ 0 & 0 & 1 \\ \frac{l_r c_r - l_f c_f}{I_z v_x} & 0 & -\frac{l_f^2 c_f + l_r^2 c_r}{I_z v_x} \end{bmatrix} \underbrace{\begin{bmatrix} v_y \\ \psi \\ \omega \end{bmatrix}}_{\text{State}} + \begin{bmatrix} \frac{c_f}{m} \\ 0 \\ \frac{c_f}{I_z} l_f \end{bmatrix} \underbrace{\delta}_{\text{Input}}$$

Can be augmented by the global position to a nonlinear state space model

Summary

- A vehicle can be modeled as a rigid body
- It is subject to holonomic and non-holonomic constraints
- The bicycle model approximates the vehicle using 2 wheels
- The kinematic bicycle model assumes no wheel slip (low speeds)
- However, modeling tires requires to consider slip
- Sliding friction is smaller than static friction
- We want to operate in the static friction area of the force curve
- The circle of forces tells us that lat. and long. forces are dependent
- The dynamic bicycle model takes into account tire foces and wheel slip