The Simplex Method

Simplex Method

- The graphical approach can be used for two-variable LP problems
- Unfortunately, most real-life LPs problems require a method to find optimal solutions capable of dealing with several variables: **the simplex algorithm**

Simplex Method
Formulation

Simplex Method - Formulation

In LP problem, the decision maker usually wants to:

maximize (usually revenue or profit)
minimize (usually costs)

the **objective function (Z)** is expressed by a set of **decision variables**

Certain limitations are often imposed to these decision variables (expressed in the form of ≤, = or ≥).

These restrictions are called **constraints**

Problem

Max: $Z = 90\, x_1 + 120\, x_2$ (€/yr)

Subject to:

$x_1 \leq 40$ (ha of pine)

$x_2 \leq 50$ (ha of eucalypt)

$2x_1 + 3x_2 \leq 180$ (days of work)

and $x_1 \geq 0; \quad x_2 \geq 0$

Simplex Method - Formulation

The Simplex algorithm is an **algebraic procedure** to solve LP problems **based on geometric concepts** that requires LP problems to be presented in the **standard form:**

- 1) Objective function is **maximized**
- 2) **Constraints** in the form of **≤ inequalities**
- 3) All **values on the right handside** are **≥**
- 4) All **variables** are **nonnegative (≥)**

Max: $Z = 90\,x_1 + 120\,x_2$

Subject to:

$$x_1 \leq 40$$

$$x_2 \leq 50$$

$$2x_1 + 3x_2 \leq 180$$

and $x_1 \geq 0; \quad x_2 \geq 0$

Simplex Method - Formulation

The **Simplex algorithm** is an algebraic procedure to solve LP problems based on **geometric concepts** that must be translated into algebraic language to allow solving systems of equations.

1st - transform all inequalities into equalities by introducing one additional variable to each constraint (the slack variables: S_1, S_2, S_3).

Original form:

Max: $Z = 90\, x_1 + 120\, x_2$

Subject to:

$$\begin{aligned} x_1 \quad\quad\quad & \le 40 \\ x_2 & \le 50 \\ 2x_1 + 3x_2 & \le 180 \end{aligned}$$

and $x_1\ ,\ x_2 \geq 0$

Standard or augmented form:

Max: $Z = 90\, x_1 + 120\, x_2$

Subject to:

$$\begin{aligned} x_1 \quad\quad + S_1 \quad\quad\quad\quad & = 40 \\ x_2 \quad\quad + S_2 \quad\quad & = 50 \\ 2x_1 + 3x_2 \quad\quad\quad\quad + S_3 & = 180 \end{aligned}$$

and $x_1\ ,\ x_2\ ,\ S_1\ ,\ S_2\ ,\ S_3 \geq 0$

Simplex Method - Formulation

The **Simplex algorithm** is an algebraic procedure to solve LP problems based on **geometric concepts** that must be translated into algebraic language to allow solving systems of equations.

> **1st - transform all inequalities into equalities** by introducing one additional variable to each constraint (the slack variables: S_1, S_2, S_3).
> **2nd - transform the objective function into an additional constraint**

Max: $Z = 90\,x_1 + 120\,x_2$

Subject to:

$$x_1 + S_1 = 40$$
$$x_2 + S_2 = 50$$
$$2x_1 + 3x_2 + S_3 = 180$$

and $x_1, x_2, S_1, S_2, S_3 \geq 0$

$$Z - 90\,x_1 - 120\,x_2 = 0$$
$$x_1 + S_1 = 40$$
$$x_2 + S_2 = 50$$
$$2x_1 + 3x_2 + S_3 = 180$$

Simplex Method - Formulation

The **Simplex algorithm** is an algebraic procedure to solve LP problems based on **geometric concepts** that must be translated into algebraic language to allow solving systems of equations.

1st - transform all inequalities into equalities by introducing one additional variable to each constraint (the slack variables: S_1, S_2, S_3).

2nd - transform the objective function into an additional constraint

3rd - build the Simplex tabular form where only the essential information is recorded

basic var.	coefficients of:						right side
	Z	x1	x2	S1	S2	S3	
Z	1	-90	-120	0	0	0	0
S1	0	1	0	1	0	0	40
S2	0	0	1	0	1	0	50
S3	0	2	3	0	0	1	180

$$
\begin{aligned}
Z - 90\,x_1 - 120\,x_2 &= 0 \\
x_1 + S_1 &= 40 \\
x_2 + S_2 &= 50 \\
2x_1 + 3x_2 + S_3 &= 180
\end{aligned}
$$

Simplex Method - Formulation

The **Simplex algorithm** is an algebraic procedure to solve LP problems based on **geometric concepts** that must be translated into algebraic language to allow solving systems of equations.

1st - transform all inequalities into equalities by introducing one additional variable to each constraint (the slack variables: S_1, S_2, S_3).

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basic var.	**coefficients of:**						**right side**
	Z	**x1**	**x2**	**S1**	**S2**	**S3**	
Z	1	-90	-120	0	0	0	0
S1	0	1	0	1	0	0	40
S2	0	0	1	0	1	0	50
S3	0	2	3	0	0	1	180

Non-basic variables

Basic variables

Each basic feasible solution has ***basic*** or ***non-basic*** variables

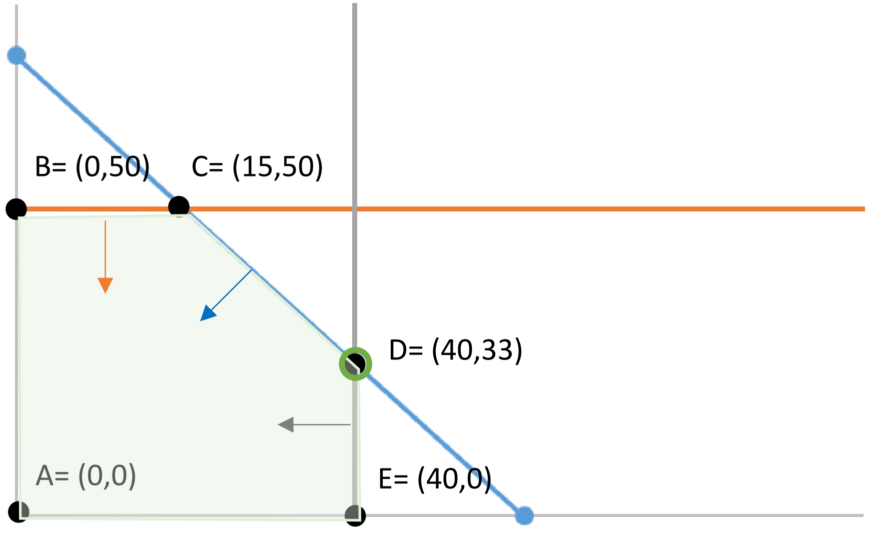
- ***non-basic* variables** are set to ZERO
- ***basic variables*** are directly obtained from the table

initialize the procedure setting $x_1 = x_2 = 0$

$(X_1, X_2, S_1, S_2, S_3) = (0, 0, 40, 50, 180)$

Simplex Method - Graphical analysis

- The Simplex algorithm is a search procedure that:
  - shifts through the set of basic feasible solutions, one at a time, until the optimal basic feasible solution (whenever it exists) is identified.
  - the method is an efficient implementation the Corner Points Procedure.



Corner point feasible solutions – vertices of the feasible region

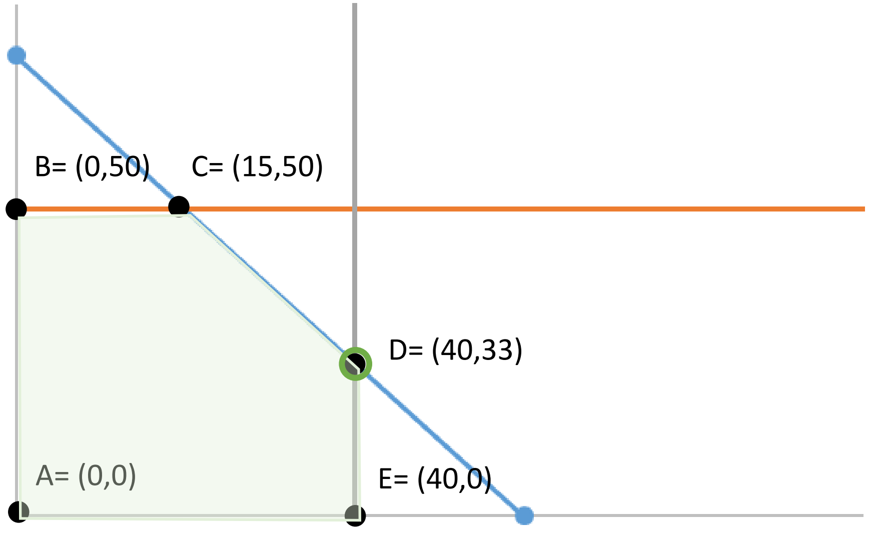
Optimal solution(s) – vertice(s) of the feasible region that maximize Z, ie solution that gives the best favorable value to the objective function

Simplex Method - Graphical analysis

- The Simplex algorithm is a search procedure that:

  - shifts through the set of basic feasible solutions, one at a time, until the optimal basic feasible solution (whenever it exists) is identified.

  - the method is an efficient implementation the Corner Points Procedure.



Replacing X_1 and X_2 by the values of A, B, C, D and E in the objective function:

$Z_A = 0$

$Z_B = 6000$

$Z_C = 7350$

$\mathbf{Z_D = 7600}$

$Z_E = 3600$

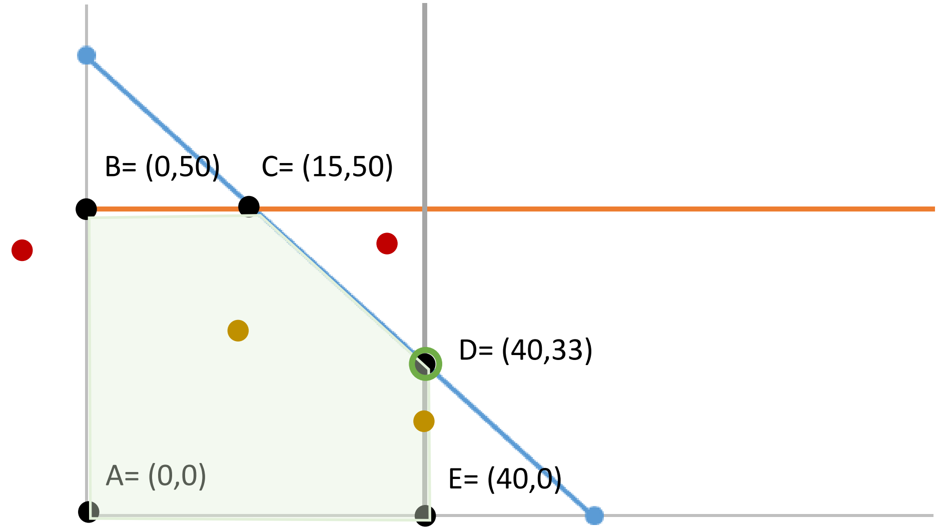
$$Z = 90\,x_1 + 120\,x_2$$

Simplex Method - Graphical analysis

- The Simplex algorithm is a search procedure that:

  - shifts through the set of basic feasible solutions, one at a time, until the optimal basic feasible solution (whenever it exists) is identified.

  - the method is an efficient implementation the Corner Points Procedure.



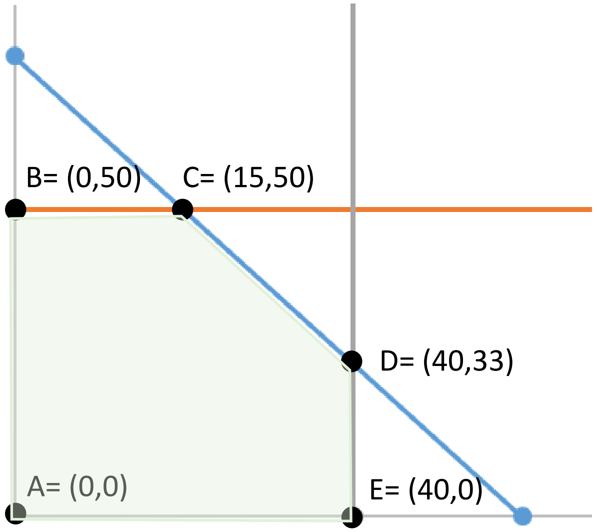
Feasible solutions – within or on the border of the feasible region ie solutions for which the constraints are satisfied

Infeasible solution – outside the feasible region, ie solution for which at least one constraint is violated

Simplex Method - Formulation

Bring the LP problem to the standard form -> obtain a BFS *ie set A= (x1, x2) = (0, 0)*

Optimality check — **No** — Find another feasible solution



Find in which direction to move towards the algebraic equivalent of an extreme point ie a Basic Feasible Solution <u>with a single different basic variable</u>

$A = (X_1, X_2, S_1, S_2, S_3) = (0, 0, 40, 50, 180)$

$B = (X_1, X_2, S_1, S_2, S_3) = (0, 50, 40, 0, 30)$

$C = (X_1, X_2, S_1, S_2, S_3) = (15, 50, 0, 25, 0)$

A is adjacent to B but not to C
B is adjacent to both A and C

	A	B	C
basic	S_1, S_2, S_3	S_1, X_2, S_3	X_1, X_2, S_2
non-basic	X_1, X_2	X_1, S_2	S_1, S_3

Simplex Method
Procedure

Simplex Method - Procedure

basic var.	coefficients of: Z	x1	x2	S1	S2	S3	right side
Z	1	-90	-120	0	0	0	0
S1	0	1	0	1	0	0	40
S2	0	0	1	0	1	0	50
S3	0	2	3	0	0	1	180

Bring the LP problem to the standard form -> obtain a BFS *ie set (x1, x2) = (0, 0)*

Optimality check:
The current BFS is optimal (in a max LP) if **every coefficient in Row 0 is ≥ 0**.

No

Find another feasible solution

Entering variable: Choose the entering variable (in a max problem) to be the NBV with the **most negative coefficient in Row 0.** Ties may be broken in an arbitrary fashion.

Leaving BV: apply minimum ratio test - identify the **row with the smallest ratio RHS /aij** (the most restrictive Row); the BV for this row is the leaving BV (it becomes nonbasic).

Apply Gauss-Jordan elimination procedure to solve the system of linear equations.

Yes

Optimal feasible solution found – **STOP SIMPLEX**

Simplex Method - Procedure

basic var.	coefficients of: Z	x1	x2	S1	S2	S3	right side
Z	1	-90	-120	0	0	0	0
S1	0	1	0	1	0	0	40
S2	0	0	1	0	1	0	50
S3	0	2	3	0	0	1	180

Row	basic var.	coefficients of: Z	x1	x2	S1	S2	S3	right side
R0	Z	1	-90	-120	0	0	0	0
R1	S1	0	1	0	1	0	0	40
R2	x2	0	0	1	0	1	0	50
R3	S3	0	2	3	0	0	1	180

-120 -> 0

3 -> 0

R0 **R0-(-120)*R2** (1+120*0)

1

Simplex Method - Procedure

basic var.	coefficients of: Z	x1	x2	S1	S2	S3	right side
Z	1	-90	-120	0	0	0	0
S1	0	1	0	1	0	0	40
S2	0	0	1	0	1	0	50
S3	0	2	3	0	0	1	180

Row	basic var.	coefficients of: Z	x1	x2	S1	S2	S3	right side	
R0	Z	1	-90	-120	0	0	0	0	-120 -> 0
R1	S1	0	1	0	1	0	0	40	
R2	x2	0	0	1	0	1	0	50	
R3	S3	0	2	3	0	0	1	180	3 -> 0

R0 **R0-(-120)*R2** (1+120*0) (-90+120*0)

1 -90

Simplex Method - Procedure

basic var.	coefficients of: Z	x1	x2	S1	S2	S3	right side
Z	1	-90	-120	0	0	0	0
S1	0	1	0	1	0	0	40
S2	0	0	1	0	1	0	50
S3	0	2	3	0	0	1	180

Row	basic var.	coefficients of: Z	x1	x2	S1	S2	S3	right side	
R0	Z	1	-90	-120	0	0	0	0	-120 -> 0
R1	S1	0	1	0	1	0	0	40	
R2	x2	0	0	1	0	1	0	50	
R3	S3	0	2	3	0	0	1	180	3 -> 0

R0 **R0-(-120)*R2** (1+120*0) (-90+120*0) (-120+120*1)

1 -90 0

Simplex Method - Procedure

basic var.	coefficients of: Z	x1	x2	S1	S2	S3	right side
Z	1	-90	-120	0	0	0	0
S1	0	1	0	1	0	0	40
S2	0	0	1	0	1	0	50
S3	0	2	3	0	0	1	180

Row	basic var.	coefficients of: Z	x1	x2	S1	S2	S3	right side
R0	Z	1	-90	-120	0	0	0	0
R1	S1	0	1	0	1	0	0	40
R2	x2	0	0	1	0	1	0	50
R3	S3	0	2	3	0	0	1	180

-120 -> 0

3 -> 0

R0 **R0-(-120)*R2** (1+120*0) (-90+120*0) (-120+120*1) (0+120*0)

1 -90 0 0

Simplex Method - Procedure

basic var.	Z	x1	x2	S1	S2	S3	right side
Z	1	-90	-120	0	0	0	0
S1	0	1	0	1	0	0	40
S2	0	0	1	0	1	0	50
S3	0	2	3	0	0	1	180

Row	basic var.	coefficients of: Z	x1	x2	S1	S2	S3	right side	
R0	Z	1	-90	-120	0	0	0	0	-120 -> 0
R1	S1	0	1	0	1	0	0	40	
R2	x2	0	0	1	0	1	0	50	
R3	S3	0	2	3	0	0	1	180	3 -> 0

R0 **R0-(-120)*R2** (1+120*0) (-90+120*0) (-120+120*1) (0+120*0) (0+120*1)

1 -90 0 0 120

Simplex Method - Procedure

basic var.	coefficients of: Z	x1	x2	S1	S2	S3	right side
Z	1	-90	-120	0	0	0	0
S1	0	1	0	1	0	0	40
S2	0	0	1	0	1	0	50
S3	0	2	3	0	0	1	180

Row	basic var.	coefficients of: Z	x1	x2	S1	S2	S3	right side
R0	Z	1	-90	-120	0	0	0	0
R1	S1	0	1	0	1	0	0	40
R2	x2	0	0	1	0	1	0	50
R3	S3	0	2	3	0	0	1	180

-120 -> 0

3 -> 0

R0	**R0-(-120)*R2**	(1+120*0)	(-90+120*0)	(-120+120*1)	(0+120*0)	(0+120*1)	(0+120*0)
		1	-90	0	0	120	0

Simplex Method - Procedure

basic var.	coefficients of: Z	x1	x2	S1	S2	S3	right side
Z	1	-90	-120	0	0	0	0
S1	0	1	0	1	0	0	40
S2	0	0	1	0	1	0	50
S3	0	2	3	0	0	1	180

Row	basic var.	coefficients of: Z	x1	x2	S1	S2	S3	right side
R0	Z	1	-90	-120	0	0	0	0
R1	S1	0	1	0	1	0	0	40
R2	x2	0	0	1	0	1	0	50
R3	S3	0	2	3	0	0	1	180

-120 -> 0

3 -> 0

R0	**R0-(-120)*R2**	(1+120*0)	(-90+120*0)	(-120+120*1)	(0+120*0)	(0+120*1)	(0+120*0)	(0+120*50)
		1	-90	0	0	120	0	6000

Simplex Method - Procedure

basic var.	coefficients of: Z	x1	x2	S1	S2	S3	right side
Z	1	-90	-120	0	0	0	0
S1	0	1	0	1	0	0	40
S2	0	0	1	0	1	0	50
S3	0	2	3	0	0	1	180

Row	basic var.	coefficients of: Z	x1	x2	S1	S2	S3	right side	
R0	Z	1	-90	-120	0	0	0	0	-120 -> 0
R1	S1	0	1	0	1	0	0	40	
R2	x2	0	0	1	0	1	0	50	
R3	S3	0	2	3	0	0	1	180	3 -> 0

R0	**R0-(-120)*R2**	(1+120*0)	(-90+120*0)	(-120+120*1)	(0+120*0)	(0+120*1)	(0+120*0)	(0+120*50)
		1	-90	0	0	120	0	6000
R1		0	1	0	1	0	0	40

Simplex Method - Procedure

basic var.	coefficients of: Z	x1	x2	S1	S2	S3	right side
Z	1	-90	-120	0	0	0	0
S1	0	1	0	1	0	0	40
S2	0	0	1	0	1	0	50
S3	0	2	3	0	0	1	180

Row	basic var.	coefficients of: Z	x1	x2	S1	S2	S3	right side	
R0	Z	1	-90	-120	0	0	0	0	-120 -> 0
R1	S1	0	1	0	1	0	0	40	
R2	x2	0	0	1	0	1	0	50	
R3	S3	0	2	3	0	0	1	180	3 -> 0

R0	**R0-(-120)*R2**	(1+120*0)	(-90+120*0)	(-120+120*1)	(0+120*0)	(0+120*1)	(0+120*0)	(0+120*50)
		1	-90	0	0	120	0	6000
R1		0	1	0	1	0	0	40
R2		0	0	1	0	1	0	50

Simplex Method - Procedure

basic var.	coefficients of: Z	x1	x2	S1	S2	S3	right side
Z	1	-90	-120	0	0	0	0
S1	0	1	0	1	0	0	40
S2	0	0	1	0	1	0	50
S3	0	2	3	0	0	1	180

Row	basic var.	coefficients of: Z	x1	x2	S1	S2	S3	right side	
R0	Z	1	-90	-120	0	0	0	0	-120 -> 0
R1	S1	0	1	0	1	0	0	40	
R2	x2	0	0	1	0	1	0	50	
R3	S3	0	2	3	0	0	1	180	3 -> 0

R0	**R0-(-120)*R2**	(1+120*0)	(-90+120*0)	(-120+120*1)	(0+120*0)	(0+120*1)	(0+120*0)	(0+120*50)
		1	-90	0	0	120	0	6000
R1		0	1	0	1	0	0	40
R2		0	0	1	0	1	0	50
R3	**R3-(3)*R2**							

Simplex Method - Procedure

basic var.	coefficients of: Z	x1	x2	S1	S2	S3	right side
Z	1	-90	-120	0	0	0	0
S1	0	1	0	1	0	0	40
S2	0	0	1	0	1	0	50
S3	0	2	3	0	0	1	180

Row	basic var.	coefficients of: Z	x1	x2	S1	S2	S3	right side
R0	Z	1	-90	-120	0	0	0	0
R1	S1	0	1	0	1	0	0	40
R2	x2	0	0	1	0	1	0	50
R3	S3	0	2	3	0	0	1	180

-120 -> 0

3 -> 0

R0	R0-(-120)*R2	(1+120*0)	(-90+120*0)	(-120+120*1)	(0+120*0)	(0+120*1)	(0+120*0)	(0+120*50)
		1	-90	0	0	120	0	6000
R1		0	1	0	1	0	0	40
R2		0	0	1	0	1	0	50
R3	R3-(3)*R2	(0-3*0)						
		0						

Simplex Method - Procedure

basic var.	Z	x1	x2	S1	S2	S3	right side
	coefficients of:						
Z	1	-90	-120	0	0	0	0
S1	0	1	0	1	0	0	40
S2	0	0	1	0	1	0	50
S3	0	2	3	0	0	1	180

Row	basic var.	Z	x1	x2	S1	S2	S3	right side
		coefficients of:						
R0	Z	1	-90	-120	0	0	0	0
R1	S1	0	1	0	1	0	0	40
R2	x2	0	0	1	0	1	0	50
R3	S3	0	2	3	0	0	1	180

-120 -> 0

3 -> 0

R0	**R0-(-120)*R2**	(1+120*0)	(-90+120*0)	(-120+120*1)	(0+120*0)	(0+120*1)	(0+120*0)	(0+120*50)
		1	-90	0	0	120	0	6000
R1		0	1	0	1	0	0	40
R2		0	0	1	0	1	0	50
R3	**R3-(3)*R2**	(0-3*0)	(2-3*0)					
		0	2					

Simplex Method - Procedure

basic var.	Z	x1	x2	S1	S2	S3	right side
Z	1	-90	-120	0	0	0	0
S1	0	1	0	1	0	0	40
S2	0	0	1	0	1	0	50
S3	0	2	3	0	0	1	180

Row	basic var.	Z	x1	x2	S1	S2	S3	right side	
R0	Z	1	-90	-120	0	0	0	0	-120 -> 0
R1	S1	0	1	0	1	0	0	40	
R2	x2	**0**	**0**	**1**	**0**	**1**	**0**	**50**	
R3	S3	0	2	3	0	0	1	180	3 -> 0

R0	**R0-(-120)*R2**	(1+120*0)	(-90+120*0)	(-120+120*1)	(0+120*0)	(0+120*1)	(0+120*0)	(0+120*50)
		1	-90	0	0	120	0	6000
R1		0	1	0	1	0	0	40
R2		0	0	1	0	1	0	50
R3	**R3-(3)*R2**	(0-3*0)	(2-3*0)	(3-3*1)				
		0	2	0				

Simplex Method - Procedure

basic var.	coefficients of: Z	x1	x2	S1	S2	S3	right side
Z	1	-90	-120	0	0	0	0
S1	0	1	0	1	0	0	40
S2	0	0	1	0	1	0	50
S3	0	2	3	0	0	1	180

Row	basic var.	coefficients of: Z	x1	x2	S1	S2	S3	right side
R0	Z	1	-90	-120	0	0	0	0
R1	S1	0	1	0	1	0	0	40
R2	x2	0	0	1	0	1	0	50
R3	S3	0	2	3	0	0	1	180

-120 -> 0

3 -> 0

R0	**R0-(-120)*R2**	(1+120*0)	(-90+120*0)	(-120+120*1)	(0+120*0)	(0+120*1)	(0+120*0)	(0+120*50)
		1	-90	0	0	120	0	6000
R1		0	1	0	1	0	0	40
R2		0	0	1	0	1	0	50
R3	**R3-(3)*R2**	(0-3*0)	(2-3*0)	(3-3*1)	(0-3*0)			
		0	2	0	0			

Simplex Method - Procedure

basic var.	Z	x1	x2	S1	S2	S3	right side
Z	1	-90	-120	0	0	0	0
S1	0	1	0	1	0	0	40
S2	0	0	1	0	1	0	50
S3	0	2	3	0	0	1	180

Row	basic var.	coefficients of: Z	x1	x2	S1	S2	S3	right side	
R0	Z	1	-90	-120	0	0	0	0	-120 -> 0
R1	S1	0	1	0	1	0	0	40	
R2	x2	0	0	1	0	1	0	50	
R3	S3	0	2	3	0	0	1	180	3 -> 0

R0	**R0-(-120)*R2**	(1+120*0)	(-90+120*0)	(-120+120*1)	(0+120*0)	(0+120*1)	(0+120*0)	(0+120*50)
		1	-90	0	0	120	0	6000
R1		0	1	0	1	0	0	40
R2		0	0	1	0	1	0	50
R3	**R3-(3)*R2**	(0-3*0)	(2-3*0)	(3-3*1)	(0-3*0)	(0-3*1)	(1-3*0)	(180-3*50)
		0	2	0	0	-3	1	30

Simplex Method - Procedure

basic var.	Z	x1	x2	S1	S2	S3	right side
	coefficients of:						
Z	1	-90	-120	0	0	0	0
S1	0	1	0	1	0	0	40
S2	0	0	1	0	1	0	50
S3	0	2	3	0	0	1	180

Row	basic var.	coefficients of:						right side
		Z	x1	x2	S1	S2	S3	
R0	Z	1	-90	0	0	120	0	6000
R1	S1	0	1	0	1	0	0	40
R2	x2	0	0	1	0	1	0	50
R3	S3	0	2	0	0	-3	1	30

R0	**R0-(-120)*R2**	(1+120*0)	(-90+120*0)	(-120+120*1)	(0+120*0)	(0+120*1)	(0+120*0)	(0+120*50)
		1	-90	0	0	120	0	6000
R1		0	1	0	1	0	0	40
R2		0	0	1	0	1	0	50
R3	**R3-(3)*R2**	(0-3*0)	(2-3*0)	(3-3*1)	(0-3*0)	(0-3*1)	(1-3*0)	(180-3*50)
		0	2	0	0	-3	1	30

Simplex Method - Procedure

basic var.	Z	x1	x2	S1	S2	S3	right side
Z	1	-90	-120	0	0	0	0
S1	0	1	0	1	0	0	40
S2	0	0	1	0	1	0	50
S3	0	2	3	0	0	1	180

Row	basic var.	coefficients of:						right side
		Z	x1	x2	S1	S2	S3	
R0	Z	1	-90	0	0	120	0	6000
R1	S1	0	1	0	1	0	0	40
R2	x2	0	0	1	0	1	0	50
R3	S3	0	2	0	0	-3	1	30

Z = 6000 X1 = 0
S1 = 40 S2 = 0
X2 = 50
S3 = 30

(x1, x2) = (0,50)

(x1, x2, S1, S2, S3) = (0, 50, 40, 0, 30)

Original form:

Max: $Z = 90 x_1 + 120 x_2$

Subject to:

$$x_1 \le 40$$
$$x_2 \le 50$$
$$2x_1 + 3x_2 \le 180$$

and $x_1 \; x_2 \ge 0$

Standard or augmented form:

Max: $Z = 90 x_1 + 120 x_2$

Subject to:

$$x_1 + S_1 = 40$$
$$x_2 + S_2 = 50$$
$$2x_1 + 3x_2 + S_3 = 180$$

and $x_1 \; x_2 \; S_1 \; S_2 \; S_3 \ge 0$

X1 = 0
X2 = 50
S1 = 40
S2 = 0
S3 = 30

Simplex Method - Procedure

basic var.	coefficients of: Z	x1	x2	S1	S2	S3	right side
Z	1	-90	-120	0	0	0	0
S1	0	1	0	1	0	0	40
S2	0	0	1	0	1	0	50
S3	0	2	3	0	0	1	180

(x1, x2) = (0,0)

(x1, x2, S1, S2, S3) = (0, 0, 40, 50, 180)

Row	basic var.	coefficients of: Z	x1	x2	S1	S2	S3	right side
R0	Z	1	-90	0	0	120	0	6000
R1	S1	0	1	0	1	0	0	40
R2	x2	0	0	1	0	1	0	50
R3	S3	0	2	0	0	-3	1	30

Z = 6000 X1 = 0
S1 = 40 S2 = 0
X2 = 50
S3 = 30

(x1, x2) = (0,50)

(x1, x2, S1, S2, S3) = (0, 50, 40, 0, 30)

Original form:

Max: $Z = 90\, x_1 + 120\, x_2$

Subject to:

$$x_1 \leq 40$$
$$x_2 \leq 50$$
$$2x_1 + 3x_2 \leq 180$$

and $x_1 \quad x_2 \geq 0$

Standard or augmented form:

Max: $Z = 90\, x_1 + 120\, x_2$

Subject to:

$$x_1 + S_1 = 40$$
$$x_2 + S_2 = 50$$
$$2x_1 + 3x_2 + S_3 = 180$$

and $x_1 \quad x_2 \quad S_1 \quad S_2 \quad S_3 \geq 0$

The basic variables in these solutions differ in one single variable (S1 and S3 are maintained as basic variables)

These are adjacent solutions

Simplex Method - Procedure

basic var.	coefficients of: Z	x1	x2	S1	S2	S3	right side
Z	1	-90	-120	0	0	0	0
S1	0	1	0	1	0	0	40
S2	0	0	1	0	1	0	50
S3	0	2	3	0	0	1	180

(x1, x2) = (0,0)

(x1, x2, S1, S2, S3) = (0, 0, 40, 50, 180)

Row	basic var.	coefficients of: Z	x1	x2	S1	S2	S3	right side
R0	Z	1	-90	0	0	120	0	6000
R1	S1	0	1	0	1	0	0	40
R2	x2	0	0	1	0	1	0	50
R3	S3	0	2	0	0	-3	1	30

Z = 6000

S1 = 40

X2 = 50

S3 = 30

X1 = 0

S2 = 0

(x1, x2) = (0,50)

(x1, x2, S1, S2, S3) = (0, 50, 40, 0, 30)

Original form:

Max: $Z = 90\, x_1 + 120\, x_2$

Subject to:

$$x_1 \le 40$$
$$x_2 \le 50$$
$$2x_1 + 3x_2 \le 180$$

and $x_1 \quad x_2 \ge 0$

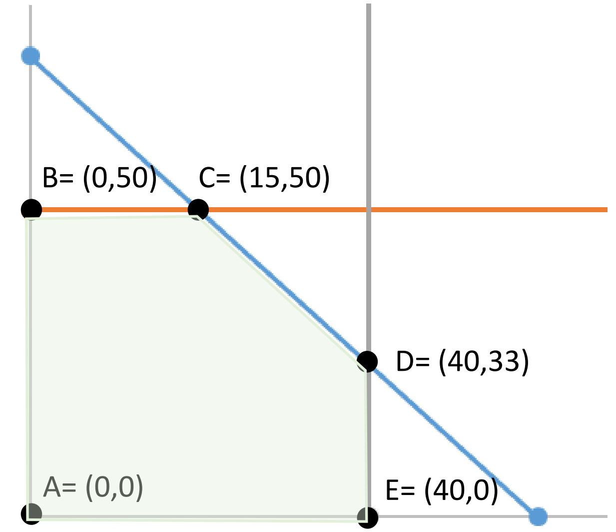
Standard or augmented form:

Max: $Z = 90\, x_1 + 120\, x_2$

Subject to:

$$x_1 + S_1 = 40$$
$$x_2 + S_2 = 50$$
$$2x_1 + 3x_2 + S_3 = 180$$

and $x_1 \quad x_2 \quad S_1 \quad S_2 \quad S_3 \ge 0$



Simplex Method - Procedure

Optimality check:
The current BFS is optimal (in a max LP) if **every coefficient in Row 0 is ≥ 0**.

Row	basic var.	coefficients of: Z	x1	x2	S1	S2	S3	right side	ratio	
R0	Z	1	-90	0	0	120	0	6000		
R1	S1	0	1	0	1	0	0	40	40/1=	40
R2	x2	0	0	1	0	1	0	50	-	
R3	S3	0	2	0	0	-3	1	30	30/2=	15

X1 will become **basic**
S3 will become **non-basic variable**

(X1 column will have to take the shape of S3: (0, 0, 0, 1)

Simplex Method - Procedure

Row	basic var.	coefficients of:						right side
		Z	x1	x2	S1	S2	S3	
R0	Z	1	-90	0	0	120	0	6000
R1	S1	0	1	0	1	0	0	40
R2	x2	0	0	1	0	1	0	50
R3	S3	0	2	0	0	-3	1	30

Row	basic var.	coefficients of:						right side
		Z	x1	x2	S1	S2	S3	
R0	Z	1	-90	0	0	120	0	6000
R1	S1	0	1	0	1	0	0	40
R2	x2	0	0	1	0	1	0	50
R3	S3	0	2	0	0	-3	1	30

R3	**R3*(1/2)**	(0*(1/2))	(2*(1/2))	(0*(1/2))	(0*(1/2))	(-3*(1/2))	(1*(1/2))	(30*(1/2))
		0	**1**	**0**	**0**	**-1.5**	**0.5**	**15**

Simplex Method - Procedure

Row	basic var.	coefficients of:						right side
		Z	x1	x2	S1	S2	S3	
R0	Z	1	-90	0	0	120	0	6000
R1	S1	0	1	0	1	0	0	40
R2	x2	0	0	1	0	1	0	50
R3	S3	0	2	0	0	-3	1	30

Row	basic var.	coefficients of:						right side
		Z	x1	x2	S1	S2	S3	
R0	Z	1	-90	0	0	120	0	6000
R1	S1	0	1	0	1	0	0	40
R2	x2	0	0	1	0	1	0	50
R3	x1	0	1	0	0	-1.5	0.5	15

-90 -> 0

1 -> 0

R0	**R0-(-90)*R3**	(1+90*0)	(-90+90*1)	(0+90*0)	(0+90*0)	(120+90*-1.5)	(0+90*0.5)	(6000+90*15)
		1	0	0	0	-15	45	7350
R1	**R1-(1)*R3**	(0-1*0)	(1-1*1)	(0-1*0)	(1-1*0)	(0-1*-1.5)	(0-1*0.5)	(40-1*40)
		0	0	0	1	1.5	-0.5	25

Simplex Method - Procedure

Row	basic var.	Z	x1	x2	S1	S2	S3	right side
R0	Z	1	-90	0	0	120	0	6000
R1	S1	0	1	0	1	0	0	40
R2	x2	0	0	1	0	1	0	50
R3	S3	0	2	0	0	-3	1	30

Row	basic var.	Z	x1	x2	S1	S2	S3	right side
R0	Z	1	0	0	0	-15	45	7350
R1	**S1**	**0**	**0**	**0**	**1**	**1.5**	**-0.5**	**25**
R2	x2	0	0	1	0	1	0	50
R3	x1	0	1	0	0	-1.5	0.5	15

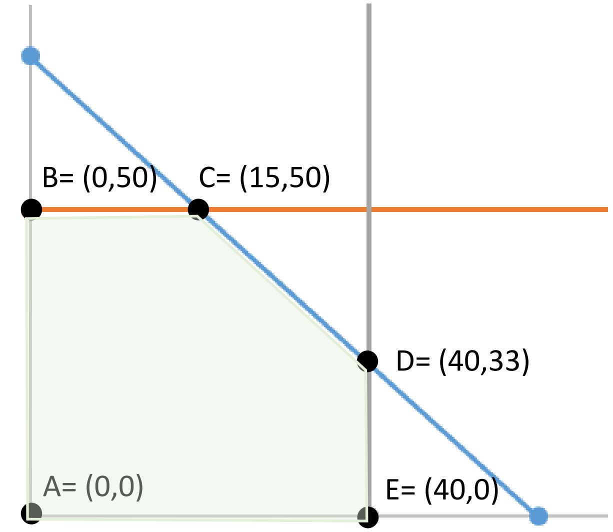
Z = 7350
S1 = 25
X2 = 50
x1 = 15

S2 = 0
S3 = 0

(x1, x2) = (0,0) **(x1, x2, S1, S2, S3) = (0, 0, 40, 50, 180)** **z=0** *(A)*

(x1, x2) = (0,50) **(x1, x2, S1, S2, S3) = (0, 50, 40, 0, 30)** **z=6000** *(B)*

(x1, x2) = (15,50) **(x1, x2, S1, S2, S3) = (15, 50, 25, 0, 0)** **z=7350** *(C)*



X1 = 15
X2 = 50
S1 = 25
S2 = 0
S3 = 0

Simplex Method - Procedure

Optimality check:
The current BFS is optimal (in a max LP) if **every coefficient in Row 0 is ≥ 0**.

Row	basic var.	coefficients of: Z	x1	x2	S1	S2	S3	right side	ratio	
R0	Z	1	0	0	0	-15	45	7350		
R1	S1	0	0	0	1	**1.5**	-0.5	25	25/1.5=	17
R2	x2	0	0	1	0	1	0	50	-	
R3	x1	0	1	0	0	-1.5	0.5	15	15/-1.5=	-10

S2 will become **basic**
S1 will become **non-basic variable**

(S2 column will have to take the shape of S1: (0, 1, 0, 0)

Entering variable: the **most negative coefficient in Row 0**

Leaving BV: **the smallest positive ratio RHS /aij**

R1	**R1*(1/1.5)**	(0*(1/1.5))	(0*(1/1.5))	(0*(1/1.5))	(1*(1/1.5))	(1.5*(1/1.5))	(-0.5*(1/1.5))	(25*(1/1.5))
		0	**0**	**0**	**0.67**	**1**	**-0.33**	**16.67**

Simplex Method - Procedure

Optimality check:
The current BFS is optimal (in a max LP) if **every coefficient in Row 0 is ≥ 0**.

Row	basic var.	coefficients of:						right side
		Z	x1	x2	S1	S2	S3	
R0	Z	1	0	0	0	-15	45	7350
R1	S2	**0**	**0**	**0**	**0.67**	**1**	**-0.33**	**16.67**
R2	x2	0	0	1	0	1	0	50
R3	x1	0	1	0	0	-1.5	0.5	15

S2 will become **basic**
S1 will become **non-basic variable**

(S2 column will have to take the shape of S1: (0, 1, 0, 0)

R0	**R0-(-15)*R1**	(1+15*0)	(0+15*0)	(0+15*0)	(0+15*0.67)	(-15+15*1)	(45+15*-0.33)	(7350+15*16.67)
		1	0	0	10	0	40	7600
R2	**R2-(1)*R1**	(0-1*0)	(0-1*0)	(1-1*0)	(0-1*0.67)	(1-1*1)	(0-1*-0.33)	(50-1*16.67)
		0	0	1	-0.67	0	0.33	33.33
R3	**R3-(-1.5)*R1**	(0+1.5*0)	(1+1.5*0)	(0+1.5*0)	(0+1.5*0.67)	(-1.5+1.5*1)	0.5+1.5*-0.33	(15+1.5*16.67)
		0	1	0	1	0	0	40

Simplex Method - Procedure

Optimality check:
The current BFS is optimal (in a max LP) if **every coefficient in Row 0 is ≥ 0**.

OPTIMAL SOLUTION!

Row	basic var.	coefficients of:						right side
		Z	x1	x2	S1	S2	S3	
R0	Z	1	0	0	10	0	40	7600
R1	S2	0	0	0	0.67	1	-0.33	16.67
R2	x2	0	0	1	-0.67	0	0.33	33.33
R3	x1	0	1	0	1	0	0	40

Z = 7600
S2 = 16.67
X2 = 33.33
x1 = 40

S1 = 0
S3 = 0

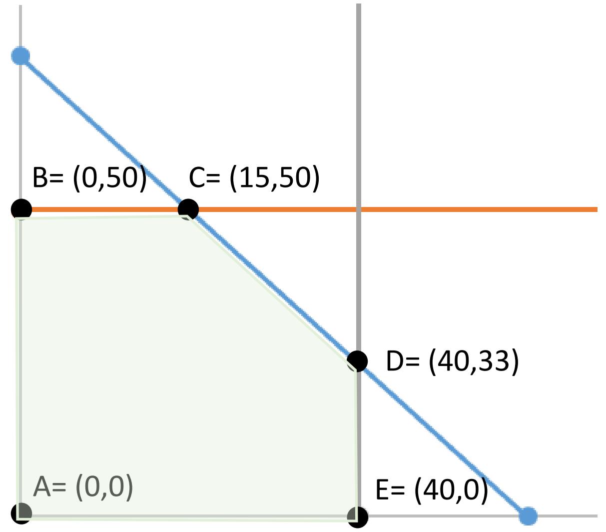
(x1, x2) = (0,0) **(x1, x2, S1, S2, S3) = (0, 0, 40, 50, 180)** **z=0** *(A)*

(x1, x2) = (0,50) **(x1, x2, S1, S2, S3) = (0, 50, 40, 0, 30)** **z=6000** *(B)*

(x1, x2) = (15,50) **(x1, x2, S1, S2, S3) = (15, 50, 25, 0, 30)** **z=7350** *(C)*

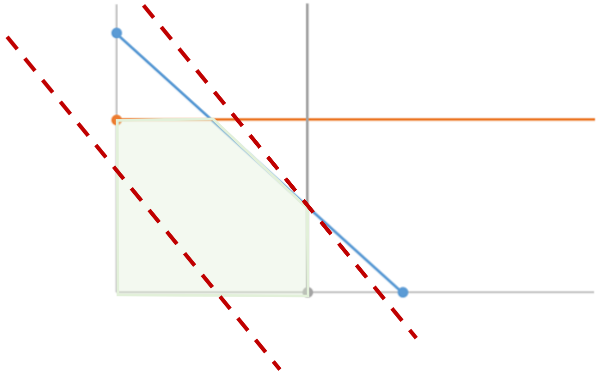
(x1, x2) = (40,33.33) **(x1, x2, S1, S2, S3) = (40, 33.33, 0, 16.67, 0)** **z=7600** *(D)*

X1 = 40
X2 = 33.33
S1 = 0
S2 = 16.67
S3 = 0



Simplex Method – Graphical approach

Graphical Method



Max: $Z = 90\,x_1 + 120x_2$

Subject to:

$x_1 \leq 40$

$x_2 \leq 50$

$2x_1 + 3x_2 \leq 180$

And $x_1 \geq 0$; $x_2 \geq 0$

- Replace each inequality by an equality
- Find the set of points satisfying the equality (allows to draw a line that cuts the plane into 2 half-planes)
- Find which half-plane satisfies the inequality
- Intercept all the half-plane areas to find the feasible region (FR) – feasible solutions = (x1, x2) corners
- Draw iso-lines for the objective function to find the optimal solution: (x1, x2) corner point of the FR

Simplex Method

Row	Algebraic form	basic var.	coefficients of:						right side	ratio
			Z	x1	x2	S1	S2	S3		
R0	$Z - 90\,x_1 - 120\,x_2 + S1$		1	-90	-120	0	0	0	0	
R1	$x_1 + S1$	S1	0	1	0	1	0	0	40	-
R2	$x_2 + S2$	S2	0	0	1	0	1	0	50	50/1 = 50
R3	$2\,x_1 + 3\,x_2 + S3$	S3	0	2	3	0	0	1	180	180/3 = 60

- Replace each inequality by an equality adding a slack variable
- Transform the objective function into an equality
- Build a table for the constraints only specifying the coefficients
- Set x1 and x2 to ZERO =>$\underbrace{x1=0;\ x2=0}_{\text{Non-basic variables}}$; $\underbrace{S1=40;\ S2=50;\ S3=180}_{\text{Basic variables}}$
- Test different combinations of basic variables
  - Select the non-basic var that results in a bigger increase in Z (the smallest coefficient in R0)
  - Select the basic var that guarantees the biggest increase in Z without leaving the feasible region and that all basic variables are nonnegative (smallest positive ratio)
  - Gaussian elimination so that the new basic var. only has: 0,1
  - Test optimality: all coeff. in R0 >=0**? If not, test new combination**

Simplex Method
Particular cases

Simplex Method – Particular cases

- **Tie for the Entering BV:**
  - **Entering variable**: Choose the entering variable (in a max problem) to be the NBV with the most negative coefficient in Row 0.
  - What to do when there is a tie for the entering basic variable ? Selection made *arbitrarily*.

Row	basic var.	coefficients of:						right side
		Z	x1	x2	S1	S2	S3	
R0	Z	1	-3	-3	0	0	0	0
R1	S1	0	1	0	1	0	0	4
R2	S2	0	0	2	0	1	0	12
R3	S3	0	3	2	0	0	1	18

Simplex Method – Particular cases

- **Tie for the Leaving BV - Degenerate:**
  - **Leaving BV**: apply minimum ratio test - identify the row with the smallest positive ratio bi /aij (the most restrictive Row); the BV for this row is the leaving BV (it becomes nonbasic).

- Choose the leaving variable arbitrary

Row	basic var.	coefficients of: Z	x1	x2	S1	S2	S3	S4	right side	
R0	Z	1	-3	-4	0	0	0	0	0	
R1	S1	0	1	1	1	0	0	0	10	10 / 1 = 10
R2	S2	0	2	3	0	1	0	0	18	18 / 3 = 6
R3	S3	0	1	0	0	0	1	0	8	-
R4	S4	0	0	1	0	0	0	1	6	**6 / 1 = 6**
R0	Z	1	-3	0	0	0	0	4	24	
R1	S1	0	1	0	1	0	0	-1	4	
R2	S2	0	2	0	0	1	0	-3	0	
R3	S3	0	1	0	0	0	1	0	8	
R4	X2	0	0	1	0	0	0	1	6	

Simplex Method – Particular cases

- **No leaving BV – Unbounded Z:**

Occurs if all the coefficients in the pivot column (where the entering basic variable is) are either negative or zero (excluding row 0)

No solution – when the constraints do not prevent improving the objective function indefinitely

Row	basic var.	coefficients of:						right side		
		Z	x1	x2	S1	S2	S3			
R0	Z	1	0	-1	1	0	0	10		
R1	x1	0	1	0	1	0	0	10	-	
R2	S2	0	0	-3	-1	1	0	5	5 / -3	< 0
R3	S3	0	0	-1	-1	0	1	10	10/ -1	< 0