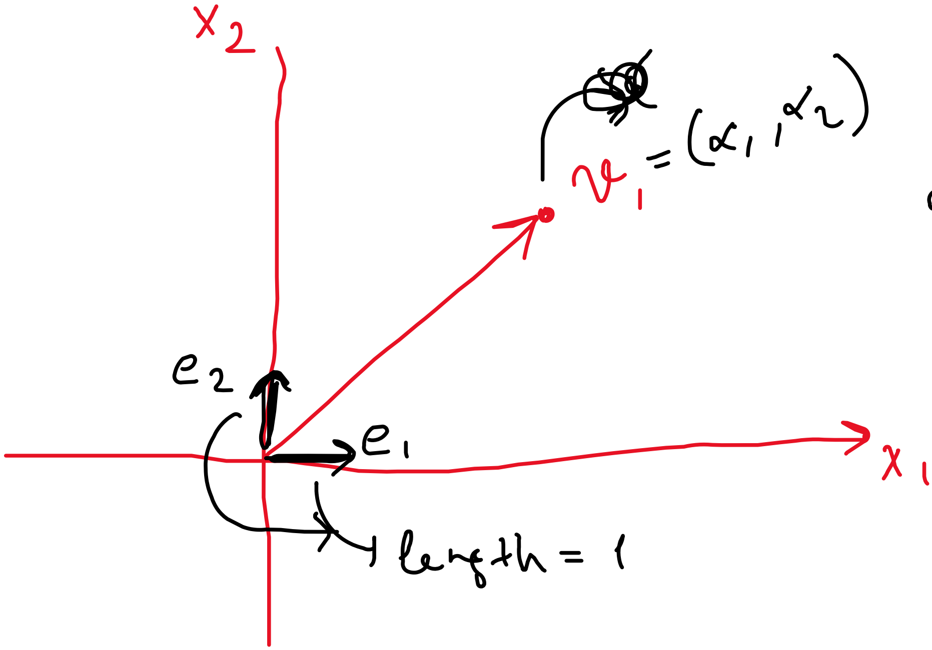
FOURIER SERIES



$$v_1 = \alpha_1 e_1 + \alpha_2 e_2$$

What is periodic function?

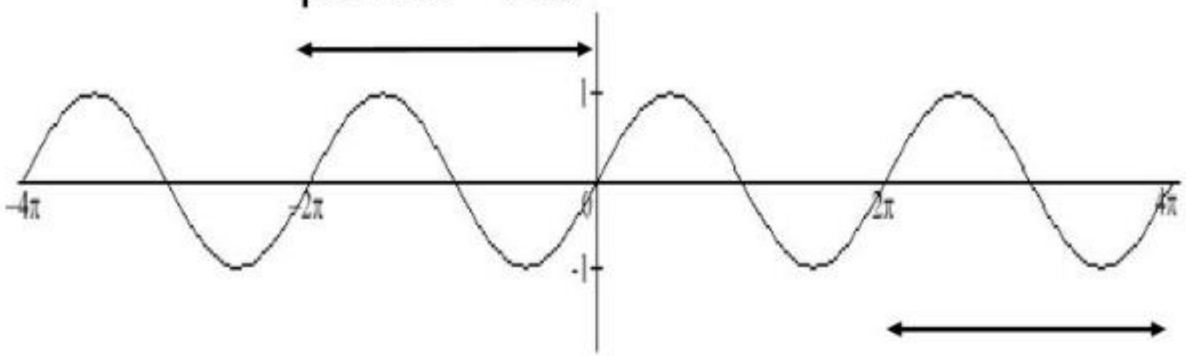
A function f is said periodic if there exists a smallest positive number T such that

$$f(x+T) = f(x)$$ for all x in the domain.

period of f

Illustration

period $= 2\pi$



Introduction

- Let f be a periodic function of period T

- It can be proved that f can be expressed as the sum of an infinite number of sine and/or cosine functions.

$$Sf(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{2n\pi t}{T} + b_n \sin\frac{2n\pi t}{T}\right)$$

constant ← DC value

- This infinite sum is known as a **Fourier Series**.

Introduction

The coefficients $a_0, a_n, b_n,\ \ n = 1,2,\ldots$

are known as **Fourier coefficients.**

They can be obtained by

$$a_0 = \frac{2}{T}\int_0^T f(t)\,dt$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos\frac{2n\pi t}{T}\,dt$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin\frac{2n\pi t}{T}\,dt$$

Introduction

If we write out the first few terms of the infinite series explicitly, we find:

$$Sf(t) = \frac{a_0}{2} + a_1 \cos\frac{2\pi t}{T} + b_1 \sin\frac{2\pi t}{T} + a_2 \cos\frac{4\pi t}{T} + b_2 \sin\frac{4\pi t}{T} \ldots.$$

Introduction

- a_n are the amplitudes of the cosine terms in the series,
- b_n are the amplitudes of the sine terms in the series.

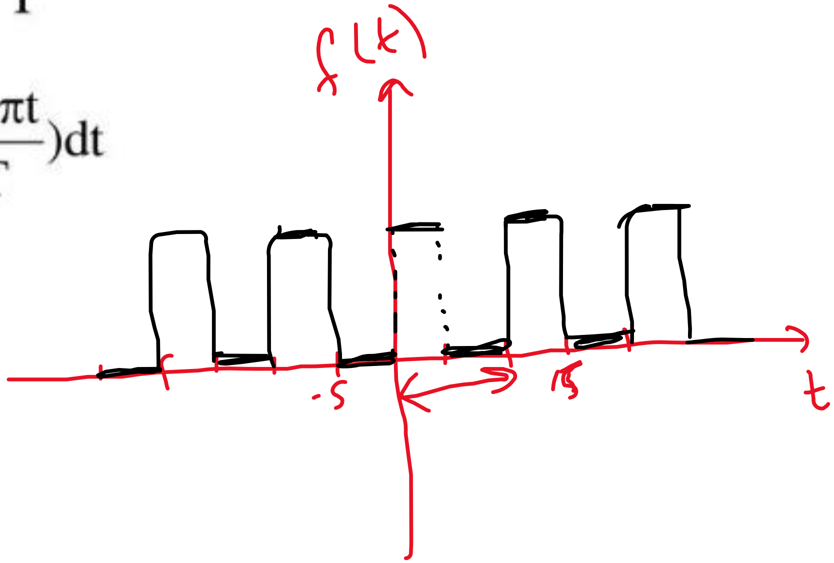
Example1

$$f(t)=\begin{cases}0 & -5<t<0\\ 1 & 0<t<5\end{cases} \qquad period\ \ T=10,$$

$$Evaluate\ \ a_0=\frac{2}{T}\int_0^T f(t)dt$$

$$a_n=\frac{2}{T}\int_0^T f(t)\cos\left(\frac{2n\pi t}{T}\right)dt$$

$$b_n=\frac{2}{T}\int_0^T f(t)\sin\left(\frac{2n\pi t}{T}\right)dt$$



Solution 1

$$f(t)=\begin{cases}0 & -5<t\le 0\\ 1 & 0<t<5\end{cases}\qquad T=10,$$

$$a_0=\frac{2}{10}\int_0^{10} f(t)\,dt$$

$$a_0=\frac{1}{5}\left(\int_0^{5} f(t)\,dt+\int_5^{10} f(t)\,dt\right)$$

$$a_0=\frac{1}{5}\left(\int_0^{5} 1\,dt+\int_5^{10} 0\,dt\right)=1$$

Solution 1

$$f(t)=\begin{cases}0 & -5<t<0\\ 1 & 0<t<5\end{cases} \qquad T=10,$$

$$a_n=\frac{2}{T}\int_0^T f(t)\cos\left(\frac{2n\pi t}{T}\right)dt$$

$$a_n=\frac{1}{5}\int_0^5 \cos\left(\frac{n\pi t}{5}\right)dt$$

$$a_n=\frac{1}{5}\frac{5}{n\pi}\sin\left(\frac{n\pi t}{5}\right)\Big|_0^5=\frac{\sin(n\pi)}{n\pi}=0$$

$t=5 \Rightarrow \sin(n\pi)=0$

$t=0 \Rightarrow \sin(0)=0$

$0-0=0$

t

$\frac{1}{2}$

Solution 1

$$f(t)=\begin{cases}0 & -5<t<0\\ 1 & 0<t<5\end{cases} \qquad T=10,$$

$$b_n=\frac{2}{T}\int_0^T f(t)\sin\left(\frac{2n\pi t}{T}\right)dt$$

$$b_n=\frac{1}{5}\int_0^5 \sin\left(\frac{n\pi t}{5}\right)dt$$

$$b_n=-\frac{1}{5}\frac{5}{n\pi}\cos\left(\frac{n\pi t}{5}\right)\Bigg|_0^5=\frac{1-\cos(n\pi)}{n\pi}=\frac{1-(-1)^n}{n\pi}$$

$$=-\frac{1}{n\pi}\left\{\cos(n\pi)-\underbrace{\cos(0)}_{1}\right\}$$

$$=\frac{-1}{n\pi}\left((-1)^n-1\right)$$

$n=2,4,6,\ldots n$
$\cos(n\pi)=1=(-1)^n$

$n=1,3,5$
$\cos(n\pi)=-1$
$=(-1)^n$

Fourier Series when T=2L

- Let f(x) be defined in the interval (c, c+2L).
- It can be represented in the given interval by the following Fourier Series:

$$f(x) = \frac{a_0}{2} + \sum_{1}^{\infty}\left(a_n \cos(\frac{\pi n}{L}x) + b_n \sin(\frac{\pi n}{L}x)\right)$$

$$a_0 = \frac{1}{L}\int_{C}^{C+2L} f(x)dx$$

$$b_n = \frac{1}{L}\int_{C}^{C+2L} f(x)\sin(\frac{n\pi}{L}x)dx$$

$$a_n = \frac{1}{L}\int_{C}^{C+2L} f(x)\cos(\frac{n\pi}{L}x)dx$$

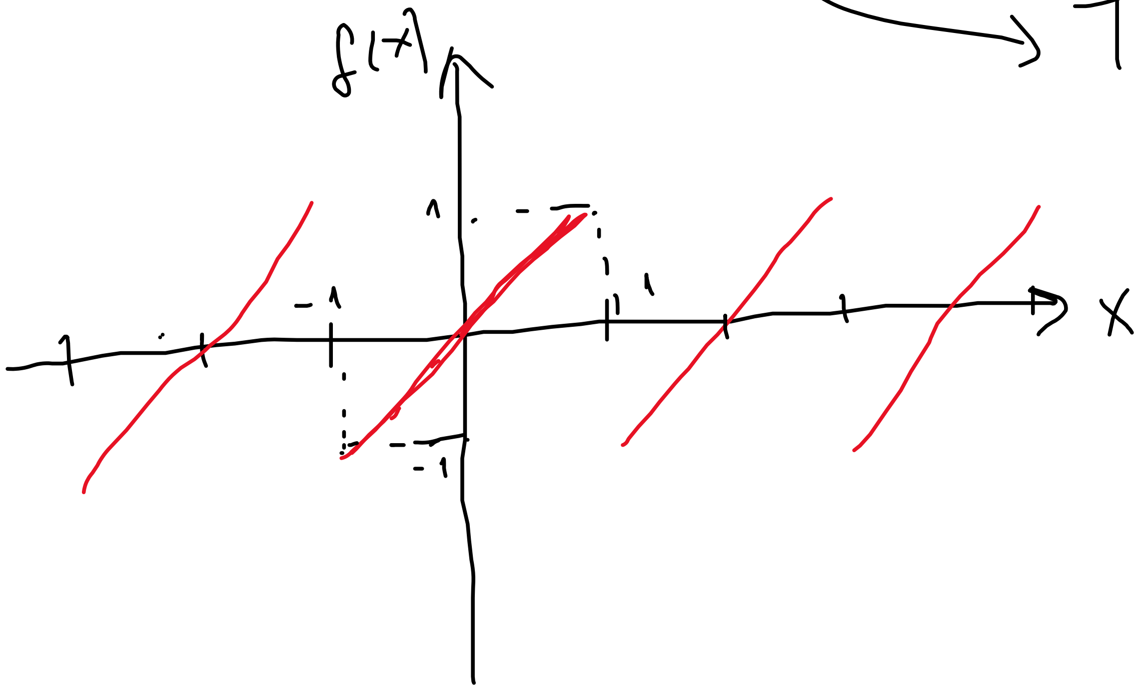
Exercise 2:

Find the Fourier Series of the function of period 2 :

c

$\downarrow$

$$f(x) = x, \quad -1 < x < 1.$$



$T = (1) - (-1) = 2$

$= 2L$

$\Rightarrow L = 1$

$C = 1$

Solution 2

Here C = -1 and L = 1.

Thus the Fourier Series is

$$Sf(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\pi x + b_n \sin n\pi x)$$

Solution 2

$1/L = 1$

$$a_0 = \int_{-1}^{1} f(x)dx = \int_{-1}^{1} x dx$$

$$= \left[\frac{x^2}{2}\right]_{-1}^{1} = \frac{1}{2} - \frac{1}{2} = 0$$

Solution 2

$$a_n = \int_{-1}^{1} \overset{u}{x}\,\underbrace{\cos(n\pi x)dx}_{dv} = \left[x\frac{1}{n\pi}\sin(n\pi x)\right]_{-1}^{1} - \int_{-1}^{1} \underbrace{\frac{1}{n\pi}\sin(n\pi x)}_{v}\underbrace{dx}_{du}$$

$u \cdot v$

$$= 0 - \int_{-1}^{1} \frac{1}{n\pi}\sin(n\pi x)dx = \frac{1}{(n\pi)^2}\left[\cos(n\pi x)\right]_{-1}^{1}$$

$$= \frac{\cos(n\pi) - \cos(-n\pi)}{(n\pi)^2} = 0$$

$x=1$

$\sin(n\pi x) = \sin(n\pi) = 0$

$\sin(n\pi x) = \sin(-n\pi) = 0$

$x=-1$

$\int u \cdot dv = \underline{u \cdot v} - \int v\, du$

$\cos(\theta) = \cos(-\theta)$

Solution 2

$$b_n = \int_{-1}^{1} \overset{u}{x}\,\overbrace{\sin(n\pi x)dx}^{dv} = \overbrace{\left[-x\frac{1}{n\pi}\cos(n\pi x)\right]_{-1}^{1}}^{u\cdot v} + \overbrace{\int_{-1}^{1}\frac{1}{n\pi}\cos(n\pi x)dx}^{v\,du}$$

$$= \left[-\frac{x}{n\pi}\cos(n\pi x)\right]_{-1}^{1} + \left[\frac{1}{(n\pi)^2}\sin(n\pi x)\right]_{-1}^{1}$$

$\sin(n\pi) = 0$
$\sin(-n\pi) = 0$

$$= \frac{-\cos(n\pi) - \cos(-n\pi)}{(n\pi)} + 0 = \frac{-2\cos(n\pi)}{(n\pi)} = \frac{-2(-1)^n}{(n\pi)}$$

$$= \frac{2(-1)^{n+1}}{(n\pi)}$$

n

$b_1, b_2, b_3, - - —$

Solution 2

$$\text{Thus,} \quad a_0 = 0, \; a_n = 0, \; b_n = \frac{2}{n\pi}(-1)^{n+1}$$

$$When \quad -1 < x < 1, \; the\ Fourier\ Series\ is$$

$$S f(x) = \sum_{n=1}^{\infty}\left[\frac{2}{n\pi}(-1)^{n+1}\sin(n\pi x)\right]$$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}}{n}\sin(n\pi x)\right]$$

$$= \frac{2}{\pi}\left[\sin \pi x - \frac{\sin 2\pi x}{2.} + \frac{\sin 3\pi x}{3} - + \cdots\right]$$

Exercise 3:

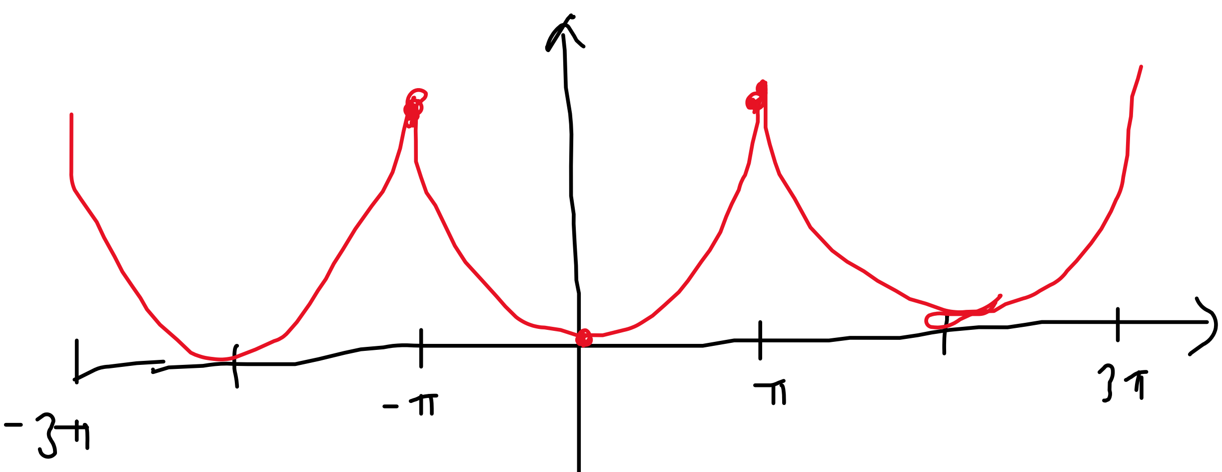
Represent the following function by a Fourier Series:

$f(x) = x^2, \quad -\pi < x < \pi \quad T = 2\pi = 2L$

Solution 3:

$$C = -\pi, \ L = \pi$$

$$\Rightarrow Sf(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$



Solution 3:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{3\pi}[\pi^3 - (-\pi^3)] = \frac{2\pi^2}{3}$$

Solution 3:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \overbrace{x^2}^{u}\,\overbrace{\cos(nx)\,dx}^{dv}$$

$\Rightarrow v = \frac{1}{n}\sin(nx)$

$\Rightarrow du = 2x\,dx$

$$=\frac{1}{\pi}\left[\cancelto{0}{\overbrace{\left[x^2(1/n)\sin(nx)\right]_{-\pi}^{\pi}}^{u\cdot v}} - \underbrace{\int_{-\pi}^{\pi} 2x\underbrace{(1/n)\sin(nx)}_{v}\,dx}_{v\cdot du}\right]$$

$$=-\frac{2}{n\pi}\int_{-\pi}^{\pi} \overset{u}{x}\,\underbrace{\sin(nx)\,dx}_{dv}$$

$$=-\frac{2}{n\pi}\left[\left[\underset{u}{x}\underbrace{(-1/n)\cos(nx)}_{v}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\underbrace{(-1/n)\cos(nx)}_{v}\,\underbrace{dx}_{du}\right]$$

Solution 3:

$$= -\frac{2}{n\pi}\left[\left[x(-1/n)\cos(nx)\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}(-1/n)\cos(nx)\,dx\right]$$

$$= -\frac{2}{n\pi}\left[-\frac{2\pi}{n}\cos\,n\pi + \frac{1}{n}\left(\frac{1}{n}\sin nx\right)_{-\pi}^{\pi}\right]$$

$$= \frac{4}{n^2}\cos n\pi = (-1)^n\frac{4}{n^2}$$

Solution 3:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \overset{f(x)=u}{x^2}\, \underbrace{\sin nx\, dx}_{dv}$$

$$= \frac{1}{\pi}\left[\left[x^2(-1/n)\cos nx\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \overbrace{2x\,\overbrace{(-1/n)\cos nx}^{v}\, dx}^{du}\right]$$

$$= \frac{2}{n\pi}\int_{-\pi}^{\pi} x \cos nx\, dx = \ldots\ldots = 0$$

Solution 3:

$$a_0 = \frac{2}{3}\pi^2, \ a_n = (-1)^n \frac{4}{n^2} \text{ and } b_n = 0$$

Thus when $-\pi < x < \pi$, the Fourier Series for x^2, is

$$x^2 = \frac{1}{2}\left(\frac{2}{3}\pi^2\right) + \sum_{n=1}^{\infty}\left((-1)^n \frac{4}{n^2}\cos nx\right)$$

$$= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\left((-1)^n \frac{\cos nx}{n^2}\right)$$

$$= \frac{\pi^2}{3} + 4(-\cos x + \frac{\cos 2x}{4} - \frac{\cos 3x}{9} + - \cdots)$$

Convergence and Sum of a Fourier Series

Not all periodic functions , defined over any type of interval, can be represented by Fourier series.

For a function, defined over an interval, to be represented by Fourier Series, it must satisfy certain conditions, known as Dirichlet Conditions.

Dirichlet Conditions

1. f must be well-defined at every x in the interval.

2. f must be continuous or have a finite number of finite discontinuities within the interval.

3. f' must be continuous or piecewise continuous within the interval

Examples:

1. Both the functions

   $f(x) = x, -1<x<1$ and

   $f(x) = x^2, -2<x<2$

   satisfy Dirichlet's conditions.

2. $$f(x) = \frac{1}{x}, \quad -\pi < x < \pi$$

   infinite discontinuity at $x = 0$.

   Dirichlet conditions not satisfied

3. $f(x) = \dfrac{1}{x-5}, -\pi < x < \pi$

Although the function has an infinite discontinuity at x = 5, this point is not in the given interval. Conditions are satisfied.

4. $f(x) = \tan x, -\pi < x < \pi$

Infinite discontinuities at x = $-\dfrac{\pi}{2}$ and x = $\dfrac{\pi}{2}$
Conditions not satisfied.

Value of the FS at a point of continuity

If *f* satisfies Dirichlet conditions and let S be the Fourier series of f at $x = a$.

If f is <u>continuous</u> at *a* then

$$Sf(a) = f(a)$$

Value of the FS at a point of discontinuity

If f is discontinuous at *a:*

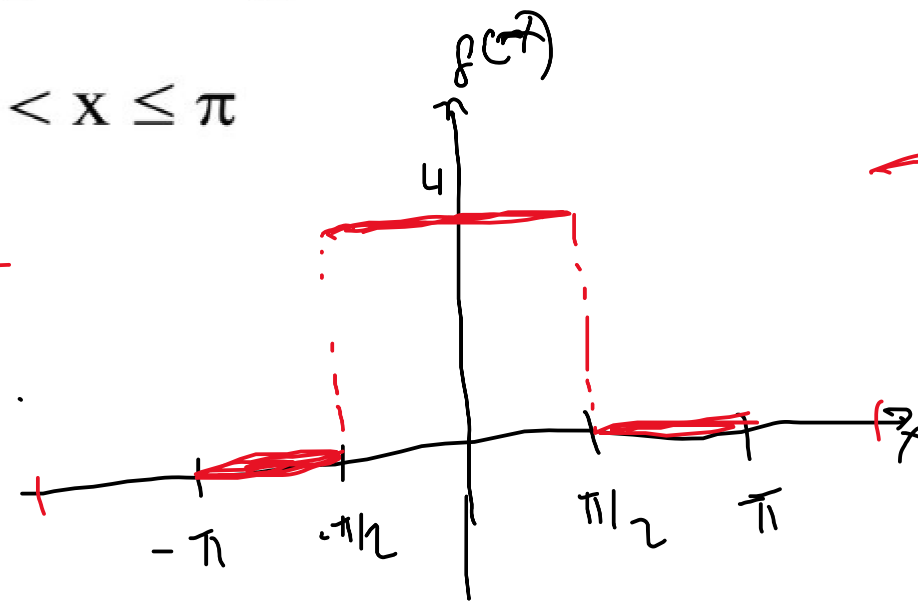
Let $f(a^-)$ and $f(a^+)$ be the left and right limits respectively.
Then the value of the Fourier Series at x = a is:

$$\boxed{Sf(a) = \frac{f(a^-) + f(a^+)}{2}}$$

Exercise 4:

Find the Fourier Series for the following 2π-periodic function in the given interval:

$$f(x)=\begin{cases} 0, & \text{if } -\pi < x \le -\frac{\pi}{2} \\ 4, & \text{if } -\frac{\pi}{2} < x \le \frac{\pi}{2} \\ 0, & \text{if } \frac{\pi}{2} < x \le \pi \end{cases}$$



Solution 4:

$\pi = 2\pi = 2L$

$C = -\pi$ and $L = \pi$

$$Sf(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx$$

Since the function is defined in three pieces over the interval (-Pi, Pi), each integral has to be evaluated separately over the three pieces and the results summed up

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{-\frac{\pi}{2}} 0dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4dx + \int_{\frac{\pi}{2}}^{\pi} 0dx\right]$$

$$= \frac{1}{\pi}(0 + 4\pi + 0) = 4$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{-\frac{\pi}{2}} 0\cos(nx)\,dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4\cos(nx)\,dx + \int_{\frac{\pi}{2}}^{\pi} 0\cos(nx)\,dx\right]$$

$$= \frac{1}{\pi}\left[0 + \frac{4}{n}\left(\sin(nx)\right)_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + 0\right] = \frac{1}{\pi}\frac{4}{n}\left[2\sin\frac{n\pi}{2}\right]$$

$$= \frac{8}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ \sin nx\, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{-\frac{\pi}{2}} 0 \sin nx\, dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4 \sin\ nx\, dx + \int_{\frac{\pi}{2}}^{\pi} 0 \sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[0 + 4\left(-\frac{\cos(nx)}{n}\right)_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + 0\right]$$

$$= \frac{1}{\pi}[0 + 0 + 0] = 0$$

Thus the Fourier Series representing the given function is:

$$Sf(x) = \frac{4}{2} + \sum_{n=1}^{\infty} \frac{8}{n\pi} \sin\frac{n\pi}{2} \cos nx$$

$$= 2 + \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{\sin\frac{n\pi}{2} \cos nx}{n}$$

0

$$= 2 + \frac{8}{\pi} \left(\sin\frac{\pi}{2} \cos x + \frac{\sin \pi \cos 2x}{2} + \frac{\sin\frac{3\pi}{2} \cos 3x}{3} + \cdots \right)$$

$$= 2 + \frac{8}{\pi} \left[\cos x - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} - \frac{\cos 7x}{7} + - \cdots \right]$$

The function is discontinuous at $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$

At $x = -\frac{\pi}{2}$, $\lim_{x \to -\frac{\pi}{2}^-} f(x) = 0$ and $\lim_{x \to -\frac{\pi}{2}^+} f(x) = 4$

At $x = \frac{\pi}{2}$, $\lim_{x \to \frac{\pi}{2}^-} f(x) = 4$ and $\lim_{x \to \frac{\pi}{2}^+} f(x) = 0$

$$S(-\frac{\pi}{2}) = \frac{0+4}{2} = 2 \qquad S(\frac{\pi}{2}) = \frac{4+0}{2} = 2$$

We can plot the graphs of f(x) and its representation by Fourier Series and see how close the approximation is.

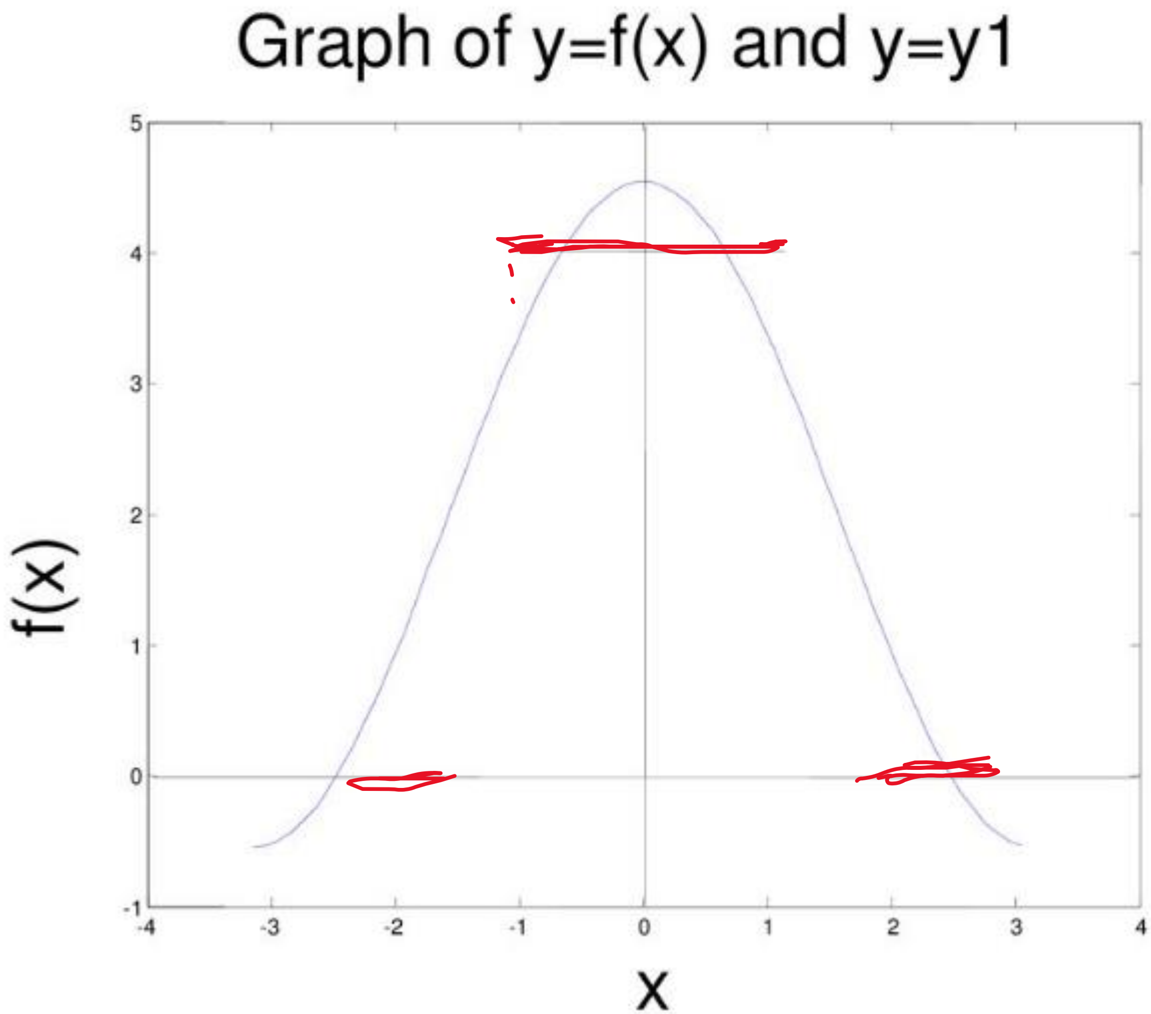
Let

$$y = f(x)$$

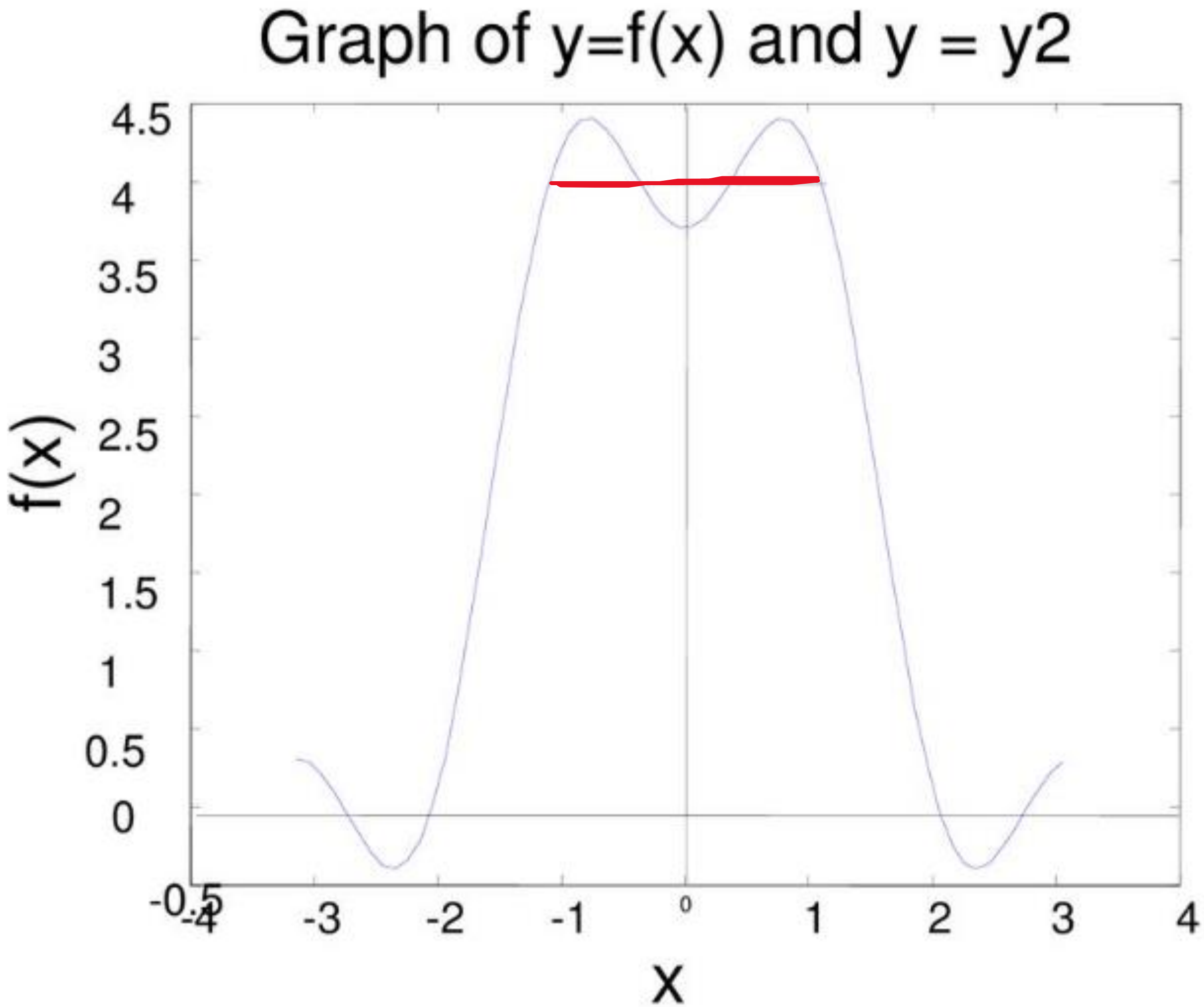
$$y_1 = 2 + \left(\frac{8}{\pi}\right)\cos x$$

$$y_2 = 2 + \left(\frac{8}{\pi}\right)\left(\cos x - \frac{\cos(3x)}{3}\right)$$

$$y_3 = 2 + \left(\frac{8}{\pi}\right)\left(\cos x - \frac{\cos(3x)}{3} + \frac{\cos(5x)}{5} - \frac{\cos(7x)}{7} + \frac{\cos(9x)}{9}\right)$$



Graph of y=f(x) and y=y1
f(x)
x
5
4
3
2
1
0
-1
-4
-3
-2
-1
0
1
2
3
4


Graph of y=f(x) and y = y2
f(x)
x
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
-0.5
-4
-3
-2
-1
0
1
2
3
4

Graph of y =f(x) and y = y3