① Determine whether v_1, v_2, and v_3 are linearly independent.

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Solution:

if there exist α and β where

$$\alpha v_i + \beta v_j = 0 \qquad \alpha, \beta \in \mathbb{R} \text{ and } \alpha \neq \beta \neq 0$$

then v_i and v_j are linearly dependent.

v_1, v_2? $\quad \alpha \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} \overset{?}{=} 0 \qquad \left.\begin{aligned} \alpha + 2\beta &= 0 \\ 2\alpha + 4\beta &= 0 \\ \alpha + 2\beta &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} \alpha &= -2 \\ \beta &= 1 \end{aligned}$

$v_1, v_2 \Rightarrow$ lin. dependent.

v_1, v_3? $\quad \alpha \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \overset{?}{=} 0 \qquad \left.\begin{aligned} \alpha + \beta &= 0 \\ 2\alpha &= 0 \\ \alpha + \beta &= 0 \end{aligned}\right\} \Rightarrow$ No solution other than $\alpha = 0, \beta = 0$

$\Rightarrow v_1, v_3 \Rightarrow$ lin. independent

② Let $x = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \quad y = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

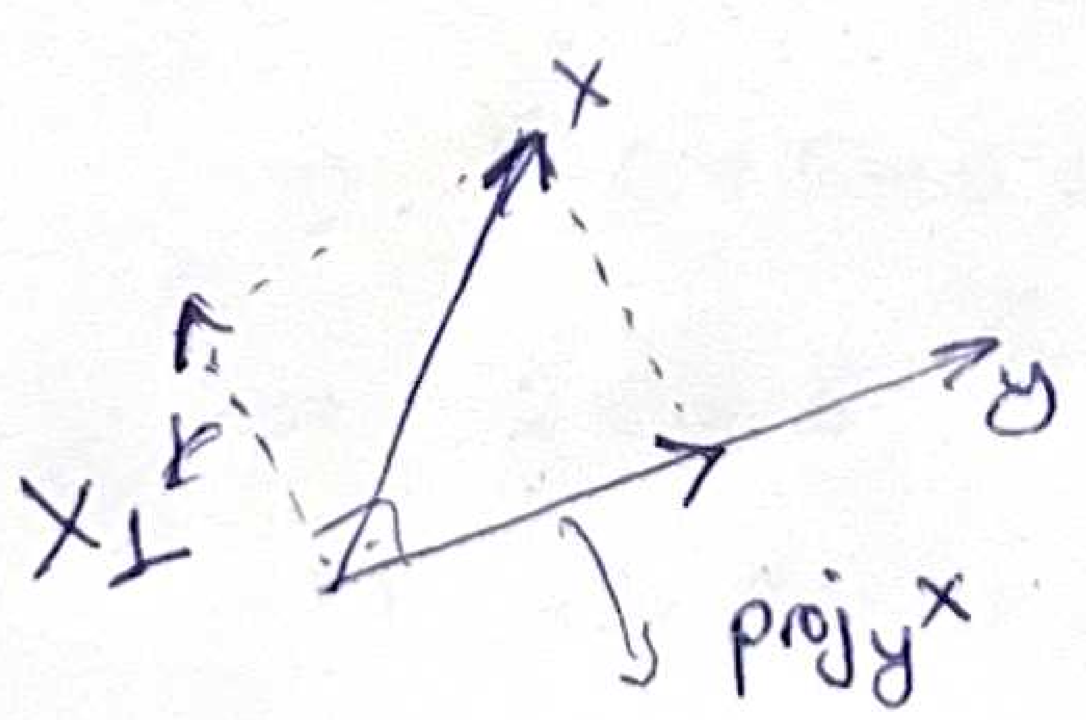
Find the orthogonal projection of x onto y, and then compute the component of x orthogonal to y.

Solution:

$$\text{proj}_y x = \frac{x \cdot y}{y \cdot y} y = \frac{2.1 - 1.1 + 2.0}{1.1 + 1.1 + 0.0} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix}$$

$$\text{orthogonal component } (x_\perp \text{ to } y) = x - \text{proj}_y x$$

$$= \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3/2 \\ -3/2 \\ 2 \end{bmatrix}$$



(3) Determine whether the system below is consistent. If it is consistent find all solutions.

$$Ax = b \quad , \quad A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ 1 & 1 & 0 \end{bmatrix} \quad , \quad b = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Solution

Find row-echelon form of $A|b$

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -2 & 2 \\ 1 & 1 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{ since } \operatorname{rank}(A) = \operatorname{rank}(A|b) = 2$$

system is consistent.

let $x_3 = t$

$\Rightarrow x_2 - x_3 = 0 \quad \Rightarrow \quad x_2 = t$

$\Rightarrow x_1 + 2x_2 - x_3 = 1$

$\Rightarrow x_1 + 2t - t = 1 \quad \Rightarrow x_1 = 1 - t$

$$\Rightarrow \quad x = \begin{bmatrix} 1-t \\ t \\ t \end{bmatrix}$$

(4) Find c so that $w - cv$ is perpendicular to v for the vectors v, w below.

$v = (1,1) \qquad w = (1,5)$

Solution $\quad v \cdot (w - cv) = 0 \quad \Rightarrow (1,1) \cdot (1-c, 5-c) = 0$

$\Rightarrow 1(1-c) + 1(5-c) = 0$

$\Rightarrow 1 - c + 5 - c = 0$

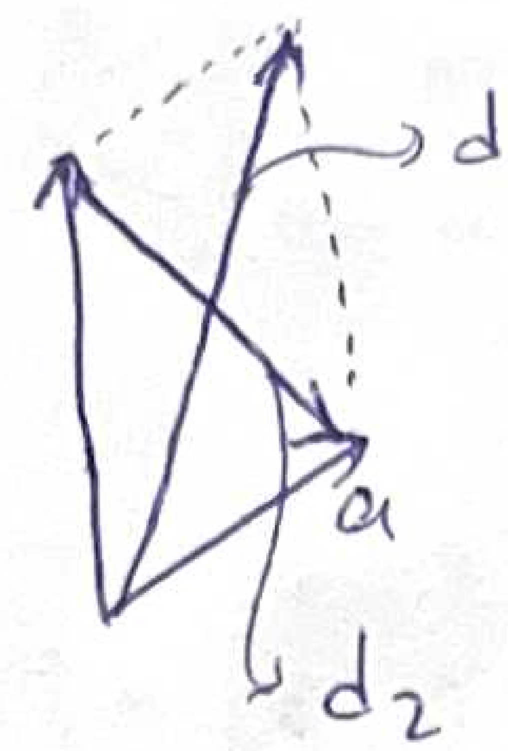
$\Rightarrow 2c = 6$

$\Rightarrow c = 3$

(5) What are the lengths of the diagonals of the parallelogram determined by the vectors $a=(4,2)$ and $b=(0,4)$?

Solution

Let



$d_1 = a+b = (4,2)+(0,4) = (4,6)$

$d_2 = a-b = (4,2)-(0,4) = (4,-2)$

$\Rightarrow \|d_1\| = \sqrt{4.4+6.6} = \sqrt{52} = 2\sqrt{13}$

$\|d_2\| = \sqrt{4.4+(-2).(-2)} = \sqrt{20} = 2\sqrt{5}$

(6) Find c, if the columns of matrix A are linearly dependent, where

$$A = \begin{bmatrix} c & c & c \\ 2 & 1 & 5 \\ 3 & 3 & 6 \end{bmatrix}$$

Solution: if columns are linearly dependent then rank of A is less than 3.

So, first obtain row-echelon form

$$\begin{vmatrix} c & c & c \\ 2 & 1 & 5 \\ 3 & 3 & 6 \end{vmatrix} \Rightarrow \begin{vmatrix} 3 & 3 & 6 \\ 2 & 1 & 5 \\ c & c & c \end{vmatrix} \rightarrow \begin{vmatrix} 1 & 1 & 2 \\ 2 & 1 & 5 \\ c & c & c \end{vmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & -c \end{bmatrix}$$

$\rightarrow R2 = R2 - 2R1$
$R3 = R3 - cR1$

Then if $\text{rank}(A) < 3$, c should be zero.

In this case column $\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 5 \\ 6 \end{bmatrix}$ are lin. dependent.

(7) Find matrix P that multiplies (x,y,z) to give (y,z,x).

Solution

$$\begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y \\ z \\ x \end{bmatrix}$$

$\Rightarrow P_{11}x + P_{12}y + P_{13}z = y \Rightarrow P_{11} = 0$, $P_{12} = 1$, $P_{13} = 0$

$\Rightarrow P_{21}x + P_{22}y + P_{23}z = z \Rightarrow P_{21} = 0$, $P_{22} = 0$, $P_{23} = 1$

$\Rightarrow P_{31}x + P_{32}y + P_{33}z = x$

$\Rightarrow P_{31} = 1$, $P_{32} = 0$, $P_{33} = 0$

Then $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

(8) For which three numbers a will elimination fail to give three pivots? or A will be singular?

$$A = \begin{bmatrix} a & 2 & 3 \\ a & a & 4 \\ a & a & a \end{bmatrix}$$

Solution

$$\begin{bmatrix} a & 2 & 3 \\ a & a & 4 \\ a & a & a \end{bmatrix} \Rightarrow \begin{bmatrix} a & 2 & 3 \\ 0 & a-2 & 1 \\ 0 & a-2 & a-3 \end{bmatrix} \Rightarrow \begin{bmatrix} a & 2 & 3 \\ 0 & a-2 & 1 \\ 0 & 0 & a-4 \end{bmatrix}$$

$R_2 = R_2 - R_1$
$R_3 = R_3 - R_1$

$R_3 = R_3 - R_2$.

$a = 0$
$a = 2$
$a = 4$

(9) The parabola $y = a + bx + cx^2$ goes through the points $(x, y) = (1, 4)$ and $(2, 8)$ and $(3, 14)$. Find and solve a matrix equation for the unknowns (a, b, c)

Solution

Substitute the points into the parabola equation:

$$\left.\begin{aligned} a + b\cdot 1 + c\cdot 1^2 &= 4 \\ a + b\cdot 2 + c\cdot 2^2 &= 8 \\ a + b\cdot 3 + c\cdot 3^2 &= 14 \end{aligned}\right\} \Rightarrow \begin{aligned} a + b + c &= 4 \\ a + 2b + 4c &= 8 \\ a + 3b + 9c &= 14 \end{aligned} \Rightarrow \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 14 \end{bmatrix}}_{\text{Matrix Equation.}}$$

Then, do the elimination $\boxed{A|b}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 14 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 3 & 4 \\ 0 & 2 & 8 & 10 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 2 & 2 \end{array}\right]$$

b b' b'

$2c = 2 \Rightarrow c = 1$

$b + 3c = 4 \Rightarrow b = 1$

$a + b + c = 4 \Rightarrow a = 2$

(10) Write the problem below in a matrix form $Ax = b$

problem: x is twice as old as y, and their ages add to 33.

Solution first eqn $\Rightarrow$ $x = 2y$ $\Rightarrow$ $x - 2y = 0$
Second eqn $\Rightarrow$ $x + y = 33$ $\Rightarrow$ $x + y = 33$

$$\Rightarrow \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 33 \end{bmatrix}$$

(11) Find the inverse of the matrix A below.

$$A = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

Solution : Start with $A | I$ end up with $I \; A^{-1}$

$$\left[\begin{array}{ccc|ccc} 0 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 1 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 1 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1/4 & 0 & 0 \end{array}\right] \Rightarrow A^{-1} = \begin{bmatrix} 0 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/4 & 0 & 0 \end{bmatrix}$$

(the right block is A^{-1})

(12) For which values of a and b, the matrix below is invertible?

$$A = \begin{bmatrix} a & b & b \\ a & a & b \\ a & a & a \end{bmatrix}$$

Solution

if $\text{rank}(A) = 3$ then A is invertible.

Use elimination methods to find rank of A.

$$\begin{bmatrix} a & b & b \\ a & a & b \\ a & a & a \end{bmatrix} \xrightarrow[R_3 = R_3 - R_1]{R_2 = R_2 - R_1} \begin{bmatrix} a & b & b \\ 0 & a-b & 0 \\ 0 & a-b & a-b \end{bmatrix} \xrightarrow{R_3 = R_3 - R_2} \begin{bmatrix} a & b & b \\ 0 & a-b & 0 \\ 0 & 0 & a-b \end{bmatrix}$$

In order to the rank of A be 3, a should not be zero. Otherwise first column will be all zero and rank(A) will be 2. Also $a-b$ should not be zero, otherwise second or third column will be zero.

Then $a \neq 0$ and $a - b \neq 0$ or

$a \neq 0$ and $a \neq b$

(13) Find LU decomposition of the matrix A in the system given below. Then use LU matrices to solve the system. $\boxed{6}$

$$2x+3y-z=5$$
$$+x-y+z=2 \Rightarrow Ax=b \Rightarrow \begin{bmatrix} 2 & 3 & -1 \\ 1 & -1 & 1 \\ +1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix}$$
$$+x+0.y+z=4$$

Solution

First, do the elimination to obtain upper triangular matrix U, and use coefficients to obtain L while obtaining U.

$$\begin{bmatrix} 2 & 3 & -1 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow[R_3=R_3-\frac{1}{2}R_1]{R_2=R_2-\frac{1}{2}R_1} \begin{bmatrix} 2 & 3 & -1 \\ 0 & -2{,}5 & 1{,}5 \\ 0 & -1{,}5 & 1{,}5 \end{bmatrix} \xrightarrow{R_3=R_3-\left(\frac{3}{5}\right)R_2} \underbrace{\begin{bmatrix} 2 & 3 & -1 \\ 0 & -2{,}5 & 1{,}5 \\ 0 & 0 & 0{,}6 \end{bmatrix}}_{U}$$

$$L=\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

Since $R_2=R_2-\frac{1}{2}R_1$, then $l_{21}=\frac{1}{2}$

$R_3=R_3-\frac{1}{2}R_1$, then $l_{31}=\frac{1}{2}$

$R_3=R_3-\frac{3}{5}R_2$, then $l_{32}=\frac{3}{5}$

$$\Rightarrow L=\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/2 & 3/5 & 1 \end{bmatrix}$$

Then $Ax=b$ and $A=LU$

$\Rightarrow LUx=b$

let $Ux=y \Rightarrow Ly=b \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/2 & 3/5 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}=\begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix}$

forward substitution

$\Rightarrow y_1=5 \quad y_2=2-\frac{1}{2}5=-\frac{1}{2}$

$y_3=4-\frac{1}{2}5-\frac{3}{5}\left(\frac{-1}{2}\right)=\frac{18}{10}$

$$\Rightarrow Ux=y \Rightarrow \begin{bmatrix} 2 & 3 & -1 \\ 0 & -2{,}5 & 1{,}5 \\ 0 & 0 & 0{,}6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 5 \\ -1/2 \\ 18/10 \end{bmatrix}$$

$\Rightarrow x_3=\frac{18}{10}\div 0{,}6=3$

$x_2=\left(\frac{-1}{2}-1{,}5.3\right)\div(-2{,}5)=2$

$x_1=(5-3.2+1.3)\div 2=1$

backward substitution

(14) Find LU decomposition of the matrix below. $\boxed{7}$

$$\begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

Solution: Do elimination as in problem 13

$$\begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix} \xrightarrow[\substack{R_2=R_2-R_1 \\ R_3=R_3-R_1 \\ R_4=R_4-R_1}]{} \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ 0 & b-a & c-a & d-a \end{bmatrix} \xrightarrow[\substack{R_3=R_3-R_2 \\ R_4=R_4-R_2}]{}$$

$$\Rightarrow \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & c-b & d-b \end{bmatrix} \xrightarrow[R_4=R_4-R_3]{} \underbrace{\begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}}_{U}$$

$\Rightarrow l_{21}=1 \quad l_{32}=1$

$l_{31}=1 \quad l_{42}=1$

$l_{41}=1 \quad l_{43}=1$

$$\Rightarrow L=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

(15) Find determinant of the matrix below

$$A=\begin{bmatrix} 1 & t & t^2 \\ t & 1 & t \\ t^2 & t & 1 \end{bmatrix}$$

Solution: Use elimination method and make A upper triangular matrix. Then multiply diagonal elements.

$$\begin{bmatrix} 1 & t & t^2 \\ t & 1 & t \\ t^2 & t & 1 \end{bmatrix} \xrightarrow[\substack{R_2=R_2-tR_1 \\ R_3=R_3-t^2R_1}]{} \begin{bmatrix} 1 & t & t^2 \\ 0 & 1-t^2 & t-t^3 \\ 0 & t-t^3 & 1-t^4 \end{bmatrix} \xrightarrow[R_3=R_3-tR_2]{} \begin{bmatrix} 1 & t & t^2 \\ 0 & 1-t^2 & t-t^3 \\ 0 & 0 & 1-t^2 \end{bmatrix}$$

$$\Rightarrow \det(A)=u_{11}\cdot u_{22}\cdot u_{33}=1\cdot(1-t^2)(1-t^2)=(1-t^2)^2$$

(16) Find eigenvalues and eigenvectors of A and A^2.

$$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} \qquad A^2 = \begin{bmatrix} 7 & -3 \\ -2 & 6 \end{bmatrix}$$

Solution: solve characteristic equation for A.

$$\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}\right) = \det\left(\begin{bmatrix} \lambda+1 & -3 \\ -2 & \lambda \end{bmatrix}\right)$$

$$\det(\lambda I - A) = \lambda(\lambda+1) - (-3)(-2) = \lambda^2 + \lambda - 6 = (\lambda+3)(\lambda-2) = 0$$

$$\Rightarrow \lambda_1 = -3 \quad \lambda_2 = 2$$

Then for $\lambda_1 = -3$

$$A v_1 = \lambda_1 v_1 \Rightarrow \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{v_1} = -3 \begin{bmatrix} a \\ b \end{bmatrix} \Rightarrow \underbrace{\begin{array}{l} -a + 3b = -3a \\ 2a = -3b \end{array}}_{a=3 \quad b=-2}$$

can be one of the solutions

$$\Rightarrow v_1 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

for $\lambda_2 = 2$

$$A v_1 = 2 v_1 \Rightarrow \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 2 \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\Rightarrow \begin{array}{l} -a + 3b = 2a \\ 2a = 2b \end{array} \Rightarrow \begin{array}{l} a = 1 \\ b = 1 \end{array}$$ can be a solution $\Rightarrow v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

We know that eigenvectors of A^2 is equal to A, and eigenvalues of A^2 is squares of eigenvalues of A.

Then eigenvalues of A^2 are $(-3)^2$ and $(2)^2$ which are 9 and 4

(17) Diagonalize A as $A = X \Lambda X^{-1}$ given in problem 16.

Solution: For this diagonalization Λ is the diagonal matrix where the diagonal elements are the eigenvalues. These eigenvalues are found to be (-3) and (2) in the previous problem. Then

$$\Lambda = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$$

The orders of the columns in X are important: The eigenvectors should be placed into the columns with the same order of eigenvalues in Λ.
First find eigenvector using the equation: $A v = \lambda v$
Then find normalized eigenvectors.

$$v_1 = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \Rightarrow v_{1n} = \begin{bmatrix} 3/\sqrt{13} \\ -2/\sqrt{13} \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow v_{2n} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$X^{-1} = \begin{bmatrix} \sqrt{13}/5 & -\sqrt{13}/5 \\ 2\sqrt{2}/5 & 3\sqrt{2}/5 \end{bmatrix}$$

$$\Rightarrow A = X \Lambda X^{-1} = \begin{bmatrix} 3/\sqrt{13} & 1/\sqrt{2} \\ -2/\sqrt{13} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \; \\ \; \end{bmatrix}$$

(18) Find A^{10} given in problem 16.

Solution: $A = X \Lambda X^{-1} \Rightarrow A^2 = \underbrace{X \Lambda X^{-1}}_{A} \underbrace{X \Lambda X^{-1}}_{A} = X \Lambda \Lambda X^{-1} = X \Lambda^2 X^{-1}$ Since $X^{-1} X = I$

$$\Rightarrow A^3 = A \cdot A^2 = X \Lambda \underbrace{X^{-1} X}_{I} \Lambda^2 X^{-1} = X \Lambda \Lambda^2 X^{-1} = X \Lambda^3 X^{-1}$$

Then one can easily see that

$$A^n = X \Lambda^n X^{-1} \Rightarrow A^{10} = X \Lambda^{10} X^{-1}$$

And we know that powers of a diagonal matrix is the same powers of diagonal elements. Then

$$\Lambda^{10} = \begin{bmatrix} (-3)^{10} & 0 \\ 0 & (2)^{10} \end{bmatrix} \Rightarrow A^{10} = \begin{bmatrix} 3/\sqrt{13} & 1/\sqrt{2} \\ -2/\sqrt{13} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} (-3)^{10} & 0 \\ 0 & (2)^{10} \end{bmatrix} \begin{bmatrix} 3/\sqrt{13} & -2/\sqrt{13} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

(19)

Two basis vector sets are given below:

$$V = (v_1, v_2) = \left(\begin{bmatrix} 35 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 10 \end{bmatrix} \right), \quad W = (w_1, w_2) = \left(\begin{bmatrix} 11 \\ 8 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \end{bmatrix} \right)$$

A vector x has coefficients w.r.t V as:

$$[x]_V = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Use change of basis matrix (B) to find the coordinates of x w.r.t W.

Solution

Use one of the methods below to find B.

1) Write v_1 and v_2 using w_1 and w_2.

$$v_1 = \alpha_1 w_1 + \alpha_2 w_2 \Rightarrow \begin{bmatrix} 35 \\ 0 \end{bmatrix} = \alpha_1 \begin{bmatrix} 11 \\ 8 \end{bmatrix} + \alpha_2 \begin{bmatrix} 6 \\ -2 \end{bmatrix} \Rightarrow \left.\begin{matrix} \alpha_1 = 1 \\ \alpha_2 = 4 \end{matrix}\right\} \text{These will be the first column of } B.$$

$$v_2 = \beta_1 w_1 + \beta_2 w_2 \Rightarrow \begin{bmatrix} 5 \\ 10 \end{bmatrix} = \beta_1 \begin{bmatrix} 11 \\ 8 \end{bmatrix} + \beta_2 \begin{bmatrix} 6 \\ -2 \end{bmatrix} \Rightarrow \left.\begin{matrix} \beta_1 = 1 \\ \beta_2 = -1 \end{matrix}\right\} \text{These will be the second column of } B$$

Then $B = \begin{bmatrix} 1 & 1 \\ 4 & -1 \end{bmatrix}$

or, 2) $B = W^{-1} V$ $\qquad W = \begin{bmatrix} 11 & 6 \\ 8 & -2 \end{bmatrix}, \quad V = \begin{bmatrix} 35 & 5 \\ 0 & 10 \end{bmatrix}$

$$W^{-1} = \frac{1}{\det(W)} \begin{bmatrix} -2 & -6 \\ -8 & 11 \end{bmatrix} = \frac{-1}{70} \begin{bmatrix} -2 & -6 \\ -8 & 11 \end{bmatrix}$$

$$\Rightarrow B = W^{-1} V = \frac{-1}{70} \begin{bmatrix} -2 & -6 \\ -8 & 11 \end{bmatrix} \begin{bmatrix} 35 & 5 \\ 0 & 10 \end{bmatrix} = \frac{-1}{70} \begin{bmatrix} -70 & -70 \\ -280 & 70 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & -1 \end{bmatrix}$$

Then, we can use B to find the coordinates w.r.t W. by multiplying B with coordinate vector of X in V.

$$\text{So, } [X]_W = B . [X]_V = \begin{bmatrix} 1 & 1 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

Note: You can check whether the coordinates of x in standard basis are the same starting with $[X]_V$ and $[X]_W$. They should be equal and are $\begin{bmatrix} 85 \\ 30 \end{bmatrix}$