LAPLACE TRANSFORM

The Laplace transform

the definition of the **Laplace transform**. Given a signal $x(t)$ and a complex number $s = \sigma + j\omega$, we define

$$X(s) \triangleq \int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}dt.$$

The above integral may diverge for some values of s. The set of all complex numbers s for which the above integral converges is called the **region of convergence (ROC)** of the Laplace transform of $x(t)$.

This is called the **bilateral** (or **two-sided**) Laplace transform. We can also define the **unilateral** (or **one-sided**) Laplace transform:

$$X(s) = \int_{0}^{\infty} x(t)e^{-st}dt.$$

We will deal with the one-sided Laplace transform, because that will allow us to deal conveniently with systems that have nonzero initial conditions.

The Laplace transform: an example

As an example, let us consider $x(t) = e^{-bt}u(t)$, where b is real or complex.

$$\begin{aligned} X(s) &= \int_0^\infty x(t)e^{-st}dt \\ &= \int_0^\infty e^{-(b+s)t}dt \\ &= \lim_{T\to\infty}\left[-\frac{1}{b+s}e^{-(b+s)t}\right]_0^T \\ &= \frac{1}{b+s} - \frac{1}{b+s}\lim_{T\to\infty}\left[e^{-(b+s)T}\right]. \end{aligned}$$

The above limit will be equal to 0 if $\mathrm{Re}(b+s) > 0$; otherwise, it does not exist. Hence,

$$X(s) = \frac{1}{s+b}, \qquad \text{if } \mathrm{Re}(b+s) > 0.$$

Thus, the ROC is the set of all complex numbers s, such that $\mathrm{Re}\, s > -\mathrm{Re}\, b$.

Properties of the Laplace transform

Just as the Fourier transform, the Laplace transform has a number of useful properties which we will learn to exploit. We will look at and prove a few of these.

We will use the notation $x(t) \leftrightarrow X(s)$ to denote the fact that $X(s)$ is the Laplace transform of $x(t)$; we will also use the notation $X(s) = \mathcal{L}[x(t)]$. First of all, the Laplace transform is linear: if $X_1(s) = \mathcal{L}[x_1(t)]$ and $X_2(s) = \mathcal{L}[x_2(t)]$, then for any constants a_1 and a_2

$$\mathcal{L}[a_1 x_1(t) + a_2 x_2(t)] = a_1 \mathcal{L}[x_1(t)] + a_2 \mathcal{L}[x_2(t)] = a_1 X_1(s) + a_2 X_2(s).$$

This is obvious from definitions.

Properties: right shift in time

IF $X(s) = \mathcal{L}[x(t)]$, then for any real number $c > 0$ we have

$$x(t-c)u(t-c) \leftrightarrow e^{-cs}X(s).$$

Proof:

$$\begin{aligned}
\mathcal{L}[x(t-c)u(t-c)] &= \int_0^\infty x(t-c)u(t-c)e^{-st}dt \\
&= \int_c^\infty x(t-c)e^{-st}dt \\
&= \int_0^\infty x(t)e^{-s(t+c)}dt \\
&= e^{-cs}\int_0^\infty x(t)e^{-st}dt \\
&= e^{-cs}X(s).
\end{aligned}$$

Right shift in time: example

Ex.: consider the rectangular pulse

$$x(t) = \begin{cases} 1, & 0 \le t < c \\ 0, & \text{otherwise} \end{cases}$$

We can write

$$x(t) = u(t) - u(t-c).$$

Then, using linearity and shift property we get

$$X(s) = \frac{1}{s} - \frac{e^{-cs}}{s} = \frac{1-e^{-cs}}{s}.$$

The ROC is $\{s \in \mathbb{C} : \mathrm{Re}\, s > 0\}$.

Properties: multiplication by a power of t

If $X(s) = \mathcal{L}[x(t)]$, then

$$\mathcal{L}[t^N x(t)] = (-1)^N \frac{d^N}{ds^N} X(s)$$

Proof: similar to the corresponding property of the Fourier transform.

Ex.: $x(t) = t^N u(t)$.

$$X(s) = (-1)^N \frac{d^N}{ds^N} \left(\frac{1}{s}\right) = \frac{N!}{s^{N+1}}.$$

In particular, for the unit ramp $x(t) = tu(t)$ we have

$$X(s) = \frac{1}{s^2}.$$

Ex.: $x(t) = te^{-bt}u(t)$.

$$X(s) = (-1)\frac{d}{ds}\left(\frac{1}{s+b}\right) = \frac{1}{(s+b)^2}.$$

Properties: multiplication by an exponential

If $X(s) = \mathcal{L}[x(t)]$, then

$$\mathcal{L}[e^{at}x(t)] = X(s-a)$$

for any a, real or complex.

Proof:

$$\begin{aligned}
\mathcal{L}[e^{at}x(t)] &= \int_0^\infty x(t)e^{at}e^{-st}dt \\
&= \int_0^\infty x(t)e^{-(s-a)t}dt \\
&= X(s-a).
\end{aligned}$$

Ex.: $x(t) = [u(t) - u(t-c)]e^{at}$, where $c > 0$ and a is a real number. Then

$$X(s) = \frac{1 - e^{-c(s-a)}}{s-a}.$$

Laplace transform of cosine and sine

1. $x(t) = \cos(\omega_0 t)u(t) = \frac{1}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right]u(t)$

$$
\begin{aligned}
X(s) &= \frac{1}{2}\left[\frac{1}{s - j\omega_0} + \frac{1}{s + j\omega_0}\right] \\
&= \frac{1}{2} \cdot \frac{2s}{s^2 + \omega_0^2} \\
&= \frac{s}{s^2 + \omega_0^2}.
\end{aligned}
$$

2. $x(t) = \sin(\omega_0 t)u(t) = \frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right]$

$$
\begin{aligned}
X(s) &= \frac{1}{2j}\left[\frac{1}{s - j\omega_0} - \frac{1}{s + j\omega_0}\right] \\
&= \frac{1}{2j} \cdot \frac{2j\omega_0}{s^2 + \omega^2 + 0} \\
&= \frac{\omega_0}{s^2 + \omega_0^2}.
\end{aligned}
$$

Exponentially damped sinusoids

1. $x(t) = e^{-bt}\cos(\omega_0 t)u(t)$

$$X(s) = \frac{s+b}{(s+b)^2+\omega_0^2}$$

2. $x(t) = e^{-bt}\sin(\omega_0 t)u(t)$

$$X(s) = \frac{\omega_0}{(s+b)^2+\omega_0^2}$$

Properties: differentiation in time domain

If $X(s) = \mathcal{L}[x(t)]$, then

$$\mathcal{L}\left[\frac{d}{dt}x(t)\right] = sX(s) - x(0),$$

(when $x(t)$ is discontinuous at $t = 0$ or if it contains an impulse or the derivative of an impulse at $t = 0$, $x(0)$ should be replaced by $x(0^-)$).

Proof: assume that, as $t \to \infty$, $x(t)$ decays faster than an exponential, i.e., $|x(t)| < Ce^{-at}$ for t large enough and for some $C \geq 0$ and $a > 0$. Then we integrate by parts to get

$$\begin{aligned}
\mathcal{L}\left[\frac{d}{dt}x(t)\right] &= \int_0^\infty \left(\frac{d}{dt}x(t)\right) e^{-st}dt \\
&= \left[x(t)e^{-st}\right]_0^\infty + s\int_0^\infty x(t)e^{-st}dt \\
&= -x(0) + sX(s),
\end{aligned}$$

where we have used the exponential decay of $x(t)$.

Differentiation in time domain

Ex.: $x(t) = \delta(t)$. Since $\delta(t) = (d/dt)u(t)$, we have

$$\mathcal{L}[\delta(t)] = s \cdot \frac{1}{s} - u(0^-) = 1.$$

For the second derivative, we have

$$\mathcal{L}\left[\frac{d^2}{dt^2}x(t)\right] = s^2X(s) - sx(0) - x'(0),$$

where $x(0)$ and $x'(0)$ are, respectively, the values of $x(t)$ and $x'(t)$ at $t = 0$.

Properties: integration in time domain

If $X(s) = \mathcal{L}[x(t)]$, then

$$\mathcal{L}\left[\int_0^\infty x(\lambda)d\lambda\right] = \frac{1}{s}X(s).$$

Proof: define the function

$$v(t) = \begin{cases} \int_0^t x(\lambda)d\lambda, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Then $v'(t) = x(t)$ for $t \geq 0$; since we are dealing with the one-sided Laplace transform, we assume that $x(t) = 0$ for $t < 0$. Hence, by the differentiation in time domain property we get

$$X(s) = sV(s),$$

so $V(s) = (1/s)X(s)$.

Properties: convolution in time domain

If $X(s) = \mathcal{L}[x(t)]$ and $V(s) = \mathcal{L}[v(t)]$, then

$$\mathcal{L}[x(t) \star v(t)] = X(s)V(s).$$

Ex.: let $x(t) = v(t) = u(t) - u(t-1)$. Then

$$X(s) = \frac{1-e^{-s}}{s}$$

and

$$\mathcal{L}[x(t) \star x(t)] = X^2(s) = \left(\frac{1-e^{-s}}{s}\right)^2 = \frac{1-2e^{-s}+e^{-2s}}{s^2}.$$

We can actually invert this Laplace transform using its properties:

$$x(t) = tu(t) - 2(t-1)u(t-1) + (t-2)u(t-2).$$

Properties: initial-value theorem

Given a signal $x(t)$ with the Laplace transform $X(s)$, we can compute the initial values $x(0)$ and $x'(0)$ via

$$x(0) = \lim_{s \to \infty} sX(s)$$

and

$$x'(0) = \lim_{s \to \infty} [s^2 X(s) - sx(0)]$$

This property is called the **Initial-Value Theorem** (IVT).

Ex.: suppose the signal $x(t)$ has the Laplace transform

$$X(s) = \frac{-s^2 + 1}{2s^3 + 3s^2 + 1}.$$

Then

$$x(0) = \lim_{s \to \infty} sX(s) = \lim_{s \to \infty} \frac{-s^3 + s}{2s^3 + 3s^2 + 1} = -\frac{1}{2}.$$

Properties: final-value theorem

Given a signal $x(t)$, such that the limit

$$x(\infty) \triangleq \lim_{t \to \infty} x(t)$$

(called the **final value of** $x(t)$) exists, we have

$$x(\infty) = \lim_{s \to 0} sX(s).$$

Note that the limit on the right-hand side may exist, but it will only equal $x(\infty)$ if the latter limit is well-defined. This property is called the **Final-Value Theorem** (FVT).

Ex.: suppose

$$X(s) = \frac{1}{s^2 + 1}.$$

Then

$$\lim_{s \to 0} sX(s) = \lim_{s \to 0} \frac{s}{s^2 + 1} = 0.$$

However, $X(s) = \mathcal{L}[\sin(t)u(t)]$, and $\sin(t)u(t)$ does not have a final value.