COMPLEX FOURIER SERIES

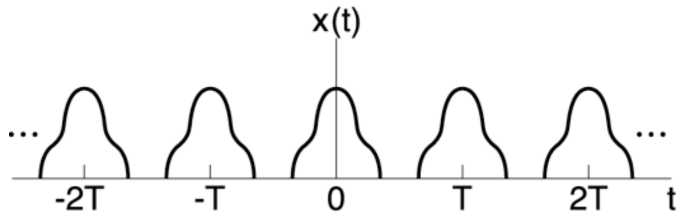
(Complex) Fourier Series Representations

- **What types of signals can be represented as sums of complex exponentials?**
  - **CT:** $s = j\omega$ **(Fourier Transform)** ➢ **signals of the form** $e^{j\omega t}$
  - **DT:** $z = e^{j\omega}$ **(z-Transform)** ➢ **signals of the form** $e^{j\omega n}$
- **These representations form the basis of the Fourier Series and Transform.**
- **Consider a periodic signal:**

$$x(t) = x(t+T) \quad \textit{for all } T$$

$$\textit{smallest such } T \textit{ is the fundamental period}$$

$$\omega_0 = \frac{2\pi}{T} \textit{ is the fundamental frequency (radians)}$$



- **Consider representing a signal as a sum of these exponentials:**

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{jk(2\pi/T)t} = x(t+T)$$

Notes:

- **Periodic with period T**
- $\{c_k\}$ **are the (complex) Fourier series coefficients**
- $k = 0$ **corresponds to the DC value;** $k = 1$ **is the first harmonic; …**

Alternate Fourier Series Representations

- **For real, periodic signals:**

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \qquad \text{"trigonometric Fourier series"}$$

$$or,$$

$$x(t) = a_0 + \sum_{k=1}^{\infty} \gamma_k \cos(k\omega_0 t + \phi_k) \qquad \text{"cosine wth phase"}$$

$$or,$$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \qquad \text{"complex"}$$

- **These are essentially interchangeable representations. For example, note:**

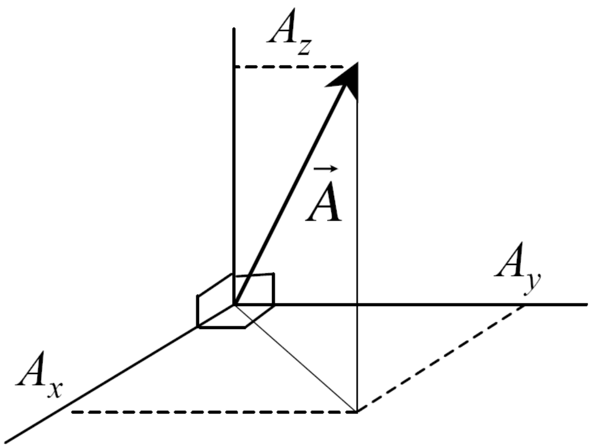
$$e^{jk\omega_0 t} = \cos(k\omega_0 t) + j\sin(k\omega_0 t)$$

- **The complex Fourier series is used for most engineering analyses.**
- **How do we compute the Fourier series coefficients?**
  - **There are several ways to arrive at the equations for estimating the coefficients. Most are based on concepts of orthogonal functions and vector space projections.**

Vector Space Projections

- **How can we compute the Fourier series coefficients?**
  - **One approach is to use the concept of a vector projection:**

$$\vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z} \quad where \quad \hat{x}, \hat{y}, \hat{z} \text{ are unit vectors}$$



- **How do we find the components:**

$$A_x = \vec{A} \bullet \hat{x},\ A_y = \vec{A} \bullet \hat{y},\ A_z = \vec{A} \bullet \hat{z}$$

We project the vector onto the corresponding axis using a dot product.

- **Note that this works because the three axes are orthogonal:**

$$\hat{x} \bullet \hat{y} = \hat{y} \bullet \hat{z} = \hat{x} \bullet \hat{z} = 0$$

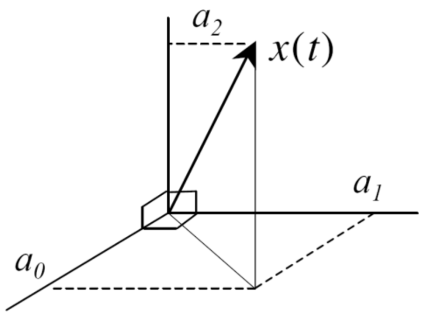
- **We can apply the same concept to signals using the notion of a Hilbert space in which each axis represents an eigenfunction (e.g., cos() and sin()):**

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

$$x(t) \leftrightarrow \vec{A}$$

$$e^{jk\omega_0 t} \leftrightarrow \hat{x}, \hat{y}, \hat{z}$$

$$c_k \leftrightarrow A_x, A_y, A_z$$



Computation of the Coefficients

- **We can make use of the principle of orthogonality in the Hilbert space:**

$$\frac{1}{T}\int_T e^{jk\omega_0 t}\cdot e^{-jn\omega_0 t}dt = \frac{1}{T}\int_T e^{j(k-n)\omega_0 t}dt = \begin{cases} 1, & k = n \\ 0, & k \neq n \end{cases} = \delta(k-n)$$

(This can be thought of as an "inner product.")

- **We can apply this to our Fourier series:**

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

- **Using the inner product:**

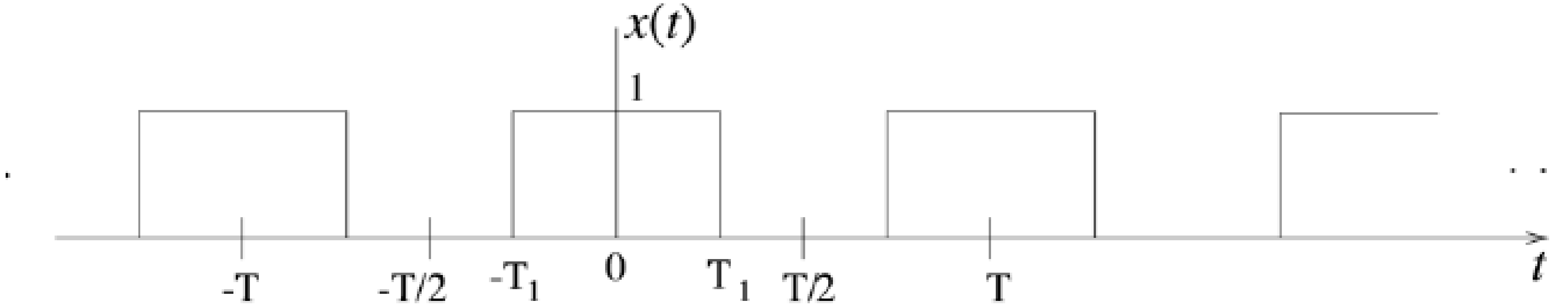
$$\frac{1}{T}\int_T x(t)\cdot e^{-jn\omega_0 t}dt = \frac{1}{T}\int_T \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}e^{-jn\omega_0 t}dt = \frac{1}{T}\sum_{k=-\infty}^{\infty} c_k \int_T e^{j(k-n)\omega_0 t}dt = c_n$$

- **This gives us our Fourier series pair (*ω_0=2ω/T*):**

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$ **(synthesis)**

$$c_k = \frac{1}{T}\int_T x(t)\cdot e^{-jk\omega_0 t}dt$$ **(analysis)**

Example: Periodic Pulse Train



$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-T_1}^{T_1} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\left[\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right]\Bigg|_{-T_1}^{T_1} = \frac{1}{T}\left[\frac{e^{-jk\omega_0 T_1}}{-jk\omega_0} - \frac{e^{-jk\omega_0(-T_1)}}{-jk\omega_0}\right]$$

$$= \frac{2}{k\omega_0 T}\left[\frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{2j}\right] = \frac{2}{k(2\pi/T)T}\left[\sin\left(k\omega_0 T_1\right)\right]$$

$$= \frac{\sin\left(k\omega_0 T_1\right)}{k\pi}$$