(20) Diagonalize matrix S below. Then write the difference from the matrix asked in problem 17.

$$S = \begin{bmatrix} 3/2 & -1/2 & 0 \\ -1/2 & 3/2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Solution: First, find the eigenvalues and then normalized eigenvector.

$\det(A - \lambda I) = 0 \Rightarrow$ characteristic equation.

$$\det\left(\begin{bmatrix} 3/2 & -1/2 & 0 \\ -1/2 & 3/2 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right) = 0$$

Use 3rd row.

$$\det\left(\begin{bmatrix} 3/2-\lambda & -1/2 & 0 \\ -1/2 & 3/2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix}\right) = 0 \Rightarrow (3-\lambda)\left[\left(\tfrac{3}{2}-\lambda\right)\left(\tfrac{3}{2}-\lambda\right) - \left(-\tfrac{1}{2}\right)\left(-\tfrac{1}{2}\right)\right] = 0$$

$$\Rightarrow (3-\lambda)\left[\tfrac{9}{4} - \tfrac{3}{2}\lambda - \tfrac{3}{2}\lambda + \lambda^2 - \tfrac{1}{4}\right] = 0 \Rightarrow (3-\lambda)(\lambda^2 - 3\lambda + 2) = 0$$

$$\Rightarrow (3-\lambda)(\lambda-2)(\lambda-1) = 0 \Rightarrow \lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3 \Rightarrow \text{eigenvalues.}$$

Find eigenvector u_1 related to $\lambda_1 = 1$. Use $Au_1 = \lambda_1 u_1$

$$\Rightarrow \begin{bmatrix} 3/2 & -1/2 & 0 \\ -1/2 & 3/2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{u_1} = \underbrace{1}_{\lambda_1} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow \begin{pmatrix} z = 0 \\ y = 1 \\ x = 1 \end{pmatrix} \text{ is a solution, so, } u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Now, use

$$Au_2 = \lambda_2 u_2 \Rightarrow \begin{bmatrix} 3/2 & -1/2 & 0 \\ -1/2 & 3/2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda_2 \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow u_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

and use

$$Au_3 = \lambda_3 u_3 \Rightarrow u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Then normalize u_1, u_2, u_3 by $\frac{u_1}{\|u_1\|}, \frac{u_2}{\|u_2\|}, \frac{u_3}{\|u_3\|}$ to get normalized eigenvectors

$$\Rightarrow \text{Normalized } u_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix},\ u_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix},\ u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\Rightarrow P = [u_1\ u_2\ u_3] = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ which is an orthonormal matrix. Therefore $P^{-1} = P^T$

$$\Rightarrow S = PDP^{-1} = PDP^T \text{ where } D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\Rightarrow S = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

In this question S is a symmetric matrix, unlike problem 17. So, the eigenvectors are orthogonal.

(21) Let 3 eigenvectors are found to be as below for a given matrix while diagonalizing. Check if the ~~ei~~ eigenvectors are mutually orthogonal. If not find a new set of eigenvectors which are ~~m~~ mutually orthogonal.

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ v_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix},\ v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Solution.

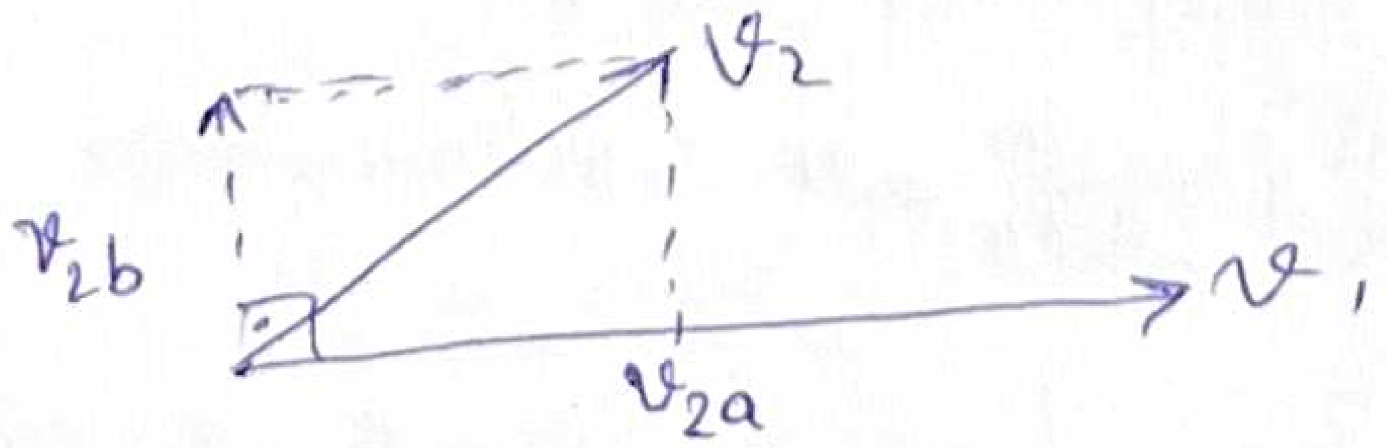
To check orthogonality of two vectors we need to check if the dot products of these vectors are equal to zero or not.

$v_1 \cdot v_2 = v_1^T v_2 = [1\ 0\ 0] \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \neq 0$ then v_1 and v_2 are not orthogonal

$v_1 \cdot v_3 = v_1^T v_3 = 0 \Rightarrow v_1$ and v_3 are orthogonal.

$v_2 \cdot v_3 = v_2^T v_3 = 0 \Rightarrow v_2$ and v_3 are orthogonal.

So, we need to find another vector which will be obtained from v_2, instead of v_2, which is orthogonal to v_1.



$$v_2 = v_{2a} + v_{2b}$$

v_{2b} is perpendicular to v_1. So, we need to find v_{2b}.

First find v_{2a} so that we can find v_{2b} as $v_2 - v_{2a}$.

v_{2a} is the projection of v_2 onto v_1.

$$v_{2a} = \frac{v_2 \cdot v_1}{v_1 \cdot v_1} v_1 \Rightarrow v_{2a} = \frac{v_2^T v_1}{v_1^T v_1} v_1$$

$$\Rightarrow v_{2a} = \frac{[2\ 2\ 0]\begin{bmatrix}1\\0\\0\end{bmatrix}}{[1\ 0\ 0]\begin{bmatrix}1\\0\\0\end{bmatrix}} \cdot \begin{bmatrix}1\\0\\0\end{bmatrix} = \frac{2}{1}\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}2\\0\\0\end{bmatrix}$$

$$\Rightarrow v_{2b} = v_2 - v_{2a} = \begin{bmatrix}2\\2\\0\end{bmatrix} - \begin{bmatrix}2\\0\\0\end{bmatrix} = \begin{bmatrix}0\\2\\0\end{bmatrix}$$ so new set of vectors are $\{v_1, v_{2b}, v_3\}$

$$\Rightarrow \left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\2\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\} \Rightarrow \text{are mutually orthogonal.}$$

(22) Find spectral decomposition of matrix $A = \begin{bmatrix}5 & 3\\3 & 5\end{bmatrix}$

Solution: First find eigenvalues, then eigenvector, then normalize eigenvectors, then write $A = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T$, where λ_1, and λ_2 are the eigenvalues and u_1 and u_2 are the eigenvectors

$\det(A - \lambda I) = 0 \Rightarrow$ characteristic equation.

$$A - \lambda I = \begin{bmatrix}5 & 3\\3 & 5\end{bmatrix} - \begin{bmatrix}\lambda & 0\\0 & \lambda\end{bmatrix} = \begin{bmatrix}5-\lambda & 3\\3 & 5-\lambda\end{bmatrix} \Rightarrow \det(A - \lambda I) = 25 - 10\lambda + \lambda^2 - 9 = 0$$

$$\Rightarrow \lambda^2 - 10\lambda + 16 = 0 \Rightarrow (\lambda-2)(\lambda-8) = 0$$

$$\Rightarrow \lambda_1 = 2, \quad \lambda_2 = 8$$

Use $Au_i = \lambda u_i$ to find u_i's.

$$Au_1 = \lambda_1 u_1 \Rightarrow \begin{bmatrix}5 & 3\\3 & 5\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = 2\begin{bmatrix}x\\y\end{bmatrix} \Rightarrow x = 1,\ y = -1 \Rightarrow u_1 = \begin{bmatrix}1\\-1\end{bmatrix}$$

$$Au_2 = \lambda_2 u_2 \Rightarrow \begin{bmatrix}5 & 3\\3 & 5\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = 8\begin{bmatrix}x\\y\end{bmatrix} \Rightarrow x = 1,\ y = 1 \Rightarrow u_2 = \begin{bmatrix}1\\1\end{bmatrix}$$

$\Rightarrow$ Normalized $u_1 = \frac{u_1}{\|u_1\|} = \begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix}$ and Normalized $u_2 = \frac{u_2}{\|u_2\|} = \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$

$$\Rightarrow A = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T = 2 \cdot \begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix}\begin{bmatrix}1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix} + 8\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}$$

$$= 2 \cdot \begin{bmatrix}1/2 & -1/2\\-1/2 & 1/2\end{bmatrix} + 8\begin{bmatrix}1/2 & 1/2\\1/2 & 1/2\end{bmatrix}$$

23 Explain spectral value decomposition (SVD).

Answer: SVD is a technique that factorizes a matrix into three components:

$$A_{m \times n} = U \Sigma V^T$$, A does not need to be symmetric

where U and V are orthogonal matrices on Σ is a diagonal matrix containing the singular values. These singular values are the square roots of eigenvalues of $A^T A$.

$A^T A$ is a symmetrical matrix. Therefore the parallel procedure is done as in problem 22, not for matrix A, but matrix $A^T A$.

Construct V as $[v_1 \dots v_2 \dots v_n]$ where v_1, v_2, and v_n are the eigenvectors of $A^T A$, related with eigenvalues in decreasing order.

Σ has the same size with A and $\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$

where D is a diagonal matrix which has eigenvalue on the diagonal.

$U = [u_1 \dots u_m]$ where $u_i = \frac{1}{\sigma_i} A v_i$

σ_i → singular values.

SVD is used for dimensionality reduction, noise reduction, text processing, solving ill-conditioned systems etc.

SVD decomposes a matrix into orthogonal directions that explain the variance in the data. It identifies the directions (basis vectors) where data has the most information. This can be helpful for understanding patterns, compressing data, and improving numerical stability.

24 During a braking test, the position of a car is recorded every 0,5 seconds after the driver hits the brakes, as below

Time (s)	0,0	0,5	1,0	1,5	2,0	2,5
Position (m)	0,0	13,5	25,0	33,0	37,5	39

i) Construct a first-order difference matrix D_1 for equally measurements ($\Delta t = 0{,}5$ s). Use it to compute the velocity vector $v = Dx$

ii) Construct a second order difference matrix D_2. Use it to compute the accelaration vector $a = D_2 x$.

iii) Interpret the results:
- What do the velocity and acceleration values tell you about the braking process?
- Is the braking smooth or abrupt?

Solution: Given position data: $x = [0{,}0 \;\; 13{,}5 \;\; 25 \;\; 33 \;\; 37{,}5 \;\; 39]^T$

and $\Delta t = 0{,}5$

i) First order difference matrix (velocity)

$$D_1 = \frac{1}{\underset{0{,}5}{\Delta t}} \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \Rightarrow v = D_1 x = \begin{bmatrix} 27 \\ 23 \\ 16 \\ 9 \\ 3 \end{bmatrix} \text{ m/s.}$$

The car is slowing down and the velocity decreases steadily

ii) Second order difference matrix (acceleration)

$$D_2 = \frac{1}{(\Delta t)^2} \begin{bmatrix} 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{bmatrix} \Rightarrow a = D_2 x = \begin{bmatrix} -8 \\ -14 \\ -14 \\ -12 \end{bmatrix} \text{ m/s}^2$$

iii) The car is slowing down. Acceleration is negative and roughly consistent. The braking is smooth

(25) A computer system distributes its computational load among three main components as:

$x =$ CPU usage

$y =$ GPU usage

$z =$ memory usage.

The total system cost function is modeled as:

$$C(x,y,z) = x^2 + 2y^2 + 3z^2 + xy + yz - 10x - 8y - 6z + 50$$

i) Analyze how the system can operate most efficiently.

ii) - Evaluate the gradient at $(x,y,z) = (1,1,1)$

- What does the direction of gradient indicate?
- In which direction should each resource be adjusted to reduce the cost?

iii) Starting from $(x,y,z) = (1,1,1)$, compute the new resource values after one step, if the system is updated its parameters using gradient descent algorithm with a learning rate $\eta = 0.1$.

Solution:

i) Gradient of C is $\nabla C = \left[\frac{\partial C}{\partial x} \quad \frac{\partial C}{\partial y} \quad \frac{\partial C}{\partial z}\right]$

$\Rightarrow \nabla C = [2x+y-10 \quad 4y+x+z-8 \quad 6z+y-6]$

$\Rightarrow$ Solution of $\nabla C = 0$ will be the best (minimum) point to operate $\Rightarrow$ ~~2x+y-10=0~~ ~~x+4y+~~

$$\left.\begin{aligned} 2x+y &= 10 \\ x+4y+z &= 8 \\ y+6z &= 6 \end{aligned}\right\} \Rightarrow \begin{aligned} x &= 4{,}7 \\ y &= 0{,}6 \\ z &= 0{,}9 \end{aligned}$$

$\Rightarrow$ optimum point $(x,y,z) = (4{,}7 \quad 0{,}6 \quad 0{,}9)$

ii) $\rightarrow \nabla C(1,1,1) = [2.1+1-10 \quad 4.1+1+1-8 \quad 6.1+1-6] = [-7 \quad -2 \quad 1]$

$\rightarrow$ Direction of gradient indicates the increase in C.

$\rightarrow$ to reduce the cost increase x (CPU usage), increase y (GPU usage) and decrease z (memory usage)

iii) Gradient descent method $\Rightarrow (x,y,z)_{new} = (x,y,z) - \eta \nabla C(x,y,z)$

$\Rightarrow (x,y,z)_{new} = (1,1,1) - \underset{\eta}{0{,}1}(-7 \quad -2 \quad 1) = (1,1,1) + (0{,}7 \quad 0{,}2 \quad -0{,}1)$

$= (1{,}7 \quad 1{,}2 \quad 0{,}9)$