Homework 3: Ordinary Differential Equations (ODEs): physics at work

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Assignment

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1. Consider the problem of planetary motion around the Sun, as discussed during lecture 8.

   (a) Write a C++ code, using Runge-Kutta at fourth order (RK4) to describe the motion of Earth and Jupiter around the Sun. You will use the RK4 library provided in class.

   - First discuss the trajectories when Jupiter and Earth do not interact. (Hint: you will have 2 bodies, 2 spatial variables per body and no cross-terms)
   - Then, turn on the Jupiter-Earth gravitational interaction. (Hint: same as before but with a cross-term between Jupiter and Earth)
   - In both cases, use realistic numbers for the orbits (size of the orbit, velocity at aphelion or perihelion, etc...). Verify Kepler's laws and compute periodicity.
   - Repeat your calculations, by artificially increasing the mass of Jupiter by a factor of 1,000. Discuss the effect on the orbits. Make sure you make clear plots of the orbits.

   Note: it is a good idea to use astronomical units.

   (b) Study the precession of the perihelion of Mercury due to general relativity corrections to the $1/r^2$ classical gravitational law. Use the following equation of the modified force:

   $$F_G = \frac{GM_S M_M}{r^2}(1 + \frac{\alpha}{r^2})$$

   where M_M and M_S correspond to the mass of Mercury and the Sun, respectively. Use $\alpha \sim 10^{-8}$ as a small coefficient accounting to relativity corrections.

   - Plot the trajectory of Mercury for a given $\alpha = 0.01AU^2$. Draw the line between the Sun and the closest approach of Mercury for a few trajectories. The change in direction indicates the change of orientation of the perihelion.
   - Plot the orbit's orientation change with time for $\alpha = 0.0008AU^2$.

   In all these calculations, do not consider the effect of the planets on the Sun (Sun is static); choose initial conditions properly, test your step size h carefully.

2. Select a problem of your choice from any physics class (or book) where ODEs cannot be solved analytically. Present the physics of the problem carefully and make a case for the solution you obtained, in light of the conditions you selected.

Make sure your report is self-contained with sufficient details and clear plots. Feel free to add listings of your codes (or use pseudo-code).

Assignment 1

Problem 1: Planetary Motion

1.1 Algorithmic Consideratoins

The problem of planetary motion can be solved by using Newton's law of universal gravitation and the second law of motion. It is possible to simulate planets that follow trajectories very similar to the real paths of the planets. What happens when these planets interact with each other? are their orbits different if they do not interact? to answer these questions we must first establish equations to describe theses systems. First let us write the equation of motion of both Jupiter and Earth in the absence of any interactions between them. The equations of motion are as follows:

$$\frac{d^2x}{dt^2} = -GM_s\frac{x}{r^3} \tag{1.1}$$

$$\frac{d^2y}{dt^2} = -GM_s\frac{y}{r^3} \tag{1.2}$$

Where M_s is the mass of the sun and G is the gravitational constant. These two differential equations, when coupled, describe the motion of an object in the presence of the sun. To simulate the orbits of two bodies that interat with each other than we need to include some cross terms. The following equations arise from such considerations:

$$\frac{d^2x}{dt^2} = -GM_s\frac{x_e}{r_e^3} - GM_j\frac{x_j - x_e}{\left((x_j - x_e)^2 + (y_j - y_e)^2\right)^{3/2}} \tag{1.3}$$

$$\frac{d^2y}{dt^2} = -GM_s\frac{y_e}{r_e^3} - GM_j\frac{y_j - y_e}{\left((x_j - x_e)^2 + (y_j - y_e)^2\right)^{3/2}} \tag{1.4}$$

where M_j, x_j and y_j are the mass, x, and y coordinates of Jupiter, respectively. The same notation is used for the Earth coordinates. These equations only represent the forces acting on Earth by both the sun and Jupiter. For jupiter, similar equations are written. The difference lays mainly on the second cross term, where the vector that specifies the direction of the force are negative and the mass of the Earth is used instead. Finally we can represent the precession due to relativity by modifying the force term as follows:

$$\frac{d^2x}{dt^2} = -GM_s\frac{x}{r^3}(1+\frac{\alpha}{r^2}) \tag{1.5}$$

$$\frac{d^2y}{dt^2} = -GM_s\frac{y}{r^3}(1+\frac{\alpha}{r^2}) \tag{1.6}$$

Where alpha is a corrective constant that is included to modify the orbit. We will use equations 1.5 and 1.6 on the orbit of Mercury in the simulation and observe how much Mercury's orbit changes. All of the above equations can be solved if we reduce them to a system of equations of first order. The end result is called Canonical form and it is written as follows:

$$\begin{aligned}
\frac{dy^{(0)}}{dt} &= f^{(0)}(t, y^{(i)}) \\
\frac{dy^{(1)}}{dt} &= f^{(1)}(t, y^{(i)}) \\
&\vdots \\
\frac{dy^{(N-1)}}{dt} &= f^{(N-1)}(t, y^{(i)})
\end{aligned}$$

When implementing this system of equations, it is important to take into account our definitions for $y^{(0)}$ and $y^{(1)}$ as these will determine if the system of equations will be solvable. We must also understand how many equations we will need to find a solution. For our above problems, some will require as little as a system of four equations, and some will need as many as 8. It all depends on the complexity and, in our case, the number of interactions that we wish to take into account.

1.2 Implementation

Implementation of the equations introduced in the previous sections involved the use of mathematical libraries that would allow us to make the neccessary calculations. These libraries were rk4.h and nr3.h, both of which are necessary to apply what is called fourth order Runge-Kutta. To use these libraries we must first change the equations into a system of linear equations. This is done by defining the following:

$$y^{(0)}(t) = x(t) \tag{1.7}$$

$$y^{(1)}(t) = \frac{dx}{dt} = \frac{dy^{(0)}}{dt} \tag{1.8}$$

using the above definitions we can transform the differential equation for gravitational force into a system of four equations:

$$\begin{aligned}
\frac{dy^{(0)}}{dt} &= y^{(1)} \\
\frac{dy^{(1)}}{dt} &= -M_s G \frac{y^{(0)}}{((y^{(0)})^2 + (y^{(2)})^2)^{3/2}} \\
\frac{dy^{(2)}}{dt} &= y^{(3)} \\
\frac{dy^{(3)}}{dt} &= -M_s G \frac{y^{(2)}}{((y^{(0)})^2 + (y^{(2)})^2)^{3/2}}
\end{aligned}$$

Where $y^{(0)}$ and $y^{(2)}$ represent the planet's horizontal and vertical position, respectively. Despite us using y for horizontal positions, it is important to remember that it is simply a variable that holds a value in the algorithm. It's definition must be remembered and noted in the code so as to not confuse it with it's usual definition. The same, but independent, system of equations is used for the motion of Jupiter in the problem. We can clearly see that these systems of equations have no other interactions other thant the sun, and even though these two planetary objects may be in the same space, they will not demonstrate effects present if small cross interactions are present. Including these interactions requires us to combine these two systems into one system, where factors such as position of one of the planets plays a key role in the forces acting on another. To obtain this we must first make definitions for our position and velocity variables:

$$\begin{aligned}
y^{(0)}(t) &= x_e(t) \\
y^{(1)}(t) &= \frac{dx_e(t)}{dt} \\
y^{(2)}(t) &= y_e(t) \\
y^{(3)}(t) &= \frac{dy_e(t)}{dt} \\
y^{(4)}(t) &= x_j(t) \\
y^{(5)}(t) &= \frac{dx_j(t)}{dt} \\
y^{(6)}(t) &= y_j(t) \\
y^{(7)}(t) &= \frac{dy_j(t)}{dt}
\end{aligned}$$

Each coordinate and velocity is related to a subscript: e for earth and j for jupiter. We use these subscripts to identify what value the variable corresponds to. Applying these definitions to our reduction method gives us the following system of equations:

$$
\begin{aligned}
\frac{dy^{(0)}}{dt} &= y^{(1)} \\
\frac{dy^{(1)}}{dt} &= -GM_s\frac{y^{(0)}}{((y^{(0)})^2+(y^{(2)})^2)^{3/2}} - GM_j\frac{y^{(0)}-y^{(4)}}{((y^{(0)}-y(4))^2+(y^{(2)}-y^{(6)})^2)^{3/2}} \\
\frac{dy^{(2)}}{dt} &= y^{(3)} \\
\frac{dy^{(3)}}{dt} &= -GM_s\frac{y^{(2)}}{((y^{(0)})^2+(y^{(2)})^2)^{3/2}} - GM_j\frac{y^{(2)}-y^{(6)}}{((y^{(0)}-y(4))^2+(y^{(2)}-y^{(6)})^2)^{3/2}} \\
\frac{dy^{(4)}}{dt} &= y^{(5)} \\
\frac{dy^{(5)}}{dt} &= -GM_s\frac{y^{(4)}}{((y^{(4)})^2+(y^{(6)})^2)^{3/2}} - GM_e\frac{y^{(4)}-y^{(0)}}{((y^{(4)}-y(0))^2+(y^{(6)}-y^{(2)})^2)^{3/2}} \\
\frac{dy^{(6)}}{dt} &= y^{(7)} \\
\frac{dy^{(7)}}{dt} &= -GM_s\frac{y^{(6)}}{((y^{(4)})^2+(y^{(6)})^2)^{3/2}} - GM_e\frac{y^{(6)}-y^{(2)}}{((y^{(4)}-y(0))^2+(y^{(6)}-y^{(2)})^2)^{3/2}}
\end{aligned}
$$

The above array, despite being filled with cross terms, is quite simple to arrive to. Imput into code must be done with care to avoid any mistakes. The last part of the problem, simulating the relativistic effects on mercury's orbit, is very similar to our first part of the problem, and follows the same method. The resulting system of equations is as follows:

$$
\begin{aligned}
\frac{dy^{(0)}}{dt} &= y^{(1)} \\
\frac{dy^{(1)}}{dt} &= -M_sG\frac{y^{(0)}}{((y^{(0)})^2+(y^{(2)})^2)^{3/2}}\left(1+\frac{\alpha}{(y^{(0)})^2+(y^{(2)})^2}\right) \\
\frac{dy^{(2)}}{dt} &= y^{(3)} \\
\frac{dy^{(3)}}{dt} &= -M_sG\frac{y^{(2)}}{((y^{(0)})^2+(y^{(2)})^2)^{3/2}}\left(1+\frac{\alpha}{(y^{(0)})^2+(y^{(2)})^2}\right)
\end{aligned}
$$

1.3 Results

After we determinte the form of the system of equations for the various situations then all that is left is to graph these solutions and interpret them. First is the graph of earth and jupiter's orbit around the sun. Here we do not take into consideration their interactions:

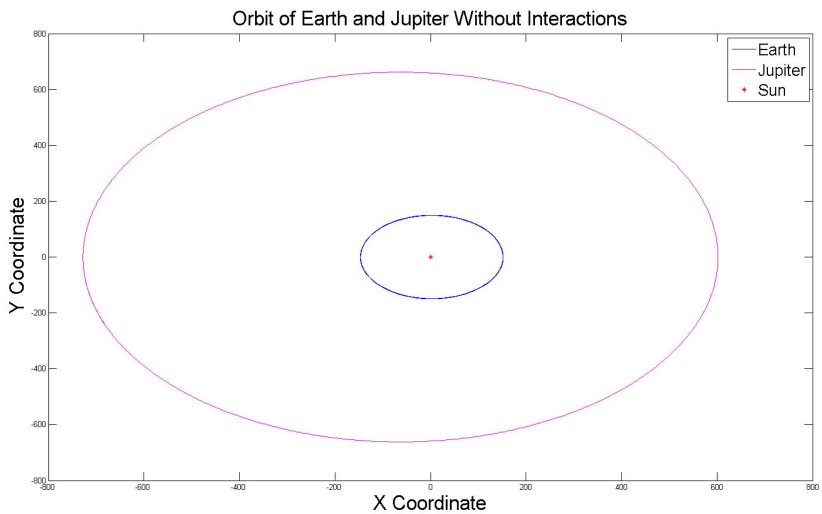

Figure 1.1: Earth and Jupiter's orbit around the sun. Here they do not interact with each other.

Now let's consider interactions:

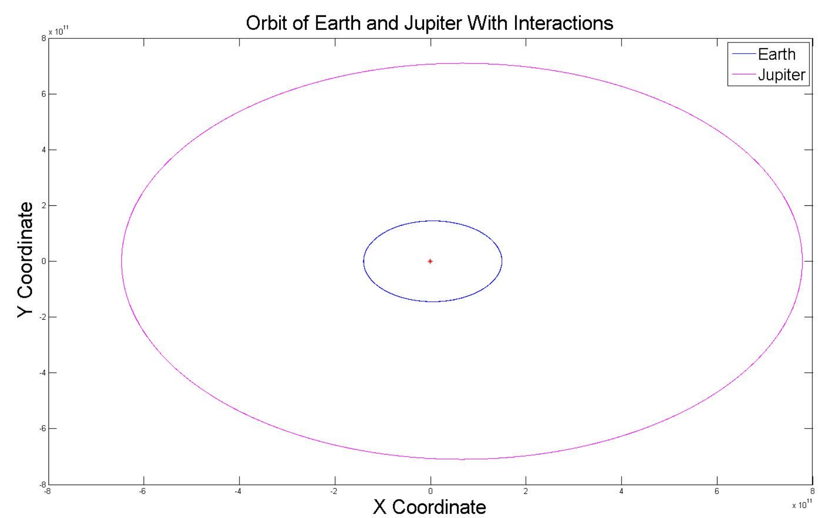

Figure 1.2: Earth and Jupiter's orbit around the sun in the presence of interactions between them.

At first we aren't able to see anything, but if we were to zoom in their individual orbits, we can see small deviations from their orbits. They are small but still present:

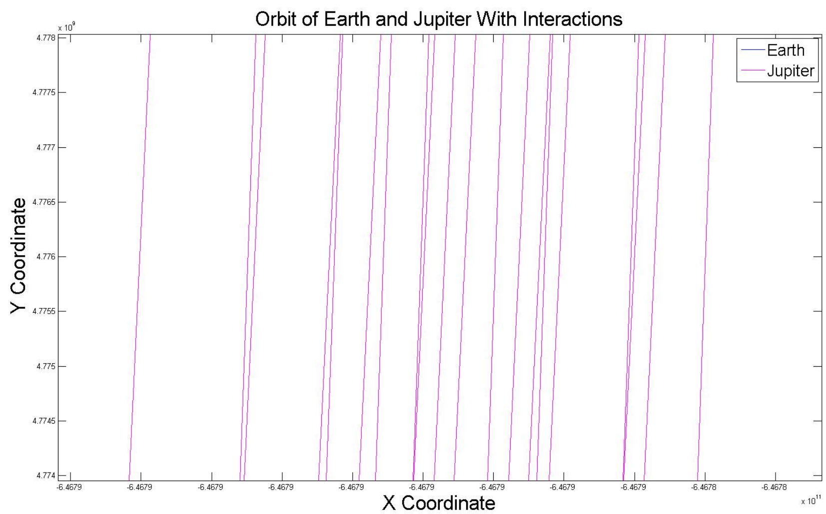

Figure 1.3: A zoom in of Jupiter's orbit. We can clearly see that there are small deviations of what would otherwise be a "pefect" orbit.

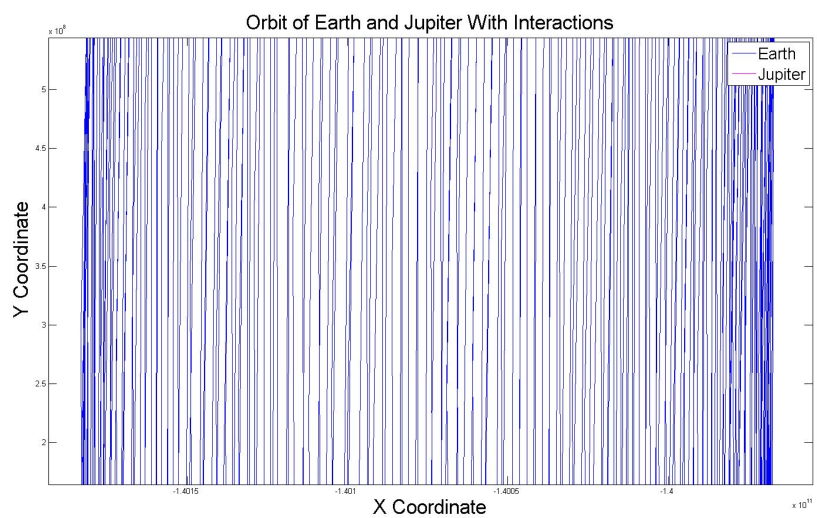

Figure 1.4: Enalrged image of earth's orbit. Like Jupiter, Earth's orbit also deviates slightly, even more so than jupiter.

We can see from the scales that earth is affected much more than jupiter in terms of orbit trajectory. Now what if jupiter was 1000 times it's current mass? with the current initial conditions we get a very unstable Earth orbit:

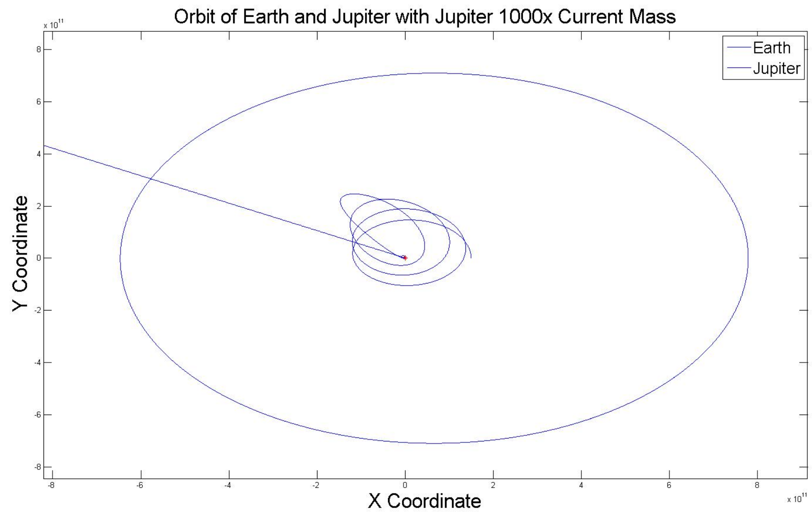

Figure 1.5: Earth's orbit around the sun when Jupiter is 1000x it's current mass. We can see how unstable it is, and with every pass around the sun, it's eccentricity increases, and it coincides with the location of jupiter.

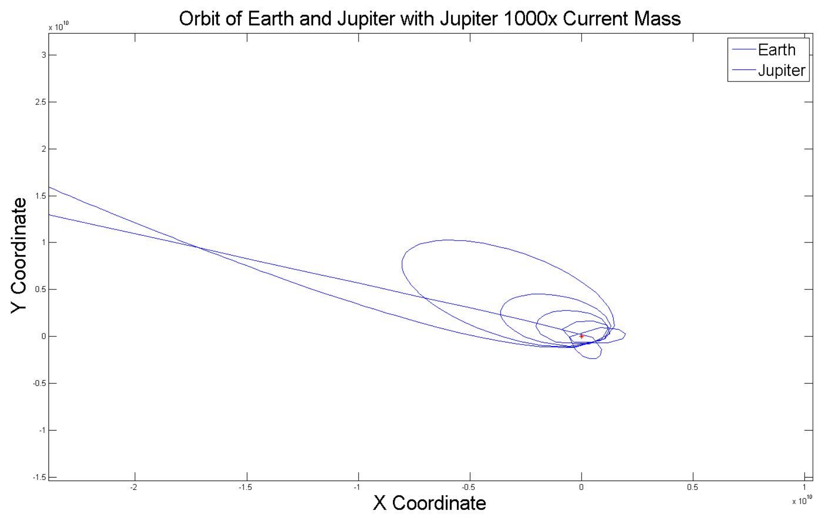

Figure 1.6: This is a zoom in of the previous image. Here earth is still orbiting very closely to the sun.

Finally we have precessions of mercury's orbit around the sun:

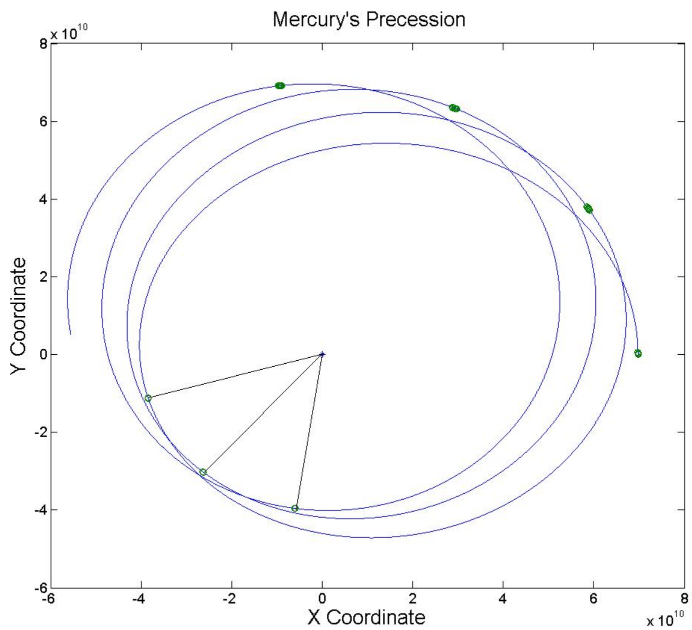

Figure 1.7: This graph shows the trajectory of mercury around the sun with $\alpha = 0.01AU^2$. The lines from the sun to the trajectory mark the direction of the perihelion of the orbit. The other circles mark the locations where the the aphelion would be located.

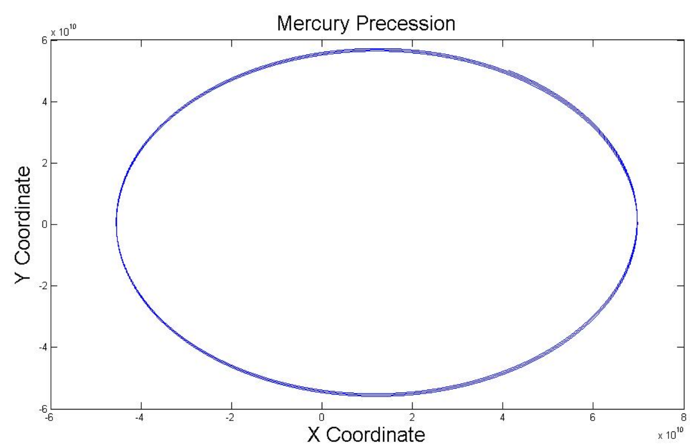

Figure 1.8: Mercury's trajectory for $\alpha = 0.0008AU^2$. Compared to higher values of α, this demonstrates a much slower precessoin around the sun.

1.4 Discussion

These problems show us the complexity of interactions of bodies in space due to gravity. This shows the usefulness of these methods of solving ODEs where pen and paper would not be enough to obtain a solution. When picking the intitial conditions such as the mass of the sun and the distances of the planets from the star, it was neccessary to choose appropriate units for these calculations. One possible outcome when not using the right units is the absence of an orbit altogether! Care must be taken when dealing with values as large as the ones used for this program. It is also interesting to note that the mass of the object does not matter when calculating their orbits. This can easily be noticed when looking at the equation of motion for the earth without interactions. Another way of interpreting this is by remembering that "All objects fall at the same rate, no matter how heavy/massive they are". With respect to the sun, a heavier earth would follow the same path as a lighter earth, assuming their initial conditions such as initial distance and velocity remain the same for both bodies.

Assignment 2

Problem 2: Lorent'z Forces

2.1 Algorithmic Considerations

The differential equations that govern the motion of an electron are very similar to the gravitational differential equations used on previous problems. Here the mass of the particle does matter. We will exlore the motion an electron around an elemntary positive electric charge using classical equations of motion for the fores on a charge in a presence of a magnetic field. Though charges might behave almost like planets in the absence of a magnetic field, their paths change drastically when there is a magnetic field present. The Lorentz force on an electron is given by:

$$\vec{F} = q_e\vec{E} + q(\vec{v_e} \times \vec{B}) \tag{2.1}$$

Applying newton's third law and expanding the cross product leaves us with a system of three differential equations:

$$\frac{d^2x}{dt^2} = \frac{q_e}{m_e}E_x + \frac{q_e}{m_e}(B_z\frac{dy}{dt} - B_y\frac{dz}{dt}) \tag{2.2}$$

$$\frac{d^2y}{dt^2} = \frac{q_e}{m_e}E_y + \frac{q_e}{m_e}(B_z\frac{dx}{dt} - B_x\frac{dz}{dt}) \tag{2.3}$$

$$\frac{d^2z}{dt^2} = \frac{q_e}{m_e}E_z + \frac{q_e}{m_e}(B_y\frac{dx}{dt} - B_x\frac{dy}{dt}) \tag{2.4}$$

$$\tag{2.5}$$

The above equations work for a charge in the presence of a static electric and magnetic field. Unfortunately these do not include quantum effects or relativistic effects, as we will only be considering the behaviour of the particle using classical equations. In this problem we will also keep the magnetic field at constant values, as their values cannot be yet calculated with the current tools that we have.

2.2 Implementation

We will implement the same methods discussed in the earlier problem to transform these equations into a computational form. In the end we obtain 9 equations, and with this we end up having to define 9 different initial conditions: 3 coordinates of position and three of velocity:

$$
\begin{aligned}
\frac{dy^{(0)}}{dt} &= y^{(1)} \\
\frac{dy^(1)}{dt} &= \frac{q_e}{m_e}\frac{q_p sin(\phi)cos(\theta)}{r} + \frac{q_e}{m_e}(B_z\frac{dy^{(2)}}{dt} - B_y\frac{dy^{(4)}}{dt}) \\
\frac{dy^{(2)}}{dt} &= y^{(3)} \\
\frac{dy^(3)}{dt} &= \frac{q_e}{m_e}\frac{q_p sin(\phi)sin(\theta)}{r} + \frac{q_e}{m_e}(B_z\frac{dy^{(0)}}{dt} - B_y\frac{dy^{(4)}}{dt}) \\
\frac{dy^{(4)}}{dt} &= y^{(5)} \\
\frac{dy^(5)}{dt} &= \frac{q_e}{m_e}\frac{q_p cos(\phi)}{r} + \frac{q_e}{m_e}(B_z\frac{dy^{(0)}}{dt} - B_y\frac{dy^{(2)}}{dt})
\end{aligned}
\tag{2.6}
$$

where we defined earlier in the program what $cos(\theta)$, $cos(\phi)$, $sin(\theta)$, $sin(\phi)$, and r are in terms of the variables $y^{(i)}$.

2.3 Results

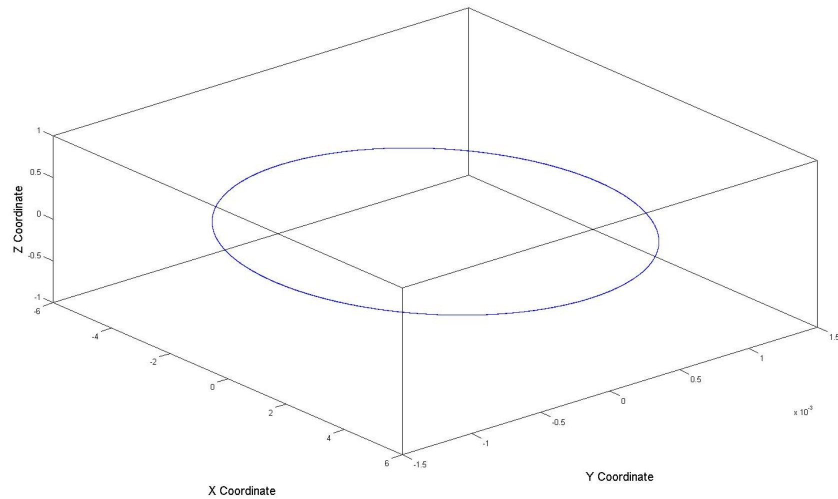

Figure 2.1: This is the orbit of a negative charge around a positive charge in the absence of a magnetic field. As you can see, the initial conditions here result in an orbit that closely resembles that of a motion of a planet around it's star. The initial conditions place the negative charge on the x axis with a purely y velocity.

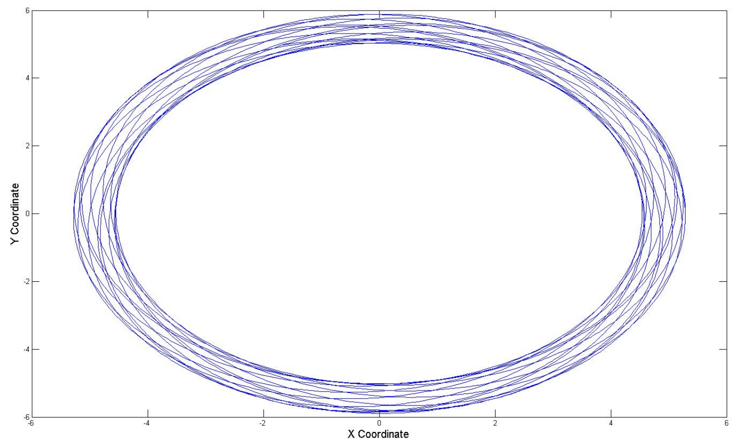

Figure 2.2: This shows how much the orbit of an electron is changed by the presence of a magnetic field in the system. Here we have only a tangential magnetic field B_z, while the other components are zero. We can see an interesting effect with the limits of the charge's orbit.

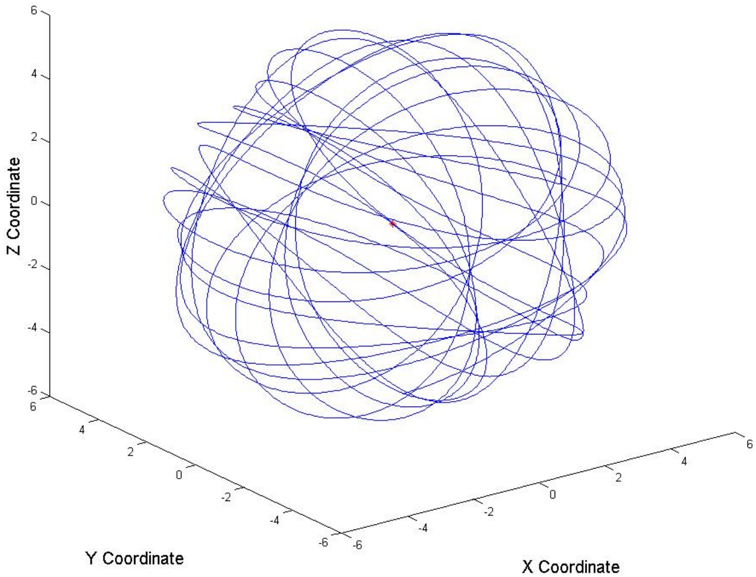

Figure 2.3: The negative charge seems to interact in more ways than one in this image. The magnetic fiels seems to resonate somehow with the the initial conditions to give a more stable orbit.

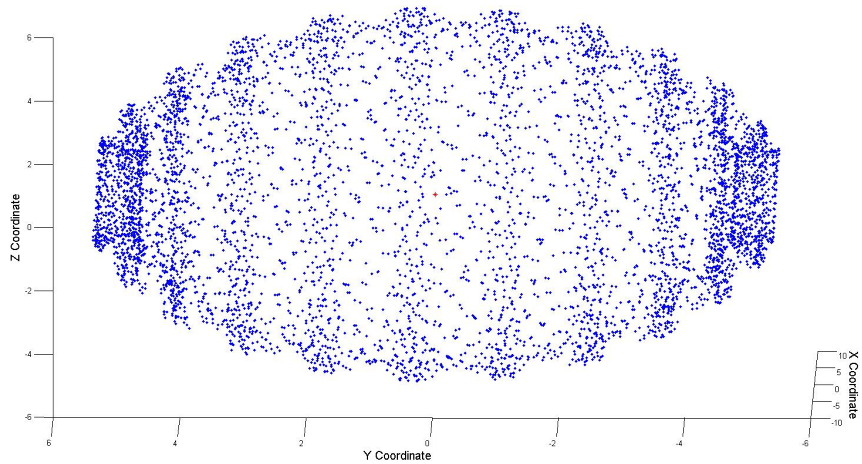

Figure 2.4: This shows the same initial conditions and magnetic field as before, except that we run this for longer periods of times and the lines connecting each point are removed for rendering purposes. The result is a very neat pattern of motion observed laterally.

2.4 Discussion

The above shows some of the many effects that magentic fields can have on moving electrons. Despite using Lorentz forces, we are still far from simulating what we would consider an atom or the motion of a particle. This is due to the fact that we did not consider quantum or relativistic effects. Nevertheless further effects should be studied when additional tools are available, such as methods of solving partial differential equations.

2.5 References

Planetary Fact Sheet(http://nssdc.gsfc.nasa.gov/planetary/factsheet/) wikipedia article on electric natural units(http://en.wikipedia.org/wiki/Lorentz