Procrustean Photogrammetry: From Exterior Orientation to Bundle Adjustment

Fabio Crosilla, Eleonora Maset and Andrea Fusiello

Abstract This work reviews the anisotropic row-scaling variant of the Procrustes analysis algorithms applied to develop new analytical tools for solving classical photogrammetric and Computer Vision problems. In [10], the anisotropic row-scaling Procrustes analysis was first applied to perform the *exterior orientation* of one image. Moreover, Fusiello and Crosilla [6] provided a Procrustean formulation of the photogrammetric *bundle block adjustment* problem. Procrustean methods do not require any linearization nor approximated values of the unknown parameters and the results obtained are comparable in terms of accuracy with those given by the state-of-the-art methods.

1 Introduction

Procrustes analysis is a well known least squares technique used to directly perform transformations among corresponding point coordinates belonging to a generic k-dimensional space. Applied at first in multifactorial analysis, shape analysis and geodesy [4], in the last decade it was also proposed in Photogrammetry [1], e.g., for matching different 3D object models from images [5]. In this case the authors applied the so called Extended Orthogonal Procrustes Analysis (EOPA) model, that considers the direct similarity transformation between two matrices, and the Generalized Procrustes Analysis (GPA) to simultaneously match more than two coordinate matrices of corresponding points expressed in different reference systems. These models will be briefly reviewed in Sec. 2.

The anisotropic row-scaling variant of the Procrustes Analysis represents instead the basic tool for an alternative solution of other classical photogrammetric problems. The analytical solution of the Anisotropic Extended Orthogonal Procrustes Analysis (AEOPA) with row-scaling, recalled in Sec. 2.3, was first derived by Garro

All the authors are with the DPIA - University of Udine, Via Delle Scienze, 208 - 33100 Udine, Italy, e-mail: (fabio.crosilla; andrea.fusiello)@uniud.it, maset.eleonora@spes.uniud.it

et al. [10], who applied it to perform the exterior orientation of one image without any a priori information. In [8] the point-line registration problem, which generalizes absolute orientation to point-line matching, was formulated as an instance of the AEOPA model, and its solution was derived. The same formulation solves the Non-Perspective-n-Point camera pose problem, that in turn generalizes exterior orientation to non-central cameras, i.e., generalized cameras where projection rays do not meet in a single point. A generalized version of AEOPA leads instead to the Procrustean solution of the classical bundle block adjustment, derived by Fusiello and Crosilla [6], also in its robust version [7]. These algorithms will be described in detail in Sec. 3.

The main advantage of the Procrustes methods is that they furnish a simple and compact solution without any linearization of the original equations, and without any approximate value of the unknown parameters.

2 The Mathematical Tool: Procrustes Analysis

Let us start this section by summarily recalling the solutions of the isotropic Procrustes analysis and of its generalized version, which are helpful to introduce the anisotropic variant, on which the entire Procrustean Photogrammetry is based.

2.1 Extended Orthogonal Procrustes Analysis (EOPA)

EOPA allows to directly recover the least squares similarity transformation between two point sets. Let us consider two matrices P and S containing two sets of numerical data, e.g., the coordinates of n points of $\mathbb{R}^3$ by rows. EOPA allows to directly estimate the unknown 3×3 rotation matrix R, the translation vector $\mathbf{c}$ and the global scale factor ζ that solves:

$$\underset{\zeta,\mathbf{c},R}{\text{minimize}} \quad \|S - \zeta P R - \mathbf{1}\mathbf{c}^\mathsf{T}\|_F^2 \tag{1}$$

subject to the orthogonality constraint $R^\mathsf{T} R = R R^\mathsf{T} = I$, where $\mathbf{1}$ is the all-ones vector and $\|\cdot\|_F$ denotes the Frobenius norm. As demonstrated in [12], the rotation matrix that solves problem (1) is given by $R = U \operatorname{diag}(1,1,\det(UV^\mathsf{T}))\, V^\mathsf{T}$, where U and V are determined from the SVD decomposition $P^\mathsf{T}(I - \mathbf{1}\mathbf{1}^\mathsf{T}/n)\, S = UDV^\mathsf{T}$. The $\det(UV^\mathsf{T})$ normalization guarantees that R is not only orthogonal but has positive determinant. Then the scale factor can be determined by:

$$\zeta = \frac{\operatorname{tr}(R^\mathsf{T} P^\mathsf{T}(I - \mathbf{1}\mathbf{1}^\mathsf{T}/n)\, S)}{\operatorname{tr}(P^\mathsf{T}(I - \mathbf{1}\mathbf{1}^\mathsf{T}/n)\, P)} \tag{2}$$

and finally the translation can be computed as $\mathbf{c} = (S - \zeta P R)^\mathsf{T}\, \mathbf{1}/n$.

2.2 Generalized Procrustes Analysis (GPA)

Generalized Procrustes Analysis (GPA) [11] is a well-known technique that generalizes the classical EOPA [12] to the alignment of more than two point sets, represented as matrices. It minimizes the following least squares objective function:

$$\underset{\zeta_i,\mathbf{c}_i,R_i}{\text{minimize}} \quad \sum_{i=1}^{m} \sum_{j=i+1}^{m} \|(\zeta_i P_i R_i + \mathbf{1}\mathbf{c}_i^\mathsf{T}) - (\zeta_j P_j R_j + \mathbf{1}\mathbf{c}_j^\mathsf{T})\|_F^2 \tag{3}$$

subject to $R_i^\mathsf{T} R_i = R_i R_i^\mathsf{T} = I$, where $P_1, P_2, \ldots, P_m$ are the m matrices that contain (by rows) the same set of n points in m different coordinate systems. The degenerate solution $\zeta_i = 0\ \forall i$ must be avoided by imposing some constraints on ζ_i.

The GPA objective function has an alternative formulation in terms of the centroid. Let $P_i' = \zeta_i P_i R_i + \mathbf{1}\mathbf{c}_i^\mathsf{T}$, the following equivalence holds [3]:

$$\sum_{i<j}^{m} \|P_i' - P_j'\|_F^2 = m \sum_{i=1}^{m} \|P_i' - \hat{S}\|_F^2, \tag{4}$$

where $\hat{S} = \frac{1}{m} \sum_{i=1}^{m} P_i'$ is the centroid of the group of matrices, or mean shape: its rows are the coordinates of the geometric centroid of corresponding transformed points.

By comparing formula (4) with the objective function (3), it is possible to define a solving criterion based on iterative computation of the centroid, that leads to great advantages in terms of simplicity and efficiency. In summary, these are the only two steps that are repeated until stabilization of the centroid $\hat{S}$:

a) solve for the similarity transformations for each matrix P_i' with respect to the centroid $\hat{S}$;
b) compute the centroid $\hat{S}$ following the sequential updating of matrices P_i'.

Step a) is a simple EOPA model, whose solution was formulated in Sec.2.1.

2.3 Anisotropic EOPA (AEOPA)

A first extension of the EOPA with anisotropic scaling along space dimensions was proposed by [2]. Another variant is the one known as AEOPA with row scaling, where each data point or measurement can be scaled independently of the others. In this case, the isotropic scale factor that characterizes EOPA is substituted by an anisotropic scaling represented by a $n \times n$ diagonal matrix Z of different scale values.

The model is the following

$$S = ZPR + \mathbf{1}\mathbf{c}^\mathsf{T} \tag{5}$$

and the unknowns R, $\mathbf{c}$ and Z can be retrieved solving the problem:

$$\underset{Z,\mathbf{c},R}{\text{minimize}} \quad \|S - ZPR - \mathbf{1}\mathbf{c}^\mathsf{T}\|_F^2 \tag{6}$$

under the orthogonality constraint $R^\mathsf{T}R = RR^\mathsf{T} = I$. Whereas in the classical solution of the EOPA problem one can recover first R, that does not depend on the other unknowns, then the isotropic scale ζ and finally $\mathbf{c}$, in the anisotropic case the unknowns are entangled in such a way that one must resort to the so called *block relaxation* scheme, where each variable is alternatively estimated while keeping the others fixed. The algorithm can be therefore seen as iterating between two stages, namely:

- assuming known Z, apply the EOPA solution to find $R = U\,\mathrm{diag}\,(1,1,\det(UV^\mathsf{T}))\,V^\mathsf{T}$, with $ZP^\mathsf{T}\,(I - \mathbf{1}\mathbf{1}^\mathsf{T}/n)\,S = UDV^\mathsf{T}$, and $\mathbf{c} = (S - ZPR)^\mathsf{T}\,\mathbf{1}/n$;
- given R and $\mathbf{c}$, solve for

$$Z = \left(I \circ PP^\mathsf{T}\right)^{-1} \left(PR(S^\mathsf{T} - \mathbf{c}\mathbf{1}^\mathsf{T}) \circ I\right) \tag{7}$$

  where $\circ$ represents the Hadamard, or element-wise, product.

The rigorous derivation of the above formulas can be found in [10], where problem (6) is rewritten in terms of a Lagrangian function F and solved by setting to zero the partial derivatives of F with respect to the unknowns.

However, expression (7) for the depth matrix Z can be derived in an alternative and more intuitive way, as follows. Assuming that R and $\mathbf{c}$ are known and defining $Y = R\,(S^\mathsf{T} - \mathbf{c}\mathbf{1}^\mathsf{T})$, (5) becomes

$$P^\mathsf{T}Z = Y. \tag{8}$$

Exploiting the Khatri-Rao product, denoted with $\odot$, and its property involving diagonal matrices and the vec operator[1], Eq. (8) can be written as

$$(I \odot P)\,\mathrm{diag}^{-1}(Z) = \mathrm{vec}\,Y \tag{9}$$

where diag^{-1} returns a vector containing the diagonal elements of its argument. In the over-determined case, the least squares solution of (9) is given by

$$\mathrm{diag}^{-1}(Z) = \left[(I \odot P)^\mathsf{T}\,(I \odot P)\right]^{-1} (I \odot P)^\mathsf{T}\,\mathrm{vec}\,Y \tag{10}$$

which is equivalent to the following more compact formulation applying the Hadamard product:

$$\mathrm{diag}^{-1}(Z) = \left(I \circ PP^\mathsf{T}\right)^{-1} \mathrm{diag}^{-1}(PY). \tag{11}$$

It is easy to see that (11) corresponds to the original solution (7) reported in [10].

[1] The vec operator transforms a matrix into a vector by stacking its columns.

3 Procrustean Photogrammetry

Some of the most common problems in Photogrammetry, such as the *exterior orientation* and the *bundle block adjustment*, can be formulated in terms of an instance of AEOPA and of its generalized version, as shown in [10] and [6]. In this section, the Procrustes solutions of these problems will be reviewed, highlighting the advantages of Procrustes methods.

3.1 Exterior Orientation

Given at least three control points and their projections, the *exterior orientation* problem, a.k.a. the *Perspective-n-Point (PnP)* camera pose problem in Computer Vision, requires to find a rotation matrix R and a vector $\mathbf{c}$ (specifying attitude and position of the camera) such that the vector form of collinearity equations:

$$\mathbf{p}_j = \zeta_j^{-1} R(\mathbf{s}_j - \mathbf{c}) \tag{12}$$

is satisfied for some positive scalar ζ_j, where $\mathbf{s}_j$ is the coordinate vector of the j-th control point in the external system, ζ_j is a positive scalar proportional to the "depth" of the point, i.e., the distance from the j-th control point to the plane containing the projection center and parallel to the image plane, $\mathbf{p}_j$ is the coordinate vector of the j-th control point in the camera system, where the third component is equal to $-c$, the principal distance or focal length.

Expressing (12) with respect to $\mathbf{s}_j$ and extending it to n control points $\mathbf{s}_1 \ldots \mathbf{s}_n$, one obtains $S = ZPR + \mathbf{1}\mathbf{c}^\mathsf{T}$, where P is the matrix by rows of image point coordinates defined in the camera frame, S is the matrix by rows of point coordinates defined in the external system and Z is the diagonal (positive) depth matrix. One can recognize an instance of the AEOPA model (5) that can be solved using the procedure described in Sec. 2.3. The *Procrustean PnP (PPnP)* algorithm can be therefore summarized as in Alg. 1.

Algorithm 1 PPnP

Input: control points S and their image coordinates P
Output: position $\mathbf{c}$ and attitude R of the camera

1. Start with any $Z > 0$
2. Compute $R = U \operatorname{diag}(1,1,\det(UV^\mathsf{T}))V^\mathsf{T}$
   with $UDV^\mathsf{T} = P^\mathsf{T} Z (I - \mathbf{1}\,\mathbf{1}^\mathsf{T}/n) S$
3. Compute $\mathbf{c} = (S - ZPR)^\mathsf{T}\mathbf{1}/n$
4. Compute $Z = (I \circ PP^\mathsf{T})^{-1}(PR(S^\mathsf{T} - \mathbf{c}\mathbf{1}^\mathsf{T}) \circ I)$
5. Iterate from step 2 until convergence.

We compared the *PPnP* algorithm to the classical exterior orientation solution, that minimizes the sum of squared image coordinate residuals of the collinearity equations. To carry out the simulation, $n = 30$ 3D points were randomly distributed in a sphere of unit radius centered on the origin. The camera was positioned at distances of 5 meters from the origin and the focal length was chosen so as to yield a view angle of 60° with an image size of 1000 x 1000 pixels. Different values of noise $\sigma_P = \{0, ..., 5\}$ [pixel] were added to the image coordinates obtained from the projection of the 3D points. For each setting the test was run 100 times and the mean error norm was computed. In all the experiments the initial depths were set to one. In Fig. 1 (left) the rotation errors are shown, which are computed as $\|\log(R^\mathsf{T}\hat{R})\|_F$, where R is the ground truth, $\hat{R}$ is the actual rotation and $\|\cdot\|_F$ is the Frobenius norm. Results show that the mean error for *PPnP* is only slightly higher than the error achieved by the classical exterior orientation.

Fusiello et al. [9] provided also a robust version of the *PPnP* algorithm based on Forward Search, which proved to be highly effective and accurate in detecting outliers, even for small data size or high outliers contamination.

3.2 Bundle Block Adjustment

In the *bundle block adjustment* problem it is required to simultaneously find the exterior orientation parameters of multiple images and the tie-points 3-D coordinates that minimize a geometric error. Classical photogrammetric bundle block adjustment minimizes residuals in the image plane. Formulating instead the problem in a Procrustes framework ([6], [7]), the geometric error to be minimized belongs to the 3D space and can be expressed as follows:

$$\sum_{i=1}^{m} \|S - Z_i P_i R_i - \mathbf{1}\mathbf{c}_i^\mathsf{T}\|_F^2 \tag{13}$$

where each term of the sum represents the difference vector between 3D tie-points (S) and the back-projected 2D points (P_i) based on their estimated depths (Z_i) and the estimated image attitude and position ($R_i, \mathbf{c}_i$) of image i.

The objective function (13) is very similar to the objective function of GPA (4), modulo the substitution of the scalar ζ with the corresponding depth matrix Z. As a consequence, its solution, reported in Alg. 2, is very similar to the GPA one, with the difference that the isotropic scale factor computation (Sec. 2.2) is substituted with the anisotropic scale formula borrowed from the AEOPA (Sec. 2.3). For this reason, the algorithm is called *Anisotropic Generalized Procrustes Analysis (AGPA)* [7].

In [6], both synthetic and real experiments aimed at assessing the accuracy of the *AGPA* algorithm were performed. For the synthetic case, $n = 96$ 3D tie-points were distributed on the scene, and in each of the $m = 16$ cameras, $p = 36$ points were visible. Increasing random noise with standard deviation $\sigma =$

Algorithm 2 AGPA

Input: a set of 3-D models $P_i \quad i = 1\dots m$
Output: translation $\mathbf{c}_i$, attitude R_i and scale Z_i of each model

1. Initialize $Z_i = I$ and $P'_i = P_i \quad \forall i$
2. Compute centroid $\hat{S} = \frac{1}{m}\sum_{i=1}^{m} P'_i$
3. Register each model P_i to $\hat{S}$:
   a. Compute $R_i = U\,\mathrm{diag}\,(1,1,\det(UV^\top))\,V^\top$ with $UDV^\top = P_i^\top Z_i\,(I - \mathbf{1}\mathbf{1}^\top/n)\,\hat{S}$
   b. Compute $\mathbf{c}_i = \left(\hat{S} - Z_i P_i R_i\right)^\top \mathbf{1}/n$
   c. Compute $Z_i = (I \circ P_i P_i^\top)^{-1}(P_i R_i(\hat{S}^\top - \mathbf{c}_i \mathbf{1}^\top) \circ I)$
   d. Update $P'_i = Z_i P_i R_i + \mathbf{1}\mathbf{c}_i^\top$
4. Iterate from step 2. until convergence of $\sum_{i=1}^{m} \|P'_i - \hat{S}\|_F^2$.

$\{0, 0.5, 1, 1.5, 2, 2.5, 3, 3.5\}$ [pixel] was added to image points coordinates and 25 trials for each noise level were averaged. In each trial first the Procrustean algorithm was run and then the photogrammetric bundle block adjustment, starting from the output of the former. The output of both methods were compared after a least-squares alignment with the ground-truth 3D points. Results reported in Fig. 1 (right) demonstrate that this new approach to bundle block adjustment based on AEOPA allows to obtain practically the same accuracy of the classical least squares solution, having the advantage of converging to the correct solution in most cases, with zero-information initialization. However, a theoretical proof of convergence is still unknown.

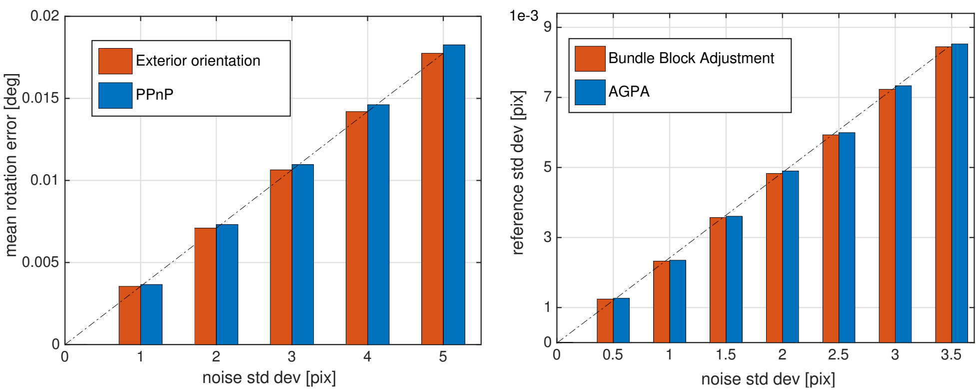

Fig. 1 (Left) Mean rotation error obtained in the exterior orientation of one image vs standard deviation of random noise added to image coordinate. (Right) Root of reference variance of the image coordinate residuals in bundle block adjustment vs standard deviation of random noise added to image tie-point. Orange is the error obtained by classical photogrammetric methods, while blue is the error achieved by Procrustean solutions. There is no substantial difference in the results obtained by Procrustean and classical photogrammetric methods.

Furthermore, a robust version of the algorithm was introduced in [7]. This new scheme, based on Iteratively Reweighted Least Squares (IRLS), achieves reliable results also in the presence of a percentage up to 10% of outliers.

4 Conclusions

Procrustes analysis was proved to be a useful tool to solve photogrammetric problems. Whereas classical OPA seeks transformations in 3D space, the anisotropic extension allows to deal with problems involving a projection to 2D space by introducing an auxiliary unknown, the depth of the points, which makes it possibile to back-project 2D points into 3D space, thereby restoring the 3D problem. AEOPA leads to a least squares solution without any linearization of the original equations and it does not require any approximate value of the unknown parameters. Results obtained with Procrustean algorithms in Photogrammetry are comparable in terms of accuracy with those given by state-of-the-art methods.

References

1. Akca, D.: Generalized procrustes analysis and its applications in photogrammetry. Internal Technical Report at the Institute of Geodesy and Photogrammetry - ETH, Zurich (2003)
2. Bennani Dosse, M., Ten Berge, J.: Anisotropic orthogonal procrustes analysis. Journal of Classification **27**(1), 111–128 (2010)
3. Commandeur, J.J.F.: Matching configurations. DSWO Press (1991)
4. Crosilla, F.: Procrustes analysis and geodetic sciences. In: Quo vadis geodesia...?, vol. 1, pp. 69–78. Department of Geodesy and GeoInformatics, University of Stuttgart (1999)
5. Crosilla, F., Beinat, A.: Use of generalised procrustes analysis for the photogrammetric block adjustment by independent models. ISPRS Journal of Photogrammetry & Remote Sensing **56**(3), 195–209 (2002)
6. Fusiello, A., Crosilla, F.: Solving bundle block adjustment by generalized anisotropic procrustes analysis. ISPRS Journal of Photogrammetry and Remote Sensing **102**, 209–221 (2015). DOI 10.1016/j.isprsjprs.2015.02.002
7. Fusiello, A., Crosilla, F.: Fast and resistant procrustean bundle adjustment. ISPRS Annals of Photogrammetry, Remote Sensing and Spatial Information Sciences **III-3**, 35–41 (2016). DOI 10.5194/isprs-annals-III-3-35-2016
8. Fusiello, A., Crosilla, F., Malapelle, F.: Procrustean point-line registration and the npnp problem. In: 3D Vision (3DV), 2015 International Conference on, pp. 250–255. IEEE (2015)
9. Fusiello, A., Maset, E., Crosilla, F.: Reliable exterior orientation by a robust anisotropic orthogonal procrustes algorithm. In: Proceedings of the ISPRS Workshop 3D-ARCH 2013: 3D Virtual Reconstruction and Visualization of Complex Architectures, *ISPRS Archives*, vol. XL-5/W1, pp. 81–86 (2013)
10. Garro, V., Crosilla, F., Fusiello, A.: Solving the pnp problem with anisotropic orthogonal procrustes analysis. In: Second Joint 3DIM/3DPVT Conference: 3D Imaging, Modeling, Processing, Visualization and Transmission (2012)
11. Gower, J.: Generalized procrustes analysis. Psychometrika **40**(1), 33–51 (1975)
12. Schönemann, P., Carroll, R.: Fitting one matrix to another under choice of a central dilation and a rigid motion. Psychometrika **35**(2), 245–255 (1970)