EP Homework 8

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1 Complementary Observables

When two observable operators do not commute, they are called complementary observables. So since the orthogonal components of $\hat{L}$ are all complementary, i.e, they do not commute ($\left[\hat{L}_x, \hat{L}_y\right] \neq 0$), they can not be simulatenously known, except for special cases like when $\hat{L}_x = \hat{L}_y = \hat{L}_z = 0$. In order to show this, we should prove that $[\hat{L}_y, \hat{L}_z] \neq 0$ when $\hat{L} = \hat{r} \times \hat{p}$.

$$[\hat{L}_y, \hat{L}_z] = [z\partial_x - x\partial_z, x\partial_y - y\partial_x] = z[\partial_x, x]\partial_y + [x, \partial_x]\partial_z y = \frac{\hbar}{i}(z\partial_y - y\partial_z) = i\hbar\hat{L}_x \neq 0 \tag{1}$$

Eq: 1 can be further generalized to the equation:

$$[\hat{L}_i, \hat{L}_j] = i\hbar\epsilon_{ijk}\hat{L}_k \tag{2}$$

2 Eigenfunctions

If ψ is an eigenfunction for $\hat{A}$ with eigenstate $|\psi\rangle$ then it would be also an eigenfunction and eigenstate for some operator $\hat{B}$ if $[\hat{A}, \hat{B}] = 0$. We can say this is true because if if $\hat{A}\psi = a\Psi \Rightarrow \hat{B}\hat{A}\psi = \hat{B}a\psi = a\hat{B}\psi$, which since $[\hat{A}, \hat{B}] = 0$, we know $\hat{B}\hat{A}\psi = \hat{A}\hat{B}\psi$, thus $\hat{A}(\hat{B}\psi) = a(\hat{B}\psi)$ meaning $\hat{B}\psi \propto \psi$, thus ψ is an eigenfunction of $\hat{B}$. So in conclusion, since $\hat{L}_z$ and $\hat{L}_x$ do not commute, ψ is not an eigenfunction for $\hat{L}_x$.

3 Unique Complementariness of Angular Momentum Operator

I say unique because to me it is quite unique that this is possible. Although necessary, still it is unique.

$$[\hat{L}_z, \hat{L}_x^2 + \hat{L}_y^2] = \hat{L}_x[\hat{L}_z, \hat{L}_x] + [\hat{L}_z, \hat{L}_y]\hat{L}_y = i\hbar\hat{L}_x\hat{L}_y - i\hbar\hat{L}_x\hat{L}_y = 0$$

Where we used the generalization from equation 2.

4 Ladder Operators

We know that $hatL^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$ and with the definition given of the raising and lowering operators as: $\hat{L}_+ = \hat{L}_x + i\hat{L}_y$ and $\hat{L}_- = \hat{L}_x - i\hat{L}_y$, we see that

$$\hat{L}_-\hat{L}_+ = (\hat{L}_x - i\hat{L}_y)(\hat{L}_x + i\hat{L}_y) = \hat{J}_x^2\hat{J}_y^2 + i[\hat{L}_x, \hat{L}_y] = \hat{L}^2 - \hat{L}_z^2 - \hbar\hat{L}_z \tag{3}$$

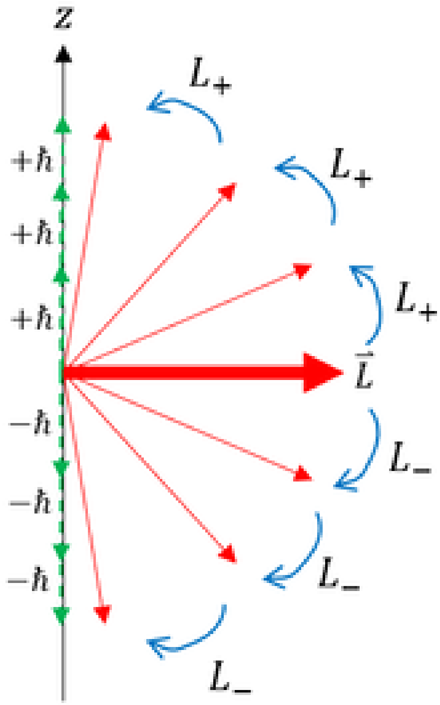

Figure 1: The ladder operator acting on the wave function raises or lowers the z-component of teh angular momentum by a value of $\hbar$, which can also be seen as increasing or decreasing the scalar quantity quantum number m by 1.

$$\Rightarrow \hat{L}^2 = \hat{L}_-\hat{L}_+ + \hat{L}_z^2 + \hbar L_z$$

or likewise the inverse can also be done:

$$\hat{L}_+\hat{L}_- = (\hat{L}_x + i\hat{L}_y)(\hat{L}_x - i\hat{L}_y) = \hat{J}_x^2\hat{J}_y^2 - i[\hat{L}_x, \hat{L}_y] = \hat{L}^2 - \hat{L}_z^2 + \hbar\hat{L}_z \tag{4}$$

$$\Rightarrow \hat{L}^2 = \hat{L}_+\hat{L}_- + \hat{L}_z^2 - \hbar L_z$$

We can use these relation sot prove that the eigenvalues of $\hat{L}^2$ are $\hbar^2 l(l+1)$. To do so we need to first state some properties of the ladder fucntions and their effect on eigenvalues. If we say that the momentum squared operator has an eigenvalue defined as: $\hat{L}^2\Psi = L^2\Psi$, then the ladder operator on a wave function is an eigenfunction of the angular momentum squared operator, and in fact the same eigenvalue of L^2 since: $\hat{L}^2(\hat{L}_\pm\Psi) = \hat{L}_\pm(\hat{L}^2\Psi) = \hat{L}_\pm(L^2\Psi)$. So even before the ladder operator touched the system, the angular momentum was found to be L, and yet after the operator acted on it, the result is the same. Thus the ladder operators do not change the angular momentum of the system. In addition, we can note that the ladder operator is merely rotating the angular momentum, as seen in Figure 1 From Figure 1, we know also that the boundrary coundition $|\hat{L}_z^2 \leq |\hat{L}^2|$ and $\hat{L}_+\Psi_{top} = \hat{L}_-\Psi_{bottom} = 0$.

Meaning that if we were to rotate a wave function that is already rotated to be aligned with some axis, then it will be zero.

Using this boundrary condition and equation 3 we can see that:

$$\hat{L}^2\Psi_{top} = (\hat{L}_-\hat{L}_+ + \hbar\hat{L}_z + \hat{L}_z^2)\Psi_{top}$$

The ladder operator becomes 0 against the wave function becasue of the boundrary condition.

$$\Rightarrow = (0 + \hbar\hat{L}_z + \hat{L}_z^2)\Psi_{top}$$

Plugging in eigenvalues for the $\hat{L}_z = \hbar L$

$$\Rightarrow = (0 + \hbar^2 L + \hbar^2 L^2)\Psi_{top} = (\hbar^2 L(L+1))\Psi_{top}$$