Dynamics of Double Pendulums & Stability of Integration Methods

Abstract

This report will first of all discuss integration methods and their suitability towards solving a single pendulum system. Then will move on to discussing the dynamics of the double pendulum. The equations of motion used for the double pendulum has been simplified using small angle approximations, causing the non-linear chaotic behaviour to vanish. In part I of the report, it is concluded that the fourth order turns out to be the best method both in terms of stability of the solution and efficiency of the algorithm. The double pendulum appears to behave like a coupled oscillater with beating. The ratio of the masses affects the motion by inverting the frequencies of the large oscillations and small oscillations as the ratio changes. It is suggested that this occurs because the oscillations of the smaller mass is affected by the larger mass more than the other way round. The smaller mass in the system tends to lag behind the larger mass and is pulled by the larger mass.

Part I

Single Pendulum

I. Aims & Introduction

This report will discuss three main methods of integrating the system for the trajectory of a single pendulum. Namely, the Euler method, Leapfrog method and finally the fourth order Runge-Kutta method. Each method will be analysed for how stable their solutions are.

II. Methods

I. Integration Methods

[1] Using the coupled differential equations for a single pendulum with small angle in natural units, simply implement the algorithms for each of the methods:

- Euler method:

$$\theta_{i(n+1)} = (\delta_{ij} + hA_{ij})\theta_{j(n)}$$

  where $\underline{\underline{A}} := \begin{pmatrix} 0 & 1 \\ -1 & -\tilde{\gamma} \end{pmatrix}$ and h is the step size.

- Leapfrog Method:

$$\underline{\theta}_{n+1} = 2h\underline{\underline{A}}\underline{\theta}_n + \underline{\theta}_{n-1}$$

- Fourth Order Runge-Kutta:

$$\theta_{i(n+1)} = \sum_{n=0}^{4} \frac{1}{n!}(hA_{ij})^n \theta_{j(n)}$$

II. Stability Analysis

[2] Stability analysis can be done for the single step methods however with the leapfrog method, the derivation for stability conditions do not apply. Firstly considering the single step methods where stability analysis can be done analytically, resulting in the following conditions:

- Euler method:

$$|1 + \lambda_i| < 1$$

  where λ_i is the i^{th} eigenvalue of the matrix $h\underline{\underline{A}}$.

- Fourth Order Runge-Kutta:

$$\left| \sum_{k=0}^{4} \frac{1}{k!}\lambda_i^k \right| < 1$$

For the Leapfrog method, stability can either be determined by looking at the difference between the computed solution and the analytic solution, or by whether the energies of the analytic solution and the computed solution

[1] See appendix section 2

[2] see appendix section 3

increase with each time step. The Hamiltonian for the system is given by:

$$\mathcal{H} = mgl \left(\frac{1}{2} \left(\frac{\mathrm{d}\theta}{\mathrm{d}\tilde{t}} \right)^2 - \cos\theta \right)$$

This trivially can be put into natural units, using $\widetilde{\mathcal{H}} := \mathcal{H}/mgl$:

$$\widetilde{\mathcal{H}} = \frac{1}{2} \left(\frac{\mathrm{d}\theta}{\mathrm{d}\tilde{t}} \right)^2 - \cos\theta$$

Hamiltonian formalism of classical mechanics only works for conservative fields, hence if you include damping terms, hamilton's equations cannot be used. The value of $\widetilde{\mathcal{H}}$ is simply the total energy.

III. Results

The results for the critical step size, as a function of gamma:

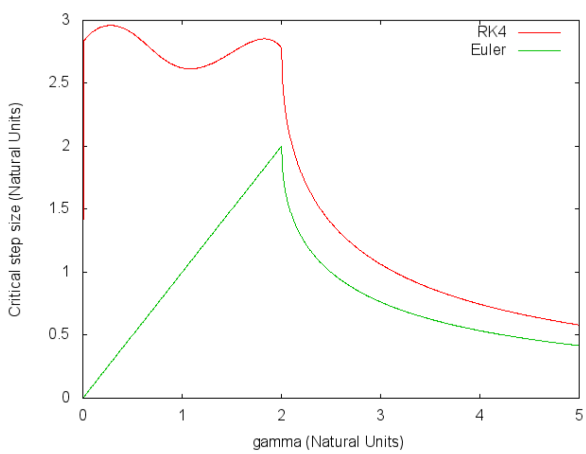

Figure 1: *The critical step sizes as a function of gamma in natural units*

The results show that in the under-damped, critically damped and over-damped cases, the fourth order Runge-Kutta method has a higher critical step size, and hence its' solution will not diverge from the analytic solution with time. Without any damping, the Euler method is unconditionally unstable.

The results for the energies for the undamped case, the energies leapfrog method give diverge by a constant amount from the energies given by the analytic solution, until the step-size goes beyond 1.0. So for the undamped case the critical step size is 1.0. With higher under-damping the critical step size required is much lower than the single step methods, so as well as requiring more computations per step, the stability conditions are also more strict than with Euler and Runge-Kutta methods. For the over damped case, there do appear to be solutions which converge for the leapfrog method, however the critical step size is still orders of magnitude below the other methods of integrating for the trajectory of the system.

The absolute differences between each solution and the analytic solution was also carried out in order to check the consistency, the differences we're sufficiently small to conclude that the stable results are also convergent.

IV. Discussion

From the previous results it appears that the fourth order Runge-Kutta is the best solution, since it has the highest values for the critical step size, and also since is also a single step method, so once the update matrix is calculated, the number of computations is of the same order as the Euler method. The Leapfrog method was not tested as extensively as the other two methods since the critical step sizes for the Leapfrog method is calculated numerically, it would have been difficult to find the values for the critical step size as a function of gamma.

V. Conclusion

In conclusion the most suitable method to use both in terms of how computationally expensive the solution is, and also the stability of the solution is the fourth order Runge-Kutta method, as this method is the most stable across all regimes: underdamped, undamped, critically damped and over-damped. Furthermore the efficiency is equal to lower order methods such as the Euler method.

Part II

Double Pendulum

VI. Aims & Introduction

The main focus of the investigation of the double pendulum is to see how the dynamics are affect by the ratio of the masses. The stability of the solution was simply determined by simply judging whether the dynamics of the system is non-physical for a given step size, for example, the angles exponentially increasing to infinity, would obviously be non-physical. Further details of how stability could be handled better is also discussed. For the following, section the ratio R is defined as the bottom mass divided by the top mass. G is the damping constant in natural units. Furthermore, ϕ is the angle from the vertical axis of the lower pendulum, θ is the angle from the vertical axis of the upper pendulum, $\nu := \dot{\phi}$ and $\omega := \dot{\theta}$.

VII. Methods

The differential equations can be derived using the Euler-Lagrange equations, differentiating the Lagrangian by the coordinates ϕ (angle from vertical of the upper pendulum), and θ (angle from vertical of the lower pendulum) to obtain generalised forces, and differentiating the Lagrangian by the time derivative of those coordinates to give generalised momenta π_ϕ and π_θ. In this case small angle approximation is used again.[Leonard Susskind, 2011] Then subbing in the momenta and coordinates, and writing in matrix form, will yield the derivative matrix needed to calculate the update matrices for n^{th} order Runge-Kutta methods.

The fourth order Runge-Kutta method is used, as this was the most stable solution for the single pendulum. The system is then solved numerically for different values of R, the ratio of the masses, both for undamped and damped cases.

VIII. Results

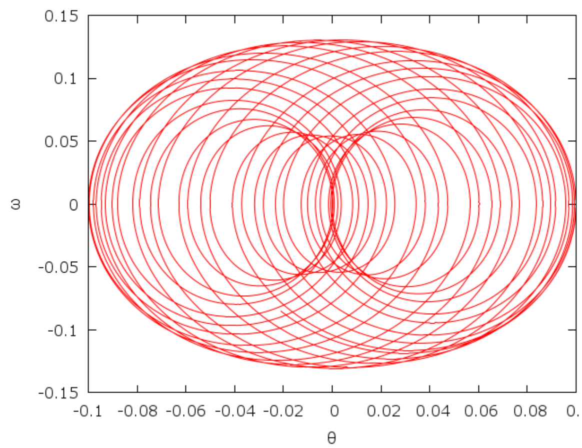

Figure 2: *Undamped trajectory of the upper pendulum in phase space.*

The trajectory in phase space forms a closed curve as expected since the non-linear terms have been linearised. The same phase space trajectories can be seen for the ϕ values, with different amplitudes. [Leonard Susskind, 2011]

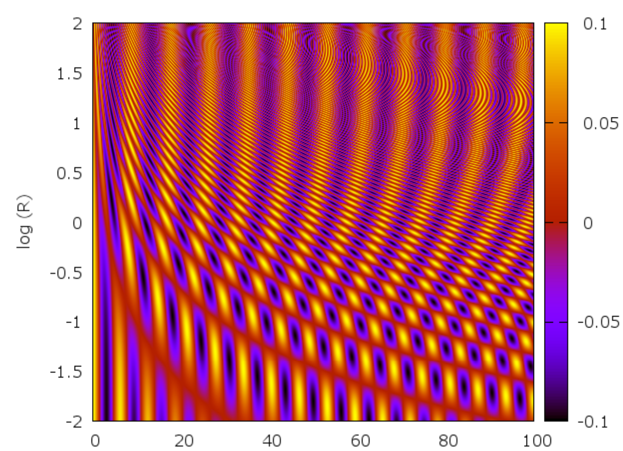

Figure 3: *Undamped oscillations, colour map representing the amplitude of theta, with time in natural units along the x axis.*

The large oscillations modulating smaller oscillations is caused by beating of the coupled system. As the ratio get's smaller, the frequency of the fast oscillations decreases and the frequency of the slow oscillations increases.

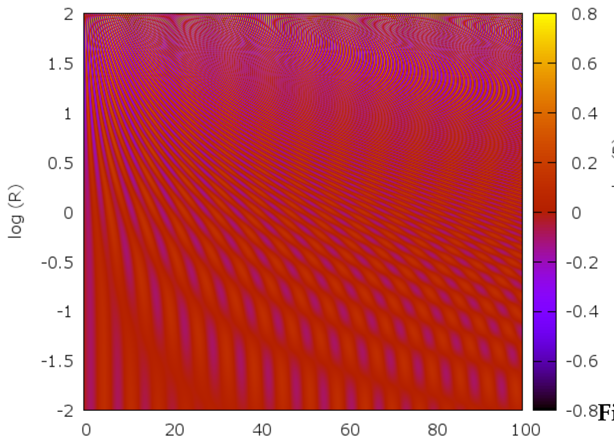

Figure 4: *Undamped oscillations, colour map representing the amplitude of omega, with time in natural units along the x axis.*

The patterns in the angular velocity is the same, however the amplitude is lower and the phase is shifted by $\pi/2$.

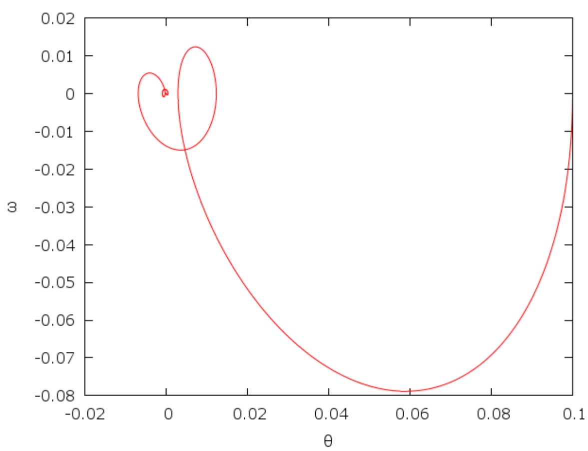

Figure 5: *Damped trajectory of the upper pendulum in phase space.*

This time the trajectory in phase space spirals to a point with zero angular velocity, because the damping reduces the energy.

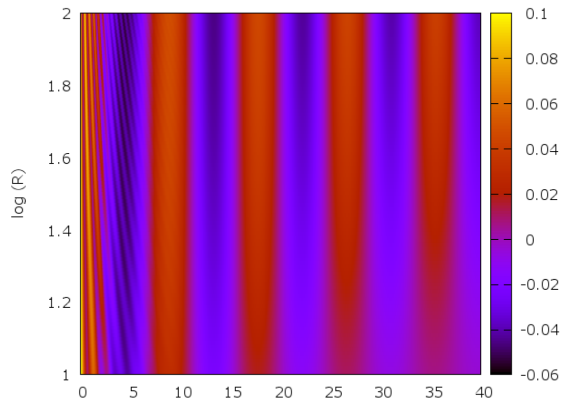

Figure 6: *Oscillations with damping, colour map representing the amplitude of theta, with time in natural units along the x axis.*

For the damped system the oscillations also exhibit beating, and decrease of frequency for lower mass ratio as before. This time the large oscillations are unaffected by the change in mass ratio, instead they simply decrease in amplitude.

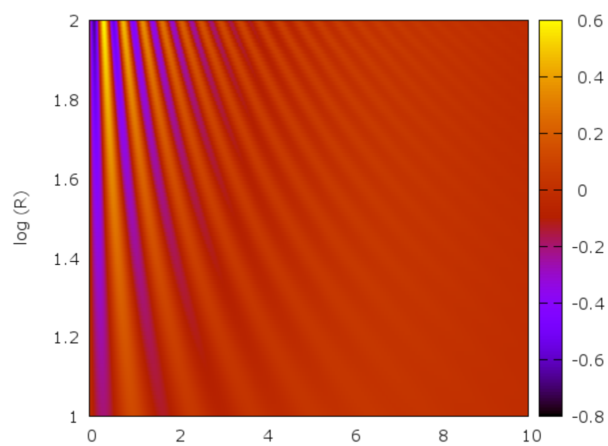

Figure 7: *Oscillations with damping, Colour map representing the amplitude of omega, with time in natural units along the x axis.*

The larger oscillations in angular velocity do not occur in the case of damping, so eventually they disappear from the *theta* trajectory.

IX. Discussion

I. Dynamics

The motion of the lower pendulum exhibits the same patterns, with a phase difference of $\pi/2$ which means that the lower pendulum is dragged by the upper pendulum, however it lags behind. The phase difference could be due to the starting conditions, the system starts off with a phase difference of $\pi/2$.

II. Stability

It was found that the critical step size for the stability of the solution depended heavily on the ratio of the masses, since the solution increased exponentially if the ratios were too small. A smaller step size was chosen until steady state solutions were found for the undamped oscillations and solutions which went to zero were found for damped oscillations. These are the only solutions which could make physical sense for the given parameters so it's obvious when the solution is unstable judging by how it behaves.

The stability was not rigourously checked as this would either solving the system analytically or eigendecomposition. The characteristic equation for the update matrix is a 4th order polynomial which would either need to be solved analytically or using numerical methods. How one might approach this method of stability analysis would be to use the formula for a general 4th order polynomial, and find each of the four values scanning through different values R and G, then checking whether the modulus is less than one. Since these formulas are long, it would be very computationally expensive to carry this out.

It is also possible to ensure stability by using implicit methods, which simply finds the inverse of the update matrix (if the rank of the matrix is equal to the number of columns). This would most easily be done using Gauss-Jordan elimination. Each row operation in Gauss-Jordan elimination on the matrix can be written as an identity matrix with an extra lower-triangular part, then left multiplying all the lower-triangular matrices which represent the row operations in the correct order will give the inverse.

X. Conclusion

The beating motion appears to reverse as the ratio changes, this could be due to the large mass being largely unaffected by the smaller mass. As the ratios reverse, the upper pendulum get's pulled by the lower pendulum as opposed to the other way around.

Other investigations could include the non-linear terms added back to the differential equation. The phase space trajectories would be no longer closed, the effect would be analogous to the trajectory of the Lorenz attractor. According to Liouville's theorem, areas in phase space remain constant. In chaotic systems, it is analytically unpredictable where a point a phase space will end up. It would be interesting to see many trajectories in phase space starting out in some area evolve in time. This would then spread out into a statistical fractal-like structure.[Leonard Susskind, 2011]

Approximate word count: 1700

XI. Appendices

The appendices mostly contains mathematical derivation and tricks used to simplify the equations.

I. Single Pendulum

1. Natural units:

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}\tilde{t}^2} + \tilde{\gamma}\frac{\mathrm{d}\theta}{\mathrm{d}\tilde{t}} + \sin\theta = 0$$

$$\tilde{t} := t\sqrt{g/l} \quad \tilde{\gamma} := \gamma/m\sqrt{gl}$$

where g is the acceleration due to gravity, l is the length of the pendulum and m is the mass of the pendulum.

2. Separated first order equations:

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} = \begin{pmatrix}0 & 1\\ -1 & -\tilde{\gamma}\end{pmatrix}\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}$$

where $\theta_1 := \theta$, $\theta_2 := \frac{\mathrm{d}\theta}{\mathrm{d}\tilde{t}}$

II. Integration Methods

1. Euler method:[Robert Kingham, 2013]

$$\theta_{i(n+1)} = (\delta_{ij} + hA_{ij})\theta_{j(n)}$$

where

$$\underline{\underline{A}} := \frac{\mathrm{d}}{\mathrm{d}\tilde{t}}$$

2. Leapfrog Method:[Robert Kingham, 2013]

$$\underline{\theta}_{n+1} = 2h\underline{\underline{A}}\underline{\theta}_n + \underline{\theta}_{n-1}$$

3. Runge-Kutta 4:[Robert Kingham, 2013]

$$\underline{\theta}_{n+1} = \underline{\theta}_n + \frac{1}{6}(\underline{k}_1 + 2\underline{k}_2 + 2\underline{k}_3 + \underline{k}_4)$$

$$\underline{k}_1 = h\frac{\mathrm{d}\theta}{\mathrm{d}\tilde{t}}(\underline{\theta}_n)$$

$$\underline{k}_2 = h\frac{\mathrm{d}\theta}{\mathrm{d}\tilde{t}}(\underline{\theta}_n + \frac{1}{2}\underline{k}_1)$$

$$\underline{k}_3 = h\frac{\mathrm{d}\theta}{\mathrm{d}\tilde{t}}(\underline{\theta}_n + \frac{1}{2}\underline{k}_2)$$

$$\underline{k}_4 = h\frac{\mathrm{d}\theta}{\mathrm{d}\tilde{t}}(\underline{\theta}_n + \underline{k}_3)$$

Define new terms:

$$\underline{\phi} := \underline{\theta}_{n+1} \qquad \underline{\theta} := \underline{\theta}_n$$

The algorithm reduces to the equation:

$$\phi_i = [\delta_{ij} + hA_{ij} + \frac{h^2}{2}A_{ij}^2 + \frac{h^3}{6}A_{ij}^3 + \frac{h^4}{24}A_{ij}^4]\theta_j$$
$$= \sum_{n=0}^{4}\frac{1}{n!}(hA_{ij})^n\theta_j$$

III. Stability:

Methods require that the modulus of the amplification factor is less than 1 for the solution to be stable:

$$g := \left|\frac{\epsilon_{n+1}}{\epsilon_n}\right| \leq 1$$

Where ϵ is the error for set n .

Finite difference methods can be written as:

$$\tilde{y}_{i(n+1)} = T_{ij}\tilde{y}_{j(n)} + q_i(\tilde{t})$$

q_i for inhomogeneous does not influence the numerical stability[Robert Kingham, 2013], and can be ignored. Adding error to the solution on both sides gives:

$$\tilde{y}_{i(n+1)} + \epsilon_{i(n+1)} = T_{ij}(\tilde{y}_{j(n)} + \epsilon_{j(n)})$$

Taylor expanding $\tilde{y}_{i(n+1)}$ gives:

$$\tilde{y}_{i(n+1)} = \tilde{y}_{i(n)} + \tilde{y}'_{i(n)}h + \mathcal{O}(h^2) \approx T_{ij}y_{j(n)}$$

Which is true since the update matrix for an n^{th}-order Runge-Kutta method is simply the taylor series of the actual solution. The higher the order of the runge-kutta method, the higher the order of error this is. Subbing into the previous equation, and ignoring errors of second order gives:

$$\boxed{\epsilon_{i(n+1)} = T_{ij}\epsilon_{j(n)}}$$

Then eigendecomposition gives $T_{ij} = R_{ik}D_{kl}R_{lj}^{-1}$ where R is a matrix whose columns are eigenvectors of T_{ij}. Subbing in the diagonalised form into the error propagation formula gives:

$$\zeta_{i(n+1)} = D_{ij}\zeta_{j(n)} = \lambda_i\zeta_{i(n)}$$

where ζ is the vector right-multiplied by the inverse of R, which rotates the coupled vectors onto an uncoupled basis. We impose

$$|g_i| = \left|\frac{\zeta_{i(n+1)}}{\zeta_{i(n)}}\right| = |\lambda_i| \leq 1$$

Since the coupled and uncoupled bases are related by a linear transformation, any constraint applied in one basis is equivalent to one applied in the other.[Robert Kingham, 2013]

1. Euler Method Stability

$$\underline{\underline{U}} = \underline{\underline{A}}^0 + h\underline{\underline{A}}$$

Where $\underline{\underline{U}}$ is the update matrix for the euler method and $\underline{\underline{A}}$ is the matrix that represents the derivative on the coupled system. h is step size.
let: $\underline{\underline{B}} = h\underline{\underline{A}}$, $\underline{\underline{B}} = \underline{\underline{D}}\underline{\underline{\Lambda}}\underline{\underline{D}}^{-1}$.

$$\underline{\underline{U}} = (\underline{\underline{D}}\underline{\underline{\Lambda}}^0\underline{\underline{D}}^{-1} + \underline{\underline{D}}\underline{\underline{\Lambda}}\underline{\underline{D}}^{-1})$$

$$\underline{\underline{U}} = \underline{\underline{D}}\left(\underline{\underline{\Lambda}}^0 + \underline{\underline{\Lambda}}\right)\underline{\underline{D}}^{-1}$$

$$\implies \left|\underline{\underline{\Lambda}}^0 + \underline{\underline{\Lambda}}\right|_{ii} < 1$$

$$\implies |1 + \lambda_i| < 1$$

2. Fourth Order Runge-Kutta Method Stability

$$\begin{aligned}
\underline{\underline{U}} =& \underline{\underline{A}}^0 + h\underline{\underline{A}} + \frac{h^2}{2}\underline{\underline{A}}^2 + \frac{h^3}{6}\underline{\underline{A}}^3 + \frac{h^4}{24}\underline{\underline{A}}^4 \\
=& \underline{\underline{D}}\underline{\underline{\Lambda}}^0\underline{\underline{D}}^{-1} + \underline{\underline{D}}\underline{\underline{\Lambda}}\underline{\underline{D}}^{-1} + \frac{1}{2}\underline{\underline{D}}\underline{\underline{\Lambda}}^2\underline{\underline{D}}^{-1} \\
&+ \frac{1}{3!}\underline{\underline{D}}\underline{\underline{\Lambda}}^3\underline{\underline{D}}^{-1} + \frac{1}{4!}\underline{\underline{D}}\underline{\underline{\Lambda}}^4\underline{\underline{D}}^{-1} \\
\implies & \underline{\underline{U}} = \underline{\underline{D}}\left(\sum_{i=0}^{4}\frac{1}{i!}\underline{\underline{\Lambda}}^i\right)\underline{\underline{D}}^{-1} \\
\implies & \left|\sum_{k=0}^{4}\frac{1}{k!}\Lambda_{ii}^k\right| < 1 \\
\implies & \left|\sum_{k=0}^{4}\frac{1}{k!}\lambda_i^k\right| < 1
\end{aligned}$$

References

[Robert Kingham, 2013] Imperial College London. Computational Physics Lecture notes
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[Leonard Susskind, 2011] Stanford University Leonard Susskind's Theoretical Minimum lecture course, Classical Mechanics.