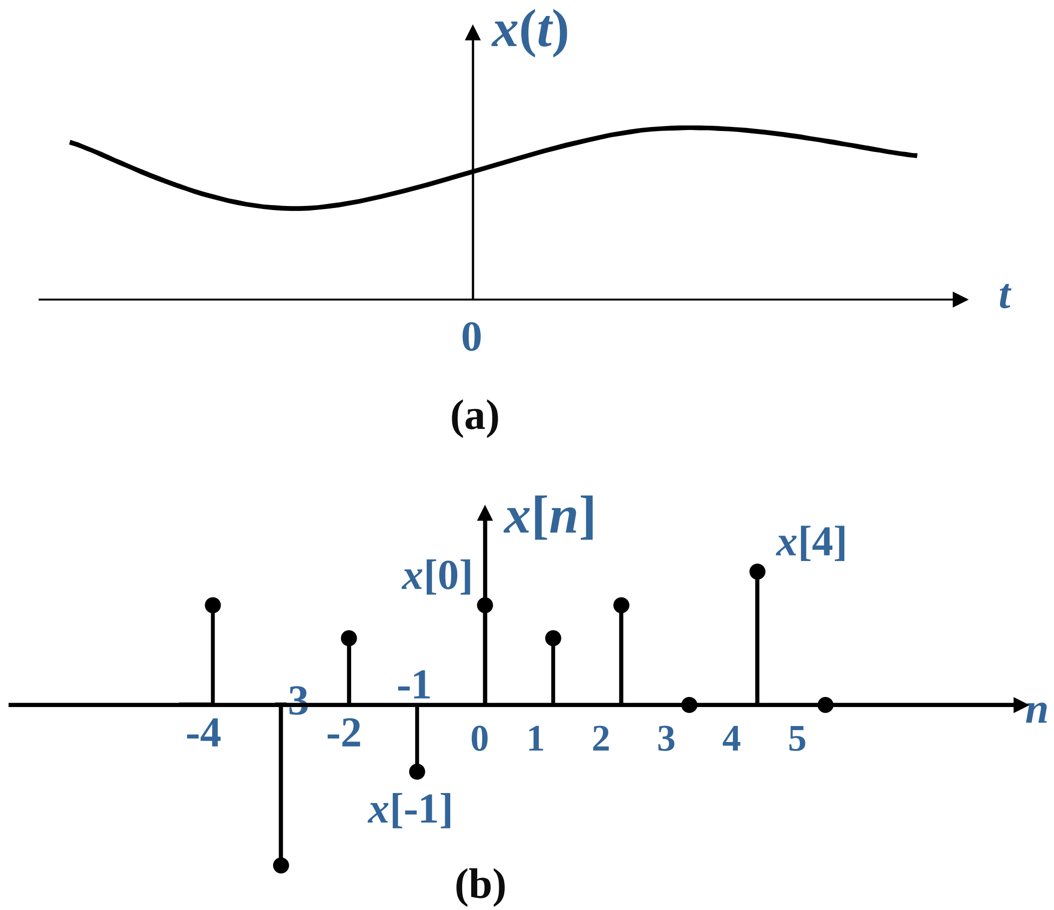
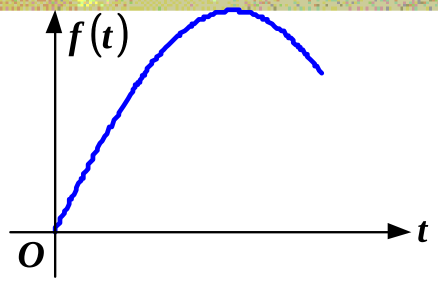
Chapter 3

Time-Domain Analysis of Discrete-Time Systems



(a) continuous-time signal (b) discrete-time signal

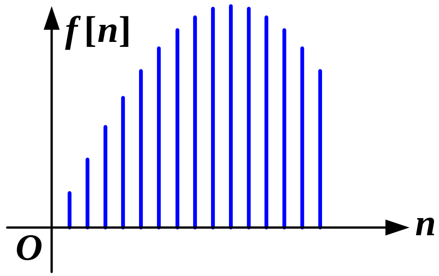
Analog Signal



The classification depends on domain (time) and codomain (amplitude) of the signal.

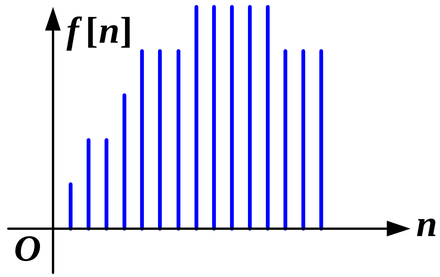
continuous time and amplitude

Sampling Signal



discrete time and continuous amplitude

Digital Signal



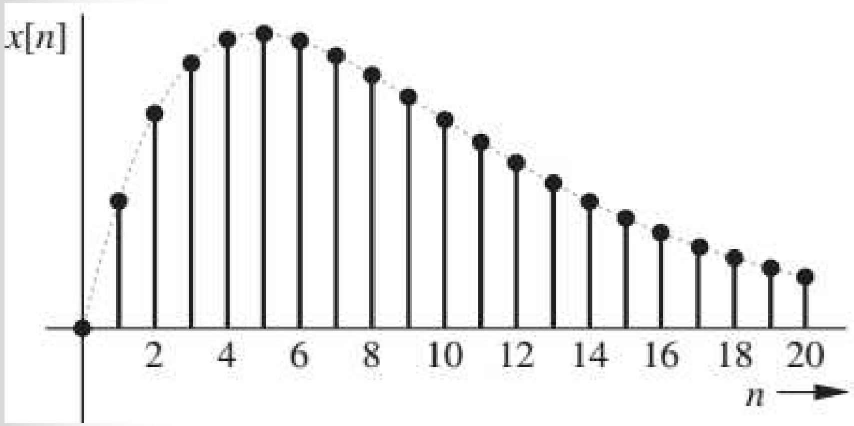
discrete time and amplitude

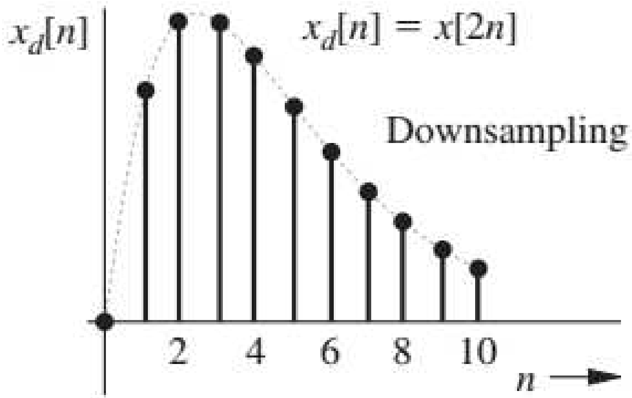
Sampling rate alteration: decimation and interpolation

Decimation

$x_d[n]=x[Mn]$, M **must be integer values**

$x[Mn]$ **selects every** M**th sample of** $x[n]$ **and deletes all the samples in between.**





In the CT case, time compression merely speeds up the signal without loss of any data.

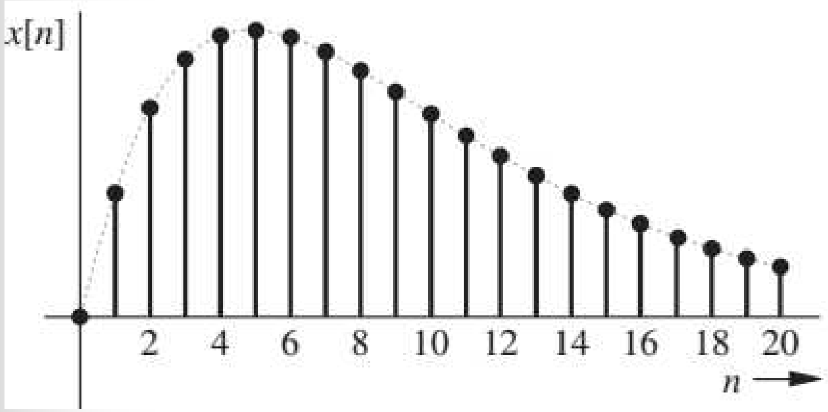
Decimation generally causes loss of data.

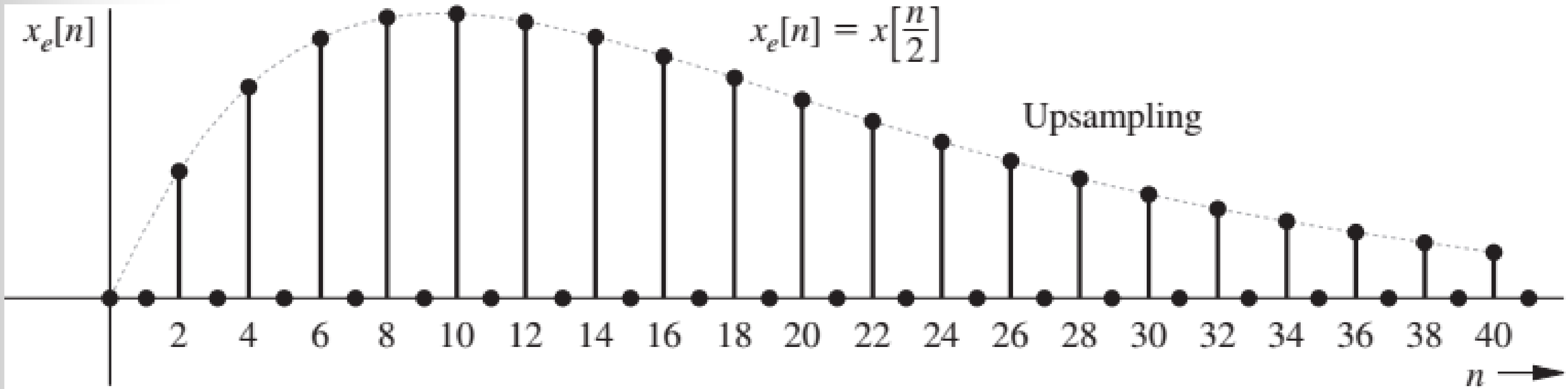
Interpolation

1) Expanding

$$x_e[n]=\begin{cases} x[n/L] & n=0,\pm L,\pm 2L,\ldots \\ 0 & otherwise \end{cases}$$

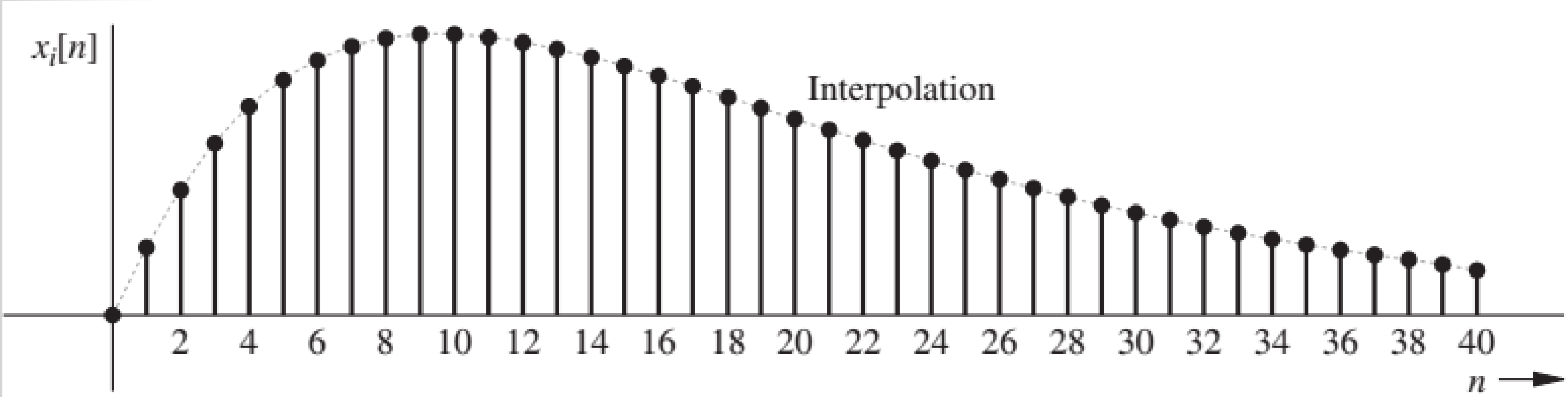
$$x_e[n]:\quad x[0],\underbrace{0,0,\ldots,0,0}_{L-1\text{ zeros}},x[1],\underbrace{0,0,\ldots,0,0}_{L-1\text{ zeros}},x[2],\underbrace{0,0,\ldots,0,0}_{L-1\text{ zeros}},\ldots$$





2) Interpolation

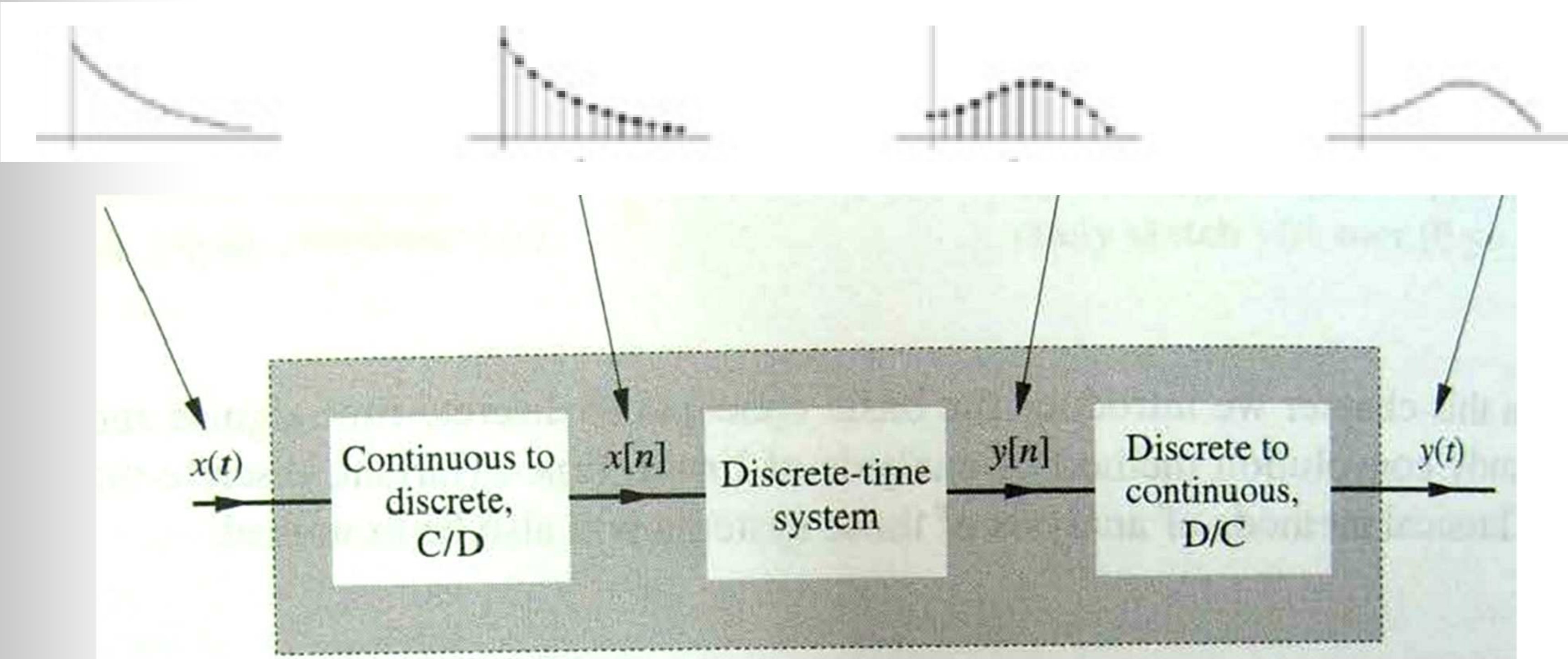
Use suitable interpolation formula to reconstruct the zero-valued samples from the non-zero-valued samples.



$x_i[n]$ is obtained by using an interpolating filer.

The optimum interpolating filter is usually an ideal lowpass filter, which is realizable only approximately.

Discrete-time system: **transform discrete-time inputs into discrete-time outputs.**



Examples of DT systems

- **Digital differentiator**

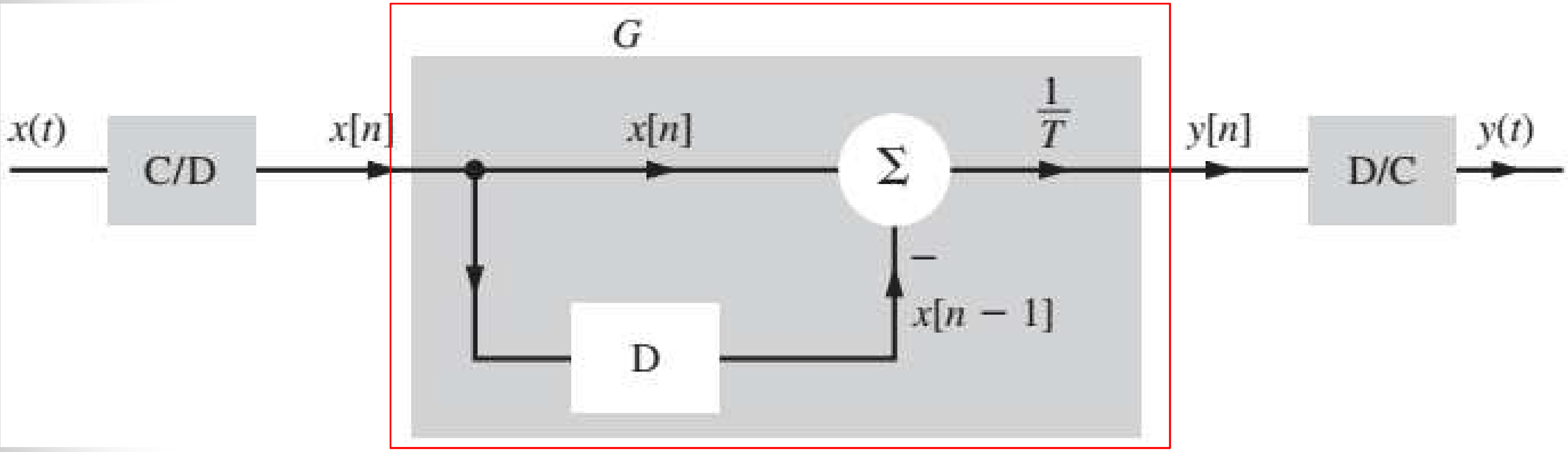
$$y(t)=\frac{dx}{dt} \Rightarrow y(nT)=\left.\frac{dx}{dt}\right|_{t=nT}=\lim_{T\to 0}\frac{1}{T}\{x(nT)-x[(n-1)T]\}$$

$$\because x[n]=x(nT) \quad and \quad y[n]=y(nT)$$

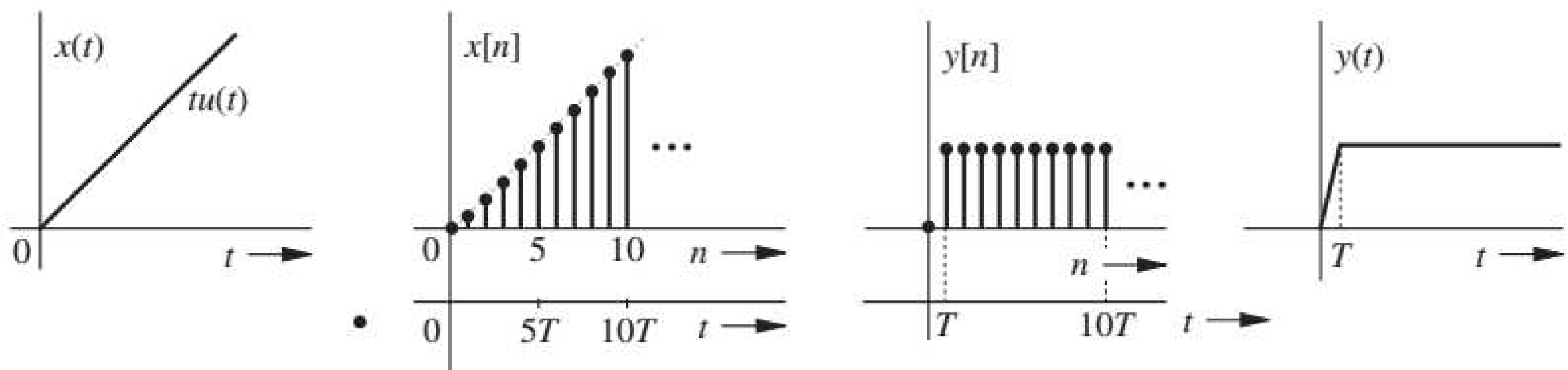
$$\therefore y[n]=\lim_{T\to 0}\frac{1}{T}\{x[n]-x[n-1]\}$$

In practice, the sampling interval *T* cannot be zero. Assuming *T* to be sufficiently small, we obtain

$$y[n]=\frac{1}{T}\{x[n]-x[n-1]\}$$



Digital differentiator $$y[n]=\frac{1}{T}\{x[n]-x[n-1]\}$$



$$y[n] = \frac{1}{T}\{x[n] - x[n-1]\}$$ **Backward difference form**

$x[n]$ is the present sample value

$x[n$-1$]$ is the preceding (backward) sample value

$$y[n] = \frac{1}{T}\{x[n+1] - x[n]\}$$ **Forward difference form**

$x[n]$ is the present sample value

$x[n$+1$]$ is the next (forward) sample value

- **Digital integrator**

$$y(t)=\int_{-\infty}^{t}x(\tau)d\tau \quad \Longrightarrow \quad y(nT)=\lim_{T\to 0}\sum_{k=-\infty}^{n}x(kT)T$$

$$\because x[n]=x(nT) \quad and \quad y[n]=y(nT)$$

$$\therefore y[n]=\lim_{T\to 0}T\sum_{k=-\infty}^{n}x[k]$$

In practice, the sampling interval *T* cannot be zero. Assuming *T* to be sufficiently small, we obtain

$$y[n]=T\sum_{k=-\infty}^{n}x[k] \quad \text{or} \quad y[n]-y[n-1]=Tx[n]$$

Accumulator system **Recursive form**

Order of a difference equation:

The highest-order difference of the output signal or the input signal, whichever is higher represents the order of the difference equation.

Advantages of digital signal processing

1. Digital systems operation can tolerate considerable variation in signal values, and, hence, are less sensitive to changes in the component parameter values caused by temperature variation, aging and other factors. This results in greater degree of precision and stability. Since they are binary circuits, their accuracy can be increased by using more complex circuitry to increase word length, subject to cost limitations.

2. Digital systems do not require any factory adjustment and can be easily duplicated in volume without having to worry about precise component values. They can be fully integrated, and even highly complex systems can be placed on a single chip by using *VLSI* (very-large-scale integrated) circuits.

3. Digital filters are more flexible. Their characteristics can be easily altered simply by changing the program. Digital hardware implementation permits the use of microprocessors, miniprocessors, digital switching, and large-scale integrated circuits.

4. A greater variety of filters can be realized by digital systems.

5. Digital signals can be stored easily and inexpensively on magnetic tapes or disks without deterioration of signal quality. It is also possible to search and select information from distant electronic storehouses.

6. Digital signals can be coded to yield extremely low error rates and high fidelity, as well as privacy. Also, more sophisticated signal processing algorithms can be used to process digital signals.

7. Digital filters can be easily time-shared and therefore can serve a number of inputs simultaneously. Moreover, it is easier and more efficient to multiplex several digital signals on the same channel.

8. Reproduction with digital messages is extremely reliable without deterioration. Analog messages such as photocopies and films, for example, lose quality at each successive stage of reproduction and have to be transported physically from one distant place to another, often at relatively high cost.

One must weigh these advantages against such disadvantages, as increased system complexity due to use of A/D and D/A interfaces, limited range of frequencies available in practice (about tens of megahertz), and use of more power than is needed for the passive analog circuits. Digital systems use power-consuming active devices.

DT system equations

- **A general form of a linear difference equation**

Using the delay operator form

$$a_0 y[n] + a_1 y[n-1] + \cdots + a_N y[n-N] =$$
$$b_0 x[n] + b_1 x[n-1] + \cdots + b_M x[n-M]$$

Using the advance operator form

$$a_0 y[n+N] + a_1 y[n+N-1] + \cdots + a_N y[n] =$$
$$b_0 x[n+M] + b_1 x[n+M-1] + \cdots + b_M x[n]$$

Causality condition: $M \le N$

Recursive solution of difference equation

$$a_0 y[n] + a_1 y[n-1] + \cdots + a_N y[n-N] = b_0 x[n] + b_1 x[n-1] + \cdots + b_M x[n-M]$$

$$\Downarrow$$

$$y[n] = -a_1 y[n-1] - \cdots - a_N y[n-N] + b_0 x[n] + b_1 x[n-1] + \cdots + b_M x[n-M]$$

a_0, the coefficient of $y[n]$, is normalized to unity.

Knowing the N initial conditions, $y[-1]$, $y[-2]$,…, $y[-N]$, and the input, we can determine the entire output $y[0]$, $y[1]$, $y[2]$,…, recursively, one value at a time.

Example: Solve iteratively $y[n]-2y[n-1]=x[n]$

with initial condition $y[-1]=0$ and an input $x[n]=u[n]$.

$$y[n]=x[n]+2y[n-1]$$

$$y[0]=x[0]+2y[-1]=1$$

$$y[1]=x[1]+2y[0]=3$$

$$y[2]=x[2]+2y[1]=7$$

$$\cdots\cdots$$

$$y[n]=x[n]+2y[n-1]=2^{n+1}-1$$

Therefore, we can get the closed-form solution.

$$y[n]=(2^{n+1}-1)u[n]$$

Example: Solve iteratively

$$y[n+2]-y[n+1]+0.24y[n]=x[n+2]-2x[n+1]$$

With initial condition $y[-1]=2$, $y[-2]=1$ and an input $x[n]=nu[n]$.

$$y[n+2]=y[n+1]-0.24y[n]+x[n+2]-2x[n+1]$$

$$y[0]=2-0.24\times 1+0-0=1.76$$

$$y[1]=1.76-0.24\times 2+1-0=2.28$$

$$y[2]=2.28-0.24\times 1.76+2-2\times 1=1.8576$$

……

Therefore, we cannot get the closed-form solution.

Operational notation

In DT systems, we use the operator E to denote the operation for advancing a sequence by one time unit.

$$Ex[n] \equiv x[n+1]$$

$$E^2 x[n] \equiv x[n+2]$$

$$\vdots$$

$$E^N x[n] \equiv x[n+N]$$

$$a_0 y[n+N] + a_1 y[n+N-1] + \cdots + a_N y[n] =$$
$$b_0 x[n+M] + b_1 x[n+M-1] + \cdots + b_M x[n]$$

$$\Downarrow$$

$$(a_0 E^N + a_1 E^{N-1} + \cdots + a_{N-1} E + a_N) y[n] = (b_0 E^M + b_1 E^{M-1} + \cdots + b_{N-1} E + b_N) x[n]$$

$$Q[E] y[n] = P[E] x[n]$$

➢The total response of an LTID system

total response = zero-input response + zero-state response

- **The zero-input response is the system response when the input $x[n]=0$.**

- **The zero-state response is the system response to the external input $x[n]$ when the system is in zero state.**

System response to internal conditions: the zero-input response

When the input $x[n]$=0, the zero-input response $y_0[n]$ is the solution of the equation $Q[E]y[n]=P[E]x[n]$

$$Q[E]y_0[n]=0$$

$$\Downarrow$$

$$(E^N+a_1E^{N-1}+\cdots+a_{N-1}E+a_N)y_0[n]=0$$

The corresponding characteristic equation is

$$Q[\gamma]=0 \quad \Longrightarrow \quad \gamma^N+a_1\gamma^{N-1}+\cdots+a_{N-1}\gamma+a_N=0$$

The roots $\gamma_i(i=1,2,...,N)$ are the characteristic roots.

The exponentials $\gamma_i^n(i=1,2,...,N)$ are the characteristic modes.

$y_0[n]$ is a linear combination of $\gamma_i^n(i=1,2,...,N)$.

The characteristic roots of an LTID system:

- There are *n* distinct roots

$$y_0[n] = c_1\gamma_1^n + c_2\gamma_2^n + \cdots + c_N\gamma_N^n = \sum_{i=1}^{N} c_i\gamma_i^n$$

- There are *u* distinct roots and *p* k_jth-order repeated roots

$$y_0[n] = \sum_{i=1}^{u} c_i\gamma_i^n + \sum_{j=1}^{p}\sum_{l=1}^{k_j} d_{j,l}\, n^{k_j-l}\gamma_j^n$$

- There are complex roots

Complex roots occur in pairs of conjugates if the system equation coefficients are real. If $\gamma=|\gamma|e^{j\beta}$ is a characteristic root, $\gamma^*=|\gamma|e^{-j\beta}$ must also be a characteristic root.

The zero-input response corresponding to this pair of complex roots is

$$y_0[n] = c_1\gamma^n + c_2(\gamma^*)^n = c_1|\gamma|^n e^{j\beta n} + c_2|\gamma|^n e^{-j\beta n}$$

For a real system, the response must also be real. This is possible only if c_1 and c_2 are conjugates. Let

$$c_1 = \frac{c}{2}e^{j\theta}, c_2 = \frac{c}{2}e^{-j\theta}$$

This yields

$$y_0[n] = \frac{c}{2}|\gamma|^n[e^{j(\beta n+\theta)} + e^{-j(\beta n+\theta)}]$$

$$= c|\gamma|^n \cos(\beta n + \theta)$$

Example: **Find the zero-input response $y_0[n]$ for an LTID system described by the following difference equation.**

$$y[n+2]-0.6y[n+1]-0.16y[n]=5x[n+2]$$

when the initial conditions are $y_0[-1]=0$, $y_0[-2]=25/4$.

The characteristic equation is

$$\gamma^2-0.6\gamma-0.16=0$$

The characteristic roots of the system are $\gamma_1=-0.2$，$\gamma_2=0.8$

The characteristic modes of the system are $(-0.2)^n$ and 0.8^n

Therefore, $y_0[n]=c_1(-0.2)^n+c_2(0.8)^n$

Substituting the initial conditions, we obtain

$$\begin{cases} -5c_1+\dfrac{5}{4}c_2=0 \\ 25c_1+\dfrac{25}{16}c_2=\dfrac{25}{4} \end{cases} \Longrightarrow \quad c_1=1/5,\ \ c_2=4/5$$

Therefore, $y_0[n]=1/5(-0.2)^n+4/5(0.8)^n,\quad n\geq 0$

Example: Find the zero-input response $y_0[n]$ for an LTID system described by the following difference equation.

$$y[n+2]+6y[n+1]+9y[n]=2x[n+2]+6x[n+1]$$

when the initial conditions are $y_0[-1]=-1/3$, $y_0[-2]=-2/9$.

The characteristic equation is

$$\gamma^2+6\gamma+9=0$$

The characteristic roots of the system are $\gamma_1=\gamma_2=-3$

The characteristic modes of the system are $(-3)^n$ and $n(-3)^n$

Therefore, $y_0[n]=c_1(-3)^n+c_2n(-3)^n$

Substituting the initial conditions, we obtain

$$\begin{cases} -\frac{1}{3}c_1+\frac{1}{3}c_2=-\frac{1}{3} \\ \frac{1}{9}c_1-\frac{2}{9}c_2=-\frac{2}{9} \end{cases} \Longrightarrow \quad c_1=4,\ \ c_2=3$$

Therefore, $y_0[n]=(4+3n)(-3)^n, \quad n\geq 0$

Example: Find the zero-input response $y_0[n]$ for an LTID system described by the following difference equation.

$$y[n+2]-1.56y[n+1]+0.81y[n]=x[n+1]+3x[n]$$

when the initial conditions are $y_0[-1]=2$, $y_0[-2]=1$.

The characteristic equation is

$$\gamma^2-1.56\gamma+0.81=0$$

The characteristic roots of the system are $\gamma_1=0.78+j0.45=0.9e^{j\pi/6}$, $\gamma_2=0.78-j0.45==0.9e^{-j\pi/6}$

The characteristic modes of the system are $(0.9e^{j\pi/6})^n$ and $(0.9e^{-j\pi/6})^n$

Therefore, $y_0[n]=c(0.9e^{j\pi/6})^n + c^*(0.9e^{j\pi/6})^n = c(0.9)^n\cos(\pi n/6+\theta)$

Substituting the initial conditions, we obtain

$$\begin{cases} \dfrac{c}{0.9}\cos(-\dfrac{\pi}{6}+\theta)=\dfrac{c}{0.9}[\dfrac{\sqrt{3}}{2}\cos\theta+\dfrac{1}{2}\sin\theta]=2 \\ \dfrac{c}{0.9^2}\cos(-\dfrac{\pi}{3}+\theta)=\dfrac{c}{0.81}[\dfrac{1}{2}\cos\theta+\dfrac{\sqrt{3}}{2}\sin\theta]=1 \end{cases} \Rightarrow \begin{matrix} c\cos\theta=2.308, \\ c\sin\theta=-0.397 \end{matrix}$$

$\tan\theta=-0.172 \qquad \theta=\tan^{-1}(-0.172)=-0.17\text{rad}$

Therefore, $y_0[n]= 2.34(0.9)^n\cos(\pi n/6-0.17), \quad n\geq 0$

The unit impulse response $h[n]$

The impulse response $h[n]$ is the system zero-state response to an impulse input $\delta[n]$ applied at n=0.

An Nth-order difference equation is

$$(E^N + a_1 E^{N-1} + \cdots + a_{N-1} E + a_N) y[n]$$

$$= (b_0 E^N + b_1 E^{N-1} + \cdots + b_{N-1} E + b_N) x[n]$$

$$Q(E) y[n] = P(E) x[n]$$

When the input $x[n]=\delta[n]$， the output $y[n]=h[n]$

$$(E^N + a_1 E^{N-1} + \cdots + a_{N-1} E + a_N) h[n]$$

$$= (b_0 E^N + b_1 E^{N-1} + \cdots + b_{N-1} E + b_N) \delta[n]$$

$$Q(E) h[n] = P(E) \delta[n]$$

Subject to the initial conditions $h[-1]=h[-2]=\ldots=h[-N]=0$

➢ Iterative determination of $h[n]$

Example: Find $h[n]$ of a system described by the equation

$$y[n]+0.2y[n-1]=x[n]$$

$$h[n]+0.2h[n-1]=\delta[n]$$

$$h[n]=\delta[n]-0.2h[n-1]$$

Substituting $h[-1]=0$ in the equation, we obtain

$$h[0]=\delta[0]-0.2h[-1]=1$$

$$h[1]=\delta[1]-0.2h[0]=-0.2$$

$$h[2]=\delta[2]-0.2h[1]=(-0.2)^2$$

……

$$h[n]=\delta[n]-0.2h[n-1]=(-0.2)^n$$

Therefore, $h[n]=(-0.2)^n u[n]$

➢ The closed-form solution of $h[n]$

$h[n]$ is the system response to input $\delta[n]$, which is zero for $n>0$. When the input is zero, only the characteristic modes can be sustained by the system.

Therefore, $h[n]$ must be made up of characteristic modes for $n>0$. At $n=0$, it may have some nonzero value A_0.

$$h[n]=A_0\delta[n]+y_c[n]u[n]$$

Where $y_c[n]$ is a linear combination of the characteristic modes.

Substituting the above equation in $Q(E)h[n]=P(E)\delta[n]$, we obtain

$$Q(E)(A_0\delta[n]+y_c[n]u[n])=P(E)\delta[n]$$

$$A_0Q(E)\delta[n]=P(E)\delta[n]$$

$$A_0(\delta[n+N]+a_1\delta[n+N-1]+...+a_N\delta[n])=b_0\delta[n+N]+...+b_N\delta[n]$$

Setting *n*=0 in this equation, we obtain

$$A_0 a_N = b_N$$

$$\therefore A_0 = \frac{b_N}{a_N}$$

Hence, $$h[n] = \frac{b_N}{a_N}\delta[n] + y_c[n]u[n]$$

If a_N=0, the method introduced in Section 3.12 of the textbook should be used.

Example: Find $h[n]$ of a system described by the equation
$y[n]-0.6y[n-1]-0.16y[n-2]=5x[n]$

This equation can be expressed in the advance operator form as $y[n+2]-0.6y[n+1]-0.16y[n]=5x[n+2]$

We have $a_N=-0.16$, $b_N=0$ ⇒ $A_0= b_N/a_N=0$

The characteristic equation is

$$\gamma^2 - 0.6\gamma - 0.16 = 0$$

The characteristic roots of the system are $\gamma_1=-0.2$, $\gamma_2=0.8$.

The characteristic modes of the system are $(-0.2)^n$ and $(0.8)^n$

Therefore, $y_c[n]=c_1(-0.2)^n + c_2(0.8)^n$

$$h[n]=[c_1(-0.2)^n + c_2(0.8)^n]u(n)$$

To determine c_1 and c_2, we need to find two values of $h[n]$ iteratively.

$$h[0]=0.6h[-1]+0.16h[-2]+5\delta[0]=5$$
$$h[1]=0.6h[0]+0.16h[-1]+5\delta[1]=3$$

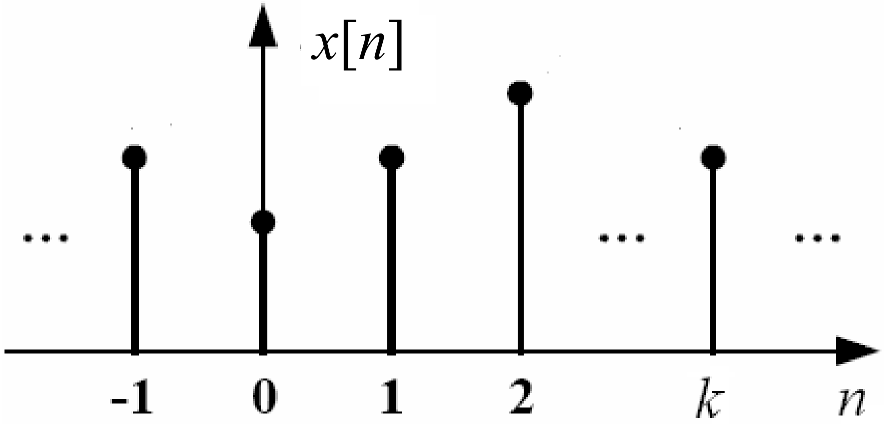
Substituting the initial conditions, we obtain

$$\begin{cases} c_1+c_2=5 \\ -0.2c_1+0.8c_2=3 \end{cases} \Rightarrow \quad c_1=1, \quad c_2=4$$

Therefore, $h[n]=[(-0.2)^n + 4(0.8)^n]u[n]$

System response to external input: zero-state response

1、Representation of DT Signals in Terms of Impulses



$$x[n] = \cdots + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \cdots$$

$$= \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]$$

Example: $x[n] = u[n]$

$$u[n] = \sum_{k=-\infty}^{+\infty} u[k]\delta[n-k] = \sum_{k=0}^{+\infty} \delta[n-k]$$

2、The Convolution Sum Representation of LTI Systems

$$x[n]=\delta[n] \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n]=h[n]$$

Unit Impulse Response *h*[*n*] :

Response of the LTI system to the unit impulse $\delta[n]$.

Question:

$$x[n] \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n]=?$$

Solution:

$$\delta[n] \longrightarrow h[n]$$

$$\delta[n-k] \longrightarrow h[n-k]$$

$$x[k]\delta[n-k] \longrightarrow x[k]h[n-k]$$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \longrightarrow \boxed{\sum_{k=-\infty}^{\infty} x[k]h[n-k] = y[n]}$$

The response $y[n]$ to $x[n]$ is the sum of weighted and shifted unit impulse responses.

Convolution Sum:

$$y[n]=\sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

$$y[n]=x[n]*h[n]$$

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]=x[n]*h[n]$$

- **The unit impulse response *h*[*n*] fully characterizes an LTI system.**

- **Method 1** **Example:** $u[n]*u[n]$

$$Solution:\quad u[n]*u[n]=\sum_{k=-\infty}^{\infty} u[k]u[n-k]$$

$$=\sum_{k=0}^{n} 1=(n+1)u[n]$$

Example: $f[n]*h[n]$, $f[n]=b^n u[n]$ $h[n]=a^n u[n]$

$$y[n]=f[n]*h[n]=\sum_{k=-\infty}^{\infty}b^k u[k]a^{n-k}u[n-k]=\sum_{k=0}^{n}b^k a^{n-k}$$

$$=\sum_{k=0}^{n}a^n\left(\frac{b}{a}\right)^k=\begin{cases}a^n\dfrac{1-\left(\frac{b}{a}\right)^{n+1}}{1-\frac{b}{a}}=\dfrac{a^{n+1}-b^{n+1}}{a-b}, & a\neq b\\ a^n(n+1), & a=b\end{cases}$$

Method 2 The convolution table

No.	$x_1[n]$	$x_2[n]$	$x_1[n]*x_2[n]=x_2[n]*x_1[n]$
1	$\delta[n-k]$	$x[n]$	$x[n-k]$
2	$\gamma^n u[n]$	$u[n]$	$\left[\frac{1-\gamma^{n+1}}{1-\gamma}\right]u[n]$
3	$u[n]$	$u[n]$	$(n+1)u[n]$
4	$\gamma_1^n u[n]$	$\gamma_2^n u[n]$	$\left[\frac{\gamma_1^{n+1}-\gamma_2^{n+1}}{\gamma_1-\gamma_2}\right]u[n] \qquad \gamma_1 \neq \gamma_2$
5	$u[n]$	$nu[n]$	$\frac{n(n+1)}{2}u[n]$
6	$\gamma^n u[n]$	$nu[n]$	$\left[\frac{\gamma(\gamma^n-1)+n(1-\gamma)}{(1-\gamma)^2}\right]u[n]$
7	$nu[n]$	$nu[n]$	$\frac{1}{6}n(n-1)(n+1)u[n]$
8	$\gamma^n u[n]$	$\gamma^n u[n]$	$(n+1)\gamma^n u[n]$
9	$n\gamma_1^n u[n]$	$\gamma_2^n u[n]$	$\frac{\gamma_1\gamma_2}{(\gamma_1-\gamma_2)^2}\left[\gamma_2^n-\gamma_1^n+\frac{\gamma_1-\gamma_2}{\gamma_2}n\gamma_1^n\right]u[n] \qquad \gamma_1 \neq \gamma_2$
10	$\lvert\gamma_1\rvert^n \cos(\beta n+\theta)u[n]$	$\lvert\gamma_2\rvert^n u[n]$	$\frac{1}{R}[\lvert\gamma_1\rvert^{n+1}\cos[\beta(n+1)+\theta-\phi]-\lvert\gamma_2\rvert^{n+1}\cos(\theta-\phi)]u[n]$ $R=[\lvert\gamma_1\rvert^2+\lvert\gamma_2\rvert^2-2\lvert\gamma_1\rvert\lvert\gamma_2\rvert\cos\beta]^{1/2}$ $\phi=\tan^{-1}\left[\frac{(\lvert\gamma_1\rvert\sin\beta)}{(\lvert\gamma_1\rvert\cos\beta-\lvert\gamma_2\rvert)}\right]$
11	$\gamma_1^n u[-(n+1)]$	$\gamma_2^n u[n]$	$\frac{\gamma_2}{\gamma_1-\gamma_2}\gamma_2^n u[n]+\frac{\gamma_1}{\gamma_1-\gamma_2}\gamma_1^n u[-(n+1)] \qquad \lvert\gamma_1\rvert > \lvert\gamma_2\rvert$

Example: Find the zero-state response of a system described by the equation $y[n]-0.6y[n-1]-0.16y[n-2]=5x[n]$ if the input $x[n]=4^{-n}u[n]$.

We have obtained $h[n]=[(-0.2)^n + 4(0.8)^n]u[n]$ in the foregoing example.

Therefore, $y[n]=x[n]*h[n]$

$$=(0.25)^n u[n] *[(-0.2)^n + 4(0.8)^n]u[n]$$

$$=(0.25)^n u[n]*(-0.2)^n u[n]+(0.25)^n u[n]*4(0.8)^n u[n]$$

Using pair 4 in Table, we obtain

$$y[n]=\left[\frac{(0.25)^{n+1}-(-0.2)^{n+1}}{0.25-(-0.2)}+4\frac{(0.25)^{n+1}-(0.8)^{n+1}}{0.25-0.8}\right]u[n]$$

$$=[-1.26(4)^{-n}+0.444(-0.2)^n+5.81(0.8)^n]u[n]$$

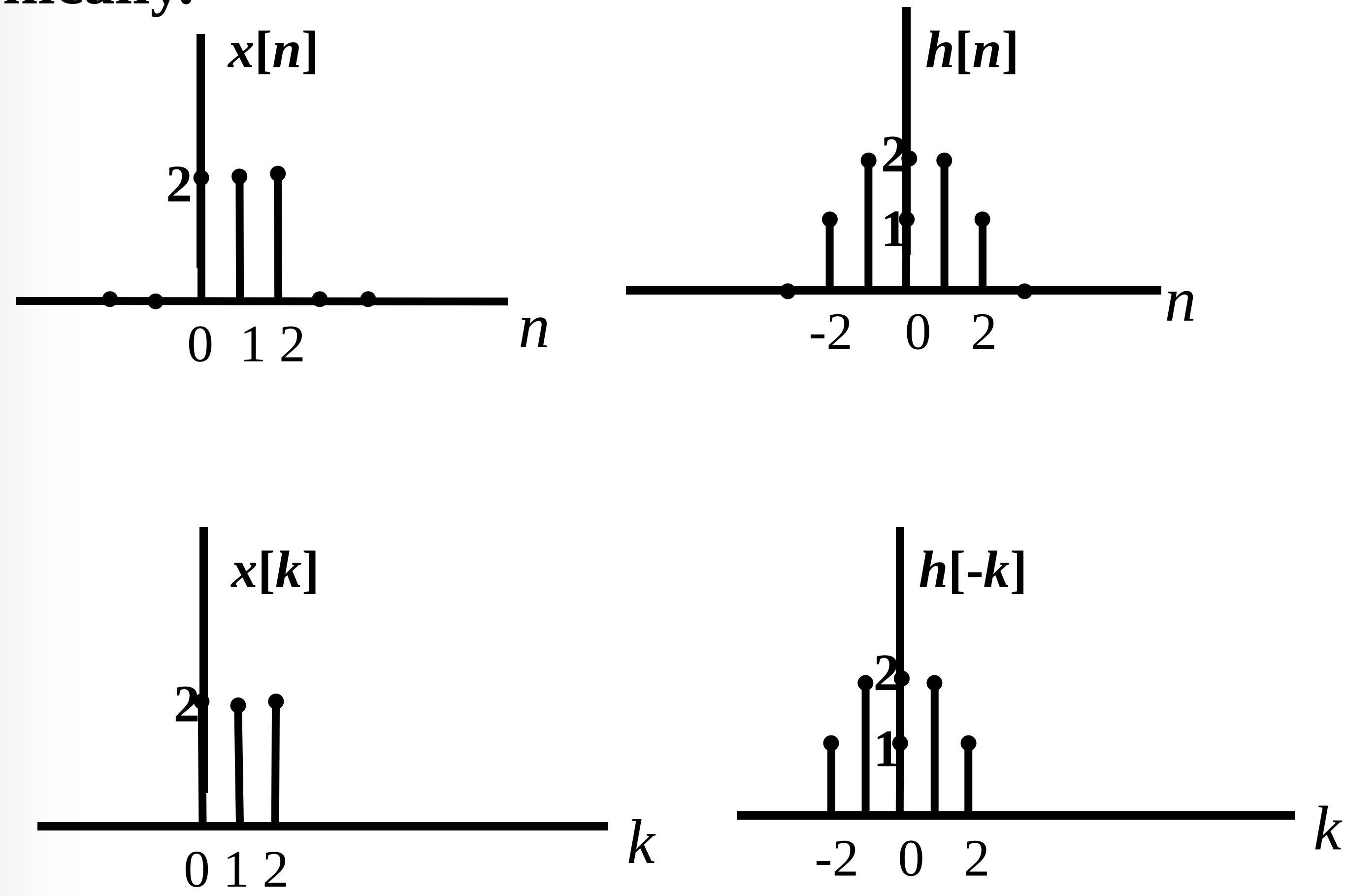
Method 3

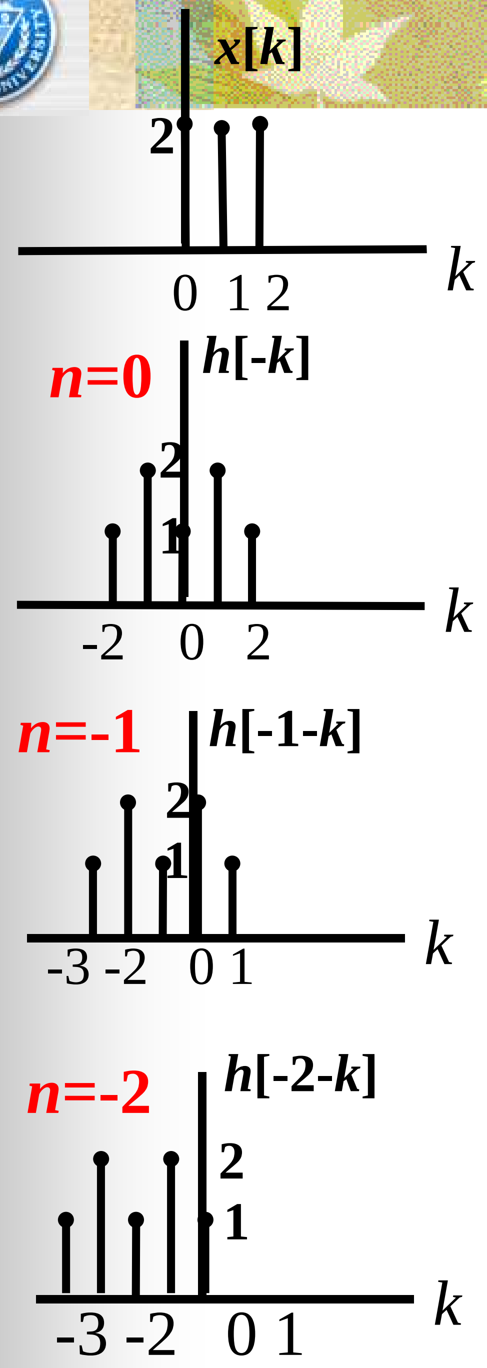
Graphical interpretation of calculating the convolution sum

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

1. **Variable substitution:** $n \longrightarrow k$
2. **Time Reversal:** $h[k] \longrightarrow h[-k]$
3. **Time Shift:** $h[-k] \longrightarrow h[n-k]$
4. **Multiplication:** $x[k]h[n-k]$
5. **Summation:** $\sum_{k=-\infty}^{\infty} x[k]h[n-k]$

Example: **Consider an LTI system with $h[n]$ and input $x[n]$. Calculate the convolution sum of these two sequences graphically.**





$$y[n]=\sum_{k=-\infty}^{+\infty}x[k]h[n-k]$$

$$y[0]=\sum_{k=-\infty}^{\infty}x[k]h[-k]$$

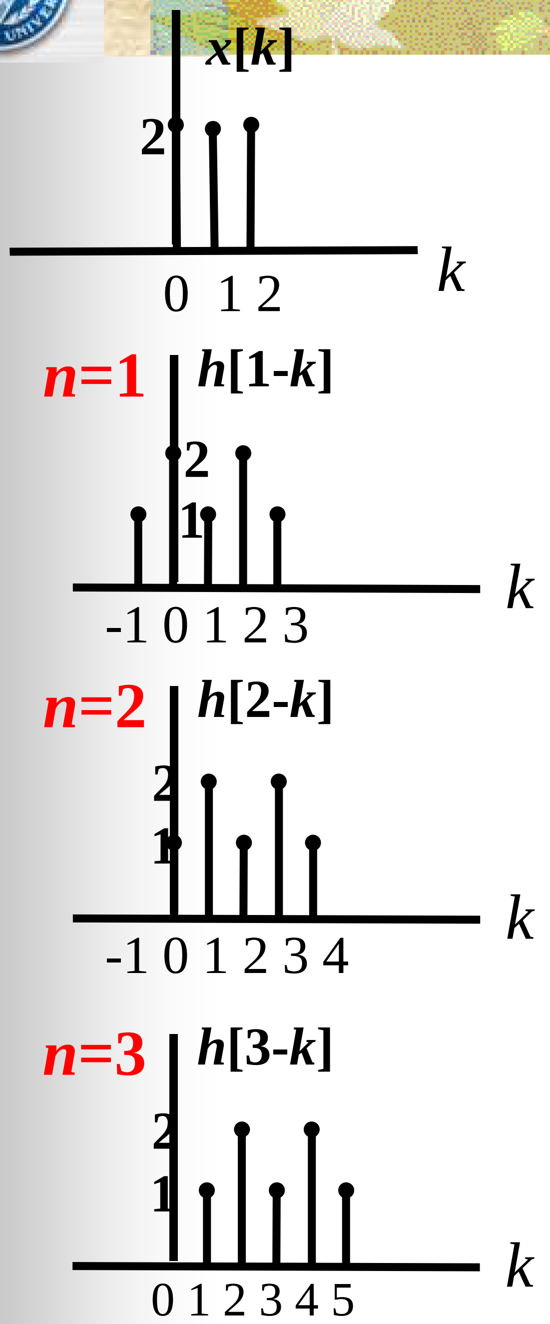
$$=x[0]h[-0]+x[1]h[-1]+x[2]h[-2]=8$$

$$y[-1]=\sum_{k=-\infty}^{\infty}x[k]h[-1-k]$$

$$=x[0]h[-1]+x[1]h[-2]=6$$

$$y[-2]=\sum_{k=-\infty}^{+\infty}x[k]h[-2-k]=x[0]h[-2]=2$$

$$y[-3]=\sum_{k=-\infty}^{+\infty}x[k]h[-3-k]=0$$



$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

$$y[1] = \sum_{k=-\infty}^{\infty} x[k]h[1-k]$$

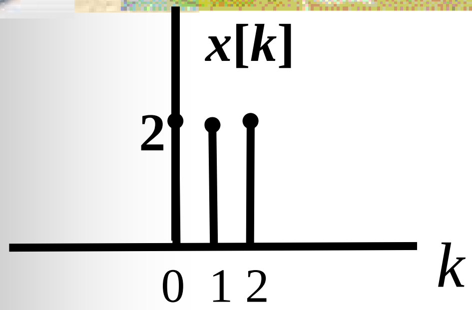
$$= x[0]h[1] + x[1]h[0] + x[2]h[-1] = 10$$

$$y[2] = \sum_{k=-\infty}^{+\infty} x[k]h[2-k]$$

$$= x[0]h[2] + x[1]h[1] + x[2]h[0] = 8$$

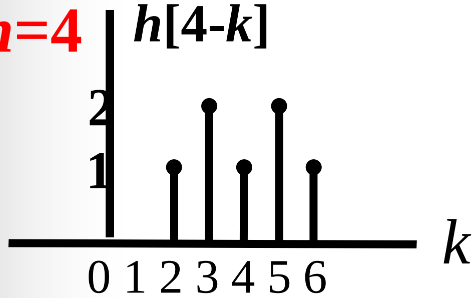
$$y[3] = \sum_{k=-\infty}^{+\infty} x[k]h[3-k]$$

$$= x[1]h[2] + x[2]h[1] + x[3]h[0] = 6$$



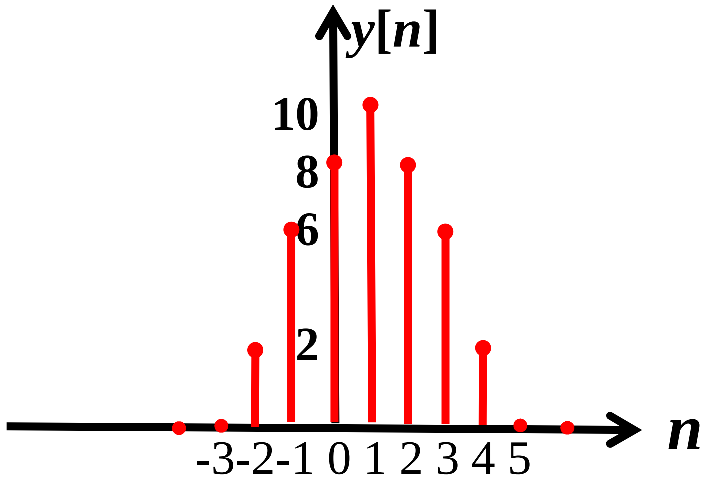
$$y[n]=\sum_{k=-\infty}^{+\infty}x[k]h[n-k]$$

n=4



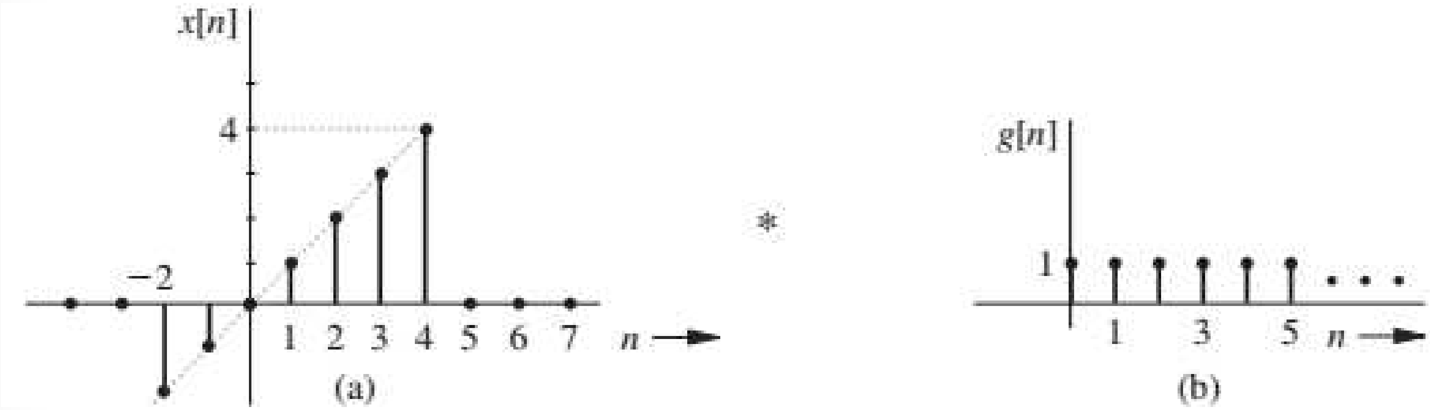
$$y[4]=\sum_{k=-\infty}^{+\infty}x[k]h[4-k]=x[2]h[2]=2$$

$$y[5]=\sum_{k=-\infty}^{+\infty}x[k]h[5-k]=0$$

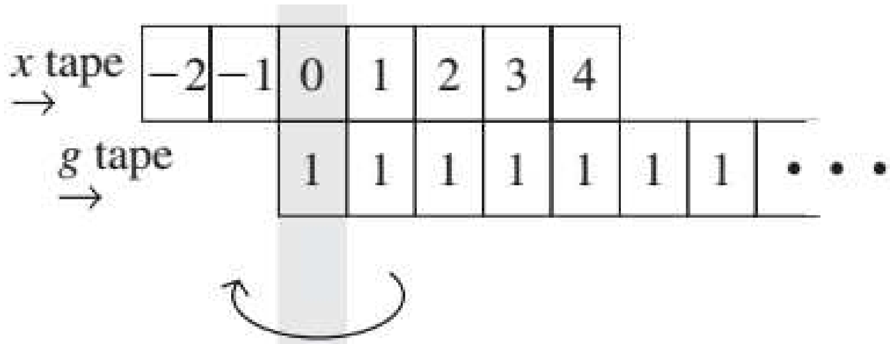


An alternative form : the sliding-tape method

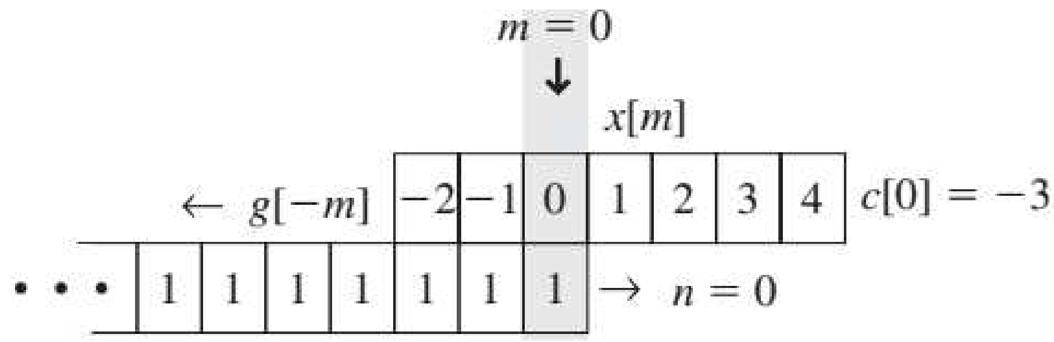
Example: **Use the sliding-tape method to convolve the two sequences depicted in the following figures.**



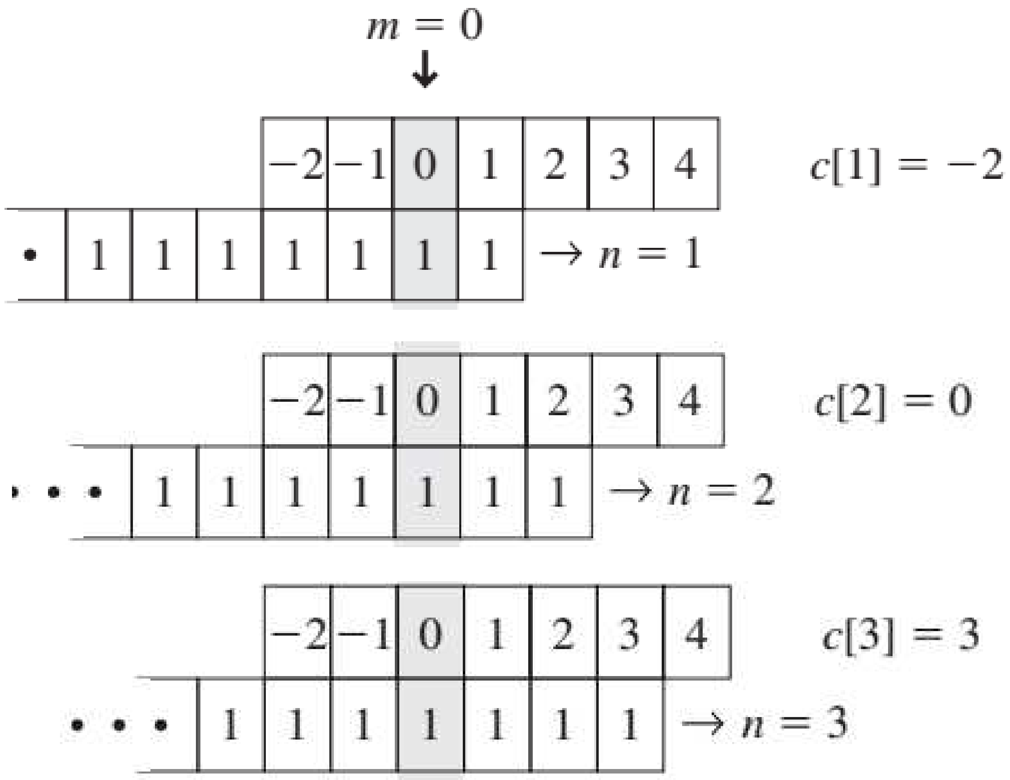
Display each signal as a sequence of numbers on tapes .

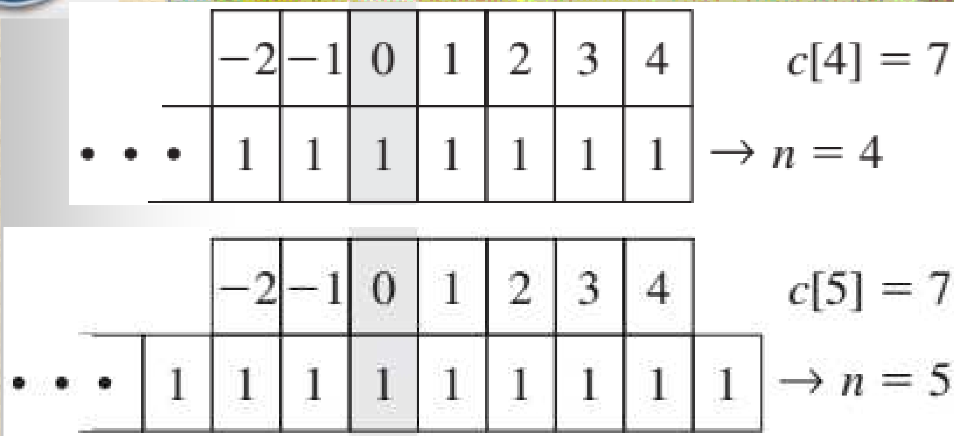


Rotate the *g* tape about the vertical axis.

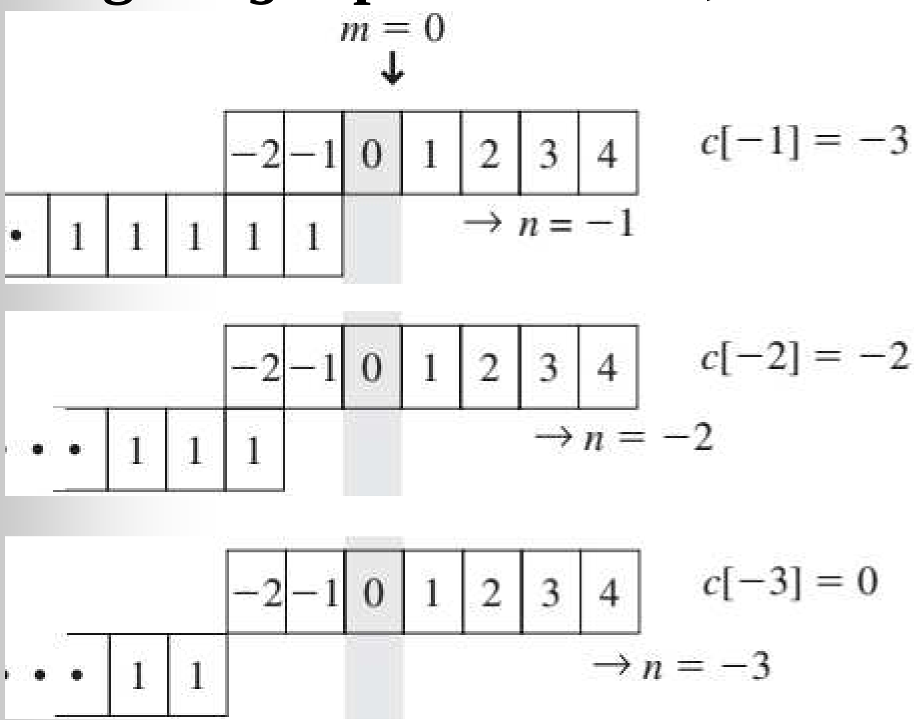


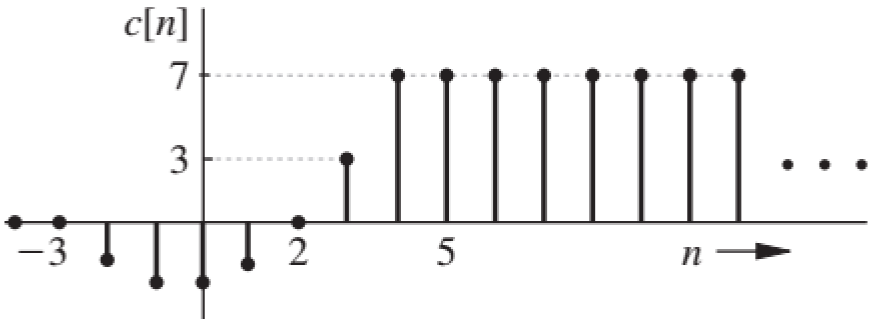
Sliding the *g* tape forward, we obtain





Sliding the *g* tape backward, we obtain





Method 4

Example:

Let
$$x_1[n] = 4\delta[n] + 3\delta[n-1] + 2\delta[n-2] + \delta[n-3],$$
$$x_2[n] = 3\delta[n] + 2\delta[n-1] + \delta[n-2],$$

determine: $y[n] = x_1[n] * x_2[n]$

Steps:

- Right alignment
- Multiplication without carry
- Addition without carry

	$x_1[n]$	:			4 ↑ $n=0$	3	2	1	
×	$x_2[n]$	:				3 ↑ $n=0$	2	1	
					4	3	2	1	
				8	6	4	2		
+			12	9	6	3			
	$y[n]$	:	12 ↑ $n=0$	17	16	10	4	1	

$$y[n]=12\delta[n]+17\delta[n-1]+16\delta[n-2]+10\delta[n-3]+4\delta[n-4]+\delta[n-5]$$

Properties

***h*[*n*] completely characterizes an LTID system**

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

The Commutative Property

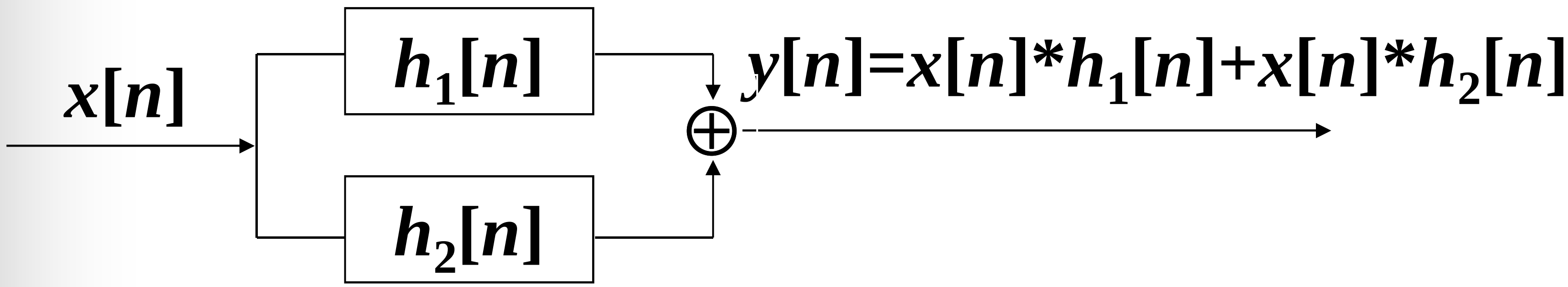
$$x[n]*h[n]=h[n]*x[n]$$

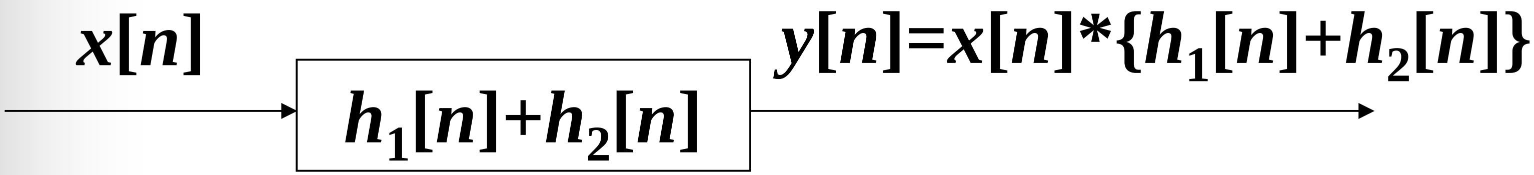
$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]=x[n]*h[n]$$

$$h[n] \longrightarrow \boxed{x[n]} \longrightarrow y[n]=h[n]*x[n]$$

The Distributive Property

$$x[n]*\{h_1[n]+h_2[n]\}=x[n]*h_1[n]+x[n]*h_2[n]$$





The Associative Property

$$x[n]*\{h_1[n]*h_2[n]\}=\{x[n]*h_1[n]\}*h_2[n]$$

$$x[n] \longrightarrow \boxed{h_1[n]} \longrightarrow \boxed{h_2[n]} \longrightarrow y[n]=x[n]*h_1[n]*h_2[n]$$

$$x[n] \longrightarrow \boxed{h_2[n]} \longrightarrow \boxed{h_1[n]} \longrightarrow y[n]=x[n]*h_2[n]*h_1[n]$$

$$x[n] \longrightarrow \boxed{h_1[n]*h_2[n]} \longrightarrow y[n]=x[n]*\{h_1[n]*h_2[n]\}$$

The Shifting Property

If $x_1[n]*x_2[n]=c[n]$, then

$$x_1[n]*x_2[n-k]= x_1[n-k]*x_2[n]= c[n-k]$$

$$x_1[n-m]*x_2[n-p]= c[n-m-p]$$

The Width Property

If the lengths of $x_1[n]$ and $x_2[n]$ are finite, given by N and M, respectively, then the length of $x_1[n]*x_2[n]$ is $N+M-1$.

➢ The Convolution with an Impulse

$$f[n]*\delta[n]=f[n]$$

$$f[n]*\delta[n-k]=f[n-k]$$

$$f[n-k_1]*\delta[n-k_2]=f[n-k_1-k_2]$$

➢ LTID System with and without Memory

Memoryless system:

DT: $y[n]=kx[n]$, $h[n]=?$

$$\delta[n] \longrightarrow \boxed{h[n]} \longrightarrow h[n]=k\delta[n]$$

Invertibility of LTID system

Inverse system

$$x[n] \longrightarrow \boxed{h_1[n]} \longrightarrow y[n] \longrightarrow \boxed{h_2[n]} \longrightarrow x[n]$$

$$x[n] \longrightarrow \boxed{\begin{array}{c} h[n]=\delta[n] \\ \text{(Identity system)} \end{array}} \longrightarrow x[n] = x[n]*\delta[n]$$

$$h_1[n]*h_2[n]=\delta[n]$$

➢ Causality for LTID system

DT system : $h[n]=0$ for $n<0$

LTID system is causal $\Leftrightarrow$ It's unit impulse response is a causal signal

Example:

System: $y[n]=x[n]+4x[n+2]$ Not causal

$h[n]=\delta[n]+4\delta[n+2]$

System: $y[n]=x[n-5]+2x[n]$ Causal

$h[n]=\delta[n-5]+2\delta[n]$

Stability for LTID system

Bounded input produces bounded output.

DT system:

If $|x[n]|<B, \left|y[n]\right| = \left|\sum_{k=-\infty}^{\infty} h[k]x[n-k]\right| \leq \sum_{k=-\infty}^{\infty} \left|h[k]\right|\left|x[n-k]\right| \leq B \sum_{k=-\infty}^{\infty} \left|h[k]\right|$

LTID system is stable $\Leftrightarrow$ $\sum_{k=-\infty}^{+\infty} |h[k]| < +\infty$

Example:

$y[n] = ax[n] + b$

$h[n] = a\delta[n] + b$

Stable

$y[n] = x[n-n_0]$

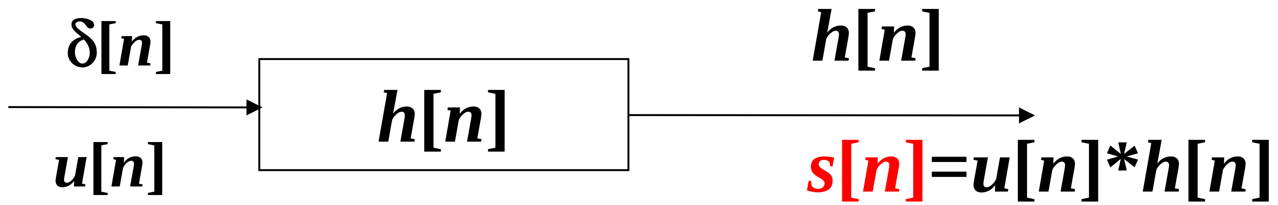
$h[n] = \delta[n-n_0]$

Stable

Not linear. The first method is valid.

The Unit Step Response of LTID system

The unit step response: $s[n]$



$$u[n]=\sum_{k=-\infty}^{n}\delta[k]$$

$$s[n]=\sum_{k=-\infty}^{+\infty}h[k]u[n-k]=\sum_{k=-\infty}^{n}h[k]$$

$$\delta[n]=u[n]-u[n-1]$$

$$h[n]=s[n]-s[n-1]$$

A very special function for LTID systems: The everlasting exponential z^n

The system response to an everlasting exponential z^n is

$$y[n] = h[n] * z^n = \sum_{m=-\infty}^{\infty} h[m] z^{n-m} = z^n \sum_{m=-\infty}^{\infty} h[m] z^{-m}$$

Assuming the sum converges

$$H[z] = \sum_{m=-\infty}^{\infty} h[m] z^{-m}$$

$$\therefore\ y[n] = H[z] z^n$$

The input and the output are the same (within a multiplicative constant) for the everlasting exponential signal.

The equation of a system is

$$Q[E]y[n] = P[E]x[n]$$

Substitution of this $x[n] = z^n$ and $y[n] = H(z)z^n$ in the above equation yields

$$H(z)\{Q[E]z^n\} = P[E]z^n$$

Moreover $E^k z^n = z^{n+k} = z^k z^n$ $\Rightarrow$

$$Q[E]z^n = Q[z]z^n$$

$$P[E]z^n = P[z]z^n$$

Therefore, the transfer function of the system is

$$H[z] = \frac{P[z]}{Q[z]}$$

Total response

$$y[n] = \underbrace{y_0[n]}_{\text{zero-input response}} + \underbrace{x[n]*h[n]}_{\text{zero-state response}}$$

Example: **Determine the total response of the system described by**

$$y[n+2]-0.6y[n+1]-0.16y[n]=5x[n+2]$$

with input $x[n]=(4)^{-n}u[n]$ and initial conditions $y[-1]=0$, $y[-2]=25/4$.

$$y[n]=\underbrace{0.2(-0.2)^n+0.8(0.8)^n}_{\text{zero-input response}}+\underbrace{0.444(-0.2)^n+5.81(0.8)^n-1.26(4)^{-n}}_{\text{zero-state response}}$$

$$n\geq 0$$

$$y[n]=\underbrace{0.644(-0.2)^n+6.61(0.8)^n}_{\text{natural response}}\underbrace{-1.26(4)^{-n}}_{\text{forced response}} \qquad n\geq 0$$

Systems stability

- The BIBO stability of a system is an external stability criterion.
- The BIBO stability of a system may not be correct indication of the internal stability.
- The internal stability is all-inclusive, and the external stability can always be determined from the internal stability.

1、Internal (Asymptotic) stability

The internal stability is defined in terms of the zero-input response of a system. The zero-input response consists of the characteristic modes, γ^n.

To be more general, let γ be complex so that

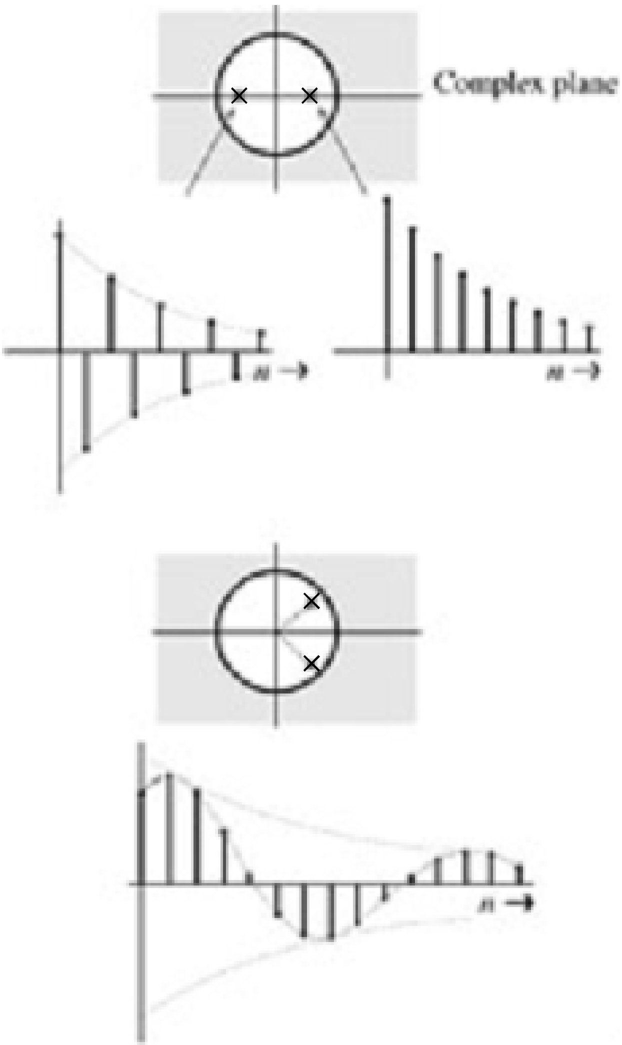
$$\gamma=|\gamma|e^{j\beta} \text{ and } \gamma^n=|\gamma|^n e^{j\beta n}$$

$$\text{If } |\gamma|<1,\ \gamma^n \to 0 \text{ as } n\to\infty$$

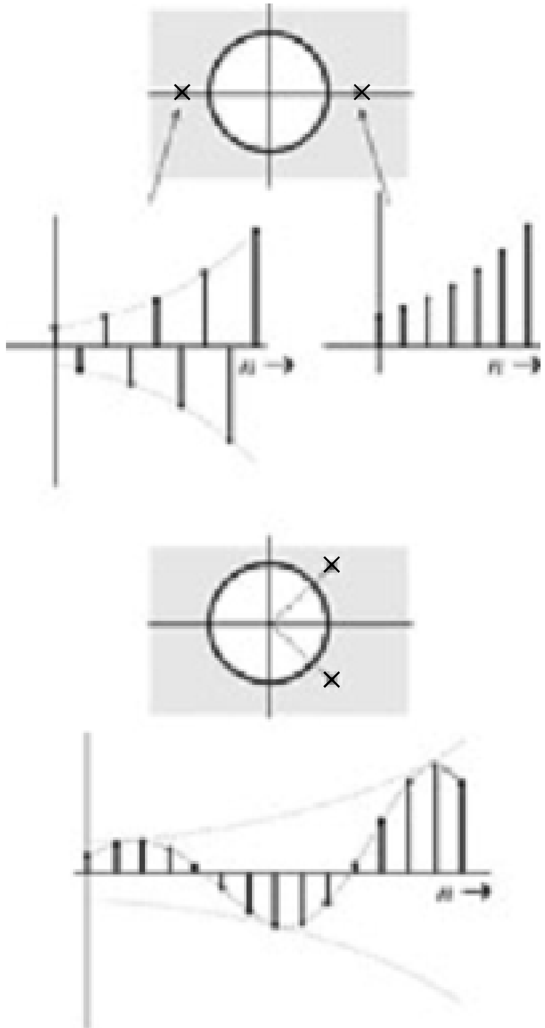
$$\text{If } |\gamma|>1,\ \gamma^n \to \infty \text{ as } n\to\infty$$

$$\text{If } |\gamma|=1,\ |\gamma|^n=1 \text{ for all } n$$

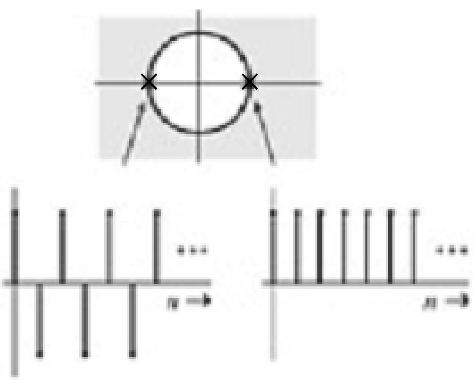
All roots lie inside the unit circle. The system is asymptotically stable.

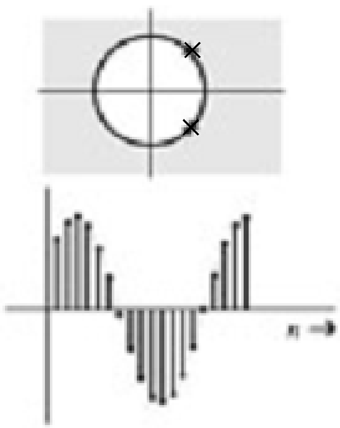


Even if one root lies outside the unit circle. The system is unstable.

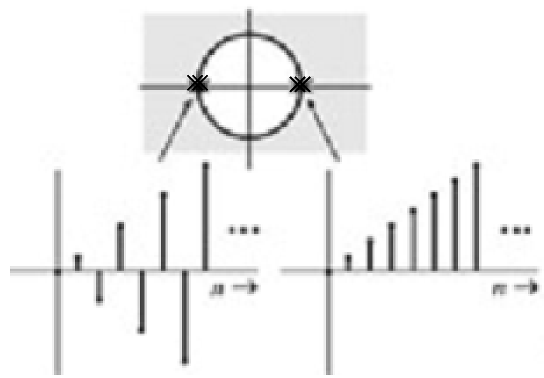


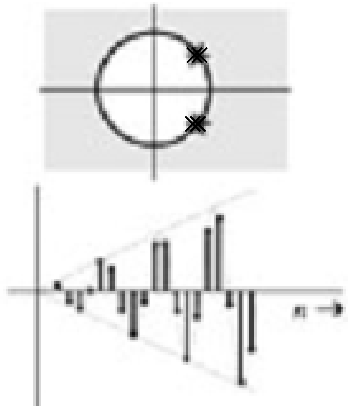
Some simple roots lie on the unit circle. The system is marginally stable.





If two or more roots coincide on the unit circle. The system is unstable.





The internal stability criterion:

- **An LTID system is asymptotically stable if, and only if, all the characteristic roots are inside the unit circle. The roots may be simple (unrepeated) or repeated.**

- **An LTID system is unstable if, and only if, one or both of the following conditions exist: (i) at least one root is outside the unit circle; (ii) there are repeated roots on the unit circle.**

- **An LTID system is marginally stable if, and only if, there are no roots outside the unit circle, and there are some unrepeated roots on the unit circle.**

2、Relationship between BIBO and Asymptotic stability

- **An asymptotically stable system is BIBO stable.**

- **A marginally stable or asymptotically unstable system is BIBO unstable.**

- **BIBO stability cannot assure internal stability.**

- **When a system is controllable and observable, its external and internal descriptions are equivalent.**

Example: An LTID system consists of two subsystems in cascade. $h_1[n]=4\delta[n]-3(0.5)^n u[n]$ and $h_2(n)=2^n u[n]$.

$$x[n] \longrightarrow \boxed{S_1} \longrightarrow \boxed{S_2} \longrightarrow y[n]$$

$$h[n]=h_1[n]*h_2[n]=\{4\delta[n]-3(0.5)^n u[n]\}*2^n u[n]$$

$$=4(2)^n u[n]-3\left[\frac{2^{n+1}-0.5^{n+1}}{2-0.5}\right]u[n]$$

$$=(0.5)^n u[n]$$

$$\sum_{k=-\infty}^{+\infty}|h[k]|<+\infty \Rightarrow$$ The composite system is BIBO stable

S_2 has a characteristic root 2. $\Rightarrow$ S_2 is asymptotically unstable

BIBO stability cannot assure internal stability.

Example: Investigate the asymptotic and the BIBO stability of LTID system described by the following equations.

	The characteristic roots	
1) $y[n+2]+2.5y[n+1]+y[n]=x[n+1]-2\,x[n]$	-0.5, -2	Unstable in both senses
2) $y[n]-y[n-1]+0.21y[n-2]=2x[n-1]-2\,x[n]$	0.3, 0.7	Stable in both senses
3) $y[n+3]+2y[n+2]+3y[n+1]/2+y[n]/2=x[n+1]$	-1, $-0.5\pm j0.5$	Marginally stable BIBO unstable
4) $(E^2-E+1)^2y[n]=(2E+4)x[n]$	$e^{\pm j(\pi/3)}$, $e^{\pm j(\pi/3)}$	Unstable in both senses