Chapter 2

Time-Domain Analysis of Continuous-Time Systems

Linear constant coefficient differential equations are used to describe LTIC systems.

$$\frac{d^n y(t)}{dt^n} + a_1 \frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_{n-1} \frac{dy(t)}{dt} + a_n y(t)$$

$$= b_{n-m} \frac{d^m x(t)}{dt^m} + b_{n-m+1} \frac{d^{m-1} x(t)}{dt^{m-1}} + \cdots + b_{n-1} \frac{dx(t)}{dt} + b_n x(t)$$

Using operational notation D to represent d/dt, we can get

$$(D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n) y(t) = (b_{n-m} D^m + b_{n-m+1} D^{m-1} + \cdots + b_{n-1} D + b_n) x(t)$$

or $$Q(D) y(t) = P(D) x(t)$$

where

$$Q(D) = D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n$$

$$P(D) = b_{n-m} D^m + b_{n-m+1} D^{m-1} + \cdots + b_{n-1} D + b_n$$

Practical systems generally use $m \le n$.

The total response of an LTIC system

total response = zero-input response + zero-state response

- The zero-input response is the system response when the input $x(t)=0$.

- The zero-state response is the system response to the external input $x(t)$ when the system is in zero state.

System response to internal conditions: the zero-input response

When the input $x(t)$=0, the zero-input response $y_0(t)$ is the solution of the equation $Q(D)y(t) = p(D)x(t)$

$$Q(D)y_0(t) = 0$$

$$\Downarrow$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) y_0(t) = 0$$

It is possible if and only if $y_0(t)$ and all its N successive derivatives are of the same form.

Only an exponential function has this property.

Assume $y_0(t) = ce^{\lambda t}$

Then $Dy_0(t) = \frac{dy_0}{dt} = c\lambda e^{\lambda t}$

$$\vdots$$

$$D^N y_0(t) = \frac{d^N y_0}{dt^N} = c\lambda^N e^{\lambda t}$$

Substituting these results in the preceding equation, we obtain

$$c(\lambda^N + a_1\lambda^{N-1} + \cdots + a_{N-1}\lambda + a_N)e^{\lambda t} = 0$$

$$\Downarrow$$

$$\lambda^N + a_1\lambda^{N-1} + \cdots + a_{N-1}\lambda + a_N = 0$$

$Q(\lambda)$ is the **characteristic polynomial**.

The corresponding **characteristic equation is**

$$Q(\lambda)=0 \quad \Longrightarrow \quad \lambda^N + a_1\lambda^{N-1} + \cdots + a_{N-1}\lambda + a_N = 0$$

The roots $\lambda_i (i = 1, 2, ..., N)$ are the **characteristic roots**.

The exponentials $e^{\lambda_i t} (i = 1, 2, ..., N)$ are the **characteristic modes**.

$y_0(t)$ is a linear combination of $e^{\lambda_i t} (i = 1, 2, ..., N)$.

The characteristic roots of an LTIC system:

- **There are N distinct roots**

$$y_0(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \cdots + c_N e^{\lambda_N t} = \sum_{i=1}^{N} c_i e^{\lambda_i t}$$

- **There are u distinct roots and p k_jth-order repeated roots**

$$y_0(t) = \sum_{i=1}^{u} c_i e^{\lambda_i t} + \sum_{j=1}^{p} \sum_{l=1}^{k_j} d_{j,l} t^{k_j - l} e^{\lambda_j t}$$

- **There are complex roots**

Complex roots occur in pairs of conjugates if the system equation coefficients are real. If $\alpha+j\beta$ is a characteristic root, $\alpha-j\beta$ must also be a characteristic root.

The zero-input response corresponding to this pair of complex roots is

$$y_0(t) = c_1 e^{(\alpha + j\beta)t} + c_2 e^{(\alpha - j\beta)t}$$

For a real system, the response must also be real. This is possible only if c_1 and c_2 are conjugates. Let

$$c_1 = \frac{c}{2} e^{j\theta}, c_2 = \frac{c}{2} e^{-j\theta}$$

This yields

$$\begin{aligned} y_0(t) &= \frac{c}{2} e^{j\theta} e^{(\alpha + j\beta)t} + \frac{c}{2} e^{-j\theta} e^{(\alpha - j\beta)t} \\ &= \frac{c}{2} e^{\alpha t} [e^{j(\beta t + \theta)} + e^{-j(\beta t + \theta)}] \\ &= c e^{\alpha t} \cos(\beta t + \theta) \end{aligned}$$

Example: Find the zero-input response $y_0(t)$ for an LTIC system described by the following differential equation.

$$\frac{d^2y(t)}{dt^2}+3\frac{dy(t)}{dt}+2y(t)=\frac{dx(t)}{dt}+2x(t)$$

when the initial conditions are $y_0(0)=1$, $\left.\frac{dy_0(t)}{dt}\right|_{t=0}=1$

The characteristic equation is

$$\lambda^2+3\lambda+2=0$$

The characteristic roots of the system are $\lambda_1=-1$， $\lambda_2=-2$

The characteristic modes of the system are e^{-t} and e^{-2t}

Therefore, $y_0(t)=c_1e^{-t}+c_2e^{-2t}$ $\dot{y}_0(t)=-c_1e^{-t}-2c_2e^{-2t}$

Substituting the initial conditions, we obtain

$$\begin{cases} c_1+c_2=1 \\ -c_1-2c_2=1 \end{cases} \Rightarrow c_1=3,\quad c_2=-2$$

Therefore, $y_0(t)=3e^{-t}-2e^{-2t}, \quad t\geq 0$

Example: **Find the zero-input response $y_0(t)$ for an LTIC system described by the following differential equation.**

$$\frac{d^2y(t)}{dt^2}+6\frac{dy(t)}{dt}+9y(t)=\frac{dx(t)}{dt}+2x(t)$$

when the initial conditions are $y_0(0)=3$, $\left.\frac{dy_0(t)}{dt}\right|_{t=0}=-7$.

The characteristic equation is

$$\lambda^2+6\lambda+9=0$$

The characteristic roots of the system are $\lambda_1=\lambda_2=-3$

The characteristic modes of the system are e^{-3t} **and** te^{-3t}

Therefore, $y_0(t)=c_1e^{-3t}+c_2te^{-3t}$

$$\dot{y}_0(t) = -3c_1e^{-3t} + c_2e^{-3t} - 3c_2te^{-3t}$$

Substituting the initial conditions, we obtain

$$\begin{cases} c_1=3 \\ -3c_1+c_2=-7 \end{cases} \Rightarrow \quad c_1=3, \quad c_2=2$$

Therefore, $y_0(t)=(3+2t)e^{-3t}, \quad t\geq 0$

Example: Find the zero-input response $y_0(t)$ for an LTIC system described by the following differential equation.

$$\frac{d^2y(t)}{dt^2}+4\frac{dy(t)}{dt}+40y(t)=\frac{dx(t)}{dt}+2x(t)$$

when the initial conditions are $y_0(0)=2, \frac{dy_0(t)}{dt}\Big|_{t=0}=16.78$

The characteristic equation is

$$\lambda^2+4\lambda+40=0$$

The characteristic roots of the system are $\lambda_1=-2+j6$, $\lambda_2=-2-j6$

The characteristic modes of the system are $e^{(-2+j6)t}$ and $e^{(-2-j6)t}$

Therefore, $y_0(t)=ce^{-2t}\cos(6t+\theta)$

$$\dot{y}_0(t)=-2ce^{-2t}\cos(6t+\theta)-6ce^{-2t}\sin(6t+\theta)$$

Substituting the initial conditions, we obtain

$$\begin{cases} c\cos\theta=2 \\ -2c\cos\theta-6c\sin\theta=16.78 \end{cases} \Rightarrow \begin{matrix} c\cos\theta=2 \\ c\sin\theta=-3.463 \end{matrix}$$

Squaring and then adding the two sides of the above equations

yields $c^2=(2)^2+(-3.463)^2 \Rightarrow c=4$

Dividing $c\sin\theta$ by $c\cos\theta$ yields $\tan\theta=-3.463/2 \Rightarrow \theta=-\pi/3$

Therefore, $y_0(t)=4e^{-2t}\cos(6t-\pi/3),\quad t\geq 0$

Practical initial conditions and the meaning of 0^- and 0^+

- In practical problems, we must derive initial conditions from the physical situation.
- The conditions immediately before t=0 (just before the input is applied) are the conditions at t=0^-.
- The conditions immediately after t=0 (just after the input is applied) are the conditions at t=0^+.
- At t=0^-, the input has not started yet.

$$y(0^-) = y_0(0^-), \qquad \dot{y}(0^-) = \dot{y}_0(0^-), \qquad \cdots$$

- At t=0^+, the input does not affect $y_0(t)$.

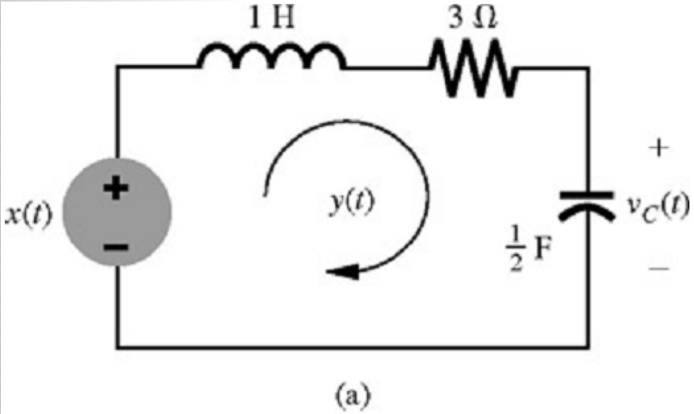
$$y_0(0^-) = y_0(0^+), \qquad \dot{y}_0(0^-) = \dot{y}_0(0^+), \qquad \cdots$$

In general,

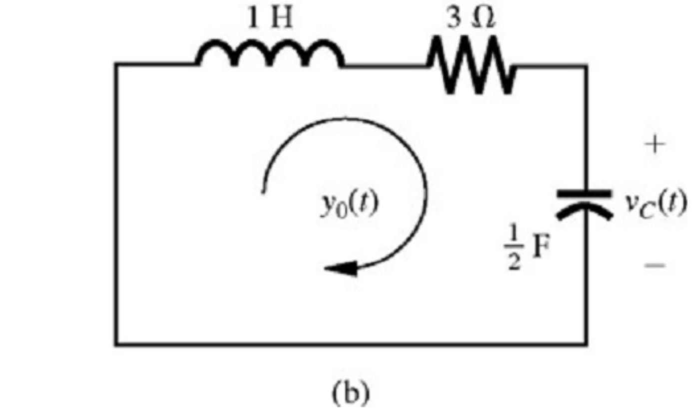
$$y(0^-) \neq y(0^+), \qquad \dot{y}(0^-) \neq \dot{y}(0^+), \qquad \cdots$$

Example: A voltage $x(t)=10e^{-3t}u(t)$ is applied at the input of the RLC circuit. Find the initial conditions of the system with $y(0^-)=0$ and $v_c(0^-)=5$.

$$L\frac{dy(t)}{dt}+Ry(t)+v_c(t)=x(t)$$



(a)



(b)

1) Find the initial conditions for the zero-input response

$$\dot{y}_0(0)+3y_0(0)+v_c(0)=0$$

The inductor current and the capacitor voltage cannot change instantaneously. Therefore,

$y_0(0)=0$ and $v_c(0)=5$ $\Rightarrow$ $\dot{y}_0(0)=-5$

The initial conditions are

$$y_0(0)=0,\ \dot{y}_0(0)=-5$$

2) Find the initial conditions for the total response

At t=0$^-$, $x(t)$=0. $$\dot{y}(0^-) + 3y(0^-) + v_c(0^-) = 0$$

At t=0$^+$, $x(t)$=10. $$\dot{y}(0^+) + 3y(0^+) + v_c(0^+) = 10$$

$y(0^+)=y(0^-)=0$ and $v_c(0^+)=v_c(0^-)=5$

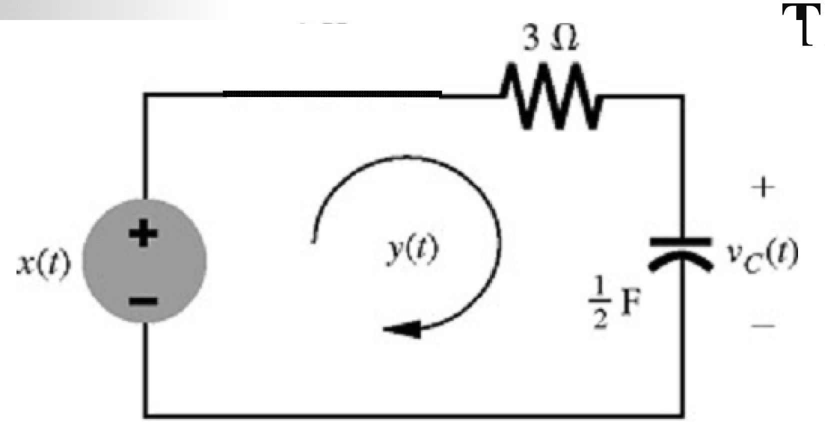
$$\Downarrow$$

$$\dot{y}(0^-) = -5, \dot{y}(0^+) = 5$$

The initial conditions are

$$y(0^-) = 0, y(0^+) = 0, \dot{y}(0^-) = -5, \dot{y}(0^+) = 5$$

Example: **In the circuit, $v_c(0)$=30. Find the zero-input component of the loop current.**



The loop equation for the system is

$$3y(t)+\frac{1}{\frac{1}{2}}\int y(t)dt = x(t)$$

$$(3D+2)y(t)=Dx(t)$$

$$(3D+2)y_0(t)=0$$

$$\lambda=-\frac{2}{3} \Rightarrow y_0(t)=ce^{-\frac{2}{3}t}$$

$$3y_0(0)+v_c(0)=0$$

$$v_c(0)=30$$

$$\Rightarrow 3c+30=0 \Rightarrow c=-10$$

Therefore, $y_0(t)=-10e^{-\frac{2}{3}t}$

The unit impulse response $h(t)$

The impulse response $h(t)$ is the system zero-state response to an impulse input $\delta(t)$ applied at t=0.

An Nth-order differential equation is

$$\frac{d^n y(t)}{dt^n}+a_1\frac{d^{n-1}y(t)}{dt^{n-1}}+\cdots+a_{n-1}\frac{dy(t)}{dt}+a_n y(t)$$

$$Q(D)y(t)=P(D)x(t)$$

$$=b_0\frac{d^m x(t)}{dt^m}+b_1\frac{d^{m-1}x(t)}{dt^{m-1}}+\cdots+b_{m-1}\frac{dx(t)}{dt}+b_m x(t)$$

When the input $x(t)$=$\delta(t)$， the output $y(t)$=$h(t)$

$$\frac{d^n h(t)}{dt^n}+a_1\frac{d^{n-1}h(t)}{dt^{n-1}}+\cdots+a_{n-1}\frac{dh(t)}{dt}+a_n h(t)$$

$$Q(D)h(t)=P(D)\delta(t)$$

$$=b_0\frac{d^m \delta(t)}{dt^m}+b_1\frac{d^{m-1}\delta(t)}{dt^{m-1}}+\cdots+b_{m-1}\frac{d\delta(t)}{dt}+b_m \delta(t)$$

The impulse input $\delta(t)$ creates nonzero initial conditions instantaneously within the system at $t=0^+$. The system will have a response generated by these newly created initial conditions.

Therefore, $h(t)$ must consist of the system's characteristic modes for $t \geq 0^+$.

$$h(t)=\text{characteristic mode terms} \qquad t \geq 0^+.$$

This response is valid for $t > 0$. What happens at $t = 0$?

At a single moment $t = 0$, there can at most be an impulse.

$$h(t)=b_0\delta(t)+\text{characteristic mode terms}$$

b_0 is the coefficient of the highest-order derivative term on the right-hand side of the system equation, when $m = n$.

$b_0=0$, when $m < n$.

Simplified impulse matching method

$$h(t)=b_0\delta(t)+[P(D)y_n(t)]u(t)$$

Where $y_n(t)$ is a linear combination of the characteristic modes of the system subject to the following initial conditions:

$$y_n(0)=\dot{y}_n(0)=\ddot{y}_n(0)=\ldots=y_n^{N-2}(0)=0 \quad \text{and} \quad y_n^{N-1}(0)=1$$

Where $y^k{}_n(0)$ is the value of the kth derivative of $y_n(t)$ at t=0.

$$N=1: y_n(0)=1$$

$$N=2: y_n(0)=0, \dot{y}_n(0)=1$$

$$N=3: y_n(0)=\dot{y}_n(0)=0, \ddot{y}_n(0)=1$$

…

Example: Find the impulse response $h(t)$ for a system specified by $(D^2+5D+6)y(t)=(D+1)x(t)$.

The characteristic equation is $\lambda^2+5\lambda+6=0$

The characteristic roots of the system are $\lambda_1=-2$, $\lambda_2=-3$

The characteristic modes of the system are e^{-2t} and e^{-3t}

Therefore, $y_n(t)=c_1e^{-2t}+c_2e^{-3t}$ $\quad \because y_n(0)=0, \dot{y}_n(0)=1$

$$\therefore \begin{array}{l} c_1+c_2=0 \\ -2c_1-3c_2=1 \end{array} \Rightarrow \begin{array}{l} c_1=1 \\ c_2=-1 \end{array} \Rightarrow y_n(t)=e^{-2t}-e^{-3t}$$

For $m<n$, $b_0=0$ $\quad h(t)=[P(D)y_n(t)]u(t)$ and $P(D)=D+1$

Therefore, $h(t)=[P(D)(e^{-2t}-e^{-3t})]u(t)=(-e^{-2t}+2e^{-3t})u(t)$

Example: Find the impulse response $h(t)$ for a system described by $(D+2)y(t)=(3D+5)x(t)$.

The characteristic equation is $\lambda+2=0$

The characteristic root of the system is $\lambda=-2$

The characteristic mode of the system is e^{-2t}

Therefore, $y_n(t)=ce^{-2t}$ $\quad \because y_n(0)=1$

$\therefore c=1 \Longrightarrow y_n(t)=e^{-2t}$

$\because m=n \Rightarrow b_0=3$ $\quad h(t)=3\delta(t)+[P(D)y_n(t)]u(t)$ and $P(D)=3D+5$

Therefore, $h(t)=3\delta(t)+[P(D)(e^{-2t})]u(t)=3\delta(t)-e^{-2t}u(t)$

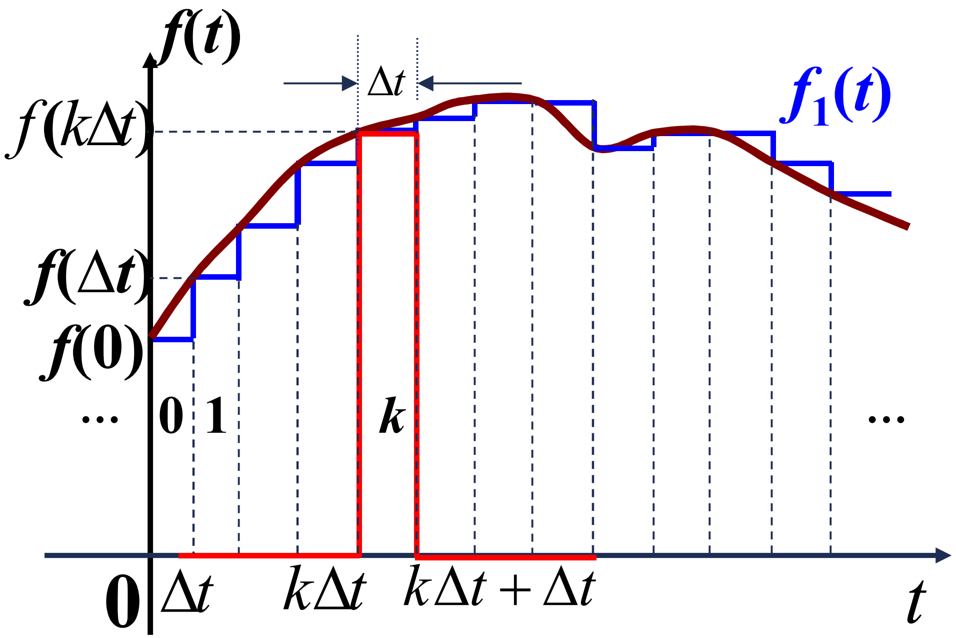
In above discussion, we have assumed $m \le n$.

The expression for $h(t)$ applicable to all possible values of m and n is given by

$$h(t)=P(D)[y_n(t)u(t)]$$

System response to external input: zero-state response

1、Representation of CT Signals in Terms of Impulses

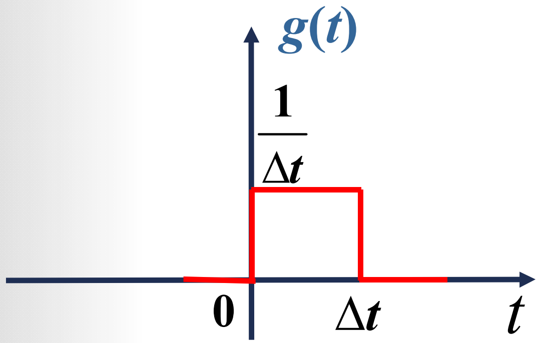


***f*(*t*) can be approximated as :**

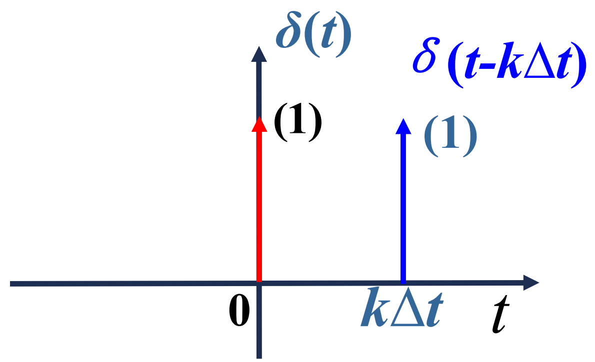
$$f(t) \approx f_1(t) = \sum_{k=-\infty}^{\infty} f(k\Delta t)[u(t-k\Delta t)-u(t-k\Delta t-\Delta t)]$$

$$f(t)=\lim_{\Delta t\to 0}\sum_{k=-\infty}^{\infty} f(k\Delta t)\boxed{\frac{u(t-k\Delta t)-u(t-k\Delta t-\Delta t)}{\Delta t}}\Delta t$$

$$\delta(t-k\Delta t)$$



$\Delta t \to 0$



$$f(t) = \lim_{\Delta t \to 0} \sum_{k=-\infty}^{\infty} f(k\Delta t)\delta(t - k\Delta t)\Delta t$$

When $\Delta t \to 0$

$$k\Delta t \to \tau, \quad \Delta t \to d\tau, \quad \sum_{-\infty}^{\infty} \to \int_{-\infty}^{\infty}$$

$$\boxed{f(t) = \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)d\tau}$$

2、The CT Unit Impulse Response and the Convolution Integral Representation of LTI Systems

Unit Impulse Response $h(t)$:

Response of the LTI system to the unit impulse $\delta(t)$.

Question:

$$f(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow y(t)=?$$

Solution:

$$\delta(t) \Longrightarrow h(t)$$

$$\delta(t-\tau) \Longrightarrow h(t-\tau)$$

$$f(\tau)\,\delta(t-\tau) \Longrightarrow f(\tau)\,h(t-\tau)$$

$$\underbrace{\int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)\,\mathrm{d}\tau}_{=\,f(t)} \Longrightarrow \boxed{\underbrace{\int_{-\infty}^{\infty} f(\tau)h(t-\tau)\,\mathrm{d}\tau}_{=\,y(t)}}$$

Convolution Integral

Convolution Integral

$$f(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)=\int_{-\infty}^{+\infty} f(\tau)h(t-\tau)d\tau$$

$$= f(t) * h(t)$$

- An LTIC system is completely characterized by its unit impulse response $h(t)$.

Method 1

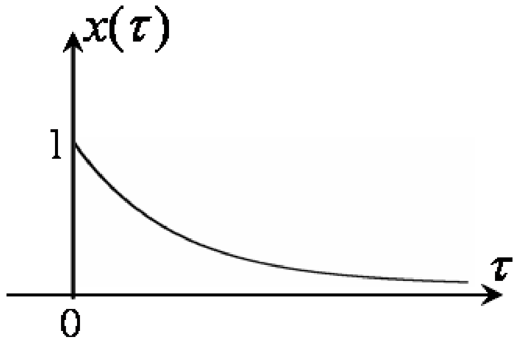
Example: **Let the input** $x(t)=e^{-at}u(t)$ $a>0$ **and** $h(t)=u(t)$,

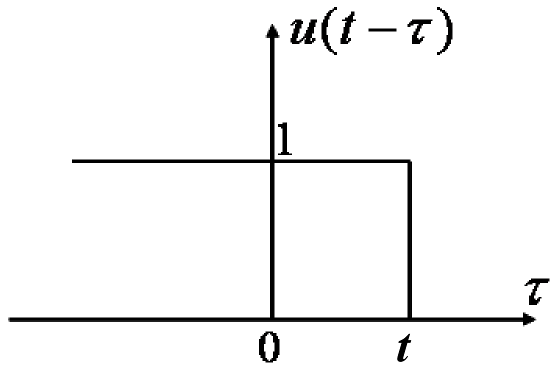
determine $y(t)=x(t)*h(t)$.

Solution: $y(t)=x(t)*h(t)=\int_{-\infty}^{\infty}x(\tau)h(t-\tau)\mathrm{d}\tau$

$$=\int_{-\infty}^{\infty}e^{-a\tau}u(\tau)u(t-\tau)\mathrm{d}\tau$$

$$=\int_{0}^{t}e^{-a\tau}\mathrm{d}\tau=\frac{1}{a}(1-e^{-at})u(t)$$





Method 2

The convolution table

No.	$x_1(t)$	$x_2(t)$	$x_1(t)*x_2(t) = x_2(t)*x_1(t)$
1	$x(t)$	$\delta(t-T)$	$x(t-T)$
2	$e^{\lambda t}u(t)$	$u(t)$	$\frac{1-e^{\lambda t}}{-\lambda}u(t)$
3	$u(t)$	$u(t)$	$tu(t)$
4	$e^{\lambda_1 t}u(t)$	$e^{\lambda_2 t}u(t)$	$\frac{e^{\lambda_1 t}-e^{\lambda_2 t}}{\lambda_1-\lambda_2}u(t) \quad \lambda_1 \neq \lambda_2$
5	$e^{\lambda t}u(t)$	$e^{\lambda t}u(t)$	$te^{\lambda t}u(t)$
6	$te^{\lambda t}u(t)$	$e^{\lambda t}u(t)$	$\frac{1}{2}t^2e^{\lambda t}u(t)$
7	$t^N u(t)$	$e^{\lambda t}u(t)$	$\frac{N!\,e^{\lambda t}}{\lambda^{N+1}}u(t)-\sum_{k=0}^{N}\frac{N!\,t^{N-k}}{\lambda^{k+1}(N-k)!}u(t)$
8	$t^M u(t)$	$t^N u(t)$	$\frac{M!N!}{(M+N+1)!}t^{M+N+1}u(t)$
9	$te^{\lambda_1 t}u(t)$	$e^{\lambda_2 t}u(t)$	$\frac{e^{\lambda_2 t}-e^{\lambda_1 t}+(\lambda_1-\lambda_2)te^{\lambda_1 t}}{(\lambda_1-\lambda_2)^2}u(t)$
10	$t^M e^{\lambda t}u(t)$	$t^N e^{\lambda t}u(t)$	$\frac{M!\,N!}{(N+M+1)!}t^{M+N+1}e^{\lambda t}u(t)$
11	$t^M e^{\lambda_1 t}u(t)$ $\lambda_1 \neq \lambda_2$	$t^N e^{\lambda_2 t}u(t)$	$\sum_{k=0}^{M}\frac{(-1)^k M!(N+k)!\,t^{M-k}e^{\lambda_1 t}}{k!(M-k)!(\lambda_1-\lambda_2)^{N+k+1}}u(t)$ $+\sum_{k=0}^{N}\frac{(-1)^k N!(M+k)!\,t^{N-k}e^{\lambda_2 t}}{k!(N-k)!(\lambda_2-\lambda_1)^{M+k+1}}u(t)$
12	$e^{-\alpha t}\cos(\beta t+\theta)u(t)$	$e^{\lambda_1 t}u(t)$	$\frac{\cos(\theta-\phi)e^{\lambda t}-e^{-\alpha t}\cos(\beta t+\theta-\phi)}{\sqrt{(\alpha+\lambda)^2+\beta^2}}u(t)$
13	$e^{\lambda_1 t}u(t)$	$e^{\lambda_2 t}u(-t)$	$\frac{e^{\lambda_1 t}u(t)+e^{\lambda_2 t}u(-t)}{\lambda_2-\lambda_1} \quad \mathrm{Re}\,\lambda_2 > \mathrm{Re}\,\lambda_1$
14	$e^{\lambda_1 t}u(-t)$	$e^{\lambda_2 t}u(-t)$	$\frac{e^{\lambda_1 t}-e^{\lambda_2 t}}{\lambda_2-\lambda_1}u(-t)$

Example: Find the response of an LTIC system $y(t)$ for the input $x(t)=10e^{-3t}u(t)$ and the impulse response $h(t)=(2e^{-2t}-e^{-t})u(t)$.

$$y(t)=x(t)*h(t)=10e^{-3t}u(t)*(2e^{-2t}-e^{-t})u(t)$$

Using the distributive property of the convolution

$x(t)*\{h_1(t)+h_2(t)\}=x(t)*h_1(t)+x(t)*h_2(t)$, we obtain

$$y(t)=10e^{-3t}u(t)*2e^{-2t}u(t)-10e^{-3t}u(t)*e^{-t}u(t)$$

Using the pair 4 in the convolution table, we obtain

$$y(t)=\frac{20}{-3-(-2)}(e^{-3t}-e^{-2t})u(t)-\frac{10}{-3-(-1)}(e^{-3t}-e^{-t})u(t)$$

$$=-20(e^{-3t}-e^{-2t})u(t)+5(e^{-3t}-e^{-t})u(t)$$

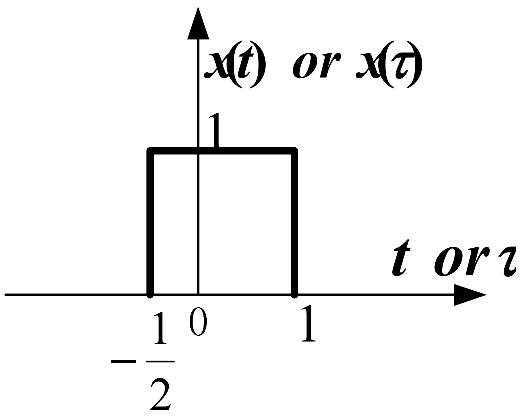
$$=(-5e^{-t}+20e^{-2t}-15e^{-3t})u(t)$$

Method 3

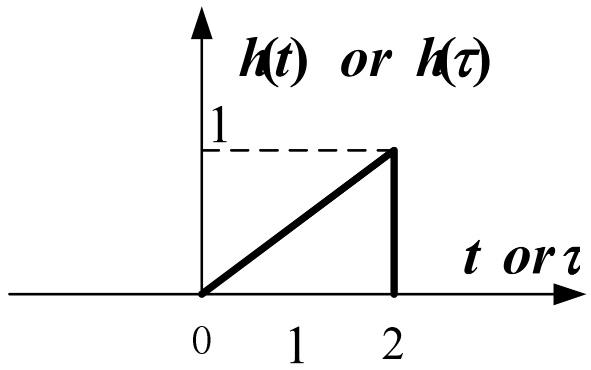
Graphical interpretation of calculating the convolution Integral

1. **Change independent variable:** $t \longrightarrow \tau$
2. **Time Reversal:** $h(\tau) \longrightarrow h(-\tau)$
3. **Time Shifting:** $h(-\tau) \longrightarrow h(t-\tau)$
4. **Multiplication:** $x(\tau)h(t-\tau)$
5. **Integration :** $y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$

Example: **Determine the response of the system when the input $x(t)$ and the impulse response $h(t)$ are shown in figure (a) and (b) .**



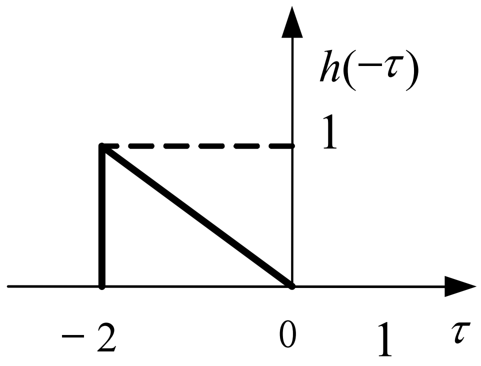
(a)



(b)

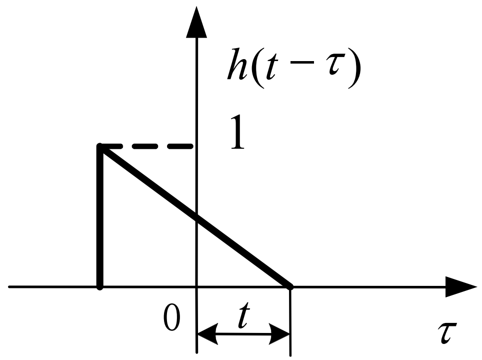
Solution: (1) $t \longrightarrow \tau$

(2) $h(\tau) \longrightarrow h(-\tau)$



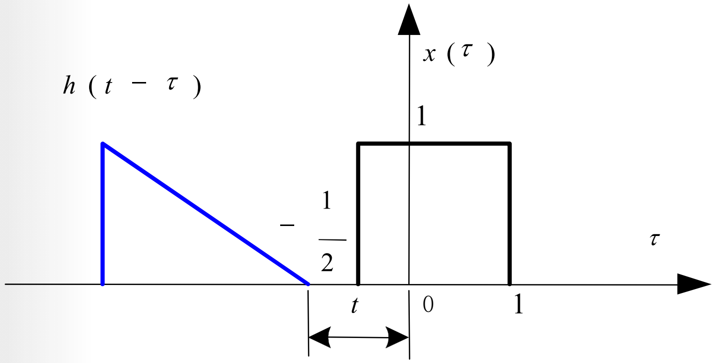
(c)

(3) $h(-\tau) \longrightarrow h(t-\tau)$



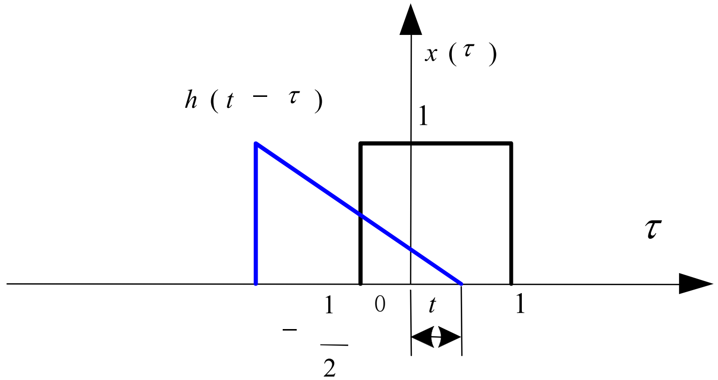
(d)

(4) $x(\tau)h(t-\tau)$



(e) $-\infty < t \le -\frac{1}{2}$

(5) Integration

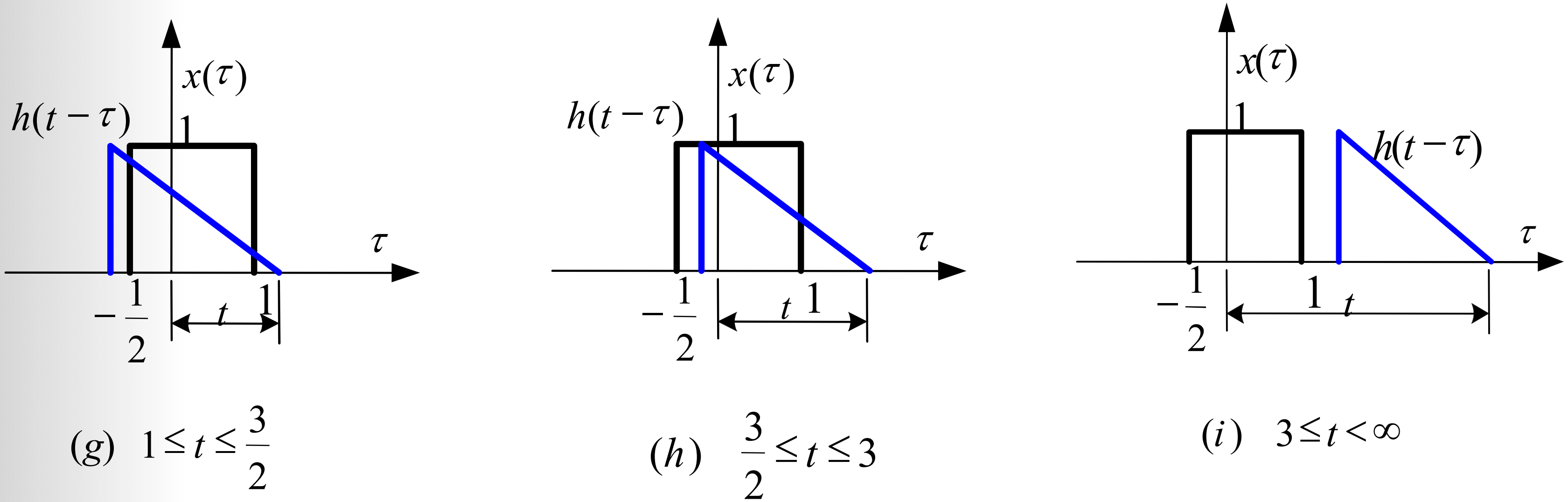


(f) $-\frac{1}{2} \le t \le 1$

$$\text{When} -\infty < t \le -\frac{1}{2}: x(t) * h(t) = 0$$

$$\text{When} -\frac{1}{2} \le t \le 1: x(t) * h(t) = \int_{-\frac{1}{2}}^{t} 1 \times \frac{1}{2}(t-\tau)d\tau$$

$$= \frac{t^2}{4} + \frac{t}{4} + \frac{1}{16}$$



$$\textbf{When } 1 \le t \le \frac{3}{2} : x(t) * h(t) = \int_{-\frac{1}{2}}^{1} 1 \times \frac{1}{2}(t-\tau)d\tau = \frac{3}{4}t - \frac{3}{16}$$

$$\textbf{When } \frac{3}{2} \le t \le 3 : x(t) * h(t) = \int_{t-2}^{1} 1 \times \frac{1}{2}(t-\tau)d\tau$$

$$= -\frac{t^2}{4} + \frac{t}{2} + \frac{3}{4}$$

$$\textbf{When } 3 \le t < \infty : x(t) * h(t) = 0$$

Properties

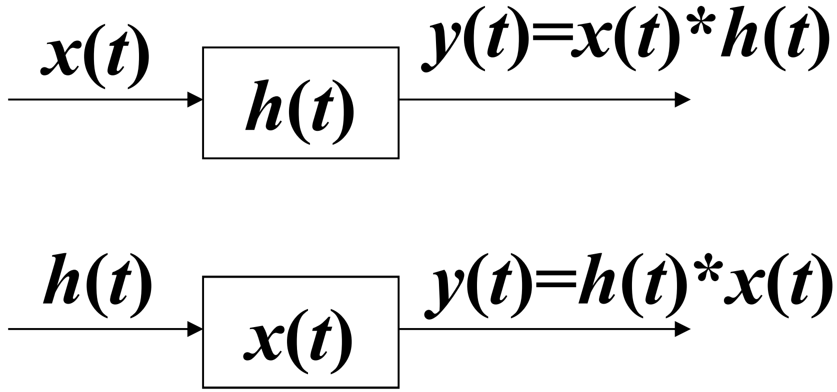
***h*(*t*) completely characterizes an LTIC system**

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow$$

$$y(t) = x(t) * h(t)$$
$$= \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

The Commutative Property

$$x(t)*h(t)=h(t)*x(t)$$

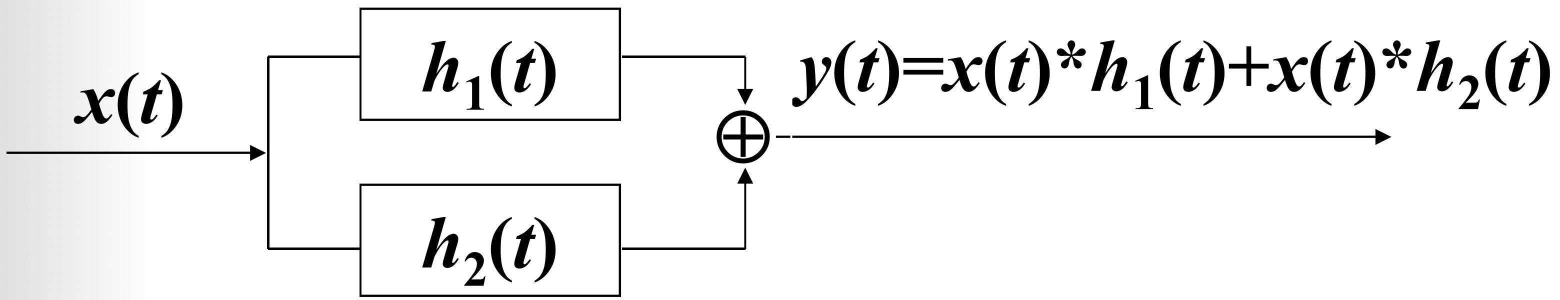


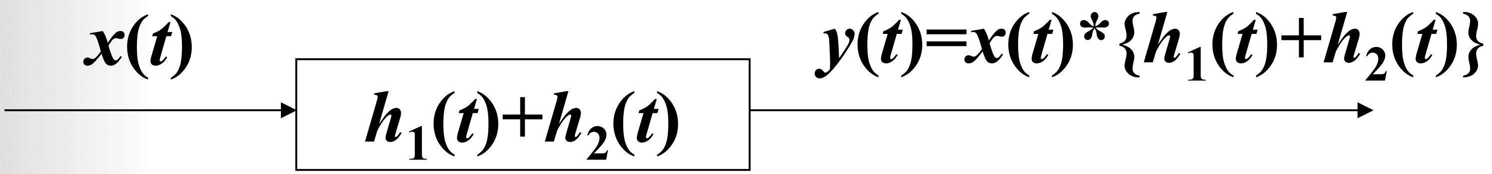
$$x(t)*h(t)=\int_{-\infty}^{\infty}x(\tau)h(t-\tau)d\tau \;\underline{\underline{\eta=t-\tau}}\; \int_{\infty}^{-\infty}x(t-\eta)h(\eta)d(-\eta)$$

$$=\int_{-\infty}^{\infty}x(t-\eta)h(\eta)d\eta = h(t)*x(t)$$

The Distributive Property

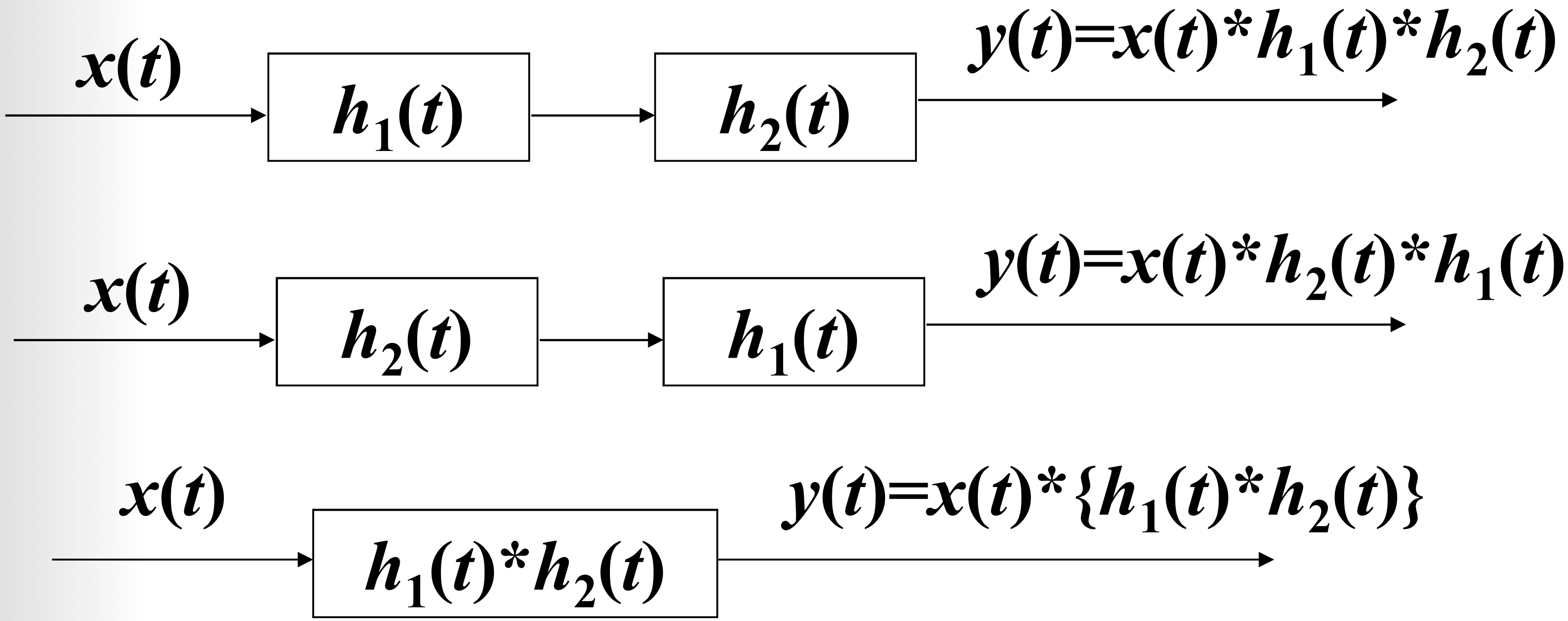
$$x(t)*\{h_1(t)+h_2(t)\}=x(t)*h_1(t)+x(t)*h_2(t)$$





➢ The Associative Property

$$x(t)*\{h_1(t)*h_2(t)\}=\{x(t)*h_1(t)\}*h_2(t)$$



The Shift Property

If $x_1(t)*x_2(t)=c(t)$, then

$$x_1(t)*x_2(t-T)= x_1(t-T)*x_2(t)= c(t-T)$$

$$x_1(t-T_1)*x_2(t-T_2)= c(t-T_1-T_2)$$

The Width Property

If the durations of $x_1(t)$ and $x_2(t)$ are finite, given by T_1 and T_2, respectively, then the duration of $x_1(t)*x_2(t)$ is $T_1 + T_2$.

$$x(t)*\delta(t)=x(t)$$

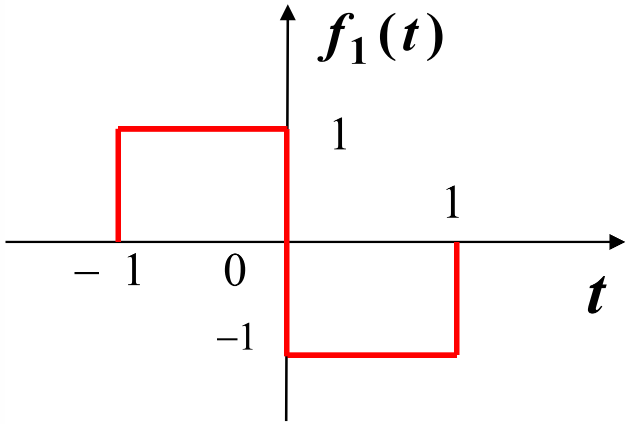
$$x(t)*\delta(t-t_0)=x(t-t_0)$$

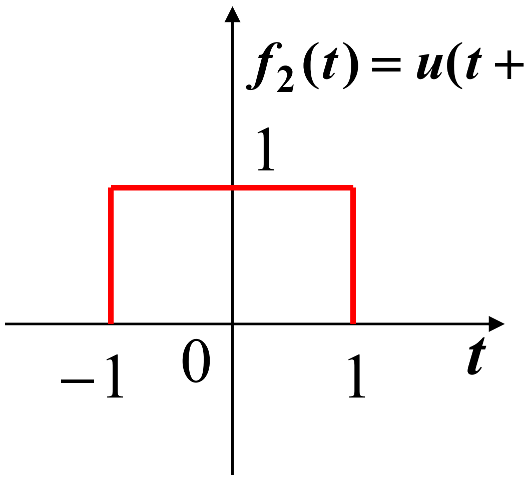
$$x(t-t_1)*\delta(t-t_2)=x(t-t_1-t_2)$$

$$x(t)*\delta'(t)=x'(t)$$

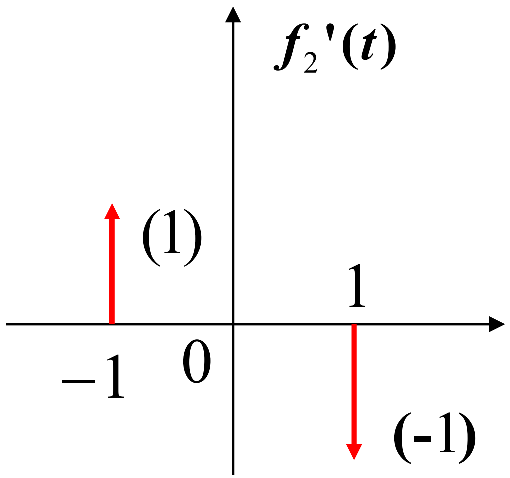
$$x(t)*u(t)=\int_{-\infty}^{t}x(\tau)d\tau$$

Example: **Determine** $y(t) = f_1(t) * f_2(t) * \delta'(t)$ **when** $f_1(t)$ **and** $f_2(t)$ **are as shown in the following figures.**

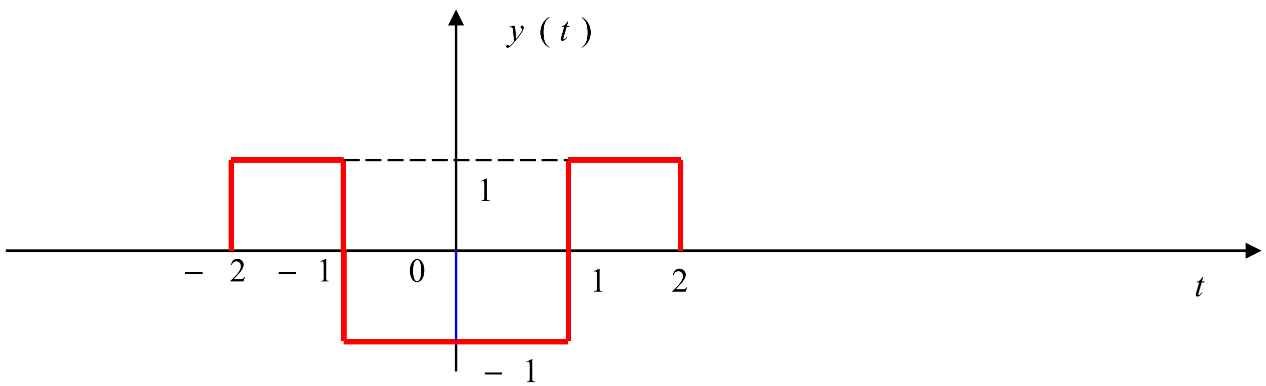




Solution:



$$
\begin{aligned}
y(t) &= f_1(t) * f_2(t) * \delta'(t) \\
&= f_1(t) * [f_2(t) * \delta'(t)] \\
&= f_1(t) * f_2'(t) \\
&= f_1(t) * [\delta(t+1) - \delta(t-1)] \\
&= f_1(t+1) - f_1(t-1)
\end{aligned}
$$

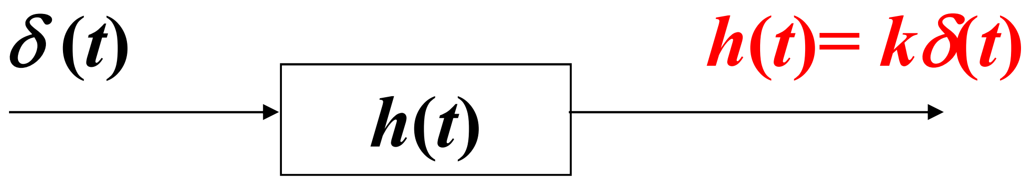


➢ LTIC System with and without Memory

An LTIC system is memoryless:

$$h(t)=0 \text{ for } t\neq 0$$

$$y(t)=kx(t)$$



➢ Invertibility of LTIC system

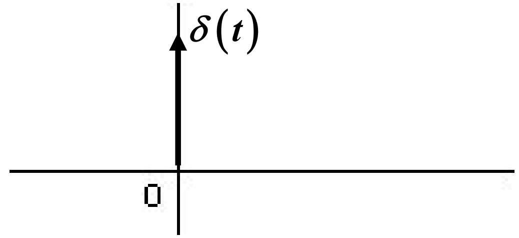
Inverse system

$$x(t) \longrightarrow \boxed{h_1(t)} \longrightarrow y(t) \longrightarrow \boxed{h_2(t)} \longrightarrow x(t)$$

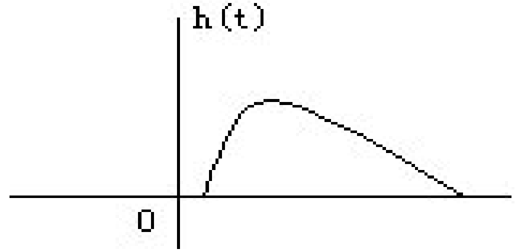
$$x(t) \longrightarrow \boxed{\begin{array}{c} h(t) = \delta(t) \\ \text{(Identity system)} \end{array}} \longrightarrow x(t) = x(t) * \delta(t)$$

$$h_1(t) * h_2(t) = \delta(t)$$

➢ Causality for LTIC system



CT system: $h(t)=0$ for $t<0$



An LTI system is causal $\Leftrightarrow$ It's unit impulse response is a **causal signal**

Example: $System: \quad y(t)=x(t)+x(t-2)$ **causal**

$$h(t)=\delta(t)+\delta(t-2)$$

$System: \quad y(t)=x(t)+x(t+2)$ **not causal**

$$h(t)=\delta(t)+\delta(t+2)$$

➢ Stability for LTIC system

Bounded input produces bounded output.

LTIC system:

If $|x(t)|<B$, $|y(t)| = \left|\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau\right| \le \int_{-\infty}^{\infty} |h(\tau)||x(t-\tau)|d\tau \le B\int_{-\infty}^{\infty} |h(\tau)|d\tau$

An LTIC system is stable $\Leftrightarrow$ $\int_{-\infty}^{+\infty} |h(\tau)|d\tau < +\infty$

Examples:

$y(t) = x(t-t_0)$

$h(t) = \delta(t-t_0)$

stable

$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$

$h(t) = \int_{-\infty}^{t} \delta(\tau)d\tau = u(t)$

not stable

➢ The Unit Step Response of LTIC system

The unit step response: $s(t)$

$$\delta(t) \longrightarrow \boxed{h(t)} \longrightarrow h(t)$$

$$u(t) \longrightarrow \boxed{h(t)} \longrightarrow s(t)=u(t)*h(t)$$

$$u(t)=\int_{-\infty}^{t}\delta(\tau)d\tau$$

$$s(t)=\int_{-\infty}^{+\infty}h(\tau)u(t-\tau)d\tau=\int_{-\infty}^{t}h(\tau)d\tau$$

$$\delta(t)=\frac{du(t)}{dt}=u'(t)$$

$$h(t)=\frac{ds(t)}{dt}=s'(t)$$

A very special function for LTIC systems: The everlasting exponential e^{st}

The system response to an everlasting exponential e^{st} is

$$y(t) = h(t) * e^{st} = \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau = e^{st} \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$$

$$H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$$

$$y(t) = H(s) e^{st}$$

The input and the output are the same (within a multiplicative constant) for the everlasting exponential signal.

The equation of a system is

$$Q(D)y(t) = P(D)x(t)$$

Substitution of this $x(t) = e^{st}$ and $y(t) = H(s)e^{st}$
in the above equation yields

$$H(s)[Q(D)e^{st}] = P(D)e^{st}$$

Moreover $D^r e^{st} = \dfrac{d^r e^{st}}{dt^r} = s^r e^{st}$ $\Rightarrow$

$$Q(D)e^{st} = Q(s)e^{st}$$

$$P(D)e^{st} = P(s)e^{st}$$

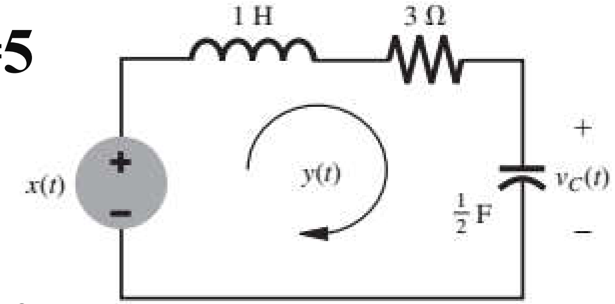
Therefore, the transfer function of the system is

$$H(s) = \frac{P(s)}{Q(s)}$$

Total response

$$y(t) = \underbrace{y_0(t)}_{\text{zero-input response}} + \underbrace{x(t) * h(t)}_{\text{zero-state response}}$$

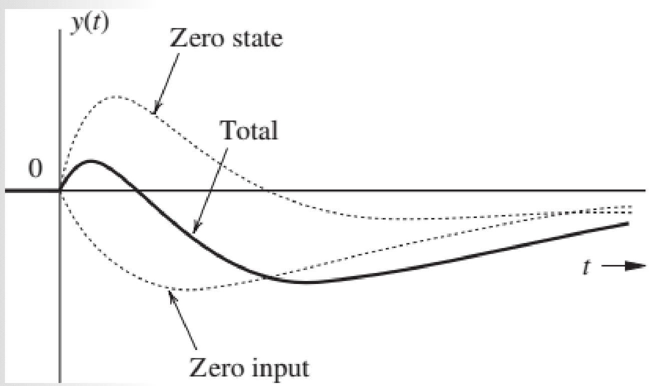
Example: $x(t)=10e^{-3t}u(t)$, $y(0^-)=0$ and $v_c(0^-)=5$

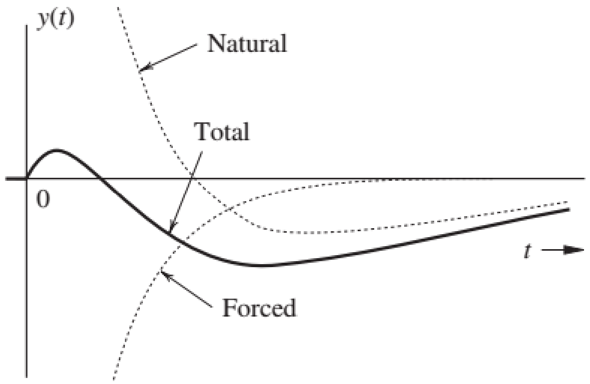


$$(D^2 + 3D + 2)\,y(t) = Dx(t)$$

$$y(t) = \underbrace{(-5e^{-t} + 5e^{-2t})}_{\text{zero-input response}} + \underbrace{(-5e^{-t} + 20e^{-2t} - 15e^{-3t})}_{\text{zero-state response}} \qquad t \geq 0$$

$$y(t) = \underbrace{(-10e^{-t} + 25e^{-2t})}_{\text{natural response}} + \underbrace{(-15e^{-3t})}_{\text{forced response}} \qquad t \geqslant 0$$





System stability

1、External (BIBO) stability

- **If every bounded input produces bounded output, the system is BIBO stable. Otherwise, it is BIBO unstable.**
- **BIBO stability is an external stability also known as the zero-state stability. It is determined by applying the external input with zero initial conditions.**

For an LTIC system

$$y(t)=h(t)*x(t)$$

An LTIC system is BIBO stable $\Leftrightarrow \int_{-\infty}^{+\infty}|h(\tau)|\,d\tau<+\infty$

2、Internal (Asymptotic) stability

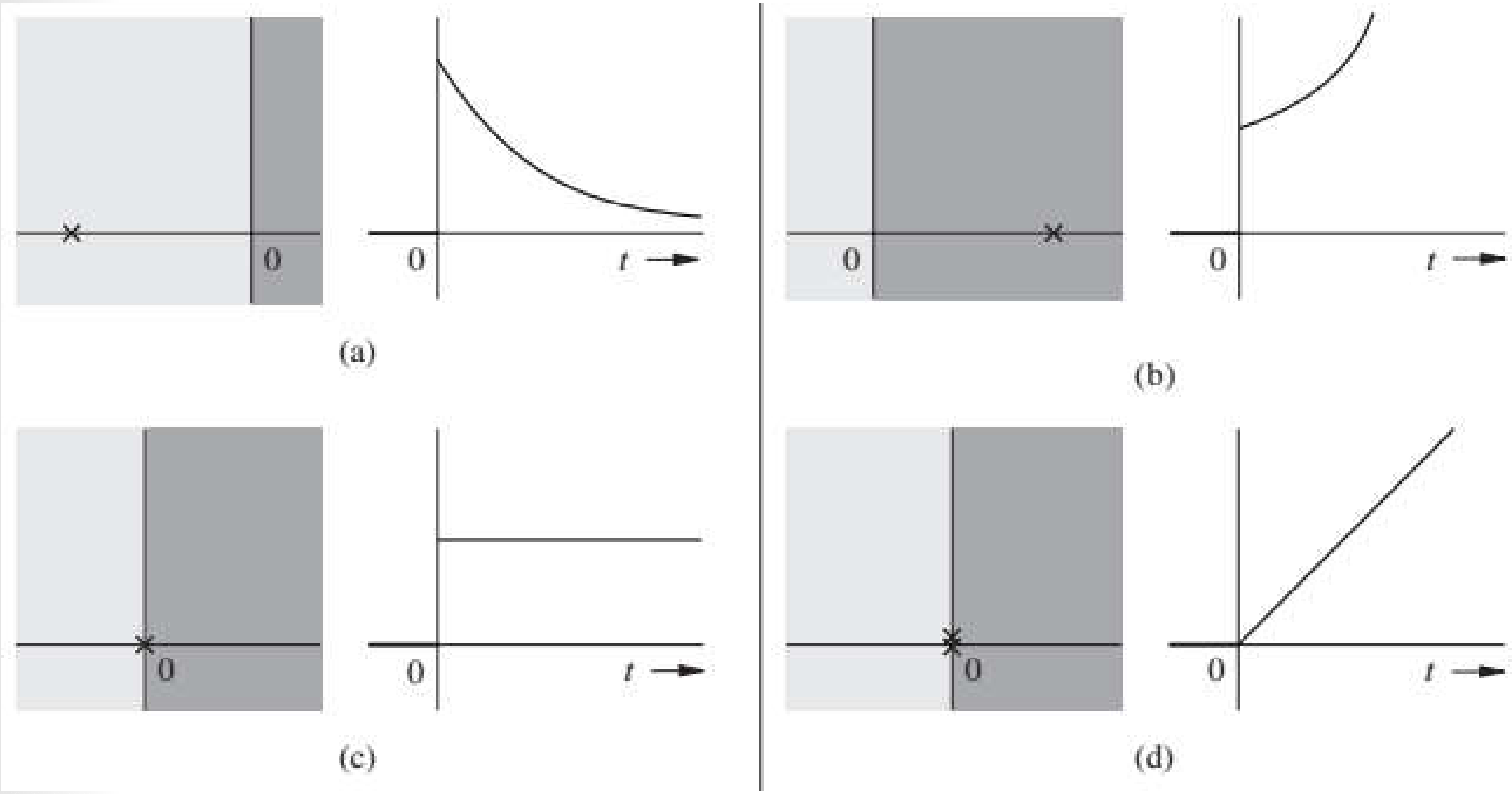
Asymptotic stability is an internal stability also known as the zero-input stability. It is determined by applying the nonzero initial conditions and no external input.

- Asymptotic stable: When some initial conditions are applied to a system in zero state, the system eventually returns to zero state.

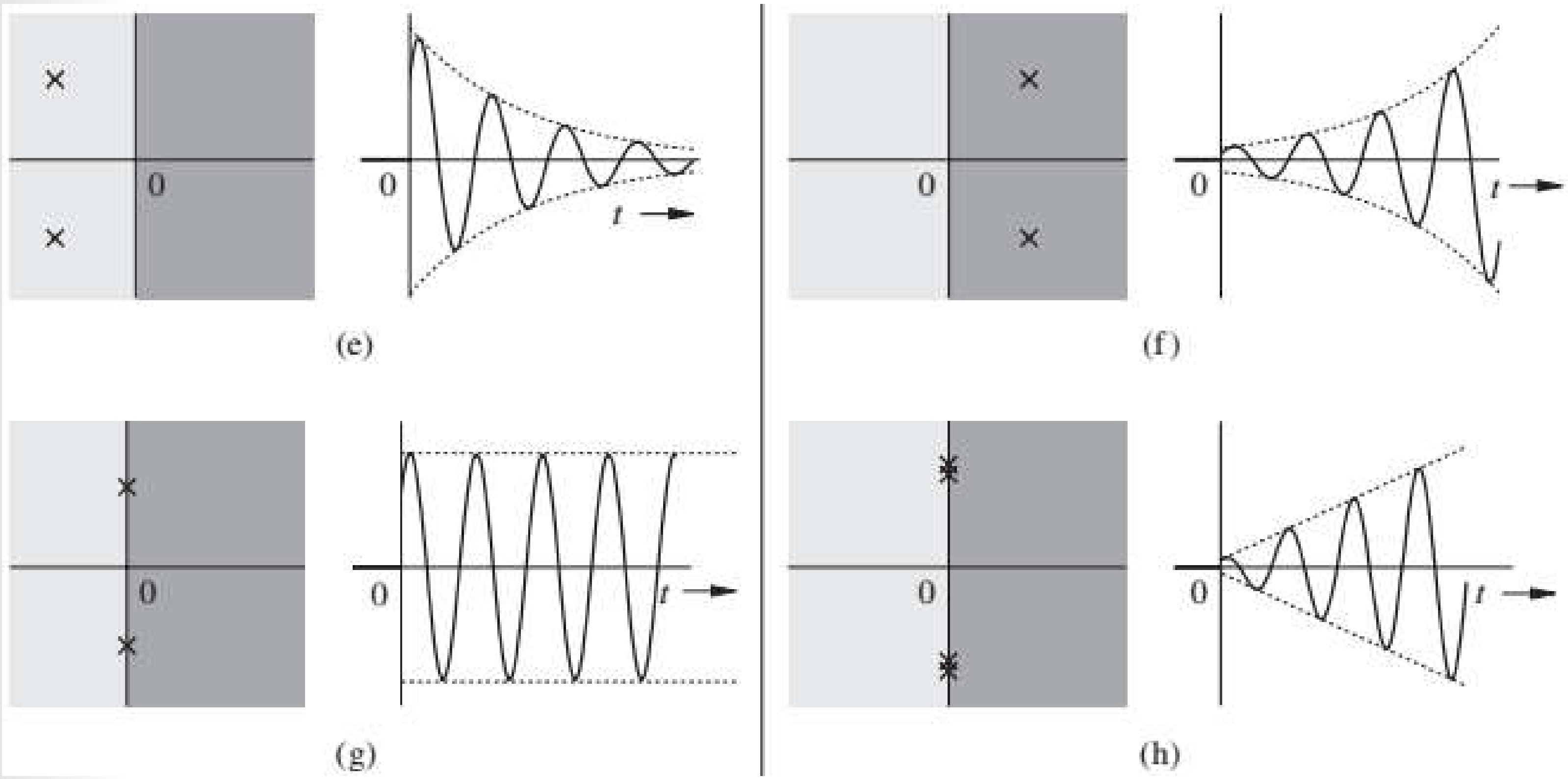
- Unstable: When some initial conditions are applied to a system in zero state, the system's response increases without bound.

- Marginally stable: When some initial conditions are applied to a system in zero state, the system does not go to zero state and the response dose not increase indefinitely.

Characteristic root location | **Zero-input response** | **Characteristic root location** | **Zero-input response**



Characteristic root location	Zero-input response	Characteristic root location	Zero-input response



$$\lim_{t\to\infty} e^{\lambda t} = \lim_{t\to\infty} e^{(\alpha+j\beta)t} = \lim_{t\to\infty} e^{\alpha t} e^{j\beta t} = \begin{cases} 0 & \alpha < 0 \\ \infty & \alpha > 0 \end{cases}$$

Also valid for the terms of the form $t^r e^{\lambda t}$

The internal stability criterion:

- **An LTIC system is asymptotically stable if, and only if, all the characteristic roots are in the LHP. The roots may be simple (unrepeated) or repeated.**

- **An LTIC system is unstable if, and only if, one or both of the following conditions exist: (i) at least one root is in the RHP; (ii) there are repeated roots on the imaginary axis.**

- **An LTIC system is marginally stable if, and only if, there are no roots in the RHP, and there are some unrepeated roots on the imaginary axis.**

3、Relationship between BIBO and Asymptotic stability

- An asymptotically stable system is BIBO stable.

- A marginally stable or asymptotically unstable system is BIBO unstable.

- BIBO stability cannot assure internal stability.

- When a system is controllable and observable, its external and internal descriptions are equivalent.

Example: An LTIC system consists of two subsystems in cascade. $h_1(t)=\delta(t)-2e^{-t}u(t)$ and $h_2(t)=e^{t}u(t)$.

$$x(t) \longrightarrow \boxed{S_1} \longrightarrow \boxed{S_2} \longrightarrow y(t)$$

$$h(t)=h_1(t)*h_2(t)=[\delta(t)-2e^{-t}u(t)]*e^{t}u(t)$$

$$=e^{t}u(t)-2[\frac{e^{-t}-e^{t}}{-2}]u(t)$$

$$=e^{-t}u(t)$$

$\int_{-\infty}^{+\infty}|h(\tau)|d\tau<+\infty$ $\Rightarrow$ The composite system is BIBO stable

S_2 has a characteristic root 1. $\Rightarrow$ S_2 is asymptotically unstable

BIBO stability cannot assure internal stability.

Example: Investigate the asymptotic and the BIBO stability of LTIC system described by the following equations.

	The characteristic roots	
1) $D(D+2)y(t)=3x(t)$	0, -2	Marginally stable BIBO unstable
2) $D^2(D+3)y(t)=(D+5)x(t)$	0, 0, -3	Unstable in both senses
3) $(D+1)(D+2)y(t)=(2D+3)x(t)$	-1, -2	Stable in both senses
4) $(D^2+1)(D^2+9)y(t)=(D^2+2D+4)x(t)$	$\pm j, \pm 3j$	Marginally stable BIBO unstable
5) $(D+1)(D^2-4D+9)y(t)=(D+7)x(t)$	-1, $2\pm\sqrt{5}j$	Unstable in both senses