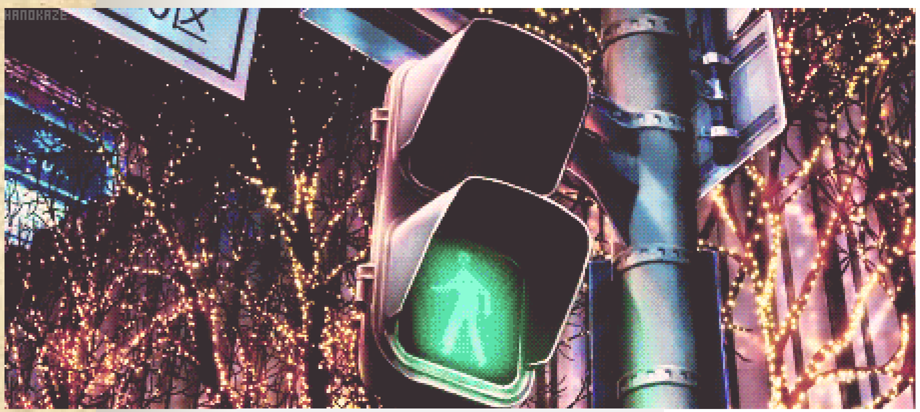
Chapter 1

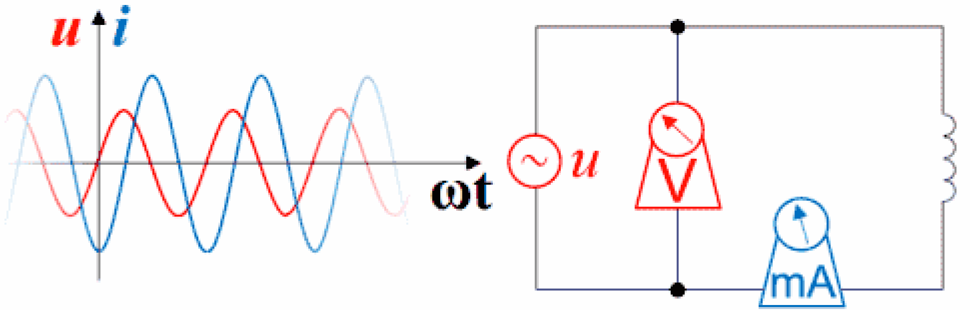
Signals and Systems

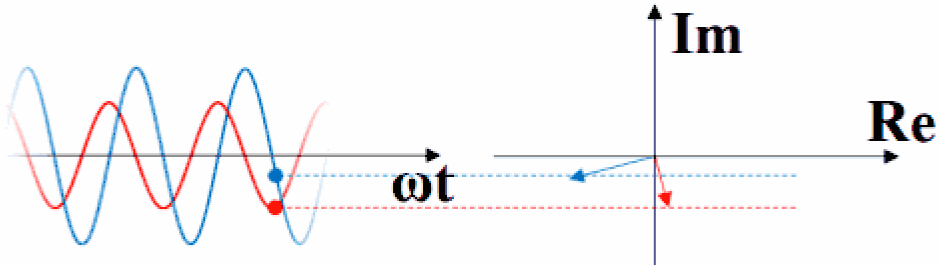
Definition of Signals and Systems

Signals :

- Functions of one or more independent variables.
- Carriers that typically carry some type of information.

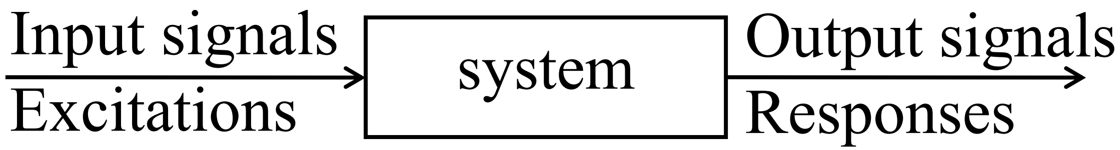


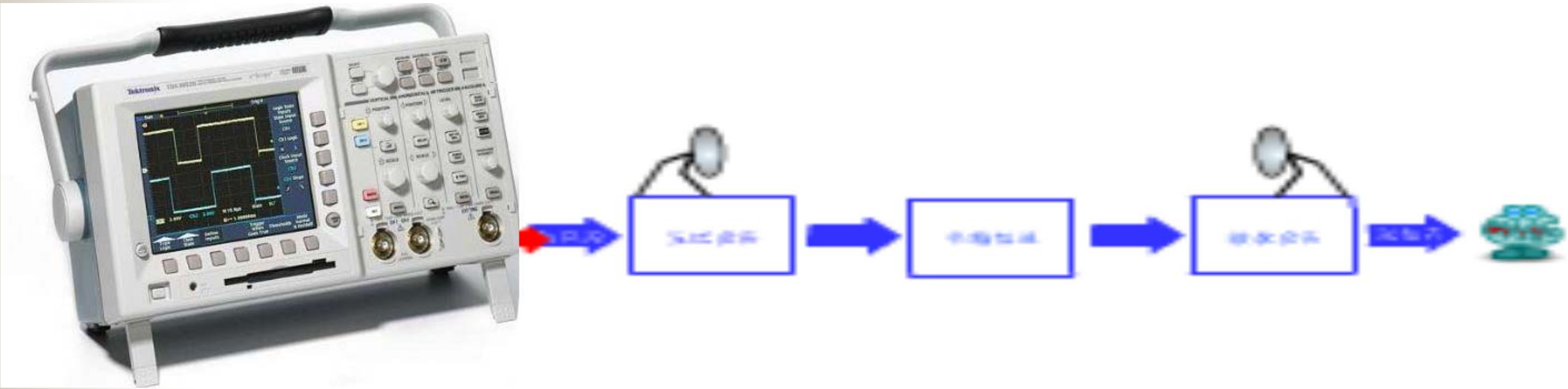




Systems :

- **Settings in which signals can be processed.**



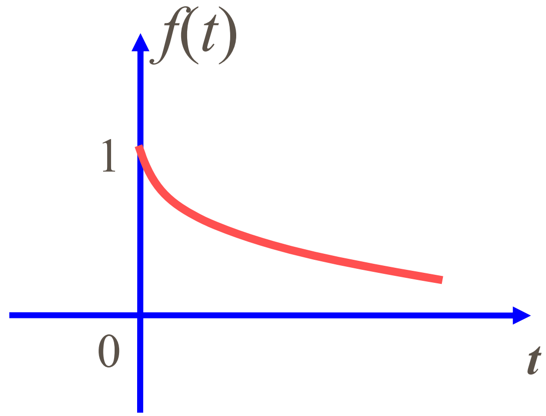


Representation of Signals

Example:

$$f(t)=\begin{cases}0 & t<0\\ e^{-at} & t\geq 0\end{cases}$$

$$a>0$$



Classification of Signals

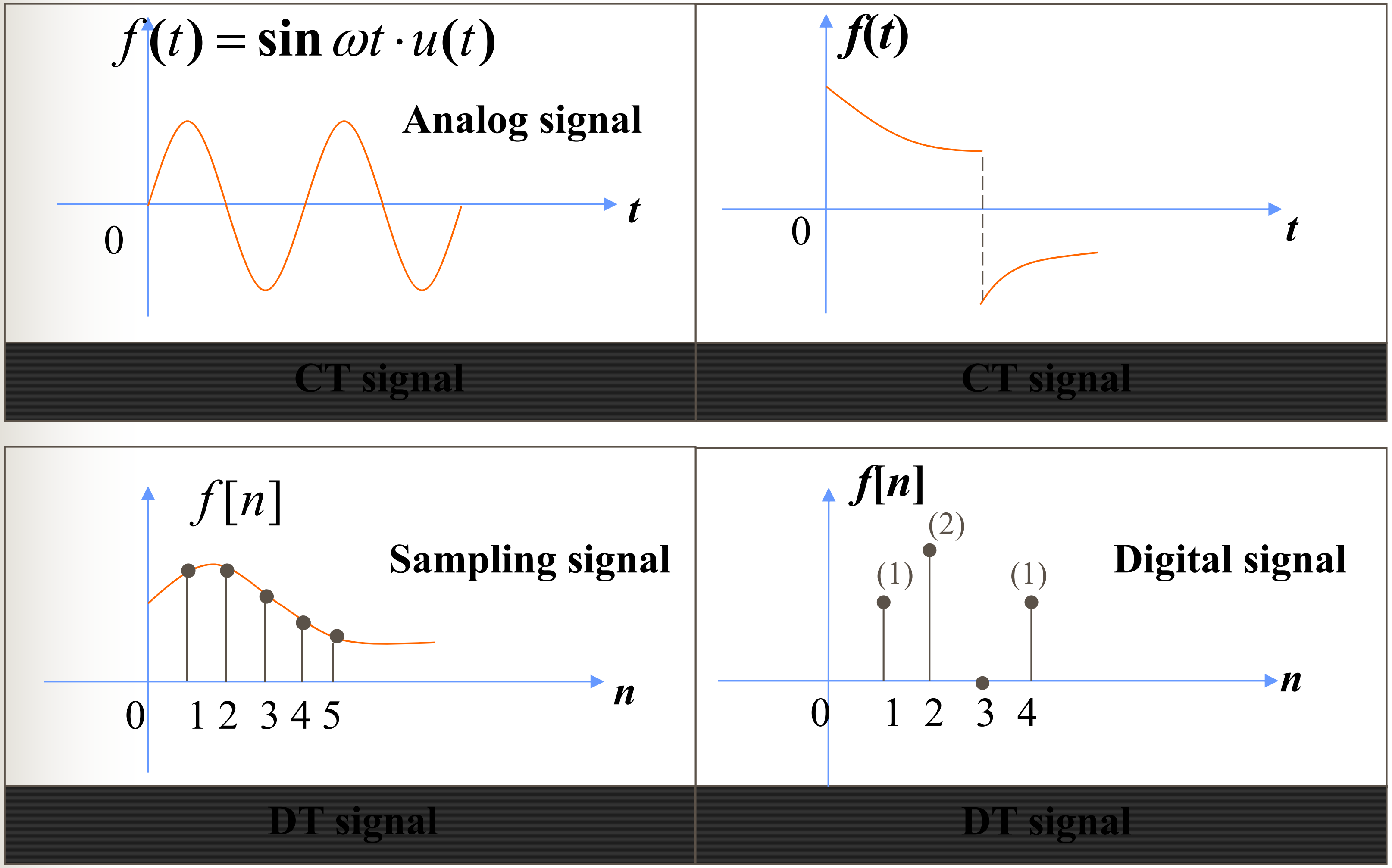
Continuous-time (CT) and Discrete-time (DT) Signals

- **CT signals:**

  The independent variable of CT signals is continuous : $x(t)$

- **DT signals:**

  DT signals are defined only at discrete times (only for integer values of the independent variable) : $x[n]$



$f(t) = \sin \omega t \cdot u(t)$
Analog signal
0
t
[redacted]
f(t)
0
t
[redacted]
f[n]
Sampling signal
0 1 2 3 4 5
n
[redacted]
f[n]
(2)
(1)
(1)
Digital signal
0 1 2 3 4
n
[redacted]

Periodic and aperiodic signals

- **Periodic signal:** **A signal, defined in (-∞， ∞), has the property that it is unchanged by a time shift, namely period *T* or *N*。**

*** If, for all values of *t*, a continuous-time signal *f*(*t*) can be expressed as:**

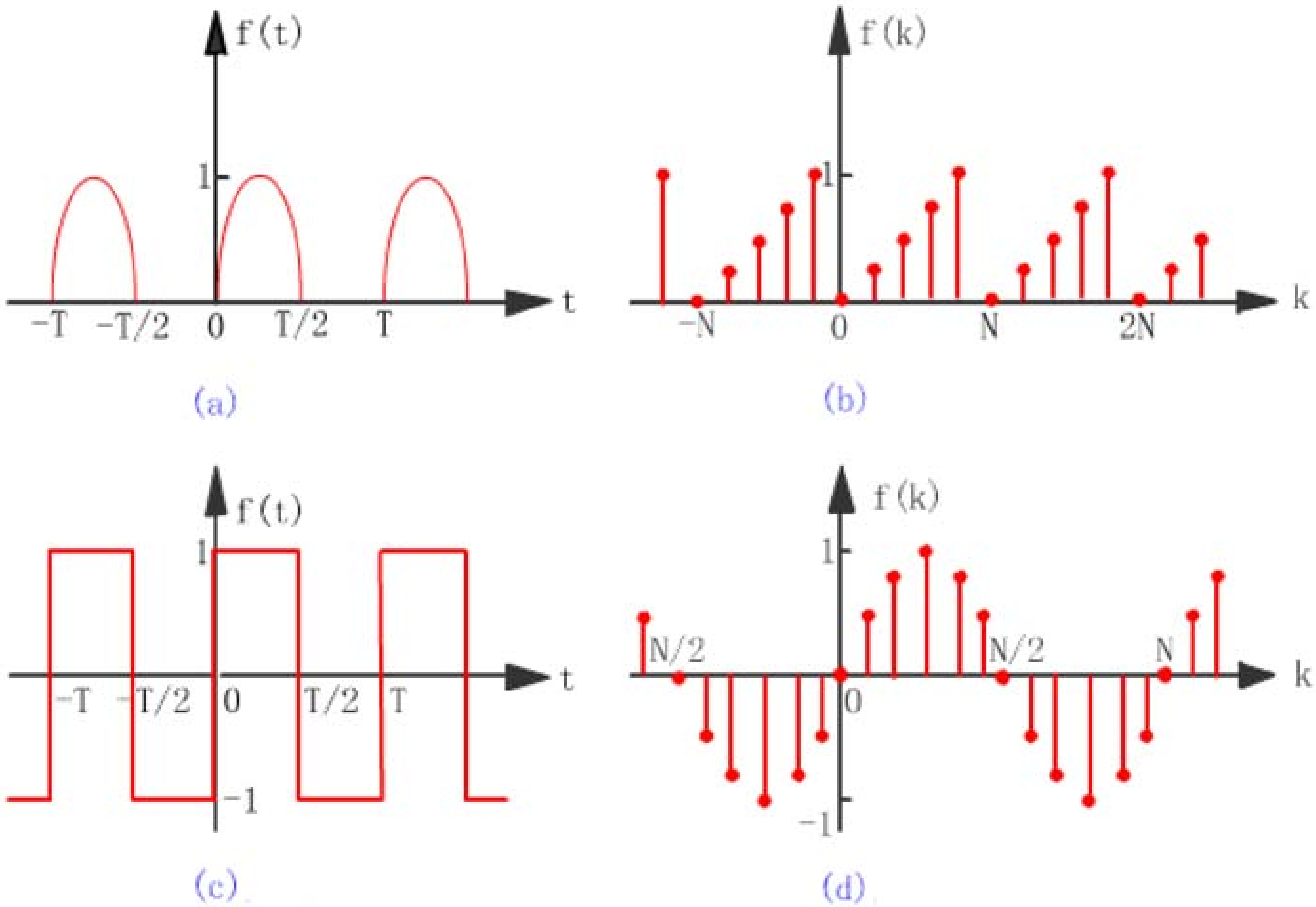
$$f(t) = f(t + mT),\quad m = 0, \pm 1, \pm 2, \ldots$$

*** If, for all values of *n*, a discrete-time signal *f*[*n*] can be expressed as:**

$$f[n] = f[n + mN],\quad m = 0, \pm 1, \pm 2, \ldots$$

then the smallest positive, nonzero value of *T* or *N* (*N* is a positive integer) is the fundamental period.

Examples :



- **Aperiodic signal:** **A signal that is not periodic.**

Example: **Determine whether or not each of the following CT signals is periodic.**

(1) $f_1(t) = \sin(2t) + \cos(3t)$ (2) $f_2(t) = \cos(2t) + \sin(\pi t)$

Consider two periodic signals $x(t)$ and $y(t)$, with fundamental period T_1 and T_2 respectively.

- **If T_1/T_2 is a rational number, the signal $x(t)+y(t)$ is periodic and its fundamental period is the least common multiple of T_1 and T_2.**
- **If T_1/T_2 is an irrational number, then the signal $x(t)+y(t)$ is not periodic.**

(1) sin(2t) is periodic with fundamental frequency ω_1= 2 rad/s, and fundamental period T_1= 2π/ω_1= π s

Similarly, cos(3t) is periodic with fundamental frequency ω_2= 3 rad/s, and fundamental period T_2= 2π/ω_2= (2π/3) s

Here, $f_1(t)$ is periodic since T_1/T_2= 3/2. Its fundamental period is the least common multiple of T_1 and T_2, namely 2π.

(2) The period of cos(2t), T_1 , is πs and the period of sinπt, T_2 , is 2s. Since T_1/T_2 is an irrational number, we conclude that $f_2(t)$ is not periodic.

Example: **Determine whether or not the following discrete-time signal is periodic. If the signal is periodic, determine its fundamental period.**

$$f[n] = \sin(\omega n)$$

$$\because \quad f[n] = \sin(\omega n) = \sin(\omega n + 2m\pi), \quad m = 0, \pm 1, \pm 2, \ldots$$

$$= \sin[\omega(n + m\frac{2\pi}{\omega})] = \sin[\omega(n + mN)]$$

1） **When** $2\pi/\omega$ **is a rational number，** $f[n]$ **is periodic with fundamental period** $N = M\frac{2\pi}{\omega}$ **，and** M **is the smallest positive integer which results in** N **to be an integer number.**

When $2\pi/\omega$ **is an integer, the fundamental period is** $N = \frac{2\pi}{\omega}$ **.**

2） **When** $2\pi/\omega$ **is an irrational number,** $f[n]$ **is not periodic.**

Example: **Determine whether or not each of the following discrete-time signals is periodic. If the signal is periodic, determine its fundamental period.**

(1) $f_1[n] = \sin(3\pi n/4) + \cos(\pi n/2)$

- **The fundamental frequency of $\sin(3\pi n/4)$, ω_1, equals to $3\pi/4$ rad and $2\pi/\omega_1$ equals to 8/3 which is a rational number，therefore the signal $\sin(3\pi n/4)$ is periodic and its fundamental period is 8.**
- **The fundamental frequency of $\cos(\pi n/2)$, ω_2 equals to $\pi/2$ rad， and $2\pi/\omega_2$ equals to 4 which is an integer，therefore the signal $\cos(\pi n/2)$ is periodic and its fundamental period is 4.**
- **Finally, we conclude that $f_1[n]$ is a periodic signal with fundamental period 8.**

(2) $f_2[n] = \cos(2n)$

The fundamental frequency of cos(2n), ω_1, equals to 2 rad and 2π/ω_1 equals to π which is an irrational number, therefore the signal cos(2n) is not periodic.

Causal, noncausal and anticausal signals

- **Causal signals:** **A signal that does not start before t=0 (n=0) is a causal signal.**
- **Noncausal signals:** **A signal that starts before t=0 (n=0) is a noncausal signal.**
- **Anticausal signals:** **A signal that is zero for all $t \geq 0$ ($n \geq 0$) is an anticausal signal.**

Energy and power signals

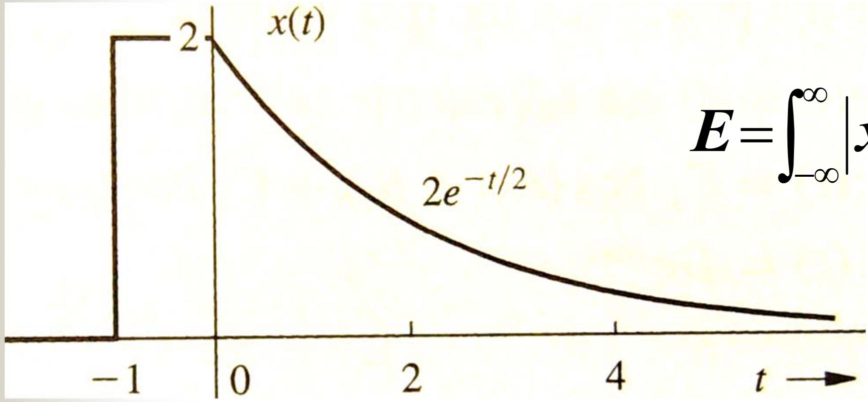
If a continuous-time signal $f(t)$, the voltage across a resistor with resistance 1Ω，the corresponding instantaneous power is $|f(t)|^2$.

The total energy and the average power expanded over an infinite time interval (–∞ , ∞) are

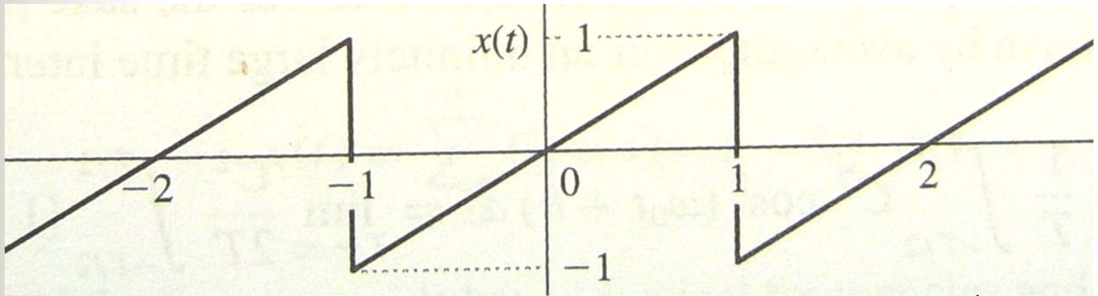
- **Total energy:** $$E=\int_{-\infty}^{\infty}\left|f(t)\right|^2 dt$$
- **Average power:** $$P=\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}|f(t)|^2\,dt$$

Example:

Determine the suitable measures of the following signals.



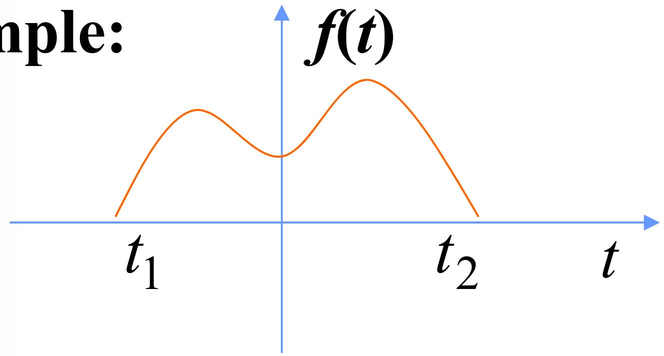
$$E=\int_{-\infty}^{\infty}\left|x(t)\right|^{2}dt=\int_{-1}^{0}(2)^{2}dt+\int_{0}^{\infty}4e^{-t}dt=4+4=8$$



$$P=\frac{1}{2}\int_{-1}^{1}x(t)^{2}dt=\frac{1}{2}\int_{-1}^{1}t^{2}dt=\frac{1}{3}$$

- **Energy signal :** The signal $f(t)$ has finite total energy, namely $E < \infty$. In this case, $P = 0$.

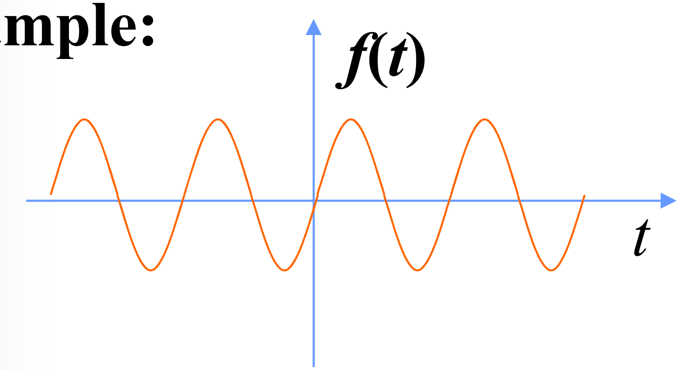
Example:



$f(t)$ has nonzero values over the time interval $t_1 \le t \le t_2$.

- **Power signal:** The signal $f(t)$ has finite average power, namely $P < \infty$. In this case, $E = \infty$.

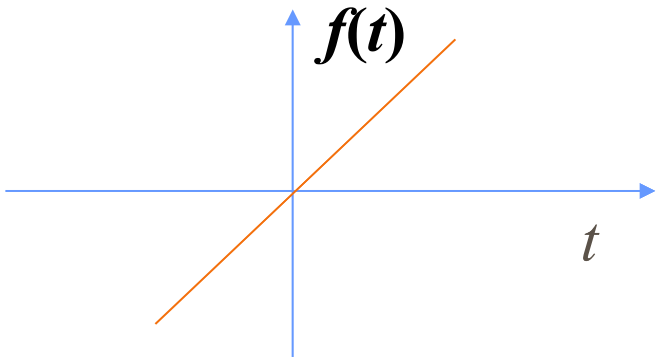
Example:



$f(t)$ is periodic.

- **There are also signals for which neither *P* nor *E* are finite.**

Example:



$$f(t) = t$$

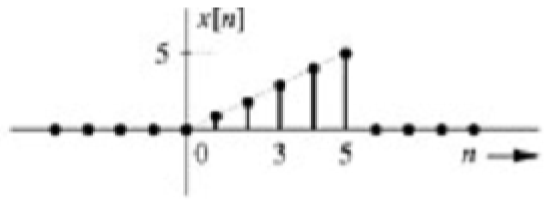
In discrete time, we can also identify three important classes of signals.

Energy signal: $$E=\sum_{n=-\infty}^{\infty}\left|f[n]\right|^{2}<\infty$$

Power signal: $$P=\lim_{N\to\infty}\frac{1}{N}\sum_{n=-N/2}^{N/2}\left|f[n]\right|^{2}<\infty$$

The third class of signals: $E=\infty$ and $P=\infty$

Example: **Find the energy of the signal shown in Fig.(a) and the power of the signal shown in Fig.(b) .**



(a)

(b)

$$(a) \qquad E=\sum_{n=0}^{5} n^2 = 55$$

(b) The signal is periodic and the fundamental period is 6. Its power can be found by averaging its energy over one period.

$$P=\frac{1}{6}\sum_{n=0}^{5} n^2 = \frac{55}{6}$$

Example: Show that the signal $x[n]=a^n u[n]$ is an energy signal of energy $E=1/(1-a^2)$ if $|a|<1$. It is a power signal of power $P=0.5$ if $|a|=1$. It is neither an energy signal nor a power signal if $|a|>1$.

$$(a) \qquad E=\sum_{n=-\infty}^{\infty}\left|x[n]\right|^2=\sum_{n=0}^{\infty}a^{2n}$$

$$E<\infty \Rightarrow |a|<1 \Rightarrow E=\frac{1}{1-a^2},\left|a\right|<1$$

$$(b) \qquad P=\lim_{N\to\infty}\frac{1}{N}\sum_{n=-\frac{N}{2}}^{\frac{N}{2}}\left|x[n]\right|^2=\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{\frac{N}{2}}1^{2n}=\lim_{N\to\infty}\frac{\frac{N}{2}+1}{N}=0.5$$

(c) if $|a|>1$, $E=\infty$ and $P=\infty$

Deterministic and Random Signals

- **Deterministic signal:**

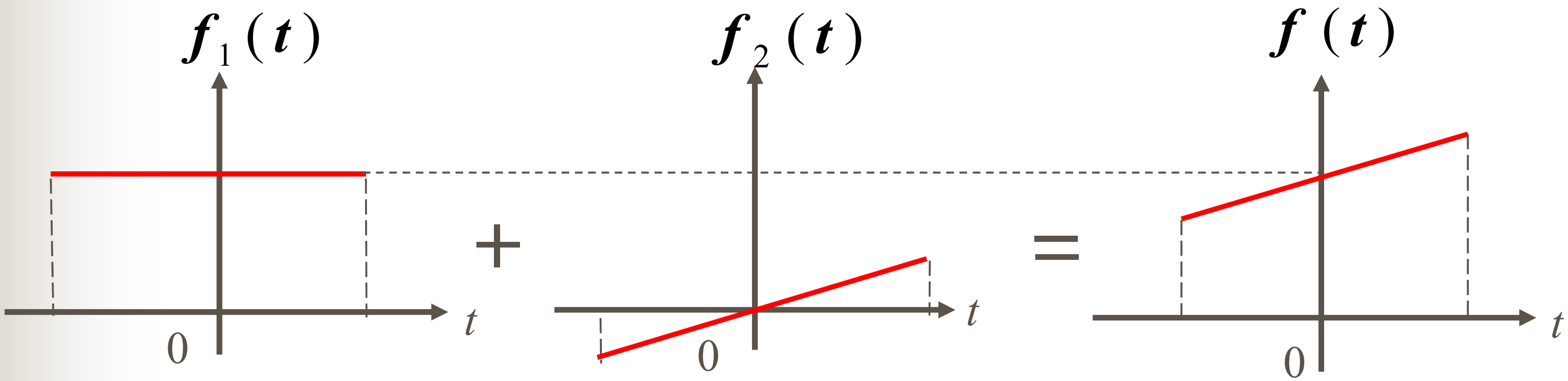
  The physical description of the signal is known completely, whether in mathematical or graphical form.

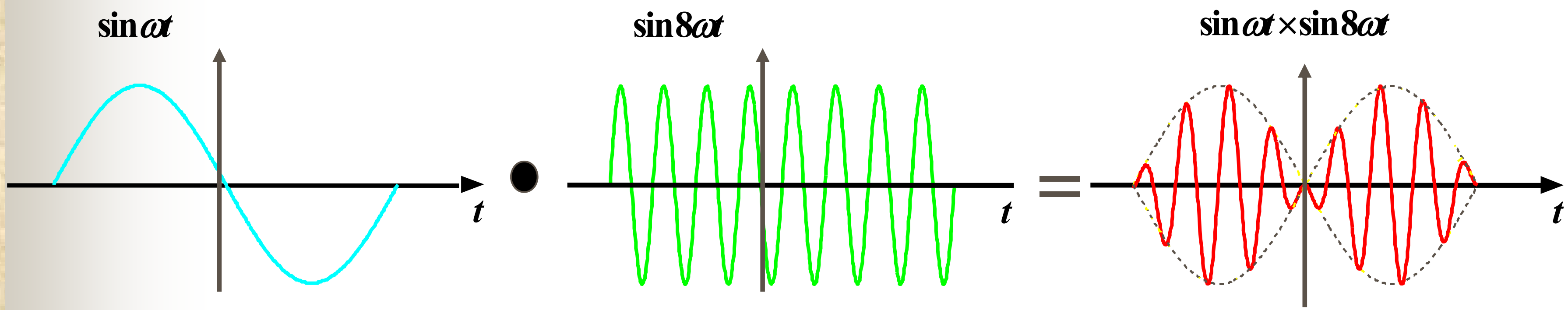
- **Random signal:**

  The signal cannot be represented precisely.

Elementary signal operations

1、+, −, × operations





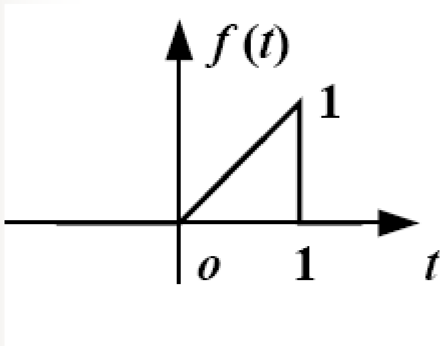
$$\because \quad f_1[k]=\begin{cases}2, k=-1\\ 3, k=0\\ 6, k=1\\ 0, others\end{cases} \qquad f_2[k]=\begin{cases}3, k=0\\ 2, k=1\\ 4, k=2\\ 0, others\end{cases}$$

$$\therefore \quad f_1[k]+f_2[k]=\begin{cases}2, k=-1\\ 6, k=0\\ 8, k=1\\ 4, k=2\\ 0, others\end{cases} \qquad f_1[k]\times f_2[k]=\begin{cases}9, k=0\\ 12, k=1\\ 0, others\end{cases}$$

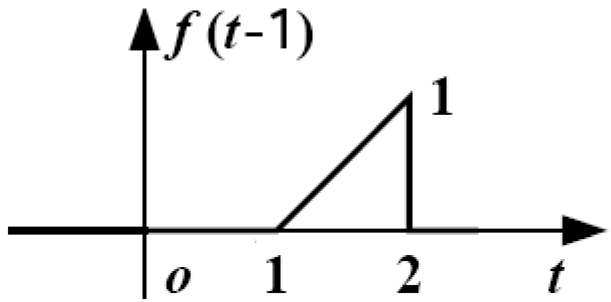
2、Time shifting

$$f(t) \to f(t-t_0),\ f[n] \to f[n-n_0]$$

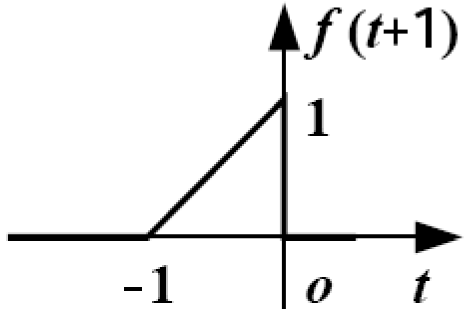
If $t_0>0$ ($n_0>0$), $f(t-t_0)$ ($f[n-n_0]$) represents a delayed signal.
If $t_0<0$ ($n_0<0$), $f(t-t_0)$ ($f[n-n_0]$) represents an advanced signal.



$t \to t-1$



$t \to t+1$

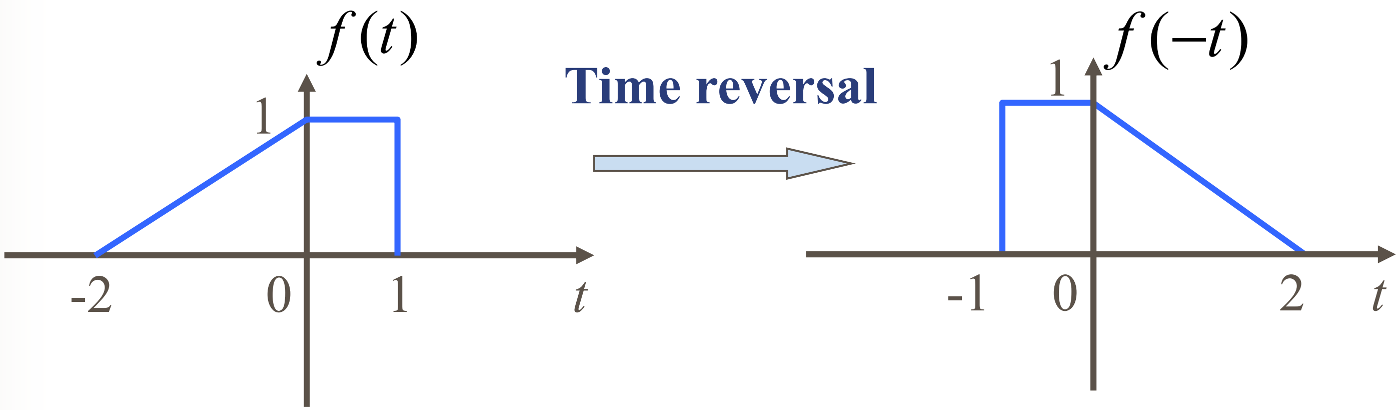


3、Time reversal

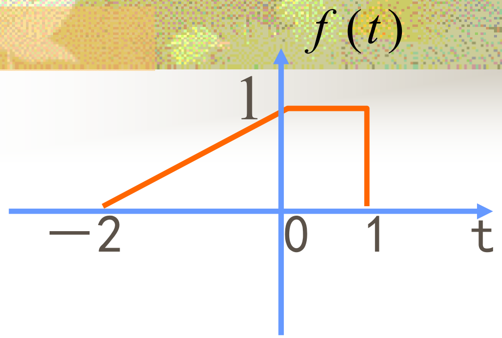
$$f(t) \to f(-t), f[n] \to f[-n]$$

f **(-*t*) is obtained by reversing *f*** **(*t*) about *t*=0.**

f **[-*n*] is obtained by reversing *f*** **[*n*] about *n*=0.**

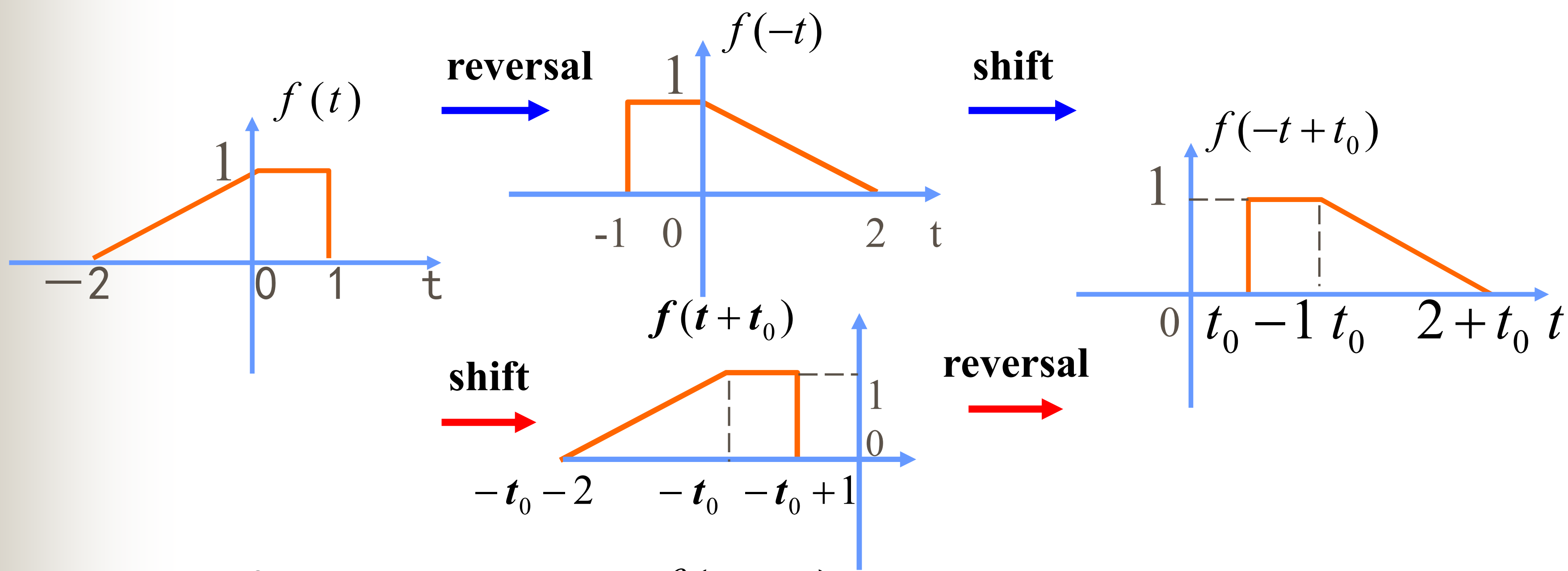


Example: Sketch $f(-t+t_0)$ based on $f(t)$.



Method 1: Time reversal－Time shifting **Method 2**: Time shifting－Time reversal

$$f(t) \longrightarrow f(-t) \longrightarrow f(-t+t_0) = f[-(t-t_0)]$$

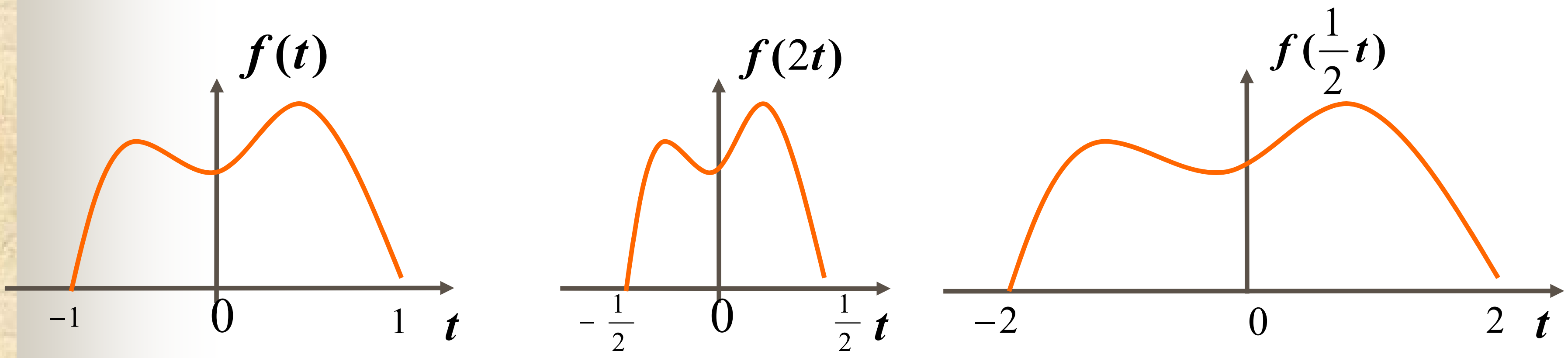


$$f(t) \longrightarrow f(t+t_0) \longrightarrow f(-t+t_0)$$

4、Time scaling

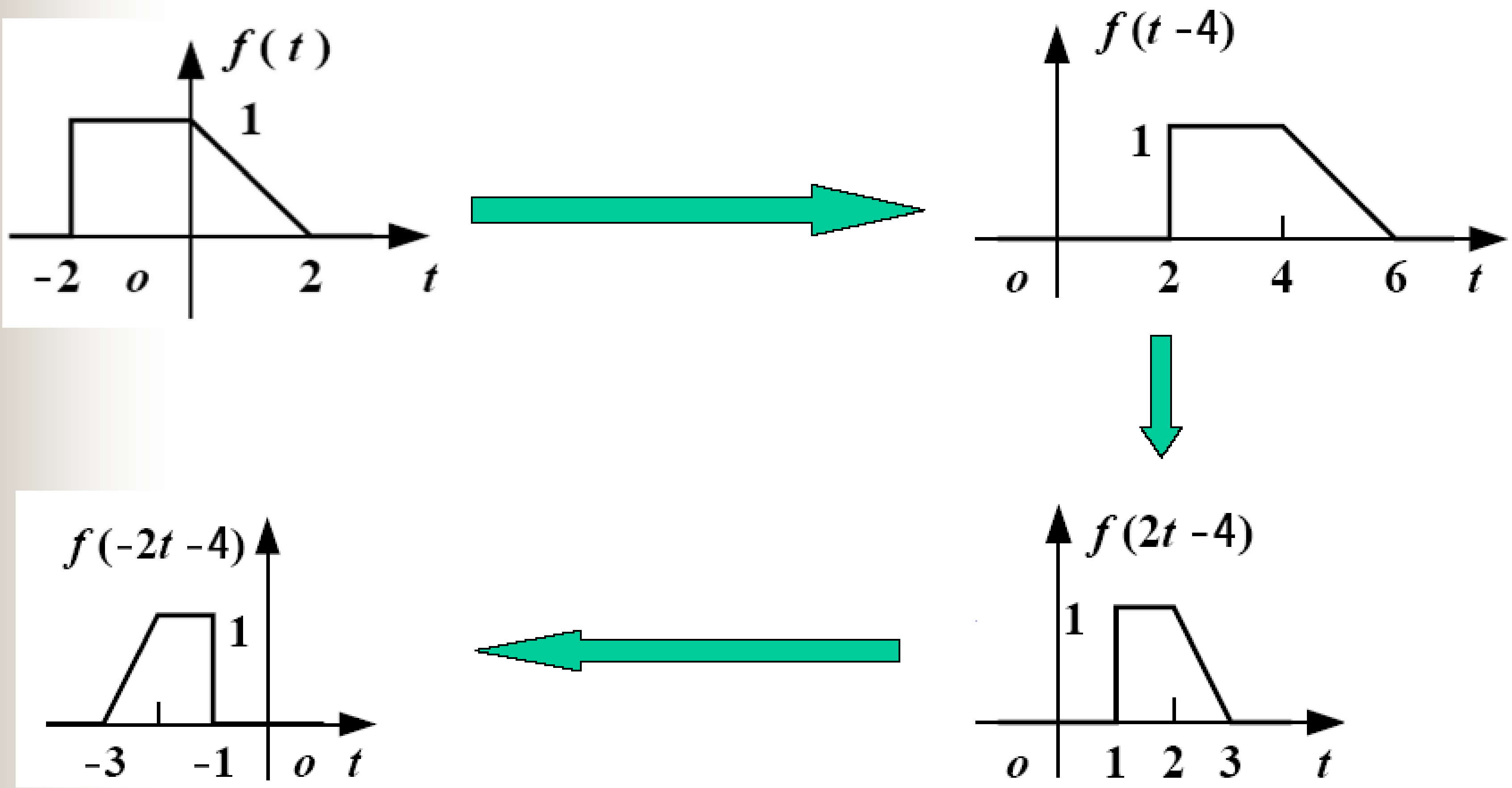
$$f(t) \to f(at), \quad a > 0$$

If $0 < a < 1$, $f(at)$ is linearly expanded (stretched).
If $a > 1$, $f(at)$ is linearly compressed.

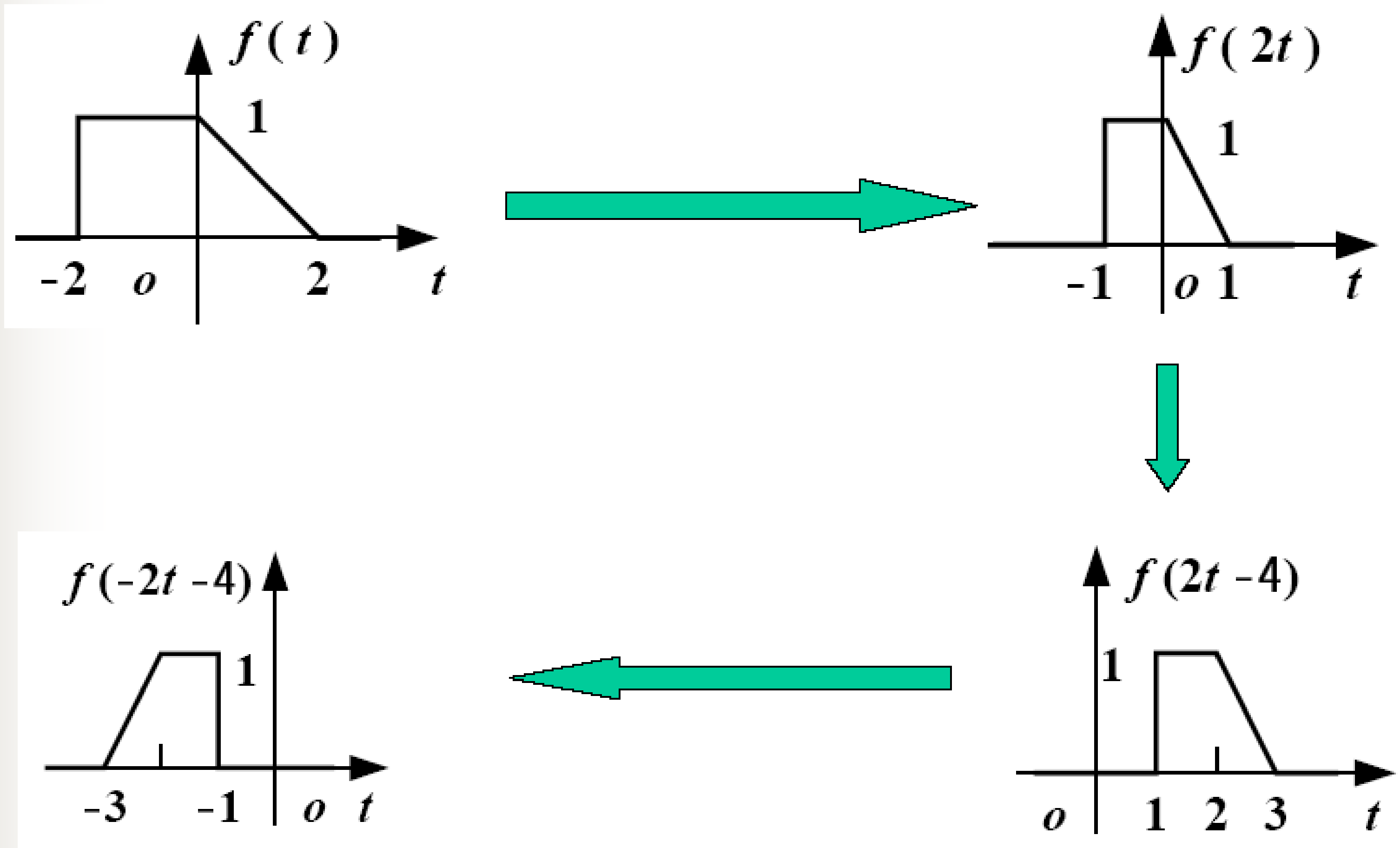


Example: Sketch $f(-4-2t)$ based on $f(t)$.

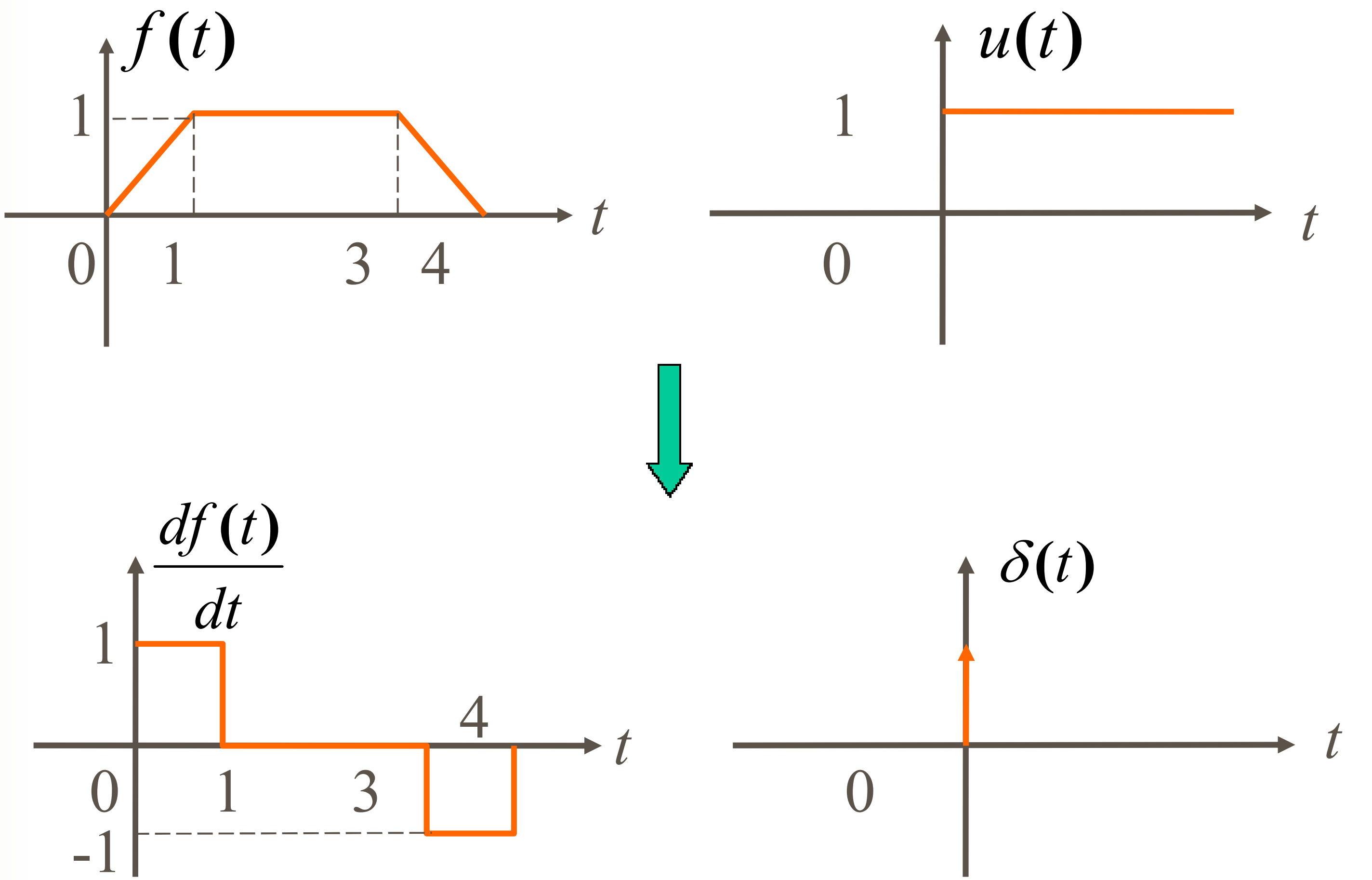
Method 1: Time shifting －Time scaling－Time reversal



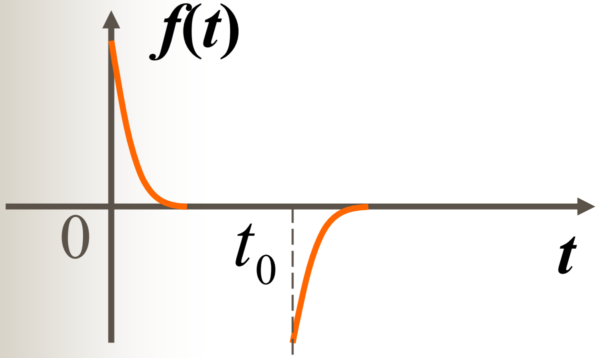
Method 2: Time scaling－Time shifting －Time reversal



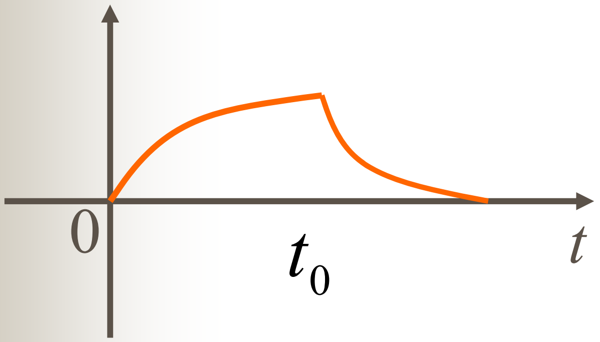
5、Differential operation



6、Integral operation



$$f(t)=\begin{cases} e^{-\alpha t} & 0<t<t_0 \\ e^{-\alpha t}-e^{-\alpha(t-t_0)} & t_0\le t<\infty \end{cases} \qquad t_0 \gg \frac{1}{\alpha}$$



$$\int_{-\infty}^{t} f(\tau)d\tau=\begin{cases} \frac{1}{\alpha}(1-e^{-\alpha t}) & 0<t<t_0 \\ \frac{1}{\alpha}(1-e^{-\alpha t})-\frac{1}{\alpha}[1-e^{-\alpha(t-t_0)}] & (t_0\le t<\infty) \end{cases}$$

Basic CT and DT signals

1、 CT signals

Exponential signal

- **Real exponential signal** $f(t) = Ae^{\alpha t}$

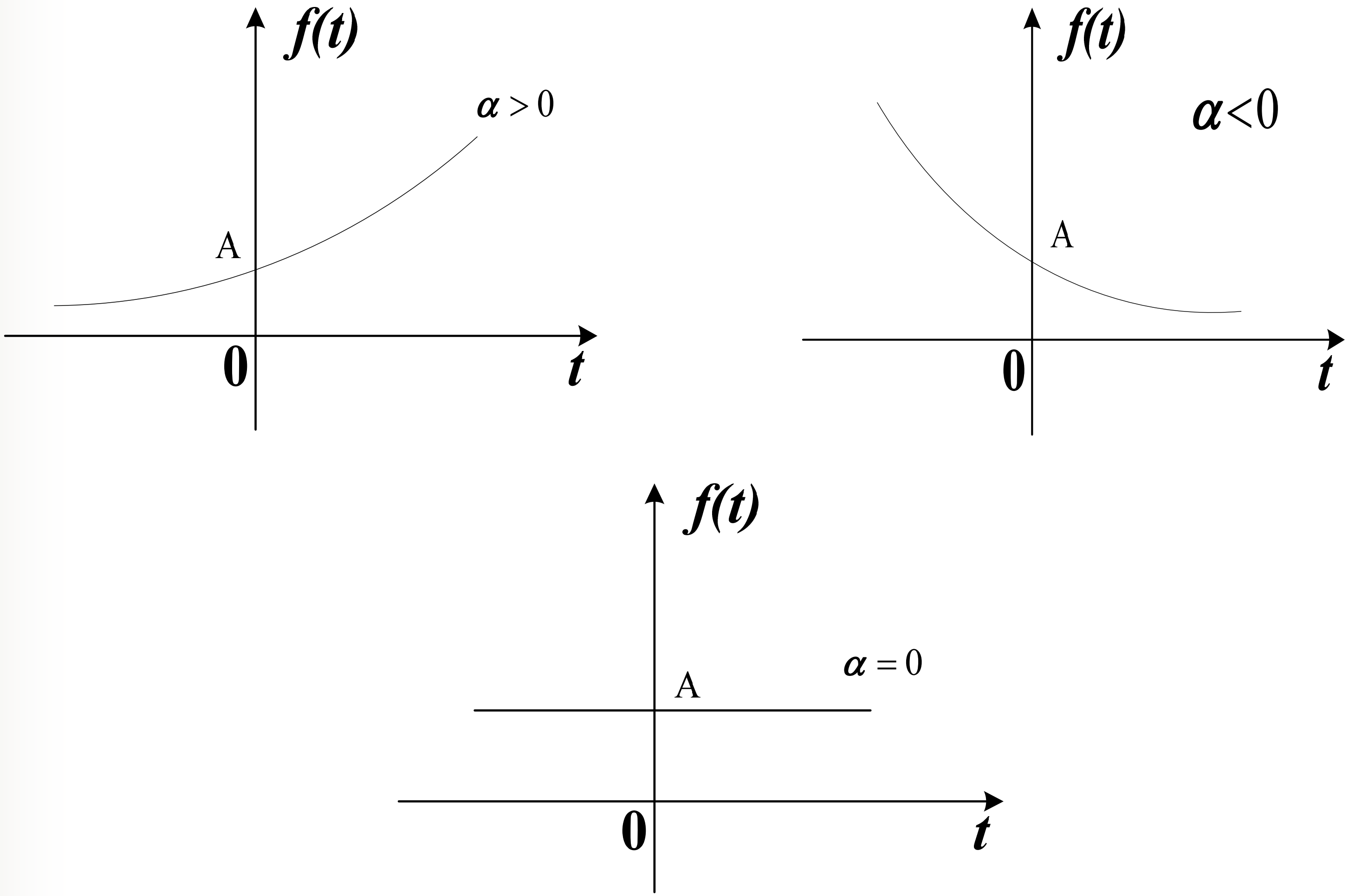
where A and α are real numbers.

Property：

The integral and differential results of the signals are still exponential signals.

Time shift⇔ Scale change

Examples:



- **Complex exponential signal**

$$f(t)=Ae^{st}=Ae^{(\sigma+j\omega)t}$$

Using Euler's formula，the following representation can be obtained

$$f(t)=Ae^{\sigma t}(\cos\omega t+j\sin\omega t)$$

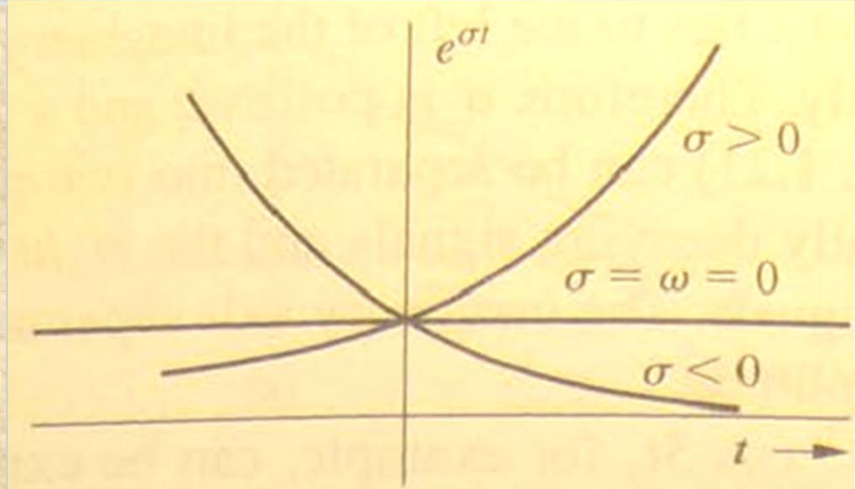
Depending on the values of these parameters, the complex exponential can represent several different signals.

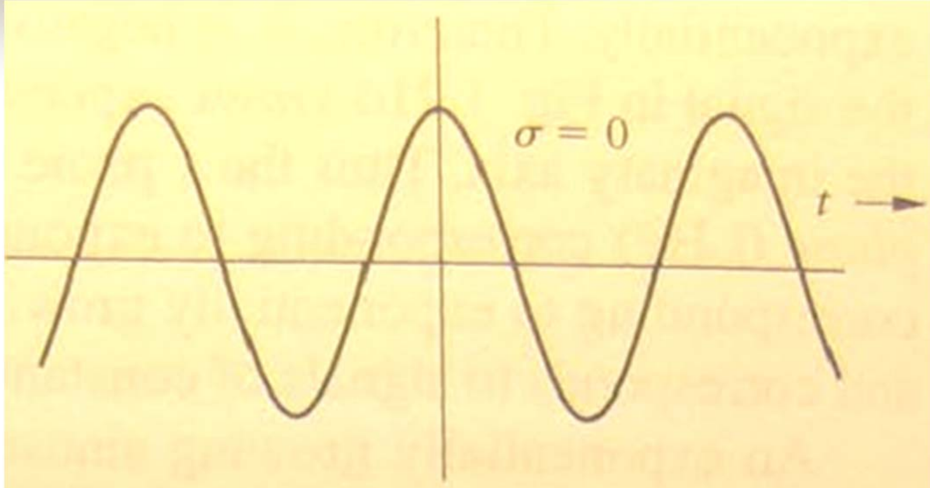
1)A DC signal ($\sigma=\omega=0$)

2)A real exponential signal ($\sigma\neq 0$, $\omega=0$)

3)The real and imaginary parts are sinusoidal ($\sigma=0$, $\omega\neq 0$)

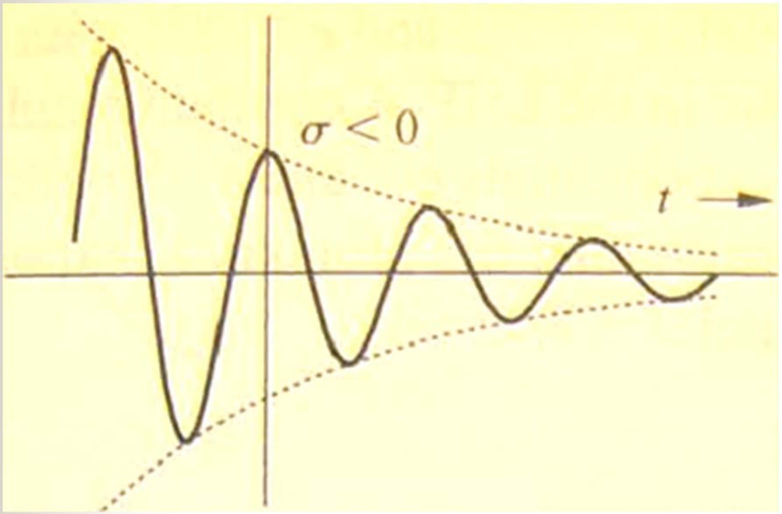
4)An exponentially varying sinusoid ($\sigma\neq 0$, $\omega\neq 0$)

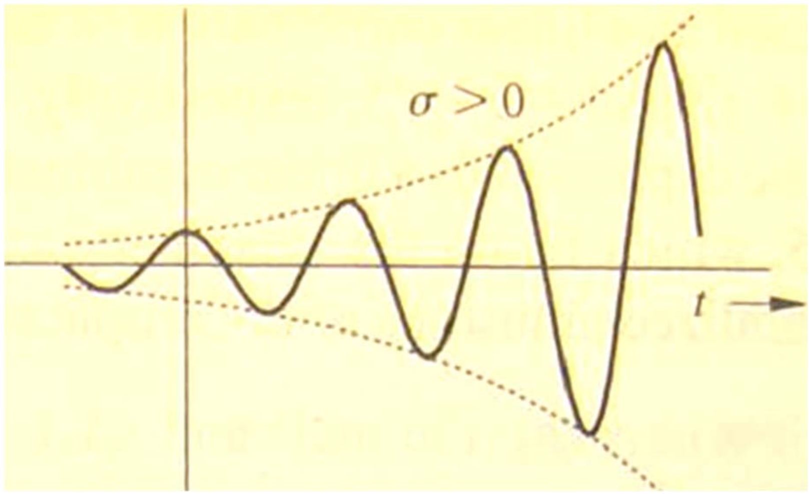




ω =0

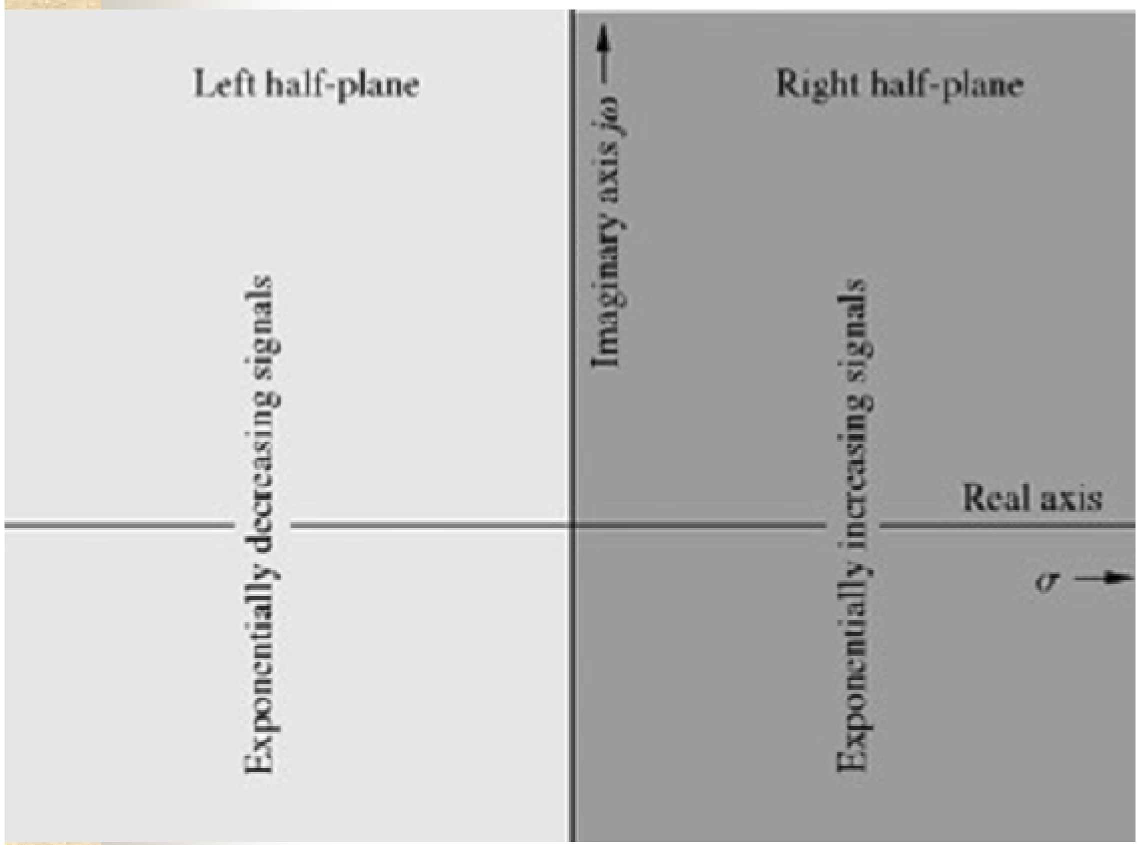
σ =0





$\sigma \neq 0$, $\omega \neq 0$

Complex frequency plane



- The left half-plane (LHP) corresponds to exponentially decaying signals.
- The right half-plane (RHP) corresponds to exponentially growing signals.
- The imaginary axis separates the two regions and corresponds to signals of constant amplitude.

➢ Sinusoidal signal

$$f(t) = C\sin(\omega t + \theta)$$

where C is a real number.

The sinusoidal signal is a periodic signal with the fundamental period $T = \dfrac{2\pi}{\omega}$.

Property:

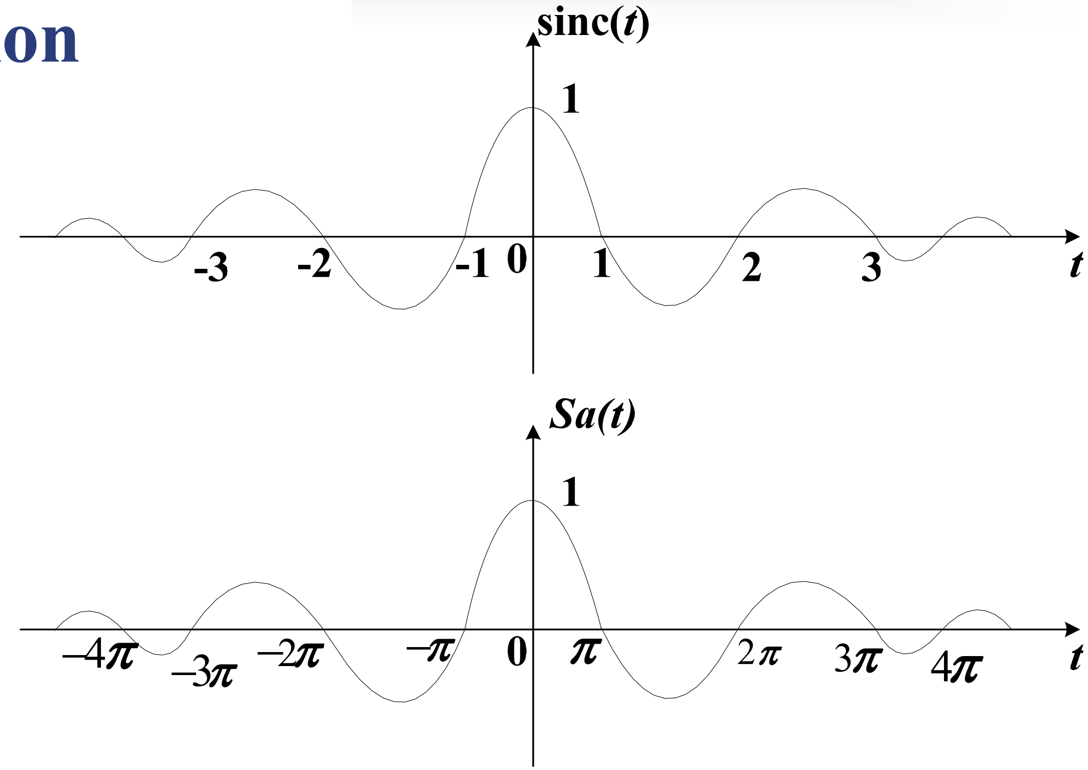
The integral and differential results of the signals are still the sinusoidal signals。

Time shift⇔ Phase change

Interpolation function

$$\mathrm{sinc}(t) = \frac{\sin \pi t}{\pi t}$$

$$\mathrm{Sa}(t) = \frac{\sin t}{t}$$



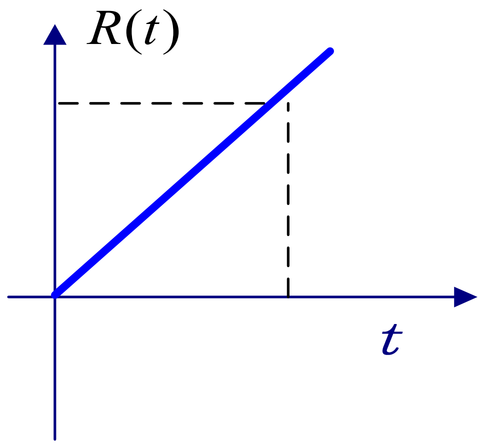
Property:

$$\int_0^{\infty} \mathrm{Sa}(t)dt = \pi / 2$$

$$\int_{-\infty}^{\infty} \mathrm{Sa}(t)dt = \pi$$

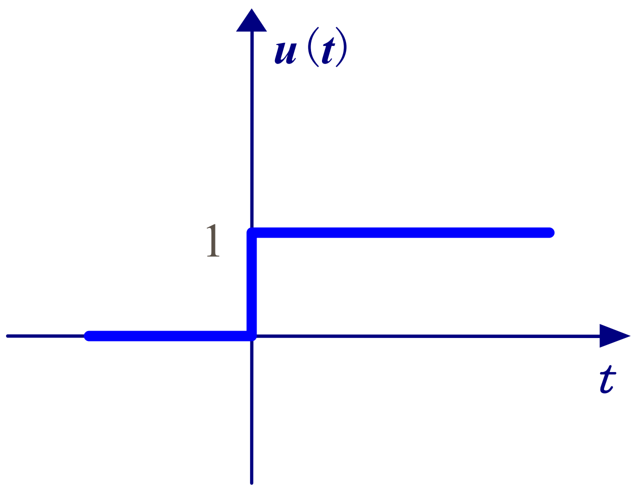
Ramp signal

$$R(t)=\begin{cases} t, & t\geq 0 \\ 0, & t<0 \end{cases}$$



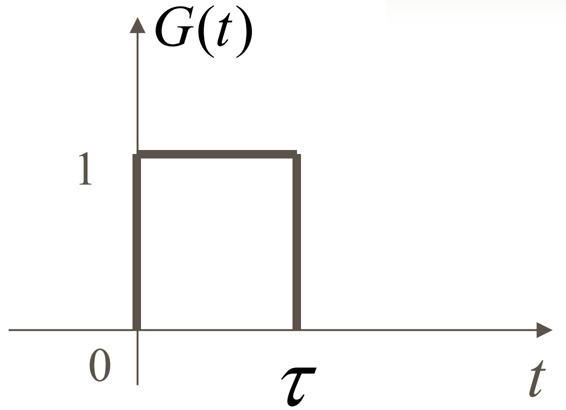
Unit step function

$$u(t)=\begin{cases}0 & t<0\\ 1 & t\geq 0\end{cases}$$



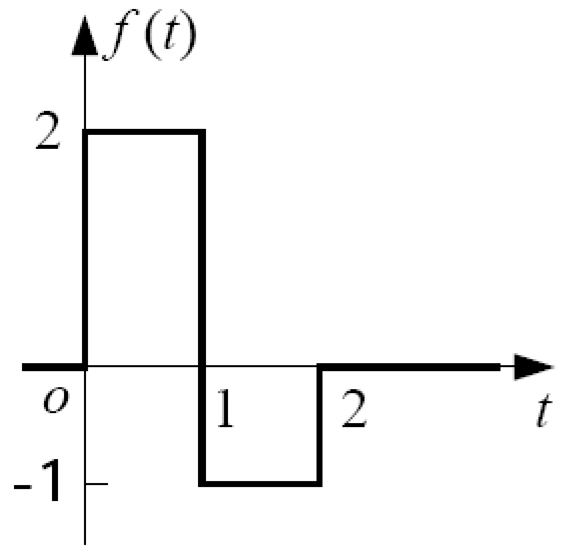
Properties:

1） The relationship between the rectangular pulse signal and *u*(*t*)



$$G(t)=u(t)-u(t-\tau)$$

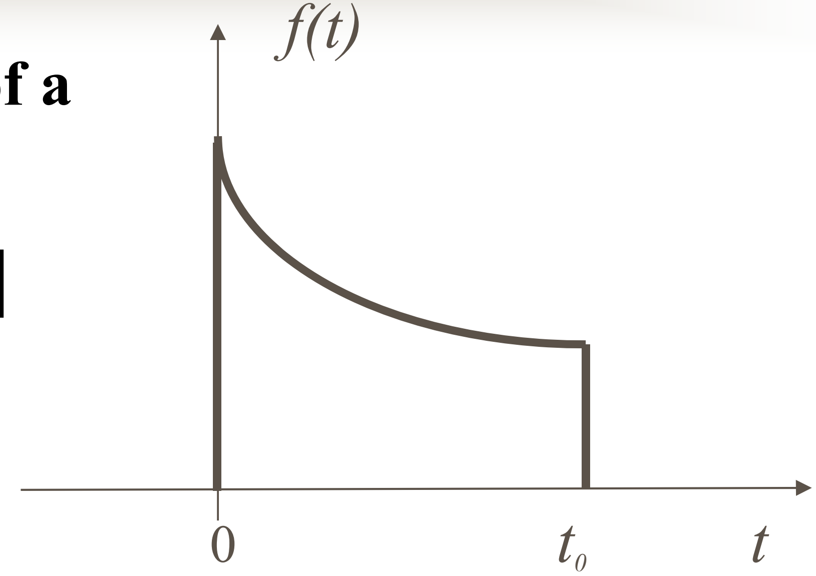
2） To represent other signals



$$f(t)=2u(t)-3u(t-1)+u(t-2)$$

3）To define the time interval of a signal

$$f(t) = e^{-t}[u(t) - u(t - t_0)]$$

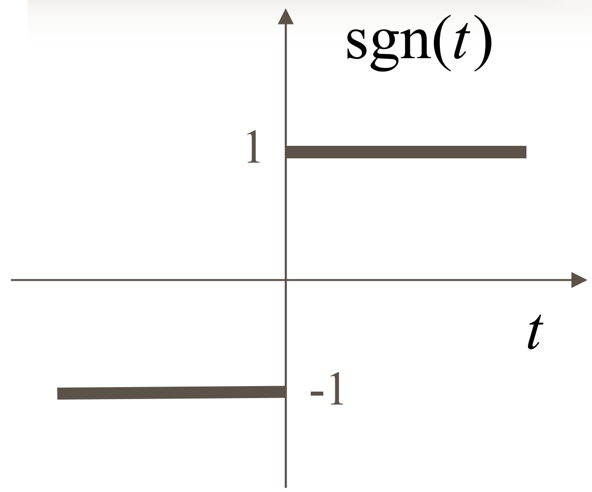


4）The relationship between the ramp signal and *u*(*t*)

$$\int_{-\infty}^{t} u(\tau)d\tau = tu(t) \qquad u(t) = \frac{dR(t)}{dt}$$

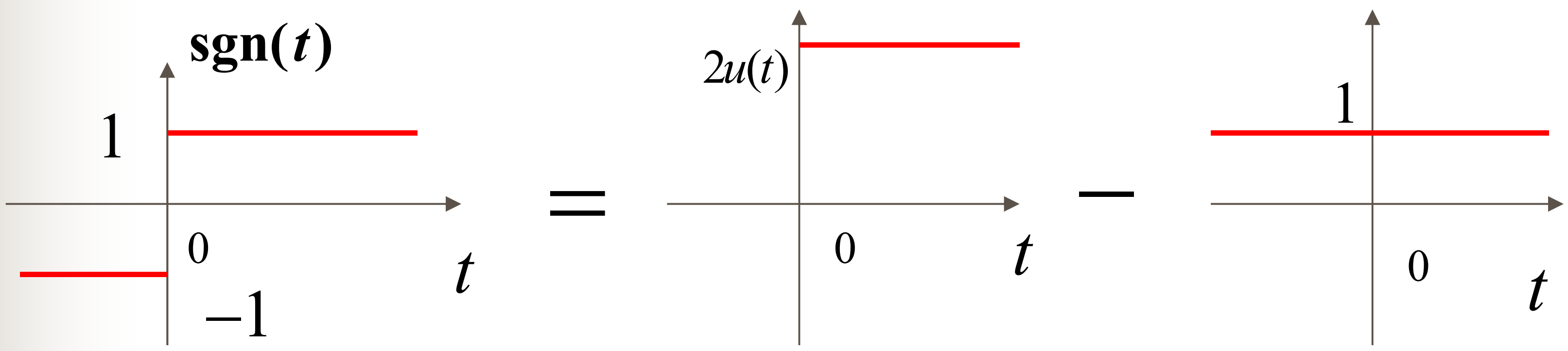
Sign function

$$\mathrm{sgn}(t) = \begin{cases} 1 & t > 0 \\ -1 & t < 0 \end{cases}$$



The sign function can also be expressed as

$$\mathrm{sgn}(t) = 2u(t) - 1$$

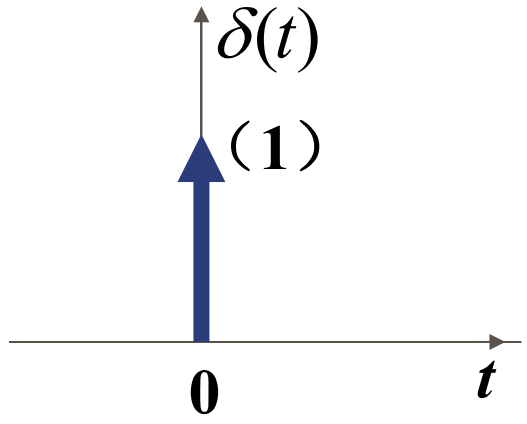


➢ Unit impulse function

The unit impulse function, $\delta(t)$, can be thought as an idealization of the short pulse as its duration becomes insignificant.

Definition 1:

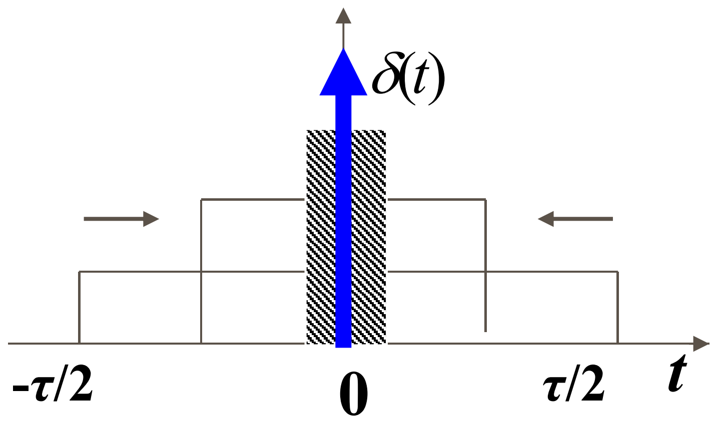
$$\begin{cases} \int_{-\infty}^{+\infty} \delta(t)\,dt = 1 \\ \delta(t) = 0,\ t \neq 0 \end{cases}$$



Definition 2:

δ (t) can be defined as the limiting form of a short pulse, of duration τ and with unit area for any value of τ. As $\tau \to 0$, the pulse becomes narrower and higher, maintaining its unit area.

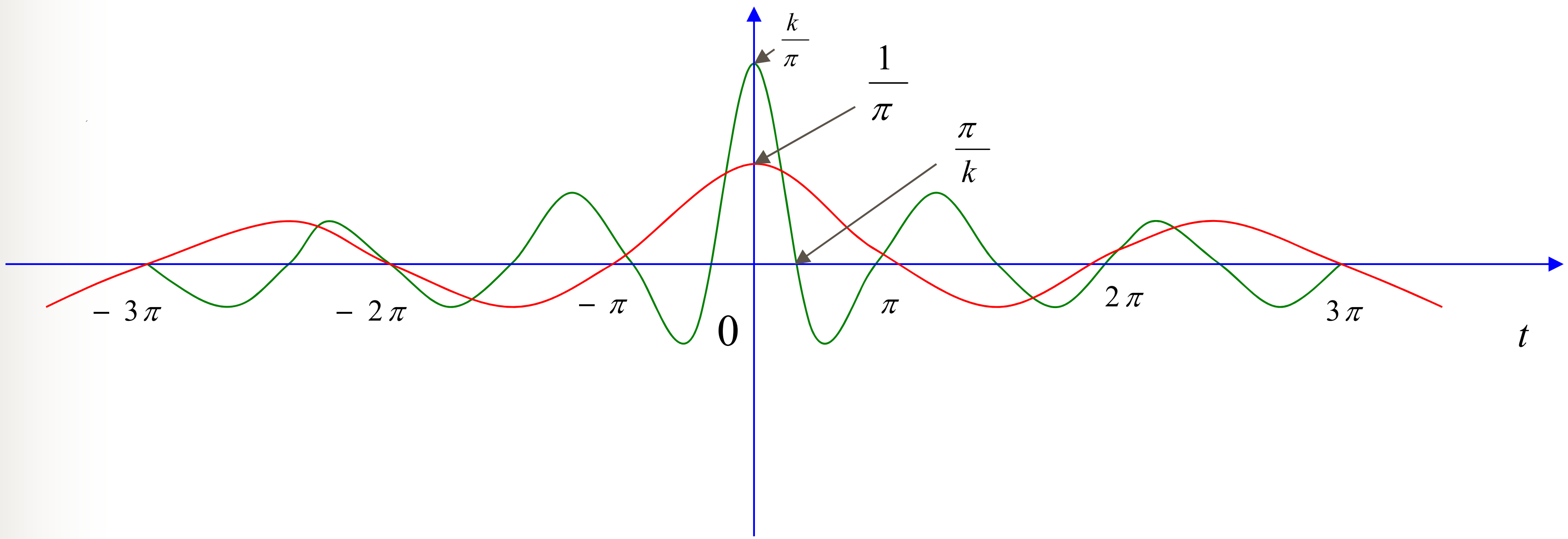
$$\delta(t) = \lim_{\tau \to 0} \frac{1}{\tau}\left[u(t+\tfrac{\tau}{2}) - u(t-\tfrac{\tau}{2})\right]$$



Definition 3:

$\delta(t)$ can be defined as the limiting form of the Sa function .

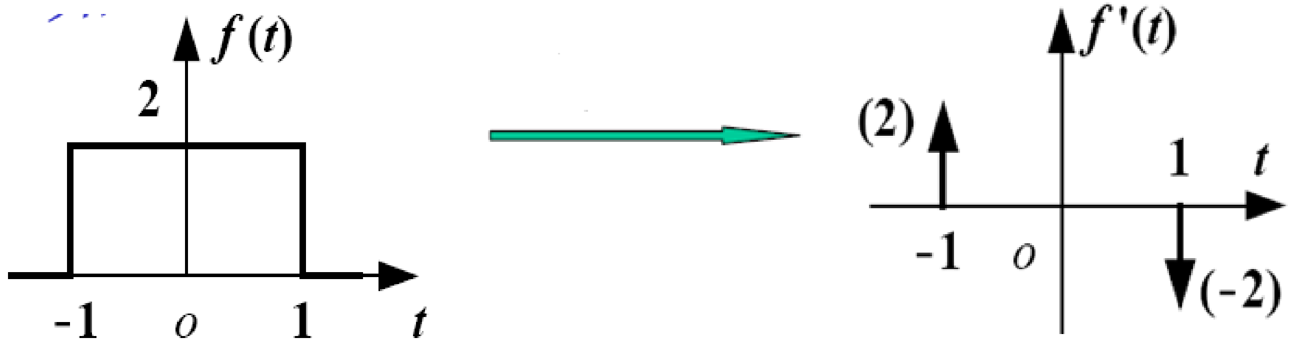
$$\delta(t)=\lim_{k\to\infty}\left[\frac{k}{\pi}\mathrm{Sa}(kt)\right],\qquad \textbf{\textit{where}}\quad \mathrm{Sa}(t)=\frac{\sin t}{t}$$



Properties:

1） $\delta(t) = \delta(-t)$

2） $\int_{-\infty}^{t} \delta(\tau)d\tau = u(t) \qquad \frac{d}{dt}u(t) = \delta(t)$



Although $f(t)$ is discontinuous at t=-1 and t=1, it is still differentiable after introducing $\delta(t)$.

3） $\delta(at) = \frac{1}{|a|}\delta(t)$

$$\because \int_{-\infty}^{\infty} \delta(at)dt = \begin{cases} \frac{1}{a}\int_{-\infty}^{\infty} \delta(at)d(at) = \frac{1}{a}\int_{-\infty}^{\infty} \delta(\tau)d\tau & a > 0 \\ \frac{1}{a}\int_{\infty}^{-\infty} \delta(at)d(at) = \frac{1}{|a|}\int_{-\infty}^{\infty} \delta(\tau)d\tau & a < 0 \end{cases}$$

$$\therefore \delta(at) = \frac{1}{|a|}\delta(t)$$

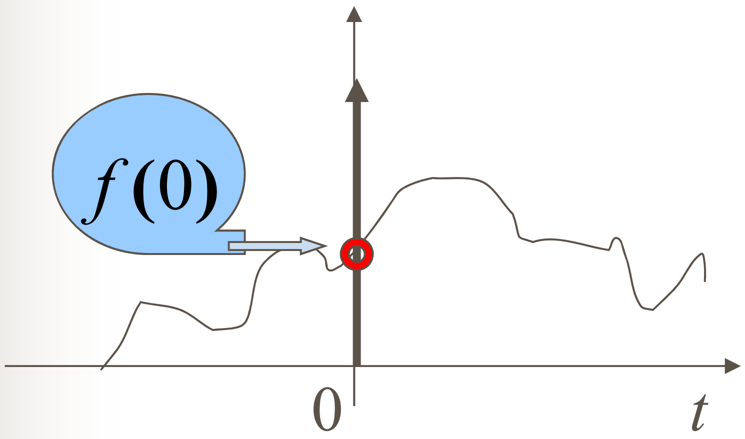
4) $f(t)\delta(t) = f(0)\delta(t)$

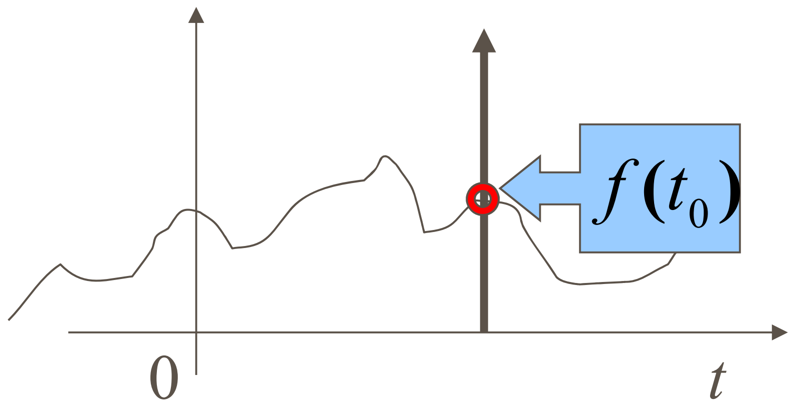
$$f(t)\delta(t-t_0) = f(t_0)\delta(t-t_0)$$

5）Sifting property

$$\int_{-\infty}^{+\infty}\delta(t)f(t)dt=\int_{-\infty}^{+\infty}\delta(t)f(0)dt=f(0)\int_{-\infty}^{+\infty}\delta(t)dt=f(0)$$

$$\int_{-\infty}^{+\infty}\delta(t-t_0)f(t)dt=\int_{-\infty}^{+\infty}f(t_0)\delta(t-t_0)dt=f(t_0)$$





Examples: **Simplify or evaluate the following expressions.**

$$(t-2)^2\delta(t) = (-2)^2\delta(t) = 4\delta(t)$$

$$\int_{-\infty}^{\infty}(t-2)^2\delta(t-4)dt = (4-2)^2 = 4$$

$$\int_{-9}^{0}(t-2)^2\delta(t-1)dt = (1-2)^2\underline{\int_{-9}^{0}\delta(t-1)dt} = 0$$

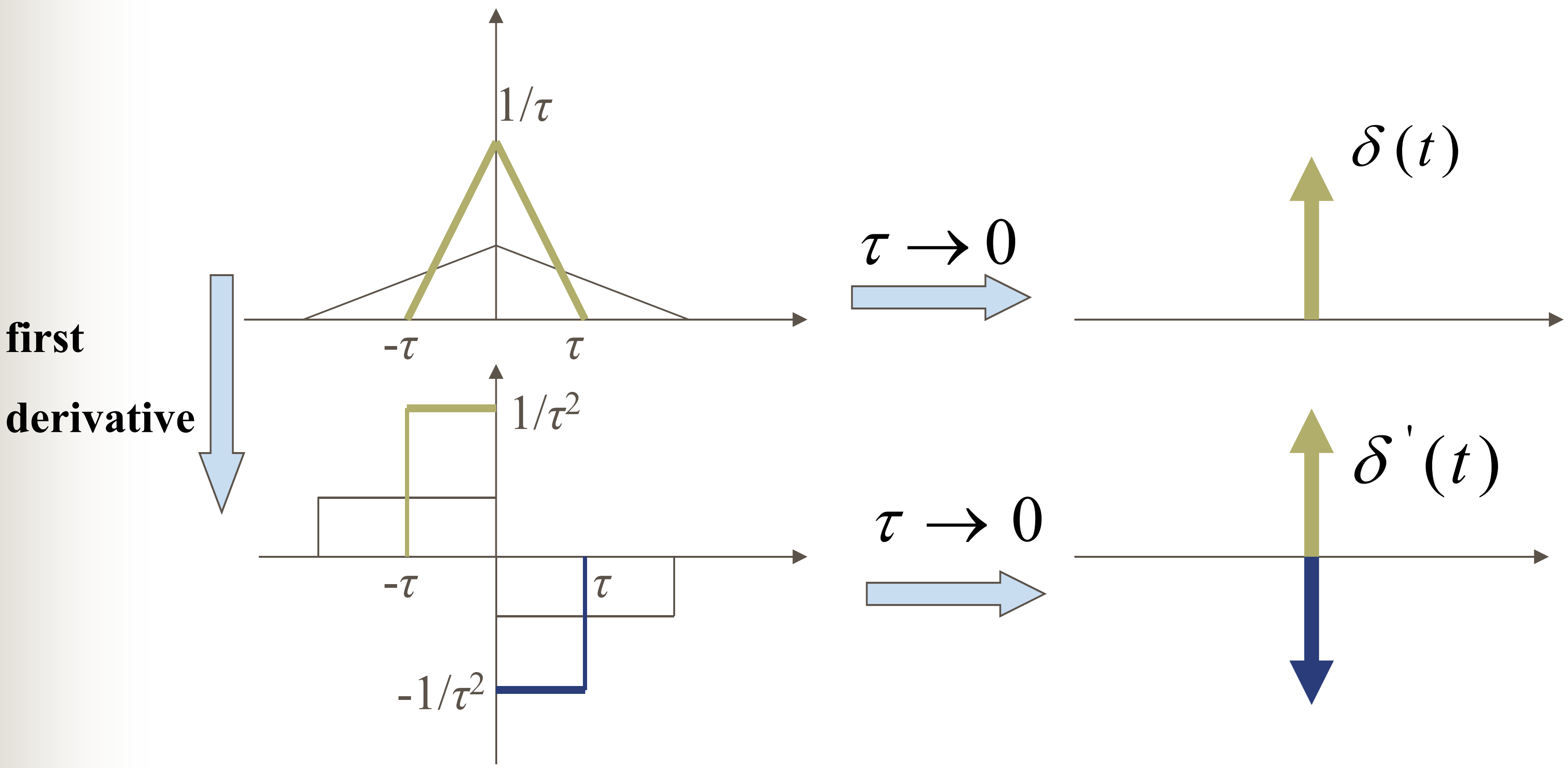
$$\int_{-9}^{3}(t-2)^2\delta(t-1)dt = (1-2)^2\underline{\int_{-9}^{3}\delta(t-1)dt} = 1$$

$$\int_{-3}^{3}(\tau-2)^2\delta(\tau-t)d\tau = (t-2)^2\underline{\int_{-3}^{3}\delta(\tau-t)d\tau}$$

$$= \begin{cases} (t-2)^2, & -3 \le t \le 3 \\ 0, & t < -3 \quad or \quad t > 3 \end{cases}$$

Unit doublets

$$\delta'(t) = \frac{d}{dt}\delta(t)$$



Properties:

1) $$\int_{-\infty}^{\infty} \delta'(t)dt = 0$$

2) $$\int_{-\infty}^{\infty} \delta'(t-t_0)f(t)dt = -f'(t_0)$$

$$\int_{-\infty}^{\infty} f(t)\delta'(t-t_0)dt = f(t)\delta(t-t_0)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(t)\delta(t-t_0)dt$$
$$= -f'(t_0)$$

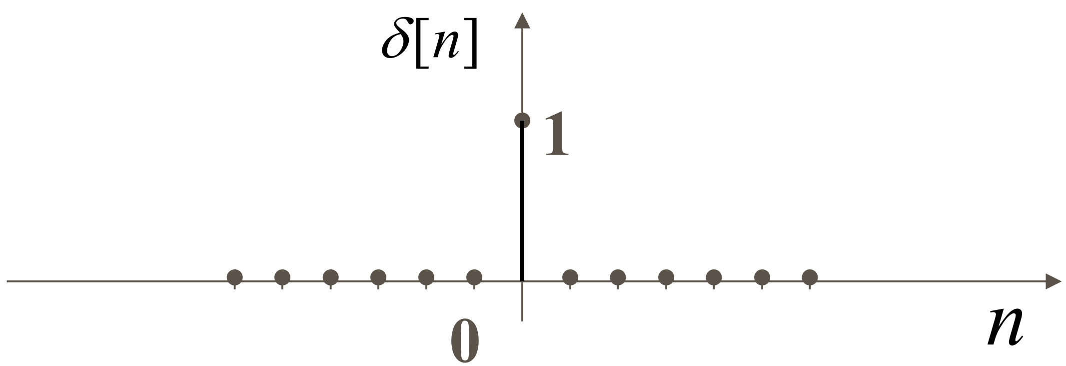
Singularity Functions:

Ramp signal, Unit step function, sign function, Unit impulse funciton, Unit doublets

2、 DT signals

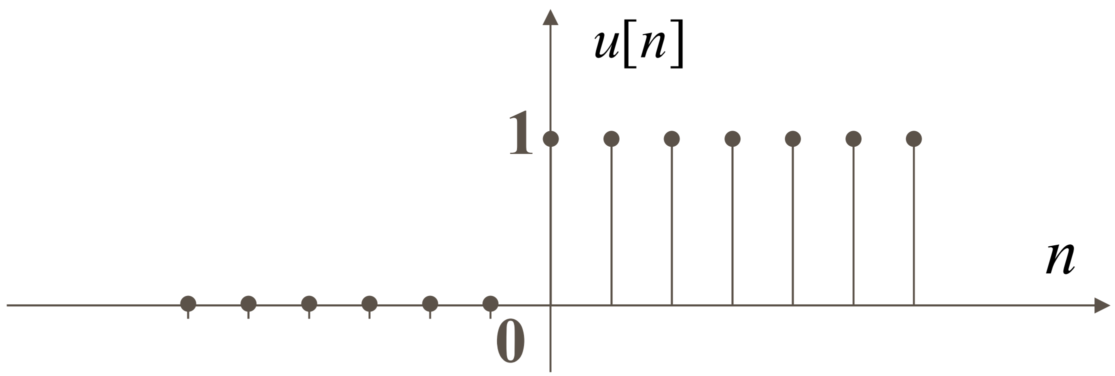
- **DT unit impulse**

$$\delta[n]=\begin{cases}1, & n=0\\ 0, & n\neq 0\end{cases}$$



➢ DT unit step sequence

$$u[n]=\begin{cases}1, & n\geq 0\\ 0, & n<0\end{cases}$$



- **Relationship between unit impulse and unit step sequence**

1) the unit impulse is the first difference of the unit step sequence

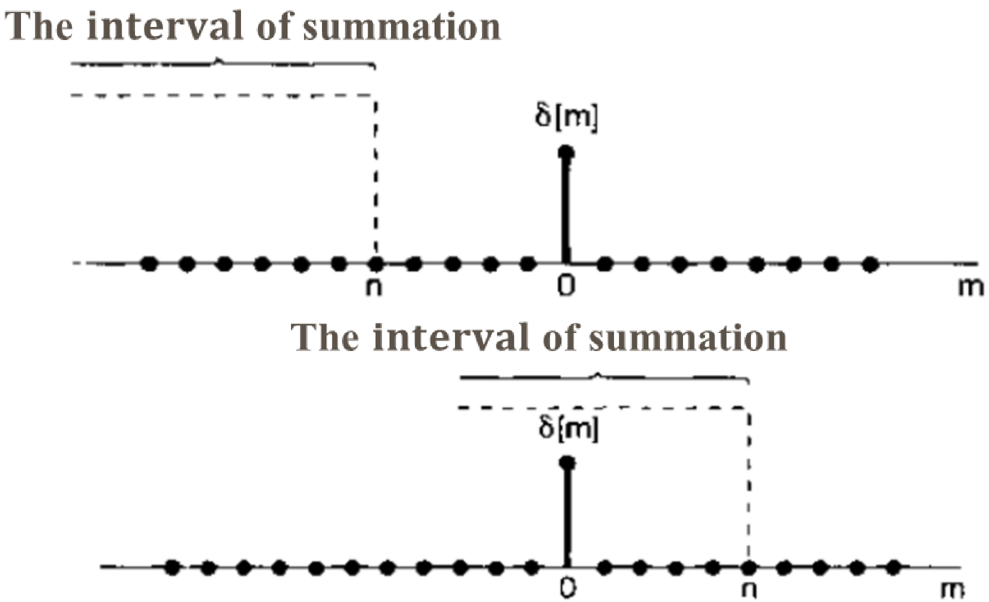
$$\delta[n] = u[n] - u[n-1]$$

2) the unit step sequence is the running sum of the unit impulse

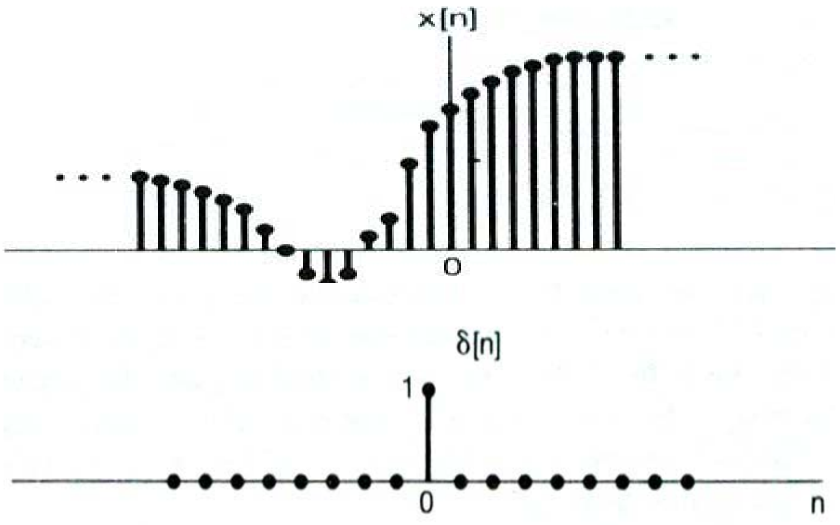
$$u[n] = \sum_{m=-\infty}^{n} \delta[m]$$

or

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k]$$



Sampling Property of Unit Sample



$$x[n]\delta[n] = x[0]\delta[n]$$

$$\sum_{n=-\infty}^{\infty} x[n]\delta[n] = x[0]$$

$$x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]$$

Any DT signal can be expressed in terms of delayed unit sample as :

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]$$

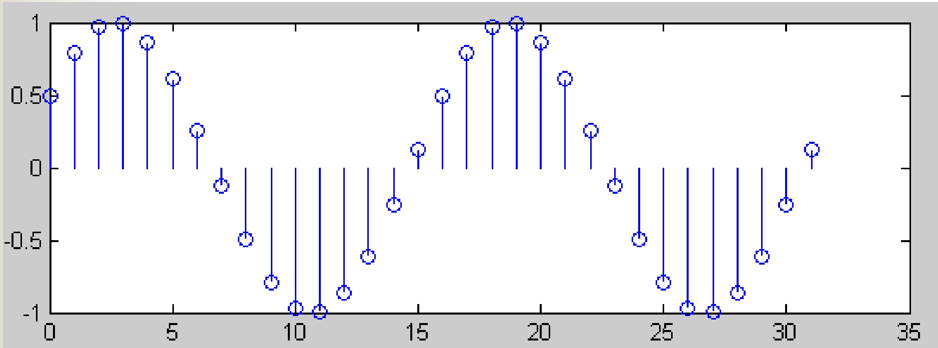
Sinusoidal sequence

$$f[n] = A\sin(\omega_0 n + \varphi)$$

Where A is the amplitude, φ is the phase in radians, and the dimensions of the frequency ω_0 are radians per sample.

A CT sinusoidal signal is definitely periodic, but a sinusoidal sequence is not necessarily periodic.

When $2\pi/\omega_0$ is a rational number, $f[n]$ is periodic, otherwise it is not periodic.



Example：

$$f[n] = \sin(\pi n/8 + \pi/6)$$

➢ Exponential sequence

- **Real Exponential Signals:**

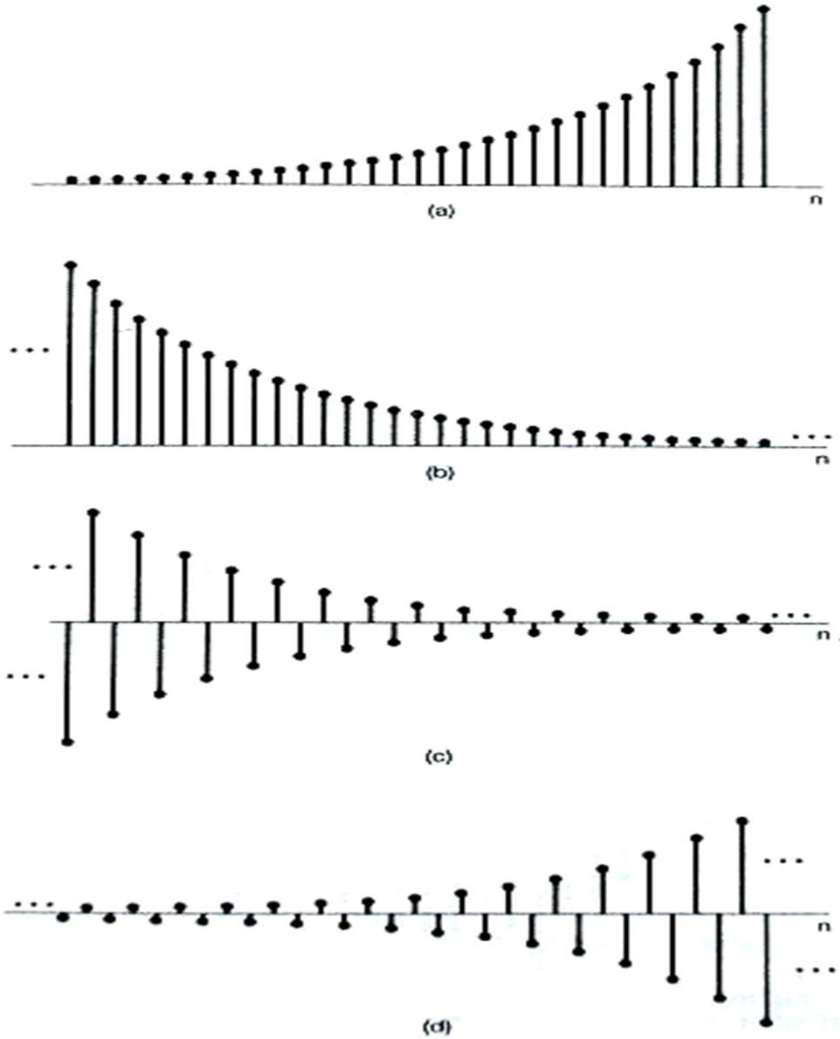
$$f[n] = \gamma^n$$

(a) $\gamma > 1$

(b) $0 < \gamma < 1$

(c) $-1 < \gamma < 0$

(d) $\gamma < -1$



- **Complex Exponential Signals:**

The discrete-time exponential can also be expressed by using a natural base, as

$$\gamma^n = e^{\lambda n}$$

1) λ in the RHP: The signal $e^{\lambda n}$ grows exponentially.

2) λ in the LHP: The signal $e^{\lambda n}$ decays exponentially.

3) λ on the imaginary axis: The signal $e^{\lambda n}$ is constant or oscillates with constant amplitude.

The λ plane, the γ plane, and their mapping

$$\lambda = a + jb \to \gamma = e^{a+jb} = e^{a} e^{jb}$$

- $a = 0 \to |\gamma| = |e^{jb}| = 1$

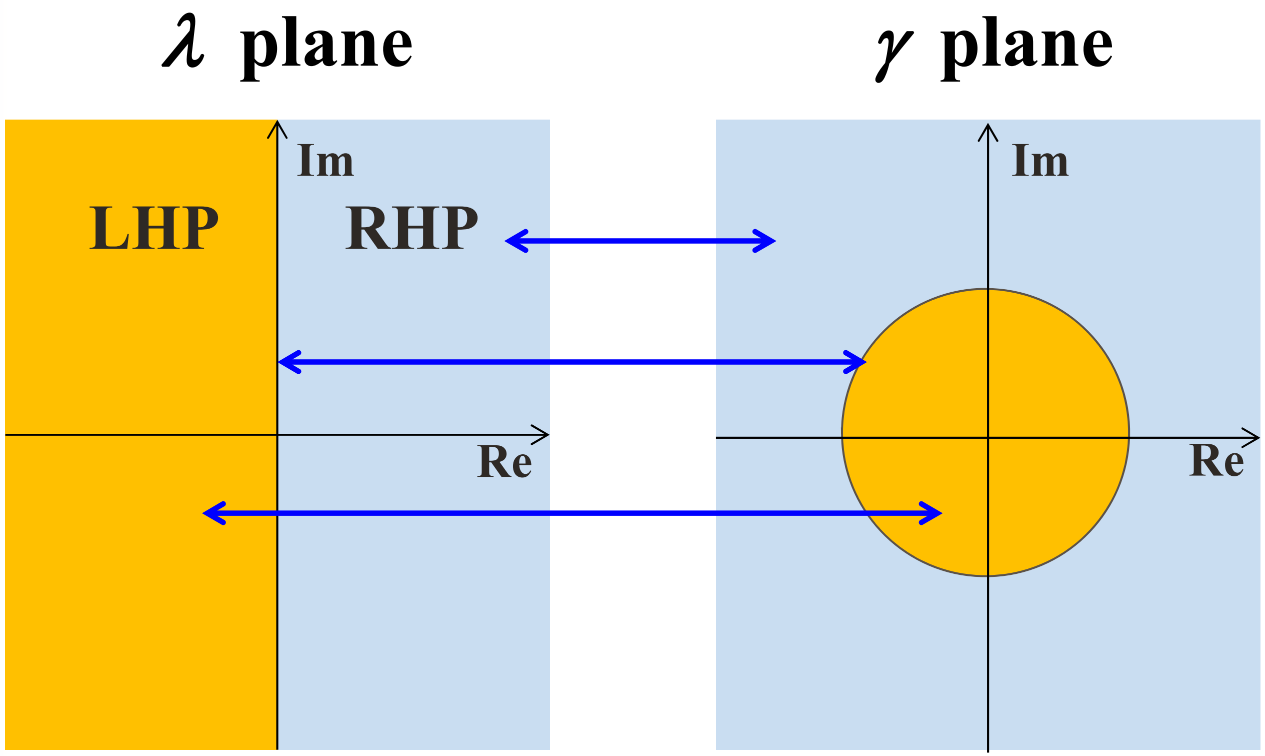
The imaginary axis in the λ plane maps into the unit circle in the γ plane.

- $a < 0 \to |\gamma| = e^{a} < 1$

The LHP in the λ plane maps into the inside of the unit circle in the γ plane.

- $a > 0 \to |\gamma| = e^{a} > 1$

The RHP in the λ plane maps into the outside of the unit circle in the γ plane.



λ plane
γ plane
Im
LHP
RHP
Re
Im
Re

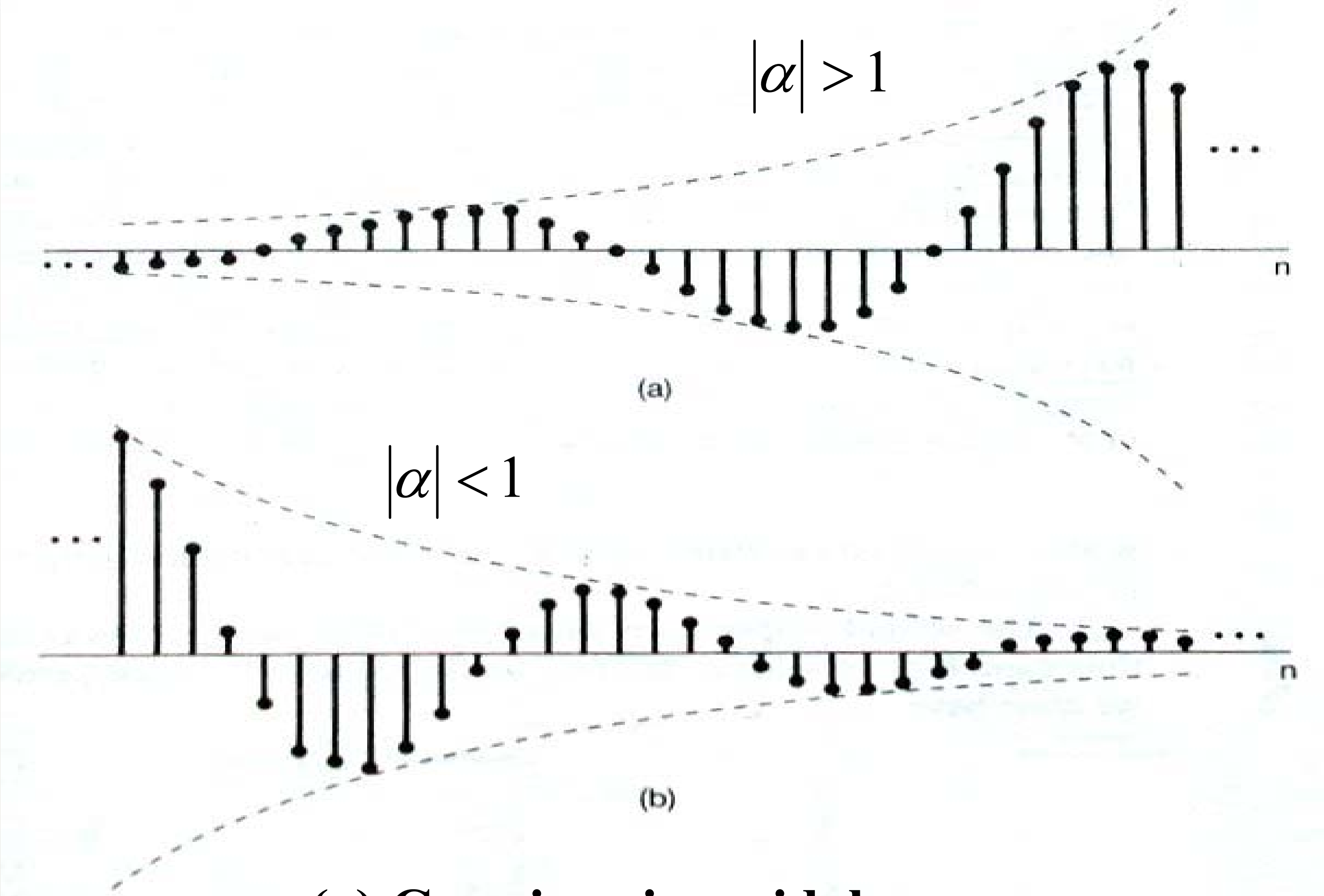
- **General Complex Exponential Signals:**

$$f[n] = C\alpha^n$$

in which $C = |C|e^{j\theta}$, $\alpha = |\alpha|e^{j\omega}$

then

$$f[n] = |C||\alpha|^n\cos(\omega n+\theta)+j\,|C||\alpha|^n\sin(\omega n+\theta)$$



(a) Growing sinusoidal sequence
(b) Decaying sinusoidal sequence

- **Periodicity Properties of DT Complex Exponentials**

$$e^{j\omega t} \quad \text{and} \quad e^{j\omega n}$$

Are they the same? **No!**

- **For $e^{j\omega t}$, it has two properties:**

  1) The larger the magnitude of ω, the higher the rate of oscillation in the signal;

  2) $e^{j\omega t}$ is periodic for any value of ω.

- For $e^{j\omega n}$, we need to consider a frequency interval of 2π.

$$0 \le \omega < 2\pi, \quad \text{or} \quad -\pi \le \omega < \pi$$

$$e^{j(\omega+2\pi)n} = e^{j\omega n} e^{j2\pi n} = e^{j\omega n}$$

- Periodicity of $e^{j\omega n}$

CT: $e^{j\omega t}$, $T=2\pi/\omega$ **DT:** $e^{j\omega n}$, $N=?$

Calculate period:

By definition: $e^{j\omega n} = e^{j\omega(n+N)}$

thus $e^{j\omega N} = 1$ or $\omega N = 2\pi m$

So $$\boxed{N = \left(\frac{2\pi}{\omega}\right)m}$$

Condition of periodicity: $2\pi/\omega$ is rational.

Even and Odd Signals

- **Even signal:** $x(-t) = x(t)$ or $x[-n] = x[n]$
- **Odd signal :** $x(-t) = -x(t)$ or $x[-n] = -x[n]$

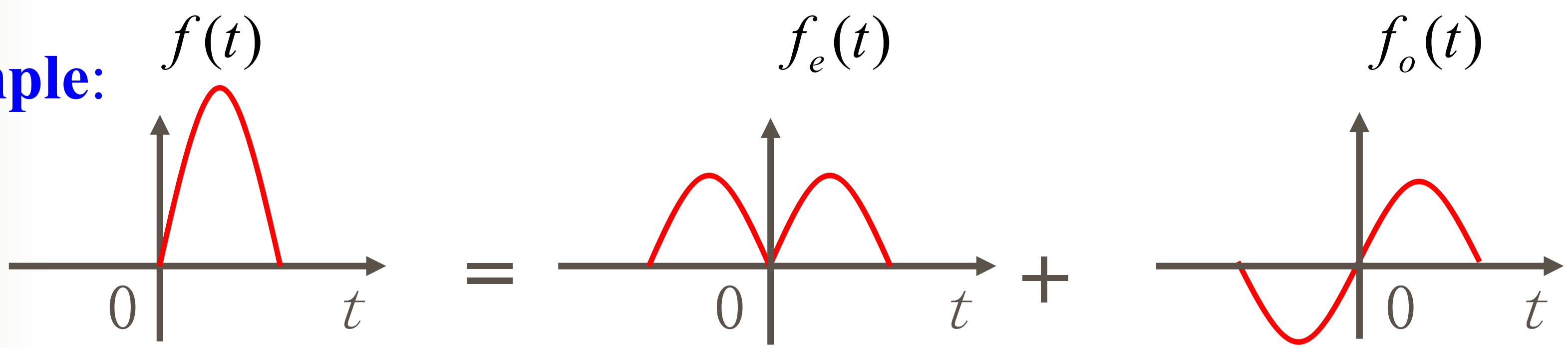
Even-Odd Decomposition:

$$x(t) = x_e(t) + x_o(t)$$

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)] \qquad x_e[n] = \frac{1}{2}\{x[n] + x[-n]\}$$

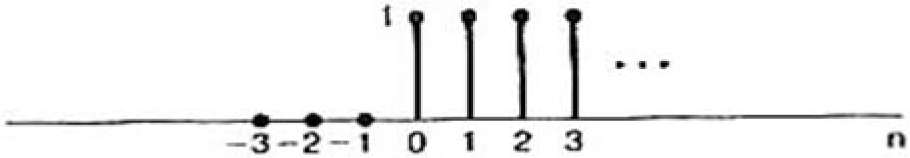
$$x_o(t) = \frac{1}{2}[x(t) - x(-t)] \qquad x_o[n] = \frac{1}{2}\{x[n] - x[-n]\}$$

Example:

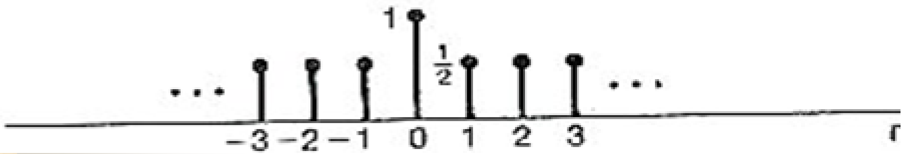


Example: **Find the even and odd components of the following signal.**

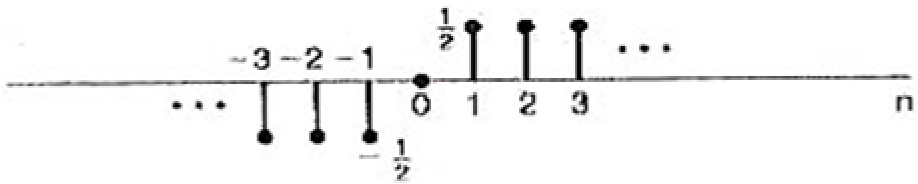
$$x[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



$$\mathcal{E}v\{x[n]\} = \begin{cases} \frac{1}{2}, & n < 0 \\ 1, & n = 0 \\ \frac{1}{2}, & n > 0 \end{cases}$$



$$\mathcal{O}d\{x[n]\} = \begin{cases} -\frac{1}{2}, & n < 0 \\ 0, & n = 0 \\ \frac{1}{2}, & n > 0 \end{cases}$$



Systems

• Representation of Systems

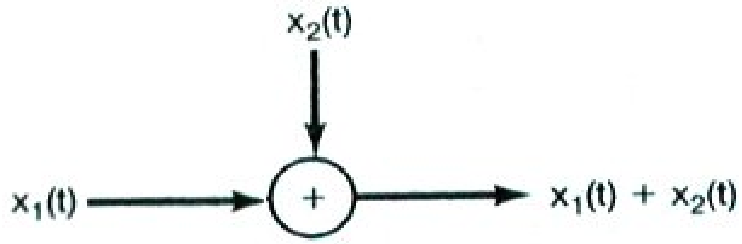
1、Mathematic model

Input-output equation、State equation

2、Block diagram

CT system Basic elements

Adder

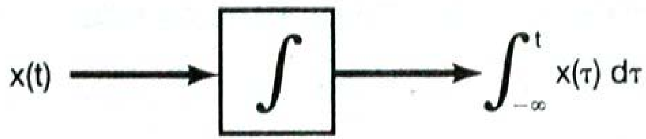


(a)

Coefficient multiplier

(b)

Integrator

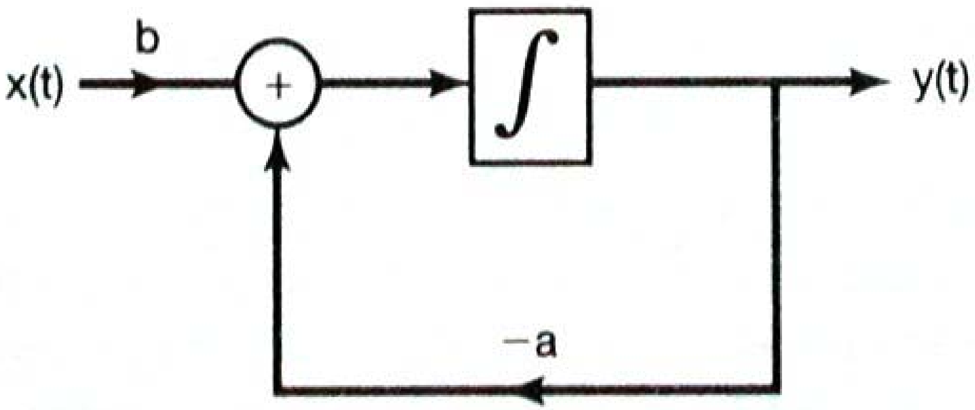


(c)

Example:

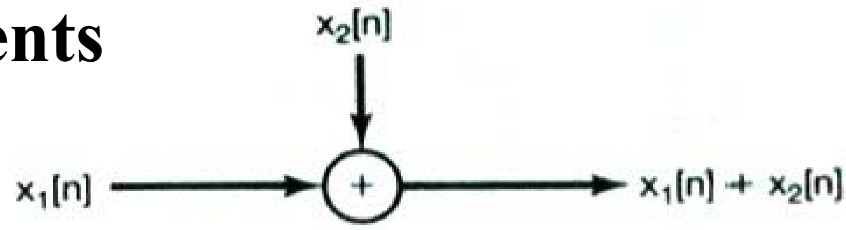
$$\frac{dy(t)}{dt} + ay(t) = bx(t)$$

$$y(t) = \int_{-\infty}^{t} [bx(\tau) - ay(\tau)] d\tau$$



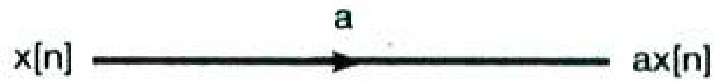
DT system Basic elements

Adder



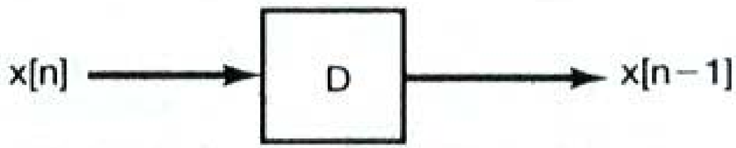
(a)

Coefficient Multiplier



(b)

Unit delay

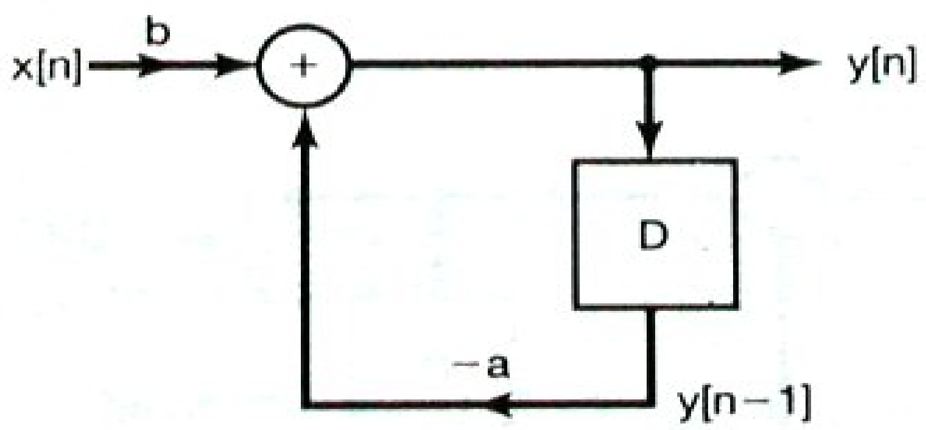


(c)

Example:

$$y[n] + ay[n-1] = bx[n]$$

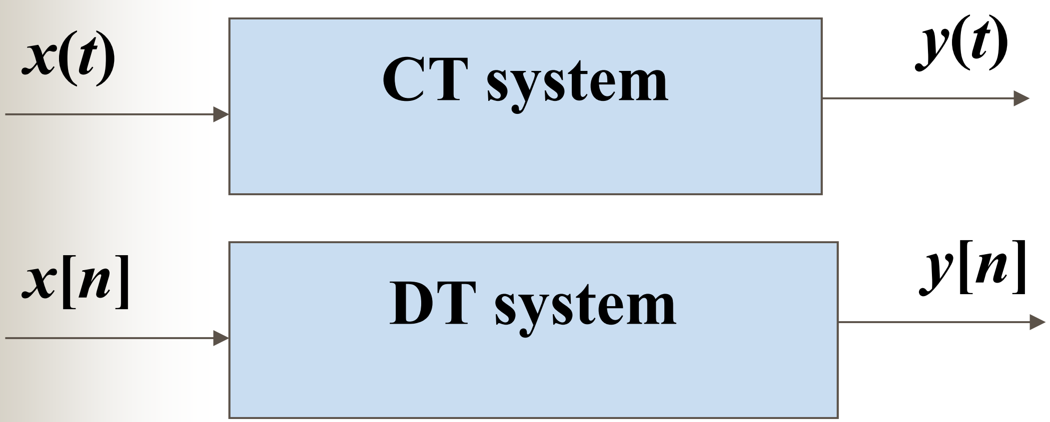
addition **delay** **multiplication**



CT and DT Systems

- **CT system: both input and output signals are CT signals.**
- **DT system: transform DT inputs into DT outputs.**

Pictorial Representation



Relation by the notation

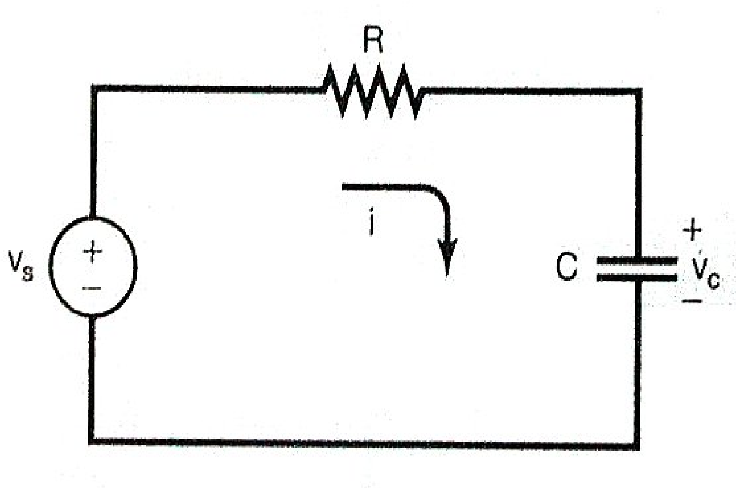
$$x(t) \longrightarrow y(t)$$
$$x[n] \longrightarrow y[n]$$

$$y(t) = T[x(t)]$$
$$y[n] = T[x[n]]$$

Simple Examples of Systems

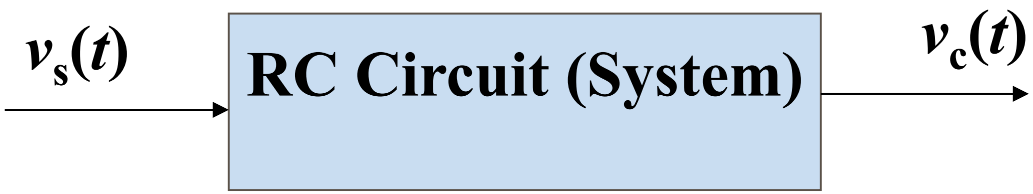
Example:

RC Circuit :



$$\frac{dv_c(t)}{dt} + \frac{1}{RC} v_c(t) = \frac{1}{RC} v_s(t)$$

Linear Constant Coefficient Differential Equation



Example:

Balance in a bank account from month to month

balance --- $y[n]$

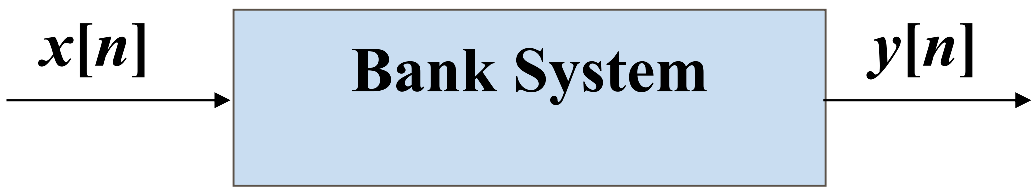
net deposit --- $x[n]$

interest rate--- 1%

Therefore, $y[n]=y[n-1]+1\%y[n-1]+x[n]$

or $y[n]-1.01y[n-1]=x[n]$

Linear Constant Coefficient Difference Equation



Linear and nonlinear systems

A system is linear if it has the additivity property and the scaling or homogeneity property:

If $x_1(\bullet) \to y_1(\bullet)$, $x_2(\bullet) \to y_2(\bullet)$

Then $x_1(\bullet)+x_2(\bullet) \to y_1(\bullet)+y_2(\bullet)$ **additivity property**

$k_1 x_1(\bullet) \to k_1 y_1(\bullet)$ **scaling property**

The two properties can be combined into the superposition property:

$$k_1 x_1(\bullet)+k_2 x_2(\bullet) \to k_1 y_1(\bullet)+k_2 y_2(\bullet)$$

Example： Is the system, T[$x(t)$]=$ax(t)+b=y(t)$, a linear system？

$$\because T[k_1x_1(t)+k_2x_2(t)]=a[k_1x_1(t)+k_2x_2(t)]+b$$

and $k_1y_1(t)+k_2y_2(t)=k_1T[x_1(t)]+k_2T[x_2(t)]$

$$=k_1[ax_1(t)+b]+k_2[ax_2(t)+b]$$

$$=a[k_1x_1(t)+k_2x_2(t)]+bk_1+bk_2$$

Therefore $T[k_1x_1(t)+k_2x_2(t)]\neq k_1y_1(t)+k_2y_2(t)$

We conclude that the system is not linear.

Example: Is the following system linear ?

$$\frac{dy(t)}{dt}+a_0y(t)=b_0x(t)+b_1\frac{dx(t)}{dt}$$

Assume $x_1(t)\rightarrow y_1(t),\ \ x_2(t)\rightarrow y_2(t);$

Therefore $k_1[\frac{d}{dt}y_1(t)+a_0y_1(t)]=k_1[b_0x_1(t)+b_1\frac{d}{dt}x_1(t)]----(1)$

$$k_2[\frac{d}{dt}y_2(t)+a_0y_2(t)]=k_2[b_0x_2(t)+b_1\frac{d}{dt}x_2(t)]----(2)$$

(1)+(2):
$$\frac{d}{d(t)}[k_1y_1(t)+k_2y_2(t)]+a_0[k_1y_1(t)+k_2y_2(t)]=$$

$$b_0[k_1x_1(t)+k_2x_2(t)]+b_1\{\frac{d}{dt}[k_1x_1(t)+k_2x_2(t)]\}$$

We have $k_1x_1(t)+k_2x_2(t)\longrightarrow k_1y_1(t)+k_2y_2(t)$

We conclude that the system is linear.

Response of a linear system

- total response = zero-input response + zero-state response
- The zero-input component must obey the principle of superposition.
- The zero-state component must obey the principle of superposition.

Example: **Are the following systems linear？**

$$(1)\,y(t)=5y(0)+y(0)x(t)+4x(t)$$

$$(2)\,y(t)=3y^2(0)+4x(t)$$

$$(3)\,y(t)=7y(0)+2x^2(t)$$

$$(1)\because\ y_{zi}(t)=5y(0),\quad y_{zs}(t)=4x(t)$$

$$\therefore\ y(t)\neq y_{zi}(t)+y_{zs}(t)$$

The system is not linear.

$$(2)\quad ay_{zi}(t)\neq 3[ay(0)]^2$$

The system is not linear.

$$(3)\quad ay_{zs}(t)\neq 2[ax(t)]^2$$

The system is not linear.

Time-invariant and time-varying systems

- **Informally, a system is time-invariant if its behavior does not depend on what time it is.**

Mathematically:

- **A CT system is time-invariant for any input and any time shift.**

$$\textit{if} \qquad T[x(t)] = y(t)$$

$$\textit{then} \qquad T[x(t-t_0)] = y(t-t_0)$$

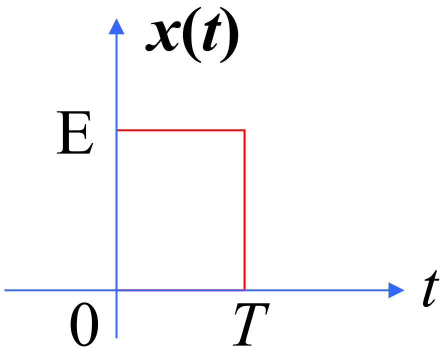
- **A DT system is time-invariant for any input and any time shift.**

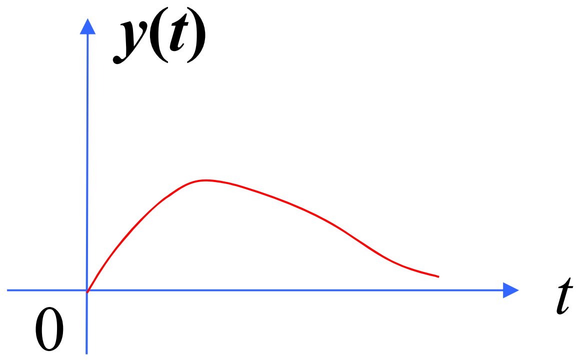
$$\textit{if} \qquad T[x[n]] = y[n]$$

$$\textit{then} \qquad T[x[n-n_0]] = y[n-n_0]$$

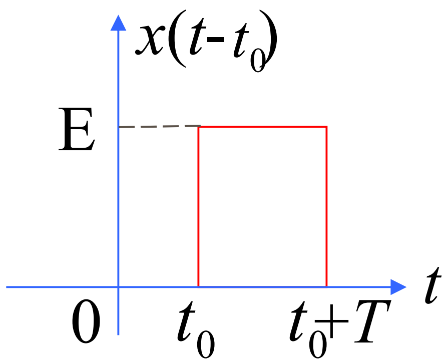
Examples:

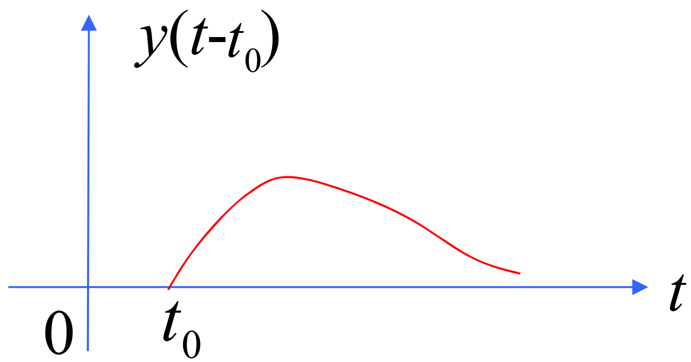
if





then





Examples: **Which of the systems is/are time-invariant?**

$(1)\ y(t)=atx(t)$ $(2)\ y(t)=ax(t)$ $(3)\ y[n]=x[n]x[n-1]$

$(4)\ y(t)=x(-t)$ $(5)\ y(t)=x(at)$

（1） $\because\ y(t-t_0)=a(t-t_0)x(t-t_0)$

$$T[x(t-t_0)]=atx(t-t_0)$$

$\therefore\ y(t-t_0)\neq T[x(t-t_0)]$ **Time-varying**

（2） $\because\ y(t-t_0)=ax(t-t_0)$

$$T[x(t-t_0)]=ax(t-t_0)$$

$\therefore\ y(t-t_0)=T[x(t-t_0)]$ **Time-invariant**

（3）$\because\ y[n-n_0]=x[n-n_0]x[n-n_0-1]$

$$T[x[n-n_0]]=x[n-n_0]x[n-n_0-1]$$

$\therefore\ y[n-n_0]=T[x[n-n_0]]$ **Time-invariant**

(4) Assume $x_1(t) = x(t - t_0)$

$\because \quad T[x_1(t)] = x_1(-t) = x(-t - t_0)$

$y(t - t_0) = x[-(t - t_0)] = x(-t + t_0)$

Time-varying

$\therefore T[x_1(t)] \neq y(t - t_0)$

(5) Assume $x_1(t) = x(t - t_0)$

$\because T[x_1(t)] = x_1(at) = x(at - t_0)$

Time-varying

$y(t - t_0) = x[a(t - t_0)] = x(at - at_0)$

$\therefore T[x_1(t)] \neq y(t - t_0)$

Instantaneous and dynamic Systems

- A system is instantaneous (or **memoryless**): Its output is dependent only on the input at that same time.
- A system is dynamic (or a system **with memory**): It depends not only on the present input, but also on **past** or **future** input and output values.

Examples: Which of the systems is/are memoryless?

$$y(t) = x(t)$$

$$y[n] = C\,x[n]$$

memoryless system

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau$$

$$y[n] = x[n-1]$$

$$y[n] = \sum_{k=-\infty}^{n} x[k] = \cdots + x[n-2] + x[n-1] + x[n]$$

memory system

Causal and noncausal systems

- **A system is causal if the output does not anticipate future values of the input, i.e., if the output at any time depends only on values of the input up to that time.**
- **All real-time physical systems are causal, because time only moves forward. Effect occurs after cause.**

Examples: **Which of the systems is/are causal?**

$y(t) = x(t-2)$ causal

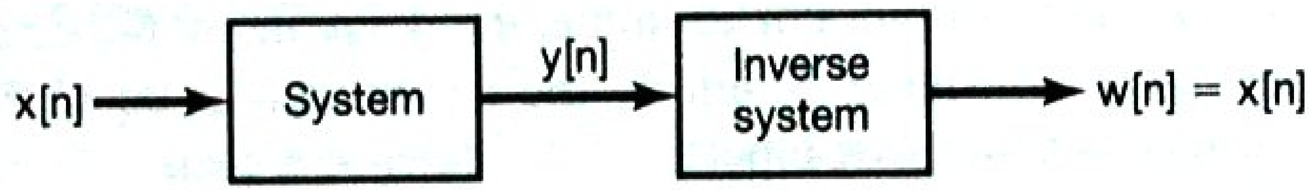
$y[n] = x[n+2]$ noncausal

$y(t) = x(t)\cos(t+1)$ causal

Invertible and noninvertible Systems

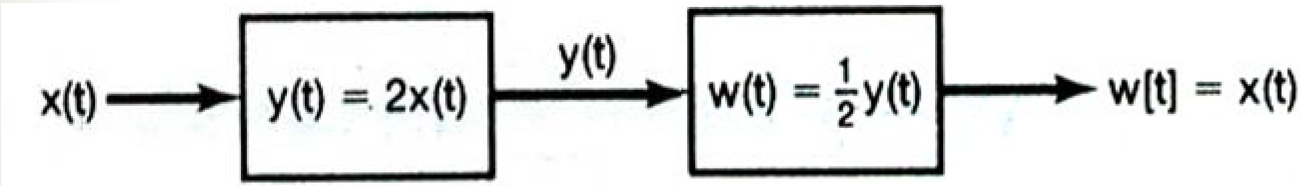
Invertible： Distinct inputs lead to distinct outputs.

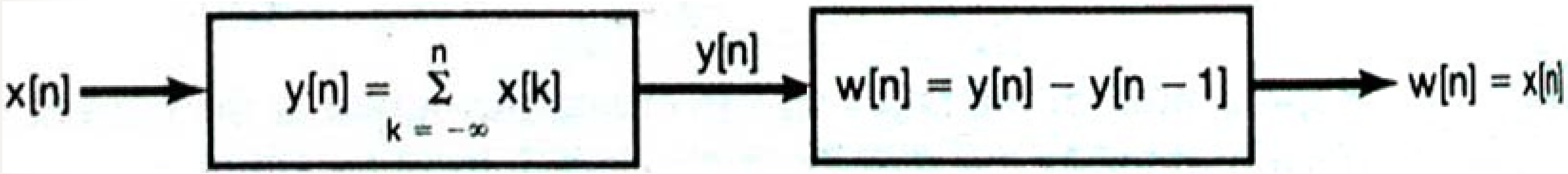
- **If a system is invertible then an inverse system exists.**
- **An inverse system cascaded with its original system, yields an output equal to the input.**



Examples: **Which of the systems is/are invertible?**

invertible system：





noninvertible system：

$$y[n]=0$$

$$y(t)=x^2(t)$$

Stable and unstable systems

A system is stable if the bounded input leads to the bounded output.

Mathematically: A system is stable.

If $|x(\bullet)| < \infty$ **then** $|y(\bullet)| < \infty$

Examples: Which of the systems is/are stable?

$$y[n] = x[n] + x[n-1]$$ **stable**

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$ **unstable**

Linear Time-Invariant (LTI） systems

- **Focus of most of this course**
- **A basic fact: If we know the response of an LTI system to some inputs, we actually know the response to many inputs.**

- **LTIC systems also have the following properties:**

1)Differential property：

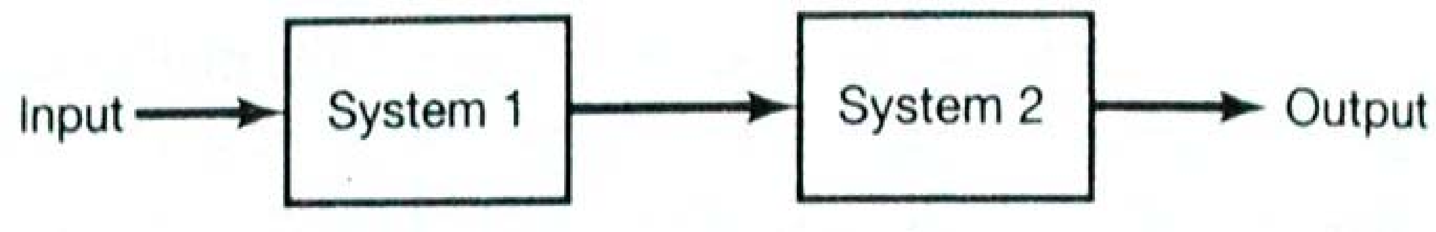
$$\text{If} \quad x(t) \to y(t), \qquad \text{then} \quad x'(t) \to y'(t)$$

2)Integral property：

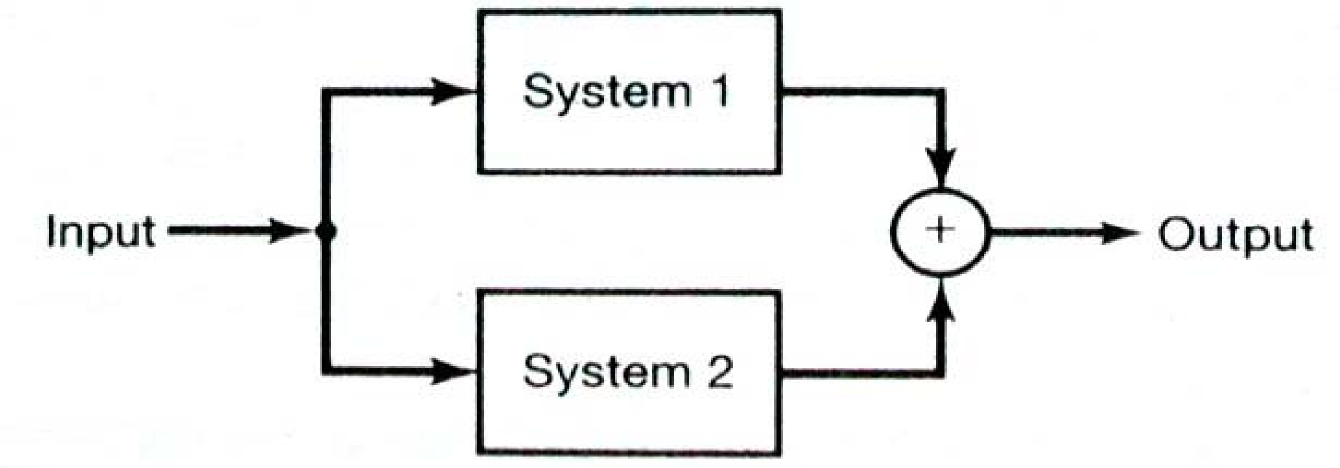
$$\text{If} \quad x(t) \to y(t), \qquad \text{then} \quad \int_{-\infty}^{t} x(\tau)d\tau \to \int_{-\infty}^{t} y(\tau)d\tau$$

Interconnections of System

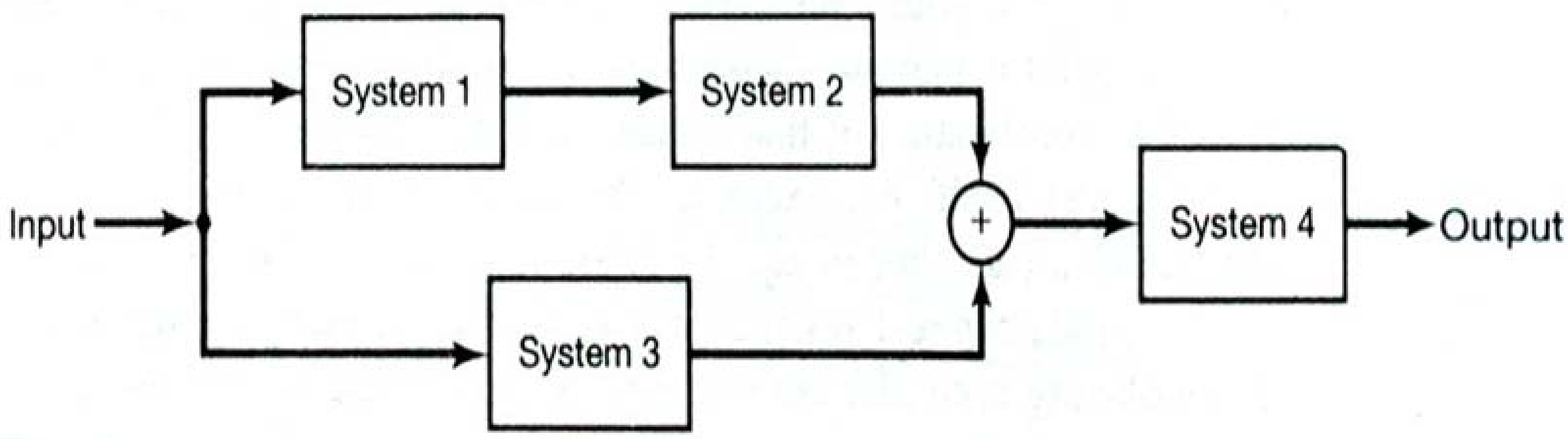
- **Series (cascade) interconnection:**



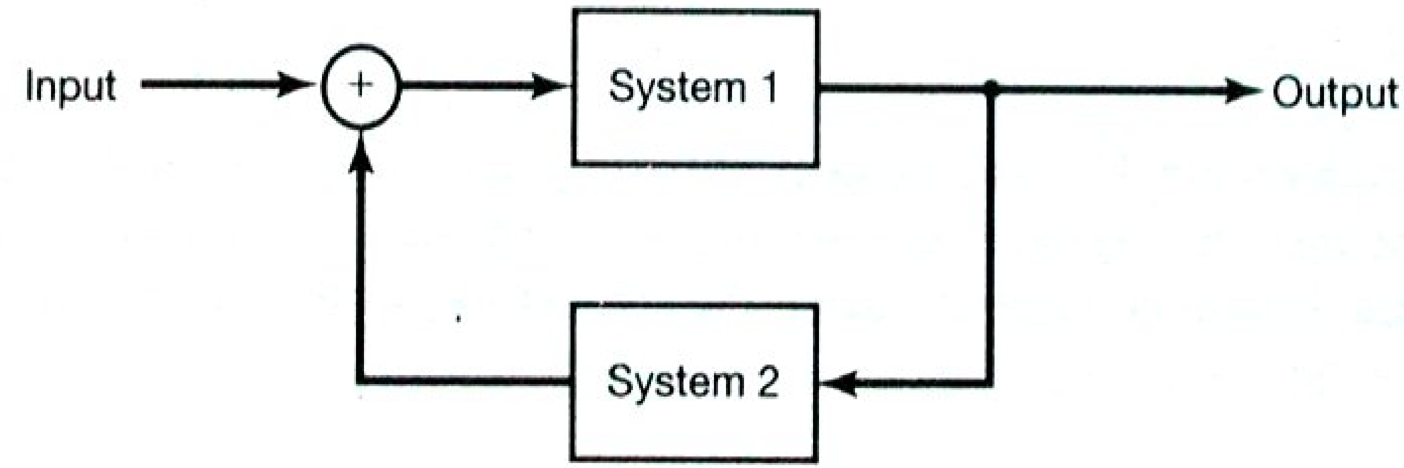
- **Parallel interconnection:**



- **Series-Parallel interconnection:**



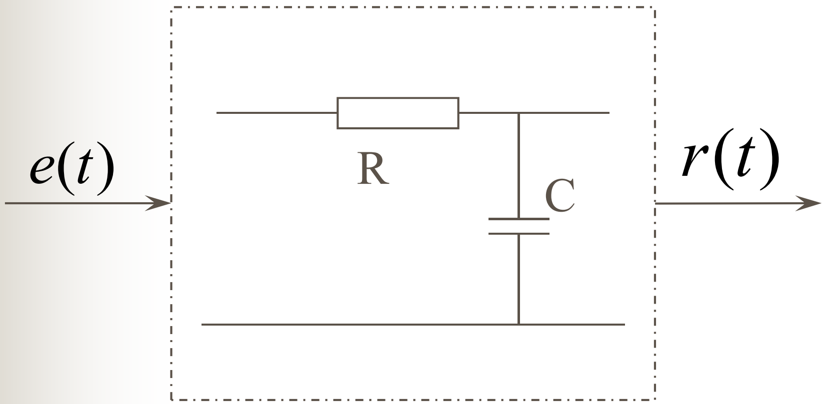
- **Feedback interconnection:**



System model: input-output description

Example: Find the input-output equation relating the input voltage *e*(*t*) to the output capacitor voltage *r*(*t*). The loop current is *i*(*t*).

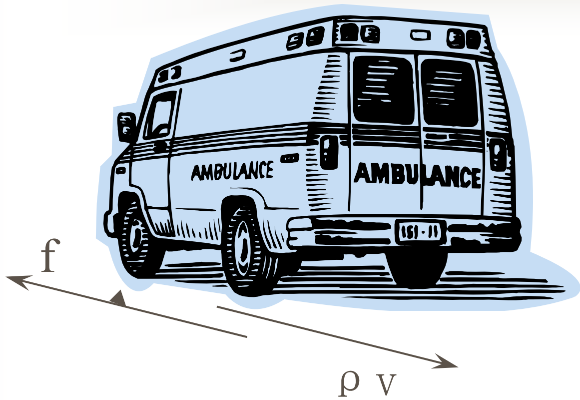
$$i(t) = \frac{e(t) - r(t)}{R} = C\frac{dr(t)}{dt}$$

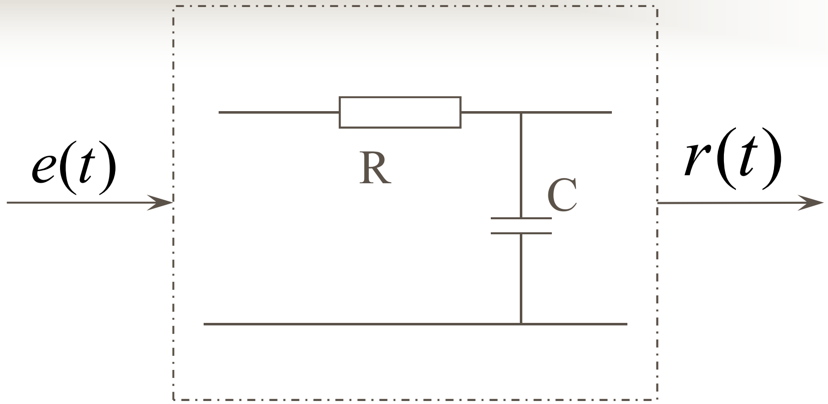


Therefore,

$$\frac{dr(t)}{dt} + \frac{1}{RC}r(t) = \frac{1}{RC}e(t)$$

This differential equation is the input-output relationship between the output *r*(*t*) and the input *e*(*t*).





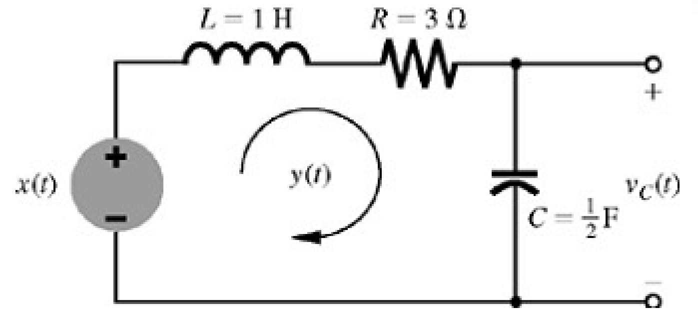
Example: **Find the input-output equation relating the input force $f(t)$ to the output speed $v(t)$. m is the mass and $\rho v(t)$ is the friction that is proportional to the speed.**

From Newton's law of motion,

$$\frac{dv(t)}{dt} = \frac{1}{m}[f(t) - \rho v(t)]$$

Therefore,

$$\frac{dv(t)}{dt} + \frac{\rho}{m} v(t) = \frac{1}{m} f(t)$$



Example: For the series RLC circuit, find the input-output equation relating the input voltage $x(t)$ to the output current (loop current) $y(t)$.

Application of Kirchhoff's voltage law around the loop yields

$$v_L(t) + v_R(t) + v_C(t) = x(t)$$

$$\frac{dy(t)}{dt} + 3y(t) + 2\int_{-\infty}^{t} y(\tau)d\tau = x(t)$$

Differentiating both sides of this equation, we obtain

$$\frac{d^2y(t)}{dt^2} + 3\frac{y(t)}{dt} + 2y(t) = \frac{dx(t)}{dt}$$

It is convenient to use a compact notation *D* for the differential operator d/d*t*. Thus

$$\frac{dy(t)}{dt} \equiv Dy(t) \qquad \frac{d^2 y(t)}{dt^2} \equiv D^2 y(t)$$

Therefore, $$(D^2 + 3D + 2)y(t) = Dx(t)$$

The differential operator is the inverse of the integral operator, so we can use the operator 1/*D* to represent integration.

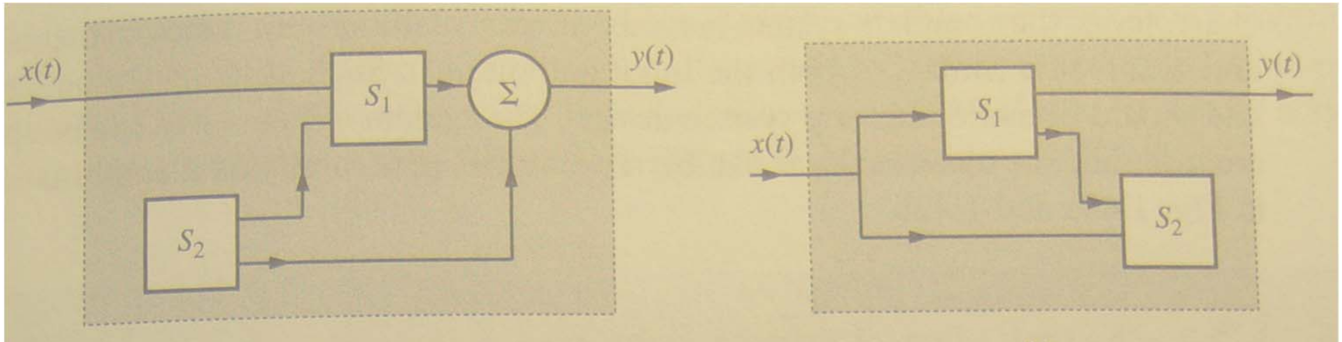
$$\int_{-\infty}^{t} y(\tau)d\tau \equiv \frac{1}{D} y(t)$$

$$(D + 3 + \frac{2}{D})y(t) = x(t)$$

Internal and external description of a system

- An internal description is a model derived from a knowledge of the internal structure of the system.
- An external description is a representation of a system as seen from its input and output terminals.

Example:



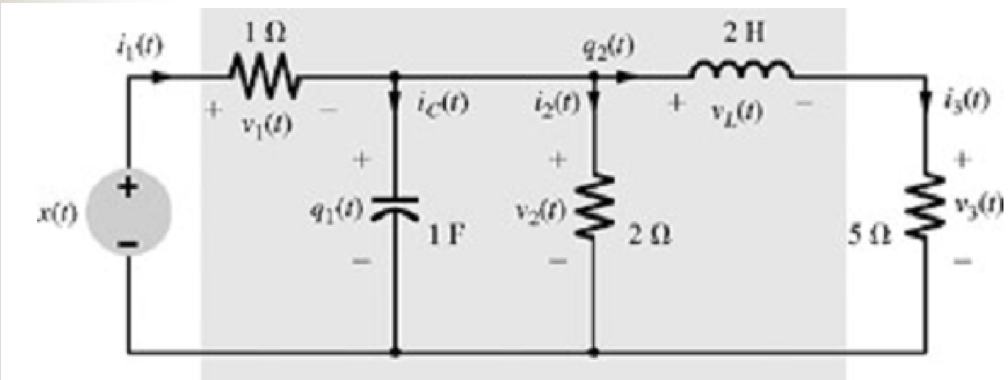
(a) (b)

In the majority of practical systems, an external description of a system is equivalent to its internal description.

Internal description: the state-space description

A system may also be described in terms of certain set of key variables called state variables. An *N*th-order system can be characterized by a set of *N* simultaneous first-order differential equations in *N* state variables.

Example:



There are two state variables: The capacitor voltage q_1 and the inductor current q_2.

$\dot{q}_1$ is the current through the capacitor.

$$\begin{aligned}\dot{q}_1 &= i_c = i_1 - i_2 - q_2 \\ &= (x - q_1) - 0.5q_1 - q_2 \\ &= -1.5q_1 - q_2 + x\end{aligned}$$

$2\dot{q}_2$ is the voltage across the inductor.

$$2\dot{q}_2 = q_1 - v_3 = q_1 - 5q_2$$

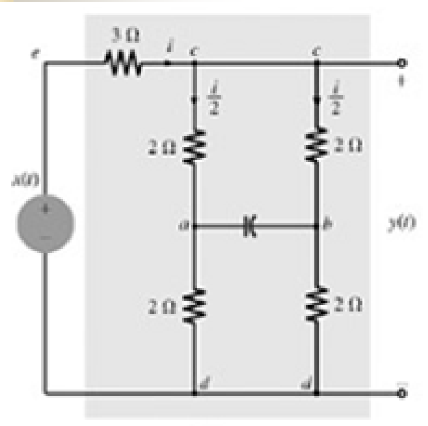
$$\dot{q}_2 = 0.5q_1 - 2.5q_2$$

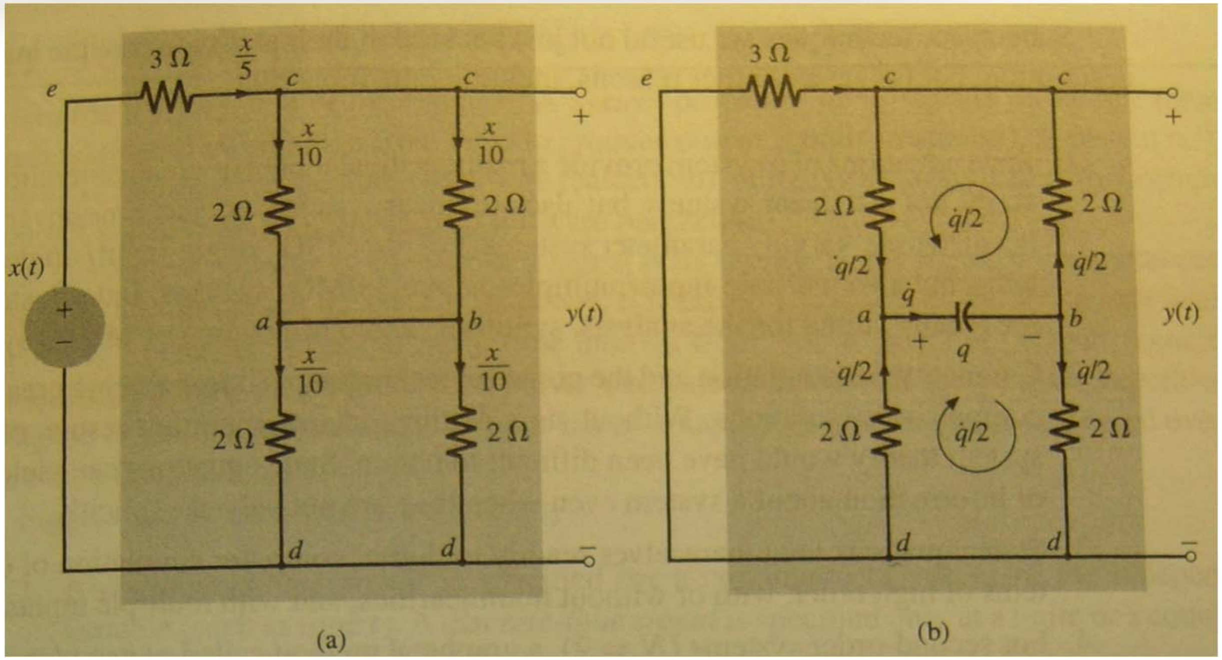
The state equations are

$$\dot{q}_1 = -1.5q_1 - q_2 + x$$

$$\dot{q}_2 = 0.5q_1 - 2.5q_2$$

Example:





(a) (b)

There is only one state variable, the capacitor voltage *q*.

The equation around the loop *acba* is $$q = 2[-\frac{x}{10} - \frac{\dot{q}}{2}] + 2[\frac{x}{10} - \frac{\dot{q}}{2}] = -2\dot{q}$$

The output is given by $$y(t) = 2[\frac{x}{10} - \frac{\dot{q}}{2}] + 2[\frac{x}{10} + \frac{\dot{q}}{2}] = \frac{2}{5}x(t)$$

The system state cannot be controlled by the input and be observed from the output. Hence, The system is uncontrollable and unobservable.