Chapter 7

Continuous-Time Signal Analysis: The Fourier Transform

- *Introduction*
- *Representation of Aperiodic Signals: CT Fourier Transform*
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- *Some Properties of the Fourier transform*
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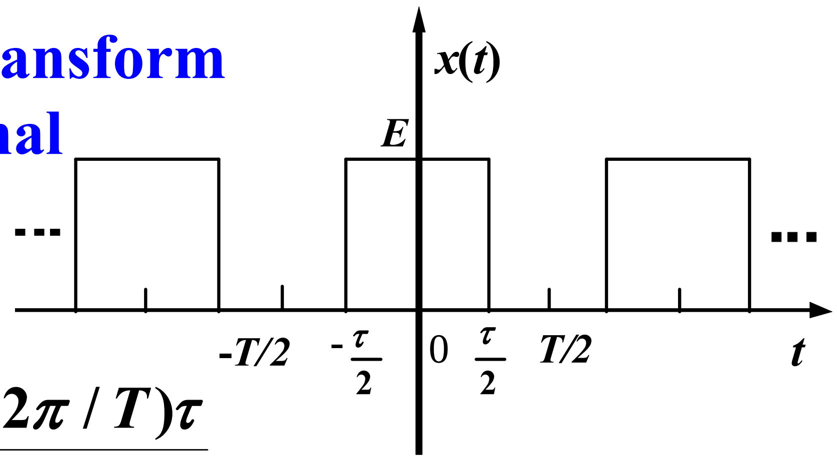
Representation of Aperiodic Signals: CT Fourier Transform

1、 Development of the Fourier transform representation of an aperiodic signal



$x(t)$ is a square wave.

$$D_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-jn(2\pi/T)t}dt = \frac{2E}{T}\frac{\sin\frac{n(2\pi/T)\tau}{2}}{n(2\pi/T)}$$

$$\because \omega_0 = 2\pi/T$$

$$\therefore D_n = \frac{E\tau}{T}\frac{\sin\frac{n\omega_0\tau}{2}}{\frac{n\omega_0\tau}{2}} = \frac{E\tau}{T}Sa\left(\frac{n\omega_0\tau}{2}\right) \qquad n = 0, \pm 1, \pm 2, \ldots$$

$$D_n = \frac{E\tau}{T}\frac{\sin\frac{n\omega_0\tau}{2}}{\frac{n\omega_0\tau}{2}} = \frac{E\tau}{T}Sa\left(\frac{n\omega_0\tau}{2}\right)$$

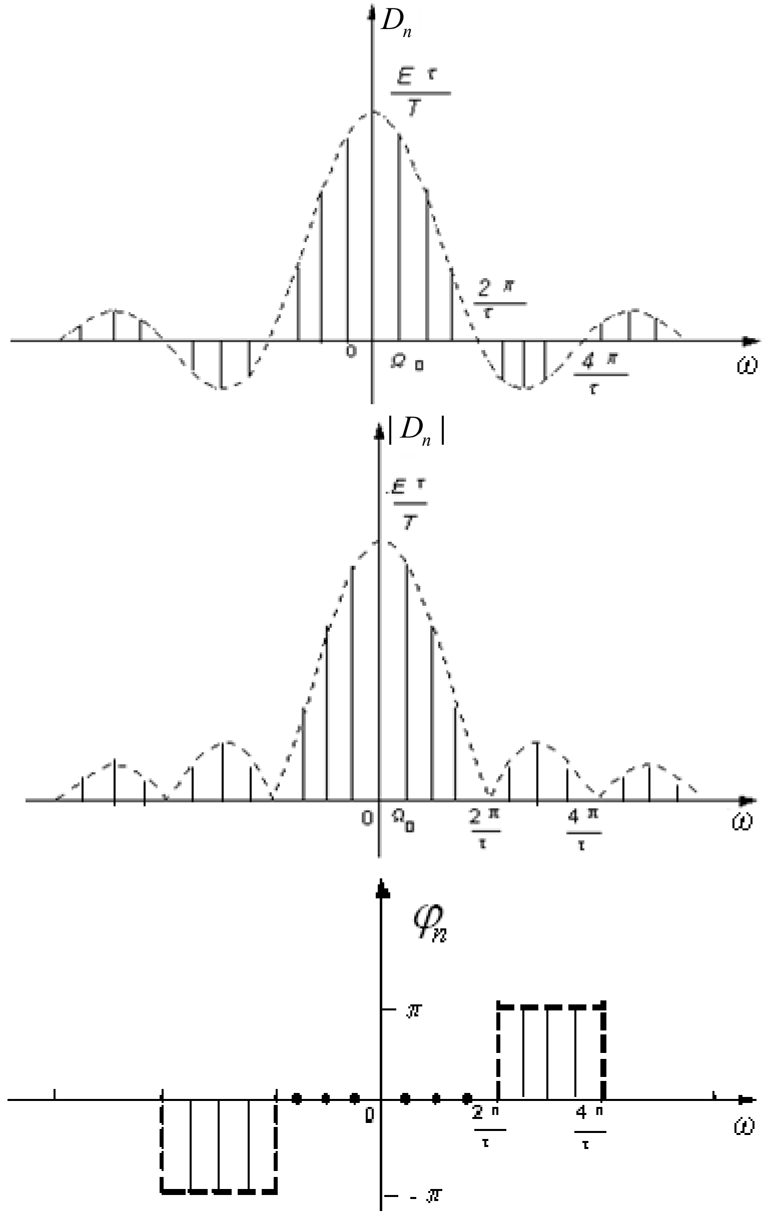
$$n = 0, \pm 1, \pm 2, \ldots$$

$$D_n = |D_n| e^{j\theta_n}$$

a) the envelope of D_n : Sa function;

b) discrete spectrum: equally spaced samples, only at $\omega = n\,\omega_0$

c) zeros at: $\pm\frac{2\pi}{\tau}, \pm\frac{4\pi}{\tau}, \ldots$

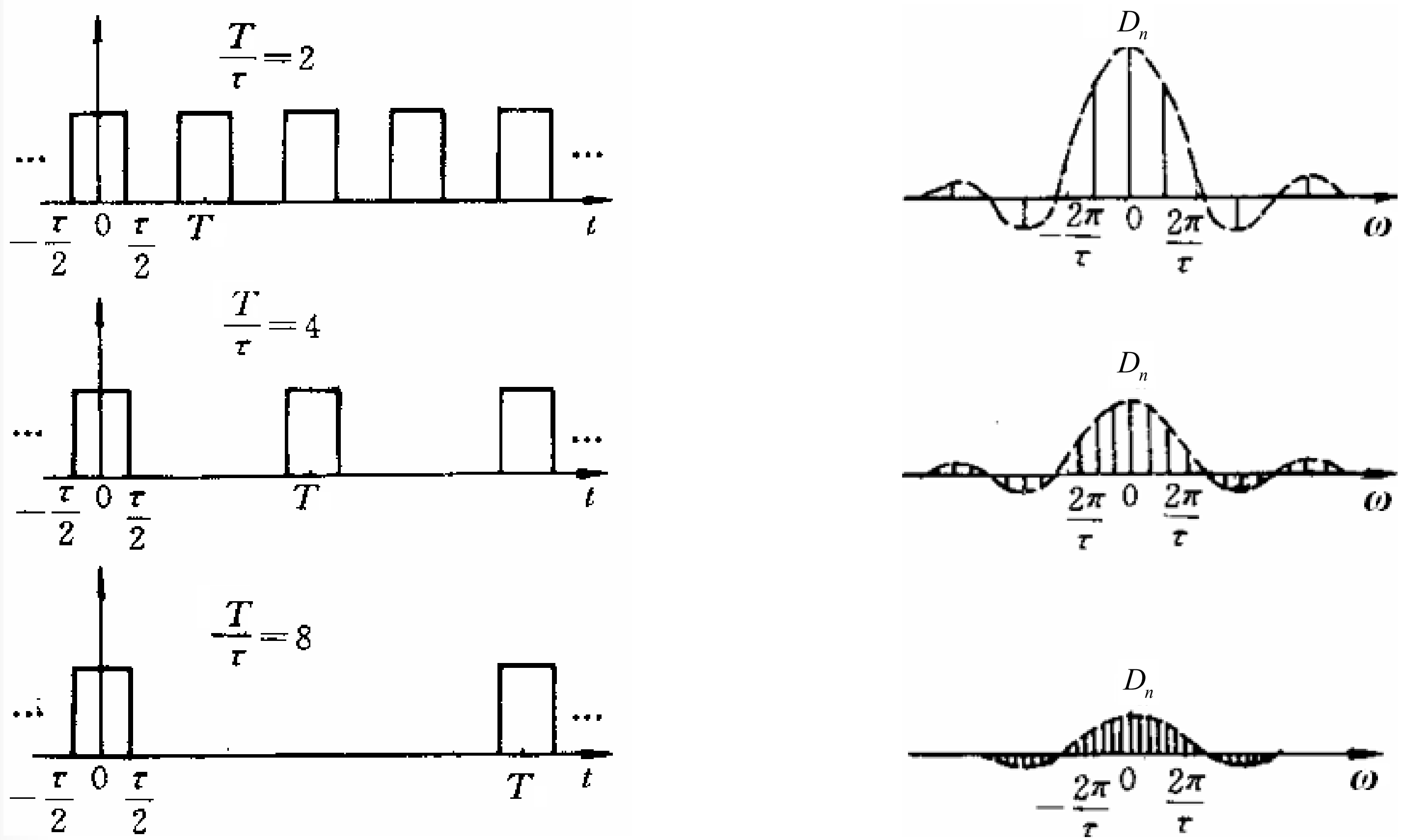


$$D_n = \frac{E\tau}{T}\frac{\sin\frac{n\omega_0\tau}{2}}{\frac{n\omega_0\tau}{2}} = \frac{E\tau}{T}Sa\left(\frac{n\omega_0\tau}{2}\right) \qquad n = 0, \pm 1, \pm 2, \ldots$$

The spectrum has some relationship with the parameters τ and T.

Questions:

1) **What will happen to the spectrum when τ decreases while T remains the same?**
2) **What will happen to the spectrum when T increases while τ remains the same?**

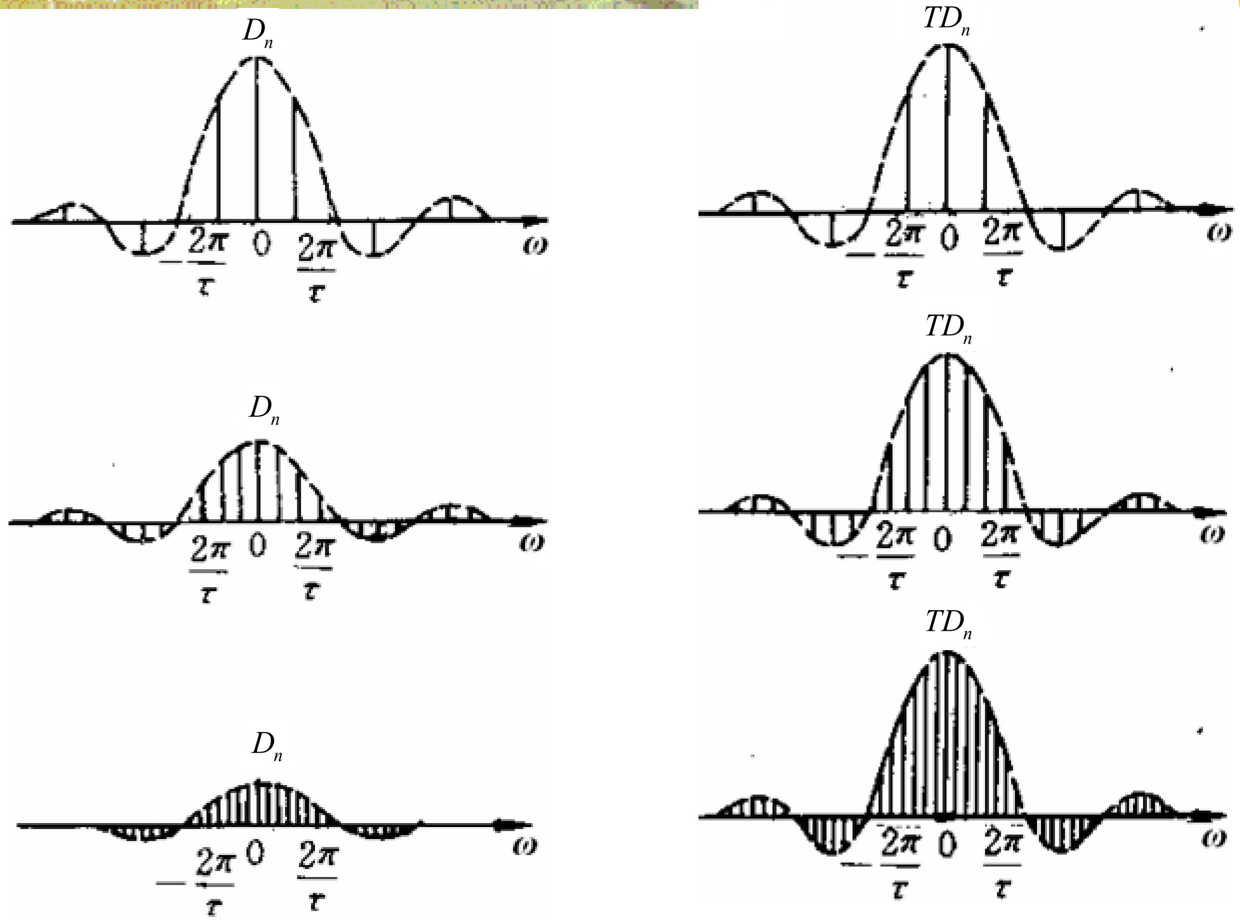


When $T\to\infty$:

$x(t)$: a periodic signal → an aperiodic signal;

spectrum: $D_n \to 0$; frequency interval: $\omega_0 \to 0$

Therefore, a discrete spectrum → a continuous spectrum, but $D_n \to 0$



TD_n does not decrease as T increases.

$$X(\omega) = \lim_{T\to\infty} TD_n = \lim_{\omega_0\to 0} E\tau Sa\left(\frac{n\omega_0\tau}{2}\right) = E\tau Sa\left(\frac{\omega\tau}{2}\right)$$

Basic idea behind Fourier's development of a representation for aperiodic signals:

An aperiodic signal can be thought as the limit of a periodic signal as the period becomes arbitrarily large. Then, we can get the Fourier transform representation for aperiodic signals by examining the limiting behavior of the Fourier series.

$$X(\omega) = \lim_{T \to \infty} TD_n = \lim_{\omega_0 \to 0} \frac{2\pi D_n}{\omega_0}$$

$X(\omega)$ is called spectrum-density function or simply as "spectrum".

$X(\omega)$ is a continuous function of ω, therefore, it is a continuous spectrum.

Fourier series pair:

$$D_n T = \int_T x(t) e^{-jn\omega_0 t} dt \qquad x(t) = \sum_{n=-\infty}^{\infty} D_n T e^{jn\omega_0 t} \frac{1}{T}$$

When $T \to \infty$:

$$\omega_0 \to d\omega \ ; \quad n\omega_0 \to \omega \ ; \qquad \frac{1}{T} = \frac{\omega_0}{2\pi} \to \frac{d\omega}{2\pi}; \quad \Sigma \to \int$$

$$\boxed{X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt}$$

Fourier transform

$$\boxed{x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega}$$

Inverse Fourier transform

The transform can also be expressed as:

$$\boxed{X(\omega) = \mathrm{F}[x(t)] \qquad x(t) = \mathrm{F}^{-1}[X(\omega)]}$$

Fourier transform pair :

$$\begin{cases} X(\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt \\ x(t) = \dfrac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)e^{j\omega t}d\omega \end{cases}$$

$$x(t) \xleftrightarrow{F} X(\omega)$$

- The aperiodic signals still can be represented as a linear combination of complex exponentials $e^{j\omega t}$. The magnitude of component with frequency ω is $\dfrac{X(\omega)d\omega}{2\pi}$.
- Although $X(\omega)$ is often abbreviated as "spectrum", it is different from D_n, which is the spectrum of periodic signals.

2、Convergence of Fourier transform

Dirichlet conditions:

(1) $x(t)$ is absolutely integrable. $\boxed{\int_{-\infty}^{\infty} |x(t)|\,dt < \infty}$

(2) $x(t)$ have a finite number of maxima and minima within any finite interval.

(3) $x(t)$ have a finite number of discontinuity within any finite interval. Furthermore, each of these discontinuities must be finite.

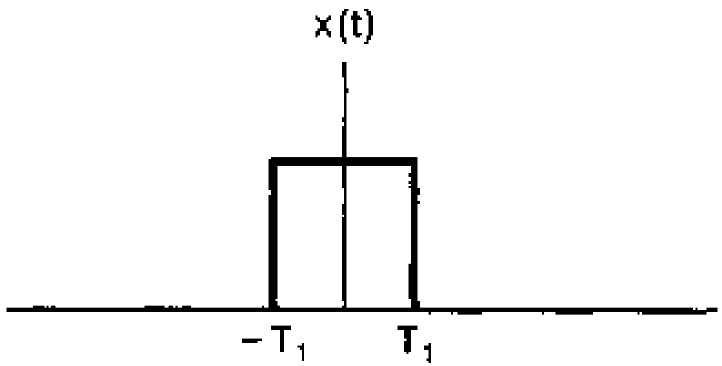
The Dirichlet conditions are sufficient for the existence and pointwise convergence of the Fourier transform, but they are not necessary.

If impulse functions are permitted in the transform, some signals which are not absolutely integrable over an infinite interval, can be considered to have Fourier transforms.

Transforms of Some Useful Functions

1）

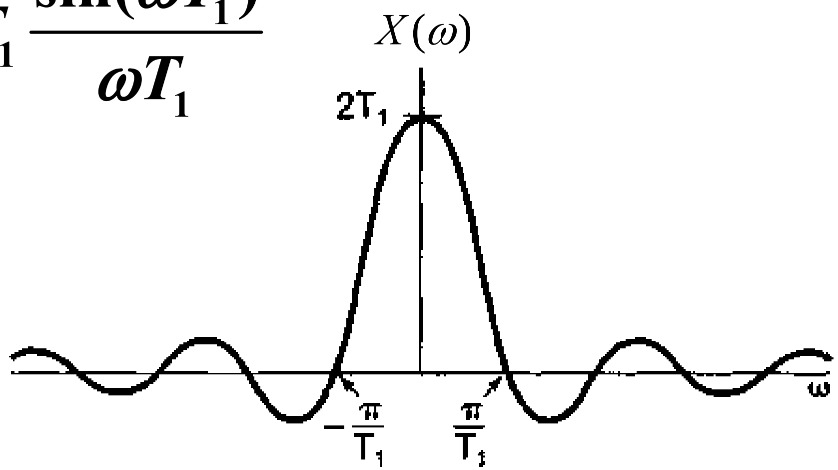
$$x(t)=\begin{cases} E & |t|<T_1 \\ 0 & |t|>T_1 \end{cases}$$



$$X(\omega)=\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt=\int_{-T_1}^{T_1} Ee^{-j\omega t}dt$$

$$=2E\frac{\sin(\omega T_1)}{\omega}=2ET_1\frac{\sin(\omega T_1)}{\omega T_1}$$

$$=2ET_1Sa(\omega T_1)$$



Consider the signal *x*(*t*) whose Fourier transform is

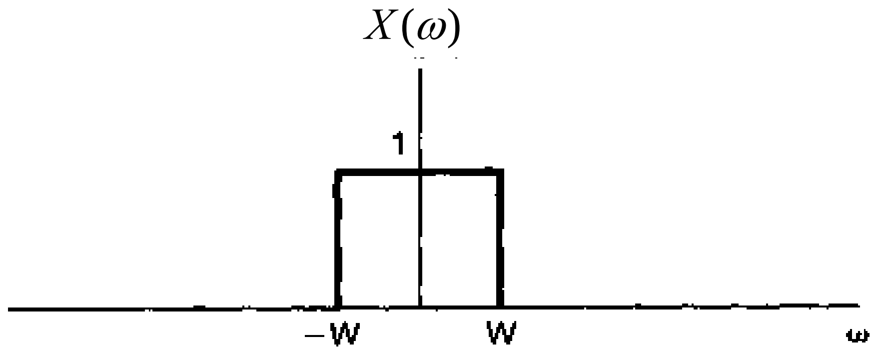
$$X(\omega)=\begin{cases}1, & |\omega|<W\\ 0, & |\omega|>W\end{cases}$$

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\omega)e^{j\omega t}d\omega$$

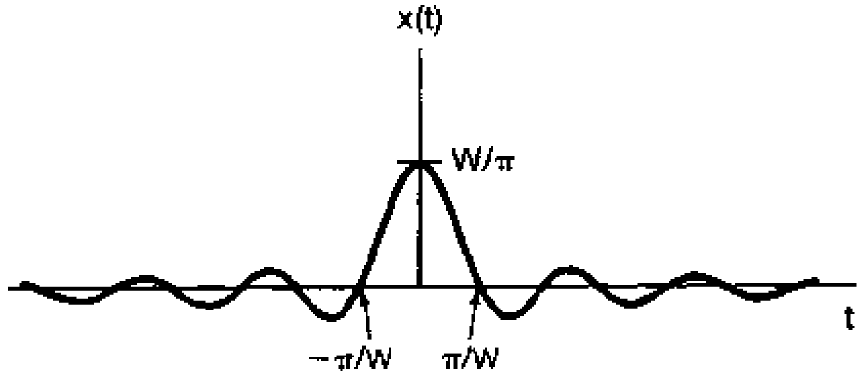
$$=\frac{1}{2\pi}\int_{-W}^{W}e^{j\omega t}d\omega$$

$$=\frac{\sin(Wt)}{\pi t}$$

$$x(t)=\frac{W}{\pi}Sa(Wt)$$



(a)



(b)

2） $x(t)=e^{-at}u(t) \qquad a>0$

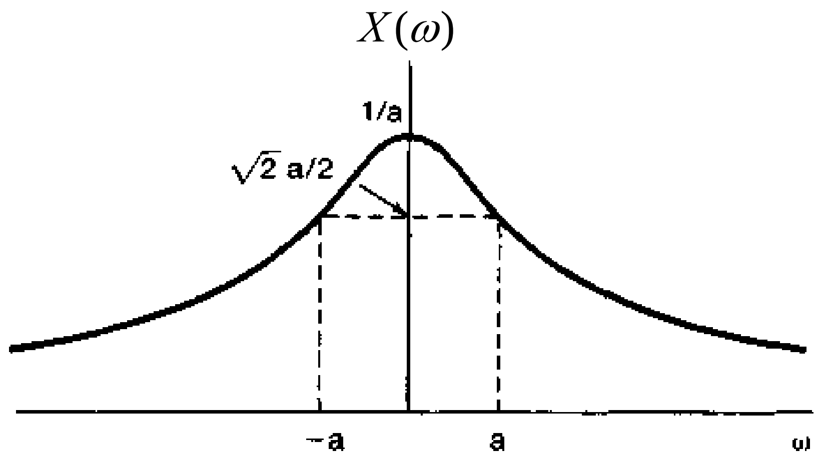
$$X(\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt=\int_{0}^{\infty}e^{-at}e^{-j\omega t}dt$$

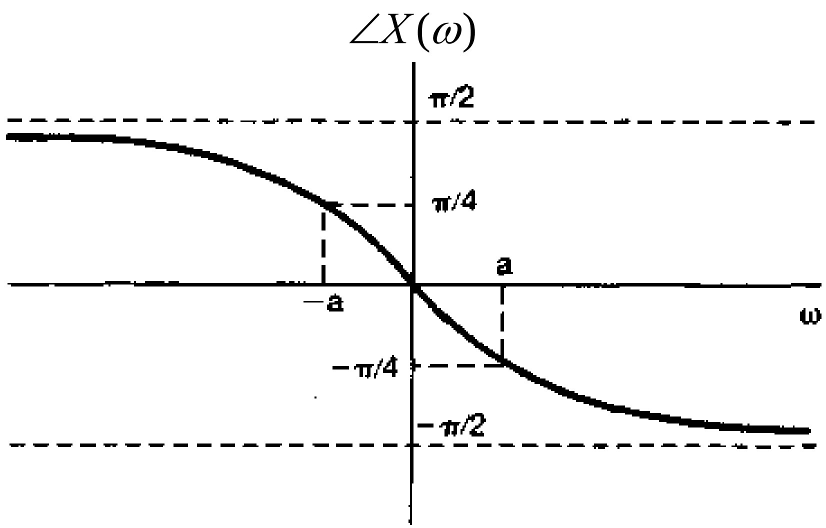
$$=\frac{-1}{a+j\omega}e^{-(a+j\omega)t}\Big|_{0}^{\infty}$$

$$=\frac{1}{a+j\omega}$$

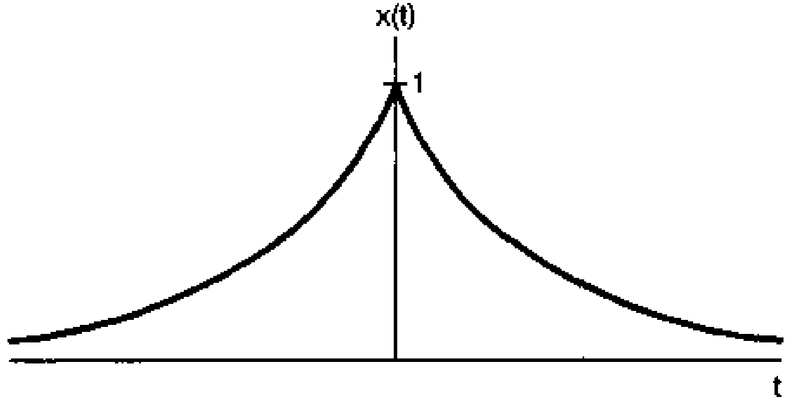
$$|X(\omega)|=\frac{1}{\sqrt{a^2+\omega^2}}$$

$$\angle X(\omega)=-\arctan(\frac{\omega}{a})$$





3） $x(t)=e^{-\alpha|t|} \qquad \alpha>0$



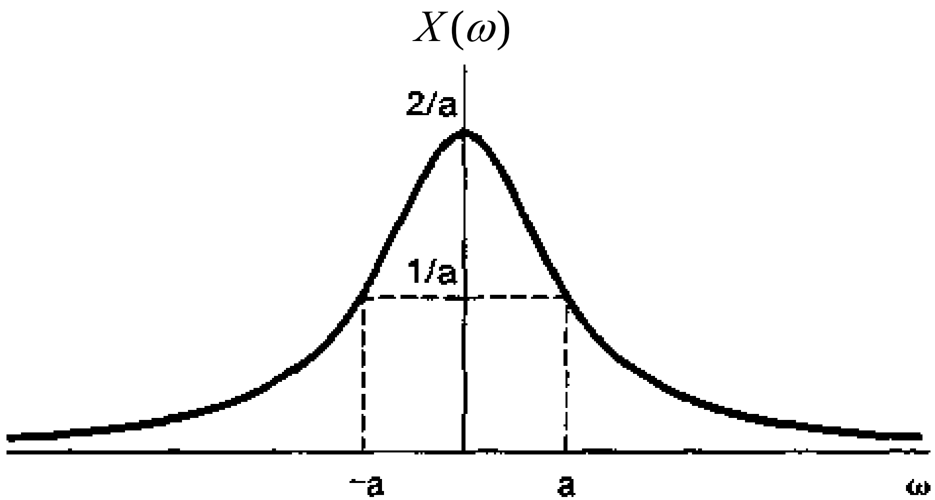
$$X(\omega)=\int_{-\infty}^{\infty} e^{-\alpha|t|}\cdot e^{-j\omega t}dt$$

$$=\frac{2\alpha}{\alpha^2+\omega^2}$$

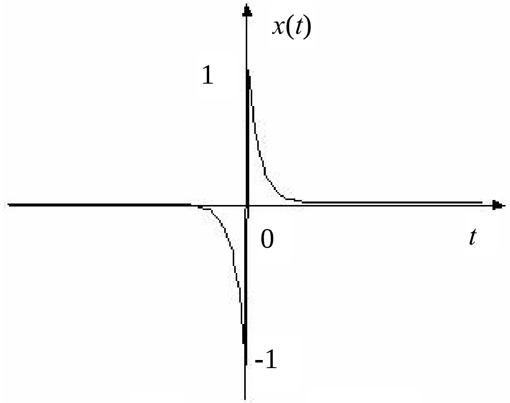
Therefore,

$$|X(\omega)|=\frac{2\alpha}{\alpha^2+\omega^2}$$

$$\angle X(\omega)=0$$

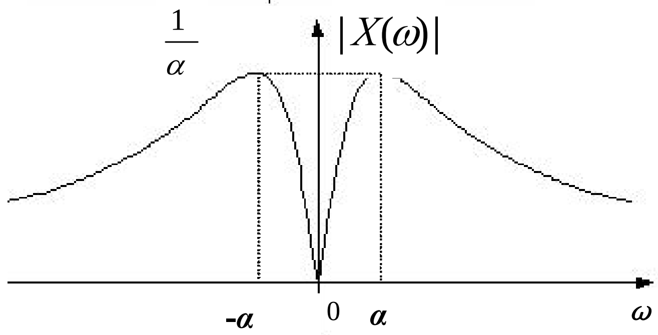


4） $$x(t)=\begin{cases}-e^{\alpha t} & t<0\\ e^{-\alpha t} & t>0\end{cases}\qquad \alpha>0$$



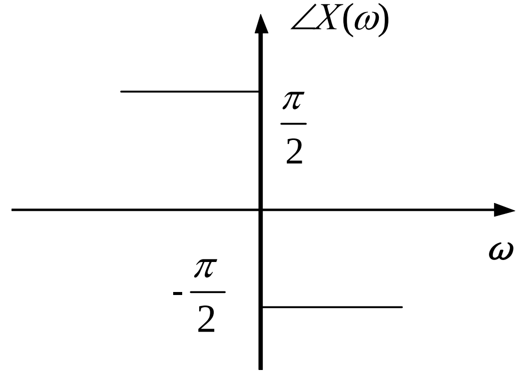
$$X(\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt$$

$$=\int_{-\infty}^{0}-e^{\alpha t}e^{-j\omega t}dt+\int_{0}^{\infty}e^{-\alpha t}e^{-j\omega t}dt$$



$$=-j\frac{2\omega}{\alpha^2+\omega^2}$$

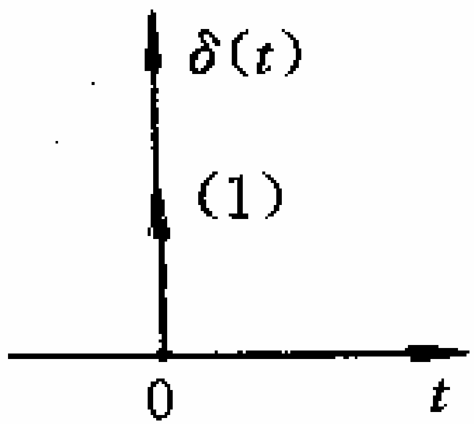
$$|X(\omega)|=\frac{2|\omega|}{\alpha^2+\omega^2}\qquad \angle X(\omega)=\begin{cases}\dfrac{\pi}{2} & \omega<0\\ -\dfrac{\pi}{2} & \omega>0\end{cases}$$

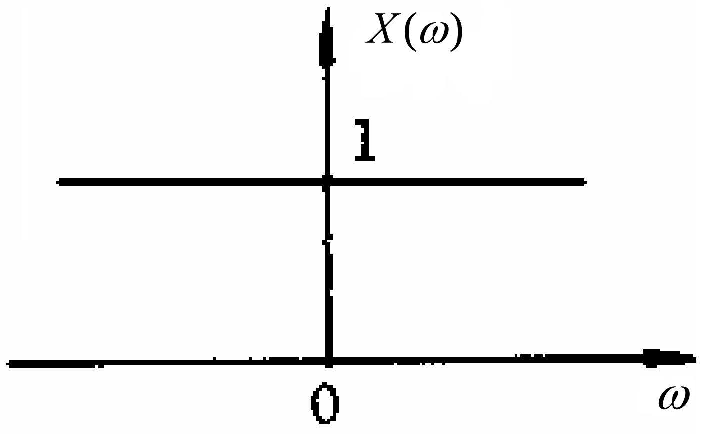


5） $x(t) = \delta(t)$

$$X(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$

The unit impulse has a Fourier transform consisting of equal contribution at all frequencies. This spectrum is referred to as white-spectrum (because the white color has the same spectrum).





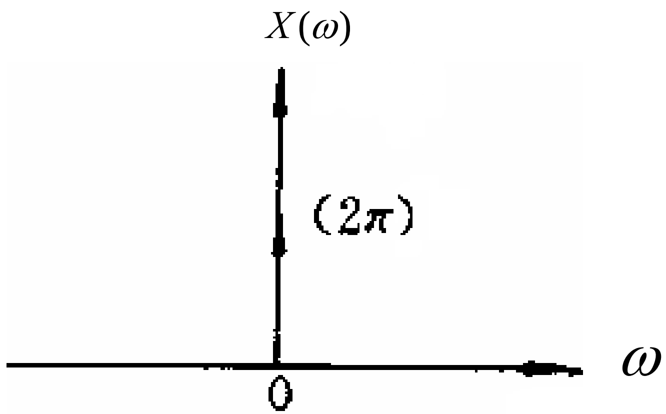
6) $$x(t)=1 \qquad -\infty < t < \infty$$

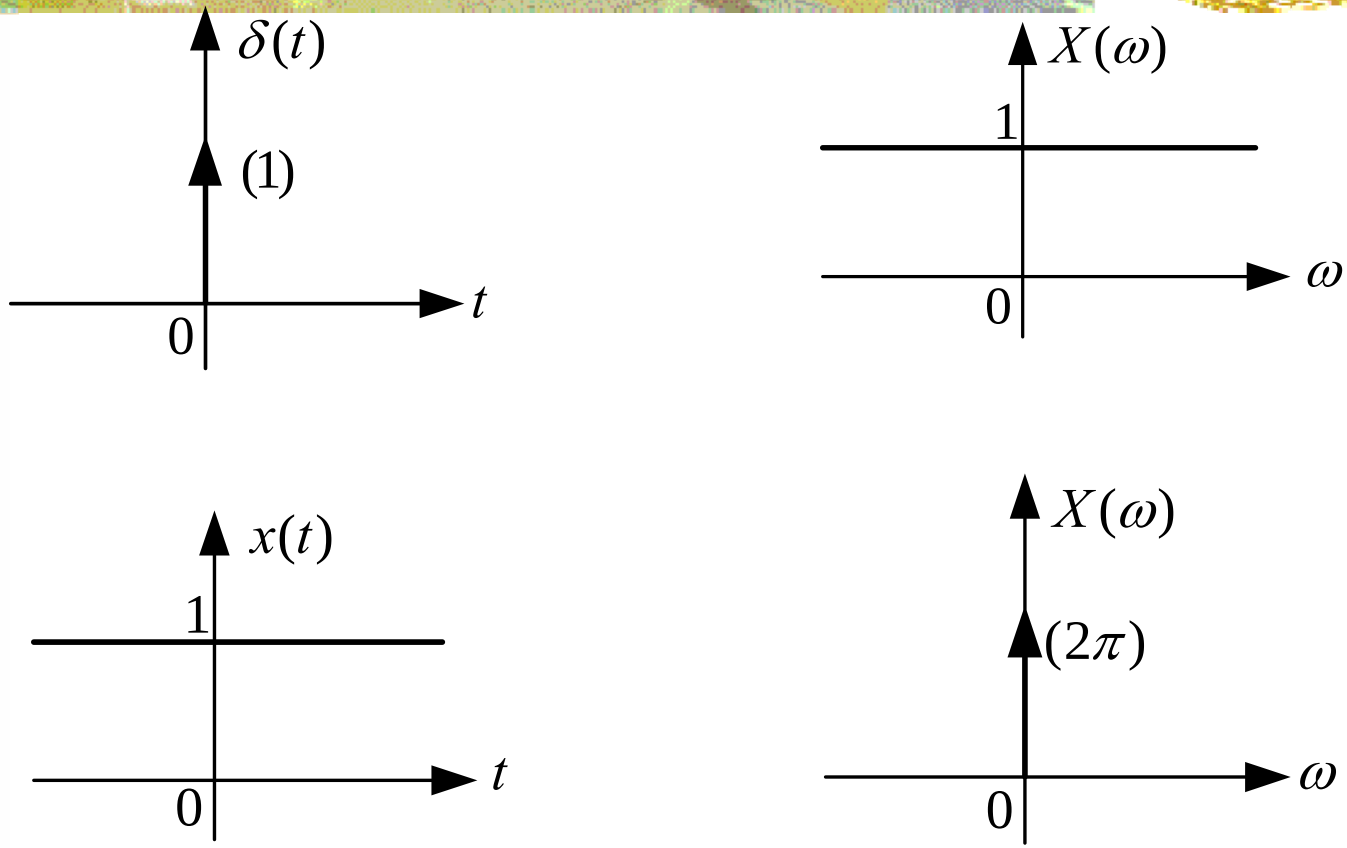
Obviously, $x(t)$ is not absolutely integrable. But it can be considered as the limit of a rectangular impulse signal as the impulse width becomes arbitrarily large.

$$X(\omega)=\lim_{\tau\to\infty}\left[\tau Sa\left(\frac{\omega\tau}{2}\right)\right]=2\pi\lim_{\tau\to\infty}\left[\frac{\tau}{2\pi}Sa\left(\frac{\omega\tau}{2}\right)\right]$$

$$\because\ \delta(t)=\lim_{k\to\infty}\left[\frac{k}{\pi}Sa(kt)\right]$$

Therefore, $X(\omega)=2\pi\delta(\omega)$





The narrower the width of a signal is in the time domain, the wider the amplitude spectrum is in the frequency domain.

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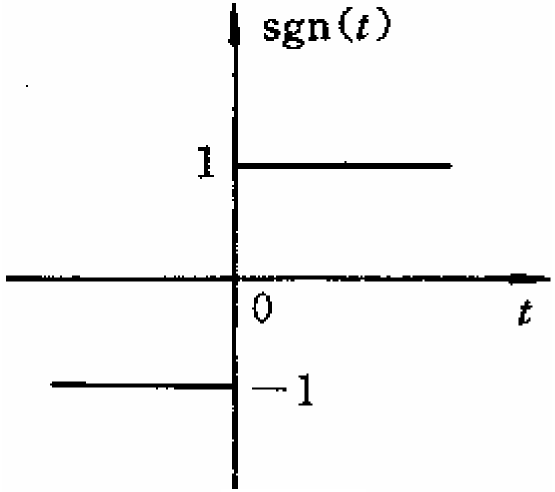
7） $$\mathrm{sgn}(t)=\begin{cases}-1 & t<0\\ 1 & t>0\end{cases}$$



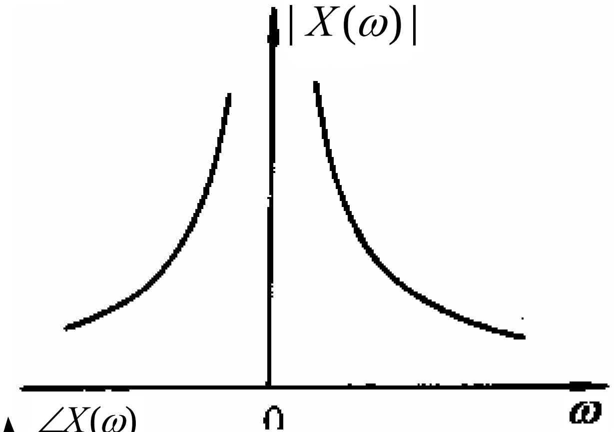
sgn(*t*) can be thought as the limit of bilateral odd exponential signal as $\alpha\to 0$.

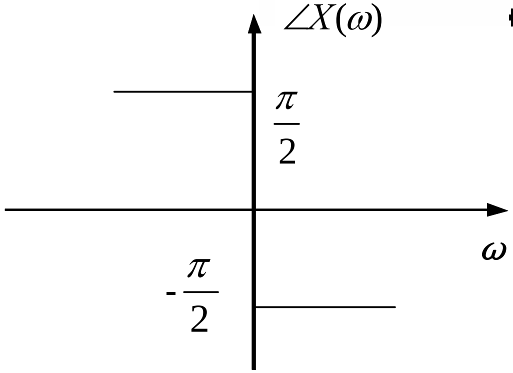
$$F[\mathrm{sgn}(t)]=\lim_{\alpha\to 0}\left(-j\frac{2\omega}{\alpha^2+\omega^2}\right)=\frac{2}{j\omega}$$

Therefore,

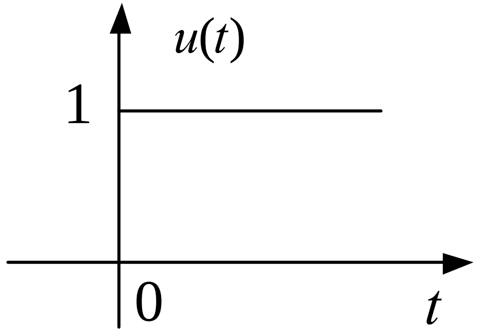
$$|X(\omega)|=\frac{2}{|\omega|}$$

$$\angle X(\omega)=\begin{cases}-\frac{\pi}{2} & \omega>0\\ \frac{\pi}{2} & \omega<0\end{cases}$$





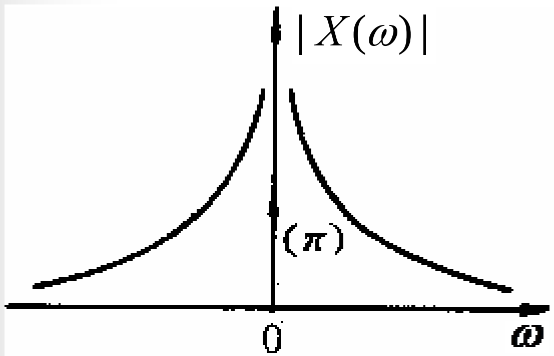
8） $u(t)=\frac{1}{2}+\frac{1}{2}\mathrm{sgn}(t)$

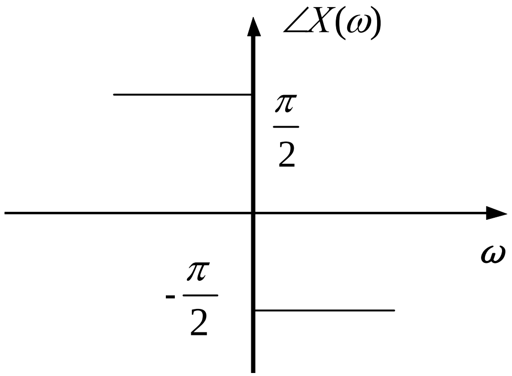


$$F[u(t)]=F[\frac{1}{2}]+F[\frac{1}{2}\mathrm{sgn}(t)]=\pi\delta(\omega)+\frac{1}{j\omega}$$

$$|X(\omega)|=\pi\delta(\omega)+\frac{1}{|\omega|}$$

$$\angle X(\omega)=\begin{cases}-\frac{\pi}{2} & \omega>0\\ \frac{\pi}{2} & \omega<0\end{cases}$$





Some Properties of CT Fourier Transform

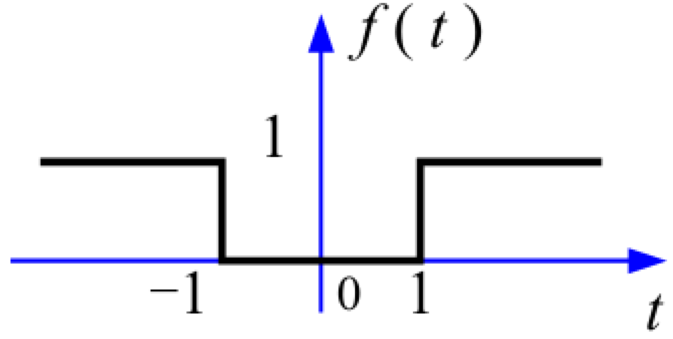
1、Linearity

If $$x(t) \xleftrightarrow{F} X(\omega),\ y(t) \xleftrightarrow{F} Y(\omega)$$

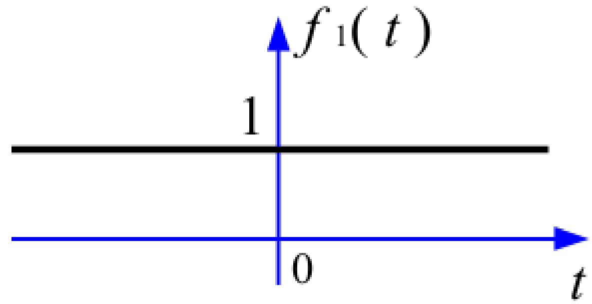
then $$ax(t) + by(t) \xleftrightarrow{F} aX(\omega) + bY(\omega)$$

Example:

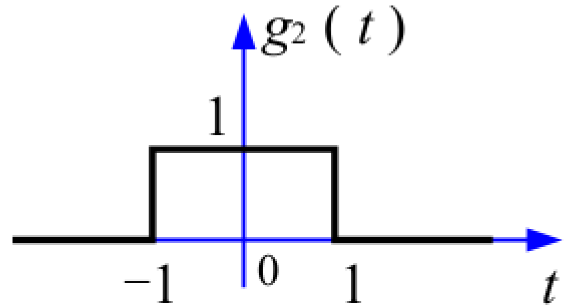
$$f(t) \xleftrightarrow{F} F(\omega) = ?$$



$$f_1(t) = 1 \xleftrightarrow{F} 2\pi\delta(\omega)$$



$$g_2(t) \xleftrightarrow{F} 2Sa(\omega)$$



$$\therefore\ f(t) \xleftrightarrow{F} 2\pi\delta(\omega) - 2Sa(\omega)$$

2、Conjugation and Conjugate Symmetry

If $x(t) \xleftrightarrow{F} X(\omega)$ then $x^*(t) \xleftrightarrow{F} X^*(-\omega)$

Proof:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$X^*(\omega) = [\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt]^*$$

$$= \int_{-\infty}^{\infty} x^*(t)e^{j\omega t}dt$$

$\omega \to -\omega$

$$X^*(-\omega) = \int_{-\infty}^{\infty} x^*(t)e^{-j\omega t}dt = F\left[x^*(t)\right]$$

If $x(t)$ is real, then $X(\omega)$ has conjugate symmetry.

$$X(-\omega) = X^*(\omega)$$

Example:

$$x(t) = e^{-at}u(t) \xleftrightarrow{F} X(\omega) = \frac{1}{a + j\omega}$$

$$\therefore \quad X(-\omega) = X^*(\omega) = \frac{1}{a - j\omega}$$

Applications of conjugate symmetry

$$X(-\omega) = X^*(\omega)$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} x(t)\cos(\omega t)dt - j\int_{-\infty}^{\infty} x(t)\sin(\omega t)dt$$

$$= \mathrm{Re}\{X(\omega)\} + j\,\mathrm{Im}\{X(\omega)\} = |X(\omega)| e^{j\angle X(\omega)}$$

$$\mathrm{Re}\{X(\omega)\} = \int_{-\infty}^{\infty} x(t)\cos(\omega t)dt \qquad \mathrm{Im}\{X(\omega)\} = -\int_{-\infty}^{\infty} x(t)\sin(\omega t)dt$$

$$|X(\omega)| = \sqrt{\mathrm{Re}^2\{X(\omega)\} + \mathrm{Im}^2\{X(\omega)\}} \qquad \angle X(\omega) = \arctan\frac{\mathrm{Im}\{X(\omega)\}}{\mathrm{Re}\{X(\omega)\}}$$

1) If $x(t)$ is real,

- $X(\omega) = \mathrm{Re}\{X(\omega)\} + j\,\mathrm{Im}\{X(\omega)\}$

$$\mathrm{Re}\{X(\omega)\} = \mathrm{Re}\{X(-\omega)\}$$ even function of ω

$$\mathrm{Im}\{X(\omega)\} = -\mathrm{Im}\{X(-\omega)\}$$ odd function of ω

- $X(\omega) = |X(\omega)| e^{j\angle X(\omega)}$

$$|X(\omega)| = |X(-\omega)|$$ even function of ω

$$\angle X(\omega) = -\angle X(-\omega)$$ odd function of ω

2) If $x(t)$ is real and even, $X(\omega)$ will also be real and even.

If $x(t)$ is real and odd, $X(\omega)$ is purely imaginary and odd.

3) If $x(t)$ is real

$$x(t) = x_e(t) + x_o(t)$$

$$\mathcal{E}v\{x(t)\} \xleftrightarrow{F} \mathcal{R}e\{X(\omega)\} \qquad \mathcal{O}d\{x(t)\} \xleftrightarrow{F} j\,\mathcal{I}m\{X(\omega)\}$$

3、Time Scaling

If $x(t) \xleftrightarrow{F} X(\omega)$

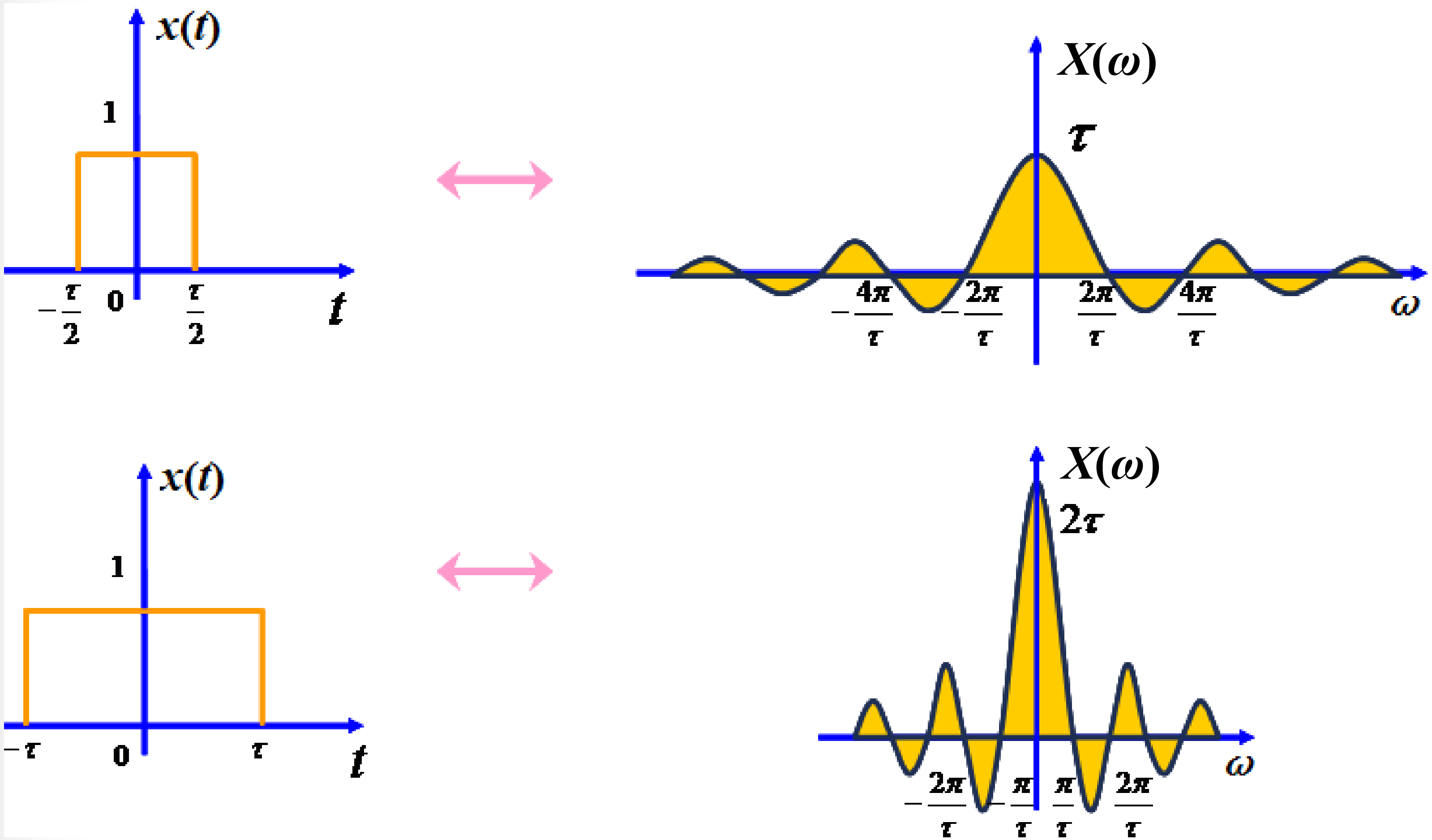
then $x(at) \xleftrightarrow{F} \dfrac{1}{|a|} X(\dfrac{\omega}{a})$, a is a nonzero real constant.

$$F[x(at)] = \int_{-\infty}^{\infty} x(at) e^{-j\omega t} dt \ \underline{\underline{\tau = at}} \int_{-\infty}^{\infty} x(\tau) e^{-j\frac{\omega\tau}{a}} \frac{d\tau}{a}$$

$$= \begin{cases} \dfrac{1}{a} \displaystyle\int_{-\infty}^{\infty} x(\tau) e^{-j\frac{\omega\tau}{a}} d\tau & a > 0 \\ -\dfrac{1}{a} \displaystyle\int_{-\infty}^{\infty} x(\tau) e^{-j\frac{\omega\tau}{a}} d\tau & a < 0 \end{cases}$$

$$= \frac{1}{|a|} X(\frac{\omega}{a})$$

Especially, $x(-t) \xleftrightarrow{F} X(-\omega)$



x(t)
1
−τ/2
0
τ/2
t
X(ω)
τ
−4π/τ
−2π/τ
2π/τ
4π/τ
ω
x(t)
1
−τ
0
τ
t
X(ω)
2τ
−2π/τ
−π/τ
π/τ
2π/τ
ω

4、Time Shifting

If $x(t) \xleftrightarrow{F} X(\omega)$

then $x(t-t_0) \xleftrightarrow{F} e^{-j\omega t_0} X(\omega)$

Proof:

$$F[x(t-t_0)] = \int_{-\infty}^{\infty} x(t-t_0) e^{-j\omega t} dt$$

assume $\tau = t - t_0$

$$F[x(\tau)] = \int_{-\infty}^{\infty} x(\tau) e^{-j\omega(\tau + t_0)} d\tau$$

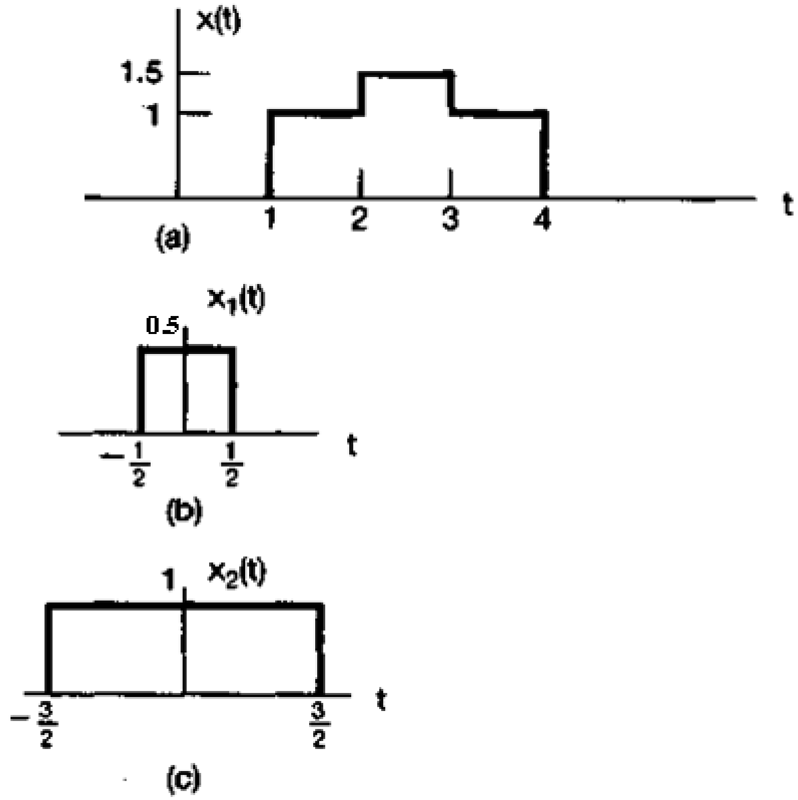
$$= e^{-j\omega t_0} \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} d\tau = e^{-j\omega t_0} X(\omega)$$

Example: $x(t) \xleftrightarrow{F} ?$

$$x(t) = x_1(t-2.5) + x_2(t-2.5)$$

$$x_1(t) \xleftrightarrow{F} 0.5Sa(0.5\omega)$$

$$x_2(t) \xleftrightarrow{F} 3Sa(1.5\omega)$$



$$X(\omega) = 0.5Sa(0.5\omega)e^{-j2.5\omega} + 3Sa(1.5\omega)e^{-j2.5\omega}$$

Example： $f(at-t_0)\xleftrightarrow{F}F[f(at-t_0)]=?$

Solution：

1） $f(at)\xleftrightarrow{F}\frac{1}{|a|}F(\frac{\omega}{a})$

$$f(at-t_0)\xleftrightarrow{F}\frac{1}{|a|}F(\frac{\omega}{a})e^{-j\omega\frac{t_0}{a}}$$

$$\|$$

$$f[a(t-\frac{t_0}{a})]$$

2) $f(t-t_0)\xleftrightarrow{F}F(\omega)e^{-j\omega t_0}$

$$f(at-t_0)\xleftrightarrow{F}\frac{1}{|a|}F(\frac{\omega}{a})e^{-j\omega\frac{t_0}{a}}$$

5、Duality

If $$x(t) \xleftrightarrow{\text{F}} X(\omega)$$

then $$X(t) \xleftrightarrow{\text{F}} 2\pi x(-\omega)$$

Proof: $$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)e^{j\omega t}d\omega$$

exchange t and ω:

$$x(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(t)e^{j\omega t}dt$$

$$2\pi\, x(-\omega) = \int_{-\infty}^{+\infty} X(t)e^{-j\omega t}dt$$

$$\therefore\ X(t) \xleftrightarrow{F} 2\pi\, x(-\omega)$$

Example:

Let us consider using duality to find the Fourier transform $G(\omega)$ of the signal

$$g(t)=\frac{2}{1+t^2}$$

$$\because \quad e^{-\alpha|t|} \xleftrightarrow{F} \frac{2\alpha}{\alpha^2+\omega^2} \qquad (\alpha>0)$$

We know

$$x(t)=e^{-|t|} \xleftrightarrow{F} X(\omega)=\frac{2}{1+\omega^2}$$

From duality property

$$g(t)=\frac{2}{1+t^2} \xleftrightarrow{F} G(\omega)=2\pi e^{-|-\omega|}=2\pi e^{-|\omega|}$$

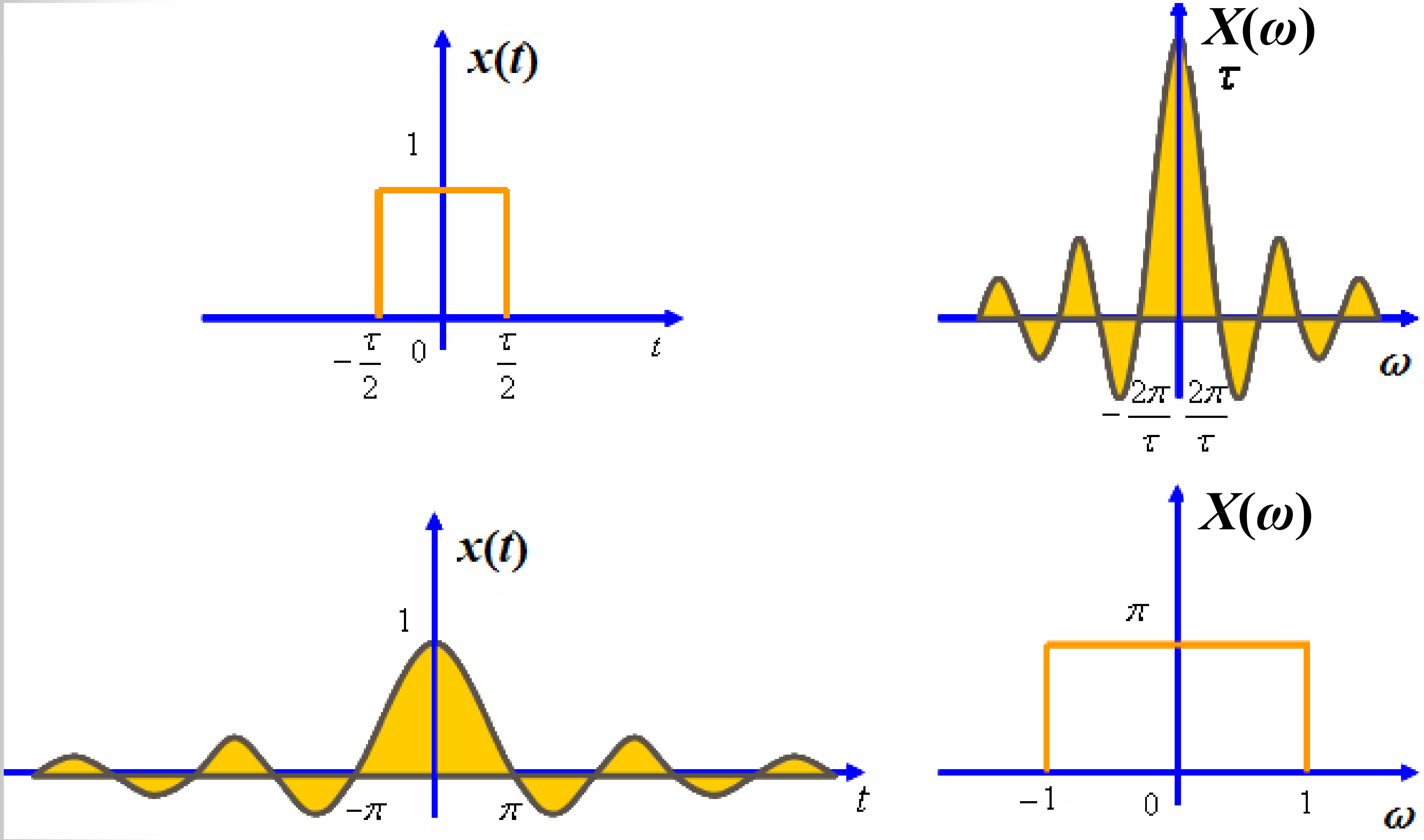
Example： **Determine the inverse Fourier transform of $u(\omega)$.**

Solution： $$u(t) \xleftrightarrow{F} \pi\delta(\omega) + \frac{1}{j\omega}$$

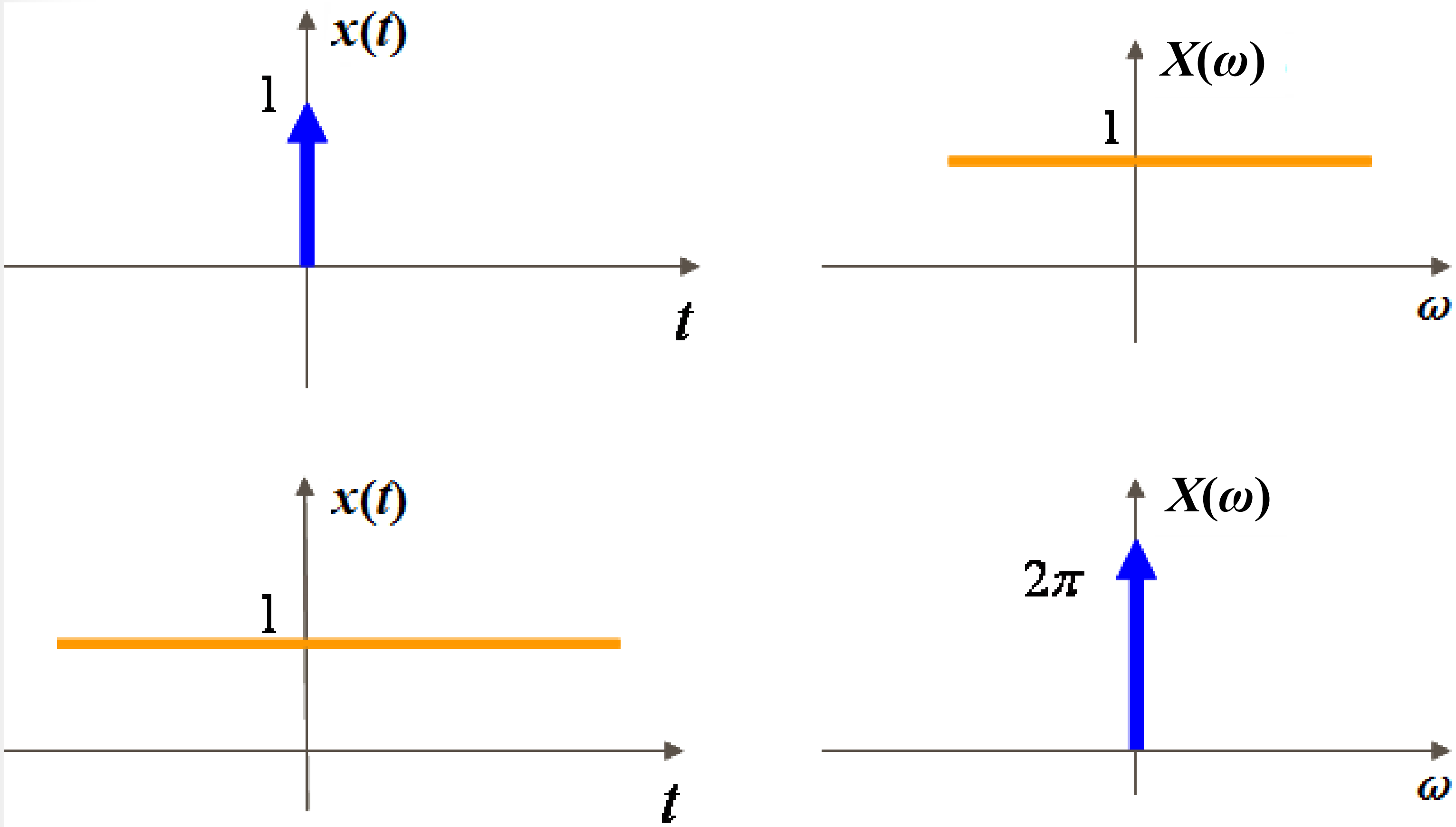
$$\pi\delta(t) + \frac{1}{jt} \xleftrightarrow{F} 2\pi u(-\omega)$$

$$\pi\delta(-t) + \frac{1}{-jt} \xleftrightarrow{F} 2\pi u(\omega)$$

$$\therefore \quad \frac{\delta(t)}{2} + \frac{j}{2\pi t} \xleftrightarrow{F} u(\omega)$$


x(t)
1
−τ/2
0
τ/2
t
X(ω)
τ
ω
−2π/τ
2π/τ
x(t)
1
−π
π
t
X(ω)
π
−1
0
1
ω

$$\delta(t) \xleftrightarrow{F} 1 \quad \text{and} \quad 1 \xleftrightarrow{F} 2\pi\delta(\omega)$$



6、Frequency Shifting

If $x(t) \xleftrightarrow{\mathrm{F}} X(\omega)$

then $x(t)e^{j\omega_0 t} \xleftrightarrow{\mathrm{F}} X(\omega-\omega_0)$

Proof:

$$\mathrm{F}[x(t)e^{j\omega_0 t}] = \int_{-\infty}^{\infty} x(t)e^{j\omega_0 t}e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} x(t)e^{-j(\omega-\omega_0)t}d\tau = X(\omega-\omega_0)$$

Example:

$$x(t) = e^{-j2t} \xleftrightarrow{F} X(\omega) = ?$$

Solution:

$$1 \xleftrightarrow{F} 2\pi\delta(\omega)$$

$$1 \times e^{-j2t} \xleftrightarrow{F} 2\pi\delta(\omega + 2)$$

Example:

$$F\{\cos(\omega_0 t)\} = ?$$

Solution:

$$\cos\omega_0 t = \frac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})$$

$$\cos(\omega_0 t) \xleftrightarrow{F} \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

Example: If $f(t) \xleftrightarrow{F} F(\omega)$, determine $F\{f(t)\cos(\omega_c t)\} = ?$

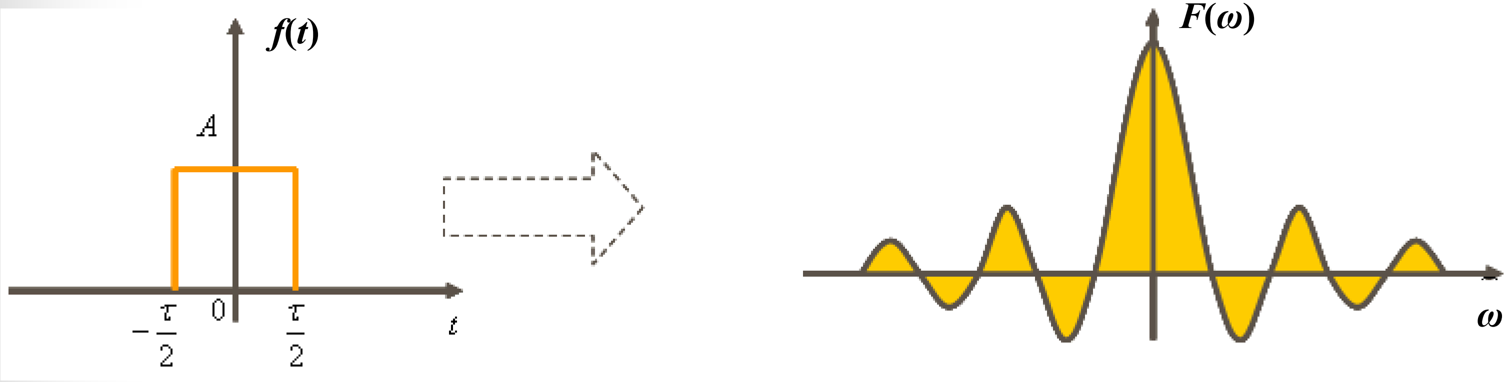
Solution: $\cos(\omega_c t) = \frac{1}{2}(e^{j\omega_c t} + e^{-j\omega_c t})$

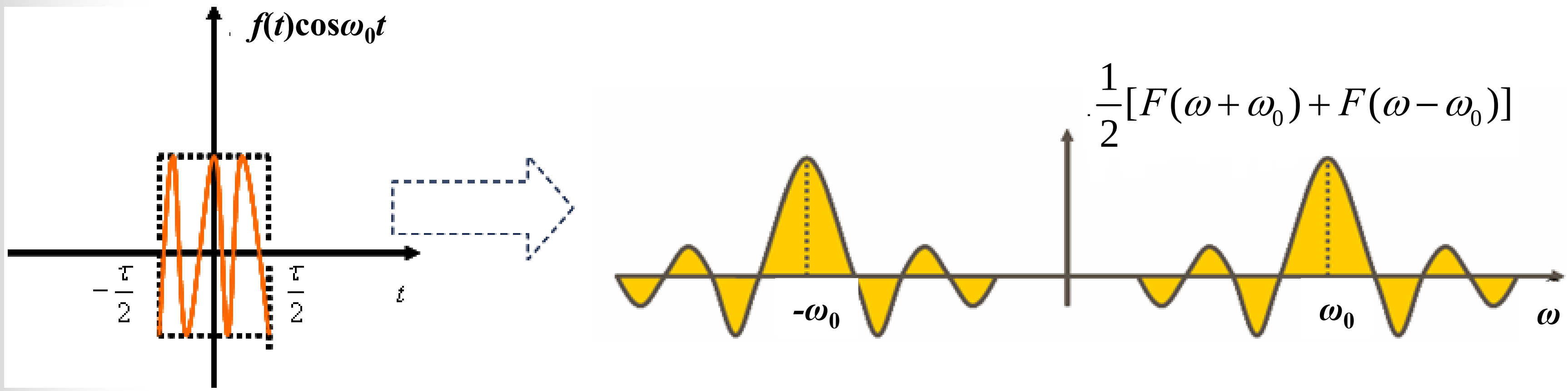
$$f(t)\cos(\omega_c t) \xleftrightarrow{F} \frac{1}{2}\left[F(\omega+\omega_c) + F(\omega-\omega_c)\right]$$

Example: $f(t)\sin(\omega_c t) \xleftrightarrow{F} ?$

$$f(t)\sin(\omega_c t) \xleftrightarrow{F} \frac{j}{2}\left[F(\omega+\omega_c) - F(\omega-\omega_c)\right]$$

Spectral shifting: This procedure shifts the signal spectrum to its allocated band.

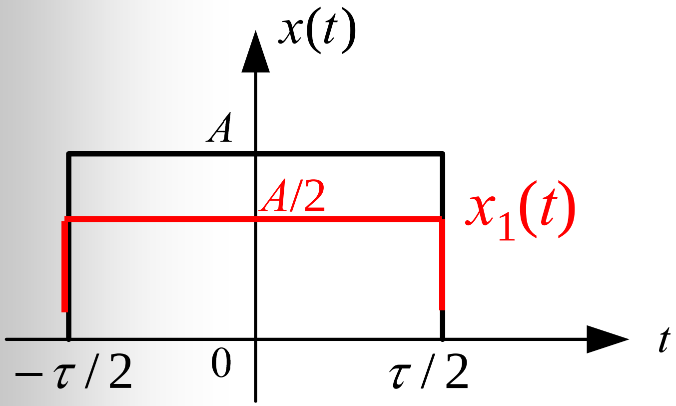


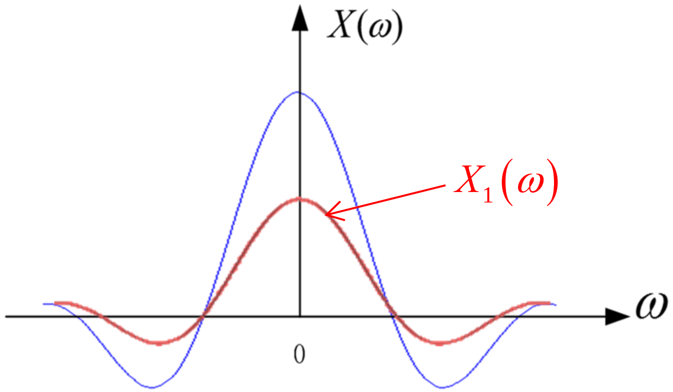


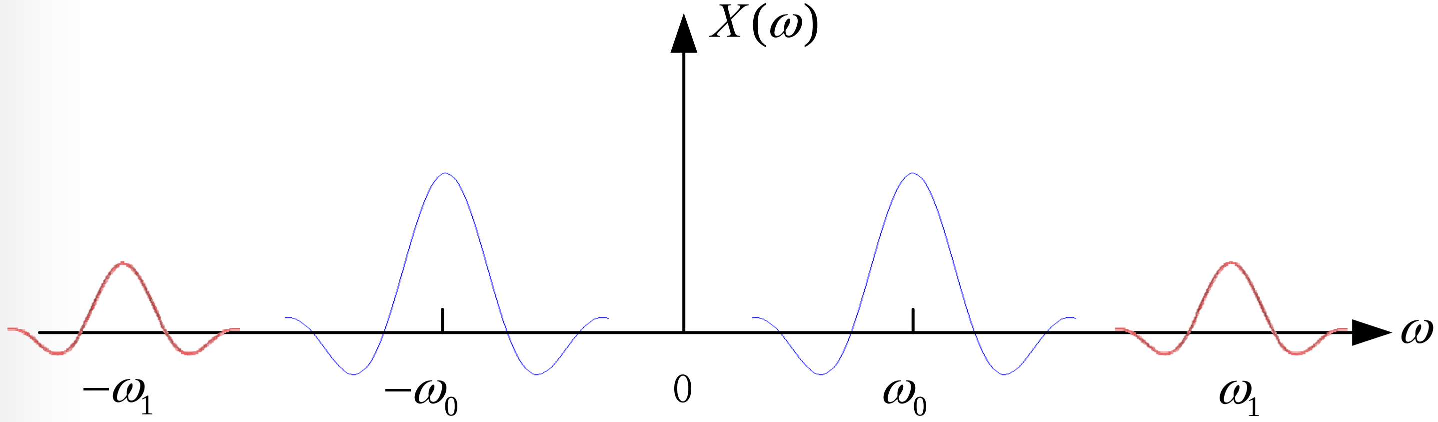
APPLICATION OF MODULATION

If several signals, all occupying the same frequency band, are transmitted simultaneously over the same transmission medium, they will all interfere. It will be impossible to separate or retrieve them at a receiver.

Problem: How are they transmitted without interference?







7、Time Differentiation and Integration

If $x(t) \xleftrightarrow{F} X(\omega)$ then

$$\frac{dx(t)}{dt} \xleftrightarrow{F} j\omega X(\omega)$$

$$\frac{d^n x(t)}{dt^n} \xleftrightarrow{F} (j\omega)^n X(\omega)$$

If $x(t) \xleftrightarrow{F} X(\omega)$ then

$$\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{F} \frac{1}{j\omega} X(\omega) + \pi X(0)\delta(\omega)$$

Example :

Determine the Fourier transform of the unit step $x(t) = u(t)$.

$$g(t) = \delta(t) \xleftrightarrow{F} G(\omega) = 1$$

$$\because x(t) = \int_{-\infty}^{t} g(\tau) d\tau$$

$$\therefore x(t) \xleftrightarrow{F} X(\omega) = \frac{G(\omega)}{j\omega} + \pi G(0)\delta(\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$$

$$u(t) \xleftrightarrow{F} \frac{1}{j\omega} + \pi\delta(\omega)$$

$$\delta(t) = \frac{du(t)}{dt} \xleftrightarrow{F} j\omega[\frac{1}{j\omega} + \pi\delta(\omega)] = 1 + j\omega\pi\delta(\omega) = 1$$

Summary:

If $\dfrac{df^n(t)}{dt^n} \xleftrightarrow{F} F_n(\omega)$ and $f(-\infty)+f(\infty)=0$

Then $f(t) \xleftrightarrow{F} F(\omega) = F_n(\omega)/(j\omega)^n$

Example: $\mathrm{sgn}(t) \xleftrightarrow{F} \mathrm{F}[\mathrm{sgn}(t)] = ?$

Solution : $\dfrac{d\,\mathrm{sgn}(t)}{dt} = 2\delta(t)$

$$\frac{d\,\mathrm{sgn}(t)}{dt} \xleftrightarrow{F} j\omega \mathrm{F}[\mathrm{sgn}(t)] \qquad 2\delta(t) \xleftrightarrow{F} 2$$

$$\therefore \mathrm{F}[\mathrm{sgn}(t)] = \frac{2}{j\omega}$$

8、Parseval's Relation

$$\int_{-\infty}^{+\infty} \left|x(t)\right|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left|X(\omega)\right|^2 d\omega$$

Proof : $\int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} x(t)x^*(t)dt$

$$= \int_{-\infty}^{+\infty} x(t)[\frac{1}{2\pi}\int_{-\infty}^{+\infty} X^*(\omega)e^{-j\omega t}d\omega]dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X^*(\omega)[\int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt]d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X^*(\omega)X(\omega)d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(\omega)|^2 d\omega$$

Energy-density spectrum $\varepsilon(\omega) = \left|X(\omega)\right|^2$

9、Time and Frequency Convolution Property

If $x_1(t) \xleftrightarrow{F} X_1(\omega)$ $\quad x_2(t) \xleftrightarrow{F} X_2(\omega)$

Then $\boxed{x_1(t) * x_2(t) \xleftrightarrow{F} X_1(\omega)\cdot X_2(\omega)}$

Proof:

$$\mathcal{F}[x_1(t) * x_2(t)] = \int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} x_1(\tau)x_2(t-\tau)d\tau\right]e^{-j\omega t}dt$$

$$= \int_{-\infty}^{+\infty} x_1(\tau)\underbrace{\left[\int_{-\infty}^{+\infty} x_2(t-\tau)e^{-j\omega t}dt\right]}_{e^{-j\omega\tau}X_2(\omega)}d\tau$$

$$= \int_{-\infty}^{+\infty} x_1(\tau)e^{-j\omega\tau}X_2(\omega)d\tau$$

$$= X_2(\omega)\underbrace{\int_{-\infty}^{+\infty} x_1(\tau)e^{-j\omega\tau}d\tau}_{X_1(\omega)} = X_1(\omega)X_2(\omega)$$

If $x_1(t) \xleftrightarrow{F} X_1(\omega)$ $\quad x_2(t) \xleftrightarrow{F} X_2(\omega)$

Then $x_1(t) \cdot x_2(t) \xleftrightarrow{F} \dfrac{1}{2\pi}[X_1(\omega) * X_2(\omega)]$

Proof: $F[x_1(t) \cdot x_2(t)] = \int_{-\infty}^{\infty} [\boxed{x_1(t)} \cdot x_2(t)] e^{-j\omega t} dt$

$$= \int_{-\infty}^{\infty} x_2(t) e^{-j\omega t} \left[\boxed{\frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\omega_0) e^{j\omega_0 t} d\omega_0} \right] dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\omega_0) \cdot \left[\int_{-\infty}^{\infty} x_2(t) e^{-j(\omega - \omega_0)t} dt \right] d\omega_0$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\omega_0) X_2(\omega - \omega_0) d\omega_0 = \frac{1}{2\pi}[X_1(\omega) * X_2(\omega)]$$

Example: $$\left(\frac{\sin t}{t}\right)^2 \xleftrightarrow{F} ?$$

Solution : $$g_2(t) \xleftrightarrow{F} 2Sa(\omega)$$

$$2Sa(t) \xleftrightarrow{F} 2\pi g_2(-\omega)$$

$$Sa(t) \xleftrightarrow{F} \pi g_2(\omega)$$

$$Sa(t)\cdot Sa(t) \xleftrightarrow{F} \frac{1}{2\pi}\pi g_2(\omega) * [\pi g_2(\omega)]$$

$$\left(\frac{\sin t}{t}\right)^2 \xleftrightarrow{F} \frac{\pi}{2}(2-|\omega|)[u(\omega+2)-u(\omega-2)]$$

The Fourier Transform for Periodic Signals

For an arbitrary periodic signal $x(t)$, representing $x(t)$ with the Fourier series as

$$x(t)=\sum_{k=-\infty}^{+\infty} D_k e^{jk\omega_0 t}$$

$$e^{j\omega_0 t} \xleftrightarrow{\text{F}} 2\pi\delta(\omega-\omega_0)$$

$$X(\omega)=F[\sum_{k=-\infty}^{+\infty} D_k e^{jk\omega_0 t}]=\sum_{k=-\infty}^{+\infty} D_k F[e^{jk\omega_0 t}]$$

$$=\sum_{k=-\infty}^{+\infty} D_k 2\pi\delta(\omega-k\omega_0)=2\pi\sum_{k=-\infty}^{+\infty} D_k\delta(\omega-k\omega_0)$$

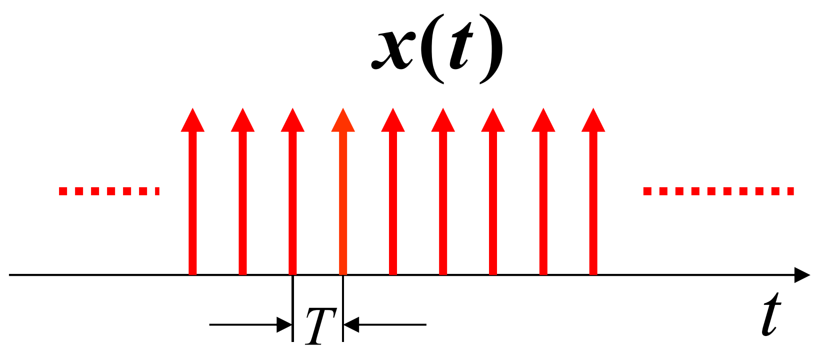
$$\boxed{x(t)=\sum_{k=-\infty}^{+\infty} D_k e^{jk\omega_0 t} \overset{F}{\leftrightarrow} X(\omega)=2\pi\sum_{k=-\infty}^{+\infty} D_k\delta(\omega-k\omega_0)}$$

- The Fourier transform of a periodic signal with Fourier series coefficients $\{D_k\}$ can be interpreted as a train of impulses occurring at the harmonically related frequencies;

- The area of the impulse at the *kth* harmonic frequency $k\omega_0$ is 2π times the *kth* Fourier series coefficient D_k.

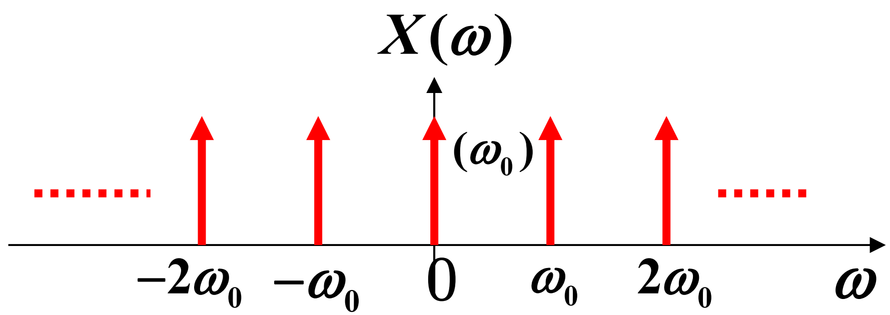
Example:

$$x(t)=\sum_{k=-\infty}^{+\infty}\delta(t-kT)$$

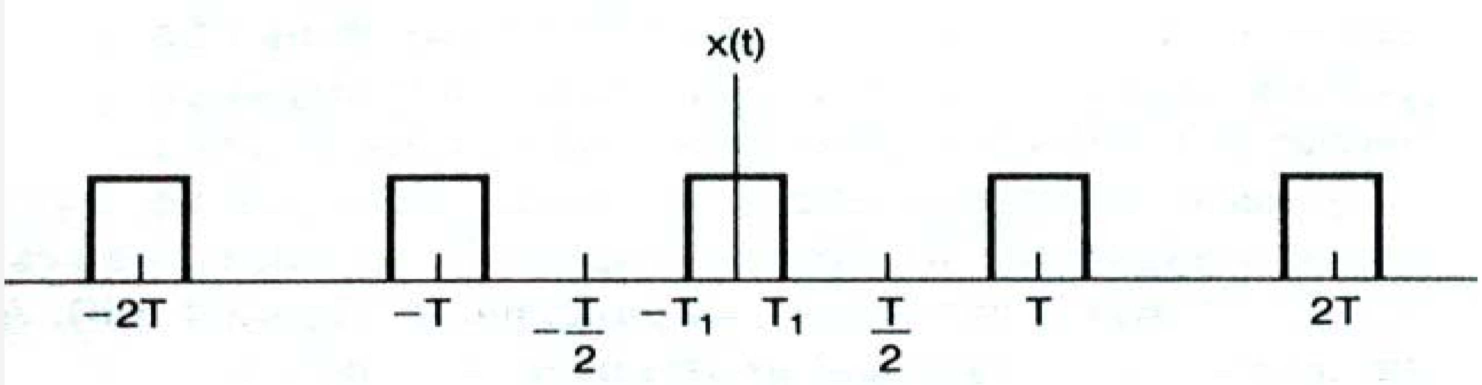


$$D_k=\frac{1}{T}\int_{-T/2}^{T/2}\delta(t)e^{-jk\omega_0 t}dt=\frac{1}{T}$$

$$X(\omega)=\frac{2\pi}{T}\sum_{k=-\infty}^{+\infty}\delta(\omega-\frac{2k\pi}{T})=\omega_0\sum_{k=-\infty}^{+\infty}\delta(\omega-k\omega_0)$$



Example:



Its Fourier series coefficients are

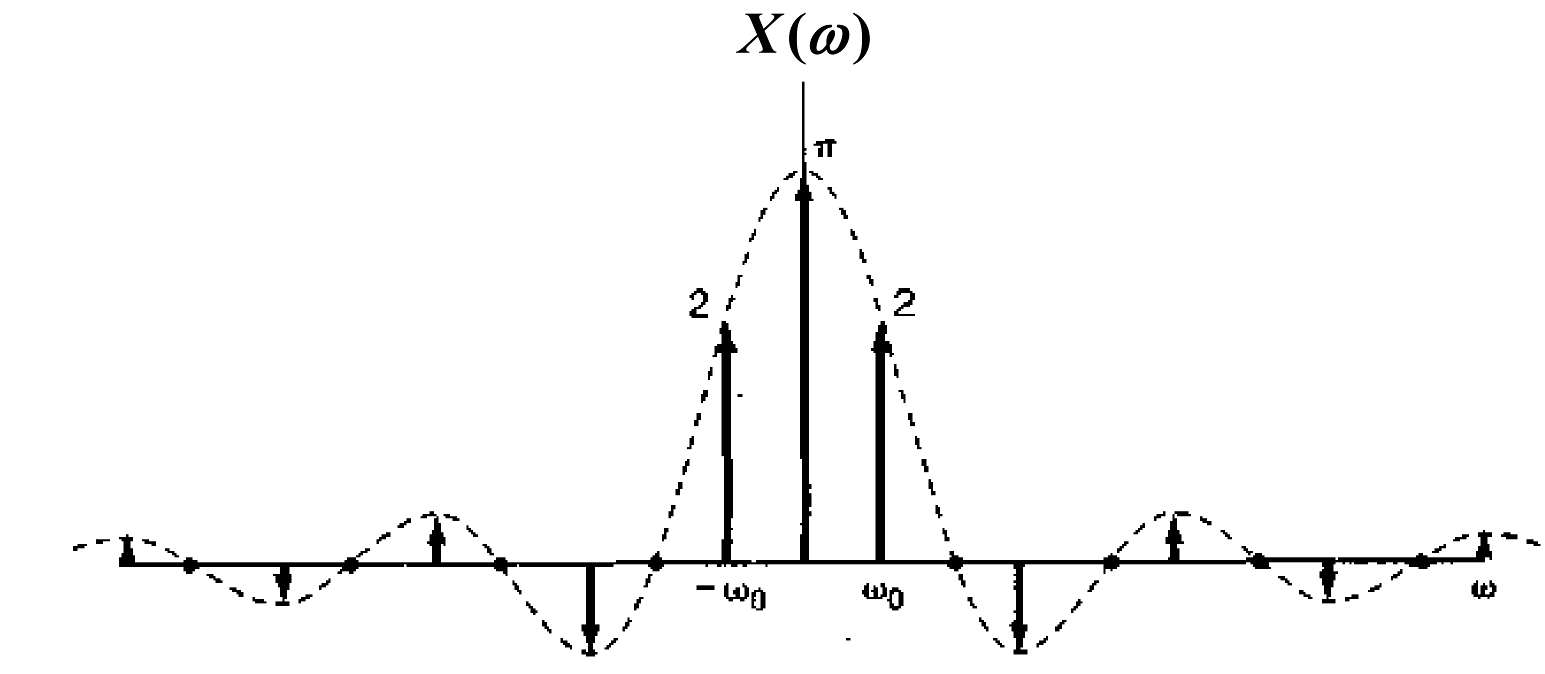
$$D_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} = \frac{\sin(k\omega_0 T_1)}{k\pi}$$

then its Fourier transform is

$$X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi D_k \delta(\omega - k\omega_0)$$

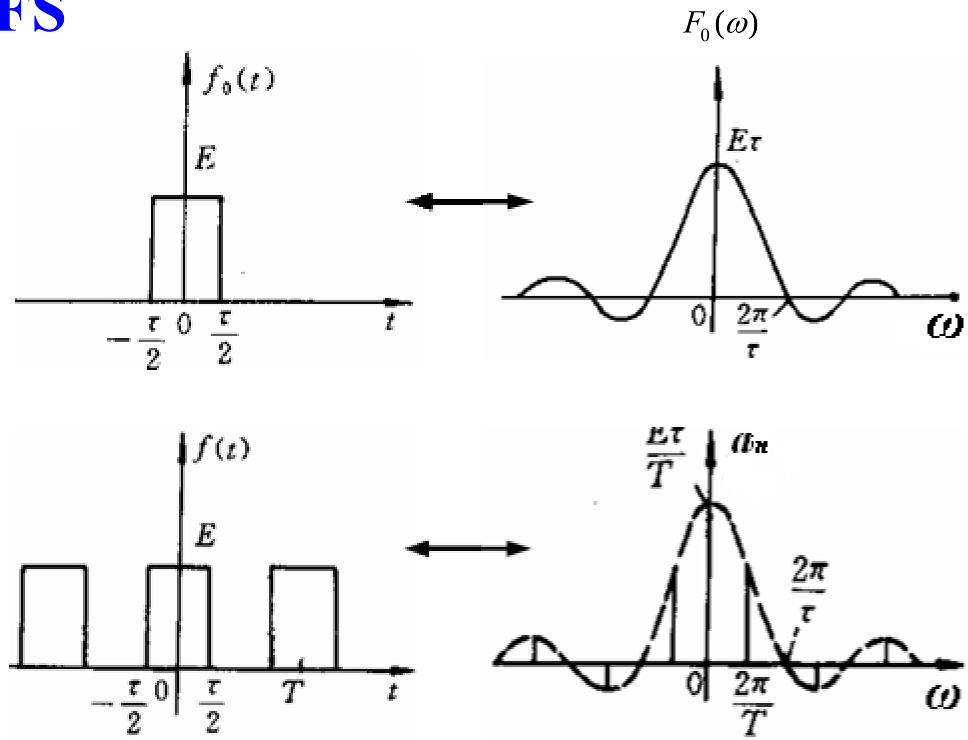
$$= \sum_{k=-\infty}^{\infty} \frac{2\sin(k\omega_0 T_1)}{k} \delta(\omega - k\omega_0)$$

$$X(\omega)=\sum_{k=-\infty}^{\infty}\frac{2\sin(k\omega_0 T_1)}{k}\delta(\omega-k\omega_0)$$



Relationship between FT and FS

$f_0(t)$ is one period of periodic signal $f(t)$.



$$F_0(\omega) = F[f_0(t)]$$

$$= \int_{-\infty}^{\infty} f_0(t)e^{-j\omega t}dt = \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-j\omega t}dt$$

$$D_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-jn\omega_0 t}dt$$

$F_0(\omega)$ is a continuous spectrum and D_n is a discrete spectrum。

Their relationship can be expressed as:

$$\boxed{D_n = \frac{1}{T}F_0(\omega)\Big|_{\omega=n\omega_0}}$$

LTIC system analysis in frequency domain

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

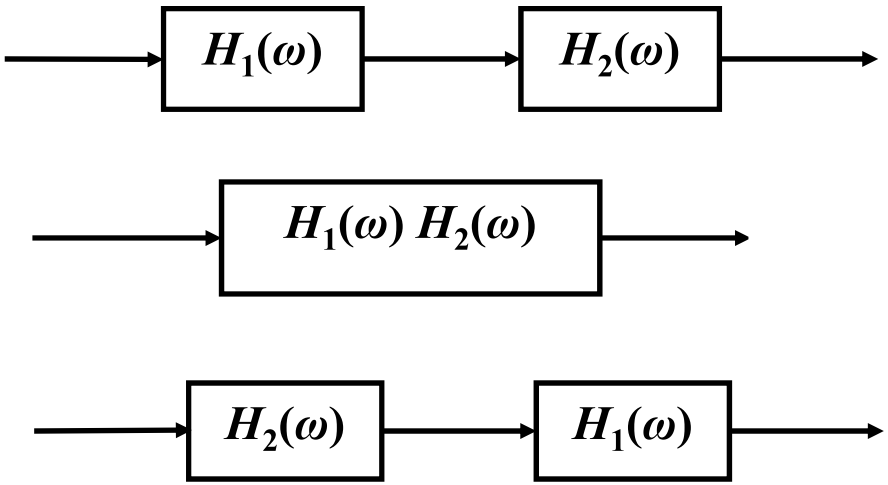
$$y(t) = h(t) * x(t) \leftrightarrow H(\omega)X(\omega) = Y(\omega)$$

$$h(t) \xleftrightarrow{F} H(\omega)$$

frequency response

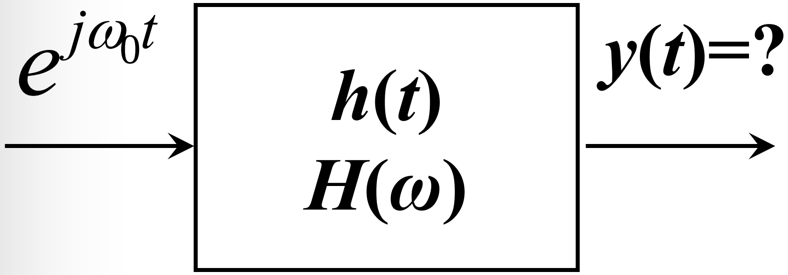
- Two methods can be used to determine the zero-state response of an LTIC system.
- One is performed in time domain, while the other is performed in frequency domain.
- The time convolution property forms the bridge between the two methods.

$$\boxed{y(t) = h(t) * x(t) \xleftrightarrow{F} Y(\omega) = H(\omega)X(\omega)}$$



Three equivalent LTI systems

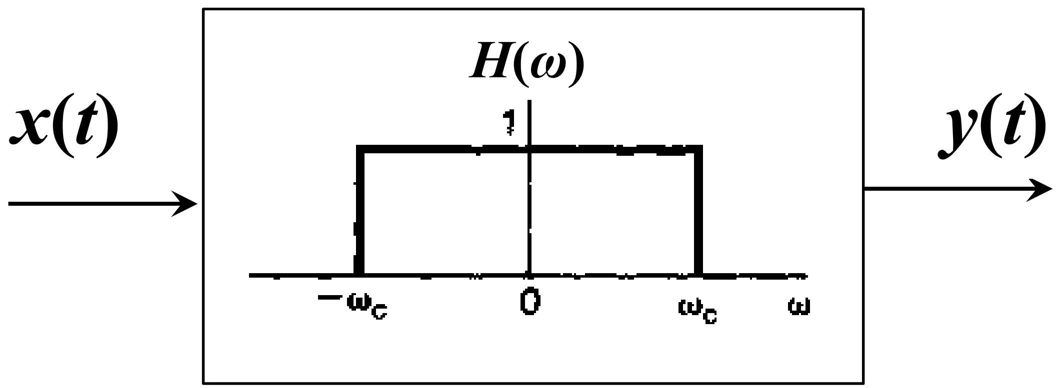
Example: **Response to a complex exponential**



$$y(t) = H(\omega_0)e^{j\omega_0 t}$$

The response of an LTI system to a complex exponential input is the same complex exponential with only a change in amplitude.

Filtering



- The frequency response cannot be defined for every LTIC system.
- If an LTIC system is stable, then, its impulse response is absolutely integrable

$$\int_{-\infty}^{+\infty} |h(t)|\, dt < \infty$$

- Since all physical or practical signals satisfy the last two conditions in Dirichlet conditions, the condition of absolutely integrable becomes the determining factor. That is, only a stable LTIC system has a frequency response $H(\omega)$.

- For an unstable LTIC system, we will develop a generalization of the continuous-time Fourier transform, the Laplace transform.

Example: Determine the response of an ideal low-pass filter to an input signal

$$x(t) = \frac{\sin \omega_i t}{\pi t}$$

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

The impulse response of the ideal low-pass filter : $h(t) = \dfrac{\sin \omega_c t}{\pi t}$

$$X(\omega) = \begin{cases} 1 & |\omega| \le \omega_i \\ 0 & elsewhere \end{cases}$$

$$H(\omega) = \begin{cases} 1 & |\omega| \le \omega_c \\ 0 & elsewhere \end{cases}$$

Therefore,

$$Y(\omega) = \begin{cases} 1 & |\omega| \le \min(\omega_i, \omega_c) \\ 0 & elsewhere \end{cases}$$

$$y(t) = \begin{cases} \dfrac{\sin \omega_c t}{\pi t} & if\ \omega_c \le \omega_i \\ \dfrac{\sin \omega_i t}{\pi t} & if\ \omega_i \le \omega_c \end{cases}$$

The time-frequency duality of linear system response

➢ For the time-domain case

$\delta(t) \Rightarrow h(t)$ shows the impulse response of the system is $h(t)$

$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$ expresses $x(t)$ as a sum of impulse components

$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$ expresses $y(t)$ as a sum of responses to impulse components of the input $x(t)$

➢ For the frequency-domain case

$e^{j\omega t} \Rightarrow H(\omega)e^{j\omega t}$ shows the system response to $e^{j\omega t}$ is $H(\omega)e^{j\omega t}$

$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$ shows $x(t)$ as a sum of everlasting exponential components

$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)H(\omega)e^{j\omega t}d\omega$ expresses $y(t)$ is a sum of responses to exponential components

The Fourier transform of an impulse $\delta(t-\tau)$ is $F[\delta(t-\tau)]=e^{-j\omega\tau}$

Time-frequency duality

The Fourier transform of $e^{j\omega_0 t}$ is $F[e^{j\omega_0 t}]=2\pi\delta(\omega-\omega_0)$

Systems Characterized by Linear Constant-Coefficient Differential Equations

(Assuming that the system is stable)

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

$$\mathcal{F}\left[\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k}\right] = \mathcal{F}\left[\sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}\right]$$

$$\sum_{k=0}^{N} a_k \mathcal{F}\left[\frac{d^k y(t)}{dt^k}\right] = \sum_{k=0}^{M} b_k \mathcal{F}\left[\frac{d^k x(t)}{dt^k}\right]$$

differentiation property

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(\omega)$$

$$Y(\omega)\left[\sum_{k=0}^{N} a_k (j\omega)^k\right] = X(\omega)\left[\sum_{k=0}^{M} b_k (j\omega)^k\right]$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

- **$H(\omega)$ is a ratio of polynomials in ($j\omega$) .**

- **coefficients of the numerator polynomial = coefficients appearing on the right side of the differential equation.**

- **coefficients of the denominator polynomial = coefficients appearing on the left side of the differential equation.**

Example: **Consider a stable LTIC system, determine its impulse response.**

$$\frac{d^2y(t)}{dt^2}+4\frac{dy(t)}{dt}+3y(t)=\frac{dx(t)}{dt}+2x(t)$$

$$(j\omega)^2Y(\omega)+4(j\omega)Y(\omega)+3Y(\omega)=(j\omega)X(\omega)+2X(\omega)$$

The frequency response is $$H(\omega)=\frac{j\omega+2}{(j\omega)^2+4(j\omega)+3}$$

Using partial-fraction expansion $$H(\omega)=\frac{\frac{1}{2}}{j\omega+1}+\frac{\frac{1}{2}}{j\omega+3}$$

The impulse response is $$h(t)=\frac{1}{2}e^{-t}u(t)+\frac{1}{2}e^{-3t}u(t)$$

Example: **Determine the output of the system in the previous example, and suppose that the input is**

$$x(t) = e^{-t}u(t)$$

$$Y(j\omega) = H(j\omega)X(j\omega) = \left[\frac{j\omega+2}{(j\omega+1)(j\omega+3)}\right] \cdot \left[\frac{1}{j\omega+1}\right]$$

$$= \frac{j\omega+2}{(j\omega+1)^2(j\omega+3)}$$

partial-fraction expansion $Y(j\omega) = \frac{A_{11}}{(j\omega+1)^2} + \frac{A_{12}}{j\omega+1} + \frac{A_2}{j\omega+3}$

$$Y(j\omega) = \frac{\frac{1}{2}}{(j\omega+1)^2} + \frac{\frac{1}{4}}{j\omega+1} + \frac{-\frac{1}{4}}{j\omega+3}$$

$$y(t) = \left[\frac{1}{2}te^{-t} + \frac{1}{4}e^{-t} - \frac{1}{4}e^{-3t}\right]u(t)$$

➢ The steps of determining the zero-state response

1、Find the Fourier transform of $x(t)$, i.e., $X(\omega)$

2、Find the frequency response of a system, $H(\omega)$

3、Find the zero-state response in the frequency domain $Y_{zs}(\omega)=X(\omega)H(\omega)$

4、Find the inverse Fourier transform of $Y_{zs}(\omega)$, $y_{zs}(t)=F^{-1}[X(\omega)H(\omega)]$.

➢ The methods to obtain $H(\omega)$

1、If $h(t)$ is given, then find the Fourier transform of $h(t)$.

2、 $H(\omega) = Y(\omega)/X(\omega)$

1) If $y(t)$ and $x(t)$ are given, find their Fourier transforms, $Y(\omega)$ and $X(\omega)$, and then obtain $H(\omega)$.

2) If the linear differential equation is given, taking the Fourier transform of both sides, and then obtain $H(\omega)$.

Distortionless Transmission and ideal filters

- Distortionless Transmission
- Frequency responses of ideal filters
- Ideal lowpass filter

For a system with frequency response $H(\omega)$, if $X(\omega)$ and $Y(\omega)$ are the spectra of the input and the output signals, respectively, then

$$Y(\omega) = X(\omega)H(\omega)$$

In polar form,

$$|Y(\omega)|e^{j\angle Y(\omega)} = |X(\omega)||H(\omega)|e^{j[\angle X(\omega)+\angle H(\omega)]}$$

$$|Y(\omega)| = |X(\omega)||H(\omega)|$$

$$\angle Y(\omega) = \angle X(\omega) + \angle H(\omega)$$

The input signal's amplitude spectrum $|X(\omega)|$ is changed to the product of the input amplitude and the amplitude of the frequency response, $|X(\omega)|\,|H(\omega)|$

The input signal's phase spectrum $\angle X(\omega)$ is also changed to the sum of the input phase and the phase of the frequency response, $\angle X(\omega) + \angle H(\omega)$

Distortionless Transmission

In distortionless transmission, the input $x(t)$ and the output $y(t)$ satisfy the condition

$$y(t) = G_0 \cdot x(t - t_d)$$

G_0 is a constant, t_d is a time delay.

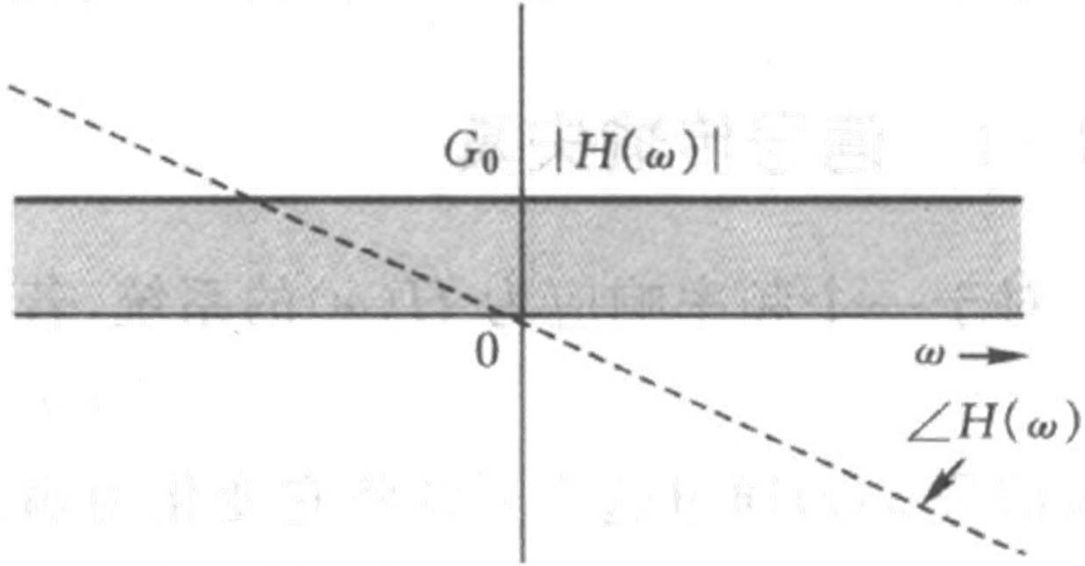
The Fourier transform of this equation yields

$$Y(\omega) = G_0 X(\omega) e^{-j\omega t_d} = X(\omega) H(\omega)$$

frequency response $H(\omega) = G_0 \cdot e^{-j\omega t_d}$

the amplitude response and the phase response

$$|H(\omega)| = G_0 \qquad \angle H(\omega) = -\omega\, t_d$$



Two conditions for distortionless transmission:

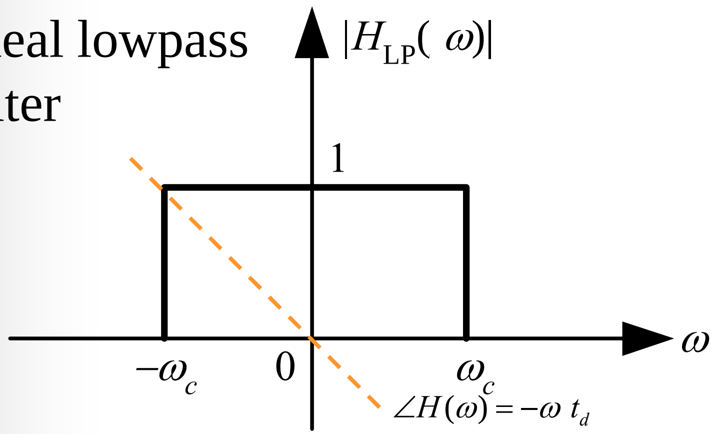
1) The amplitude response $|H(\omega)|$ must be a constant.

2) The phase response $\angle H(\omega)$ must be a linear function of ω with slope $-t_d$, where t_d is the delay of the output with respect to the input.

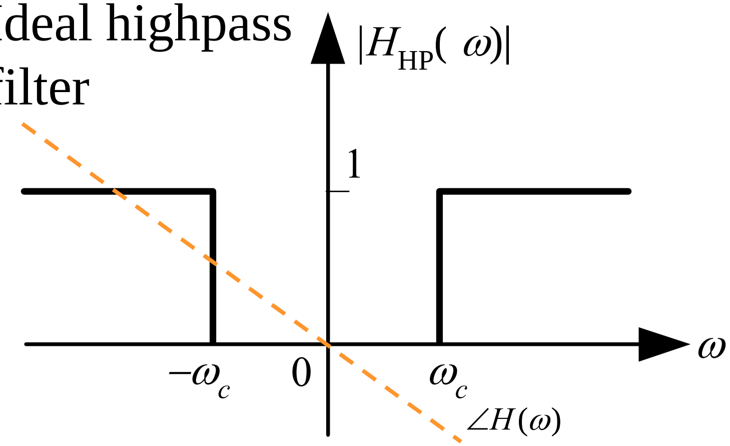
Frequency responses of ideal filters

An ideal filter is a system which allows distortionless transmission of a certain band of frequencies and completely suppress the remaining frequencies.

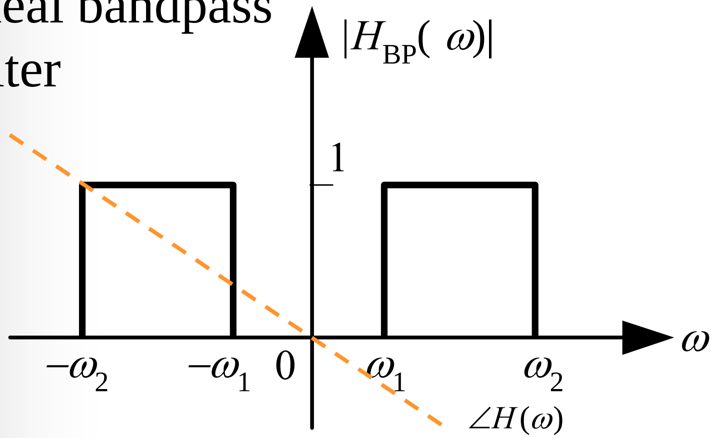
Ideal lowpass filter



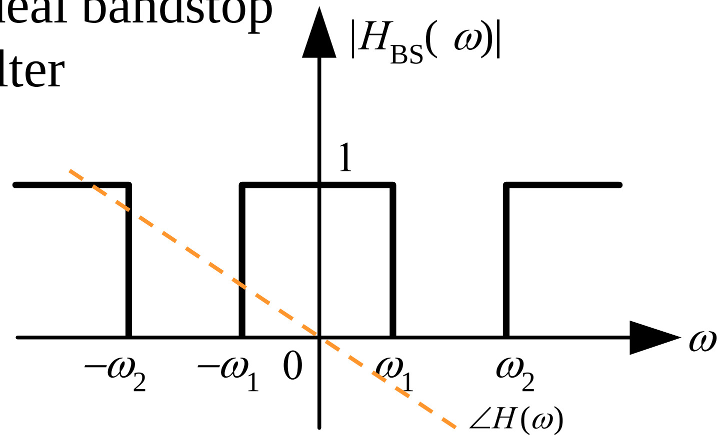
Ideal highpass filter



Ideal bandpass filter

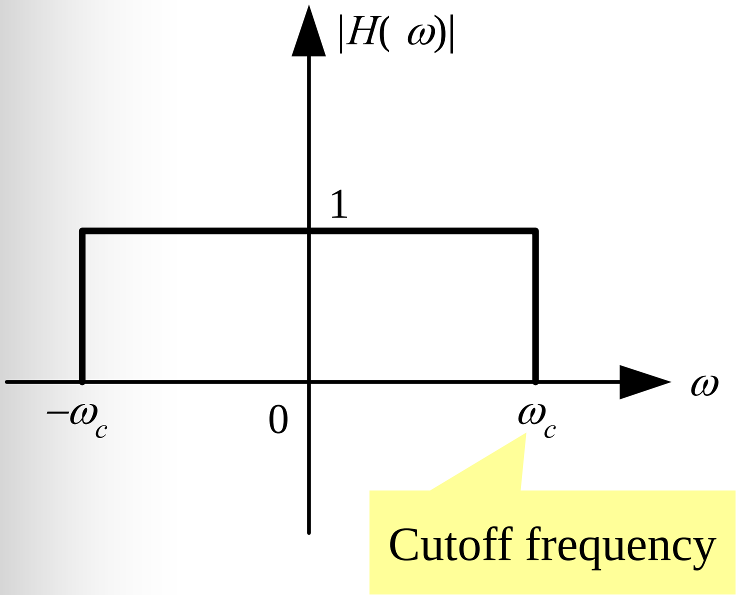


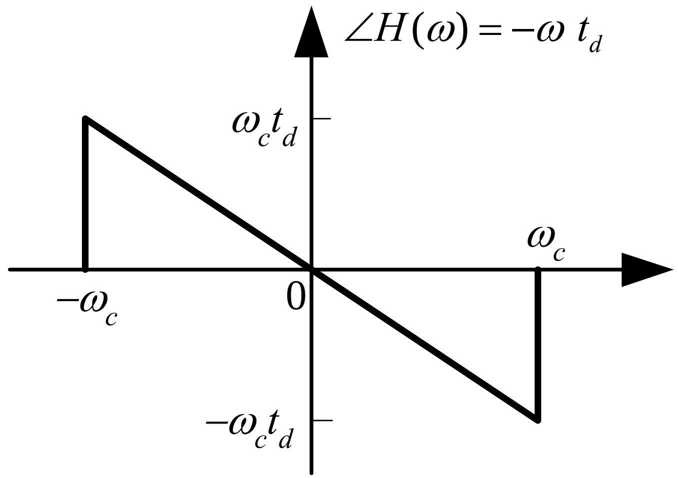
Ideal bandstop filter



Ideal lowpass filter

$$H(\omega) = |H(\omega)| e^{j\angle H(\omega)} = \mathrm{rect}\left(\frac{\omega}{2\omega_c}\right) e^{-j\omega t_d}$$



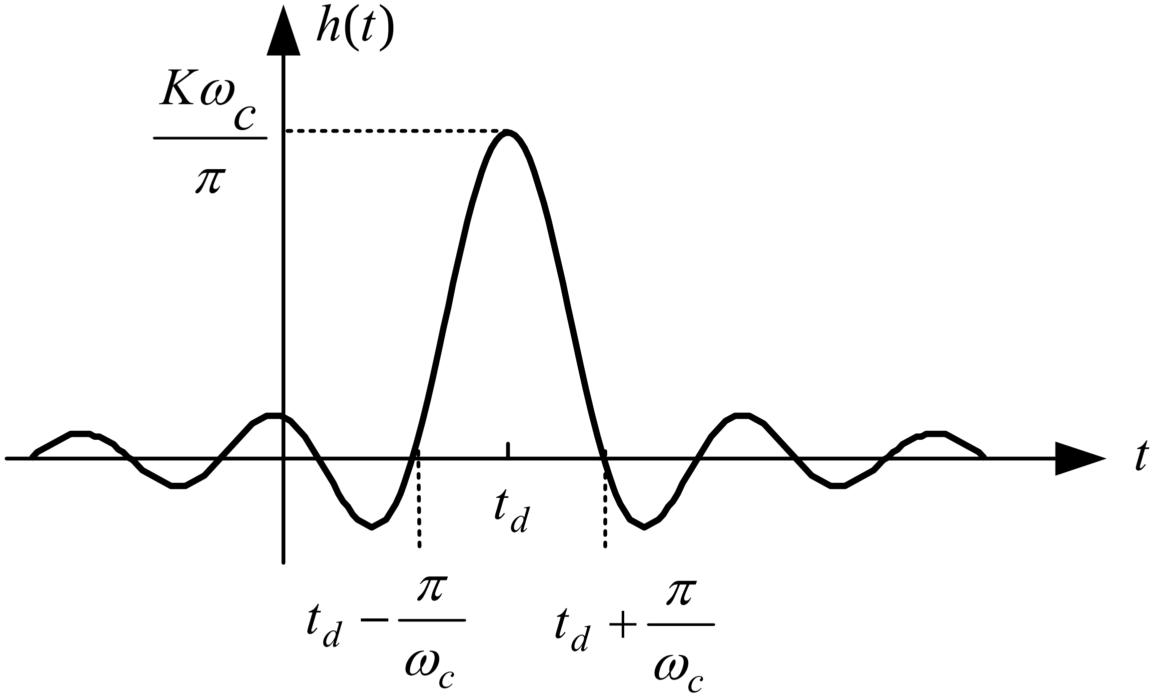


The amplitude response $|H(\omega)|$ is one over the interval $[-\omega_c, \omega_c]$ and zero outside this interval.

The phase response $\angle H(\omega)$ is a linear function of ω in the interval $[-\omega_c, \omega_c]$.

Impulse response of an ideal lowpass filter

$$h(t) = F^{-1}\left[H(\omega)\right] = F^{-1}\left[\operatorname{rect}\left(\frac{\omega}{2\omega_c}\right)e^{-j\omega t_d}\right] = \frac{\omega_c}{\pi}\mathrm{Sa}[\omega_c(t-t_d)]$$



Note: The lowpass filter is noncausal and physically unrealizable, because the response $h(t)$ begins even before the input is applied (at $t = 0$). Similarly, other ideal filters are also physically unrealizable due to the same reason.