1.1 Introduction to magnetic circuit

The complete, detailed solution for magnetic fields in most situations of practical engineering interest involves the solution of Maxwell's equations and requires a set of constitutive relationships to describe material properties. Although in practice exact solutions are often unattainable, various simplifying assumptions permit the attainment of useful engineering solutions.

$$\oint_c \vec{H}\mathrm{d}\vec{l} = \int_S \vec{J}\mathrm{d}\vec{a} \tag{1}$$

$$\oint_S \vec{B}\mathrm{d}\vec{a} = 0 \tag{2}$$

Equation 1 frequently referred to as *Ampere's Law*, states that the line integral of the tangential component of the magnetic field intensity $\vec{H}$ (磁场强度，是一个空间矢量) around a closed contour c is equal to the total current passing through any surface S linking that contour. From the Eq.1 we see that the source of $\vec{H}$ is the current density $\vec{J}$. Eq.2, frequently referred to as *Gauss's Law for magnetic fields*, states that magnetic flux density $\vec{B}$ is conserved (守恒的), i.e., that no net flux enters or leaves a closed surface (this is equivalent to saying that there exist no monopolar sources of magnetic fields). From these equations we see that the magnetic field quantities can be determined solely from the instantaneous values of the source currents and hence that time variations of the magnetic fields follow directly from time variations of the sources.

A second simplifying assumption involves the concept of a *magnetic circuit*. It is extremely difficult to obtain the general solution for the magnetic field intensity $\vec{H}$ and the magnetic flux density $\vec{B}$ in a structure of complex geometry. However, in many practical applications, including the analysis of many types of electric machines, a three-dimensional problem can often be approximated by what is essentially a one-dimensional circuit equivalent, yielding solutions of acceptable engineering accuracy.

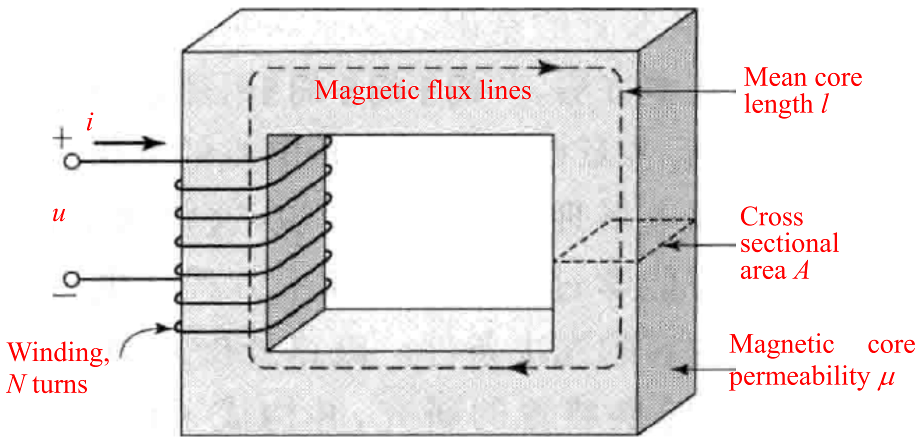


Figure 1.1 Simple magnetic circuit.

A magnetic circuit consists of a structure composed for the most part of high-permeability magnetic material. The presence of high-permeability material tends to cause magnetic flux to be confined to the paths defined by the structure, much as currents are confined to the conductors of an electric circuit. In the simplest definition, magnetic permeability can be thought of as the ratio of the magnitude of the magnetic flux density B to the magnetic field intensity H. Use of this concept of the magnetic circuit is illustrated in this section and will be seen to apply quite well to many situations in this book.

A simple example of a magnetic circuit is shown in Fig.1.1. The core is assumed to be composed of magnetic material whose *magnetic permeability* μ is much greater than that of the surrounding air ($\mu \gg \mu_0$)where $\mu_0 = 4\pi \times 10^{-7}$H/m is the magnetic permeability of free space. The core is of uniform cross section and is excited by a winding of N turns carrying a current of i amperes. This winding produces a magnetic field in the core, as shown in the figure.

Because of the high permeability of the magnetic core, an exact solution would show that the magnetic flux is confined almost entirely to the core, with the field lines following the path defined by the core, and that the flux density is essentially uniform over a cross section because the cross-sectional area is uniform. The magnetic field can be visualized in terms of flux lines which form closed loops interlinked with the winding.

As applied to the magnetic circuit of Fig.1, the source of the magnetic field in the core is the ampere-turn product Ni. In the magnetic circuit terminology Ni is the *magnetomotive force* (mmf) F acting on the magnetic circuit. Although Fig.1.1 shows only a single winding, transformers and most rotating machines typically have at least two windings, and Ni must be replaced by the algebraic sum of the ampere-turns of all the windings.

The net magnetic flux ϕ crossing a surface S is the surface integral of the normal component of $\vec{B}$; thus

$$\phi = \int_S \vec{B} \cdot \mathrm{d}\vec{a} \tag{3}$$

In SI units, the unit of ϕ is *weber* (Wb).

Equation 2 states that the net magnetic flux entering or leaving a closed surface (equal to the surface integral of $\vec{B}$ over that closed surface) is zero. This is equivalent to saying that all the flux which enters the surface enclosing a volume must leave that volume over some other portion of that surface because magnetic flux lines form closed loops. Because little flux "leaks" out the sides of the magnetic circuit of Fig.1, this result shows that the net flux is the same through each cross section of the core.

For a magnetic circuit of this type, it is common to assume that the magnetic flux density (and correspondingly the magnetic field intensity) is uniform across the cross section and throughout the core. In this case Eq.3 reduces to the simple scalar equation

$$\phi_c = B_c A_c \tag{4}$$

where

ϕ_c= core flux

B_c = core flux density

A_c = core cross-sectional area

From Eq.1, the relationship between the mmf acting on a magnetic circuit and the magnetic field intensity in that circuit is. In general, the mmf drop across any segment of a magnetic circuit can be calculated as $\oint \vec{H} d\vec{l}$ over that portion of the magnetic circuit.

$$F = Ni = \oint_c \vec{H} d\vec{l} \tag{5}$$

The core dimensions are such that the path length of any flux line is close to the mean core length l_c. As a result, the line integral of Eq.5 becomes simply the scalar product $H_c l_c$ of the magnitude of $\vec{H}$ and the mean flux path length l_c. Thus,the relationship between the mmf and the magnetic field intensity can be written inmagnetic circuit terminology as

$$F = Ni = H_c l_c \tag{6}$$

where H_c is average magnitude of $\vec{H}$ in the core.

The direction of H_c in the core can be found from the *right-hand rule*, which can be stated in two equivalent ways. (1) Imagine a current-carrying conductor held in the right hand with the thumb pointing in the direction of current flow; the fingers then point in the direction of the magnetic field created by that current. (2) Equivalently, if the coil in Fig.1 is grasped in the right hand (figuratively speaking) with the fingers pointing in the direction of the current, the thumb will point in the direction of the magnetic fields.

The relationship between the magnetic field intensity $\vec{H}$ and the magnetic flux density $\vec{B}$ is a property of the material in which the field exists. It is common to assume a linear relationship; thus

$$\vec{B} = \mu \vec{H} \tag{7}$$

where μ is known as the material's magnetic permeability. In SI units, $\vec{H}$ is measured in units of amperes per meter, $\vec{B}$ is in *webers per square meter*, also known as *terlas*(T), and μ is in *webers per ampere-turn-meter*, or equivalently *henrys per meter*, In SI units the permeability of free space is $\mu_0 = 4\pi \times 10^{-7}$ henrys per meter. The permeability of liner magnetic material can be expressed in terms of its *relative permeability* μ_r, its value relative to that of free space: $\mu = \mu_r \mu_0$. Typical values of μ_r range from 2,000 to 80,000 for materials used in transformers and rotating machines. For the present we assume that μ_r is a known constant, although it actually varies appreciably with the magnitude of the magnetic flux density.

Transformers are wound on closed cores like that of Fig.1. However, energy conversion devices which incorporate a moving element must have air gaps in their magnetic circuits. A magnetic circuit with an air gap is shown in Fig.2. When the air-gap length g is much smaller than the dimensions of the adjacent core faces, the core flux ϕ_c will follow the path defined by the core and the air gap and the techniques of magnetic-circuit analysis can be used. If the air-gap length becomes excessively large, the flux will be observed to "leak out" of the sides of the air gap and the techniques of magnetic-circuit analysis will no longer be strictly applicable.

Thus, provided the air-gap length g is sufficiently small, the configuration of Fig.1.2 can be analyzed as a magnetic circuit with two series components both carrying the same flux ϕ: a magnetic core of permeability μ, cross sectional area A_c and mean length l_c, and an air gap of permeability μ_0, cross-sectional area A_g and length g. In the core

$$B_c=\phi/A_c \tag{8}$$

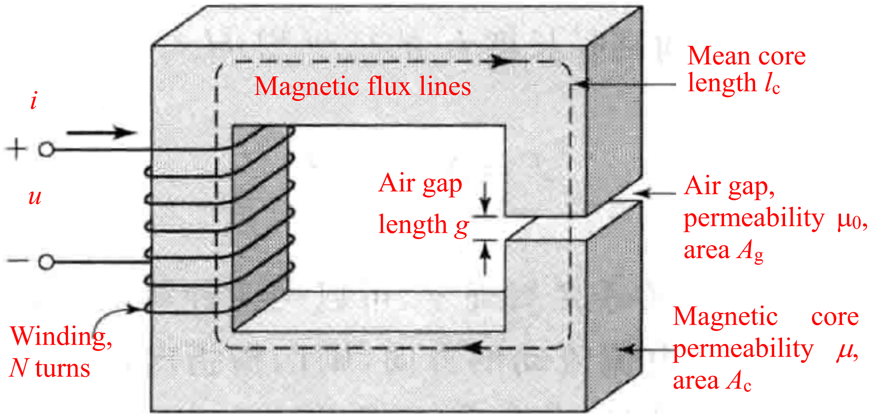


Figure 2 Magnetic circuit with air gap

and in the air gap

$$B_g=\phi/A_g \tag{9}$$

Application of Eq.5 to this magnetic circuit yields

$$F=H_c l_c+H_g g \tag{10}$$

and using the linear B-H relationship of Eq.7 gives

$$F=\frac{B_c}{\mu}l_c+\frac{B_g}{\mu_0}g \tag{11}$$

Here the $F = Ni$ is the mmf applied to the magnetic circuit. From Eq.1.10 we see that a portion of the mmf, $F_c = H_c l_c$, is required to produce magnetic field in the core while the remainder, $F_g = H_g g$ produces magnetic field in the air gap.

For practical magnetic materials (as is discussed in Sections 1.3 and 1.4). B_c and H_c are not simply related by a known constant permeability μ as described by Eq.7. In fact, B_c is often a nonlinear, multi-valued function of H_c. Thus, although Eq.10 continues to hold, it does not lead directly to a simple expression relating the mmf and the flux densities, such as that of Eq.11. Instead the specifics of the nonlinear B_c-H_c relation must be used, either graphically or analytically. However, in many cases, the concept of constant material permeability gives results of acceptable engineering accuracy and is frequently used.

From Eqs.8 and 9, Eq.11 can be rewritten in terms of the flux ϕ_c as

$$F=\phi\left(\frac{l_c}{\mu A_c}+\frac{g}{\mu_0 A_g}\right) \tag{12}$$

The terms that multiply the flux in this equation are known as the *reluctance* (R) of the core and air gap, respectively,

$$R_c = \frac{l_c}{\mu A_c} \tag{13}$$

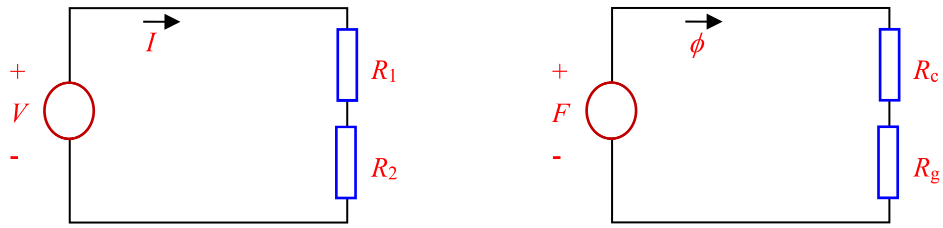
$$R_g = \frac{g}{\mu_0 A_g} \tag{14}$$

and thus

$$F = \phi(R_c + R_g) \tag{15}$$

Finally, Eq.15 can be inverted to solve for the flux

$$\phi = F/(R_c + R_g) \tag{16}$$



(a) $I = \dfrac{V}{R_1 + R_2}$ (b) $\phi = \dfrac{F}{R_c + R_g}$

Figure 3 Analogy between electric and magnetic circuits
(a) Electric circuit. (b) Magnetic circuit.

In general, for any magnetic circuit of total reluctance R_{tot}, the flux can be found as $\phi = F/R_{tot}$.

The term which multiplies the mmf is known as the *permeance* Λ (磁导)and is the inverse of the reluctance; thus, for example, the total permeance of a magnetic circuit is $\Lambda_{tot}=1/ R_{tot}$.

Note that Eqs.15 and 16 are analogous to the relationships between the current and voltage in an electric circuit. This analogy is illustrated in Fig.3. Figure 3a shows an electric circuit in which a voltage V drives a current I through resistors R_1 and R_2. Figure 3b shows the schematic equivalent representation of the magnetic circuit of Fig.2. Here we see that the mmf F (analogous to voltage in the electric circuit) drives a flux Φ (analogous to the current in the electric circuit) through the combination of the reluctances of the core R_c and the air gap R_g. This analogy between the solution of electric and magnetic circuits can often be exploited to produce simple solutions for the fluxes in magnetic circuits of considerable complexity.

The fraction of the mmf required to drive flux through each portion of the magnetic circuit, commonly referred to as the *mmf drop across* that portion of the magnetic circuit, varies in proportion to its reluctance (directly analogous to the voltage drop across a resistive element in an electric circuit). Consider the magnetic circuit of Fig 2. From Eq.13 we see that high material permeability can result in low core reluctance, which can often be made much smaller than that of the air gap: i.e., for $(\mu A_c/l_c) \gg (\mu_0 A_g/g)$, $R_c \ll R_g$ and thus $R_{tot} \approx R_g$. In this case, the reluctance of the core can be neglected and the flux can be found from Eq. 16 in terms of F and the air-gap properties alone:

$$\phi \approx \frac{F}{R_{\mathrm{g}}} = \frac{F\mu_0 A_{\mathrm{g}}}{g} = Ni\frac{\mu_0 A_{\mathrm{g}}}{g} \tag{17}$$

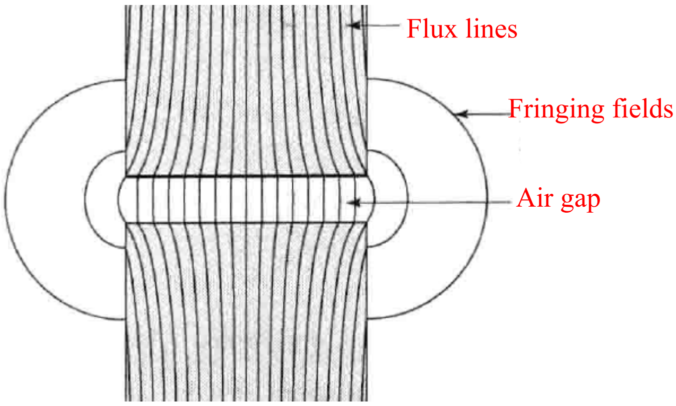


Figure 4 Air-gap fringing fields

As will be seen in Section 1.3, practical magnetic materials have permeabilities which are not constant but vary with the flux level. From Eqs.13 to 16 we see that as long as this permeability remains sufficiently large, its variation will not significantly affect the performance of a magnetic circuit in which the dominant reluctance is that of an air gap.

In practical systems, the magnetic field lines "fringe" outward somewhat as they cross the air gap, as illustrated in Fig.4. Provided this fringing effect is not excessive, the magnetic-circuit concept remains applicable. The effect of these *fringing fields* is to increase the effective cross-sectional area A_{g} of the air gap. Various empirical methods have been developed to account for this effect. A correction for such fringing fields in short air gaps can be made by adding the gap length to each of the two dimensions making up its cross-sectional area. In this book the effect of fringing fields is usually ignored. If fringing is neglected, $A_{\mathrm{g}} = A_{\mathrm{c}}$.

In general, magnetic circuits can consist of multiple elements in series and parallel. To complete the analogy between electric and magnetic circuits, we can generalize Eq.5 as

$$F = Ni = \oint_c \vec{H} \mathrm{d}\vec{l} = \sum_k F_k = \sum_k H_k l_k \tag{18}$$

where F is the mmf (total ampere-turns) acting to drive flux through a closed loop magnetic circuit, and $F_k = H_k l_k$ is the mmf drop across the k'th element of that loop. This is directly analogous to Kirchoff's voltage law for electric circuits consisting of voltage sources and resistors

$$V = \sum_k R_k i_k \tag{19}$$

where V is the source voltage driving current around a loop and $R_k i_k$ is the voltage drop across the k'th resistive element of that loop.

Similarly, the analogy to Kirchoff's current law

$$\sum_n i_n = 0 \tag{20}$$

which says that the net current, i.e. the sum of the currents, into a node in an electric circuit equals zero is

$$\sum_n \phi_n = 0 \tag{21}$$

which states that the net flux into a node in a magnetic circuit is zero.

We have now described the basic principles for reducing a magneto-quasi-static field problem with simple geometry to a *magnetic circuit model*. Our limited purpose in this section is to introduce some of the concepts and terminology used by engineers in solving practical design problems. We must emphasize that this type of thinking depends quite heavily on engineering judgment and intuition. For example, we have tacitly assumed that the permeability of the "iron" parts of the magnetic circuit is a constant known quantity, although this is not true in general (see Section 1.3), and that the magnetic field is confined solely to the core and its air gaps. Although this is a good assumption in many situations, it is also true that the winding currents produce magnetic fields outside the core. As we shall see, when two or more windings are placed on a magnetic circuit, as happens in the case of both transformers and rotating machines, these fields outside the core, referred to as *leakage fields*, cannot be ignored and may significantly affect the performance of the device.

1.2 Flux linkage, Inductance and energy

$$e = -N\frac{\mathrm{d}\phi}{\mathrm{d}t} = -\frac{\mathrm{d}\psi}{\mathrm{d}t} \tag{22}$$

Where ψ is the flux linkage of the winding and is defined as $\psi = N\phi$. Flux linkage is measured in units of webers (or equivalently weber-turns). Note that we have chosen the symbol ψ to indicate the instantaneous value of a time-varying flux.

In general the flux linkage of a coil is equal to the surface integral of the normal component of the magnetic flux density integrated over any surface spanned by that coil. Note that the direction of the induced voltage e is defined by Eq.22 so that if the winding terminals were short-circuited, a current would flow in such a direction as to oppose the change of flux linkage.

For a magnetic circuit composed of magnetic material of constant magnetic permeability or which includes a dominating air gap, the relationship between ψ and i will be linear and we can define the *inductance* L as $L=\psi/i$.

$$L=\psi/i \text{ and } \phi = F/R_{\mathrm{tot}} = F\Lambda_{\mathrm{tot}} \quad \Rightarrow \quad L=N\phi/i = N(Ni\Lambda_{\mathrm{tot}})/i = N^2\Lambda_{\mathrm{tot}} \tag{23}$$

From which we see that the inductance of a winding in a magnetic circuit is proportional to the square of the turns and the *permeance* of the magnetic circuit (or inversely proportional to the reluctance of the magnetic circuit) associated with that winding.

For example, from Eq.17, under the assumption that the reluctances of the core is negligible as compared to that of the air gap, the inductance of the winding in Fig.2 is equal to

$$L = \frac{N^2}{(g/\mu_0 A_{\mathrm{g}})} = \frac{N^2 \mu_0 A_{\mathrm{g}}}{g} \tag{24}$$

Inductance is measured in *henrys* (H) or *weber-turns per ampere*. It is must be emphasized that strictly speaking, the concept of inductance requires a linear relationship between flux and mmf. Thus, it cannot be rigorously applied in situations where the nonlinear characteristics of magnetic materials, as discussed in Section 1.3 and 1.4, dominate the performance of the magnetic system. However, in many situations of practical interest, the reluctance of the system is dominated by that of an air gap (which is of course linear) and the non-linear effects of the magnetic material can be ignored.

The power at the terminals of a winding on a magnetic circuit is a measure of the rate of energy flow into the circuit through that particular winding. The *power*, *p*, is determined from the product of the voltage and the current

$$p = -ie = i\frac{\mathrm{d}\psi}{\mathrm{d}t} \tag{25}$$

and its unit is *watts*(W), or *joules per second*. Thus the change in *magnetic stored energy* ΔW in the circuit in the time interval t_1 and t_2 is

$$\Delta W = \int_{t_1}^{t_2} p \cdot \mathrm{d}t = \int_{t_1}^{t_2} -ie \cdot \mathrm{d}t = \int_{t_1}^{t_2} i\frac{\mathrm{d}\psi}{\mathrm{d}t} \cdot \mathrm{d}t = \int_{\psi_1}^{\psi_2} i\mathrm{d}\psi \tag{26}$$

In SI units, the magnetic stored energy W is measured in *joules* (J).

For a single-winding system of constant inductance, the change in magnetic stored energy as the flux level is changed from ψ_1 to ψ_2 can be written as

$$\Delta W = \int_{\psi_1}^{\psi_2} i\mathrm{d}\psi = \int_{\psi_1}^{\psi_2} \frac{\psi}{L}\mathrm{d}\psi = \frac{1}{2L}(\psi_2^2 - \psi_1^2) \tag{27}$$

The total magnetic stored energy at any given value of ψ can be found from setting ψ_1 equal to zero:

$$W = \frac{1}{2L}\psi^2 = \frac{L}{2}i^2 \tag{28}$$