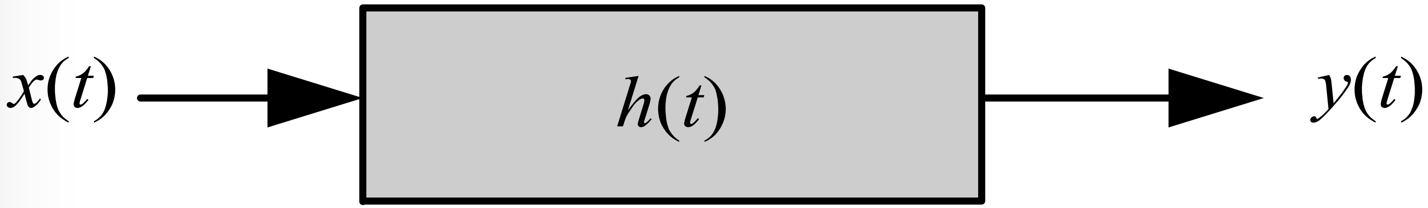
Chapter 6

Continuous-Time Signal Analysis: The Fourier Series

- *Introduction*
- *CT Fourier Series of Periodic Signals*
- *The Effects of Symmetry*
- *The Fourier Spectrum and properties*
- *Summary*

Why do we learn Fourier analysis?



Question:

when a signal $x(t)$ passes through a system, which frequency components of this signal are boosted and which ones are attenuated?

J.B.J. Fourier
(1768 –1830)

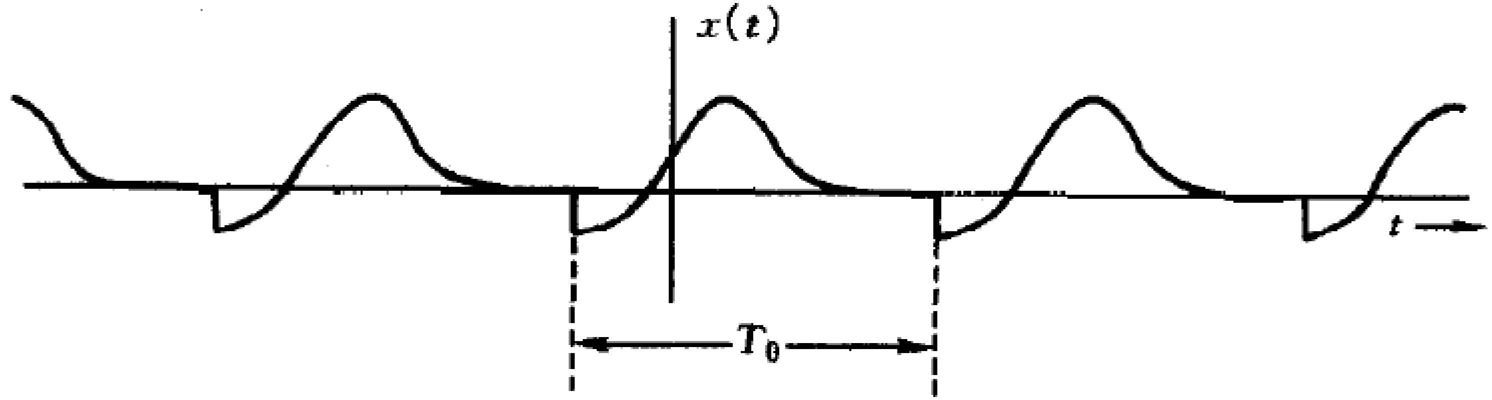
- Born in 1768, in France.
- In 1807, Fourier claimed that an arbitrary function defined in a finite interval can always be expressed as a sum of sinusoids.
- Fourier's paper was rejected because of vehement opposition by Lagrange.
- In 1822, fifteen years later, Fourier published the results in a book.
- In 1829, Dirichlet proved Fourier’s claim.

Continuous-time Fourier Series

- Periodic signal: a signal, $x(t)$, defined in $(-\infty, \infty)$, satisfies the following equation

$$x(t) = x(t + T_0) \quad for\ all\ \ t$$

- The smallest positive nonzero value of T_0 is the fundamental period.



- Trigonometric Fourier Series
- Compact Form of Fourier Series
- Exponential Fourier Series

Trigonometric Fourier Series

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \qquad (6.3)$$

$$T_0 = \frac{1}{f_0} = \frac{2\pi}{\omega_0} \text{ and } \omega_0 = 2\pi / T_0 = 2\pi f_0 \qquad \boxed{\{1, \sin(n\omega_0 t), \cos(n\omega_0 t), n = 1,2,\ldots\}}$$

ω_0 (f_0) is called the fundamental frequency.

A sinusoid of frequency $n\omega_0$ (nf_0) is the nth harmonic of the sinusoid of frequency $n\omega_0$ (nf_0).

a_0 is called the direct current (DC) component.

An extremely important property:

$x(t)$ in Eq. (6.3) is a periodic signal with period T_0, regardless of the values of the amplitudes a_n and b_n .

$$x(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) = x(t+T_0)$$

Proof: **the periodicity of *x*(*t*)**

$$x(t+T_0) = a_0 + \sum_{n=1}^{\infty}\{a_n \cos[n\omega_0(t+T_0)] + b_n \sin[n\omega_0(t+T_0)]\}$$

$$= a_0 + \sum_{n=1}^{\infty}[a_n \cos(n\omega_0 t + n\omega_0 T_0) + b_n \sin(n\omega_0 t + n\omega_0 T_0)]\left(\because n\omega_0 T_0 = 2\pi n\right)$$

$$= a_0 + \sum_{n=1}^{\infty}[a_n \cos(n\omega_0 t + 2\pi n) + b_n \sin(n\omega_0 t + 2\pi n)]$$

$$= a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

$$= x(t)$$

$$x(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \qquad (6.3)$$

A periodic signal $x(t)$ with period T_0 can be expressed as a sum of a sinusoid of frequency f_0 (f_0=1/T_0) and all its harmonics.

Questions:

$a_0 = ?$ $\qquad$ $a_n = ?$ $\qquad$ $b_n = ?$

Computing the Coefficients of a Fourier Series

Consider an integral I defined as

$$I = \int_{T_0} \cos n\omega_0 t \cos m\omega_0 t dt$$

$$= \frac{1}{2}\left[\int_{T_0} \cos[(n+m)\omega_0 t]dt + \int_{T_0} \cos[(n-m)\omega_0 t]dt\right]$$

$$\int_{T_0} \cos n\omega_0 t \cos m\omega_0 t dt = \begin{cases} 0 & n \neq m \\ \dfrac{T_0}{2} & n = m \neq 0 \end{cases}$$

$$\int_{T_0} \sin n\omega_0 t \sin m\omega_0 t dt = \begin{cases} 0 & n \neq m \\ \dfrac{T_0}{2} & n = m \neq 0 \end{cases}$$

$$\int_{T_0} \sin n\omega_0 t \cos m\omega_0 t dt = 0$$ for all n and m

To determine a_0, integrating both sides of Eq. (6.3) over one period T_0 yields

$$\int_{T_0} x(t)dt = a_0\int_{T_0} dt + \sum_{n=1}^{\infty}\left[a_n\int_{T_0}\cos n\omega_0 tdt + b_n\int_{T_0}\sin n\omega_0 tdt\right]$$

$$= a_0\int_{T_0} dt = a_0 T_0$$

$$\boxed{a_0 = \frac{1}{T_0}\int_{T_0} x(t)dt}$$

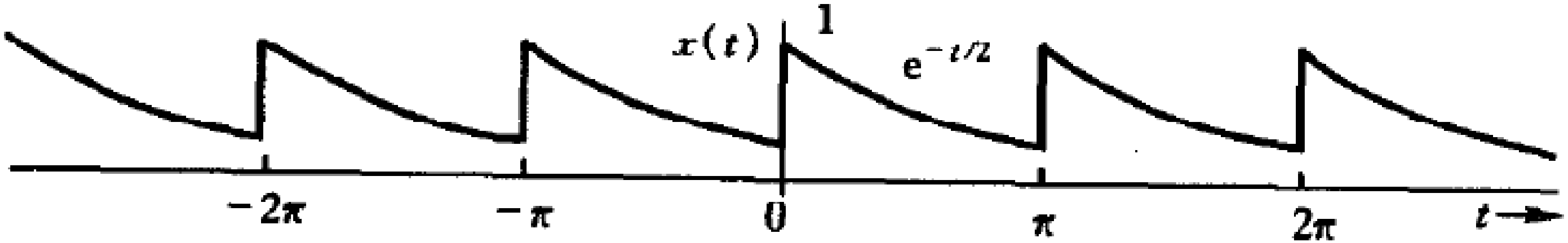
To determine a_n, multiplying both sides of Eq. (6.3) by $\cos m\omega_0 t$ and integrating the resulting equation over an interval T_0 yields

$$\int_{T_0} x(t)\cos m\omega_0 tdt = a_0\int_{T_0}\cos m\omega_0 tdt + \sum_{n=1}^{\infty}\left[a_n\int_{T_0}\cos n\omega_0 t\cos m\omega_0 tdt + b_n\int_{T_0}\sin n\omega_0 t\cos m\omega_0 tdt\right]$$

$$\boxed{a_n = \frac{2}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\cos n\omega_0 tdt}$$

$$\boxed{b_n = \frac{2}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\sin n\omega_0 tdt}$$

Example: **Find the trigonometric Fourier series for the periodic signal $x(t)$ shown in the following figure.**



$$x(t) = e^{-t/2}, 0 \le t \le \pi$$
$$x(t) = x(t+\pi)$$

$$T_0 = \pi, f_0 = \frac{1}{\pi}$$
$$\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{\pi} = 2rad/s$$

Therefore :

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos 2nt + b_n \sin 2nt)$$

where :

$$a_0 = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)dt = \frac{1}{\pi}\int_{T_0} x(t)dt = \frac{1}{\pi}\int_0^{\pi} e^{-t/2}dt = \frac{-2}{\pi}e^{-t/2}\Big|_0^{\pi} = 0.504$$

$$a_n = \frac{2}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\cos n\omega_0 t dt$$

$$= \frac{2}{\pi}\int_{T_0} x(t)\cos 2nt dt = \frac{2}{\pi}\int_0^{\pi} e^{-t/2}\cos 2nt dt = 0.504(\frac{2}{1+16n^2})$$

$$b_n = \frac{2}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\sin n\omega_0 t dt$$

$$= \frac{2}{\pi}\int_{T_0} x(t)\sin 2nt dt = \frac{2}{\pi}\int_0^{\pi} e^{-t/2}\sin 2nt dt = 0.504(\frac{8n}{1+16n^2})$$

Compact Form of Fourier Series

$$x(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n) \Leftrightarrow$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

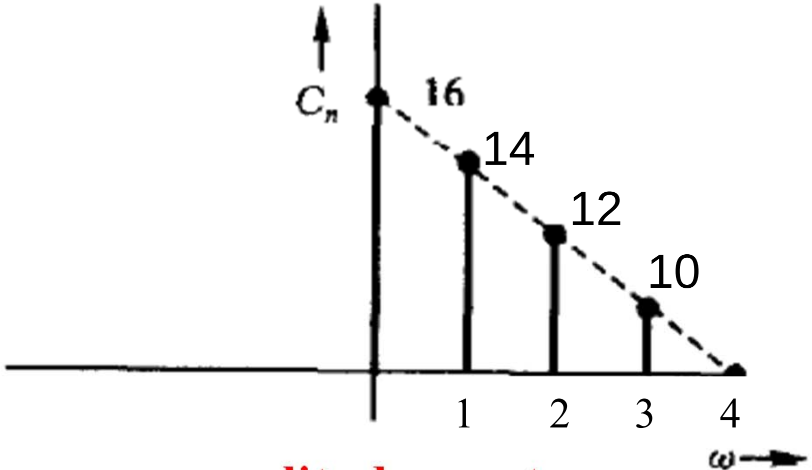
where C_n and θ_n are related to a_n and b_n

$$\begin{cases} C_0 = a_0, \quad C_n = \sqrt{a_n^2 + b_n^2} \\ \theta_n = \arctan(-\dfrac{b_n}{a_n}) \end{cases}, (n=1,2,\cdots)$$

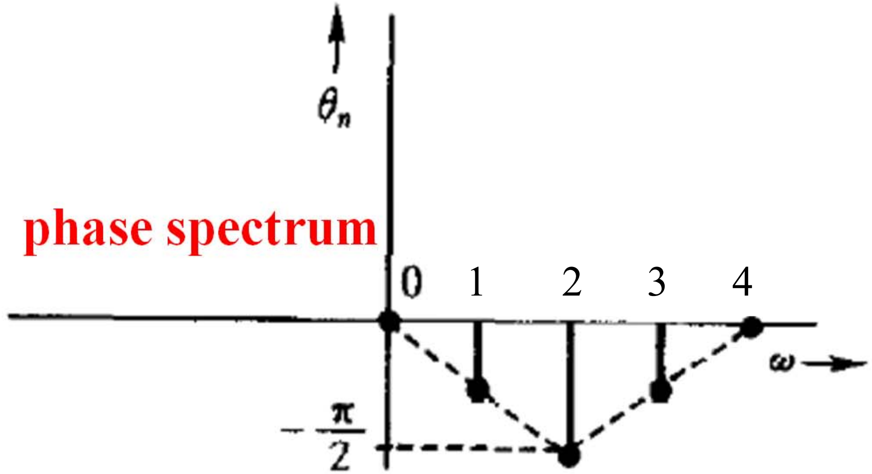
The Fourier Spectrum

$$x(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n) \begin{cases} \text{amplitude spectrum} \Rightarrow C_n \text{ vs. } n\omega_0 \\ \text{phase spectrum} \Rightarrow \theta_n \text{ vs. } n\omega_0 \end{cases}$$

- The plot of amplitude C_n versus $n\omega_0$ is the amplitude spectrum
- The plot of phase θ_n versus $n\omega_0$ is called the phase spectrum.
- The two plots together are the frequency spectra of $x(t)$.

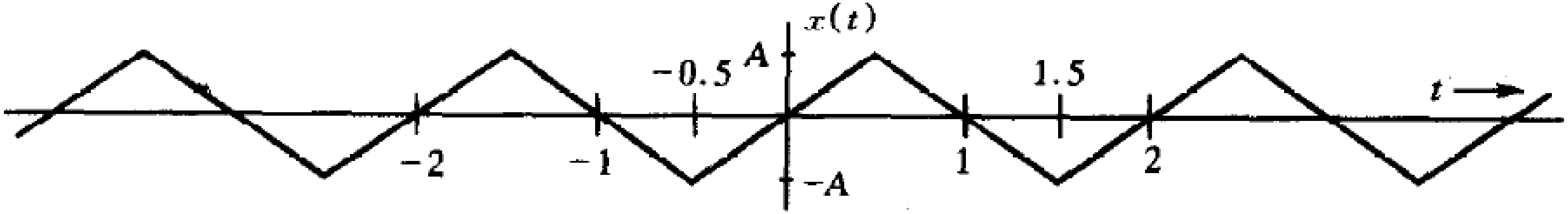


amplitude spectrum



phase spectrum

Example: **Find the compact trigonometric Fourier series for the triangular periodic signal *x*(*t*) shown in Fig (a), and sketch the amplitude and phase spectra for *x*(*t*).**



(a) A triangular periodic signal

$$x(t)=\begin{cases}2At & |t|<\frac{1}{2}\\ 2A(1-t) & \frac{1}{2}<t<\frac{3}{2}\end{cases}$$

$$x(t)=x(t+2)$$

$$T_0=2$$

$$\omega_0=\frac{2\pi}{T_0}=\frac{2\pi}{2}=\pi rad/s$$

$$x(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\pi t + b_n \sin n\pi t)$$

$where:$

$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)dt = 0, \text{ because the average value of } x(t) \text{ is zero}$$

$$a_n = \frac{2}{T_0}\int_{T_0} x(t)\cos n\pi t dt$$

$$= \frac{2}{2}\int_{-1/2}^{1/2} 2At\cos n\pi t dt + \frac{2}{2}\int_{1/2}^{3/2} 2A(1-t)\cos n\pi t dt = 0$$

$$b_n = \frac{2}{T_0}\int_{T_0} x(t)\sin n\pi t dt$$

$$= \frac{2}{2}\int_{-1/2}^{1/2} 2At\sin n\pi t dt + \frac{2}{2}\int_{1/2}^{3/2} 2A(1-t)\sin n\pi t dt = \frac{8A}{n^2\pi^2}\sin(\frac{n\pi}{2})$$

$$b_n = \frac{8A}{n^2\pi^2}\sin(\frac{n\pi}{2}) = \begin{cases} 0 & \text{n=even} \\ \frac{8A}{n^2\pi^2} & \text{n=1,5,9,13,...} \\ -\frac{8A}{n^2\pi^2} & \text{n=3,7,11,15,...} \end{cases}$$

$$\therefore x(t) = \frac{8A}{\pi^2}[\sin \pi t - \frac{1}{3^2}\sin 3\pi t + \frac{1}{5^2}\sin 5\pi t - \frac{1}{7^2}\sin 7\pi t + \cdots]$$

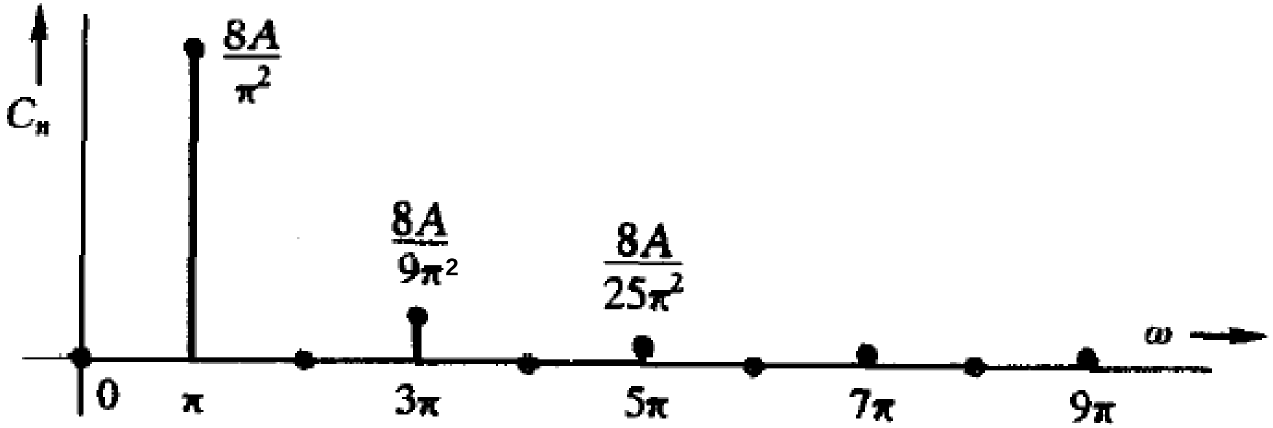
$$\because \quad \sin kt = \cos(kt - 90^\circ), -\sin kt = \cos(kt + 90^\circ)$$

$$x(t) = \frac{8A}{\pi^2}[\sin \pi t - \frac{1}{3^2}\sin 3\pi t + \frac{1}{5^2}\sin 5\pi t - \frac{1}{7^2}\sin 7\pi t + \cdots]$$

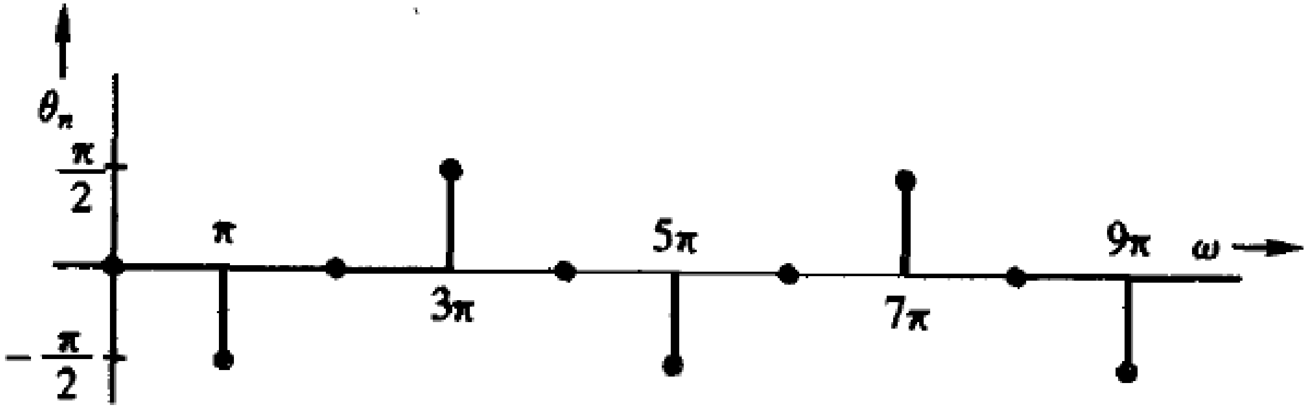
$$= \frac{8A}{\pi^2}[\cos(\pi t - 90^\circ) + \frac{1}{3^2}\cos(3\pi t + 90^\circ) + \frac{1}{5^2}\cos(5\pi t - 90^\circ) + \frac{1}{7^2}\cos(7\pi t + 90^\circ) + \cdots]$$

In this series, all the even harmonics are missing. The phases of the odd harmonics alternate from -90° to 90°.

$$x(t)=\frac{8A}{\pi^2}[\cos(\pi t-90^\circ)+\frac{1}{9}\cos(3\pi t+90^\circ)$$
$$+\frac{1}{25}\cos(5\pi t-90^\circ)+\frac{1}{49}\cos(7\pi t+90^\circ)+\cdots]$$

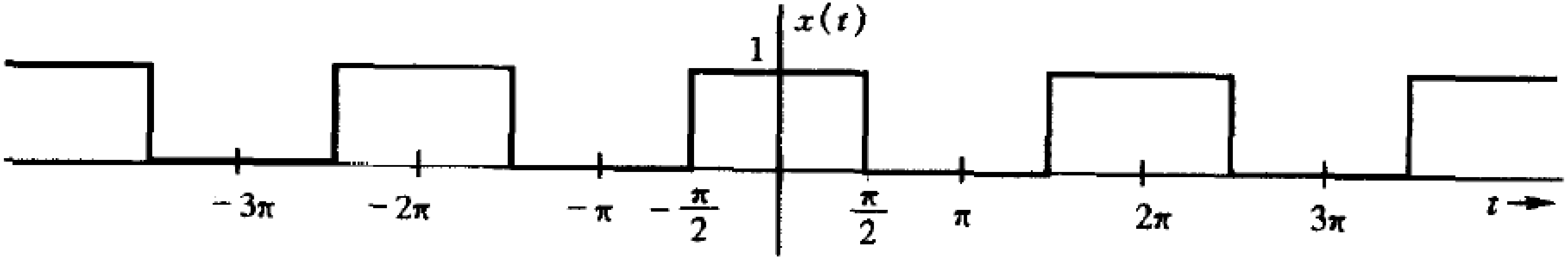


(b) amplitude spectrum of $x(t)$.



(c) phase spectrum of $x(t)$.

Example: **Find the compact trigonometric Fourier series for the square-pulse periodic signal shown below and sketch its amplitude and phase spectra.**



(a) A square pulse periodic signal

$$x(t)=\begin{cases}1 & |t|\le\frac{\pi}{2}\\ 0 & \frac{\pi}{2}<t<\frac{3\pi}{2}\end{cases}$$

$$x(t)=x(t+2\pi)$$

$$T_0=2\pi$$

$$\omega_0=\frac{2\pi}{T_0}=\frac{2\pi}{2\pi}=1rad/s$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

$where:$

$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)dt = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} 1dt = \frac{1}{2}$$

$$a_n = \frac{2}{T_0}\int_{T_0} x(t)\cos ntdt = \frac{2}{2\pi}\int_{-\pi/2}^{\pi/2} 1\cos ntdt = \frac{2}{n\pi}\sin(\frac{n\pi}{2})$$

$$= \begin{cases} 0 & n\text{=even} \\ \dfrac{2}{n\pi} & n\text{=1,5,9,13,...} \\ -\dfrac{2}{n\pi} & n\text{=3,7,11,15,...} \end{cases}$$

$$b_n = \frac{2}{T_0}\int_{T_0} x(t)\sin ntdt = 0$$

$$\therefore x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

$$= \frac{1}{2} + \frac{2}{\pi}(\cos t - \frac{1}{3}\cos 3t + \frac{1}{5}\cos 5t - \frac{1}{7}\cos 7t + \cdots)$$

$$\because \quad -\cos x = \cos(x - \pi)$$

Therefore:

$$x(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$$

$$= \frac{1}{2} + \frac{2}{\pi}[\cos t + \frac{1}{3}\cos(3t - \pi) + \frac{1}{5}\cos 5t + \frac{1}{7}\cos(7t - \pi) + \cdots]$$

$$x(t)=\frac{1}{2}+\frac{2}{\pi}[\cos t+\frac{1}{3}\cos(3t-\pi)+\frac{1}{5}\cos 5t+\frac{1}{7}\cos(7t-\pi)+\cdots]$$

The amplitudes and phases of *x*(*t*) are:

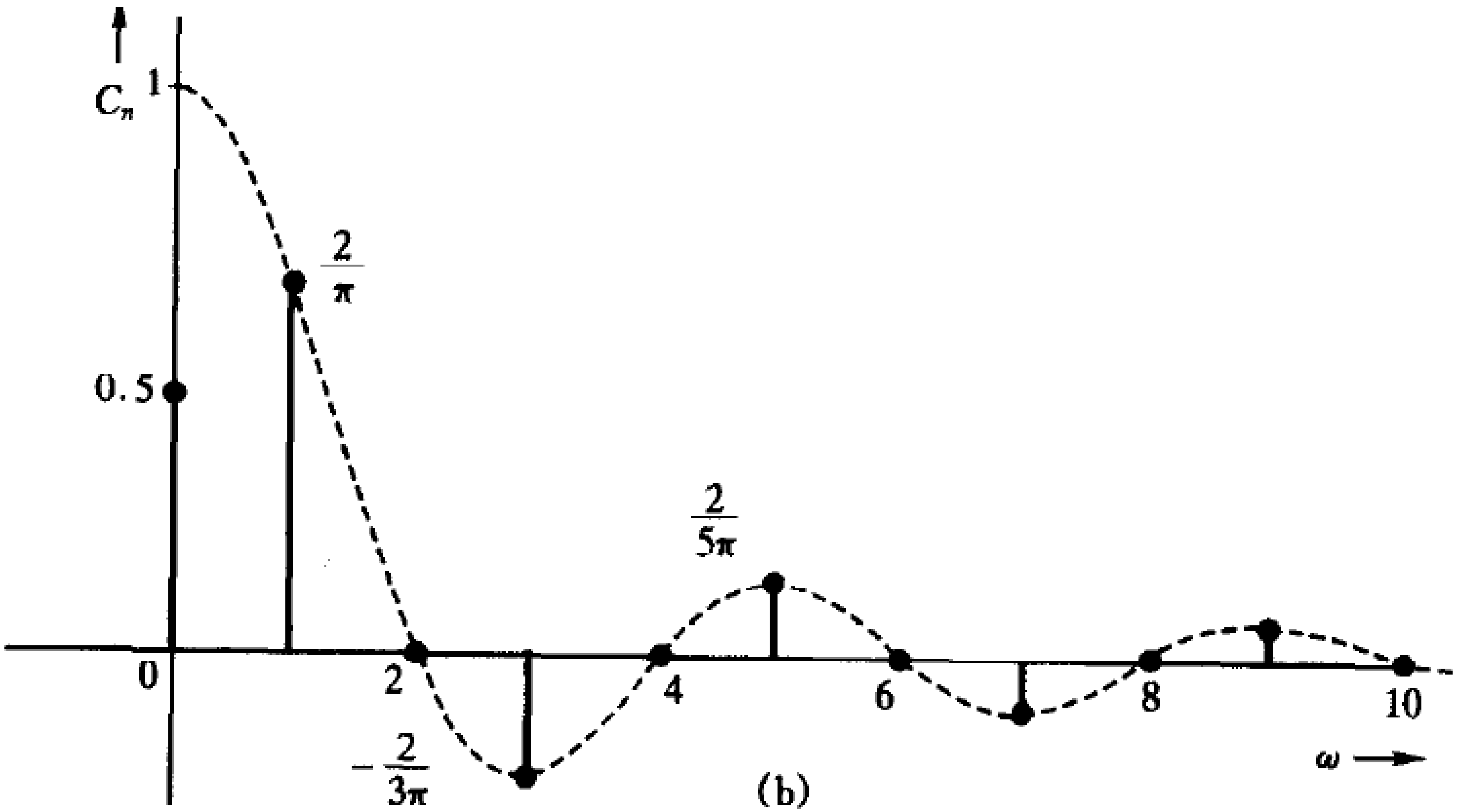
$$C_0=\frac{1}{2}$$

$$C_n=\begin{cases}0 & n=\text{even}\\ \dfrac{2}{n\pi} & n=\text{odd}\end{cases}$$

$$\theta_n=\begin{cases}-\pi & n=3,7,11,15,...\\ 0 & \text{for all } n\neq 3,7,11,15,...\end{cases}$$

We might use these values to plot amplitude and phase spectra.

$$x(t)=\frac{1}{2}+\frac{2}{\pi}[\cos t+\frac{1}{3}\cos(3t-\pi)+\frac{1}{5}\cos 5t+\frac{1}{7}\cos(7t-\pi)+\cdots]$$

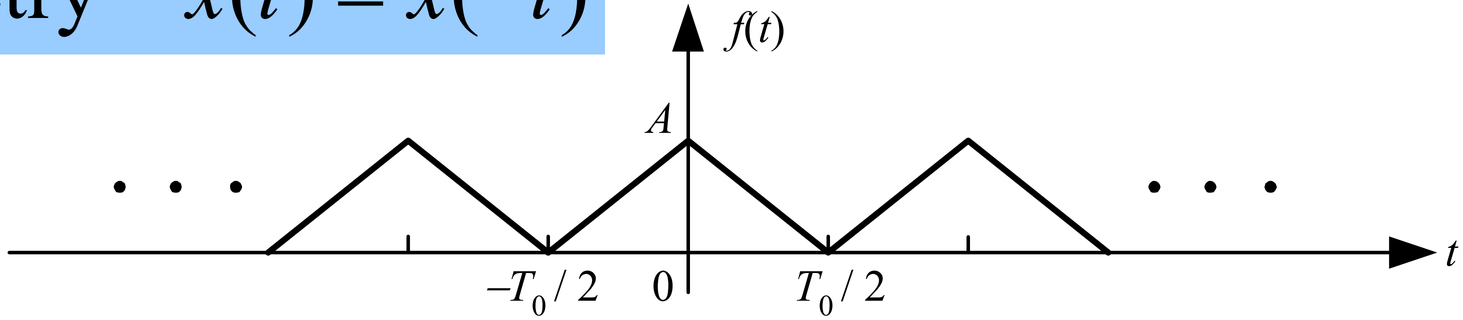


(b) its Fourier spectra

The effect of Symmetry

- Even Symmetry
- Odd Symmetry
- Half-wave Symmetry

Even symmetry $\quad x(t) = x(-t)$



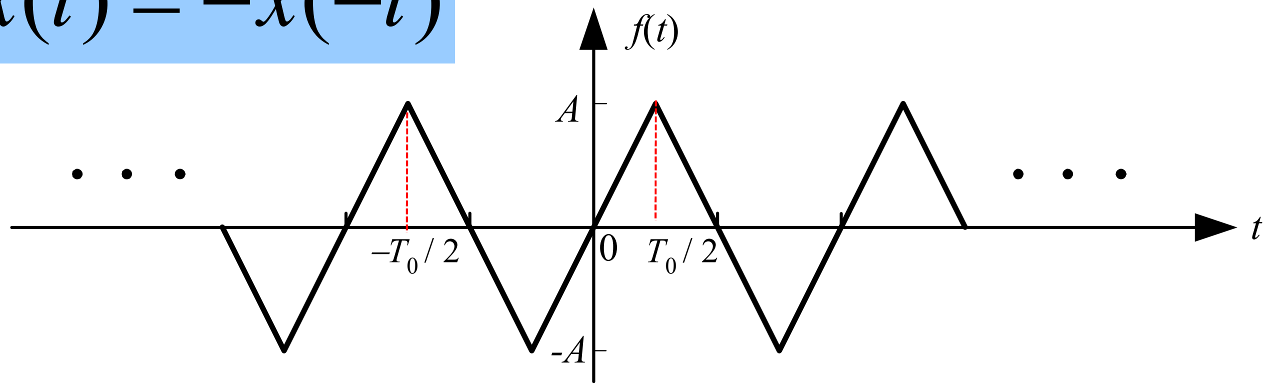
$$a_0 = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)dt = \frac{2}{T_0}\int_0^{\frac{T_0}{2}} x(t)dt$$

$$a_n = \frac{2}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\cos n\omega_0 t dt = \frac{4}{T_0}\int_0^{\frac{T_0}{2}} x(t)\cos n\omega_0 t dt$$

$$b_n = \frac{2}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\sin n\omega_0 t dt = 0$$

Any even periodic function *x*(*t*) only consists of cosine terms.

Odd symmetry $x(t) = -x(-t)$



$$a_0 = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)dt = 0$$

$$a_n = \frac{2}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\cos n\omega_0 t dt = 0$$

$$b_n = \frac{2}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\sin n\omega_0 t dt = \frac{4}{T_0}\int_0^{\frac{T_0}{2}} x(t)\sin n\omega_0 t dt$$

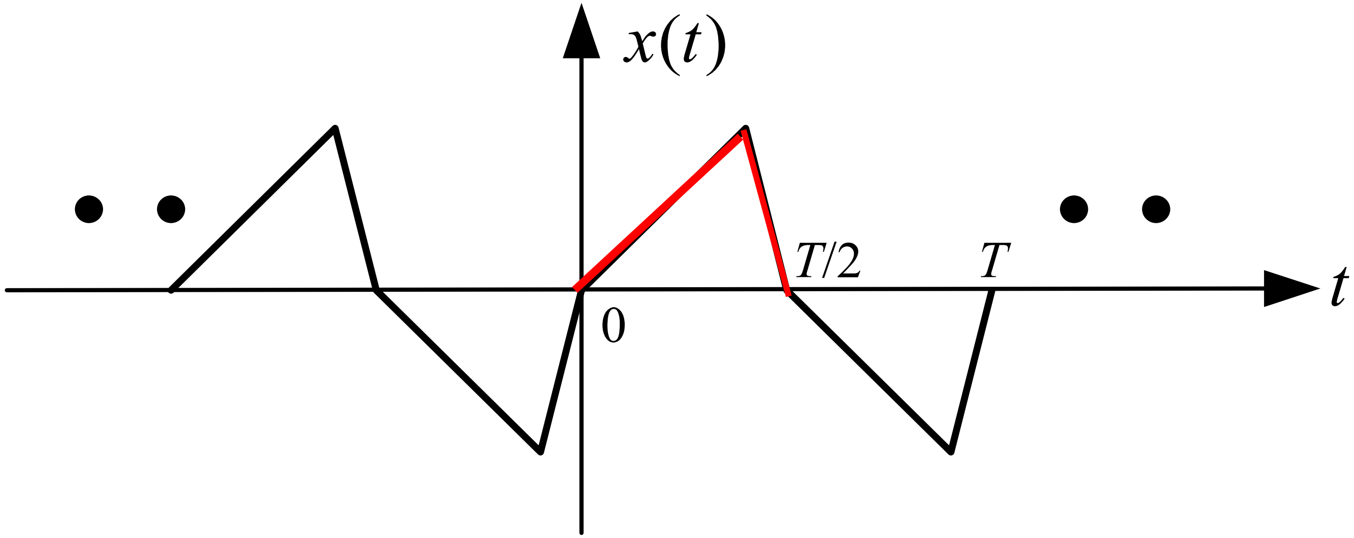
Any odd periodic function *x*(*t*) only consists of sine terms.

Half-wave Symmetry

$$x(t) = -x(t \pm \frac{T_0}{2})$$

$$a_n = \frac{2}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\cos n\omega_0 t dt \quad (n=1,3,\cdots,2n+1)$$

$$b_n = \frac{2}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\sin n\omega_0 t dt \quad (n=1,3,\cdots,2n+1)$$



For a signal with a half-wave symmetry, only the odd-numbered harmonics remain, but all the even-numbered harmonics vanish.

Exponential Fourier Series

The exponential Fourier series can be expressed as

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_o t}$$

$$\boxed{\left\{e^{jn\omega_0 t}, n = 0, \pm 1, \pm 2, \ldots\right\}}$$

$$where \;\; D_n = \frac{1}{T_0}\int_{T_0} x(t) e^{-jn\omega_0 t} dt$$

The principal virtue of the exponential Fourier series:

- **The exponential form of the Fourier series and the expressions for the series coefficients are more compact.**
- **The response of LTIC systems to an exponential input is much simpler than that for a sinusoidal input.**

Relationships among coefficients

$$x(t)=a_0+\sum_{n=1}^{\infty}(a_n\cos n\omega_0 t+b_n\sin n\omega_0 t),\quad x(t)=C_0+\sum_{n=1}^{\infty}C_n\cos(n\omega_0 t+\theta_n),$$

$$x(t)=\sum_{n=-\infty}^{\infty}D_n e^{jn\omega_o t}$$

$$D_0=C_0=a_0$$

$$|D_n|=|D_{-n}|=\frac{1}{2}C_n=\frac{1}{2}\sqrt{a_n^2+b_n^2}$$

$$D_n=|D_n|e^{j\angle D_n}=\frac{1}{2}(a_n-jb_n)$$

$$D_{-n}=|D_{-n}|e^{-j\angle D_n}=\frac{1}{2}(a_n+jb_n)$$

$$C_n^2=a_n^2+b_n^2=4D_nD_{-n}$$

$$|D_n|+|D_{-n}|=C_n,\ D_n+D_{-n}=a_n,\ j(D_n-D_{-n})=b_n$$

Exponential Fourier Spectra

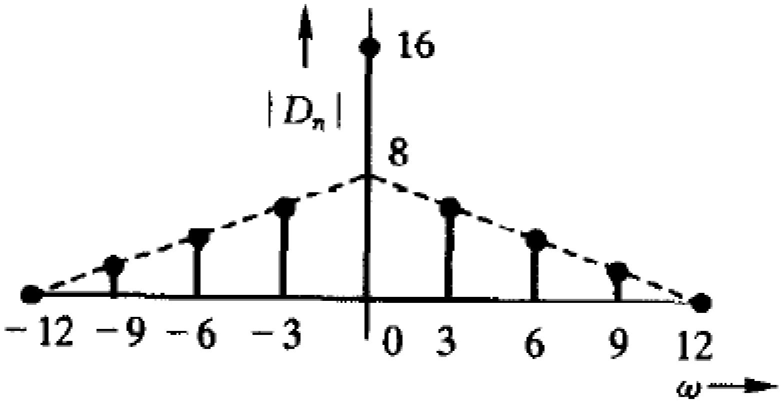
$$x(t)=\sum_{n=-\infty}^{\infty} D_n e^{jn\omega_o t},$$

$$D_n=|D_n|e^{j\angle D_n}$$

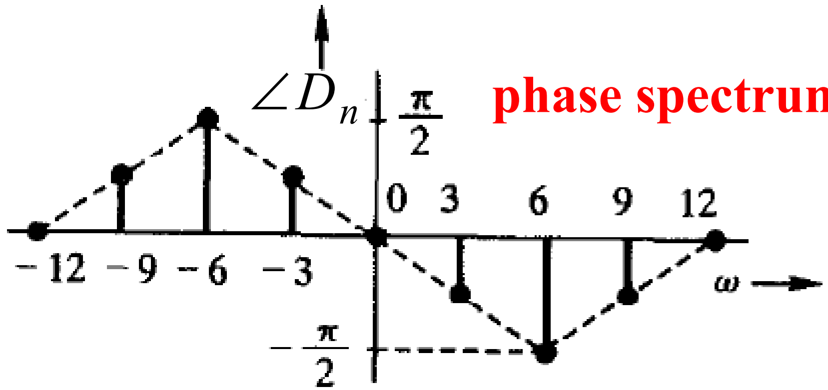
- The frequency of exponentials $e^{\pm j\omega_0 t}$ is indeed $|\omega_0|$.
- Existence of the spectrum at $\omega=-n\omega_0$ is merely an indication that an exponential component $e^{-j\omega_0 t}$ exists in the series.

The plot of $|D_n|$ versus $n\omega_0$ is the amplitude spectrum.
The plot of $\angle D_n$ versus $n\omega_0$ is called the phase spectrum.

The two plots together are the frequency spectra of $x(t)$.

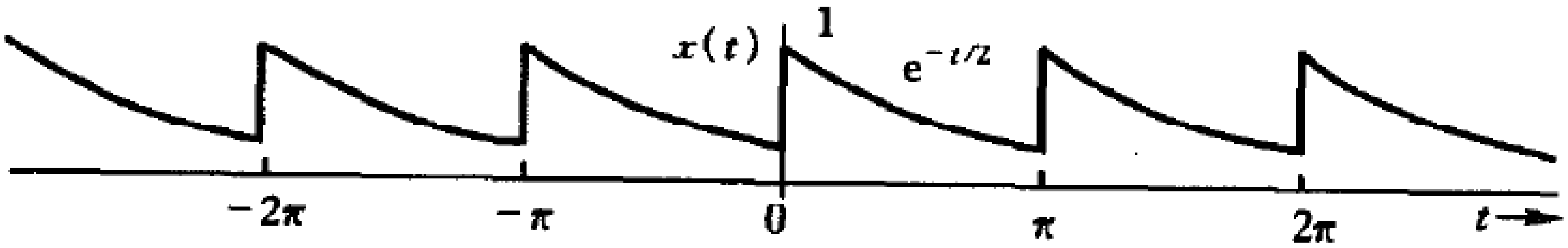


amplitude spectrum



phase spectrum

Example: **Find the exponential Fourier series for the signal shown below and sketch the amplitude and phase spectra for *x*(*t*).**



$$x(t) = e^{-t/2}, 0 \le t \le \pi$$

$$x(t) = x(t+\pi)$$

$$T_0 = \pi, f_0 = \frac{1}{\pi}$$

$$\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{\pi} = 2rad/s$$

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_o t} = \sum_{n=-\infty}^{\infty} D_n e^{j2nt}$$

where

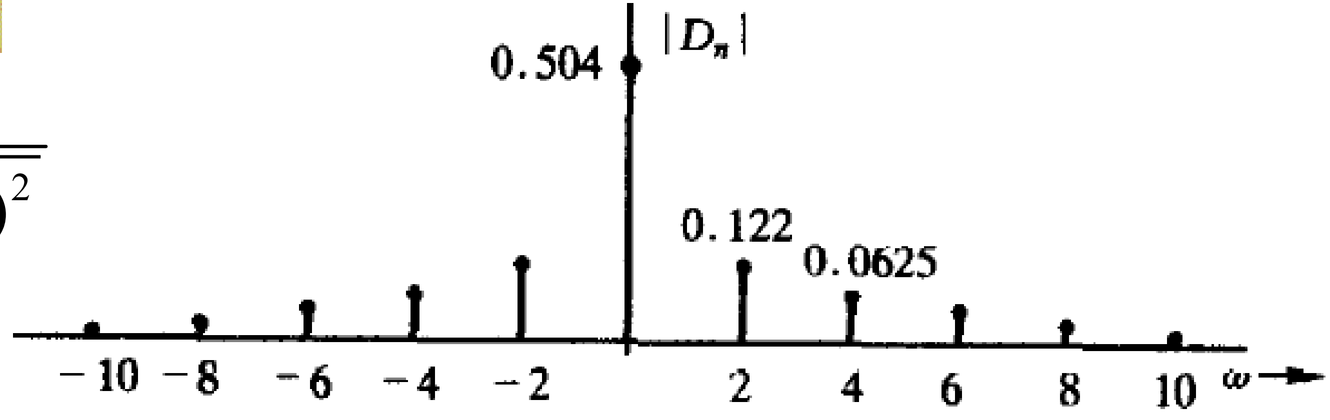
$$D_n = \frac{1}{T_0}\int_{T_0} x(t)e^{-j2nt}dt = \frac{1}{\pi}\int_0^{\pi} e^{-t/2}e^{-j2nt}dt$$

$$= \frac{1}{\pi}\int_0^{\pi} e^{-(1/2+j2n)t}dt$$

$$= \frac{-1}{\pi(\frac{1}{2}+j2n)} e^{-(1/2+j2n)t} \Big|_0^{\pi} = \frac{0.504}{1+j4n} = \frac{0.504}{\sqrt{1+(4n)^2}} e^{j\arctan(-4n)}$$

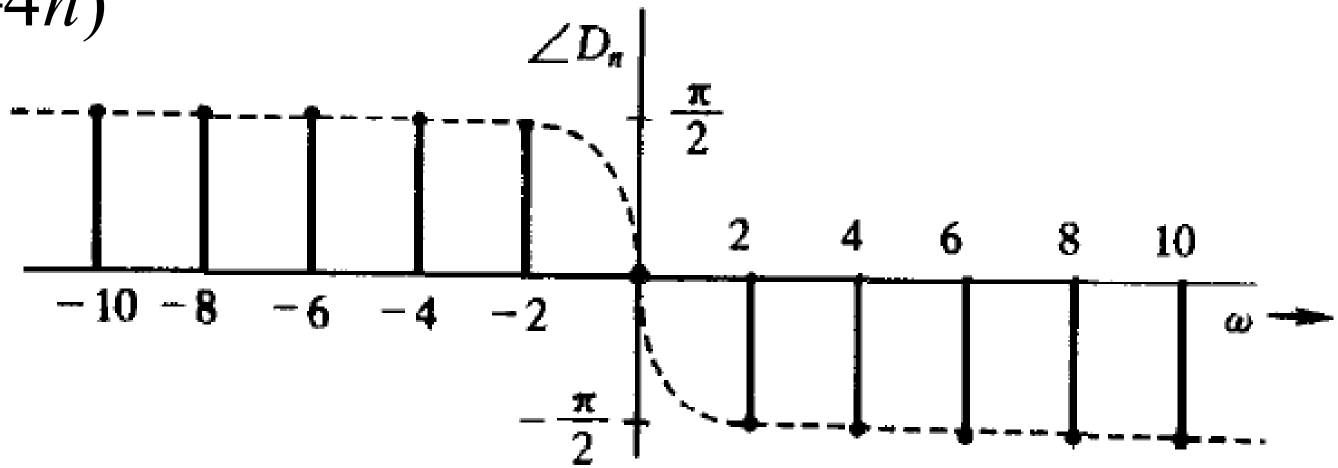
Therefore， $x(t) = \sum_{n=-\infty}^{\infty} \frac{0.504}{1+j4n} e^{j2nt}$

$$\left|D_n\right| = \frac{0.504}{\sqrt{1+(4n)^2}}$$



(a) **amplitude spectrum**

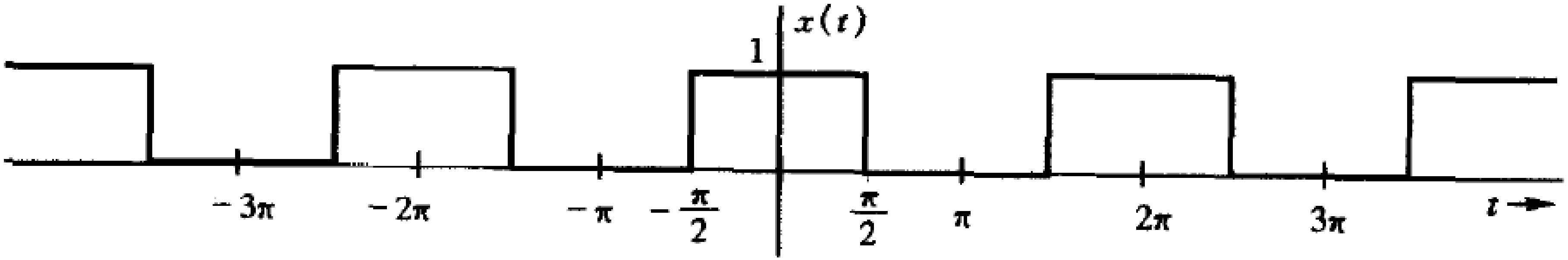
$$\angle D_n = \arctan(-4n)$$



(b) **phase spectrum**

- The spectra exist for positive as well as negative values of ω.
- The amplitude spectrum is an even function of ω and the phase spectrum is an odd function of ω.

Example: **Find the exponential Fourier series for the square-pulse periodic signal shown below and sketch its amplitude and phase spectra.**



(a) A square pulse periodic signal

$$x(t) = \begin{cases} 1 & |t| \le \dfrac{\pi}{2} \\ 0 & \dfrac{\pi}{2} < t < \dfrac{3\pi}{2} \end{cases}$$

$$x(t) = x(t + 2\pi)$$

$$T_0 = 2\pi$$

$$\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{2\pi} = 1 rad / s$$

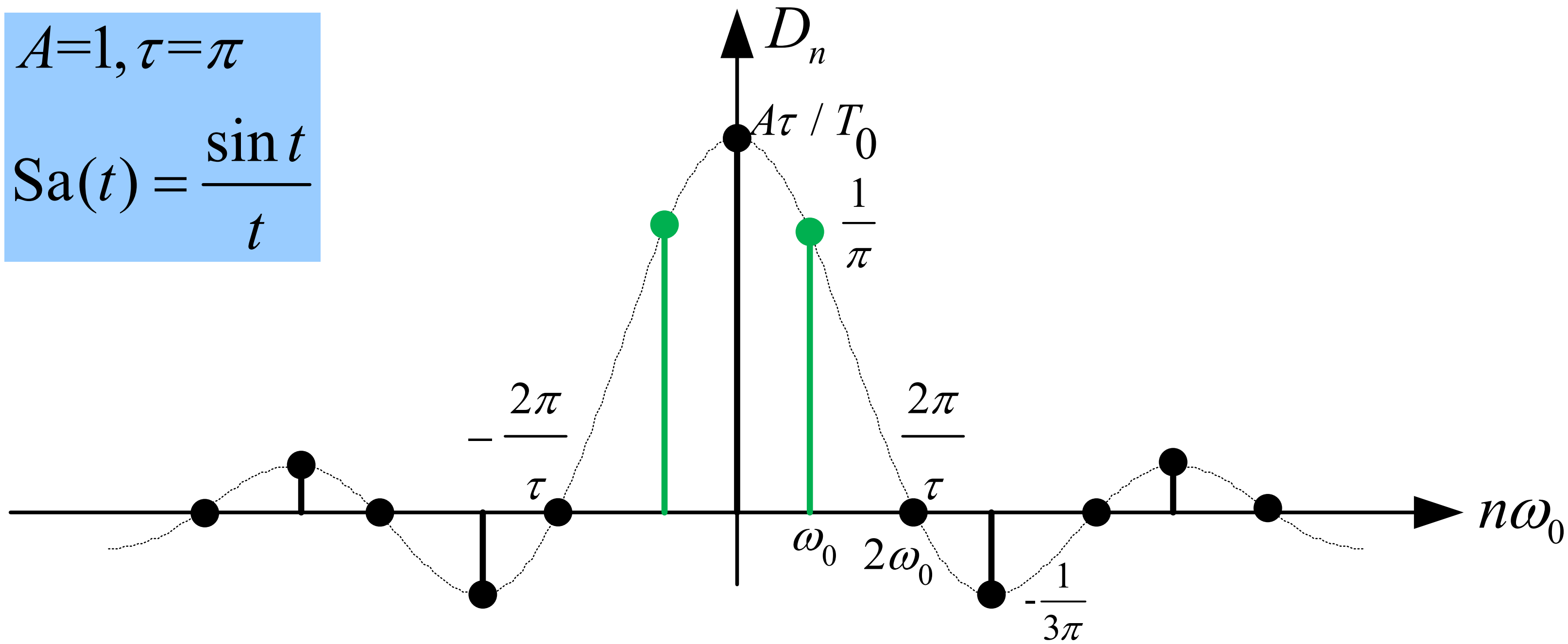
$$D_n = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)e^{-jn\omega_0 t}dt = \frac{1}{T_0}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} Ae^{-jn\omega_0 t}dt = \frac{A}{T_0(-jn\omega_0)}e^{-jn\omega_0 t}\Bigg|_{t=-\frac{\tau}{2}}^{t=\frac{\tau}{2}}$$

$$= \frac{A\tau\sin\left(\frac{n\omega_0\tau}{2}\right)}{T_0\frac{n\omega_0\tau}{2}}$$

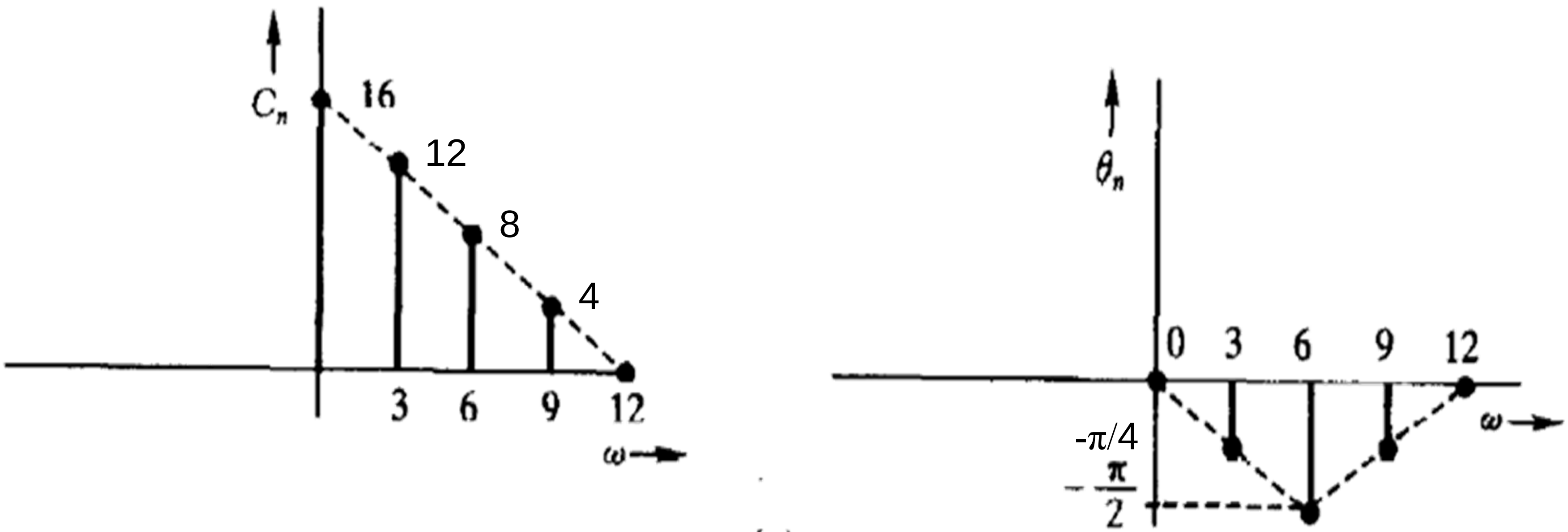
$$x(t) = \sum_{n=-\infty}^{\infty} D_n\, e^{jn\omega_0 t} = \frac{A\tau}{T_0}\sum_{n=-\infty}^{\infty} Sa\left(\frac{n\omega_0\tau}{2}\right)e^{jn\omega_0 t}$$

$A=1, \tau=\pi$

$$\mathrm{Sa}(t) = \frac{\sin t}{t}$$



Example: **The compact trigonometric Fourier spectra of a periodic signal $x(t)$ are shown below. By inspecting these spectra, sketch the corresponding exponential Fourier spectra and verify your results analytically.**



(a)

$x(t)$ has four spectral components of frequencies 0, 3, 6, and 9. The dc component is 16.

$$x(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$$

$$= 16 + 12\cos(3t - \frac{\pi}{4}) + 8\cos(6t - \frac{\pi}{2}) + 4\cos(9t - \frac{\pi}{4})$$

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_o t} = \sum_{n=-\infty}^{\infty} |D_n| e^{j\theta_n} e^{jn\omega_o t} = \sum_{n=-\infty}^{\infty} |D_n| e^{j(n\omega_o t + \theta_n)}$$

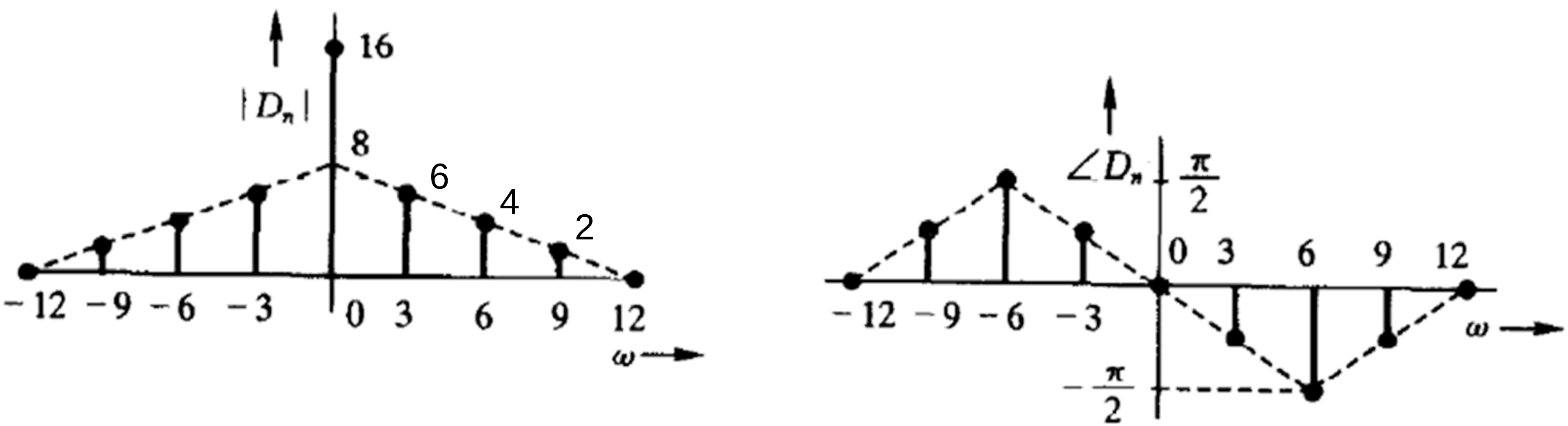
$$D_0 = C_0$$

$$|D_n| = |D_{-n}| = \frac{1}{2} C_n$$

The exponential spectral components exist at 0 (dc), ±3, ±6, and ±9. The dc component is D_0 = 16.

$$x(t) = 16 + 6\left[e^{j(3t-\pi/4)} + e^{-j(3t-\pi/4)}\right]$$

$$+4\left[e^{j(6t-\pi/2)} + e^{-j(6t-\pi/2)}\right] + 2\left[e^{j(9t-\pi/4)} + e^{-j(9t-\pi/4)}\right]$$



(b)

Fourier Series Representation of a Periodic Signal of Period T_0

Series Form	Coefficient Computation	Conversion Formulas
Trigonometric	$a_0 = \frac{1}{T_0}\int_{T_0} f(t)\,dt$	$a_0 = C_0 = D_0$
$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$	$a_n = \frac{2}{T_0}\int_{T_0} f(t)\cos n\omega_0 t\,dt$	$a_n - jb_n = C_n e^{j\theta_n} = 2D_n$
	$b_n = \frac{2}{T_0}\int_{T_0} f(t)\sin n\omega_0 t\,dt$	$a_n + jb_n = C_n e^{-j\theta_n} = 2D_{-n}$
Compact trigonometric	$C_0 = a_0$	$C_0 = D_0$
$f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$	$C_n = \sqrt{a_n{}^2 + b_n{}^2}$	$C_n = 2\|D_n\| \quad n \geq 1$
	$\theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right)$	$\theta_n = \angle D_n$
Exponential		
$f(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$	$D_n = \frac{1}{T_0}\int_{T_0} f(t)e^{-jn\omega_0 t}\,dt$	

Existence and Convergence of the Fourier Series

Existence :

Coefficients must be finite for the existence of the Foutier series.

The existence condition of these finite coefficients:

$$\int_T |x(t)|\, dt < \infty$$

Convergence :

- **Converge uniformly**

A series guarantees convergence within an arbitrarily small error at every value of *t* using a fixed number of terms.

- **Converge pointwise**

A series converges at every value of *t*, but to guarantee convergence within a given error requires different number of terms at different *t*.

- **Converge in the mean**

A series refuses to converge at some t, no matter how many terms are added. But the energy of the difference E_N between $x(t)$ and the corresponding finite term series $x_N(t)$ approaches zero as the number of terms approaches infinity.

An approximation of $x(t)$: $$x_N(t) = \sum_{k=-N}^{+N} a_k e^{jk\omega_0 t}$$ **(Finite series)**

Approximation error: $$e_N(t) = x(t) - x_N(t) = x(t) - \sum_{k=-N}^{+N} a_k e^{jk\omega_0 t}$$

The energy of the error over one period T: $$E_N = \int_T |e_N(t)|^2 dt$$

$$E_N \to 0 \quad as \quad N \to \infty$$

Condition: **The signals which have finite energy over a single period are representable through the Fourier series.**

$$\int_T |x(t)|^2\, dt < \infty$$

When this condition is satisfied, E_N converges to 0 as $N \to \infty$.

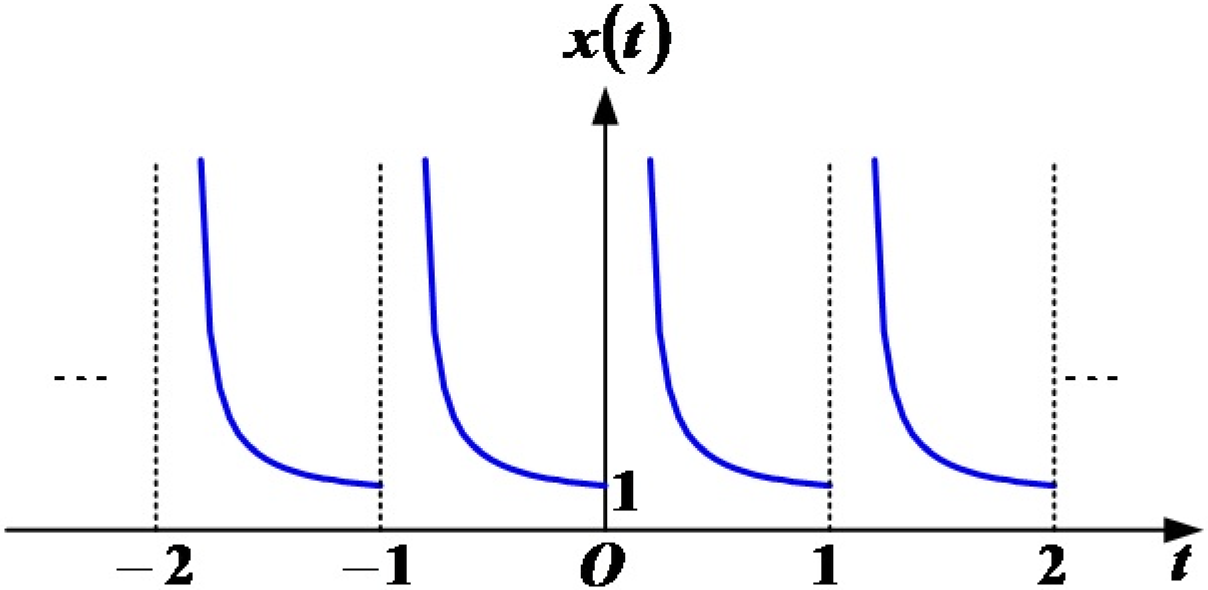
The corresponding Fourier Series converges in the mean.

This condition guarantees that the Fourier coefficients are finite.

Dirichlet conditions:

Condition 1: Over any period, $x(t)$ must be absolutely integrable.

$$\int_T |x(t)|\, dt < \infty$$

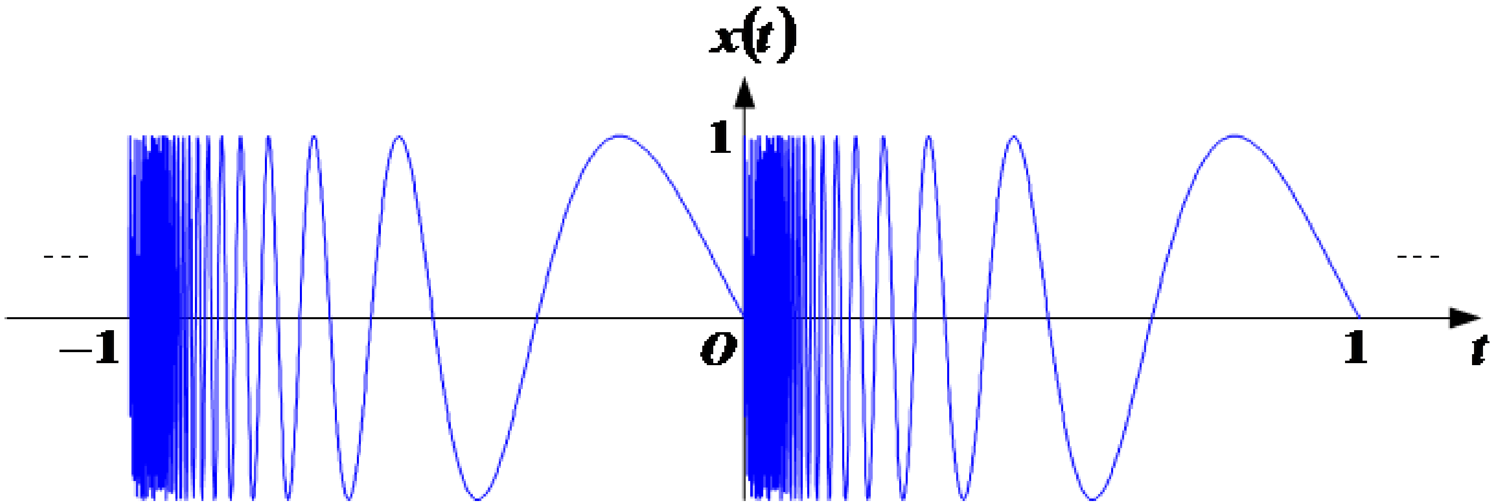


$$x(t) = \frac{1}{t}$$

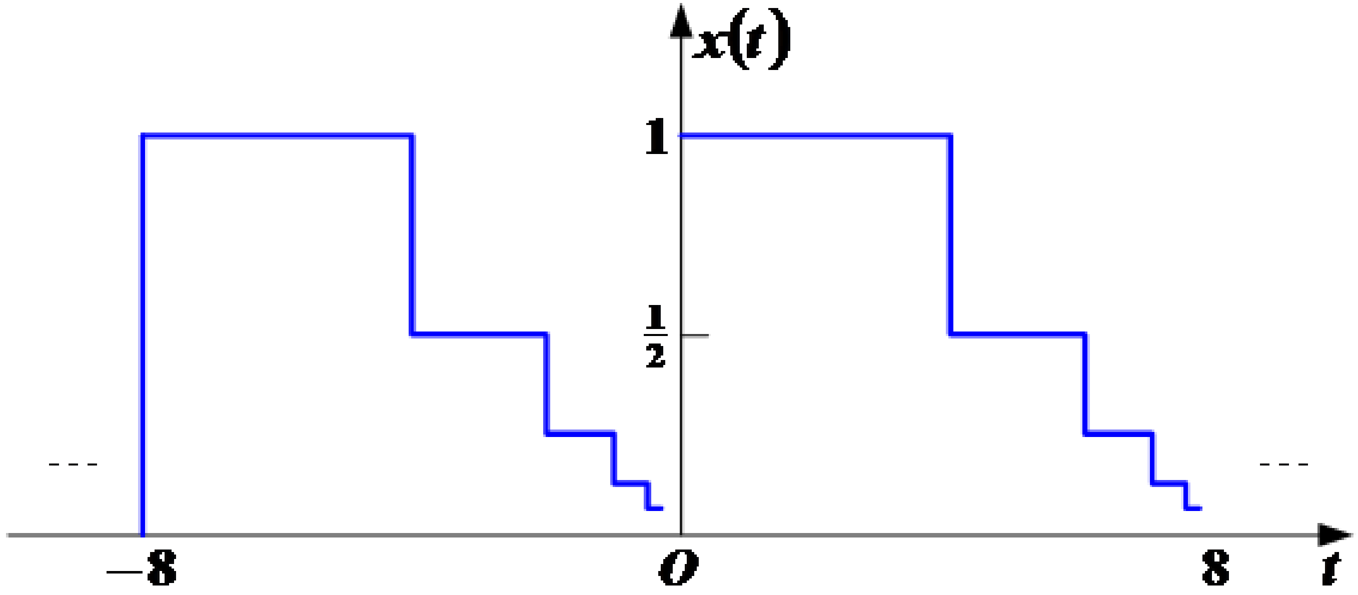
$$0 < t \le 1$$

Condition 2: $x(t)$ must contain only a finite number of maxima and minima in one period.

$$x(t)=\sin\left(\frac{2\pi}{t}\right),(0<t\le 1)$$



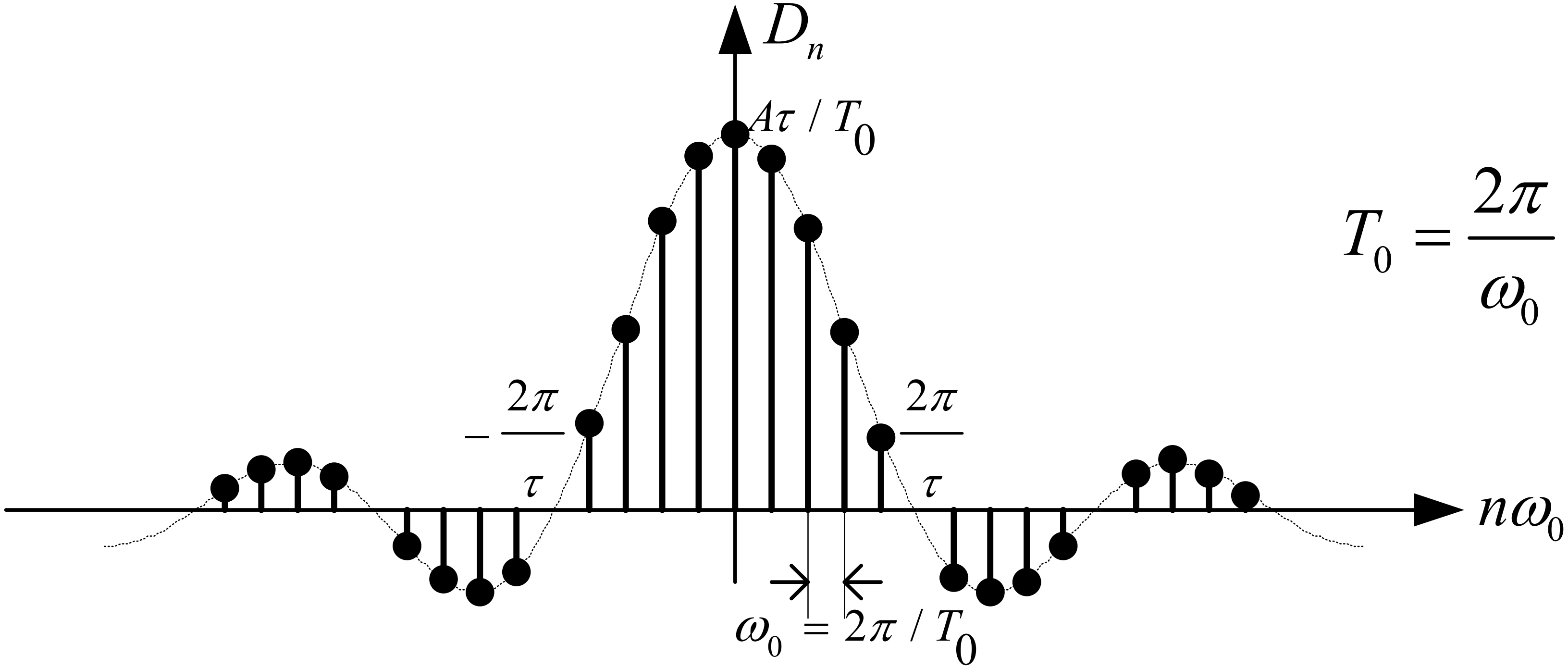
Condition 3: $x(t)$ must have only a finite number of finite discontinuities in one period.



When this condition is satisfied, the corresponding Fourier Series is guaranteed to converge pointwise at all points where $x(t)$ is continuous. Moreover, at the points of discontinuities, $x(t)$ converges to the value midway between the two values of $x(t)$ on either sides of the discontinuity.

Note: The Dirichlet conditions are sufficient for the existence and convergence of the Fourier series, but they are not necessary.

Properties of Fourier Spectra



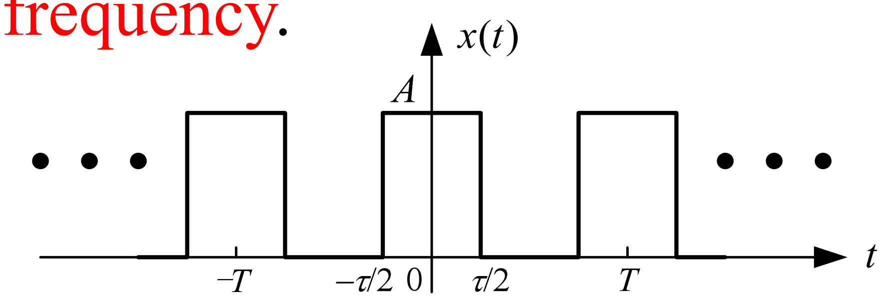
- The spectra of a continuous-time periodic signal consist of many discrete lines that only appear on $n\omega_0$.
- The length of the interval between two lines is $\omega_0=2\pi/T_0$.
- The amplitude spectrum indicates the amounts (amplitudes) of various frequency components of $x(t)$.

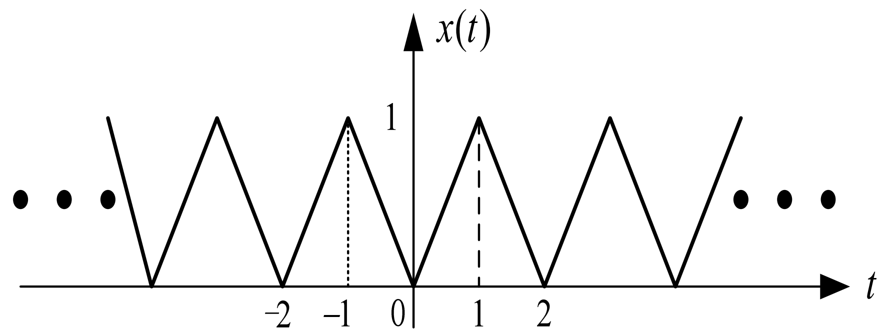
Asymptotic rate of amplitude spectrum decay

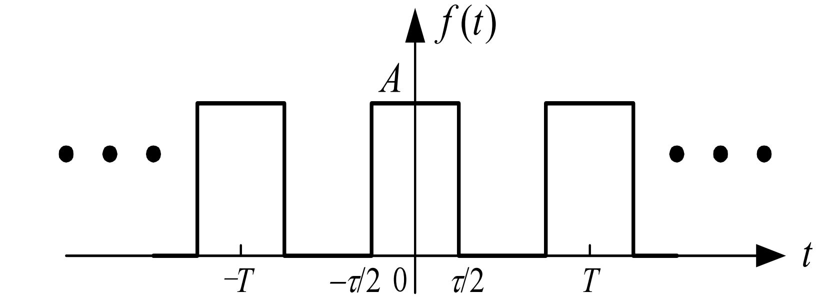
➤ Lower frequencies in the Fourier series affect the large-scale behavior of $x(t)$, whereas the higher frequencies determine the details of the shape of a signal.

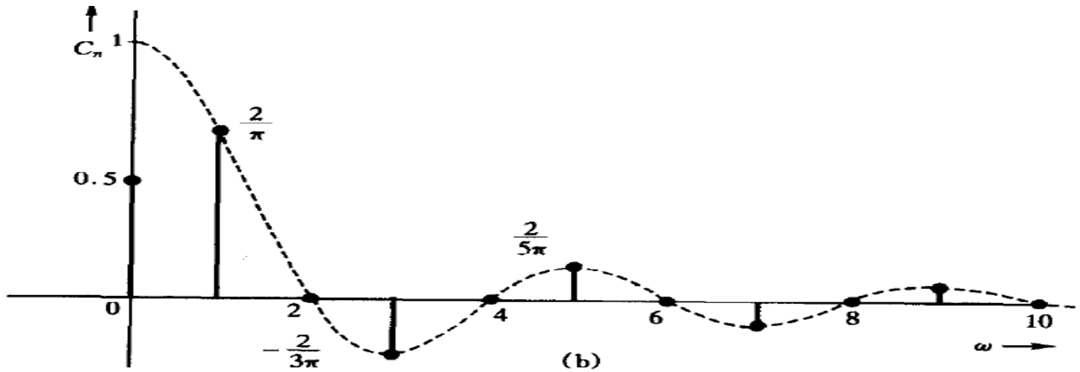
➤ If $x(t)$ is a smooth function, its synthesis requires predominantly lower-frequency sinusoids and relatively small amounts of higher-frequency sinusoids. The amplitude spectrum of such a function would decay swiftly with frequency.

➤ If there is a signal with sharp changes, its synthesis requires relatively large amount of high frequency components. The amplitude spectrum of such a signal would decay slowly with frequency.

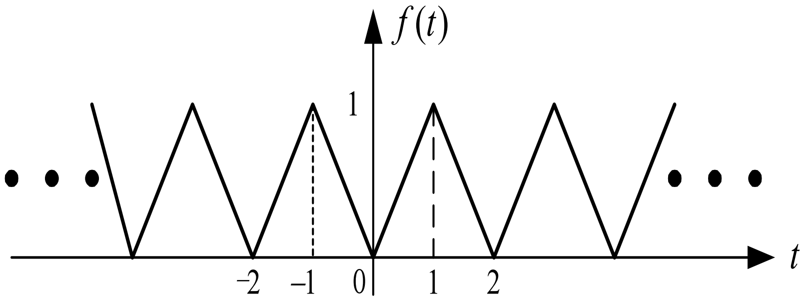


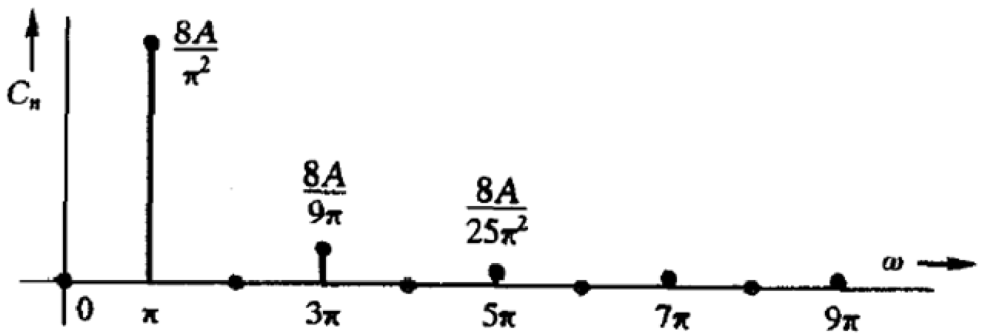






➤ The square wave $x(t)$ is a signal with jump discontinuities, and therefore its amplitude spectrum decays rather slowly, as $1/n$.

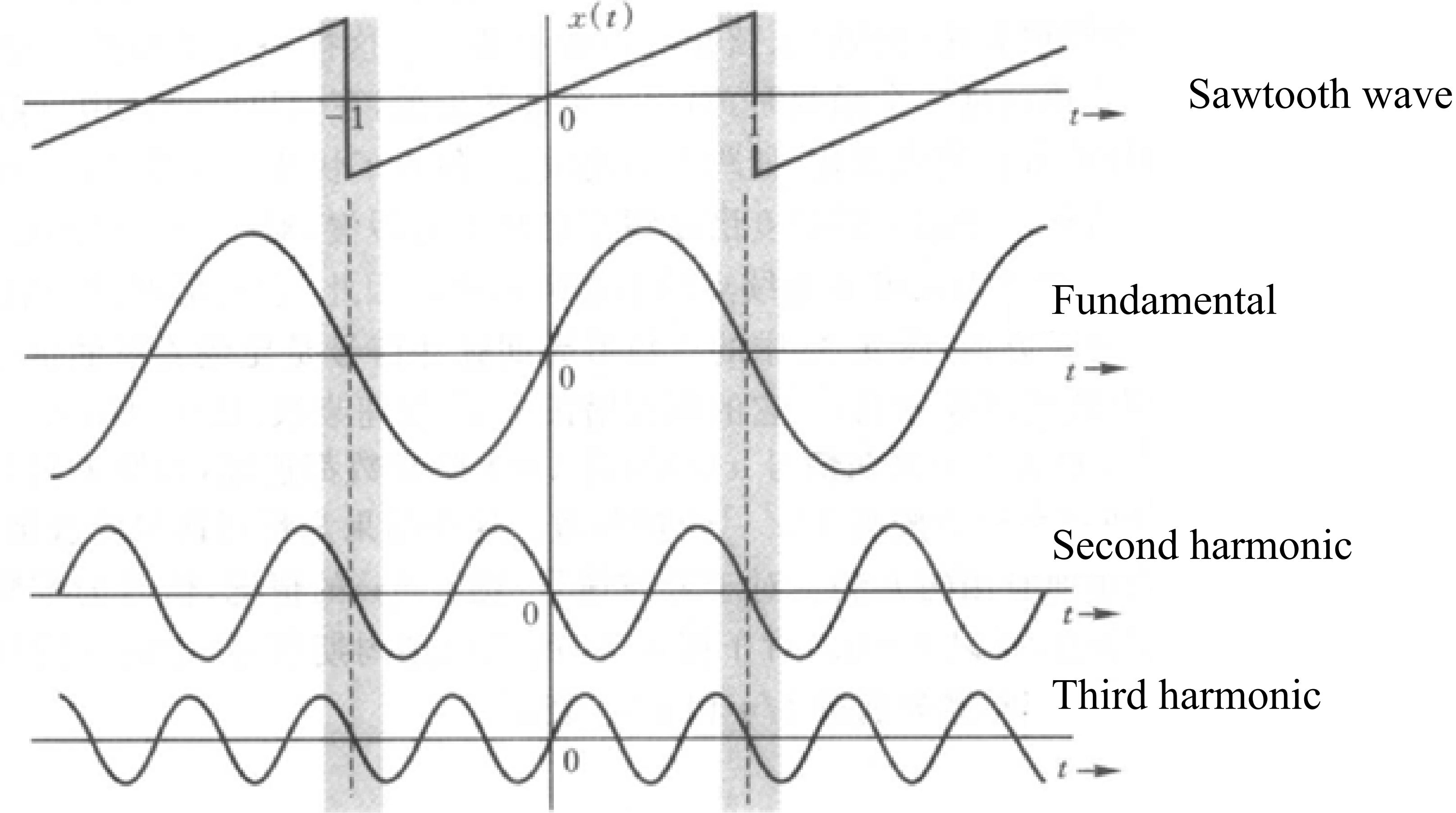




➤ The triangle wave is smoother because it is a signal without any jump discontinuities. Its spectrum decays rapidly with frequency as $1/n^2$.

Effects of phase spectrum in waveshaping

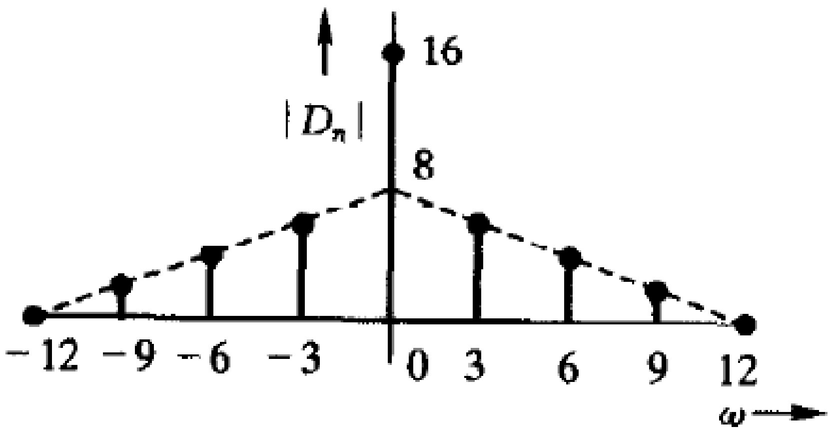
Considering a signal $x(t)$ that has jump discontinuities

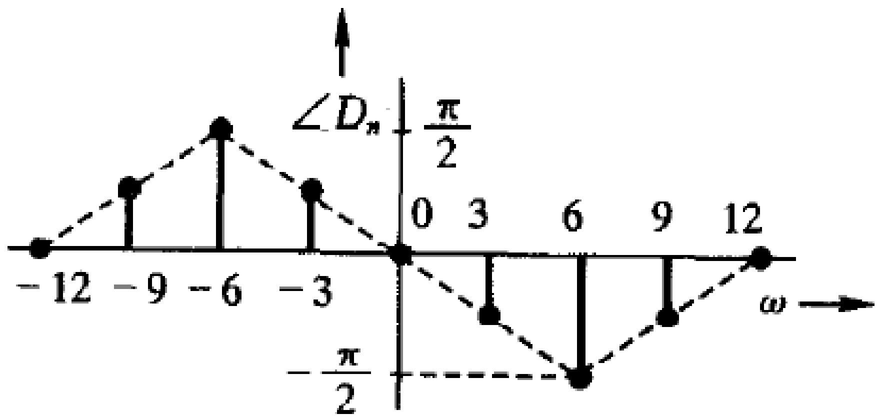


All of the harmonic components have one sign before the discontinuity and the opposite sign after the discontinuity. This sign change in all the harmonics adds up to produce very nearly a jump discontinuity.

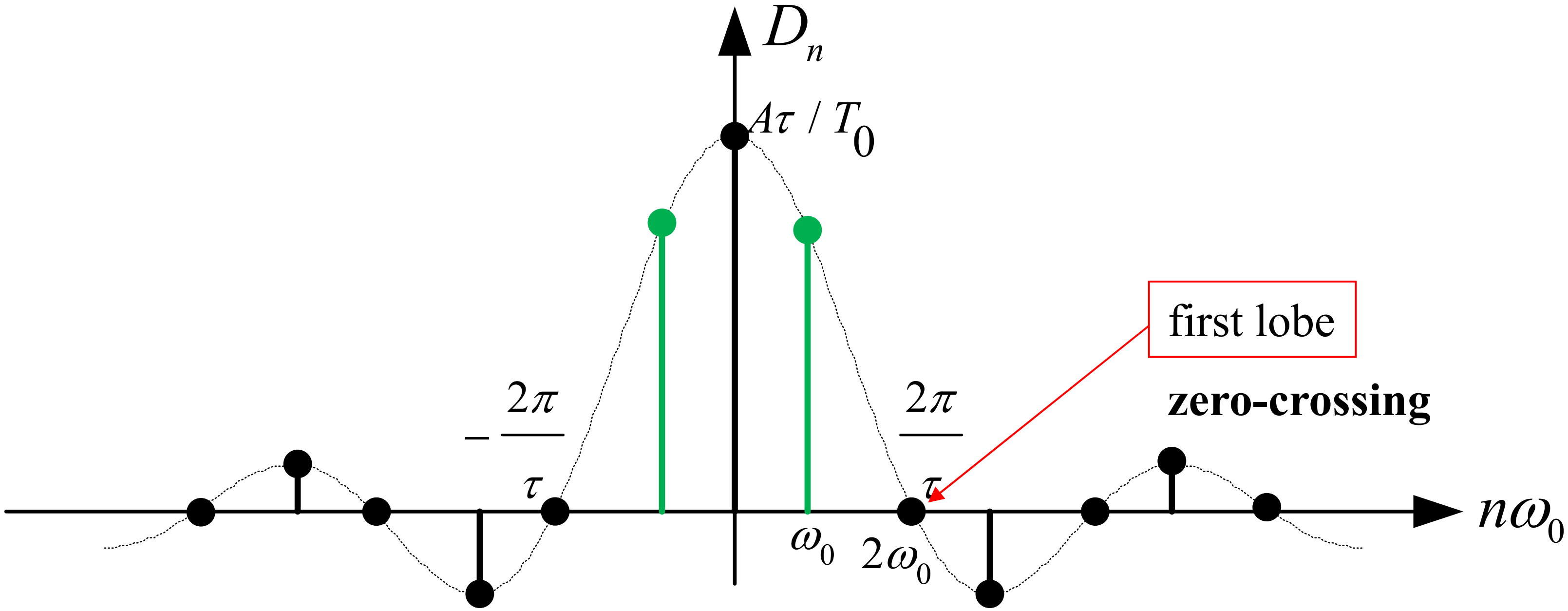
Bandwidth of a Signal

The difference between the highest and the lowest frequencies of the spectral components of a signal is the *bandwidth of the signal*.





The highest and lowest frequencies are 9 and 0, respectively. Therefore the bandwidth of this signal is 9.

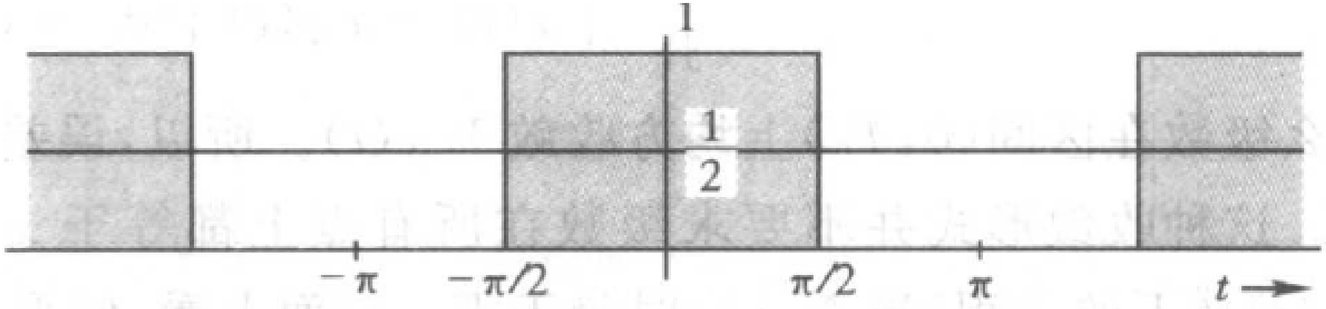


What is the bandwidth of this signal?

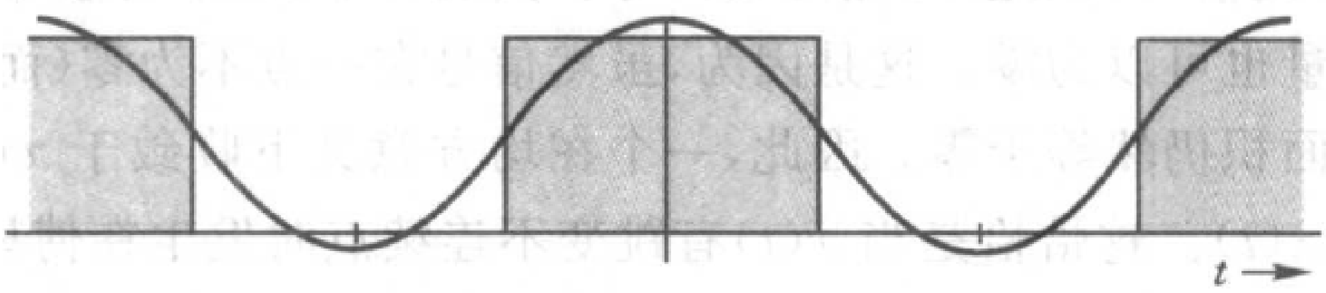
The spectrum extends from 0 to ∞, so the bandwidth is ∞.

Most of the spectrum is concentrated within the first lobe (from ω=0 to ω=2π/τ). Therefore, a rough estimate of the bandwidth of the signal is 2π/τ rad/s.

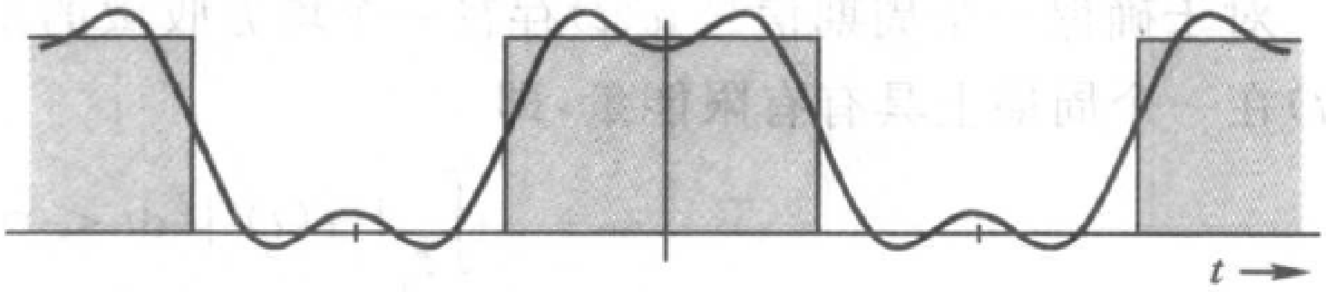
Gibbs phenomenon



(a) One term



(b) Two terms



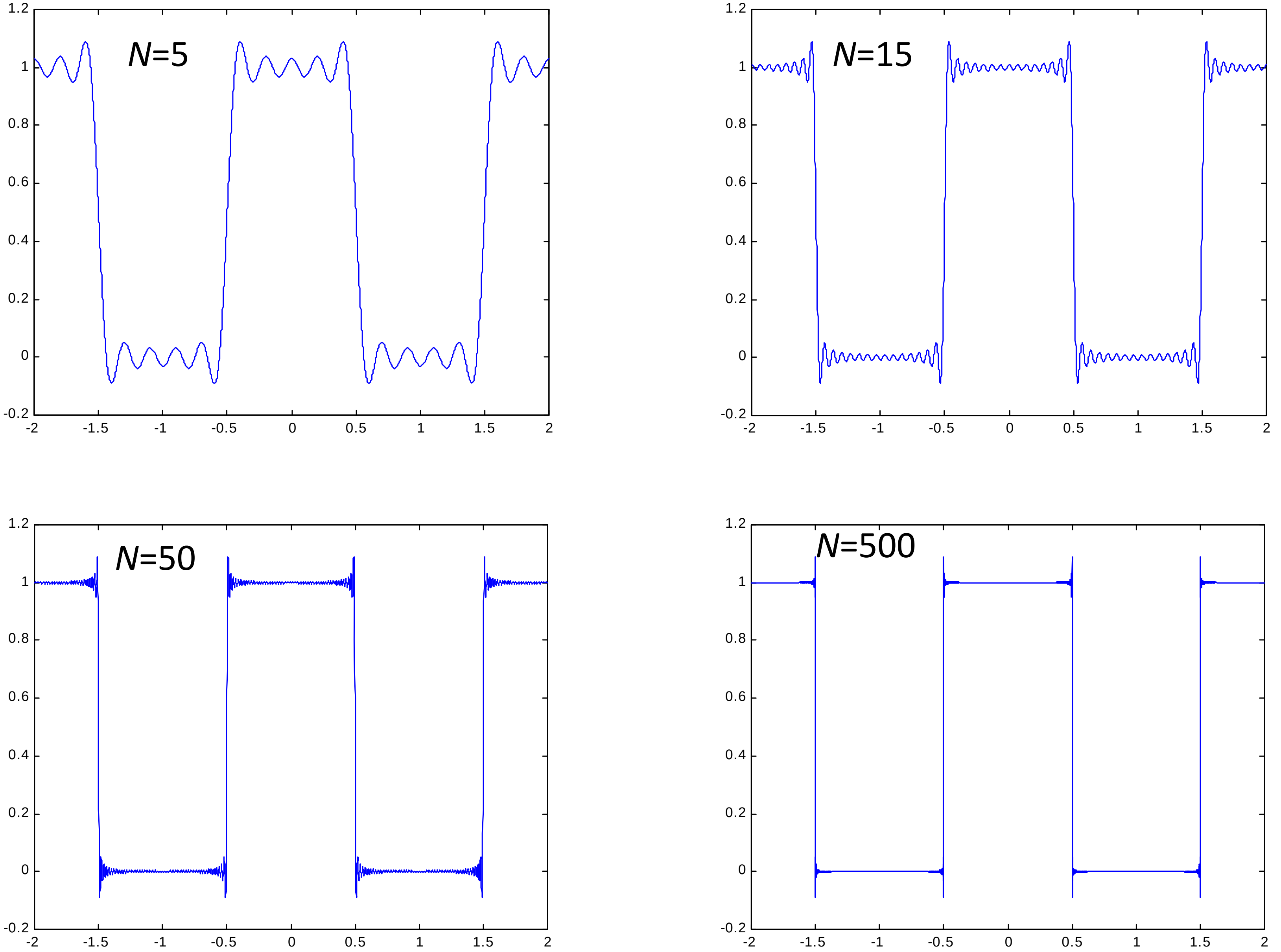
(c) Three terms

$x(t)$ is a square wave

$$x(t) = \frac{1}{2} + \frac{2}{\pi}[\cos t + \frac{1}{3}\cos(3t-\pi) + \frac{1}{5}\cos 5t + \frac{1}{7}\cos(7t-\pi) + \cdots]$$

If using the partial sum of the first N terms, $x_N(t)$ to synthesize $x(t)$, as $N \to \infty$,

$$x_N(t) \xrightarrow{?} x(t)$$


N=5
N=15
N=50
N=500
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

- If $x(t)$ is a signal with discontinuities, there is an overshoot of about 9% in the vicinity of the discontinuity at the nearest peak of oscillation, which does not vanish as the number of terms increases.
- If $x(t)$ is a signal without discontinuities, no *Gibbs phenomenon* appears.

The reason for *Gibbs phenomenon*

The Fourier series converges to the mean of the left-hand and right-hand limits of $x(t)$ at the discontinuities.