Journal of Quantitative Spectroscopy & Radiative Transfer 274 (2021) 107866

Contents lists available at ScienceDirect

Journal of Quantitative Spectroscopy & Radiative Transfer

journal homepage: www.elsevier.com/locate/jqsrt

View factors of spherical, conic, and cylindrical spiral surfaces

Vladimir A. Lebedev[a], Vladimir P. Solovjov[b], Brent W. Webb[b,*]

[a] S.S. Kutateladze Institute of Thermophysics, Russian Academy of Sciences, 1, Acad. Lavrentyev Ave., Novosibirsk 630090, Russia
[b] Brigham Young University, EB-360, Provo, UT 84602, USA

A R T I C L E I N F O

Article history:
Received 1 July 2021
Revised 2 August 2021
Accepted 2 August 2021
Available online 5 August 2021

Keywords:
View factors
Spherical spirals
Conic spirals
Cylindrical spirals
Surface parameterization

A B S T R A C T

The view factors of a family of spiral surfaces of different type are derived using parameterization and differential geometry methods. Analytical results are presented for the view factors of the conic, spherical and cylindrical spiral surfaces.

1. Introduction

The view factor between two surfaces is defined as a fraction of the total radiant energy emitted/reflected diffusely by one surface and intercepted by the second surface. The view factor is a purely geometrical parameter, and includes no radiative property information for the surfaces [1]. Classically, view factors are needed for radiation heat transfer analysis in systems consisting of radiatively interacting surfaces which enclose non-participating media, for computer graphics and architectural design, etc. There is a long history in the development of view factors for different geometries and configurations. The most extensive collection of view factors is given as supplemental material in Howell at al. [1], and it is available for access online [2]. Despite the considerable body of cases presented in the online catalog, there are still some important and relevant configurations for which view factors have not developed. With the exception of a single cylindrical spiral [C-78a, 2], view factors of different types of spirals have not yet been studied. In the present paper, we investigate three spiral surfaces which have relevance in different technical applications and are thus of practical interest. An analytical approach based on view factor algebra, invariant principles, and similarity properties of view factors [3–6] is used for development of the view factors in the present work. Rigorous mathematical definition of all surfaces considered with the help of parametric equations is employed. All of the parameterizations used in the derivation of the view factors for these spiral configurations are original to this work. Some details related to the calculation of the surface areas using differential geometry analysis are given in the Appendix.

2. Spherical spirals

There are many different types of the spherical spirals such as those shown in the Fig. 1 which are of technical relevance. For example, a balanced ball antenna using a spherical helix [7], a spherical photon cage, and 3D optical micro-resonators [8] all feature geometric representations similar to what is shown. Different parameterizations for the surfaces are possible depending on the type of spherical spiral.

Consider the following geometric parameters defining the spherical spiral:

- a_1 total interior surface area of the spiral surface;
- $A = 4\pi R^2$ total interior surface area of the sphere;
- $a_2 = A - a_1$ interior area of the sphere not covered by the spiral;
- h_1 spiral band width (constant or variable as in the case of a loxodromic spiral);
- h_2 width of the gap between the spiral bands.

To determine the view factors for this configuration we need to know the radius of the sphere R and the surface area of the spiral a_1. If the spiral is defined parametrically, the area can be calculated exactly by integration as shown in Appendix A.1. An alternative approach is for the case when the spiral band and the gap between bands are geometrically similar. Two objects are geometrically similar if they have the same shape, and therefore, one can be obtained from the other by scaling, possibly with additional

* Corresponding author.
E-mail address: webb@byu.edu (B.W. Webb).

https://doi.org/10.1016/j.jqsrt.2021.107866
0022-4073/

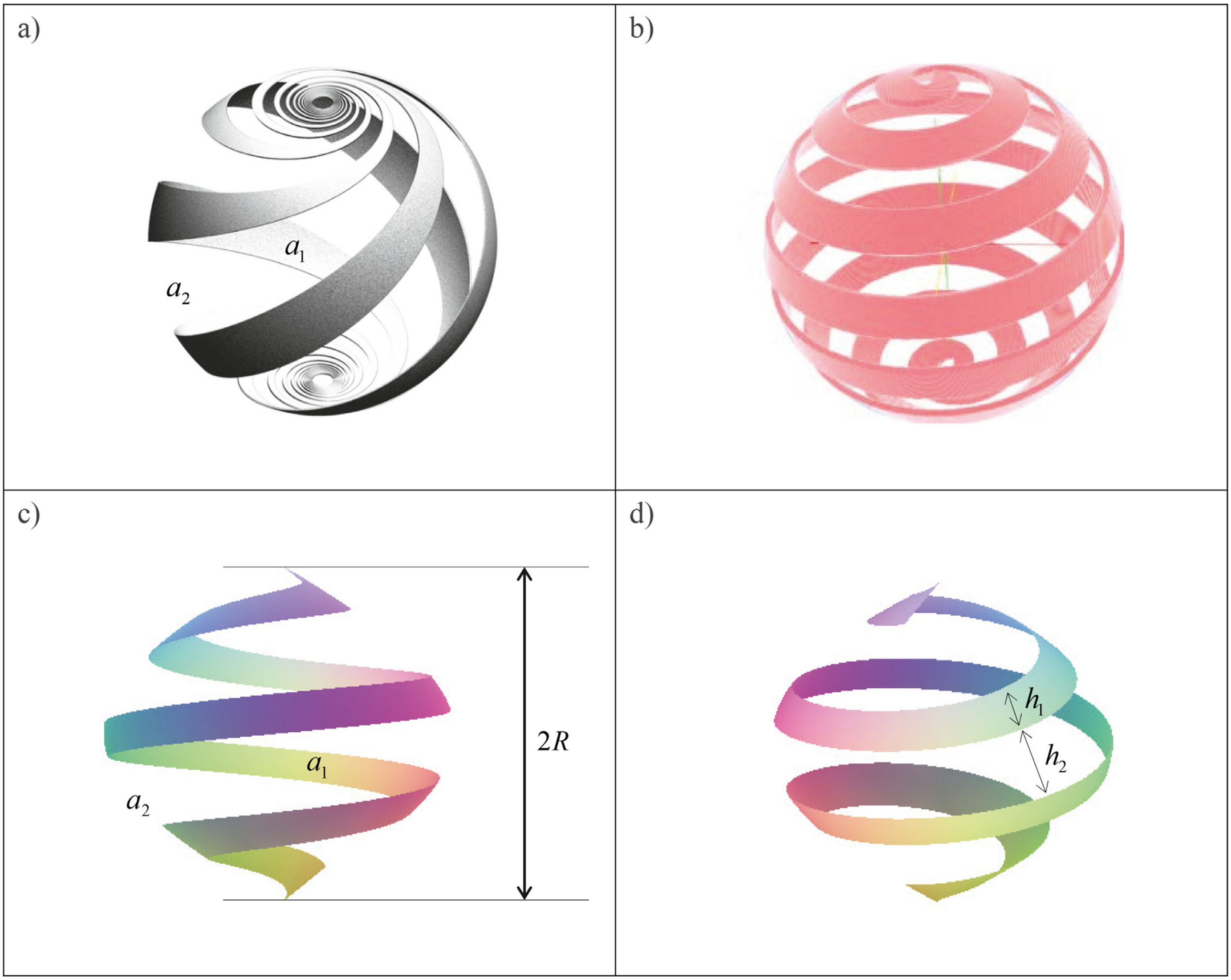


Fig. 1. Geometry of the spherical spirals: (a) logarithmic spirals (loxodromic sphere), (b) spherical photon cage, (c) dimension and surfaces of the spherical spiral, (d) Euclidian spherical spiral with a uniform spiral band.

translation and rotation. For example, when the spiral is formed by rotation of a uniform band of width h_1 leaving uniform band gaps of width h_1, then the areas a_1 and a_2 are proportional to the ratio of the band widths

$$\frac{a_1}{a_2} = \frac{h_1}{h_2} \tag{1}$$

and the areas a_1 and a_2 can be calculated by proportional subdivision of the total area A of the sphere as

$$a_1 = \frac{h_1}{h_1 + h_2} A, a_2 = \frac{h_2}{h_1 + h_2} A \tag{2}$$

There are two relevant surface areas in the spherical system considered, a_1 and a_2. These two sub-surfaces form part of the same spherical surface $A = a_1 + a_2$. Then, according to the inside sphere method [1], the view factor between two arbitrary parts of the spherical surface a_1 and a_2 is equal to the ratio of the area of the of the receiving surface a_2 to the total area of the sphere

$$F_{a_1-a_2} = \frac{a_2}{A} \tag{3}$$

Note that the view factor $F_{a_1-a_2}$ does not depend on the area of the radiating surface a_1 or the location of surfaces on the sphere. The summation rule may then be exercised to find $F_{a_1-a_1}$ and the two remaining view factors can be found by the same consideration as

$$F_{a_1-a_1} = \frac{a_1}{A} \tag{4}$$

$$F_{a_2-a_1} = \frac{a_1}{A} \tag{5}$$

$$F_{a_2-a_2} = \frac{a_2}{A} \tag{6}$$

Therefore, the required view factors associated with a single spiral surface of arbitrary shape on the sphere are found.

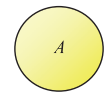
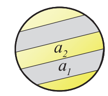
As stated previously, if the spiral band a_1 and the gap between the spiral bands a_2 possess geometric similarity, as shown in the examples of logarithmic or Euclidian spirals in Fig. 1, from the results obtained the following similarity property for the view factors can be derived

$$\frac{F_{a_1-a_1}}{F_{a_2-a_2}} = \frac{a_1}{a_2} \tag{7}$$

In a case of bands of uniform width h_1 and h_2, the similarity property is

$$\frac{F_{a_1-a_1}}{F_{a_2-a_2}} = \frac{a_1}{a_2} = \frac{h_1}{h_2} \tag{8}$$

Table 1
Spherical spiral summary.

Area	Arbitrary Spiral	Uniform Euclidian Spiral
$4\pi R^2$	$4\pi R^2$	
a_1	Eq. (A.5)	$\frac{h_1}{h_1+h_2}A$
a_2	$A-a_1$	$\frac{h_2}{h_1+h_2}A$ 

View Factor	Arbitrary Spiral	Uniform Euclidian Spiral
F_{A-A}	1	1
$F_{A_1-a_1}$	$\frac{a_1}{A}$	$\frac{h_1}{h_1+h_2}$
F_{A-a_2}	$\frac{a_2}{A}$	$\frac{h_2}{h_1+h_2}$
F_{a_1-A}	1	1
$F_{a_1-a_1}$	$\frac{a_1}{A}$	$\frac{h_1}{h_1+h_2}$
$F_{a_1-a_2}$	$\frac{a_2}{A}$	$\frac{h_2}{h_1+h_2}$
$F_{a_2-a_2}$	$\frac{a_2}{A}$	$\frac{h_2}{h_1+h_2}$
$F_{a_2-a_1}$	$\frac{a_1}{A}$	$\frac{h_1}{h_1+h_2}$

Using the similarity property Eq. (7) and view factor algebra (summation and reciprocity rules), all view factors Eqs. (3)–(6) of the spherical spiral system of two surfaces a_1 and a_2 may be found without knowing the result Eq. (3). Note that in the case of a spherical spiral, the similarity property of Eq. (7) is exact. The results for the view factors associated with the spherical spirals are summarized in Table 1.

Example 1 Euclidian spherical spiral – spherical photon cage

As an example of a spherical spiral, consider the spiral surface obtained by N full rotations of a band of uniform width h_1 over a sphere of radius R, as shown in Fig. 2. This geometry corresponds to a spherical photon cage [8]. Let $R = 10$, $h_1 = 2$, and $N = 10$. We may calculate the following parameters. The supplemental angles are

$$\alpha_1 = \frac{h_1}{R}, \alpha_2 = \frac{\pi - \alpha_1}{N} - \alpha_1 \tag{9}$$

and the width of the uniform gaps between the spiral bands is

$$h_2 = \alpha_2 R \tag{10}$$

Using these parameters, the Euclidian spherical spiral may be defined by the following parameterization:

$$\mathbf{\Phi}(u,v) = \begin{bmatrix} R\sin[N(\alpha_1+\alpha_2)u+\alpha_1 v]\cos(2N\pi u) \\ R\sin[N(\alpha_1+\alpha_2)u+\alpha_1 v]\sin(2N\pi u) \\ R\cos[N(\alpha_1+\alpha_2)u+\alpha_1 v] \end{bmatrix},$$

$$\text{for} \quad 0 \le v \le 1, \quad 0 \le u \le 1 \tag{11}$$

The exact area of this spiral is calculated using Eq. (A.5) as $a_1 = 851.0$. The interior-facing area of the spiral band can be approximated as

$$a_1 \approx \frac{h_1}{h_1 + h_2} A = 854.0$$

Fig. 2. Euclidian spherical spiral defined by parameterization with $R = 10$, $h_1 = 2$, $N = 10$.

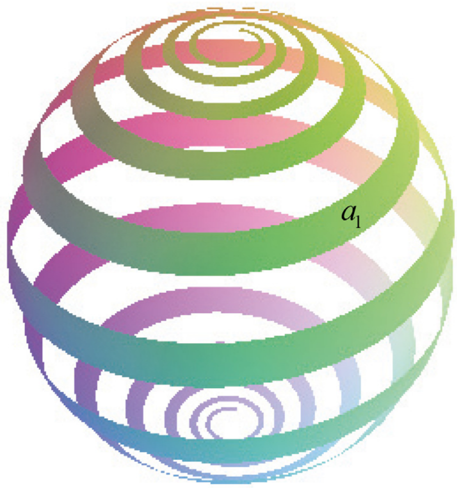


Fig. 3. Logarithmic spherical spiral defined by parameterization with $R = 10$, $h_1 = 2$, $N = 6$.

The relative error in this approximation is about 0.3%. The view factors for this example may then be determined as $F_{1-1} = 0.677$ and $F_{1-2} = 0.323$.

Example 2 Logarithmic spherical spiral of the single lane loxodromic type

Now consider a logarithmic spherical spiral defined by the following parameterization:

$$\mathbf{\Phi}(u,v) = \begin{bmatrix} R\cos(2N\pi u)/\cosh[N(\alpha_1+\alpha_2)u+\alpha_1 v] \\ R\sin(2N\pi u)/\cosh[N(\alpha_1+\alpha_2)u+\alpha_1 v] \\ R\coth[N(\alpha_1+\alpha_2)u+\alpha_1 v] \end{bmatrix},$$

$$\text{for} \quad 0 \le v \le 1, \quad -1 \le u \le 1 \tag{12}$$

where $\alpha_1 = h_1/R$ and $\alpha_2 = (\pi - \alpha_1)/N - \alpha_1$ are the supplemental angles, h_1 is the maximum width of the spiral band, and $2N$ is the number of full rotations in the sphere. Note that for this case, the parameter u changes in the interval $-1 \le u \le 1$. An example of such spiral is shown in Fig. 3.

The spiral band width in this example is not constant. However, the similarity property is still valid for this case. The view factors for this spiral are defined by Eqs. (3)–(6), where the area a_1 can be found using Eq. (A.4) with the parameterization Eq. (12) the same way as the area a_1 is calculated in Appendix A.1, and $a_2 = A - a_1$ is the area of the sphere not covered by the spiral.

3. Conic spiral (conical spiral of Pappus)

Consider a spiral band wrapped around a right, circular cone as shown in Fig. 4. Let the geometry of a right circular conic spiral be defined by the following parameters:

A_1 area of the lateral surface of the cone, $A_1 = \pi R\sqrt{R^2+H^2}$;
A_2 area of the cone circular base of radius R, $A_2 = \pi R^2$;
a_1 area of the interior surface of the spiral;
a_2 area of the lateral surface of the cone not covered by the spiral;
h_1 width of the spiral band;
h_2 width between the spiral lanes.

Consider such a conic spiral defined by the following parametric equations:

$$\mathbf{\Phi}(u,v) = \begin{bmatrix} R_0[N(h_1+h_2)u+h_1 v]\cos(2N\pi u) \\ R_0[N(h_1+h_2)u+h_1 v]\sin(2N\pi u) \\ N(h_1+h_2)u+h_1 v \end{bmatrix},$$

$$\text{for} \quad 0 \le v \le 1, \quad 0 \le u \le 1 \tag{13}$$

where N is the number of full rotations of the spiral band around cylinder, as shown in Fig. 4. If the dimensions R, H, h_1, and h_2 are prescribed, then the required number of full rotations should be

$$N = \frac{H-h_1}{h_1+h_2} \tag{14}$$

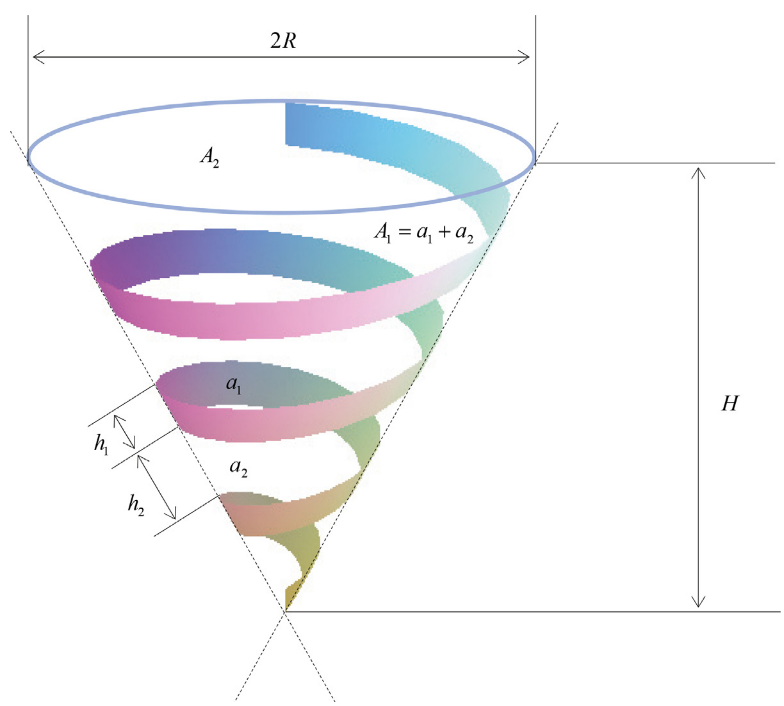


Fig. 4. Conic spiral with a uniform band of constant width defined by parameterization Eq. (13).

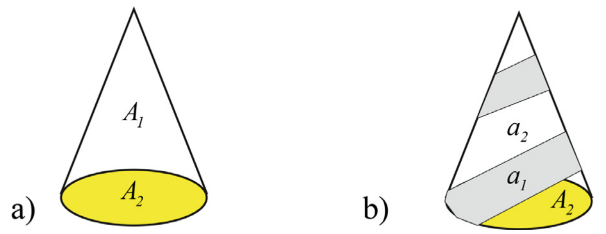


Fig. 5. (a) Two-surface circular cone, and b) three-surface cone with spiral.

Note that N is not necessarily an integer, and the parameter R_0 defining the radius R of the cone base is

$$R_0 = \frac{R}{(h_1 + h_2)N + h_1} \tag{15}$$

The exact area of the spiral defined by the parameterization of Eq. (13) may be determined as outlined in Appendix A.2, yielding

$$\begin{aligned} a_1 &= R_0 h_1 N \pi \sqrt{1 + R_0^2}[N(h_1 + h_2) + h_1] \\ &= \pi R \frac{h_1}{h_1 + h_2}\left(1 - \frac{h_1}{H}\right)\sqrt{H^2 + R^2} = \frac{h_1}{h_1 + h_2}\left(1 - \frac{h_1}{H}\right)A_1 \end{aligned} \tag{16}$$

In the case of a long thin spiral ($h_1 \ll H$), the following approximation can be used

$$a_1 = \pi R \frac{h_1}{h_1 + h_2}\sqrt{H^2 + R^2} = \frac{h_1}{h_1 + h_2}A_1 \tag{17}$$

Then assuming that $A_1 = a_1 + a_2$, the exact lateral area of the cone not covered by the spiral band is

$$a_2 = A_1 - a_1 = A_1 - \frac{h_1}{h_1 + h_2}\left(1 - \frac{h_1}{H}\right)A_1 \tag{18}$$

and the approximate area a_2 in a case of a long spiral band is

$$a_2 = \frac{h_2}{h_1 + h_2}A_1 \tag{19}$$

View factors for the two-surface circular cone with surfaces A_1 and A_2, shown in Fig. 5a, are found elsewhere [2], as:

$$F_{A_1-A_1} = 1 - \frac{A_2}{A_1}, F_{A_1-A_2} = \frac{A_2}{A_1} \tag{20}$$

$$F_{A_2-A_1} = 1, F_{A_2-A_2} = 0 \tag{21}$$

where

$$\frac{A_1}{A_2} = \frac{\pi R\sqrt{R^2 + H^2}}{\pi R^2} = \frac{\sqrt{R^2 + H^2}}{R}, \frac{A_2}{A_1} = \frac{R}{\sqrt{R^2 + H^2}} \tag{22}$$

View factors for the three-surface conic spiral shown in Fig. 5b with the surfaces a_1, a_2, and A_2 obey the composite wall rule:

$$F_{A_2-a_1} + F_{A_2-a_2} = F_{A_2-A_1} = 1 \tag{23}$$

The assumption of geometric similarity of surfaces a_1 and a_2 yields the similarity relation

$$\frac{F_{A_2-a_1}}{F_{A_2-a_2}} = \frac{a_1}{a_2} \tag{24}$$

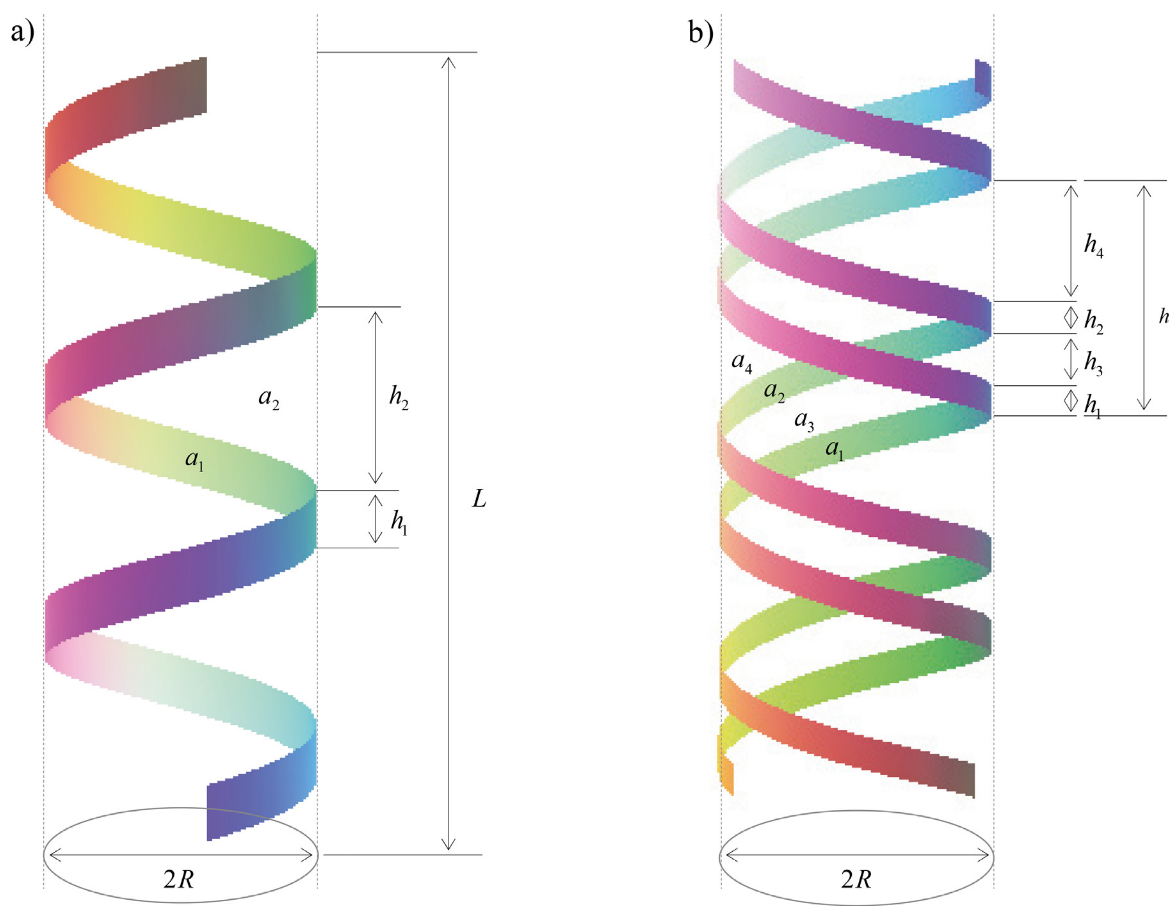
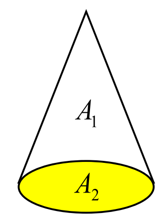
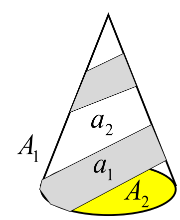


Fig. 6. (a) Long flat cylindrical spiral surface (helix), and b) long double helix.

Table 2
Conic spiral summary.

Area	Exact	Approximate
a_1	$\frac{h_1}{h_1+h_2}\left(1-\frac{h_1}{H}\right)A_1$	$\frac{h_1}{h_1+h_2}A_1$

View Factor	Exact	Approximate
$F_{A_1-A_1}$	$1-\frac{A_1}{A_2}$	
$F_{A_1-A_2}$	$\frac{A_2}{A_1}$	
$F_{A_2-A_1}$	1	
$F_{A_2-A_2}$	0	
$F_{a_1-A_2}$	$\frac{A_2}{A_1}$	
$F_{a_2-A_2}$	$\frac{A_2}{A_1}$	
$F_{A_2-a_1}$	$\frac{a_1}{A_1}$	$\frac{h_1}{h_1+h_2}$
$F_{A_2-a_2}$	$\frac{a_2}{A_1}$	$\frac{h_2}{h_1+h_2}$
$F_{a_1-a_1}$	$\frac{a_1}{A_1}\left(1-\frac{A_2}{A_1}\right)=\frac{a_1}{A_1}\left(1-\frac{R}{\sqrt{R^2+H^2}}\right)$	$\left(\frac{h_1}{h_1+h_2}\right)\left(1-\frac{A_2}{A_1}\right)=\left(\frac{h_1}{h_1+h_2}\right)\left(1-\frac{R}{\sqrt{R^2+H^2}}\right)$
$F_{a_2-a_2}$	$\frac{a_2}{A_1}\left(1-\frac{A_2}{A_1}\right)=\frac{a_2}{A_1}\left(1-\frac{R}{\sqrt{R^2+H^2}}\right)$	$\left(\frac{h_2}{h_1+h_2}\right)\left(1-\frac{A_2}{A_1}\right)=\left(\frac{h_2}{h_1+h_2}\right)\left(1-\frac{R}{\sqrt{R^2+H^2}}\right)$
$F_{a_1-a_2}$	$1-F_{a_1-a_1}-\frac{A_2}{A_1}$	
$F_{a_2-a_1}$	$1-\frac{a_2}{a_1}F_{a_1-a_1}-\frac{A_2}{A_1}$	$1-\frac{h_2}{h_1}F_{a_1-a_1}-\frac{A_2}{A_1}$

which, for the case of a long spiral band using Eqs. (17) and (19), is reduced to an approximate relation

$$\frac{F_{A_2-a_1}}{F_{A_2-a_2}}=\frac{a_1}{a_2}\approx\frac{h_1}{h_2} \tag{25}$$

Using view factor algebra, all other view factors for this configuration can now be determined. The results for the view factors associated with the conic spirals are summarized in Table 2, with both exact relations and corresponding approximate expressions for the long thin spiral ($h_1 \ll H$).

4. Long, flat cylindrical spirals – spherical helix and double spherical helix

Consider a flat right cylindrical spiral band surface obtained by rotating the flat thin strip of width h_1 over an imaginary cylinder of radius R with a gap h_2 between strips, as shown in Fig. 6. Such a spiral surface can be defined by the following parametric equations

$$\mathbf{\Phi}(u,v)=\begin{bmatrix} R\cos(2\pi Nu) \\ R\sin(2\pi Nu) \\ (h_1+h_2)u+h_1 v \end{bmatrix}, \text{for} \quad 0\le v\le 1, \quad 0\le u\le 1 \tag{26}$$

where N is the number of full rotations of the spiral around the cylinder. The total length of the spiral band is

$$L = (h_1 + h_2)N \tag{27}$$

and the surface area of each side of the spiral band may be calculated following the approach outlined in Appendix A.3, yielding

$$a_1 = 2\pi R h_1 N \tag{28}$$

We now denote a_2 as the area of the gaps between the spiral strips.

The view factors corresponding to finite and infinitely long spirals are studied in [5,6]. The view factor of the interior surface of the spiral a_1 to the surroundings defined by the imaginary surface a_2 for the case of sufficiently long spiral, $R \ll L$ and when the radiation losses through the top and the bottom openings of the cylinder can be neglected, is given in Lebedev and Solovjov [6] and it is included into the Catalog of view factors [2] as C-78a. (It is noted, however, that this entry is not in the table of contents of the Catalog):

$$F_{1-1} = \frac{h_1}{h_1 + h_2} \tag{29}$$

$$F_{1-2} = \frac{h_2}{h_1 + h_2} \tag{30}$$

4.1. Long double helix

The pair of two or more long cylindrical spiral surfaces can be studied as well. (The case of two identical spirals form a configuration which resembles a DNA molecule, as shown in Fig. 5b). This configuration is often used in convective heating applications. Noting that

$$h = h_1 + h_2 + h_3 + h_4 \tag{31}$$

the similarity assumption for the case of multiple geometrically similar bands can be formulated as

$$\frac{F_{i-i}}{F_{k-k}} = \frac{a_i}{a_k} \tag{32}$$

where $i = 1,...,4$, $k = 1,...,4$

In a case of a uniform band width h_i, Eq. (32) is written as

$$\frac{F_{i-i}}{F_{k-k}} = \frac{h_i}{h_k}$$

where $i = 1,...,4$, $k = 1,...,4$

Then, considerations similar to the previous spiral cases, together with the results of [6] yield the following results for the full entries of the matrix of the view factors:

$$\begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ F_{21} & F_{22} & F_{23} & F_{24} \\ F_{31} & F_{32} & F_{33} & F_{34} \\ F_{41} & F_{42} & F_{43} & F_{44} \end{bmatrix} = (F_{ik})_{4\times 4} \tag{33}$$

where $F_{ik} = \frac{h_k}{h}, \ i = 1, \dots, 4, \ k = 1, \dots, 4$

It should be noted that while for the spherical spirals, Eq. (32) holds for any geometric configuration of spiral, for the conical and cylindrical spirals, the equation is valid only for the case of geometrically similar spiral bands and the gaps between bands.

5. Conclusions

The view factors of a family of spiral surfaces of different type are derived, and the analytical results are presented for the view factors of the conic, spherical and cylindrical spiral surfaces.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix. Area of parameterized surfaces in differential geometry

Let a surface S be defined by a vector function $\Phi(u, v) : D \subset \mathbb{R}^2 \to \mathbb{R}^3$ which maps a region D in the plane $\mathbb{R}^2$ onto a surface in $\mathbb{R}^3$, as illustrated in Fig. A.1. Then the vector defining parametrically the surface may be written

$$\mathbf{\Phi}(u, v) = \begin{bmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{bmatrix}, (u, v) \in D \subset \mathbb{R}^2 \tag{A.1}$$

For simplicity, in all parameterizations employed here, the domain D is chosen to be a unit square $D = [0, 1] \times [0, 1]$. Defined parametrically, surface S is a simple smooth regular surface, if the functions $x(u, v)$, $y(u, v)$, and $z(u, v)$ are differentiable with respect to u and v in D, and

$$\frac{\partial}{\partial u}\mathbf{\Phi}(u, v) \times \frac{\partial}{\partial v}\mathbf{\Phi}(u, v) \neq \mathbf{0} \tag{A.2}$$

for all points u,v in D, where the partial derivatives of the vector function $\mathbf{\Phi}$ are

$$\frac{\partial}{\partial u}\mathbf{\Phi}(u, v) = \begin{bmatrix} \frac{\partial}{\partial u}x(u, v) \\ \frac{\partial}{\partial u}y(u, v) \\ \frac{\partial}{\partial u}z(u, v) \end{bmatrix}, \frac{\partial}{\partial v}\mathbf{\Phi}(u, v) = \begin{bmatrix} \frac{\partial}{\partial v}x(u, v) \\ \frac{\partial}{\partial v}y(u, v) \\ \frac{\partial}{\partial v}z(u, v) \end{bmatrix} \tag{A.3}$$

The area of the parametrically defined regular surface is calculated by a double integration over the region D as follows:

$$A = \iint_D \left\| \frac{\partial}{\partial u}\mathbf{\Phi}(u, v) \times \frac{\partial}{\partial v}\mathbf{\Phi}(u, v) \right\| du\, dv \tag{A.4}$$

This indicates how much $\mathbf{\Phi}$ is stretching D as it maps D onto the surface S.

A.1. Area of Spherical Spiral

Consider parameterization of the Euclidian spherical spiral given by Eq. (11). The partial derivatives are

$$\frac{\partial}{\partial u}\mathbf{\Phi}(u, v) = RN \begin{bmatrix} (\alpha_1 + \alpha_2)\cos[N(\alpha_1 + \alpha_2)u + \alpha_1 v]\cos(2N\pi u) \\ -2\pi \sin[N(\alpha_1 + \alpha_2)u + \alpha_1 v]\sin(2N\pi u) \\ (\alpha_1 + \alpha_2)\cos[N(\alpha_1 + \alpha_2)u + \alpha_1 v]\sin(2N\pi u) \\ +2\pi \sin[N(\alpha_1 + \alpha_2)u + \alpha_1 v]\cos(2N\pi u) \\ -(\alpha_1 + \alpha_2)\cos[N(\alpha_1 + \alpha_2)u + \alpha_1 v] \end{bmatrix}$$

$$= RN \begin{bmatrix} (\alpha_1 + \alpha_2)\cos[N(\alpha_1 + \alpha_2)u + \alpha_1 v]\cos(2N\pi u) \\ -2\pi \sin[N(\alpha_1 + \alpha_2)u + \alpha_1 v]\sin(2N\pi u) \\ (\alpha_1 + \alpha_2)\cos[N(\alpha_1 + \alpha_2)u + \alpha_1 v]\sin(2N\pi u) \\ +2\pi \sin[N(\alpha_1 + \alpha_2)u + \alpha_1 v]\cos(2N\pi u) \\ -(\alpha_1 + \alpha_2)\cos[N(\alpha_1 + \alpha_2)u + \alpha_1 v] \end{bmatrix}$$

$$\frac{\partial}{\partial v}\mathbf{\Phi}(u, v) = R \begin{bmatrix} \alpha_1 \cos[N(\alpha_1 + \alpha_2)u + \alpha_1 v]\cos(2N\pi u) \\ \alpha_1 \cos[N(\alpha_1 + \alpha_2)u + \alpha_1 v]\sin(2N\pi u) \\ -\alpha_1 \sin[N(\alpha_1 + \alpha_2)u + \alpha_1 v] \end{bmatrix}$$

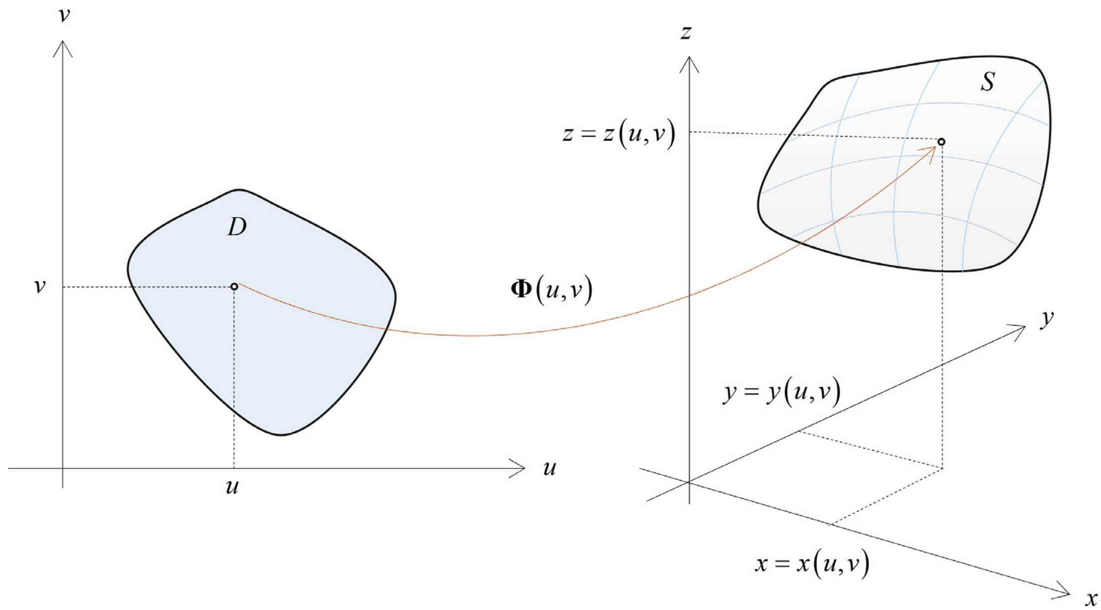


Fig. A.1. Parametric definition of a surface with a help of the vector function $\mathbf{\Phi}(u, v)$.

The norm of the vector product is

$$\left\| \frac{\partial}{\partial u}\mathbf{\Phi}(u,v) \times \frac{\partial}{\partial v}\mathbf{\Phi}(u,v) \right\| = \alpha_1 N R^2 \sqrt{s_1^2 + s_2^2 + s_3^2}$$

where

$$\begin{aligned} s_1 = &-(\alpha_1+\alpha_2)\cos[N(\alpha_1+\alpha_2)u + \alpha_1 v]\sin[N(\alpha_1+\alpha_2)u \\ &+\alpha_1 v]\sin(2N\pi u) \\ &-2\pi\sin^2[N(\alpha_1+\alpha_2)u+\alpha_1 v]\cos(2N\pi u) \\ &+(\alpha_1+\alpha_2)\cos^2[N(\alpha_1+\alpha_2)u+\alpha_1 v]\sin(2N\pi u) \end{aligned}$$

$$\begin{aligned} s_2 = &(\alpha_1+\alpha_2)\cos[N(\alpha_1+\alpha_2)u + \alpha_1 v]\sin[N(\alpha_1+\alpha_2)u \\ &+\alpha_1 v]\cos(2N\pi u) \\ &-2\pi\sin^2[N(\alpha_1+\alpha_2)u+\alpha_1 v]\sin(2N\pi u) \\ &-(\alpha_1+\alpha_2)\cos^2[N(\alpha_1+\alpha_2)u+\alpha_1 v]\cos(2N\pi u) \end{aligned}$$

$$s_3 = -2\pi\sin[N(\alpha_1+\alpha_2)u+\alpha_1 v]\cos[N(\alpha_1+\alpha_2)u+\alpha_1 v]$$

Then the surface area of the spherical spiral can be determined by numerical integration of

$$a_1 = \alpha_1 N R^2 \int_0^1\int_0^1 \sqrt{s_1^2+s_2^2+s_3^2}\,du\,dv \tag{A.5}$$

A.2. Area of Conical Spiral

Consider the conic spiral defined by parameterization Eq. (13), for which the partial derivatives are

$$\frac{\partial}{\partial u}\mathbf{\Phi}(u,v) = \begin{bmatrix} R_0 N(h_1+h_2)\cos(2N\pi u) \\ -2N\pi R_0[N(h_1+h_2)u+h_1 v]\sin(2N\pi u) \\ R_0 N(h_1+h_2)\sin(2N\pi u) \\ +2N\pi R_0[N(h_1+h_2)u+h_1 v]\cos(2N\pi u) \\ N(h_1+h_2) \end{bmatrix}$$

$$\frac{\partial}{\partial v}\mathbf{\Phi}(u,v) = \begin{bmatrix} R_0 h_1\cos(2N\pi u) \\ R_0 h_1 \sin(2N\pi u) \\ h_1 \end{bmatrix}$$

The norm of the vector product Eq. (A.2) is found as

$$\left\| \frac{\partial}{\partial u}\mathbf{\Phi}(u,v) \times \frac{\partial}{\partial v}\mathbf{\Phi}(u,v) \right\|$$
$$= 2R_0 h_1 N\pi\sqrt{1+R_0^2}[N(h_1+h_2)u+h_1 v]$$

Then the double integral defining the area is

$$a_1 = \int_0^1\int_0^1 2R_0 h_1 N\pi\sqrt{1+R_0^2}[N(h_1+h_2)u+h_1 v]\,du\,dv$$

Evaluation of this double integral yields the exact area of the conic spiral defined by parameterization Eq. (13)

$$a_1 = Nh_1[N(h_1+h_2)+h_1]\pi R_0\sqrt{1+R_0^2}$$

For prescribed R, H, h_1, and h_2, the required number of full rotations N and parameter R_0 are

$$N = \frac{H-h_1}{h_1+h_2}$$

$$R_0 = \frac{R}{(h_1+h_2)N+h_1}$$

Then, the equation for the area becomes

$$a_1 = \frac{h_1}{h_1+h_2}\left(1-\frac{h_1}{H}\right)\pi R\sqrt{H^2+R^2} \tag{A.6}$$

For a long spiral, if $h_1 \ll H$, then $h_1/H \approx 0$, and this formula reduces to the approximate expression for the area

$$a_1 = \frac{h_1}{h_1+h_2}\left(1-\frac{h_1}{H}\right)\pi R\sqrt{H^2+R^2} \approx \frac{h_1}{h_1+h_2}\pi R\sqrt{H^2+R^2} \tag{A.7}$$

A.3. Area of Cylindrical Spiral

Consider a single cylindrical spiral defined by parameterization:

$$\mathbf{\Phi}(u,v) = \begin{bmatrix} R\cos(2N\pi u) \\ R\sin(2N\pi u) \\ N(h_1+h_2)u + h_1 v \end{bmatrix}, \text{ for } 0 \le v \le 1,\ \ 0 \le u \le 1 \tag{A.8}$$

The derivatives of this vector function are

$$\frac{\partial}{\partial u}\mathbf{\Phi}(u,v)=\begin{bmatrix}-2N\pi R\sin(2N\pi u)\\ 2N\pi R\cos(2N\pi u)\\ N(h_1+h_2)\end{bmatrix}$$

$$\frac{\partial}{\partial v}\mathbf{\Phi}(u,v)=\begin{bmatrix}0\\ 0\\ h_1\end{bmatrix}$$

and the vector product is

$$\begin{aligned}&\left\|\frac{\partial}{\partial u}\mathbf{\Phi}(u,v)\times\frac{\partial}{\partial v}\mathbf{\Phi}(u,v)\right\|\\ &=\left\|\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -2N\pi R\sin(2N\pi u) & 2N\pi R\cos(2N\pi u) & N(h_1+h_2)\\ 0 & 0 & h_1\end{vmatrix}\right\|\\ &=\| h_1 2N\pi R\cos(2N\pi u)\mathbf{i}+h_1 2N\pi R\sin(2N\pi u)\mathbf{j}\,\|\\ &=h_1 2N\pi R\end{aligned}$$

Then the area of the cylindrical spiral is defined by equation

$$A=\int_0^1\int_0^1 h_1 2N\pi R\,du\,dv=(Nh_1)\,(2\pi R) \tag{A.9}$$

The area of a spiral defined by Eq. (A.9) is equivalent to the lateral area of the circular cylinder of the radius R and the height Nh_1.

CRediT authorship contribution statement

Vladimir A. Lebedev: Conceptualization, Formal analysis. **Vladimir P. Solovjov:** Conceptualization, Validation, Writing – original draft. **Brent W. Webb:** Conceptualization, Methodology, Software, Validation, Formal analysis, Writing – original draft.

References

[1] Howell JR, Mengüç MP, Daun K, Siegel R. Thermal radiation heat transfer. CRC Press; 2021. 7th Ed.
[2] Howell JR, A catalog of radiation heat transfer configuration factors, 3rd Ed. ThermalRadiation.net, University of Texas at Austin, www.thermalradiation.net/indexCat.html.
[3] Rubtsov NA, Lebedev VA. Geometric Invariants of Radiation (in Russian), 1989. Inst. of Thermal Physics, Siberian Branch of Academy of Sciences of USSR.
[4] Lebedev VA. Geometric invariants of radiation of spiral heaters. Thermophys. Aeromech. 2003;10(1):101–8.
[5] Lebedev VA. Radiation configuration factors for a flat cylindrical spiral. Thermophys. Aeromech. 2000;7(3):447–50.
[6] Lebedev VA, Solovjov VP. View factors of cylindrical spiral surfaces. J. Quant. Spectr. Radiat. Transf. 2016;171:1–3.
[7] Brister JA, Edwards RM. Design of a balanced ball antenna using a spherical helix wound over a full sphere. In: Proceedings of the loughborough antennas and propagation conference; 2011. p. 1–4. doi:10.1109/LAPC.2011. 6114128.
[8] Sieutat C, et al. 3D optical micro-resonators by curving nanostructures using intrinsic stress. In: Proceedings of the SPIE - The International Society for Optical Engineering, 8425. Bruxelles: SPIE Photonics Europe; 2012. p. 28. doi:10.1117/12.922375.