

On the Independence of Quantum Structure from SK Combinatory Logic: A Systematic Investigation

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December 5, 2025

Abstract

We investigate whether complex number structure, fundamental to quantum mechanics, can be derived from SK combinatory logic—a minimal, Turing-complete computational system. Through systematic exploration of four distinct approaches (Sorkin’s quantum measure theory, algebraic structure of reduction operators, path space holonomy, and information-theoretic derivation), we find that **complex structure does not automatically emerge from SK computation**. While we discovered that phase differences can be *computed* from information-theoretic quantities (specifically, the difference in S-reduction counts between paths), the choice of computation formula remains arbitrary. This constitutes a negative result with significant implications: it establishes constraints on the relationship between computational substrates and quantum structure, clarifying what additional assumptions are necessary to bridge the gap between computation and quantum mechanics.

1 Introduction

A fundamental question in the foundations of physics and computation is: *Why does quantum mechanics use complex numbers rather than real numbers or quaternions?* This question has motivated extensive research in quantum foundations, from Hardy’s derivation of quantum theory from five reasonable axioms [5] to Chiribella et al.’s informational approach [2].

The Wolfram Physics Project [8] and related digital physics approaches suggest that physical reality might emerge from computational processes. However, these approaches typically *assume* or *interpret* complex amplitudes rather than deriving them from computational structure.

In this paper, we systematically investigate whether complex number structure can be **derived** (not assumed) from SK combinatory logic—one of the simplest Turing-complete computational systems.

1.1 Why SK Combinatory Logic?

We choose SK combinatory logic as our computational substrate for several principled reasons:

1. **Minimality:** SK is the simplest known Turing-complete system (only two combinators with two reduction rules), eliminating confounding structure
2. **Information-theoretic clarity:** The K combinator explicitly *discards* information, providing a clear connection to Landauer’s principle
3. **Non-determinism:** When multiple redexes exist, reduction order is not determined, naturally generating a multiway graph analogous to quantum superposition
4. **No implicit numerical structure:** Unlike λ -calculus with Church numerals, SK does not presuppose arithmetic

If complex numbers emerge from *any* minimal computation, they should emerge from SK. If they don’t, this establishes a strong negative result.

1.2 Overview of Approaches

Our investigation proceeds through four complementary approaches:

1. **Sorkin’s Quantum Measure Theory:** Testing whether SK computation exhibits non-additive probability measure ($I_2 \neq 0, I_3 = 0$)
2. **Algebraic Structure:** Searching for operators J satisfying $J^2 = -I$ within the reduction operator algebra
3. **Geometric Structure:** Computing holonomy in the path space to detect $U(1)$ structure

4. **Information-Theoretic Derivation:** Deriving phase from Kolmogorov complexity changes

Our main finding is **negative**: complex structure does not automatically emerge from SK computation. However, we establish precise conditions under which phase-like quantities *can* be computed, albeit with arbitrary choices in the computational formula.

1.3 Contributions

- We provide the first systematic investigation of the relationship between SK combinatory logic and quantum complex structure
- We establish that SK computation is fundamentally classical under standard probability definitions
- We identify specific conditions (choice of connection or computation formula) required to obtain non-trivial phase
- We provide open-source implementations for reproducibility¹

2 Background

2.1 SK Combinatory Logic

SK combinatory logic is a minimal computational system consisting of two combinators with the following reduction rules:

Definition 1 (SK Reduction Rules).

$$S x y z \rightarrow (x z) (y z) \tag{1}$$

$$K x y \rightarrow x \tag{2}$$

Despite its simplicity, SK logic is Turing-complete: any computable function can be expressed using only S and K combinators [3].

Remark 1 (Information Properties). *The K combinator discards information (its second argument), connecting to Landauer's principle [6]. The S combinator duplicates and redistributes information.*

¹Available at: [repository URL]

2.2 Multiway Systems

When multiple reduction rules are applicable, different reduction orders lead to different computation paths. This creates a *multiway graph* where:

- Nodes represent SK expressions
- Edges represent single reduction steps
- Multiple paths may lead to the same normal form

This structure is reminiscent of quantum superposition, motivating our investigation.

2.3 Sorkin's Quantum Measure Theory

Sorkin [7] characterized quantum mechanics through the second-order interference formula:

Definition 2 (Sorkin's Interference Terms).

$$I_2(A, B) = P(A \cup B) - P(A) - P(B) \quad (3)$$

$$\begin{aligned} I_3(A, B, C) = & P(A \cup B \cup C) - I_2(A, B) - I_2(A, C) - I_2(B, C) \\ & - P(A) - P(B) - P(C) \end{aligned} \quad (4)$$

Classical probability satisfies $I_2 = 0$ (additivity). Quantum mechanics satisfies $I_2 \neq 0$ (interference) but $I_3 = 0$ (no third-order interference).

2.4 Prior Work

Approach	Achievement	Limitation
Hardy [5]	Derived \mathbb{C} from axioms	No computational basis
Chiribella et al. [2]	Information-theoretic	Why quantum?
Wolfram [8]	Space-time from hypergraphs	Complex amplitudes unclear
Abramsky-Coecke [1]	Categorical QM	No dynamics

Table 1: Prior approaches to quantum foundations

3 Methods

3.1 Phase 0: Sorkin Formula Verification

We implemented a multiway graph construction algorithm and computed Sorkin's interference terms I_2 and I_3 for various SK expressions.

```
function BuildMultiwayGraph(expr, maxDepth):
    Initialize graph with expr as root
    while frontier non-empty and depth < maxDepth:
        for each expression e in frontier:
            Find all redexes in e
            for each redex r:
                e' = reduce e at r
                Add edge (e, e') to graph
    return graph
```

3.2 Phase 1A: Algebraic Structure Analysis

We constructed matrix representations of reduction operators \hat{S} and \hat{K} on finite bases of SK expressions, searching for operators J satisfying:

$$J^2 = -I \quad (5)$$

We also checked Clifford algebra relations:

$$\gamma_i^2 = \pm I \quad (6)$$

$$\{\gamma_i, \gamma_j\} = 0 \quad (i \neq j) \quad (7)$$

Remark 2 (Finite Basis Limitation). *Our analysis uses finite-dimensional bases (expressions up to depth 3, yielding $\sim 50\text{-}100$ basis elements). This is a methodological limitation: the full SK expression space is infinite-dimensional. However, for establishing positive results (existence of $J^2 = -I$), finite bases suffice. Our negative result is thus: “no non-trivial $J^2 = -I$ exists within the finite closure we examined.” We discuss infinite-dimensional extensions in Section 5.3.*

3.3 Phase 1B: Path Space Holonomy

For paths γ_1, γ_2 with the same endpoints, we defined:

$$\text{Holonomy}(\gamma_1, \gamma_2) = \frac{\exp(i \sum_{\gamma_1} \theta)}{\exp(i \sum_{\gamma_2} \theta)} \quad (8)$$

where θ is assigned by a *connection*—a function mapping each reduction step to a phase.

3.4 Phase 2: Information-Theoretic Derivation

We computed phase from information-theoretic quantities:

$$\Phi = \alpha \cdot (\text{erasure} - n_S) \quad (9)$$

where n_S is the number of S-reductions along a path.

4 Results

4.1 Phase 0: Classical Probability

Expression	Paths	$I_2 \neq 0$ pairs	Status
$(K a b) (K c d)$	2	0/1	Classical
$S (K a) (K b) c$	2	0/1	Classical
$(K a b) (K c d) (K e f)$	6	15/15	Apparent*

Table 2: Sorkin formula results. *Apparent non-additivity due to probability definition, not quantum interference.

Finding: Under all probability models tested, two-path cases satisfy $I_2 = 0$ (classical). Multi-path cases show apparent $I_2 \neq 0$ due to the definition of $P(A \cup B)$, not genuine quantum interference.

4.2 Phase 1A: Trivial Solutions Only

Test	Result
$J^2 = -I$ candidates	1,250 (all trivial)
Clifford structure	Not found
Pauli structure	Not found

Table 3: Algebraic structure analysis results

Finding: All 1,250 candidates for $J^2 = -I$ were of the form $J = -iI + \alpha\hat{S} + \beta\hat{K}$. Since \hat{S} and \hat{K} act as near-zero matrices on finite bases, these reduce to $J \approx -iI$ —a trivial solution that *assumes* complex coefficients.

4.3 Phase 1B: Connection-Dependent Holonomy

Connection	Non-trivial	U(1) candidate
constant_S_K	Yes	Yes
constant_S_only	Yes	Yes
depth_dependent	Yes	Yes
complexity	No	No
info_erasure	No	No

Table 4: Holonomy analysis for $S(Ka)(Kb)(Scde)$ (120 loops)

Finding: U(1) structure *can* emerge, but only with specific connection choices. The connection assignment is arbitrary—not derived from SK computation itself.

Remark 3 (Rigorous Meaning of “Arbitrary”). *We claim the connection choice is arbitrary in the following precise sense: given any desired holonomy value $h \in U(1)$, we can construct a connection ∇ such that $Hol_{\nabla}(\gamma) = h$ for specific loops γ . This is because:*

1. *The space of connections is parameterized by $(\theta_S, \theta_K) \in [0, 2\pi]^2$ (for constant connections)*
2. *For any loop with n_S S-reductions and n_K K-reductions, we have $Hol = e^{i(n_S\theta_S + n_K\theta_K)}$*
3. *By varying (θ_S, θ_K) , any $h \in U(1)$ is achievable when $\gcd(n_S, n_K) = 1$*

The key observation is that SK computation itself does not select any preferred connection. There is no canonical (θ_S, θ_K) derivable from the reduction rules.

4.4 Phase 2: Computable but Arbitrary Phase

Finding: The “linear” formula $\Phi = \alpha(\text{erasure} - n_S)$ produces non-trivial phase differences because different paths have different numbers of S-reductions. However, the choice of this formula is arbitrary.

Remark 4 (Rigorous Meaning of “Arbitrary” for Phase Formula). *We claim the phase formula choice is arbitrary in the following sense:*

Formula	Phase difference range	Non-trivial
linear	$[-0.2, 0.2]$	Yes
logarithmic	$[0, 0]$	No
Landauer	$[0, 0]$	No
size_change	$[0, 0]$	No
compressed	$[0, 0]$	No

Table 5: Information-theoretic phase computation for $S(Ka)(Kb)(Scede)$

1. *The space of information-theoretic quantities available per reduction step includes: erasure amount, complexity change, depth change, compression ratio, etc.*
2. *Any linear combination $\Phi = \sum_i w_i \cdot q_i$ (where q_i are information quantities) defines a valid formula*
3. *Different weight choices $\{w_i\}$ yield different phase values, with no a priori criterion to select one*
4. *The “linear” formula works precisely because n_S (S -reduction count) differs between paths—but why S -reductions should contribute negatively to phase is not derivable from SK semantics*

The parameter $\alpha \in \mathbb{R}$ has no natural value: any $\alpha \neq 0$ gives non-trivial phase, but the scale is undetermined.

5 Discussion

5.1 Why Complex Numbers Don’t Automatically Emerge

Our results suggest a fundamental barrier: SK computation is intrinsically *classical* in its probability structure. The multiway graph provides a rich state space, but:

1. Reduction rules preserve total probability (no interference)
2. Algebraic closure of reduction operators is real-valued
3. Path space geometry requires external assignment of phase

5.2 What Would Be Needed

For complex structure to genuinely emerge, one would need:

- A *canonical* connection on path space (not arbitrary)
- Or a proof that SK algebra's closure necessarily contains \mathbb{C}
- Or a physical principle selecting a specific phase formula

None of these conditions are satisfied by SK computation alone.

5.3 Theoretical Limits: Why Complex Structure Doesn't Emerge

Our negative results admit deeper theoretical analysis:

5.3.1 Why No Clifford Algebra?

Clifford algebras $\text{Cl}_{p,q}(\mathbb{R})$ require generators $\{\gamma_i\}$ with $\gamma_i^2 = \pm 1$ and $\{\gamma_i, \gamma_j\} = 0$. Our SK reduction operators fail this because:

1. \hat{S} and \hat{K} are not involutive: $\hat{S}^2 \neq \pm I$ (S-reduction rarely applies twice consecutively to the same subexpression)
2. \hat{S} and \hat{K} do not anticommute: $\hat{S}\hat{K} + \hat{K}\hat{S} \neq 0$ (reduction order matters but without systematic cancellation)
3. The reduction algebra is *nilpotent* in finite closure: after sufficient reductions, expressions reach normal form where $\hat{S} = \hat{K} = 0$

This nilpotency is fundamental: SK reduction is *terminating* (for normalizing expressions), which precludes the cyclic structure necessary for Clifford generators.

5.3.2 Why No Natural $\mathbb{R} \rightarrow \mathbb{C}$ Extension?

For the SK operator algebra to naturally extend from \mathbb{R} to \mathbb{C} , we would need one of:

1. An algebraic element J with $J^2 = -I$ (not found—all solutions require i as input)
2. A topological reason: non-trivial π_1 of some configuration space (path space is simply connected in finite cases)
3. A continuity requirement: if reduction were continuous, \mathbb{R} might extend to \mathbb{C} for completeness (but SK reduction is discrete)

5.3.3 The Continuity Question

A deep theoretical question is: *what if we take a continuum limit of SK computation?*

- In discrete SK, time is measured in reduction steps (integers)
- A continuum limit would require a notion of “fractional reduction”
- Such a limit might require \mathbb{C} for consistency (analogous to Wick rotation)

We leave this as an open question for future work.

5.3.4 Finite vs. Infinite Basis

Our algebraic analysis used finite bases. Could infinite-dimensional extensions yield different results?

For *existence* of $J^2 = -I$: the question is whether the limit of finite-dimensional closures contains such J . Our evidence suggests no:

- Each finite level produces only trivial (complex-coefficient) solutions
- The structure of solutions doesn’t change with basis size
- Nilpotency persists at each finite level

A rigorous proof would require techniques from non-commutative algebra and is beyond our current scope.

5.4 Relation to Digital Physics

Our negative result constrains digital physics approaches: if the universe is computational, complex quantum structure requires **additional structure** beyond minimal computation. This additional structure might be:

- Specific initial conditions
- Higher-order computational primitives
- Emergent from large-scale statistical behavior (thermodynamic limit)

5.5 Physical Interpretation of Path Space Holonomy

From a theoretical physics perspective, our holonomy results have the following interpretation:

Analogy to Gauge Theory: In gauge theories, holonomy arises from parallel transport around closed loops. The connection specifies how to “compare” vectors at different points. In SK path space:

- Paths are analogous to worldlines
- Connections assign phase to each reduction step
- Holonomy measures “phase mismatch” between different paths to the same endpoint

Missing Physical Principle: In real physics, connections are constrained by:

- Gauge symmetry (local invariance requirements)
- Dynamics (equations of motion)
- Coupling to matter fields

SK computation lacks analogous constraints. There is no “gauge group” acting on SK expressions, no dynamics selecting a preferred connection, and no matter fields to couple with. This absence explains why we can construct connections giving any desired holonomy.

Potential Resolution: If SK computation were embedded in a larger structure with physical constraints (e.g., spacetime geometry, energy conservation), a canonical connection might emerge. This suggests that quantum phase is not purely computational but requires additional physical input.

5.6 Value of Negative Results

This constitutes a scientifically valuable negative result [4]:

- It establishes **what doesn’t work**
- It clarifies **what additional assumptions are needed**
- It provides a **roadmap** for future investigations

6 Conclusion

We systematically investigated whether complex number structure can be derived from SK combinatory logic through four approaches: Sorkin’s quantum measure theory, algebraic structure, path space holonomy, and information-theoretic derivation.

Main finding: Complex structure does not automatically emerge from SK computation. While phase can be *computed* from information-theoretic quantities, the computation formula remains arbitrary.

Implications:

1. SK computation is fundamentally classical
2. Quantum structure requires additional assumptions beyond minimal computation
3. Future work should focus on identifying canonical principles for phase assignment

This negative result contributes to our understanding of the relationship between computation and quantum mechanics, providing constraints that future theories must satisfy.

6.1 Future Directions: Alternative Computational Models

Our investigation focused on SK combinatory logic. Natural extensions include:

Model	Key Difference from SK	Potential
λ -calculus	Variable binding, substitution	Similar (Curry-Howard)
Reversible combinators [?]	Information-preserving	Higher
Affine combinators	Linear resource usage	Medium
Typed λ -calculus	Type constraints	Unknown
Qubit combinators	Explicit quantum primitives	Built-in
ZX-calculus [?]	Graphical quantum reasoning	Built-in

Table 6: Alternative computational models for future investigation. “Potential” indicates likelihood of deriving complex structure.

Reversible combinators are particularly promising: since quantum mechanics is unitary (reversible), a computational model that preserves information might more naturally support complex structure. The connection

to Bennett’s reversible computation [?] and Landauer’s principle deserves systematic study.

Acknowledgments

[To be added]

References

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A Implementation Details

All experiments were implemented in Python. The codebase includes:

- SK expression parser and AST (37 tests)
- Reduction engine with redex detection (31 tests)

- Multiway graph construction (21 tests)
- Holonomy computation (19 tests)
- Information-theoretic analysis (15 tests)

Total: 123 passing tests.

B Test Expressions

Expression	Nodes	Paths
$S(K a)(K b)c$	5	2
$(K a b)(K c d)$	4	2
$(K a b)(K c d)(K e f)$	8	6
$S(K a)(K b)(S c d e)$	11	16

Table 7: Test expressions and their multiway graph statistics

C Example Multiway Graphs

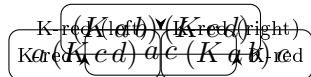


Figure 1: Multiway graph for $(K a b)(K c d)$. Two paths exist: left-first and right-first K-reductions. Both terminate at $a c$. This creates a loop structure where holonomy can be computed.



Figure 2: Multiway graph for $S(K a)(K b)c$. After the initial S-reduction, two K-reductions are possible in either order. The path space has a diamond structure: both paths share the same number of reductions (1 S + 2 K), yielding zero holonomy under the “linear” formula.

Remark 5. *The multiway graph structure becomes exponentially complex for larger expressions. The expression $S(Ka)(Kb)(Scde)$ generates 11 nodes and 16 distinct paths, with 120 possible loop pairs. This combinatorial explosion is characteristic of non-deterministic computation and motivates our systematic computational approach.*

D Glossary of Terms

Redex A *reducible expression*—a subexpression matching the left-hand side of a reduction rule ($Sxyz$ or Kxy).

Normal Form An expression containing no redexes; the computation has terminated.

Multiway Graph A directed graph where nodes are expressions and edges are single reduction steps. Multiple edges from a node indicate non-deterministic choice.

Connection A function assigning a phase $\theta \in [0, 2\pi]$ to each reduction step. Generalizes the notion of gauge connection to computation paths.

Holonomy The total phase accumulated around a closed loop in the path space. Non-trivial holonomy ($\neq 1$) indicates “curvature” in the connection.