IT309 Digital Signal Processing
Lecture 05 Fourier Transform

Adding sine waves

Adding together sine waves of equal frequency, but arbitrary amplitude and phase, results in another sine wave of the same frequency:

$$A_1 \cdot \sin(\omega t + \varphi_1) + A_2 \cdot \sin(\omega t + \varphi_2) \; = \; A \cdot \sin(\omega t + \varphi)$$

Why?

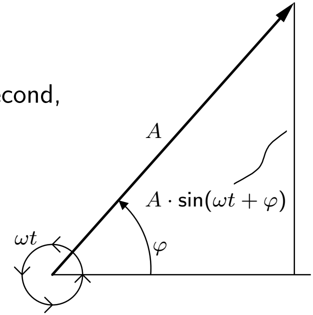
Adding sine waves

Adding together sine waves of equal frequency, but arbitrary amplitude and phase, results in another sine wave of the same frequency:

$$A_1 \cdot \mathsf{sin}(\omega t + \varphi_1) + A_2 \cdot \mathsf{sin}(\omega t + \varphi_2) \;=\; A \cdot \mathsf{sin}(\omega t + \varphi)$$

Why?

Think of $A \cdot \mathsf{sin}(\omega t + \varphi)$ as the height of an arrow of length A, rotating $\frac{\omega}{2\pi}$ times per second, with start angle φ (radians) at $t = 0$.



Adding sine waves

Adding together sine waves of equal frequency, but arbitrary amplitude and phase, results in another sine wave of the same frequency:

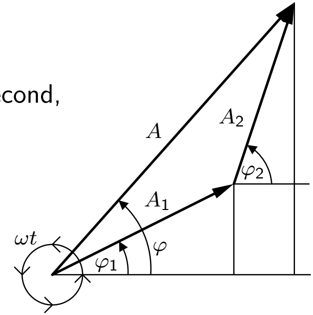
$$A_1 \cdot \sin(\omega t + \varphi_1) + A_2 \cdot \sin(\omega t + \varphi_2) \;=\; A \cdot \sin(\omega t + \varphi)$$

Why?

Think of $A \cdot \sin(\omega t + \varphi)$ as the height of an arrow of length A, rotating $\frac{\omega}{2\pi}$ times per second, with start angle φ (radians) at $t = 0$.

Consider two more such arrows, of length A_1 and A_2, with start angles φ_1 and φ_2.

A_1 and A_2 stuck together are as high as A, all three rotating at the same frequency.



Adding sine waves

Adding together sine waves of equal frequency, but arbitrary amplitude and phase, results in another sine wave of the same frequency:

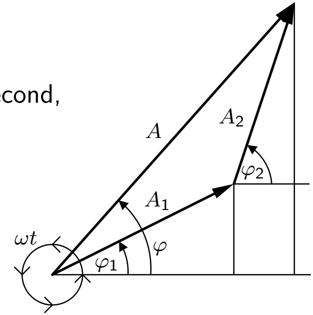
$$A_1 \cdot \sin(\omega t + \varphi_1) + A_2 \cdot \sin(\omega t + \varphi_2) \; = \; A \cdot \sin(\omega t + \varphi)$$

Why?

Think of $A \cdot \sin(\omega t + \varphi)$ as the height of an arrow of length A, rotating $\frac{\omega}{2\pi}$ times per second, with start angle φ (radians) at $t = 0$.

Consider two more such arrows, of length A_1 and A_2, with start angles φ_1 and φ_2.

A_1 and A_2 stuck together are as high as A, all three rotating at the same frequency.



But adding sine waves as vectors (A_1, φ_1) and (A_2, φ_2) in polar coordinates is cumbersome:

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\varphi_2 - \varphi_1)}, \quad \tan\varphi = \frac{A_1 \sin\varphi_1 + A_2 \sin\varphi_2}{A_1 \cos\varphi_1 + A_2 \cos\varphi_2}$$

Cartesian coordinates for sine waves

Sine waves of any amplitude A and phase (start angle) φ can be represented as linear combinations of $\sin(\omega t)$ and $\cos(\omega t)$:

$$A \cdot \sin(\omega t + \varphi) = x \cdot \sin(\omega t) + y \cdot \cos(\omega t)$$

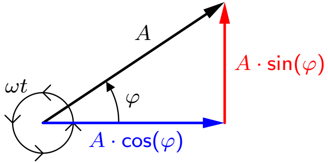
$\cos(\omega t) = \sin(\omega t + 90^\circ)$

where

$$x = A \cdot \cos(\varphi), \quad y = A \cdot \sin(\varphi)$$

and

$$A = \sqrt{x^2 + y^2}, \quad \tan\varphi = \frac{y}{x}.$$



Base: two rotating arrows with start angles 0° [height $= \sin(\omega)$] and 90° [height $= \cos(\omega)$].

Adding two sine waves as vectors in Cartesian coordinates is simple:

$$\begin{aligned} f_1(t) &= x_1 \cdot \sin(\omega) + y_1 \cdot \cos(\omega) \\ f_2(t) &= x_2 \cdot \sin(\omega) + y_2 \cdot \cos(\omega) \\ f_1(t) + f_2(t) &= (x_1 + x_2) \cdot \sin(\omega) + (y_1 + y_2) \cdot \cos(\omega) \end{aligned}$$

Why are sine waves useful?

1) Sine-wave sequences form a family of discrete sequences that is closed under convolution with arbitrary sequences.

Convolution of a discrete sequence $\{x_n\}$ with another sequence $\{h_n\}$ is nothing but adding together scaled and delayed copies of $\{x_n\}$.
Think again of $\{h_n\}$ as decomposed into a sum of impulses:

$$\{x_n\} * \{h_n\} = \{x_n\} * \sum_k h_k \cdot \{\delta_{n-k}\}_n = \sum_k h_k \cdot (\{x_n\} * \{\delta_{n-k}\}_n)$$
$$= \sum_k h_k \cdot \{x_{n-k}\}_n$$

If $\{x_n\}$ is a sampled sine wave of frequency f, i.e.

$$x_n = A_x \cdot \sin(2\pi ft + \phi_x)$$

then $\{y_n\} = \{x_n\} * \{h_n\} = \sum_k h_k \cdot \{x_{n-k}\}_n$ is another sampled sine wave of frequency f, i.e. for each $\{h_n\}$ there exists a pair (A_y, ϕ_y) with

$$y_n = A_y \cdot \sin(2\pi ft + \phi_y)$$

The equivalent applies for continuous sine waves and convolution.

2) Sine waves are orthogonal to each other

The term "orthogonal" is used here in the context of an (infinitely dimensional) vector space, where the "vectors" are functions of the form $f : \mathbb{R} \to \mathbb{R}$ (or $f : \mathbb{R} \to \mathbb{C}$) and the scalar product is defined as

$$f \cdot g = \int f(t) \cdot g(t) \, \mathrm{d}t.$$

Over integer (half-)periods:

$$m, n \in \mathbb{N}, m \neq n \qquad \Rightarrow \qquad \int_0^\pi \sin(nt) \sin(mt) \mathrm{d}t = 0$$

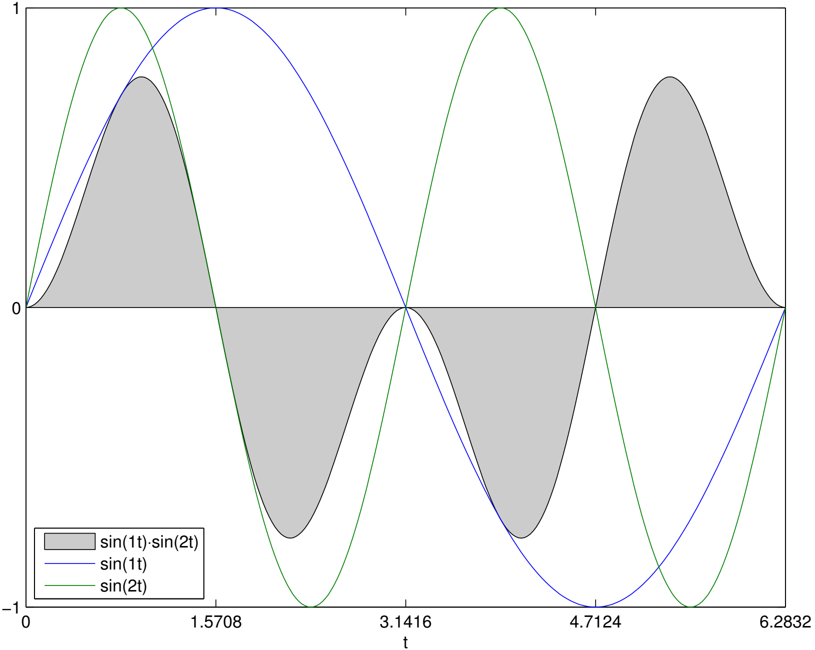
$$m, n \in \mathbb{N} \qquad \Rightarrow \qquad \int_{-\pi}^\pi \sin(nt) \cos(mt) \mathrm{d}t = 0$$

We can even (with some handwaving) extend this to improper integrals:

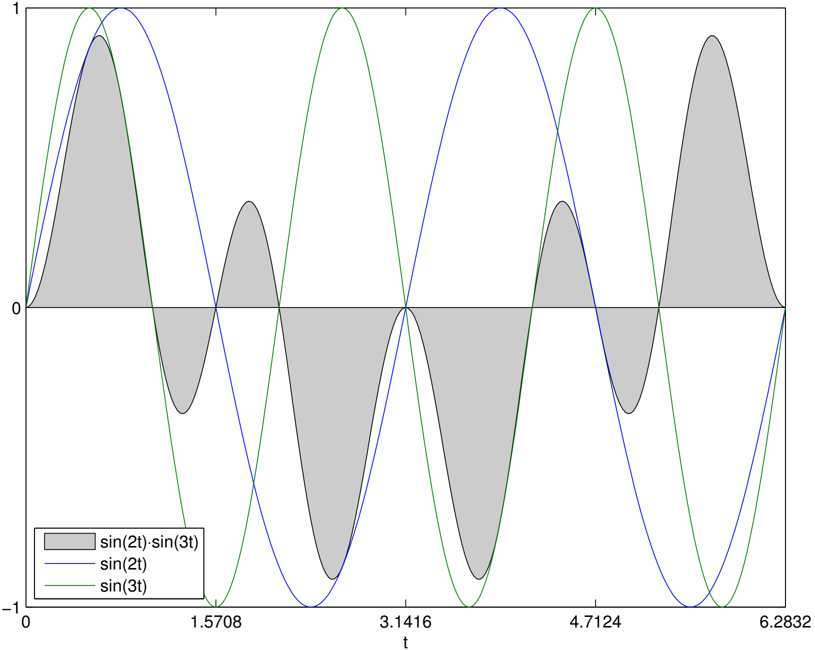
$$\int_{-\infty}^\infty \sin(\omega_1 t + \varphi_1) \cdot \sin(\omega_2 t + \varphi_2) \, \mathrm{d}t \text{ ``$=$'' } 0$$

$$\iff \quad \omega_1 \neq \omega_2 \quad \vee \quad \varphi_1 - \varphi_2 = (2k+1)\pi/2 \quad (k \in \mathbb{Z})$$

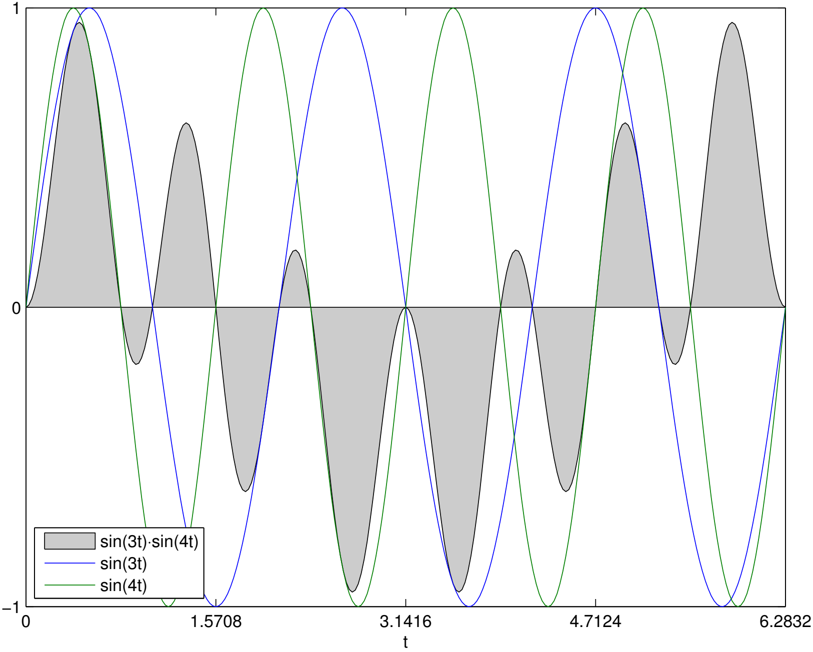
They can be used to form an orthogonal function basis for a transform.

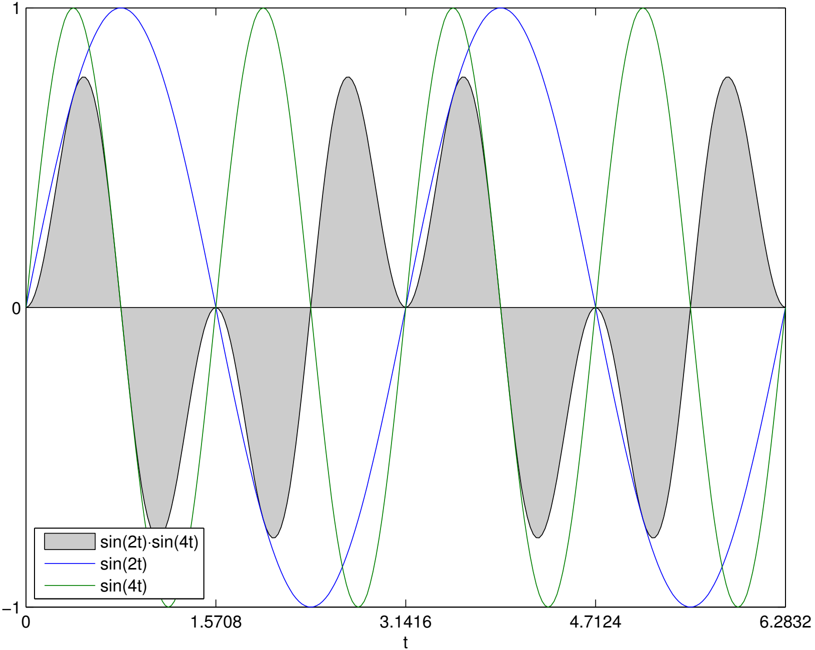


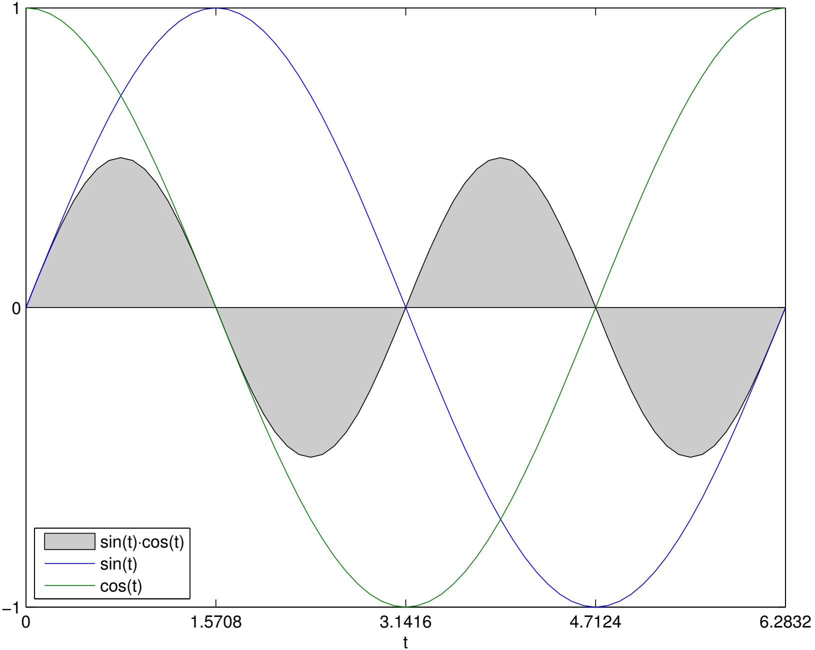
1
0
−1
0
1.5708
3.1416
4.7124
6.2832
t
sin(1t)·sin(2t)
sin(1t)
sin(2t)



1
0
−1
0
1.5708
3.1416
4.7124
6.2832
t
sin(2t)·sin(3t)
sin(2t)
sin(3t)


1
0
−1
0
1.5708
3.1416
4.7124
6.2832
t
sin(3t)·sin(4t)
sin(3t)
sin(4t)


1
0
−1
0
1.5708
3.1416
4.7124
6.2832
t
sin(2t)·sin(4t)
sin(2t)
sin(4t)


1
0
−1
0
1.5708
3.1416
4.7124
6.2832
t
sin(t)·cos(t)
sin(t)
cos(t)

Why are exponential functions useful?

Adding together two exponential functions with the same base z, but different scale factor and offset, results in another exponential function with the same base:

$$\begin{aligned} A_1 \cdot z^{t+\varphi_1} + A_2 \cdot z^{t+\varphi_2} &= A_1 \cdot z^t \cdot z^{\varphi_1} + A_2 \cdot z^t \cdot z^{\varphi_2} \\ &= (A_1 \cdot z^{\varphi_1} + A_2 \cdot z^{\varphi_2}) \cdot z^t = A \cdot z^t \end{aligned}$$

Likewise, if we convolve a sequence $\{x_n\}$ of values

$$\ldots, z^{-3}, z^{-2}, z^{-1}, 1, z, z^2, z^3, \ldots$$

$x_n = z^n$ with an arbitrary sequence $\{h_n\}$, we get $\{y_n\} = \{z^n\} * \{h_n\}$,

$$y_n = \sum_{k=-\infty}^{\infty} x_{n-k} \cdot h_k = \sum_{k=-\infty}^{\infty} z^{n-k} \cdot h_k = z^n \cdot \sum_{k=-\infty}^{\infty} z^{-k} \cdot h_k = z^n \cdot H(z)$$

where $H(z)$ is independent of n.

Exponential sequences are closed under convolution with arbitrary sequences.

The same applies in the continuous case.

Why are complex numbers so useful?

1) They give us all n solutions ("roots") of equations involving polynomials up to degree n (the " $\sqrt{-1} = \mathrm{j}$ " story).

2) They give us the "great unifying theory" that combines sine and exponential functions:

$$\begin{aligned} \cos(\theta) &= \frac{1}{2}\left(\mathrm{e}^{\mathrm{j}\theta} + \mathrm{e}^{-\mathrm{j}\theta}\right) \\ \sin(\theta) &= \frac{1}{2\mathrm{j}}\left(\mathrm{e}^{\mathrm{j}\theta} - \mathrm{e}^{-\mathrm{j}\theta}\right) \end{aligned}$$

or

$$\cos(\omega t + \varphi) = \frac{1}{2}\left(\mathrm{e}^{\mathrm{j}(\omega t+\varphi)} + \mathrm{e}^{-\mathrm{j}(\omega t+\varphi)}\right)$$

or

$$\begin{aligned} \cos(\dot{\omega} n + \varphi) &= \Re(\mathrm{e}^{\mathrm{j}(\dot{\omega} n+\varphi)}) = \Re[(\mathrm{e}^{\mathrm{j}\dot{\omega}})^n \cdot \mathrm{e}^{\mathrm{j}\varphi}] \\ \sin(\dot{\omega} n + \varphi) &= \Im(\mathrm{e}^{\mathrm{j}(\dot{\omega} n+\varphi)}) = \Im[(\mathrm{e}^{\mathrm{j}\dot{\omega}})^n \cdot \mathrm{e}^{\mathrm{j}\varphi}] \end{aligned}$$

Notation: $\Re(a + \mathrm{j}b) := a$, $\Im(a + \mathrm{j}b) := b$ and $(a + \mathrm{j}b)^* := a - \mathrm{j}b$, where $\mathrm{j}^2 = -1$ and $a, b \in \mathbb{R}$.
Then $\Re(x) = \frac{1}{2}(x + x^*)$ and $\Im(x) = \frac{1}{2\mathrm{j}}(x - x^*)$ for all $x \in \mathbb{C}$.

We can now represent sine waves as projections of a rotating complex vector. This allows us to represent sine-wave sequences as exponential sequences with basis $\mathrm{e}^{\mathrm{j}\dot\omega}$.

A phase shift in such a sequence corresponds to a rotation of a complex vector.

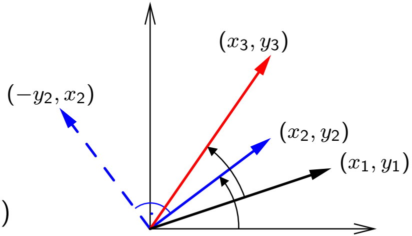
3) Complex multiplication allows us to modify the amplitude and phase of a complex rotating vector using a single operation and value.

Rotation of a 2D vector in (x, y)-form is notationally slightly messy, but fortunately $\mathrm{j}^2 = -1$ does exactly what is required here:

$$\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_2 & -y_2 \\ y_2 & x_2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_1x_2 - y_1y_2 \\ x_1y_2 + x_2y_1 \end{pmatrix}$$

$$z_1 = x_1 + \mathrm{j}y_1, \quad z_2 = x_2 + \mathrm{j}y_2$$

$$z_1 \cdot z_2 = x_1x_2 - y_1y_2 + \mathrm{j}(x_1y_2 + x_2y_1)$$



Complex phasors

Amplitude and phase are two distinct characteristics of a sine function that are inconvenient to keep separate notationally.

Complex functions (and discrete sequences) of the form

$$(A \cdot \mathrm{e}^{\mathrm{j}\varphi}) \cdot \mathrm{e}^{\mathrm{j}\omega t} = A \cdot \mathrm{e}^{\mathrm{j}(\omega t + \varphi)} = A \cdot [\cos(\omega t + \varphi) + \mathrm{j} \cdot \sin(\omega t + \varphi)]$$

(where $\mathrm{j}^2 = -1$) are able to represent both amplitude $A \in \mathbb{R}^+$ and phase $\varphi \in [0, 2\pi)$ in one single algebraic object $A \cdot \mathrm{e}^{\mathrm{j}\varphi} \in \mathbb{C}$.

Thanks to complex multiplication, we can also incorporate in one single factor both a multiplicative change of amplitude and an additive change of phase of such a function. This makes discrete sequences of the form

$$x_n = \mathrm{e}^{\mathrm{j}\dot{\omega} n}$$

eigensequences with respect to an LTI system T, because for each $\dot{\omega}$, there is a complex number (eigenvalue) $H(\dot{\omega})$ such that

$$T\{x_n\} = H(\dot{\omega}) \cdot \{x_n\}$$

In the notation of slide 14, where the argument of H is the base, we would write $H(\mathrm{e}^{\mathrm{j}\dot{\omega}})$.

Recall: Fourier transform

We define the Fourier integral transform and its inverse as

$$\mathcal{F}\{g(t)\}(f) = G(f) = \int_{-\infty}^{\infty} g(t) \cdot \mathrm{e}^{-2\pi \mathrm{j} f t} \, \mathrm{d}t$$

$$\mathcal{F}^{-1}\{G(f)\}(t) = g(t) = \int_{-\infty}^{\infty} G(f) \cdot \mathrm{e}^{2\pi \mathrm{j} f t} \, \mathrm{d}f$$

Many equivalent forms of the Fourier transform are used in the literature. There is no strong consensus on whether the forward transform uses $\mathrm{e}^{-2\pi \mathrm{j} f t}$ and the backwards transform $\mathrm{e}^{2\pi \mathrm{j} f t}$, or vice versa. The above form uses the *ordinary frequency* f, whereas some authors prefer the *angular frequency* $\omega = 2\pi f$:

$$\mathcal{F}\{h(t)\}(\omega) = H(\omega) = \alpha \int_{-\infty}^{\infty} h(t) \cdot \mathrm{e}^{\mp \mathrm{j}\omega t} \, \mathrm{d}t$$

$$\mathcal{F}^{-1}\{H(\omega)\}(t) = h(t) = \beta \int_{-\infty}^{\infty} H(\omega) \cdot \mathrm{e}^{\pm \mathrm{j}\omega t} \, \mathrm{d}\omega$$

This substitution introduces factors α and β such that $\alpha\beta = 1/(2\pi)$. Some authors set $\alpha = 1$ and $\beta = 1/(2\pi)$, to keep the convolution theorem free of a constant prefactor; others prefer the unitary form $\alpha = \beta = 1/\sqrt{2\pi}$, in the interest of symmetry.

Properties of the Fourier transform

If

$$x(t) \quad \bullet\!\!-\!\!\circ \quad X(f) \qquad \text{and} \qquad y(t) \quad \bullet\!\!-\!\!\circ \quad Y(f)$$

are pairs of functions that are mapped onto each other by the Fourier transform, then so are the following pairs.

Linearity:

$$ax(t) + by(t) \quad \bullet\!\!-\!\!\circ \quad aX(f) + bY(f)$$

Time scaling:

$$x(at) \quad \bullet\!\!-\!\!\circ \quad \frac{1}{|a|} X\left(\frac{f}{a}\right)$$

Frequency scaling:

$$\frac{1}{|a|} x\left(\frac{t}{a}\right) \quad \bullet\!\!-\!\!\circ \quad X(af)$$

Time shifting:

$$x(t - \Delta t) \quad \bullet\!\!-\!\!\circ \quad X(f) \cdot \mathrm{e}^{-2\pi \mathrm{j} f \Delta t}$$

Frequency shifting:

$$x(t) \cdot \mathrm{e}^{2\pi \mathrm{j} \Delta f t} \quad \bullet\!\!-\!\!\circ \quad X(f - \Delta f)$$

Time reversal:

$$x(-t) \quad \bullet\!\!-\!\!\circ \quad X(-f)$$

Complex conjugate:

$$\begin{aligned} x^*(t) \quad &\bullet\!\!-\!\!\circ \quad X^*(-f) \\ x^*(-t) \quad &\bullet\!\!-\!\!\circ \quad X^*(f) \end{aligned}$$

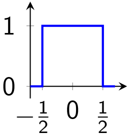
Parseval's theorem (total energy):

$$\int_{-\infty}^{\infty} |x(t)|^2 \mathrm{d}t \quad = \quad \int_{-\infty}^{\infty} |X(f)|^2 \mathrm{d}f$$

Fourier transform example: rect and sinc

The Fourier transform of the "rectangular function"

$$\mathrm{rect}(t) = \begin{cases} 1 & \text{if } |t| < \frac{1}{2} \\ \frac{1}{2} & \text{if } |t| = \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$



is the "(normalized) sinc function"

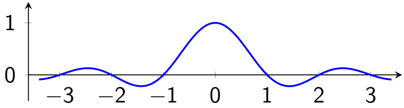
$$\mathcal{F}\{\mathrm{rect}(t)\}(f) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{e}^{-2\pi \mathrm{j} f t} \mathrm{d}t = \frac{\sin \pi f}{\pi f} = \mathrm{sinc}(f)$$

and vice versa

$$\mathcal{F}\{\mathrm{sinc}(t)\}(f) = \mathrm{rect}(f).$$

Some noteworthy properties of these functions:

- $\int_{-\infty}^{\infty} \mathrm{sinc}(t)\,\mathrm{d}t = 1 = \int_{-\infty}^{\infty} \mathrm{rect}(t)\,\mathrm{d}t$
- $\mathrm{sinc}(0) = 1 = \mathrm{rect}(0)$
- $\forall n \in \mathbb{Z} \setminus \{0\} : \mathrm{sinc}(n) = 0$



Convolution theorem

Convolution in the time domain is equivalent to (complex) scalar multiplication in the frequency domain:

$$\mathcal{F}\{(f * g)(t)\} = \mathcal{F}\{f(t)\} \cdot \mathcal{F}\{g(t)\}$$

Proof: $z(r) = \int_s x(s)y(r-s)\mathrm{d}s \quad\Longleftrightarrow\quad \int_r z(r)\mathrm{e}^{-\mathrm{j}\omega r}\mathrm{d}r = \int_r \int_s x(s)y(r-s)\mathrm{e}^{-\mathrm{j}\omega r}\mathrm{d}s\mathrm{d}r =$
$\int_s x(s) \int_r y(r-s)\mathrm{e}^{-\mathrm{j}\omega r}\mathrm{d}r\mathrm{d}s = \int_s x(s)\mathrm{e}^{-\mathrm{j}\omega s} \int_r y(r-s)\mathrm{e}^{-\mathrm{j}\omega(r-s)}\mathrm{d}r\mathrm{d}s \overset{t:=r-s}{=}$
$\int_s x(s)\mathrm{e}^{-\mathrm{j}\omega s} \int_t y(t)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t\mathrm{d}s = \int_s x(s)\mathrm{e}^{-\mathrm{j}\omega s}\mathrm{d}s \cdot \int_t y(t)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t.$

Convolution in the frequency domain corresponds to scalar multiplication in the time domain:

$$\mathcal{F}\{f(t) \cdot g(t)\} = \mathcal{F}\{f(t)\} * \mathcal{F}\{g(t)\}$$

This second form is also called "modulation theorem", as it describes what happens in the frequency domain with amplitude modulation of a signal (see slide 53).

The proof is very similar to the one above.

Both equally work for the inverse Fourier transform:

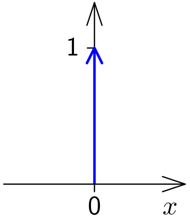
$$\mathcal{F}^{-1}\{(F * G)(f)\} = \mathcal{F}^{-1}\{F(f)\} \cdot \mathcal{F}^{-1}\{G(f)\}$$
$$\mathcal{F}^{-1}\{F(f) \cdot G(f)\} = \mathcal{F}^{-1}\{F(f)\} * \mathcal{F}^{-1}\{G(f)\}$$

Dirac delta function

The continuous equivalent of the impulse sequence $\{\delta_n\}$ is known as Dirac delta function $\delta(x)$. It is a generalized function, defined such that

$$\delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(x)\,\mathrm{d}x = 1$$



and can be thought of as the limit of function sequences such as

$$\delta(x) = \lim_{n\to\infty} \begin{cases} 0, & |x| \geq 1/n \\ n/2, & |x| < 1/n \end{cases}$$

or

$$\delta(x) = \lim_{n\to\infty} \frac{n}{\sqrt{\pi}}\,\mathrm{e}^{-n^2x^2}$$

The delta function is mathematically speaking not a function, but a *distribution*, that is an expression that is only defined when integrated.

Some properties of the Dirac delta function:

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\delta(x-a)\,\mathsf{d}x &= f(a) \\
\int_{-\infty}^{\infty} \mathsf{e}^{\pm 2\pi \mathsf{j} xa} \mathsf{d}x &= \delta(a) \\
\sum_{i=-\infty}^{\infty} \mathsf{e}^{\pm 2\pi \mathsf{j} ixa} &= \frac{1}{|a|} \sum_{i=-\infty}^{\infty} \delta(x - i/a) \\
\delta(ax) &= \frac{1}{|a|}\delta(x)
\end{aligned}$$

Fourier transform:

$$\begin{aligned}
\mathcal{F}\{\delta(t)\}(f) &= \int_{-\infty}^{\infty} \delta(t) \cdot \mathsf{e}^{-2\pi \mathsf{j} ft}\,\mathsf{d}t = \mathsf{e}^0 = 1 \\
\mathcal{F}^{-1}\{1\}(t) &= \int_{-\infty}^{\infty} 1 \cdot \mathsf{e}^{2\pi \mathsf{j} ft}\,\mathsf{d}f = \delta(t)
\end{aligned}$$

Linking the Dirac delta with the Fourier transform

The Fourier transform of 1 follows from the Dirac delta's ability to sample inside an integral:

$$\begin{aligned}
g(t) &= \mathcal{F}^{-1}(\mathcal{F}(g))(t) \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(s) \cdot \mathrm{e}^{-2\pi \mathrm{j} f s} \cdot \mathrm{d}s \right) \cdot \mathrm{e}^{2\pi \mathrm{j} f t} \cdot \mathrm{d}f \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \mathrm{e}^{-2\pi \mathrm{j} f s} \cdot \mathrm{e}^{2\pi \mathrm{j} f t} \cdot \mathrm{d}f \right) \cdot g(s) \cdot \mathrm{d}s \\
&= \int_{-\infty}^{\infty} \underbrace{\left(\int_{-\infty}^{\infty} \mathrm{e}^{-2\pi \mathrm{j} f (s-t)} \cdot \mathrm{d}f \right)}_{\delta(s-t)} \cdot g(s) \cdot \mathrm{d}s
\end{aligned}$$

So if δ has the property

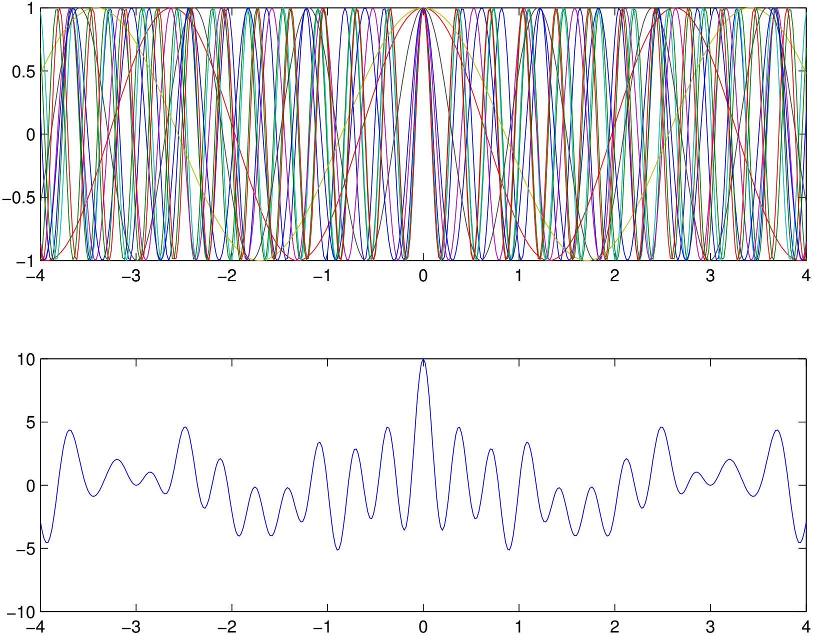
$$g(t) = \int_{-\infty}^{\infty} \delta(s-t) \cdot g(s) \cdot \mathrm{d}s$$

then

$$\int_{-\infty}^{\infty} \mathrm{e}^{-2\pi \mathrm{j} f (s-t)} \, \mathrm{d}f = \delta(s-t)$$

$\int_{-\infty}^{\infty} \mathrm{e}^{2\pi \mathrm{j} t f} \mathrm{d}f = \delta(t)$ $\qquad$ $\sum_{i=1}^{10} \cos(2\pi f_i t) \approx \delta(t)$

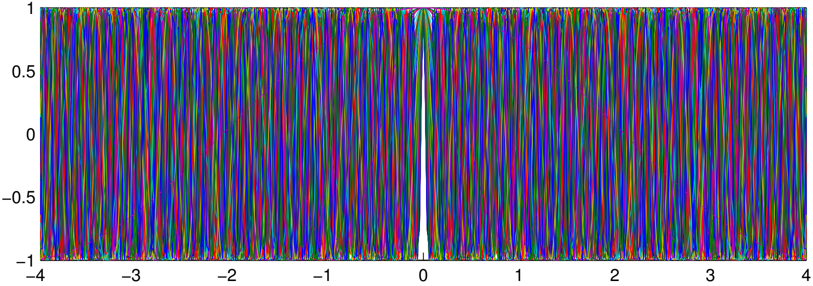
$f_1, \ldots, f_{10} \in [0, 3]$ chosen uniformly at random

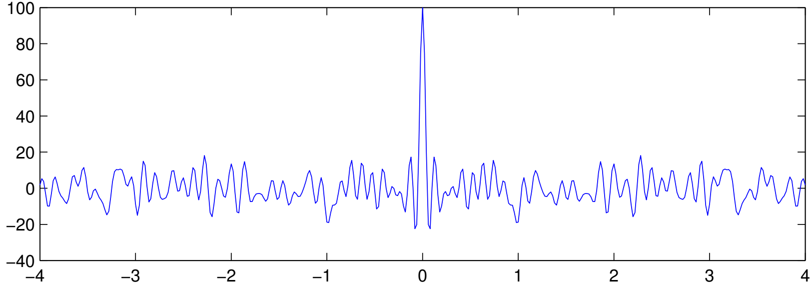


$$\int_{-\infty}^{\infty} e^{2\pi j t f} df = \delta(t)$$

$\sum_{i=1}^{100} \cos(2\pi f_i t) \approx \delta(t)$

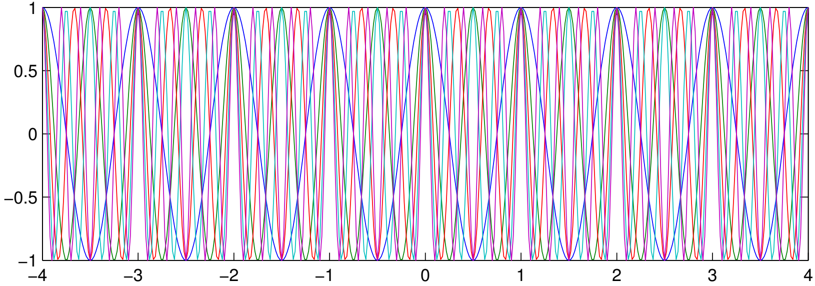
$f_1, \dots, f_{100} \in [0, 10]$ chosen uniformly at random

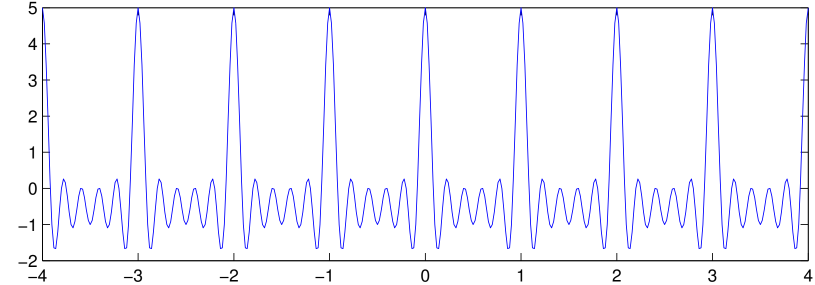




$$\sum_{n=-\infty}^{\infty} \mathsf{e}^{\pm 2\pi \mathsf{j} n t} = \sum_{n=-\infty}^{\infty} \delta(t-n)$$

$$\sum_{n=1}^{5} \cos(2\pi nt) \approx \sum_{n=-\infty}^{\infty} \delta(t-n)$$



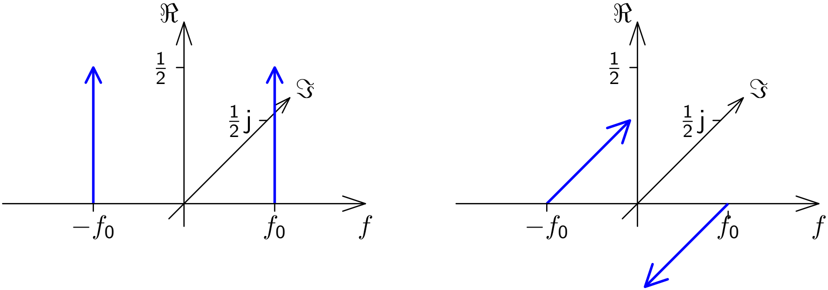


Sine and cosine in the frequency domain

$$\cos(2\pi f_0 t) = \frac{1}{2}\,\mathrm{e}^{2\pi \mathrm{j} f_0 t} + \frac{1}{2}\,\mathrm{e}^{-2\pi \mathrm{j} f_0 t} \qquad \sin(2\pi f_0 t) = \frac{1}{2\mathrm{j}}\,\mathrm{e}^{2\pi \mathrm{j} f_0 t} - \frac{1}{2\mathrm{j}}\,\mathrm{e}^{-2\pi \mathrm{j} f_0 t}$$

$$\mathcal{F}\{\cos(2\pi f_0 t)\}(f) = \frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$$

$$\mathcal{F}\{\sin(2\pi f_0 t)\}(f) = -\frac{\mathrm{j}}{2}\delta(f - f_0) + \frac{\mathrm{j}}{2}\delta(f + f_0)$$



As any $x(t) \in \mathbb{R}$ can be decomposed into sine and cosine functions, the spectrum of any real-valued signal will show the symmetry $X(-f) = [X(f)]^*$, where * denotes the complex conjugate (i.e., negated imaginary part).

Fourier transform symmetries

We call a function $x(t)$

$$\begin{aligned} \text{odd if } x(-t) &= -x(t) \\ \text{even if } x(-t) &= x(t) \end{aligned}$$

and $\cdot^*$ is the complex conjugate, such that $(a + \mathrm{j}b)^* = (a - \mathrm{j}b)$.

Then

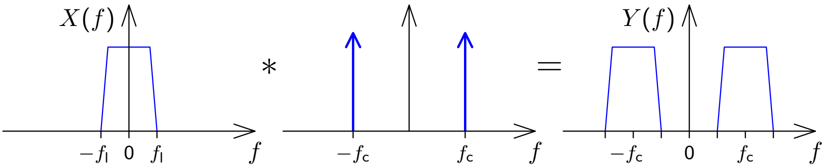
$$\begin{array}{lcl}
x(t) \text{ is real} & \Leftrightarrow & X(-f) = [X(f)]^* \\
x(t) \text{ is imaginary} & \Leftrightarrow & X(-f) = -[X(f)]^* \\
x(t) \text{ is even} & \Leftrightarrow & X(f) \text{ is even} \\
x(t) \text{ is odd} & \Leftrightarrow & X(f) \text{ is odd} \\
x(t) \text{ is real and even} & \Leftrightarrow & X(f) \text{ is real and even} \\
x(t) \text{ is real and odd} & \Leftrightarrow & X(f) \text{ is imaginary and odd} \\
x(t) \text{ is imaginary and even} & \Leftrightarrow & X(f) \text{ is imaginary and even} \\
x(t) \text{ is imaginary and odd} & \Leftrightarrow & X(f) \text{ is real and odd}
\end{array}$$

Example: amplitude modulation

Communication channels usually permit only the use of a given frequency interval, such as 300–3400 Hz for the analog phone network or 590–598 MHz for TV channel 36. Modulation with a carrier frequency f_c shifts the spectrum of a signal $x(t)$ into the desired band.

Amplitude modulation (AM):

$$y(t) = A \cdot \cos(2\pi t f_c) \cdot x(t)$$



The spectrum of the baseband signal in the interval $-f_l < f < f_l$ is shifted by the modulation to the intervals $\pm f_c - f_l < f < \pm f_c + f_l$.

How can such a signal be demodulated?