5

Discrete-time Fourier Transform

5.1 INTRODUCTION

A continuous-time signal can be represented in the frequency domain using Laplace transform or continuous-time Fourier transform (CTFT). Similarly, a discrete-time signal can be represented in the frequency domain using Z-transform or discrete-time Fourier transform. The Fourier transform of a discrete-time signal is called discrete-time Fourier transform (DTFT). DTFT is very popular for digital signal processing because of the fact that using this the complicated operation of convolution of two sequences in the time domain can be converted into a much simpler multiplicative operation in the frequency domain. In this chapter, we discuss about DTFT, its properties and its use in the analysis of signals.

5.2 DISCRETE-TIME FOURIER TRANSFORM (DTFT)

The Fourier transform of discrete-time signals is called the discrete-time Fourier transform (DTFT).

If $x(n)$ is the given discrete-time sequence, then $X(\omega)$ or $X(e^{j\omega})$ is the discrete-time Fourier transform of $x(n)$.

The DTFT of $x(n)$ is defined as:

$$\mathrm{F}[x(n)] = X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

The inverse DTFT of $X(\omega)$ is defined as:

$$\mathrm{F}^{-1}[X(\omega)] = x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega$$

We also refer to $x(n)$ and $X(\omega)$ as a Fourier transform pair and this relation is expressed as:

$$x(n) \xleftrightarrow{\text{FT}} X(\omega)$$

5.3 EXISTENCE OF DTFT

The Fourier transform exists for a discrete-time sequence $x(n)$ if and only if the sequence is absolutely summable, i.e. the sequence has to satisfy the condition:

$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty$$

The DTFT does not exist for the sequences that are growing exponentially (ex. $a^n\, u(n)$, $a > 1$) since they are not absolutely summable. Therefore, DTFT method of analyzing a system can be applied for a limited class of signals. Moreover this method can be applied only to asymptotically stable systems and it cannot be applied for unstable systems. That is, DTFT can be used only for the systems whose system function $H(z)$ has poles inside the unit circle.

The Fourier transform $X(\omega)$ of a signal $x(n)$ represents the frequency content of $x(n)$. We can say that, by taking Fourier transform, the signal $x(n)$ is decomposed into its frequency components. Hence $X(\omega)$ is called signal spectrum.

The difference between the Fourier transforms of a discrete-time signal and analog signal are as follows:

1. The Fourier transform of analog signals consists of a spectrum with a frequency range $-\infty$ to ∞. But the Fourier transform of discrete-time signals is unique in the frequency range $-\pi$ to π (or equivalently 0 to 2π). Also Fourier transforms of discrete-time signals are periodic with period 2π. Hence the frequency range for any discrete-time signal is limited to $-\pi$ to π (or 0 to 2π) and any frequency outside this interval has an equivalent frequency within this interval.
2. Since the analog signals are continuous, the Fourier transform of analog signals involves integration, but the Fourier transform of discrete-time signals involves summation because the signals are discrete.

5.4 RELATION BETWEEN Z-TRANSFORM AND FOURIER TRANSFORM

The Z-transform of a discrete sequence $x(n)$ is defined as:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

where z is a complex variable.

The Fourier transform of a discrete-time sequence $x(n)$ is defined as:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

The $X(z)$ can be viewed as a unique representation of the sequence $x(n)$ in the complex z-plane.

Let $z = re^{j\omega}$

$$\therefore \qquad X(z) = \sum_{n=-\infty}^{\infty} x(n)\,(re^{j\omega})^{-n}$$

$$= \sum_{n=-\infty}^{\infty} [x(n)\,r^{-n}]\,e^{-j\omega n}$$

The RHS is the Fourier transform of $x(n)\,r^{-n}$, i.e. the Z-transform of $x(n)$ is the Fourier transform of $x(n)\,r^{-n}$.

When $r = 1$,

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\,e^{-j\omega n} = X(\omega)$$

The RHS is the Fourier transform of $x(n)$. So we can conclude that the Fourier transform of $x(n)$ is same as the Z-transform of $x(n)$ evaluated along the unit circle centred at the origin of the z-plane.

$$\therefore \qquad X(\omega) = X(z)\Big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} x(n)\,z^{-n}\Big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} x(n)\,e^{-j\omega n}$$

For $X(\omega)$ to exist, the ROC must include the unit circle. Since ROC cannot contain any poles of $X(z)$ all the poles must lie inside the unit circle. Therefore, we can conclude that Fourier transform can be obtained for any sequence $x(n)$, from its Z-transform $X(z)$ if the poles of $X(z)$ are inside the unit circle.

EXAMPLE 5.1 Find the DTFT of the following sequences:

(a) $\delta(n)$ (b) $u(n)$

(c) $\delta(n - m)$ (d) $u(n - m)$

(e) $a^n u(n)$ (f) $-a^n u(-n - 1)$

(g) $\delta(n + 3) - \delta(n - 3)$ (h) $u(n + 3) - u(n - 3)$

Solution:

(a) Given $\qquad x(n) = \delta(n)$

$$\delta(n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$$

$$X(\omega) = \mathrm{F}\{\delta(n)\} = \sum_{n=-\infty}^{\infty} \delta(n)\,e^{-j\omega n}\Big|_{n=0} = 1$$

$$\therefore \qquad \mathrm{F}\{\delta(n)\} = 1$$

$$\boxed{\delta(n) \xleftrightarrow{\text{FT}} 1}$$

(b) Given $x(n) = u(n)$

$$u(n) = \begin{cases} 1 & \text{for } n \ge 0 \\ 0 & \text{for } n < 0 \end{cases}$$

$$X(\omega) = \text{F}\{u(n)\} = \sum_{n=-\infty}^{\infty} u(n)\, e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (1)\, e^{-j\omega n} = \frac{1}{1-e^{-j\omega}}$$

$\therefore$
$$\text{F}\{u(n)\} = \frac{1}{1-e^{-j\omega}}$$

$$\boxed{u(n) \xleftrightarrow{\text{FT}} \frac{1}{1-e^{-j\omega}}}$$

(c) Given $x(n) = \delta(n-m)$

$$\delta(n-m) = \begin{cases} 1 & \text{for } n = m \\ 0 & \text{for } n \ne m \end{cases}$$

$$X(\omega) = \text{F}\{\delta(n-m)\} = \sum_{n=-\infty}^{\infty} \delta(n-m)\, e^{-j\omega n} = e^{-j\omega n}\Big|_{n=m} = e^{-j\omega m}$$

$\therefore$
$$\text{F}\{\delta(n-m)\} = e^{-j\omega m}$$

$$\boxed{\delta(n-m) \xleftrightarrow{\text{FT}} e^{-j\omega m}}$$

(d) Given $x(n) = u(n-m)$

$$u(n-m) = \begin{cases} 1 & \text{for } n \ge m \\ 0 & \text{for } n < m \end{cases}$$

$$X(\omega) = \text{F}\{u(n-m)\} = \sum_{n=-\infty}^{\infty} u(n-m)\, e^{-j\omega n} = \sum_{n=m}^{\infty} (1)\, e^{-j\omega n}$$

$$= e^{-j\omega m} + e^{-j\omega(m+1)} + e^{-j\omega(m+2)} + \cdots$$

$$= e^{-j\omega m}(1 + e^{-j\omega} + e^{-j2\omega} + \cdots)$$

$$= \frac{e^{-j\omega m}}{1-e^{-j\omega}}$$

$\therefore$
$$\text{F}\{u(n-m)\} = \frac{e^{-j\omega m}}{1-e^{-j\omega}}$$

$$\boxed{u(n-m) \xleftrightarrow{\text{FT}} \frac{e^{-j\omega m}}{1-e^{-j\omega}}}$$

(e) Given $x(n) = a^n u(n)$

$$X(\omega) = \mathrm{F}\{a^n u(n)\} = \sum_{n=-\infty}^{\infty} a^n u(n)\, e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1 - ae^{-j\omega}}$$

$$\therefore \quad \mathrm{F}\{a^n u(n)\} = \frac{1}{1 - ae^{-j\omega}}$$

$$\boxed{a^n u(n) \xleftrightarrow{\text{FT}} \frac{1}{1 - ae^{-j\omega}}}$$

(f) Given $x(n) = -a^n u(-n-1)$

$$X(\omega) = \mathrm{F}\{-a^n u(-n-1)\}$$

$$= \sum_{n=-\infty}^{\infty} -a^n u(-n-1)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{-1} -a^n e^{-j\omega n} = -\sum_{n=1}^{\infty} a^{-n} e^{j\omega n} = -\sum_{n=1}^{\infty} (a^{-1}e^{j\omega})^n$$

$$= -\left[a^{-1}e^{j\omega} + (a^{-1}e^{j\omega})^2 + (a^{-1}e^{j\omega})^3 + \cdots\right]$$

$$= -a^{-1}e^{j\omega}\left[1 + (a^{-1}e^{j\omega})^1 + (a^{-1}e^{j\omega})^2 + \cdots\right]$$

$$= \frac{-a^{-1}e^{j\omega}}{1 - a^{-1}e^{j\omega}}$$

$$= \frac{1}{1 - ae^{-j\omega}}$$

$$\therefore \quad \mathrm{F}\{-a^n u(-n-1)\} = \frac{1}{1 - ae^{-j\omega}}$$

$$\boxed{-a^n u(-n-1) \xleftrightarrow{\text{FT}} \frac{1}{1 - ae^{-j\omega}}}$$

(g) Given $x(n) = \delta(n+3) - \delta(n-3)$

$$X(\omega) = \mathrm{F}\{\delta(n+3) - \delta(n-3)\}$$

$$= \sum_{n=-\infty}^{\infty} \{\delta(n+3) - \delta(n-3)\}\, e^{-j\omega n}$$

$$= e^{-j\omega n}\Big|_{n=-3} - e^{-j\omega n}\Big|_{n=3}$$

$$= e^{j3\omega} - e^{-j3\omega} = 2j \sin 3\omega$$

(h) Given $x(n) = u(n+3) - u(n-3)$

$$X(\omega) = \mathrm{F}\{u(n+3) - u(n-3)\}$$

$$= \sum_{n=-\infty}^{\infty} \{u(n+3) - u(n-3)\}\, e^{-j\omega n} = \sum_{n=-3}^{\infty} (1)\, e^{-j\omega n} - \sum_{n=3}^{\infty} (1)\, e^{-j\omega n}$$

$$= e^{j3\omega} + e^{j2\omega} + e^{j\omega} + 1 + e^{-j\omega} + e^{-j2\omega} + \cdots - e^{-j3\omega} - e^{-j4\omega} - \cdots$$

$$= e^{j3\omega} + e^{j2\omega} + e^{j\omega} + 1 + e^{-j\omega} + e^{-j2\omega}$$

EXAMPLE 5.2 Find the DTFT of:

(a) $x(n) = \{1, -2, 2, 3\}$

(b) $x(n) = 3^n u(n)$

(c) $x(n) = (0.5)^n u(n) + 2^n u(-n-1)$

(d) $x(n) = \left(\dfrac{1}{4}\right)^n u(n+1)$

(e) $x(n) = \begin{cases} n, & -4 \le n \le 4 \\ 0, & \text{otherwise} \end{cases}$

(f) $x(n) = \begin{cases} 1, & 0 \le n \le 3 \\ 0, & \text{otherwise} \end{cases}$

(g) $x(n) = a^{|n|}$

Solution:

(a) Given $x(n) = \{1, -2, 2, 3\}$

$$X(\omega) = \mathrm{F}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$= x(0) + x(1)\, e^{-j\omega} + x(2)\, e^{-j2\omega} + x(3)\, e^{-j3\omega}$$

$$= 1 - 2e^{-j\omega} + 2e^{-j2\omega} + 3e^{-j3\omega}$$

(b) Given $x(n) = 3^n u(n)$. The given sequence is not absolutely summable. Therefore, its DTFT does not exist.

(c) Given $x(n) = (0.5)^n u(n) + 2^n u(-n-1)$

$$X(\omega) = \mathrm{F}\{x(n)\} = \sum_{n=-\infty}^{\infty} \{(0.5)^n u(n) + 2^n u(-n-1)\}\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \{(0.5)^n u(n)\}\, e^{-j\omega n} + \sum_{n=-\infty}^{\infty} \{2^n u(-n-1)\}\, e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (0.5)^n e^{-j\omega n} + \sum_{n=-\infty}^{-1} 2^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (0.5e^{-j\omega})^n + \sum_{n=1}^{\infty} (2^{-1} e^{j\omega})^n$$

$$= \frac{1}{1 - 0.5e^{-j\omega}} + \frac{2^{-1} e^{j\omega}}{1 - 2^{-1} e^{j\omega}}$$

$$= \frac{1}{1 - 0.5e^{-j\omega}} - \frac{1}{1 - 2e^{-j\omega}}$$

(d) Given $$x(n) = \left(\frac{1}{4}\right)^n u(n+1)$$

$$X(\omega) = \mathrm{F}\left\{\left(\frac{1}{4}\right)^n u(n+1)\right\} = \sum_{n=-\infty}^{\infty} \left(\frac{1}{4}\right)^n u(n+1)\, e^{-j\omega n}$$

$$= \sum_{n=-1}^{\infty} \left(\frac{1}{4}\right)^n e^{-j\omega n} = \sum_{n=-1}^{\infty} \left(\frac{1}{4} e^{-j\omega}\right)^n$$

$$= \left(\frac{1}{4} e^{-j\omega}\right)^{-1} \left[\sum_{n=0}^{\infty} \frac{1}{4} e^{-j\omega}\right] = 4e^{j\omega} \left[\frac{1}{1-(1/4)\, e^{-j\omega}}\right]$$

$$= \frac{4e^{j\omega}}{1-(1/4)\, e^{-j\omega}}$$

(e) Given $$x(n) = \begin{cases} n, & -4 \le n \le 4 \\ 0, & \text{otherwise} \end{cases}$$

$$X(\omega) = \mathrm{F}\{x(n)\} = \sum_{n=-4}^{4} n e^{-j\omega n}$$

$$= -4e^{j4\omega} - 3e^{j3\omega} - 2e^{j2\omega} - e^{j\omega} + e^{-j\omega} + 2e^{-j2\omega} + 3e^{-j3\omega} + 4e^{-4j\omega}$$

$$= -2j\,\{4 \sin 4\omega + 3 \sin 3\omega + 2 \sin 2\omega + \sin \omega\}$$

(f) Given $$x(n) = \begin{cases} 1, & 0 \le n \le 3 \\ 0, & \text{otherwise} \end{cases}$$

By definition of Fourier transform,

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} = \sum_{n=0}^{3} (1)\, e^{-j\omega n}$$

$$= \frac{1-e^{-j4\omega}}{1-e^{-j\omega}} = \frac{1-e^{-j2\omega}e^{-j2\omega}}{1-e^{(-j\omega/2)}e^{(-j\omega/2)}}$$

$$= \frac{e^{-j2\omega}\{e^{j2\omega} - e^{-j2\omega}\}}{e^{(-j\omega/2)}\{e^{(j\omega/2)} - e^{(-j\omega/2)}\}} = \left[\frac{2j \sin 2\omega}{2j \sin(\omega/2)}\right] e^{-j2\omega + j\omega/2}$$

$$= \frac{\sin 2\omega}{\sin(\omega/2)}\, e^{-j(3/2)\omega}$$

(g) Given $$x(n) = a^{|n|}$$

$$X(\omega) = \mathrm{F}\{a^{|n|}\} = \sum_{n=-\infty}^{\infty} a^{|n|}\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{-1} a^{-n}\, e^{-j\omega n} + \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=1}^{\infty} (ae^{j\omega})^n + \sum_{n=0}^{\infty} (ae^{-j\omega})^n$$

$$= \frac{ae^{j\omega}}{1-ae^{j\omega}} + \frac{1}{1-ae^{-j\omega}} = \frac{1-a^2}{1-2a\cos\omega + a^2}$$

EXAMPLE 5.3 Find the DTFT of the following sequences:

(a) $\sin\left(\dfrac{n\pi}{2}\right)u(n)$ (b) $\cos\left(\dfrac{n\pi}{3}\right)u(n)$

(c) $\left(\dfrac{1}{2}\right)^n \sin\left(\dfrac{n\pi}{4}\right)u(n)$ (d) $\left(\dfrac{1}{2}\right)^{n-2} u(n-2)$

(e) $\cos(\omega_0 n)\, u(n)$ (f) $\sin(\omega_0 n)\, u(n)$

Solution:

(a) Given $\qquad x(n) = \sin\left(\dfrac{n\pi}{2}\right)u(n)$

$$X(\omega) = \mathrm{F}\left\{\sin\left(\frac{n\pi}{2}\right)u(n)\right\} = \sum_{n=-\infty}^{\infty}\left\{\sin\left(\frac{n\pi}{2}\right)u(n)\right\}e^{-j\omega n} = \sum_{n=0}^{\infty}\sin\left(\frac{n\pi}{2}\right)e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty}\frac{e^{j(n\pi/2)} - e^{-j(n\pi/2)}}{2j}e^{-j\omega n} = \frac{1}{2j}\left[\sum_{n=0}^{\infty}e^{j[(\pi/2)-\omega]n} - \sum_{n=0}^{\infty}e^{-j[(\pi/2)+\omega]n}\right]$$

$$= \frac{1}{2j}\left[\frac{1}{1-e^{j[(\pi/2)-\omega]}} - \frac{1}{1-e^{-j[(\pi/2)+\omega]}}\right]$$

$$= \frac{1}{2j}\left[\frac{1-e^{-j(\pi/2)}e^{-j\omega} - 1 + e^{j(\pi/2)}e^{-j\omega}}{1+e^{-j2\omega} - e^{-j\omega}[e^{j(\pi/2)} + e^{-j(\pi/2)}]}\right]$$

$$= \frac{e^{-j\omega}\sin(\pi/2)}{1+e^{-j2\omega}} = \frac{e^{-j\omega}}{1+e^{-j2\omega}}$$

(b) Given $\qquad x(n) = \cos\left(\dfrac{n\pi}{3}\right)u(n)$

$$X(\omega) = \mathrm{F}\left\{\cos\left(\frac{n\pi}{3}\right)u(n)\right\} = \sum_{n=-\infty}^{\infty}\left\{\cos\left(\frac{n\pi}{3}\right)u(n)\right\}e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty}\left[\frac{e^{j(n\pi/3)} + e^{-j(n\pi/3)}}{2}u(n)\right]e^{-j\omega n} = \frac{1}{2}\left[\sum_{n=0}^{\infty}e^{j[(\pi/3)-\omega]n} + \sum_{n=0}^{\infty}e^{-j[(\pi/3)+\omega]n}\right]$$

$$= \frac{1}{2}\left[\frac{1}{1-e^{j[(\pi/3)-\omega]}} + \frac{1}{1-e^{-j[(\pi/3)+\omega]}}\right]$$

$$= \frac{1}{2}\left[\frac{1-e^{-j(\pi/3)}e^{-j\omega} + 1 - e^{j(\pi/3)}e^{-j\omega}}{1+e^{-j2\omega} - e^{-j\omega}[e^{j(\pi/3)} + e^{-j(\pi/3)}]}\right]$$

$$= \frac{1}{2}\left[\frac{2-e^{-j\omega}}{1-e^{-j\omega}+e^{-j2\omega}}\right]$$

(c) Given $$x(n) = \left(\frac{1}{2}\right)^n \sin\left(\frac{n\pi}{4}\right) u(n)$$

$$X(\omega) = \mathrm{F}\left\{\left(\frac{1}{2}\right)^n \sin\left(\frac{n\pi}{4}\right) u(n)\right\}$$

$$= \sum_{n=-\infty}^{\infty} \left\{\left(\frac{1}{2}\right)^n \sin\left(\frac{n\pi}{4}\right) u(n)\right\} e^{-j\omega n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \sin\left(\frac{n\pi}{4}\right) e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \left\{\frac{e^{j(n\pi/4)} - e^{-j(n\pi/4)}}{2j}\right\} e^{-j\omega n}$$

$$= \frac{1}{2j}\left[\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n e^{j[(\pi/4)-\omega]n} - \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n e^{-j[(\pi/4)+\omega]n}\right]$$

$$= \frac{1}{2j}\left[\sum_{n=0}^{\infty} \left(\frac{1}{2} e^{j[(\pi/4)-\omega]}\right)^n - \sum_{n=0}^{\infty} \left(\frac{1}{2} e^{-j[(\pi/4)+\omega]}\right)^n\right]$$

$$= \frac{1}{2j}\left[\frac{1}{1-(1/2)\, e^{j[(\pi/4)-\omega]}} - \frac{1}{1-(1/2)\, e^{-j[(\pi/4)+\omega]}}\right]$$

$$= \frac{(1/2)\, e^{-j\omega} \sin(\pi/4)}{1+(1/4)\, e^{-j2\omega} - e^{-j\omega} \cos(\pi/4)}$$

$$= \frac{(1/2\sqrt{2})\, e^{-j\omega}}{1-(1/\sqrt{2})\, e^{-j\omega} + (1/4) e^{-j2\omega}}$$

(d) Given $$x(n) = \left(\frac{1}{2}\right)^{n-2} u(n-2)$$

$$X(\omega) = \mathrm{F}\{x(n)\} = \sum_{n=-\infty}^{\infty} \left[\left(\frac{1}{2}\right)^{n-2} u(n-2)\right] e^{-j\omega n}$$

$$= \sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^{n-2} e^{-j\omega n}$$

$$= e^{-j2\omega} + \frac{1}{2} e^{-j3\omega} + \left(\frac{1}{2}\right)^2 e^{-j4\omega} + \cdots$$

$$= e^{-j2\omega}\left[1 + \frac{1}{2} e^{-j\omega} + \left(\frac{1}{2}\right)^2 e^{-j2\omega} + \cdots\right]$$

$$= \frac{e^{-j2\omega}}{1-(1/2)\, e^{-j\omega}}$$

(e) Given $x(n) = \cos(\omega_0 n)\ u(n)$

$$X(\omega) = \text{F}\{x(n)\} = \sum_{n=-\infty}^{\infty} \{\cos(\omega_0 n)\ u(n)\}\ e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left[\frac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2}\right] e^{-j\omega n}$$

$$= \frac{1}{2}\left\{\sum_{n=0}^{\infty} [e^{j(\omega_0-\omega)}]^n + \sum_{n=0}^{\infty} [e^{-j(\omega_0+\omega)}]^n\right\}$$

$$= \frac{1}{2}\left[\frac{1}{1-e^{j(\omega_0-\omega)}} + \frac{1}{1-e^{-j(\omega_0+\omega)}}\right]$$

$$= \frac{1}{2}\left[\frac{1-e^{-j(\omega_0+\omega)} + 1 - e^{j(\omega_0-\omega)}}{1+e^{-j2\omega} - e^{-j\omega}(e^{j\omega_0} + e^{-j\omega_0})}\right]$$

$$= \frac{1-e^{-j\omega}\cos\omega_0}{1-2e^{-j\omega}\cos\omega_0 + e^{-j2\omega}}$$

(f) Given $x(n) = \sin(\omega_0 n)\ u(n)$

$$X(\omega) = \text{F}\{x(n)\} = \sum_{n=-\infty}^{\infty} \{\sin(\omega_0 n)\ u(n)\}\ e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left\{\frac{e^{j\omega_0 n} - e^{-j\omega_0 n}}{2j}\right\} e^{-j\omega n} = \sum_{n=0}^{\infty} \left\{\frac{e^{j(\omega_0-\omega)n} - e^{-j(\omega_0+\omega)n}}{2j}\right\}$$

$$= \frac{1}{2j}\left[\frac{1}{1-e^{j(\omega_0-\omega)}} - \frac{1}{1-e^{-j(\omega_0+\omega)}}\right]$$

$$= \frac{1}{2j}\left[\frac{1-e^{-j(\omega_0+\omega)} - 1 + e^{j(\omega_0-\omega)}}{1+e^{-j2\omega} - e^{-j\omega}(e^{j\omega_0} + e^{-j\omega_0})}\right]$$

$$= \frac{e^{-j\omega}\sin\omega_0}{1-2e^{-j\omega}\cos\omega_0 + e^{-j2\omega}}$$

EXAMPLE 5.4 Find the DTFT of the rectangular pulse sequence:

$$x(n) = \begin{cases} A, & |n| \le N \\ 0, & |n| > N \end{cases}$$

Solution: Given

$$x(n) = \begin{cases} A, & |n| \le N \\ 0, & |n| > N \end{cases}$$

$$X(\omega) = \sum_{n=-N}^{N} Ae^{-j\omega n} = \sum_{n=-N}^{-1} Ae^{-j\omega n} + \sum_{n=0}^{N} Ae^{-j\omega n}$$

$$= \sum_{n=1}^{N} Ae^{j\omega n} + \sum_{n=0}^{N} Ae^{-j\omega n} = Ae^{j\omega}\sum_{n=0}^{N-1} e^{j\omega n} + A\sum_{n=0}^{N} e^{-j\omega n}$$

$$= Ae^{j\omega}\left[\frac{1-e^{j\omega N}}{1-e^{j\omega}}\right] + A\left[\frac{1-e^{-j\omega(N+1)}}{1-e^{-j\omega}}\right]$$

$$= A\left[\frac{e^{j\omega} - e^{j\omega(N+1)}}{1-e^{j\omega}}\right] + A\left[\frac{1-e^{-j\omega(N+1)}}{1-e^{-j\omega}}\right]$$

$$= A\left[\frac{e^{j\omega} - 1 - e^{j\omega(N+1)} + e^{j\omega N} + 1 - e^{j\omega} - e^{-j\omega(N+1)} + e^{-j\omega N}}{1+1-e^{j\omega} - e^{-j\omega}}\right]$$

$$= A\left[\frac{(e^{j\omega N} + e^{-j\omega N}) - (e^{j\omega(N+1)} + e^{-j\omega(N+1)})}{2-(e^{j\omega} + e^{-j\omega})}\right]$$

$$= A\left[\frac{2\cos\omega N - 2\cos\omega(N+1)}{2-2\cos\omega}\right]$$

$$= A\left[\frac{2\sin\omega[N+(1/2)]\sin(\omega/2)}{2\sin^2(\omega/2)}\right] = \frac{A\sin\omega[N+(1/2)]}{\sin(\omega/2)}$$

5.5 INVERSE DISCRETE-TIME FOURIER TRANSFORM

The process of finding the discrete-time sequence $x(n)$ from its frequency response $X(\omega)$ is called the inverse discrete-time Fourier transform.

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega$$

The integral solution of the above equation for $x(n)$, i.e, for the inverse Fourier transform is useful for analytic purpose, but it is usually very difficult to evaluate for typical functional forms of $X(\omega)$. An alternate and more useful method of determining the values of $x(n)$ follows directly from the definition of the Fourier transform.

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} = \cdots + x(-2)\, e^{j2\omega} + x(-1)\, e^{j\omega} + x(0) + x(1)\, e^{-j\omega} + x(2)\, e^{-j2\omega} + \cdots$$

From the defining equation of $X(\omega)$ we can say that, if $X(\omega)$ can be expressed as a series of complex exponentials as shown in the above equation for $X(\omega)$, then $x(n)$ is simply the coefficient of $e^{-j\omega n}$. Inverse Fourier transform can be obtained by using the partial fraction method or by using the convolution theorem.

EXAMPLE 5.5 Determine the signal $x(n)$ for the given Fourier transforms:

(a) $X(\omega) = e^{-j\omega}$ for $-\pi \le \omega \le \pi$ (b) $X(\omega) = e^{-j\omega}(1 + \cos\omega)$

Solution:

(a) Given $X(\omega) = e^{-j\omega}$

$$x(n) = \mathrm{F}^{-1}\{X(\omega)\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\omega}\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-1)}\, d\omega = \frac{1}{2\pi}\left[\frac{e^{j\omega(n-1)}}{j(n-1)}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\pi(n-1)} - e^{-j\pi(n-1)}}{j(n-1)}\right] = \frac{1}{\pi(n-1)}\left[\frac{e^{j\pi(n-1)} - e^{-j\pi(n-1)}}{2j}\right]$$

$$= \frac{\sin \pi(n-1)}{\pi(n-1)}$$

(b) Given $X(\omega) = e^{-j\omega}(1 + \cos\omega)$

$$= e^{-j\omega}\left(1 + \frac{e^{j\omega} + e^{-j\omega}}{2}\right)$$

$$= e^{-j\omega} + 0.5 + 0.5e^{-j2\omega}$$

$$= x(0) + x(1)\, e^{-j\omega} + x(2)e^{-j2\omega}$$

Therefore, $x(0) = 0.5,\ x(1) = 1,\ x(2) = 0.5$

$x(n) = 0$, otherwise

i.e. $x(n) = \{0.5,\ 1,\ 0.5\}$

EXAMPLE 5.6 Obtain the impulse response of the system described by

$$H(\omega) = \begin{cases} 1, & \text{for } |\omega| \le \omega_0 \\ 0, & \text{for } \omega_0 \le |\omega| \le \pi \end{cases}$$

Solution: Given

$$H(\omega) = \begin{cases} 1, & \text{for } |\omega| \le \omega_0 \\ 0, & \text{for } \omega_0 \le |\omega| \le \pi \end{cases}$$

The impulse response $h(n)$ is given by

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\omega)\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} (1)\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\left(\frac{e^{j\omega n}}{jn}\right)_{-\omega_0}^{\omega_0}$$

$$= \frac{1}{2\pi}\left(\frac{e^{j\omega_0 n} - e^{-j\omega_0 n}}{jn}\right) = \frac{\sin(\omega_0 n)}{\pi n}$$

EXAMPLE 5.7 Find the inverse Fourier transform of the following:

$$X(\omega) = \begin{cases} 1, & \dfrac{\pi}{3} \le |\omega| \le \dfrac{2\pi}{3} \\ 0, & \dfrac{2\pi}{3} \le |\omega| \le \pi \end{cases}$$

Solution: The Fourier transform $X(\omega)$ is given. Then, the inverse Fourier transform of $X(\omega)$ is given by

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-2\pi/3}^{-\pi/3} 1\, e^{j\omega n}\, d\omega + \int_{\pi/3}^{2\pi/3} 1\, e^{j\omega n}\, d\omega\right] = \frac{1}{2\pi}\left[\left(\frac{e^{j\omega n}}{jn}\right)_{-2\pi/3}^{-\pi/3} + \left(\frac{e^{j\omega n}}{jn}\right)_{\pi/3}^{2\pi/3}\right]$$

$$= \frac{1}{n\pi}\left[\frac{e^{-jn(\pi/3)} - e^{-jn(2\pi/3)} + e^{jn(2\pi/3)} - e^{jn(\pi/3)}}{2j}\right]$$

$$= \frac{1}{n\pi}\left[\frac{e^{jn(2\pi/3)} - e^{-jn(2\pi/3)}}{2j} - \frac{e^{jn(\pi/3)} - e^{-jn(\pi/3)}}{2j}\right] = \frac{1}{n\pi}\left[\sin n\frac{2\pi}{3} - \sin n\frac{\pi}{3}\right]$$

EXAMPLE 5.8 Find the inverse Fourier transform of

$$X(\omega) = 2 + e^{-j\omega} + 3e^{-j3\omega} + 4e^{-j4\omega}$$

Solution: Given $\quad X(\omega) = 2 + e^{-j\omega} + 3e^{-j3\omega} + 4e^{-j4\omega}$

We know that $\quad X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$

$$= \cdots + x(-2)\, e^{j2\omega} + x(-1)\, e^{j\omega} + x(0) + x(1)\, e^{-j\omega} + x(2)\, e^{-j2\omega} + x(3)\, e^{-j3\omega} + x(4)\, e^{-j4\omega} + \cdots$$

Comparing the above two values of $X(\omega)$, we get

i.e. $\quad x(0) = 2,\ x(1) = 1,\ x(2) = 0,\ x(3) = 3,\ x(4) = 4$

$$x(n) = \{2, 1, 0, 3, 4\}$$

5.6 PROPERTIES OF DISCRETE-TIME FOURIER TRANSFORM

5.6.1 Linearity Property

The linearity property of DTFT states that

If $\quad \mathrm{F}[x_1(n)] = X_1(\omega)$ and $\mathrm{F}[x_2(n)] = X_2(\omega)$

Then $\quad \mathrm{F}[ax_1(n) + bx_2(n)] = aX_1(\omega) + bX_2(\omega)$

Proof:
$$F\{ax_1(n)+bx_2(n)\} = \sum_{n=-\infty}^{\infty} [ax_1(n)+bx_2(n)]\, e^{-j\omega n}$$
$$= \sum_{n=-\infty}^{\infty} ax_1(n)\, e^{-j\omega n} + \sum_{n=-\infty}^{\infty} bx_2(n)\, e^{-j\omega n} = aX_1(\omega)+bX_2(\omega)$$

5.6.2 Periodicity Property

The periodicity property of DTFT states that the DTFT $X(\omega)$ is periodic in ω with period 2π, i.e.

$$X(\omega+2n\pi) = X(\omega)$$

Implication: We need only one period of $X(\omega)$ {i.e., $\omega \in (0, 2\pi)$ or $(-\pi, \pi)$} for analysis and not the whole range $-\infty < \omega < \infty$.

5.6.3 Time Shifting Property

The time shifting property of DTFT states that

If $$F[x(n)] = X(\omega)$$

Then $F[x(n - m)] = e^{-j\omega m}\, X(\omega)$ where m is an integer.

Proof:
$$F\{x(n-m)\} = \sum_{n=-\infty}^{\infty} x(n-m)\, e^{-j\omega n}$$

Let $n - m = p$

$\therefore$ $$n = p + m$$

$\therefore$
$$F\{x(n-m)\} = \sum_{p=-\infty}^{\infty} x(p)\, e^{-j\omega(p+m)}$$
$$= e^{-j\omega m} \sum_{p=-\infty}^{\infty} x(p)\, e^{-j\omega p} = e^{-j\omega m}\, X(\omega)$$

This result shows that the time shifting of a signal by m units does not change its amplitude spectrum but the phase spectrum is changed by $-\omega m$.

5.6.4 Frequency Shifting Property

The frequency shifting property of DTFT states that

If $$F\{x(n)\} = X(\omega)$$

Then $$F\{x(n)\, e^{j\omega_0 n}\} = X(\omega-\omega_0)$$

Proof:
$$F\{x(n)\, e^{j\omega_0 n}\} = \sum_{n=-\infty}^{\infty} \{x(n)\, e^{j\omega_0 n}\}\, e^{-j\omega n}$$
$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{-j(\omega-\omega_0)n} = X(\omega-\omega_0)$$

This property is the dual of the time shifting property.

5.6.5 Time Reversal Property

The time reversal property of DTFT states that

If $$\text{F}\{x(n)\} = X(\omega)$$

Then $$\text{F}\{x(-n)\} = X(-\omega)$$

Proof:

$$\text{F}\{x(-n)\} = \sum_{n=-\infty}^{\infty} x(-n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{-j(-\omega)n}$$

$$= X(-\omega)$$

That is, folding in the time domain corresponds to the folding in the frequency domain.

5.6.6 Differentiation in the Frequency Domain Property

The differentiation in the frequency domain property of DTFT states that

If $$\text{F}\{x(n)\} = X(\omega)$$

Then $$\text{F}\{n\,x(n)\} = j\frac{d}{d\omega}[X(\omega)]$$

Proof:

$$\text{F}\{x(n)\} = X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

Differentiating both sides w.r.t. ω, we get

$$\frac{d}{d\omega}\{X(\omega)\} = \frac{d}{d\omega}\left\{\sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}\right\}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\frac{d}{d\omega} e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\,(-jn)\, e^{-j\omega n}$$

$$= -j\left\{\sum_{n=-\infty}^{\infty} n\,x(n)\, e^{-j\omega n}\right\}$$

$$\therefore \qquad \sum_{n=-\infty}^{\infty} nx(n)\, e^{-j\omega n} = \text{F}\{n\,x(n)\} = j\frac{d}{d\omega}[X(\omega)]$$

5.6.7 Time Convolution Property

The time convolution property of DTFT states that

If $$F\{x_1(n)\} = X_1(\omega) \text{ and } F\{x_2(n)\} = X_2(\omega)$$

Then $$F\{x_1(n) * x_2(n)\} = X_1(\omega)\, X_2(\omega)$$

Proof: $$x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)$$

$\therefore$ $$F\{x_1(n) * x_2(n)\} = \sum_{n=-\infty}^{\infty} [x_1(n) * x_2(n)]\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left\{ \sum_{k=-\infty}^{\infty} [x_1(k)\, x_2(n-k)] \right\} e^{-j\omega n}$$

Interchanging the order of summations, we get

$$F\{x_1(n) * x_2(n)\} = \sum_{k=-\infty}^{\infty} x_1(k) \sum_{n=-\infty}^{\infty} x_2(n-k)\, e^{-j\omega n}$$

Put $n - k = p$ in the second summation.

$\therefore$ $$n = p + k$$

$$F\{x_1(n) * x_2(n)\} = \sum_{k=-\infty}^{\infty} x_1(k) \sum_{p=-\infty}^{\infty} x_2(p)\, e^{-j\omega(p+k)}$$

$$= \sum_{k=-\infty}^{\infty} x_1(k)\, e^{-j\omega k} \sum_{p=-\infty}^{\infty} x_2(p)\, e^{-j\omega p}$$

$$= X_1(\omega)\, X_2(\omega)$$

That is, the convolution of the signals in the time domain is equal to multiplying their spectra in the frequency domain.

5.6.8 Frequency Convolution Property

The frequency convolution property of DTFT states that

If $$F\{x_1(n)\} = X_1(\omega) \text{ and } F\{x_2(n)\} = X_2(\omega)$$

Then $$F\{x_1(n)\, x_2(n)\} = X_1(\omega) * X_2(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\theta)\, X_2(\omega - \theta)\, d\theta$$

Proof: $$F[x_1(n)\, x_2(n)] = \sum_{n=-\infty}^{\infty} x_1(n)\, x_2(n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\theta)\, e^{j\theta n}\, d\theta \right] e^{-j\omega n}\, x_2(n)$$

Interchanging the order of summation and integration, we get

$$\mathrm{F}\{x_1(n)\,x_2(n)\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\theta)\left[\sum_{n=-\infty}^{\infty} x_2(n)\,e^{-j(\omega-\theta)n}\right]d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\theta)\,X_2(\omega-\theta)\,d\theta$$

This operation is known as periodic convolution because it is the convolution of two periodic functions $X_1(\omega)$ and $X_2(\omega)$.

5.6.9 The Correlation Theorem

The correlation theorem of DTFT states that

If $\qquad \mathrm{F}\{x_1(n)\} = X_1(\omega)$ and $\mathrm{F}\{x_2(n)\} = X_2(\omega)$

Then $\qquad \mathrm{F}\{R_{x_1x_2}(l)\} = X_1(\omega)\,X_2(-\omega) = \Gamma_{x_1x_2}(\omega)$

The function $\Gamma_{x_1x_2}(\omega)$ is called the cross energy spectrum of the signals $x_1(n)$ $x_2(n)$.

5.6.10 The Modulation Theorem

If $\qquad \mathrm{F}\{x(n)\} = X(\omega)$

Then $\qquad \mathrm{F}\{x(n)\cos\omega_0 n\} = \frac{1}{2}\{X(\omega+\omega_0) + X(\omega-\omega_0)\}$

Proof:

$$\mathrm{F}\{x(n)\cos\omega_0 n\} = \sum_{n=-\infty}^{\infty} x(n)\,\frac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2}\,e^{-j\omega n}$$

$$= \frac{1}{2}\left\{\sum_{n=-\infty}^{\infty} x(n)\,e^{-j(\omega-\omega_0)n} + \sum_{n=-\infty}^{\infty} x(n)\,e^{-j(\omega+\omega_0)n}\right\}$$

$$= \frac{1}{2}\{X(\omega-\omega_0) + X(\omega+\omega_0)\}$$

5.6.11 Parseval's Theorem

If $\qquad \mathrm{F}\{x(n)\} = X(\omega)$

Then

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2\,d\omega$$

Proof:

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \sum_{n=-\infty}^{\infty} x(n)\, x^*(n)$$

$$= \sum_{n=-\infty}^{\infty} x(n) \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega \right\}^*$$

$$= \sum_{n=-\infty}^{\infty} x(n) \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} X^*(\omega)\, e^{-j\omega n}\, d\omega \right\}$$

Interchanging the order of summation and integration, we get

$$E = \frac{1}{2\pi} \int_{-\pi}^{\pi} X^*(\omega) \left\{ \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} \right\} d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X^*(\omega)\, X(\omega)\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2\, d\omega$$

5.6.12 Symmetry Properties

The DTFT $X(\omega)$ is a complex function of ω and can be expressed as:

$$X(\omega) = X_R(\omega) + jX_I(\omega)$$

where $X_R(\omega)$ is real part and $X_I(\omega)$ is imaginary part of $X(\omega)$ respectively. We have

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) \cos \omega n - j \sum_{n=-\infty}^{\infty} x(n) \sin \omega n$$

i.e.

$$X_R(\omega) + jX_I(\omega) = \sum_{n=-\infty}^{\infty} x(n) \cos \omega n - j \sum_{n=-\infty}^{\infty} x(n) \sin \omega n$$

Comparing LHS and RHS, we have

$$X_R(\omega) = \sum_{n=-\infty}^{\infty} x(n) \cos \omega n$$

$$X_I(\omega) = -\sum_{n=-\infty}^{\infty} x(n) \sin \omega n$$

Since $\cos(-\omega)n = \cos\omega n$, and $\sin(-\omega)n = -\sin\omega n$

$$X_R(-\omega) = \sum_{n=-\infty}^{\infty} x(n)\cos(-\omega)n = \sum_{n=-\infty}^{\infty} x(n)\cos\omega n = X_R(\omega)$$

i.e. $$X_R(-\omega) = X_R(\omega) \qquad \text{(Even symmetry)}$$

$$X_I(-\omega) = -\sum_{n=-\infty}^{\infty} x(n)\sin(-\omega)n = \sum_{n=-\infty}^{\infty} x(n)\sin\omega n = -X_I(\omega)$$

i.e. $$X_I(-\omega) = -X_I(\omega) \qquad \text{(Odd symmetry)}$$

Therefore, $X_R(\omega)$ is an even function of ω and $X_I(\omega)$ is an odd function of ω. We can write $X_R(\omega)$ in the polar form as:

$$X(\omega) = |X(\omega)|e^{j\theta(\omega)}$$

where $|X(\omega)|$ is the magnitude and $\theta(\omega)$ is the phase of $X(\omega)$.

Expanding $$X(\omega) = |X(\omega)|\{\cos\theta(\omega) + j\sin\theta(\omega)\}$$

i.e. $$X_R(\omega) + jX_I(\omega) = |X(\omega)|\cos\theta(\omega) + j|X(\omega)|\sin\theta(\omega)$$

Comparing LHS and RHS, we get

$$X_R(\omega) = |X(\omega)|\cos\theta(\omega)$$

$$X_I(\omega) = |X(\omega)|\sin\theta(\omega)$$

$$|X(\omega)|^2 = \{X_R(\omega)\}^2 + \{X_I(\omega)\}^2$$

i.e. $$X(\omega) = \sqrt{\{X_R(\omega)\}^2 + \{X_I(\omega)\}^2}$$

and $$\tan\theta(\omega) = \frac{X_I(\omega)}{X_R(\omega)}$$

or $$\theta(\omega) = \tan^{-1}\frac{X_I(\omega)}{X_R(\omega)}$$

Similarly, $$|X(-\omega)| = \sqrt{\{X_R(-\omega)\}^2 + \{X_I(-\omega)\}^2}$$

$$= \sqrt{|X_R(\omega)|^2 + |X_I(\omega)|^2}$$

$$= |X(\omega)|$$

Therefore, $X(\omega)$ is an even function of ω,

$$\theta(-\omega) = \tan^{-1}\frac{X_I(-\omega)}{X_R(-\omega)}$$

$$= \tan^{-1}\frac{-X_I(\omega)}{X_R(\omega)}$$

$$= -\tan^{-1}\frac{X_I(\omega)}{X_R(\omega)} = -\theta(\omega)$$

$$\therefore \qquad \theta(-\omega) = -\theta(\omega)$$

That is
$$\underline{|X(-\omega)} = -\underline{|X(\omega)}$$

Therefore $\underline{|X(\omega)}$ is an odd function of ω.

TABLE 5.1 Properties of DTFT

Property	*Sequence*	DTFT
	$x(n)$	$X(\omega)$
	$x_1(n)$	$X_1(\omega)$
	$x_2(n)$	$X_2(\omega)$
Linearity	$ax_1(n) + bx_2(n)$	$aX_1(\omega) + bX_2(\omega)$
Time shifting	$x(n - m)$	$e^{-j\omega m} X(\omega)$
Time reversal	$x(-n)$	$X(-\omega_0)$
Frequency shifting	$x(n)e^{j\omega 0n}$	$X(\omega - \omega)$
Differentiation in frequency domain	$nx(n)$	$j\dfrac{d}{d\omega}[X(\omega)]$
Convolution	$x_1(n) * x_2(n)$	$X_1(\omega)\, X_2(\omega)$
Multiplication	$x_1(n)\, x_2(n)$	$X_1(\omega) * X_2(\omega)$
Correlation	$R_{x_1x_2}(l)$	$X_1(\omega)\, X_2(-\omega)$
Modulation theorem	$x(n) \cos \omega_0 n$	$\dfrac{1}{2}\{X(\omega+\omega_0) + X(\omega-\omega_0)\}$
Parseval's theorem	$\sum\limits_{n=-\infty}^{\infty} \lvert x(n)\rvert^2$	$\dfrac{1}{2\pi}\int\limits_{-\pi}^{\pi} \lvert X(\omega)\rvert^2\, d\omega$
Symmetry property	$x^*(n)$	$X(-\omega)$
	$x^*(-n)$	$X^*(\omega)$
	$x_R(n)$	$X_e(\omega)$
	$jx_I(n)$	$X_0(\omega)$
	$x_e(n)$	$X_R(\omega)$
	$x_0(n)$	$jX_I(\omega)$

EXAMPLE 5.9 Using properties of DTFT, find the DTFT of the following:

(a) $\left(\dfrac{1}{4}\right)^{|n-2|}$

(b) $\left(\dfrac{1}{3}\right)^{|n-3|} u(n-3)$

(c) $\delta(n-2) - \delta(n+2)$

(d) $u(n+1) - u(n+2)$

(e) $n2^n\, u(n)$

(f) $u(-n)$

(g) $n3^{-n}\, u(-n)$

(h) $e^{3n}\, u(n)$

Solution:

(a) Using the time shifting property, we have

$$\mathrm{F}\left\{\left(\frac{1}{4}\right)^{|n-2|}\right\} = e^{-j2\omega}\ \mathrm{F}\left\{\left(\frac{1}{4}\right)^{|n|}\right\}$$

$$\mathrm{F}\left\{\left(\frac{1}{4}\right)^{|n|}\right\} = \sum_{n=-\infty}^{\infty}\left(\frac{1}{4}\right)^{|n|} e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{-1}\left(\frac{1}{4}\right)^{-n} e^{-j\omega n} + \sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^{n} e^{-j\omega n}$$

$$= \sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^{n} e^{j\omega n} + \sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^{n} e^{-j\omega n}$$

$$= \frac{1}{4}e^{j\omega}\frac{1}{1-(1/4)\,e^{j\omega}} + \frac{1}{1-(1/4)\,e^{-j\omega}} = \frac{15/16}{(17/16)-(1/2)\cos\omega}$$

$$\therefore\quad \mathrm{F}\left\{\left(\frac{1}{4}\right)^{|n-2|}\right\} = e^{-j2\omega}\frac{15/16}{(17/16)-(1/2)\cos\omega}$$

(b) Using the time shifting property, we have

$$\mathrm{F}\left\{\left(\frac{1}{3}\right)^{|n-3|} u(n-3)\right\} = e^{-j3\omega}\ \mathrm{F}\left\{\left(\frac{1}{3}\right)^{n} u(n)\right\} = e^{-j3\omega}\left\{\frac{1}{1-(1/3)\,e^{-j\omega}}\right\}$$

(c) Using the time shifting property, we have

$$\mathrm{F}\{\delta(n-2)-\delta(n+2)\} = \mathrm{F}\{\delta(n-2)\} - \mathrm{F}\{\delta(n+2)\}$$

$$= e^{-j2\omega}\ \mathrm{F}\{\delta(n)\} - e^{j2\omega}\ \mathrm{F}\{\delta(n)\}$$

$$= e^{-j2\omega} - e^{j2\omega} = -2j\sin 2\omega$$

(d) Using the time shifting property, we have

$$\mathrm{F}\{u(n+1)-u(n+2)\} = \mathrm{F}\{u(n+1)\} - \mathrm{F}\{u(n+2)\}$$

$$= e^{j\omega}\ \mathrm{F}\{u(n)\} - e^{j2\omega}\ \mathrm{F}\{u(n)\}$$

$$= \frac{e^{j\omega}}{1-e^{-j\omega}} - \frac{e^{j2\omega}}{1-e^{-j\omega}}$$

(e) Using differentiation in the frequency domain property, we have

$$\mathrm{F}\left\{n\left(\frac{1}{2}\right)^{n} u(n)\right\} = j\frac{d}{d\omega}\left[\mathrm{F}\left\{\left(\frac{1}{2}\right)^{n} u(n)\right\}\right]$$

$$= j\frac{d}{d\omega}\left[\frac{1}{1-(1/2)\,e^{-j\omega}}\right]$$

$$= j\frac{\{-[-(1/2)\,e^{-j\omega}(-j)]\}}{\{1-(1/2)\,e^{-j\omega}\}^2} = \frac{(1/2)\,e^{-j\omega}}{\{1-(1/2)\,e^{-j\omega}\}^2}$$

(f) Using the time reversal property, we have

$$\text{F}\{u(-n)\} = \text{F}\{u(n)\}\big|_{\omega=-\omega} = \left\{\frac{1}{1-e^{-j\omega}}\right\}_{\omega=-\omega} = \frac{1}{1-e^{j\omega}}$$

(g) Using differentiation in frequency domain and time reversal properties, we have

$$\text{F}\{n3^{-n}\,u(-n)\} = j\frac{d}{d\omega}[\text{F}\{3^{-n}\,u(-n)\}]$$

$$= j\frac{d}{d\omega}[\text{F}\{3^{n}\,u(n)\}]_{\omega=-\omega} = j\frac{d}{d\omega}\left[\frac{1}{1-3e^{-j\omega}}\right]_{\omega=-\omega}$$

$$= j\frac{d}{d\omega}\left[\frac{1}{1-3e^{j\omega}}\right] = j\frac{-[-3e^{j\omega}(j)]}{[1-3e^{j\omega}]^2} = \frac{-3e^{j\omega}}{\{1-3e^{j\omega}\}^2}$$

(h) Using the frequency shifting property, we have

$$\text{F}\{e^{3n}\,u(n)\} = \text{F}\{u(n)\}\big|_{\omega=\omega-3}$$

$$= \left\{\frac{1}{1-e^{-j\omega}}\right\}_{\omega=\omega-3} = \frac{1}{1-e^{-j(\omega-3)}}$$

EXAMPLE 5.10 Find the inverse Fourier transform for the first order recursive filter

$$H(\omega) = (1 - ae^{-j\omega})^{-1}$$

Solution: Given $\quad H(\omega) = (1-ae^{-j\omega})^{-1} = \dfrac{1}{1-ae^{-j\omega}}$

$$= 1 + ae^{-j\omega} + a^2e^{-j2\omega} + a^3e^{-j3\omega} + \cdots$$

Let $h(n)$ be the inverse Fourier transform of $H(\omega)$.

$$\therefore\quad H(\omega) = \sum_{n=-\infty}^{\infty} h(n)\,e^{-j\omega n} = \cdots + h(-2)\,e^{j2\omega} + h(-1)\,e^{j\omega} + h(0) + h(1)\,e^{-j\omega} + h(2)\,e^{-j2\omega} + \cdots$$

On comparing the two expressions for $H(\omega)$, we can say that the samples of $h(n)$ are the coefficients of $e^{-j\omega n}$.

$\therefore$ $$h(n) = \{1, a, a^2, \ldots, a^k, \ldots\}$$

i.e. $$h(n) = \begin{cases} a^n, & n \ge 0 \\ 0, & n < 0 \end{cases} \quad \text{or } h(n) = a^n\,u(n)$$

EXAMPLE 5.11 Determine the output sequence from the output spectrum:

$$Y(\omega) = \frac{1}{4}\frac{e^{j2\omega} + 1 + e^{-j2\omega}}{1 - ae^{-j\omega}}$$

Solution: Given
$$Y(\omega) = \frac{1}{4}\frac{e^{j2\omega} + 1 + e^{-j2\omega}}{1 - ae^{-j\omega}}$$

The output sequence $y(n)$ is the inverse Fourier transform of $Y(\omega)$.

$$\therefore \qquad y(n) = \mathrm{F}^{-1}\left\{\frac{1}{4}\frac{e^{j2\omega} + 1 + e^{-j2\omega}}{1 - ae^{-j\omega}}\right\}$$

$$= \frac{1}{4}\left[\mathrm{F}^{-1}\left\{\frac{e^{j2\omega}}{1 - ae^{-j\omega}}\right\} + \mathrm{F}^{-1}\left\{\frac{1}{1 - ae^{-j\omega}}\right\} + \mathrm{F}^{-1}\left\{\frac{e^{-j2\omega}}{1 - ae^{-j\omega}}\right\}\right]$$

Using the time shifting property, we have

$$\mathrm{F}^{-1}\left\{\frac{e^{j2\omega}}{1 - ae^{-j\omega}}\right\} = \mathrm{F}^{-1}\left\{\frac{1}{1 - ae^{-j\omega}}\right\}\Bigg|_{n=n+2} = a^n u(n)\Big|_{n=n+2} = a^{n+2}\, u(n+2)$$

Also, we know that

$$\mathrm{F}^{-1}\left\{\frac{1}{1 - ae^{-j\omega}}\right\} = a^n u(n)$$

Using the time shifting property, we have

$$\mathrm{F}^{-1}\left\{\frac{e^{-j2\omega}}{1 - ae^{-j\omega}}\right\} = \mathrm{F}^{-1}\left\{\frac{1}{1 - ae^{-j\omega}}\right\}\Bigg|_{n=n-2} = a^{n-2}\, u(n-2)$$

$$\therefore \qquad y(n) = \frac{1}{4}\{a^{n+2}\, u(n+2) + a^n u(n) + a^{n-2}\, u(n-2)\}$$

EXAMPLE 5.12 The impulse response of a LTI system is $h(n) = \{1, 2, 1, -2\}$. Find the response of the system for the input $x(n) = \{1, 3, 2, 1\}$.

Solution: The response of the system $y(n)$ for an input $x(n)$ and impulse response $h(n)$ is given by

$$y(n) = x(n) * h(n)$$

Using the convolution property of Fourier transform, we get

$$Y(\omega) = X(\omega)\, H(\omega)$$

$$\therefore \qquad y(n) = \mathrm{F}^{-1}\{X(\omega)\, H(\omega)\}$$

Given
$$x(n) = \{1, 3, 2, 1\}$$

$\therefore$ $$X(\omega) = 1 + 3e^{-j\omega} + 2e^{-j2\omega} + e^{-j3\omega}$$

Given $$h(n) = \{1, 2, 1, -2\}$$

$\therefore$ $$H(\omega) = 1 + 2e^{-j\omega} + e^{-j2\omega} - 2e^{-j3\omega}$$

$$Y(\omega) = X(\omega)\, H(\omega) = (1 + 3e^{-j\omega} + 2e^{-j2\omega} + e^{-j3\omega})(1 + 2e^{-j\omega} + e^{-j2\omega} - 2e^{-j3\omega})$$

$$= 1 + 5e^{-j\omega} + 9e^{-j2\omega} + 6e^{-j3\omega} - 2e^{-j4\omega} - 3e^{-j5\omega} - 2e^{-j6\omega}$$

Taking inverse Fourier transform on both sides, we get

$$y(n) = 1 + 5\delta(n-1) + 9\delta(n-2) + 6\delta(n-3) - 2\delta(n-4) - 3\delta(n-5) - 2\delta(n-6)$$

or $$y(n) = \{1, 5, 9, 6, -2, -3, -2\}$$

EXAMPLE 5.13 Find the convolution of the signals given below using Fourier transform:

$$x_1(n) = \left(\frac{1}{2}\right)^n u(n)$$

$$x_2(n) = \left(\frac{1}{3}\right)^n u(n)$$

Solution: Given $$x_1(n) = \left(\frac{1}{2}\right)^n u(n)$$

$\therefore$ $$X_1(\omega) = \frac{1}{1 - (1/2)\, e^{-j\omega}}$$

$$x_2(n) = \left(\frac{1}{3}\right)^n u(n)$$

$\therefore$ $$X_2(\omega) = \frac{1}{1 - (1/3)\, e^{-j\omega}}$$

Using the convolution property of Fourier transform, we get

$$\text{F}[x_1(n) * x_2(n)] = X_1(\omega)\, X_2(\omega)$$

$$= \left[\frac{1}{1 - (1/2)\, e^{-j\omega}}\right]\left[\frac{1}{1 - (1/3)\, e^{-j\omega}}\right]$$

$\therefore$ $$x_1(n) * x_2(n) = \text{F}^{-1}\left\{\left[\frac{1}{1 - (1/2)e^{-j\omega}}\right]\left[\frac{1}{1 - (1/3)e^{-j\omega}}\right]\right\}$$

Let $$X(\omega) = \left(\frac{1}{1 - (1/2)e^{-j\omega}}\right)\left(\frac{1}{1 - (1/3)e^{-j\omega}}\right) = \left[\frac{e^{j\omega}}{e^{j\omega} - (1/2)}\right]\left[\frac{e^{j\omega}}{e^{j\omega} - (1/3)}\right]$$

$$\therefore \qquad \frac{X(\omega)}{e^{j\omega}} = \frac{e^{j\omega}}{[e^{j\omega}-(1/2)][e^{j\omega}-(1/3)]}$$

$$= \frac{A}{e^{j\omega}-(1/2)} + \frac{B}{e^{j\omega}-(1/3)} = \frac{3}{e^{j\omega}-(1/2)} + \frac{-2}{e^{j\omega}-(1/3)}$$

$$\therefore \qquad X(\omega) = \frac{3e^{j\omega}}{e^{j\omega}-(1/2)} - \frac{2e^{j\omega}}{e^{j\omega}-(1/3)}$$

$$= 3\frac{1}{1-(1/2)e^{-j\omega}} - 2\frac{1}{1-(1/3)e^{-j\omega}}$$

Taking inverse Fourier transform on both sides, we have

$$x(n) = x_1(n) * x_2(n) = 3\left(\frac{1}{2}\right)^n u(n) - 2\left(\frac{1}{3}\right)^n u(n)$$

EXAMPLE 5.14 Consider a discrete-time LTI system with impulse response $h(n) = \left(\frac{1}{2}\right)^n u(n)$. Use Fourier transform to determine the response to the signal $x(n) = \left(\frac{3}{4}\right)^n u(n)$.

Solution: Given the impulse response $h(n)$ and the input $x(n)$ to the system, the response $y(n)$ is given by

$$y(n) = x(n) * h(n)$$

Using the convolution property of Fourier transform, we have

$$y(n) = x(n) * h(n) = \mathrm{F}^{-1}[X(\omega)\, H(\omega)]$$

where $X(\omega)$ and $H(\omega)$ are the Fourier transforms of $x(n)$ and $h(n)$, respectively.

$$x(n) = \left(\frac{3}{4}\right)^n u(n)$$

$$\therefore \qquad X(\omega) = \frac{1}{1-(3/4)\,e^{-j\omega}}$$

$$h(n) = \left(\frac{1}{2}\right)^n u(n)$$

$$\therefore \qquad H(\omega) = \frac{1}{1-(1/2)\,e^{-j\omega}}$$

$$\therefore \qquad Y(\omega) = X(\omega)\, H(\omega) = \left[\frac{1}{1-(3/4)\, e^{-j\omega}}\right]\left[\frac{1}{1-(1/2)\, e^{-j\omega}}\right]$$

$$= \left[\frac{e^{j\omega}}{e^{j\omega}-(3/4)}\right]\left[\frac{e^{j\omega}}{e^{j\omega}-(1/2)}\right]$$

$$\therefore \qquad \frac{Y(\omega)}{e^{j\omega}} = \frac{e^{j\omega}}{[e^{j\omega}-(3/4)]\,[e^{j\omega}-(1/2)]}$$

$$= \frac{A}{e^{j\omega}-(3/4)} + \frac{B}{e^{j\omega}-(1/2)} = \frac{3}{e^{j\omega}-(3/4)} + \frac{-2}{e^{j\omega}-(1/2)}$$

$$\therefore \qquad Y(\omega) = \frac{3e^{j\omega}}{e^{j\omega}-(3/4)} - \frac{2e^{j\omega}}{e^{j\omega}-(1/2)} = \frac{3}{1-(3/4)\, e^{-j\omega}} - \frac{2}{1-(1/2)\, e^{-j\omega}}$$

Taking inverse Fourier transform on both sides, we get the response

$$y(n) = 3\left(\frac{3}{4}\right)^n u(n) - 2\left(\frac{1}{2}\right)^n u(n)$$

5.7 TRANSFER FUNCTION

If $X(\omega)$ is the Fourier transform of the input signal $x(n)$, and $Y(\omega)$ is the Fourier transform of the output signal $y(n)$, then $H(\omega)$, the transfer function of the system is given by

$$H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

i.e. the transfer function of an LTI system is defined as the ratio of the Fourier transform of the output to the Fourier transform of the input. It is also defined as the Fourier transform of the impulse response $h(n)$ of the system. It can be obtained as follows:

If the input to the system is of the form $e^{j\omega n}$, then the output $y(n)$ is given by

$$y(n) = h(n) * x(n) = \sum_{k=-\infty}^{\infty} h(k)\, e^{j\omega(n-k)} = e^{j\omega n} H(\omega)$$

i.e. $\qquad y(n) = x(n)\; H(\omega)$

We have $\qquad x(n) = \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega$

$$y(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega\, H(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, H(\omega)\, e^{j\omega n}\, d\omega$$

$y(n)$ in terms of $Y(\omega)$ is given by

$$y(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} Y(\omega)\, e^{j\omega n}\, d\omega$$

Comparing the above two expressions for $y(n)$, we get

$$Y(\omega) = X(\omega)\, H(\omega)$$

$$\therefore \qquad H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

where $H(\omega)$ is known as the transfer function of the system.

5.8 FREQUENCY RESPONSE OF DISCRETE-TIME SYSTEMS

The frequency response of a linear time-invariant discrete-time system can be obtained by applying a spectrum of input sinusoids to the system. The frequency response gives the gain and phase response of the system to the input sinusoids at all frequencies.

Let $h(n)$ be the impulse response of an LTI discrete system, and let the input $x(n)$ to the system be a complex exponential $e^{j\omega n}$.

The output of the system $y(n)$ can be obtained by using convolution sum.

$$y(n) = h(n) * x(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$

For $x(n) = e^{j\omega n}$,

$$\begin{aligned} y(n) &= \sum_{k=-\infty}^{\infty} h(k)\, e^{j\omega(n-k)} \\ &= e^{j\omega n} \sum_{k=-\infty}^{\infty} h(k)\, e^{-j\omega k} \\ &= \underbrace{e^{j\omega n}}_{\text{Input}} \quad \underbrace{H(\omega)}_{\text{Frequency response}} \end{aligned}$$

That is, if we force the system with a complex exponential $e^{j\omega n}$, then the output is of the form $H(\omega)e^{j\omega n}$. Therefore, the output of the system is identical to the input modified in amplitude and phase by $H(\omega)$. The quantity $H(\omega)$ is the frequency response of the system.

It is same as the transfer function $H(\omega)$ in the frequency domain. The frequency response $H(\omega)$ is complex and can be expressed in the polar form as:

$$H(\omega) = |H(\omega)|\, e^{j\angle H(\omega)}$$

where the magnitude of $H(\omega)$, i.e. $|H(\omega)|$ is called the magnitude response and phase angle of $H(\omega)$, i.e. $\angle H(\omega)$ is called the phase response. The plot of $|H(\omega)|$ versus ω is called the magnitude response plot and the plot of $\angle H(\omega)$ versus ω is called the phase response plot.

Properties of frequency response

Frequency response is a complex function that describes the magnitude and phase shift of a filter over a range of frequencies. If $h(n)$ is a real sequence, the frequency response $H(\omega)$ has the following properties:

1. $H(\omega)$ takes on values for all ω, i.e. on a continuum of ω.
2. $H(\omega)$ is periodic in ω, with period of 2π.
3. The magnitude response $|H(\omega)|$ is an even function of ω and symmetrical about π.
4. The phase response $\angle H(\omega)$ is an odd function of ω and antisymmetrical about π.

EXAMPLE 5.15 Write a difference equation that characterizes a system whose frequency response is:

$$H(\omega) = \frac{1 - e^{-j\omega} - 3e^{-j2\omega}}{1 + (1/3)\,e^{-j\omega} + (1/6)\,e^{-j2\omega}}$$

Solution: Given $\quad H(\omega) = \dfrac{Y(\omega)}{X(\omega)} = \dfrac{1 - e^{-j\omega} - 3e^{-j2\omega}}{1 + (1/3)\,e^{-j\omega} + (1/6)\,e^{-j2\omega}}$

On cross multiplication, we get

$$Y(\omega) + \frac{1}{3}e^{-j\omega}Y(\omega) + \frac{1}{6}e^{-j2\omega}Y(\omega) = X(\omega) - e^{-j\omega}X(\omega) - 3e^{-j2\omega}X(\omega)$$

Taking inverse Fourier transform on both sides, we get the difference equation:

$$y(n) + \frac{1}{3}y(n-1) + \frac{1}{6}y(n-2) = x(n) - x(n-1) - 3x(n-2)$$

EXAMPLE 5.16 Find the frequency response of the following causal systems:

(a) $y(n) - y(n-1) + \dfrac{3}{16}y(n-2) = x(n) - \dfrac{1}{2}x(n-1)$

(b) $y(n) - \dfrac{1}{4}y(n-1) - \dfrac{3}{8}y(n-2) = x(n) + x(n-1)$

Solution: The frequency response of a system is given by $H(\omega) = [Y(\omega)/X(\omega)]$, where $Y(\omega)$ and $X(\omega)$ are the Fourier transforms of output and input signals respectively.

(a) Given $\quad y(n) - y(n-1) + \dfrac{3}{16}y(n-2) = x(n) - \dfrac{1}{2}x(n-1)$

Taking Fourier transform on both sides, we have

$$Y(\omega) - e^{-j\omega}Y(\omega) + \frac{3}{16}e^{-j2\omega}Y(\omega) = X(\omega) - \frac{1}{2}e^{-j\omega}X(\omega)$$

i.e. $\quad Y(\omega)\left(1 - e^{-j\omega} + \dfrac{3}{16}e^{-j2\omega}\right) = X(\omega)\left(1 - \dfrac{1}{2}e^{-j\omega}\right)$

$\therefore \quad H(\omega) = \dfrac{Y(\omega)}{X(\omega)} = \dfrac{1 - (1/2)\,e^{-j\omega}}{1 - e^{-j\omega} + (3/16)\,e^{-j2\omega}} = \dfrac{e^{j\omega}\,[e^{j\omega} - (1/2)]}{e^{j2\omega} - e^{j\omega} + (3/16)}$

(b) Given $$y(n)-\frac{1}{4}y(n-1)-\frac{3}{8}y(n-2)=x(n)+x(n-1)$$

Taking Fourier transform on both sides, we have

$$Y(\omega)-\frac{1}{4}e^{-j\omega}\,Y(\omega)-\frac{3}{8}e^{-j2\omega}\,Y(\omega)=X(\omega)+e^{-j\omega}\,X(\omega)$$

i.e. $$Y(\omega)\left(1-\frac{1}{4}e^{-j\omega}-\frac{3}{8}e^{-j2\omega}\right)=X(\omega)\,(1+e^{-j\omega})$$

$$\therefore\ H(\omega)=\frac{Y(\omega)}{X(\omega)}=\frac{1+e^{-j\omega}}{1-(1/4)\,e^{-j\omega}-(3/8)\,e^{-j2\omega}}=\frac{e^{j\omega}(e^{j\omega}+1)}{e^{j2\omega}-(1/4)\,e^{j\omega}-(3/8)}$$

EXAMPLE 5.17 A discrete system is given by the following difference equation:

$$y(n)-5y(n-1)=x(n)+4x(n-1)$$

where $x(n)$ is the input and $y(n)$ is the output. Determine its magnitude and phase response.

Solution: Given $$y(n)-5y(n-1)=x(n)+4x(n-1)$$

Taking Fourier transform on both sides, we have

$$Y(\omega)-5e^{-j\omega}\,Y(\omega)=X(\omega)+4e^{-j\omega}\,X(\omega)$$

i.e. $$Y(\omega)\,\{1-5e^{-j\omega}\}=X(\omega)\,\{1+4e^{-j\omega}\}$$

The transfer function of the system, i.e. the frequency response of the system is:

$$\frac{Y(\omega)}{X(\omega)}=H(\omega)=\frac{1+4\,e^{-j\omega}}{1-5e^{-j\omega}}=\frac{e^{j\omega}+4}{e^{j\omega}-5}$$

$$=\frac{\cos\omega+j\sin\omega+4}{\cos\omega+j\sin\omega-5}$$

The magnitude response of the system is:

$$\left|H(\omega)\right|=\frac{\sqrt{(4+\cos\omega)^2+(\sin\omega)^2}}{\sqrt{(\cos\omega-5)^2+(\sin\omega)^2}}=\frac{\sqrt{17+8\cos\omega}}{\sqrt{26-10\cos\omega}}$$

The phase response of the system is:

$$\underline{|H(\omega)}=\tan^{-1}\left\{\frac{\sin\omega}{4+\cos\omega}\right\}-\tan^{-1}\left\{\frac{\sin\omega}{\cos\omega-5}\right\}$$

EXAMPLE 5.18 The output $y(n)$ for a linear shift-invariant system, with input $x(n)$ is given by

$$y(n)=x(n)-2x(n-1)+x(n-2)$$

Determine the magnitude and phase response of the system.

Solution: Given $y(n) = x(n) - 2x(n-1) + x(n-2)$

Taking Fourier transform on both sides, we have

$$Y(\omega) = X(\omega) - 2e^{-j\omega} X(\omega) + e^{-j2\omega} X(\omega)$$

The transfer function of the system, i.e. the frequency response of the system is:

$$\begin{aligned} H(\omega) &= \frac{Y(\omega)}{X(\omega)} = 1 - 2e^{-j\omega} + e^{-j2\omega} \\ &= 1 - 2[\cos\omega - j\sin\omega] + \cos 2\omega - j\sin 2\omega \\ &= [1 - 2\cos\omega + \cos 2\omega] + j[2\sin\omega - \sin 2\omega] \end{aligned}$$

The magnitude response of the system is:

$$|H(\omega)| = \sqrt{[1 - 2\cos\omega + \cos 2\omega]^2 + [2\sin\omega - \sin 2\omega]^2}$$

The phase response of the system is given as:

$$\underline{|H(\omega)} = \tan^{-1}\left\{\frac{2\sin\omega - \sin 2\omega}{1 - 2\cos\omega + \cos 2\omega}\right\}$$

EXAMPLE 5.19 The impulse response of a system is:

$$h(n) = \begin{cases} 1 & \text{for } 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

Find the transfer function, frequency response, magnitude response and phase response.

Solution: The transfer function $H(\omega)$ is obtained by taking the Fourier transform of $h(n)$.

$$\therefore \qquad H(\omega) = \sum_{n=-\infty}^{\infty} h(n)\, e^{-j\omega n} = \sum_{n=0}^{N-1} (1)\, e^{-j\omega n}$$

$$= \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}}$$

The frequency response of the system is same as the transfer function, i.e.

$$H(\omega) = \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}}$$

The magnitude function is given as:

$$\begin{aligned} |H(\omega)| &= \{H(\omega)\, H^*(\omega)\}^{1/2} \\ &= \left\{\left[\frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}}\right]\left[\frac{1 - e^{j\omega N}}{1 - e^{j\omega}}\right]\right\}^{1/2} \end{aligned}$$

$$= \left\{ \frac{1 + 1 - e^{j\omega N} - e^{-j\omega N}}{1 + 1 - e^{j\omega} - e^{-j\omega}} \right\}^{1/2}$$

$$= \left\{ \frac{2 - (e^{j\omega N} + e^{-j\omega N})}{2 - (e^{j\omega} + e^{-j\omega})} \right\}^{1/2} = \left\{ \frac{1 - \cos \omega N}{1 - \cos \omega} \right\}^{1/2}$$

In order to determine the phase function, the real and imaginary parts of $H(\omega)$ have to be separated.

$$\therefore \quad H(\omega) = \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}} \times \frac{1 - e^{j\omega}}{1 - e^{j\omega}} = \frac{1 - e^{j\omega} - e^{-j\omega N} + e^{-j\omega(N-1)}}{2 - (e^{j\omega} + e^{-j\omega})}$$

$$= \frac{1 - (\cos \omega + j \sin \omega) - (\cos \omega N - j \sin \omega N) + [\cos \omega(N-1) - j \sin \omega(N-1)]}{2 - 2 \cos \omega}$$

Now,

$$H_R(\omega) = \frac{1 - \cos \omega - \cos \omega N + \cos \omega(N-1)}{2 - 2\cos \omega}$$

$$H_I(\omega) = \frac{-\sin \omega + \sin \omega N - \sin \omega(N-1)}{2 - 2 \cos \omega}$$

$$\therefore \quad \underline{|H(\omega)} = \tan^{-1} \frac{H_I(\omega)}{H_R(\omega)}$$

$$= \tan^{-1} \left\{ \frac{-\sin \omega + \sin \omega N - \sin \omega(N-1)}{1 - \cos \omega - \cos \omega N + \cos \omega(N-1)} \right\}$$

EXAMPLE 5.20 The impulse response of a LTI system is given by $h(n)$. Find the frequency response, magnitude and phase response.

Solution: Given $\qquad h(n) = 0.6^n\, u(n)$

The frequency response of the system is:

$$H(\omega) = \mathrm{F}\{0.6^n\, u(n)\} = \frac{1}{1 - 0.6e^{-j\omega}}$$

Here $H(\omega)$ is a complex function of frequency. To separate the real and imaginary parts of $H(\omega)$, multiply the numerator and denominator by the complex conjugate of the denominator. Thus,

$$H(\omega) = \frac{1}{1 - 0.6e^{-j\omega}} \times \frac{1 - 0.6e^{j\omega}}{1 - 0.6e^{j\omega}} = \frac{1 - 0.6e^{j\omega}}{1 - 0.6e^{j\omega} - 0.6e^{-j\omega} + 0.36}$$

$$= \frac{1 - 0.6 \cos \omega}{1.36 - 1.2 \cos \omega} - j \frac{0.6 \sin \omega}{1.36 - 1.2 \cos \omega}$$

$$\therefore \quad H_R(\omega) = \frac{1 - 0.6 \cos \omega}{1.36 - 1.2 \cos \omega} \quad \text{and} \quad H_I(\omega) = \frac{-0.6 \sin \omega}{1.36 - 1.2 \cos \omega}$$

The magnitude function of $H(\omega)$ is:

$$|H(\omega)| = \{H_R(\omega)^2 + H_I(\omega)^2\}^{1/2}$$

$$= \left\{\left[\frac{1-0.6\cos\omega}{1.36-1.2\cos\omega}\right]^2 + \left[\frac{-0.6\sin\omega}{1.36-1.2\cos\omega}\right]^2\right\}^{1/2}$$

$$= \frac{1}{(1.36-1.2\cos\omega)^{1/2}}$$

The phase function of $H(\omega)$ is:

$$\underline{|H(\omega)} = \tan^{-1}\frac{H_I(\omega)}{H_R(\omega)} = \tan^{-1}\left\{\frac{-0.6\sin\omega}{1-0.6\cos\omega}\right\}$$

The magnitude function $|H(\omega)|$ can be obtained as follows:

$$|H(\omega)| = \{H(\omega)\,H^*(\omega)\}^{1/2} = \{H(\omega)\,H(-\omega)\}^{1/2}$$

$$= \left(\frac{1}{1-0.6e^{-j\omega}} \times \frac{1}{1-0.6e^{j\omega}}\right)^{1/2} = \left[\frac{1}{1+0.36-0.6\,(e^{j\omega}+e^{-j\omega})}\right]^{1/2}$$

$$= \frac{1}{(1.36-1.2\cos\omega)^{1/2}}$$

EXAMPLE 5.21 A causal LTI system is described by the difference equation:

$$y(n) - ay(n-1) = bx(n) + x(n-1)$$

where a is real and less than 1 in magnitude. Find a value of $b(b \neq a)$ such that the frequency response of the system satisfies $|H(\omega)| = 1$ for all ω (an all pass system, the magnitude of the frequency response is constant independent of frequency).

Solution: Given $\qquad y(n) - ay(n-1) = bx(n) + x(n-1)$

Taking Fourier transform on both sides, we have

$$Y(\omega)\,[1-ae^{-j\omega}] = X(\omega)\,[b+e^{-j\omega}]$$

$$\therefore \qquad H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{b+e^{-j\omega}}{1-ae^{-j\omega}} = \frac{b+\cos\omega - j\sin\omega}{1-a\cos\omega + ja\sin\omega}$$

$$|H(\omega)|^2 = \frac{(b+\cos\omega)^2 + \sin^2\omega}{(1-a\cos\omega)^2 + a^2\sin^2\omega} = \frac{1+b^2+2b\cos\omega}{1+a^2-2a\cos\omega}$$

For magnitude to be independent of frequency,

$$\frac{d}{d\omega}|H(\omega)|^2 = 0$$

i.e. $$\frac{d}{d\omega}\left(\frac{1+b^2+2b\cos\omega}{1+a^2-2a\cos\omega}\right)=0$$

Simplifying, we get $b = 1/a$.

EXAMPLE 5.22 A causal and stable LTI system has the property that

$$\left(\frac{4}{5}\right)^n u(n) \longrightarrow n\left(\frac{4}{5}\right)^n u(n)$$

(a) Determine the frequency response $H(\omega)$ for the system.
(b) Determine a difference equation relating any input $x(n)$ and the corresponding output $y(n)$.

Solution: Given $$x(n)=\left(\frac{4}{5}\right)^n u(n)$$

$\therefore$ $$X(\omega)=\frac{1}{1-(4/5)e^{-j\omega}}$$

$$y(n)=n\left(\frac{4}{5}\right)^n u(n)$$

$\therefore$ $$Y(\omega)=j\frac{d}{d\omega}[X(\omega)]=\frac{(4/5)\,e^{-j\omega}}{[1-(4/5)\,e^{-j\omega}]^2}$$

$\therefore$ $$Y(\omega)=\frac{(4/5)\,e^{-j\omega}}{[1-(4/5)\,e^{-j\omega}]}\,\frac{1}{[1-(4/5)\,e^{-j\omega}]}=\frac{(4/5)\,e^{-j\omega}}{[1-(4/5)\,e^{-j\omega}]}X(\omega)$$

Therefore, the frequency response is:

$$H(\omega)=\frac{Y(\omega)}{X(\omega)}=\frac{(4/5)\,e^{-j\omega}}{[1-(4/5)\,e^{-j\omega}]}$$

Cross multiplying, we get

$$Y(\omega)-\frac{4}{5}e^{-j\omega}Y(\omega)=\frac{4}{5}e^{-j\omega}X(\omega)$$

Taking inverse Fourier transform, we get the difference equation:

$$y(n)-\frac{4}{5}y(n-1)=\frac{4}{5}x(n-1)$$

EXAMPLE 5.23 Determine the impulse response of all the four types of ideal filters shown in Figure 5.1.

Solution:

(a) For an ideal low-pass filter shown in Figure 5.1(a),

$$H(\omega)=\begin{cases}1, & \text{for } 0\le|\omega|\le\omega_c\\ 0, & \text{otherwise}\end{cases}$$

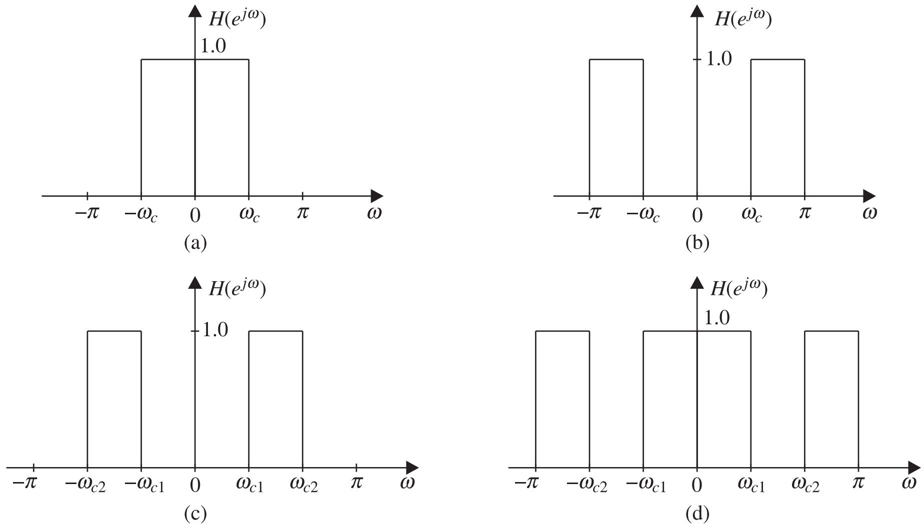


Figure 5.1 Frequency response of ideal filters, (a) low-pass filter, (b) high-pass filter, (c) band pass filter, and (d) band stop filter.

We have $$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\omega)\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}\, d\omega = \frac{e^{j\omega_c n} - e^{-j\omega_c n}}{\pi n(2j)} = \frac{\sin \omega_c n}{\pi n}$$

i.e. $$h(n) = \frac{\sin \omega_c n}{\pi n} \quad -\infty \le n \le \infty$$

(b) For an ideal high-pass filter shown in Figure 5.1(b),

$$H(\omega) = \begin{cases} 0, & \text{for } 0 \le |\omega| \le \omega_c \\ 1, & \text{for } \omega_c \le |\omega| \le \pi \end{cases}$$

We have $$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\omega)\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{-\omega_c} e^{j\omega n}\, d\omega + \int_{\omega_c}^{\pi} e^{j\omega n}\, d\omega\right]$$

$$= \frac{1}{2\pi jn}[e^{-j\omega_c n} - e^{-j\pi n} + e^{j\pi n} - e^{j\omega_c n}]$$

$$= \frac{1}{\pi n}\left[\left(\frac{e^{j\pi n} - e^{-j\pi n}}{2j}\right) - \left(\frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2j}\right)\right]$$

$$= \frac{1}{\pi n}[\sin \pi n - \sin \omega_c n]$$

$$= -\frac{\sin \omega_c n}{\pi n}$$

i.e. $$h(n) = -\frac{\sin \omega_c n}{\pi n} \quad -\infty \le n \le \infty$$

(c) For an ideal band pass filter shown in Figure 5.1(c),

$$H(\omega) = \begin{cases} 1, & \text{for } \omega_{c1} \le |\omega| \le \omega_{c2} \\ 0, & \text{otherwise} \end{cases}$$

We have $$h(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(\omega)\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi} \left(\int_{-\omega_{c2}}^{-\omega_{c1}} e^{j\omega n}\, d\omega + \int_{\omega_{c1}}^{\omega_{c2}} e^{j\omega n}\, d\omega \right)$$

$$= \frac{1}{2\pi jn}[e^{-j\omega_{c1}n} - e^{-j\omega_{c2}n} + e^{j\omega_{c2}n} - e^{j\omega_{c1}n}]$$

$$= \frac{1}{\pi n}\left[\left(\frac{e^{j\omega_{c2}n} - e^{-j\omega_{c2}n}}{2j}\right) - \left(\frac{e^{j\omega_{c1}n} - e^{-j\omega_{c1}n}}{2j}\right)\right]$$

$$= \frac{1}{\pi n}[\sin \omega_{c2}n - \sin \omega_{c1}n]$$

i.e. $$h(n) = \frac{\sin \omega_{c2}n - \sin \omega_{c1}n}{\pi n} \quad -\infty \le n \le \infty$$

(d) For an ideal band stop filter shown in Figure 5.1(d),

$$H(\omega) = \begin{cases} 1, & \text{for } |\omega| \le \omega_{c1} \text{ and } \omega_{c2} \le |\omega| \le \pi \\ 0, & \text{for } \omega_{c1} \le |\omega| \le \omega_{c2} \end{cases}$$

We have $$h(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(\omega) e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi} \left(\int_{-\pi}^{-\omega_{c2}} e^{j\omega n}\, d\omega + \int_{-\omega_{c1}}^{\omega_{c1}} e^{j\omega n}\, d\omega + \int_{\omega_{c2}}^{\pi} e^{j\omega n}\, d\omega \right)$$

$$= \frac{1}{2\pi jn}[e^{-j\omega_{c2}n} - e^{-j\pi n} + e^{j\omega_{c1}n} - e^{-j\omega_{c1}n} + e^{j\pi n} - e^{j\omega_{c2}n}]$$

$$= \frac{1}{\pi n}\left[-\left(\frac{e^{j\omega_{c2}n} - e^{-j\omega_{c2}n}}{2j}\right) + \left(\frac{e^{j\omega_{c1}n} - e^{-j\omega_{c1}n}}{2j}\right) + \left(\frac{e^{j\pi n} - e^{-j\pi n}}{2j}\right)\right]$$

$$= \frac{1}{\pi n}[\sin \omega_{c1}n - \sin \omega_{c2}n]$$

i.e. $$h(n) = \frac{\sin \omega_{c1}n - \sin \omega_{c2}n}{\pi n} \quad -\infty \le n \le \infty$$

SHORT QUESTIONS WITH ANSWERS

1. Define Fourier transform of a discrete-time signal.

Ans. The Fourier transform of a discrete-time signal $x(n)$ is defined as:

$$\text{F}\{x(n)\} = X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

The Fourier transform exists only if $\sum_{n=-\infty}^{\infty} |x(n)| < \infty$, i.e. only if the sequence is absolutely summable.

2. Define inverse discrete-time Fourier transform.

Ans. The inverse discrete-time Fourier transform of $X(\omega)$ is defined as:

$$\text{F}^{-1}\{X(\omega)\} = x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega$$

3. Why the Fourier transform of a discrete-time signal is called signal spectrum?

Ans. By taking Fourier transform of a discrete-time signal $x(n)$ it is decomposed into its frequency components. Hence the Fourier transform is called signal spectrum.

4. List the differences between Fourier transform of discrete-time signal and analog signal.

Ans.

1. The Fourier transform of analog signal consists of a spectrum with frequency range $-\infty$ to ∞, but the Fourier transform of a discrete-time signal is unique in the range $-\pi$ to π (or 0 to 2π), and also it is periodic with periodicity of 2π.
2. The Fourier transform of analog signal involves integration, but Fourier transform of discrete-time signal involves summation.

5. Give some applications of discrete-time Fourier transform.

Ans. Some applications of discrete-time Fourier transform are:

1. The frequency response of LTI system is given by the Fourier transform of the impulse response of the system.

2. The ratio of the Fourier transform of output to the Fourier transform of input is called the transfer function of the system in the frequency domain.
3. The response of an LTI system can be easily computed using the convolution property of Fourier transform.

6. What is the relation between discrete-time Fourier transform and Z-transform?

Ans. The Fourier transform of the discrete-time sequence $x(n)$ is the Z-transform of the sequence $x(n)$ evaluated along the unit circle centred at the origin of the z-plane.

$$X(\omega) = X(z)\Big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}\Big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

7. Write the properties of frequency response of LTI system.

Ans. The properties of frequency response of LTI system are:

1. The frequency response is periodic function of ω with a period of 2π.
2. If $h(n)$ is real, then $|H(\omega)|$ is symmetric and $\angle H(\omega)$ is antisymmetric.
3. If $h(n)$ is complex, then the real part of $H(\omega)$ is symmetric and the imaginary part of $H(\omega)$ is antisymmetric over the interval $0 \le \omega \le 2\pi$.
4. The frequency response is a continuous function of ω.

8. What is frequency response of LTI systems?

Ans. The Fourier transform of the impulse response $h(n)$ of the system is called frequency response of the system. It is denoted by $H(\omega)$. $\{F[h(n)] = H(\omega)\}$. The frequency response has two components: magnitude function $|H(\omega)|$ and phase function $\angle H(\omega)$.

9. What is the sufficient condition for the existence of DTFT?

Ans. The sufficient condition for the existence of DTFT for a sequence $x(n)$ is:

$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty$$

i.e. the sequence $x(n)$ must be absolutely summable.

10. State Parseval's energy theorem for discrete-time aperiodic signals.

Ans. The Parseval's energy theorem for discrete-time aperiodic signals is given by

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2\, d\omega$$

11. Define DTFT pair.

Ans. The Fourier transform pair of a discrete-time signal is:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

and

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega$$

12. If $F\{x(n)\} = X(\omega)$, what is the $F\{x(n-m)\}$?

Ans. $$F\{x(n-m)\} = e^{-j\omega m} X(\omega)$$

13. If $F\{x(n)\} = X(\omega)$, what is the $F\{e^{j\omega_0 n} x(n)\}$?

Ans. $$F\{e^{j\omega_0 n} x(n)\} = X(\omega - \omega_0)$$

14. If $F\{x(n)\} = X(\omega)$, what is the $F\{x(-n)\}$?

Ans. $$F\{x(-n)\} = X(-\omega)$$

15. If $F\{x(n)\} = X(\omega)$, what is the $F\{nx(n)\}$?

Ans. $$F\{nx(n)\} = j\frac{d}{d\omega}[X(\omega)]$$

16. If $F\{x_1(n)\} = X_1(\omega)$ and $F\{x_2(n)\} = X_2(\omega)$, what is the $F\{x_1(n) * x_2(n)\}$?

Ans. $$F\{x_1(n) * x_2(n)\} = X_1(\omega)\, X_2(\omega)$$

17. If $F\{x_1(n)\} = X_1(\omega)$ and $F\{x_2(n)\} = X_2(\omega)$, what is the $F\{x_1(n)\, x_2(n)\}$?

Ans. $$F\{x_1(n)\, x_2(n)\} = X_1(\omega) * X_2(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\theta)\, X_2(\omega - \theta)\, d\theta$$

18. If $F\{x(n)\} = X(\omega)$, what is the $F\{x(n)\cos\omega_0 n\}$?

Ans. $$F\{x(n)\cos\omega_0 n\} = \frac{1}{2}\{X(\omega - \omega_0) + X(\omega + \omega_0)\}$$

REVIEW QUESTIONS

1. State and prove the time shifting and frequency shifting properties of DTFT.
2. State and prove the time reversal and differentiation in frequency domain properties of DTFT.
3. State and prove the time convolution and frequency convolution properties of DTFT.
4. State and prove the modulation theorem and Parseval's theorem.
5. Compare the Fourier transforms of discrete-time signal and analog signal.
6. What are the applications of DTFT?

FILL IN THE BLANKS

1. The DTFT of $x(n)$ is defined as _______.
2. The DTFT exists only if _______.
3. The FT of a discrete-time signal is called _______.
4. The FT of a discrete-time signal is periodic with period _______.

5. The FT of an analog signal involves ______, but the FT of discrete-time signal involves ______.
6. The FT of analog signals consists of a spectrum with a frequency range ______, but the FT of a discrete-time signal is unique in the range ______.
7. The inverse Fourier transform of $X(\omega)$ is defined as ______.
8. The FT of $x(n)$ is nothing but the Z-transform of $x(n)$ evaluated along the ______ centred at the origin of z-plane.
9. The relation between DTFT $X(\omega)$ and Z-transform $X(z)$ is ______.
10. The FT of a discrete and aperiodic sequence is ______.
11. The frequency response of LTI system is given by the FT of the ______ of the system.
12. The impulse response is the inverse Fourier transform of the ______ of the system.
13. The ratio of the FT of the output to the FT of the input is called the ______ or ______ of the system.
14. The frequency response is a ______ function of ω.
15. The frequency response has two components: 1. ______ 2. ______.
16. If $h(n)$ is real, then $H(\omega)$ is ______ $\underline{|H(\omega)}$ ______.
17. If $h(n)$ is complex, then ______ part of $H(\omega)$ is symmetric and the ______ part of $H(\omega)$ is antisymmetric over the interval $0 \le \omega \le 2\pi$.
18. The Fourier transform of ______ is equal to the product $X(\omega)\, H(\omega)$.

OBJECTIVE TYPE QUESTIONS

1. The DTFT of a sequence $x(n)$ is defined as $X(\omega) =$

   (a) $\sum_{n=-\infty}^{\infty} x(n)\, e^{j\omega n}$ (b) $\sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$

   (c) $\sum_{n=0}^{\infty} x(n)\, e^{j\omega n}$ (d) $\sum_{n=0}^{\infty} x(n)\, e^{-j\omega n}$

2. The inverse DTFT of $X(\omega)$ is defined as $x(n) =$

   (a) $\frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, e^{-j\omega n}\, d\omega$ (b) $\frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega$

   (c) $\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{-j\omega n}\, d\omega$ (d) $\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega n}\, d\omega$

3. The FT of a discrete-time signal is periodic with period

   (a) 2π (b) π (c) ∞ (d) finite

4. The relation between DTFT and Z-transform is $X(\omega) =$

(a) $X(z)\big|_{z=e^{j\omega}}$ (b) $X(z)\big|_{z=e^{-j\omega}}$ (c) $X(z)\big|_{z=j\omega}$ (d) $X(z)\big|_{z=\omega}$

5. The frequency response of LTI system is given by the FT of the ________ of the system.

(a) transfer function (b) output
(c) impulse response (d) input

6. The Fourier transform of $x(n) * h(n)$ is equal to

(a) $X(\omega)\, H(\omega)$ (b) $X(\omega) * H(\omega)$ (c) $X(\omega)\, H(-\omega)$ (d) $X(\omega) * H(-\omega)$

7. The FT of analog signal consists of a spectrum with frequency range

(a) $-\pi$ to π (b) 0 to 2π (c) 0 to ∞ (d) $-\infty$ to ∞

8. The FT of a discrete-time signal is unique in the range

(a) $-\infty$ to ∞ (b) 0 to ∞ (c) $-\pi$ to π (d) 0 to π

9. The FT of $\delta(n)$ is

(a) 0 (b) 1 (c) ∞ (d) not defined

10. The FT of $u(n)$ is

(a) $\dfrac{1}{1-e^{-j\omega}}$ (b) $\dfrac{1}{1-e^{j\omega}}$ (c) $\dfrac{1}{1-\omega}$ (d) $\dfrac{1}{1-j\omega}$

11. The FT of $a^n u(n)$ is

(a) $\dfrac{1}{1-ae^{j\omega}}$ (b) $\dfrac{1}{1-ae^{-j\omega}}$ (c) $\dfrac{1}{1-ja\omega}$ (d) $\dfrac{1}{1+ja\omega}$

12. The FT of $-a^n u(-n-1)$ is

(a) $\dfrac{1}{1-ae^{j\omega}}$ (b) $\dfrac{1}{1-ae^{-j\omega}}$ (c) $\dfrac{1}{1-ja\omega}$ (d) $\dfrac{1}{1+aj\omega}$

13. The FT of $\left(\dfrac{1}{2}\right)^{n-1} u(n-1)$ is

(a) $\dfrac{e^{-j\omega}}{1-(1/2)e^{-j\omega}}$ (b) $\dfrac{e^{j\omega}}{1-(1/2)e^{j\omega}}$ (c) $\dfrac{e^{-j\omega}}{1-(1/2)e^{j\omega}}$ (d) $\dfrac{e^{j\omega}}{1-(1/2)e^{-j\omega}}$

14. The FT of $2^n u(n)$ is

(a) $\dfrac{1}{1-2e^{-j\omega}}$ (b) $\dfrac{1}{1-2e^{j\omega}}$ (c) $\dfrac{1}{1+2e^{j\omega}}$ (d) does not exist

15. The FT of $\delta(n+2) - \delta(n-2)$ is

(a) $2j \sin 2\omega$ (b) $2 \cos 2\omega$
(c) $\sin 2\omega$ (d) $\cos 2\omega$

PROBLEMS

1. Find the DTFT of

   (a) $x(n) = \{2, -1, 3, 2\}$ (b) $\left(\frac{1}{4}\right)^n u(n+2)$

   (c) $x(n) = (0.2)^n u(n) - 2^n u(-n-1)$ (d) $x(n) = a^n \cos \omega_0 n$

2. Using properties of DTFT, find the FT of the following:

   (a) $\left(\frac{1}{2}\right)^{|n-3|}$ (b) $\left(\frac{1}{2}\right)^{n-4} u(n-4)$

   (c) $nu(-n)$ (d) $e^{j2n}\, u(n)$

   (e) $n3^{-n}\, u(-n)$

3. The impulse response of an LTI system is $h(n) = \{1, 2, 1, -1\}$. Find the response of the system for the input $x(n) = \{1, 3, 2, 1\}$.

4. Find the convolution of the sequences $x_1(n) = x_2(n) = \{1, \underset{\uparrow}{1}, 1\}$

5. Find the frequency response of $x(n) = \{2, 1, 2\}$.

6. Determine the output sequence from the spectrum $Y(\omega)$.

$$Y(\omega) = \frac{1}{3}\,\frac{e^{j\omega} + 1 + e^{-j\omega}}{1 - ae^{-j\omega}}$$

7. A system has unit sample response $h(n)$ given by

$$h(n) = -\frac{1}{4}\delta(n+1) + \frac{1}{2}\delta(n) - \frac{1}{4}\delta(n-1)$$

   Find the frequency response.

8. Determine the frequency response, magnitude response and phase response of the LTI system governed by the difference equation:

$$y(n) = x(n) + 0.81x(n-1) + 0.81x(n-2) - 0.45y(n-2)$$

MATLAB PROGRAMS

Program 5.1

% Fourier transform and Inverse Fourier transform of a given sequence

```
clc; clear all; close all;
syms x;
f = exp(–x^2);
disp('The input equation is')
disp(f)
a=fourier(f);
disp('The fourier transform of the input equation is')
disp(a)
b=ifourier(a);
disp('The Inverse fourier transform is')
disp(b)
```

Output:

```
The input equation is
exp(–x^2)

The fourier transform of the input equation is
pi^(1/2)*exp(–w^2/4)

The Inverse fourier transform is
exp(–x^2)
```

Program 5.2

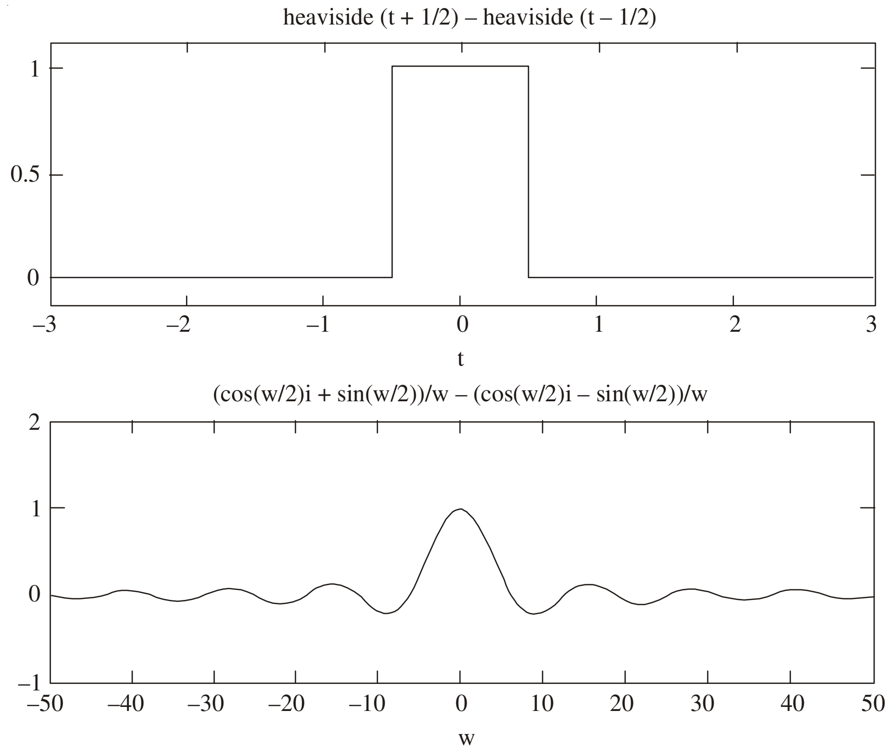
% Fourier transform of a signal
% u(t+0.5)–u(t–0.5)

```
clc; close all; clear all;
syms t w
a = heaviside(t + 0.5) – heaviside(t – 0.5);
subplot(2,1,1),ezplot(a, [–3 3]);
```

```
b = fourier(a)
subplot(2,1,2);
ezplot(b, [–50 50]);
axis([–50 50 –1 2])

b =
(cos(w/2)*i + sin(w/2))/w – (cos(w/2)*i – sin(w/2))/w
```

Output:

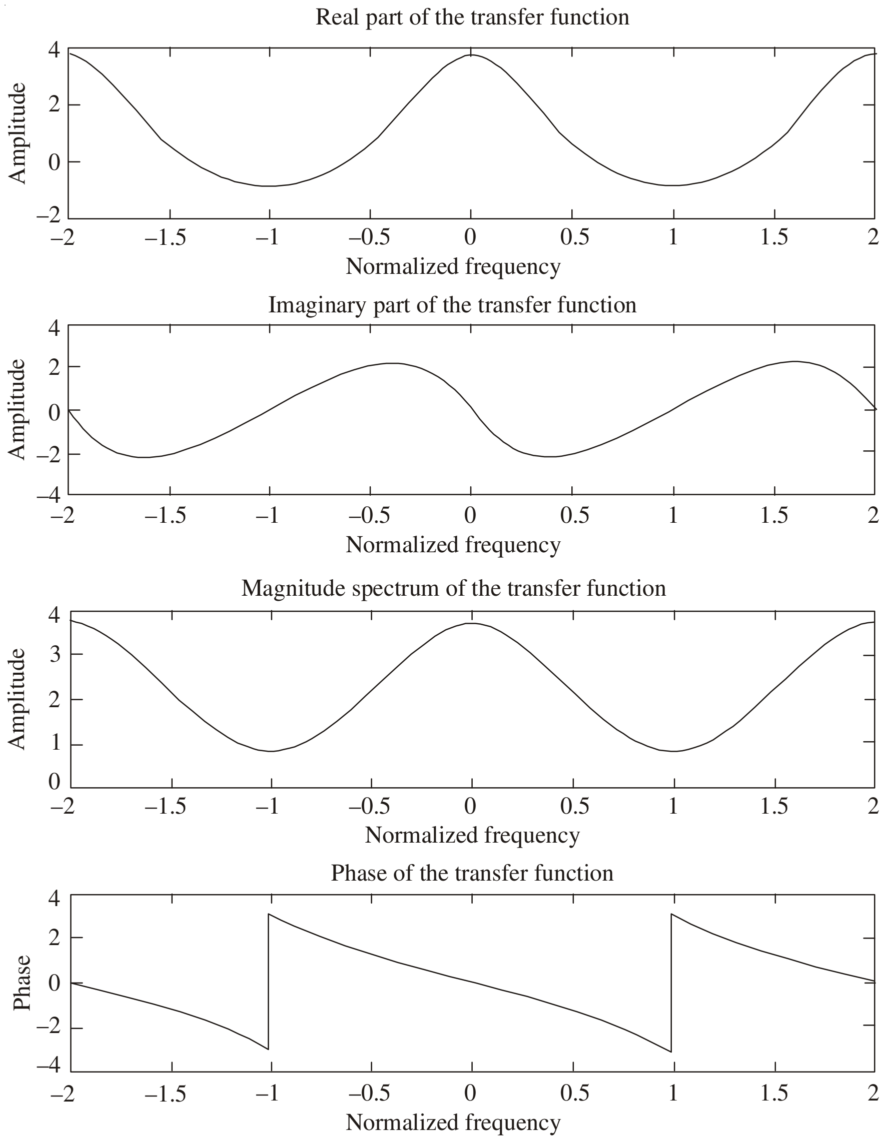


Program 5.3

```
% Evaluation and plotting of DTFT of the transfer function of the form a=e^(–jw)
% h(e)=[1+2*a^(–1)]/[1–0.2*a^(–1)]

clc; clear all; close all;
w=–2*pi:8*pi/511:2*pi;
num=[1 2];den=[1 –0.2];
h=freqz(num,den,w);
subplot(2,1,1);plot(w/pi,real(h));
xlabel('Normalized frequency')
ylabel('Amplitude')
title('Real part of the transfer function')
subplot(2,1,2);plot(w/pi,imag(h));
xlabel('Normalized frequency')
ylabel('Amplitude')
title('Imaginary part of the transfer function')
figure;
subplot(2,1,1);plot(w/pi,abs(h));
xlabel('Normalized frequency')
ylabel('Amplitude')
title('Magnitude spectrum of the transfer function')
subplot(2,1,2);plot(w/pi,angle(h));
xlabel('Normalized frequency')
ylabel('phase')
title('phase of the transfer function')
```

Output:

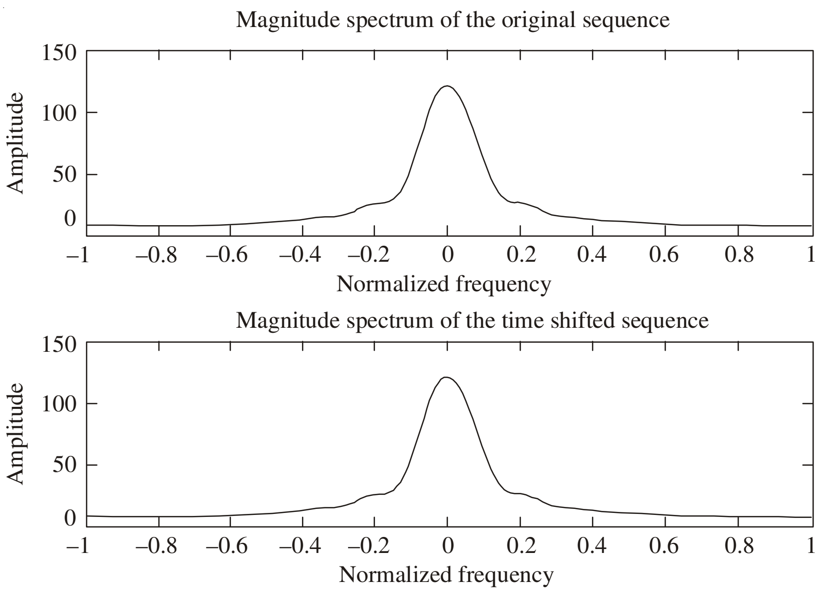


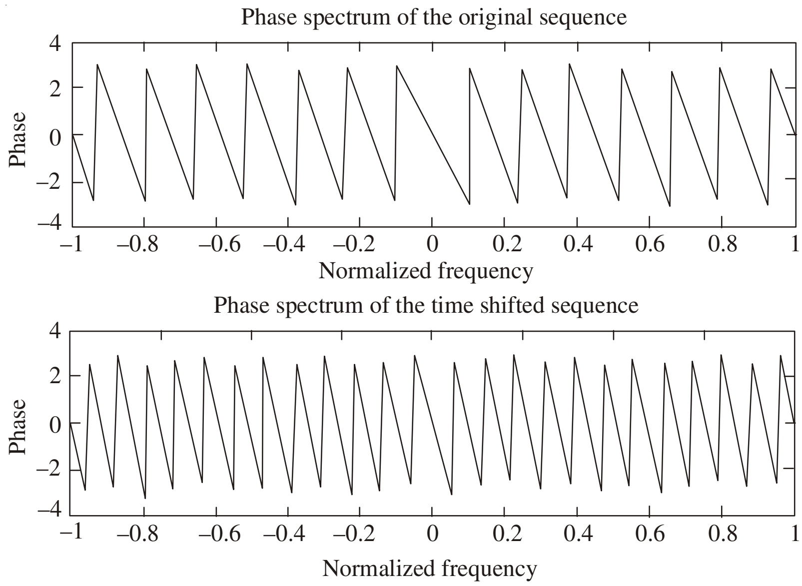
Program 5.4

```
% Time shifting property of DTFT
clc; clear all; close all
w=-pi:2*pi/255:pi;
d=10;num=1:15;
h1=freqz(num,1,w);
a=[zeros(1,d) num];%shifting
```

```
h2=freqz(a,1,w);
subplot(2,1,1);plot(w/pi,abs(h1));
xlabel('Normalized frequency')
ylabel('Amplitude')
title('magnitude spectrum of the original sequence')
subplot(2,1,2); plot(w/pi,abs(h2));
xlabel('Normalized frequency')
ylabel('Amplitude')
title('magnitude spectrum of the time shifted sequence')
figure;
subplot(2,1,1);plot(w/pi,angle(h1));
xlabel('Normalized frequency')
ylabel('phase')
title('phase spectrum of the original sequence')
subplot(2,1,2),;plot(w/pi,angle(h2));
xlabel('Normalized frequency')
ylabel('phase')
title('phase spectrum of the time shifted sequence')
```

Output:



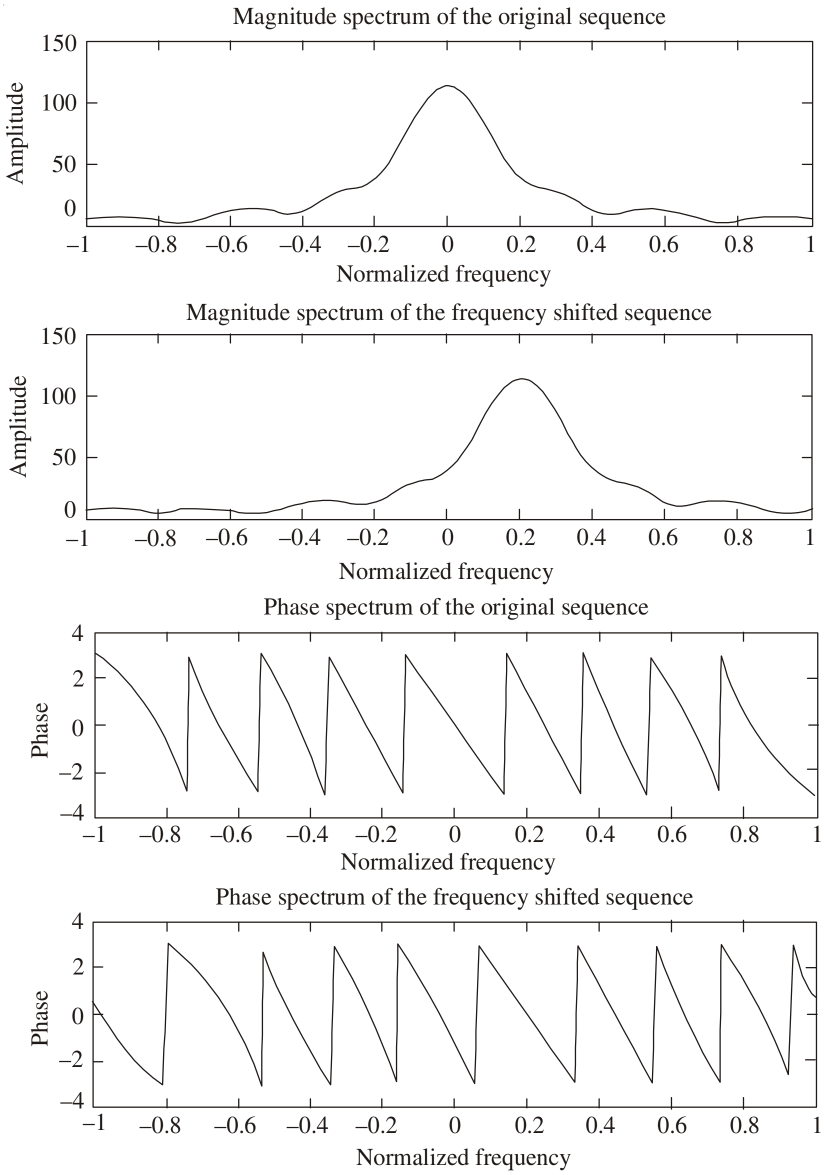


Program 5.5

```
% Frequency shifting property of DTFT
clc; clear all; close all;
w=–pi:2*pi/255:pi;
wo=0.2*pi;
num1=[1 3 5 7 5 11 13 17 18 21 12];
l=length(num1);
h1=freqz(num1,1,w);
n=0:l–1;
num2=exp(wo*i*n).*num1;
h2=freqz(num2,1,w);
subplot(2,1,1);plot(w/pi,abs(h1));
xlabel('Normalized frequency')
ylabel('Amplitude')
title('magnitude spectrum of the original sequence')
subplot(2,1,2);plot(w/pi,abs(h2));
xlabel('Normalized frequency')
ylabel('Amplitude')
title('magnitude spectrum of the Frequency shifted sequence')
figure;
subplot(2,1,1);plot(w/pi,angle(h1));
```

```
xlabel('Normalized frequency')
ylabel('phase')
title('phase spectrum of the original sequence')
subplot(2,1,2);plot(w/pi,angle(h2));
xlabel('Normalized frequency')
ylabel('phase')
title('phase spectrum of the Frequency shifted sequence')
```

Output:

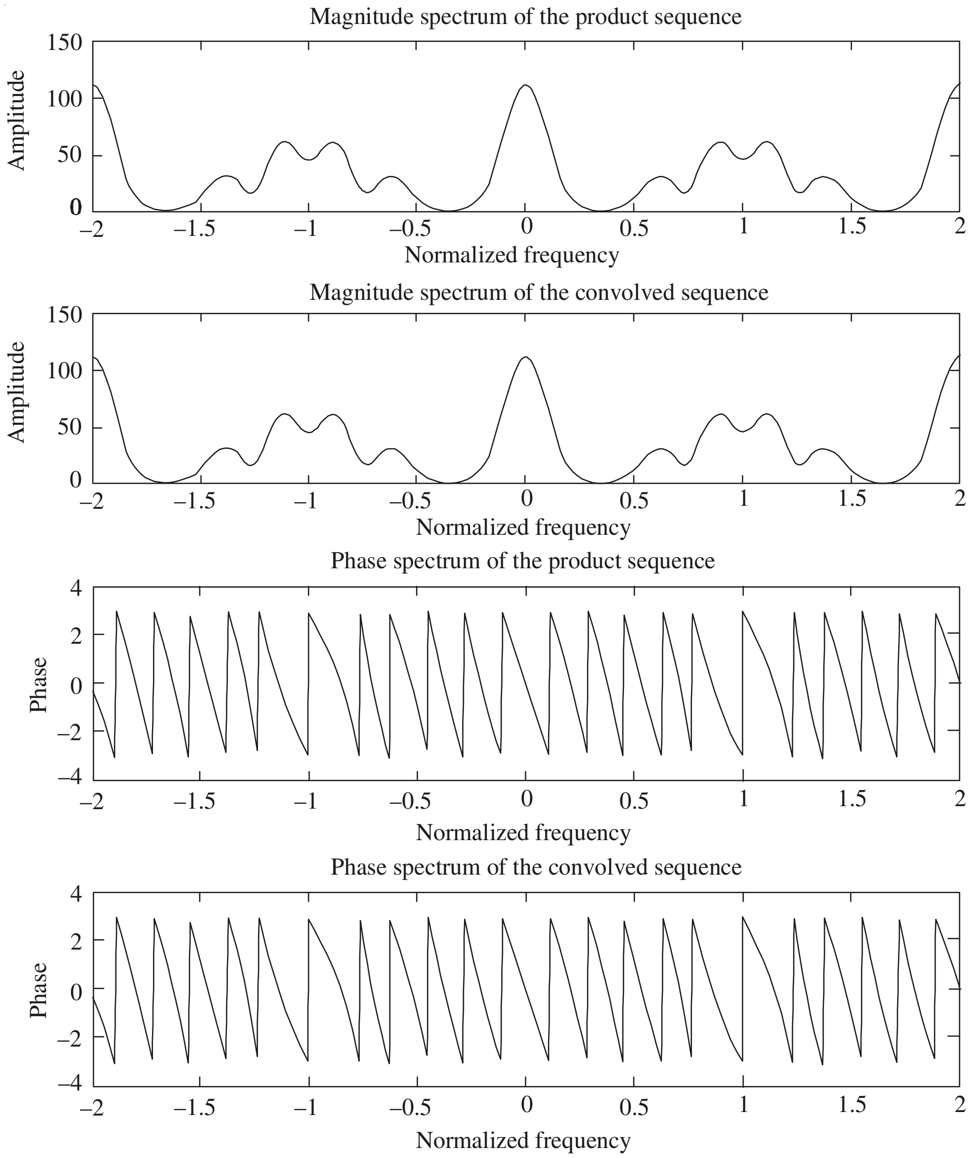


Program 5.6

% Time convolution property of DTFT

```
clc; clear all; close all;
w=–2*pi:2*pi/255:2*pi;
x1=[1 3 5 7 5 11 13 17 18 21 12];
x2=[1 –2 3 –2 1];
y=conv(x1,x2);
h1=freqz(x1,1,w);
h2=freqz(x2,1,w);
h=h1.*h2;
h3=freqz(y,1,w);
subplot(2,1,1); plot(w/pi,abs(h));
xlabel('Normalized frequency')
ylabel('Amplitude')
title('magnitude spectrum of the product sequence')
subplot(2,1,2); plot(w/pi,abs(h3));
xlabel('Normalized frequency')
ylabel('Amplitude')
title('magnitude spectrum of the convolved sequence')
figure
subplot(2,1,1),;plot(w/pi,angle(h));
xlabel('Normalized frequency')
ylabel('phase')
title('phase spectrum of the product sequence')
subplot(2,1,2);plot(w/pi,angle(h3));
xlabel('Normalized frequency')
ylabel('phase')
title('phase spectrum of the convolved sequence')
```

Output:

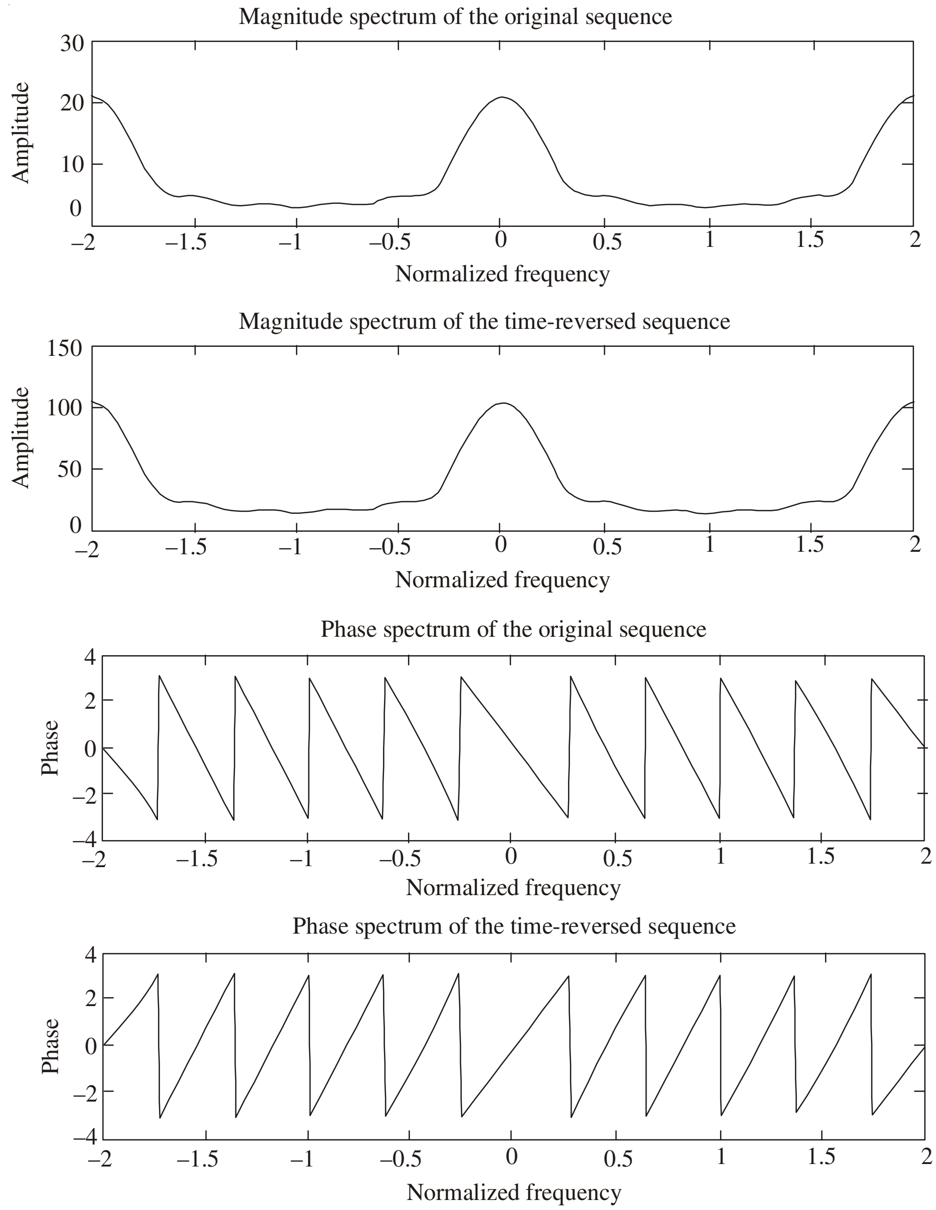

Magnitude spectrum of the product sequence
Amplitude
150
100
50
0
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2
Normalized frequency
Magnitude spectrum of the convolved sequence
Amplitude
150
100
50
0
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2
Normalized frequency
Phase spectrum of the product sequence
Phase
4
2
0
−2
−4
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2
Normalized frequency
Phase spectrum of the convolved sequence
Phase
4
2
0
−2
−4
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2
Normalized frequency

Program 5.7

% Time reversal property of DTFT

```
clc; clear all; close all
w=–2*pi:2*pi/255:2*pi;
num=[1 2 3 4 5 6];
l=length(num)–1;
h1=freqz(num,1,w);
h2=freqz(fliplr(num),1,w);
h3=exp(w*l*i).*h2;
subplot(2,1,1); plot(w/pi,abs(h1));
xlabel('Normalized frequency')
ylabel('Amplitude')
title('magnitude spectrum of the original sequence')
subplot(2,1,2);plot(w/pi,abs(h3));
xlabel('Normalized frequency')
ylabel('Amplitude')
title('magnitude spectrum of the time–reversed sequence')
figure
subplot(2,1,1); plot(w/pi,angle(h1));
xlabel('Normalized frequency')
ylabel('phase')
title('phase spectrum of the original sequence')
subplot(2,1,2); plot(w/pi,angle(h3));
xlabel('Normalized frequency')
ylabel('phase')
title('phase spectrum of the time–reversed sequence')
```

Output:

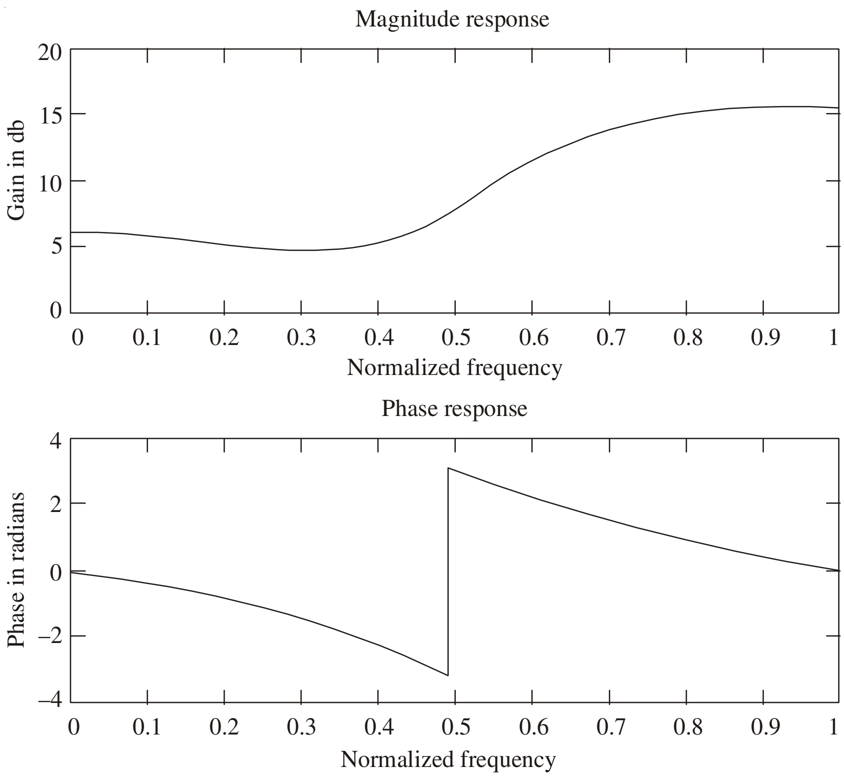

Magnitude spectrum of the original sequence
Amplitude
Normalized frequency
Magnitude spectrum of the time-reversed sequence
Amplitude
Normalized frequency
Phase spectrum of the original sequence
Phase
Normalized frequency
Phase spectrum of the time-reversed sequence
Phase
Normalized frequency

Program 5.8

Frequency response of the given system

```
clc; clear all; close all;
num=[1 –1 3];
den=[1 1/3 1/6];
[h,om]=freqz(num,den);
subplot(2,1,1);plot(om/pi,20*log10(abs(h)));
xlabel('normalized frequency')
ylabel('gain in db')
title('magnitude response')
subplot(2,1,2);plot(om/pi,angle(h));
xlabel('Normalized Frequency')
ylabel('phase in radians')
title('phase response')
```

Output:



Program 5.9

% Periodicity property of DTFT

```
clc; clear all; close all;
n=1:10;
x=(0.9*exp(i*pi/3)).^n;
k=–200:200;
w=(pi/100)*k;
x1=x*exp(–i*pi/100).^(n'*k);
subplot(2,1,1); plot(w/pi,abs(x1));
xlabel('Normalized frequency')
ylabel('Amplitude')
title('magnitude spectrum ')
subplot(2,1,2); plot(w/pi,angle(x1));
xlabel('Normalized frequency')
ylabel('phase')
title('phase spectrum ')
```

Output: