Digital Signal Processing

SECTION 6

CONVOLUTION

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What is Convolution ?

- Convolution is a mathematical process used in signal processing and engineering mathematics.
- In the context of signal processing, this is used for many purposes such as applying filters and analyzing signals.

USE CONVOLUTION

- Naturally, Here are some practical examples of how the Convolution Process has been used in various fields:
  - Applying filters in image processing:
    - In image processing, convolution is used to apply filters to images to enhance or modify them. For example, the Sharpening Filter uses convolution to improve image sharpness.
  - Use in Neural Networks:
    - In machine learning, convolution is widely used in neural networks, especially in image analysis and shape recognition. It is used to extract features from images and identify patterns.
  - Audio signal processing:
    - In signal processing, convolution is used to apply filters to audio signals, enabling noise to be removed or quality improved.

Methods To Compute The Convolution Sum of Two Sequences $x(n)$ And $h(n)$

Method 1 Linear Convolution Using Graphical Method

Step 1: Choose the starting time n for evaluating the output sequence $y(n)$. If $x(n)$ starts at $n = n_1$ and $h(n)$ starts at $n = n_2$, then $n = n_1 + n_2$ is a good choice.

Step 2: Express both the sequences $x(n)$ and $h(n)$ in terms of the index k.

Step 3: Fold $h(k)$ about $k = 0$ to obtain $h(-k)$ and shift by n to the right if n is positive and to the left if n is negative to obtain $h(n - k)$.

Step 4: Multiply the two sequences $x(k)$ and $h(n - k)$ element by element and sum the products to get $y(n)$.

Step 5: Increment the index n, shift the sequence $h(n - k)$ to the right by one sample and perform Step 4.

Step 6: Repeat Step 5 until the sum of products is zero for all remaining values of n.

Method 2 Linear Convolution Using Tabular Array

Let $x_1(n)$ and $x_2(n)$ be the given N sample sequences. Let $x_3(n)$ be the N sample sequence obtained by linear convolution of $x_1(n)$ and $x_2(n)$. The following procedure can be used to obtain one sample of $x_3(n)$ at $n = q$:

Step 1: Change the index from n to k, and write $x_1(k)$ and $x_2(k)$.
Step 2: Represent the sequences $x_1(k)$ and $x_2(k)$ as two rows of tabular array.
Step 3: Fold one of the sequences. Let us fold $x_2(k)$ to get $x_2(-k)$.

Step 4: Shift the sequence $x_2(-k)$, q times to get the sequence $x_2(q-k)$. If q is positive, then shift the sequence to the right and if q is negative, then shift the sequence to the left.
Step 5: The sample of $x_3(n)$ at $n = q$ is given by

$$x_3(q) = \sum_{k=0}^{N-1} x_1(k)\,x_2(q-k)$$

Determine the product sequence $x_1(k)x_2(q-k)$ for one period.
Step 6: The sum of the samples of the product sequence gives the sample $x_3(q)$ [i.e. $x_3(n)$ at $n = q$].

The above procedure is repeated for all possible values of n to get the sequence $x_3(n)$.

Method 3 Linear Convolution Using Tabular Method

Given $x(n) = \{x_1, x_2, x_3, x_4\}$, $h(n) = \{h_1, h_2, h_3, h_4\}$

The convolution of $x(n)$ and $h(n)$ can be computed as per the following procedure.

Step 1: Write down the sequences $x(n)$ and $h(n)$ as shown in Table 2.1.
Step 2: Multiply each and every sample in $h(n)$ with the samples of $x(n)$ and tabulate the values.
Step 3: Group the elements in the table by drawing diagonal lines as shown in table.
Step 4: Starting from the left sum all the elements in each strip and write down in the same order.

$$y(n) = x_1h_1,\ x_1h_2 + x_2h_1,\ x_1h_3 + x_2h_2 + x_3h_1,\ x_1h_4 + x_2h_3 + x_3h_2 + x_4h_1,\ x_2h_4 + x_3h_3 + x_4h_2,\ x_3h_4 + x_4h_3,\ x_4h_4$$

Step 5: Mark the symbol $\uparrow$ at time origin ($n = 0$).

TABLE 2.1 Table for Computing $y(n)$

$h(n)$ \ $x(n)$	x_1	x_2	x_3	x_4
h_1	x_1h_1	x_2h_1	x_3h_1	x_4h_1
h_2	x_1h_2	x_2h_2	x_3h_2	x_4h_2
h_3	x_1h_3	x_2h_3	x_3h_3	x_4h_3
h_4	x_1h_4	x_2h_4	x_3h_4	x_4h_4

Method 4 Linear Convolution Using Matrices

If the number of elements in $x(n)$ are N_1 and in $h(n)$ are N_2, then to find the convolution of $x(n)$ and $h(n)$ form the following matrices:

1. Matrix H of order $(N_1 + N_2 - 1) \times N_1$ with the elements of $h(n)$
2. A column matrix X of order $(N_1 \times 1)$ with the elements of $x(n)$
3. Multiply the matrices H and X to get a column matrix Y of order $(N_1 + N_2 - 1)$ that has the elements of $y(n)$, the convolution of $x(n)$ and $h(n)$.

$$\underbrace{\begin{bmatrix} h(0) & 0 & \cdots & 0 \\ h(1) & h(0) & \cdots & 0 \\ \vdots & \vdots & \cdots & 0 \\ h(N_2-1) & h(N_2-2) & \cdots & h(0) \\ 0 & h(N_2-1) & \cdots & h(1) \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & h(N_2-1) \end{bmatrix}}_{H} \underbrace{\begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ x(N_1-1) \end{bmatrix}}_{X} \underset{=}{=} \underbrace{\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ y(N_1+N_2-1) \end{bmatrix}}_{Y}$$

EXAMPLE 2.11 Determine the convolution sum of two sequences:

$$x(n) = \{4, 2, 1, 3\}, \qquad h(n) = \left\{\begin{matrix}1, 2, 2, 1\\ \uparrow\end{matrix}\right\}$$

Solution:

$x(n)$ starts at $n_1 = 0$ and $h(n)$ starts at $n_2 = -1$. Therefore, the starting sample of $y(n)$ is at

$$n = n_1 + n_2 = 0 - 1 = -1$$

$x(n)$ has 4 samples, $h(n)$ has 4 samples. Therefore, $y(n)$ will have $N = 4 + 4 - 1 = 7$ samples, i.e., from $n = -1$ to $n = 5$.

Method 1 Graphical Method

We know that $$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

From Figure 2.6, we get

For $n = -1$ $$y(-1) = \sum_{k=-\infty}^{\infty} x(k)h(-1-k) = 4\cdot1 = 4$$

For $n = 0$ $$y(0) = \sum_{k=-\infty}^{\infty} x(k)h(-k) = 4\cdot2 + 2\cdot1 = 10$$

For $n = 1$ $$y(1) = \sum_{k=-\infty}^{\infty} x(k)h(1-k) = 4\cdot2 + 2\cdot2 + 1\cdot1 = 13$$

For $n = 2$ $$y(2) = \sum_{k=-\infty}^{\infty} x(k)h(2-k) = 4\cdot1 + 2\cdot2 + 1\cdot2 + 3\cdot1 = 13$$

For $n = 3$ $$y(3) = \sum_{k=-\infty}^{\infty} x(k)h(3-k) = 2\cdot1 + 1\cdot2 + 3\cdot2 = 10$$

For $n = 4$ $$y(4) = \sum_{k=-\infty}^{\infty} x(k)h(4-k) = 1\cdot1 + 3\cdot2 = 7$$

For $n = 5$ $\quad y(5) = \sum_{k=-\infty}^{\infty} x(k)\,h(5-k) = 3\cdot 1 = 3$

$\therefore \quad y(n) = \left\{ \begin{matrix} 4, 10, 13, 13, 10, 7, 3 \\ \uparrow \end{matrix} \right\}$

To check the correctness of the result sum all the samples in $x(n)$ and multiply with the sum of all samples in $h(n)$. This value must be equal to sum of all samples in $y(n)$.

In the given problem, $\sum_n x(n) = 10$, $\sum_n h(n) = 6$ and $\sum_n y(n) = 60$

This shows $\sum_n x(n) \cdot \sum_n h(n) = \sum_n y(n)$ (proved). Therefore, the result is correct.

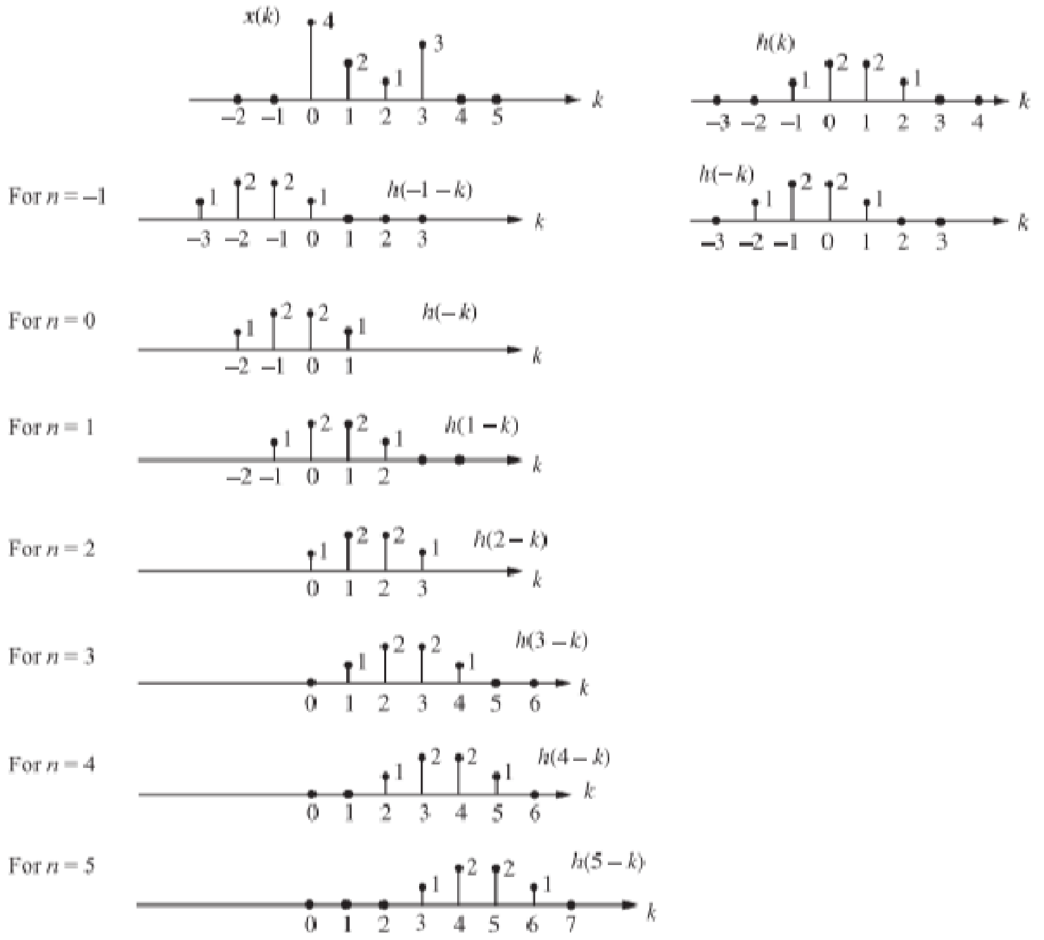

Figure 2.6 Operation on signals $x(n)$ and $h(n)$ to compute convolution.

Method 2 Tabular Array

Tabulate the sequence $x(k)$ and shifted version of $h(k)$ as shown in Table 2.2.

TABLE 2.2 Table for computing $y(n)$.

k		-4	-3	-2	-1	0	1	2	3	4	5	6	7
$x(k)$		–	–	–	–	4	2	1	3	–	–	–	–
$h(-k)$		–	–	1	2	2	1	–	–	–	–	–	–
$n = -1$	$h(-1-k)$	–	1	2	2	1	–	–	–	–	–	–	–
$n = 0$	$h(-k)$	–	–	1	2	2	1	–	–	–	–	–	–
$n = 1$	$h(1-k)$	–	–	–	1	2	2	1	–	–	–	–	–
$n = 2$	$h(2-k)$	–	–	–	–	1	2	2	1	–	–	–	–
$n = 3$	$h(3-k)$	–	–	–	–	–	1	2	2	1	–	–	–
$n = 4$	$h(4-k)$	–	–	–	–	–	–	1	2	2	1	–	–
$n = 5$	$h(5-k)$	–	–	–	–	–	–	–	1	2	2	1	–

The starting value of $n = -1$. From the table, we can see that

For $n = -1$ $$y(-1) = \sum_{k=-\infty}^{\infty} x(k)h(-1-k) = 4 \cdot 1 = 4$$

For $n = 0$ $$y(0) = \sum_{k=-\infty}^{\infty} x(k)h(-k) = 4 \cdot 2 + 2 \cdot 1 = 10$$

For $n = 1$ $$y(1) = \sum_{k=-\infty}^{\infty} x(k)h(1-k) = 4 \cdot 2 + 2 \cdot 2 + 1 \cdot 1 = 13$$

For $n = 2$ $$y(2) = \sum_{k=-\infty}^{\infty} x(k)h(2-k) = 4 \cdot 1 + 2 \cdot 2 + 1 \cdot 2 + 3 \cdot 1 = 13$$

For $n = 3$ $$y(3) = \sum_{k=-\infty}^{\infty} x(k)h(3-k) = 2 \cdot 1 + 1 \cdot 2 + 3 \cdot 2 = 10$$

For $n = 4$ $$y(4) = \sum_{k=-\infty}^{\infty} x(k)h(4-k) = 1 \cdot 1 + 3 \cdot 2 = 7$$

For $n = 5$ $$y(5) = \sum_{k=-\infty}^{\infty} x(k)h(5-k) = 3 \cdot 1 = 3$$

$\therefore$ $$y(n) = \left\{ 4, \underset{\uparrow}{10}, 13, 13, 10, 7, 3 \right\}$$

Method 3 Tabular Method

Given $x(n) = \{4, 2, 1, 3\}$, $h(n) = \{1, \underset{\uparrow}{2}, 2, 1\}$

The convolution of $x(n)$ and $h(n)$ can be computed as shown in Table 2.3.

TABLE 2.3 Table for computing $y(n)$.

		$x(n)$			
		4	2	1	3
$h(n)$	1	4	2	1	3
	2	8	4	2	6
	2	8	4	2	6
	1	4	2	1	3

$$y(n) = 4,\ 8 + 2,\ 8 + 4 + 1,\ 4 + 4 + 2 + 3,\ 2 + 2 + 6,\ 1 + 6,\ 3$$
$$= 4,\ 10,\ 13,\ 13,\ 10,\ 7,\ 3$$

The starting value of n is equal to -1, mark the symbol $\uparrow$ at time origin ($n = 0$).

$$\therefore \quad y(n) = \{4, \underset{\uparrow}{10}, 13, 13, 10, 7, 3\}$$

Method 4 Matrices Method

The given sequences are: $x(n) = \{x(0), x(1), x(2), x(3)\} = \{4, 2, 1, 3\}$

and $h(n) = \{h(0), h(1), h(2), h(3)\} = \{1, \underset{\uparrow}{2}, 2, 1\}$

The sequence $x(n)$ is starting at $n = 0$ and the sequence $h(n)$ is starting at $n = -1$. So the sequence $y(n)$ corresponding to the linear convolution of $x(n)$ and $h(n)$ will start at $n = 0 + (-1) = -1$. $x(n)$ is of length 4 and $h(n)$ is also of length 4. So length of $y(n) = 4 + 4 - 1 = 7$. Substituting the sequence values in matrix form and multiplying as shown below, we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 1 & 2 & 2 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ 13 \\ 13 \\ 10 \\ 7 \\ 3 \end{bmatrix}$$

$$y(n) = x(n) * h(n) = [4 \;\; \underset{\uparrow}{10} \;\; 13 \;\; 13 \;\; 10 \;\; 7 \;\; 3]$$

Thank You