Digital Signal Processing

SECTION 5

OPERATIONS ON MATRIX

LECTURE 2 EXERCISES

ASMAA FATHY

Energy and Power Signals

- Signals may also be classified as energy signals and power signals. However, there are some signals which can neither be classified as energy signals nor power signals.
- The total energy E of a discrete-time signal $x(n)$ is defined as:

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

and the average power P of a discrete-time signal $x(n)$ is defined as:

$$P = \lim_{n \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

and

$$P = \frac{1}{N} \sum_{n=0}^{n-1} |x(n)|^2$$

for a digital signal with $x(n) = 0$ for $n < 0$

EXAMPLE 1.6

- Find whether the signal

$$x(n) = \begin{cases} n^2 & 0 \le n \le 3 \\ 10 - n & 4 \le n \le 6 \\ n & 7 \le n \le 9 \\ 0 & \text{otherwise} \end{cases}$$

is a power signal or an energy signal. Also find the energy and power of the signal.

Solution

Energy of the signal $E = \sum_{n=-\infty}^{\infty} |x(n)|^2$

$$= \sum_{n=0}^{3} (n^2)^2 + \sum_{n=4}^{6} (10-n)^2 + \sum_{n=7}^{9} (n)^2$$

$$= \sum_{n=0}^{3} n^4 + \sum_{n=4}^{6} (100 + n^2 - 20n) + \sum_{n=7}^{9} n^2$$

$$= (0+1+16+81) + (36+25+16) + (49+64+81)$$

$$= 369 < \infty \text{ joule}$$

Power of the signal $P = \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$

$$= \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} \left[\sum_{n=0}^{3} (n^2)^2 + \sum_{n=4}^{6} (10-n)^2 + \sum_{n=7}^{9} (n)^2 \right]$$

$$= \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} [369] = 0$$

Here energy is finite and power is zero. So it is an energy signal.

Causal and Non- Causal Signals

- A discrete-time signal $x(n)$ is said to be causal if $x(n) = 0$ for $n < 0$, otherwise the signal is non-causal. A discrete-time signal $x(n)$ is said to be anti-causal if $x(n) = 0$ for $n > 0$.
- A causal signal does not exist for negative time and an anti-causal signal does not exist for positive time. A signal which exists in positive as well as negative time is called a non-casual signal.

EXAMPLE 1.7

- Find whether the signal causal or non-causal

(a) $x(n) = u(n+4) - u(n-2)$

Solution

(a) Given $x(n) = u(n + 4) - u(n - 2)$
The given signal exists from $n = -4$ to $n = 1$. Since $x(n) \neq 0$ for $n < 0$, it is non-causal.

Even and Odd Signals

Any signal $x(n)$ can be expressed as sum of even and odd components. That is

$$x(n) = x_e(n) + x_o(n)$$

where $x_e(n)$ is even components and $x_o(n)$ is odd components of the signal.

Even (symmetric) signal

A discrete-time signal $x(n)$ is said to be an even (symmetric) signal if it satisfies the condition:

$$x(n) = x(-n) \quad \text{for all } n$$

Odd (anti-symmetric) signal

A discrete-time signal $x(n)$ is said to be an odd (anti-symmetric) signal if it satisfies the condition:

$$x(-n) = -x(n) \quad \text{for all } n$$

Evaluation of even and odd parts of signal

We have
$$x(n) = x_e(n) + x_o(n)$$

$$\therefore \quad x(-n) = x_e(-n) + x_o(-n) = x_e(n) - x_o(n)$$

$$x(n) + x(-n) = x_e(n) + x_o(n) + x_e(n) - x_o(n) = 2x_e(n)$$

$$\therefore \quad x_e(n) = \frac{1}{2}[x(n) + x(-n)]$$

$$x(n) - x(-n) = [x_e(n) + x_o(n)] - [x_e(n) - x_o(n)] = 2x_o(n)$$

$$\therefore \quad x_o(n) = \frac{1}{2}[x(n) - x(-n)]$$

EXAMPLE 1.8

- Find the even and odd components of the following signals:

(a) $x(n)=\left\{-3, 1, \underset{\uparrow}{2}, -4, 2\right\}$

(b) $x(n)=\left\{-2, 5, \underset{\uparrow}{1}, -3\right\}$

Solution

(a) $x(n)=\left\{-3, 1, \underset{\uparrow}{2}, -4, 2\right\}$

Given $\quad x(n)=\left\{-3, 1, \underset{\uparrow}{2}, -4, 2\right\}$

$\therefore \quad x(-n)=\left\{2, -4, \underset{\uparrow}{2}, 1, -3\right\}$

$$x_e(n)=\frac{1}{2}[x(n)+x(-n)]$$

$$=\frac{1}{2}[-3+2, 1-4, 2+2, -4+1, 2-3]$$

$$=\left\{-0.5, -1.5, \underset{\uparrow}{2}, -1.5, -0.5\right\}$$

$$x_o(n)=\frac{1}{2}[x(n)-x(-n)]$$

$$=\frac{1}{2}[-3-2, 1+4, 2-2, -4-1, 2+3]$$

$$=\left\{-2.5, 2.5, \underset{\uparrow}{0}, -2.5, 2.5\right\}$$

Solution

(b) $x(n) = \left\{ -2, 5, \underset{\uparrow}{1}, -3 \right\}$

Given $x(n) = \left\{ -2, 5, \underset{\uparrow}{1}, -3 \right\}$

$$x(-n) = \left\{ -3, \underset{\uparrow}{1}, 5, -2 \right\}$$

$$x_e(n) = \frac{1}{2}[x(n) + x(-n)]$$

$$= \frac{1}{2}\left[-2 + 0, 5 - 3, 1 + 1, -3 + 5, 0 - 2\right]$$

$$= \left\{ -1, 1, \underset{\uparrow}{1}, 1, -1 \right\}$$

$$x_o(n) = \frac{1}{2}[x(n) - x(-n)]$$

$$= \frac{1}{2}\left[-2 - 0, 5 + 3, 1 - 1, -3 - 5, 0 + 2\right]$$

$$= \left\{ -1, 4, \underset{\uparrow}{0}, -4, 1 \right\}$$

Static and Dynamic Systems

A system is said to be static or memoryless if the response is due to present input alone, i.e., for a static or memoryless system, the output at any instant n depends only on the input applied at that instant n but not on the past or future values of input or past values of output.

For example, the systems defined below are static or memoryless systems.

$$y(n) = x(n)$$

$$y(n) = 2x^2(n)$$

In contrast, a system is said to be dynamic or memory system if the response depends upon past or future inputs or past outputs. A summer or accumulator, a delay element is a discrete-time system with memory.

For example, the systems defined below are dynamic or memory systems.

$$y(n) = x(2n)$$
$$y(n) = x(n) + x(n-2)$$
$$y(n) + 4y(n-1) + 4y(n-2) = x(n)$$

Any discrete-time system described by a difference equation is a dynamic system.

A purely resistive electrical circuit is a static system, whereas an electric circuit having inductors and/or capacitors is a dynamic system.

EXAMPLE 1.12 Find whether the following systems are dynamic or not:

(a) $y(n) = x(n + 2)$ (b) $y(n) = x^2(n)$

(c) $y(n) = x(n - 2) + x(n)$

Solution:

(a) Given $$y(n) = x(n + 2)$$

The output depends on the future value of input. Therefore, the system is dynamic.

(b) Given $$y(n) = x^2(n)$$

The output depends on the present value of input alone. Therefore, the system is static.

(c) Given $$y(n) = x(n - 2) + x(n)$$

The system is described by a difference equation. Therefore, the system is dynamic.

Causal and Non-Causal Systems

A system is said to be causal (or non-anticipative) if the output of the system at any instant n depends only on the present and past values of the input but not on future inputs, i.e., for a causal system, the impulse response or output does not begin before the input function is applied, i.e., a causal system is non anticipatory.

Causal systems are real time systems. They are physically realizable.

The impulse response of a causal system is zero for $n < 0$, since $\delta(n)$ exists only at $n = 0$,

i.e.
$$h(n) = 0 \quad \text{for} \quad n < 0$$

The examples for causal systems are:

$$y(n) = nx(n)$$

$$y(n) = x(n-2) + x(n-1) + x(n)$$

A system is said to be non-causal (anticipative) if the output of the system at any instant n depends on future inputs. They are anticipatory systems. They produce an output even before the input is given. They do not exist in real time. They are not physically realizable.

A delay element is a causal system, whereas an image processing system is a non-causal system.

The examples for non-causal systems are:

$$y(n) = x(n) + x(2n)$$

$$y(n) = x^2(n) + 2x(n+2)$$

EXAMPLE 1.13 Check whether the following systems are causal or not:

(a) $y(n) = x(n) + x(n-2)$ (b) $y(n) = x(2n)$

(c) $y(n) = \sin[x(n)]$ (d) $y(n) = x(-n)$

Solution:

(a) Given $y(n) = x(n) + x(n-2)$

For $n = -2$ $y(-2) = x(-2) + x(-4)$

For $n = 0$ $y(0) = x(0) + x(-2)$

For $n = 2$ $y(2) = x(2) + x(0)$

For all values of n, the output depends only on the present and past inputs. Therefore, the system is causal.

(b) Given $y(n) = x(2n)$

For $n = -2$ $y(-2) = x(-4)$

For $n = 0$ $y(0) = x(0)$

For $n = 2$ $y(2) = x(4)$

For positive values of n, the output depends on the future values of input. Therefore, the system is non-causal.

(c) Given $y(n) = \sin[x(n)]$

For $n = -2$ $y(-2) = \sin[x(-2)]$

For $n = 0$ $y(0) = \sin[x(0)]$

For $n = 2$ $y(2) = \sin[x(2)]$

For all values of n, the output depends only on the present value of input. Therefore, the system is causal.

(d) Given $y(n) = x(-n)$

For $n = -2$ $y(-2) = x(2)$

For $n = 0$ $y(0) = x(0)$

For $n = 2$ $y(2) = x(-2)$

For negative values of n, the output depends on the future values of input. Therefore, the system is non-causal.

Linear and Non-Linear Systems

$$T(ax_1(n) + bx_2(n)] = aT[x_1(n)] + bT[x_2(n)]$$

- Simply we can say that a system is linear if the output due to weighted sum of inputs is equal to the weighted sum of outputs.
- In general, if the describing equation contains square or higher order terms of input and/or output and/or product of input/output and its difference or a constant, the system will definitely be non-linear.

EXAMPLE 1.14 Check whether the following systems are linear or not:

(a) $y(n) = n^2 x(n)$

(b) $y(n) = x(n) + \dfrac{1}{2x(n-2)}$

(c) $y(n) = 2x(n) + 4$

(d) $y(n) = x(n) \cos \omega n$

(e) $y(n) = |x(n)|$

(f) $y(n) = \dfrac{1}{N} \sum_{k=0}^{N-1} x(n-k)$

Solution:

(a) Given $$y(n) = n^2 x(n)$$

$$y(n) = T[x(n)] = n^2 x(n)$$

Let an input $x_1(n)$ produce an output $y_1(n)$.

$\therefore$ $$y_1(n) = T[x_1(n)] = n^2 x_1(n)$$

Let an input $x_2(n)$ produce an output $y_2(n)$.

$\therefore$ $$y_2(n) = T[x_2(n)] = n^2 x_2(n)$$

The weighted sum of outputs is:

$$ay_1(n) + by_2(n) = a[n^2 x_1(n)] + b[n^2 x_2(n)] = n^2[ax_1(n) + bx_2(n)]$$

The output due to weighted sum of inputs is:

$$y_3(n) = T[ax_1(n) + bx_2(n)] = n^2[ax_1(n) + bx_2(n)]$$

$$y_3(n) = ay_1(n) + by_2(n)$$

The weighted sum of outputs is equal to the output due to weighted sum of inputs. The superposition principle is satisfied. Therefore, the given system is linear.

Shift-invariant and Shift varying Systems

Time-invariance is the property of a system which makes the behaviour of the system independent of time. This means that the behaviour of the system does not depend on the time at which the input is applied. For discrete-time systems, the time invariance property is called shift invariance.

A system is said to be shift-invariant if its input/output characteristics do not change with time, i.e., if a time shift in the input results in a corresponding time shift in the output as shown in Figure 1.23, i.e.

If $$T[x(n)] = y(n)$$

Then $$T[x(n - k)] = y(n - k)$$

A system not satisfying the above requirements is called a time-varying system (or shift-varying system). A time-invariant system is also called a fixed system.

The time-invariance property of the given discrete-time system can be tested as follows:

Let $x(n)$ be the input and let $x(n - k)$ be the input delayed by k units.

$y(n) = T[x(n)]$ be the output for the input $x(n)$.

EXAMPLE 1.15 Determine whether the following systems are time-invariant or not:

(a) $y(n) = x(n/2)$ (b) $y(n) = x(n)$

(c) $y(n) = x^2(n-2)$ (d) $y(n) = x(n) + nx(n-2)$

Solution:

(a) Given
$$y(n) = x\left(\frac{n}{2}\right)$$
$$y(n) = T[x(n)] = x\left(\frac{n}{2}\right)$$

The output due to input delayed by k units is:
$$y(n, k) = T[x(n-k)] = y(n)\big|_{x(n)=x(n-k)} = x\left(\frac{n}{2} - k\right)$$

The output delayed by k units is:
$$y(n-k) = y(n)\big|_{n=n-k} = x\left(\frac{n-k}{2}\right)$$
$$y(n, k) \neq y(n-k)$$

i.e. the delayed output is not equal to the output due to delayed input. Therefore, the system is time-variant.

(b) Given
$$y(n) = x(n)$$
$$y(n) = T[x(n)] = x(n)$$

The output due to input delayed by k units is:
$$y(n, k) = T[x(n-k)] = y(n)\big|_{x(n)=x(n-k)} = x(n-k)$$

The output delayed by k units is:
$$y(n-k) = y(n)\big|_{n=n-k} = x(n-k)$$
$$y(n, k) = y(n-k)$$

i.e. the delayed output is equal to the output due to delayed input. Therefore, the system is time-invariant.

Stable and Unstable Systems

A bounded signal is a signal whose magnitude is always a finite value, i.e. $|x(n)| \le M$, where M is a positive real finite number. For example a sinewave is a bounded signal. A system is said to be bounded-input, bounded-output (BIBO) stable, if and only if every bounded input produces a bounded output. The output of such a system does not diverge or does not grow unreasonably large.

Let the input signal $x(n)$ be bounded (finite), i.e.,

$$|x(n)| \le M_x < \infty \quad \text{for all } n$$

where M_x is a positive real number. If

$$|y(n)| \le M_y < \infty$$

i.e. if the output $y(n)$ is also bounded, then the system is BIBO stable. Otherwise, the system is unstable. That is, we say that a system is unstable even if one bounded input produces an unbounded output.

It is very important to know about the stability of the system. Stability indicates the usefulness of the system. The stability can be found from the impulse response of the system which is nothing but the output of the system for a unit impulse input. If the impulse response is absolutely summable for a discrete-time system, then the system is stable.

EXAMPLE 1.18 Check the stability of the system defined by

(a) $y(n) = ax(n-7)$ (b) $y(n) = x(n) + \frac{1}{2}x(n-1) + \frac{1}{4}x(n-2)$

(c) $h(n) = a^n$ for $0 < n < 11$ (d) $h(n) = 2^n u(n)$

(e) $h(n) = u(n)$

Solution:

(a) Given $y(n) = ax(n-7)$

Let $x(n) = \delta(n)$

Then $y(n) = h(n)$

$\therefore$ $h(n) = a\delta(n-7)$

$\therefore$ $h(n) = a$ for $n = 7$

$= 0$ for $n \neq 7$

A system is stable if its impulse response $h(n)$ is absolutely summable.

i.e. $$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

In this case,

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=-\infty}^{\infty} a\delta(n-7) = a$$

Hence the given system is stable if the value of a is finite.

(b) Given $$y(n) = x(n) + \frac{1}{2}x(n-1) + \frac{1}{4}x(n-2)$$

Let $$x(n) = \delta(n)$$

Then $$y(n) = h(n)$$

$$\therefore \quad h(n) = \delta(n) + \frac{1}{2}\delta(n-1) + \frac{1}{4}\delta(n-2)$$

A discrete-time system is stable if

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

The given $h(n)$ has a value only at $n = 0$, $n = 1$ and $n = 2$. For all other values of n from $-\infty$ to ∞, $h(n) = 0$.

At $n = 0$, $h(0) = \delta(0) + \frac{1}{2}\delta(0-1) + \frac{1}{4}\delta(0-2) = \delta(0) + \frac{1}{2}\delta(-1) + \frac{1}{4}\delta(-2) = 1$

At $n = 1$, $h(1) = \delta(1) + \frac{1}{2}\delta(1-1) + \frac{1}{4}\delta(1-2) = \delta(1) + \frac{1}{2}\delta(0) + \frac{1}{4}\delta(-2) = \frac{1}{2}$

At $n = 2$, $h(2) = \delta(2) + \frac{1}{2}\delta(2-1) + \frac{1}{4}\delta(2-2) = \delta(2) + \frac{1}{2}\delta(1) + \frac{1}{4}\delta(0) = \frac{1}{4}$

$$\therefore \quad \sum_{n=-\infty}^{\infty} |h(n)| = 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4} < \infty \quad \text{a finite value.}$$

Hence the system is stable.

Thank You