**Trevor Leong** **21-241 Fall 2019**
**Homework 4** **Due Friday, September 20**

## Required Problems

1. (Strang 3.2.32, see Page 144 for picture and matrix $A$) Kirchoff's Current Law $A^T\mathbf{y} = \mathbf{0}$ says that *current in = current out* at every node. At node 1 this is $y_3 = y_1 + y_4$. Write the four equations for Kirchoff's Law at the four nodes (arrows show the positive direction of each $y$). Reduce $A^T$ to $R$ and find three special solutions in the nullspace of $A^T$ (4 by 6 matrix). *Solution:*

$$\begin{array}{c}
\text{Node 1: } y_3 = y_1 + y_4 \\
\text{Node 2: } y_1 = y_2 + y_5 \\
\text{Node 3: } y_2 = y_3 + y_6 \\
\text{Node 4: } y_6 = y_4 + y_5 \\
\text{RREF}(A^T)
\end{array}$$

$$A^T = \begin{bmatrix} -1 & 0 & 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2+R_1} A^T = \begin{bmatrix} -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & -1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{R_4+R_2} A^T = \begin{bmatrix} -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3+R_2} A^T = \begin{bmatrix} -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & -2 & -2 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{R_4/2} A^T = \begin{bmatrix} -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_4-2R_1} A^T = \begin{bmatrix} -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{swapR_3,R_4} A^T = \begin{bmatrix} -1 & 0 & 1 & -1 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{-1*R_1,-1*R_2} A^T = \begin{bmatrix} 1 & 0 & -1 & 0 & -1 & -1 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We can see that the free columns are the 3rd,5th and 6th ones. Thus, the special solutions can be found from the original columns corresponding to these. The basis of the null space is thus formed by combinations of the three in the form of, $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, and \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

2. (from Strang 3.3.1) Find the RREF for the matrix $A$. Then describe the column space and null space for $A$. Finally, describe a complete solution to $A\mathbf{x} = \mathbf{b}$.

$$A = \begin{bmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$$

*Solution:*

$$A = \begin{bmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 2 & 4 & 6 & 4 \\ 0 & 1 & 1 & 2 \\ 2 & 3 & 5 & 2 \end{bmatrix}$$
$$\xrightarrow{R_3 - R_1} \begin{bmatrix} 2 & 4 & 6 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & -1 & -1 & -2 \end{bmatrix} \xrightarrow{R_3 + R_2} \begin{bmatrix} 2 & 4 & 6 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\xrightarrow{R_1 - 4R_2} \begin{bmatrix} 2 & 0 & 2 & -4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From this, we can see that the pivot columns are 1 and two, while the free columns are 2 and 3. From this, we can conclude that the null-space consists of solution in the form of: $x_h = s_1 + s_2,\ where\ s_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}\ and\ s_2 = \begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \end{bmatrix}$. We can also see that the column space is a hyper plane in $\mathbb{R}$ from the pivot columns. Now, to see if a particular solution exists, an augmented matrix was made using the solution and the original matrix to determine whether a certain combination of b's in the last row would evaluate to 0. The combination turned out to be: $-2b_1 + b_2 + b_3 = 0$. Plugging out original values in, we should get: $-2(4) + 3 + 5 = 0 \implies 0 = 0$. Thus, our particular solution should exist. Thus, our solution found using the augmented matrix, $x_p = \begin{bmatrix} 4 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ exists. Now, for he complete solution, I have: $x_h + x_p = \begin{bmatrix} 4 \\ -1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \end{bmatrix}$

3. (Strang 3.3.6) What conditions on $b_1$, $b_2$, $b_3$, $b_4$ make each system solvable? Find $\mathbf{x}$ in that case:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 2 & 5 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 5 & 7 \\ 3 & 9 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

*Solution:* (a)

$$\begin{bmatrix} A & \mid & b \end{bmatrix} = \begin{bmatrix} 1 & 2 & b_1 \\ 2 & 4 & b_2 \\ 2 & 5 & b_3 \\ 3 & 9 & b_4 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 2 & b_1 \\ 2 & 4 & b_2 \\ 0 & 1 & b_3 - b_2 \\ 3 & 9 & b_4 \end{bmatrix}$$

$$\xrightarrow{R_1 - 2R_3} \begin{bmatrix} 1 & 0 & b_1 - 2b_3 + 2b_2 \\ 2 & 4 & b_2 \\ 0 & 1 & b_3 - b_2 \\ 3 & 9 & b_4 \end{bmatrix} \xrightarrow{R_1 - 2R_3} \begin{bmatrix} 1 & 0 & b_1 - 2b_3 + 2b_2 \\ 0 & 4 & -2b_1 + 4b_3 - 3b_2 \\ 0 & 1 & b_3 - b_2 \\ 3 & 9 & b_4 \end{bmatrix}$$

$$\xrightarrow{R_4 - 3R_1} \begin{bmatrix} 1 & 0 & b_1 - 2b_3 + 2b_2 \\ 0 & 4 & -2b_1 + 4b_3 - 3b_2 \\ 0 & 1 & b_3 - b_2 \\ 0 & 9 & b_4 - 3b_1 + 6b_3 - 6b_2 \end{bmatrix} \xrightarrow{R_4 - 9R_3} \begin{bmatrix} 1 & 0 & b_1 - 2b_3 + 2b_2 \\ 0 & 4 & -2b_1 + 4b_3 - 3b_2 \\ 0 & 1 & b_3 - b_2 \\ 0 & 0 & b_4 - 3b_1 - 3b_3 + 3b_2 \end{bmatrix}$$

$$\xrightarrow{R_2 - 4R_3} \begin{bmatrix} 1 & 0 & b_1 - 2b_3 + 2b_2 \\ 0 & 0 & -2b_1 + b_2 \\ 0 & 1 & b_3 - b_2 \\ 0 & 0 & b_4 - 3b_1 - 3b_3 + 3b_2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & b_1 - 2b_3 + 2b_2 \\ 0 & 1 & b_3 - b_2 \\ 0 & 0 & -2b_1 + b_2 \\ 0 & 0 & b_4 - 3b_1 - 3b_3 + 3b_2 \end{bmatrix}$$

$$x = \begin{bmatrix} b_1 - 2b_3 + 2b_2 \\ b_3 - b_2 \end{bmatrix}.$$

$$(b) \begin{bmatrix} A & \mid & b \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 2 & 4 & 6 & b_2 \\ 2 & 5 & 7 & b_3 \\ 3 & 9 & 12 & b_4 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 2 & 5 & 7 & b_3 \\ 3 & 9 & 12 & b_4 \end{bmatrix}$$

$$\xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 1 & 1 & b_3 - 2b_1 \\ 3 & 9 & 12 & b_4 \end{bmatrix} \xrightarrow{R_4 - 3R_1} \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 1 & 1 & b_3 - 2b_1 \\ 0 & 3 & 3 & b_4 - 3b_1 \end{bmatrix} \xrightarrow{R_4 - 3R_3} \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 1 & 1 & b_3 - 2b_1 \\ 0 & 0 & 0 & b_4 + 3b_1 - 3b_3 \end{bmatrix}$$

$$\xrightarrow{R_1 - 2R_1} \begin{bmatrix} 1 & 0 & 1 & 5b_1 - 2b_3 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 1 & 1 & b_3 - 2b_1 \\ 0 & 0 & 0 & b_4 + 3b_1 - 3b_3 \end{bmatrix} \xrightarrow{R_1 - 2R_1} \begin{bmatrix} 1 & 0 & 1 & 5b_1 - 2b_3 \\ 0 & 1 & 1 & b_3 - 2b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_4 + 3b_1 - 3b_3 \end{bmatrix}$$

$$x = \begin{bmatrix} 5b_1 - 2b_3 \\ b_2 - 2b_1 \\ 3b_1 - 2b_3 + b_2 \end{bmatrix}$$

4. (Strang 3.4.1) Show that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are independent but $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are dependent:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

Solve $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 = \mathbf{0}$ or $A\mathbf{x} = \mathbf{0}$. The $\mathbf{v}$'s go in the columns of $A$.

*Solution:*

$$A = \begin{bmatrix} c_1 & c_2 & c_3 & 2c_4 \\ 0 & c_2 & c_3 & 3c_4 \\ 0 & 0 & c_3 & 4c_4 \end{bmatrix} \xrightarrow{R_1 - R_2} = \begin{bmatrix} c_1 & 0 & 0 & -c_4 \\ 0 & c_2 & c_3 & 3c_4 \\ 0 & 0 & c_3 & 4c_4 \end{bmatrix}$$

$$\xrightarrow{R_2 - R_3} = \begin{bmatrix} c_1 & 0 & 0 & -c_4 \\ 0 & c_2 & 0 & -c_4 \\ 0 & 0 & c_3 & 4c_4 \end{bmatrix} \xrightarrow{R_1/c_1} \xrightarrow{R_2/c_2} \xrightarrow{R_3/c_3} = \begin{bmatrix} 1 & 0 & 0 & -c_4/c_1 \\ 0 & 1 & 0 & -c_4.c_2 \\ 0 & 0 & 1 & 4c_4/c_1 \end{bmatrix}$$

From the reduced row echelon form of $\begin{bmatrix} I \mid F \end{bmatrix}$, it can be seen that there are 3 pivot columns ,and one free columns. This indicates that the first three vectors, $v_1, v_2, v_3$ are linearly independent, as the pivot variables correspond to the first three columns in the original matrix. $v4$ however, is not linearly independent and can be formed by multiplying column 1 by $-c_4/c_1$ and column 2 $-c_4/c_2$ and column 3 $-4c_4/c_1$.

5. (Strang 3.5.23) $U$ comes from $A$ by subtracting row 1 from row 3:

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Find bases for the two column spaces. Find bases for the two row spaces. Find bases for the two nullspaces. Which spaces stay fixed in elimination?

*Solution:*

$$\text{c(A)} = span(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix})$$

$$\text{c(U)} = span(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix})$$

note: if you do $R_1 - 2R_2$, you will notice that the middle column is a linear combination of the first and thrid columns.

$$c(A^T) = span(\begin{bmatrix} 1 & 3 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \end{bmatrix})$$

$$c(U^T) = span(\begin{bmatrix} 1 & 3 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \end{bmatrix})$$

$$RREF(A) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}$$

$$N(A) = x_2 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \quad N(U) = x_2 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

In conclusion, the column space changes during elimination, while the row and column space do not.

6. Let $A$ be a matrix and $\mathbf{x}$ and $\mathbf{y}$ be vectors.

   (a) Prove that $A\mathbf{x} = A\mathbf{y}$ if and only if $(\mathbf{x} - \mathbf{y}) \in N(A)$.

   (b) Use Part (a) to explain why if $A\mathbf{x}_p = \mathbf{b}$, then any other solution to the matrix equation $A\mathbf{x} = \mathbf{b}$ has the form $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_n$, where $\mathbf{x}_n$ is in $N(A)$.

*Solution:*

(a) let A be an arbitrary nxn matrix, and v,y be arbitrary vectors of size n.

$$\mathbf{v} = \left[x_1, x_2, ...., x_n\right], \mathbf{y} = \left[y_1, y_2, ...., y_n\right]$$

$$(Ax)_i = A_{i*}\mathbf{x}, (Ay)_i = A_{i*}\mathbf{y}$$

$$\text{Ax} = \text{Ay}$$

$$\implies Ax - Ay = \mathbf{0}$$

$$\implies (A\mathbf{x})_i - (A\mathbf{y})_i = \mathbf{0}$$

$$\implies A_{i*}\mathbf{x} - A_{i*}\mathbf{y} = \mathbf{0}$$

$$\implies A_{i*}(\mathbf{x} - \mathbf{y}) = \mathbf{0}\text{(distributive property of matrices)}$$

$$\implies \mathbf{x} - \mathbf{y} \in N(A)\text{(definition of a vector in Null space)}$$

We can now conclude that Ax = Ay $\implies \mathbf{x} - \mathbf{y} \in N(A)$, as the expression above illustrates the definition of vectors in the null space. It is also a biimplication, because all steps involve simple algebra steps that implicate both ways.

(b) From Part(a), we found that if the difference of two vectors was in the Null space, then they give the same result when transformed by the matrix. From this, we can further say that x and y are also in the null space , as their difference is a linear combination, and the null space consists of all linear combinations of a certain set of vectors. Since their difference was in the Null space, they must also be in the Null space.

Now, assuming that:

$$Ax_p = \mathbf{b}, Ax_n = \mathbf{0}$$

$$A(\mathbf{x}_p + \mathbf{x}_n) = A\mathbf{x}_p + A\mathbf{x}_n = \mathbf{b} + \mathbf{0} = \mathbf{b}$$

Knowing that $x_n$ is in the null space, every other vector in the null space can be found as a linear combination of $x_n$ and some other vector, even their difference. Thus, for every vector $\mathbf{v}$ in the null space, $\mathbf{x}_n - \mathbf{v} \in N(A)$, which means that $A\mathbf{x}_n = A\mathbf{v} = \mathbf{0}$ and finally, $A(\mathbf{x}_p + \mathbf{v}) = A\mathbf{x}_p + A\mathbf{v} = \mathbf{b} + \mathbf{0} = \mathbf{b}$

7. In this problem we'll carefully prove an important property of rank.

   (a) Let $a$ be a scalar. Prove that $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ is a linearly independent set of vectors if and only if the set $\{\mathbf{w}_1 + a\mathbf{w}_2, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ is also linearly independent.

   (b) Let $a \neq 0$ be a scalar. Prove that $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ is a linearly independent set of vectors if and only if the set $\{a\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ is also linearly independent.

   (c) Use Parts (a) and (b) to explain why the number of linearly independent rows in a matrix is not changed by row operations.

   (d) Use Part (c) to explain why the rank of a matrix is equal to the number of linearly independent rows.

*Solution:* (a)

$$\begin{aligned}
Let:\ & c_1, c2, ...c_n, d_1, d_2, d_3...d_n \in \mathbb{R} \\
Assume: & \ c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \ldots + c_n\mathbf{w}_m = 0 \\
& \longleftrightarrow c_1, c_2, c_3.....c_n = 0 \\
W.T.S: & \ d_1\mathbf{w}_1 + a\mathbf{w}_2 + d_2\mathbf{w}_2 + \ldots + c_n\mathbf{w}_m = 0 \\
& \implies d_1\mathbf{w}_1 + (a + d_2)\mathbf{w}_2 + \ldots + c_n\mathbf{w}_m = 0 \\
& \implies d_1, a + d_2, d_3, ....d_m = 0(original\ assumption)
\end{aligned}$$

We can say this implication because we assumed that initially that there was no linear combination of vectors $\{c_1w_1 + c_2w_2, .....c_mw_m = 0\}$ unless all scalars were 0. Such is the case at which we arrived at here. Since the same logic can be applied in the reverse direction, thus showing the biimplication.

(b)

$$\begin{aligned}
Let:\ & c_1, c2, ...c_n, d_1, d_2, d_3...d_n \in \mathbb{R} \\
Assume: & \ c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \ldots + c_n\mathbf{w}_m = 0 \\
& \longleftrightarrow c_1, c_2, c_3.....c_n = 0 \\
W.T.S: & \ d_1\mathbf{w}_1 + ad_2\mathbf{w}_2 + \ldots + c_n\mathbf{w}_m = 0 \\
& \implies d_1\mathbf{w}_1 + ad_2)\mathbf{w}_2 + \ldots + c_n\mathbf{w}_m = 0 \\
& \implies d_1, ad_2, d_3, ....d_m = 0(original\ assumption)
\end{aligned}$$

We can say this implication because we assumed that initially that there was no linear combination of vectors $\{c_1w_1 + c_2w_2, .....c_mw_m = 0\}$ unless all scalars were 0. Such is the case at which we arrived at here. Since the same logic can be applied in the reverse direction, thus showing the biimplication.

(c)

The previous parts have shown that the linear independence of a set of vectors is preserved even through vector addition and scalar multiplication. These actions are are exactly what occur during row operations. Thus, if a set of rows were linearly independent, they would be linearly independent after row operations, as we have shown that they preserve linear independence.

(d)

The previous part has shown that row operations preserve linearly independent rows. The definition of the rank is the number of nonzero rows in the rref of a matrix. By definition, a zero row is not linearly independent with other zero rows. Thus, since only lineally independent rows are preserved through row operations, all other vectors will eventually decompose to 0. Thus, no matter what row operations occur, the linearly independent rows will be preserved, meaning that they will remain as the only non-zero rows, and the rank,which is equal to the number of non-zero rows, will be equivalent to the number of linearly independent rows, as they are the only nonzero rows.

## Optional Problems

8. Prove any linearly independent set of vectors in $\mathbb{R}^n$ can be expanded to a basis for $\mathbb{R}^n$.

9. Prove any set of vectors whose span is $\mathbb{R}^n$ contains a basis for $\mathbb{R}^n$.

10. Prove that the set of solutions to any matrix equation $A\mathbf{x} = \mathbf{b}$ is a set of all affine combinations of some set of vectors.