Name:

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Please read all of the following information.

Do not open the test until permission is given. You have 50 minutes to complete your answers.

There are three problems, and each problem is worth 20 points (total is 60).

Please put all notes, calculators, and electronic devices away.

Grading will be done with Gradescope, so please write answers only on the pages assigned to the appropriate question. If you need to put work on a different page, please write a note on the page where the answer appears.

For full credit, your work must be shown clearly and legibly, with appropriate justifications.

You do not need to simplify numerical expressions involving vectors and matrices for full credit.

You do not have to complete every part of every problem perfectly to do well. Be sure to give yourself time to work on every problem.

Good luck!

1. Let $A = \begin{bmatrix} .4 & .5 \\ .6 & .5 \end{bmatrix}$.

   (a) Find the characteristic polynomial for A.

   (b) Use the characteristic polynomial to verify that 1 and $-.1$ are the eigenvalues of A.

   (c) Verify that $(5, 6)$ and $(1, -1)$ are eigenvectors for A.

   (d) What are eigenvalues and eigenvectors for A^{-1}? (Do not compute A^{-1}.)

   (e) What are eigenvalues and eigenvectors for A^{10}? (Do not compute A^{10}.)

   (f) Suppose the matrix A describes a two state Markov chain for yearly population migration between countries C and D. Describe the population migration using one sentence for each country.

   (g) In the long run, what proportion of people will live in country C?

Solution:

(a) The characteristic polynomial is

$$\det(A - \lambda I) = (.4 - \lambda)(.5 - \lambda) - (.5)(.6) = \lambda^2 - .9\lambda - .1.$$

(b) To show that 1 and $-.1$ are the eigenvalues of A, we check that 1 and $-.1$ are roots of the characteristic polynumial:

$$1^2 - .9(1) - .1 = 1 - .9 - .1 = 0 \text{ and } (-.1)^2 - .9(-.1) - .1 = .01 + .09 - .1 = 0.$$

(c)

$$\begin{bmatrix} .4 & .5 \\ .6 & .5 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} .4(5) + .5(6) \\ .6(5) + .5(6) \end{bmatrix} = \begin{bmatrix} 2 + 3 \\ 3 + 3 \end{bmatrix} = 1 \begin{bmatrix} 5 \\ 6 \end{bmatrix}.$$

and

$$\begin{bmatrix} .4 & .5 \\ .6 & .5 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} .4(1) + .5(-1) \\ .6(1) + .5(-1) \end{bmatrix} = \begin{bmatrix} .4 - .5 \\ .6 - .5 \end{bmatrix} = (-.1) \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

(d) The matrix A^{-1} has eigenvectors $(5, 6)$ and $(1, -1)$ (same as A) with eigenvalues 1 and -10 (reciprocals of eigenvalues of A).

(e) The matrix A^{-1} has eigenvectors $(5, 6)$ and $(1, -1)$ (same as A) with eigenvalues 1 and $(-.1)^{10}$ (eigenvalues of A to the tenth power).

(f) Each year forty percent of people in country C remain in country C while sixty percent move to country D. Fifty percent of people in country D remain in country D, while the other fifty percent move to country C.

(g) Since A is a Markov matrix with a single eigenvalue with $|\lambda| = 1$, the eigenvector $(5, 6)$ with $\lambda = 1$ is an attracting steady state. Thus in the long run we expect $5/11$ of the people to live in country C and $6/11$ of the people to live in country D, regardless of the initial distribution.

2. Suppose a 4×4 matrix A has eigenvalues $\lambda_1 = 1$, $\lambda_2 = -3$, $\lambda_3 = -5$, and $\lambda_4 = 2$, with corresponding eigenvectors $\mathbf{x}_1 = (1, 0, 0, 1)$, $\mathbf{x}_2 = (0, 1, 0, 0)$, $\mathbf{x}_3 = (0, 0, 1, 0)$, and $\mathbf{x}_4 = (-1, 0, 0, 1)$.

   (a) Define what it means for a matrix A to be diagonalizable.

   (b) Explain why $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ are orthogonal. (You do not need to show the computations, just indicate why it's true).

   (c) Find an orthonormal basis of eigenvectors for A, and use this basis to give a diagonalization for A.

   (d) What are the determinants of each matrix in the diagonalization?

   (e) What is $\det A$?

   (f) Is A invertible?

   (g) Does A preserve or reverse orientation?

   (h) Write $\mathbf{v} = (4, -1, 3, 4)$ as a linear combination of $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, and $\mathbf{x}_4$.

   (i) If $\mathbf{v} = (4, -1, 3, 4)$, what is $A^{50}\mathbf{v}$?

Solution:

(a) A matrix A is diagonalizable if it is similar to a diagonal matrix.

(b) The vectors $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ are orthogonal because for all $i \neq j$, $x_i^T x_j = 0$.

(c) An orthonormal basis of eigenvectors for A is

$$\{(1/\sqrt{2}, 0, 0, 1/\sqrt{2}), (0, 1, 0, 0), (0, 0, 1, 0), (-1/\sqrt{2}, 0, 0, 1/\sqrt{2})\}.$$

Thus

$$A = X\Lambda X^{-1} = \begin{bmatrix} 1/\sqrt{2} & 0 & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \end{bmatrix}.$$

Note that $X^{-1} = X^T$ since X is orthogonal (orthonormal columns).

(d) $\det X = \det X^{-1} = (1/\sqrt{2}(1)(1)(1/\sqrt{2}) - (1/\sqrt{2}(1)(1)(-1/\sqrt{2}) = 1$, and $\det \Lambda = (1)(-3)(-5)(2) = 30$.

(e) $\det A = \det(X\Lambda X^{-1}) = \det X \det \Lambda \det X^{-1} = (1)(30)(1) = 30$.
Or: $\det A$ is the product of its eigenvalues, so $\det A = (1)(-3)(-5)(2) = 30$.

(f) Since $\det A \neq 0$, A is invertible.

(g) Since $\det A > 0$, A preserves orientation.

(h) Since $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ are orthogonal,

$$\begin{aligned} \mathbf{v} &= \frac{\mathbf{v}^T\mathbf{x}_1}{\mathbf{x}_1^T\mathbf{x}_1}\mathbf{x}_1 + \frac{\mathbf{v}^T\mathbf{x}_2}{\mathbf{x}_2^T\mathbf{x}_2}\mathbf{x}_2 + \frac{\mathbf{v}^T\mathbf{x}_3}{\mathbf{x}_3^T\mathbf{x}_3}\mathbf{x}_3 + \frac{\mathbf{v}^T\mathbf{x}_4}{\mathbf{x}_4^T\mathbf{x}_4}\mathbf{x}_4 \\ &= \frac{8}{2}\mathbf{x}_1 + \frac{-1}{1}\mathbf{x}_2 + \frac{3}{1}\mathbf{x}_3 + \frac{0}{2}\mathbf{x}_4 \\ &= 4\mathbf{x}_1 - \mathbf{x}_2 + 3\mathbf{x}_3 + 0\mathbf{x}_4 \end{aligned}$$

(i)

$$\begin{aligned}
A^{50}\mathbf{v} &= A^{50}(4\mathbf{x}_1 - \mathbf{x}_2 + 3\mathbf{x}_3) \\
&= 4A^{50}\mathbf{x}_1 - A^{50}\mathbf{x}_2 + 3A^{50}\mathbf{x}_3) \\
&= 4(1^{50})\mathbf{x}_1 - (-3)^{50}\mathbf{x}_2 + 3(-5)^{50}\mathbf{x}_3) \\
&= (4, -3^{50}, 3 \cdot 5^{50}, 4)
\end{aligned}$$

3. Suppose $A = U\Sigma V^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix}$.

(a) How many rows and columns does A have? (Do not compute A!)

(b) Find an orthonormal basis for each of the four fundamental subspaces for A.

(c) What is the dimension of each of the four fundamental subspaces for A?

(d) What is the rank r of A?

(e) Write the rank r format singular value decomposition for A.

(f) Write the rank 1 approximation A_1 for A given by the SVD (If you write it as a product, you do not need to multiply).

(g) Suppose $\mathbf{x} = (1, 1, 1, 1)$. What is $A\mathbf{x}$?

(h) Write diagonalizations for $A^T A$ and AA^T.

(i) Suppose $\mathbf{v}$ is a unit vector. What is the largest possible value of $\|A\mathbf{v}\|^2$?

Solution:

(a) The matrix A has 3 rows and 4 columns.

(b)

Subspace	Basis
$C(A)$	$\{(1,0,0), (0, 1/\sqrt{2}, 1/\sqrt{2})\}$
$N(A^T)$	$\{(0, -1/\sqrt{2}, 1/\sqrt{2})\}$
$C(A^T)$	$\{(1/2, 1/2, 1/2, 1/2), (1/2, 1/2, -1/2, -1/2)\}$
$N(A)$	$\{(1/2, -1/2, 1/2, -1/2), (1/2, -1/2, -1/2, 1/2)\}$

(c) $\dim C(A) = 2$, $\dim N(A^T) = 1$, $\dim C(A^T) = 2$, $\dim N(A) = 2$.

(d) A has rank $r = 2$.

(e) The rank $r = 2$ format singular value decomposition for A is

$$A = U_2 \Sigma_2 V_2^T = \begin{bmatrix} 1 & 0 \\ 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \end{bmatrix}$$

(f) The rank 1 approximation A_1 for A given by the SVD is

$$A = \sigma_1 u_1 v_1^T = 4 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \end{bmatrix}$$

(g) Since $\mathbf{x} = (1, 1, 1, 1) = 2\mathbf{v}_1$, $A\mathbf{x} = 2\sigma_1 \mathbf{u}_1 = (8, 0, 0)$.

(h)

$$\begin{aligned}
A^TA &= (U\Sigma V^T)^T U\Sigma V^T\\
&= V\Sigma^T U^T U\Sigma V^T\\
&= V\Sigma^T\Sigma V^T\\
&= \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2\\ 1/2 & 1/2 & -1/2 & -1/2\\ 1/2 & -1/2 & 1/2 & -1/2\\ 1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 16 & 0 & 0 & 0\\ 0 & 4 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2\\ 1/2 & 1/2 & -1/2 & -1/2\\ 1/2 & -1/2 & 1/2 & -1/2\\ 1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
AA^T &= U\Sigma V^T (U\Sigma V^T)^T\\
&= U\Sigma V^T V\Sigma^T U^T\\
&= U\Sigma\Sigma^T U^T\\
&= \begin{bmatrix} 1 & 0 & 0\\ 0 & 1/\sqrt{2} & -1/\sqrt{2}\\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 16 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0\\ 0 & 1/\sqrt{2} & 1/\sqrt{2}\\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}
\end{aligned}$$

Note that A^TA and AA^T both have positive eigenvalues which are the squares of the singular values, with their remaining eigenvalues 0.

(i) Let $\mathbf{v}$ be a unit vector. Note that $\|A\mathbf{v}\|^2 = (A\mathbf{v})^T(A\mathbf{v}) = \mathbf{v}^T A^T A\mathbf{v}$.
Since $A^TA = Q\Lambda Q^T$, where Q is orthogonal, and $\|Q\mathbf{v}\| = \|\mathbf{v}\|$, letting $\mathbf{y} = Q^T\mathbf{v}$, we seek the unit vector $\mathbf{y}$ that maximizes the value of $\mathbf{y}^T\Lambda\mathbf{y}$.
If $\mathbf{y} = (y_1, y_2, y_3, y_4)$, then

$$\mathbf{y}^T\Lambda\mathbf{y} = \begin{bmatrix} y_1 & y_2 & y_3 & y_4 \end{bmatrix} \begin{bmatrix} 16 & 0 & 0 & 0\\ 0 & 4 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1\\ y_2\\ y_3\\ y_4 \end{bmatrix} = 16y_1^2 + 4y_2^2.$$

Since $y_1^2 + y_2^2 \le 1$, the value of $\mathbf{y}^T\Lambda\mathbf{y}$ is maximized (maximum value 16) when $\mathbf{y} = (1,0,0,0)$, and if $\mathbf{y} = (1,0,0,0)$, then

$$\mathbf{v} = Q^T\mathbf{y} = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2\\ 1/2 & 1/2 & -1/2 & -1/2\\ 1/2 & -1/2 & 1/2 & -1/2\\ 1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1\\ 0\\ 0\\ 0 \end{bmatrix} = \begin{bmatrix} 1/2\\ 1/2\\ 1/2\\ 1/2 \end{bmatrix}.$$

So the maximum value of $\|A\mathbf{v}\|^2$ is obtained when $\mathbf{v} = \mathbf{v}_1$, the first singular vector, and the maximum value is

$$\|A\mathbf{v}_1\|^2 = \|(4,0,0)\|^2 = 16 = \sigma_1^2.$$

Geometrically, this corresponds to the observation that U and V^T are orthogonal transformations (preserve lengths), and the diagonal transformation Σ has greatest stretch $\sigma_1 = 4$. Thus the unit vector which ends up the longest after being transformed by A is $\mathbf{v}_1$, which is mapped to $4\mathbf{u}_1$, length 4.