Trevor Leong **21-241 Fall 2019**
Homework 7 **Due Friday, October 18**

Required Problems

1. (Strang 5.1.8) Prove that every orthogonal matrix ($Q^T Q = I$) has determinant 1 or -1.

   (a) Use the product rule $\det AB = \det A \det B$ and the transpose rule $\det Q = \det Q^T$.

   (b) Use only the product rule. If $|\det Q| > 1$, then $\det Q^n = (\det Q)^n$ blows up. How do you know this can't happen to Q^n?

Solution: (a)

$$\begin{gathered}
\text{Assume:} \\
QQ^T = I \\
\implies det(QQ^T) = det(I) \\
\implies det(QQ^T) = 1(\text{axiom states that det(I) = 1}) \\
\implies det(Q)det(Q^T) = 1(\text{multiplicity of matrices}) \\
\implies det(Q)^2 = 1(\text{determinant of transpose is identical to original}) \\
\implies det(Q) = 1 \vee det(Q) = -1
\end{gathered}$$

(b)

$$\begin{gathered}
\text{Assume for sake of contradiction:} \\
|\det Q| > 1 \\
\text{Begin with:} \\
\det Q^n = (\det Q)^n \\
\implies \det(Q^{n-2}Q^2) = (\det Q)^n \\
\implies \det(Q^{n-2}Q^TQ) = (\det Q)^n \\
\implies \det(Q^{n-2})\det((I)) = (\det Q)^n(\text{property of orthogonal matrix}) \\
\implies \det(Q)^{n-2}(1) = (\det Q)^n \\
1 = \det(Q)^2(\text{assumed det(Q) > 1}) \\
\implies \det(Q) = 1 \vee \det(Q) = -1
\end{gathered}$$

This is a contradiction, because we assumed that $|\det Q| > 1$.
Since we have proven the assumption wrong, it is safe to assume that the negation is true.
We know that this can't happen to Q, because Q^TQ is always I,and the only way that this is true is when $\det(Q) = 1$

2. (Strang 5.1.10) If the entries in every row of A add to zero, solve $A\mathbf{x} = \mathbf{0}$ to prove $\det A = 0$. If those entries add to one, show that $\det(A - I) = 0$. Does this mean $\det A = 1$?

Solution:
(a) If the entries in every row sum to zero, then this implies that a vector of all ones will satisfy $A\mathbf{x} = 0$. This implies that there is a non-zero solution in the Null space. This then implies that the matrix is not invertible, and thus implies that $\det(A) = 0$ (b)

The definition of the identity matrix is that the only nonzero element in every row is 1.
Knowing this, the sum of the ith row of A-I can be expressed as:

$\sum_{j=1}^{n} a_{ij} - 1$

Since it was given that $\sum_{j=1}^{n} a_{ij} = 1$, the expression for the sum of each row is now:
1-1 = 0
So, this means that the sum of each row will be 0. From the previous part, we proved that if the rows sum to zero then the determinant is 0. Using this fact, we have proven that $\det(A - I) = 0$.
(c)
This does not imply that det(A) = 1, as A could be $\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$
which has a determinant of 2, but still satisfies the conditions above.

3. (Strang 5.1.23) From $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$, find A^2 and A^{-1} and $A - \lambda I$ and their determinants. Which two numbers λ lead to $\det(A - \lambda I) = 0$?

Solution:

$$(a) A^2 = A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 18 & 7 \\ 14 & 11 \end{bmatrix}$$

$$\det(\begin{bmatrix} 18 & 7 \\ 14 & 11 \end{bmatrix}) = 18(11) - 7(14) = 198 - 98 = 100$$

$$(b) A^-1 = \frac{1}{\det A} \begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} \frac{3}{10} & -\frac{1}{10} \\ -\frac{2}{10} & \frac{4}{10} \end{bmatrix}$$

$$\det(\begin{bmatrix} \frac{3}{10} & -\frac{1}{10} \\ -\frac{2}{10} & \frac{4}{10} \end{bmatrix}) = \frac{12}{100} - \frac{2}{100} = \frac{10}{100} = \frac{1}{10}$$

$$(c) A - \lambda I = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 4 - \lambda & 1 \\ 2 & 3 - \lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (4 - \lambda)(3 - \lambda) - 2$$
$$= \lambda^2 - 7\lambda + 12 - 2 = \lambda^2 - 7\lambda + 10$$

$$\det(A - \lambda I) = 0 \text{ When } \lambda = 5 \vee \lambda = 2$$

4. Strang (5.2.4) Find two ways to choose nonzeros from four different rows and columns:

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 3 & 4 & 5 \\ 5 & 4 & 0 & 3 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

(B has the same zeros as A). Is $\det A$ equal to $1+1$ or $1-1$ or $-1-1$? What is $\det B$?

Solution:

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 4 & 0 \\ 0 & 4 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix},$$

$\det(A) = 0$ because Rows 1 and 4 are identical, so by a certain axiom, the determinant must be 0.

Alternatively, we can say that the only nonzero cofactors of A will be the two configurations above. Since the first cofactor has an odd number of swaps to get to identity, the cofactor it corresponds to is -1, and since the other one is the identity, the result is 1-1 = 0.

Since adding multiples of rows don't change the determinant, $\det(B) = \det(B)$ with $R_1 - 2R_3$.

B will then become: $\begin{bmatrix} -3 & 0 & 0 & 0 \\ 0 & 3 & 4 & 5 \\ 5 & 4 & 0 & 3 \\ 2 & 0 & 0 & 1 \end{bmatrix}$

$$\implies \det(B) = -3\det(\begin{bmatrix} 3 & 4 & 5 \\ 4 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}) + 0 + 0 + 0$$

= -3(-16) = 48

Alternatively, we could say that by the same logic as previously, the determinant of b is 16 + 32 = 48

5. (From Strang 5.2.12) Find the cofactor matrix C and multiply A times C^T. Compare C^T with A^{-1}:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \quad A^{-1} = \frac{1}{4} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

Solution:
The Cofactor Matrix is the matrix such that each entry a_{ij} is the determinant of matrix formed by the elements not in the ith row or jth column multiplied by $(-1)^{i+j}$.
Thus:

$$\text{C} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

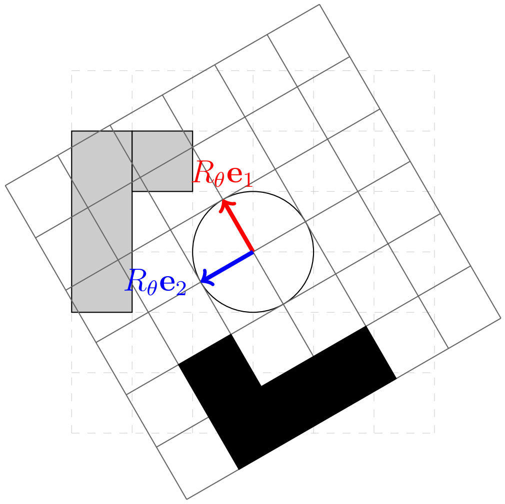
.

$$C^T = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

$$AC^T = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

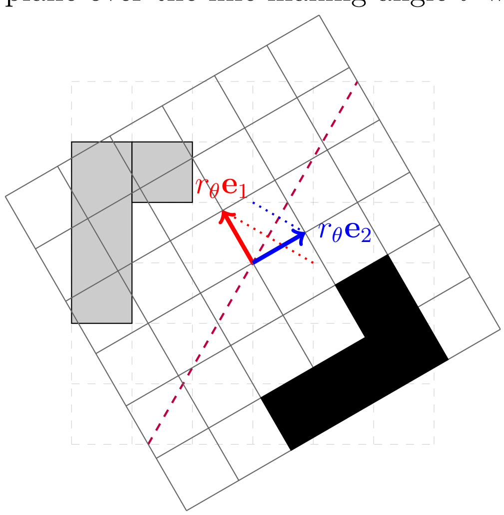
From the results, it appears that $A^{-1} = \frac{1}{\det(A)} C^T$

6. (a) The rotation matrix $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ rotates each point in the the plane counterclockwise around the origin by angle θ.



- What is $\det R_\theta$?
- How much is area scaled by R_θ?
- Does R_θ reverse orientation?
- Find the inverse of R_θ.
- Verify that R_θ is an orthogonal matrix.
- Where is the point $(7, 3)$ mapped by $R_{3\pi/4}$?

(b) The reflection matrix $r_\theta = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$ reflects each point in the plane over the line making angle θ with the x-axis.



- What is $\det r_\theta$?
- How much is area scaled by r_θ?
- Does r_θ reverse orientation?
- Find the inverse of r_θ.
- Verify that r_θ is an orthogonal matrix.

- Where is the point $(7,3)$ mapped by $r_{3\pi/4}$?

Solution: (a)

- $\det(R_\theta) = cos^2\theta + sin^2\theta = 1$
- Since $\det(R_\theta) = 1$, the area is scaled by 1 and therefore unchanged
- No, since the determinant is positive
- $R_\theta^{-1} = \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix}$
- ($cos\theta$,$sin\theta$) $\cdot$(-sinθ,$cos\theta$) $= -sin\theta cos\theta + sin\theta cos\theta$ =0
- $\begin{bmatrix} \cos\frac{3\pi}{4} & -\sin\frac{3\pi}{4} \\ \sin\frac{3\pi}{4} & \cos\frac{3\pi}{4} \end{bmatrix} \begin{bmatrix} 7 \\ 3 \end{bmatrix} = \begin{bmatrix} \cos 135 & -\sin 135 \\ \sin 135 & \cos 135 \end{bmatrix} \begin{bmatrix} 7 \\ 3 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 7 \\ 3 \end{bmatrix} =$
  $\begin{bmatrix} -5\sqrt{2} & 2\sqrt{2} \end{bmatrix}$
  .

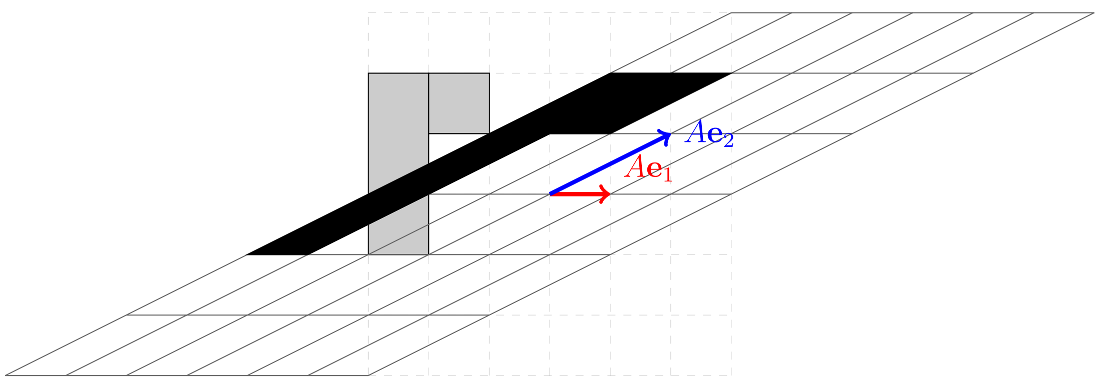
(b)

- $\det(r_\theta) = -\cos^2 2\theta - \sin^2 2\theta = -1$
- The area is scaled by 1
- The orientation is reversed
- $r_\theta^{-1} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$
- $(\cos 2\theta, -\sin 2\theta) \cdot (\sin 2\theta, -\cos 2\theta) = -\cos 2\theta \sin 2\theta + \sin 2\theta \cos 2\theta = 0$
- $\begin{bmatrix} \cos\frac{3\pi}{2} & -\sin\frac{3\pi}{2} \\ \sin\frac{3\pi}{2} & \cos\frac{3\pi}{2} \end{bmatrix} \begin{bmatrix} 7 \\ 3 \end{bmatrix} = \begin{bmatrix} \cos 270 & -\sin 270 \\ \sin 270 & \cos 270 \end{bmatrix} \begin{bmatrix} 7 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ -7 \end{bmatrix}$

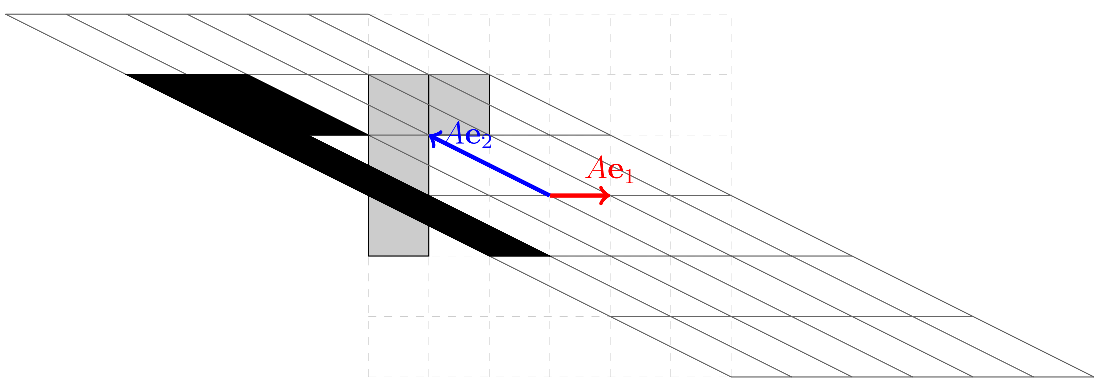
7. Each part below shows the image of the grid where $-3 \leq x \leq 3$ and $-3 \leq y \leq 3$ after being transformed by a 2×2 matrix A. The original grid is shown lightly in the background. The gray area is mapped to the black area.. For each image,

- Find the matrix A.
- Find the determinant of A.
- Find the area of one parallelogram in the transformed grid.
- State whether A preserves or reverses orientation.

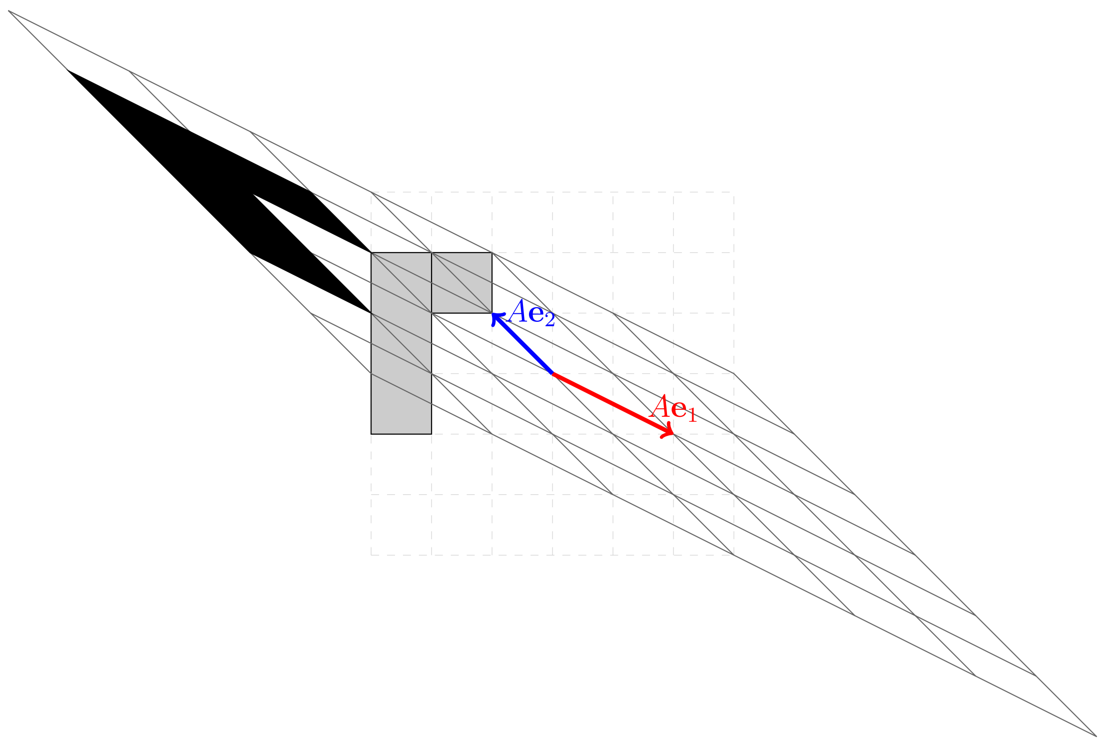
(a)



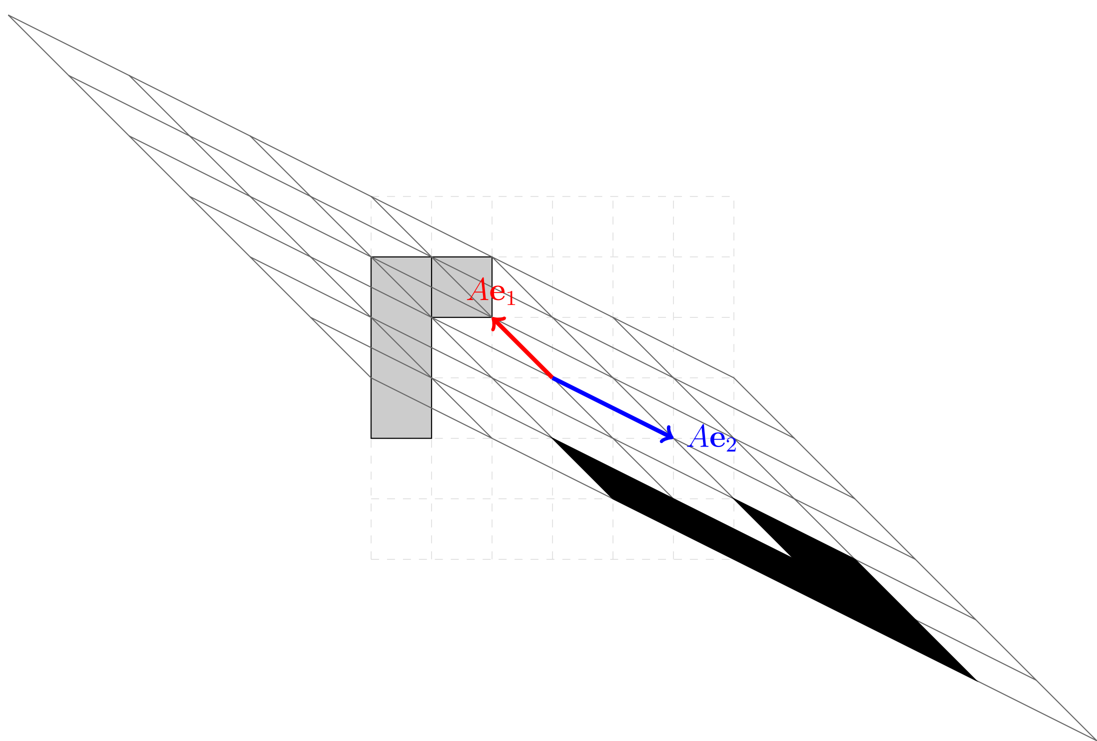
(b)



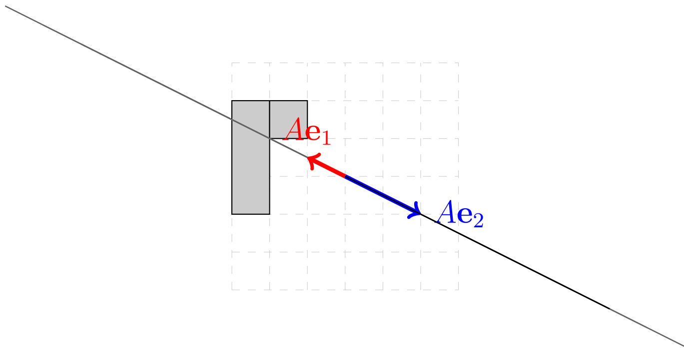
(c)



(d)

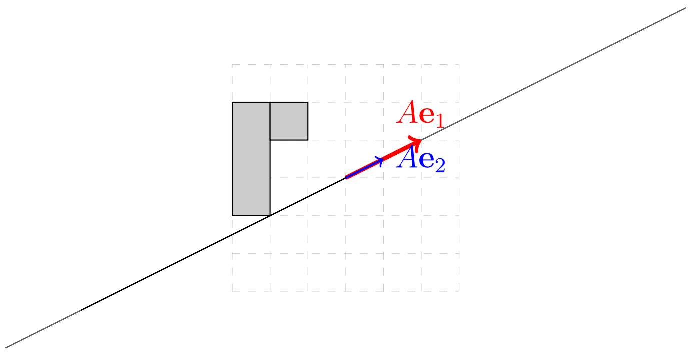


(e)



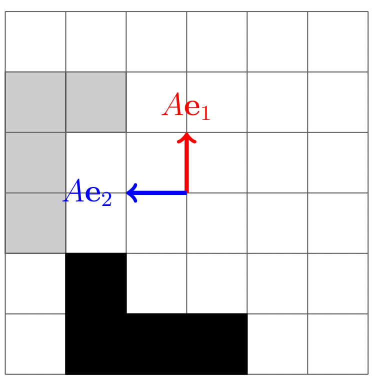
$A\mathbf{e}_1$
$A\mathbf{e}_2$

(f)



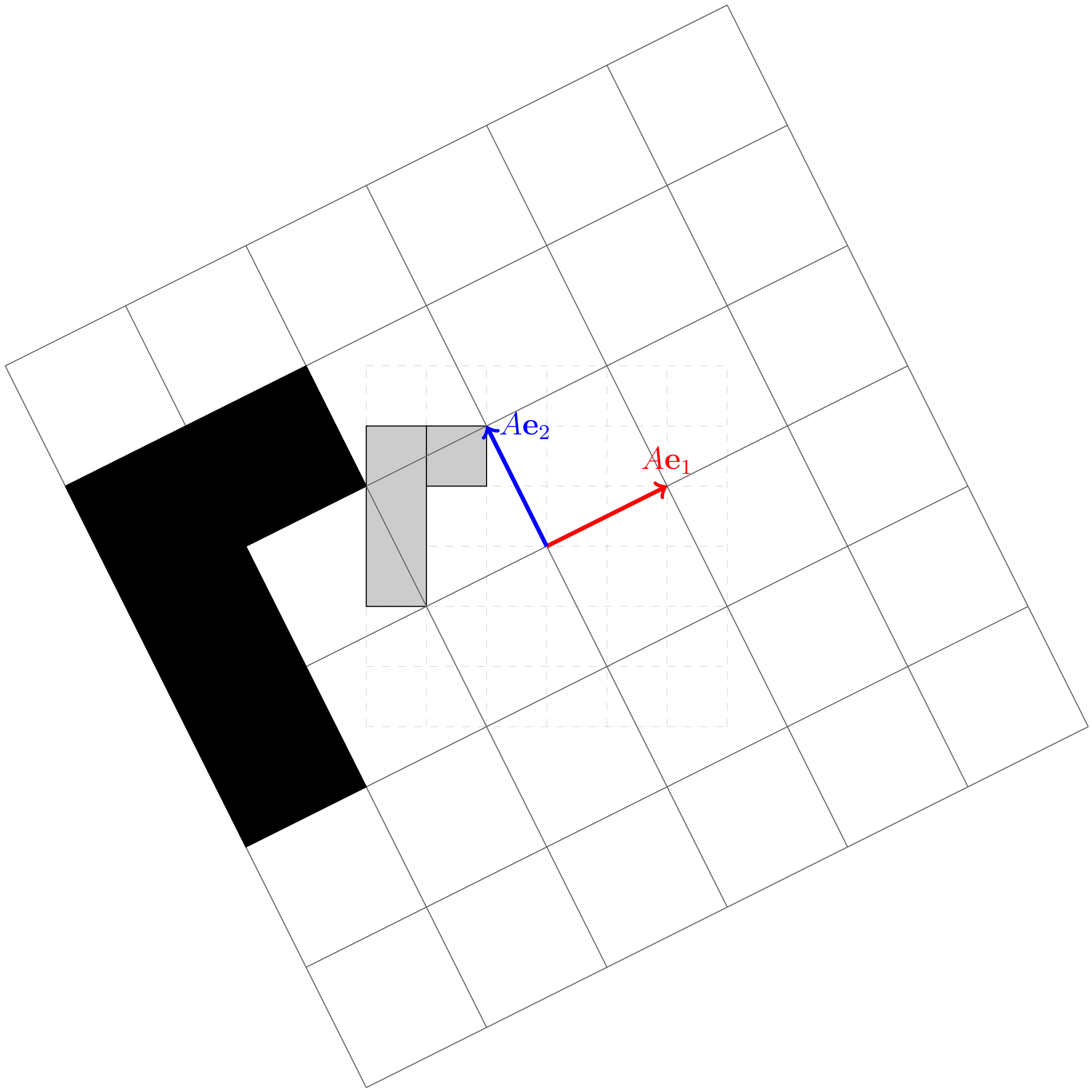
$A\mathbf{e}_1$
$A\mathbf{e}_2$

(g)

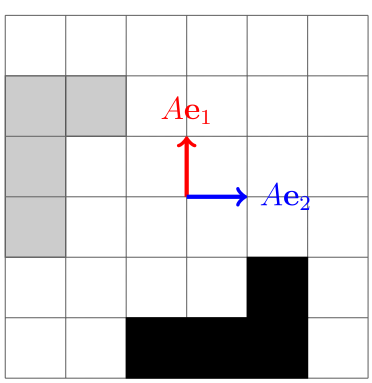


$A\mathbf{e}_1$
$A\mathbf{e}_2$

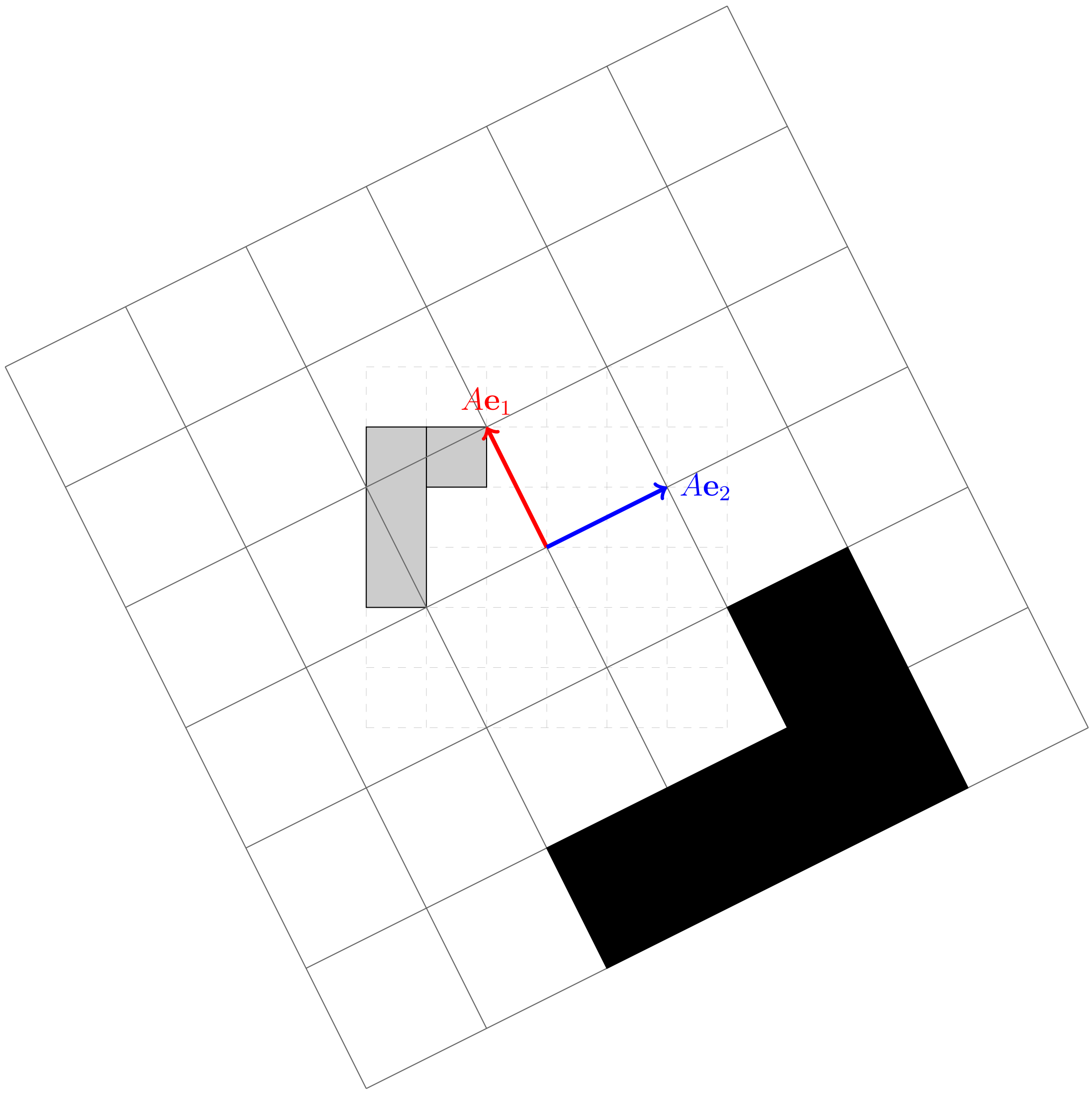
(h)



(i)



(j)



Solution: (a)

Appears that the new vector space is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

- A $= \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$
- $\det(A) = 0 - 1 = -1$
- $area = \mid \det(A) \mid = 1$
- since $\det(A) < 0$, it reverses

(b)

Appears that new vector space is: $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

- A $= \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}$

- $\det(A) = 0 - 1 = -1$
- $area = \mid \det(A) \mid = 1$
- since $\det(A) < 0$, it reverses

(c)

Appears that new vector space is: $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

- A $= \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$
- $\det(A) = 1 - 2 = 1$
- $area = \mid \det(A) \mid = 1$
- since $\det(A) > 0$, it does not reverse

(d)

Appears that new vector space is: $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

- A $= \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$
- $\det(A) = 1 - 2 = -1$
- $area = \mid \det(A) \mid = 1$
- since $\det(A) < 0$, it reverses

(e)

Appears that new vector space is: $\begin{bmatrix} -1 \\ \frac{1}{2} \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

- A $= \begin{bmatrix} -1 & 2 \\ \frac{1}{2} & -1 \end{bmatrix}$
- $\det(A) = 1 - 1 = 0$
- $area = \mid \det(A) \mid = 0$
- since $\det(A) = 0$, it is not invertible , it has no orientation

(f)

Appears that new vector space is: $\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

- A $= \begin{bmatrix} 1 & 2 \\ \frac{1}{2} & 1 \end{bmatrix}$
- $\det(A) = 1 - 1 = 0$
- $area = \mid \det(A) \mid = 0$
- since $\det(A) = 0$, it is not invertible , it has no orientation

(g)

Appears that new vector space is: $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$

- A = $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
- $\det(A) = 1 - 1 = 1$
- $area = \mid \det(A) \mid = 1$
- since $\det(A) > 0$, it remains the same orientation

(h)

Appears that new vector space is: $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

- A = $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$
- $\det(A) = 4 + 1 = 5$
- $area = \mid \det(A) \mid = 5$
- since $\det(A) > 0$, orientation remains the same

(i)

Appears that new vector space is: $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

- A = $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
- $\det(A) = 0 - 1 = -1$
- $area = \mid \det(A) \mid = 1$
- since $\det(A) < 0$, it reverses orientation

(j)

Appears that new vector space is: $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

- A = $\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$
- $\det(A) = -1 - 4 = -5$
- $area = \mid \det(A) \mid = 5$
- since $\det(A) < 0$, it reverses orientation

Optional Problems

8. Prove that any method for transforming a given permutation matrix P to the identity matrix by row swaps will always do so with an even number of swaps or an odd number of swaps (depending on P). In other words, while different methods of transforming P to I may take different numbers of swaps, the parity will be the same.

9. Using the notation for rotation and reflection matrices in Problem 6,

   (a) Verify that $R_\theta R_\phi = R_{\theta+\phi}$.

   (b) What type of transformation is $R_\theta r_\phi$?

   (c) What type of transformation is $r_\theta R_\phi$?

   (d) What type of transformation is $r_\theta r_\phi$?

10. Prove that any orthogonal 2×2 matrix is either a rotation matrix or a reflection matrix.