Required Problems

1. Suppose m, n, and p are distinct positive integers, and

   - A is an $m \times n$ matrix,
   - B is an $n \times p$ matrix,
   - C is an $m \times p$ matrix,
   - x is a vector with n coordinates, and
   - y is a vector with p coordinates.

   For each of the following expressions, describe what kind of object is produced, or if the expression is not well-defined, explain why. You may wish to experiment with some random matrices and vectors in Julia to verify your answers.

   (a) Ax
   (b) xA
   (c) AB
   (d) AC
   (e) BC
   (f) $Ax + Bx$
   (g) $By + Cy$
   (h) $(B + C)y$
   (i) $(AB)y$
   (j) $A(By)$
   (k) $3A$
   (l) $-5y$
   (m) $x + y$
   (n) $x \cdot y$
   (o) AA^T
   (p) $A^T A$

   Solution:
   (a) An m-dimensional vector with each element v_i equal to the dot product of row i of A and vector $\overrightarrow{x}$

   (b) This expression is not well defined, as multiplication is defined for when the number columns of the first object is equivalent to the number of rows in the second object. Since $\overrightarrow{x}$ is of dimension n x 1 and A is of dimension m x n, and m $\neq$ 1 for all but one case, the expression is not well defined.

(c) A matrix with m rows and p columns with each element a_{ij} equal to the dot product of row i of A and column j of B

(d) This expression is not well defined, as multiplication is defined for when the number columns of the first object is equivalent to the number of rows in the second object. Since A is of dimension m x n and C is of dimension m x p, and n and m are distinct, the expression is not well defined.

(e) This expression is not well defined, as multiplication is defined for when the number columns of the first object is equivalent to the number of rows in the second object. Since B is of dimension n x p, C is of dimension m x p, and p m are distinct, the expression is not well defined.

(f) This expression is not well defined, as addition is defined for when both objects have the same dimension. Since Ax is of dimension m x 1, C is of dimension n x 1, and m and n are distinct, the expression is not well defined.

(g) This expression is not well defined, as addition is defined for when both objects have the same dimension. Since Ax is of dimension n x 1, C is of dimension m x 1, and n m are distinct, the expression is not well defined.

(h) This expression is not well defined, as addition is defined for when both objects have the same dimension. Since B is of dimension n x p, C is of dimension m x p, and n and m are distinct, the expression is not well defined.

(i) The resulting object would be an m-dimensional vector where each element v_i is the dot product of row i of the matrix D and $\overrightarrow{y}$ with D being the result of the product of A and B

(j) The resulting object would be an m-dimensional vector where each element v_i is the dot product of row i of the matrix A and $\overrightarrow{w}$ with $\overrightarrow{w}$ being the result of the product of B and $\overrightarrow{y}$

(k) The resulting matrix would be of dimension m x n with each value a_{ij} being the corresponding value in matrix A multiplied by 3

(l) The resulting vector would be of dimension p with each value v_i being the corresponding value y_i in $\rightarrow y$ multiplied by -5

(m) This expression is not well defined, as addition is defined for when both objects have the same dimension. Since $\overrightarrow{x}$ is of dimension n, and $\overrightarrow{y}$ is of dimension p the expression is not well defined.

(n) This expression is not well defined, as dot product is defined for when both objects have the same dimension. Since $\overrightarrow{x}$ is of dimension n, and $\overrightarrow{y}$ is of dimension p the expression is not well defined.

(o) The resulting object would be an m x m where each element v_i is the square of the corresponding element in A

(p) This expression is not well defined, as multiplication is defined for when the number columns of the first object is equivalent to the number of rows in the second object. Since A is of dimension n x p, A^T is of dimension n x m, and m and p are distinct, the expression is not well defined.

2. Prove that scalar multiplication distributes over vector addition. That is, prove that if $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^n$, and a is a real number, that

$$a(\mathbf{v} + \mathbf{w}) = a\mathbf{v} + a\mathbf{w}.$$

Solution:

$$\begin{aligned}
let\ \mathbf{v} = (v_1, v_2......v_n)\ and\ \mathbf{w} = (w_1, w_2......w_n) \\
\mathbf{v} + \mathbf{w} = (v_1 + w_1, v_2 + w_2, ..., v_n + w_n) \\
a(\mathbf{v} + \mathbf{w}) = (a(v_1 + w_1), a(v_2 + w_2), ..., a(v_n + w_n)) \\
Since\ scalar\ addition\ is\ distributive \\
a(\mathbf{v} + \mathbf{w}) = (av_1 + aw_1, av_2 + aw_2, ..., av_n + aw_n) \\
a\mathbf{v} = (av_1, av_2......av_n)\ and\ a\mathbf{w} = (aw_1, aw_2......aw_n) \\
a\mathbf{v} + a\mathbf{w} = (av_1 + aw_1, av_2 + aw_2, ..., av_n + aw_n) \\
Since\ a(\mathbf{v} + \mathbf{w}) = (a(v_1 + w_1), a(v_2 + w_2), ..., a(v_n + w_n)) \\
a(\mathbf{v} + \mathbf{w}) = a\mathbf{v} + a\mathbf{w}
\end{aligned}$$

3. Let A, B, and C be matrices, x and y be vectors, and c and d be scalars. Use Julia to experiment with some random matrices, vectors, and scalars. Which of the following equations seem to always be true? (Assume in each case that the dimensions of the matrices and vectors are such that the expressions are well-defined)

   (a) $(AB)C = A(BC)$
   (b) $AB = BA$
   (c) $A(x + y) = Ax + Ay$
   (d) $A(Bx) = (AB)x$
   (e) $A(cx + dy) = c(Ax) + d(Ay)$
   (f) $c(A + B) = cA + cB$

(g) $c(AB) = (cA)(cB)$

(h) $A(B+C) = AB + AC$

(i) $(A+B)C = AC + BC$

(j) $(A+B) + C = A + (B+C)$

(k) $A + B = B + A$

Solution:
(a) True
(b) False
(c) True
(d) True
(e) True
(f) True
(g) False
(h) True
(i) True
(j) True
(k) True

4. An $n \times n$ *permutation matrix* is a matrix with exactly one 1 in each row and column, and zeros everywhere else. The *standard basis* (standard set of basis vectors) for $\mathbb{R}^n$ is the set of vectors $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$, where $\mathbf{e}_i$ is the vector with n coordinates, all zero except the ith coordinate is equal to 1. For example the standard basis for $\mathbb{R}^4$ is

$$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\} = \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}.$$

(a) Compute the dot product $\mathbf{e}_i \cdot \mathbf{e}_j$

- if $i = j$,
- if $i \neq j$.

(b) Find the product $P_1 P_2$ of the following two permutation matrices.

$$P_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, P_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) How are the rows of a $4 \times n$ matrix A permuted by multiplication on the left by P_1, P_2, and $P_1 P_2$ (i.e. which rows of A correspond to which rows of $P_1 A$, etc.)?

(d) For any $n \times n$ permutation matrix P, explain why $PP^T = I$, where I is the $n \times n$ identity matrix.

Solution:
(a)

$$
\begin{aligned}
if\ i &= j \\
e_i \cdot e_j &= 1 \\
\\
if\ i &\neq j \\
e_i \cdot e_j &= 0
\end{aligned}
$$

(b)

$$
\begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
$$

(c)

$$
\begin{gathered}
\text{For } P_1A \\
\text{the 1st row of A corresponds to the 3rd row of } P_1A \\
\text{the 2nd row of A corresponds to the 1st row of } P_1A \\
\text{the 3rd row of A corresponds to the 4th row of } P_1A \\
\text{the 4rd row of A corresponds to the 2nd row of } P_1A \\
\text{For } P_2A \\
\text{the 1st row of A corresponds to the 3rd row of } P_2A \\
\text{the 2nd row of A corresponds to the 1st row of } P_2A \\
\text{the 3rd row of A corresponds to the 2nd row of } P_2A \\
\text{the 4rd row of A corresponds to the 4th row of } P_2A \\
\text{For } P_1P_2A \\
\text{the 1st row of A corresponds to the 4th row of } P_1P_2A \\
\text{the 2nd row of A corresponds to the 3rd row of } P_1P_2A \\
\text{the 3rd row of A corresponds to the 1st row of } P_1P_2A \\
\text{the 4rd row of A corresponds to the 2nd row of } P_1P_2A
\end{gathered}
$$

(d) The transpose of matrix P^T is defined such that each element $a^*_{i\,j}$ corresponds to a_{ij}. Thus, each element e_{ij} in PP^T can be expressed as

$$
\sum_{k=1}^{n} (a_{kj})(a^*_{i\,k})
$$

We can simply this down to

$$
\sum_{k=1}^{n} (a_{kj})(a_{ki})
$$

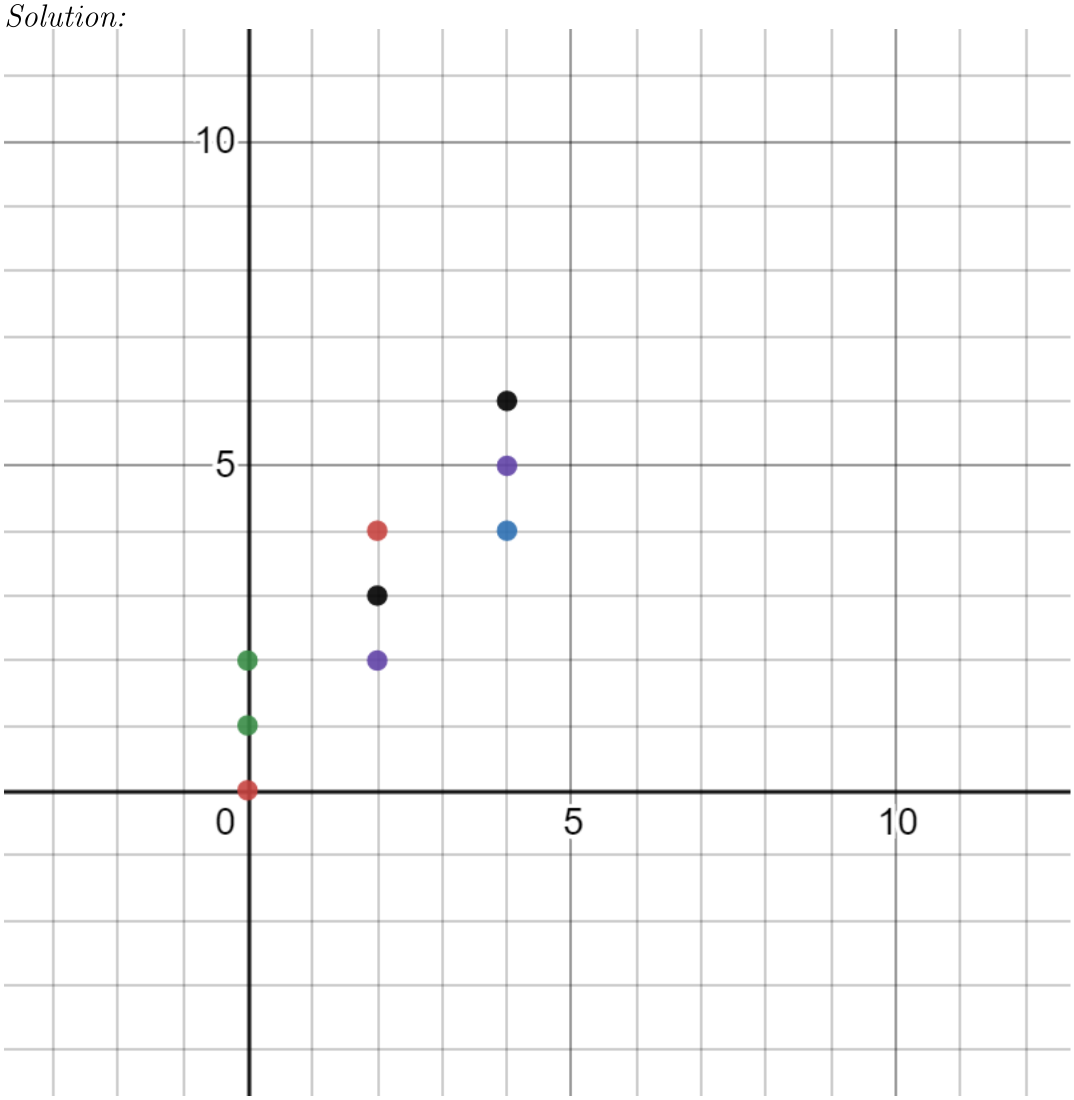
knowing that a^*_{ij} corresponds to a_{ji}.

From this expression, we can see that the element will only be a 1 if both parts are 1, as there are only 1's and 0's in a permutation matrix. Further, since both parts of the product are from the same row and matrix, it means that i must be equal to j, as there is only one 1 per row. Since the only case where i = j is the diagonal, only elements on the diagonal will be 1. Thus, we end up with a diagonal matrix with 1's, which is the identity matrix.

5. (Strang 1.1.7) In the xy plane mark all nine of these linear combinations:

$$c\begin{bmatrix}2\\2\end{bmatrix}+d\begin{bmatrix}0\\1\end{bmatrix} \quad \text{with } c=0,1,2 \text{ and } d=0,1,2.$$

Solution:



6. (Strang 1.1.10, see Page 9 of the textbook for a picture) Which point of the cube is $\mathbf{i}+\mathbf{j}$? Which point is the vector sum of $\mathbf{i} = (1, 0, 0)$ and $\mathbf{j} = (0, 1, 0)$ and $\mathbf{k} = (0, 0, 1)$? Describe all points (x, y, z) in the cube.

   Solution:
   (a) $i + j$ is the bottom right point of the bottom face of the cube
   (b) $i + j + k$ is the bottom right point of the top face of the cube
   (c) All points on the cube are the linear combinations of i, j, k represented by $ai + bj + ck$ where a =0,1 $\underline{=}$ 0,1 and c = 0,1

7. (Strang 1.1.13a,b, see Page 9 of the textbook for a picture)

   (a) What is the sum $\mathbf{V}$ of the twelve vectors that go from the center of a clock to the hours 1:00, 2:00, ... , 12:00?

   (b) If the 2:00 vector is removed, why do the 11 remaining vectors add to 8:00?

   Solution:
   (a) The sum would be $\overrightarrow{0}\, in\ \mathbb{R}^2$
   (b) When all the vectors that go from the center of of a clock to each hour are added up, we get the zero vector, as each vector has a vector going in the opposite direction with equal magnitude and it sums with its counterpart to add to the zero vector. When one vector is taken away, in this case, the vector pointing to 2, all other vectors cancel out, except for the counterpart of two, which is the vector pointing to 8. Thus, that vector becomes V. Since the 8 vector was the counterpart of two, the sum becomes the 8 vector.

Optional Problems

8. Verify that the field $\mathbb{F}_2$ satisfies all nine field axioms. Show that set of numbers $\{0, 1, 2, 3\}$ with addition and multiplication $\mod 4$ is not a field. For which positive integers n is the set of numbers $\{0, 1, 2, \dots, n\}$ with addition and multiplication $\mod n$ a field?

9. How many $n \times n$ permutation matrices are there?

the field of 0,1,2,3 mod 4 is not a field because there is no multiplicative inverse for 2 or 3.

n!

For a permutation matrix, each row and column must have only one 1. Thus,when we place a 1 at the first i and first j, we are unable to place a 1 in that row or column, restricting us to another row and column.Thus,there are 4 possibilities, as there are four possible rows we can place the first 1 in. After this first 1, there are n-1 possibilities,as the next column is only restricted by the first one. The next one is n-2. As this continues, you can see that you will end up with n*(n-1)*(n-2)*(n-3)...*1, which is n!.

10. Write code in Julia to multiply two matrices.