21-241 Matrices and Linear Transformations **Fall 2019**
Review of Topics **Friday, November 8**

1 Definitions

You should be able to define each of the following terms

- Error correcting code
- message
- (valid) codeword
- Hamming distance
- redundancy (of a code)
- directed graph
- incidence matrix for a directed graph
- spanning tree
- cycle basis
- determinant
- (d-dimensional) unit cube
- parallelepiped
- signed volume
- orientation (chirality)
- cofactor
- rotation matrix
- reflection matrix
- eigenvector
- eigenvalue
- characteristic polynomial
- algebraic multiplicity (of an eigenvalue)
- geometric multiplicity (of an eigenvalue)
- similar matrices

- diagonalizable matrix
- Markov matrix
- Markov chain
- transition probabilities (in a Markov chain)
- steady state
- attracting steady state
- complex number
- complex conjugate
- magnitude (of a complex number)
- positive definite matrix
- positive semidefinite matrix
- Singular value decomposition (rank r and full format)
- singular values
- singular vectors
- Frobenius norm of a matrix

2 Computations

You should be able to

- Construct the parity check matrix and generator matrix for a Hamming code.
- Encode messages and decode codewords (with error correction) using a Hamming code.
- Find a spanning tree for a graph using a basis for the row space of its incidence matrix.
- Find a cycle basis for a graph using a basis for the left null space of its incidence matrix.
- Interpret determinant of A as ratio of signed volume of a region before and after transformation by A.
- Compute the determinant of a square matrix using LU decomposition.
- Compute the determinant of an $n \times n$ matrix using $n!$ terms, one for each $n \times n$ permutation matrix.

- Compute the determinant of a square matrix using cofactor formulas.
- Find the inverse of a matrix using cofactors.
- Find the matrix for a linear tranformation based on the image of the standard basis vectors.
- Find the matrix for rotation or reflection transformations of the plane.
- Find the characteristic polynomial $\det(A - \lambda I)$ of a square matrix.
- Use the characteristic polynomial to find eigenvalues and eigenvectors of a matrix.
- Use a basis of eigenvectors to diagonalize a matrix.
- Compute large powers of a diagonalizable matrix.
- Use powers of a matrix and eigenvectors to analyze dynamical systems and recurrences.
- Formulate dynamical systems like population, weather, and random walks as Markov chains, with corresponding Markov matrices.
- Use Markov matrices to analyze long term behavior of a Markov chain.
- Find a "square root" for a positive definite (or semidefinite) matrix.
- Find singular values, singular vectors, and singular value decomposition of any matrix.
- Use SVD to interpret a linear transformation as a composition of rotations or reflections with dilation (stretch).
- Use SVD of A to write A as a sum of rank 1 matrices.

3 Theorems and proofs

You should be able to prove or explain why

- A Hamming code can correct one error but not two.
- Valid codewords for a Hamming code are at Hamming distance 3 or more from each other.
- Hamming codes can be constructed with redundancy arbitrarily close to 1.
- A set of rows forming a basis for the row space of an incidence matrix of a graph corresponds to a spanning tree.
- A basis for the left nullspace of an incidence matrix of a graph corresponds to a cycle basis for the graph.

- Adding a scalar multiple of one row to another leaves the determinant unchanged.
- The determinant of a triangular matrix is the product of its diagonal entries.
- The determinant of an invertible matrix is the product of its pivots.
- The determinant function is multiplicative ($\det(AB) = \det(A)\det(B)$).
- The determinant of a permutation matrix is $(-1)^x$, where x is the number of row swaps required to obtain the identity matrix.
- A is invertible if and only if $\det A \neq 0$.
- If A is invertible, $\det(A^{-1}) = 1/\det A$.
- The determinant of a matrix and its transpose are the same.
- Every statement about determinants and rows has a corresponding statement about columns.
- If Q is an orthogonal matrix, then $\det Q = 1$ or -1.
- Rotation and reflection matrices are orthogonal.
- If A is invertible and C is its matrix of cofactors, then $A^{-1} = \frac{1}{\det A} C^T$.
- The standard basis vectors are eigenvectors of a diagonal matrix.
- The eigenvalues of a matrix are the roots of its characteristic polynomial.
- Similar matrices have the same eigenvalues.
- If $A = BCB^{-1}$ and C has the eigenvector $\mathbf{x}$, then A has the eigenvector $B\mathbf{x}$.
- If an $n \times n$ matrix A has n independent eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$, then $A = X\Lambda X^{-1}$, where Λ is the diagonal matrix of eigenvalues.
- If A is invertible, then the eigenvalues of A^{-1} are the reciprocals of the eigenvalues of A.
- If A is invertible, then A and A^{-1} have the same eigenvectors.
- If two eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$ have the same eigenvalue λ, then any linear combination of $\mathbf{x}_1$ and $\mathbf{x}_2$ is also an eigenvector with eigenvalue λ.
- The determinant of a matrix is the product of its eigenvalues.
- If A and B have the same eigenvalues and same corresponding eigenvectors, then $A = B$.
- If $\mathbf{x}$ is an eigenvector of A with eigenvalue λ, then for all $c \neq 0$, $c\mathbf{x}$ is an eigenvector of A with eigenvalue λ.

- If $\mathbf{x}$ is an eigenvector of A with eigenvalue λ, then for $k \geq 1$, $\mathbf{x}$ is an eigenvector of A^k with eigenvalue λ^k.
- If $A = X\Lambda X^{-1}$ then A and Λ are the same transformation with respect to different bases. X is the "change of basis" matrix.
- Every Markov matrix has eigenvalue $\lambda = 1$.
- If λ is an eigenvalue of a Markov matrix, then $|\lambda| \leq 1$.
- A and A^T have the same eigenvalues.
- If A and B are Markov matrices, then so is AB. Any power of a Markov matrix is Markov.
- If a diagonalizable $n \times n$ Markov matrix has eigenvector $\mathbf{x}_1$ with $\lambda_1 = 1$, and $|\lambda_i| < 1$ for all $i \geq 2$, then $\mathbf{x}_i$ is an attracting steady state vector.
- For all matrices (not just Markov), the largest eigenvector determines behavior of A^k when k is large.
- Spectral theorem: Every real symmetric matrix is orthogonally diagonalizable with real eigenvalues. $S = Q\Lambda Q^{-1} = Q\Lambda Q^T$.
- The conjugate of a sum is the sum of the conjugates.
- The conjugate of a product is the product of the conjugates.
- A complex number is real if and only if it is equal to its conjugate.
- A symmetric matrix S has all eigenvalues > 0 (≥ 0) if and only if for all $\mathbf{x}$, $\mathbf{x}^T S\mathbf{x} > 0$ (≥ 0).
- If S is a symmetric matrix, then $S = A^T A$ for a matrix A with independent columns (some dependent columns) if and only if for all $\mathbf{x}$, $\mathbf{x}^T S\mathbf{x} > 0$ (≥ 0).
- The singular vectors $\mathbf{v}_1, \ldots \mathbf{v}_r$ for A are eigenvectors of $A^T A$.
- The singular vectors $\mathbf{u}_1, \ldots \mathbf{u}_r$ for A are eigenvectors of AA^T.
- The singular values $\sigma_1, \ldots, \sigma_r$ for A are the square roots of the positive eigenvalues of $A^T A$ and AA^T.
- The singular vectors $\mathbf{v}_1, \ldots \mathbf{v}_r$ for A are an orthonormal basis for $C(A^T)$.
- The singular vectors $\mathbf{u}_1, \ldots \mathbf{u}_r$ for A are an orthonormal basis for $C(A)$.
- For a matrix A with singular value decomposition $A = U\Sigma V^T$, $A\mathbf{v}_i = \sigma_i \mathbf{u}_i$.
- For a matrix $A = U\Sigma V^T$, $A = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \cdots + \sigma_1 \mathbf{u}_r \mathbf{v}_r^T$.

4 Equivalent statements about square matrices

Suppose A is an $n \times n$ matrix. Then each column contains equivalent statements.

A ia nonsingular	A is singular
A is invertible	A is not invertible
The columns are independent	The columns are dependent
The rows are independent	The rows are dependent
$A\mathbf{x} = \mathbf{0}$ has one solution, $\mathbf{x} = \mathbf{0}$	$A\mathbf{x} = \mathbf{0}$ has many solutions
$A\mathbf{x} = \mathbf{b}$ has one solution $\mathbf{x} = A^{-1}\mathbf{b}$	$A\mathbf{x} = \mathbf{b}$ has no solution or infinitely many
A has n (nonzero) pivots	A has $r < n$ pivots
A has full rank $r = n$	A has rank $r < n$
The reduced row echelon form is $R = I$	R has at least one zero row
The column space is all of $\mathbb{R}^n$	The column space has dimension $r < n$
The row space is all of $\mathbb{R}^n$	The row space has dimension $r < n$
The determinant is not zero	The determinant is zero
All eigenvalues are nonzero	Zero is an eigenvalue of A
$A^T A$ is symmetric positive definite	$A^T A$ is only semidefinite
A has n (positive) singular values	A has $r < n$ singular values