Required Problems

1. Let x and y be complex numbers, and let $\overline{x}$ denote the complex conjugate of x.
   (a) Prove that $(\overline{x})(\overline{y}) = \overline{xy}$.
   (b) Prove that $\overline{x} + \overline{y} = \overline{x + y}$.

   Solution:

   (a) **Proof:** Let $x = a + bi$ and $y = c + di$, where $a, b, c, d \in \mathbb{R}$. Then

   $$\overline{xy} = \overline{(a + bi)(c + di)} = \overline{ac + adi + bci + bdi^2} = \overline{(ac - bd) + (ad + bc)i} = (ac-bd)-(ad+bc)i.$$

   Since $\overline{x} = a - bi$ and $\overline{y} = c - di$,

   $$(\overline{x})(\overline{y}) = (a - bi)(c - di) = ac - adi - bci + bdi^2 = (ac - bd) - (ad + bc)i.$$

   Thus $(\overline{x})(\overline{y}) = \overline{xy}$. ∎

   (b) **Proof:** Let $x = a + bi$ and $y = c + di$, where $a, b, c, d \in \mathbb{R}$. Then

   $$\overline{x + y} = \overline{(a + bi) + (c + di)} = \overline{(a + c) + (b + d)i} = (a + c) - (b + d)i.$$

   Since $\overline{x} = a - bi$ and $\overline{y} = c - di$,

   $$\overline{x} + \overline{y} = (a - bi) + (c - di) = (a + c) - (b + d)i.$$

   Thus $\overline{x} + \overline{y} = \overline{x + y}$. ∎

2. (Strang 6.4.1) Which of these matrices ASB will be symmetric with eigenvalues 1 and -1?

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$B = A^T$ doesn't do it. $B = A^{-1}$ doesn't do it. $B = \quad = \quad$ will succeed. So B must be an matrix.

Solution: The first ASB has $B = A^T$, and is symmetric, but does not have eigenvalues 1 and -1 (Its eigenvalues are λ_1 and λ_2 from HW 9 Q6: This is the matrix that encodes the Fibonacci recurrence).

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

The second ASB has $B = A^{-1}$, and has eigenvalues 1 and -1, but is not symmetric.

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$$

The third ASB has $B = A^{-1} = A^T$, has eigenvalues 1 and -1, and is symmetric.

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$B = A^T$ doesn't do it. $B = A^{-1}$ doesn't do it. $B = A^T = A^{-1}$ will succeed. So B must be an **orthogonal** matrix.

3. (Strang 6.4.13) Write S and B in the form $\lambda_1 \mathbf{x}_1 \mathbf{x}_1^T + \lambda_2 \mathbf{x}_2 \mathbf{x}_2^T$ of the Spectral theorem $Q \Lambda Q^T$:

$$S = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} \quad \text{(keep } \|\mathbf{x}_1\| = \|\mathbf{x}_2\| = 1\text{)}.$$

Solution: We use eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 4$ and eigenvectors $\mathbf{x}_1 = \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$ to diagonalize S:

$$S = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

Thus

$$S = \lambda_1 \mathbf{x}_1 \mathbf{x}_1^T + \lambda_2 \mathbf{x}_2 \mathbf{x}_2^T = 2 \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} + 4 \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}.$$

We use eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 25$ and eigenvectors $\mathbf{x}_1 = \begin{bmatrix} -.8 \\ .6 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} .6 \\ .8 \end{bmatrix}$ to diagonalize B:

$$B = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} = \begin{bmatrix} -.8 & .6 \\ .6 & .8 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 25 \end{bmatrix} \begin{bmatrix} -.8 & .6 \\ .6 & .8 \end{bmatrix}$$

Thus

$$B = \lambda_1 \mathbf{x}_1 \mathbf{x}_1^T + \lambda_2 \mathbf{x}_2 \mathbf{x}_2^T = 0 \begin{bmatrix} -.8 \\ .6 \end{bmatrix} \begin{bmatrix} -.8 & .6 \end{bmatrix} + 25 \begin{bmatrix} .6 \\ .8 \end{bmatrix} \begin{bmatrix} .6 & .8 \end{bmatrix}.$$

Note that B has rank 1, so one of the terms is 0.

4. (Strang 6.5.7) Test to see if $A^T A$ is positive definite in each case: A needs independent columns.

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \text{ and } A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix} \text{ and } A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

Solution: If $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$, then $A^T A = \begin{bmatrix} 1 & 2 \\ 2 & 13 \end{bmatrix}$.

As expected, since A has independent columns, A^TA is positive definite. For example, its upper left determinants are 1 and 9, both positive.

If $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix}$, then $A^TA = \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix}$.

As expected, since A has independent columns, A^TA is positive definite. For example, its upper left determinants are 6 and 11, both positive.

If $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$, then $A^TA = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 5 & 6 \\ 4 & 6 & 8 \end{bmatrix}$.

As expected, since A has dependent columns, A^TA is not positive definite–it is only positive semidefinite. For example, it has determinant 0.

5. (Strang 6.5.22) From $S = Q\Lambda Q^T$ compute the positive definite symmetric square root $Q\sqrt{\Lambda}Q^T$ of each matrix. Check that this square root gives $A^TA = S$:

$$S = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}.$$

Solution: **First matrix:**

We use eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 9$ and eigenvectors $\mathbf{x}_1 = \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$ to diagonalize $S = Q\Lambda Q^T$:

$$S = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}.$$

Since $\begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$, we find the positive definite symmetric square root $Q\sqrt{\Lambda}Q^T$ of S.

$$Q\sqrt{\Lambda}Q^T = \begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Since

$$S = \begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}.$$

then if $A = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 \\ 3\sqrt{2}/2 & 3\sqrt{2}/2 \end{bmatrix}$, then $S = A^TA$.

Another acceptable answer is $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Since A is symmetric and $A^2 = S$, $A^TA = S$.

Second matrix: We use eigenvalues $\lambda_1 = 4$ and $\lambda_2 = 16$ and eigenvectors $\mathbf{x}_1 = \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$ to diagonalize $S = Q\Lambda Q^T$:

$$S = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix} = \begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 16 \end{bmatrix} \begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}.$$

Since $\begin{bmatrix} 4 & 0 \\ 0 & 16 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$, we find the positive definite symmetric square root $Q\sqrt{\Lambda}Q^T$ of S.

$$Q\sqrt{\Lambda}Q^T = \begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

Since

$$S = \begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}.$$

then if $A = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} -\sqrt{2} & \sqrt{2} \\ 2\sqrt{2} & 2\sqrt{2} \end{bmatrix}$, then $S = A^T A$.

Another acceptable answer is $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$. Since A is symmetric and $A^2 = S$, $A^T A = S$.

6. (Strang 7.2.1) Find the eigenvalues of these matrices. Then find singular values from $A^T A$:

$$A = \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}.$$

For each A, construct V from the eigenvectors of $A^T A$ and U from the eigenvectors of AA^T. Check that $A = U\Sigma V^T$.

Solution:

First matrix: Note that in this example, even though A is not diagonalizable, it has a singular value decomposition.

Both eigenvalues of $A = \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix}$ are 0 (Note the characteristic polynomial is $\det(A - \lambda I) = \lambda^2$). But there is only one independent eigenvector, $(1, 0)$. The eigenvalue 0 has algebraic multiplicity 2 and geometric multiplicity 1, and A is not diagonalizable.

Since

$$A^T A = \begin{bmatrix} 0 & 0 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 16 \end{bmatrix},$$

A^TA has eigenvalues $\lambda_1 = 16$ and $\lambda_2 = 0$, with eigenvectors $\mathbf{v}_1 = (0,1)$ and $\mathbf{v}_2 = (1,0)$. (Note that since A does not have independent columns, A^TA is positive semidefinite, but not positive definite.)

Similarly,

$$AA^T = \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 16 & 0 \\ 0 & 0 \end{bmatrix},$$

so AA^T has eigenvalues $\lambda_1 = 16$ and $\lambda_2 = 0$ (same as A^TA), with eigenvectors $\mathbf{u}_1 = (1,0)$ and $\mathbf{u}_2 = (0,1)$.

Thus letting $\sigma_1 = \sqrt{\lambda_1} = \sqrt{16} = 4$ and $\sigma_2 = \sqrt{\lambda_2} = \sqrt{0} = 0$, we obtain the singular value decomposition of A:

$$A = U\Sigma V^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Second matrix: In this example, A has a diagonalization and an SVD, but since A is not symmetric, they are different.

The eigenvalues of $A = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}$ are 2 and -2 (Note the characteristic polynomial is $\det(A - \lambda I) = \lambda^2 - 4$), with eigenvectors $\frac{1}{\sqrt{5}}(2,1)$ and $\frac{1}{\sqrt{5}}(-2,1)$. (Since A has a basis of eigenvectors, it can be diagonalized as $A = X\Lambda X^{-1}$:

$$\begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} & -2/\sqrt{5} \\ 1/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} \sqrt{5}/4 & -\sqrt{5}/2 \\ \sqrt{5}/4 & \sqrt{5}/2 \end{bmatrix}.)$$

Since

$$A^TA = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 16 \end{bmatrix},$$

A^TA has eigenvalues $\lambda_1 = 16$ and $\lambda_2 = 1$, with eigenvectors $\mathbf{v}_1 = (0,1)$ and $\mathbf{v}_2 = (1,0)$. (Note that since A has independent columns, A^TA is positive definite.)

Similarly,

$$AA^T = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 16 & 0 \\ 0 & 1 \end{bmatrix},$$

so AA^T has eigenvalues $\lambda_1 = 16$ and $\lambda_2 = 1$ (same as A^TA), with eigenvectors $\mathbf{u}_1 = (1,0)$ and $\mathbf{u}_2 = (0,1)$.

Thus letting $\sigma_1 = \sqrt{\lambda_1} = \sqrt{16} = 4$ and $\sigma_2 = \sqrt{\lambda_2} = \sqrt{1} = 1$, we obtain the singular value decomposition of A:

$$A = U\Sigma V^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Note that the SVD $A = U\Sigma V^T$ is different from the diagonalization $A = X\Lambda X^{-1}$.

7. (Strang 7.2.4) Compute A^TA and AA^T and their eigenvalues and unit eigenvectors for V and U.

$$\textbf{Rectangular matrix} \quad A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Check $AV = U\Sigma$ (this decides $\pm$ signs in U). Σ has the same shape as A: 2×3.

Solution:

Since

$$A^TA = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

A^TA has eigenvalues $\lambda_1 = 3$, $\lambda_2 = 1$, and $\lambda_3 = 0$ with eigenvectors $\mathbf{v}_1 = \frac{1}{\sqrt{6}}(1, 2, 1)$, $\mathbf{v}_2 = \frac{1}{\sqrt{2}}(-1, 0, 1)$, and $\mathbf{v}_3 = \frac{1}{\sqrt{3}}(1, -1, 1)$. (Note that since A has dependent columns, A^TA is positive semidefinite, but not positive definite.)

Similarly,

$$AA^T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},$$

so AA^T has eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 1$ (same as A^TA), with eigenvectors $\mathbf{u}_1 = \frac{1}{\sqrt{2}}(1, 1)$ and $\mathbf{u}_2 = \frac{1}{\sqrt{2}}(-1, 1)$.

Thus letting $\sigma_1 = \sqrt{\lambda_1} = \sqrt{3}$ and $\sigma_2 = \sqrt{\lambda_2} = \sqrt{1} = 1$, we obtain the singular value decomposition $AV = U\Sigma$:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

8. (Strang 7.2.5)

   (a) The row space of $A = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$ is 1-dimensional. Find $\mathbf{v}_1$ in the row space and $\mathbf{u}_1$ in the column space. What is σ_1? Why is there no σ_2?

   (b) Choose $\mathbf{v}_2$ and $\mathbf{u}_2$ in V and U. Then $A = U\Sigma V^T = \mathbf{u}_1\sigma_1\mathbf{v}_1^T$ (one term only).

Solution:

(a) If $\mathbf{v}_1 = \frac{1}{\sqrt{2}}(1, 1)$ and $\mathbf{u}_1 = \frac{1}{\sqrt{10}}(1, 3)$, then $\mathbf{v}_1$ is a unit vector in the row space and $\mathbf{u}_1$ us a unit vector in the column space of $A = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$.

The singular values of A are the square roots of the eigenvalues of $A^TA = \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 10 & 10 \\ 10 & 10 \end{bmatrix}$.

Since A^TA has eigenvlaues $\lambda_1 = 20$ and $\lambda_2 = 0$, there is only one positive singular value, $\sigma_1 = \sqrt{10}$. There no σ_2 because A has dependent columns, so A^TA is positive semidefinite, but not positive definite, and has an eigenvalue of 0.

(b) We can choose unit vectors $\mathbf{v}_2 = \frac{1}{\sqrt{2}}(-1, 1)$ orthogonal to $\mathbf{v}_1$ and $\mathbf{u}_2 = \frac{1}{\sqrt{10}}(-3, 1)$ orthogonal to $\mathbf{u}_1$. Note that $\{\mathbf{v}_2\}$ is a basis for the null space of A, and $\{\mathbf{u}_2\}$ is a basis for the left null space of A (each is 1-dimensional). Then

$$\begin{aligned} A &= U\Sigma V^T \\ &= \begin{bmatrix} 1/\sqrt{10} & -3/\sqrt{10} \\ 3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix} \begin{bmatrix} \sqrt{20} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \\ &= \sqrt{20} \begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} + 0 \begin{bmatrix} -3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \\ &= \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T. \end{aligned}$$

Optional Problems

9. Let p be a polynomial with real coefficients. Prove that $p(\overline{x}) = 0$ if and only if $p(x) = 0$. Conclude that every real polynomial with odd degree has at least one real root.

10. Prove that every entry on the diagonal of a positive definite matrix must be positive.