YOUR NAME GOES HERE **21-241 Fall 2019**
Homework 10 **Due Friday, November 8**

Required Problems

1. Let x and y be complex numbers, and let $\overline{x}$ denote the complex conjugate of x.

   (a) Prove that $(\overline{x})(\overline{y}) = \overline{xy}$.

   (b) Prove that $\overline{x} + \overline{y} = \overline{x + y}$.

   Solution:

   (a) : Let $a, b, c, d \in \mathbb{Z}, \overline{x} = a + bi, \overline{y} = c + di$

   $$(\overline{x})(\overline{y}) = (a - bi)(c - di) = ac - adi - bci - bd = ac - bd - (ad - bc)i$$

   On the right side, we have:

   $$\overline{xy} = \overline{(a + bi)(c + di)} = \overline{ac - bd + (ad - bc)i} = ac - bd - (ad - bci)$$

   Therefore:

   $$(\overline{x})(\overline{y}) = \overline{xy}$$

   (b) Let $a, b, c, d \in \mathbb{Z}, \overline{x} = a + bi, \overline{y} = c + di$

   $$\overline{x} + \overline{y} = (a - bi) + (c - di) = a + c - (b + d)i$$

   On the right side, we have:

   $$\overline{x + y} = \overline{(a + bi) + (c + di)} = \bar{(}a + c) + (b + d)i = a + c - (b + d)i$$

   Therefore:

   $$\overline{x} + \overline{y} = \overline{x + y}$$

2. (Strang 6.4.1) Which of these matrices ASB will be symmetric with eigenvalues 1 and -1?

$$\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix} \quad \begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\-1 & 1\end{bmatrix} \quad \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}$$

$B = A^T$ doesn't do it. $B = A^{-1}$ doesn't do it. $B = \quad = \quad$ will succeed. So B must be an matrix.

Solution:

(a)
$$\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}$$
will be symmetric with eigenvalues 1, and -1. Because it is in the form of the diagonalization of a symmetric matrix, as said by the spectral theorem.

(b) $1.B = A^T = A^{-1}, 2.$Orthonormal

3. (Strang 6.4.13) Write S and B in the form $\lambda_1 \mathbf{x}_1 \mathbf{x}_1^T + \lambda_2 \mathbf{x}_2 \mathbf{x}_2^T$ of the Spectral theorem $Q\Lambda Q^T$:

$$S = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} \quad (\text{keep } \|\mathbf{x}_1\| = \|\mathbf{x}_2\| = 1).$$

Solution:

(a) $S = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

Solve characteristic polynomial, $\det S - \lambda I$ to get eigenvalues. Jbox tells us that:

$$\lambda_1 = 2, \ and\ \lambda_2 = 4$$

Solving for $N(S - \lambda I)$, we get the eigenvalues. Jbox tells us that:

$$\mathbf{v_1} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Normalizing this, we get:

$$\mathbf{v_1} = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

So, in the form of $\lambda_1 \mathbf{x}_1 \mathbf{x}_1^T + \lambda_2 \mathbf{x}_2 \mathbf{x}_2^T$:

$$2 \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} + 4 \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

(b) $B = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$

Solve characteristic polynomial, $\det S - \lambda I$ to get eigenvalues. Jbox tells us that:

$$\lambda_1 = 25, \ and\ \lambda_2 = 0$$

Solving for $N(B - \lambda I)$, we get the eigenvalues. Jbox tells us that:

$$\mathbf{v_1} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

Normalizing this, we get:

$$\mathbf{v_1} = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} \frac{-4}{5} \\ \frac{3}{5} \end{bmatrix}$$

So, in the form of $\lambda_1 \mathbf{x}_1 \mathbf{x}_1^T + \lambda_2 \mathbf{x}_2 \mathbf{x}_2^T$:

$$25 \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix} \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \end{bmatrix} + 0 \begin{bmatrix} \frac{-4}{5} \\ \frac{3}{5} \end{bmatrix} \begin{bmatrix} \frac{-4}{5} & \frac{3}{5} \end{bmatrix}$$

4. (Strang 6.5.7) Test to see if $A^T A$ is positive definite in each case: A needs independent columns.

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \text{ and } A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix} \text{ and } A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

Solution:

(a) A $= \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

$$A^-1 = \frac{1}{3} \begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix}$$

Since the inverse exists, we know that A has independent columns, and $A^T A$ is positive Definite.

(b) $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix}$

$$A^T A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix}$$

Solving for $det(A - \lambda I)$ yields the eigenvalues:

$$\lambda_1 = 11, \lambda_2 = 1$$

Since $A^T A$ has all eigenvalues ¿ 0, it is positive definite.

(c) $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ In this case, $A^T A$ can never be positive definite, as A does not have independent columns because it has 3 vectors in R^2. It

5. (Strang 6.5.22) From $S = Q\Lambda Q^T$ compute the positive definite symmetric square root $Q\sqrt{\Lambda}Q^T$ of each matrix. Check that this square root gives $A^TA = S$:

$$S = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}.$$

Solution:

(a) S $= \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$

We start by finding the eigenvectors by solving $\det(S - \lambda I) = 0$. Using Juliabox, we get:

$$\lambda_1 = 9, \lambda_2 = 1$$

We can then solve $N(S - \lambda I)$ to get the eigenvectors:

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Normalizing these vectors, we get:

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, Q^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Since Λ is diagonal, we simply take the square root of the elemnts in to get:

$$\sqrt{\Lambda} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

Together, $Q\sqrt{\lambda}Q^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

$$Check: \; A^TA = (Q\sqrt{\lambda}Q^T)(Q\sqrt{\lambda}Q^T) =$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= Q\Lambda Q^T = S$$

(b) $S = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}$

We can follow a similar procedure as before to see that the eignevalues are:

$$\lambda_1 = 16, \lambda_2 = 4$$

Similarly, the normalized eigenvectors are:

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, Q^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Since Λ is diagonal, we simply take the square root of the elemnts in to get:

$$\sqrt{\Lambda} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$$

Together, $Q\sqrt{\lambda}Q^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

$$Check: \; A^T A = (Q\sqrt{\lambda}Q^T)(Q\sqrt{\lambda}Q^T) =$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 16 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= Q\Lambda Q^T = S$$

6. (Strang 7.2.1) Find the eigenvalues of these matrices. Then find singular values from $A^T A$:

$$A = \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}.$$

For each A, construct V from the eigenvectors of $A^T A$ and U from the eigenvectors of AA^T. Check that $A = U\Sigma V^T$.

Solution:

(a) $A = \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix}$

Solving the Characteristic equation yields:

$$\lambda = 0$$

$A^T A$ yields:

$$A^T A = \begin{bmatrix} 0 & 0 \\ 0 & 16 \end{bmatrix}$$

So the eigenvalues are:

$$\lambda_1 = 16, \lambda_2 = 0$$

The eigenvectors from $A^T A$ turn out to be :

$$\mathbf{u_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{u_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Doing the same for AA^T yields the eigenvectors forming U:

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

This shows us that $A = U\Sigma V^T$ is:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Check: Since This is in the form: IAI, it is A.

(b) $A = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}$

Solving the Characteristic equation yields:

$$\lambda_1 = 2, \lambda_2 = -2$$

$A^T A$ yields:

$$A^T A = \begin{bmatrix} 1 & 0 \\ 0 & 16 \end{bmatrix}$$

So the eigenvalues are:

$$\lambda_1 = 16, \lambda_2 = 1$$

The eigenvectors from $A^T A$ turn out to be :

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Doing the same for AA^T yields the eigenvectors forming U:

$$\mathbf{u_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{u_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

This shows us that $A = U\Sigma V^T$ is:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Check: Since This is in the form: $I(PA)(P^{-1}I)$, it is A.

7. (Strang 7.2.4) Compute A^TA and AA^T and their eigenvalues and unit eigenvectors for V and U.

$$\textbf{Rectangular matrix} \quad A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Check $AV = U\Sigma$ (this decides $\pm$ signs in U). Σ has the same shape as A: 2×3.

Solution: $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

Start by finding AA^T:

$$AA^T = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

The eigenvalues are:

$$\lambda_1 = 3, \lambda_2 = 1$$

The normalized eigenvectors are:

$$\mathbf{u_1} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \mathbf{u_2} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Now for A^TA:

$$A^TA = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

The eigenvalues are:

$$\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 3$$

The normalized eigenvectors are:

$$\mathbf{v_1} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \mathbf{v_3} = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

Putting these all together, we get:

$$A = U\Sigma V^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

Checking this, we get:

$$AV = U\Sigma$$

$$AV = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{6}} & -\frac{1}{2} & 0 \\ \frac{3}{\sqrt{6}} & \frac{1}{2} & 0 \end{bmatrix}$$

$$U\Sigma = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{6}} & -\frac{1}{2} & 0 \\ \frac{3}{\sqrt{6}} & \frac{1}{2} & 0 \end{bmatrix}$$

8. (Strang 7.2.5)

(a) The row space of $A = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$ is 1-dimensional. Find $\mathbf{v}_1$ in the row space and $\mathbf{u}_1$ in the column space. What is σ_1? Why is there no σ_2?

(b) Choose $\mathbf{v}_2$ and $\mathbf{u}_2$ in V and U. Then $A = U\Sigma V^T = \mathbf{u}_1\sigma_1\mathbf{v}_1^T$ (one term only).

Solution: To find $\mathbf{u_1}$ *and* $\mathbf{v_1}$ we will find eigenvalues and vectors of AA^T *and* A^TA

The eigenvalues and eigenvectors of V are:

$$\lambda_1 = 20, \lambda_2 = 0, \mathbf{v_1} = \begin{bmatrix} \frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{bmatrix}, \mathbf{v_2} = -\begin{bmatrix} \frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix}$$

The eigenvalues and eigenvectors of U are:

$$\lambda_1 = 20, \lambda_2 = 0, \mathbf{u_1} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \mathbf{u_2} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

From this info,we can see that:

$$\mathbf{v_1} = \begin{bmatrix} \frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{bmatrix} = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

And This:

$$\mathbf{u_1} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\sigma_1 = \sqrt{2}$

There is no σ_2 because the original matrix had rank 1. Therefore, the eigenspace could then only be rank 1. This reflects why σ_2 is technically, 0, as all vectors not formed by the first eigenvector, will map to 0, as they are in the null space.

(b)

Let these values be v_2, u_2

$$\mathbf{v_2} = \begin{bmatrix} -\frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix}, \mathbf{u_2} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

With this,we have that:

$$A = U\Sigma V^T = \mathbf{u}_1\sigma_1\mathbf{v}_1^T = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} 20 \begin{bmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix}$$

Optional Problems

9. Let p be a polynomial with real coefficients. Prove that $p(\overline{x}) = 0$ if and only if $p(x) = 0$. Conclude that every real polynomial with odd degree has at least one real root.

10. Prove that every entry on the diagonal of a positive definite matrix must be positive.