Trevor Leong **21-241 Fall 2019**
Homework 3 **Due Friday, September 13**

Required Problems

1. (Strang 2.5.7) If A has row 1 + row 2 = row 3, show that A is not invertible:

   (a) Explain why $A\mathbf{x} = (0, 0, 1)$ cannot have a solution. Add eqn 1 + eqn 2.

   (b) Which right sides (b_1, b_2, b_3) might allow a solution to $A\mathbf{x} = \mathbf{b}$?

   (c) In elimination, what happens to equation 3?

Solution: (a)

Assuming that row1 + row 2 = row3, the conditions for $A\mathbf{x} = (0, 0, 1)$ mus be such that $A_{1*}\mathbf{x} = 0, A_{2*}\mathbf{x} = 0, A_{3*}\mathbf{x} = 1$. However, since $A_{1*} + A_{2*} = A_{3*}$, Then $A_{3*}\mathbf{x} = (A_{1*} + A_{2*})\mathbf{x} = A_{1*}\mathbf{x} + A_{2*}\mathbf{x} = 0 \neq 1$. Thus, these conditions contradict each other, and therefore cannot be a solution.

(b)

If the right side (b_1, b_2, b_3), is in the form of $(b_1, b_2, b_1 + b_2)$. This will allow for there to be solutions. This is because each b_n in $\mathbf{b}$ must be equal to $A_{n*}\mathbf{x}$,which means that $b_3 = A_{3*}\mathbf{x} = A_{3*}\mathbf{x} = (A_{1*} + A_{2*})\mathbf{x} = A_{1*}\mathbf{x} + A_{2*}\mathbf{x} = b_1 + b_2$. Additionally, when A is put into RREF, there will be a zero row at the bottom,and that is only possible, if some combination of the previous two elements adds up to the third.

(c)

During elimination, row 3 will become all 0's, leaving equation 3 to equal $0x + 0y + 0z = b_3$, thus giving no information. If $b_3! = 0$, then there will be no solutions. If $b_3 = 0$, then there may be 1 solution.

2. (Strang 2.5.8) If A has column 1 + column 2 = column 3, show that A is not invertible:

   (a) Find a nonzero solution $\mathbf{x}$ to $A\mathbf{x} = \mathbf{0}$. The matrix is 3 by 3.

   (b) Elimination keeps column 1 + column 2 = column 3. Explain why there is no third pivot.

Solution: (a)

Need to find vector such that multiplying a matrix by that vector yields 0.
Since $c_1 + c_2 = c_3$

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

This works because each entry

$$b_i = \mathbf{x}_i a_{1i} + \mathbf{x}_i a_{2i} + \mathbf{x}_i a_{3i} = \mathbf{x}_i(a_{1i} + a_{2i}) + \mathbf{x}_i a_{3i},$$

which is 0 when x is the specified vector above.

(b)

As elimination proceeds, the goal is to have three pivot values, such that they are unique to their rows and columns. If there was a third pivot variable in a 3x3 matrix, this would imply that the reduced row echelon form would have rows with two zeroes and a no-zero value. This matrix cannot exist, as two zeros cannot sum to a non-zero number, and you cannot get to zero by adding zero to a non-zero number.

3. (Strang 2.5.25) Find A^{-1} and B^{-1} (if they exist) by elimination on $[A|I]$ and $[B|I]$:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

Solution: (a)

$$A = \begin{bmatrix} 2 & 1 & 1 & | & 1 & 0 & 0 \\ 1 & 2 & 1 & | & 0 & 1 & 0 \\ 1 & 1 & 2 & | & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1-R_2} \begin{bmatrix} 1 & -1 & 0 & | & 1 & -1 & 0 \\ 1 & 2 & 1 & | & 0 & 1 & 0 \\ 1 & 1 & 2 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2-R_3} \begin{bmatrix} 1 & -1 & 0 & | & 1 & -1 & 0 \\ 0 & 1 & -1 & | & 0 & 1 & -1 \\ 1 & 1 & 2 & | & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3-R_1} \begin{bmatrix} 1 & -1 & 0 & | & 1 & -1 & 0 \\ 0 & 1 & -1 & | & 0 & 1 & -1 \\ 0 & 2 & 2 & | & -1 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3-2R_2} \begin{bmatrix} 1 & -1 & 0 & | & 1 & -1 & 0 \\ 0 & 1 & -1 & | & 0 & 1 & -1 \\ 0 & 0 & 4 & | & -1 & -1 & 3 \end{bmatrix} \xrightarrow{4R_2+R_3} \begin{bmatrix} 1 & -1 & 0 & | & 1 & -1 & 0 \\ 0 & 4 & 0 & | & -1 & 3 & -1 \\ 0 & 0 & 4 & | & -1 & -1 & 3 \end{bmatrix}$$

$$\xrightarrow{4R_1-R_2} \begin{bmatrix} 4 & 0 & 0 & | & 3 & -1 & -1 \\ 0 & 4 & 0 & | & -1 & 3 & -1 \\ 0 & 0 & 4 & | & -1 & -1 & 3 \end{bmatrix} \begin{matrix} \xrightarrow{\frac{1}{4}R_1} \\ \xrightarrow{\frac{1}{4}R_2} \\ \xrightarrow{\frac{1}{4}R_3} \end{matrix} \begin{bmatrix} 1 & 0 & 0 & | & 0.75 & -0.25 & -0.25 \\ 0 & 1 & 0 & | & -0.25 & 0.75 & -0.25 \\ 0 & 0 & 1 & | & -0.25 & -0.25 & 0.75 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & | & 0.75 & -0.25 & 0.25 \\ 0 & 1 & 0 & | & -0.25 & 0.75 & -0.25 \\ 0 & 0 & 1 & | & -0.25 & -0.25 & 0.75 \end{bmatrix}$$

(b)

$$B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

$$-column1 + -column2 = column3$$

Since the column space of B does not span $\mathbb{R}^3$, due to the fact that the column vectors are linearly dependent, it is impossible for there to be only one solution and thus A is not invertible. Therefore, B^{-1} cannot exist.

4. (Strang 3.1.10) Which of the following subsets of $\mathbb{R}^3$ are actually subspaces?

(a) The plane of vectors (b_1, b_2, b_3) with $b_1 = b_2$.
(b) The plane of vectors with $b_1 = 1$.
(c) The vectors with $b_1 b_2 b_3 = 0$.
(d) All linear combinations of $\mathbf{v} = (1, 4, 0)$ and $\mathbf{w} = (2, 2, 2)$.
(e) All vectors that satisfy $b_1 + b_2 + b_3 = 0$.
(f) All vectors with $b_1 \leq b_2 \leq b_3$.

Solution:
(a)

This is a subset:
1. Contains 0, because multiplying any vecto by 0, gives the 0 vector
2. Scalar multiplication is closed, because if $b_1 = b_2$ then $ab_1 = ab_2$,which still stasifies the conditions
3. vector addition is closed, because any vectors $\mathbf{v} = (v_1, v_1, v_2)$ and $\mathbf{v} = (v_1, v_1, v_2)$ will add such that $\mathbf{v} + \mathbf{w} = (v_1, v_1, v_2) + (w_1, w_1, w_2) = (v_1 + w1, v_1 + w_1, v_2 + w_2)$, which still sastisfies the condition of equality for the first two components.

(b)

This subset of $\mathbb{R}$ is not a subspace,because addition is not closed. Let $\mathbf{v} = (1, v_1, v_2)$ and $\mathbf{2} = (1, w_1, w_2)$ be two vectors in the subset. The result of $\mathbf{v} + \mathbf{w}$ is $(2, v_1 + w_1, v_2 + w_2)$, which does not have $b_1 = 1$, and thus not in the subset

(c)

This subset of $\mathbb{R}$ is not a subspace,because addition is not closed. Let $\mathbf{v} = (0, v_1, v_2)$ and $\mathbf{2} = (w_1, 0, w_2)$ be two vectors in the subset where each only has one zero component. Both vectors are valid, because there is at least one component which is 0, making their product 0. The result of $\mathbf{v} + \mathbf{w}$ is (w_1, v_1, v_2), which does not have $b_1 b_2 b_3 = 0$, and thus not in the subset.

(d)

This is a subset:
1. Contains 0, because $0\mathbf{v} + 0\mathbf{w} = 0$
2. Scalar multiplication is closed, because multiplying either vector by a scalar is contained in a linear combination when the other vector is multiplied by 0
3. vector addition is closed, because any vectors $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ will add such that $\mathbf{v} + \mathbf{w} = (v_1, v_1, v_2) + (w_1, w_1, w_2) = (v_1 + w1, v_1 + w_1, v_2 + w_2)$, which still satisfies the condition of equality for the first two components.
Also, all 3 conditions are automatically satisfied for all linear combinations

(e)

This is a subset:
1. Contains 0, because $0\mathbf{v} = 0$
2. Scalar multiplication is closed, because multiplying either vector by a scalar is contained in a linear combination when the other vector is multiplied by 0
3. vector addition is closed, because any vectors $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ will add such that $\mathbf{v} + \mathbf{w} = (v_1, v_2, v_3) + (w_1, w_1, w_2) = (v_1 + w1, v_2 + w_2, v_3 + w_3)$, which when summed, comes out to $v_1 + w_1 + v_2 + w_2, v_3 + w_3 = (v_1 + v_2 + v_3) + (w_1 + w_2 + w_3) = 0 + 0 = 0$ which still satisfies the condition of all elements summing to 0.

(f)

This is a subset:
1. Contains 0, because $0\mathbf{v} = 0$
2. Scalar multiplication is closed, because multiplying either vector by a scalar is contained in a linear combination when the other vector is multiplied by 0
3. vector addition is closed, because any vectors $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ will add such that $\mathbf{v} + \mathbf{w} = (v_1, v_2, v_3) + (w_1, w_1, w_2) = (v_1 + w1, v_2 + w_2, v_3 + w_3)$. From this we can conclude that $v_1 + w1 < v_2 + w_2$, as $v_1 \leq v_2 \; and \; w_1 \leq w_2$. The same logic applies for the second and third elements. So, $\mathbf{v} + \mathbf{w}$ still satisfies the condition of all elements summing to 0.

5. (Strang 3.1.19) Describe the column spaces (lines or planes) of these particular matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Solution: (a)

The column space of matrix A is the span of the vectors $(1, 0, 0) and (2, 0, 0)$. However, since $(2, 0, 0) = 2(1, 0, 0)$, It ends up being all scalar multiples of $(1, 0, 0)$, which is the x-axis.

(b)

The column space of matrix A is the span of the vectors $(1, 0, 0) and (0, 2, 0)$. However, since $a(1, 0, 0) + b(0, 2, 0) = (a, 2b, 0)$, where we can express any number b, as $\frac{1}{2}k$, making it$a(1, 0, 0) + b(0, 1, 0) = (a, b, 0)$. This leaves all points with a third-component of 0, meaning that it is a hyperplane. Looking from cartesian coordinates, it is the xy-pane.

(c)

The column space of matrix A is the span of the vectors $(1, 0, 0) and (0, 0, 0)$. However, since $a(1, 2, 0) + b(0, 0, 0) = (a, 2a, 0)$, it is in reality, only all scalar multiples of $(1, 2, 0)$. This leaves us with a line going through the origin in $\mathbb{R}^3$ such that y = 2x.

6. (Strang 3.1.23) The columns of AB are combinations of the columns of A. This means: *The column space of AB is contained in (possibly equal to) the column space of A.* Give an example where the column spaces of A and AB are not equal.

   Solution:

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$
$$AB = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

   The column space of A is the xy-plane, as the two vectors are linearly independent and span $\mathbb{R}^2$. The column space of AB is the line y = 2x, which does not span $\mathbb{R}^2$.

7. Let U and V be two subspaces of $\mathbb{R}^n$.

   (a) Prove that the intersection $U \cap V$ is also a subspace of R^n.

   (b) Prove that if neither subspace contains the other, then the union $U \cup V$ is not a subspace of $\mathbb{R}^n$

Solution: (a)

Assume for the sake of contradiction that $U \cap V$ is not a subspace of $\mathbb{R}^n$

Let $\mathbf{c}, \mathbf{d} \in U \cap \mathbf{c}, \mathbf{d} \in V$

This would mean that one of the following conditions should be true:

1. $0 \notin \mathbb{R}^n$
2. $\mathbf{c} + \mathbf{d} \notin \mathbb{R}^n$
3. $a\mathbf{c} \notin \mathbb{R}^n$

The first condition cannot be true, as one $\mathbf{c} \in (U \cap V)$, must be zero, as it was assumed that U and V are sub spaces, meaning that they both must contain 0. Thus, their conjunction must also contain 0.

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The second condition cannot be true, as every vector in U and V must only contain real components to be subspace in $\mathbb{R}^n$. This means that any sum of vectors must also only obtain reals, as reals can only sum to other reals. Thus, the statement cannot be true.(addition is closed)

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The third condition also cannot be true, as every vector in U and V must only contain real components to be subspace in $\mathbb{R}^n$. This means that any scalar multiple of vectors must also only contain reals, as the product of any two reals is always real.(Multiplication is closed)

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Since none of these conditions are true, our original assertion, $U \cap V$ is not a subspace of $\mathbb{R}^n$ is false, meaning that $U \cap V$ is a subspace of $\mathbb{R}^n$ is a subspace is true.

Optional Problems

8. (Strang 2.5.33) Find and check the inverses (assuming they exist) of these block matrices:

$$\begin{bmatrix} I & 0 \\ C & I \end{bmatrix} \quad \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \quad \begin{bmatrix} 0 & I \\ I & D \end{bmatrix}$$

9. (Strang 3.1.23) If we add an extra column $\mathbf{b}$ to a matrix A, then the column space gets larger unless   . Give an example where the column space gets larger and an example where it doesn't. Why is $A\mathbf{x} = \mathbf{b}$ solvable exactly when the column space *doesn't* get larger–it is the same for A and $[A|\mathbf{b}]$?

10. Prove that if A is an invertible matrix, that $(A^T)^{-1} = (A^{-1})^T$.

11. Let M_n denote the set of $n \times n$ matrices with entries in the real numbers.

    (a) Prove that, with usual rules of matrix addition and scalar multiplication, M_n is a vector space. What is its dimension?

    (b) Prove that the set of upper triangular matrices is a subspace of M_n. What is its dimension?

    (c) Prove that the set of diagonal matrices is a subspace of M_n. What is its dimension?

12. Prove that in the vector space $\mathbb{F}_2^n$, there are exactly 2^k vectors in the span of any k linearly independent vectors.