Trevor Leong **21-241 Fall 2019**
Homework 2 **Due Friday, September 6**

Required Problems

1. (Strang 2.3.10)

   (a) What 3 by 3 matrix E_{13} will add row 3 to row 1?

   (b) What matrix adds row 1 to row 3 and *at the same time* row 3 to row 1?

   (c) What matrix adds row 1 to row 3 and *then* adds row 3 to row 1?

   Solution: (a)E_{13}= $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (b) E_{13}= $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

   (c) $\mathrm{E}_{13} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

2. (Strang 2.6.5) What matrix E puts A into a triangular form $EA = U$? Multiply by $E^{-1} = L$ to factor A into LU:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{bmatrix}$$

] [

Solution: $\mathrm{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$

$$EA = U$$
$$E^{-1}EA = E^{-1}U$$

$$E^-1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$
$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$

3. (Strang 2.7.22) Find the $PA = LU$ factorizations (and check them) for

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix} \text{ and } A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Solution: (a)
It can clearly be seen that the first and second rows must be switched, as the first row has a zero pivot.

$$P_{13} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$PA = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \xrightarrow{R_3 - 2R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 2 \end{bmatrix} \xrightarrow{R_3 - 3R_2}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

$PA = LU$ *where*

$$L = E_{31}^{-1} E_{32}^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

So, $PA = LU$ *is*

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\text{Check: } PA : \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$$

$$LU: \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$$

(b)

It can be seen that the second and third rows must be switched.

$$P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$PA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & 4 & 1 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 2 & 4 & 1 \end{bmatrix} \xrightarrow{R_3 - 2R_1}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 2 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$PA = LU \ where$$

$$L = E_{31}^{-1} E_{21}^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$So, \ PA = LU \ is$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Check:

$$PA: \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & 4 & 1 \end{bmatrix} \ LU: \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & 4 & 1 \end{bmatrix}$$

4. (Strang 2.6.10) L and U for the symmetric matrix A:

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

Solution: $E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$ $E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

First step: Find combinations of $E = E_{11}E_{21}....E2n$ such that $EA = U$
Let us begin by subtracting the third row from the fourth, the second row from the third row and finally the third row from the first.

Using the following matrix: $E_1E_2E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix} = \begin{bmatrix} a & a & a & a \\ a-a & b-a & b-a & b-a \\ a-a & b-b & c-b & c-b \\ a-a & b-b & c-c & d-c \end{bmatrix} = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}$$

$$L = E^{-1} = E_3^{-1}E_2^{-1}E_1^{-1}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$So, LU =$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}$$

5. (Strang 2.6.13) Solve $L\mathbf{c} = \mathbf{b}$ to find $\mathbf{c}$. Then solve $U\mathbf{x} = \mathbf{c}$ to find $\mathbf{x}$. What was A?

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

Solution:

$$Lc = b$$

let c = $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$

let x = $\begin{bmatrix} d \\ e \\ f \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ Then, multiply by inverse on both sides.

$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} a \\ b-a \\ c-b \end{bmatrix}$

$\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} a \\ b-a \\ c-b \end{bmatrix}$

$a = 4, \; b = 9, \; = 15$

$c = \begin{bmatrix} 4 \\ 9 \\ 15 \end{bmatrix}$

$$U\mathbf{x} = \mathbf{c}$$

$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 15 \end{bmatrix}$

$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 9 \\ 15 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} -5 \\ -6 \\ 15 \end{bmatrix}$

$$x = \begin{bmatrix} -5 \\ -6 \\ 15 \end{bmatrix}$$

$$A = LU$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

6. An **affine combination** of a set of vectors $\mathbf{v}_1, \mathbf{v_2}, \ldots, \mathbf{v_n}$ is a linear combination $a_1\mathbf{v}_1 + a_2\mathbf{v_2} + \cdots + a_n\mathbf{v_n}$ where $\sum_{i=1}^{n} a_i = 1$, i.e. the sum of the coefficients is equal to 1.

   (a) Suppose A is a matrix and $\mathbf{b}$, $\mathbf{x}$, and $\mathbf{y}$ are vectors such that $\mathbf{x} \neq \mathbf{y}$, $A\mathbf{x} = \mathbf{b}$, and $A\mathbf{y} = \mathbf{b}$. Prove that if $\mathbf{v}$ is an affine combination of $\mathbf{x}$ and $\mathbf{y}$, then $A\mathbf{v} = b$.

   You may use the facts that if c is a scalar, $\mathbf{x}$ and $\mathbf{y}$ are vectors, and A is a matrix, then

   - $A(c\mathbf{x})) = c(A\mathbf{x})$, and
   - $A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y}$.

   (b) Use Part (a) to explain why a system of linear equations cannot have exactly two solutions

   (c) Use Julia to generate some random vectors in $\mathbb{R}^2$, and plot affine combinations of them. What geometric object is formed by the set of all affine combinations of two vectors?

Solution: (a) Let b x and y be an arbitrary vector with n components and A be an n x n matrix such that:

$$\text{b} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_n \end{bmatrix}, \text{x} = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix}, \text{y} = \begin{bmatrix} y_1 \\ y_2 \\ . \\ . \\ y_n \end{bmatrix}, A = \begin{bmatrix} a_{11} & a_{12} & ......a_{1_n} \\ a_{21} & a_{22} & ......a_{2n} \\ ... & .. & .. \\ ...n1 & an2 & ......a_{nn} \end{bmatrix}$$

$$A\mathbf{x} = \begin{bmatrix} a_{11} & a_{12} & ......a_{1_n} \\ a_{21} & a_{22} & ......a_{2n} \\ ... & .. & .. \\ ...n1 & an2 & ......a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix}, = \begin{bmatrix} A_{1*} \cdot \mathbf{x} \\ A_{2*} \cdot \mathbf{x} \\ . \\ . \\ . \\ A_{n*} \cdot \mathbf{x} \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2... + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2... + a_{2n}x_n \\ ..... \\ a_{11}x_1 + a_{12}x_2... + a_{1n}x_n \end{bmatrix}$$

$$A\mathbf{y} = \begin{bmatrix} b_{11} & a_{12} & ......a_{1_n} \\ a_{21} & a_{22} & ......a_{2n} \\ ... & .. & .. \\ ...n1 & an2 & ......a_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ . \\ . \\ y_n \end{bmatrix}, = \begin{bmatrix} A_{1*} \cdot \mathbf{y} \\ A_{2*} \cdot \mathbf{y} \\ . \\ . \\ . \\ A_{n*} \cdot \mathbf{x} \end{bmatrix} = \begin{bmatrix} a_{11}y_1 + a_{12}y_2... + a_{1n}y_n \\ a_{21}y_1 + a_{22}y_2... + a_{2n}y_n \\ ..... \\ a_{11}y_1 + a_{12}y_2... + a_{1n}y_n \end{bmatrix}$$

$$Since\ \mathbf{b} = A\mathbf{y} = A\mathbf{x}$$

$$\begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2... + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2... + a_{2n}x_n \\ ..... \\ a_{11}x_1 + a_{12}x_2... + a_{1n}x_n \end{bmatrix} = \begin{bmatrix} a_{11}y_1 + a_{12}y_2... + a_{1n}y_n \\ a_{21}y_1 + a_{22}y_2... + a_{2n}y_n \\ ..... \\ a_{11}y_1 + a_{12}y_2... + a_{1n}y_n \end{bmatrix}$$

Let $v = c\mathbf{x} + d\mathbf{y}$ where c+d = 1.

$$v = \text{c} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix} + \text{d} \begin{bmatrix} y_1 \\ y_2 \\ . \\ . \\ y_n \end{bmatrix}$$

$$A\mathbf{v} = \begin{bmatrix} a_{11} & a_{12} & ......a_{1_n} \\ a_{21} & a_{22} & ......a_{2n} \\ ... & .. & .. \\ ...n1 & an2 & ......a_{nn} \end{bmatrix} \left(c \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix} + d \begin{bmatrix} y_1 \\ y_2 \\ . \\ . \\ y_n \end{bmatrix} \right) \text{ note: These are parenthesis}$$

By the rule stating that $A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y}$, We now have $A\mathbf{v} =$

$$\begin{bmatrix} a_{11} & a_{12} & ......a_{1_n} \\ a_{21} & a_{22} & ......a_{2n} \\ ... & .. & .. \\ ...n1 & an2 & ......a_{nn} \end{bmatrix} \text{c} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & ......a_{1_n} \\ a_{21} & a_{22} & ......a_{2n} \\ ... & .. & .. \\ ...n1 & an2 & ......a_{nn} \end{bmatrix} \text{d} \begin{bmatrix} y_1 \\ y_2 \\ . \\ . \\ y_n \end{bmatrix}$$ Using the rule that $A(c\mathbf{x})) = c(A\mathbf{x})$, we now have $A\mathbf{v} =$:

$$\text{c} \left(\begin{bmatrix} a_{11} & a_{12} & ......a_{1_n} \\ a_{21} & a_{22} & ......a_{2n} \\ ... & .. & .. \\ ...n1 & an2 & ......a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix} \right) +$$

$$\text{d} \left(\begin{bmatrix} a_{11} & a_{12} & ......a_{1_n} \\ a_{21} & a_{22} & ......a_{2n} \\ ... & .. & .. \\ ...n1 & an2 & ......a_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ . \\ . \\ y_n \end{bmatrix} \right)$$

$$=$$

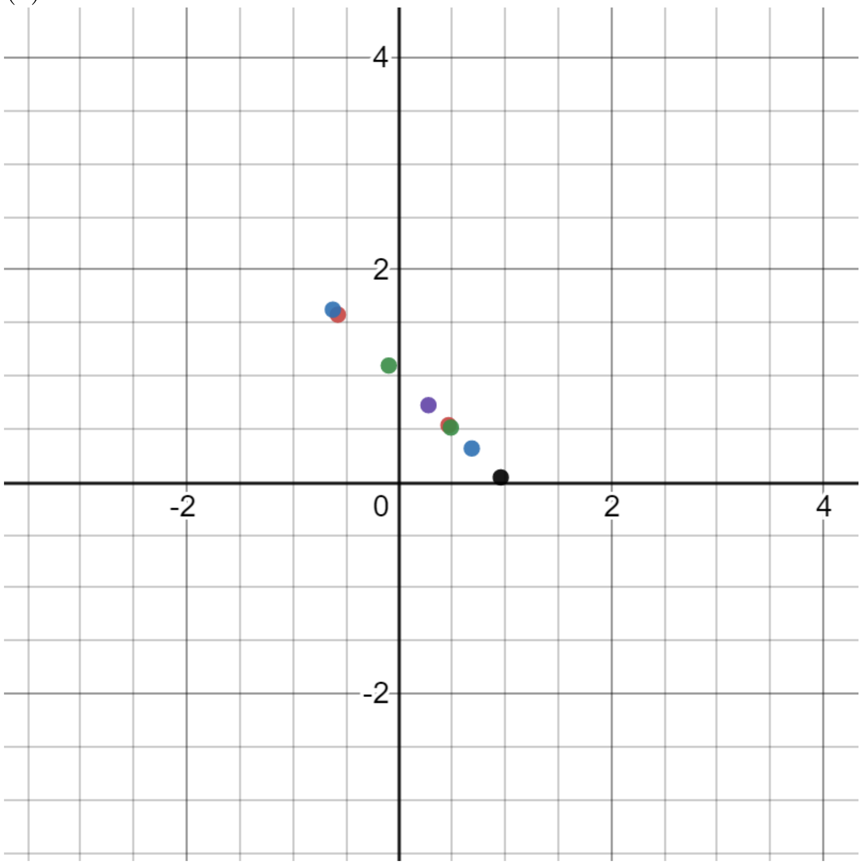
$$\text{c} \begin{bmatrix} a_{11}x_1 + a_{12}x_2... + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2... + a_{2n}x_n \\ ..... \\ a_{11}x_1 + a_{12}x_2... + a_{1n}x_n \end{bmatrix} + \text{d} \begin{bmatrix} a_{11}y_1 + a_{12}y_2... + a_{1n}y_n \\ a_{21}y_1 + a_{22}y_2... + a_{2n}y_n \\ ..... \\ a_{11}y_1 + a_{12}y_2... + a_{1n}y_n \end{bmatrix} =$$

$c\mathbf{b} + d\mathbf{b} = (\mathbf{c} + \mathbf{d})b = (1)\mathbf{b} = \mathbf{b}$ Thus, $Av = \mathbf{b}$

(b)

In part a, it was shown that if there existed two such solutions that satisfied $Ax = b$, then they could be combined in a linear combination such that the coefficients that multiplied them added up to 1. Such logic can be applied to any two distinct solutions, as if there are only 2 solutions, then you can find an affine combination of such two that is distinct and also a solution, contradicting the statement that there were only two solutions.

(c)



As we can see, the geometric object formed is a straight line going through the positions of both vectors.

7. A square matrix A is **symmetric** if for all i and j, $A_{ij} = A_{ji}$. Prove that for any $m \times n$ matrix A, AA^T is a symmetric $m \times m$ matrix. *Solution:* Let us prove this statement by determining what each element a_{ij} is and a_{ji} is for A and A^T :*Let a be an element in A, a' be anelement in*^{T} *s.t.*a$_{ij}$ is the element in the ith row and jth column of A and a'_{ij} is the element in the ith row and jth column of A^T.
   Since A and A^T are symmetric, it is true that: $a_{ij} = a_{ji}$ *and* $a'_{ij} = a'_{ji}$
   Let us now define a matrix C, such that $C = AA^T$ By the rules of matrix multiplication, each element c_{ij} can be expressed as:
   $$\sum_{k=1}^{n} a_{ik}a'_{kj}$$
   In contrast, each element c_{ji} can be expressed as:
   $$\sum_{k=1}^{n} a_{jk}a'_{ik}$$
   Due to the properties of symmetric matrices, we can rewrite c_{ji} as:
   $$\sum_{k=1}^{n} a_{kj}a'_{ki}$$
   Due to the fact that in symmetric matrices, $a_{ij} = a_{ji}$ *and* $a'_{ij} = a'_{ji}$
   This is identical to the expression for c_{ij}
   Since $c_{ij} = c_{ji}$, we can say that AA^T is symmetric by definition.

Optional Problems

8. Prove that the identity matrix is unique, i.e. if some $n \times n$ matrix A has the property that $A\mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$, then $A = I$.

9. Explain why it is always possible to change one element of a nonsingular matrix so that the result is singular. Give an example to show the converse is not true, i.e. give an example of a singular matrix where no matter what element is changed, the result is still singular.

10. Coding problems

    (a) Write Julia code that will return the LU decomposition of a square nonsingular matrix A.

    (b) Write Julia code that, given b and a $PA = LU$ decomposition for the square matrix A, will return the solution x to the matrix equation $Ax = b$.