YOUR NAME GOES HERE **21-241 Fall 2019**
Homework 6 **Due Friday, October 11**

Required Problems

1. (Strang 4.4.18) Find orthogonal vectors **A**, **B**, **C** by Gram-Schmidt from **a**, **b**, **c**:

$$\mathbf{a} = (1, -1, 0, 0) \quad \mathbf{b} = (0, 1, -1, 0) \quad \mathbf{c} = (0, 0, 1, -1).$$

A, **B**, **C** and **a**, **b**, **c** are bases for the vector space perpendicular to $\mathbf{d} = (1, 1, 1, 1)$.

Solution:

$$A = \mathbf{a}$$

$$B = \mathbf{b} - \frac{a^T b}{a^T a} A = \mathbf{b} + \frac{1}{2} A = \begin{bmatrix} 0 + \frac{1}{2} \\ 1 - \frac{1}{2} \\ -1 + 0 \\ 0 + 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \\ 0 \end{bmatrix}$$

$$C = \mathbf{c} - \frac{a^t c}{a^t a} A - \frac{B^t B}{B^t B} B = \mathbf{c} - 0A + \frac{2}{3} B = \mathbf{c} - \mathbf{0} + \frac{2}{3} B = \begin{bmatrix} 0 + \frac{1}{3} \\ 0 + \frac{1}{3} \\ 1 - \frac{2}{3} \\ -1 + 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ -1 \end{bmatrix}$$

2. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$. Write A in the form QR, where Q has orthonormal columns, and R is upper triangular.

Hint: Use results from Problem 1.

Solution:

$$\text{A} = \text{QR}$$

$$q_1 = \frac{A}{||A||} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{bmatrix}, \; q_2 = \frac{B}{||B||} = \begin{bmatrix} \frac{\sqrt{2}}{2\sqrt{3}} \\ \frac{\sqrt{2}}{2\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ 0 \end{bmatrix}, \; q_3 = \frac{C}{||C||} = \begin{bmatrix} \frac{\sqrt{3}}{6} \\ \frac{\sqrt{3}}{6} \\ \frac{\sqrt{3}}{6} \\ -\frac{\sqrt{3}}{2} \end{bmatrix}$$

$$\text{Q} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{2\sqrt{3}} & \frac{\sqrt{3}}{6} \\ -\frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{2\sqrt{3}} & \frac{\sqrt{3}}{6} \\ 0 & -\frac{1}{\sqrt{3}} & \frac{\sqrt{3}}{6} \\ 0 & 0 & -\frac{\sqrt{3}}{2} \end{bmatrix}$$

$$\text{R} = \begin{bmatrix} q_1^T\mathbf{a} & q_1^T\mathbf{b} & q_1^T\mathbf{c} \\ 0 & q_2^T\mathbf{b} & q_2^T\mathbf{c} \\ 0 & 0 & q_3^T\mathbf{c} \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{\sqrt{6}}{2} & -\frac{1}{\sqrt{3}} \\ 0 & 0 & \frac{2}{\sqrt{3}} \end{bmatrix}$$

$$\text{A} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{2\sqrt{3}} & \frac{\sqrt{3}}{6} \\ -\frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{2\sqrt{3}} & \frac{\sqrt{3}}{6} \\ 0 & -\frac{1}{\sqrt{3}} & \frac{\sqrt{3}}{6} \\ 0 & 0 & -\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{\sqrt{6}}{2} & -\frac{1}{\sqrt{3}} \\ 0 & 0 & \frac{2}{\sqrt{3}} \end{bmatrix}$$

3. (From Strang 4.4.10) In this problem you will prove that orthonormal vectors are automatically linearly independent in two ways (Note they are really the same proof, just written differently! –DO).

   (a) Vector proof: When $c_1\mathbf{q}_1 + c_2\mathbf{q}_2 + \cdots + c_n\mathbf{q}_n = \mathbf{0}$, what dot products lead to $c_1 = 0$? Similarly $c_2 = 0$, and so forth. Thus the $\mathbf{q}$'s are independent.

   (b) Matrix proof: Show that $Q\mathbf{x} = \mathbf{0}$ leads to $\mathbf{x} = \mathbf{0}$. Since Q may be rectangular, you can use Q^T but not Q^{-1}.

Solution:
(a) W.T.S:
only solution to $c_1\mathbf{q}_1 + c_2\mathbf{q}_2 + \cdots + c_n\mathbf{q}_n = \mathbf{0}$ is when:
$c_1 = c_2 = c_3 = ...c_n = 0$

let q_x be an arbitrary orthonormal vector, $1 \geq x \geq n$.

$$\text{If } c_1\mathbf{q}_1 + c_2\mathbf{q}_2 + \cdots + c_n\mathbf{q}_n = \mathbf{0}$$
$$\implies \mathbf{q_x}(c_1\mathbf{q}_1 + c_2\mathbf{q}_2 + \cdots + c_n\mathbf{q}_n) = \mathbf{q_x}(\mathbf{0})$$
$$\mathbf{q_x}c_1\mathbf{q}_1 + \mathbf{q_x}c_2\mathbf{q}_2 + \cdots + \mathbf{q_x}c_n\mathbf{q}_n = \mathbf{q_x}(\mathbf{0})(\text{distributivity of dot product})$$
$$c_1\mathbf{q_x}^T\mathbf{q}_1 + c_2\mathbf{q_x}^T\mathbf{q}_2 + \cdots + c_n\mathbf{q_x}^T\mathbf{q}_n = (\mathbf{0})(\text{associativity of dot product})$$
$$c_1\mathbf{0} + c_2\mathbf{0} + \cdots + c_x(\mathbf{q_x}^2) + \cdots c_n\mathbf{0} = \mathbf{0}(\text{property of orthonormal vectors})$$
$$\implies c_x(\mathbf{q_x}^2) = \mathbf{0}$$

$\implies c_x = 0$ Since we declared q_x to be an arbitrary vector, this implies that the only solution to $c_1\mathbf{q}_1 + c_2\mathbf{q}_2 + \cdots + c_n\mathbf{q}_n = \mathbf{0}$ is when:

$$c_1 = c_2 = c_3 = ...c_n = 0$$

(b)

Let Q be an m x n matrix with the columns being orthonormal vectors, m $\geq$ n.

W.T.S Only solution to $Q\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$

Claim $Q^TQ = I$.

$$a_{ij} = row_i(Q^T) \cdot col_j(Q)$$
$$row_i(Q^T) = \mathbf{q}_i, col_j = \mathbf{q}_j$$
$$\implies a_{ij} = \mathbf{q}_i\mathbf{q}_j$$

case1: i = j(diagonals):

$$a_{ij} = \mathbf{q}_i\mathbf{q}_j = 1(\text{orthonormal vectors are unit vectors})$$

case2: i $\neq$ j(non-diagonals):

$$a_{ij} = \mathbf{q}_i\mathbf{q}_j = 0(\text{dot product of orthogonal vectors is 0})$$

Thus, the resulting product Q^TQ has its diagonals as 1's and everywhere else is 0, which is the identity matrix, proving our claim. It is important to note that this works for non-square matrices, as Q^TQ is always a square matrix, and we have proven that the property that it has to have its diagonal elements as 1, and everywhere else 0.

$$Q\mathbf{x} = \mathbf{0}$$
$$Q^TQ\mathbf{x} = Q^T\mathbf{0}$$
$$I\mathbf{x} = \mathbf{0}(\text{proven above})$$
$$\mathbf{x} = \mathbf{0}$$

4. Use the (7,4) Hamming code defined in class, with

$$H = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

to answer the following questions.

(a) Suppose Alice wishes to encode the message $(0, 0, 1, 0)$. What is C_A?

(b) Suppose Bob receives the message $C_B = (0, 0, 1, 0, 0, 1, 1)$. What was the original message M?

(c) Suppose Bob receives the message $C_B = (0, 0, 1, 0, 1, 1, 1)$. What was the original message M?

(d) Suppose Bob receives the message $C_B = (0, 0, 1, 0, 1, 0, 1)$. What was the original message M?

Solution: (a)

$$C_A = GM = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

(b)

$$e_i = HC_B = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = col_2(G)$$

$$\implies \textit{Error in column 2 of } H$$

$$\implies C_A = C_B + e_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \textit{ Message is first 4 elements}: \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

(c)

$$e_i = HC_B = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = col_6(H)$$

$$\implies Error\ in\ column\ 6\ of\ H$$

$$C_A = C_B + e_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} Message\ is\ first\ 4\ elements: \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

(d)

$$e_i = HC_B = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$No\ error$$

$$C_A = C_B$$

$$Message\ is\ first\ 4\ elements: \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

5. Define the *Hamming distance* between two vectors in $\mathbb{F}_2^n$ to be the number of components in which they differ. For example, the Hamming distance between $(0,1,0,0,1,1,1,0)$ and $(1,1,1,0,1,1,0,0)$ is three (components 1, 3, and 7 are different). In this problem, we show that any two codewords in the Hamming code must have Hamming distance at least 3 from each other.

   (a) Show that two codewords in the (7,4) Hamming code cannot be at Hamming distance 1 from each other.

   Hint: Use the fact that if C_1 and C_2 are two codewords at distance 1 then for some i, $C_1 = C_2 + e_i$.

   (b) Show that two codewords in the (7,4) Hamming code cannot be at Hamming distance 2 from each other.

   Hint: Use the fact that if C_1 and C_2 are two codewords at distance 2 then for some i, j, $C_1 = C_2 + e_i + e_j$.

Solution: (a)

$$\text{Let } C_b \; and \; D_b \; be \; codewords \; , C_b = D_b + e_i, C_a = C_b, H(C_i) = 0$$
$$\implies H(C_b) = H(D_b + e_i) = H(D_b) + H(e_i) = 0(\text{Linearity})$$
$$\implies H(C_b) = H(-e_i)$$
$$\implies H(C_j) = H(e_i)(e_i = -e_i \; in \; \mathbb{F}^2)$$
$$\implies D_a = D_b + e_i(H(e_i \; isguaranteedtobeinH, ase_i \in \mathbb{R}^3 \text{ are in H})$$
$$\implies D_a = C_b = C_a$$

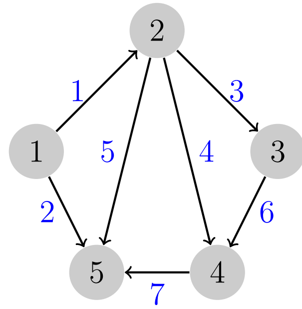
Since we have shown that all if two code words are at hamming distance once, they must code for the same message, which proves that two different messages must have codewords that are grater than 1 hamming distance away.

(b)

$$C_b \; and \; D_b \; be \; codewords \; , C_b = D_b + e_i + e_j, C_a = C_b, H(C_i) = 0$$
$$\implies H(C_b) = H(D_b + e_i + e_j) = H(D_b) + H(e_i) + h(e_j) = 0(\text{Linearity})$$
$$\implies H(C_b) = H(-e_i) + H(-e_j)$$
$$\implies H(C_j) = H(e_i) + H(e_j)(e_i = -e_i \; in \; \mathbb{F}^2)$$
$$\implies D_a = D_b + e_i + e_j(\text{Similar to prior case, all } e_i + e_j \in \mathbb{R}^3 \text{ are in H})$$
$$\implies D_a = C_b = C_a$$

Since we have shown that all if two code words are at hamming distance once, they must code for the same message, which proves that two different messages must have codewords that are greater than 2 hamming distance away.

6. Consider the following directed graph D.



(a) What is the incidence matrix A corresponding to D? For grading purposes, please use the node and arc labels in order as given in the figure.

(b) Use RREF to find a basis for the row space of A. You do not need to show each step of finding the RREF if you do not want to.

(c) What is the spanning tree of D corresponding to the basis in Part (b)?

Solution: (a)

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

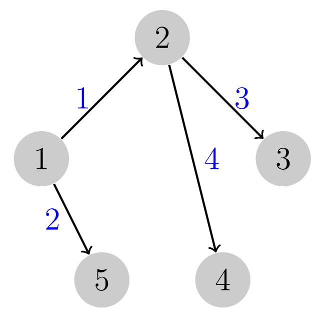
(b)

$$RREF(A^T) = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Form this, we can see that the first 4 rows of A form a basis, as the pivot columns in A^T are the first 4.

$$\text{Basis}(\text{c}(A^T)) = \{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \} .$$

(c)


2
1
3
1
3
4
2
5
4

7. Consider the same digraph D and adjacency matrix A from Problem 6.

   (a) Find the RREF for A^T, and use it to find a basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for $N(A^T)$.

   (b) Find cycles C_1, C_2, and C_3 in D corresponding to $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, respectively.

   (c) Show that the cycle with arcs 1, 3, 6, 7, and 2 can be obtained from a linear combination of the vectors in the cycle basis. Write the corresponding equation in terms of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

   (d) Show that the cycle with arcs 5, 3, 6, and 7 can be obtained from a linear combination of the vectors in the cycle basis. Write the corresponding equation in terms of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

   (e) Let T be the spanning tree from Problem 6. Show that each cycle in the cycle basis is obtained by taking the union of an arc e not in T with the arcs in T connecting the endpoints of e.

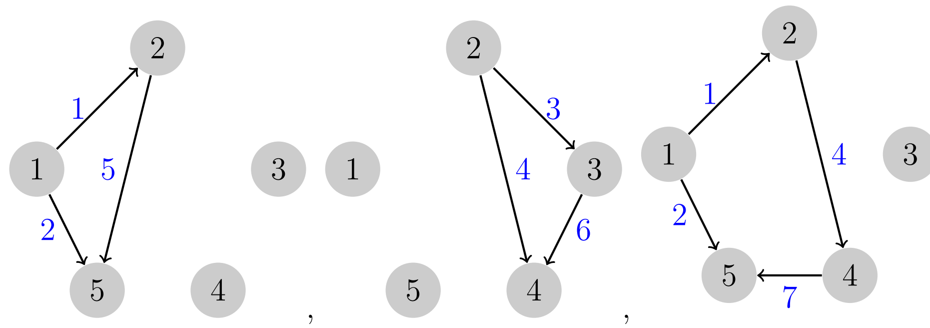
   [This is an example of a general principle for generating a cycle basis of a connected graph: Given a spanning tree T, the cycles formed by adding a non-tree edge to T form a cycle basis for the graph. Note that this is consistent with the fact that the dimension of $N(A^T)$ is $n - m + 1$ (Since the rank of A is $n - 1$, the dimension of the column space is $n - 1$, so by the Rank-nullity theorem, $\dim N(A^T) = m - n + 1$.)]

Solution: (a)

From before: $RREF(A^T) = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ Fromt Gilbert's special solutions, we can see that the vectors in N(A^T) are:

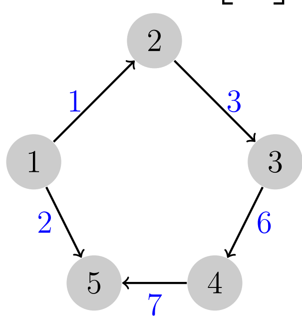
$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(b)

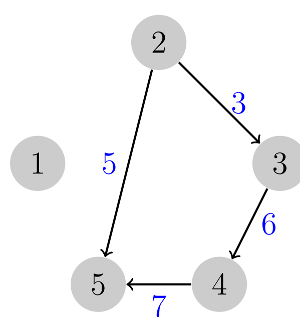


(c)

Try $\mathbf{v_2} + \mathbf{v_3} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$



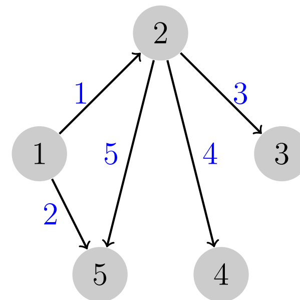
(d)



Consider $\mathbf{v_1} - \mathbf{v_2} - \mathbf{v_3}$

(e)

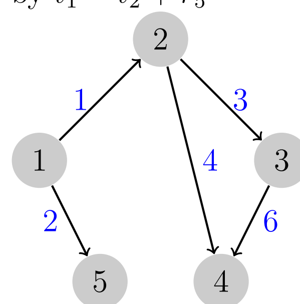
$$T = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$



$$T \cup r_5 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Let t_i be the ith vector in T.

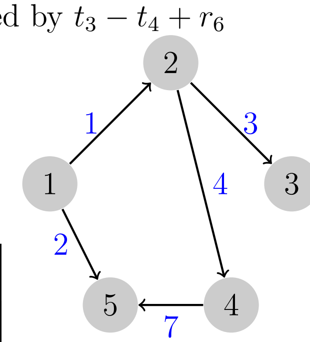
v_1 shows s that a cycle can be formed by $t_1 - t_2 + r_5$



$$T \cup r_6 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

Let t_i be the ith vector in T.

v_2 shows us that a cycle can be formed by $t_3 - t_4 + r_6$



$$T \cup r_7 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

Let t_i be the ith vector in T.

v_3 shows us that a cycle can be formed by $t_1 - t_2 + t_4 + r_7$

Optional Problems

8. Let A be an $m \times n$ matrix with rank r. Suppose $V = \{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is an orthonormal basis for $C(A^T)$ and $W = \{\mathbf{w}_1, \ldots, \mathbf{w}_{n-r}\}$ is an orthonormal basis for $N(A)$. Use Problem 3 to give a simple proof that $V \cup W$ is a basis for $\mathbb{R}^n$.

9. Write Julia code to perform the Gram-Schmidt algorithm.

10. (Strang 4.4.33) Find all matrices that are both orthogonal and lower triangular.