Required Problems

1. Suppose m, n, and p are distinct positive integers, and

   - A is an $m \times n$ matrix,
   - B is an $n \times p$ matrix,
   - C is an $m \times p$ matrix,
   - x is a vector with n components, and
   - y is a vector with p components.

   For each of the following expressions, describe what kind of object is produced, or if the expression is not well-defined, explain why. You may wish to experiment with some random matrices and vectors in Julia to verify your answers.

   (a) Ax
   (b) xA
   (c) AB
   (d) AC
   (e) BC
   (f) $Ax + Bx$
   (g) $By + Cy$
   (h) $(B + C)y$
   (i) $(AB)y$
   (j) $A(By)$
   (k) $3A$
   (l) $-5y$
   (m) $x + y$
   (n) $x \cdot y$
   (o) AA^T
   (p) $A^T A$

   Solution:

   (a) Ax is a vector with m components.
   (b) xA is not defined unless $m = 1$.
   (c) AB is an $m \times p$ matrix
   (d) AC is not defined because $n \neq m$

(e) BC is not defined because $p \neq m$

(f) $Ax + Bx$ is not defined because Bx is not defined.

(g) $By + Cy$ is not defined: By is a vector with n components but Cy is a vector iwth m components

(h) $(B + C)y$ is not defined: B and C are not the same size.

(i) $(AB)y$ is a vector with m components.

(j) $A(By)$ is a vector with m components.

(k) $3A$ is an $m \times n$ matrix.

(l) $-5y$ is a vector with p components

(m) $x + y$ is not defined because the vectors have different numbers of components.

(n) $x \cdot y$ is not defined because the vectors have different numbers of components.

(o) AA^T is an $m \times m$ matrix.

(p) $A^T A$ is an $n \times n$ matrix.

2. Prove that scalar multiplication distributes over vector addition. That is, prove that if $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^n$, and a is a real number, that

$$a(\mathbf{v} + \mathbf{w}) = a\mathbf{v} + a\mathbf{w}.$$

Solution: **Proof:** Let $\mathbf{v} = (c_1, c_2, \ldots, c_n)$ and $\mathbf{w} = (b_1, b_2, \ldots, b_n)$ be vectors in $\mathbb{R}^n$ and let a be a scalar.

Then

$$\begin{aligned}
a(\mathbf{v} + \mathbf{w}) &= a((c_1, c_2, \ldots, c_n) + (b_1, b_2, \ldots, b_n)) \\
&= a(c_1 + b_1, c_2 + b_2, \ldots, c_n + b_n) && \text{def. of vect. addition} \\
&= (a(c_1 + b_1), a(c_2 + b_2), \ldots, a(c_n + b_n)) && \text{def. of scalar multiplication} \\
&= (ac_1 + ab_1, ac_2 + ab_2, \ldots, ac_n + ab_n) && \text{Axiom (D) for real numbers} \\
&= (ac_1, ac_2, \ldots, ac_n) + (ab_1, ab_2, \ldots, ab_n) && \text{def. of vect. addition} \\
&= a(c_1, c_2, \ldots, c_n) + a(b_1, b_2, \ldots, b_n) && \text{def. of scalar multiplication} \\
&= a\mathbf{v} + a\mathbf{w}.
\end{aligned}$$

$\blacksquare$

3. Let A, B, and C be matrices, x and y be vectors, and c and d be scalars. Use Julia to experiment with some random matrices, vectors, and scalars. Which of the following equations seem to always be true? (Assume in each case that the dimensions of the matrices and vectors are such that the expressions are well-defined)

(a) $(AB)C = A(BC)$

(b) $AB = BA$

(c) $A(x + y) = Ax + Ay$

(d) $A(Bx) = (AB)x$

(e) $A(cx + dy) = c(Ax) + d(Ay)$

(f) $c(A + B) = cA + cB$

(g) $c(AB) = (cA)(cB)$

(h) $A(B + C) = AB + AC$

(i) $(A + B)C = AC + BC$

(j) $(A + B) + C = A + (B + C)$

(k) $A + B = B + A$

Solution:

(a) $(AB)C = A(BC)$ is true.

(b) $AB = BA$ is FALSE!

(c) $A(x + y) = Ax + Ay$ is true.

(d) $A(Bx) = (AB)x$ is true.

(e) $A(cx + dy) = c(Ax) + d(Ay)$ is true.

(f) $c(A + B) = cA + cB$ is true.

(g) $c(AB) = (cA)(cB)$ is FALSE!

(h) $A(B + C) = AB + AC$ is true

(i) $(A + B)C = AC + BC$ is true

(j) $(A + B) + C = A + (B + C)$ is true.

(k) $A + B = B + A$ is true.

4. An $n \times n$ *permutation matrix* is a matrix with exactly one 1 in each row and column, and zeros everywhere else. The *standard basis* (standard set of basis vectors) for $\mathbb{R}^n$ is the set of vectors $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$, where $\mathbf{e}_i$ is the vector with n coordinates, all zero except the ith coordinate is equal to 1. For example the standard basis for $\mathbb{R}^4$ is

$$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\} = \{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}.$$

(a) Compute the dot product $\mathbf{e}_i \cdot \mathbf{e}_j$

- if $i = j$,
- if $i \neq j$.

(b) Find the product $P_1 P_2$ of the following two permutation matrices.

$$P_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, P_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) How are the rows of a $4 \times n$ matrix A permuted by multiplication on the left by P_1, P_2, and $P_1 P_2$ (i.e. which rows of A correspond to which rows of $P_1 A$, etc.)?

(d) For any $n \times n$ permutation matrix P, explain why $PP^T = I$, where I is the $n \times n$ identity matrix.

Solution:

(a)
- If $i = j$, then the dot product contains $n-1$ $0 \cdot 0$ terms, and one $1 \cdot 1$ term, so $\mathbf{e}_i \cdot \mathbf{e}_j = 1$.
- If $i \neq j$, then the dot product contains $n-2$ $0 \cdot 0$ terms and two $0 \cdot 1$ terms, so $\mathbf{e}_i \cdot \mathbf{e}_j = 0$.

(b)

$$P_1P_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

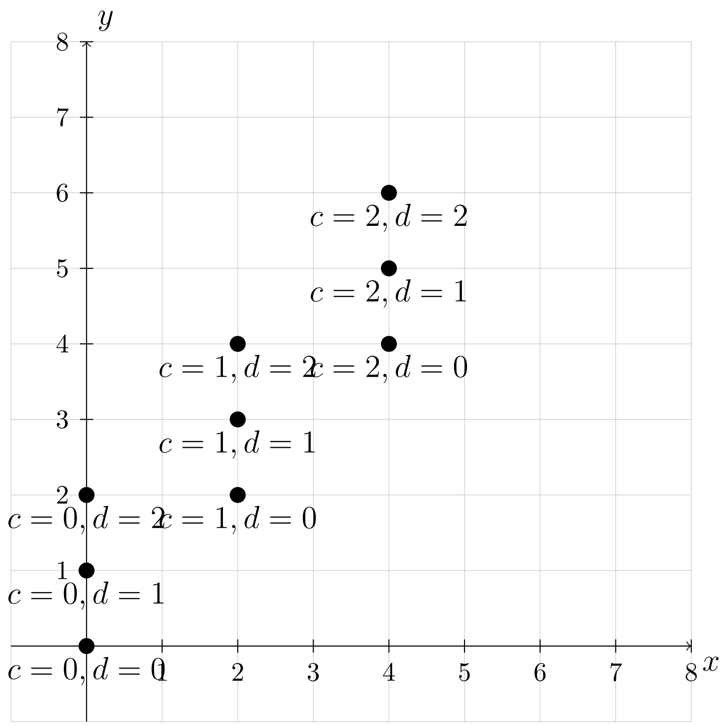
(c) If the rows of A are labeled (top to bottom) R1, R2, R3, and R4, then
- The rows of P_1A are (top to bottom) R2, R4, R1, R3.
- The rows of P_2A are (top to bottom) R2, R3, R1, R4.
- The rows of P_1P_2A are (top to bottom) R3, R4, R2, R1.

(d) Each row of a permutation matrix is a different one of the standard basis vectors $\mathbf{e}_i$. The jth row of P is the jth column of P^T. By Part (a) $(PP^T)_{ij}$ is equal to 1 if and only if $i = j$, since the dot product of two standard basis vectors is 1 if they are the same vector, and 0 otherwise.

5. (Strang 1.1.7) In the xy plane mark all nine of these linear combinations:

$$c \begin{bmatrix} 2 \\ 2 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ with } c = 0, 1, 2 \text{ and } d = 0, 1, 2.$$

Solution:



6. (Strang 1.1.10, see Page 9 of the textbook for a picture) Which point of the cube is $\mathbf{i}+\mathbf{j}$? Which point is the vector sum of $\mathbf{i} = (1,0,0)$ and $\mathbf{j} = (0,1,0)$ and $\mathbf{k} = (0,0,1)$? Describe all points (x,y,z) in the cube.

   Solution:

   - $\mathbf{i}+\mathbf{j} = (1,0,0)+(0,1,0) = (1,1,0)$. This is the point in the lower right of the image.
   - $\mathbf{i}+\mathbf{j}+\mathbf{k} = (1,0,0)+(0,1,0)+(0,0,1) = (1,1,1)$. This is the point one unit above $(1,1,0)$ in the picture.
   - The points (x,y,z) in the cube are those with $0 \le x \le 1$, $0 \le y \le 1$, and $0 \le z \le 1$.

7. (Strang 1.1.13a,b, see Page 9 of the textbook for a picture)

   (a) What is the sum $\mathbf{V}$ of the twelve vectors that go from the center of a clock to the hours 1:00, 2:00, ..., 12:00?

   (b) If the 2:00 vector is removed, why do the 11 remaining vectors add to 8:00?

   Solution:

   (a) Since every vector and its opposite are in the sum, the sum is the vector $(0,0)$.

   (b) In this case every vector is being summed with its opposite, except 8:00, which is the opposite of 2:00.

Optional Problems

8. Verify that the field $\mathbb{F}_2$ satisfies all nine field axioms. Show that set of numbers $\{0, 1, 2, 3\}$ with addition and multiplication $\mod 4$ is not a field. For which positive integers n is the set of numbers $\{0, 1, 2, \ldots, n-1\}$ with addition and multiplication $\mod n$ a field?

9. How many $n \times n$ permutation matrices are there?