**Trevor Leong** **21-241 Fall 2019**
**Homework 5** **Due Friday, September 27**

## Required Problems

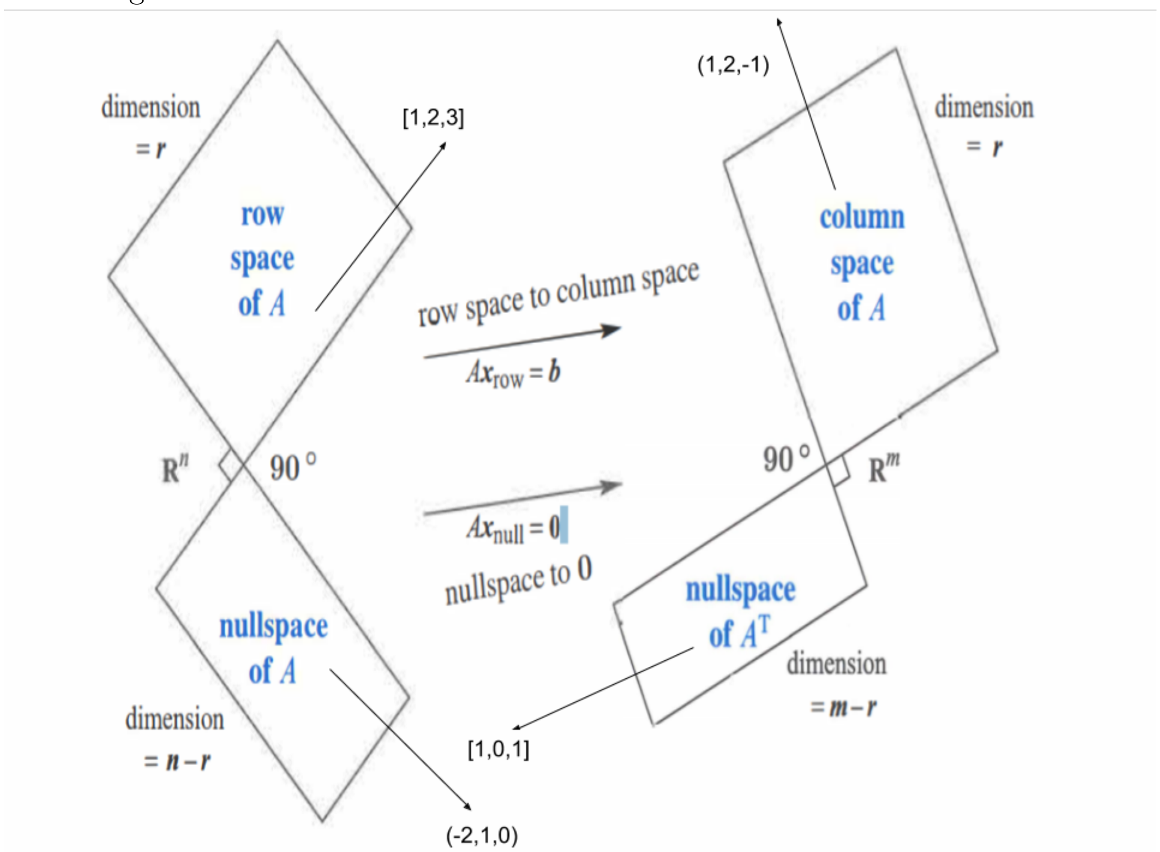
1. (Strang 4.1.1) Construct any 2 by 3 matrix of rank one. Copy Figure 4.2 and put one vector in each subspace (and put two in the nullspace). Which vectors are orthogonal? [Use a different matrix than the one we used in class.]

   *Solution:*

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -1 & -2 & -3 \end{bmatrix}$$

The vector in the row space, $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$, and the vector in the Null space $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ are orthogonal.

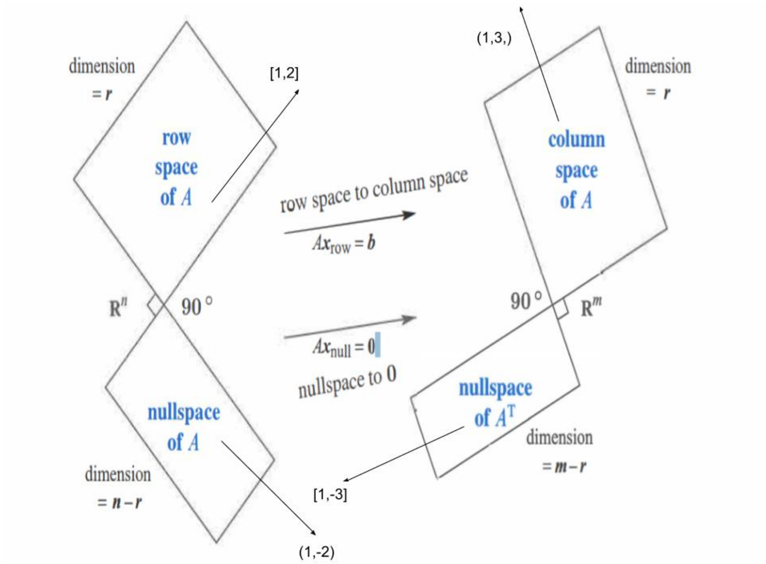
The vector in the column space, $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ ,and the vector in the left null space, $\begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$, are orthogonal.



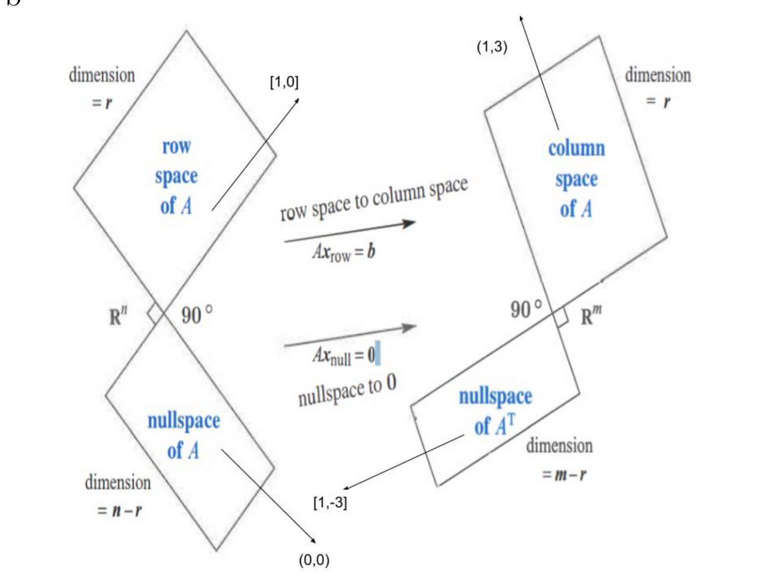
2. (Strang 4.1.11) Draw Figure 4.2 to show each subspace correctly for

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}.$$

*Solution:* a




b



3. (Strang 4.1.17) If $S$ is the subspace of $\mathbb{R}^3$ containing only the zero vector, what is $S^{\perp}$? If $S$ is spanned by $(1,1,1)$, what is $S^{\perp}$? If $S$ is spanned by $(1,1,1)$ and $(1,1,-1)$, what is a basis for $S^{\perp}$?

*Solution:*

(a) If S only contains the zero vector, then $S^{\perp}$ is $\mathbb{R}^3$

.

(b)If S is spanned by $(1,1,1)$:

$$RREF(\begin{bmatrix}1 & 1 & 1\end{bmatrix}) = \begin{bmatrix}1 & 1 & 1\end{bmatrix}$$
$$N(A) = (-1,1,0),(-1,0,1)$$
$$N(A) = C(A)^{\perp}$$

Finally, we have that $S^{\perp} = span((-1,1,0),(-1,0,1))$

.

(c)If S is Spanned by $(1,1,1)$ and $(1,1,-1)$,

$$RREF(\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & -1\end{bmatrix}) = \begin{bmatrix}1 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$
$$N(A) = (-1,1,0)$$
$$N(A) = C(A)^{\perp}$$

Finally, we have that $S^{\perp} = span((-1,1,0)$

4. (from Strang 4.2.6) Project $\mathbf{b} = (1,0,0)$ onto the lines through $\mathbf{a}_1 = (-1,2,2)$, $\mathbf{a}_2 = (2,2,-1)$, and $\mathbf{a}_3 = (2,-1,2)$. Add up the three projections $\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3$. [What do you notice? Note the $\mathbf{a}$'s are orthogonal.]

*Solution:*

$$proj_b(a_1):$$

$$\hat{x}_{a1}a_1 = \frac{a^T b}{a^T a}a = \frac{\begin{bmatrix}-1 & 2 & 2\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}}{\begin{bmatrix}-1 & 2 & 2\end{bmatrix}\begin{bmatrix}-1\\2\\2\end{bmatrix}}\begin{bmatrix}-1\\2\\2\end{bmatrix}$$

$$= \frac{-1}{9}\begin{bmatrix}-1\\2\\2\end{bmatrix} = \begin{bmatrix}\frac{1}{9}\\-\frac{2}{9}\\-\frac{2}{9}\end{bmatrix}.$$

$$\hat{x}_{a2}a_2 = \frac{a^T b}{a^T a}a = \frac{\begin{bmatrix}2 & 2 & -1\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}}{\begin{bmatrix}2 & 2 & -1\end{bmatrix}\begin{bmatrix}2\\2\\-1\end{bmatrix}}\begin{bmatrix}2\\2\\-1\end{bmatrix}$$

$$= \frac{2}{9}\begin{bmatrix}2\\2\\-1\end{bmatrix} = \begin{bmatrix}\frac{4}{9}\\\frac{4}{9}\\-\frac{2}{9}\end{bmatrix}.$$

$$\hat{x}_{a3}a_3 = \frac{a^T b}{a^T a}a = \frac{\begin{bmatrix}2 & -1 & 2\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}}{\begin{bmatrix}2 & -1 & 2\end{bmatrix}\begin{bmatrix}2\\-1\\2\end{bmatrix}}\begin{bmatrix}2\\-1\\2\end{bmatrix}$$

$$= \frac{2}{9}\begin{bmatrix}2\\-1\\2\end{bmatrix} = \begin{bmatrix}\frac{4}{9}\\-\frac{2}{9}\\\frac{4}{9}\end{bmatrix}.$$

$$p_1 + p_2 + p_3 = \begin{bmatrix}\frac{1}{9}\\-\frac{2}{9}\\-\frac{2}{9}\end{bmatrix} + \begin{bmatrix}\frac{4}{9}\\\frac{4}{9}\\-\frac{2}{9}\end{bmatrix} + \begin{bmatrix}\frac{4}{9}\\-\frac{2}{9}\\\frac{4}{9}\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$Notice: span(a_1, a_2, a_3) = \mathbb{R}^3$$

5. (from Strang 4.2.8) Project the vector $\mathbf{b} = (1,1)$ onto the lines through $\mathbf{a}_1 = (1,0)$ and $\mathbf{a}_2 = (1,2)$. Draw the projections $\mathbf{p}_1$ and $\mathbf{p}_2$ and add $\mathbf{p}_1 + \mathbf{p}_2$. The projections do not add to $\mathbf{b}$ because the $\mathbf{a}$'s are not orthogonal.

*Solution:* $\text{proj}_b(a_1)$ :

$$\hat{x}_{a1}a_1 = \frac{a^T b}{a^T a} a = \frac{\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}}{\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}}\begin{bmatrix}1\\0\end{bmatrix}$$

$$= \frac{1}{1}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$$

.

$proj_b(a_2)$ :

$$\hat{x}_{a1}a_1 = \frac{a^T b}{a^T a} a = \frac{\begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}}{\begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}}\begin{bmatrix}1\\2\end{bmatrix}$$

$$= \frac{3}{5}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}\frac{3}{5}\\\frac{6}{5}\end{bmatrix}$$

.

$$p_1 + p_2 = \begin{bmatrix}\frac{3}{5}\\\frac{6}{5}\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\frac{8}{5}\\\frac{6}{5}\end{bmatrix} \neq \begin{bmatrix}1\\1\end{bmatrix}$$

6. (Strang 4.2.21) Multiply the matrix $P = A(A^TA)^{-1}A^T$ by itself. Cancel to prove that $P^2 = P$. Explain why $P(P\mathbf{b})$ always equals $P\mathbf{b}$: The vector $P\mathbf{b}$ is in the column space of $A$ so its projection onto that column space is ...

   *Solution:*

$$
\begin{aligned}
P &= A(A^TA)^{-1}A^T \\
P^2 &= A(A^TA)^{-1}A^TA(A^TA)^{-1}A^T \\
&= A(A^TA)^{-1}IA^T(associative) \\
&= A(A^TA)^{-1}A^T \\
&= P
\end{aligned}
$$

   The vector $Pb$ is in the column space of $A$, so its projection onto that column space has no component perpendicular to itself. Thus, only subtracting out the zero vector would project the vector onto itself. Any non-zero value will cause $Pb$ to not be on the column space, thus violating the condition of a projection.

7. (Strang 4.3.1. See Page 228 for a picture.) With $\mathbf{b} = 0, 8, 8, 20$ at $t = 0, 1, 3, 4$, set up and solve the normal equations $A^T A\hat{\mathbf{x}} = A^T\mathbf{b}$. For the best straight line in Figure 4.9a, find its four heights $p_i$ and four errors $e_i$. What is the minimum value $E = e_1^2 + e_2^2 + e_3^2 + e_4^2$?

*Solution:*

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}, A^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 4 & 8 \\ 8 & 16 \end{bmatrix}, A^T b = \begin{bmatrix} 36 \\ 112 \end{bmatrix}$$

$$A^T A\hat{\mathbf{x}} = A^T\mathbf{b}$$

$$\begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \end{bmatrix}$$

$$\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$Line\ Of\ Best\ Fit:\ 1 + 4t$$

$$p_1 = 1, p_2 = 5, p_3 = 13, p_4 = 17$$

$$error = actual - acquired$$

$$error = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix} - \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix}$$

$$error = \begin{bmatrix} -1 \\ 3 \\ 5 \\ 3 \end{bmatrix}$$

$$E = e_1^2 + e_2^2 + e_3^2 + e_4^2 = error^T error$$

$$= \begin{bmatrix} -1 & 3 & 5 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ 5 \\ 3 \end{bmatrix} = 44$$

8. (Strang 4.3.9) For the closest parabola $b = C + Dt + Et^2$ to the same four points (as in Strang 4.3.1), write down the unsolvable equations $A\mathbf{x} = \mathbf{b}$ in three unknowns $\mathbf{x} = (C, D, E)$. Set up the three normal equations $A^TA\hat{\mathbf{x}} = A^T\mathbf{b}$ (solution not required). In Figure 4.9a you are now fitting a parabola to four points. What is happening in Figure 4.9b?

*Solution:*

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix}, A^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 1 & 9 & 16 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 1 & 9 & 16 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 26 \\ 8 & 26 & 92 \\ 26 & 92 & 338 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 1 & 9 & 16 \end{bmatrix} \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \\ 400 \end{bmatrix}$$

$$A^TA\hat{\mathbf{x}} = A^T\mathbf{b}$$

$$\begin{bmatrix} 4 & 8 & 26 \\ 8 & 26 & 92 \\ 26 & 92 & 338 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \\ 400 \end{bmatrix}.$$

In figure 4.9a, 4 coordinates were being shown to be fit to a parabola. In 4b, the points were assumed to be vectors in $\mathbb{R}^4$. While the illustrations are different, they are both solving the same problem of fitting a point/vector to a gievn subspace.

## Optional Problems

9. (Strang 4.2.2) *Draw* the projection of $\mathbf{b}$ onto $\mathbf{a}$ and also compute it from $\mathbf{p} = \hat{\mathbf{x}}\mathbf{a}$:

$$\text{(a) } \mathbf{b} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \text{ and } \mathbf{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{(b) } \mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \mathbf{a} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

10. Let $\mathbf{v}, \mathbf{w}, \mathbf{x} \in \mathbb{R}^n$, $c, \in \mathbb{R}$.
    (a) Prove that $(c\mathbf{v})^T\mathbf{w} = c(\mathbf{v}^T\mathbf{w})$.
    (b) Prove that $(\mathbf{v} + \mathbf{x})^T\mathbf{w} = \mathbf{v}^T\mathbf{w} + \mathbf{x}^T\mathbf{w}$.

11. Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$. Prove the *Triangle inequality*:

$$\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|.$$

    What does this inequality have to do with triangles?

12. Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$. Prove the *Cauchy-Schwarz inequality*:

$$|\mathbf{v} \cdot \mathbf{w}| \leq \|\mathbf{v}\|\|\mathbf{w}\|.$$

13. Let $A$ be an $m \times n$ matrix. Prove that if $V = \{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is a basis for $C(A^T)$ and $W = \{\mathbf{w}_1, \ldots, \mathbf{w}_{n-r}\}$ is a basis for $N(A)$, then $V \cup W$ is a basis for $\mathbb{R}^n$.

14. Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$.
    (a) Prove the Pythagorean theorem.
    (b) Prove that

$$\|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 = \|\mathbf{v} - \mathbf{w}\|^2$$

    if and only if $\mathbf{v} \perp \mathbf{w}$.

15. Let $\mathbf{v} \in \mathbb{R}^n$. Prove that if $\mathbf{v}^T\mathbf{v} = 0$ then $\mathbf{v} = \mathbf{0}$. Show that this is not true in $\mathbb{C}^n$. Hint: Consider the vector $\mathbf{v} = (1, i) \in \mathbb{C}^2$.