YOUR NAME GOES HERE **21-241 Fall 2019**
Homework 9 **Due Friday, November 1**

Required Problems

1. (Strang 8.2.15)

   (a) What matrix M transforms $(1,0)$ and $(0,1)$ into (r,t) and (s,u)?

   (b) What matrix N transforms (a,c) and (b,d) into $(1,0)$ and $(0,1)$?

   (c) What conditions on a,b,c,d will make part (b) impossible?

   Solution:

   (a) $M\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}r & s\\ t & u\end{bmatrix}$

   $N = \begin{bmatrix}r & s\\ t & u\end{bmatrix}$

   (b) $N\begin{bmatrix}a & b\\ c & d\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

   $N = \begin{bmatrix}r & s\\ t & u\end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$

   (c) if ad-bc = 0, then part (b) will be impossible

2. (Strang 10.3.5) Every year 2% of young people become old and 3% of old people become dead. (No births.) Find the steady state for

$$\begin{bmatrix} \text{young} \\ \text{old} \\ \text{dead} \end{bmatrix}_{k+1} = \begin{bmatrix} .98 & .00 & 0 \\ .02 & .97 & 0 \\ .00 & .03 & 1 \end{bmatrix} \begin{bmatrix} \text{young} \\ \text{old} \\ \text{dead} \end{bmatrix}_{k}$$

Solution:

To find steady state, the eigenvector with a corresponding eigenvalue of 1 must be found.

$$N(A{-}I\lambda) = N(a{-}I(1)) = N(\begin{bmatrix} .98-1 & .00 & 0 \\ .02 & .97-1 & 0 \\ .00 & .03 & 1-1 \end{bmatrix}) = N(\begin{bmatrix} -0.02 & .00 & 0 \\ .02 & -0.03 & 0 \\ .00 & .03 & 0 \end{bmatrix})$$

Now that we have the matrix, we can row reduce in jbox to find the Null space:

$$N(\begin{bmatrix} -0.02 & .00 & 0 \\ .02 & -0.03 & 0 \\ .00 & .03 & 0 \end{bmatrix}) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \ \textit{and} \ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

However, since the steady state cannot contain negative values, the steady state must be:

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Where everybody is dead

3. Roots of a diagonalizable matrix. Suppose A is diagonalizable with nonnegative eignevalues ($A = X\Lambda X^{-1}$), and let $\sqrt{\Lambda}$ be the matrix each of whose entries is the square root of the corresponding entry in Λ. Let $B = X\sqrt{\Lambda}X^{-1}$.

   (a) Show that $B^2 = A$.

   (b) Diagonalize the matrix $C = \begin{bmatrix} 51 & -21 \\ 70 & -26 \end{bmatrix}$.

   (c) Find a matrix D such that $D^2 = C$.

   Solution:

   (a) $B^2 = BB = (X\sqrt{\Lambda}X^{-1})(X\sqrt{\Lambda}X^{-1})$

   $= X\sqrt{\Lambda}X^{-1}X\sqrt{\Lambda}X^{-1}$

   $= X\sqrt{\Lambda}I\sqrt{\Lambda}X^{-1}$

   $= X\sqrt{\Lambda}\sqrt{\Lambda}X^{-1}$

   Claim:$\sqrt{\Lambda}\sqrt{\Lambda} = \Lambda$

   let a_{ij} be an element in A, and $\sqrt{a_{ij}}$ be the corresponding element in $\sqrt{\Lambda}$

   We know:

   $$a_{ij} = \sum_{k=1}^{n} a'_{kj}a'_{ik}$$

   We also know that since Λ is a diagonal, the only non-zero entries are where i = j. This now yields:

   $$a_{ii} = a'_{ii}a'_{ii} = \sqrt{a_{ij}}\sqrt{a_{ij}} = a_{ii}$$

   We can now see that: $\sqrt{\Lambda}\sqrt{\Lambda} = \Lambda$

   Wiht this information, we see that:

   $$B^2 = X\Lambda X^{-1} = A$$

   And we are done

   (b) To diagonalize C, we must find the eigenvalues and eigenvectors for it. We begin by solving $\det(A - I\lambda) = 0$

   $$A - I\lambda = \begin{bmatrix} 51-\lambda & -21 \\ 70 & -26-\lambda \end{bmatrix}$$

   After jbox does the work of solving the characteristic polynomial, We see that the eigenvalues are $\lambda_1 = 9 \; and \; \lambda_2 = 16$. We now solve$N(A - I\lambda)$ to get the eigenvalues, which turn out to be:

   $$\begin{bmatrix} 0.5 \\ 1 \end{bmatrix} \; and \; \begin{bmatrix} 0.6 \\ 1 \end{bmatrix}$$

Now, with these values, we can now say:

$$X = \begin{bmatrix} 0.5 & 0.6 \\ 1 & 1 \end{bmatrix}, X^{-1} = \begin{bmatrix} -10 & 6 \\ 10 & -5 \end{bmatrix}, \Lambda = \begin{bmatrix} 9 & 0 \\ 0 & -16 \end{bmatrix}$$

Combining these, we get that:

$$A = X^{-1} = \begin{bmatrix} 0.5 & 0.6 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & 16 \end{bmatrix} \begin{bmatrix} -10 & 6 \\ 10 & -5 \end{bmatrix}$$

(c) From part (a) we can easily find D:

$$D = X\sqrt{\Lambda}X^{-1} = \begin{bmatrix} 0.5 & 0.6 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} -10 & 6 \\ 10 & -5 \end{bmatrix}$$

4. Consider the permutation matrix

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Find the characteristic polynomial for P. [Hint: You can do this easily by looking at how many ways there are to choose a nonzero entry from each row and column of $P - \lambda I$. Alternatively you could use the Row 4 cofactor formula on $P - \lambda I$ and then use the fact that the determinant of a triangular matrix is the product of the entries on the diagonal.]

(b) Find the steady state vector $\mathbf{v}$ corresponding to the eigenvalue $\lambda = 1$.

(c) Let $\mathbf{u}_0 = (1, 0, 0, 0)$. If $\mathbf{u}_{k+1} = P\mathbf{u}_k$, what are $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, and $\mathbf{u}_4$? Does the sequence $\{\mathbf{u}_k\}$ converge to $\mathbf{v}$?

(d) What are the four eigenvalues for P?

Solution:

(a)

$$\det(P - \lambda I) = \det(\begin{bmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 0 & -\lambda & 1 \\ 1 & 0 & 0 & -\lambda \end{bmatrix})$$

The only ways that a non-zero entry can be chosen in each row and column is when:

$$\begin{bmatrix} -\lambda & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

So, the characteristic polynomial will be:

$$\det(\begin{bmatrix} -\lambda & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{bmatrix}) + \det(\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}) = (-\lambda)^4 - 1 = \lambda^4 - 1$$

(b) To calculate the eigenvector corresponding to 1, we can solve $N(A-I)$ with jbox, yielding:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

(c)

$$\mathbf{u_1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

using similar logic, we can see that:

$$\mathbf{u_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{u_3} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{u_4} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

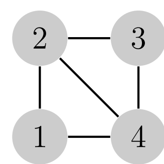
From these results, it appears that the sequence will not converge to P

(d) We can find the eigenvalues by solving $\lambda^4 - 1 = 0$ and see that:

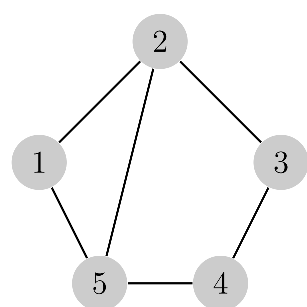
$$\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = i, \lambda_4 = -i$$

5. A random walk on a graph. Suppose a person walks from node to node on a graph, at each time step choosing randomly to move to a neighboring node, each with equal probability. For each of the graphs below, what Markov matrix describes the transition probabilities for the random walk? [For grading purposes, please let Row i and column i correspond to node i, as labeled.] What proportion of the time would we expect to find the random walker at each node?

(a)



(b)



Do you notice any similarity between the steady states for the graphs?

Solution:

(a) Disclaimer: I know $.33 \neq \frac{1}{3}$, but the second is annoying to type.

We can see that the markov matrix for the first one is:

$$\begin{bmatrix} 0 & 0.33 & 0 & 0.33 \\ 0.5 & 0 & 0.5 & 0.33 \\ 0 & 0.33 & 0 & 0.33 \\ 0.5 & 0.33 & 0.5 & 0 \end{bmatrix}$$

Using jbox to solve $N(A - I)$ to find the eigenvector for 1, we get:

$$\mathbf{v_1} = \begin{bmatrix} \frac{2}{3} \\ 1 \\ \frac{2}{3} \\ 1 \end{bmatrix}$$

However, since the sum of these is greater than 1, we can divide by the sum to obtain the true probabilities to see that :

$$probabilities = \begin{bmatrix} 0.2 \\ 0.3 \\ 0.2 \\ 0.3 \end{bmatrix} \implies Prob_1 = 0.2, Prob_2 = 0.3, Prob_3 = 0.2, Prob_4 = 0.3$$

(b) We can see that the markov matrix for the first one is:

$$\begin{bmatrix} 0 & 0.33 & 0 & 0 & 0.33 \\ 0.5 & 0 & 0.5 & 0 & 0.33 \\ 0 & 0.33 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0.33 \\ 0.5 & 0.33 & 0 & 0.5 & 0 \end{bmatrix}$$

Using jbox to solve $N(A - I)$ to find the eigenvector for 1, we get:

$$\mathbf{v_1} = \begin{bmatrix} \frac{2}{3} \\ 1 \\ \frac{2}{3} \\ \frac{2}{3} \\ 1 \end{bmatrix}$$

However, since the sum of these is greater than 1, we can divide by the sum to obtain the true probabilities to see that :

$$probabilities = \begin{bmatrix} \frac{1}{6} \\ 0.25 \\ \frac{1}{6} \\ \frac{1}{6} \\ 0.25 \end{bmatrix} \implies Prob_1 = \frac{1}{6}, Prob_2 = 0.25, Prob_3 = \frac{1}{6}, Prob_4 = \frac{1}{6}, Prob_5 = 0.25$$

(c) After seeing the two steady states, the similarities lie in the presence of the $\frac{2}{3}$. It appears that nodes with an edge connecting them to nodes that are not 1 above or below them have a higher chance of being arrived at. This makes sense, as having an extra edge does not affect the chances of leaving a node, but increase the likelihood of others coming to that node.

6. In this problem we will see how calculating powers of (non-Markov) matrices can help understand recurrence relations. We will use the Fibonacci numbers as an example. The Fibonacci numbers are defined recursively as $F_0 = 0$, $F_1 = 1$, and for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$.

   (a) Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and $\mathbf{u}_0 = (1, 0)$. Note that $\mathbf{u}_0 = (F_1, F_0)$. Prove that $A^k \mathbf{u}_0 = (F_{k+1}, F_k)$.

   (b) Verify that the eigenvalues of A are $\lambda_1 = (1 + \sqrt{5})/2$ and $\lambda_2 = (1 - \sqrt{5})/2$, with corresponding eigenvectors $\mathbf{x}_1 = (\lambda_1, 1)$ and $\mathbf{x}_2 = (\lambda_2, 1)$.

   (c) Verify that $\mathbf{u}_0 = \frac{1}{\lambda_1 - \lambda_2}(\mathbf{x}_1 - \mathbf{x}_2)$. [Note you have expressed $\mathbf{u}_0$ in the basis of eigenvectors.]

   (d) Write a simple expression for $\mathbf{u}_k = A^k \mathbf{u}_0$ in terms of $\mathbf{x}_1$ and $\mathbf{x}_2$. [Hint: Use the expression for $\mathbf{u}_0$ in Part (c).]

   (e) Since F_k is the second component of $\mathbf{u}_k$ and the second components of $\mathbf{x}_1$ and $\mathbf{x}_2$ are 1, write a simple expression for F_k in terms of λ_1 and λ_2.

   (f) Explain why F_{100} is extremely close to $\frac{1}{\sqrt{5}}\lambda_1^{100}$. (Note that $\lambda_1 - \lambda_2 = \sqrt{5}$),

Solution:

(a) Prove using induction:

Base case: k = 0

$$A^0 \mathbf{u_0} = I \mathbf{u_0} = (F_1, F_0)$$

Induction Step: Assume $\forall 0 \geq k \leq n$.

W.T.S:

$$A^{k+1} \mathbf{u_0} = (F_{k+2}, F_{k+1})$$

We know by algebra that:

$$A^{k+1} \mathbf{u_0} = A A^k \mathbf{u_0}$$

We can now invoke our induction hypothesis to see that:

$$A A^k \mathbf{u_0} = A \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$$

Multiplying these two yields:

$$\begin{bmatrix} F_{k+1} + F_k \\ F_{k+1} \end{bmatrix}$$

We can then use the definition of fibonacci numbers to see that:

$$\begin{bmatrix} F_{k+1} + F_k \\ F_{k+1} \end{bmatrix} = (F_{k+2}, F_{k+1})$$

Which is what we wanted. By the induction hypothesis, we are done

(b) To find the eigenvalues, we will solve: $\det(A - I\lambda) = 0$ Plugging these into jbox, we see that the eigenvalues are indeed:

$$\lambda_1 = \frac{1}{2}(1+\sqrt{5}), \lambda_2 = \frac{1}{2}(1-\sqrt{5})$$

and that the Solutions to $N(A - \lambda I)$ are:

$$\mathbf{x_1} = \begin{bmatrix} \frac{1}{2}(1+\sqrt{5}) \\ 1 \end{bmatrix}, \mathbf{x_2} = \begin{bmatrix} \frac{1}{2}(1-\sqrt{5}) \\ 1 \end{bmatrix}$$

Which are in the form: $(\lambda_1, 1)$ *and* $(\lambda_2, 1)$

(c)

$$\frac{1}{\lambda_1 - \lambda_2}(\mathbf{x}_1 - \mathbf{x}_2) = \frac{1}{\frac{1}{2}(1+\sqrt{5}) - \frac{1}{2}(1+\sqrt{5})} \begin{bmatrix} \frac{1}{2}(1+\sqrt{5}) - \frac{1}{2}(1-\sqrt{5}) \\ 1-1 \end{bmatrix}$$

$$= \frac{1}{2(\frac{\sqrt{5}}{2})} \begin{bmatrix} 2(\frac{\sqrt{5}}{2}) \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \mathbf{u_0}$$

(d) $\mathbf{u}_k = A^k \mathbf{u}_0 = A^k \frac{1}{\lambda_1 - \lambda_2}(\mathbf{x}_1 - \mathbf{x}_2) = A^k \frac{1}{\lambda_1 - \lambda_2}\mathbf{x_1} - A^k \frac{1}{\lambda_1 - \lambda_2}\mathbf{x_2}$

$$= \frac{1}{\lambda_1 - \lambda_2} A^k \mathbf{x_1} - \frac{1}{\lambda_1 - \lambda_2} A^k \mathbf{x_2}$$

$$= \frac{1}{\lambda_1 - \lambda_2} \lambda_1^k \mathbf{x_1} - \frac{1}{\lambda_1 - \lambda_2} \lambda_2^k \mathbf{x_2}$$

$$= \frac{\lambda_1^k}{\lambda_1 - \lambda_2} \mathbf{x_1} - \frac{\lambda_2^k}{\lambda_1 - \lambda_2} \mathbf{x_2}$$

(e)

$$F_k = \frac{\lambda_1^k}{\lambda_1 - \lambda_2} - \frac{\lambda_2^k}{\lambda_1 - \lambda_2}$$

$$= \frac{1}{\lambda_1 - \lambda_2}(\lambda_1^k - \lambda_2^k)$$

(f) From before, we know that $F_k = \frac{(\frac{1}{2}(1+\sqrt{5}))^k}{\sqrt{5}} - \frac{(\frac{1}{2}(1-\sqrt{5}))^k}{\sqrt{5}}$ The expression on the left,

$$\frac{(\frac{1}{2}(1+\sqrt{5}))^k}{\sqrt{5}}$$

Will not go to 0, as the numerator > 1 . On the other hand, the expression on the right,

$$\frac{(\frac{1}{2}(1-\sqrt{5}))^k}{\sqrt{5}}$$

Approaches 0 ,as $\mid (\frac{1}{2}(1-\sqrt{5})) \mid < 1$. So, since 100 is large, our original expression gets close to:

$$\frac{1}{\sqrt{5}} \lambda_1^{100} - 0$$

7. Let M be an $n \times n$ positive Markov matrix, with n independent eigenvectors $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$, and n eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, where $\lambda_1 = 1$ and $|\lambda_i| < 1$ when $i \geq 2$.

   (a) Verify that $\mathbf{1} = (1, 1, \ldots, 1)$ is in the left nullspace of $M - I$.

   (b) Verify that $\mathbf{x}_i$ is in the column space of $M - I$ if $i \geq 2$.

   (c) Fill in the blank: Since the column space of $(M - I)$ is __________ the left nullspace of $(M - I)$, $\mathbf{x}_i$ is __________ $(1, 1, \ldots, 1)$ if $i \geq 2$.

   (d) If $i \geq 2$, explain why the components of $\mathbf{x}_i$ sum to zero. [Hint: Use Part (c) and the definition of orthogonal.]

   (e) For any $\mathbf{v} \in \mathbb{R}^n$, let $\mathbf{v} = c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n$. Explain why the coefficient c_1 is the same for all vectors $\mathbf{v}$ whose components have a given sum.

Solution:

(a) Since the columns of of a markov matrix M sum to 1, this means that the rows of M^T sum to 1. This means that $(M^T - I^T)\mathbf{1} = 0$ and therefore $(M - I)^T\mathbf{1} = 0$, which means that $\mathbf{1}$ is in $LN(M^T)$

(b) Since we know that $\lambda_i, i \geq 2$, we know that the only nonzero vector in $N(M - I)$ is associated with $\lambda_1 = 1$. Since i must be > 1, the eigenvector must be different. That being said, since M-I and $(M-I)^T$ share the same eigenvalues, the only nonzero vector in $N(M-I)^T$ must be associated with λ_1. Thus, since it is not associated with λ_i, the vector$\mathbf{x_i}$ is not in the left null space and therefore in the column space.

(c) first blank: is orthogonal, second blank: is also orthogonal

(d) Since the left Null space and column space are orthogonal, the dot product of a vector in the column space and a vector in the null space has to be 0. Thus, since we saw in part(a) that $\mathbf{1}$ was in the left null space, and we proved in part(b) that $\mathbf{x_i}$ was in the column space, $\mathbf{1} \cdot \mathbf{x_i} = 0$. The only wasy this is true is if the components of $\mathbf{x_i}$ sum to 0.

(e) if:

$$\mathbf{v} = c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n$$

Then

$$\mathbf{1} \cdot \mathbf{v} = \mathbf{1} \cdot (c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n)$$

And

$$\mathbf{1} \cdot \mathbf{v} = \mathbf{1} \cdot c_1\mathbf{x}_1 + \cdots + \mathbf{1} \cdot c_n\mathbf{x}_n$$

But from the previous problem, we know that the components of $x_2.....x_n$ sum to 0. This means that:

$$\mathbf{1} \cdot \mathbf{v} = \mathbf{1} \cdot c_1\mathbf{x}_1 + \cdots + 0 + 0 + 0 \cdot c_n\mathbf{x}_n$$

Which means that

$$\mathbf{1} \cdot \mathbf{v} = \mathbf{1} \cdot c_1\mathbf{x}_1$$

But since $\mathbf{x_1} is\ fixed$ We can say:

$$\mathbf{1} \cdot \mathbf{v} = \mathbf{1} \cdot c_1 \mathbf{x_1}$$

Which means that for a given sum of coefficients, there is only one possible c_1 that will satisfy the above equation.

Optional Problems

8. Prove that if M is a Markov matrix, then the sum of the components of a vector $\mathbf{v}$ is the same as the sum of the components of $M\mathbf{v}$.

9. Prove that if M is a positive Markov matrix, the columns of M^k converge to the eigenvector $\mathbf{x}_1$ with eigenvalue 1, whose components sum to 1.

10. Consider a random walk on the integers from 1 to n, where a random walker on i moves to $i+1$ or $i-1$ with equal probability unless they are at 1 or n, in which case they move to 2 or $n-1$ with probability 1. Find the stationary distribution for various values of n. Can you make a conjecture for the form of the stationary distribution for large values of n?