YOUR NAME GOES HERE | **21-241 Fall 2019**
Homework 8 | **Due Friday, October 25**

Required Problems

1. (Strang 6.1.3) Compute the eigenvalues and eigenvectors of A and A^{-1}.

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \text{ and } A^{-1} = \begin{bmatrix} -1/2 & 1 \\ 1/2 & 0 \end{bmatrix}.$$

A^{-1} has the   eigenvectors as A. When A has eigenvalues λ_1 and λ_2, its inverse has eigenvalues   .

Solution: eigenvalues of A consists of solving the characteristic polynomial $\det(A - \lambda I)$

$$\det(A - \lambda I) = (-\lambda)(1 - \lambda) - 2 = \lambda^2 - \lambda - 2$$

We now have

$$\lambda_1 = 2, \lambda_2 = -1$$

To solve for eigenvectors, we need to find solutions to:

$$N(A - \lambda I)$$

Calculations then yield,

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{v_1} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

eigenvalues of A^{-1} consists of solving the characteristic polynomial

$$\det(A^{-1} - \lambda I)$$

$$\det(A^{-1} - \lambda I) = (-\frac{1}{2} - \lambda)(-\lambda) - \frac{1}{2} = \lambda^2 + \frac{1}{2}\lambda - \frac{1}{2}$$

We now have

$$\lambda_1 = \frac{1}{2}, \lambda_2 = -1$$

To solve for eigenvectors, we need to find solutions to:

$$N(A^{-1} - \lambda I)$$

Calculations then yield,

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{v_1} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

We can now complete the statement above:

A^{-1} has the SAME eigenvectors as A
A^{-1} has eigenvalues $\frac{1}{\lambda}$

2. (Strang 6.1.12) Find three eigenvectors for this matrix P (Projection matrices have $\lambda = 1$ and 0.):

$$\textbf{Projection matrix} \qquad P = \begin{bmatrix} .2 & .4 & 0 \\ .4 & .8 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If two eigenvectors share the same λ, so do all of their linear combinations. Find an eigenvector of P with no zero components.

Solution:
Assuming the only two eigenvalues are 0 and 1:
.
Solve for $N(A - \lambda I)$:
.

$$\begin{bmatrix} .2-\lambda & .4 & 0 \\ .4 & .8-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} .2 & .4 & 0 \\ .4 & .8 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies \mathbf{v_1} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -.8 & .4 & 0 \\ .4 & -.2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies \mathbf{v_2} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{v_2} + \mathbf{v_3} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

Since all linear combinations of eigenvectors with the same eigenvalue are also eigenvectors of P, then adding the two above yields an eigenvector of P with no 0 compnents.

3. (Strang 6.1.16) The determinant of A equals the product $\lambda_1\lambda_2\cdots\lambda_n$. Start with the polynomial $\det(A-\lambda I)$ separated into its n factors (always possible). Then set $\lambda = 0$:

$$\det(A-\lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda) \text{ so } \det A =$$

Check this rule in Example 1 where the Markov matrix has $\lambda = 1$ and $1/2$.

Solution:

$$\det(A-\lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda)$$

=

$$(\lambda_1 - 0)(\lambda_2 - 0)\cdots(\lambda_n - 0)$$

=

$$(\lambda_1)(\lambda_2)(\lambda_3)(\lambda_4)\cdots(\lambda_n)$$

=

$$\prod_{i=0}^{n} \lambda_i$$

$$A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}$$

$$\det A = ad - bc = .56 - 0.06 = 0.5 = \frac{1}{2}$$

$$(\lambda_1)(\lambda_2) = (1)(\frac{1}{2}) = \frac{1}{2}$$

4. (Strang 6.1.25) Suppose A and B have the same eigenvalues $\lambda_1, \ldots, \lambda_n$ with the same independent eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$. Then $A = B$. Reason: Any vector $\mathbf{x}$ is a linear combination $c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n$. What is $A\mathbf{x}$? What is $B\mathbf{x}$?

Solution:

.

Assume A and B have identical eigenvalues and eigenvalues

Since x can be written as a linear combination of eigenvectors, we can express it as

$$\mathbf{x} = c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n$$

$$A\mathbf{x} = A(c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n)$$

$$= Ac_1\mathbf{x_1} + Ac_2\mathbf{x_2} + \cdots Ac_n\mathbf{x_n}(linearity)$$

$$= c_1(A\mathbf{x_1}) + c_2(A\mathbf{x_2}) + \cdots c_n(A\mathbf{x_n})(associativity))$$

$$= c_1(\lambda_1\mathbf{x_1}) + c_2(\lambda_2\mathbf{x_2}) + \cdots c_n(\lambda_n\mathbf{x_n})(property\ of\ eigenvalues)$$

$$= c_1(B\mathbf{x_1}) + c_2(B\mathbf{x_2}) + \cdots c_n(B\mathbf{x_n})(Assumption)$$

$$= Bc_1\mathbf{x_1} + Bc_2\mathbf{x_2} + \cdots Bc_n\mathbf{x_n}(associativity)$$

$$= B(c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n)$$

$$= B\mathbf{x}$$

$$A\mathbf{x} = B\mathbf{x}$$

$$A = B$$

5. (Strang 6.2.16) Find Λ and X to diagonalize $A = \begin{bmatrix} .6 & .9 \\ .4 & .1 \end{bmatrix}$ (in Strang 6.2.15). What is the limit of Λ^k as $k \to \infty$? What is the limit of $X\Lambda^k X^{-1}$? In the columns of this limiting matrix you see the ______.

Solution:

To find Λ, we must first find the eigenvalues of A and solve $\det A - \lambda I$

$$\det A - \lambda I = (.6 - \lambda)(.1 - \lambda) - (.9)(.4) = \lambda^2 - .7\lambda - .3$$

$$\lambda_1 = 1, \lambda_2 = -\frac{3}{10}$$

Plugging this back to find our eigenvectors to solve $N(A - \lambda I)$

$$\begin{bmatrix} .6 - \lambda & .9 \\ .4 & .1 - \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{v_1} = \begin{bmatrix} 9 \\ 4 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

We now know that:

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & -.3 \end{bmatrix}$$

With this we can say that

$$\lim_{k->\infty} \Lambda^k = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

We can apply this information to find $\lim_{k->\infty} X\Lambda^k X^{-1}$

$$\lim_{k->\infty} X\Lambda^k X^{-1} = \begin{bmatrix} 9 & -1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{13} & \frac{1}{13} \\ -\frac{4}{13} & \frac{9}{13} \end{bmatrix} = \begin{bmatrix} 9 & -1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{13} & \frac{1}{13} \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{9}{13} & \frac{9}{13} \\ \frac{4}{13} & \frac{4}{13} \end{bmatrix} = \frac{1}{13} \begin{bmatrix} 9 & 9 \\ 4 & 4 \end{bmatrix}$$

In the columns of the limiting matrix, we see the eigenvector corresponding to eigenvalue 1.

6. (Strang 6.2.29) Suppose the same X diagonalizes both A and B. They have the same eigenvectors in $A = X\Lambda_1 X^{-1}$ and $B = X\Lambda_2 X^{-1}$. Prove that $AB = BA$.

   Solution:
   Begin with:

$$\begin{aligned} AB &= (X\Lambda_1 X^{-1})(X\Lambda_2 X^{-1}) \\ &= X\Lambda_1 (X^{-1}X)\Lambda_2 X^{-1} \\ &= X\Lambda_1 I \Lambda_2 X^{-1} \\ &= X\Lambda_1 \Lambda_2 X^{-1} \end{aligned}$$

   And Since A and B have the same X, $\Lambda_1 = \Lambda_2$

$$= X\Lambda^2 X^{-1}$$

   Similarly:

$$\begin{aligned} BA &= (X\Lambda_1 X^{-1})(X\Lambda_2 X^{-1}) \\ &= X\Lambda_1 (X^{-1}X)\Lambda_2 X^{-1} \\ &= X\Lambda_1 I \Lambda_2 X^{-1} \\ &= X\Lambda_1 \Lambda_2 X^{-1} \\ &= X\Lambda^2 X^{-1} \end{aligned}$$

   Finally, we have that

$$\begin{aligned} X\Lambda^2 X^{-1} &= X\Lambda^2 X^{-1} \\ AB &= BA \end{aligned}$$

7. Let A be a matrix, and $\mathbf{x}$ and $\mathbf{y}$ be eigenvectors for A. Prove or disprove each of the following statements.

   (a) For all scalars $c \neq 0$, the vector $c\mathbf{x}$ is an eigenvector for A.

   (b) For all integers $k \geq 1$, x is an eigenvector for A^k.

   (c) The vector $\mathbf{x} + \mathbf{y}$ is always an eigenvector for A.

   Solution:

   (a) True
   An eigenvector must satisfy $A\mathbf{x} = \lambda\mathbf{x}$
   Checking $c\mathbf{x}$, We then have $Ac\mathbf{x}$ *and* $\lambda c\mathbf{x}$

   By associativiy we have:
   $c(A\mathbf{x})$ *and* $c(\lambda\mathbf{x})$

   And since $A\mathbf{x} = \lambda\mathbf{x}$
   We have that $Ac\mathbf{x} = \lambda c\mathbf{x}$

   (b) True
   Proof by induction:

   Base case n = 1:

   True, because it was given that x was an eigenvalue of A

   Induction Step:
   $A^n\mathbf{x} = \lambda^n\mathbf{x}$(Induction hypothesis)

   $= A(A^n)\mathbf{x} = A\lambda^n\mathbf{x}$(Induction Hypothesis)

   $A^{n+1}\mathbf{x} = \lambda^n(A\mathbf{x})$

   $A^{n+1}\mathbf{x} = \lambda^n(\lambda\mathbf{x})$
   $A^{n+1}\mathbf{x} = \lambda^{n+1}\mathbf{x}$

   Since there exists a lambda, s.t. $A^{n+1}\mathbf{x} = \lambda\mathbf{x}$, by the induction hypothesis, our proof is complete.

   (c) False
   Consider
   $$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$
   This matrix has eigenvalues 2,3 and eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

   The sum of these 2, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not an eigenvector, as $A\mathbf{x} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, and there is no such scalar λ that exists which when multipiled by 1 gives 2 and 3.

Optional Problems

8. Prove that if A_1 is similar to A_2 and A_2 is similar to A_3, then A_1 is similar to A_3.

9. Prove or disprove:

   (a) If $\mathbf{x}$ is an eigenvector for A and B, then $\mathbf{x}$ is an eigenvector for AB and BA.

   (b) If λ is an eigenvalue for A and B, then λ^2 is an eigenvalue for AB and BA.

10. List all matrices that are similar to the identity matrix.

11. Prove that the eigenvalues of a triangular matrix are the entries on the diagonal.

12. The trace of a matrix is the sum of the diagonal entries. Prove that the sum of the eigenvalues is equal to the trace.

13. Suppose $\mathbf{x}_1$ and $\mathbf{x}_2$ are eigenvectors for A with eigenvalues λ_1 and λ_2. Under what conditions on λ_1 and λ_2 is $\mathbf{x}_1 + \mathbf{x}_2$ an eigenvector for A?

14. We have seen how it is possible to find eigenvalues and eigenvectors of a matrix by finding roots of its characteristic polynomial. In this problem you will show how to do the reverse: You can find the roots of a polynomial by finding the eigenvectors of its "companion matrix." Let p be the degree n polynomial $p(z) = c_0 + c_1 z + c_2 z^2 + \cdots + z^n$.

    Note that the coefficient of z^n is 1.

    Define the companion matrix for p to be the $n \times n$ matrix

$$C = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 & 1 \\ -c_0 & -c_1 & \dots & -c_{n-2} & -c_{n-1} \end{bmatrix}.$$

   (a) Show that $\det(C - \lambda I) = p(\lambda)$.

   (b) Prove that z is a root of p if and only if it is an eigenvalue of C with eigenvector $(1, z, z^2, \dots, z^{n-1})$.

   (c) Explain how to determine the roots of any degree n polynomial (even if its leading coefficient is not 1) if you know how to find eigenvectors for a matrix. [This is actually how some polynomial solvers proceed: Rather than solving the polynomial they instead find the eigenvectors of its companion matrix.]

15. Write Julia code to recursively calculate the determinant of any $n \times n$ matrix using cofactors.