Required Problems

1. (Strang 6.1.3) Compute the eigenvalues and eigenvectors of A and A^{-1}.

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \text{ and } A^{-1} = \begin{bmatrix} -1/2 & 1 \\ 1/2 & 0 \end{bmatrix}.$$

A^{-1} has the eigenvectors as A. When A has eigenvalues λ_1 and λ_2, its inverse has eigenvalues .

2. (Strang 6.1.12) Find three eigenvectors for this matrix P (Projection matrices have $\lambda = 1$ and 0.):

$$\textbf{Projection matrix} \qquad P = \begin{bmatrix} .2 & .4 & 0 \\ .4 & .8 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If two eigenvectors share the same λ, so do all of their linear combinations. Find an eigenvector of P with no zero components.

3. (Strang 6.1.16) The determinant of A equals the product $\lambda_1 \lambda_2 \cdots \lambda_n$. Start with the polynomial $\det(A - \lambda I)$ separated into its n factors (always possible). Then set $\lambda = 0$:

$$\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda) \text{ so } \det A =$$

Check this rule in Example 1 where the Markov matrix has $\lambda = 1$ and $1/2$.

4. (Strang 6.1.25) Suppose A and B have the same eigenvalues $\lambda_1, \ldots, \lambda_n$ with the same independent eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$. Then $A = B$. Reason: Any vector $\mathbf{x}$ is a linear combination $c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n$. What is $A\mathbf{x}$? What is $B\mathbf{x}$?

5. (Strang 6.2.16) Find Λ and X to diagonalize $A = \begin{bmatrix} .6 & .9 \\ .4 & .1 \end{bmatrix}$ (in Strang 6.2.15). What is the limit of Λ^k as $k \to \infty$? What is the limit of $X\Lambda^k X^{-1}$? In the columns of this limiting matrix you see the .

6. (Strang 6.2.29) Suppose the same X diagonalizes both A and B. They have the same eigenvectors in $A = X\Lambda_1 X^{-1}$ and $B = X\Lambda_2 X^{-1}$. Prove that $AB = BA$.

7. Let A be a matrix, and $\mathbf{x}$ and $\mathbf{y}$ be eigenvectors for A. Prove or disprove each of the following statements.

   (a) For all scalars $c \neq 0$, the vector $c\mathbf{x}$ is an eigenvector for A.
   (b) For all integers $k \geq 1$, x is an eigenvector for A^k.
   (c) The vector $\mathbf{x} + \mathbf{y}$ is always an eigenvector for A.

Optional Problems

8. Prove that if A_1 is similar to A_2 and A_2 is similar to A_3, then A_1 is similar to A_3.

9. Prove or disprove:

   (a) If $\mathbf{x}$ is an eigenvector for A and B, then $\mathbf{x}$ is an eigenvector for AB and BA.

   (b) If λ is an eigenvalue for A and B, then λ^2 is an eigenvalue for AB and BA.

10. List all matrices that are similar to the identity matrix.

11. Prove that the eigenvalues of a triangular matrix are the entries on the diagonal.

12. The trace of a matrix is the sum of the diagonal entries. Prove that the sum of the eigenvalues is equal to the trace.

13. Suppose $\mathbf{x}_1$ and $\mathbf{x}_2$ are eigenvectors for A with eigenvalues λ_1 and λ_2. Under what conditions on λ_1 and λ_2 is $\mathbf{x}_1 + \mathbf{x}_2$ an eigenvector for A?

14. We have seen how it is possible to find eigenvalues and eigenvectors of a matrix by finding roots of its characteristic polynomial. In this problem you will show how to do the reverse: You can find the roots of a polynomial by finding the eigenvectors of its "companion matrix." Let p be the degree n polynomial $p(z) = c_0 + c_1 z + c_2 z^2 + \cdots + z^n$.

    Note that the coefficient of z^n is 1.

    Define the companion matrix for p to be the $n \times n$ matrix

$$C = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 & 1 \\ -c_0 & -c_1 & \dots & -c_{n-2} & -c_{n-1} \end{bmatrix}.$$

   (a) Show that $\det(C - \lambda I) = p(\lambda)$.

   (b) Prove that z is a root of p if and only if it is an eigenvalue of C with eigenvector $(1, z, z^2, \dots, z^{n-1})$.

   (c) Explain how to determine the roots of any degree n polynomial (even if its leading coefficient is not 1) if you know how to find eigenvectors for a matrix. [This is actually how some polynomial solvers proceed: Rather than solving the polynomial they instead find the eigenvectors of its companion matrix.]

15. Write Julia code to recursively calculate the determinant of any $n \times n$ matrix using cofactors.