1. Algebraic theory - basic terms

- **Basic terms**

- Positive powers of z

$$S(z) = \frac{b(z)}{a(z)} = \frac{b_m z^m + b_{m-1} z^{m-1} + \cdots + b_0}{z^n + a_{n-1} z^{n-1} + \cdots + a_0}$$

Causality: Discrete system is causal if the degree of numerator polynomial $b(z)$ is lower or equal to the degree of denominator polynomial $a(z)$, $m \leq n$.

- Negative powers of z^{-1}

$d = z^{-1}$

$$S(z^{-1}) = \frac{b(z^{-1})}{a(z^{-1})} = \frac{b_m z^{m-n} + b_{m-1} z^{m-1-n} + \cdots + b_0 z^{-n}}{1 + a_{n-1} z^{-1} + \cdots + a_0 z^{-n}}$$

The operator z^{-1} is often replaced by operator d

$$S(d) = \frac{b(d)}{a(d)} = \frac{b_m d^{n-m} + \cdots + b_0 d^n}{1 + a_{n-1} d + \cdots + a_1 d^{n-1} + a_0 d^n}$$

Causality: Discrete system $S(z^{-1})$ in negative powers of z (z^{-1} or d) is causal when the absolute term of the denominator is non-zero.

Characteristic polynomial Δ - denominator polynomial of the closed loop transfer function in positive powers of z. $\quad S(z) = \frac{b(z)}{a(z)} \Rightarrow \Delta = a(z)$

The absolute value of roots gives the decision on stability.

Pseudocharacteristic polynomial – denominator polynomial of the closed-loop transfer function in negative powers of z (z^{-1} or d). The roots are inverse of the roots of the characteristic polynomial. Roots from origin shift away to infinity.

Degree of polynomial:

$$a = a_0 + a_1 d + \cdots + a_n d^n \Rightarrow n = \deg a$$

degree of the polynomial is equal to n.

- **Divide polynomials:**

The polynomial b is the divisor of polynomial a if there exist polynomial c such that $a = bc$ (marked also as $a|b$ (b divides a)).

For given polynomials a, b we can always find polynomials u, v such that the following holds

$$a = bu + v\ ,$$

where $\deg v < \deg b$, u is quotient and v is the reminder.

Example 1: Divide polynomials a, b where $a = 2 + 3d + d^2$ a $b = 4 + d$.
Determine the reminder v.

$$\begin{array}{l} d^2 + 3d + 2 \ \div d + 4 = d - 1 + \frac{6}{d+4} \\ -d^2 - 4d \\ \qquad -d + 2 \\ \qquad\ \ d + 4 \\ \qquad\qquad 6 \end{array} \qquad\qquad \begin{array}{l} u = d - 1 \\ v = 6 \end{array}$$

- **Polynomials - GCD and LCM**

For two arbitrary polynomials a, b exist Greatest Common Divisor (GCD) g and Least Common Multiple (LCM) l.

GCD

- the expression can be expressed as a product of the smallest divisors = the numbers decompose into prime numbers,
- from them select the prime factors in the maximum common power

<u>Find greatest common divisor GCD of numbers: 78; 130; 132</u>

78 = <u>2</u> * 3 * <u>13</u>
130 = <u>2</u> * 5 * <u>13</u>
182 = <u>2</u> * 7 * <u>13</u>

GCD (78; 130; 182) = 2 * 13 = 26

LCM

- the expression can be expressed as a product of the smallest divisors = the numbers decompose into prime numbers,
- from them select the prime factors in the maximum power

<u>Find least common multiple LCM of numbers: 6; 12; 14; 35</u>

6 = 2 * <u>3</u>
12 = <u>2</u> * <u>2</u> * 3
14 = 2 * <u>7</u>
35 = <u>5</u> * 7

LCM (6; 12; 14; 35) = 3 * 2 * 2 * 7 * 5 = 420

Example 2: Find GCD g and LCM l of the polynomials a, b.
$a = 1 + 3d + 3d^2 + d^3$
$b = 2 + 3d + d^2$

Hint: $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
$(x + r)(x + s) = x^2 + (s + r)x + r \cdot s$

$a = (d + 1)^3$
$b = (d + 1)(d + 2)$

$g = d + 1$
$l = (d + 1)^3(d + 2)$

- **Coprime polynomials**

For two arbitrary polynomials a, b exist two couples of coprime polynomials p, q and r, s such that

$ap + bq = g$
$ar + bs = 0$
$l = ar = -bs$

In matrix representation:

$$\begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} g \\ 0 \end{bmatrix}$$

Bezout's identity - Test for coprime polynomials a and b.

$$ap + bq = 1 = g$$

Extended Euclidean algorithm for computation of GCD and of two couples of polynomials p, q and r, s.

$$\begin{bmatrix} a & b \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} g & 0 \\ p & r \\ q & s \end{bmatrix}$$

Matrix on the left can be using allowed column operations converted to the matrix on the right.

Allowed column operations:

- columns can be exchanged
- arbitrary column can be multiplied with non-zero constant
- any column can be multiplied with an arbitrary polynomial and the result can be added to the second column

Example 3: Find GCD of polynomials a, b.

$a = 3 + 13d + 6d^2$

$b = 2 + d$

Using the extended Euclidean algorithm, we can write

$$\begin{bmatrix} 3+13d+6d^2 & 2+d \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \sim (1c. -2c.* 6d) \sim \begin{bmatrix} 3+d & 2+d \\ 1 & 0 \\ -6d & 1 \end{bmatrix} \sim (1c. -2c.) \sim$$

$$\sim \begin{bmatrix} 1 & 2+d \\ 1 & 0 \\ -6d-1 & 1 \end{bmatrix} \sim (2c. -1c.* (2+d)) \sim \begin{bmatrix} 1 & 0 \\ 1 & -2-d \\ -6d-1 & 6d^2+13d+3 \end{bmatrix}$$

$g = 1 \Rightarrow$ Polynomials do not have common factors.

- **Stability of polynomial**

The polynomial $a(z)$ is stable when its roots α_i are located inside the unit circle $|\alpha_i| < 1$. The roots of polynomial $a(d)$ are $\lambda_i = \frac{1}{\alpha_i}$, the polynomial $a(d)$ is stable when roots λ_i are located outside the unit circle $|\lambda_i| > 1$.

Example 4: Determine the stability of a.

$a = 3d^2 + 4d + 1$

$$\lambda_{1,2} = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a} = \frac{-4 \mp \sqrt{4^2 - 4 * 3 * 1}}{2 * 3} = \frac{-4 \mp 2}{6}$$

$\lambda_1 = -1$

$\lambda_2 = -\frac{1}{3}$

$|\lambda_i| \leq 1$ - polynomial is not stable

- **Reciprocal polynomial** $\overline{a} = a_n + a_{n-1}d + \cdots + a_0 d^n$

Roots of polynomial $\overline{\lambda_i} = 1/\lambda_i$, where λ_i are original roots of polynomial a.

- **Factorization of polynomial** a - consists in finding coprime polynomials a^+, a^-

$$a = a^+ a^-$$

where a^+ is the stable part of polynomial and a^- is the unstable part of polynomial a.

(decompose the expression into the form of multiples of the smallest divisors of the expression = decompose the numbers into prime numbers - find out the stability of each element)

Example 5: Write the reciprocal polynomial of a polynomial a and determine the roots
$a = 3d^2 + 4d + 1$
$\overline{a} = 3 + 4d + d^2$
$\overline{\lambda_1} = -1$
$\overline{\lambda_2} = -3$

Example 6: Realize factorization of the polynomial $a = 3d^2 + 4d + 1$ from the previous example.
$a^+ = 1$
$a^- = 3d^2 + 4d + 1$

Example 7: Realize factorization of the polynomial $a = d^2(d-1)(d+3)$

Roots: $\lambda_{1,2} = 0$, λ_3 =1, $\lambda_4 = -3$

$a^+ = d + 3$
$a^- = d^2(d-1)$

- **Solution of polynomial equations** (Diophantine equation)

$$ax + by = c$$

where a, b, c are known polynomials and x, y are unknown polynomials. The polynomial equation has a solution if GCD of a, b divides the polynomial c.

$$(a, b)|c$$

If polynomial g is GCD of polynomials a, b, then $a = ga^0$, $b = gb^0$, where a^0, b^0 are coprime polynomials.
The polynomial equation can be given as:

$$g(a^0x + b^0y) = c \quad \rightarrow \quad a^0x + b^0y = \frac{c}{g} = c^0$$

If x^0 and y^0 is the arbitrary solution of the Diophantine equation, then the general solution is given as

$$x = x^0 - b^0h$$
$$y = y^0 + a^0h$$

where h is arbitrary polynomial.

The solution of the polynomial equation can be obtained by the extended Euclidean algorithm for calculating GCD. For the given polynomials a, b we calculate the polynomials p, q, r, s, g according to equations

$$ap + bq = g$$
$$ar + bs = 0$$

Subsequently, the obtained polynomials can be substituted p, q, r, s, g into the general solution of the Diophantine equation.

$$x = p\frac{c}{g} + rh$$

$$y = q\frac{c}{g} + sh$$

where h is arbitrary polynomial.

Example 8: Find the solution of the polynomial equation $ax + by = c$
$a = d^2 + 3d + 2$
$b = d^2 + d$
$c = d^2 + 4d + 3$
Using the extended Euclidean algorithm, we can write

$$\begin{bmatrix} d^2+3d+2 & d^2+d \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 2d+2 & 2d^2+2d \\ 1 & 0 \\ -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 2d+2 & 0 \\ 1 & -d \\ -1 & 2+d \end{bmatrix}$$

$g = 2(d+1)$
$p = 1$
$q = -1$
$r = -d$
$s = 2 + d$

We can substitute p, q to $ap + bq = g$ and r, s to $ar + bs = 0$

$d^2 + 3d + 2 - d^2 - d = 2d + 2 = 2(d+1) = g$
$-d(d^2 + 3d + 2) + (d^2 + d)(d + 2) = (d + 2)(d + 1)(-d) + (d + 2)(d + 1)d = 0$

General solution:

$x = \frac{d^2+4d+3}{2(d+1)} - dh = \frac{(d+3)(d+1)}{2(d+1)} - dh = \frac{1}{2}(d+3) - dh$
$y = -\frac{d^2+4d+3}{2(d+1)} + (d+2)h = -\frac{(d+3)(d+1)}{2(d+1)} + (d+2)h = -\frac{1}{2}(d+3) + (d+2)h$

For $h = 1$ we can write one particular solution
$x = -\frac{d}{2} + \frac{3}{2}$
$y = -\frac{d}{2} - \frac{3}{2} + 2 + d = \frac{d}{2} + \frac{1}{2}$

Test of correctness: by placing the solution x, y into the equation $ax + by = c$
$\left[\frac{(d^2+3d+2)(-d+3)}{2} + \frac{(d^2+d)(d+1)}{2}\right] =$
$= \frac{1}{2}[-d^3 - 3d^2 - 2d + 3d^2 + 9d + 6 + d^3 + d^2 + d] = \frac{1}{2}(2d^2 + 8d + 6) = d^2 + 4d + 3$

Example 9: Find the solution of the polynomial equation from the previous example which minimizes the degree of the polynomial x
$x = \frac{d+3}{2} - dh$
$y = -\frac{d+3}{2} + (2 + \mathrm{d})\mathrm{h}$
Solution:
$h = \frac{1}{2}$
$x = \frac{3}{2}$
$y = -\frac{3}{2} + 1 = -\frac{1}{2}$

Test of correctness:

$$ax + by = c$$

$$(d^2 + 3d + 2)\frac{3}{2} - (d^2 + d)\frac{1}{2} = 2[3d^2 - d^2 + 9d - d + 6] = 2[2d^2 + 8d + 6] = d^2 + 4d + 3$$

- **Solution which minimizes the degree of polynomial x:**

If the degree of polynomial x^0 is higher than the degree of polynomial b^0

$$\frac{x^0}{b^0} = u + \frac{v}{b^0} \qquad\qquad x = x^0 - b^0 h$$
$$x^0 = b^0 u + v \qquad\qquad y = y^0 + a^0 h$$
$$x = v + b^0(u - h)$$

We choose $h = u$ and then a solution with a minimum degree of polynomial x is given as

$$x = v$$
$$y = y^0 + a^0 u$$

Example 10: Find the solution of the polynomial equation from the previous example which minimizes the degree of the polynomial x

We know that: $x = x^0 - b^0 h$
$y = y^0 + a^0 h$

$$x = d^3 + 4d^2 + 2d + 1 + (d + 5)h$$
$$y = d^2 + 4d + 3 + (d + 2)h$$

$$x^0 = d^3 + 4d^2 + 2d + 1$$
$$b^0 = -(d + 5)$$

$$d^3 + 4d^2 + 2d + 1 \div -(d + 5) = -d^2 + d - 7 = u$$
$$-d^3 - 5d^2$$
$$-d^2 + 2d + 1$$
$$d^2 + 5d$$
$$7d + 1$$
$$-7d - 35$$
$$v = -34$$

$$x = d^3 + 4d^2 + 2d + 1 + (d + 5)(-d^2 + d - 7) = -34$$
$$y = d^2 + 4d + 3 + (d + 2)(-d^2 + d - 7) = -d^3 - d - 11$$

References: Havlena, V., Štecha, J., Moderní teorie řízení, ČVUT,Praha 1999