Modeling of Uncertainties

RAL - Robust control

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Parametric uncertainties - Interval uncertainties

Working with interval polynomials

Interval polynomial (IP) can be generally written as

$$p(s,q) = \sum_{i=0}^{n} [q_i^-, q_i^+] s^i$$

Example of second order IP where all the coefficients are given as intervals:

$$p(s,q) = \overset{s^0}{[1,2]} + [3,5]s + [10,12]s^2$$

Each interval is defined by its upper and lower bound $[q^-, q^+]$

Interval addition and multiplication

Addition (subtraction): Add lower bound to the other lower bound and upper bound to the other upper bound.

$$[1,2] + [3,4] = [1+3, 2+4] = [4,6]$$

Multiplication: Multiply each bound with all the others, smallest result is the new lower bound, largest number the new upper bound.

$$[1,2] * [3,4] \rightarrow 1*3, 2*3, 1*4, 2*4 \rightarrow 3,6,4,8 \rightarrow [3,8]$$

$$[-1,1] * [-2,3] \rightarrow -1*-2, 1*-2, -1*3, 1*3 \rightarrow 2,-2,-3,3 \rightarrow [-3,3]$$

Stability of interval polynomials

Kharitonov polynomials are commonly used to test the stability of interval polynomials. The interval polynomial is stable, if and only if all four Kharitonov polynomials are stable.

$$
\begin{aligned}
K_1 &= q_0^- + q_1^- s + q_2^+ s^2 + q_3^+ s^3 + q_4^- s^4 + q_5^- s^5 + q_6^+ s^6 \ldots \\
K_2 &= q_0^+ + q_1^+ s + q_2^- s^2 + q_3^- s^3 + q_4^+ s^4 + q_5^+ s^5 + q_6^- s^6 \ldots \\
K_3 &= q_0^+ + q_1^- s + q_2^- s^2 + q_3^+ s^3 + q_4^+ s^4 + q_5^- s^5 + q_6^- s^6 \ldots \\
K_4 &= q_0^- + q_1^+ s + q_2^+ s^2 + q_3^- s^3 + q_4^- s^4 + q_5^+ s^5 + q_6^+ s^6 \ldots
\end{aligned}
$$

Different tools can be used to determine stability of each Kharitonov polynomial. For example the Hurwitz criteria can be used on polynomial in a form $\Delta = a_n p^n + a_{n-1} p^{n-1} + ... + a_1 p + a_0$ under the assumption that $a_0, a_1, ..., a_n > 0$. Then the following Hurwitz matrix can be constructed using the polynomial coeficients:

$$
H = \begin{bmatrix} a_{n-1} & a_{n-3} & a_{n-5} & \ldots & 0 \\ a_n & a_{n-2} & a_{n-4} & \ldots & 0 \\ 0 & a_{n-1} & a_{n-3} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & a_0 \end{bmatrix}
$$

Size of the Hurwitz matrix is the same as the order of the polynomial. Determinant of the Hurwitz matrix is $det|H|$ and all the subdeterminants (principal diagonal minors) are denoted as Δ_i.

$$
\Delta_1 = a_{n-1}, \Delta_2 = \begin{vmatrix} a_{n-1} & a_{n-3} \\ a_n & a_{n-2} \end{vmatrix}, \Delta_3 = \begin{vmatrix} a_{n-1} & a_{n-3} & a_{n-5} \\ a_n & a_{n-2} & a_{n-4} \\ 0 & a_{n-1} & a_{n-3} \end{vmatrix}
$$

Using Hurwitz criteria the polynomial is stable, if:

$$
\Delta_{n-1} > 0, \Delta_{n-2} > 0, \ldots, \Delta_1 > 0
$$

If any of the determinants is zero or negative, the polynomial is unstable. Notice that we don't need to compute Δ_n since $\Delta_n = a_0 \Delta_{n-1}$ and $a_0 > 0$ was assumed at the beginning.

Task 1

Determine the stability of polynomial $\Delta = s^3 + [8, 12]s^2 + [25, 39]s + [38, 58]$

$$
\begin{aligned}
K_1 &= s^3 + 12s^2 + 25s + 38 \\
K_2 &= s^3 + 8s^2 + 39s + 58 \\
K_3 &= s^3 + 8s^2 + 25s + 58 \\
K_4 &= s^3 + 12s^2 + 39s + 38
\end{aligned}
$$

$$K_1 = \begin{bmatrix} 12 & 38 \\ 1 & 25 \end{bmatrix}, \Delta_2 > 0, \Delta_1 > 0$$
$$K_2 = \begin{bmatrix} 8 & 58 \\ 1 & 39 \end{bmatrix}, \Delta_2 > 0, \Delta_1 > 0$$
$$K_3 = \begin{bmatrix} 8 & 58 \\ 1 & 25 \end{bmatrix}, \Delta_2 > 0, \Delta_1 > 0$$
$$K_4 = \begin{bmatrix} 12 & 38 \\ 1 & 39 \end{bmatrix}, \Delta_2 > 0, \Delta_1 > 0$$

All Kharitonov polynomials are stable - interval polynomial is stable.

Task 2

Determine the stability of the following system

$$F_R(s) = \frac{a_1}{s}, F_s(s) = \frac{(s + a_2)}{(s + a_3)(s + a_4)}$$

$$a_1 \in [1, 2], a_2 \in [2, 3], a_3 \in [3, 4], a_4 \in [4, 5]$$

$$F = \frac{F_R F_s}{1 + F_R F_s}$$

$$F = \frac{a_1 s + a_2 a_1}{s^3 + s^2(a_4 + a_3) + a_4 a_3 s + a_1 s + a_1 a_2} = \frac{a_1 s + a_2 a_1}{s^3 + s^2(a_4 + a_3) + (a_4 a_3 + a_1)s + a_1 a_2}$$

$$a_4 + a_3 = [4, 5] + [3, 4] = [7, 9]$$

$$a_4 a_3 + a_1 = [4, 5][3, 4] + [1, 2] = [12, 20] + [1, 2] = [13, 22]$$

$$a_1 a_2 = [1, 2][2, 3] = [2, 6]$$

$$\Delta = s^3 + [7, 9]s^2 + [13, 22]s + [2, 6]$$

$$\begin{aligned} K_1 = &\quad s^3 + 9s^2 + 13s + 2 \\ K_2 = &\quad s^3 + 7s^2 + 22s + 6 \\ K_3 = &\quad s^3 + 7s^2 + 13s + 6 \\ K_4 = &\quad s^3 + 9s^2 + 22s + 2 \end{aligned}$$

$$K_1 = \begin{bmatrix} 9 & 2 \\ 1 & 13 \end{bmatrix}, D_1 = +, D_2 = +$$
$$K_2 = \begin{bmatrix} 7 & 6 \\ 1 & 22 \end{bmatrix}, D_1 = +, D_2 = +$$
$$K_3 = \begin{bmatrix} 7 & 6 \\ 1 & 13 \end{bmatrix}, D_1 = +, D_2 = +$$
$$K_4 = \begin{bmatrix} 9 & 2 \\ 1 & 22 \end{bmatrix}, D_1 = +, D_2 = +$$

All Khariton polynomials are stable, the interval polynomial is stable.

Kharitonov rectangles, Mikhailov plot

Kharitonov polynomials are considered stable if the the Mikhailov plot of each Kharitonov polynomial starts on the positive real axis for the zero frequency and with rising frequency circles the origin in counterclockwise direction until its phase is $n\pi/2$. The edges defined by each polynomial at each frequency should never cross the origin of the complex plane.

Task 3

Write Kharitonov polynomials for the following interval polynomial:

$$\Delta = s^3 + [8, 12]s^2 + [25, 39]s + [38, 58]$$

and enumerate all of them for frequencies $\omega = 0 : 0.1 : 5$ by replacing s with $j\omega$.

$$\begin{aligned} K_1 &= s^3 + 12s^2 + 25s + 38 \\ K_2 &= s^3 + 8s^2 + 39s + 58 \\ K_3 &= s^3 + 8s^2 + 25s + 58 \\ K_4 &= s^3 + 12s^2 + 39s + 38 \end{aligned}$$

Example for $\omega = 1$ and $\omega = 2$:

$$\begin{aligned} K_1(\omega = 1) &= -j - 12 + 25j + 38 = 26 + j24 \\ K_2(\omega = 1) &= -j - 8 + 39j + 58 \quad = 50 + j38 \\ K_3(\omega = 1) &= \ldots \qquad\qquad\qquad\quad\;\; = 50 + j24 \\ K_4(\omega = 1) &= \ldots \qquad\qquad\qquad\quad\;\; = 26 + j38 \end{aligned}$$

$$\begin{aligned} K_1(\omega = 2) &= -j8 - 48 + 50j + 38 = -10 + j42 \\ K_2(\omega = 2) &= -j8 - 32 + 78j + 58 = 26 + j70 \\ K_3(\omega = 2) &= \ldots \qquad\qquad\qquad\qquad\; = 26 + j42 \\ K_4(\omega = 2) &= \ldots \qquad\qquad\qquad\qquad\; = -10 + j70 \end{aligned}$$

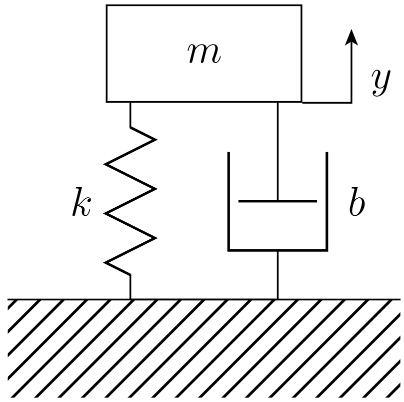


Figure 1: Mass/Spring/Damper System

Use Matlab to plot the Kharitonov rectangles. Each Kharitonov polynomial will give you a complex number. You can use *real* and *imag* matlab functions to split the complex number and get x-y coordinates for each frequency point. All 4 polynomials will form a rectangle in complex plane. You can use either *line* or *rectangle* to connect four corners in order to display the rectangles. You can use the following code to generate set of colors for the plot:

```
w = %frequency vector
colors = jet(length(w));
for i = 1:length(w);
    line(X,Y,'Color',colors(i,:));
end
```

Task 4

Modify the script for the following interval polynomial:

$$\Delta = s^6 + [3.95, 4.05]s^5 + [3.95, 4.05]s^4 + [5.95, 6.05]s^3 + [2.95, 3.05]s^2 + [1.95, 2.05]s + [0.45, 0.55]$$

Use the following frequencies: $\omega = 0 : 0.001 : 1$.

Unstructured uncertainties - conversion from parametric uncertainties

Mass/Spring/Damper System

The process of converting parametric uncertainties to unstructured ones is demonstrated using a simple example of Mass/Spring/Damper System. The real plant is composed of a mass connected to a spring and dampener connected in parallel (Fig. 1).

Mass damper spring system can be described by the following second order differential equation:

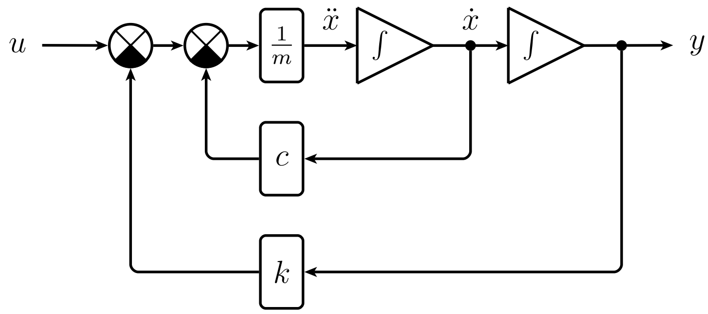


Figure 2: Diagram of the state space model

$$m\ddot{x} + c\dot{x} + x = u$$

Where m is the mass, k spring stiffness, c - damping coefficient

Using $y = x$, $x_1 = x$, $\dot{x}_1 = x_2$ the following state space representation is given:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{1}{m}(-cx_2 - kx_1 + u) \\
y &= x_1
\end{aligned}$$

Uncertainties of the system

It can be assumed that the real coefficients m, k, c cannot be precisely measured or they change significantly during the control process. These real-valued coefficients can be referred to as parametric uncertainties. The actual value of each uncertainty can be expressed by its nominal value, its maximal deviation from the nominal value and generalized parameter $\delta =< -1, 1 >$, which determines the how far is the parameter from its nominal value.

$$k = \bar{k}(1 + p_k\delta_k) \quad c = \bar{c}(1 + p_c\delta_c) \quad m = \bar{m}(1 + p_m\delta_m)$$

Now it is necessary to modify each uncertain parameter so that the generalized parameter δ can reside outside the state space diagram. The goal is to get the description in Fig. 3, where $\Delta = diag(\delta_k, \delta_c, \delta_m)$. Changing the value of exogenous input w now determines deviation of the parameter from its nominal value. The process of extraction of each individual generalized parameter δ can be observe on Fig.4. Then all the modified uncertain parameters can be placed back into the original state space diagram (Fig. 5). It is recommended to clean up the diagram by collecting all the exogenous inputs/outputs at the top of the diagram and all other inputs/outputs on the bottom of the diagram.

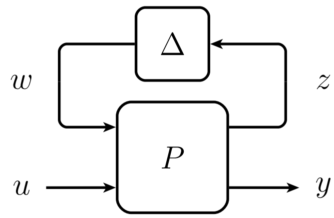


Figure 3: General uncertain plant

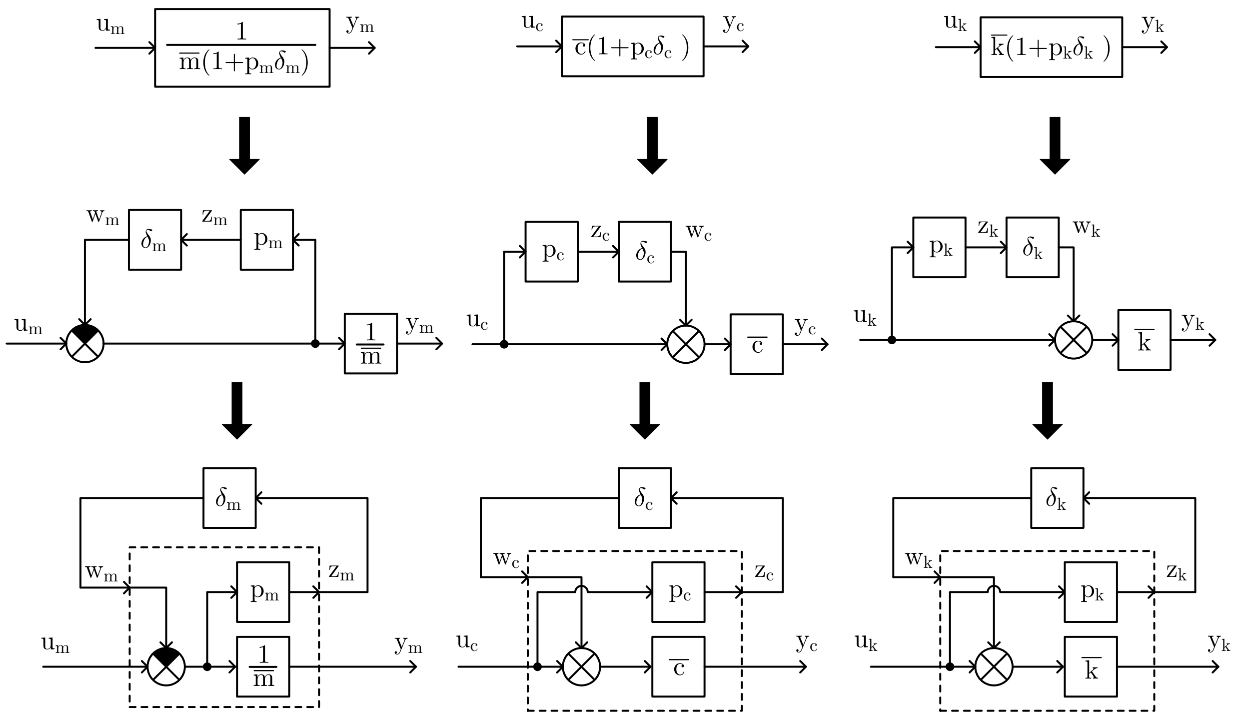
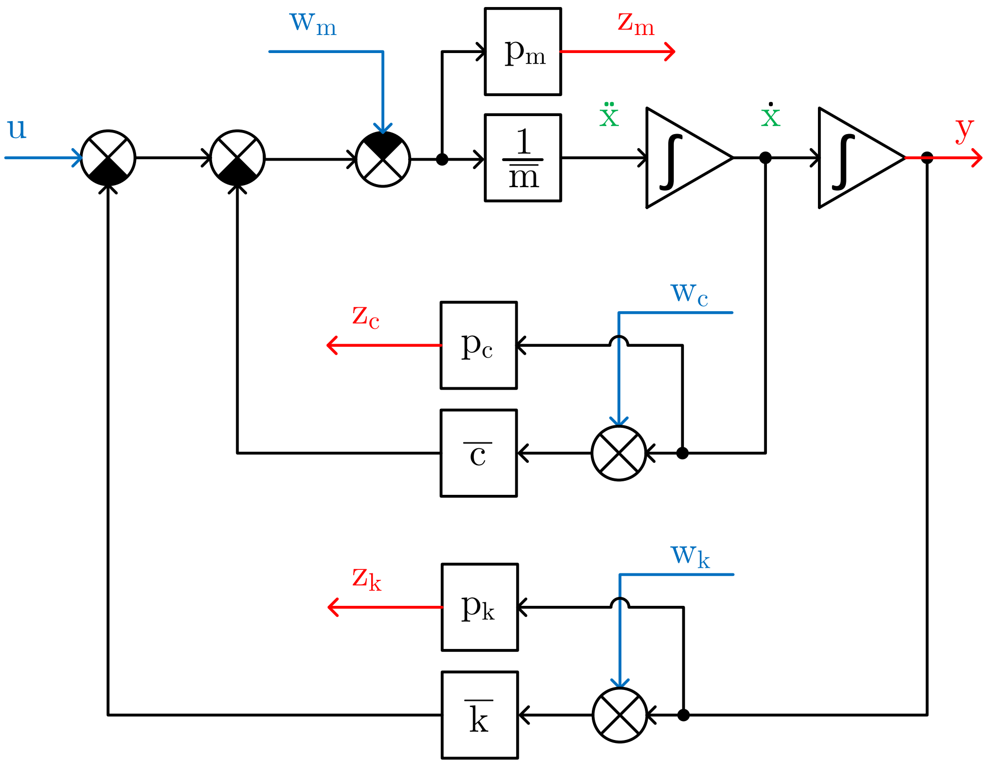


Figure 4: Step by step modification of uncertain parameters



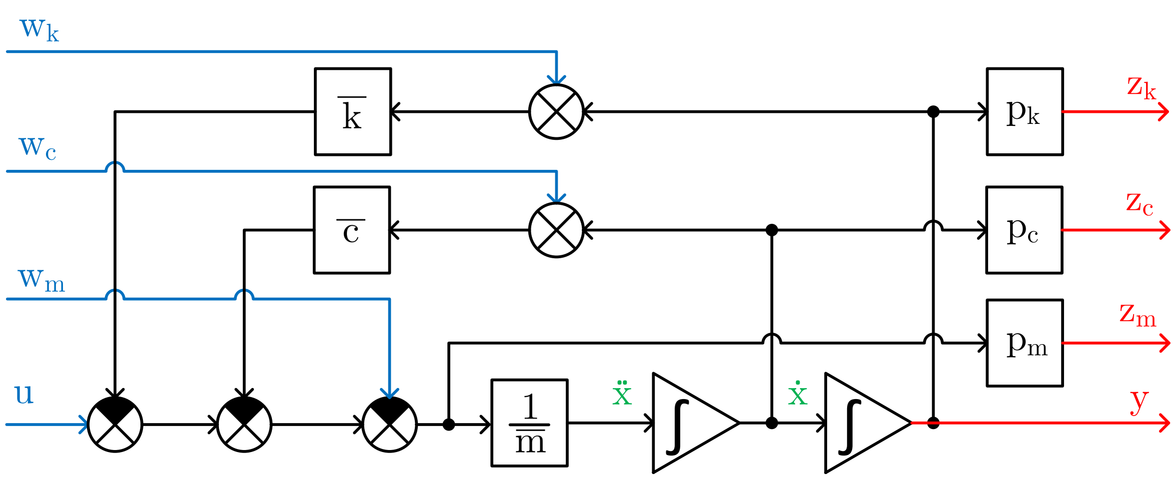


Figure 5: State space diagram with modified uncertainties

The state space diagram from Fig. 5 can be described by following set of state space equations:

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{1}{\bar{m}}[-\bar{k}(w_k + x_1) - \bar{c}(w_c + x_2) - w_m + u] \\
z_k &= p_k x_1 \\
z_c &= p_c x_2 \\
z_m &= p_m[-\bar{k}(w_k + x_1) - \bar{c}(w_c + x_2) - w_m + u] \\
y &= x_1
\end{aligned}
$$

From these we can extract following system matrices:

$$
\begin{aligned}
A &= \begin{bmatrix} 0 & 1 \\ -\frac{\bar{k}}{\bar{m}} & -\frac{\bar{c}}{\bar{m}} \end{bmatrix} &
B_w &= \begin{bmatrix} 0 & 0 & 0 \\ -\frac{\bar{k}}{\bar{m}} & -\frac{\bar{c}}{\bar{m}} & -\frac{1}{\bar{m}} \end{bmatrix} &
B_u &= \begin{bmatrix} 0 \\ \frac{1}{\bar{m}} \end{bmatrix} \\
C_z &= \begin{bmatrix} p_k & 0 \\ 0 & p_c \\ -p_m\bar{k} & -p_m\bar{c} \end{bmatrix} &
D_{wz} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -p_m\bar{k} & -p_m\bar{c} & -p_m \end{bmatrix} &
D_{uz} &= \begin{bmatrix} 0 \\ 0 \\ p_m \end{bmatrix} \\
C_y &= \begin{bmatrix} 1 & 0 \end{bmatrix} &
D_{wy} &= \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} &
D_{uy} &= \begin{bmatrix} 0 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) \;\; + B_w\mathbf{w}(t) \;\; + B_u\mathbf{u}(t) \\
\mathbf{z}(t) &= C_z\mathbf{x}(t) + D_{wz}\mathbf{w}(t) + D_{uz}\mathbf{u}(t) \\
\mathbf{y}(t) &= C_y\mathbf{x}(t) + D_{wy}\mathbf{w}(t) + D_{uy}\mathbf{u}(t)
\end{aligned}
$$

This description follows the structure showed on Fig. 3. This structure can be further extended to a general feedback control structure with uncertainties by adding a controller.

Task 5

Fill the template you've been given with the correctly assembled system matrices A, B_w, B_u, C_z, C_y, D_{wz}, D_{uz}, D_{wy}, D_{uy}. Use the pck function to create general uncertain system matrix from the system matrices so that the state space representation can be written as:

$$
\begin{bmatrix} \dot{x} \\ z \\ y \end{bmatrix} = G \begin{bmatrix} x \\ w \\ u \end{bmatrix}
$$

Test the system by ploting bode diagram using the given unctbode function:

```
omega = logspace(-1,1,100);
nunct = 3;
unctbode(G,omega,nunct)
```

This function is not part of the MATLAB, you have to download it from the exercises page unless you are typing the solution directly into the MATLAB grader Script window embedded on this side.