Exercise 2:

→ Recap gcd and lcm:

Ex: find gcd and lcm of polynomials a, b

$$a = d^2 + 3d + 2$$

$$b = d^3 + 6d^2 + 11d + 6$$

Help: write matrix $\begin{bmatrix} a & b \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ and find matrix $\begin{bmatrix} g & 0 \\ p & r \\ q & s \end{bmatrix}$ — 0 in this position

$$\begin{bmatrix} d^2+3d+2 & d^3+6d^2+11d+6 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \overset{C_2(1) = C_2(0) - d \cdot C_1(0)}{\sim} \begin{bmatrix} d^2+3d+2 & 3d^2+9d+6 \\ 1 & -d \\ 0 & 1 \end{bmatrix} \overset{C_2(2) = \frac{1}{3} C_2(1)}{\sim}$$

$$\begin{bmatrix} d^2+3d+2 & d^2+3d+2 \\ 1 & -\frac{d}{3} \\ 0 & \frac{1}{3} \end{bmatrix} \overset{C_2(3) = C_2(2) - C_1(2)}{\sim} \begin{bmatrix} d^2+3d+2 & 0 \\ 1 & -\frac{d}{3}-1 \\ 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} g & 0 \\ p & r \\ q & s \end{bmatrix}$$

$g = \gcd(a, b) = d^2 + 3d + 2 = a$ ⇒ polynomial b can be divided by a with 0 remainder

$$\frac{b}{a}: \quad (d^3 + 6d^2 + 11d + 6) : (d^2 + 3d + 2) = d + 3$$
$$-(d^3 + 3d^2 + 2d)$$
$$3d^2 + 9d + 6$$
$$-(3d^2 + 9d + 6)$$
$$0$$

$$\Rightarrow b = (d^2 + 3d + 2) \cdot (d + 3)$$

$$b = (d + 1) \cdot (d + 2) \cdot (d + 3)$$

$$\ell = \mathrm{lcm}(a, b) = a \cdot r = -\left(\frac{d}{3} + 1\right) \cdot a = -\frac{1}{3} \cdot (d + 3) \cdot a = -\frac{1}{3} \cdot b$$

$$\ell = \mathrm{lcm}(a, b) = -b \cdot s = -\frac{1}{3} \cdot b$$

→ Stability of polynomial

1) Positive powers of z, where z is the Z-transform operator

→ Discrete-time system is stable if and only if the roots of it's denominator are inside the unit circle

$$F(z) = \frac{\prod_{i=0}^{m} (z - \beta_i)}{\prod_{i=0}^{n} (z - \alpha_i)} \qquad |\alpha_i| < 1$$

→ Causality: system $F(z)$ is causal if $n \geq m$

2) Negative powers of z

Suppose: $d = z^{-1}$

$$F(z) = \frac{\prod_{i=0}^{m} (z - \beta_i)}{\prod_{i=0}^{n} (z - \alpha_i)} = \frac{z^{-n}}{z^{-n}} \frac{\prod_{i=0}^{m} (z - \beta_i)}{\prod_{i=0}^{n} (z - \alpha_i)} = \frac{z^{-(n-m)} \cdot \prod_{i=0}^{m} (1 - \beta_i \cdot z^{-1})}{\prod_{i=0}^{n} (1 - \alpha_i z^{-1})}$$

$$F(d) = \frac{d^{(n-m)} \cdot \prod_{i=0}^{m} (1 - \beta_i d)}{\prod_{i=0}^{n} (1 - \alpha_i d)} = \frac{d^{(n-m)} \cdot \prod_{i=0}^{n} \beta_i \left(\frac{1}{\beta_i} - d\right)}{\prod_{i=0}^{n} \alpha_i \left(\frac{1}{\alpha_i} - d\right)} = \frac{d^{(n-m)} \cdot \prod_{i=0}^{m} -\beta_i \left(d - \frac{1}{\beta_i}\right)}{\prod_{i=0}^{n} -\alpha_i \left(d - \frac{1}{\alpha_i}\right)}$$

What happened to the roots of denominator?

From $\prod_{i=0}^{n} (z - \alpha_i)$ we have obtained $\prod_{i=0}^{n} -\alpha_i \left(d - \frac{1}{\alpha_i}\right)$

if $|\alpha_i| < 1$ then $\frac{|\alpha_i|}{|\alpha_i|} < \frac{1}{|\alpha_i|}$ leading to $1 < \frac{1}{|\alpha_i|}$

→ Discrete-time system defined in negative powers of z $(z^{-1} = d)$ is stable if the roots of it's denominator are outside of the unit circle

→ causality: causality is determined by condition $n \geq m$. What happens to $F(d)$ if this condition is violated?

Suppose $m > n$ (noncausality):

$$F(d) = \frac{d^{-(m-n)} \prod_{i=0}^{m} -\beta_i \left(d - \frac{1}{\beta_i}\right)}{\prod_{i=0}^{n} -\alpha_i \left(d - \frac{1}{\alpha_i}\right)} = \frac{\prod_{i=0}^{m} -\beta_i \left(d - \frac{1}{\beta_i}\right)}{d^{m-n} \cdot \prod_{i=0}^{n} -\alpha_i \left(d - \frac{1}{\alpha_i}\right)}$$

Since $\prod_{i=0}^{n} -\alpha_i \left(d - \frac{1}{\alpha_i}\right) = \sum_{i=0}^{n} \xi_i \cdot d^i$

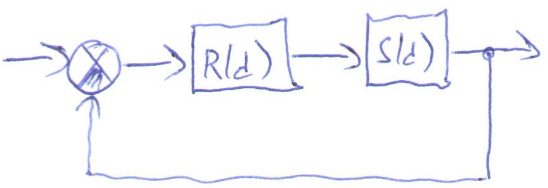
Then $d^{m-n} \cdot \prod_{i=0}^{n} -\alpha_i \left(d - \frac{1}{\alpha_i}\right) = d^{m-n} \cdot \sum_{i=0}^{n} \xi_i d^i = \sum_{i=0}^{n} \xi_i d^{i+m-n}$

$\Rightarrow$ noncausal do not have absolute term (for $i = 0$ $d^{i+m-n} \neq 1$)

Ex: System $S(d)$ is controlled by controller $R(d)$. Is the closed-loop transfer function stable?

$$S(d) = \frac{1}{d+0{,}9}$$

$$R(d) = \frac{d}{d-1}$$



$$F_w(d) = \frac{R(d) \cdot S(d)}{1 + R(d) \cdot S(d)} = \frac{\frac{d}{(d-1)\cdot(d+0{,}9)}}{1 + \frac{d}{(d-1)\cdot(d+0{,}9)}} =$$

$$= \frac{d}{(d-1)\cdot(d+0{,}9) + d} = \frac{d}{d^2 + 0{,}9d - 0{,}9}$$

$$d_{1,2} = \frac{-0{,}9 \pm \sqrt{0{,}9^2 + 4 \cdot 0{,}9}}{2} = \frac{-0{,}9 \pm \sqrt{4{,}41}}{2} =$$

$$= \frac{-0{,}9 \pm 2{,}1}{2} \quad \begin{cases} -\frac{3}{2} = -1{,}5 \\ \frac{1{,}2}{2} = 0{,}6 \end{cases}$$

$$F_w(d) = \frac{d}{(d+1{,}5)\cdot(d-0{,}6)}$$

Since $0{,}6 < 1$, closed-loop transfer function is unstable

→ Factorization of polynomial into stable and unstable part

Suppose polynomial a that contains ~~both~~ stable roots $\frac{1}{|\alpha_i|} > 1$ and unstable roots $\frac{1}{|\alpha_i|} \leq 1$. Then polynomial a can be rewrite as $a = a^+ \cdot a^-$ where $\gcd(a^+, a^-) = 1$, a^+ contains only stable roots $\frac{1}{|\alpha_i|} > 1$, and a^- contains only unstable roots $\frac{1}{|\alpha_i|} < 1$.

Ex: Factorize polynomial a to stable a^+ and unstable a^- part

$$a = (d + 1{,}1) \cdot (d - 1) \cdot (d + 0{,}9)$$

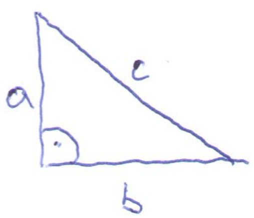
$$a^+ = d + 1{,}1$$

$$a^- = (d - 1) \cdot (d + 0{,}9)$$

→ Polynomial Diophantine equation

- In mathematics, Diophantine equation is an equation, typically in two or more unknowns with integer coefficients, such that the only solutions of interest are the integer ones.

Ex: Finding all right triangles with integer side-lengths is Diophantine equation



$$a^2 + b^2 = c^2$$

! There is ∞ valid solutions !

Some of them: $a = 3, b = 4, c = 5$; $a = 5, b = 12, c = 13$; ...

Note that $\gcd(a, b) = 1$. What happens if $\gcd(a, b) = 2$ then we can write $a = 2 \cdot a^0$ $b = 2b^0$ and the equation reads:

$$2^2 a^{0^2} + 2^2 b^{0^2} = c^2 \Rightarrow a^{0^2} + b^{0^2} = \frac{c^2}{4}$$ ⇒ The requirement on integer coefficients is fulfilled only if $c = 2 \cdot c^0$

- Polynomial Diophantine equation is an analogy to classical Diophantine equation, where the requirements on integer coefficients and integer solution lead to equation, where polynomial coefficients and polynomia solution are <u>not rational functions</u>.

Suppose linear equation:

$$ax + by = c$$

$x, y \ldots$ unknown solutions

$a, b, c \ldots$ known polynomials

The solutions to this equation read:

$$x = p \cdot \frac{c}{g} + r \cdot h$$
$$y = q \cdot \frac{c}{g} + s \cdot h$$

where $\begin{bmatrix} a & b \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \sim \ldots \sim \begin{bmatrix} g & 0 \\ p & r \\ q & s \end{bmatrix}$

$h \ldots$ arbitrary polynomial

$g \ldots$ greatest common divisor $g = \gcd(a,b)$

The solutions and equation coefficients wont be rational if and only if $\frac{c}{g} = c^0$, where c^0 is polynomial

Ex: Are defined equations solvable?

$$\rightarrow \frac{1}{d+1} x + (d+2) y = \frac{d+2}{d+1}$$

$$\rightarrow 1 \cdot x + (d+2) y = \frac{d+2}{d+1}$$

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if the first and the second equations are multiplied ~~[illegible]~~ by $(d+1)$ we obtain:

$$\rightarrow 1x + (d+1) \cdot (d+2) y = d+2$$

$$\rightarrow (d+1) x + (d+1) \cdot (d+2) y = d+2$$

$\Rightarrow$ coefficients of both equations are not rational functions

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solutions are not rational if $\frac{c}{g} = \frac{c}{\gcd(a,b)}$ is not rational

$\rightarrow g = \gcd(1, (d+1)\cdot(d+2)) = 1 \Rightarrow \frac{c}{g} = d+2$

$\rightarrow g = \gcd((d+1), (d+1)\cdot(d+2)) = d+1 \Rightarrow \frac{c}{g} = \frac{d+2}{d+1}$

! solution of the second equation is rational => equation is not solvable !

Ex: Find solution of the polynomial equation $ax+by=c$

$a = d^2+3d+2$

$b = d^2+d$

$c = d^2+4d+3$

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1) Find $\begin{bmatrix} g & 0 \\ p & r \\ q & s \end{bmatrix}$ from $\begin{bmatrix} a & b \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} d^2+3d+2 & d^2+d \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \underset{\sim}{C_1(1)=C_1(0)-C_2(0)} \begin{bmatrix} 2d+2 & d^2+d \\ 1 & 0 \\ -1 & 1 \end{bmatrix} \underset{\sim}{C_1(2)=\frac{1}{2}C_1(1)}$$

$$\begin{bmatrix} d+1 & d^2+d \\ \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 \end{bmatrix} \underset{\sim}{C_2(3)=C_2(2)-dC_1(2)} \begin{bmatrix} d+1 & 0 \\ \frac{1}{2} & -\frac{d}{2} \\ -\frac{1}{2} & 1+\frac{d}{2} \end{bmatrix} = \begin{bmatrix} g & 0 \\ p & r \\ q & s \end{bmatrix}$$

2) write solution $x = p\cdot\frac{c}{g} + r\cdot h$, $y = q\cdot\frac{c}{g} + s\cdot h$

$\frac{c}{g}$: $(d^2+4d+3) : (d+1) = d+3$

$-(d^2+d)$

$3d+3$

$-(3d+3)$

0

$x = \frac{1}{2}\cdot(d+3) - \frac{d}{2}\cdot h$

$y = -\frac{1}{2}(d+3) + (1+\frac{d}{2})\cdot h$

→ Solution which minimizes the degree of polynomial x

Suppose $\deg\left(\frac{pc}{g}\right) > \deg(r)$ for solution

$$x = p\frac{c}{g} + r \cdot h$$

then h can be selected to minimize $\deg\left(\frac{pc}{g}\right)$. If we write the general solution:

$$x = x^0 - b^0 \cdot h \qquad \text{where} \qquad x^0 = p\frac{c}{g}$$

$$b^0 = -r$$

Note: b^0 is monic polynomial - it's leading coefficient is equal to 1. Since h is arbitrary polynomial, b^0 can be always transformed into monic form by involving the value of leading coef of b^0 into h'

$$\text{Ex: } b^0 h = \left(\tfrac{1}{2}d + 1\right) \cdot h = \tfrac{1}{2}(d+2) \cdot h = (d+2) \cdot h'$$

$$b^0 h = (2d^2 + 4d + 6) \cdot h = 2(d^2 + 2d + 3) \cdot h = (d^2 + 2d + 3) h'$$

Since $\deg(x^0) > \deg(b^0)$ we can divide these polynomials

$$\frac{x^0}{b^0} = u + \frac{v}{b^0}$$

--- remainder

if we substitute into general solution:

$$x = x^0 - b^0 \cdot h$$

$$x = \frac{b^0}{b^0}\left(x^0 - b^0 \cdot h\right)$$

$$x = b^0\left(\frac{x^0}{b^0} - h\right)$$

$$x = b^0\left(u + \frac{v}{b^0} - h\right)$$

if we select the solution $u = h$

$$x = b^0 \cdot \frac{v}{b^0} = v \qquad \text{and the solution will have minimum order}$$

$$y = y^0 + a^0 \cdot u$$

Ex: Find the solution that minimizes the degree of x

$$x = \frac{1}{2}(d+3) - \frac{d}{2}\cdot h$$

#2)

$$y = -\frac{1}{2}(d+3) + \left(1+\frac{d}{2}\right)\cdot h$$

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b^0 and a^0 have to be monic:

$$x = \frac{1}{2}(d+3) - \frac{1}{2}d\cdot h = \frac{1}{2}(d+3) - d\cdot h'$$

$$y = -\frac{1}{2}(d+3) + \frac{1}{2}(d+2)\cdot h = -\frac{1}{2}(d+3) + (d+2)\cdot h'$$

$$x^0 = \frac{1}{2}(d+3)$$

$$b^0 = d$$

$\frac{x^0}{b^0}$:

$$\left(\frac{1}{2}d + \frac{3}{2}\right) : (d) = \frac{1}{2} + \frac{\frac{3}{2}}{d} \Rightarrow u = \frac{1}{2} = h'$$

$$-\left(\frac{1}{2}d\right) \qquad\qquad v = \frac{3}{2}$$

$$\frac{3}{2}$$

$$x = \frac{1}{2}(d+3) - \frac{1}{2}d = \underline{\underline{\frac{3}{2}}} = v$$

$$y = -\frac{1}{2}(d+3) + (d+2)\cdot\frac{1}{2} = -\frac{d}{2} - \frac{3}{2} + \frac{d}{2} + 1 = \underline{\underline{-\frac{1}{2}}}$$
