

RAL Ex 04

Ex. 1 Modal control (pole placement)

Design the controller $R = \frac{q_R}{p_R}$ for the system $S = \frac{b}{a} = \frac{0.2z+0.1}{z^2-0.5z+0.1}$ such that the characteristic polynomial of the closed loop system is equal to $\Delta = z^i$. The position of the poles can be set.

$$ap_R + bq_R = \Delta$$

$$\text{Generally: } ax + by = \Delta$$

The degree of the characteristic polynomial must be sufficiently high to find the solution leading to causal controller.

Solvability condition: $\deg(\Delta) \geq 2 \deg(a) - 1$ (holds only for positive powers of z)

$$i = 2 \deg(a) - 1 = 4 - 1 = 3$$

$$(z^2 - 0.5z + 0.1)p + (0.2z + 0.1)q = z^3$$

$$\begin{aligned} \begin{bmatrix} a & b \\ 1 & 0 \\ 0 & 1 \end{bmatrix} &\sim \begin{bmatrix} z^2 - \frac{1}{2}z + \frac{1}{10} & \frac{1}{5}z + \frac{1}{10} \\ \frac{1}{5}z + \frac{1}{10} & 0 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} z^2 - \frac{1}{2}z + \frac{1}{10} & z + \frac{1}{2} \\ \frac{1}{5}z + \frac{1}{10} & 0 \\ 0 & 5 \end{bmatrix} \sim \begin{bmatrix} -z + \frac{1}{10} & z + \frac{1}{2} \\ \frac{1}{5}z + \frac{1}{10} & 0 \\ -5z & 5 \end{bmatrix} \sim \begin{bmatrix} \frac{3}{5} & z + \frac{1}{2} \\ \frac{1}{5}z + \frac{1}{10} & 0 \\ -5z + 5 & 5 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & z + \frac{1}{2} \\ \frac{5}{3} & 0 \\ -\frac{25}{3}z + \frac{25}{3} & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{5}{3} & -\frac{5}{3}z \\ -\frac{25}{3}z + \frac{25}{3} & 5 + \frac{25}{3}z^2 - \frac{25}{3}z \end{bmatrix} \sim \begin{bmatrix} \frac{1}{5} & 0 \\ \frac{5}{3} & -\frac{10}{3}z - \frac{5}{3} \\ -\frac{25}{3}z + \frac{25}{3} & \frac{5}{3} + \frac{50}{3}z^2 - \frac{25}{3}z \end{bmatrix} \sim \begin{bmatrix} g_{NSD} & 0 \\ p & r \\ q & s \end{bmatrix} \end{aligned}$$

$$p_R = x = p \frac{c}{g_{NSD}} + rh$$

$$q_R = y = q \frac{c}{g_{NSD}} + sh$$

$$p_R = x = \frac{5}{3}z^3 + \left(-\frac{10}{3}z - \frac{5}{3}\right)h$$

$$q_R = y = \left(-\frac{25}{3}z + \frac{25}{3}\right)z^3 + \left(\frac{5}{3} + \frac{50}{3}z^2 - \frac{25}{3}z\right)h$$

In our controller, p_R is x and q_R is y , h is arbitrary polynomial.

Appropriate selection of the polynomial h enables us to reduce the degree of the controller. This selection of h cancels the high powers in the polynomial q_R .

$$h = h_2z^2 + h_1z + h_0$$

After the substitution h

$$p_R = \left(\frac{5}{3} - \frac{10}{3}h_2\right)z^3 + \left(-\frac{5}{3}h_2 - \frac{10}{3}h_1\right)z^2 + \left(-\frac{5}{3}h_1 - \frac{10}{3}h_0\right)z - \frac{5}{3}h_0$$

$$q_R = \left(-\frac{25}{3} + \frac{50}{3}h_2\right)z^4 + \left(\frac{25}{3} - \frac{25}{3}h_2 + \frac{50}{3}h_1\right)z^3 + \left(\frac{5}{3}h_2 - \frac{25}{3}h_1 + \frac{50}{3}h_0\right)z^2$$

$$+ \left(\frac{5}{3}h_1 - \frac{25}{3}h_0\right)z + \frac{5}{3}h_0$$

We search for the simplest controller. It is the one with lowest degree, i.e. lowest degree of the polynomial q_R . Successively we choose:

$$q_4 = 0 \rightarrow h_2 = 0.5 \rightarrow p_3 = 0$$

$$q_3 = 0 \rightarrow h_1 = -0.25 \rightarrow p_2 = 0$$

$$q_2 = 0 \rightarrow h_0 = -\frac{7}{40} \rightarrow p_1 = 1$$

$$q_1 = \frac{25}{24}$$

Witch gives: $q_R = \frac{25}{24}z - \frac{7}{24}$ $p_R = z + \frac{7}{24}$

The transfer function of the controller is $R = \frac{\frac{25}{24}z - \frac{7}{24}}{z + \frac{7}{24}}$

Test of correctness:

Closed loop transfer function is: $F_w = \frac{bq_R}{ap_R + bq_R} = \frac{\frac{5}{24}z^2 + \frac{11}{240}z - \frac{7}{240}}{z^3}$

Characteristic polynomial is as it was desired.

Ex. 2 EMMP – Exact Model Matching Problem

Design the feedback controller $R = \frac{q_R}{p_R}$ for the system $S = \frac{b}{a}$ such that the closed loop system behaves like a model with the transfer function $M = \frac{g_M}{f_M}$.

$$S = \frac{0.2z^{-1} + 0.1z^{-2}}{1 - 0.5z^{-1} + 0.1z^{-2}} \quad M = \frac{0.5z^{-1} + 0.5z^{-2}}{1} = \frac{g_M}{f_M}$$

$$\frac{Y(z)}{W(z)} = \frac{SR}{1 + SR} = \frac{bq_R}{ap_R + bq_R} = \frac{g_M}{f_M} \rightarrow g_M^0(ap_R + bq_R) = f_M b^0 q_R$$

$$p_R a g_M^0 + g_M^0 b q_R - f_M b^0 q_R = 0$$

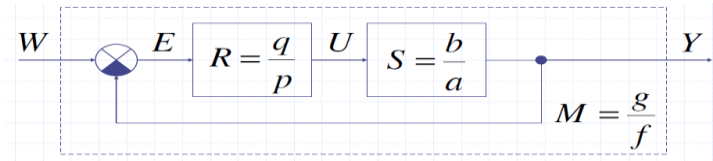
$$p_R a g_M^0 - q_R (f_M - g_M) b^0 = 0$$

$$p_R = b^0 (f_M - g_M) * x_{lib}$$

$$q_R = a g_M^0 * x_{lib}$$

x_{lib} -arbitrary polynomial

$$R = \frac{q_R}{p_R} = \frac{ag_M^0}{b^0(f_M - g_M)} = \frac{(1 - 0.5d + 0.1d^2)(0.5d + 0.5d^2)}{(0.2d + 0.1d^2)(1 - 0.5d - 0.5d^2)}$$



The goal is to find polynomials p, q such that the following equality:

$$\frac{Y(z)}{W(z)} = \frac{SR}{1 + SR} = \frac{bq_R}{ap_R + bq_R} \equiv \frac{g_M}{f_M} = M$$

Test of correctness: $\frac{SR}{1+SR} = M$

$$\begin{aligned} F_W &= \frac{\frac{0.2d + 0.1d^2}{1 - 0.5d + 0.1d^2} * \frac{(1 - 0.5d + 0.1d^2)(0.5d + 0.5d^2)}{(0.2d + 0.1d^2)(1 - 0.5d - 0.5d^2)}}{1 + \frac{0.2d + 0.1d^2}{1 - 0.5d + 0.1d^2} * \frac{(1 - 0.5d + 0.1d^2)(0.5d + 0.5d^2)}{(0.2d + 0.1d^2)(1 - 0.5d - 0.5d^2)}} = \frac{\frac{0.5d + 0.5d^2}{1 - 0.5d - 0.5d^2}}{1 + \frac{0.5d + 0.5d^2}{1 - 0.5d - 0.5d^2}} \\ &= \frac{0.5d + 0.5d^2}{1 - 0.5d - 0.5d^2 + 0.5d + 0.5d^2} = 0.5d + 0.5d^2 = M \end{aligned}$$

Ex. 3 Matlab – Modal control (pole placement)

Design the controller $R = \frac{q_R}{p_R}$ for the system $S = \frac{0.2z+0.1}{z^2-0.5z+0.1}$. Find the solution in both, positive and negative powers of z .

$$\Delta = z$$

$$\Delta = z^2$$

$$\Delta = z^3$$

$$\Delta = ap_R + bq_R = h$$

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syms d
a = 1-0.5*d+0.1*d^2;
b = 0.2*d+0.1*d^2;
c = 1;
[xmin,y] = axminbyc(a,b,c,d)
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$$p = xmin = (7*d)/24 + 1$$

$$q = y = 25/24 - (7*d)/24$$

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syms z
a = z^2-0.5*z+0.1;
b = 0.2*z+0.1;
c = z^1;
[ymin,x] = axminbyc(b,a,c,z)
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$$\begin{aligned} c &= z^1 \\ q &= ymin = (25*z)/6 + 5/6 \\ p &= x = -5/6 \end{aligned}$$

- Controller is not causal
- Degree of the numerator is higher than the degree of the denominator

$c = z^2$
 $q = y_{\min} = (35z)/12$
 $p = x = 5/12$

- Controller is not causal
- Degree of the numerator is higher than the degree of the denominator

$c = z^3$
 $q = y_{\min} = (25z)/24 - 7/24$
 $p = x = z + 7/24$