10. Resource Allocation
COMP4418 Knowledge Representation and Reasoning

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Outline

Outline

Allocation Setting

Basic Allocation Setting

- Agents $N = \{1, \dots, n\}$
- Items $O = \{o_1, \dots, o_m\}$
- Preferences (of agents) $\succsim = \{\succsim_1, \dots, \succsim_n\}$; preferences can be encoded by utility function $u = (u_1, \dots, u_n)$ over bundles of items.

An *allocation* $X = (X(1), \dots, X(n))$ assigns $X(i) \subseteq O$ to agent i.

- We will assume that $X(i) \cap X(j) = \emptyset$ for all $i, j \in N$ such that $i \neq j$.
- We will focus on allocations that allocate all the items: $\bigcup_{i \in N} X_i = O$.

Some notation

- $$A \succsim_i B$$
  (agent i prefers A at least as much as B)
- $$A \succ_i B \iff A \succsim_i B \text{ and } B \not\succsim_i A$$
  (agent i strictly prefers A over B)
- $$A \sim_i B \iff A \succsim_i B \text{ and } B \succsim_i A$$
  (agent i is indifferent between A and B).

Allocation setting: Additive Utilities

We assume additive utilities:

- $u_i : O \longrightarrow \mathbb{R}^+$ specifies the utility function of each agent i.
- $u_i(O') = \sum_{o \in O'} u_i(o)$ for any $O' \subseteq O$.

Example

	o_1	o_2	o_3	o_4
1	6	3	2	1
2	4	1	2	3

$u_1(o_1) = 6$; $u_1(o_2) = 3$; $u_1(o_3) = 2$; $u_1(o_4) = 1$.

$$u_1(\{o_1, o_2\}) > u_1(\{o_2, o_3\}).$$

$$\{o_1, o_2\} \succ_1 \{o_2, o_3\}.$$

Outline

Pareto optimality

An allocation X is *Pareto optimal* if there exists no allocation Y such that $Y(i) \succsim_i X(i)$ for all $i \in N$ and $Y(i) \succ_i X(i)$ for some $i \in N$.

An allocation X is *Pareto optimal* if there exists no allocation Y such that $u_i(Y(i)) \geq u_i(X(i))$ for all $i \in N$ and $u_i(Y(i)) > u_i(X(i))$ for some $i \in N$.

- Introduced by Italian polymath Vilfredo Pareto (1848 – 1923).

Pareto optimality

An allocation X is *Pareto optimal* if there exists no allocation Y such that $Y(i) \succsim_i X(i)$ for all $i \in N$ and $Y(i) \succ_i X(i)$ for some $i \in N$.

Example (Not Pareto optimal)

	o_1	o_2	o_3	o_4
1	6	2	3	1
2	4	1	2	3

$X(1) = \{o_1, o_3, o_4\}$, $X(2) = \{o_2\}$.

Example (Pareto optimal)

	o_1	o_2	o_3	o_4
1	6	2	3	1
2	4	1	2	3

$X(1) = \{o_1, o_2, o_3\}$, $X(2) = \{o_4\}$.

Utilitarian Social Welfare

An allocation X's *utilitarian social welfare* is

$$\sum_{i \in N} u_i(X(i))$$

Example (utilitarian welfare maximizing allocation)

	o_1	o_2	o_3	o_4
1	6	3	2	1
2	4	1	2	3

$X(1) = \{o_1, o_2, o_3\}$, $X(2) = \{o_4\}$.

- Advocated by Jeremy Bentham (1748 – 1832).

Egalitarian Social Welfare

An allocation X's *egalitarian social welfare* is

$$\min_{i \in N}\{u_i(X(i))\}$$

Example (egalitarian welfare maximizing allocation)

	o_1	o_2	o_3	o_4
1	6	2	3	1
2	4	1	2	3

$X(1) = \{o_1\}$, $X(2) = \{o_2, o_3, o_4\}$.

- Advocated by John Rawls [1971]

Lexmin Welfare

For any allocation X, let $f(X)$ be the vector that orders the utilities achieved by the agents in non-decreasing order.

An allocation X maximizes lexmin welfare if it lexicographially maximizes $f(X)$.

Example (lexicographic comparison)

$$(4,5) >_{lex} (3,8).$$

Example (lexmin welfare maximizing allocation)

	o_1	o_2	o_3	o_4
1	6	2	3	1
2	4	1	2	3

$X(1) = \{o_1\}$, $X(2) = \{o_2, o_3, o_4\}$.

Nash Product Social Welfare

An allocation X's *Nash product welfare* is

$$\prod_{i \in N} u_i(X(i))$$

Example (Nash product welfare maximizing allocation)

	o_1	o_2	o_3	o_4
1	6	2	3	1
2	4	1	2	3

$X(1) = \{o_1, o_2\}$, $X(2) = \{o_3, o_4\}$.

Nash Product Welfare

	o_1	o_2	o_3	o_4
1	6	2	3	1
2	4	1	2	3

Table: Utilitarian welfare maximizing allocation

	o_1	o_2	o_3	o_4
1	6	2	3	1
2	4	1	2	3

Table: Nash welfare maximizing allocation

	o_1	o_2	o_3	o_4
1	6	2	3	1
2	4	1	2	3

Table: Egalitarian welfare maximizing allocation

Welfare-Pareto optimality

Fact

If an allocation maximizes utilitarian welfare or Nash product welfare or is a lexmin allocation, then it is Pareto optimal.

Proof.

- Assume the allocation is not Pareto optimal.
- Then there exists another allocation in which each agent gets at least as much utility and one agents strictly more utility.
- But then the allocation does not maximize welfare.

□

Outline

Envy-freeness

An allocation X satisfies *envy-freeness* if for all $i, j \in N$

$$X(i) \succsim_i X(j)$$

$$u_i(X(i)) \geq u_i(X(j))$$

Example (Not envy-free)

	o_1	o_2	o_3	o_4
1	6	2	3	1
2	4	1	2	3

$X(1) = \{o_1, o_2, o_3\}$, $X(2) = \{o_4\}$.

Proportional

An allocation X satisfies *proportionality* if for all $i, j \in N$

$$u_i(X(i)) \geq \frac{u_i(O)}{n}$$

Example (Not proportional)

	o_1	o_2	o_3	o_4
1	6	2	3	1
2	4	1	2	3

$X(1) = \{o_1, o_2, o_3\}$, $X(2) = \{o_4\}$.

Envy-freeness implies proportionality

Fact

If an allocation is complete and utilities are additive, envy-freeness implies proportionality.

Assume that an allocation X is envy-free.

Then for each $i \in N$,

$$u_i(X(i)) \geq u_i(X(j)) \text{ for all } j \in N.$$

Thus,

$$n \cdot u_i(X(i)) \geq \sum_{j \in N} u_i(X(j)) = u_i(O).$$

Hence

$$u_i(X(i)) \geq u_i(O)/n.$$

Non-existence of envy-free or proportional allocation

Example

	o_1	o_2
1	9	1
2	9	1

Allocation of indivisible items

Theorem (Demko and Hill [1988])

For additive utilities, checking whether there exists an envy-free or proportional allocation is NP-complete.

Proof.

We present a reduction from the following NP-complete problem.
IntegerPartition
Input: A set of integers $S = \{w_1, \dots, w_m\}$ such that $\sum_{w \in S} w = 2W$.
Question: Does there exist a partiton (S', S'') of S such that $\sum_{w \in S'} w = \sum_{w \in S''} w = W$?

- Consider the setting in which two agents have identical utilities over the m items with the utility for the j-th item being w_j and the total utility of each agents over the items being $2W$.
- Then, there exists a proportional allocation iff there is a integer partition of the integers corresponding to the weights so that each partition has total weight W.

Maxmin Fair Share (MmS) Fairness

Definition (Maxmin Fair Share Fairness)

Given an instance $I = (N, O, u)$, let Π_n denote the space of all partitions of O into n sets. The *maximin share guarantee* of an agent $i \in N$ is

$$\mathsf{MmS}_i(I) = \max_{(P_1,\ldots,P_n)\in\Pi_n} \min_{j\in\{1,\ldots,n\}} u_i(P_j).$$

An allocation X is a *maximin share (MmS) allocation* if we have $u_i(X(i)) \geq \mathsf{MmS}_i(I)$ for each agent $i \in N$.

Example (Satisfies MmS Fairness)

	o_1	o_2	o_3	o_4
1	2	1	1	6
2	1	1	3	5

$\mathsf{MmS}_1(I) = 4$; $\mathsf{MmS}_2(I) = 5$

Maxmin Fair Share (MmS) Fairness

Definition (Maxmin Fair Share Fairness)

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An allocation X is a *maximin share (MmS) allocation* if we have $u_i(X(i)) \geq \mathsf{MmS}_i(I)$ for each agent $i \in N$.

Example (Satisfies MmS Fairness)

	o_1	o_2	o_3	o_4
1	2	1	1	6
2	1	1	3	5

$X(1) = \{o_1, o_2, o_3\}$, $X(2) = \{o_4\}$.

Proportionality implies MmS fairness

Fact

A proportional allocation satisfies MmS fairness.

Suppose an agent i does not get her MmS value in allocation X. Then

$$u_i(X(i)) < \mathsf{MmS}_i(I).$$

Then there exists an allocation $Y = (Y(1), \ldots, Y(n))$ such that

$$\mathsf{MmS}_i(I) \geq Y(j) \quad \forall j \in [n].$$

Note however that

$$\mathsf{MmS}_i(I) \leq u_i(O)/n.$$

Hence

$$u_i(X(i)) < \mathsf{MmS}_i(I) \leq u_i(O)/n.$$

Pareto optimality and Fairness

Fact

Pareto improvement over a proportional allocation is proportional.

Fact

Pareto improvement over a MmS fair allocation is MmS fair.

Fact

Pareto improvement over an envy-free allocation may not be envy-free.

EF1 Fairness

Definition (EF1 Fairness)

Given an instance $I = (N, O, u)$, an allocation X satisfies EF1 (envy-freeness up to 1 item) if for each $i, j \in N$, there exists some item $o \in X(j)$ such that

$$X(i) \succsim_i X(j) \setminus \{o\}.$$

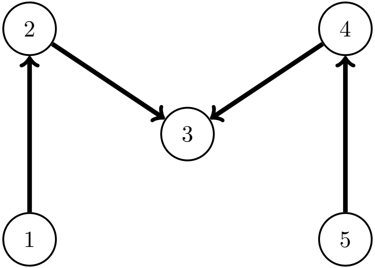
Example (Satisfies EF1 Fairness)

	o_1	o_2	o_3	o_4
1	2	1	1	6
2	1	1	3	5

$X(1) = \{o_1, o_2, o_3\}$, $X(2) = \{o_4\}$.

Algorithm for EF1 fairness (Lipton et al. (2004))

Algorithm by Lipton et al. [2004].

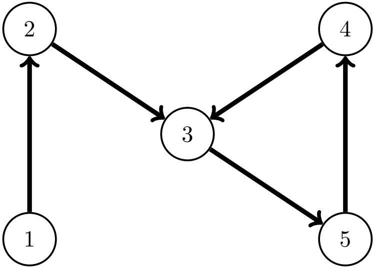


Envy graph (an agent points to another agent if she envies her).

Suppose the graph is for a partial allocation that is EF1 fair.

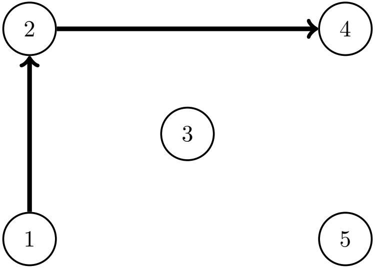
Agent 5 has no incoming arc so if she gets a new item, the allocation is still EF1 fair.

Algorithm for EF1 fairness (Lipton et al. (2004))



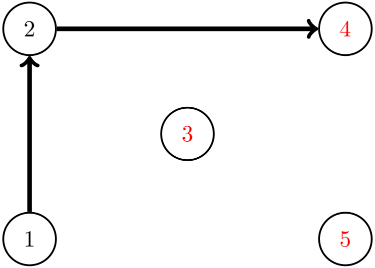
A new item is given to agent 5 who has no incoming arc. This may make some other agent envious (in this case agent 3 is now envious of agent 5).

Algorithm for EF1 fairness (Lipton et al. (2004))



We enable an exchange of allocations along the cycle which removes the cycle.

Algorithm for EF1 fairness (Lipton et al. (2004))



We enable an exchange of allocations along the cycle which removes the cycle.

Algorithm for EF1 fairness (Lipton et al. (2004))

Input : n agents, m items, and valuations $u_i(o_j) \geq 0$ for each $i \in [n]$ and $j \in O$.
Output: EF1 allocation X

Initialize allocation $X = (X(1), X(2), \ldots, X(n))$ with $X(i) = \emptyset$ for all $i \in [n]$.
for $j = 1$ to m **do**
    Construct an envy-graph $G(X) = (N, E)$ where $(i, j) \in E$ if i is envious of j's allocation wrt allocation X.
    Pick a vertex i that has no incoming edges in $G(X)$
    Update $X(i) \leftarrow X(i) \cup \{o_j\}$.
    while the $G(X)$ contains a cycle **do**
        Implement an exchange in which if i points to j in the cycle, then i gets j's allocation.
    end while
end for
Return X.

Fairness Overview

- EF implies proportionality which implies MmS fairness.
- EF implies EF1 fairness.
- EF, Proportional, and MmS fair allocations may not exist and are computationally hard to compute even if they exist.
- An EF1 allocation always exists and can be computed in polynomial time.

Outline

Strategyproofness

An allocation rule f is *strategyproof* if there exists no preference profile $\succsim$ such that

$$f(\succsim_1, \ldots, \succsim_{i-1}, \succsim_i', \succsim_{i+1}, \ldots, \succsim_n) \succ_i f(\succsim).$$

Example (Leximin Mechanism is not strategyproof)

	o_1	o_2	o_3	o_4
1	6	2	2	1
2	4	1	2	3

	o_1	o_2	o_3	o_4
1	4	2	3	2
2	4	1	2	3

Computational complexity

- We want that the solution concept should be efficiently computable
- Even if the algorithm computing the solution is manipulable, we may prefer that the algorithm is computationally hard to manipulate [Bartholdi, III et al., 1989].

Desirable properties of mechanisms: a summary

- *welfare*: utilitarian/egalitarian/Nash; Pareto optimality
- *fairness*: envy-freeness; proportionality; egalitarian equivalence; MmS; and EF1.
- *resistance to manipulation*: strategyproof; computationally hard to manipulate; rarely manipulable
- *computationally efficient*

We will also look at stability.

Outline

House Allocation

Consider the *house allocation* setting $(N, O, \succ)$ where $|N| = |O|$ and agents have strict and ordinal preferences over individual items. Each agent gets one item.

We use comma separated lists to denote the preference lists in strictly decreasing order of preferences.

Example

$$\succ_1, \succ_2: \quad o_1, o_2, o_3, o_4$$
$$\succ_3, \succ_4: \quad o_2, o_1, o_4, o_3$$

Serial Dictatorship

For a house allocation problem $(N, O, \succ)$ where $|N| = |O|$, Serial Dictatorship with respect to permutation π over N: agents get one turn each in the order of the permutation. They sequentially take their most preferred item that has not yet been allocated.

Example

$$\succ_1, \succ_2: \quad o_1, o_2, o_3, o_4 \qquad\qquad \succ_3, \succ_4: \quad o_2, o_1, o_4, o_3$$

$\pi = 1234.$

$$\text{SerialDictator}((N, O, \succ), \pi) = (\{o_1\}, \{o_2\}, \{o_4\}, \{o_3\}).$$

Serial Dictatorship

Non-bossiness: an agent cannot change her preference so that she gets the same allocation but some other agent gets a different allocation.
Neutral: the allocation does not depend on the names of the items.

Theorem (Svensson [1999])

For housing allocation problems, a mechanism is strategyproof, non-bossy and neutral if and only if it is a serial dictatorship.

Theorem (Abdulkadiroğlu and Sönmez [1998])

For housing allocation problems, an allocation is Pareto optimal iff it is a result of serial dictatorship.

Sequential Allocation where $|O|$ can be greater than $|N|$

For an assignment problem $(N, O, \succ)$, sequential allocation with respect to policy π over N: agents come in the order the policy π and sequentially take their most preferred item that has not yet been allocated.

Example

$$\succ_1: a, b, c, d$$
$$\succ_2: b, c, d, a$$

For policy: $\pi = 1212$

1. 1 takes a
2. 2 takes b
3. 1 takes c
4. 2 takes d

Sequential Allocation where $|O|$ can be greater than $|N|$

Fact (Kohler and Chandrasekaran [1971])

Sequential allocation is not strategyproof in general.

Idea: If an agent can always get a highly preferred item because no one likes it, she can take it later and first go for the highly demanded lesser preferred item.

Example

Policy: 1212

$$\succ_1 : a, b, c, d$$
$$\succ_2 : b, c, d, a$$

1 takes a; 2 takes b; 1 takes c; 2 takes d

$$\succ_1' : b, a, c, d$$
$$\succ_2 : b, c, d, a$$

1 takes b; 2 takes c; 1 takes a; 2 takes d

Outline

Housing market: model with endowments

$(N, O, \succ, \omega)$

- $|N| = |O|$
- $\omega(i) = \{o\}$ iff o is owned by $i \in N$.
- Agents have strict preferences over items
- Each agent owns and is allocated one item.

Example

Housing market $(N, O, \omega, \succ)$ such that

- $N = \{1, \ldots, 5\}$, $O = \{o_1, \ldots, o_5\}$,
- $\omega(i) = \{o_i\}$ for all $i \in \{1, \ldots, 5\}$
- and preferences $\succ$ are defined as follows:

agent	1	2	3	4	5
preferences	o_2	o_3	o_4	o_1	o_2
	o_1	o_2	o_3	o_5	o_4
				o_4	o_5

Individual rationality

An allocation X is *individually rational* if no agent minds participating in the allocation procedure:

$$\forall i \in N : \quad X(i) \succsim_i \omega(i)$$

If an agent does not have any endowment, her allocation is *individually rational* if her allocation is acceptable (at least as preferred as the empty allocation).

An agent can express an allocation or an item as unacceptable by simply not lising it in the preference list.

Allocation with endowments: Core

An allocation X is *core stable* if there exists no $S \subseteq N$ such that there exists an allocation Y of the items in $\bigcup_{i \in S} \omega(i)$ to the agents in S such that

$$\forall i \in S : \quad Y(i) \succ_i X(i)$$

Fact

A core stable allocation is individually rational.

Housing Markets: Gale's Top Trading Cycles (TTC) Algorithm

- Each item points to its owner.
- Each agent points to her most preferred item in the graph.
- Find a cycle, allocate to each agent in the cycle the item she was pointing to. Remove the agents and items in the cycle. Adjust the graph so the agents in the graph point to their most preferred item in the graph.
- Repeat until the graph is empty.

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- Each agent points to her most preferred item in the graph.
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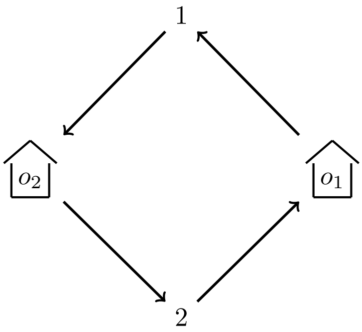
- Repeat until the graph is empty.

agents	1	2
item owned	o_1	o_2

agents	1	2
preferences	o_2	o_1
	o_1	o_2



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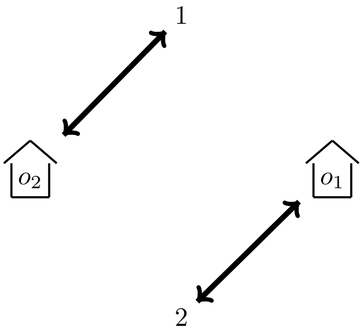
- Repeat until the graph is empty.

agents	1	2
item owned	o_1	o_2

agents	1	2
preferences	$\underline{o_2}$	$\underline{o_1}$
	o_1	o_2



Housing Market Example

Example

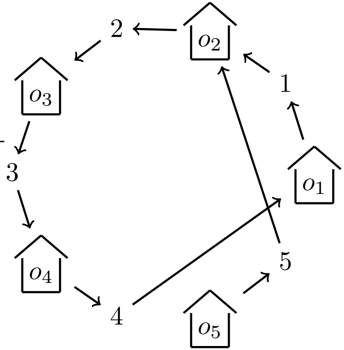
Housing market $M = (N, O, \omega, \succ)$ such that

- $N = \{1, \ldots, 5\}$, $O = \{o_1, \ldots, o_5\}$,
- $\omega(i) = \{o_i\}$ for all $i \in \{1, \ldots, 5\}$
- and preferences $\succ$ are defined as follows:

agent	1	2	3	4	5
preferences	o_2	o_3	o_4	o_1	o_2
	o_1	o_2	o_3	o_5	o_4
				o_4	o_5

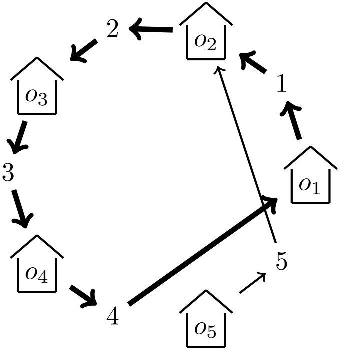
Housing Markets: Gale's Top Trading Cycles (TTC) Algorithm

agent	1	2	3	4	5
preferences	o_2	o_3	o_4	o_1	o_2
	o_1	o_2	o_3	o_5	o_4
				o_4	o_5



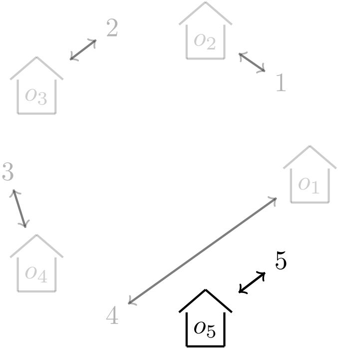
Example: TTC

agent	1	2	3	4	5
preferences	o_2	o_3	o_4	o_1	o_2
	o_1	o_2	o_3	o_5	o_4
				o_4	o_5



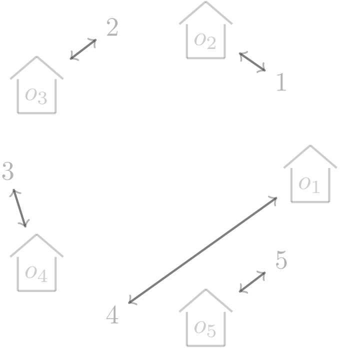
Housing Markets: Gale's Top Trading Cycles (TTC) Algorithm

agent	1	2	3	4	5
preferences	$\underline{o_2}$	$\underline{o_3}$	$\underline{o_4}$	$\underline{o_1}$	o_2
	o_1	o_2	o_3	o_5	o_4
				o_4	o_5



Housing Markets: Gale's Top Trading Cycles (TTC) Algorithm

agent	1	2	3	4	5
preferences	$\underline{o_2}$	$\underline{o_3}$	$\underline{o_4}$	$\underline{o_1}$	o_2
	o_1	o_2	o_3	o_5	o_4
				o_4	$\underline{o_5}$



TTC (Top Trading Cycles)

Theorem (Shapley and Scarf [1974] and Roth and Postlewaite [1977])

For housing markets (with strict preferences), TTC is strategyproof, individually rational, Pareto optimal and core stable.

Theorem (Ma [1994])

For housing markets (with strict preferences), a mechanism is strategyproof, individually rational and Pareto optimal iff it is TTC.

Outline

School Choice

- $N = \{1, \ldots, n\}$ set of of students/agents.
- Set of schools $S = \{s_1, \ldots, s_k\}$
- Each school $s \in S$ has $q(s)$ seats
- $\succ = (\succ_1, \ldots, \succ_n)$ preferences of agents. Each agent $i \in N$ has strict preferences over the schools.
- Each school has a strict priority $\succ_s$ order over the students.

Agents are students; school seats are items.

Student Proposing DA (Deferred Acceptance)

Agents (students) with strict preferences over schools; items (school seats with each school containing certain quota)

1. Each agent applies to her most preferred school. If a school s received at most $q(s)$ applications from the agents so far, all those agents are put on the school's waiting list. Otherwise the school puts its highest priority $q(s)$ agents among all applicants on the waiting list and rejects all remaining ones.
2. Each agent that was rejected in the previous step applies to her most preferred among the schools she has not yet applied to.
3. Steps 1 and 2 are repeated until it holds for all agents that they were either not rejected in the previous step or already applied to all schools.
4. Each school admits all the agents on its waiting list.

Student Proposing DA (Deferred Acceptance)

The quota of each school a, b, c is 1.

$1 : b, a, c$	$a : 1, 3, 2$
$2 : a, b, c$	$b : 2, 1, 3$
$3 : a, b, c$	$c : 2, 1, 3$

- 2 and 3 apply to a; 1 applies to b
- a rejects 2 in favour of 3 $\{\{1, b\}, \{3, a\}\}$
- 2 applies to b
- b rejects 1 in favour of 2 $\{\{2, b\}, \{3, a\}\}$
- 1 applies to a
- a rejects 3 in favour of 1 $\{\{2, b\}, \{1, a\}\}$
- 3 applies to b
- b rejects it in favour of 2
- 3 applies to c and gets accepted. $\{\{3, c\}, \{2, b\}, \{1, a\}\}$.

Student Proposing DA (Deferred Acceptance)

Student Proposing DA algorithm terminates in time linear in the size of the preference profile.

- In each step, the agents' potential school matches decreases (if it does not decrease each agent is matched)
- School's tentative matches keep improving (if they do not improve, it means there are no new proposals)

Student Proposing DA (Deferred Acceptance)

Justified envy-freeness: there exists no agent i who prefers another school s over her match and s admits j a lower priority agent than i.

Theorem (Roth and Sotomayor [1990])

Student Proposing DA returns an allocation that satisfies justified envy-freeness.

Suppose for contradiction that i has justified envy for j with respect to school s. Then i is matched to a less preferred school than s. Then i got rejected by s.

Case 1: If j had proposed to s at or before this time point, she would have been rejected by s as well.

Case 2: If j proposed to s after this time point, it would have been rejected as well since s has enough proposals by higher priority agents.

Student Proposing DA (Deferred Acceptance)

Justified envy-freeness: there exists no agent i who prefers another school s over her match and s had admitted j a lower priority agent than i.

Theorem (Roth and Sotomayor [1990])

Student Proposing DA is strategyproof. The resultant allocation Pareto dominates (wrt to students) all allocations that satisfy justified envy-freeness.

Two-sided matching

For details on two-sided matching, see books by Roth and Sotomayor [1990], Gusfield and Irving [1989] and Manlove [2013].

Outline

Allocation of divisible items: AW (Adjusted Winner)

Example

	o_1	o_2	o_3
1	67	6	27
2	34	5	61

- Initially, agent 1 gets 73 points; Agent 2 gets 61 points
- o_2 is given from agent 1 to agent 2
- o_1 must be partially given to agent 2. Agent 2 gets ${}^{1}/_{101}$ of o_1 and agent 1 gets ${}^{100}/_{101}$ so that both get $67 \times \frac{100}{101} \approx 66.3366337$ points.

	o_1	o_2	o_3
1	${}^{100}/_{101}$ 67	6	27
2	${}^{1}/_{101}$ 34	5	61

Allocation of divisible items: AW (Adjusted Winner)

Agent 1 and 2 are each given k points that they can use for acquiring m items.
Phase 1: Each item is assigned to the agent that values it the most. Ties are broken in favour of agent 1.
Phase 2: Some of the items are redistributed to ensure *equitability* (total points used by both agents on items they have won is same). In order for that to happen, at most one item may be required to be split.
Let x_i be the number of points used by 1 on item i and y_i be the number of points used by 2 on item i.

$$\underbrace{\frac{x_{k_1}}{y_{k_1}} \geq \frac{x_{k_2}}{y_{k_2}} \geq \cdots \geq \frac{x_{k_i}}{y_{k_i}} \geq}_{\text{Allocation of agent } 1} \Bigg| \geq \underbrace{\frac{x_{k_{i+1}}}{y_{k_{i+1}}} \geq \cdots \geq \frac{x_{k_m}}{y_{k_m}}}_{\text{Allocation of agent } 2}$$

Allocation of divisible items: AW (Adjusted Winner)

Equitability: all agents get the same utility.

Theorem (Brams and Taylor [1996])

AW is Pareto optimal, equitable, envy-free, and proportional, and requires at most one item to be split.

Theorem (Aziz et al. [2015])

For two agents, AW is the only Pareto optimal and equitable rule that requires at most one item to be split.

Allocation of divisible items: Proportional Allocation Rule

Both agents are given equal number of points that they can allocate to the items. Let x_i be the number of points used by 1 on item i and y_i be the number of points used by 2 on item i. Then agent 1 gets $\frac{x_i}{x_i+y_i}$ of the item o_i and 2 gets $\frac{y_i}{x_i+y_i}$ of the item o_i

Theorem (Brams and Taylor [1996])

The Proportional Allocation Rule is equitable and envy-free but not necessarily Pareto optimal.

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Theorem (Brams and Taylor [1996])

The Proportional Allocation Rule is equitable and envy-free but not necessarily Pareto optimal.

Argument for equitability:
Utility of agent 1 is $\sum_{i=1}^{m}(x_i \times \frac{x_i}{x_i+y_i})$. Utility of agent 2 is $\sum_{i=1}^{m}(y_i \times \frac{y_i}{x_i+y_i})$.

$$\sum_{i=1}^{m} \frac{x_i^2 - y_i^2}{x_i + y_i} = \sum_{i=1}^{m} \frac{(x_i - y_i)(x_i + y_i)}{x_i + y_i} = \sum_{i=1}^{m}(x_i - y_i) = \sum_{i=1}^{m} x_i - \sum_{i=1}^{m} y_i = 0.$$

Allocation of divisible items

Theorem (Zhou [1990])

If fractional allocations are allowed and agents have additive cardinal utilities, then strategyproofness, Pareto optimality and envy-freeness are incompatible.

Note that any two of the properties are easy to achieve:

- strategyproofness and Pareto optimality: dictatorship
- strategyproofness and envy-freeness: null allocation
- envy-freeness and Pareto optimality: Nash welfare maximizing allocation.

PA (Partial Allocation) mechanism for allocation of divisible items

1. Compute the Nash welfare maximizing allocation x^* based on the reported valuations.
2. For each agent i, remove the agent and compute the Nash welfare maximizing allocation $x^*{}_{-i}$ that would arise when i does not exist.
3. Allocate to each agent i a fraction f_i of everything i receives according to x^* where

$$f_i = \frac{\prod_{i' \neq i}[u_{i'}(x^*)]}{\prod_{i' \neq i}[u_{i'}(x^*_{-i})]}.$$

Theorem (Cole et al. [2013])

PA is strategyproof, envy-free and each agent gets $1/e$ of the utility she would get in a Nash welfare maximizing allocation.

Outline

Some notation

Feasible Outcome:

$$p = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

where $p(i)(o_j)$ is the probability of agent i getting o_j. It is the entry in the i-th row and j-th column of the matrix p.

Setting: Random Assignment Problem

An random assignment problem is a tuple $(N, O, \succ)$ where

- $N = \{1, \ldots, n\}$ is the set of agents
- $O = \{o_1, \ldots, o_n\}$ is the set of items
- $\succ_i$ is the strict and transitive preference of agent $i \in N$ over O
- agents may have private cardinal utilities

Example (Assignment problem)

$$\succ_1, \succ_2: \quad o_1, o_2, o_3, o_4$$

$$\succ_3, \succ_4: \quad o_2, o_1, o_4, o_3$$

Feasible Outcome:

$$p = \begin{pmatrix} 5/12 & 1/12 & 5/12 & 1/12 \\ 5/12 & 1/12 & 5/12 & 1/12 \\ 1/12 & 5/12 & 1/12 & 5/12 \\ 1/12 & 5/12 & 1/12 & 5/12 \end{pmatrix}.$$

where $p(i)(o_j)$ is the probability of agent i getting o_j. It is the entry in the i-th row and j-th column of the matrix p.
$p(i)$ is the **allocation** of agent i.

PS (Probabilistic Serial)

For an assignment problem $(N, O, \succ)$.

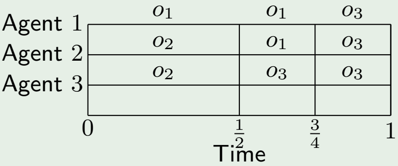
- Each item is considered to have a divisible probability weight of one, and agents simultaneously and with the same speed eat their most preferred item.
- Once an item has been eaten, the agent proceeds to eat the next most preferred item that has not been completely eaten.
- The procedure terminates after all the items have been eaten.
- The allocation of an agent by PS is the amount of each item she has eaten.

Proposed by Bogomolnaia and Moulin [2001].

PS (Probabilistic Serial)

Example (PS rule)

$$\succ_1: \quad o_1, o_2, o_3$$
$$\succ_2: \quad o_2, o_1, o_3$$
$$\succ_3: \quad o_2, o_3, o_1$$



$$PS(\succ_1, \succ_2, \succ_3) = \begin{pmatrix} 3/4 & 0 & 1/4 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1/2 & 1/2 \end{pmatrix}.$$

PS (Probabilistic Serial)

Example (PS)

Consider an assignment problem in which $N = \{1, 2, 3, 4\}$, $O = \{o_1, o_2, o_3, o_4\}$ and the preferences $\succ$ are as follows.

$$\succ_1, \succ_2\colon \quad o_1, o_2, o_3, o_4$$
$$\succ_3, \succ_4\colon \quad o_2, o_1, o_4, o_3$$

The following is the result of PS:

PS (Probabilistic Serial)

Example (PS)

Consider an assignment problem in which $N = \{1, 2, 3, 4\}$, $O = \{o_1, o_2, o_3, o_4\}$ and the preferences $\succ$ are as follows.

$$\begin{aligned} \succ_1, \succ_2&: \quad o_1, o_2, o_3, o_4 \\ \succ_3, \succ_4&: \quad o_2, o_1, o_4, o_3 \end{aligned}$$

The following is the result of PS:

$$PS(N, O, \succ) = \begin{pmatrix} 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 & 1/2 \end{pmatrix}.$$

The probability of agent 1 getting o_3 is $1/2$.

RSD (Random Serial Dictatorship)

For an assignment problem $(N, O, \succ)$, takes a permutation π uniformly at random and then applies serial dictatorship with respect to it.

Example (RSD)

Consider an assignment problem in which $N = \{1, 2, 3, 4\}$, $O = \{o_1, o_2, o_3, o_4\}$ and the preferences $\succsim$ are as follows.

$$\succ_1, \succ_2\colon \quad o_1, o_2, o_3, o_4$$

$$\succ_3, \succ_4\colon \quad o_2, o_1, o_4, o_3$$

The following is the result of RSD:

$$RSD(N, O, \succ) = \begin{pmatrix} 5/12 & 1/12 & 5/12 & 1/12 \\ 5/12 & 1/12 & 5/12 & 1/12 \\ 1/12 & 5/12 & 1/12 & 5/12 \\ 1/12 & 5/12 & 1/12 & 5/12 \end{pmatrix}.$$

The probability of agent 1 getting o_3 is $5/12$.

RSD (Random Serial Dictatorship)

Theorem (Aziz et al. [2013], Saban and Sethuraman [2013])

Checking whether an agent gets a particular item with probability at least $p \in (0,1)$ is NP-hard.

$$\succ_1, \succ_2\colon \quad o_1, o_2, o_3, o_4$$

$$\succ_3, \succ_4\colon \quad o_2, o_1, o_4, o_3$$

$$RSD(N, O, \succ) = \begin{pmatrix} 5/12 & 1/12 & 5/12 & 1/12 \\ 5/12 & 1/12 & 5/12 & 1/12 \\ 1/12 & 5/12 & 1/12 & 5/12 \\ 1/12 & 5/12 & 1/12 & 5/12 \end{pmatrix}$$

SD (Stochastic Dominance) relation between allocations

An agent **SD-prefers** one allocation over another if for each item o, the former allocation gives the agent as much probability of getting at least preferred an item as the latter allocation.

$$p(i) \succsim_i^{SD} q(i)$$

$$\iff \sum_{o_j \in \{o_k \mid o_k \succsim_i o\}} p(i)(o_j) \geq \sum_{o_j \in \{o_k \mid o_k \succsim_i o\}} q(i)(o_j) \text{ for all } o \in O.$$

Example (SD relation)

$$1: \quad o_1, o_2, o_3, o_4$$

$$\begin{pmatrix} 1/2 & 0 & 1/2 & 0 \end{pmatrix} \succ_1^{SD} \begin{pmatrix} 5/12 & 1/12 & 5/12 & 1/12 \end{pmatrix}$$

	$\sum_{i=1}^{1} p(1)(o_i)$	$\sum_{i=1}^{2} p(1)(o_i)$	$\sum_{i=1}^{3} p(1)(o_i)$	$\sum_{i=1}^{4} p(1)(o_i)$
Partial sum	$1/2$	$1/2$	1	1
Partial sum	$5/12$	$1/2$	$11/12$	1

SD (Stochastic Dominance) relation between allocations

An agent **SD-prefers** one allocation over another if for each item o, the former allocation gives the agent as much probability of getting at least preferred an item as the latter allocation.

$$p(i) \succsim_i^{SD} q(i)$$

$$\iff \sum_{o_j \in \{o_k \mid o_k \succsim_i o\}} p(i)(o_j) \geq \sum_{o_j \in \{o_k \mid o_k \succsim_i o\}} q(i)(o_j) \text{ for all } o \in O.$$

Stochastic dominance implies getting at least as much utility for all utility functions consistent with the ordinal preferences.

$$p(i) \succsim_i^{SD} q(i) \iff \sum_{o \in O} (p(i)(o)) u_i(o) \geq \sum_{o \in O} (q(i)(o)) u_i(o) \quad \forall u_i \in \mathcal{U}(\succsim_i)$$

where

$$u_i(o) \geq u_i(o') \text{ if } o \succsim_i o' \; \forall u_i \in \mathcal{U}(\succsim_i)$$

- **SD envy-freeness**: Each agent SD-prefers her allocation over allocations of other agents:

$$p(i) \succsim_i^{SD} p(j) \text{ for all } i, j \in N.$$

- **SD-efficiency**: Pareto optimality with respect to the SD relation. Assignment p is SD-efficient if there exists no q such that

$$q(i) \succsim_i^{SD} p(i) \text{ for all } i \in N$$

and

$$q(i) \succ_i^{SD} p(i) \text{ for some } i \in N.$$

- f is **SD-strategyproof** if

$$f(\succsim)(i) \succsim_i^{SD} f(\succsim_i', \succsim_{-i})(i) \text{ for } i \in N.$$

PS (Probabilistic Serial)

$\succ_1, \succ_2$: o_1, o_2, o_3, o_4

$\succ_3, \succ_4$: o_2, o_1, o_4, o_3

$$RSD(N, O, \succ) = \begin{pmatrix} 5/12 & 1/12 & 5/12 & 1/12 \\ 5/12 & 1/12 & 5/12 & 1/12 \\ 1/12 & 5/12 & 1/12 & 5/12 \\ 1/12 & 5/12 & 1/12 & 5/12 \end{pmatrix}.$$

$$PS(N, O, \succ) = \begin{pmatrix} 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 & 1/2 \end{pmatrix}.$$

Theorem (Bogomolnaia and Moulin [2001])

RSD is SD-strategyproof but not SD-efficient or SD envy-free. PS is SD-efficient and SD envy-free but not SD-strategyproof.

Survey and Further Reading

- Most relevant resource: book chapter by Bouveret et al. [2016] in the Handbook of Computational Social Choice.
  http://www.cse.unsw.edu.au/~haziz/comsoc.pdf
- Brandt et al. [2016] especially chapters 11-14
- Brams and Taylor [1996]
- Robertson and Webb [1998]
- Moulin [2003]
- Endriss [2010]
- Roth and Sotomayor [1990]
- Gusfield and Irving [1989]
- Manlove [2013]
- Chalkiadakis et al. [2011]

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