COMP4418 Review Final A

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Coalitional Game

Definition (Coalitional game)

- A **coalitional game** is a pair (N, v)
- $N = \{1, \ldots, n\}$ is the set of players
- $v : 2^N \to \mathbb{R}$ is a *valuation function* that associates with each coalition $S \subseteq N$ a value $v(S)$ where $v(\emptyset) = 0$.
- $v(S)$ can be considered as the value generated when players in coalition S cooperate.

Usual assumptions: valuations are *non-negative* and v is *monotonic* i.e., $S \subseteq T \subseteq N$ implies that $v(S) \leq v(T)$,

在合作游戏中我们并不考虑竞争的情况

Example

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	4	2	1	7	10	11	15

Coalitional Game

Payoff

Definition (Payoffs)

A *payoff vector* $(x_1, \ldots, x_n) \in \mathbb{R}^N$ specifies for each player $i \in N$ the payoff x_i which is player i's share of $v(N)$.

$v(N)$ 表示玩家合作所能达到的利益

Payoff 向量中的 x_i 表示每个玩家能从最后的盈利中获得的份额

Definition (Efficient payoffs)

A *payoff vector* $(x_1, \ldots, x_n) \in \mathbb{R}^N$ is **efficient** if $\sum_{i \in N} x_i = v(N)$, where x_i denotes player i's share of $v(N)$.
Notation: $x(S) = \sum_{i \in S} x_i$

Efficient payoffs 是指玩家的所有份额之和等于总收益（没有收益被浪费）

Definition (Individual rational payoffs)

A payoff vector $x = (x_1, \ldots, x_n)$ satisfies **individual rationality** if $x_i \geq v(\{i\})$ for all $i \in N$.

Individual rational payoffs 是指玩家合作获得份额要大于自己独自游戏所能获得的收益

Core

Definition (Core)

A payoff vector $x = (x_1, \ldots, x_n)$ is in the **core** of a coalitional game (N, v) if for all $S \subset N$, $x(S) \geq v(S)$.

Core表示了一种可以让大家都愿意合作的分配方式

在这种情况下，所有人合作要比拒绝合作（比如一个人或者小范围串联）获得的收益一样或更大

所有人都合作的情况也称为 grand coalition

A parliament is made up of four political parties, A, B, C, and D, which have 45, 25, 15, and 15 representatives, respectively. They are to vote on whether to pass a $100 million spending bill and how much of this amount should be controlled by each of the parties. A majority vote, that is, a minimum of 51 votes, is required in order to pass any legislation, and if the bill does not pass then every party gets zero to spend.

	A	B	C	D
Sharpley	50	16.67	16.67	16.67
A和B合作	75	25	0	0

Coalitional Game

Simple Game

Definition (Simple coalitional game)

- A **simple coalition game** is a monotone coalitional game (N, v) with $v : 2^N \to \{0, 1\}$ such that $v(N) = 1$.
- A coalition $S \subseteq N$ is **winning** if $v(S) = 1$ and **losing** if $v(S) = 0$.

Veto Player 拥有一票否决权的玩家

Definition (Vetoer)

A player i is a **vetoer** if $v(S) = 0$ for any $S \subseteq N \setminus \{i\}$.

Player 1 has a right hand glove, player 2 has a left hand glove and player 3 also has a left hand glove. A group of players has gets value 1 for a proper pair of gloves and 0 otherwise.

S	$\{1,2\}$	$\{1,3\}$	$\{1,2,3\}$	$\emptyset$	$\{2\}$	$\{3\}$	$\{1\}$	$\{2,3\}$
$v(S)$	1	1	1	0	0	0	0	0

Table: Glove Game

Coalitional Game

Convex Game

Definition (Convex Game)

(N, v) is **convex** if

$$v(S \cup T) \geq v(S) + v(T) - v(S \cap T)$$

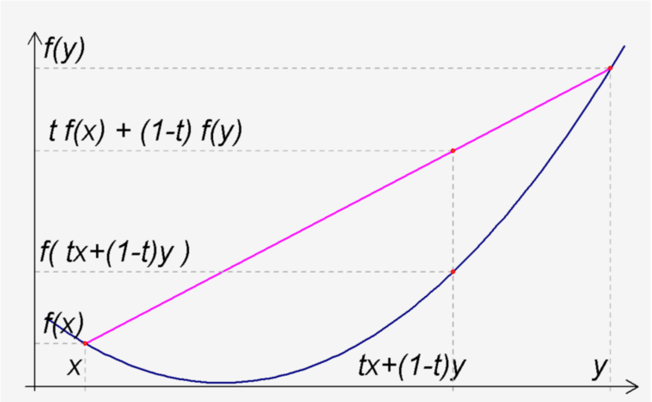
for all $S, T \subset N$.

Equivalently, (N, v) is **convex** if

$$v(A \cup \{i\}) - v(A) \geq v(B \cup \{i\}) - v(B)$$

for all $A, B \subseteq N \setminus \{i\}$ such that $B \subseteq A$.

在convex game当中，越多人合作获得的收益增幅越大
收益曲线是凸的



Core and Simple / Convex Game

一般我们关注Core的两个问题：**empty & unique**

Is Core empty?

Theorem

A simple game (N, v) has a non-empty core iff it has a vetoer. Moreover, an outcome $(x_1, \ldots, x_n)$ is in the core iff $x_i = 0$ for all non-veto players.

Theorem (Shapley, 1971)

A convex game has a non-empty core.

Is Core unique? **No!**

	A	B	C	D
票数	45	25	15	15

假如标准提高为至少获得80票的决议才能被通过，则 {A,B} 构成了Veto Player 无论A和B如何分配都属于Core

Coalitional Game

Sharpley Value

$$\phi_i = \frac{1}{|N|!} \sum_{S \subseteq N \setminus \{i\}} (|S|!)(|N| - |S| - 1)!(v(S \cup \{i\}) - v(S))$$

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{2,3\}$	$\{1,2\}$	$\{1,3\}$	$\{1,2,3\}$
$v(S)$	0	0	0	0	500	500	750	1000

- 213: $v(\{2\}) - v(\emptyset) = 0$
- 231: $v(\{2\}) - v(\emptyset) = 0$
- 123: $v(\{1,2\}) - v(\{1\}) = 500$
- 321: $v(\{3,2\}) - v(\{3\}) = 500$
- 312: $v(\{1,2,3\}) - v(\{1,3\}) = 250$
- 132: $v(\{1,2,3\}) - v(\{1,3\}) = 250$

$$\phi_2 = (500 + 500 + 250 + 250)/6 = 250.$$

$$\phi_1 = \phi_3 = 375$$

Sharpley Value

A parliament is made up of four political parties, A, B, C, and D, which have 45, 25, 15, and 15 representatives, respectively. They are to vote on whether to pass a $100 million spending bill and how much of this amount should be controlled by each of the parties. A majority vote, that is, a minimum of 51 votes, is required in order to pass any legislation, and if the bill does not pass then every party gets zero to spend.

Calculate the Sharpley Value

	A	B	C	D
Sharpley	50	16.67	16.67	16.67

Sharpley Value: 特性 Sharpley Value 满足 Efficiency

Symmetry axiom

- The **symmetry axiom** says that players which make the same contribution should get the same payoff.
  $v(S \cup \{i\}) - v(S) = v(S \cup \{j\}) - v(S)$ for all $S \subseteq N \setminus \{i, j\} \Rightarrow \phi_i = \phi_j$

Dummy player axiom

- The **dummy player axiom** says that players which make no contribution should get no payoff: if $v(S \cup \{i\}) - v(S) = 0$ for all $S \subseteq N \setminus \{i\}$, $\Rightarrow \phi_i = 0$.

Additivity axiom

- $(N, v_1 + v_2)$ is the game such that $(v_1 + v_2)(S) = v_1(S) + v_2(S)$ for all $S \subseteq N$. **Additivity axiom** says that
  $\forall i \in N$, $\phi_i(N, v_1 + v_2) = \phi_i(N, v_1) + \phi_i(N, v_2)$

Nucleolous excess vector

Definition 8.1 Let $e(x;S) = v(S) - \sum_{i \in S} x_i$ the excess of a coalition S. For every imputation we order the coalitions $S \subset N$ non-increasing. Put $\theta(x)$ such that $\theta(x)_i \geq \theta(x)_j$ for all $i \leq j$.

Thus we can see $\theta(x)$ as a 2^n-dimensional vector.

Definition 8.2 We can order two imputations x, y lexicographically. $x <_l y$ if $\theta(x)_j = \theta(y)_j$ for $1 \leq j < i$ and $\theta(x)_i < \theta(y)_i$ for a certain i.

An example: let $N = \{1,2,3\}$, $\mathcal{C} = \{1,2\}$, $v(\{1,2\}) = 8$, $x = \langle 3,2,5 \rangle$,
$e(\mathcal{C}, x) = v(\{1,2\}) - (x_1 + x_2) = 8 - (3+2) = 3.$

Excess vector可以理解为成员对分配的不满程度

Coalitional Game

Nucleolous excess vector

$$N=\{1,2,3\},\ v(\{i\})=0 \text{ for } i\in\{1,2,3\}$$
$$v(\{1,2\})=5,\ v(\{1,3\})=6,\ v(\{2,3\})=6$$
$$v(N)=8$$

Let us consider two payoff vectors $x=\langle 3,3,2\rangle$ and $y=\langle 2,3,3\rangle$. Let $e(x)$ denote the sequence of **excesses** of all coalitions at x.

$x=\langle 3,3,2\rangle$

coalition $\mathcal{C}$	$e(\mathcal{C},x)$
{1}	-3
{2}	-3
{3}	-2
{1,2}	-1
{1,3}	1
{2,3}	1
{1,2,3}	0

$y=\langle 2,3,3\rangle$

coalition $\mathcal{C}$	$e(\mathcal{C},y)$
{1}	-2
{2}	-3
{3}	-3
{1,2}	0
{1,3}	1
{2,3}	0
{1,2,3}	0

Which payoff should we prefer? x or y? Let us write the excess in the decreasing order (from the greatest excess to the smallest)

$\langle 1,1,0,-1,-2,-3,-3\rangle$ $\langle 1,0,0,0,-2,-3,-3\rangle$

- The nucleolus is in the least core.
- It is in the core if the core is non-empty.
- The nucleolus is unique [Schmeidler, 1969]

Nucleolous的计算复杂，其主要目的在于尽可能找到贡献最大的人

In the example, $e(x)=\langle -3,-3,-2,-1,1,1,0\rangle$ and then $e(x)^{\blacktriangleright}=\langle 1,1,0,-1,-2,-3,-3\rangle$.

Hence, we can say that y is better than x by writing
$$e(x)^{\blacktriangleright}\geqslant_{lex} e(y)^{\blacktriangleright}.$$

Banzhaf Index

Definition (Banzhaf index)

- A player i is **critical** in a coalition C if the player's exclusion results in C changing from winning to losing.
- **Banzhaf value** η_i of a player i is the number of coalitions for which i is critical.
- **Banzhaf index**

$$\beta_i = \frac{\eta_i}{\sum_{i \in N} \eta_i}$$

S	$\{1,2\}$	$\{1,3\}$	$\{1,2,3\}$	$\emptyset$	$\{2\}$	$\{3\}$	$\{1\}$	$\{2,3\}$
$v(S)$	1	1	1	0	0	0	0	0

Table: Game

- Banzhaf indices: ?
  $\beta_1 = 3/5$; $\beta_2 = 1/5$; $\beta_3 = 1/5$.

Coalitional Game

Weighted Voting Game

Definition (Weighted voting game)

- Players, $N = \{1, ..., n\}$ with corresponding voting weights $\{w_1, ..., w_n\}$
- **Quota**, $0 \le q \le \sum_{1 \le i \le n} w_i$
- $v(S) = 1$ if and only if $\sum_{i \in S} w_i \ge q$.
- Notation: $[q; w_1, ..., w_n]$

Example

S	$\{1,2\}$	$\{1,3\}$	$\{1,2,3\}$	$\emptyset$	$\{2\}$	$\{3\}$	$\{1\}$	$\{2,3\}$
$v(S)$	1	1	1	0	0	0	0	0

$[3; 2, 1, 1]$

所有Simple Game都可以用WVG来表示吗？ No!

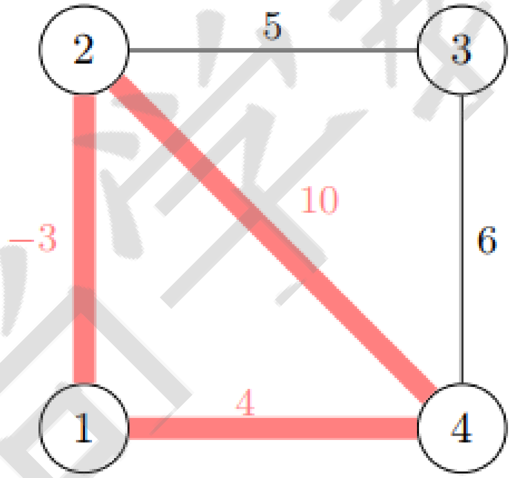
Coalitional Game

Graph Game

Definition (Graph game)

Graph game: Let $G = (V, E, w)$ be a weighted undirected graph. The **graph game** for $S \subseteq N$, corresponding to G is the coalitional game (N, v) with

- $N = V$
- for each $S \subseteq N$, the value $v(S)$ is the sum of the weight of the edges in the subgraph induced by S.



Coalitional Game

Marginal Contribution Net

Definition (Marginal Contribution Nets)

- Valuation function represented as **rules**: pattern $\rightarrow$ value.
- Pattern is conjunction of players (negation of a player is allowed).
- Value of a coalition is the sum over the values of all the rules that apply to the coalition.

Example

$x_1 \wedge x_2 \rightarrow 4$, $x_1 \rightarrow 1$, $\neg x_3 \rightarrow 2$. Then we have $v(\{1,2\}) = 4+1+2 = 7$ as all three rules apply to coalition $\{1,2\}$.

1. $x_1 \wedge x_2 \longrightarrow 5$
2. $x_2 \longrightarrow 2$
3. $x_3 \longrightarrow 4$
4. $x_2 \wedge \neg x_3 \longrightarrow -2$

- $v(\{1\}) = 0$ (no rules apply)
- $v(\{2\}) = 0$ (rules 2 and 4 apply)
- $v(\{3\}) = 4$ (rules 3 applies)
- $v(\{1,2\}) = 5$ (rules 1, 2, 4 apply)
- $v(\{1,3\}) = 4$ (rule 3 applies)
- $v(\{2,3\}) = 6$ (rules 2 and 3 apply)
- $v(\{1,2,3\}) =$ **?**

Summary

Solution concept	Existence	Uniqueness
Core	-	-
Least Core	✓	-
Nucleolus	✓	✓
Shapley value	✓	✓

Table: Solution concepts for coalitional games

- For $\epsilon > 0$, a payoff vector x is in the ϵ-**core** if for all $S \subset N$, $e(x, S) \geq -\epsilon$.
- The **least core** is the intersection of all non-empty ϵ-cores.
- The **least core** is the refinement of the ϵ-core and is the solution of the following LP:

$$\begin{aligned} \min \quad & \epsilon \\ \text{s.t.} \quad & x(S) \geq v(S) - \epsilon \quad \text{for all } S \subset N, \\ & x_i \geq 0 \text{ for all } i \in N, \\ & \textstyle\sum_{i=1,\ldots,n} x_i = v(N)\ . \end{aligned} \tag{1}$$

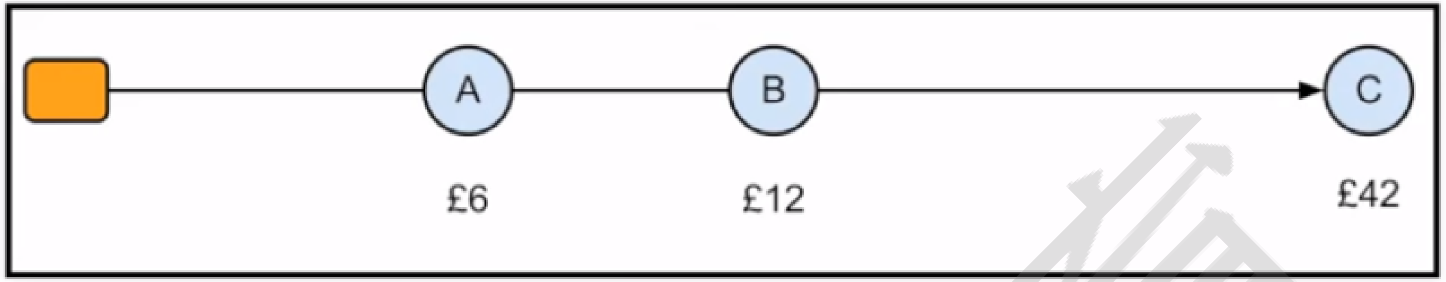
Coalitional Game

Exercise A,B,C are sharing a meal and the amount they want to pay is like below. Calculate the Sharpley value for each of them.

$$v(c) = \begin{cases} 80, & \text{if } c = \{A\} \\ 56, & \text{if } c = \{B\} \\ 70, & \text{if } c = \{C\} \\ 80, & \text{if } c = \{A, B\} \\ 85, & \text{if } c = \{A, C\} \\ 72, & \text{if } c = \{B, C\} \\ 90, & \text{if } c = \{A, B, C\} \end{cases}$$

Coalitional Game

Exercise



A,B,C are trying to catch a taxi together. The amount of fee for them to get home by taxi is given as shown above. Model the game and calculate how much they each should pay.

Non-coalitional Game

Pure Nash Equilibria

Mixed Nash Equilibria

An n-player game (N, A, u) consists of

- Set of players $N = \{1, \ldots, n\}$
- $A = A_1 \times \cdots A_n$ where A_i is the action set of player i
  - $a \in A$ is an action profile.
  - $u = (u_1, \ldots, u_n)$ specifies a utility function $u_i : A \to \mathbb{R}$ for each player.

Non-coalitional Game

囚徒困境

Both prisoners benefit if they cooperate. If one prisoner defects and the other does not, then the defecting prisoner gets scott free!

	cooperate	defect
cooperate	2,2	0,3
defect	3,0	1,1

	a_2^1	a_2^2
a_1^1	$u_1(a_1^1, a_2^1), u_2(a_1^1, a_2^1)$	$u_1(a_1^1, a_2^2), u_2(a_1^1, a_2^2)$
a_1^2	$u_1(a_1^2, a_2^1), u_2(a_1^2, a_2^1)$	$u_1(a_1^2, a_2^2), u_2(a_1^2, a_2^2)$

- Actions of player $1 = A_1 = \{a_1^1, a_1^2\}$.
- Actions of player $2 = A_2 = \{a_2^1, a_2^2\}$.

Non-coalitional Game

Pure Nash Equilibria

	Ballet	Football
Ballet	2,1	0,0
Football	0,0	1,2

Pure Nash equilibria:

- (Ballet, Ballet)
- (Football, Football)

	cooperate	defect
cooperate	2,2	0,3
defect	3,0	1,1

- The only Nash equilibrium is (defect, defect).
- The outcome of (defect,defect) is Pareto dominated by the outcome of (cooperate, cooperate).

Non-coalitional Game

Pure Nash Equilibria

	Left	Right
Left	1	-1
Right	-1	1

A pure Nash Equilibria may not exist.

单纯纳什均衡有可能不存在，或者一个，或者多个

Non-coalitional Game

Mixed Nash Equilibria

Definition (Best Response)

Best response: $s_i' \in BR(s_{-i})$ iff $\forall s_i \in S_i$, $u_i(s_i', s_{-i}) \geq u_i(s_i, s_{-i})$.

The best response of a player gives the player maximum possible utility.

Definition (Nash equilibrium)

$s = (s_1, \ldots, s_n)$ is a Nash equilibrium iff $\forall i \in N$, $s_i \in BR(s_{-i})$.

A Nash equilibrium is an action profile in which each player plays a best response.

Non-coalitional Game

Mixed Nash Equilibria

	Ballet	Football
Ballet	2,1	0,0
Football	0,0	1,2

- Let us assume that both players play their full support.
- Player 2 plays B with p and F with probability $1-p$.
- Player 1 must be indifferent between the actions it plays.

$$2(p) + 0(1-p) = 0p + 1(1-p)$$
$$p = 1/3.$$

- Player 1 plays B with q and F with probability $1-q$
- Player 2 must be indifferent between the actions it plays.

$$1(q) + 0(1-q) = 0q + 2(1-q)$$
$$q = 1/3.$$

Thus the mixed strategies $(2/3, 1/3), (1/3, 2/3)$ are in Nash equilibrium.

Exercise

		Server	
		F	B
Receiver	F	90,10	20,80
	B	30,70	60,40

In a tennis match, a server can aim the forehand or backhand of the receiver. The receiver can choose to move to its forehand or backhand.

a) What is the pure Nash equilibria? Why?

b) What is the mixed Nash equilibria for both players?

Social Choice

- Voting Rules
  - Plurality and Score rules
  - Condorcet winner
- Majority Graph
  - Copeland
  - Toy Cycle
  - Uncovered Set
  - Banks
- Restricted Domain

Social Choice – Voting Rules

- **Plurality**: alternatives that are ranked first by most voters win.

- **Borda**: Most preferred alternative gets $m-1$ points, the second

- **Plurality with runoff**: Two alternatives that are ranked first by most voters are short-listed. Then among the shortlisted alternatives, the alternative which is preferred by a majority wins.

  先选出两名最受欢迎的候选人，再进行决赛，记分方式和Plurality一样

- **Instant Runoff**: Alternatives that are ranked first by the lowest number of voters are removed from consideration. Repeat until no more alternatives can be deleted.

  每轮中最不受欢迎的候选人被淘汰出局，记分方式和Plurality一样

Social Choice – Voting Rules

$$
\begin{aligned}
33\% \text{ voters}: &\quad a \succ b \succ c \succ d \succ e \\
16\% \text{ voters}: &\quad b \succ d \succ c \succ e \succ a \\
3\% \text{ voters}: &\quad c \succ d \succ b \succ a \succ e \\
8\% \text{ voters}: &\quad c \succ e \succ b \succ d \succ a \\
18\% \text{ voters}: &\quad d \succ e \succ c \succ b \succ a \\
22\% \text{ voters}: &\quad e \succ c \succ b \succ d \succ a
\end{aligned}
$$

- Plurality winner: a
- Borda winner: c
- Plurality with runoff: e (after beating a)
- Instant Runoff: d (removal: c, b, e, a)

- **Borda** rule: PSR with scoring vector $(m-1, m-2, \ldots, 0)$
- **Plurality** rule: PSR with scoring vector $(1, 0, \ldots, 0)$

记分方式

A **positional scoring rule (PSR)** is given by a scoring vector $s = (s_1, \ldots, s_m)$ with $s_1 \geq s_2 \cdots \geq s_m$ and $s_1 > s_m$. When a voter puts alternative a in position j, a gets score s_j. Alternatives with the maximum total score win.

- **Borda** rule: PSR with scoring vector $(m-1, m-2, \ldots, 0)$
- **Plurality** rule: PSR with scoring vector $(1, 0, \ldots, 0)$
- **Antiplurality** rule: PSR with scoring vector $(1, 1, \ldots, 1, 0)$
- k **approval** rule: PSR with scoring vector $(\underbrace{1, \ldots, 1}_{k}, 0, \ldots, 0)$

Axioms of Voting Rules

- **Anonymity:** The voting rule treats voters equally: the outcome remains the same as long as the set of votes is the same.
- **Neutrality:** The voting rule treats alternatives equally: F is neutral if $F(\pi(\succ)) = \pi(F(\succ))$ where π is a permutation $\pi : A \longrightarrow A$.
- **Monotonicity:** A chosen alternative will still be chosen when it rises in individual preference rankings (while leaving everything else unchanged).
- **Strategyproof:** A voter cannot misreport his/her preference to select a more preferred alternative.
- **Pareto optimality:** An alternative will not be chosen if there exists another one that all voters prefer the latter to the former.
- **Independence of Irrelevant Alternatives (IIA):** If alternative a is socially preferred to b, then this should not change when a voter changes her ranking of $c \neq a, b$.

Social Choice – Voting Rules

Anonymity 匿名性:
如果选票情况不变，则结果不变

Example

$N = \{1, 2, 3\}$, $A = \{a, b, c, d\}$

$$1 : a \succ_1 b \succ_1 c \succ_1 d$$
$$2 : a \succ_2 c \succ_2 b \succ_2 d$$
$$3 : b \succ_3 d \succ_3 c \succ_3 a$$

$$3 : a \succ_1 b \succ_1 c \succ_1 d$$
$$2 : a \succ_2 c \succ_2 b \succ_2 d$$
$$1 : b \succ_3 d \succ_3 c \succ_3 a$$

两次选举结果应该相同，因为选票没有发生变化

Neutrality 中立性:
结果只和选票情况有关，和候选人名字无关

Example

$N = \{1, 2, 3\}$, $A = \{a, b, c, d\}$

$$1 : a \succ_1 b \succ_1 c \succ_1 d$$
$$2 : a \succ_2 c \succ_2 b \succ_2 d$$
$$3 : b \succ_3 d \succ_3 c \succ_3 a$$

$$3 : b \succ_1 c \succ_1 d \succ_1 a$$
$$2 : b \succ_2 d \succ_2 c \succ_2 a$$
$$1 : c \succ_3 a \succ_3 d \succ_3 b$$

如果第一次a获胜，那么第二次b也应该获胜

Monotonicity 单调性:
如果获胜人在某些选票中的顺位提升，他依然获胜

Example

$N = \{1,2,3\}$, $A = \{a,b,c,d\}$

$$1: a \succ_1 b \succ_1 c \succ_1 d$$
$$2: a \succ_2 c \succ_2 b \succ_2 d$$
$$3: b \succ_3 d \succ_3 c \succ_3 a$$

$$1: a \succ_1 b \succ_1 c \succ_1 d$$
$$2: a \succ_2 c \succ_2 b \succ_2 d$$
$$3: b \succ_3 d \succ_3 a \succ_3 c$$

第一次a获胜，第二次也应该获胜

Strategyproof 反策略性：
选民不能以撒谎的方式操纵选举结果（谎报自己的偏好）

Example

$N = \{1,2,3\}$, $A = \{a,b,c,d\}$

$$1: a \succ_1 b \succ_1 c \succ_1 d$$
$$2: c \succ_2 a \succ_2 b \succ_2 d$$
$$3: c \succ_3 d \succ_3 a \succ_3 b$$

$$1: a \succ_1 b \succ_1 c \succ_1 d$$
$$2: c \succ_2 b \succ_2 d \succ_2 a$$
$$3: c \succ_3 d \succ_3 a \succ_3 b$$

如果第一次a获胜，第二次c获胜，则2号选民可以操纵结果，不符合strategyproof

Social Choice – Voting Rules

Pareto Optimality 帕累托最优:

Example

$N = \{1, 2, 3\}, \ A = \{a, b, c, d\}$

$$1 : a \succ_1 b \succ_1 c \succ_1 d$$
$$2 : a \succ_2 c \succ_2 b \succ_2 d$$
$$3 : c \succ_3 d \succ_3 a \succ_3 b$$

在符合帕累托最优的选举规则下，b不可能获胜，因为a在所有选民中都优于b

Independence of Irrelevant Alternatives 无关项独立：
其他候选人的情况不影响单独比较两人时的结果

Example

$N = \{1, 2, 3\}, \ A = \{a, b, c, d\}$

$$1 : a \succ_1 b \succ_1 c \succ_1 d$$
$$2 : a \succ_2 c \succ_2 b \succ_2 d$$
$$3 : c \succ_3 d \succ_3 a \succ_3 b$$

$$1 : a \succ_1 b \succ_1 d \succ_1 c$$
$$2 : a \succ_2 c \succ_2 b \succ_2 d$$
$$3 : d \succ_3 c \succ_3 a \succ_3 b$$

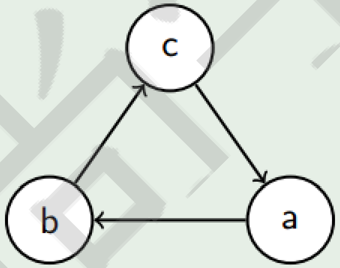
a一直优于b，d的排位变化不会影响这个结果

Condorcet winner: an alternative that is pairwise preferred by a majority of voters over every other alternative.

A Condorcet winner **may not exist – Condorcet's Paradox**

Example (Condorcet's Paradox)

$$1 : a \succ_1 b \succ_1 c$$
$$2 : b \succ_2 c \succ_2 a$$
$$3 : c \succ_3 a \succ_3 b$$



- **Condorcet-extension:** if an alternative is pairwise preferred by a majority of voters over every other alternative, then that alternative is selected.

Example

$N = \{1, 2, 3\}$, $A = \{a, b, c, d\}$

$$1 : a \succ_1 b \succ_1 c \succ_1 d$$
$$2 : a \succ_2 c \succ_2 b \succ_2 d$$
$$3 : c \succ_3 d \succ_3 a \succ_3 b$$

A Condorcet-extension voting rule should select a.

Social Choice – Majority Graph

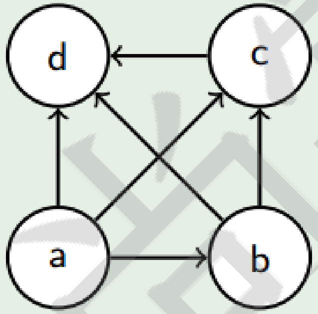
Given $(N, A \succ)$, the **corresponding majority graph** is a directed graph (V, E) in which $(x, y) \in E$ if and only if x is preferred over y by a majority of voters. If $(x, y) \in E$, we say that x *dominates* y. We will denote $D(x) = \{y \mid (x, y) \in E\}$.

这类选举比赛我们称之为 **Tournament**

Example (Tournament)

$N = \{1, 2, 3\}$, $A = \{a, b, c, d\}$.

$$1 : a \succ_1 b \succ_1 c \succ_1 d$$
$$2 : a \succ_2 c \succ_2 b \succ_2 d$$
$$3 : b \succ_3 d \succ_3 c \succ_3 a$$



Social Choice – Copeland Rule

The Copeland rule selects alternatives based on the number of other alternatives they dominate. Define the **Copeland score** of an alternative x in tournament $T = (V, E)$ as the outdegree of the alternative.

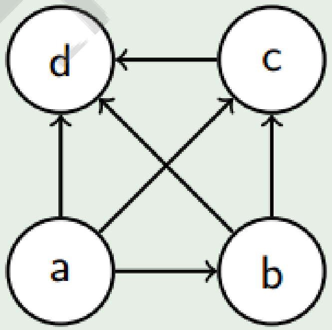
即看一个候选人节点的出度（向外指向的边的数量）

The set of **Copeland winners** $CO(T)$ then consists of all alternatives that have maximal Copeland score.

The Copeland rule is a Condorcet-extension.

- **Condorcet-extension:** if an alternative is pairwise preferred by a majority of voters over every other alternative, then that alternative is selected.

Example (Tournament)



$CO(T) = \{a\}$.

Social Choice – Top Cycle

A non-empty subset $X \subseteq V$ of alternatives in a tournament (V, E) is **dominant** if every alternative in X dominates every alternative outside X.

The **top cycle** of a tournament $T = (V, E)$, denoted by $TC(T)$, is the unique **minimal** dominant subset of V.

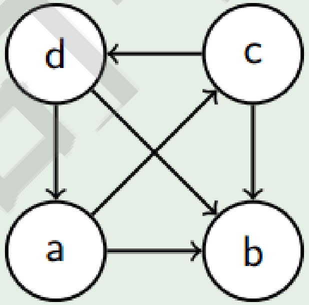
从一个点出发可以找到路径到达其他所有点

The top cycle rule is a Condorcet-extension.

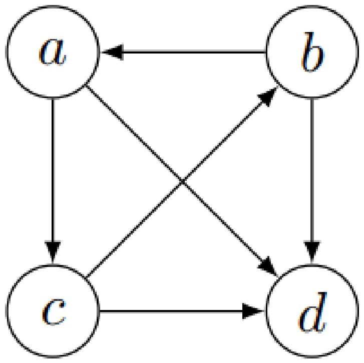
但是在超过3个候选人时，Top Cycle不是帕累托最优

Example (Tournament)

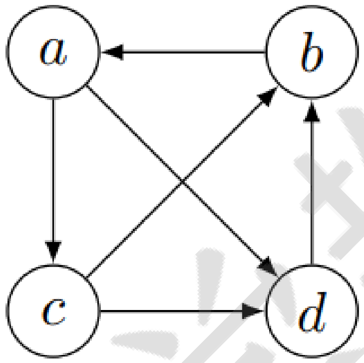
Top cycle: $\{a, c, d\}$



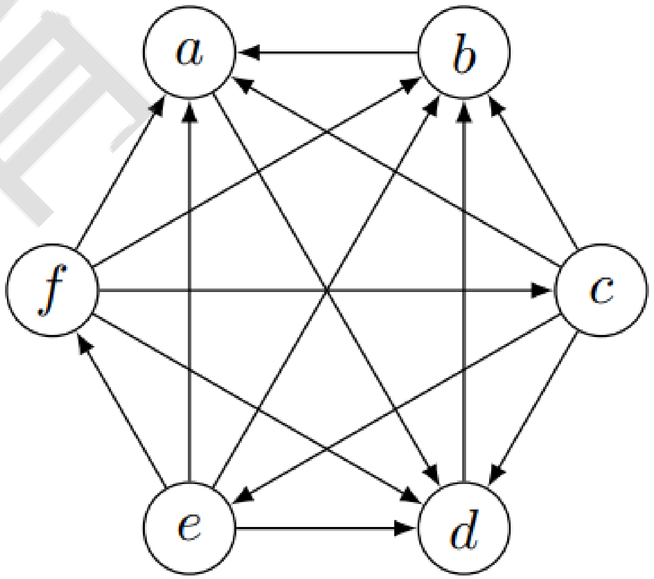
Social Choice – Top Cycle



$TC(A,R_M)=\{a,b,c\}$



$TC(A,R_M)=\{a,b,c,d\}$



$TC(A,R_M)=\{c,e,f\}$

The **Uncovered Set** of a tournament $T = (V, E)$, denoted by $UC(T)$, is the set of alternative that can reach every other alternative in at most two steps.

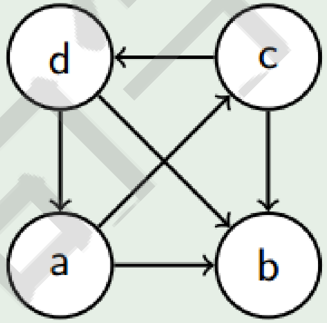
和Top Cycle的主要区别：Uncovered Set的节点要在**两步**之内指向所有其他外部节点
因此Uncovered Set是Top Cycle的子集

The alternatives in the uncovered set are also referred to as kings.

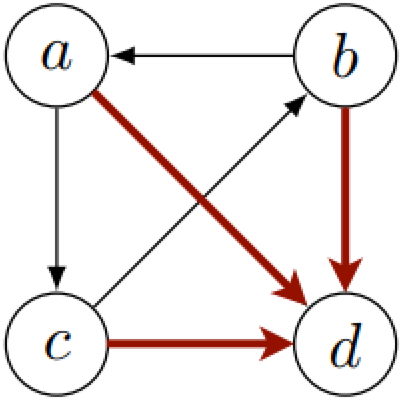
The uncovered set rule is a Condorcet-extension.

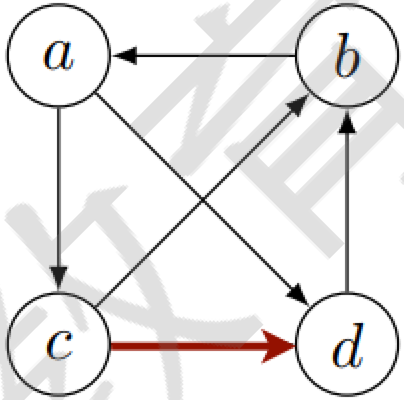
Example (Tournament)

Uncovered Set: $\{a, c, d\}$



Social Choice – Uncovered Set





$UC(A, R_M) = \{a, b, c\}$

$UC(A, R_M) = \{a, b, c\}$

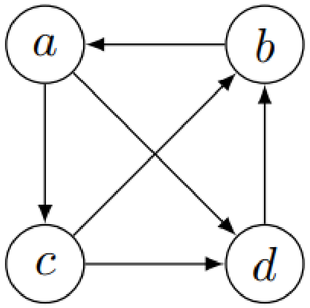
$TC(A, R_M) = \{a, b, c, d\}$

Under the Banks rule, an alternative x is a **Banks winner** if it is a top element in a maximal acyclic subgraph of the tournament. The set of Banks winners of a tournament T is denoted by $BA(T)$.

grow the set of alternative as long as the graph is acyclic. The top element of the set is a Banks winner.

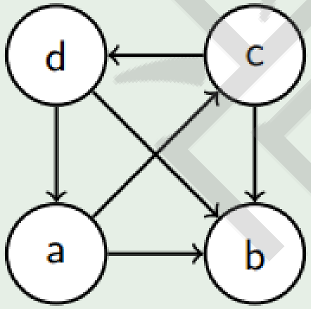
从图中找到最大无环子图，其顶点就是Banks winner
最大无环子图并不一定只有一个，所以Banks winner也不唯一

The Bank rule is a Condorcet-extension.



Example (Tournament)

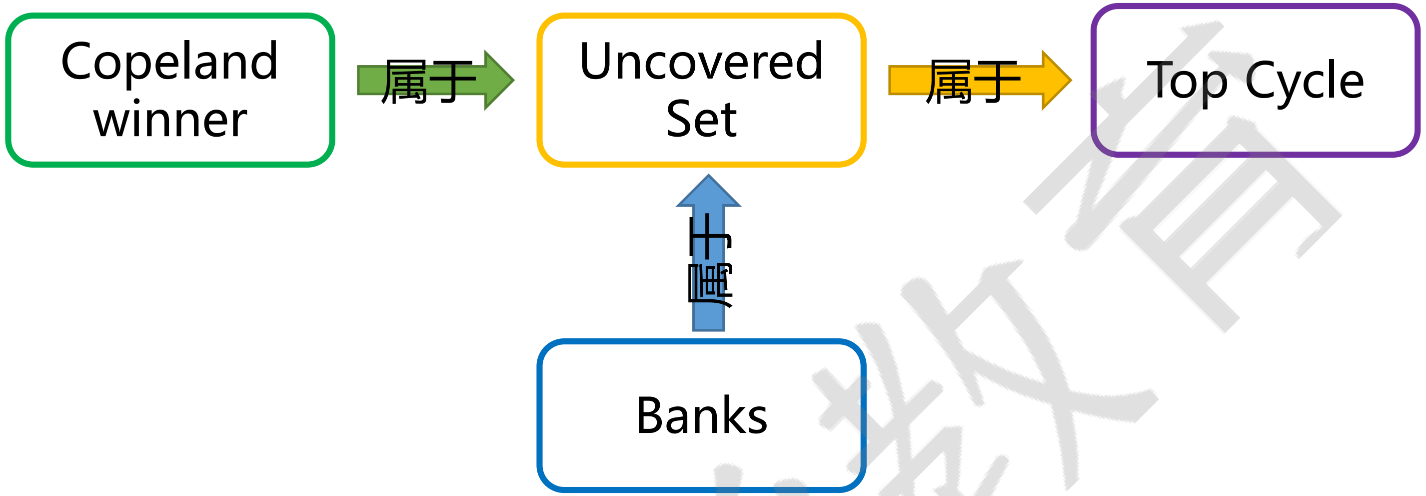
Banks winners: $\{a, c, d\}$



$UC(A,R_M)=\{a,b,c\}$

$BA(A,R_M)=\{a,b,c\}$

Social Choice – Relations



	Pareto Optimality
Copeland	Y
Top Cycle	N
Uncovered Set	Y
Bank	Y

Positional Scoring Rules are not Condorcet-extensions

Theorem (Fishburn, 1973)

No positional scoring rule is a Condorcet-extension when there are 3 or more alternatives.

$$\begin{aligned} 6 \text{ voters} &: a \succ b \succ c \\ 3 \text{ voters} &: c \succ a \succ b \\ 4 \text{ voters} &: b \succ a \succ c \\ 4 \text{ voters} &: b \succ c \succ a \end{aligned}$$

Alternative b is the winner under every PSR.
However a is the Condorcet winner.

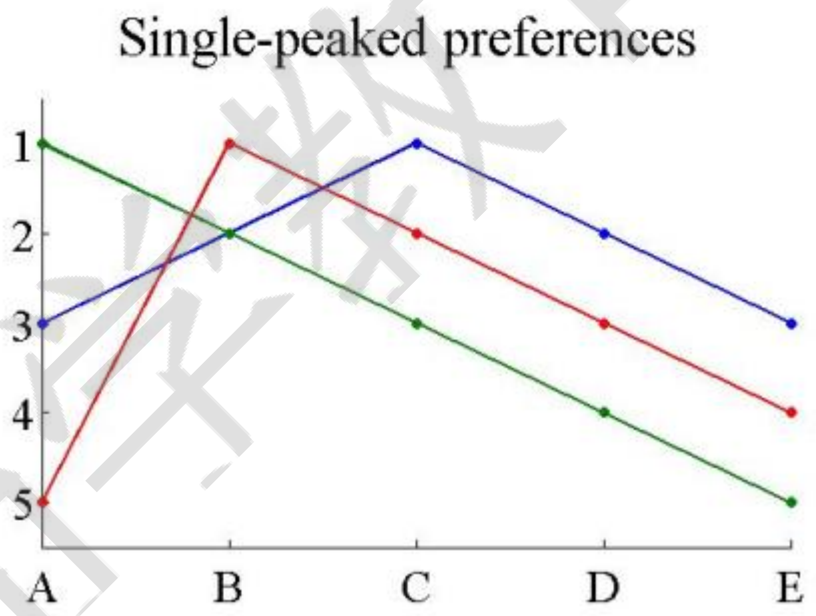
- Score of a: $6s_1 + 7s_2 + 4s_3$
- Score of b: $8s_1 + 6s_2 + 3s_3$
- Score of c: $3s_1 + 4s_2 + 10s_3$

Social Choice – Domain Restriction

Theorem (Black's Theorem, 1948)

If an odd number of voters submit single-peaked preferences, then there exists a Condorcet winner and it will get elected by the median voter rule.

$$1: A \succ B \succ C \succ D \succ E$$
$$2: B \succ C \succ D \succ E \succ A$$
$$3: C \succ B \succ D \succ A \succ E$$

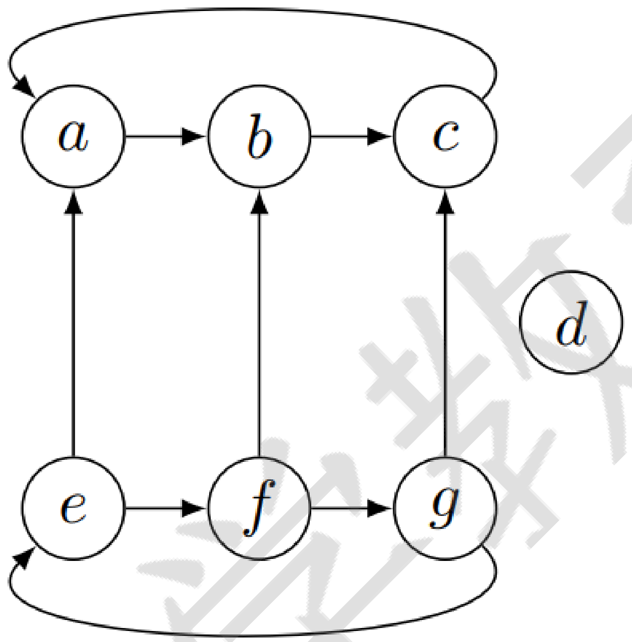


在有奇数个候选人时，排在最中间的single-peaked候选人成为winner

Social Choice

Exercise

(All missing edges are pointing downwards.)



Find the Top Cycle, Uncovered Set and Banks winner

Social Choice

Exercise

If, for a given set of voters and alternatives, there exists a Condorcet winner, then the Condorcet winner will get the highest score using the Borda count. “ True or false? Why?

But a Condorcet winner should get at least the average value of Borda score.

Utilities

Welfare

- Utilitarian
- Egalitarian
- Lexmin
- Nash Product

Envy-freeness

EF1

Resource Allocation

Gale's Top Trading Cycles

Deferred Acceptance

Adjusted Winner

We assume additive utilities:

- $u_i : O \longrightarrow \mathbb{R}^+$ specifies the utility function of each agent i.
- $u_i(O') = \sum_{o \in O'} u_i(o)$ for any $O' \subseteq O$.

Example

	o_1	o_2	o_3	o_4
1	6	3	2	1
2	4	1	2	3

$u_1(o_1) = 6$; $u_1(o_2) = 3$; $u_1(o_3) = 2$; $u_1(o_4) = 1$.

$$u_1(\{o_1, o_2\}) > u_1(\{o_2, o_3\}).$$

$$\{o_1, o_2\} \succ_1 \{o_2, o_3\}.$$

Resource Allocation – Pareto Optimality

An allocation X is *Pareto optimal* if there exists no allocation Y such that $Y(i) \succsim_i X(i)$ for all $i \in N$ and $Y(i) \succ_i X(i)$ for some $i \in N$.

An allocation X is *Pareto optimal* if there exists no allocation Y such that $u_i(Y(i)) \geq u_i(X(i))$ for all $i \in N$ and $u_i(Y(i)) > u_i(X(i))$ for some $i \in N$.

Example (Not Pareto optimal)

	o_1	o_2	o_3	o_4
1	6	2	3	1
2	4	1	2	3

$X(1) = \{o_1, o_3, o_4\}$, $X(2) = \{o_2\}$.

Example (Pareto optimal)

	o_1	o_2	o_3	o_4
1	6	2	3	1
2	4	1	2	3

$X(1) = \{o_1, o_2, o_3\}$, $X(2) = \{o_4\}$.

如果新的分配方式可以让所有人的份额都不比之前的差，有的人比之前的还好，这就说明存在当前的分配方式并不是Pareto optimal

Utilitarian Social Welfare 实用主义分配：使得所有人的价值总和最大化

An allocation X's *utilitarian social welfare* is $\sum_{i \in N} u_i(X(i))$

Example (utilitarian welfare maximizing allocation)

	o_1	o_2	o_3	o_4
1	6	3	2	1
2	4	1	2	3

$X(1) = \{o_1, o_2, o_3\}$, $X(2) = \{o_4\}$.

Egalitarian Social Welfare 平等主义分配

An allocation X's *egalitarian social welfare* is $\min_{i \in N}\{u_i(X(i))\}$

Example (egalitarian welfare maximizing allocation)

	o_1	o_2	o_3	o_4
1	6	2	3	1
2	4	1	2	3

$X(1) = \{o_1\}$, $X(2) = \{o_2, o_3, o_4\}$.

Lexmin Welfare 字典序列分配

For any allocation X, let $f(X)$ be the vector that orders the utilities achieved by the agents in non-decreasing order.

把所有人拿到的份额价值进行递增排序

Lexicographically comparison:
像字典序列一样比较每个数值，即如果第一个相同就比较第二个，以此类推

Example (lexicographic comparison)

$$(4,5) >_{lex} (3,8).$$

Example (lexmin welfare maximizing allocation)

	o_1	o_2	o_3	o_4
1	6	2	3	1
2	4	1	2	3

$X(1)=\{o_1\}$, $X(2)=\{o_2,o_3,o_4\}$.

Nash Product Social Welfare 纳什乘积分配

An allocation X's *Nash product welfare* is $\prod_{i \in N} u_i(X(i))$

Example (Nash product welfare maximizing allocation)

	o_1	o_2	o_3	o_4
1	6	2	3	1
2	4	1	2	3

$X(1) = \{o_1, o_2\}$, $X(2) = \{o_3, o_4\}$.

Resource Allocation - Welfare： Pareto Optimality

	o_1	o_2	o_3	o_4
1	6	2	3	1
2	4	1	2	3

Table: Utilitarian welfare maximizing allocation

	o_1	o_2	o_3	o_4
1	6	2	3	1
2	4	1	2	3

Table: Nash welfare maximizing allocation

	o_1	o_2	o_3	o_4
1	6	2	3	1
2	4	1	2	3

Table: Egalitarian welfare maximizing allocation

Fact

If an allocation maximizes utilitarian welfare or Nash product welfare or is a lexmin allocation, then it is Pareto optimal.

Proof.

- Assume the allocation is not Pareto optimal.
- Then there exists another allocation in which each agent gets at least as much utility and one agents strictly more utility.
- But then the allocation does not maximize welfare.

□

除了平等主义分配，其他三种都是帕累托最优

Resource Allocation – Envy Freeness

An allocation X satisfies *envy-freeness* if for all $i, j \in N$

$$X(i) \succsim_i X(j)$$

$$u_i(X(i)) \geq u_i(X(j))$$

Example (Not envy-free)

	o_1	o_2	o_3	o_4
1	6	2	3	1
2	4	1	2	3

$X(1) = \{o_1, o_2, o_3\}$, $X(2) = \{o_4\}$.

Resource Allocation – Envy Freeness

Proportional

An allocation X satisfies *proportionality* if for all $i, j \in N$

$$u_i(X(i)) \geq \frac{u_i(O)}{n}$$

Example (Not proportional)

	o_1	o_2	o_3	o_4
1	6	2	3	1
2	4	1	2	3

$X(1) = \{o_1, o_2, o_3\}$, $X(2) = \{o_4\}$.

既然我得到的是我认为最大的，那我得到的必然比我认为的大家得到的平均价值大

$u_i(X(i)) \geq u_i(X(j))$ for all $j \in N$.

$$n \cdot u_i(X(i)) \geq \sum_{j \in N} u_i(X(j)) = u_i(O).$$

$$u_i(X(i)) \geq u_i(O)/n.$$

May not exist such allocation that is envy free or proportional

Example

	o_1	o_2
1	9	1
2	9	1

Resource Allocation – Maxmin Fair Share Fairness

Definition (Maxmin Fair Share Fairness)

Given an instance $I = (N, O, u)$, let Π_n denote the space of all partitions of O into n sets. The *maximin share guarantee* of an agent $i \in N$ is

$$\mathrm{MmS}_i(I) = \max_{(P_1,\ldots,P_n)\in\Pi_n} \min_{j\in\{1,\ldots,n\}} u_i(P_j).$$

An allocation X is a *maximin share (MmS) allocation* if we have $u_i(X(i)) \geq \mathrm{MmS}_i(I)$ for each agent $i \in N$.

考虑所有可能的分配方式，对每种数量的分配取最小值（agent拿一个有多少情况，拿两个有多少情况，以此类推），总共再取最大

Example (Satisfies MmS Fairness)

	o_1	o_2	o_3	o_4
1	2	1	1	6
2	1	1	3	5

$\mathrm{MmS}_1(I) = 4$; $\mathrm{MmS}_2(I) = 5$

Fact

A proportional allocation satisfies MmS fairness.

比例分配满足MmS

Resource Allocation – EF1 Fairness

Definition (EF1 Fairness)

Given an instance $I = (N, O, u)$, an allocation X satisfies EF1 (envy-freeness up to 1 item) if for each $i, j \in N$, there exists some item $o \in X(j)$ such that

$$X(i) \succsim_i X(j) \setminus \{o\}.$$

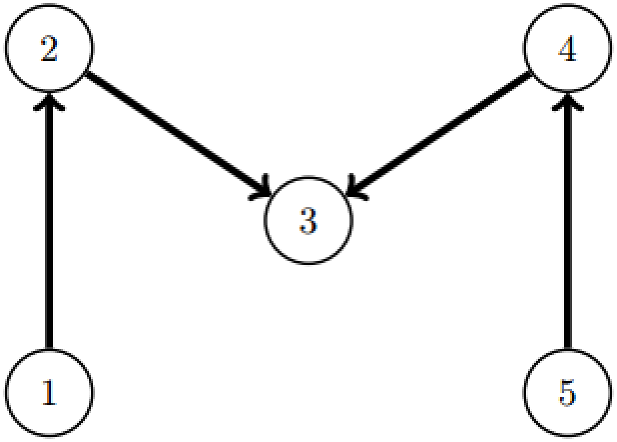
对每个agent最多允许一件导致envy的item存在

Example (Satisfies EF1 Fairness)

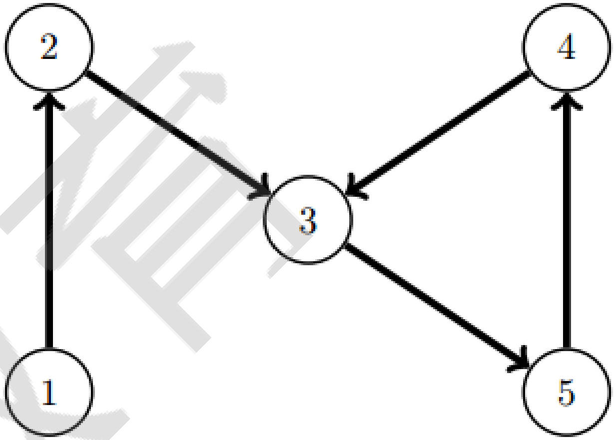
	o_1	o_2	o_3	o_4
1	2	1	1	6
2	1	1	3	5

$X(1) = \{o_1, o_2, o_3\}$, $X(2) = \{o_4\}$.

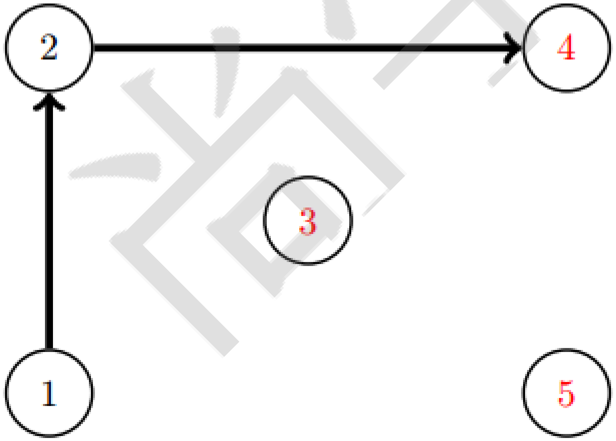
Resource Allocation – EF1 Fairness



Agent 5 没有入度边，可以把物品给他



给了之后发现agent 3 envy 5



顺着agent 3,4,5形成的环重新分配

Resource Allocation – Fairness Overview

- EF implies proportionality which implies MmS fairness.
- EF implies EF1 fairness.
- EF, Proportional, and MmS fair allocations may not exist and are computationally hard to compute even if they exist.
- An EF1 allocation always exists and can be computed in polynomial time.

Resource Allocation – Housing market: model with endowments

$(N, O, \succ, \omega)$

- $|N| = |O|$
- $\omega(i) = \{o\}$ iff o is owned by $i \in N$.
- Agents have strict preferences over items
- Each agent owns and is allocated one item.

Example

Housing market $(N, O, \omega, \succ)$ such that

- $N = \{1, \ldots, 5\}$, $O = \{o_1, \ldots, o_5\}$,
- $\omega(i) = \{o_i\}$ for all $i \in \{1, \ldots, 5\}$
- and preferences $\succ$ are defined as follows:

agent	1	2	3	4	5
preferences	o_2	o_3	o_4	o_1	o_2
	o_1	o_2	o_3	o_5	o_4
				o_4	o_5

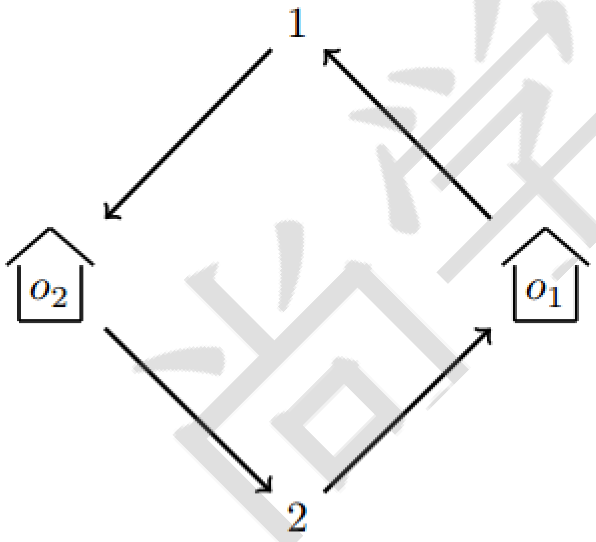
Resource Allocation – Housing market: model with endowments

Gale's Top Trading Cycles (TTC) Algorithm

- Each item points to its owner.
- Each agent points to her most preferred item in the graph.
- Find a cycle, allocate to each agent in the cycle the item she was pointing to. Remove the agents and items in the cycle. Adjust the graph so the agents in the graph point to their most preferred item in the graph.
- Repeat until the graph is empty.

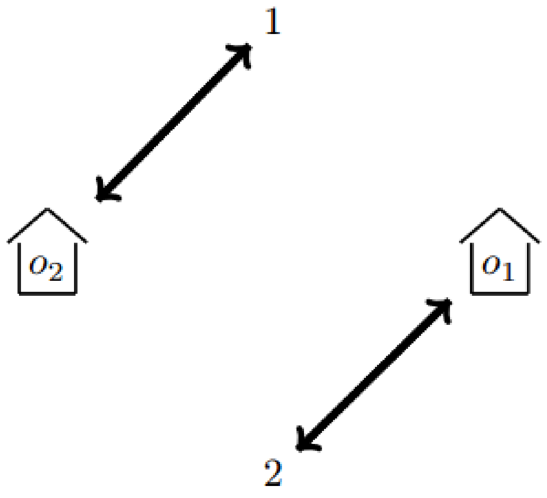
agents	1	2
item owned	o_1	o_2

agents	1	2
preferences	o_2	o_1
	o_1	o_2



agents	1	2
item owned	o_1	o_2

agents	1	2
preferences	$\underline{o_2}$	$\underline{o_1}$
	o_1	o_2



Resource Allocation – Housing market: model with endowments

Gale’s Top Trading Cycles (TTC) Algorithm

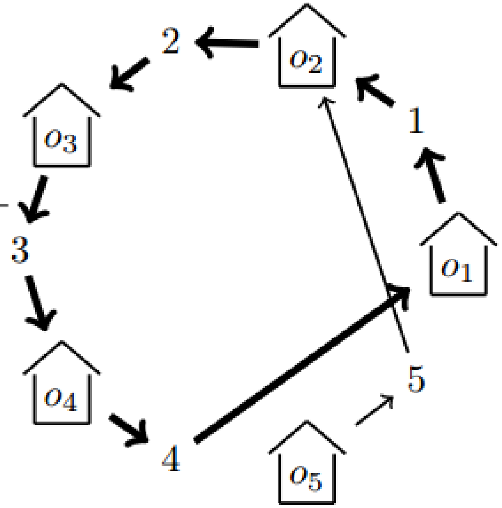
Example

Housing market $M = (N, O, \omega, \succ)$ such that

- $N = \{1, \ldots, 5\}$, $O = \{o_1, \ldots, o_5\}$,
- $\omega(i) = \{o_i\}$ for all $i \in \{1, \ldots, 5\}$
- and preferences $\succ$ are defined as follows:

agent	1	2	3	4	5
preferences	o_2	o_3	o_4	o_1	o_2
	o_1	o_2	o_3	o_5	o_4
				o_4	o_5

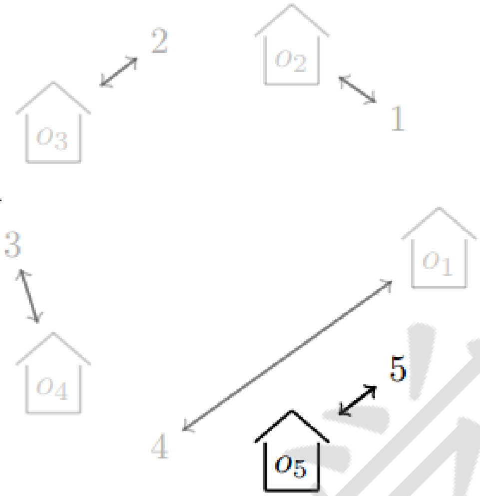
agent	1	2	3	4	5
preferences	o_2	o_3	o_4	o_1	o_2
	o_1	o_2	o_3	o_5	o_4
				o_4	o_5



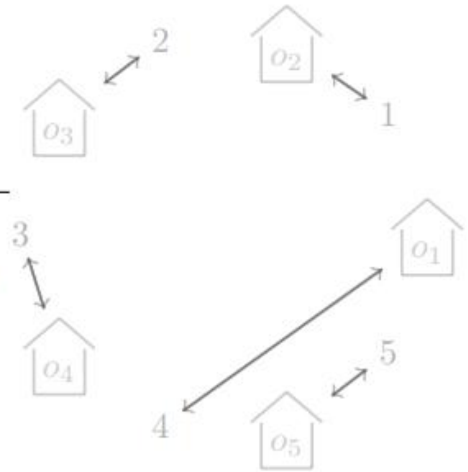
Resource Allocation – Housing market: model with endowments

Gale’ s Top Trading Cycles (TTC) Algorithm

agent	1	2	3	4	5
preferences	$\underline{o_2}$	$\underline{o_3}$	$\underline{o_4}$	$\underline{o_1}$	o_2
	o_1	o_2	o_3	o_5	o_4
				o_4	o_5



agent	1	2	3	4	5
preferences	$\underline{o_2}$	$\underline{o_3}$	$\underline{o_4}$	$\underline{o_1}$	o_2
	o_1	o_2	o_3	o_5	o_4
				o_4	$\underline{o_5}$



Resource Allocation – Student Proposing DA (Deferred Acceptance)

Agents (students) with strict preferences over schools; items (school seats with each school containing certain quota)

1. Each agent applies to her most preferred school. If a school s received at most $q(s)$ applications from the agents so far, all those agents are put on the school's waiting list. Otherwise the school puts its highest priority $q(s)$ agents among all applicants on the waiting list and rejects all remaining ones.
2. Each agent that was rejected in the previous step applies to her most preferred among the schools she has not yet applied to.
3. Steps 1 and 2 are repeated until it holds for all agents that they were either not rejected in the previous step or already applied to all schools.
4. Each school admits all the agents on its waiting list.

学校和学生各有Preferences，进行双向选择

Resource Allocation – Student Proposing DA (Deferred Acceptance)

The quota of each school a, b, c is 1.

$1 : b, a, c$	$a : 1, 3, 2$
$2 : a, b, c$	$b : 2, 1, 3$
$3 : a, b, c$	$c : 2, 1, 3$

- 2 and 3 apply to a; 1 applies to b
- a rejects 2 in favour of 3 $\{\{1, b\}, \{3, a\}\}$
- 2 applies to b
- b rejects 1 in favour of 2 $\{\{2, b\}, \{3, a\}\}$
- 1 applies to a
- a rejects 3 in favour of 1 $\{\{2, b\}, \{1, a\}\}$
- 3 applies to b
- b rejects it in favour of 2
- 3 applies to c and gets accepted. $\{\{3, c\}, \{2, b\}, \{1, a\}\}$.

Resource Allocation – Student Proposing DA (Deferred Acceptance)

Student Proposing DA algorithm terminates in time linear in the size of the preference profile.

- In each step, the agents' potential school matches decreases (if it does not decrease each agent is matched)
- School's tentative matches keep improving (if they do not improve, it means there are no new proposals)

Justified envy-freeness: there exists no agent i who prefers another school s over her match and s had admitted j a lower priority agent than i.

Theorem (Roth and Sotomayor [1990])

Student Proposing DA is strategyproof. The resultant allocation Pareto dominates (wrt to students) all allocations that satisfy justified envy-freeness.

DA算法符合帕累托最优，但注意哪边开始（学生侧开始还是学校侧）会导致结果略微不同，一般对于开始的一侧结果是最优的

Resource Allocation - Allocation of divisible items:

AW (Adjusted Winner)

Alice	Item	Bob
50	A	40
5	B	5
6	C	5
31	D	10
8	E	40

Alice	Item	Bob
50	A	40
5	B	5
6	C	5
31	D	10
8	E	40

首先双方按出价分配物品，每个物品由出价高的一方获得。出价相同时由总值较低的一方获得

然后通过重新分配来平衡双方的价值：

对胜者拥有的每个物品，根据双方给出的价值比值进行排序，从小到大选择重新分配，如果分配无法让两人的价值相等，则考虑拆分物品

Resource Allocation - Allocation of divisible items:

AW (Adjusted Winner)

Alice	Item	Bob
50	A	40
5	B	5
6	C	5
31	D	10
8	E	40
Total 87		Total 45

C的比值最低，先把C给Bob

需要重新分配的	比值
A	50/40 = 1.25
C	6/5 = 1.2
D	31/10 = 3.1

Alice	Item	Bob
50	A	40
5	B	5
6	C	5
31	D	10
8	E	40
Total 81		Total 50

如果直接把A给Bob的话，Bob的价值就超过Alice了，所以考虑分割A

$$81 - 50 * x = 50 + 40 * x$$

$$x = 31/90$$

Questions