Answer Set Programming

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COMP4418, Week 5

Non-Monotonic Reasoning

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Non-Monotonic Reasoning

$$\left.\begin{array}{l}\forall x\,(\mathrm{Car}(x) \rightarrow \neg\mathrm{Entry}(x)) \\ \forall x\,(\mathrm{Car}(x) \wedge \mathrm{Auth}(x) \rightarrow \mathrm{Entry}(x))\end{array}\right\} \models \mathrm{Car}(C) \wedge \mathrm{Auth}(C) \rightarrow \neg\mathrm{Entry}(C)$$

ASP at a Glance

- ASP $=$ Answer Set Programming
  - ASP $\neq$ Microsoft's Active Server Pages

- ASP belongs to logic programming
  - Like Prolog: $Head \leftarrow Body$ or $Head$ `:-` $Body$`.`
  - Like Prolog: negation as failure
  - Unlike Prolog: $Head$ may be empty $\Rightarrow$ constraints

- Declarative programming
  - Unlike Prolog: no procedural control
  - Order has no impact on semantics

- ASP programs compute *models*
  - Unlike Prolog: not query-oriented, no resolution
  - Unlike Prolog: not Turing-complete
  - Tool for problems in NP and $\mathrm{NP}^{\mathrm{NP}}$

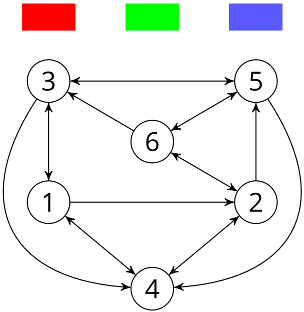
Motivation for ASP and this Lecture

- Very useful in practice!
  - Declarative problem solving
  - Very fast to write
  - Very fast to run
  - Few experts
- Interesting case study
  - Small, simple core language
  - Great expressivity by reduction to core language
- Knowing the theory is essential

Example: Graph Colouring

Definition: graph colouring problem

Input: graph with vertices V and edges $E \subseteq V \times V$, set of colors C.
Is there a mapping $m : V \to C$ with $m(x) \neq m(y)$ for all $(x, y) \in E$?



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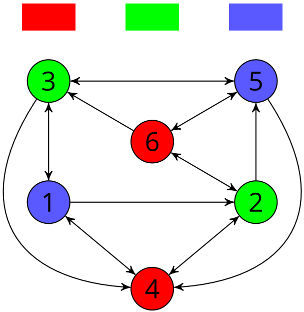

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- Graph Coulouring is NP-complete
  - NP: guess solution, verify in polynomial time
  - NP-complete: among hardest in NP
- Many applications:
  - Mapping (neighbouring countries to different colors)
  - Compilers (register allocation)
  - Scheduling (e.g., conflicting jobs to different time slots)
  - Allocation problems, Sudoku, ...

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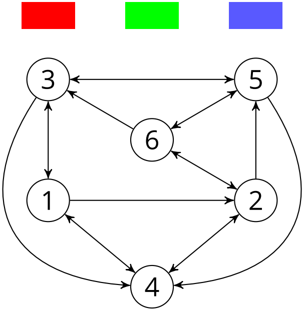

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$v(1)$. ... $v(6)$.
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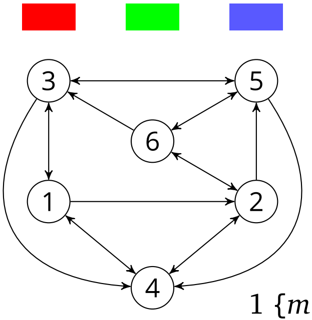

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:- $e(X,Y)$, $m(X,C)$, $m(Y,C)$. verify $m(X) \neq m(Y)$

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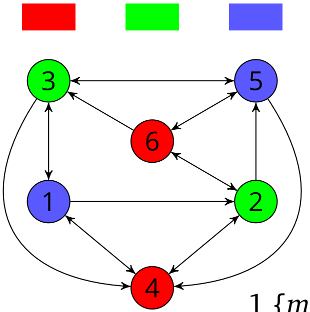

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Applications of ASP

- Automated product configuration
- Linux package manager
- Decision-support system for space shuttle
- Bioinformatics (diagnosis, inconsistency detection)
- General game playing

- Several implementations are available
- For this lecture: **Clingo** `www.potassco.org`

Overview of the Lecture

- **Semantics of ASP programs**
- Extensions of ASP programs
- Handling of variables in ASP
- ASP as modelling language

Prolog vs ASP

Consider the following logic program:

- $a.$
  $c \leftarrow a, b.$
  $d \leftarrow a, \mathrm{not}\, b.$

```
a.
c :- a, b.
d :- a, not b.
```

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  - Models: $M_1 = \begin{array}{|c|c|c|c|}\hline a & b & c & d \\ \hline 1 & 0 & 0 & 1 \\ \hline\end{array}$ $\quad M_2 = \begin{array}{|c|c|c|c|}\hline a & b & c & d \\ \hline 1 & 1 & 1 & 0 \\ \hline\end{array}$ $\quad \ldots$

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  - M_1 corresponds to Prolog, what is special about M_1?
  - M_1 is a **stable model** a.k.a. **answer set**:
    M_1 only satisfies *justified* propositions

ASP gives **semantics** to **logic programming**

Intuition

The motivating guidelines behind stable model semantics are:

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Next: formalisation of this intuition

For now: only ground programs, i.e., no variables

Syntax

Definition: normal logic program (NLP)

A **normal logic program** P is a set of (normal) rules of the form

$$A \leftarrow B_1, \ldots, B_m, \mathrm{not}\, C_1, \ldots, \mathrm{not}\, C_n.$$

where A, B_i, C_j are atomic propositions.

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For such a rule r, we define:

- $\mathrm{Head}(r) = \{A\}$
- $\mathrm{Body}(r) = \{B_1, \ldots, B_m, \mathrm{not}\, C_1, \ldots, \mathrm{not}\, C_n\}$

In code, r is written as A `:-` $B_1, \ldots, B_m,$ `not` $C_1, \ldots,$ `not` C_n`.`

Semantics: Interpretation

Definition: interpretation, satisfaction

A **interpretation** S is a set of atomic propositions.

S **satisfies** $A \leftarrow B_1, \ldots, B_m, \mathrm{not}\, C_1, \ldots, \mathrm{not}\, C_n$ iff
$A \in S$ or some $B_i \notin S$ or some $C_j \in S$.

In English:

- S satisfies rule iff S satisfies the head or falsifies the body
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Semantics without Negation

Definition: stable model for programs without negation

For P without negated literals:
S is a **stable model** of P iff
S is a minimal set (w.r.t. $\subseteq$) that satisfies all $r \in P$.

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Theorem: unique-model property

If P is negation-free (i.e., contains no $(\text{not}\, C)$), then there is exactly one stable model, which can be computed in linear time.

Semantics without Negation – Examples

Compute stable model of a negation-free P by *unit propagation*:

- $S^0 = \{\}$
- $S^{i+1} = S^i \cup \bigcup_{r \in P : S \text{ satisfies } \mathrm{Body}(r)} \mathrm{Head}(r)$ until $S^{i+1} = S^i$

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<u>Ex.</u>: $P_3 = \{a \leftarrow b. \quad b \leftarrow a. \quad a.\}$

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Semantics with Negation

Definition: reduct

The **reduct** P^S of P relative to S is the least set such that
if $A \leftarrow B_1, \ldots, B_m, \mathrm{not}\, C_1, \ldots, \mathrm{not}\, C_n \in P$ and $C_1, \ldots, C_n \notin S$
then $A \leftarrow B_1, \ldots, B_m \in P^S$.

In English: for each rule r from P,

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then $A \leftarrow B_1, \ldots, B_m \in P^S$.

In English: for each rule r from P,

- if $(\text{not}\, C) \in \text{Body}(r)$ for some $C \in S$: drop the rule
- else: remove all negated literals and add to P^S

<u>Ex.</u>: $P = \{a. \quad c \leftarrow a, b. \quad d \leftarrow a, \text{not}\, b.\}$

$$\begin{aligned}
S_1 &= \{a\} & \Rightarrow \quad P^{S_1} &= \{a. \quad c \leftarrow a, b. \quad d \leftarrow a.\} \\
S_2 &= \{a, b\} & \Rightarrow \quad P^{S_2} &= \{a. \quad c \leftarrow a, b.\} \\
S_3 &= \{a, d\} & \Rightarrow \quad P^{S_3} &= \{a. \quad c \leftarrow a, b. \quad d \leftarrow a.\}
\end{aligned}$$

Semantics with Negation

Definition: reduct

The **reduct** P^S of P relative to S is the least set such that
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Definition: stable model for programs with negation

For P with negated literals:
S is a **stable model** of P iff S is a stable model of P^S.

Semantics with Negation

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S_3 = \{a, d\} & \Rightarrow & P^{S_3} = \{a. \quad c \leftarrow a, b. \quad d \leftarrow a.\} & ✓
\end{array}
$$

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Semantics with Negation – Examples

Ex.: $P = \{a \leftarrow \mathrm{not}\, b. \quad b \leftarrow \mathrm{not}\, a.\}$

Semantics with Negation – Examples

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$S_1 = \{\} \qquad \Rightarrow \; P^{S_1} =$

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Ex.: $P = \{a \leftarrow \mathrm{not}\, b. \quad b \leftarrow \mathrm{not}\, a.\}$

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Semantics with Negation – Examples

Ex.: $P = \{a \leftarrow \text{not}\, b. \quad b \leftarrow \text{not}\, a.\}$

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$$S_2 = \{a\} \quad \Rightarrow \quad P^{S_2} =$$

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\end{aligned}
$$

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\end{array}
$$

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$$\begin{array}{llll} S_1 = \{\} & \Rightarrow & P^{S_1} = \{a. \quad b\} & ✗ \\ S_2 = \{a\} & \Rightarrow & P^{S_2} = \{a.\} & \\ S_3 = \{b\} & \Rightarrow & P^{S_3} = \{b.\} & \\ S_4 = \{a, b\} & \Rightarrow & P^{S_4} = \{\} & \end{array}$$

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\end{array}
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Two stable models!

Semantics with Negation – Examples

Ex.: $P = \{a \leftarrow \text{not}\, b. \quad b \leftarrow \text{not}\, a.\}$

$S_1 = \{\} \quad \Rightarrow \quad P^{S_1} = \{a. \quad b\}$ ✗

$S_2 = \{a\} \quad \Rightarrow \quad P^{S_2} = \{a.\}$ ✓

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$$S_2 = \{a\} \quad \Rightarrow \quad P^{S_2} = \{a.\} \qquad ✓$$
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Semantics with Negation – Examples

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\begin{array}{llll}
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S_4 = \{a, b\} & \Rightarrow & P^{S_4} = \{\} & ✗
\end{array}
$$

Two stable models!

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\begin{array}{lll}
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No stable model!

Semantics: Overview

Definition: reduct

The **reduct** P^S of P relative to S is the least set such that
if $A \leftarrow B_1, \ldots, B_m, \mathrm{not}\, C_1, \ldots, \mathrm{not}\, C_n \in P$ and $C_1, \ldots, C_n \notin S$
then $A \leftarrow B_1, \ldots, B_m \in P^S$.

Definition: stable model

If P contains no $(\mathrm{not}\, C)$:
S is a **stable model** of P iff
S is a minimal set (w.r.t. $\subseteq$) that satisfies all $r \in P$.

If P contains $(\mathrm{not}\, C)$:
S is a **stable model** of P iff S is a stable model of P^S.

Theorem: necessary satisfaction condition

If S is a stable model and $A \in S$,
then S satisfies some $r \in P$ with $A \in \mathrm{Head}(r)$.

Semantics – Examples

Ex.: $P = \{a \leftarrow a. \quad b \leftarrow \mathrm{not}\, a.\}$

S $\qquad$ P^S $\qquad$ Stable model?

Ex.: $P = \{a \leftarrow \mathrm{not}\, b. \quad b \leftarrow \mathrm{not}\, c.\}$

S $\qquad$ P^S $\qquad$ Stable model?

Example on paper

Entailment

Definition: entailment, cautious monotonicity

P **entails** a rule r iff every stable model of P satisfies r.

P is **cautiously monotonic** iff
for all rules r_1, r_2, if P entails r_1 and r_2, then $P \cup \{r_1\}$ entails r_2.

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If P is cautiously monotonic, a solver can iteratively augment it with already proved lemmas.

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Ex.: $P = \{a \leftarrow \mathrm{not}\, b. \quad b \leftarrow c, \mathrm{not}\, a. \quad c \leftarrow a.\}$

$S_1 = \{a, c\} \;\Rightarrow\; P^{S_1} = \{a. \quad c \leftarrow a.\}$ ✓

(no other stable model S: $b \notin S \Rightarrow a \in S \Rightarrow c \in S$ and $b \in S \Rightarrow c \in S \Rightarrow a \in S \Rightarrow b \notin S$)

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(no other stable model S: $b \notin S \Rightarrow a \in S \Rightarrow c \in S$ and $b \in S \Rightarrow c \in S \Rightarrow a \in S \Rightarrow b \notin S$)

$S_1 = \{a, c\} \ \Rightarrow \ (P \cup \{c.\})^{S_1} = \{a. \quad c \leftarrow a. \quad c.\}$ ✓

$S_2 = \{b, c\} \ \Rightarrow \ (P \cup \{c.\})^{S_2} = \{b \leftarrow c. \quad c \leftarrow a. \quad c.\}$ ✓

(no other stable model S: $c \in S$ and $a \notin S \Rightarrow b \in S$ and $b \notin S \Rightarrow a \in S$)

P entails $c.$ and $a.$ But $P \cup \{c.\}$ does not entail $a.$

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(no other stable model S: $b \notin S \Rightarrow a \in S \Rightarrow c \in S$ and $b \in S \Rightarrow c \in S \Rightarrow a \in S \Rightarrow b \notin S$)

$S_1 = \{a, c\} \;\Rightarrow\; (P \cup \{c.\})^{S_1} = \{a. \quad c \leftarrow a. \quad c.\}$ ✓
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(no other stable model S: $c \in S$ and $a \notin S \Rightarrow b \in S$ and $b \notin S \Rightarrow a \in S$)

P entails $c.$ and $a.$ But $P \cup \{c.\}$ does not entail $a.$

Good news: some classes of programs are cautiously monotonic.

Overview of the Lecture

- Semantics of ASP programs
- **Extensions of ASP programs**
- Handling of variables in ASP
- ASP as modelling language

Integrity Constraints

Definition: integrity constraint

An **integrity constraint** is a rule r of the form

$$\leftarrow B_1, \ldots, B_m, \text{not } C_1, \ldots, \text{not } C_n$$

S **satisfies** r iff some $B_i \notin S$ or some $C_j \in S$.

P^S contains $\leftarrow B_1, \ldots, B_m$ iff P contains r and $C_1, \ldots, C_n \notin S$.

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P^S contains $\leftarrow B_1, \ldots, B_m$ iff P contains r and $C_1, \ldots, C_n \notin S$.

Theorem: reduction to normal rules

Let P' be like P except that every integrity constraint

$$\leftarrow B_1, \ldots, B_m, \text{not } C_1, \ldots, \text{not } C_n$$

is replaced with

$$dummy \leftarrow B_1, \ldots, B_m, \text{not } C_1, \ldots, \text{not } C_n, \text{not } dummy$$

for some new atom $dummy$.

Then P and P' have the same stable models.

Proof on paper

Choice Rules

Definition: choice rule

A **choice rule** is a rule the form

$$\{A_1, \ldots, A_k\} \leftarrow B_1,\ \ldots, B_m, \mathrm{not}\, C_1,\ \ldots, \mathrm{not}\, C_n$$

which allows any subset of $\{A_1, \ldots, A_k\}$ in a stable model.

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Further extensions:

- Conditional literals: $\{A : B\}$
  Ex.: $\{m(v, C) : c(C)\}$ expands to $\{m(v, r), m(v, g), m(v, b)\}$
- Cardinality constraints: $min\ \{A_1, \ldots, A_k\}\ max$
  Ex.: $1\ \{m(v, r), m(v, g), m(v, b)\}\ 1$

Negation in the Rule Head

Definition: rules with negated head

A rule with **negated head** is of the form

$$\mathrm{not}\, A \leftarrow B_1, \ldots, B_m, \mathrm{not}\, C_1, \ldots, \mathrm{not}\, C_n$$

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Theorem: reduction to normal rules

Let P' be like P except that every rule with negated head

$$\text{not}\, A \leftarrow B_1, \ldots, B_m, \text{not}\, C_1, \ldots, \text{not}\, C_n$$

is replaced with

$$\leftarrow B_1, \ldots, B_m, \text{not}\, C_1, \ldots, \text{not}\, C_n, \text{not}\, \mathit{dummy}$$

and

$$\mathit{dummy} \leftarrow \text{not}\, A$$

for some new atom dummy.

Then P and P' have the same stable models (modulo dummy propositions).

Complexity

Theorem: complexity of NLPs without negations

Is S a stable model of a negation-free P? – **Linear time**
Does a negation-free P have a stable model? – **Constant** (yes, one)

Theorem: complexity of NLPs with negations

Is S a stable model of P? – **Linear time**
Does P have a stable model? – **NP-complete**

Note: integrity constraints, choice rules, negation in heads **preserve complexity** (program grows only polynomially)

Disjunctive Logic Programs

Definition: disjunctive rule

A **disjunctive rule** is of the form

$$A_1; \ldots; A_k \leftarrow B_1, \ldots, B_m, \text{not}\, C_1, \ldots, \text{not}\, C_n$$

and means that A_1 or A_2 or $\ldots$ or A_k is true if the body is true.

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Theorem: complexity of disjunctive logic programs

Is S a stable model of P? – **co-NP-complete**
Does P have a stable model? – **$\mathbf{NP^{NP}}$-complete**

Reason: P^S may have multiple minimal models!

We won't consider disjunctive logic problems any further

Overview of the Lecture

- Semantics of ASP programs
- Extensions of ASP programs
- **Handling of variables in ASP**
- ASP as modelling language

Programs with Variables

- Atomic propositions may now contain variables, e.g., $p(X, Z) \leftarrow e(X, Y), p(Y, Z)$.

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  - U contains all constants from P
  - U contains all $f(t_1, \ldots, t_k)$ from P if f is a k-ary function in P and U contains $t_1, \ldots, t_k$

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- ASP **grounds** variables with Herbrand universe
  - Unlike Prolog: instantiation instead of unification
  - Caution: the ground program may grow exponentially
  - Caution: function symbols make grounding Turing-complete
  - If P is finite and mentions only constants, grounding is finite

Programs with Variables

- $f(X) \leftarrow b(X), \text{not}\, a(X).$
  $a(X) \leftarrow p(X).$
  $b(\text{sam}).$
  $b(\text{tweety}).$
  $p(\text{tweety}).$

- $f(\text{sam}) \leftarrow b(\text{sam}), \text{not}\, a(\text{sam}).$
  $f(\text{tweety}) \leftarrow b(\text{tweety}), \text{not}\, a(\text{tweety}).$
  $a(\text{sam}) \leftarrow p(\text{sam}).$
  $a(\text{tweety}) \leftarrow p(\text{tweety}).$
  $b(\text{sam}).$
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- Semantics of ASP programs
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ASP Modelling

$c(r).\ c(g).\ c(b).$
$v(1).\ \ldots\ v(6).$
$e(1,2).\ e(1,3).\ e(1,4).$
$e(2,4).\ e(2,5).\ e(2,6).$
$e(3,1).\ e(3,4).\ e(3,5).$
$e(4,1).\ e(4,2).$
$e(5,3).\ e(5,4).\ e(5,6).$
$e(6,2).\ e(6,3).\ e(6,5).$

Typical ASP structure:

- Problem **instance**: a set of facts
- Problem **class**: a set of rules
  - Generator rules: often choice rules
  - Test rules: often integrity constraints

$1\ \{m(X,C) : c(C)\}\ 1\ \text{:-}\ v(X).$

$\text{:-}\ e(X,Y), m(X,C), m(Y,C).$

Ideal modeling is **uniform**: problem class encoding fits all instances

Semantically equivalent encodings may differ immensely in performance!

Example: Non-monotonic Reasoning

Tweety the penguin:

- (Normal) Birds fly.
- Penguins are abnormal.
- Tweety is a bird. So Tweety flies.
- Tweety is a penguin. So Tweety doesn't fly.

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$$U = \{f(X) \leftarrow b(X), \mathrm{not}\, a(X). \quad a(X) \leftarrow p(X). \quad b(t).\}$$
$$P = \{f(t) \leftarrow b(t), \mathrm{not}\, a(t). \quad a(t) \leftarrow p(t). \quad b(t).\}$$

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$$S_1 = \{b(t), f(t)\} \quad \Rightarrow \quad P^{S_1} = \{f(t) \leftarrow b(t)\cancel{, \mathrm{not}\, a(t)}. \quad a(t) \leftarrow p(t). \quad b(t).\} ✓$$
$$S_2 = \{a(t), b(t), p(t)\} \quad \Rightarrow \quad P^{S_2} = \{\cancel{f(t) \leftarrow b(t), \mathrm{not}\, a(t).} \quad a(t) \leftarrow p(t). \quad b(t).\} ✗$$

Tweety flies!

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Tweety flies!

$$S_1 = \{b(t), f(t)\} \quad \Rightarrow \quad (P \cup \{p(t).\})^{S_1} = P_2^{S_1} \cup \{p(t).\} \quad \text{✗}$$
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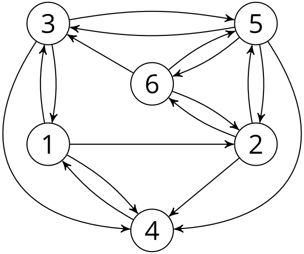
Tweety doesn't fly.

Example: Hamilton Cycle

Definition: Hamilton cycle problem

Input: graph with vertex set V and edges $E \subseteq V \times V$.
Is there a cycle that visits every vertex exactly once?



$$\{p(X,Y)\} \leftarrow e(X,Y).$$
$$r(X) \leftarrow p(1,X).$$
$$r(Y) \leftarrow r(X), p(X,Y).$$
$$\leftarrow 2\ \{p(X,Y)\}\ , v(X).$$
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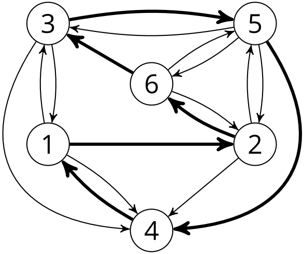

$$
\begin{aligned}
&\{p(X,Y)\} \leftarrow e(X,Y).\\
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&\leftarrow \mathrm{not}\, r(X), v(X).
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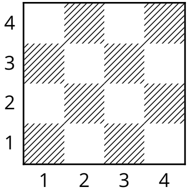
Example: N-Queens

Definition: N-queens problem

Place N queens on a $N \times N$ chessboard so that they do not attack each other, i.e., share no row, column, or diagonal.



Program on paper

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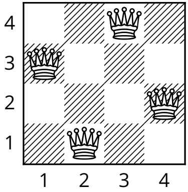

Program on paper