Satisfiability

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COMP4418, Week 6 & 7

The *Reasoning* in KRR

We've touched reasoning a few times so far:

- On a theoretical level:
  - Resolution
    - Prove that KB entails ψ by deriving contradiction from $\mathrm{KB} \wedge \neg\psi$.
    - Downside: Difficult to guide search towards goal.
  - Prolog
    - Uses restricted version of resolution.
    - Downside: More programming than reasoning.

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- From a user perspective:
  - Answer Set Programming
  - Satisfiability (in Assignment 1, Question 3)

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  - Satisfiability (in Assignment 1, Question 3)

Today: propositional reasoning

- What algorithms?
- What do real-world implementations look like?
- Which problems can be solved using it (in theory & practice)?

Satisfiability

Definition: SAT

Input: Propositional formula in CNF.
Problem: Is this formula satisfiable?

- Many problems can be reduced to SAT.
- In theory, SAT is very difficult.
- In practice, SAT is often feasible.
- SAT is extremely important for theory *and* practice.

This lecture: only propositional logic.

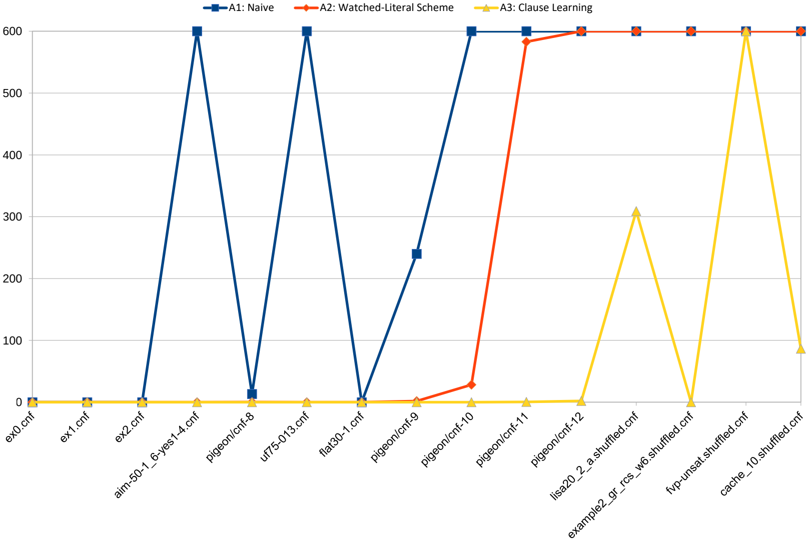
This Lecture

The goal of this lecture is twofold:

- Practical aspects:
  - What are data structures / algorithms for SAT solving?
  - What does an implementation of a SAT solver look like?

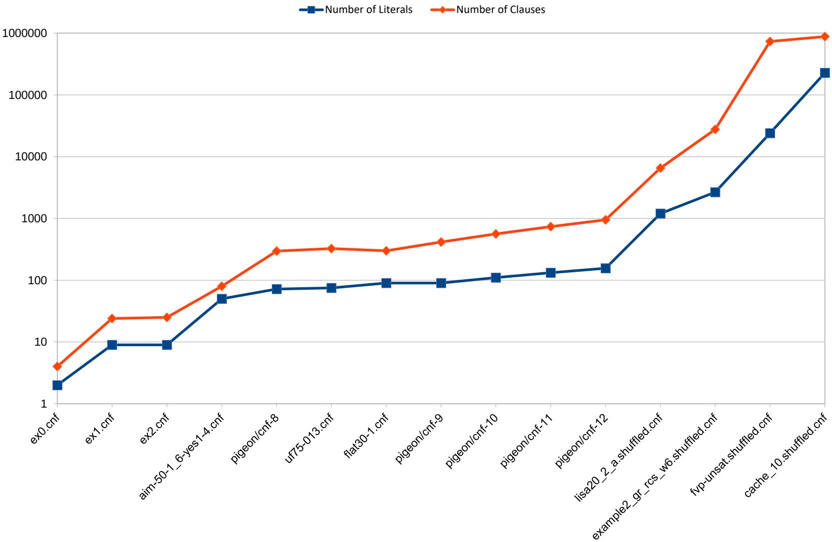
- Theoretical aspects:
  - Why is SAT hard?
  - Why is it so important?

Impact of Advanced Algorithms



Running times (in seconds) of algorithms for different instances.

Impact of Advanced Algorithms



Size of the instances.

Outline

Outline

Satisfiability and Complexity Theory

Solver Ingredient 1: Watched-Literal Scheme

Solver Ingredient 2: Conflict-Driven Clause Learning

Other Solver Ingredients

Conclusion

Beyond SAT

Propositional Logic

Syntax:

- Atomic propositions a.k.a. variables.
- Negation: $\neg\phi$
- Disjunction: $(\phi_1 \vee \phi_2)$
- Conjunction: $(\phi_1 \wedge \phi_2)$
- TRUE, FALSE, $\rightarrow$, $\leftrightarrow$ can be expressed with $\neg, \vee, \wedge$.

Semantics:

- A truth table satisfies or falsifies a formula.
- It maps each variable to true or false.
- It satisfies $\neg\phi$ iff it falsifies[1] ϕ.
- It satisfies $(\phi_1 \vee \phi_2)$ iff it satisfies ϕ_1 or ϕ_2.
- It satisfies $(\phi_1 \wedge \phi_2)$ iff it satisfies ϕ_1 and ϕ_2.

---

[1] falsify = do not satisfy

Satisfiability and Validity

- ϕ is satisfiable iff some truth table satisfies ϕ

- ϕ is valid iff all truth tables satisfy ϕ

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  iff not: all truth tables falsify ϕ

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  iff not: all truth tables satisfy ¬ϕ
  iff ¬ϕ is not valid

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  iff not: some truth table satisfies $\neg\phi$
  iff $\neg\phi$ is not satisfiable

Disjunctive Normal Form

Definition: DNF

A formula ψ is in Disjunctive Normal Form iff

ψ is of the form $(d_1 \vee \ldots \vee d_k)$, where each
d_i is of the form $(x_{i1} \wedge \ldots \wedge x_{il_i})$, where each
x_{ij} is a literal.

Ex.: $(p \wedge q) \vee (\neg p \wedge \neg q) \vee (p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge r)$

Every formula can be converted into an equivalent formula in DNF.
(Convert into NNF, then use transitivity and associativity of $\wedge$ and $\vee$.)

Note: Satisfiability for DNF is very easy (solvable in linear time).

Conjunctive Normal Form

Definition: CNF

A formula ϕ is in Conjunctive Normal Form iff
ϕ is of the form $(c_1 \wedge \ldots \wedge c_k)$, where each
c_i is of the form $(x_{i1} \vee \ldots \vee x_{il_i})$, where each
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Ex.: $(\neg p \vee \neg q) \wedge (p \vee q) \wedge (\neg p \vee q \vee \neg r) \wedge (p \vee \neg q \vee \neg r)$

Every formula can be converted into an equivalent formula in CNF.
(Convert into NNF, then use transitivity and associativity of $\wedge$ and $\vee$.)

Note: Validity for CNF is very easy (solvable in linear time).

SAT and CNF

Definition: SAT

Input: ϕ in CNF.
Problem: Is this formula satisfiable?

Why not DNF?

- Satisfiability for DNF is very easy, but . . .
- DNF of formula may grow exponentially!

Why CNF?

- Structure of CNF is much simpler than of arbitrary formulas.
- Every formula ψ can be transformed to a formula ϕ such that:
  - ϕ is satisfiable iff ψ is satisfiable.
  - The size of ϕ is linear in the size of ψ.

Tseitin transformation: Example

Input: $\psi = (\neg((p \vee q) \wedge r) \vee \neg s)$

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p

q

r

s

$\neg s$

$(p \vee q)$

$((p \vee q) \wedge r)$

$\neg((p \vee q) \wedge r)$

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$(\neg((p \vee q) \wedge r) \vee \neg s)$	
p	$x_1 \leftrightarrow p$
q	$x_2 \leftrightarrow q$
r	$x_3 \leftrightarrow r$
s	$x_4 \leftrightarrow s$
$\neg s$	$x_5 \leftrightarrow \neg s$
$(p \vee q)$	$x_6 \leftrightarrow (p \vee q)$
$((p \vee q) \wedge r)$	$x_7 \leftrightarrow ((p \vee q) \wedge r)$
$\neg((p \vee q) \wedge r)$	$x_8 \leftrightarrow \neg((p \vee q) \wedge r)$
$(\neg((p \vee q) \wedge r) \vee \neg s)$	$x_9 \leftrightarrow (\neg((p \vee q) \wedge r) \vee \neg s)$

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$(p \vee q)$	$x_6 \leftrightarrow (p \vee q)$
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$(\neg((p \vee q) \wedge r) \vee \neg s)$	$x_9 \leftrightarrow (\neg((p \vee q) \wedge r) \vee \neg s)$

Tseitin transformation: Example

Input: $\psi = (\neg((p \lor q) \land r) \lor \neg s)$

$(\neg((p \lor q) \land r) \lor \neg s)$	
p	$x_1 \leftrightarrow p$
q	$x_2 \leftrightarrow q$
r	$x_3 \leftrightarrow r$
s	$x_4 \leftrightarrow s$
$\neg s$	$x_5 \leftrightarrow \neg s$
$(p \lor q)$	$x_6 \leftrightarrow (x_1 \lor x_2)$
$((p \lor q) \land r)$	$x_7 \leftrightarrow (x_6 \land x_3)$
$\neg((p \lor q) \land r)$	$x_8 \leftrightarrow \neg x_7$
$(\neg((p \lor q) \land r) \lor \neg s)$	$x_9 \leftrightarrow (x_8 \lor x_5)$

Tseitin transformation: Example

Input: $\psi = (\neg((p \vee q) \wedge r) \vee \neg s)$

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$$\begin{aligned}
\phi = {} & \mathsf{CNF}[x_1 \leftrightarrow p] \\
& \wedge \mathsf{CNF}[x_2 \leftrightarrow q] \\
& \wedge \mathsf{CNF}[x_3 \leftrightarrow r] \\
& \wedge \mathsf{CNF}[x_4 \leftrightarrow s] \\
& \wedge \mathsf{CNF}[x_5 \leftrightarrow \neg s] \\
& \wedge \mathsf{CNF}[x_6 \leftrightarrow (x_1 \vee x_2)] \\
& \wedge \mathsf{CNF}[x_7 \leftrightarrow (x_6 \wedge x_3)] \\
& \wedge \mathsf{CNF}[x_8 \leftrightarrow \neg x_7] \\
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& \wedge x_9
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& \wedge (\neg x_4 \vee s) \wedge (x_4 \vee \neg s) \\
& \wedge (\neg x_5 \vee \neg x_4) \wedge (x_5 \vee \neg x_4) \\
& \wedge (\neg x_6 \vee x_1 \vee x_2) \wedge (x_6 \vee \neg x_1) \wedge (x_6 \vee \neg x_2) \\
& \wedge (\neg x_7 \vee x_6) \wedge (\neg x_7 \vee x_3) \wedge (x_7 \vee \neg x_6 \vee \neg x_3) \\
& \wedge (\neg x_8 \vee \neg x_7) \wedge (x_8 \vee x_7) \\
& \wedge (\neg x_9 \vee x_8 \vee x_5) \wedge (x_9 \vee \neg x_8) \wedge (x_9 \vee \neg x_5) \\
& \wedge x_9
\end{aligned}$$

Tseitin Transformation

Input: formula ψ. Output: formula ϕ in CNF.

- Let $\rho_1, \ldots, \rho_k$ be all sub-formulas of ψ, where $\rho_k = \psi$.
- Let $x_1, \ldots, x_k$ be fresh variables.
- Let $\phi = \text{CNF}[x_1 \leftrightarrow f(\rho_1)] \wedge \ldots \wedge \text{CNF}[x_k \leftrightarrow f(\rho_k)] \wedge x_k$ where
  - $f(\rho_i \wedge \rho_j) = (x_i \wedge x_j)$
  - $f(\rho_i \vee \rho_j) = (x_i \vee x_j)$
  - $f(\neg \rho_i) = \neg x_i$
  - $f(\rho_i) = x_i$ if ρ_i is a variable.

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Theorem: Tseitin transformation

ϕ is satisfiable iff ψ is satisfiable.
ϕ is in 3-CNF and its size is linear in the size of ψ.

Why? The number of subformulas of ψ is at most twice the length ψ.
The size of $f(\rho_i)$ is constant. The size of $\mathrm{CNF}[x_i \leftrightarrow f(\rho_i)]$ is constant.

Syntax and Semantics Revisited

Syntax:

- A CNF formula ϕ is of the form $(c_1 \wedge \ldots \wedge c_k)$,
  where each c_i is of the form $(x_{i1} \vee \ldots \vee x_{il_i})$
  where each x_{ij} is a variable or a negated variable.
- We identify c_i with the set $\{x_{i1}, \ldots, x_{il_i}\}$.
- We identify ϕ with the set $\{c_1, \ldots, c_k\}$.
- We write $\overline{x}$ to flip the negation of a literal x: $\overline{\neg p} = p \quad \overline{p} = \neg p$

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Semantics:

- A partial interpretation I is a consistent set of literals ($x \notin I$ or $\overline{x} \notin I$)
  It may falsify or satisfy a formula or neither (because it is partial)
- I satisfies a CNF formula ϕ iff I satisfies all clauses $c \in \phi$
  I falsifies a CNF formula ϕ iff I falsifies some clause $c \in \phi$
- I satisfies a clause c iff I satisfies some literal $x \in c$
  I falsifies a clause c iff I falsifies all literals $x \in c$
- I satisfies a literal x iff $x \in I$
  I falsifies a literal x iff $\overline{x} \in I$

SAT Algorithm 1a: Nondeterministic

```
Let $I = \{\}$. Repeat:
  1. If $I$ falsifies some $c \in \phi$:   return NO
  2. Select a variable $x$ such that $x, \bar{x} \notin I$
  3. If there is none:   return YES
  4. Let $I = I \cup \{x\}$ or $I = I \cup \{\neg x\}$
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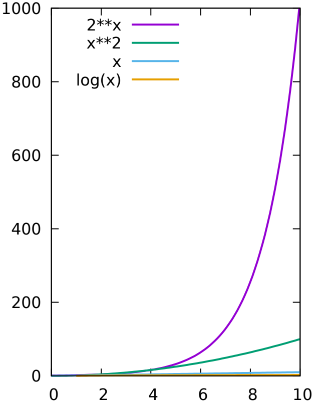
Bad news: Search space is exponential in number of variables.
(It is believed that) we can't do better.

Complexity Theory: Big O Notation

- Complexity is (usually) measured in the length n of the input:
  - $f(n) = \mathcal{O}(g(n))$ iff for some k, for large n, $f(n) \leq k \cdot g(n)$
  - Exponential complexity: $\mathcal{O}(k^n)$
  - Polynomial complexity: $\mathcal{O}(n^k)$
  - Linear complexity: $\mathcal{O}(n)$
  - Logarithmic complexity: $\mathcal{O}(\log n)$
  - Constant complexity: $\mathcal{O}(1)$



- Length of input $=$ number of symbols:
  - Length of "$\neg(p \wedge q)$" $= 6$
  - Length of 173 in decimal $= 3$
  - Length of 173 in binary $= 8$
- Time complexity: number of computation steps.
- Space complexity: amount of memory used.
- Time is upper bound for space.

Complexity Theory: P and NP

Definition: decision problem, complexity class

A decision problem is a yes/no question over a set of instances.
An instance is a finite sequence of symbols from finite alphabet.
A complexity class is a set of decision problems of related complexity.

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Definition: complexity class P, NP

$A \in$ P iff solvable in poly. time by a deterministic machine.
$A \in$ NP iff solvable in poly. time by a nondeterministic machine.

- Det. machine takes a predetermined action in each state.
- Nondet. m. takes one out of a set of possible actions in each state.
  - The machine guesses which action to take.
  - The machine guesses the shortest way to "yes" if there is one.

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Theorem: alternative characterisation of NP

$A \in$ NP iff "yes"-proof verifiable by a det. machine in poly. time.

Complexity Theory: NP-Completeness

A problem is NP-complete if it’s among the hardest problems in NP.

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Definition: reduction from A to B

$A \leqq_{\mathsf{P}} B$ iff for some function f from A-instances to B-instances:
for all instances of x of A:
$f(x)$ is computable in polynomial time and
x is a "yes"-instance of A iff $f(x)$ is a "yes"-instance of B.

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Definition: NP-hard, NP-complete

B is NP-hard iff $A \leqq_{\mathsf{P}} B$ for all $A \in$ NP.
B is NP-complete iff $B \in$ NP and B is NP-hard.

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Common hypothesis: P $\neq$ NP. (Thus SAT $\notin$ P.)

Motivation

NP-complete problems . . .

- . . . include many real-world problems.
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Modern SAT solvers are very fast for many real-world instances:

- Hardware verification
- Software verification
- Planning
- Cryptography
- Answer Set solving

SAT Algorithm 1b: Deterministic

```
Let $I = \{\}$. Repeat:
  1. If $I$ falsifies some $c \in \phi$:
      1.1 Backtrack to the last decision $\neg x$          undo last decisions
      1.2 If there is none:   return NO
      1.3 Let $I = I \cup \{x\}$                          flip last negative decision
  2. Else:
      2.1 Select a variable $x$ such that $x, \bar{x} \notin I$
      2.2 If there is none:   return YES
      2.3 Let $I = I \cup \{\neg x\}$                     make a decision
```

Outline

Satisfiability and Complexity Theory

Solver Ingredient 1: Watched-Literal Scheme

Solver Ingredient 2: Conflict-Driven Clause Learning

Other Solver Ingredients

Conclusion

Beyond SAT

Unit Propagation: Idea

Unit resolution rule: from p and q and $(\neg p \vee \neg q \vee r)$ infer r.

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In SAT, we want to find a satisfying assignment I of ϕ:

If $p, q \in I$, then I can satisfy $(\neg p \vee \neg q \vee r) \in \phi$ only if $r \in I$.

Unit Propagation

Definition: closure of I under unit propagation relative to ϕ

- Let $I^0 = I$
- Repeat for $j > 0$ until $I^j = I^{j+1}$:
  - If there is a $(x_1 \vee \ldots \vee x_k) \in \phi$ with $\bar{x}_1, \ldots, \bar{x}_k \in I^j$:
    Return conflict $(x_1 \vee \ldots \vee x_k)$
  - If there is a $(x_1 \vee \ldots \vee x_{k+1}) \in \phi$ with $\bar{x}_1, \ldots, \bar{x}_k \in I^j$:
    Let $I^{j+1} = I^j \cup \{x_{k+1}\}$
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Ex. 1: $I = \{\neg p\} \quad \phi = \{(\neg p \vee \neg q \vee s), (p \vee \neg q \vee r), (p \vee q)\}$

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Ex. 2: $I = \{\neg p\} \quad \phi = \{(\neg p \vee \neg q \vee s), (p \vee \neg q), (p \vee q)\}$

- $I^0 = \{\neg p\}$
- $I^1 = \{\neg p, q\}$
- Conflict with $(p \vee \neg q)$

SAT Algorithm 2: Adding Unit Propagation

```
Let $I = \{\}$. Repeat:
  1. Close $I$ under unit propagation relative to $\phi$
  2. If conflict:
     2.1 Backtrack to the last decision $\neg x$          undo last decisions
     2.2 If there is none:   return NO
     2.3 Let $I = I \cup \{x\}$                            flip last negative decision
  3. Else:
     3.1 Select a variable $x$ such that $x, \bar{x} \notin I$
     3.2 If there is none:   return YES
     3.3 Let $I = I \cup \{\neg x\}$                       make a decision
```

Watched-Literal Scheme: Motivation

- Algorithm 2 spends almost all its time on
    - unit propagation and
    - backtracking.

- Watched-Literal Scheme is a lazy data structure for
    - fast unit propagation and
    - very cheap backtracking.

Watched-Literal Scheme

Recall: (i) If $(x_1 \vee \ldots \vee x_k) \in \phi$ and $\bar{x}_1, \ldots, \bar{x}_k \in I$: Return conflict.

(ii) If $(x_1 \vee \ldots \vee x_{k+1}) \in \phi$ and $\bar{x}_1, \ldots, \bar{x}_k \in I$: Add x_{k+1} to I.

How to close I under UP relative to ϕ?

For every $(x_1 \vee \ldots \vee x_k) \in \phi$:

(i) If all $\bar{x}_i \in I$: Return conflict.

(ii) If all but one $\bar{x}_i \in I$: Add remaining x_j to I.

Ex.: $\phi = (\, p \;\vee\; q \;\vee\; r \;\vee\; s\,) \wedge (\, p \;\vee\; q \;\vee\; t\,) \wedge (\, u \;\vee\; v \;\vee\; w\,)$

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$I = \{\}$

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For every $c \in \phi$, select two distinct watched literals $x_c, y_c \in c$
such that $\overline{x}_c \notin I$ if possible, otherwise choose arbitrarily
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Suppose I is closed under UP relative to ϕ.
How to close $I' = I \cup \{z\}$ under UP relative to ϕ?

For every $(x_1 \vee \ldots \vee x_k) \in \phi$ that watches $\overline{z} = x_c$ (w.l.o.g.):

1. Try to update x_c.
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Suppose I is closed under UP relative to ϕ.
How to close $I' = I \cup \{z_1, \ldots, z_\ell\}$ under UP relative to ϕ?

For every $(x_1 \vee \ldots \vee x_k) \in \phi$ that watches $\overline{z}_1 = x_c$ (w.l.o.g.):

1. Try to update x_c.
2. Otherwise: If $\overline{y}_c \in I'$: Return conflict.
   If $\overline{y}_c \notin I'$: Add y_c to I'.

Now $I \cup \{z_1\}$ is closed under UP relative to ϕ. Repeat for z_2, and so on.

Ex.: $\phi = (p \vee q \vee \utilde{r} \vee \utilde{s}) \wedge (p \vee \utilde{q} \vee \utilde{t}) \wedge (\utilde{u} \vee \utilde{v} \vee w)$

$I = \{\neg p, \neg q, t\}$

Watched-Literal Scheme

For every $c \in \phi$, select two distinct watched literals $x_c, y_c \in c$

such that $\overline{x}_c \notin I$ if possible, otherwise choose arbitrarily

and $\overline{y}_c \notin I$ if possible, otherwise choose arbitrarily.

Suppose I is closed under UP relative to ϕ.
How to close $I' = I \cup \{z_1, \ldots, z_\ell\}$ under UP relative to ϕ?

For every $(x_1 \vee \ldots \vee x_k) \in \phi$ that watches $\overline{z}_1 = x_c$ (w.l.o.g.):

1. Try to update x_c.
2. Otherwise: If $\overline{y}_c \in I'$: Return conflict.
   If $\overline{y}_c \notin I'$: Add y_c to I'.

Now $I \cup \{z_1\}$ is closed under UP relative to ϕ. Repeat for z_2, and so on.

Ex.: $\phi = (p \vee q \vee \utilde{r} \vee \utilde{s}) \wedge (p \vee \utilde{q} \vee \utilde{t}) \wedge (\utilde{u} \vee \utilde{v} \vee w)$

$I = \{\neg p, \neg q, t\}$

How to backtrack? Remove literals from I. No update of x_c, y_c needed!

Watched-Literal Scheme: Example

Clauses: $p \lor q \lor s$ $\quad \neg r \lor \neg s \lor \neg t$
$t \lor \neg u$ $\quad t \lor \neg v$ $\quad t \lor \neg w$
$u \lor v \lor w \lor y$ $\quad v \lor \neg y$

Decisions: $\neg p, \neg q, r$

Watched-Literal Scheme: Example

I	Clauses and Watched Literals						
	$p \vee q \vee s$	$\bar{r} \vee \bar{s} \vee \bar{t}$	$t \vee \overline{u}$	$t \vee \overline{v}$	$t \vee \overline{w}$	$u \vee v \vee w \vee y$	$v \vee \overline{y}$
	p, q	$\bar{r}, \bar{s}$	$t, \overline{u}$	$t, \overline{v}$	$t, \overline{w}$	u, v	$v, \overline{y}$

Recall watched-literal invariant: $\bar{x}_c, \bar{y}_c \notin I$ if possible.

Watched-Literal Scheme: Example

I	Clauses and Watched Literals						
	$p \vee q \vee s$	$\bar{r} \vee \bar{s} \vee \bar{t}$	$t \vee \overline{u}$	$t \vee \bar{v}$	$t \vee \overline{w}$	$u \vee v \vee w \vee y$	$v \vee \bar{y}$
	p, q	$\bar{r}, \bar{s}$	$t, \overline{u}$	$t, \bar{v}$	$t, \overline{w}$	u, v	$v, \bar{y}$
$\bar{p}$							

Recall watched-literal invariant: $\bar{x}_c, \bar{y}_c \notin I$ if possible.

Watched-Literal Scheme: Example

I	Clauses and Watched Literals						
	$p \vee q \vee s$	$\bar{r} \vee \bar{s} \vee \bar{t}$	$t \vee \bar{u}$	$t \vee \bar{v}$	$t \vee \bar{w}$	$u \vee v \vee w \vee y$	$v \vee \bar{y}$
	p, q	$\bar{r}, \bar{s}$	$t, \bar{u}$	$t, \bar{v}$	$t, \bar{w}$	u, v	$v, \bar{y}$
$\bar{p}$	s, q						

Recall watched-literal invariant: $\bar{x}_c, \bar{y}_c \notin I$ if possible.

Watched-Literal Scheme: Example

I	Clauses and Watched Literals						
	$p \lor q \lor s$	$\bar{r} \lor \bar{s} \lor \bar{t}$	$t \lor \bar{u}$	$t \lor \bar{v}$	$t \lor \bar{w}$	$u \lor v \lor w \lor y$	$v \lor \bar{y}$
	p, q	$\bar{r}, \bar{s}$	$t, \bar{u}$	$t, \bar{v}$	$t, \bar{w}$	u, v	$v, \bar{y}$
$\bar{p}$	s, q						
$\bar{q}$							

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Watched-Literal Scheme: Example

I	Clauses and Watched Literals						
	$p \lor q \lor s$	$\bar{r} \lor \bar{s} \lor \bar{t}$	$t \lor \bar{u}$	$t \lor \bar{v}$	$t \lor \bar{w}$	$u \lor v \lor w \lor y$	$v \lor \bar{y}$
	p, q	$\bar{r}, \bar{s}$	$t, \bar{u}$	$t, \bar{v}$	$t, \bar{w}$	u, v	$v, \bar{y}$
$\bar{p}$	s, q						
$\bar{q}$	s, q						
s							

Recall watched-literal invariant: $\bar{x}_c, \bar{y}_c \notin I$ if possible.

Watched-Literal Scheme: Example

I	Clauses and Watched Literals						
	$p \vee q \vee s$	$\bar{r} \vee \bar{s} \vee \bar{t}$	$t \vee \bar{u}$	$t \vee \bar{v}$	$t \vee \bar{w}$	$u \vee v \vee w \vee y$	$v \vee \bar{y}$
	p, q	$\bar{r}, \bar{s}$	$t, \bar{u}$	$t, \bar{v}$	$t, \bar{w}$	u, v	$v, \bar{y}$
$\bar{p}$	s, q						
$\bar{q}$	s, q						
s		$\bar{r}, \bar{t}$					

Recall watched-literal invariant: $\bar{x}_c, \bar{y}_c \notin I$ if possible.

Watched-Literal Scheme: Example

I	Clauses and Watched Literals						
	$p \vee q \vee s$	$\bar{r} \vee \bar{s} \vee \bar{t}$	$t \vee \overline{u}$	$t \vee \bar{v}$	$t \vee \overline{w}$	$u \vee v \vee w \vee y$	$v \vee \bar{y}$
	p, q	$\bar{r}, \bar{s}$	$t, \overline{u}$	$t, \bar{v}$	$t, \overline{w}$	u, v	$v, \bar{y}$
$\bar{p}$	s, q						
$\bar{q}$	s, q						
s		$\bar{r}, \bar{t}$					
r							

Recall watched-literal invariant: $\bar{x}_c, \bar{y}_c \notin I$ if possible.

Watched-Literal Scheme: Example

I	Clauses and Watched Literals						
	$p \vee q \vee s$	$\bar{r} \vee \bar{s} \vee \bar{t}$	$t \vee \overline{u}$	$t \vee \overline{v}$	$t \vee \overline{w}$	$u \vee v \vee w \vee y$	$v \vee \overline{y}$
	p, q	$\bar{r}, \bar{s}$	$t, \overline{u}$	$t, \overline{v}$	$t, \overline{w}$	u, v	$v, \overline{y}$
$\overline{p}$	s, q						
$\overline{q}$	s, q						
s		$\bar{r}, \bar{t}$					
r		$\bar{r}, \bar{t}$					
$\bar{t}$							

Recall watched-literal invariant: $\bar{x}_c, \bar{y}_c \notin I$ if possible.

Watched-Literal Scheme: Example

I	Clauses and Watched Literals						
	$p \vee q \vee s$	$\bar{r} \vee \bar{s} \vee \bar{t}$	$t \vee \bar{u}$	$t \vee \bar{v}$	$t \vee \bar{w}$	$u \vee v \vee w \vee y$	$v \vee \bar{y}$
	p, q	$\bar{r}, \bar{s}$	$t, \bar{u}$	$t, \bar{v}$	$t, \bar{w}$	u, v	$v, \bar{y}$
$\bar{p}$	s, q						
$\bar{q}$	s, q						
s		$\bar{r}, \bar{t}$					
r		$\bar{r}, \bar{t}$					
$\bar{t}$			$t, \bar{u}$				
$\bar{u}$							

Recall watched-literal invariant: $\bar{x}_c, \bar{y}_c \notin I$ if possible.

Watched-Literal Scheme: Example

I	Clauses and Watched Literals						
	$p \vee q \vee s$	$\bar{r} \vee \bar{s} \vee \bar{t}$	$t \vee \bar{u}$	$t \vee \bar{v}$	$t \vee \bar{w}$	$u \vee v \vee w \vee y$	$v \vee \bar{y}$
	p, q	$\bar{r}, \bar{s}$	$t, \bar{u}$	$t, \bar{v}$	$t, \bar{w}$	u, v	$v, \bar{y}$
$\bar{p}$	s, q						
$\bar{q}$	s, q						
s		$\bar{r}, \bar{t}$					
r		$\bar{r}, \bar{t}$					
$\bar{t}$			$t, \bar{u}$	$t, \bar{v}$			
$\bar{u}$							
$\bar{v}$							

Recall watched-literal invariant: $\bar{x}_c, \bar{y}_c \notin I$ if possible.

Watched-Literal Scheme: Example

I	Clauses and Watched Literals						
	$p \vee q \vee s$	$\bar{r} \vee \bar{s} \vee \bar{t}$	$t \vee \bar{u}$	$t \vee \bar{v}$	$t \vee \bar{w}$	$u \vee v \vee w \vee y$	$v \vee \bar{y}$
	p, q	$\bar{r}, \bar{s}$	$t, \bar{u}$	$t, \bar{v}$	$t, \bar{w}$	u, v	$v, \bar{y}$
$\bar{p}$	s, q						
$\bar{q}$	s, q						
s		$\bar{r}, \bar{t}$					
r		$\bar{r}, \bar{t}$					
$\bar{t}$			$t, \bar{u}$	$t, \bar{v}$	$t, \bar{w}$		
$\bar{u}$							
$\bar{v}$							
$\bar{w}$							

Recall watched-literal invariant: $\bar{x}_c, \bar{y}_c \notin I$ if possible.

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	$p \vee q \vee s$	$\bar{r} \vee \bar{s} \vee \bar{t}$	$t \vee \bar{u}$	$t \vee \bar{v}$	$t \vee \bar{w}$	$u \vee v \vee w \vee y$	$v \vee \bar{y}$
	p, q	$\bar{r}, \bar{s}$	$t, \bar{u}$	$t, \bar{v}$	$t, \bar{w}$	u, v	$v, \bar{y}$
$\bar{p}$	s, q						
$\bar{q}$	s, q						
s		$\bar{r}, \bar{t}$					
r		$\bar{r}, \bar{t}$					
$\bar{t}$			$t, \bar{u}$	$t, \bar{v}$	$t, \bar{w}$		
$\bar{u}$						y, v	
$\bar{v}$							
$\bar{w}$							

Recall watched-literal invariant: $\bar{x}_c, \bar{y}_c \notin I$ if possible.

Watched-Literal Scheme: Example

I	Clauses and Watched Literals						
	$p \lor q \lor s$	$\bar{r} \lor \bar{s} \lor \bar{t}$	$t \lor \bar{u}$	$t \lor \bar{v}$	$t \lor \bar{w}$	$u \lor v \lor w \lor y$	$v \lor \bar{y}$
	p, q	$\bar{r}, \bar{s}$	$t, \bar{u}$	$t, \bar{v}$	$t, \bar{w}$	u, v	$v, \bar{y}$
$\bar{p}$	s, q						
$\bar{q}$	s, q						
s		$\bar{r}, \bar{t}$					
r		$\bar{r}, \bar{t}$					
$\bar{t}$			$t, \bar{u}$	$t, \bar{v}$	$t, \bar{w}$		
$\bar{u}$						y, v	
$\bar{v}$						y, v	
$\bar{w}$							
y							

Recall watched-literal invariant: $\bar{x}_c, \bar{y}_c \notin I$ if possible.

Watched-Literal Scheme: Example

I	Clauses and Watched Literals						
	$p \lor q \lor s$	$\bar{r} \lor \bar{s} \lor \bar{t}$	$t \lor \bar{u}$	$t \lor \bar{v}$	$t \lor \bar{w}$	$u \lor v \lor w \lor y$	$v \lor \bar{y}$
	p, q	$\bar{r}, \bar{s}$	$t, \bar{u}$	$t, \bar{v}$	$t, \bar{w}$	u, v	$v, \bar{y}$
$\bar{p}$	s, q						
$\bar{q}$	s, q						
s		$\bar{r}, \bar{t}$					
r		$\bar{r}, \bar{t}$					
$\bar{t}$			$t, \bar{u}$	$t, \bar{v}$	$t, \bar{w}$		
$\bar{u}$						y, v	
$\bar{v}$						y, v	$v, \bar{y}$ ϟ
$\bar{w}$							
y							

Recall watched-literal invariant: $\bar{x}_c, \bar{y}_c \notin I$ if possible.

Watched-Literal Scheme: Example

I	Clauses and Watched Literals						
	$p \vee q \vee s$	$\bar{r} \vee \bar{s} \vee \bar{t}$	$t \vee \bar{u}$	$t \vee \bar{v}$	$t \vee \bar{w}$	$u \vee v \vee w \vee y$	$v \vee \bar{y}$
	p, q	$\bar{r}, \bar{s}$	$t, \bar{u}$	$t, \bar{v}$	$t, \bar{w}$	u, v	$v, \bar{y}$
$\bar{p}$	s, q						
$\bar{q}$	s, q						
s		$\bar{r}, \bar{t}$					
r		$\bar{r}, \bar{t}$					
$\bar{t}$			$t, \bar{u}$	$t, \bar{v}$	$t, \bar{w}$		
$\bar{u}$						y, v	
$\bar{v}$						y, v	$v, \bar{y}$ ↯
$\bar{w}$							
y							

Recall watched-literal invariant: $\bar{x}_c, \bar{y}_c \notin I$ if possible.

Backtracking preserves this invariant!

Outline

Satisfiability and Complexity Theory

Solver Ingredient 1: Watched-Literal Scheme

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Conclusion

Beyond SAT

Conflict-Driven Clause Learning: Motivation

- Algorithm 2 with Watched-Literal Scheme still spends almost all its time on unit propagation.

- Suppose $I = \{x_1, \ldots, x_k\}$ leads to a conflict.

  That is: I falsifies some $c \in \phi$.

- Let's learn from the conflict to avoid similar mistakes later:
  - Find a cause $\{x_{i_1}, \ldots, x_{i_l}\} \subseteq I$ of the conflict.
  - Add learnt clause $(\overline{x}_{i_1} \vee \ldots \vee \overline{x}_{i_l})$ to avoid the conflict next time!
  - This avoids assignments $x_{i_1}, \ldots, x_{i_l}$ in the remaining search.
  - Note: must have $\phi \models (\overline{x}_{i_1} \vee \ldots \vee \overline{x}_{i_l})$.

SAT Algorithm 3: Adding Clause Learning

```
Let $I = \{\}$. Repeat:
  1. Close $I$ under unit propagation relative to $\phi$
  2. If conflict:
       2.1 Analyse conflict                                  find the cause
       2.2 Backtrack to the appropriate level                undo last decisions
       2.3 If there is none:   return NO
       2.4 Add conflict clause to $\phi$                     new decision implicitly
  3. Else:
       3.1 Select a variable $x$ such that $x, \overline{x} \notin I$
       3.2 If there is none:   return YES
       3.3 Let $I = I \cup \{\neg x\}$                       make a decision
```

Implication Graph

Maintain implication graph during unit propagation:

- For every decision x, create a node x.
- When $(x_1 \vee \ldots \vee x_{k+1})$ and $\overline{x}_1, \ldots, \overline{x}_k$ produce x_{k+1}:
  - Add a node x_{k+1}.
  - Add edges $(\overline{x}_i, x_{k+1})$ and add c to their label set.
- When $(x_1 \vee \ldots \vee x_k)$ and $\overline{x}_1, \ldots, \overline{x}_k$ produce a conflict:
  - Add a node x_k.
  - Add edges $(\overline{x}_i, x_k)$.
  - Add edges $(x_k, \perp)$ and $(\overline{x}_k, \perp)$.

Implication Graph

Maintain implication graph during unit propagation:

- For every decision x, create a node x.
- When $(x_1 \vee \ldots \vee x_{k+1})$ and $\bar{x}_1, \ldots, \bar{x}_k$ produce x_{k+1}:
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- When $(x_1 \vee \ldots \vee x_k)$ and $\bar{x}_1, \ldots, \bar{x}_k$ produce a conflict:
  - Add a node x_k.
  - Add edges $(\bar{x}_i, x_k)$.
  - Add edges $(x_k, \bot)$ and $(\bar{x}_k, \bot)$.

Find conflict clause:

- Let $\{(x_1, y_1), \ldots, (x_k, y_k)\}$ be a cut separating decisions from $\bot$.
- Then $(x_1 \wedge \ldots \wedge x_k)$ is a cause of the conflict.
- Conflict clause: $(\bar{x}_1 \vee \ldots \vee \bar{x}_k)$.

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- When $(x_1 \vee \ldots \vee x_k)$ and $\overline{x}_1, \ldots, \overline{x}_k$ produce a conflict:
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Find conflict clause:

- Let $\{(x_1, y_1), \ldots, (x_k, y_k)\}$ be a cut separating decisions from $\bot$.
- Then $(x_1 \wedge \ldots \wedge x_k)$ is a cause of the conflict.
- Conflict clause: $(\overline{x}_1 \vee \ldots \vee \overline{x}_k)$.

Good scheme is FirstUIP: Cut such that x_i are:

- The UIP: Node through which all paths from last decision to $\bot$ go.
- Literals from earlier levels if necessary.

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Variable Selection Heuristic

Recall algorithm:

3.1 Select a variable x such that $x, \bar{x} \notin I$

3.3 Let $I = I \cup \{\neg x\}$

- Variable Selection Heuristics rank variables by attractiveness:
  - Maximum Occurrence in clauses of Minimum size (MOM):
    - Prefer literals that occur often in small clauses.
  - Dynamic Largest Individual Sum (DLIS):
    - Select variable that occurs most frequently in unsatisfied clauses.
  - Variable State Independent Decaying Sum (VSIDS):
    - Score variables numerically.
    - Each time x occurs in conflict, increase its score.
    - Decay scores to increase meaning of recent conflicts.

Direction Heuristics

Recall algorithm:

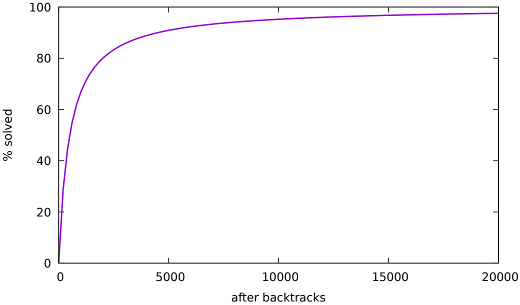
3.1 Select a variable x such that $x, \bar{x} \notin I$

3.3 Let $I = I \cup \{\neg x\}$

- Branching to $\neg x$ is better than branching to x:
  - Keeps the search in the same search space.
  - Most things in the real world are false.
- Phase-Saving Heuristic:
  - Remember the last value assigned to x.
  - Assign x this value in 3.3.

Randomized Restarts

- Heavy-tail behaviour typical:



- Restarts avoid getting stuck in long tail:
  - Restart after N conflicts.
  - Keep the learnt clauses.
  - Dynamically grow N exponentially – otherwise incomplete.
- N differs a lot between solvers:
  - Initial N may vary from high hundreds to low tens.
  - Trend is to smaller N.

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Enumerating Models

Suppose the SAT solver computes a satisfying assignment I.

How to find another one?

Satisfiability for $\leq$ 2-Clauses

Theorem: complexity of 2-SAT

2-SAT can be solved in polynomial time.

- Input: 2-CNF formula ϕ of length n.
- Let ϕ^* be the least set such that
  - $\phi \subseteq \phi^*$ and
  - if $(x \vee y),\ (\overline{x} \vee z) \in \phi^*,$ then $(y \vee z) \in \phi^*$.
- ϕ is equivalent to ϕ^*.
- ϕ is unsatisfiable iff $(x \vee x), (\neg x \vee \neg x) \in \phi^*$ for some x.
- $|\{x \mid x \text{ literal in } \phi\}| \leq n$.
- $|\{(x \vee y, \overline{x} \vee z) \mid y, z \text{ literal in } \phi\}| \leq n^2$ for each literal x.

Benchmarks

Benchmark	#Variables	#Clauses	Naive	WSL	CDCL
aim-50-1_6-yes1-4	50	80	—	3×10^{-3}	3×10^{-4}
uf75-013	75	325	—	4×10^{-4}	2×10^{-4}
pigeon/12	156	949	—	—	2.0
example2_gr_rcs_w6	2,664	27,684	—	—	3×10^{-3}
fvp-unsat	24,065	731,850	—	—	—
cache_10	227,210	879,754	—	—	86.4

Execution time (in seconds) of different algorithms:

- Naive: Slide 21
- WSL: Slide 25 with watched literal scheme
- CDCL: Slide 31 with watched literal scheme, FirstUIP clause learning

Timeout after 600 seconds.

Most of the files are from othe 2002 SAT competition.

Pigeonhole Principle

Pigeonhole principle: $k+1$ pigeons cannot be put in k holes.

$$k = 1 : \qquad \begin{aligned} &(h_1 = p_1) \wedge \\ &(h_1 = p_2) \end{aligned}$$

where we assume $p_i \neq p_j$ for $i \neq j$.

Pigeonhole Principle

Pigeonhole principle: $k+1$ pigeons cannot be put in k holes.

$$k = 2: \qquad \begin{aligned} &(h_1 = p_1 \vee h_2 = p_1) \wedge \\ &(h_1 = p_2 \vee h_2 = p_2) \wedge \\ &(h_1 = p_3 \vee h_2 = p_3) \end{aligned}$$

where we assume $p_i \neq p_j$ for $i \neq j$.

Pigeonhole Principle

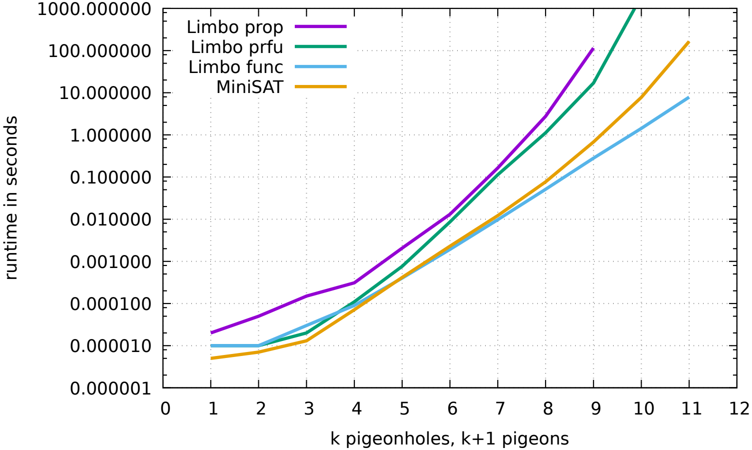
Pigeonhole principle: $k+1$ pigeons cannot be put in k holes.

$$k = 3 : \qquad \begin{aligned} &(h_1 = p_1 \vee h_2 = p_1 \vee h_3 = p_1) \wedge \\ &(h_1 = p_2 \vee h_2 = p_2 \vee h_3 = p_2) \wedge \\ &(h_1 = p_3 \vee h_2 = p_3 \vee h_3 = p_3) \wedge \\ &(h_1 = p_4 \vee h_2 = p_4 \vee h_3 = p_4) \end{aligned}$$

where we assume $p_i \neq p_j$ for $i \neq j$.

Pigeonhole Principle

Pigeonhole principle: $k + 1$ pigeons cannot be put in k holes.



Summary SAT

- SAT is provably hard:
    - SAT is NP-complete – believed to take exponential time.
    - All problems in NP can be reduced to SAT efficiently.
    - SAT is perhaps the best-understood problem of complexity theory.

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  - SAT solvers are far-developed.
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  - There are small but very hard instances though.

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- Key ingredients for a fast SAT solver:
  - Watched-Literal Scheme
  - Clause Learning
  - Variable Selection Heuristic
  - Random Restarts

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Quantified Boolean Formulas

Syntax: propositional logic $\exists x\, \phi$ $\forall x\, \phi$

Quantified Boolean Formulas

Syntax: propositional logic $\exists x\,\phi$ $\forall x\,\phi$

Semantics:

- x is true iff $x = \textsc{true}$
- $\neg\phi$ is true iff ϕ is not true
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Theorem: complexity of QBF

Deciding whether a QBF is true is PSPACE-complete.

Model Counting

How many satisfying assignment does a formula have?

- Exact solvers struggle with huge solution space.
- Approximative solvers sample solution space.
- Useful for probabilistic reasoning.

Satisfiability Modulo Theories

Combine SAT solver with a theory solver (e.g., linear equations).

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- $p \vee q$
- $r \vee \neg s$
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