NOTES ON CONTINUED FRACTIONS

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1. Paving a rectangle by squares, Euclid's algorithm for computing the greatest common divisor, and finite continued fractions

Euclid's algorithm determines that gcd(180,64) = 4 by performing the computations displayed in red in the following:

```
180 = 180 // 64 * 64 + 180 % 64 = 2 * 64 + 52

64 = 64 // 52 * 52 + 64 % 52 = 1 * 52 + 12

52 = 52 // 12 * 12 + 52 % 12 = 4 * 12 + 4

12 = 12 // 4 * 4 + 12 % 4 = 3 * 12 + 0
```

It corresponds to finding out that 4 is the size of the largest square thanks to which it is possible to pave a rectangle of size 180 by 64, based on the following geometric construction:

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	180							
64	64	64			52			
			12	12	12	12	$\frac{4}{4}$	

So when the gcd is 1, the paving of the rectangle can only be achieved with squares of size 1 by 1:

```
45 = 45 // 16 * 16 + 45 % 16 = 2 * 16 + 13

16 = 16 // 13 * 13 + 16 % 13 = 1 * 13 + 3

13 = 13 // 3 * 3 + 13 % 3 = 4 * 3 + 1

3 = 3 // 1 * 3 + 3 % 1 = 3 * 1 + 0
```

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The blue part in both previous sets of equations is the same, and the pictures illustrate that

$$\frac{180}{64} = \frac{45}{16} = 2 + \frac{1}{1 + \frac{1}{4 + \frac{1}{2}}}$$

The pictures illustrate that more generally, any rational number can be written as:

$$a_0 + 1/(a_1 + 1/(a_2 + \dots + 1/a_k) \cdot \dots)$$

where $a_0 \in \mathbb{Z}$, $k \in \mathbb{N}$, and $a_1, \ldots, a_k \in \mathbb{N} \setminus \{0\}$ with $a_k \neq 1$, which is the general form of a finite continued fraction, that it is convenient to denote by $[a_0, a_1, a_2, \ldots, a_k]$. Note that we could allow a finite continued fraction to end in 1 because for all $b \in \mathbb{N} \setminus \{0, 1\}$, $b = b - 1 + \frac{1}{1}$; that would make $[a_0, a_1, a_2, \ldots, a_k - 1, 1]$ an alternative representation to $[a_0, a_1, a_2, \ldots, a_k]$.

2. Computation of a finite continued fraction

More generally, given $k \in \mathbb{N} \setminus \{0\}$ and $r_1, \ldots, r_k \in \mathbb{R}$ with r_2, \ldots, r_k at least equal to 1, let $[r_1, \ldots, r_k]$ be defined as r_1 if k = 1, and as $r_1 + \frac{1}{[r_1, \ldots, r_k]}$ if k > 0. For all $i \in \{-1, \ldots, k\}$:

- let p_i be equal to 0 if i = -1, to 1 if i = 0, and to $r_k p_{k-1} + p_{k-2}$ if $k \ge 1$;
- let q_i be equal to 1 if i = -1, to 0 if i = 0, and to $r_k q_{k-1} + q_{k-2}$ if $k \ge 1$.

A trivial proof by induction shows that for all nonzero $j \leq k, q_k > 0$. Then:

$$[r_1, \dots, r_k] = \frac{p_k}{q_k}$$

which provides an effective method for computing $[r_1, \ldots, r_k]$.

Towards proving (1), first define for all $j \leq k$ the matrix M_j as

$$\begin{bmatrix} p_j & q_j \\ p_{j-1} & q_{j-1} \end{bmatrix}$$

It is immediately verified by induction that for all nonzero $j \leq k$,

$$M_j = \begin{bmatrix} r_j & 1\\ 1 & 0 \end{bmatrix} M_{j-1},$$

from which it follows that for all nonzero $j \leq k$,

As the transpose of the product of two matrixes A and B is the product of B by A, we have that for all nonzero $j \leq k$,

$$\begin{bmatrix} p_j & p_{j-1} \\ q_j & q_{j-1} \end{bmatrix} = \begin{bmatrix} r_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} r_j & 1 \\ 1 & 0 \end{bmatrix}$$

which implies that for all nonzero $j \leq k$,

Now proof of (1) is by induction on the length of finite continued fractions and application of (3). It is trivial that if k = 1 then (1) holds. Assume that k > 1. Denoting $[r_2, \ldots, r_k]$ by $\frac{a}{b}$, we have by induction and (3) that

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} r_2 & 1 \\ 1 & 0 \end{bmatrix} \dots \begin{bmatrix} r_k & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Set

$$\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} r_2 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} r_k & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Then $[r_1,\ldots,r_k]=r_1+\frac{1}{[r_2,\ldots,r_k]}=r_1+\frac{b}{a}=\frac{ar_1+b}{a}$. It then follows from (3) again that

$$\begin{bmatrix} p_j \\ q_j \end{bmatrix} = \begin{bmatrix} r_1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} ar_1 + b & cr_1 + d \\ a & c \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} ar_1 + b \\ a \end{bmatrix}$$

hence $[r_1, \ldots, r_k] = \frac{p_1}{q_j}$, which completes the proof of (1).

3. Infinite continued fractions

Extend the notation of the previous section with $c_j = \frac{p_j}{q_j}$ for all strictly positive $j \leq k$. Then for all $j \in \{2, \ldots, k\}$, $c_j - c_{j-1}$ is equal to $\frac{p_j q_{j-1} - p_{j-1} q_j}{q_j q_{j-1}}$. Note that for all $j \leq k$, $p_j q_{j-1} - p_{j-1} q_j$ is the determinant of the matrix M_j , and it then follows from (3) that it is equal to $(-1)^r$. Hence for all strictly positive $j \leq k$,

$$c_j - c_{j-1} = \frac{(-1)^r}{q_j q_{j-1}}$$

Moreover, it is immediately verify by induction that $(q_j)_{2 \le j \le k}$ is a strictly increasing sequence. This shows that given $a_0 \in \mathbb{Z}$ and a sequence $(a_j)_{j \in \mathbb{N} \setminus \{0\}}$ of members of $\mathbb{N} \setminus \{0\}$, the sequence $([a_0, \dots, a_j])_{j \in \mathbb{N}}$ converges; it is called an infinite continued fraction and it is denoted either as

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_2 + \dots}}}$$

or as $[a_0, a_1, a_2, a_3 \dots]$.

It follows from the previous observations that given an infinite continued fraction $[a_0, a_1, a_2, a_3...]$, $j \in \mathbb{N} \setminus \{0, 1\}$ and $n \in \mathbb{N}$, if $[a_0, \ldots, a_j]$ and $[a_0, \ldots, a_{j+1}]$ agree up to n digits after the decimal point, then $[a_0, \ldots, a_j]$ and $[a_0, a_1, a_2, a_3...]$ agree up to n digits after the decimal point. This allows one to compute exactly any approximation of $[a_0, a_1, a_2, a_3...]$.

4. Negating continued fractions

Obviously, for all $a \in \mathbb{Z}$, -[a] = -a and -[a, 2] = [-a - 1, 2].

Given $a \in \mathbb{Z}$, $b \in \mathbb{N} \setminus \{0\}$, and $N \in \mathbb{N} \setminus \{0\} \cup \{\infty\}$,

$$-\left(a+\frac{1}{b+\frac{1}{N}}\right)=-a-1+1-\frac{1}{b+\frac{1}{N}}=-a-1+\frac{b+\frac{1}{N}-1}{b+\frac{1}{N}}=-a-1+\frac{1}{\frac{b+\frac{1}{N}}{b+\frac{1}{N}-1}}=-a-1+\frac{1}{1+\frac{1}{b-1+\frac{1}{N}}}=-a-1+\frac{1$$

Given $a \in \mathbb{Z}$, $b \in \mathbb{N} \setminus \{0\}$ and $N \in \mathbb{N} \setminus \{0\} \cup \{\infty\}$,

$$a + \frac{1}{0 + \frac{1}{b + \frac{1}{N}}} = a + b + \frac{1}{N}$$

It follows that, using ... to denote the possibly missing terms of a finite or infinite continued fraction,

- for all $a \in \mathbb{Z}$ and $b \in \mathbb{N} \setminus \{0, 1, 2\}, -[a, b \dots] = [-a 1, 1, b 1 \dots];$
- for all $a \in \mathbb{Z}$ and $c \in \mathbb{N} \setminus \{0\}, -[a, 2, c \dots] = [-a 1, 1, 1, c \dots];$
- for all $a \in \mathbb{Z}$ and $c \in \mathbb{N} \setminus \{0\}, -[a, 1, c...] == [-a-1, 1+c...]$

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