

# Generalized universal instability: Transient linear amplification and subcritical turbulence

Matt Landreman,<sup>1,\*</sup> Gabriel G. Plunk,<sup>2</sup> and William Dorland<sup>1</sup>

<sup>1</sup>*Institute for Research in Electronics and Applied Physics,  
University of Maryland, College Park, MD, 20742, USA*

<sup>2</sup>*Max Planck Institute for Plasma Physics,  
EURATOM Association and Max-Planck/Princeton Research Center for Plasma Physics*

(Dated: December 27, 2014)

In this work we numerically demonstrate both significant transient (i.e. non-modal) linear amplification and sustained nonlinear turbulence in a kinetic system with no unstable eigenmodes. The particular system considered is an electrostatic slab with magnetic shear, kinetic electrons and ions, weak collisions, and a density gradient, but with no temperature gradient. In contrast to hydrodynamic examples of non-modal growth and subcritical turbulence, here there is no sheared flow in the equilibrium. Significant transient linear amplification is found when the magnetic shear and collisionality are weak. It is also demonstrated that nonlinear turbulence can be sustained if initialized at sufficient amplitude. We prove these two phenomena are related: when sustained turbulence occurs without unstable eigenmodes, states that are typical of the turbulence must yield transient linear amplification of the gyrokinetic free energy.

## I. INTRODUCTION

Despite the usefulness of linear eigenmode analysis in many contexts, it has been established that such analysis can in practice give misleading predictions about the stability of various systems [1]. In particular, turbulence may be sustained in a number of fluid systems despite the absence of any unstable eigenmodes. The first such systems known were certain sheared flow patterns in neutral fluids, such as plane Poiseuille or plane Couette flow [2–4].

---

\*mattland@umd.edu

Later, sustained turbulence was seen in fluid simulations of several plasma systems with no absolute linear instabilities [5–8]. More recently, turbulence in the absence of linear instability has been seen in kinetic plasma simulations involving sheared equilibrium flow [9, 10]. These examples demonstrate that linear eigenmode analysis can give deceiving predictions for nonlinear behavior, a conclusion supported by other studies of plasma turbulence in which linear instability exists, but the fluctuations in saturated nonlinear turbulence are different in character from the linear eigenmodes [11–13].

For the cases of turbulence without absolute linear stability in the presence of sheared flows, for both neutral fluids and plasmas, the phenomenon of transient linear amplification appears to be important [1, 3, 4, 14–18]. Also known as non-modal amplification, transient amplification is a period of growth before the exponential decay sets in. A necessary (but not sufficient) condition for transient amplification is that the linear operator must not have a complete orthogonal set of eigenvectors. Analysis of transient amplification has also been applied to plasma systems in which linear instability exists but transient linear behavior may be more important [13, 19–22].

In the present work, we consider the phenomena of transient amplification and turbulence without instability for the case of the weakly-collisional plasma drift wave driven by a density gradient [23, 24], neglecting toroidal effects. This oscillation has been called the “universal instability” as it was at first mistakenly thought to be linearly unstable in any plasma with a density gradient. The oscillation is electrostatic in nature, and linear instability can indeed be found in slab geometry without magnetic shear or temperature gradients. It was later established that when any amount of magnetic shear is included in the model, no matter how small, the eigenmodes become linearly damped for  $k_y \rho_i \ll 1$ , where  $k_y$  is the perpendicular wavenumber and  $\rho_i$  is the ion gyroradius [25–27]. However, instability can persist at  $k_y \rho_i \geq 1$  when the collisionality is very low [28]. When both magnetic shear and modest collisions are included [28], the mode becomes linearly stable at all  $k_y \rho_i$ . It is this latter situation that we will consider here.

The fluid analogue of this drift wave system was investigated in [8], using a model with three fluctuating quantities (density, electrostatic potential, and parallel velocity). In this model all linear modes were damped, yet nonlinearly, turbulence could be sustained. It was pointed out that an important secondary instability in this system is a drift wave rotated by  $\pi/2$  in the plane perpendicular to the magnetic field, since the stabilizing effects of

magnetic shear are reduced in this orientation. In this rotated drift instability, the role of the equilibrium density gradient is replaced in a sense by the perturbed density gradient, the latter represented in the nonlinear term. However, since the nonlinearity is conservative, the rotated drift wave does not inject energy into the system in the way that can be done by a perturbation feeding off the density gradient of the equilibrium. Several authors have also investigated transient linear effects in fluid drift wave models [12, 13, 19]. However, these studies did not include magnetic shear, which we will find to have a very strong effect on transient behavior.

In the following sections, we establish three results concerning the drift wave system in slab geometry with magnetic shear, extending previous work by considering a kinetic description and exploring the role of transient linear amplification. For these studies, the initial-value gyrokinetic code gs2 [29] will be used for numerical computations. First, in section III we demonstrate that significant transient linear amplification is possible in this system, as long as the magnetic shear is not too strong. Second, in section IV we demonstrate that universal-mode turbulence without linear instability, seen previously in the fluid model in [8], is also possible in the gyrokinetic model. Finally, we prove there is a relationship between transient linear amplification and nonlinear turbulence in this system, using similar arguments to the proof for hydrodynamics in [30] and arriving at similar conclusions. Specifically, if the physical parameters are such that turbulence can be sustained, typical states of the turbulence must be amplified linearly for some amount of time, even if these states eventually decay exponentially. Two more precise statements of this result and their proof will be given in section V.

## II. GYROKINETIC MODEL

We consider slab geometry with magnetic shear, so the magnetic field in Cartesian coordinates  $(x', y', z')$  is  $\mathbf{B} = [\mathbf{e}_{z'} + (x'/L_s)\mathbf{e}_{y'}]B$  for some constant  $B$  and shear length  $L_s$ . A field-aligned coordinate system is introduced:  $x = x'$ ,  $y = y' - x'z'/L_s$ ,  $z = z'$ , so  $\mathbf{B} \cdot \nabla x = \mathbf{B} \cdot \nabla y = 0$  and  $\mathbf{B} \cdot \nabla z = B$ . In this geometry, we consider the nonlinear

electrostatic gyrokinetic equation [31], dropping toroidal effects:

$$\begin{aligned} \frac{\partial h_s}{\partial t} + v_{\parallel} \frac{\partial h_s}{\partial z} + \frac{1}{B} \left[ \frac{\partial \langle \Phi \rangle \mathbf{R}_s}{\partial X_s} \frac{\partial h_s}{\partial Y_s} - \frac{\partial \langle \Phi \rangle \mathbf{R}_s}{\partial Y_s} \frac{\partial h_s}{\partial X_s} \right] - \langle C_s \{h_s\} \rangle \mathbf{R}_s \\ = \frac{q_s}{T_s} f_{Ms} \frac{\partial \langle \Phi \rangle \mathbf{R}_s}{\partial t} - \frac{f_{Ms}}{BL_n} \frac{\partial \langle \Phi \rangle \mathbf{R}_s}{\partial Y_s} \left[ 1 + \left( \mathcal{E}_s - \frac{3}{2} \right) \eta_s \right] \end{aligned} \quad (1)$$

together with the quasineutrality condition

$$-\frac{ne\Phi}{T_i} + \int d^3v \langle h_i \rangle_{\mathbf{r}} = \frac{ne\Phi}{T_e} + \int d^3v \langle h_e \rangle_{\mathbf{r}}. \quad (2)$$

Here  $s \in \{i, e\}$  subscripts denote species,  $\Phi(t, \mathbf{r})$  is the electrostatic potential,  $h_s(t, \mathbf{R}_s, \mathcal{E}_s, \mu)$  is the nonadiabatic distribution function,  $\mathcal{E}_s = m_s v^2 / (2T_s)$ ,  $\mu = v_{\perp}^2 / (2B)$ , and  $f_{Ms}$  is the leading-order Maxwellian. The scale length of the equilibrium density  $n$  is  $L_n = -n / (dn/dx)$ ,  $q_i = e = -q_e$  is the proton charge,  $T_s$  is the species temperature,  $C_s$  is the collision operator, and  $\langle \rangle_{\mathbf{R}_s}$  and  $\langle \rangle_{\mathbf{r}}$  denote gyroaverages at fixed particle position  $\mathbf{r} = (x, y, z)$  and guiding-center position  $\mathbf{R}_s = (X_s, Y_s, Z_s)$ . The distinction between  $z$  and  $Z_s$  may be neglected. Expanding the fluctuating quantities in spatial Fourier modes as  $\Phi = \sum_{k_x, k_y} \Phi_{k_x, k_y}(t, z) \exp(ik_x x + ik_y y)$  and  $h_s = \sum_{k_x, k_y} h_{s, k_x, k_y}(t, z, \mathcal{E}_s, \mu) \exp(ik_x X_s + ik_y Y_s)$ , we may replace the gyro-averages with Bessel functions:  $\langle \Phi_{k_x, k_y} \exp(ik_x x + ik_y y) \rangle_{\mathbf{R}_s} = J_{0s} \Phi_{k_x, k_y} \exp(ik_x X_s + ik_y Y_s)$  and  $\langle h_{s, k_x, k_y} \exp(ik_x X_s + ik_y Y_s) \rangle_{\mathbf{r}} = J_{0s} h_{s, k_x, k_y} \exp(ik_x x + ik_y y)$  with  $J_{0s} = J_0(k_{\perp} v_{\perp} / \Omega_s)$  and  $\Omega_s = eB / m_s$ . The perpendicular wavenumber is given by

$$k_{\perp} = \sqrt{k_y^2 + (k_x - k_y z / L_s)^2}. \quad (3)$$

The system is uniquely specified by the following dimensionless parameters:  $L_s / L_n$ ,  $\eta_i$ ,  $\eta_e$ ,  $T_e / T_i$ , and collisionality  $\nu_{ee} L_n / v_i$ , where  $\nu_{ee} = \sqrt{2} \pi n e^4 \ln \Lambda / [(4\pi \epsilon_0)^2 m_e^{1/2} T_e^{3/2}]$  is the electron-electron collision frequency and  $v_i = \sqrt{2T_i / m_i}$ .

Notice that the parallel coordinate  $z$  enters the system (1)-(3) only through (3). Also notice  $k_x$  enters the system (1)-(3) only through (3) and through the nonlinear term in (1). Therefore, a linear mode at nonzero  $k_x$  and nonzero  $k_y$  satisfies exactly the same equations as the corresponding  $k_x = 0$  mode, but translated in  $z$  by  $L_s k_x / k_y$ . As a consequence, the linear growth or damping rates of  $k_y \neq 0$  modes are independent of  $k_x$ , and the eigenmode structures are independent of  $k_x$  up to this translation in  $z$ .

One important property of the system (1)-(2) is the free energy conservation equation [32, 33] obtained from the  $\sum_s \int d^3 R_s \int d^3 v (T_s h_s / f_{Ms})(\dots)$  moment of (1):

$$\frac{dW}{dt} = S(h_i, h_e) \quad (4)$$

where

$$W = \sum_s \int d^3r \int d^3v \frac{T_s \delta f_s^2}{2f_{Ms}} \quad (5)$$

is the free energy,  $\delta f_s = h_s - q_s \Phi f_{Ms}/T_s$  is the total departure of the distribution function from a Maxwellian, and

$$\begin{aligned} S(h_i, h_e) = & \sum_s \int d^3r \int d^3v \frac{T_s h_s}{f_{Ms}} C_s \{h_s\} \\ & + \sum_s \int d^3R_s \int d^3v \frac{h_s}{B} \frac{\partial \langle \Phi \rangle_{\mathbf{R}_s}}{\partial Y_s} \frac{T_s}{n_s} \frac{dn_s}{dx} \left[ 1 + \left( \mathcal{E}_s - \frac{3}{2} \right) \eta_s \right] \end{aligned} \quad (6)$$

is the net energy source. (To derive (4)-(6), the equalities  $\int d^3R_s \int d^3v \langle \dots \rangle_{\mathbf{R}_s} = \int d^3R_s \int d^3v (\dots) \approx \int d^3r \int d^3v (\dots) = \int d^3r \int d^3v \langle \dots \rangle_{\mathbf{r}}$  are used.) The first term on the right-hand side of (6) is negative-definite (by the  $H$ -theorem). The second term has the form of the particle and heat fluxes multiplied by the density and temperature gradients, and this term increases  $W$  if the fluxes have the appropriate sign to relax the gradients. Thus, in a turbulent steady state where  $W$  is statistically constant, equations (4)-(6) indicate that energy is injected by the equilibrium gradient(s) and dissipated by collisions. Note that the operation  $\int d^3R_s$  used in the derivation of (4) annihilates the nonlinear term in (1), and so (4)-(6) apply to both the linear and nonlinear dynamics.

The system (1)-(2) may or may not possess absolute linear instability, depending on the physical parameters. For the rest of this paper, we take  $\eta_i = \eta_e = 0$  so it is certain that the ion-temperature-gradient (ITG) and electron-temperature-gradient (ETG) instabilities are suppressed. Even without temperature gradients, the system may still be unstable to the universal instability driven by the density gradient. For reference, the growth rates and frequencies of this instability in the absence of magnetic shear are given in Appendix A. As this instability is stabilized for  $k_y \rho_i \ll 1$  when even a small amount of magnetic shear is introduced [25, 26], we might expect that this system can exhibit transient linear amplification, with the number of  $e$ -foldings of amplification tending to infinity as the magnetic shear tends to zero. In this way, the apparently singular limit of zero shear may be seen as a continuous limit of the general shear case.

### III. TRANSIENT LINEAR AMPLIFICATION

Turning now to linear initial-value simulations, we indeed find that significant transient linear amplification is possible for certain parameters. Figure 1.a illustrates a linear computation for a single  $(k_x, k_y)$  in a case of very weak magnetic shear,  $L_s/L_n = 300$ . Other parameters used are  $k_y\rho_i = 1$  (where  $\rho_i = v_i/\Omega_i$ ),  $\nu_{ee}L_n/v_i = 0.05$ ,  $m_i = 3672m_e$ ,  $T_e = T_i$ , and  $\eta_i = \eta_e = 0$ . The results are independent of  $k_x$  up to a translation in  $z$ , as noted previously. Collisions are also included to eliminate absolute instability for  $k_y\rho_i \geq 0.7$ , as discussed in [28], using the collision operators detailed in Ref. [34]. As transient amplification generally depends on the choice of norm, here we compare three reasonable quadratic norms:  $W$ ,  $\int dz|\Phi|^2$ , and the maximum of  $|\Phi|^2$  along  $z$ . The curve for each norm is scaled by a constant so the initial amplitude is 1. Amplification by several orders of magnitude is observed in all norms before exponential decay sets in.

Results for the three norms shown are quite similar, though the curve for the  $W$  norm appears slightly higher than the other curves due to less decay in the initial period  $tv_i/L_n < 20$ . Indeed, we find all results described or plotted for the remainder of this paper are quite similar for all of these norms. Henceforth we restrict our attention to the  $W$  norm since it is well motivated by the conservation equation (4).

For the calculation in figure 1, the initial condition used is the standard “noise” initial condition of gs2:  $g_s = h_s - \langle\Phi\rangle \mathbf{R}_s q_s f_{Ms}/T_s$  is set to a Maxwellian velocity distribution times a random complex number at each grid point in  $z$ . (The real and imaginary parts of these complex numbers are each taken from a uniform distribution on  $[-1, 1]$ .) While techniques exist to calculate the optimal perturbation that gives maximum amplification [14, 21, 22], such techniques are numerically challenging for this kinetic model due to the high dimensionality of the problem. The noise initial condition is a convenient shortcut, ensuring energy is likely to be given to a growing perturbation if one exists. It has been shown in [19, 22] that for studies of transient linear amplification, random initial conditions are a reasonable proxy for a directly calculated optimal initial condition.

Figure 1.b shows the same computation, zooming in to the initial transient period. Before the amplification begins around  $tv_i/L_n = 20$ , there is an initial decay period due to strongly damped modes. An amplification factor may be defined relative to either to this minimum amplitude, or relative to the amplitude at  $t = 0$ . In both cases, we define the amplification

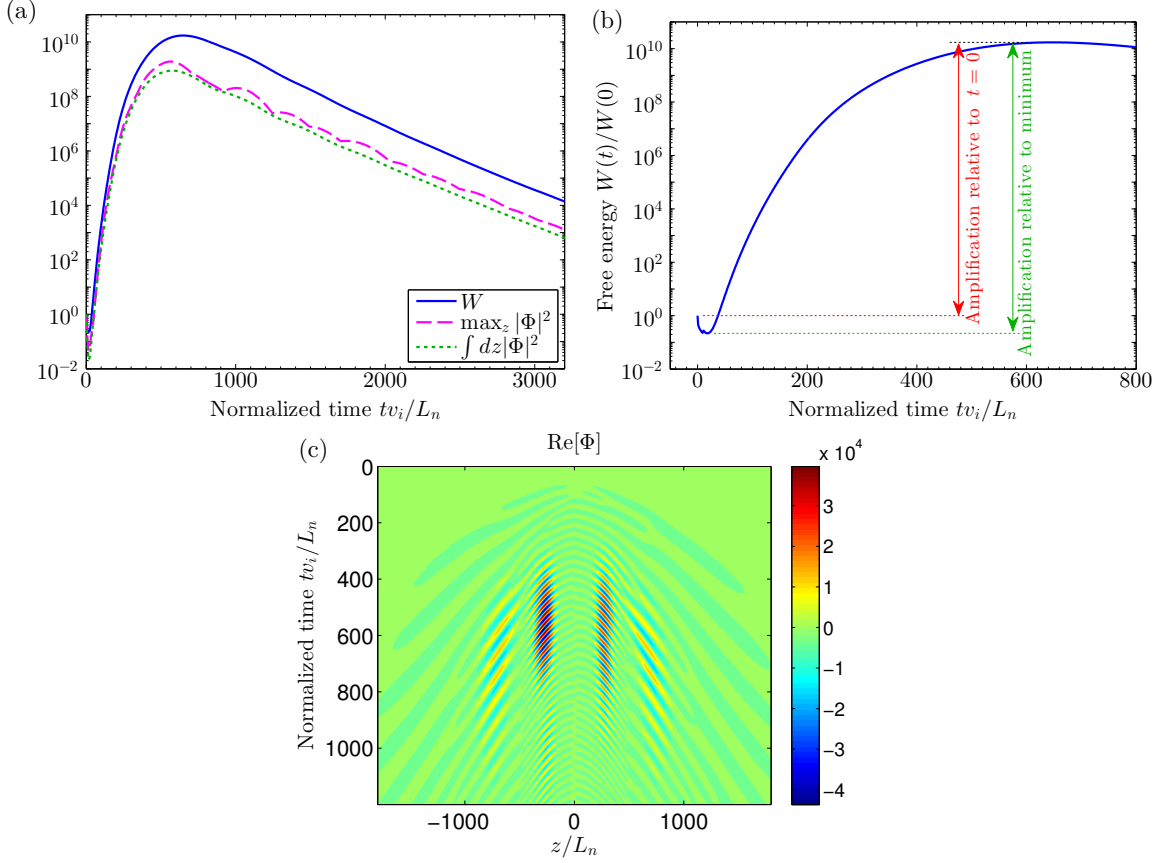


FIG. 1: (Color online) (a) Linear evolution for  $L_s/L_n = 300$ ,  $\eta_i = \eta_e = 0$ ,  $\nu_{ee} = 0.05v_i/L_n$ , and  $k_y\rho_i = 1$ , illustrating transient amplification in three norms. The initial conditions are detailed in the main text. (b) The amount of transient amplification may be defined relative to  $t = 0$  or relative to the minimum that occurs shortly thereafter. (c) The potential structure along a field line for this example.

factor to be 1 if there is no such transient amplification.

Transient amplification is exponentially reduced by increasing magnetic shear, as illustrated in figure 2. For this figure,  $k_y\rho_i$  is fixed at 2, and other parameters are the same as in figure 1. (We choose  $k_y\rho_i = 2$  here since wavenumbers near this value will turn out to be the most amplified, as we will find shortly.) There is some scatter in figure 2, associated with the random initial condition, but the exponential trend with  $L_s/L_n$  is clear. Notice that amplification only appears possible if  $L_s/L_n \gtrsim 20$ . Values of  $L_s/L_n$  far in excess of 20 are obtained in the edge of tokamak experiments. (For a high aspect ratio tokamak,  $L_s \sim qR/\hat{s}$  where  $q$  is the safety factor,  $R$  is the major radius,  $\hat{s} = (r/q)dq/dr$ , and  $r$  is

the minor radius.) Figure 2.b demonstrates that the time during which amplification occurs varies approximately linearly with  $L_s/L_n$ .

Figure 3 shows the dependence of the amplification factor on  $k_y$  at fixed  $L_s/L_n = 200$ . Significant amplification can be observed for a range of  $k_y$  surrounding  $1/\rho_i$ . Results are shown for two values of collisionality,  $\nu_{ee}L_n/v_i = 0.1$  and  $0.5$ . (When the collisionality is much smaller than these values, absolute linear instability is present [28].) It can be seen that increasing collisionality tends to limit amplification at larger  $k_y$  while having much less effect at smaller  $k_y$ . This kinetic problem is similar to conventional hydrodynamics in that for both cases, non-modal amplification decreases with increasing collisional dissipation (associated with viscosity in the Navier-Stokes equation for the hydrodynamic case.) However, the kinetic problem here is rather different from the hydrodynamic case in the sensitivity of the transient amplification to the geometric parameter  $L_s/L_n$ , which is not associated with entropy-producing collisions.

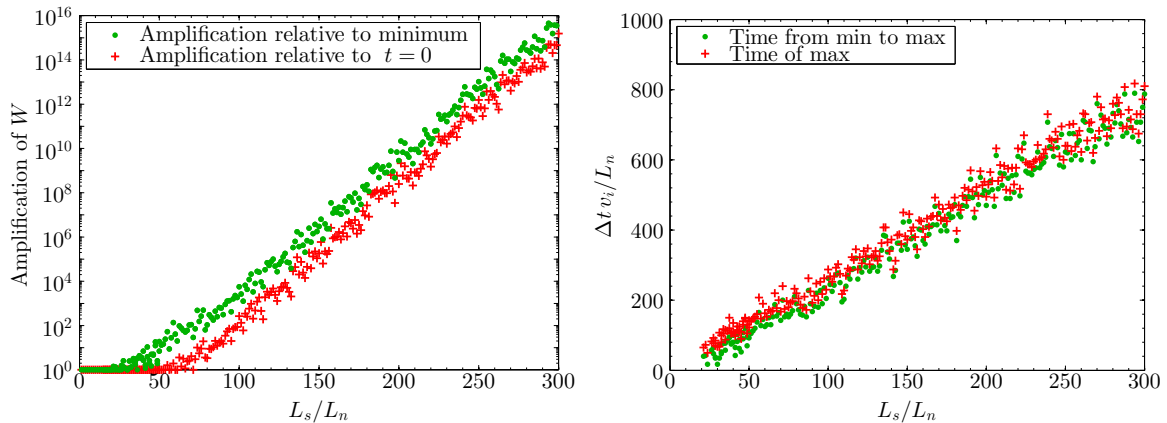


FIG. 2: (Color online) The (a) number of  $e$ -foldings of transient amplification and (b) duration of the amplification both scale with the magnetic shear length scale  $L_s$ , consistent with the appearance of absolute instability in the limit of vanishing shear ( $L_s \rightarrow \infty$ ). Parameters used are  $k_y \rho_i = 2$  and  $\nu_{ee} = 0.05 v_i / L_n$ .

#### IV. TURBULENCE

We now demonstrate that when the nonlinear term in (1) is retained, sustained turbulence is possible in this system. For the demonstration here we consider the parameters  $L_s/L_n =$



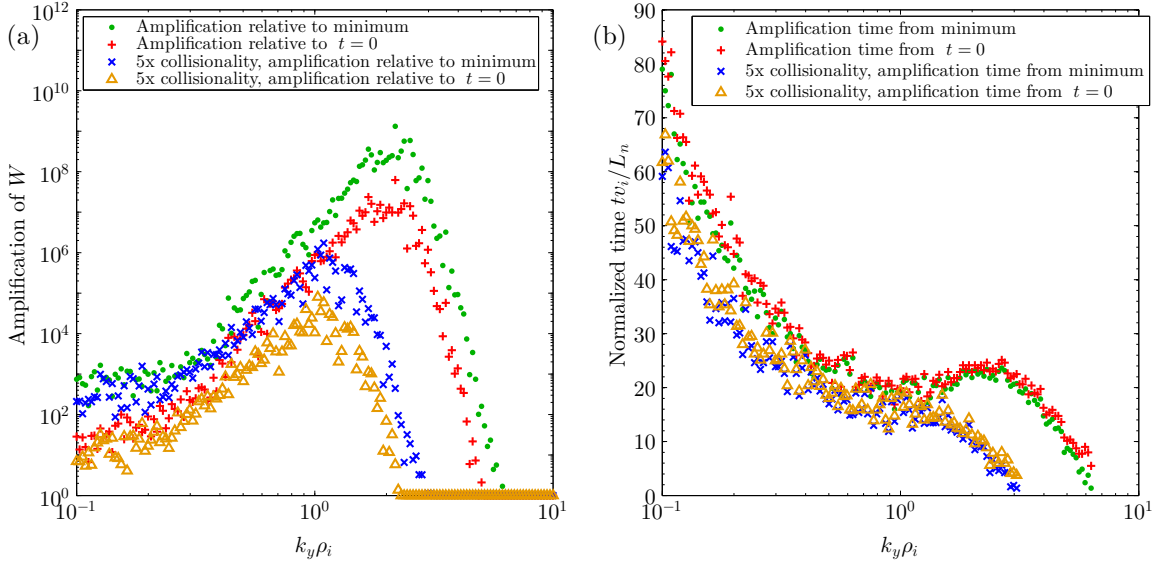


FIG. 3: (Color online) Transient amplification can be observed for a range of  $k_y$  surrounding  $1/\rho_i$ . For this figure,  $L_s/L_n = 200$ . The two collisionalities shown are  $\nu_{ee}L_n/v_i = 0.1$  and  $0.5$ .

20,  $T_e = T_i$ ,  $m_i = 3672m_e$ ,  $\eta_i = \eta_e = 0$ , and  $\nu_{ee}L_n/v_i = 0.05$ . We find this point in parameter-space is approximately on the edge of where sustained turbulence is possible: if the magnetic shear or collisionality is increased much beyond these values, turbulence cannot be sustained indefinitely (though it may persist for a finite time.) Note that  $L_s/L_n = 20$  is also approximately the maximum shear in figure 2 for which transient linear amplification is observed for random initial conditions. (While absolute linear instability was found in [28] for  $L_s/L_n \geq 17$  in the limit of vanishing collisionality, here the finite collisionality is enough to suppress this linear instability.) Before presenting nonlinear simulations, we first illustrate the linear behavior of the system using all the physical and numerical parameters that will be used for the nonlinear runs. Figure 4 shows the time evolution of  $W$  for all the dealiased  $k_y$  modes that will be used for nonlinear simulations, including twist-and-shift boundary conditions, but with the nonlinearity turned off, and using the random initial conditions described in section III. More precisely, the quantities plotted are  $W_{k_x, k_y}$ , summed over the  $k_x$  that link to  $k_x = 0$ , where

$$W_{k_x, k_y} = L_x L_y \sum_s \int dz \int d^3v \frac{T_s}{2f_{Ms}} |\delta f_{s, k_x, k_y}|^2 \quad (7)$$

are the 2D Fourier contributions to  $W = \sum_{k_x, k_y} W_{k_x, k_y}$ . Here,  $L_x$  and  $L_y$  are the sizes of the simulation domain in  $x$  and  $y$ , and  $\delta f_s = \sum_{k_x, k_y} \delta f_{s, k_x, k_y} \exp(ik_x x + ik_y y)$ . Curves for

the other  $k_x$  in the simulation look similar to those in figure 4, as expected from the  $z$ -translation symmetry of the system (1)-(2); see discussion following (3). The amplitude of each of these curves decreases for large  $t$ , confirming there is no instability for these physical and numerical parameters. The decrease is mostly monotonic, but there is typically a small amount of transient growth by a factor  $\lesssim 2\times$  for  $k_y\rho_i$  between 1-2. Thus, this point in parameter space is on the edge of the region in which transient linear amplification occurs, as anticipated from figure 2.

Throughout this section, the numerical parameters used were as follows:  $12\times 2$  grid points in pitch angle and  $\text{sgn}(v_{\parallel})$ , 12 grid points in energy, 24 grid points per cell in  $z$  (with cells at  $k_y = j\Delta k_y$  linked if separated in  $k_x$  by  $4j\Delta k_x$ ), CFL number 0.5,  $256\times 64$  modes in  $x$  and  $y$  before dealiasing, and box size  $(L_x, L_y, L_z) = (31.4\rho_i, 62.8\rho_i, 160L_n)$ . It was verified that both the linear and nonlinear results reported in this section were not significantly altered under factor-of-2 changes in any of these numerical parameters.

One noteworthy linear property apparent in figure 4 is that damping of modes with  $k_y = 0$  is very weak, nearly invisible on the scale of the figure. Linear modes with  $k_y = 0$  and  $\partial/\partial z = 0$  are damped only by collisions. (In the absence of any collisions, any  $h_e$  and  $h_i$  that depend only on  $\mathcal{E}_s$  and  $\mu$  are steady-state solutions of the linearized system.) Collisions introduce damping of these  $k_y = 0$  modes, with a damping rate that increases with  $k_x^2$  due to the classical transport terms in the gyroaveraged collision operator.

Finally, figure 5 shows two nonlinear simulations for the same parameters as figure 4, differing only in the initial amplitude of the  $k_y = 0.1$  part of the perturbation. For sufficiently large initial amplitude, turbulence is sustained, demonstrating nonlinear instability.

A variety of strategies can be used to initiate the nonlinear instability in numerical simulations. Large-amplitude noise can be used, but this method is numerically inconvenient (at least in gs2) due to the rapid drop in amplitude of strongly damped modes over the first few timesteps, which causes significant changes in the CFL-stable timestep associated with the perpendicular  $\mathbf{E}\times\mathbf{B}$  speed, thereby requiring recomputation of the implicit time-advance operator. Another method for initiating the nonlinear instability is to begin with a sufficient temperature gradient to bring about linear instability, and then to reduce the temperature gradient once the turbulence reaches a sufficient amplitude [8]. A third effective method, the one used in figure 5, is to initialize with one of the least-damped linear eigenmodes scaled up to appropriate amplitude, with noise in the other modes. We find this method to be

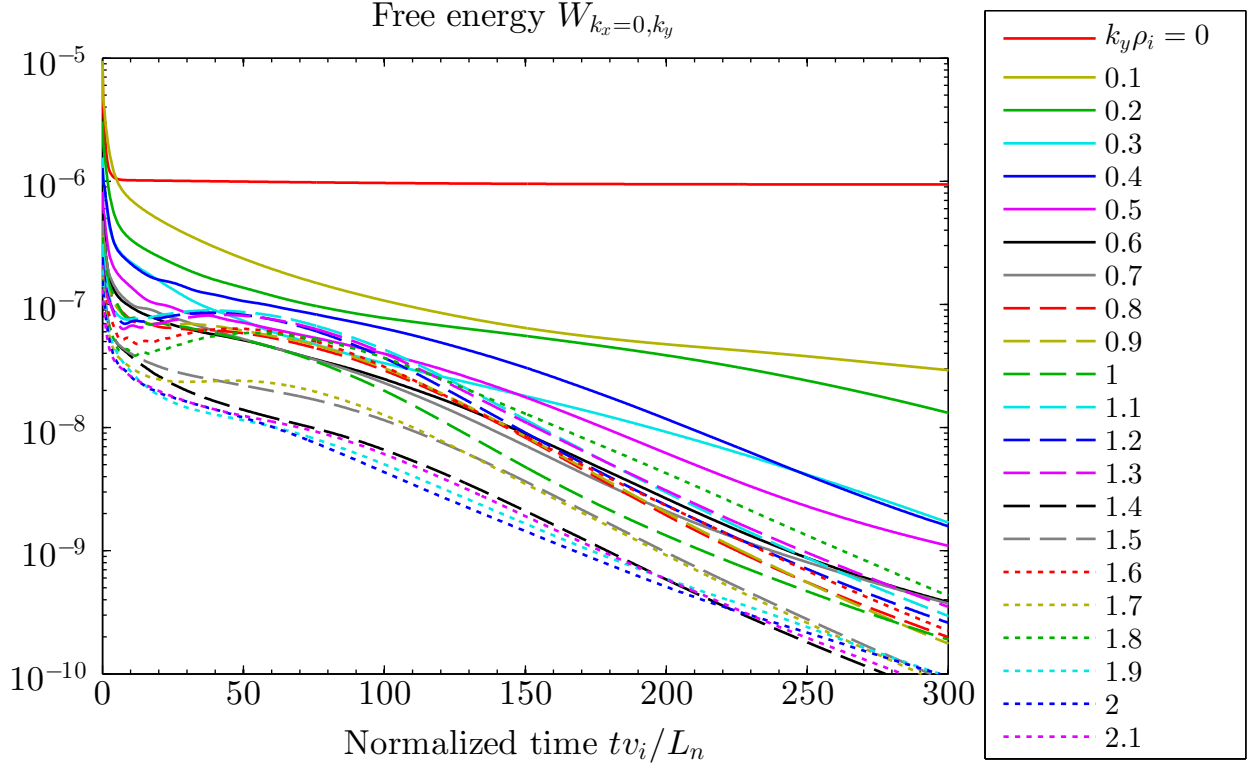


FIG. 4: (Color online) Linear dynamics for the parameters used in the nonlinear simulations of figure 5. The large- $t$  behavior demonstrates there is no absolute linear instability. A small amount of transient amplification can be seen.

particularly convenient since variation in the stable timestep is minimized.

As another demonstration that the system displays nonlinear instability without linear instability, figure 6 shows the result of turning off the nonlinear term after a period of saturated turbulence. The nonlinear term is turned off at time  $tv_i/L_n = 2064$ , shown as a vertical dotted line. All modes eventually decay. However some modes display a modest amount ( $\lesssim 3\times$ ) of transient amplification before decaying.

The particle and heat fluxes for a long nonlinear simulation at these parameters are plotted in figure 7. As the time average of  $S$  in (4)-(6) must vanish, and since the collision term is negative-definite, there must be an average particle flux in the appropriate direction to relax the equilibrium density gradient, and indeed we find such a flux. Although there is no equilibrium temperature gradient, nonzero average fluxes of both ion and electron heat are found. Both heat fluxes have the same sign as the particle flux, meaning heat flows in the  $-\nabla n$  direction.

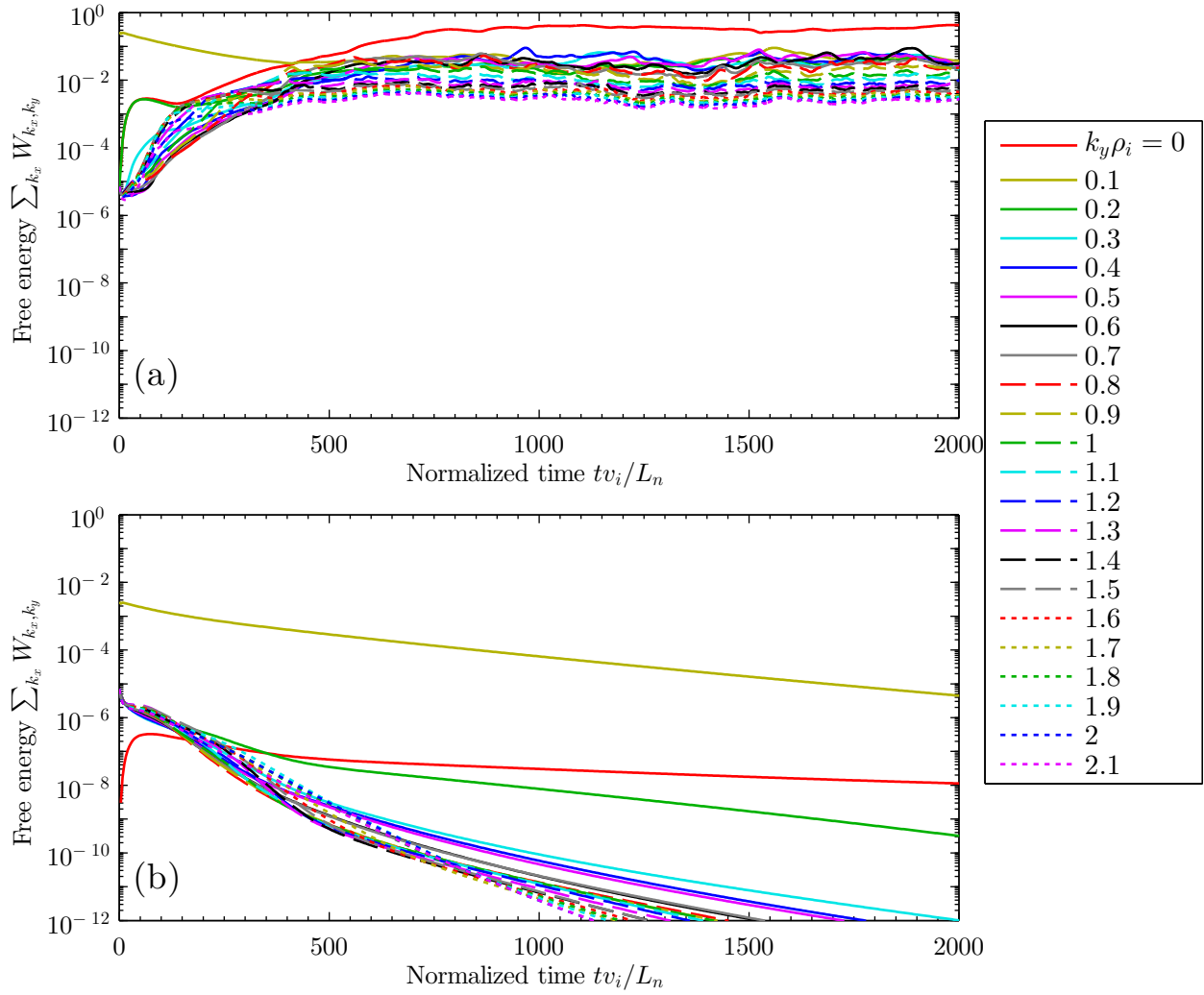


FIG. 5: (Color online) Simulations for exactly the same physical and numerical parameters as the decaying linear simulation in figure (4), but now including nonlinearity and varying the initial amplitude of the  $k_y = 0.1$  component. (a) When the initial amplitude is sufficient, turbulence grows and is sustained, whereas (b) for small initial amplitude, all modes decay as they do in a linear simulation.

The fluxes in figure 7 are normalized by gyro-Bohm values defined using the length  $L_n$  and including the 2 in the thermal speed, e.g. the particle flux is normalized by  $(\rho_i/L_n)^2 n v_i$ . Converting to dimensional units for a deuterium plasma, we obtain the following estimate for the turbulent particle diffusivity:  $D \approx (0.4 \text{ m}^2/\text{s}) (T/500 \text{ eV})^{3/2} (B/2 \text{ T})^{-2} (L_n/1 \text{ cm})^{-1}$ . This magnitude of diffusion coefficient may be large enough to be of interest experimentally. A larger diffusion coefficient is obtained if  $L_s/L_n$  is increased.

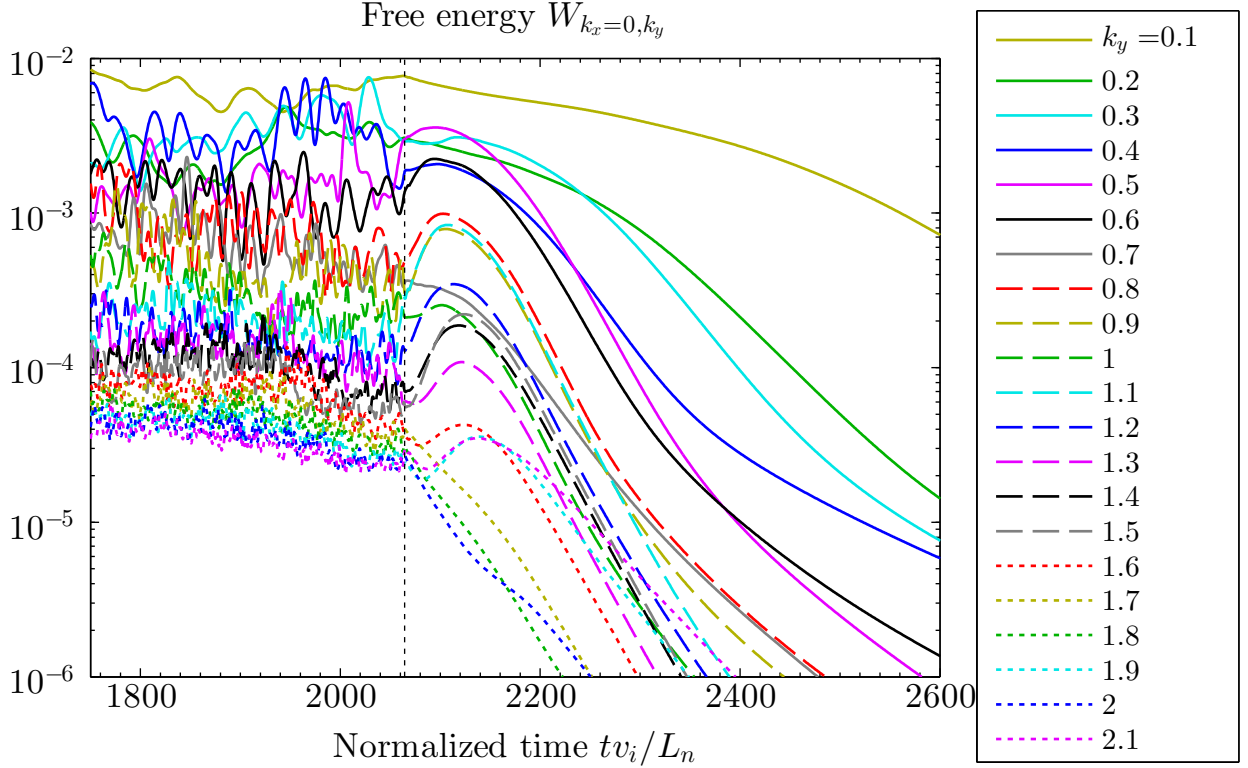


FIG. 6: (Color online) Demonstration of the concepts in section V. Curves show free energy in the 2D modes with  $k_x = 0$  (including intervals linked by the twist-and-shift parallel boundary condition) for various  $k_y$ . After a period of saturated turbulence, the nonlinear term is turned off at normalized time 2064, denoted by the vertical dashed line. All modes eventually decay, but transient amplification is apparent first for some of the 2D modes. Such amplification must occur in at least one 2D mode in such a linear “restart” from steady turbulence, as proved in section V.

## V. TURBULENCE IMPLIES TRANSIENT AMPLIFICATION OF TYPICAL STATES

It seems plausible that the previous two sections are related, that a connection exists between transient linear amplification and sustained nonlinear turbulence. Let us now prove this relationship rigorously. Here we present two such arguments, both of which yield the conclusion that non-modal amplification is a necessary condition for sustained turbulence. One argument for the case of hydrodynamics was given in [30] and section 4 of [15], and below we adapt this reasoning to gyrokinetics. Before we do so, we give an alternative argument that relies less on linear algebra. For this discussion, it is useful to define a “2D

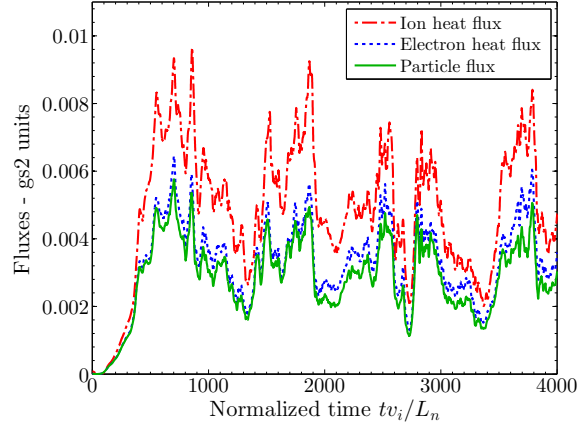


FIG. 7: (Color online) Fluxes for the simulation of Figure (5).a with  $\nu_{ee}L_n/v_i = 0.05$  and  $L_s/L_n = 20$ , approximately the maximum shear at which turbulence can be sustained at this collisionality. Even in the absence of equilibrium temperature gradients, nonzero average ion and electron heat fluxes are obtained.

mode” to be any structure with given  $k_x$  and  $k_y$ , but with no restriction on the  $z$  dependence.

*Statement 1:* Consider a turbulent solution of the nonlinear gyrokinetic system (1)-(2), i.e. a solution for which  $W$  does not decay to zero at large  $t$ . Choose any time  $t_0$  at which the free energy satisfies  $dW/dt > 0$ . Suppose at  $t_0$  we turn off the nonlinear term and re-evolve the system in time, as in figure 6. Then at least one 2D mode must grow in the  $W$  norm for some time following  $t_0$ , i.e. there must be transient linear amplification.

*Proof:* Recall that (4) is a moment of both the nonlinear and linear dynamical equations. When the nonlinear term is turned off at  $t_0$ ,  $S(h_i, h_e)$  does not immediately change, because the system state does not immediately change. Therefore,  $dW/dt$  must vary *continuously* when the nonlinear term is turned off, and so  $dW/dt$  is positive for some period following  $t_0$  under the action of the linear dynamics. Recall  $W$  is a sum of positive contributions (7) from each 2D mode:  $W = \sum_{k_x, k_y} W_{k_x, k_y}$ . Therefore  $dW_{k_x, k_y}/dt$  must be positive in the linear dynamics for at least one 2D mode, so this mode must be growing transiently in the  $W$  norm.

Roughly speaking, the upshot of this argument is that we should not be surprised that there is transient amplification after the nonlinear term is turned off in figure (6). Such amplification *must* occur. Put another way, a necessary condition for turbulence (in the absence of linear instability) is that typical states of the turbulence can be transiently

amplified. Furthermore, the energy growth in the 2D modes that are amplified must be sufficient to balance the energy decay from the damped 2D modes.

Next, we present a second argument for the necessity of transient amplification. This argument is based on linear algebra and closely follows [30]. For this second approach, we let  $\mathbf{h}$  denote a column vector that specifies the state of the system  $\{h_i, h_e\}$  in some convenient basis. For example,  $\mathbf{h}$  could represent the array of values specifying the distribution function in a code.

*Statement 2:* Given a set of physical parameters for which the nonlinear gyrokinetic system (1)-(2) permits statistically steady turbulence, there exists at least one state  $\mathbf{h}_0$  such that (1)  $dW/dt > 0$  if  $\mathbf{h} = \mathbf{h}_0$ , and (2) the squared projection of the turbulent state  $\mathbf{h}(t)$  onto  $\mathbf{h}_0$ , defined below, has a non-zero average.

*Proof:* Let  $\mathbf{L}$  denote the time-independent linear operator associated with (1)-(2) in the absence of the nonlinear term, so the linearized (1)-(2) can be written

$$d\mathbf{h}/dt = \mathbf{L}\mathbf{h}. \quad (8)$$

(Here we will take  $\mathbf{L}$  to be real since the equations (1)-(2) are real, though the following argument may be generalized to complex  $\mathbf{L}$  by replacing the transpose operation with Hermitian conjugation.) Also let  $\mathbf{E}$  be the time-independent operator that defines the free energy norm:

$$W = \mathbf{h}^T \mathbf{E} \mathbf{h} \quad (9)$$

where  $^T$  denotes the transpose. Note that  $\mathbf{E}$  is real and symmetric ( $\mathbf{E} = \mathbf{E}^T$ ). Differentiating (9),

$$dW/dt = \mathbf{h}^T [(\mathbf{E}\mathbf{L})^T + \mathbf{E}\mathbf{L}] \mathbf{h} \quad (10)$$

which is equivalent to (4). As discussed previously, the presence of the nonlinear term does not alter (4) (since the nonlinear term is annihilated in the integration over the perpendicular directions  $X_s$  and  $Y_s$ ), so (10) holds for both the linear and nonlinear gyrokinetic equations. Introducing a time average  $\langle \dots \rangle$  over statistically steady turbulence, then

$$\langle \mathbf{h}^T [(\mathbf{E}\mathbf{L})^T + \mathbf{E}\mathbf{L}] \mathbf{h} \rangle = 0. \quad (11)$$

Next, observe that the operator  $(\mathbf{E}\mathbf{L})^T + \mathbf{E}\mathbf{L}$  in (10)-(11) is real and symmetric, so it possesses a complete set of orthogonal eigenvectors with real eigenvalues. Let  $\mathbf{u}_j$  and

$\lambda_j$  denote these eigenvectors (normalized so  $\mathbf{u}_j^T \mathbf{u}_k = \delta_{j,k}$ ) and their associated eigenvalues. In the hydrodynamics literature, the eigenvectors  $\mathbf{u}_k$  are sometimes called instantaneous optimals; the  $\mathbf{u}_j$  with maximum  $\lambda_j$  is the fluctuation that maximizes the instantaneous  $dW/dt$ . We may decompose the state vector in the  $\mathbf{u}_j$  basis:  $\mathbf{h}(t) = \sum_j a_j(t) \mathbf{u}_j$  for some amplitudes  $a_j(t)$ . Using this decomposition in (11) gives

$$\sum_j \lambda_j \langle |a_j(t)|^2 \rangle = 0. \quad (12)$$

Neglecting the uninteresting case in which all terms in the sum (12) are zero, then there must be at least one positive eigenvalue  $\lambda_{j_0}$  with an associated nonzero average weight  $\langle |a_{j_0}(t)|^2 \rangle$ . Considering the associated  $\mathbf{u}_{j_0}$  in (10), then this state  $\mathbf{u}_{j_0}$  must give positive  $dW/dt$ , proving statement 2.

Thus, we have showed in two ways that transient linear amplification is a necessary condition for turbulence in the associated nonlinear system. We have also shown that the amplitude of the transiently growing state(s) must have a significant amplitude in the turbulent state, in the following senses. In the first line of argument, not only must there be at least a single growing 2D mode, but there must be enough growing 2D modes of sufficient amplitude to balance the negative  $dW_{k_x, k_y}/dt$  of all the decaying 2D modes. In the second argument, in (12) there must not only be at least one state  $j$  with an instantaneous growth rate  $\lambda_j > 0$ , but the amplitudes of states with positive and negative  $\lambda_j$  must be in balance so  $\langle dW/dt \rangle = 0$ . In a statistically steady state, there must be a balance between energy input from the instantaneously growing linear states and the instantaneously decaying linear states.

Both of the arguments above relied on the fact that the norm  $W$  satisfies the same equation (4) in both the linear and nonlinear dynamics. Thus, the “nonlinear invariant”  $W$  is a special norm, and the arguments do not generally apply to other norm-like quantities such as  $\int d^3r |\Phi|^2$  in which one might choose to measure transient amplification. (In fact for any operator with damped eigenmodes, even if transient amplification exists in one norm, another norm always exists in which transient amplification is impossible. The sum of squared eigenmode coefficients is such a norm [35].)

While the arguments here relied on the gyrokinetic conservation law (4), the same reasoning may be applied to other models and systems in which an energy conservation equation holds. For example, eq (6) in Ref. [8] gives the appropriate conservation law for the fluid



model in that work, and the first line of (6) gives the associated norm to use in place of  $W$ .

## VI. DISCUSSION AND CONCLUSIONS

The “universal mode” system (slab geometry with a density gradient, electrostatic, no sheared equilibrium flow, and weak temperature gradients) is known to have a dramatic change in linear stability as the magnetic shear is raised from zero to small finite values [25–28]. As the eigenvalues are so sensitive to magnetic shear, we might expect traditional modal analysis to give a misleading picture of typical behavior, as with other systems that have sensitive eigenvalues [1]. Indeed, here we have demonstrated that significant transient linear amplification is possible in the drift wave system even when all eigenmodes are decaying. We have demonstrated that the amount and duration of amplification scale with the magnetic shear length  $L_s$ , so these quantities naturally become unbounded in the limit of vanishing shear. The non-modal amplification also decreases with increasing collisionality, particularly at the shortest perpendicular wavelengths, and the amplification tends to be largest at wavelengths near  $k_y \rho_i \sim 1 - 2$ . In contrast to more famous examples of non-modal amplification from hydrodynamics [3], and to some recent examples from plasma physics [9, 10, 17, 18], the amplification in the plasma drift wave system here does not depend on sheared equilibrium flow.

We have also demonstrated using nonlinear gyrokinetic simulations that sustained turbulence is possible in this system even when all linear modes are damped. This phenomenon had been seen previously in fluid models [6–8]. This phenomenon, clearly visible in figure 5, is opposite to the “Dimits shift” [36] seen at different physical parameters, in which nonlinear fluctuations are suppressed despite the presence of linear instability.

In this system without unstable eigenvalues, we find that sustained nonlinear turbulence and transient linear amplification both become possible around  $L_s/L_n \gtrsim 20$  (for the collisionality  $\nu_{ee}L_n/v_i = 0.05$  and other parameters considered.) As discussed in section V, the fact that these two phenomena appear together is not a coincidence. Transient linear amplification in the free energy norm  $W$  is a necessary condition for statistically steady turbulence, and the turbulent state must have a significant projection onto states that are linearly growing in the  $W$  norm. When the nonlinear term is turned off in a simulation of turbulence, as in figure 6, amplification in  $W$  norm must occur in at least one of the 2D

modes. The amplitude of the instantaneously growing states (instantaneous optimals) must be significant, enough for the associated input of free energy to balance decay of energy from the states that are decaying.

Note that the transient linear process emphasized here and nonlinear or secondary instability are not mutually exclusive. Both are required to support turbulence. Some form of secondary instability (as discussed in [8]) must be present to transfer energy out of the transiently growing linear structures before these structures begin to decay substantially. However, nonlinear processes cannot directly contribute to the system's free energy  $W$ , due to the annihilation of the nonlinear term in deriving (4).

Several effects that may be important for realistic plasma turbulence were not included in the present work, including toroidicity and trapped particles, and electromagnetic fluctuations. Therefore, further work is required to determine how relevant the nonlinear turbulence considered here is for realistic experimental parameters. However, we at least conclude that linear stability need not preclude the possibility of significant turbulent particle and energy fluxes, and that nonlinear simulations should sometimes be initialized at finite amplitude to examine this possibility.

### **Acknowledgments**

This material is based upon work supported by the U.S. Department of Energy, Office of Science, Office of Fusion Energy Science, under Award Numbers DEFG0293ER54197 and DEFC0208ER54964. Computations were performed on the Edison system at the National Energy Research Scientific Computing Center, a DOE Office of Science User Facility supported by the Office of Science of the U.S. Department of Energy under Contract No. DE-AC02-05CH11231. Conversations with Tom Antonsen, Michael Barnes, Greg Colyer, James Drake, Edmund Highcock, Greg Hammett, Adil Hassam, and Freja Nordsiek are gratefully acknowledged.

## APPENDIX A: LINEAR STABILITY IN THE ABSENCE OF MAGNETIC SHEAR

For reference, here we present the growth/damping rates and real frequencies of the universal instability in the limit of vanishing magnetic shear and vanishing collisions. In the absence of shear, we can replace  $\partial/\partial z$  in (1) with  $ik_{\parallel}$  for some constant  $k_{\parallel}$ . Using (2), and after some algebra, the local dispersion relation may then be written

$$\begin{aligned} \frac{T_e}{T_i} - \zeta_i \frac{\omega_*}{\omega} \left\{ - \left[ \frac{\omega T_e}{\omega_* T_i} + 1 \right] Z_i \Gamma_{0i} + \eta_i Z_i \left[ \left( \frac{1}{2} + b_i \right) \Gamma_{0i} - b_i \Gamma_{1i} \right] - \eta_i \zeta_i [1 + \zeta_i Z_i] \Gamma_{0i} \right\} \\ + 1 + \zeta_e \frac{\omega_*}{\omega} \left\{ \left[ \frac{\omega}{\omega_*} - 1 \right] Z_e \Gamma_{0e} + \eta_e Z_e \left[ \left( \frac{1}{2} + b_e \right) \Gamma_{0e} - b_e \Gamma_{1e} \right] - \eta_e \zeta_e [1 + \zeta_e Z_e] \Gamma_{0e} \right\} = 0. \end{aligned} \quad (\text{A1})$$

Here,  $\zeta_s = \omega/(|k_{\parallel}| v_s)$ ,  $v_s = \sqrt{2T_s/m_s}$  is the thermal speed,  $Z_s = Z(\zeta_s)$ ,  $Z$  is the plasma dispersion function,  $\Gamma_{js} = I_j(b_s)e^{-b_s}$ ,  $I_j$  is a modified Bessel function,  $b_s = k_{\perp}^2 v_s^2/(2\Omega_s^2)$ , and  $\omega_* = k_y T_e/(eBL_n)$ . Figure 8 shows the solution of (A1) with largest imaginary part for the case  $T_e = T_i$  and the deuterium-electron mass ratio. We take  $\eta_i = \eta_e = 0$  so it is certain that the ion-temperature-gradient (ITG) and electron-temperature-gradient (ETG) modes are suppressed. Figure 8 was generated by applying nonlinear root-finding to (A1), verifying the solution agreed with initial-value gs2 simulations for several values of  $k_{\parallel}$  and  $k_{\perp}$ . It can be seen that absolute linear instability exists for sufficiently small values of  $|k_{\parallel}|$  when  $k_{\perp} \rho_i < 40$ .

- 
- [1] L. Trefethen and M. Embree, *Spectra and pseudospectra: The behavior of nonnormal matrices and operators* (Princeton University Press, Princeton, 2005).
  - [2] N. Tillmark and P. H. Alfredsson, J. Fluid Mech. **235**, 89 (1992).
  - [3] L. N. Trefethen, A. E. Trefethen, S. C. Reddy, and T. A. Driscoll, Science **261**, 578 (1993).
  - [4] S. Grossman, Rev. Mod. Phys. **72**, 603 (2000).
  - [5] R. E. Waltz, Phys. Rev. Lett. **55**, 1098 (1985).
  - [6] B. D. Scott, Phys. Rev. Lett. **65**, 3289 (1990).
  - [7] B. D. Scott, Phys. Fluids B **4**, 2468 (1992).
  - [8] J. F. Drake, A. Zeiler, and D. Biskamp, Phys. Rev. Lett. **75**, 4222 (1995).

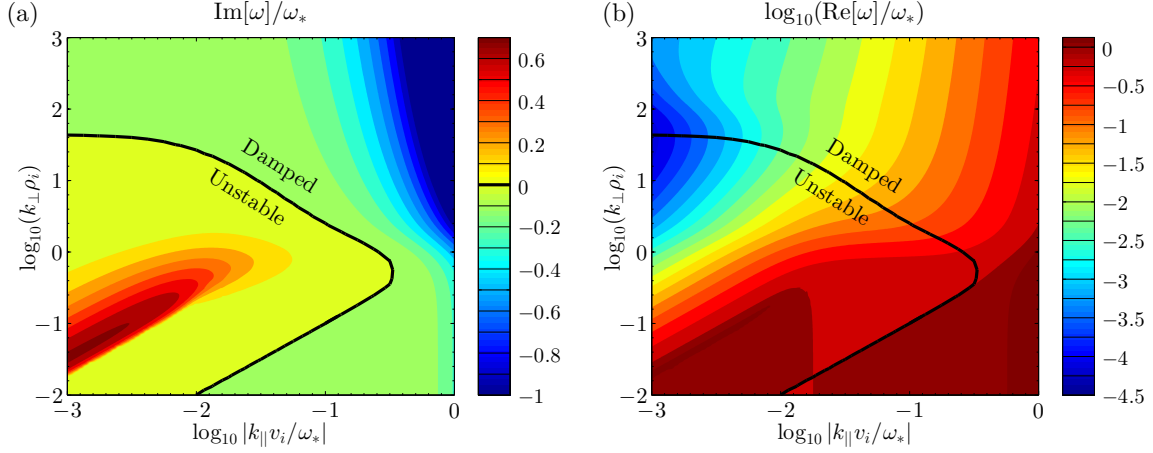


FIG. 8: (Color online) Solution of the linear dispersion relation in the zero-shear (local) limit, i.e. the root of (A1) with largest imaginary part, for  $\eta_i = \eta_e = 0$ ,  $T_e = T_i$ , and  $m_i = 3672m_e$ . The (a) imaginary and (b) real parts of the frequency are shown.

- [9] E. G. Highcock, M. Barnes, A. A. Schekochihin, F. I. Parra, C. M. Roach, and S. C. Cowley, Phys. Rev. Lett. **105**, 215003 (2010).
- [10] M. Barnes, F. I. Parra, E. G. Highcock, A. A. Schekochihin, S. C. Cowley, and C. M. Roach, Phys. Rev. Lett. **106**, 175004 (2011).
- [11] D. Biskamp and A. Zeiler, Phys. Rev. Lett. **74**, 706 (1995).
- [12] B. Friedman, T. A. Carter, M. V. Umansky, D. Schaffner, and B. Dudson, Phys. Plasmas **19**, 102307 (2014).
- [13] B. Friedman and T. A. Carter, Phys. Rev. Lett. **113**, 025003 (2014).
- [14] B. F. Farrell and P. J. Ioannou, J. Atmospheric Sci **53**, 2025 (1996).
- [15] T. DelSole, Surv. Geophys. **25**, 107 (2004).
- [16] P. J. Schmid, Ann. Rev. Fluid Mech. **39**, 129 (2007).
- [17] S. L. Newton, S. C. Cowley, and N. F. Loureiro, Plasma Phys. Controlled Fusion **52**, 125001 (2010).
- [18] A. A. Schekochihin, E. G. Highcock, and S. C. Cowley, Plasma Phys. Controlled Fusion **54**, 055011 (2012).
- [19] S. J. Camargo, M. K. Tippet, and I. L. Caldas, Phys. Rev. E **58**, 3693 (1998).
- [20] E. Camporeale, T. Passot, and D. Burgess, Astrophys. J. **715**, 260 (2010).
- [21] J. Squire and A. Bhattacharjee, Phys. Rev. Lett. **113**, 025006 (2014).

- [22] J. Squire and A. Bhattacharjee, *Astrophys. J.* **797**, 67 (2014).
- [23] A. A. Galeev, V. N. Oraevsky, and R. Z. Sagdeev, *Soviet Phys. JETP* **17**, 615 (1963).
- [24] N. A. Krall and M. N. Rosenbluth, *Phys. Fluids* **8**, 1488 (1965).
- [25] D. W. Ross and S. M. Mahajan, *Phys. Rev. Lett.* **40**, 324 (1978).
- [26] K. T. Tsang, P. J. Catto, J. C. Whitson, and J. Smith, *Phys. Rev. Lett.* **40**, 327 (1978).
- [27] T. M. Antonsen, *Phys. Rev. Lett.* **41**, 33 (1978).
- [28] M. Landreman, T. M. Antonsen Jr, and W. Dorland, *arXiv:1411.1807* (2014).
- [29] M. Kotschenreuther, G. Rewoldt, and W. M. Tang, *Comp. Phys. Comm.* **88**, 128 (1995).
- [30] T. DelSole, *J. Atmospheric Sci.* **61**, 1086 (2004).
- [31] E. A. Frieman and L. Chen, *Phys. Fluids* **25**, 502 (1982).
- [32] G. G. Howes, S. C. Cowley, W. Dorland, G. W. Hammett, E. Quataert, and A. A. Schekochihin, *Astrophys. J.* **651**, 590 (2006).
- [33] I. G. Abel, G. G. Plunk, E. Wang, M. Barnes, S. C. Cowley, W. Dorland, and A. A. Schekochihin, *Rep. Prog. Phys.* **76**, 116201 (2013).
- [34] I. G. Abel, M. Barnes, S. C. Cowley, W. Dorland, and A. A. Schekochihin, *Phys. Plasmas* **15**, 122509 (2008).
- [35] B. F. Farrell and P. J. Ioannou, *Phys. Fluids A* **5**, 2600 (1993).
- [36] A. M. Dimits, G. Bateman, M. A. Beer, B. I. Cohen, W. Dorland, G. W. Hammett, C. Kim, J. E. Kinsey, M. Kotschenreuther, A. H. Kritz, et al., *Phys. Plasmas* **7**, 969 (2000).