PHYS 342 - EM 02

Lecture 6

In Lecture 5, we have seen that electric and magnetic forces are different representations of the same phenomenon: The electromagnetic interaction.

In the spirit of that finding, we will next consider a few electric and magnetic problems back to back and in much more depth than in PHYS 242.

6.1 Weak Formulation of the Electrostatic Problem

In PHYS 242, we demonstrated that the total electrostatic energy associated with a charge distribution with volume density ρ in a region Ω of space is

$$U_e = \frac{1}{2} \iiint_{\Omega} \rho\phi dV \tag{6.1}$$

where ϕ is the electrostatic potential. Assuming ρ to be limited in space, we also demonstrated that

$$U_e = \frac{1}{2} \iiint_{\Omega_\infty} \epsilon_0 E^2 dV \tag{6.2}$$

where $\vec{E}$ is the field generated by ρ. In Eq. (6.2), Ω_∞ represents the entire Euclidean space. The region Ω where ρ is defined is thus a subspace of Ω_∞, $\Omega \subset \Omega_\infty$. Hence, (6.1) can also be written as

$$U_e = \frac{1}{2} \iiint_{\Omega_\infty} \rho\phi dV \tag{6.1'}$$

A different way to write the total electrostatic energy is then

$$U_e = \iiint_{\Omega_\infty} \rho\phi dV - \frac{1}{2}\epsilon_0 \iiint_{\Omega_\infty} E^2 dV \tag{6.3}$$

From the definition of potential, $\vec{E} = -\vec{\nabla}\phi$. Thus,

$$U_e = \iiint_{\Omega_\infty} \rho\phi dV - \frac{1}{2}\epsilon_0 \iiint_{\Omega_\infty} ||\vec{\nabla}\phi||^2 dV \tag{6.4}$$

When ϕ is a unknown arbitrary function, expression (6.4) becomes a functional of ϕ, $U_e(\phi)$. As we will show in Appendix 6.1, among all possible solutions, the special function $\bar{\phi}$ that minimizes $[-U_e(\phi)]$ (note the negative sign) is the actual electrostatic potential associated with ρ. We can conjecture that the electrostatic problem can be stated as finding the function $\bar{\phi}$ that minimizes

$$\frac{1}{2}\epsilon_0 \iiint\limits_{\Omega_\infty} ||\vec{\nabla}\phi||^2 dV - \iiint\limits_{\Omega_\infty} \rho\phi dV \tag{6.4'}$$

This is called the weak formulation of the electrostatic problem. Incidentally, the weak formulation is a special case of the least action principle!

More in general, given a so-called strong Poisson problem

$$\begin{cases} \vec{\nabla}^2 u = -f & \text{in } \Omega \\ \\ u = 0 & \text{on } \Sigma \end{cases} \tag{6.5}$$

where Σ is the surface enclosing Ω, the corresponding weak problem is written as

$$\begin{cases} \text{find } u \in H: \quad J(u) = \min J(v), \qquad v \in H \\ \\ \qquad\qquad J(v) = \dfrac{1}{2}\displaystyle\int\limits_{\Omega} ||\vec{\nabla} v||^2 dV - \int\limits_{\Omega} f v dV \end{cases} \tag{6.6}$$

In these equations, the function u can be the vertical displacement of an elastic membrane acted upon by the external force f or, as in our case, the electrostatic potential ϕ associated with a charge density $\rho = \epsilon_0 f$.

Note that the space H in (6.6) is a special Sobolev space. H is the space of functions v which are square-integrable and with square-integrable derivative. We remind that, given a domain $\Omega \subset \mathbb{R}^n$, a square-integrable function is defined in a space

$$L^2(\Omega) = \left\{ v : \Omega \to \mathbb{R} : \int\limits_{\Omega} v^2 dV < +\infty \right\}$$

Therefore

$$H = H^1(\Omega) = \left\{ v \in L^2(\Omega);\ v' \in L^2(\Omega) \right\}$$

Consider now Poisson's equation (6.5) for $u = 0$ and $f = \rho/\epsilon_0$. Multiplying both sides of (6.5) by a trial function φ, by neglecting the boundary condition $\phi = 0$ on Σ and, instead, extending Ω to Ω_∞ (more general case), and by integrating over Ω_∞, we obtain

$$\iiint\limits_{\Omega_\infty} \varphi(\vec{\nabla}^2\phi) dV + \frac{1}{\epsilon_0} \iiint\limits_{\Omega_\infty} \varphi\rho dV = 0 \tag{6.7}$$

From Green's first identity

$$\iiint\limits_{\Omega_\infty} \varphi(\vec{\nabla}^2\phi) dV + \iiint\limits_{\Omega_\infty} (\vec{\nabla}\varphi)\cdot(\vec{\nabla}\phi) dV = \oiint\limits_{\Sigma_\infty} \varphi(\vec{\nabla}\phi)\cdot\vec{n} dA \tag{6.8}$$

where Σ_∞ can be assumed to be the surface of a sphere of infinite radius, enclosing entirely Ω_∞.

The trial function φ, while, in general, is not the potential associated with ρ, can be considered to mainatin a similar structure as the actual electrostatic potential.

Thus, for a limited ρ we expect $\varphi \sim 1/r$ for large r. From definition, $\vec{\nabla}\phi = -\vec{E}$, which we expect to go as $\sim 1/r^2$ for large r. The area of Σ_∞ goes as $\sim r^2$, thus

$$\oiint\limits_{\Sigma_\infty} \varphi(\vec{\nabla}\phi) \cdot \vec{n} dA \sim \frac{1}{r^3} r^2 = \frac{1}{r} \tag{6.9}$$

for large r. For $r \to +\infty$, this integral tends to zero. Therefore,

$$\iiint\limits_{\Omega_\infty} \varphi(\vec{\nabla}^2\phi) dV = -\iiint\limits_{\Omega_\infty} (\vec{\nabla}\varphi) \cdot (\vec{\nabla}\phi) dV \tag{6.10}$$

Finally, we can re-write Eq. (6.5) as

$$\epsilon_0 \iiint\limits_{\Omega_\infty} (\vec{\nabla}\phi) \cdot (\vec{\nabla}\varphi) dV - \iiint\limits_{\Omega_\infty} \rho\varphi dV = 0 \tag{6.11}$$

The electrostatic problem, thus, can be equivalently stated as to finding the function ϕ for all possible trial functions φ that satisfies the condition of Eq. (6.11).

In one dimension and with respect to a Cartesian coordinate system, Eq. (6.11) can be written as

$$\int\limits_\lambda \phi' \varphi' dx - \int\limits_\lambda \rho\varphi dx = 0 \tag{6.11'}$$

where we assumed $\epsilon_0 = 1$ and the "primes" indicate a spatial derivative with respect to x. Similarly, Eq. (6.4) in one dimension reads

$$U_e(\phi) = \int\limits_\lambda \rho\phi dx - \frac{1}{2}\epsilon_0 \int\limits_\lambda (\phi')^2 dx \tag{6.12}$$

For pedagogical reasons, we will use this equation in the proof of Appendix 6.1. By the end of the proof, we will convince ourselves that the minimization of $[-U_e(\phi)]$ gives the actual solution to the electrostatic problem.

The fact that we need to minimize the negative of the total energy should not be a concern. In fact, we remind that ϕ can be offset by any arbitrary constant c. Thus, Eq. (6.12) can be written as

$$\begin{aligned}
&\int\limits_\lambda \rho(\phi + c) dx - \frac{1}{2}\epsilon_0 \int\limits_\lambda \left[(\phi + c)'\right]^2 dx \\
&= \int\limits_\lambda \rho c dx + \int\limits_\lambda \rho\phi dx - \frac{1}{2}\epsilon_0 \int\limits_\lambda (\phi')^2 dx \\
&= \int\limits_\lambda \rho c dx + U_e(\phi)
\end{aligned} \tag{6.12'}$$

We can then always choose $\int\limits_\lambda \rho c dx = -2U_e$ since c is an arbitrary constant, obtaining $[-U_e(\phi)]$.

The fact that Eq. (6.4) is equivalent to Eq. (6.11) has an important consequence. Take a special function $\varphi = \phi$ that satisfies Eq. (6.11). We can then write

$$\begin{aligned}\frac{1}{2}\epsilon_0 \iiint_{\Omega_\infty} ||\vec{\nabla}^2\phi|| dV &= \frac{1}{2}\epsilon_0 \iiint_{\Omega_\infty} E^2 dV \\ &= \frac{1}{2} \iiint_{\Omega_\infty} \rho\phi dV \qquad (6.13)\end{aligned}$$

This can be interpreted as the energy of the internal forces is equal to that of the external ones. In other words, this means that Eq. (6.11) is one way to write what in mechanics is called the principle of virtual works: The work due to the external forces corresponding to a virtual "displacement" φ is equal to the work due to the internal forces.

Our proof shows that the principle of virtual works, i.e., finding the special function ϕ for which Eq. (6.11) is fulfilled for all possible admissible functions φ, is the same as finding the function ϕ that minimizes the total potential energy (6.4). The principle of virtual works is a least action principle.

It is worth mentioning that weak formulations are the starting point to implement the finite element method, one of the most powerful numerical methods for solving partial differential equations.

Appendix 6.1.

Consider Eq. (6.12) and assume that ϕ is a solution that minimizes this equation. Thus, a function $\widetilde{\phi} = \phi + \delta\varphi$, with $\delta \in \mathbb{R}$, must be such that

$$U_e(\phi) \leq U_e(\phi + \delta\varphi), \qquad \forall\varphi \qquad (6.1A)$$

The function $\psi(\delta) = U_e(\phi + \delta\varphi)$ is a quadratic function of δ with minimum at $\delta = 0$. Thus,

$$\psi'(\delta)|_{\delta=0} = \frac{\partial}{\partial\delta} U_e(\phi + \delta\varphi)|_{\delta=0} = 0 \qquad (6.2A)$$

By definition

$$\frac{\partial}{\partial\delta} U_e(\phi + \delta\varphi) = \lim_{\delta\to 0} \frac{U_e(\phi + \delta\varphi) - U_e(\phi)}{\delta}, \quad \forall\varphi \qquad (6.3A)$$

Consider the term $U_e(\phi + \delta\varphi)$:

$$\begin{aligned}U_e(\phi + \delta\varphi) &= \int_\lambda \rho(\phi + \delta\varphi) dx - \frac{1}{2}\epsilon_0 \int_\lambda \left[(\phi + \delta\varphi)'\right]^2 dx \\ &= \int_\lambda \rho\phi dx + \int_\lambda \rho\delta\varphi dx - \frac{1}{2}\epsilon_0 \int_\lambda \left[(\phi')^2 + 2\delta\phi'\varphi' + (\delta\varphi')^2\right] dx \\ &= U_e(\phi) + \int_\lambda \rho\delta\varphi dx - \frac{1}{2}\epsilon_0 \int_\lambda \left[\delta^2(\varphi')^2 + 2\delta\phi'\varphi'\right] dx \qquad (6.4A)\end{aligned}$$

Thus,

$$\frac{U_e(\phi+\delta\varphi)-U_e(\phi)}{\delta} = \int_\lambda \rho\varphi dx - \frac{1}{2}\epsilon_0 \int_\lambda \left[\delta(\varphi')^2 + 2\phi'\varphi'\right] dx \tag{6.5A}$$

In the limit $\delta \to 0$ and imposing that such limit for Eq. (6.5A) is zero, we obtain (assume $\epsilon_0 = 1$)

$$\int_\lambda \phi'\varphi' dx - \int_\lambda \rho\varphi dx = 0, \qquad \forall\varphi \tag{6.6A}$$

Vice-versa, if ϕ is a solution of (6.11'), assuming $\widetilde{\varphi} = \delta\varphi$ we have

$$\int_\lambda \phi'\widetilde{\varphi}' dx - \int_\lambda \rho\widetilde{\varphi} dx = 0 \tag{6.7A}$$

or

$$\int_\lambda \phi'\delta\varphi' dx - \int_\lambda \rho\delta\varphi dx = 0 \tag{6.8A}$$

As a consequence,

$$\int_\lambda \delta(\phi'\varphi' - \rho\varphi) dx = 0 \tag{6.8A'}$$

which is true iff $\rho\varphi = \phi'\varphi'$. Thus,

$$\begin{aligned} U_e(\phi+\delta\varphi) &= \int_\lambda \rho(\phi+\delta\varphi)dx - \frac{1}{2}\int_\lambda \left[(\phi+\delta\varphi)'\right]^2 dx \\ &= \int_\lambda \rho\phi dx - \frac{1}{2}\int_\lambda (\phi')^2 dx + \int_\lambda \delta\rho\varphi dx - \int_\lambda \delta\phi'\varphi' dx - \frac{1}{2}\int_\lambda \delta^2(\varphi')^2 dx \\ &= U_e(\phi) - \frac{1}{2}\int_\lambda \delta^2(\varphi')^2 dx \end{aligned}$$

It is evident that

$$\frac{1}{2}\int_\lambda \delta^2(\varphi')^2 dx \geq 0, \qquad \forall\varphi \text{ and } \forall\delta \in \mathbb{R} \tag{6.9A}$$

This would mean that

$$U_e(\phi) \geq U_e(\phi+\delta\varphi) \tag{6.10A}$$

This result is the opposite of what stated in (6.1A). The apparent problem can be resolved by inverting the signs in Eq. (6.12) and, thus, (6.4).