ECE 106 - Tutorial

Quiz #2, Problem 2

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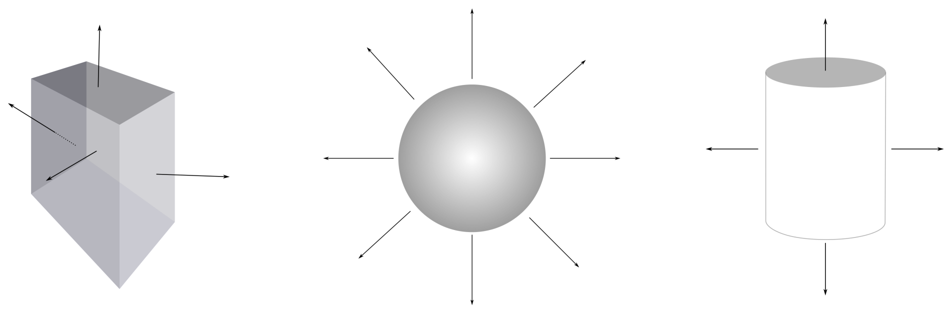

Figure 1: da vectors for many common shapes Note that the direction of the da vector depends on where on the surface you are.

Quiz 3, Pr. 1

Alright, you guys, let's go through this quiz that you did last week. It seemed to be a point of struggle for a number of you. I'll treat these problems in order and I will supply you with the problem statements that you were given on the quiz.

1a

A point charge is placed inside of a cube of side L, then inside of a sphere of radius L/2, then inside a cyliner of radius L/2 and height L. Which case will produce the highest [electric] flux through the above volumes individually?

Okay, so the trick with this problem is to understand what electric flux is. Electric flux is a measure of the amount of electric field lines that permeate a surface. For a stronger source of the electric field (like a larger charge or more charges) then the electric field will be greater at a given point in space. In order to measure the electric flux, though, we need to consider a surface.

See, the electric flux is a measure of how much electric field permeates a surface. So, we need something (a sheet, a sphere, a cube, etc.) which the electric field vectors will pierce. The electric flux through the surface will be a measure of the strength of the electric field directed through that surface. Mathematically, this is expressed as:

$$\text{Electric flux} = \oint_{\text{surface}} \vec{E} \cdot \vec{da} = \oint_{\text{surface}} |\vec{E}||\vec{da}| \cos\theta$$

What is that $\vec{da}$ quantity? Well, it's the vector that points normal to the surface at the point you are interested in (see considering is closed (that is, it is a closed volume, unlike a sheet or something with a hole in it) then the area vector is defined to point normal to the surface and exterior to that surface. That is, it points to the outside of the surface. For an open surface (like a sheet) the area vector can be defined as normal to the surface in one of two directions (up or down, for a sheet). It must be specified which direction someone is considering when they specify a solution to a problem.

See, the integral is calculated over the surface where you're looking at the electric field. The $\vec{da}$ vector is defined to be perpendicular to the surface. This might mean that it might have different $\hat{x}$ and $\hat{y}$ coordinates at different points in space (like for a sphere). But, it will always be perpendicular to the surface.

The electric field, at the surface, may not be perpendicular to the surface. So, there may be an angle between the two vectors. In this case, the dot product becomes important.

The dot product in that expression accounts for the fact that the electric field might not be piercing the surface in the same direction as the area vector. See, if the electric field is not parallel to the area vector at the surface then the amount of flux through that surface, the amount of electric field which passes through that surface is reduced. Think about it like this: If the electric field was perpendicular to the area vector at the surface then none of the electric field would pass through the surface.

An oft-cited analogy for electric flux is the water that flows through a shower head. The amount of water that flows through the head of the shower can be thought of as the amount of electric field lines that penetrate a surface. If the water flows perpendicular through the surface then there will be no "water flux". The way in which to get the largest volume of water to pass through the surface is to align the water flow perpendicular to the surface. This is the same way in which electric flux is maximized: By being perpendicular to the surface.

Okay, now that we've established what flux is qualiatively and quantitatively let's relate it to a quantity that's easier to measure: electric charge. It would be extremely difficult for most cases to calculate $\oint \vec{E} \cdot \vec{da}$. The electric field might not have a nice relationship with the area vector. The prototypical example is the case of a charge placed in the center of a cube. The electric field from the charge will be in the $\hat{r}$ direction. Thus, immediately above the charge, the electric field will point in the $\hat{z}$ direction. But, off to the side, above the charge (for non-zero x and y coordinates), the direction of the electric field will havea non-zero x and y component. The top face of the cube, however, has an area vector that always points in the $+\hat{z}$ direction. Thus, the angle between the electric field and the area vector will change over the whole surface of the cube. The expression for this, while possible to write, is difficult enough to make us want to find the magnitude of the electric flux in another way. It turns out, that the electric flux is related to the charge inside of a volume through the simple relationship:

$$\Phi = \frac{Q_{enc}}{\epsilon_0}$$

This is incredible! It tells us that if we want to calculate the flux and we happen to know the amount of enclosed charge inside the volume in which we're interested, we can just add that up and divide by ϵ_0, the permittivity of free space, and we're done! The change outside of our volume has absolutely no effect on the electric flux through our surface! (I hate explanation points, but hopefully these help drive the importance of this result to you guys.)

Now, let's think about the problem. We are asked to compare the amount of flux through each of the various surfaces. All of the closed surfaces (the cube, the sphere and the cylinder) have the exact same amount of charge enclosed inside of them. Thus, expression $\frac{Q_{enc}}{\epsilon_0}$ must be the same for all of them. Thus, the electric flux generated through each of the surfaces must be exactly the same. This is the answer.

1b

A point charge is placed at the center of a cube of side L. What is the flux through one of the faces?

Okay, the trick to this problem is to neglect the dimensions of the cube. We know that the enclosed charge determines thet total flux through the surface. Thus, we know that the total flux through the surface is $\Phi = \frac{Q}{\epsilon_0}$, where Q is the charge that is placed at the center of the cube. Now, if the charge is placed in the center, there is symmetry in the problem.

For example, consider the left and right sides of the cube (see figure 2). The amount of electric field and the direction of the electric field at the left and the right faces of the cube are exactly the same magnitude but in the opposite direction. So, $\vec{E}$ flips direction based on whether you're considering the right

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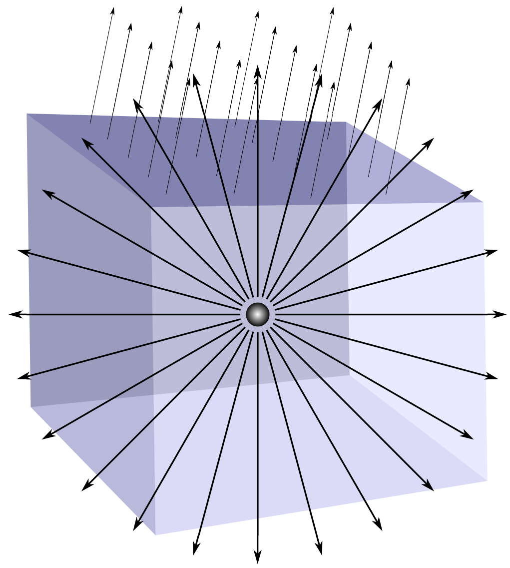

Figure 2: Figure depicting the electric field generated by a point charge in a box (area vectors of the top of the box shown above the point charge

or left side of the cube. But, so does $\vec{da}$, the vector normal to the surface of the cube. Thus, the dot product is the same: $-\vec{E} \cdot -\vec{da} = \vec{E} \cdot \vec{da}$.

Now, I could perform the same argument with the top and bottom and the front and back of the cube. Thus, there is symmetry with respect to three pairs of the cube's faces. Now, if I rotate the cube so that the right side becomes the top side I have the same electric field and the same $\vec{da}$ vectors. I can do this to show that the electric flux through each of the 6 faces of the cube is the exact same. Thus $\Phi = 6\Phi_{face} = \frac{Q}{\epsilon_0}$. So, finally:

$$\Phi_{face} = \frac{Q}{6\epsilon_0}$$

This is the answer to 1b.

1c

A point charge Q is placed inside the hemispherical Gaussian surface a distance δ from the flat face, as shown. Find the flux through the flat face in the limit as δ as goes to zero?

Let's consider this problem in the following way. There are a number of ways to justify the answer for this question. I like the following argument: The total flux through the surface is the sum of the flux through the flat face and the flux through the curved face: $\Phi = \Phi_{\text{curved face}} + \Phi_{\text{flat face}}$.

However, the amount of flux through the curved face is one half the amount of flux that would go through an entire sphere. Notice that I can only say this in the limit as the charge goes infinitesimally close to the flat face.

If the charge was far from the flat face then it would not look like the charge was at the center of a sphere to the hemisphere. If it's not at the center of the hemisphere then we can't simply say that the electric flux through the surface would be one half of that through the sphere, since the eletric field distribution with respect to the hemisphere changes drastically.

Another way to say it is that as the charge gets close to the hemisphere (far from the flat face) more electric fields go through the flat face and less goes through the hemisphere since the electric field

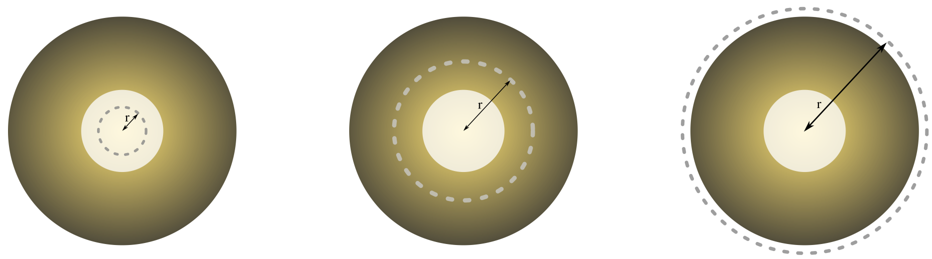

Figure 3: Hollow, charged sphere with Gaussian spheres located within the hollow (left), within the charged region (right) and encompassing all of the charge in the sphere (right)

lines generated by the point charge would not be in the same direction as the area vector on the hemisphere ($\vec{da}_{\text{hemisphere}} = da\hat{r}$).

Thus, since the entire flux through a sphere would be $\Phi = \frac{Q}{\epsilon_0}$, the flux through the hemisphere would be $\frac{Q}{2\epsilon_0}$. Thus, the flux through the flat face would be:

$$\Phi - \Phi_{\text{curved face}} = \frac{Q}{\epsilon_0} - \frac{Q}{2\epsilon_0} = \frac{Q}{2\epsilon_0}$$

This is the question to 1c.

Quiz 3, Pr. 2

Charge Q is uniformly distributed over the volume of a hollow sphere of inner radius a and outer radius b, as shown. r is the distance from the center of the hollow part (the geometrical center of the structure).

2a

Find the electric field inside the hollow part.

This is a little more challenging than the previous, problems. Now, we are dealing with a continuous distribution of charge. In order to find the electric field we must use Gauss' law: $\oint_{\text{surface}} \vec{E} \cdot \vec{da} = \frac{Q}{\epsilon_0}$. The way in which we'll use Gauss' law is we'll define a surface that takes advantage of the symmetry of the problem. We'll pick a surface whose differential area vector ($\vec{da}$) is in the same direction as the electric field, $\vec{E}$, for all the places at which we need to compute this integral. The most natural surface to use is a sphere.

The spherical charge distribution should, through symmetry considerations, generate an electric field that is radially distributed. You can justify this by imagining rotating the sphere. If the sphere is perfectly symmetrical then there must be no way that you can tell what is "up" ,what is "left", etc. Thus, the electric field must be symmetric with respect to rotations. The only distrbution of vectors that accomplishes this is a radial distribution.

Now, our surface, our "Gaussian surface", will be a sphere with a specified radius. In this part of the problem, we are only asked to compute the electric field in the region where $r < a$. That is, in the region where there is no electric charge. The way in which we'll calculate the electric field is to define a sphere of a certain radius ($r < a$) and we'll find out how much charge is enclosed inside that sphere. By relating the enclosed charge to the surface area of our sphere we can calculate the electric field. Watch:

$$\Phi = \oint_{\text{sphere}} \vec{E} \cdot \vec{da} = \frac{Q_{enc}}{\epsilon_0}$$

$$\oint_{\text{sphere}} \vec{E} \cdot \vec{da} = \frac{\oint_{\text{charge enclosed}} \rho dV}{\epsilon_0} \tag{1}$$

Note, that my Gaussian surface, this sphere, might enclose a distribution of charge. So, it's necessary for me to add up all the charge that might be enclosed by my sphere. That's the reason I have rewritten the right-hand side.

Okay, now we can simplify the expression on the left-hand side of the above expression. $\vec{E}$ is, by symmetry considerations, directed radially outward. That is $\vec{E} = E(r)\hat{r}$. Now, I can also argue that by symmetry the magnitude of the electric field should be the same at all distances r away from the origin. Thus, $E(r)$ is a constant and can be pulled out of the integral. The differential area element $\vec{da}$, by virtue of me having chosen a sphere as my Gaussian surface is $da\hat{r}$. So, it is in the same direction as the electric field. Thus, $\vec{E}(r) \cdot \vec{da} = E(r)da$, where $E(r)$ is a constant. Applying all these simplying results yields:

$$\oint_{\text{sphere}} \vec{E} \cdot \vec{da} = E(r) \oint_{\text{sphere}} da$$

The last expression is just asking for the total surface area of the sphere (the sum of all of the differential areas is the total area). Thus:

$$E(r) \oint_{\text{sphere}} da = E(r) 4\pi r^2$$

Where $r < a$ is the radius of our Gaussian surface. Now, how much charge is enclosed by our Gaussian surface for $r < a$? Nothing! There is no charge for $r < a$. Thus $\frac{\oint_{\text{charge enclosed}} \rho dV}{\epsilon_0} =$. Since $r \neq 0$ equation 1 (along with our simplifying expressions) informs us that $E(r) = 0$ everywhere inside the hollow region (for all $r < a$). Thus, the answer to 1a is $E(r < a) = 0$.

2c

Solve for the electric field for outside of the sphere where $r > b$.

Okay, this part of the quiz is easier than 2b, which is why I'm solving it first. I will start by using equation 1, which still applies.

$$\oint_{\text{sphere}} \vec{E} \cdot \vec{da} = \frac{\oint_{\text{charge enclosed}} \rho dV}{\epsilon_0}$$

Okay, by the same arguments as before, I can rewrite the left hand side as:

$$E(r) \oint_{\text{sphere}} da = \frac{\oint_{\text{charge enclosed}} \rho dV}{\epsilon_0}$$

Now, though, the charge enclosed is not zero. In fact, for $r > b$ we have enclosed all of the charge on the sphere. Can we prove this? Yes. Let's first consider how we would find the amount of enclosed charge $Q_{enc} = \oint\limits_{\text{charge enclosed}} \rho dV$ Well, how much charge is this? It's $Q = \rho * V_{\text{charged hollow sphere}}$. What is the volume of the charged hollow sphere? Well, it's the volume of a sphere of radius b minus the volume of a sphere of radius a : $V_{\text{charged hollow sphere}} = \frac{4\pi}{3}b^3 - \frac{4\pi}{3}a^3$. What is ρ? $\rho = \frac{Q}{V_{\text{charged hollow sphere}}}$. It is the amount of charge stored per volume of the charged hollow sphere.

$$\rho = \frac{Q}{\frac{4\pi}{3}b^3 - \frac{4\pi}{3}a^3}$$

Note that this is not the same thing as:

$$\rho = \frac{Q}{\frac{4\pi}{3}(b-a)^3}$$

because $(a-b)^3 \neq (a^3 - b^3)$, in general. The best way to think about the volume (to avoid algebraic mistakes) is to subtract the volumes of two spheres.

So, finally, what is the total charge enclosed by our Gaussian surface? Well, it's our charge density, ρ, times the volume of charge enclosed. Since our Gaussian surface is larger than the charged sphere, though, then $V_{\text{charged hollow sphere}} = \frac{4\pi}{3}b^3 - \frac{4\pi}{3}a^3$, the volume of the entire charged hollow sphere. And, finally, $Q_{enc} = \rho * V_{\text{charged hollow sphere}} = Q$, the total charge on the sphere.

Okay, we already said this. It makes sense. Once our Gaussian surface is bigger than the charged hollow sphere then our Gaussian surface must enclose all of the charge. But, this mode of thinking will help us, later, when we try to solve 2b. Okay, now, ρ is a constant so we can pull it out of the right-hand side of our expression for Gauss' law. Then, we're just integrating over the volume of our Gaussian surface. For a Gaussian surface with radius $r > b$, the Gaussian surface's volume is just $\frac{4\pi}{3}r^3$. Thus, our expression for Gauss' law can be reduced to:

$$E(r)4\pi r^2 = \frac{Q}{\epsilon_0}$$

Now, $E(r) = \frac{Q}{4\pi\epsilon_0 r^2} = \frac{kQ}{r^2}$. This is an important result. Outside of a spherically symmetric charge distribution, the electric field looks like exactly that of a point charge located at the origin. How big is that point charge? Well, it's as big as the amount of charge stored in that spherically symmetric charge distribution. Okay, this is the answer for 2c. Let's tackle 2b.

2b

Okay, we'll start with Gauss' law. I'll immediately apply the symmetry results to simplify the expression.

$$E(r) \oint\limits_{\text{sphere}} da = \frac{\oint\limits_{\text{charge enclosed}} \rho dV}{\epsilon_0}$$

We have the expression for ρ from before.

$$\rho = \frac{Q}{\frac{4\pi}{3}(b^3 - a^3)}$$

Now, ρ is a constant (uniform charge distribution), so I can pull this out of the integral. The integral on the left-hand side over da is just the area of our Gaussian surface $\oint da = A = 4\pi r^2$, as usual. The most difficult thing in this problem is the integral over dV in the right-hand side. This is not the volume of our Gaussian surface! We are trying to calculate the amount of enclosed charge. Thus, this volume should be the volume of the enclosed charge. At a distance r from the center, the amount of charge I have enclosed is

$$V_{\text{charge enclosed}} = \frac{4\pi}{3}r^3 - \frac{4\pi}{3}a^3$$

It's exactly the volume of my Gaussian surface **minus the volume of my Gaussian surface in which there is no charge**. This is the center of the hollow sphere. Substituting this yields:

$$E(r)4\pi r^2 = \frac{1}{\epsilon_0}\left(\frac{Q}{\frac{4\pi}{3}b^3 - \frac{4\pi}{3}a^3}\right)\left(\frac{4\pi}{3}r^3 - \frac{4\pi}{3}a^3\right)$$

After a little algebra this can be rewritten as:

$$E(r) = \frac{Q}{4\pi\epsilon_0 r^2}\frac{r^3 - a^3}{b^3 - a^3}$$

This is the answer for problem 2b.

2d

Solve for the electric field at the outer surface where $r = b$.

Now, this problem is very simple. We just have to plug in $r = b$ into our expressions for $E(r)$ from either part b or or part c. How come we can use either expression? Well, the electric field outside the sphere, for all $r > b$, is given by the answer in 2c. The electric field inside the sphere, for $a < r < b$, is given in 2b. So, taking the limit as $r \to b$ both expressions should yield the same thing, if a limit exists. And, this is a fairly simple situation, that could be constructed fairly easily, so we hope a limit exists. It's easiest to plug $r = b$ into the answer we got in 2c, but we could just as well plug it into 2b. If we do this, we obtain:

$$E(b) = \frac{kQ}{b^2}$$