PHYS 342 - EM 02

Lecture 12

In PHYS 242, we studied the problem of the electrostatic dipole. We also gave an introduction to the general method called multipole series expansion of the electrostatic potential. We will now present this method more in depth. We will then consider the same problem in magnetostatics.

12.1 Multipole Series Expansion of the Electrostatic Potential for a Continuous Volume Charge Distribution

Consider a charge distribution with continuous and limited density ρ in a region Ω of the three-dimensional Euclidean space (cf. Fig. 12.1).

The electrostatic potential at a point P outside Ω is given by

$$\phi(P) = \frac{1}{4\pi\epsilon_0} \iiint\limits_{\Omega} \frac{\rho(Q)}{r_{QP}}\, dV \qquad (12.1)$$

where r_{QP} is the absolute value of the distance between P and the arbitrary infinitesimal volume element dV centered at Q.

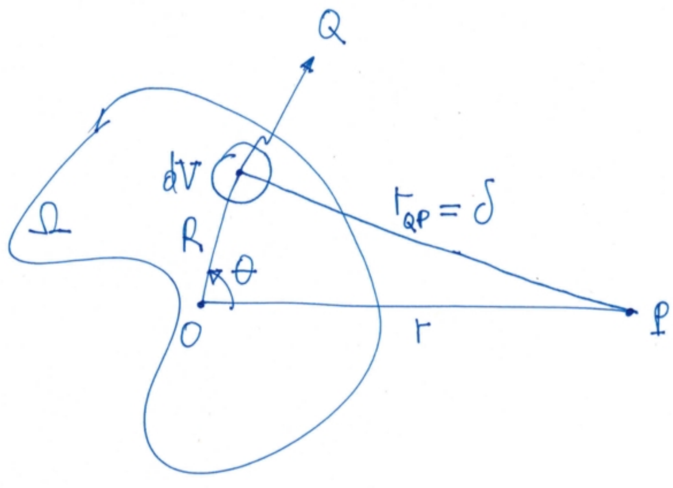

Figure 12.1

For simplicity, we can rename $r_{QP} = \delta$ and define r to be the magnitude of the distance between O and P, where O is a point inside Ω with $O \neq Q$, and R the magnitude of the distance between O and Q, where Q is also inside Ω. By calling $\theta = P\hat{O}Q$, from the theorem of cosines

$$\delta^2 = r^2 + R_2 - 2rR\cos\theta \tag{12.2}$$

Since $r \neq 0$ because P is outside Ω, (12.2) can be rewritten as

$$\delta = r\sqrt{1 + \left(\frac{R}{r}\right)^2 - 2\left(\frac{R}{r}\right)\cos\theta} \tag{12.3}$$

By substituting

$$\begin{cases} \dfrac{R}{r} = u & (12.4a) \\ \cos\theta = v & (12.4b) \end{cases}$$

we have

$$\delta = r\sqrt{1 + u^2 - 2uv} \tag{12.5}$$

Using (12.5) in (12.1), we obtain

$$\phi(P) = \frac{1}{4\pi\epsilon_0}\,\frac{1}{r}\iiint_\Omega \frac{\rho}{\sqrt{1 + u^2 - 2uv}}\,dV \tag{12.6}$$

The integrand of (12.6) can be expanded by means of a power series:

$$\frac{1}{\sqrt{1 + u^2 - 2uv}} = \sum_{n=0}^{\infty} P_n(v)u^n \tag{12.7}$$

where $P_n(v)$ is the Legendre polynomial of order n.

Note that Eq. (12.7) is one way to define the Legendre's polynomials. These can also be defined as the solutions of the ordinary differential equations with variable coefficients

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0 \tag{12.8}$$

The Legendre's polynomial can be calculated by means of the Rodriguez's identity

$$P_n(x) = \frac{1}{2^n n!}\,\frac{d^n}{dx^n}\,(x^2 - 1)^n \tag{12.9}$$

Using this identity, the first few polynomials are

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \frac{1}{2}\,(3x^2 - 1) \\ P_3(x) &= \frac{1}{2}\,(5x^3 - 3x) \end{aligned}$$

The series (12.7) converges uniformly in the range $v \in (-1, +1)$; also, $u < 1$ at each point if point P is very far from the region Ω (in which case $r \gg R$). Using (12.7) in (12.6), we have

$$\phi(P) = \frac{1}{4\pi\epsilon_0}\,\frac{1}{r}\iiint_\Omega \sum_{n=0}^{\infty} \rho P_n(v)u^n dV \tag{12.10}$$

The series in the integrand is uniformly convergent and the terms of the series are continuous functions. Thus, the sign of integral and that of series can be safely exchanged,

$$\phi(P) = \frac{1}{4\pi\epsilon_0}\frac{1}{r}\sum_{n=0}^{\infty}\iiint\limits_{\Omega}\rho P_n(v)u^n dV \tag{12.11}$$

The first few terms of this series of integrals are

$$\phi_0 = \frac{1}{4\pi\epsilon_0}\frac{1}{r}\iiint\limits_{\Omega}\rho dV = \frac{1}{4\pi\epsilon_0}\frac{q}{r} \tag{12.12a}$$

$$\phi_1 = \frac{1}{4\pi\epsilon_0}\frac{1}{r}\iiint\limits_{\Omega}\frac{\rho R\cos\theta}{r}dV \tag{12.12b}$$

$$\phi_2 = \frac{1}{4\pi\epsilon_0}\frac{1}{r}\iiint\limits_{\Omega}\frac{\rho R^2(3\cos^2\theta - 1)}{2r^2}dV \tag{12.12c}$$

where we used the Legendre's polynomials P_0, P_1, and P_2. In summary, we first found $\phi(P)$, we then showed how ϕ depends on r, and, at last, we expanded in power series the part of the potential that does not depend on r (note that this is a legitimate procedure even if r appears as a scaling factor in the definition of u).

The term ϕ_0 is the electrostatic potential at P as if the total charge q of Ω was concentrated at O.

The term

$$\phi_1 = \frac{1}{4\pi\epsilon_0}\frac{p}{r^2} \tag{12.13}$$

is the term of electrostatic dipole, where

$$p = \iiint\limits_{\Omega}\rho R\cos\theta dV \tag{12.14}$$

is the electrostatic dipole moment along the OP direction.

The next term

$$\phi_2 = \frac{1}{4\pi\epsilon_0}\frac{m}{r^3} \tag{12.15}$$

is the term of electrostatic quadrupole, where

$$m = \iiint\limits_{\Omega}\frac{1}{2}\rho R^2(3\cos^2\theta - 1)dV \tag{12.16}$$

is the electrostatic quadrupole moment.

In conclusion,

$$\phi(P) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r} + \frac{p}{r^2} + \frac{m}{r^3} + \dots\right) \tag{12.17}$$

By defining the vectors $\vec{r}$ and $\vec{R}$, another way to write the potential of (12.17) considering only the first two terms is

$$\phi(\vec{r}) \approx \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r}\iiint\limits_{\Omega}\rho(\vec{R})dV + \frac{\vec{r}}{r^3}\cdot\iiint\limits_{\Omega}\rho(\vec{R})\vec{R}dV\right] \tag{12.18}$$

where, again, we can define

$$\begin{cases} q = \iiint\limits_{\Omega} \rho(\vec{R})dV & (12.19a) \\ \vec{p} = \iiint\limits_{\Omega} \rho(\vec{R})\vec{R}dV & (12.19b) \end{cases}$$

This form unveils the vector nature of $\vec{p}$, the dipole moment. Thus,

$$\phi(\vec{r}) \approx \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r} + \frac{\vec{r}\bullet\vec{p}}{r^3}\right) \tag{12.20}$$

12.2 Ideal Electrostatic Dipole

In the case of two point-like charges $+q$ and $-q$ at a distance $\vec{d}$ from each other, from (12.19a) and (12.19b)

$$\begin{cases} q &= \iiint\limits_{\Omega} \Big[-q\delta(\vec{R}) + q\delta(\vec{R}-\vec{d})\Big]dV = 0 & (12.21a) \\ \vec{p} &= \iiint\limits_{\Omega} \Big[-q\delta(\vec{R})\vec{R} + q\delta(\vec{R}-\vec{d})\vec{R}\Big]dV & \\ &= -q\Big[\vec{R}\Big]_0 + q\Big[\vec{R}\Big]_{\vec{d}} = q\vec{d} & (12.21b) \end{cases}$$

From (12.20)

$$\phi(\vec{r}) \approx \frac{1}{4\pi\epsilon_0}\,\frac{\vec{r}\bullet\vec{p}}{r^3} \tag{12.22}$$

and

$$\vec{E}(\vec{r}) = -\vec{\nabla}\phi(\vec{r}) \approx \frac{1}{4\pi\epsilon_0}\left[\frac{3(\vec{r}\bullet\vec{p})\vec{r}}{r^5} - \frac{\vec{p}}{r^3}\right] \tag{12.23}$$

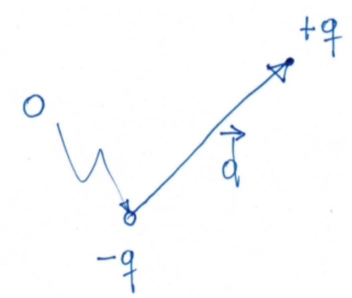

Figure 12.2

When $d \to 0$ and $qd = \text{const.}$, the electrostatic dipole is said to be ideal. In this case, the sign of approximate in Eqs. (12.22) and (12.23) can be substituted by the sign of equal (cf. PHYS 242).

12.3 Interaction Between an Ideal Electrostatic Dipole and an Electrostatic Field

When an ideal electrostatic dipole is placed in an electrostatic field generated by a set of external charge distributions, the dipole is acted upon by mechanical actions. If we assume the two point-like charges in the dipole to be rigidly attached to each other, the system can be mechanically regarded as a rigid body. With respect to a given pivot, those mechanical actions can be completely described by a net force and torque.

Net Force

Consider the ideal electrostatic dipole of Fig. 12.3.

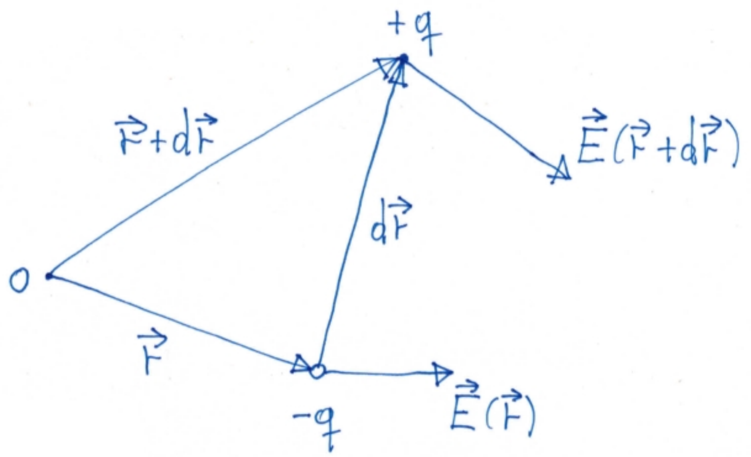

Figure 12.3

The distance between O and the point occupied by the negative charge $(-q)$ is $\vec{r}$. The electrostatic field at that point is $\vec{E}(\vec{r})$. The field at $(+q)$, which is $d\vec{r}$ apart from $(-q)$ (ideal dipole condition), is

$$\vec{E}(\vec{r}+d\vec{r}) = \vec{E}(\vec{r}) + d\vec{E} \tag{12.24}$$

where $d\vec{E}$ is the field variation corresponding to the infinitesimal displacement $d\vec{r}$.

With respect to a Cartesian coordinate system $Oxyz$,

$$d\vec{r} = dx\,\vec{u}_x + dy\,\vec{u}_y + dz\,\vec{u}_z \tag{12.25}$$

and

$$d\vec{E} = dE_x\vec{u}_x + dE_y\vec{u}_y + dE_y\vec{u}_y + dE_z\vec{u}_z \tag{12.26}$$

Thus, from simple vector calculus

$$dE_x = \frac{\partial}{\partial x}E_x dx + \frac{\partial}{\partial y}E_x dy + \frac{\partial}{\partial z}E_x dz \tag{12.27a}$$

$$dE_y = \frac{\partial}{\partial x}E_y dx + \frac{\partial}{\partial y}E_y dy + \frac{\partial}{\partial z}E_y dz \tag{12.27b}$$

$$dE_z = \frac{\partial}{\partial x}E_z dx + \frac{\partial}{\partial y}E_z dy + \frac{\partial}{\partial z}E_z dz \tag{12.27c}$$

which, in matrix form, can be written as

$$\begin{bmatrix} dE_x \\ \\ dE_y \\ \\ dE_z \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} E_x & \frac{\partial}{\partial y} E_x & \frac{\partial}{\partial z} E_x \\ \frac{\partial}{\partial x} E_y & \frac{\partial}{\partial y} E_y & \frac{\partial}{\partial z} E_y \\ \frac{\partial}{\partial x} E_z & \frac{\partial}{\partial y} E_z & \frac{\partial}{\partial z} E_z \end{bmatrix} \begin{bmatrix} dx \\ \\ dy \\ \\ dz \end{bmatrix} \tag{12.28}$$

It is worth mentioning that the field studied here is that in absence of the dipole. In fact, since we assumed the dipole to be a rigid body, the reciprocal actions of the two charges in the dipole are ineffective from a mechanical point of view.

By defining (12.28) as the gradient of the vector field $\vec{E}$ (this gradient is a second order tensor), we can write (12.24) as

$$d\vec{E} = (\vec{\nabla}\vec{E}) \cdot d\vec{r} \tag{12.29}$$

Calling the forces acting on the negative and positive charges $\vec{F}_-$ and $\vec{F}_+$, respectively we have the net force

$$\begin{aligned} \vec{F} &= \vec{F}_- + \vec{F}_+ = -q\vec{E} + q(\vec{E} + d\vec{E}) \\ &= qd\vec{E} = qd\vec{r} \cdot \vec{\nabla}\vec{E} = \vec{p} \cdot \vec{\nabla}\vec{E} \end{aligned} \tag{12.30}$$

where $\vec{p} = qd\vec{r}$ is the dipole moment.

Net Torque

We choose O in Fig. 12.3 as pivot.

Thus, the torque with respect to O is

$$\begin{aligned} \vec{\tau}_0 &= \vec{r} \times \vec{F}_- + (\vec{r} + d\vec{r}) \times \vec{F}_+ \\ &= -\vec{r} \times \vec{E}_q + (\vec{r} + d\vec{r}) \times (\vec{E} + d\vec{E})q \\ &= qd\vec{r} \times \vec{E} + q\vec{r} \times d\vec{E} + qd\vec{r} \times d\vec{E} \\ &= \vec{p} \times \vec{E} + \vec{r} \times \vec{F} + \vec{p} \times d\vec{E} \end{aligned} \tag{12.31}$$

where $\vec{F}$ is given by (12.30). The last term in (12.31) is $\sim d\vec{r} \times d\vec{E}$, i.e., is of higher order compared to the other terms. This term can thus be neglected. Hence,

$$\vec{\tau}_0 = \vec{r} \times \vec{F} + \vec{p} \times \vec{E} \tag{12.32}$$

When O is chosen to be at $(-q)$,

$$\vec{\tau}'_0 = \vec{p} \times \vec{E} \tag*{(12.32')}$$

In summary, the net force acting on an ideal electrostatic dipole is different from zero only for a non uniform electrostatic field in the neighborhood occupied by the dipole. In fact, for a uniform field $d\vec{E} = \vec{0}$ and, from (12.30), $\vec{F} = \vec{0}$. For a uniform field, the system is fully described by (12.32'). In this case, if the dipole is free from any other mechanical action, it rotates until it aligns with the field, at which point $\vec{\tau}'_0 = \vec{0}$. There are two possible equilibrium conditions for which $\vec{\tau}'_0 = \vec{0}$. One when $\vec{p}$ and $\vec{E}$ are parallel with same sign and one when they are parallel with opposite sign (antiparallel). The first position is stable, i.e., for any small rotations the dipole goes back to its original position. The second position is unstable.