PHYS 342 - EM 02

Lecture 7

In Lecture 6, we formulated the electrostatic problem as the minimization of a functional representing the total energy of a system of charges. The functional was defined as

$$U_e^*(\phi) = -U_e(\phi) = \frac{1}{2}\epsilon_0 \iiint_\Omega (\vec{\nabla}\phi)^2 dV - \iiint_\Omega \rho\phi dV \tag{7.1}$$

In PHYS 242, we defined the capacitance of a system of two conductors charged with opposite charges $+q$ and $-q$ and with potentials ϕ_1 and ϕ_2, respectively, as

$$C \equiv \frac{q}{\Delta\phi} \tag{7.2}$$

where $\Delta\phi = \phi_1 - \phi_2$.

We will now attempt to use the weak formulation (7.1) to find an approximate solution for the capacitance of a system of conductors.

7.1 Electrostatic Energy of a System of Charged Conductors

Given a charge distribution with surface density σ, the electrostatic energy associated with the distribution is

$$U_e = \frac{1}{2} \iint_\Sigma \sigma\phi dA \tag{7.3}$$

where σ is defined on the surface Σ and ϕ is the potential of the distribution.

We remind that given a charged perfect conductor in electrostatic equilibrium, the charge at each point inside the conductor is zero, i.e.,

$$\rho = 0 \qquad \forall p \in \Omega \tag{7.4}$$

where Ω is the region delimited by Σ. The charge must be distributed on the conductor surface only.

Consider now a system of N conductors Ω_i, with $i = 1 \dots N$. Each of the conductors is characterized by a surface Σ_i with charge surface density σ_i. By means of Eq. (7.3), we find the electrostatic energy of the system of conductors:

$$U_e^N = \frac{1}{2} \sum_{i=1}^{N} \iint_{\Sigma_i} \sigma_i \phi_i dA \tag{7.5}$$

Note that ϕ_i is constant on the surface of each conductor and, thus, it can be taken outside the sign of integral. Moreover, the charge on the i-th conductor can be written as

$$q_i = \iint\limits_{\Sigma_i} \sigma_i dA \tag{7.6}$$

Hence,

$$U_e^N = \frac{1}{2} \sum_{i=1}^{N} q_i \phi_i \tag{7.7}$$

In the special case of a condenser, $N = 2$ and $q_1 = +q$ and $q_2 = -q$ (which are associated with potentials ϕ_1 and ϕ_2, respectively).

Thus,

$$\begin{aligned} U_e^C &= \frac{1}{2} q\phi_1 - \frac{1}{2} q\phi_2 = \frac{1}{2} q\Delta\phi \\ &= \frac{1}{2} C(\Delta\phi)^2 = \frac{1}{2} \frac{q^2}{C} \end{aligned} \tag{7.8}$$

where we have used the definition (7.2).

7.2 The Principle of Least Action for the Capacitance

In the case of perfect conductors, Eq. (7.1) reduces to

$$U_e^*(\phi) = \frac{1}{2} \epsilon_0 \iiint\limits_{\Omega} (\vec{\nabla}\phi)^2 dV \tag{7.9}$$

because of condition (7.4). The electrostatic problem for perfect conductors can be thus stated as to finding the special potential $\bar{\phi}$ that minimizes the electrostatic energy of Eq. (7.9). Note that we could now remove the "*" from $U_e^*(\phi)$ since, in this case, the functional to be minimized coincides with the electrostatic energy and not with the opposite of the total energy.

Consider an indefinitely long cylindrical condenser, the cross-section of which is shown in Fig. 7.1.

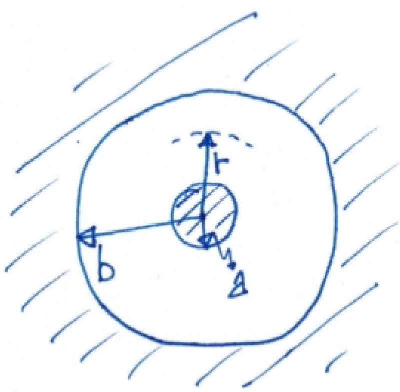

Figure 7.1

The inner conductor has radius a and potential ϕ_1 and the outer conductor radius b and potential $\phi_2 = 0$.

By using the solution $\bar{\phi}$ that minimizes the functional (7.9), we would be able to calculate the capacitance C of the condenser from

$$\frac{1}{2}CV^2 = \frac{1}{2}\epsilon_0 \iiint_{\Omega} (\vec{\nabla}\bar{\phi})^2 dV \tag{7.10}$$

If instead of $\bar{\phi}$ we were to use a wrong solution, we would find a capacitance that is larger than the correct value. In fact, in Eq. (7.10) ϕ_1 is fixed and must correspond to the potential that minimizes the electrostatic energy.

Note, however, that even if the potential ϕ used in Eq. (7.10) is a rough approximation of $\bar{\phi}$, the corresponding C will be a good approximation because the error in C is second order in the error in ϕ. Thus is one of the strengths of the weak formulation.

In PHYS 242, we calculated the capacitance of a cylindrical condenser,

$$C = \frac{2\pi\epsilon_0}{\ln \dfrac{b}{a}} \tag{7.11}$$

Assume we did not know this result and assume we did not even know the correct distribution of ϕ and $\vec{E}$ inside the condenser. We can try to use (7.10) to guess for C.

Suppose the potential goes linearly with distance r inside the condenser. This means the field is constant. This assumption is consistent with the symmetries associated with charged indefinite cylindrical structures.

In order to fulfil the boundary conditions at the inner and outer conductor, it must be

$$\phi = \phi_1 \left(1 - \frac{r-a}{b-a}\right), \qquad r \in [a, b] \tag{7.12}$$

This function is ϕ_1 at $r = a$, zero at $r = b$, and in between has a constant slope equal to $-\phi_1/(b-a)$. To use Eq.(7.10) to calculate C, we must square this gradient and integrate over all volume. Assuming a length $l = 1$ m of the cylinders, a volume element at radius r is $2\pi r dr$. We obtain,

$$\frac{1}{2}C_1\phi_1^2 = \frac{1}{2}\epsilon_0 \int_a^b \frac{\phi_1^2}{(b-a)^2} 2\pi r dr \tag{7.13}$$

which gives

$$C_1 = \pi\epsilon_0 \frac{b+a}{b-a} \tag{7.14}$$

Following Feynman $19 - 12$, Table 7.1 shows a comparison between the correct capacitance C calculated from (7.11) and the approximate capacitance C_1 calculated from (7.14) for various valves of the ration b/a.

$\frac{b}{a}$	$\frac{C}{2\pi\epsilon_0}$	$\frac{C_1}{2\pi\epsilon_0}$
2	1.4423	1.500
4	0.721	0.833
10	0.434	0.612
100	0.267	0.51
1.5	2.4662	2.50
1.1	10.492070	10.500000

Table 7.1

For a relatively big $b/a = 2$, which gives a pretty big variation in the field compared with a linearly varying field, we get a pretty good approximation. As expected, the value of C_1 is slightly larger than C. This is obviously consistent with the least action principle. In the case of a tiny wire inside a big cylinder, as for $b/a = 100$ for example, the approximation is much rougher and we are off by a factor of nearly 2. This is because the actual field has huge variations inside the two walls and a constant is a very coarse approximation. On the contrary, for conductors close to each other, e.g., for $b/a = 1.1$, we obtain a C_1 which is within 0.1% from C.

We can now try to refine our approximation by guessing a more complicated potential. For example, assume the potential is quadratic in r (i.e., the field is linear in r). The most general quadratic form that fits $\phi = 0$ at $r = b$ and $\phi = \phi_1$ at $r = a$ is

$$\phi = \phi_1 \Big[1 + \alpha \Big(\frac{r-a}{b-a} \Big) - \big(1 + \alpha \big) \Big(\frac{r-a}{b-a} \Big)^2 \Big] \tag{7.15}$$

where $r \in [a, b]$ and α is an arbitrary constant number.

In this case, the magnitude of the field is simply,

$$E = -\frac{d}{dr}\phi = -\frac{\alpha\phi_1}{b-a} + 2(1+\alpha)\frac{(r-a)\phi_1}{(b-a)^2} \tag{7.16}$$

We can now calculate the capacitance for an arbitrary α by using this E in Eq. (7.10). We obtain (cf. Feynman 19 − 13)

$$C_2 = 2\pi\epsilon_0 \frac{a}{b-a} \left[\frac{b}{a} \Big(\frac{\alpha^2}{6} + \frac{2\alpha}{3} + 1 \Big) + \frac{1}{6}\alpha^2 + \frac{1}{3} \right] \tag{7.17}$$

From the least action principle we know that among all possible capacitances for a given configuration (i.e., for given parameters a and b), the best approximation is the smallest capacitance. In general, we can compute the minimum of Eq. (7.17) by deriving with respect to α and equating to zero (the function (7.17) is definite positive; the minimum from the derivative is a global minimum). We get

$$\frac{b}{b-a}\frac{2\alpha}{6} + \frac{b}{b-a}\frac{2}{3} + \frac{a}{b-a}\frac{2}{6}\alpha = 0 \tag{7.18}$$

from which $\alpha = -2b/(b+a)$. By substituting this value into Eq. (7.17), we obtain the minimum capacitance

$$C_2 = 2\pi\epsilon_0 \frac{b^2 + 4ab + a^2}{3(b^2 - a^2)} \tag{7.19}$$

Table 7.2 shows a comparison between C from (7.11) and C_2 from (7.19) for a few values of b/a.

$\frac{b}{a}$	$\frac{C}{2\pi\epsilon_0}$	$\frac{C_1}{2\pi\epsilon_0}$
2	1.4423	1.444
4	0.721	0.733
10	0.434	0.475
100	0.267	0.346
1.5	2.4662	2.4667
1.1	10.492070	10.492065

Table 7.2

The new approximation C_2 is much better than C_1 for all values of b/a.

In this example, we knew the correct result for the capacitance. We thus used the least action principle just to confirm it is right. However, for condensers of very complicated shape it could be extremely hard to find a correct analytical solution from the strong Poisson problem. The least action principle, instead, gives a relatively easy way to guess a reasonable approximation for the capacitance.