PHYS 342 - EM 02

Lecture 4

4.1 On the Vector Potential of $\vec{B}$

In PHYS 242 we saw that a complete definition of a vector potential $\vec{A}$ for a magnetostatic field $\vec{B}$ comprises a set of two equations:

$$\vec{\nabla} \times \vec{A} = \vec{B} \tag{4.1a}$$

$$\vec{\nabla} \cdot \vec{A} = 0 \tag{4.1b}$$

To solve these equations, we must also specify how $\vec{A}$ behaves on some boundary, or at large distances. For example, given a current density $\vec{J}$ limited in space, we can assume a field that goes to zero at infinite.

4.1.1 Vector Potential for a Uniform Field $\vec{B}$

Consider a uniform magnetostatic field $\vec{B}_0$ along the z-axis of a Cartesian coordinate system. From Eq. (4.1a),

$$\begin{cases} B_x = \dfrac{\partial}{\partial y} A_z - \dfrac{\partial}{\partial z} A_y = 0 \\ B_y = \dfrac{\partial}{\partial z} A_x - \dfrac{\partial}{\partial x} A_z = 0 \\ B_z = \dfrac{\partial}{\partial x} A_y - \dfrac{\partial}{\partial y} A_x = B_0 \end{cases} \tag{4.2}$$

By inspection, we find one possible solution of these equations:

$$A_x = 0, \qquad A_y = B_0 x, \qquad A_z = 0 \tag{4.3a}$$

Another possible solution is:

$$A_x = -B_0 y, \qquad A_y = 0, \qquad A_z = 0 \tag{4.3b}$$

A linear combination of (4.3a) and (4.3b) is also a valid solution:

$$A_x = \beta(-B_0 y), \qquad A_y = \alpha B_0 x, \qquad A_z = 0 \tag{4.3c}$$

with α, $\beta \in \mathbb{R}$. For a given $\vec{B}$, $\vec{A}$ is not unique.

When $\alpha = \beta = 1/2$, from (4.3c) it follows that the x-component of $\vec{A}$ is proportional to $-y$ and the y-component to x. The parametric equations of a unit circle with center on the z-axis and parallel to the xy-plane are

$$\begin{cases} x = \cos t \\ \\ y = \sin t \end{cases}, t \in [0, 2\pi) \tag{4.4}$$

that is, a point P on the circle can be represented by the vector

$$(\cos t)\vec{u}_x + (\sin t)\vec{u}_y \tag{4.5}$$

By deriving Eqs. (4.4) with respect to t, we obtain

$$\begin{cases} \dot{x} = -\sin t \\ \\ \dot{y} = \cos t \end{cases} \tag{4.6}$$

From which it follows the tangent vector to the circle at P is

$$(-\sin t)\vec{u}_x + (\cos t)\vec{u}_y \tag{4.7}$$

For a circle of radius r, the coordinates x and y in Eqs. (4.3c) are

$$\begin{cases} x = r\cos t \\ \\ y = r\sin t \end{cases} \tag{4.8}$$

from which it follows that the components of $\vec{A}$ in (4.3c) are those of a tangent vector to the points of the circle of radius r.

As a consequence, $\vec{A}$ must be at right angles to vector $\vec{r}$ (cf. Fig. 4.1).

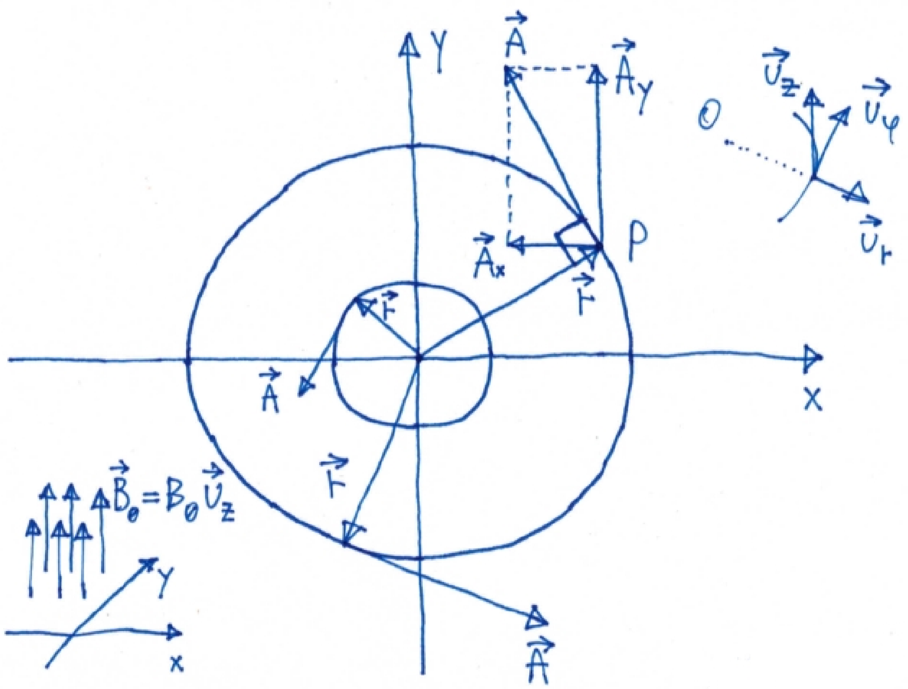


Figure 4.1

For $\alpha = \beta = 1/2$, the magnitude of $\vec{A}$ is

$$||\vec{A}|| = \frac{1}{2}\, B_0 \sqrt{x^2 + y^2} = \frac{1}{2}\, B_0\, r \tag{4.9}$$

Hence, for a uniform field $\vec{B}_0$

$$\vec{A} = \frac{1}{2}\, \vec{B}_0 \times \vec{r} = \frac{1}{2}\, B_0\, r\, \vec{u}_\varphi \tag{4.10}$$

where $\vec{u}_\varphi$ is the tangent unit vector of a cylindrical coordinate system $Or\varphi z$. The vector potential $\vec{A}$ has the magnitude $B_0\, r/2$ and rotates about the z-axis as shown in Fig. 4.1. For example, if $\vec{B}$ is the axial field inside a long, straight solenoid, $\vec{A}$ circulates in the same sense as do the currents of the solenoid.

In cylindrical coordinates

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{r}\, \frac{\partial}{\partial r}\, (r A_r) + \frac{1}{r}\, \frac{\partial}{\partial \varphi}\, A_\varphi + \frac{\partial}{\partial z}\, A_z \tag{4.11}$$

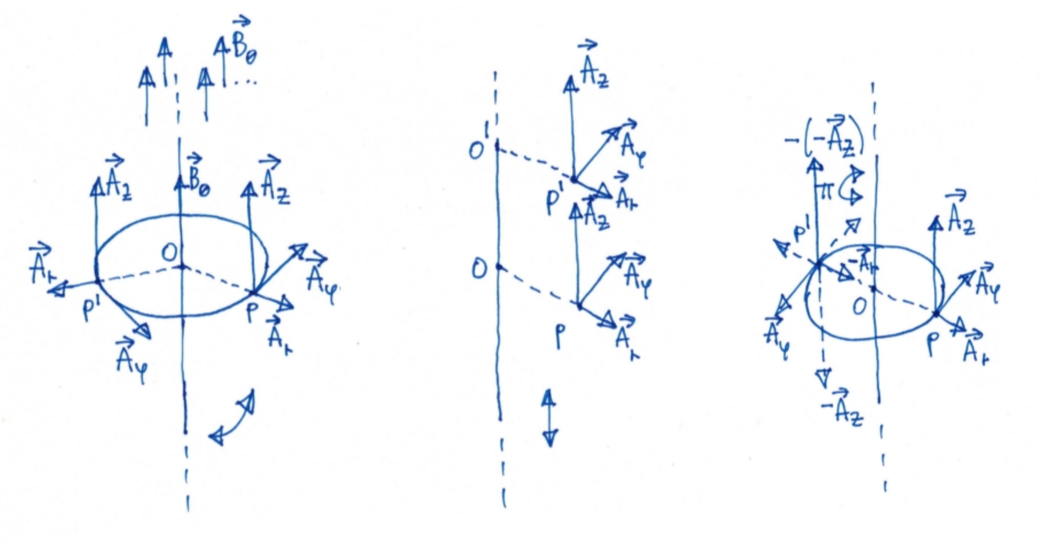
The only component of $\vec{A}$ in (4.10) is $\vec{A}_\varphi$, which does not depend on φ. Thus, the vector potential for a uniform field $\vec{B}_0$ fulfils condition (4.1b).

The vector potential for a uniform field can be obtained in another way. Given a surface Σ_γ with border γ, from Stokes' theorem

$$\begin{aligned} \Phi_\gamma = \iint\limits_{\Sigma_\gamma} \vec{B} \cdot \vec{n} dA \quad &= \iint\limits_{\Sigma_\gamma} (\vec{\nabla} \times \vec{A}) \cdot \vec{n} dA \\ &= \oint\limits_{\gamma} \vec{A} \cdot \vec{t} d\ell \end{aligned} \tag{4.12}$$

In the case of $\vec{B}_0$, we can attempt to use Eq. (4.12) to find $\vec{A}$. In general, we must assume $\vec{A} = A_r(r, \varphi, z)\vec{u}_r + A_\varphi(r, \varphi, z)\vec{u}_\varphi + A_z(r, \varphi, z)\vec{u}_z$, where we represented $\vec{A}$ in $Or\varphi z$.

The configuration of $\vec{B}_0$ gives rise to three symmetries for A: Rotational, translational, and anti-reflectional symmetry (cf. Fig. 4.2).



(a) (b) (c)

Figure 4.2

Because of the anti-reflectional symmetry shown in Fig. 4.2c, $\vec{A}_r(P)$ gets transformed into $-\vec{A}_r(P')$. This is because upon rotating the z-axis by $\mp\pi$ about O, $\vec{B}_0$ changes sign. Thus, in order to keep the same field configuration, the sign of all vectors of interest must be flipped. Applying the rotational symmetry from P' to P in Fig. 4.2c, we find that $\vec{A}_r = \vec{0}$. Note that if the pivot for the rotation of the z-axis was $0' \neq 0$, we should have applied both the rotational and translational symmetry to go back to P in Fig. 4.2c. The symmetry argument also show that the components $\vec{A}_\varphi$ and $\vec{A}_z$ can both exist and, if they do, they can only depend on r.

From Eq. (4.12), assuming γ to be a circle of radius r and center on the z-axis, we obtain

$$\begin{aligned}
\Phi_\gamma &= \iint\limits_{\Sigma_\gamma} B_0 \vec{u}_z \cdot \vec{u}_z dA = B_0 \pi r^2 \\
&= \oint\limits_{\gamma} \big(A_\varphi(r)\vec{u}_\varphi + A_z(r)\vec{u}_z\big) \cdot \vec{u}_\varphi d\ell \\
&= A_\varphi(r) 2\pi r \qquad (4.13)
\end{aligned}$$

Thus,

$$\vec{A}_\varphi = \frac{1}{2} B_0 \, r \, \vec{u}_\varphi \qquad (4.14)$$

confirming the result of Eq. (4.10). However, nothing can be said on $A_z(r)\vec{u}_z$.

We can try to impose conditions (4.1a) and (4.1b) and learn more about $\vec{A}_z$. From the definition of curl in cylindrical coordinates,

$$\begin{aligned}
\vec{\nabla} \times \vec{A} &= \left(\frac{1}{r}\frac{\partial}{\partial\varphi} A_z - \frac{\partial}{\partial z} A_\varphi\right)\vec{u}_r \\
&+ \left(\frac{\partial}{\partial z} A_r - \frac{\partial}{\partial r} A_z\right)\vec{u}_\varphi \\
&+ \frac{1}{r}\left[\frac{\partial}{\partial r}(r\, A_\varphi) - \frac{\partial}{\partial\varphi} A_r\right]\vec{u}_z \qquad (4.15)
\end{aligned}$$

from

$$\vec{A} = \frac{1}{2} B_0 \, r \, \vec{u}_\varphi + A_z(r)\vec{u}_z \qquad (4.16)$$

and from

$$\vec{\nabla} \times \vec{A} = B_0 \vec{u}_z \qquad (4.17)$$

We find

$$\begin{cases} -\dfrac{\partial}{\partial r} A_z(r)\vec{u}_\varphi = \vec{0} \\ B_0\vec{u}_z = B_0\vec{u}_z \end{cases} \qquad (4.18)$$

From which it follows that if a component A_z exists it must not depend on r. Hence, if A_z exists has to be a constant. Nothing new is learnt from Eq. (4.1b), which is fulfilled even if $A_z = \text{const}$. As always, when using symmetry arguments one has to be particularly careful when making assumptions. In the example of a uniform magnetostatic field, the symmetry arguments do not allow us to assume that $A_z = 0$.

Thus, the approach to find $\vec{A}$ based on symmetries and on Eq. (4.12) is less general than that based on solving Eq. (4.1a).

In the example we have just given, we have calculated $\vec{A}$ from $\vec{B}$. This is the opposite of what one normally does.

In PHYS 242 and in Lecture 2, we have seen that, given a current density $\vec{J}$ in τ,

$$\vec{A}(P) = \frac{\mu_0}{4\pi} \iiint\limits_{\tau} \frac{\vec{J}(Q)}{r_{QP}} \, dV \tag{4.19}$$

We remind that here $\vec{J}$ is assumed to be a stationary (or steady) current, i.e., $\vec{\nabla} \cdot \vec{J} = 0$. Equation (4.19) means that we can find each component of $\vec{A}$ by solving three imaginary electrostatic problems for the charge distributions $\rho_1(Q) = J_x(Q)/c^2$, $\rho_2(Q) = J_y(Q)/c^2$, and $\rho_3(Q) = J_z(Q)/c^2$, where $c \equiv 1/\sqrt{\mu_0 \epsilon_0}$.

4.1.2 Magnetostatic field of an infinite, straight wire revisited

We solved this problem in PHYS 242 and again in Tutorial 1, where we used Ampère's law and symmetry arguments. As before, we take an infinite, straight wire carrying a steady and uniform current I along the bottom-top direction of the wire, assumed to be an ideal filiform wire.

In order to solve this problem by means of the vector potential, at first we must assume the wire to have a finite cross-section of radius R and a cylindrical shape.

This assumption allows us to define a current density, which would by ill-defined for an ideal filiform conductor (where $R \to 0^+$). We will solve the problem for each point outside the cylinder and, at the end, impose $R \to 0^+$.

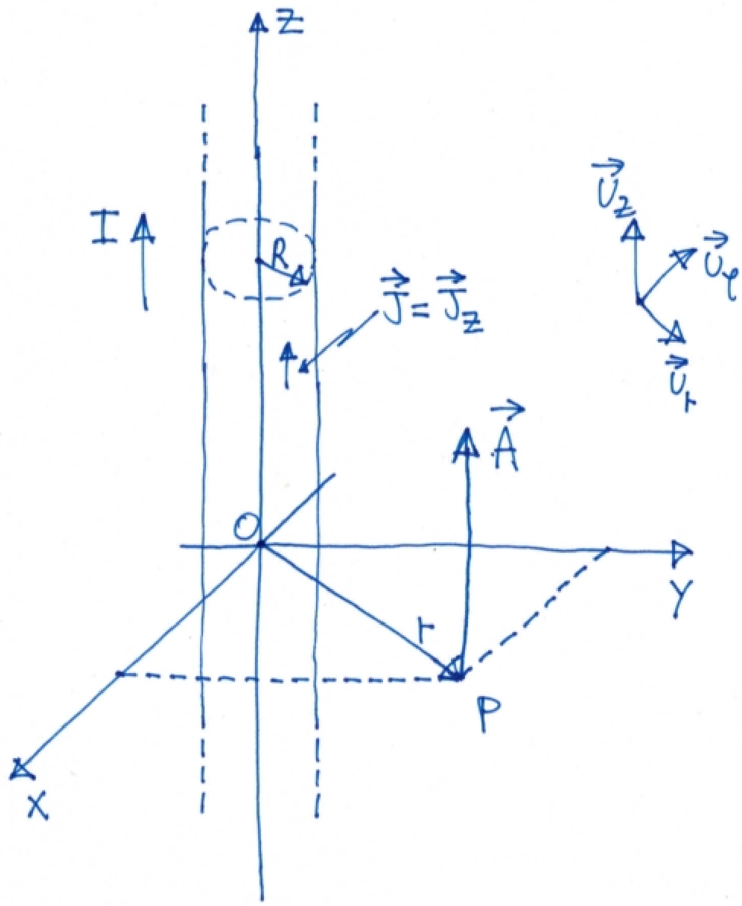


Figure 4.3

Following the notation in Fig. 4.3, and due to the uniformity assumption on I, it is easy to convince oneself that

$$\begin{cases} J_x = J_y = 0; \; J_z = \dfrac{I}{\pi R^2} \quad , \quad r \in [0, R] \\ J_x = J_y = J_z = 0 \quad , \quad r \in (R, +\infty) \end{cases} \tag{4.20}$$

with $r = \sqrt{x^2 + y^2}$. The current source points Q in Eq. (4.19) are for $r \in [0, R]$, where $\vec{J} = (I/\pi R^2)\vec{u}_z$. From Eqs. (4.20), it is straightforward to find the three components of the vector integral of Eq. (4.19) for each point P outside the cylindrical conductor, i.e., for $r \in (R, +\infty)$:

$$A_x = 0 \tag{4.21a}$$

$$A_y = 0 \tag{4.21b}$$

$$A_z = \frac{\mu_0}{4\pi} \iiint\limits_{cylinder} \frac{J_z}{r_{QP}} \, dV \tag{4.21c}$$

We encountered an integral similar to that of Eq. (4.21c) when calculating the electrostatic field $\vec{E}$ and potential ϕ of a wire as the one in Fig. 4.3, however, with a uniform charge distribution with volume density ρ. In that case, we found the solution by means of Gauss' theorem and symmetry arguments. In the special case $R \to 0^+$ (i.e., $\rho \to \lambda$, a linear charge density), we also found the solution by direct integration. We found (cf. Appendix 4.2)

$$\phi = -\frac{\lambda}{2\pi\epsilon_0} \ln r, \qquad r \in (R, +\infty) \tag{4.22}$$

where, in general, $\lambda = \pi R^2 \rho$.

The component A_z can then be obtained by assuming a fictitious volume charge density $\rho = J_z/c^2$ in (4.22). We finally find

$$A_z = -\frac{\pi R^2 J_z}{2\pi\epsilon_0 c^2} \ln r = -\frac{\mu_0 I}{2\pi} \ln r, \qquad r \in (R, +\infty) \tag{4.23}$$

Note that, assuming $R \to 0^+$ (i.e., ideal filiform condition) does not affect the last expression of Eq. (4.23), which, thus, can be used for the ideal filiform wire originally considered in this problem:

$$A_z = -\frac{\mu_0 I}{2\pi} \ln r, \qquad r \in (0, +\infty) \tag{4.23'}$$

From Eq. (4.1a), the definition of curl in Cartesian coordinates, and (4.23')

$$\begin{cases} B_x &= \dfrac{\partial}{\partial y} A_z - \dfrac{\partial}{\partial z} A_y = -\dfrac{\mu_0 I}{2\pi}\dfrac{\partial}{\partial y} \ln\sqrt{x^2+y^2} \\ &= -\dfrac{\mu_0 I}{2\pi}\dfrac{y}{r^2} \\ B_y &= \dfrac{\partial}{\partial z} A_x - \dfrac{\partial}{\partial x} A_z = \dfrac{\mu_0 I}{2\pi}\dfrac{x}{r^2} \\ B_z &= 0 \end{cases} \tag{4.24}$$

As for the case of Eqs. (4.3c) for $\alpha = \beta = 1/2$, the components B_x and B_y in Eqs. (4.24) are those of a vector tangent to each point of a circle of radius r, center on the z-axis, and parallel to the xy-plane, that is

$$\vec{B} = \frac{\mu_0 I}{2\pi}\frac{1}{r}\vec{u}_\varphi, \qquad r \in (0, +\infty) \tag{4.25}$$

This is exactly the same result found from Ampère's law and symmetry arguments.

Appendix 4.1.

The electrostatic potential at a generic point P due to a charge distribution with volume density ρ defined at each point Q in a region Ω is

$$\phi(P) = \frac{1}{4\pi\epsilon_0} \iiint_\Omega \frac{\rho(Q)}{r_{QP}} dV \tag{4.1A}$$

The three Cartesian components of the vector integral of Eq. (4.19) can be cast into Eq. (4.1A) by noting that

$$\frac{1}{4\pi\epsilon_0}\rho(Q) = \frac{\mu_0}{4\pi} J_i(Q) \tag{4.2A}$$

with $i = x, y, z$. That is

$$\rho(Q) = \mu_0 \epsilon_0 J_i(Q) = \frac{J_i(Q)}{c^2} \tag{4.3A}$$

Same equations have same solutions: Magnetostatic problems can be thus solved from known electrostatic ones.

Appendix 4.2.

Note that the electrostatic potential of Eq. (4.22) is not well-defined because the argument of the logarithm is not dimensionless. In general, we should have defined

$$\phi = -\frac{\lambda}{2\pi\epsilon_0} \, ln \, \frac{r}{r_0} \tag{4.4A}$$

with r_0 a constant with units m. This is equivalent to

$$\phi = -\frac{\lambda}{2\pi\epsilon_0} \, (ln \, r - ln \, r_0) \tag{4.4A'}$$

The constant part of this expression vanishes every time we calculate a derivative to find $\vec{B}$ from $\vec{A}$. Thus, we decided to neglect it in (4.22).