ECE 106 - Tutorial

Quiz #2, Problem 2

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June 2, 2015

Contents

Quiz 2, Problem 2 **3**
Approach 1 3
Approach 2 6

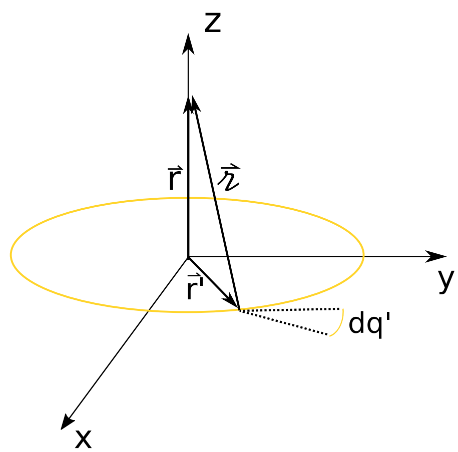

Figure 1: Ring of charge centered along z-axis

Quiz 2, Problem 2

Okay, the problem of solving for the z-component of the electric field generated by a charged disk seemed to be difficult for you guys. Let me discuss a few ways in which this problem could be solved.

The first approach that I will use will be to establish the electric field set up by a ring centered around the z-axis. Then, I will integrate over an infinite number of rings whose radius varies smoothly from 0 to the radius of the disk.

The second approach that I will use will be to establish the electric field from a disk directly using polar coordinates. I will integrate around the disk (in the angular direction) and I will integrate radially outward (from 0 to R, the radius of the disk). This involves the use of the differential area alement $\mathrm{d}a$ in polar coordinates. I will try my best to explain the origins of this differential area element. That may go in an appendix at the end.

Approach 1

Okay, so I'm going to first start by solving for the electric field established by a uniformly charged ring. The ring will have a constant charge per unit length, λ, such that the total charge on the ring is $2\pi R\lambda = Q$. My approach will be to chop up the ring into a bunch of infinitesimal point charges $\mathrm{d}q'$ and evaluate the electric field, $\mathrm{d}\vec{E}$, generated by each of these infinitesimal point charges. Once I have this contribution to the electric field I will integrate over all the infinitesimal point charges to find the net electric field $\vec{E} = \int \mathrm{d}\vec{E}$.

The key to starting to write this integral is to realize that each infinitesimal check of charge $\mathrm{d}q'$ is like a little point charge located at position r' (see figure 1) and, as such, the electric field contribution due a particular chunk of charge, $\mathrm{d}q'$, looks like:

$$\mathrm{d}\vec{E}(\vec{r}) = \frac{k\,\mathrm{d}q'}{|\vec{\mathscr{r}}|^2}\hat{\mathscr{r}}$$

Now, $\mathrm{d}q'$ really describes the position of a chunk of charge $\mathrm{d}q'$ located at $\vec{r'}$. The infinitesimal chunk of charge can be written as $\mathrm{d}q' = \lambda\,\mathrm{d}l'$, where $\mathrm{d}l'$ is an infinitesimal arc length along the path of the ring.

Thus, my expression for $\mathrm{d}\vec{E}(\vec{r})$ looks like

$$\mathrm{d}\vec{E}(\vec{r}) = \frac{k\lambda\,\mathrm{d}l'}{|\vec{\mathscr{r}}|^2}\hat{\mathscr{r}}$$

But, $\mathrm{d}l'$ just represents a little arc length. We know that an arc length s can be related to the radius of the circle and the swept (subtended) angle, θ can be written as $s = R\theta$. Thus, I will write the arc length $\mathrm{d}l'$ as $\mathrm{d}l' = R\,\mathrm{d}\theta'$ so that $\mathrm{d}\vec{E}(\vec{r})$ becomes:

$$\mathrm{d}\vec{E}(\vec{r}) = \frac{k\lambda R\,\mathrm{d}\theta'}{|\vec{\mathscr{r}}|^2}\hat{\mathscr{r}}$$

Now, our expression is looking more and more like something we can integrate. However, we should really right $\vec{\mathscr{r}}$ in terms of something we know. What is $\vec{\mathscr{r}}$? Remember, this vector points from the point where the charge that we're considering is located to the point where we would like to evaluate the electric field. Thus, in terms of $\vec{r}$, the vector that points from the origin to where we would like to evaluate the electric field and $\vec{r'}$, the vector that points from the origin to the chunk of charge we're considering $\vec{\mathscr{r}} = \vec{r} - \vec{r'}$. In order to put this into my expression for $\mathrm{d}\vec{E}$ I need to scale thetope and bottom by the magnitude of $\vec{\mathscr{r}}$ to get rid of that pesky unit vector. Doing this yields:

$$\mathrm{d}\vec{E}(\vec{r}) = \frac{k\lambda R\,\mathrm{d}\theta'}{|\vec{\mathscr{r}}|^3}\vec{\mathscr{r}}$$

Performing my substitution for $\vec{\mathscr{r}}$ yields:

$$\mathrm{d}\vec{E}(\vec{r}) = \frac{k\lambda R\,\mathrm{d}\theta'}{|\vec{r} - \vec{r'}|^3}(\vec{r} - \vec{r'})$$

Now, can we simplify this expression at all before we integrate? Yes! We know that $\vec{r} = z\hat{z}$ since we are only solving for the electric field along the z-axis. We also know that $|\vec{\mathscr{r}}|$ is $\sqrt{R^2 + z^2}$ (look at 1) Performing these substitutions yields:

$$\mathrm{d}\vec{E}(z) = \frac{k\lambda R\,\mathrm{d}\theta'}{(R^2 + z^2)^{1.5}}(z\hat{z} - \vec{r'})$$

Now, we can almost perform this integral. We just need an expression for $\vec{r'}$ in terms of θ' so that we can perform the integral over θ'. But, $\vec{r'} = R\cos(\theta')\hat{x} - R\cos(\theta')\hat{y}$. This can be seen by considering how $\vec{r'}$ must change direction as a function of θ'. Now, we just plug this into our previous expression and integrate over θ' from $0 \to \pi/2$.

$$\vec{E}(z) = \int_0^{2\pi} k\lambda R\frac{\mathrm{d}\theta'}{(R^2 + z^2)^{1.5}}(z\hat{z} - R\cos(\theta')\hat{x} + R\cos(\theta')\hat{y})$$

Now, k and λ are constants that I can pull out of the integral. The integrals over $\cos\theta'$ and $\sin\theta'$ I already know are going to go away. I know this because $\int_0^{2\pi}\cos\theta\,\mathrm{d}\theta = \int_0^{2\pi}\sin\theta\,\mathrm{d}\theta = 0$. Thus, only the $\hat{z}$ component survives the integral (like we'd expect) and the integral reduces to:

$$\vec{E}(z) = \int_0^{2\pi} k\lambda R\frac{\mathrm{d}\theta'}{(R^2 + z^2)^{1.5}}z\hat{z}$$

Notice that everything inside the integral is a constant with respect to θ' except for our integration differential $\mathrm{d}\theta'$. Thus, the integral is just:

$$\vec{E}(z) = \frac{k\lambda R 2\pi z\hat{z}}{(R^2 + z^2)^{1.5}}$$

This looks a little ugly. Once we realize that $2\pi R\lambda$ is just Q, thet total charge on the ring (the charge per unit length on the ring times the total length of the ring) we can simplify this to:

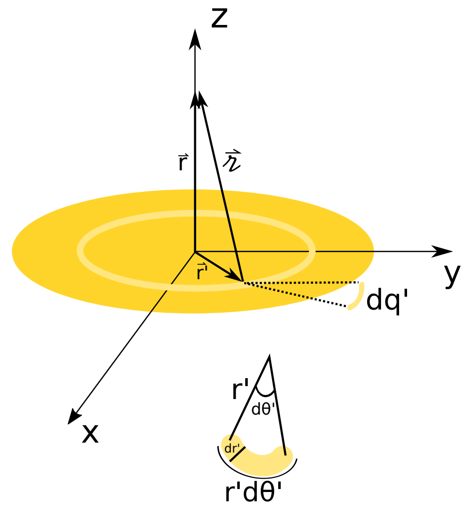

Figure 2: Disc of charge broken up into rings

$$\vec{E}(z) = \frac{kQ}{(R^2 + z^2)^{1.5}} z\hat{z}$$

Note that this expression depends on the radius of the ring. Our next goal will be to integrate over a bunch of rings with different radii to obtain the electric field along the z-axis due to a charged disc. Thus, I'm going to chop the electric field for the disc up into a bunch of rings.

$$\mathrm{d}\vec{E}(z) = \frac{kdQ}{(R^2 + z^2)^{1.5}} z\hat{z}$$

What is dQ? Well, dQ is me treating each ring as a chunk of charge that generates a little portion of the total electric field. Now, since I'm integrating over rings to form a disc I need to give these rings some infinitesimal width. The disc will be formed of a bunch of rings of infinitesimal width. A ring of infinitesimal width and of radius R' will have an amount of charge equal to $2\pi R' dR' \sigma$ (see 2 and allow $\mathrm{d}\theta'$ to be 2π - you might argue that this is not an infinitesimal quantity... well it turns out that this doesn't matter. It's still the area of that ring: $2\pi r' dr'$). What is σ? It's the surface charge density. Since we're considering areas of charge, now, we need to consider area charge densities. Think about whether a relationship can be drawn between λ and σ. Let's try to write down the integral now that we know what dQ is.

$$\vec{E}(z) = \int_0^{R_{disc}} \mathrm{d}\vec{E}(z) = \int_0^{R_{disc}} \frac{k 2\pi R' dR' \sigma}{(R'^2 + z^2)^{1.5}} z\hat{z}$$

Now, this integral is not hard to solve. I'll pull out the constants so that the math is a little more transparent.

$$\vec{E}(z) = \int_0^{R_{disc}} \mathrm{d}\vec{E}(z) = 2\pi\sigma z\hat{z} \int_0^{R_{disc}} \frac{R' dR'}{(R'^2 + z^2)^{1.5}} \tag{1}$$

Now, I solved this using WolframAlpha but this type of integral can be solved with a simple u-substitution (allow $u = R'^2 + z^2$ so that $\frac{\mathrm{d}u}{\mathrm{d}r} = 2R'dR'$).

The final result, after doing all of this is that:

$$\vec{E}(z) = k\sigma 2\pi \left(1 - \frac{z}{z^2 + R_{disc}^2}\right)\hat{z}$$

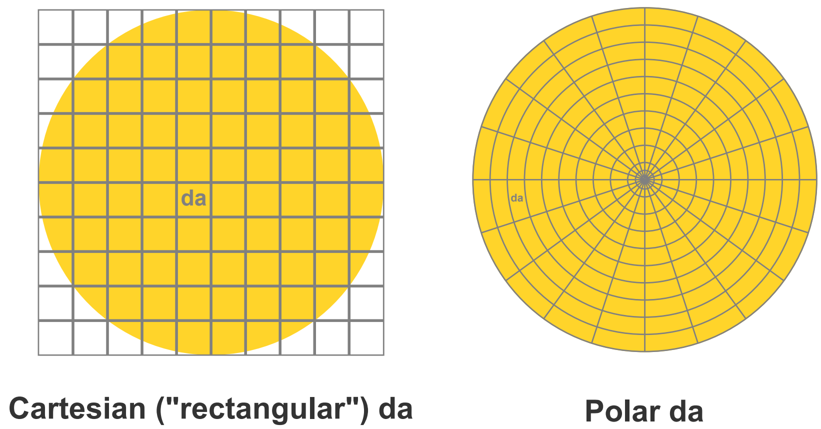

Figure 3: Differential area element in Cartesian coordinates vs Polar Coordinates

Approach 2

Now, that was super tedious. Let's try this another way. Let's now break the disc up into a little bunch of patches of area da instead of a bunch of rings of infinitesimal thickness. If we do this, we need to think about how best to think about the area of the disc.

If we break the disc up into a bunch of rectangular areas, da, like is done with Cartesian coordinates we will have a difficult time integrating over our disc (see 3. Our bounds will have to be over the area of the disc which require that we express the limits of our integral in terms of the equation of the circle. I don't really want to deal with $y = \pm\sqrt{1 - x^2}$ in my limits.

Instead, let's consider the patches of area to be little arcs of some finite thickness. See 2 for a decent visual explanation. Now, it's easy to break up the area of our circle into little chunks. We will talk about a little chunk of area $da = (R\,\mathrm{d}\theta)(\mathrm{d}R)$. That is, the little chunk will have an arclength that is $R\,\mathrm{d}\theta$ long and $\mathrm{d}R$ wide. That determines the area of this patch. You can convince yourself, easily, that this is a great way to think about areas when it comes to dealing with problems with circles in them. Let's integrate the area of a circle with these coordinates:

$$\begin{aligned}
A &= \int_0^R da \\
&= \int_0^R \int_0^{2\pi} r\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_0^R r\,\mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta \\
&= \frac{r^2}{2}\Big|_0^R \quad \theta\Big|_0^{2\pi} \\
&= \pi R^2
\end{aligned}$$

So, although this isn't a proof, it should at least be convincing evidence that maybe this is a good expression to use for the differential area element when we're dealing with circles. Now, I'm going to immediately write the integral we're going to solve since the approach is very similar to what we've already done.

$$\vec{E}(z) = \int_0^{2\pi} \int_0^R \frac{kdq'(z\hat{z} - r'\cos(\theta')\hat{x} + r'\cos(\theta')\hat{y})}{(z^2 + r'^2)^{1.5}}$$

Now, r' describes the distance the infinitesimal chunk of charge is away from the origin. θ describes the location of that chunk with respect to the x-axis. Thus, by sweeping over all $\theta' : 0 \to 2\pi$ and all $r' : 0 \to R$ we sweep over all the charge. Now, we just have to right dq' in terms of r' and θ' and we're good. But, $dq' = \sigma \,\mathrm{d}a' = \sigma r' dr' \,\mathrm{d}\theta'$, from the earlier discussion. The integral now reads:

$$\vec{E}(z) = \int_0^{2\pi} \int_0^R \frac{k\sigma r' dr' \,\mathrm{d}\theta' (z\hat{z} - r'\cos(\theta')\hat{x} + r'\cos(\theta')\hat{y})}{(z^2 + r'^2)^{1.5}}$$

Right away I'm going to realize that the integral over the $\hat{x}$ and the $\hat{y}$ components go away because I'm integrating $\sin\theta'$ and $\cos\theta'$ over one period (from 0 to 2π).

$$\vec{E}(z) = \int_0^{2\pi} \int_0^R \frac{k\sigma r' dr' \,\mathrm{d}\theta' z\hat{z}}{(z^2 + r'^2)^{1.5}}$$

Now, the only θ' in my integral is the $\mathrm{d}\theta'$. So, I can evaluate the integral over $\mathrm{d}\theta'$ very easily.

$$\vec{E}(z) = 2\pi \int_0^R \frac{k\sigma r' dr' z\hat{z}}{(z^2 + r'^2)^{1.5}}$$

I then realize that this is the same integral as the one I was doing earlier (see Equation 1). Thus, the result must be the same. This is an easier, less confusing way to get the electric field for a disc along the z-axis.
