Lecture 11

In Lecture 10, we studied the forces acting on electrically charged conductors. We will now consider a similar problem, but for current-carrying circuits in magnetic fields.

11.1 Forces on Electric Current-Carrying Circuits in an External Magnetic Field

Consider a quasi-filiform circuit carrying a current I in a region of space characterized by an external magnetic field $\vec{B}_0$. Consider a circuit element (i.e., a piece) of length $\Delta\ell$, much smaller than the total length of the circuit. The circuit element defines two cross sections with area A_1 and A_2, respectively, as shown in Fig. 11.1

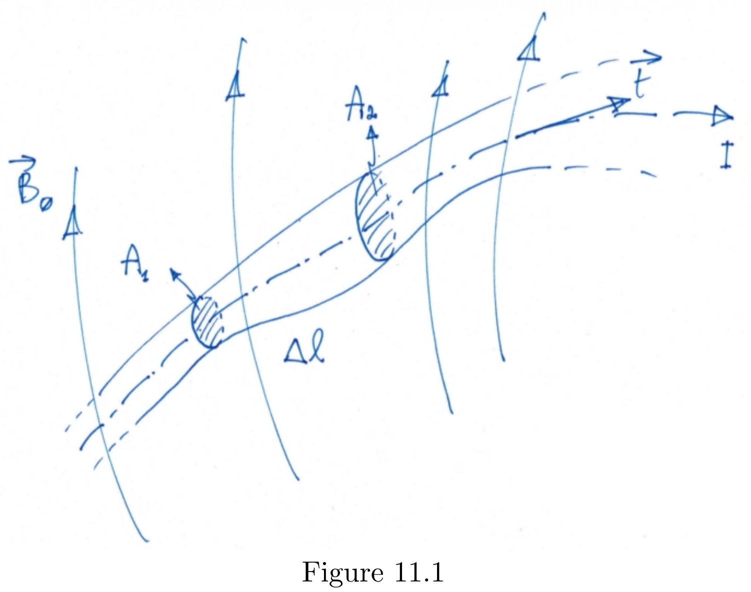

Figure 11.1

Imagine to be able to observe the motion of the charge carries in the conductor. We indicate with q the value of electric charge of each carrier and with $\vec{v}$ its velocity. Note that such a velocity corresponds to the ordered motion which determines the current. If $\vec{B}$ is the net magnetostatic field in the considered circuit element ($\vec{B}$ is due to $\vec{B}_0$ as well as I), the force acting on each moving charge carrier is (from Lorentz's force)

$$\vec{F}_q = q\vec{v} \times \vec{B} \tag{11.1}$$

As we saw in PHYS 242, this force gets transferred to the lattice of fixed charges in the conductor (i.e., $\vec{F}_q$ is a pondermotive force). Thus, $\vec{F}_q$ acts globally on the entire conductor.

We now want to calculate the force $\Delta\vec{F}$ acting on the conductor element of length $\Delta\ell$. To this end, we imagine to take a snapshot at a certain instant in time of the charge carriers existing in $\Delta\ell$. The force $\Delta\vec{F}$ is the net of the forces acting on the carriers at the given time. Assuming that all charge carriers have the same velocity $\vec{v}$ we have

$$\Delta\vec{F} = q\vec{v} \times \vec{B}\Delta n \tag{11.2}$$

where Δn is the number of carriers in $\Delta\ell$ at the considered time instant. If Δt is the time interval necessary for each carrier to travel through $\Delta\ell$, we have

$$\vec{v} = \frac{\Delta\ell}{\Delta t}\vec{t} \tag{11.3}$$

where $\vec{t}$ is the unit vector along the central longitudinal axis of the circuit (cf. Fig. 11.1). As a consequence, we can write (11.2) as

$$\Delta\vec{F}q\,\frac{\Delta\ell}{\Delta t}\vec{t} \times \vec{B}\Delta n \tag{11.4}$$

It must be that the carriers going through A_2 in the time window Δt are equal to all and only those contained within the piece of length $\Delta\ell$. Therefore, the total charge crossing A_2 during Δt is

$$\Delta q = q\Delta n \tag{11.5}$$

The current is thus

$$I = \frac{\Delta q}{\Delta t} = q\,\frac{\Delta n}{\Delta t} \tag{11.6}$$

and (11.4) becomes

$$\Delta\vec{F} = I\Delta\ell\,\vec{t} \times \vec{B} \tag{11.7}$$

In the limit for $\Delta\ell \to 0^+$ we obtain the expression for the infinitesimal force acting on an infinitesimal element $d\ell$ of a conductor carrying a current I in a magnetostatic field $\vec{B}$:

$$d\vec{F} = Id\ell\vec{t} \times \vec{B} \tag{11.8}$$

This force is normal to both the field $\vec{B}$ and the conductor itself; the force is directed according to a corkscrew rotation from $Id\ell\vec{t}$ to $\vec{B}$ by the smallest angle.

In order to obtain the net force acting on the entire current-carrying circuit, it is sufficient to sum up all contributions relative to each infinitesimal element, obtaining

$$\vec{F} = \oint_\gamma I\vec{t} \times \vec{B}d\ell \tag{11.9}$$

where γ is a closed oriented line corresponding to the central (main) longitudinal axis of the circuit (indeed, any other longitudinal axis would work as well for a quasi-filiform conductor).

Equation (11.9) can be used, for example, to determine the dynamics of a rigid circuit under the action of the magnetostatic field. In this case, the net force can be calculate from (11.9) considering only the external field $\vec{B}_0$. In fact, the field generated by I itself can only result in a zero force on the circuit. This is similar to a single charge, which cannot produce a nonzero force on itself.

The scenario changes drastically when considering a non-rigid circuit, when studying the deforming effects on the circuit due to forces of type (11.9). In this case, we must determine the distribution of forces acting on each infinitesimal circuit element, taking into account the effective field $\vec{B}$ acting on that element. Such a field is the sum of the field generated by external electric currents and that due to the current on the circuit itself.

We now want to consider the general case of a bulk current-carrying conductor. To this end, we can consider a generic piece of the bulk conductor, assuming to know the current density distribution $\vec{J}$ and the magnetostatic field $\vec{B}$ within the piece. Given a generic flux tube for vector $\vec{J}$, consider an infinitesimal element of right-angle cross-section dA and length $d\ell$. Assuming that such a flux tube is a filiform conductor carrying the infinitesimal current

$$dI = \vec{J} \cdot \vec{n} dA \tag{11.10}$$

where $\vec{n}$ is the normal unit vector to dA and is parallel to $\vec{J}$, by means of Eq. (11.8) we obtain

$$d\vec{F} = (\vec{J} \cdot \vec{n} dA) d\ell \vec{t} \times \vec{B} \tag{11.11}$$

Since $\vec{J}$ and $\vec{n}$ have the same direction and $\vec{t} = \vec{n}$, we have

$$d\vec{F} = dA d\ell \vec{J} \times \vec{B} \tag{11.12}$$

By defining $dV = dA d\ell$ the volume of the infinitesimal flux tube, we have

$$\vec{f} = \frac{d\vec{F}}{dV} = \vec{J} \times \vec{B} \tag{11.13}$$

This result gives the force per unit volume acting at a generic point in the conductor. The net force acting on a finite piece of conductor is sufficient to sum up the contributions due to each infinitesimal volume element, obtaining

$$\vec{F} = \iiint_{\tau} (\vec{J} \times \vec{B}) dV \tag{11.14}$$

where τ is the region of conductor under consideration.

11.2 Force Between Two Straight, Indefinite (Infinitely Long) Current-Carrying Conductors, Parallel to Each Other

Consider two straight and indefinite filiform conductors. Assume the conductors are parallel to each other and placed at a distance d from each other, in vacuum. Furthermore, assume they carry steady currents I_1 and I_2, respectively. Figure 11.2 shows a sketch of the problem under consideration.

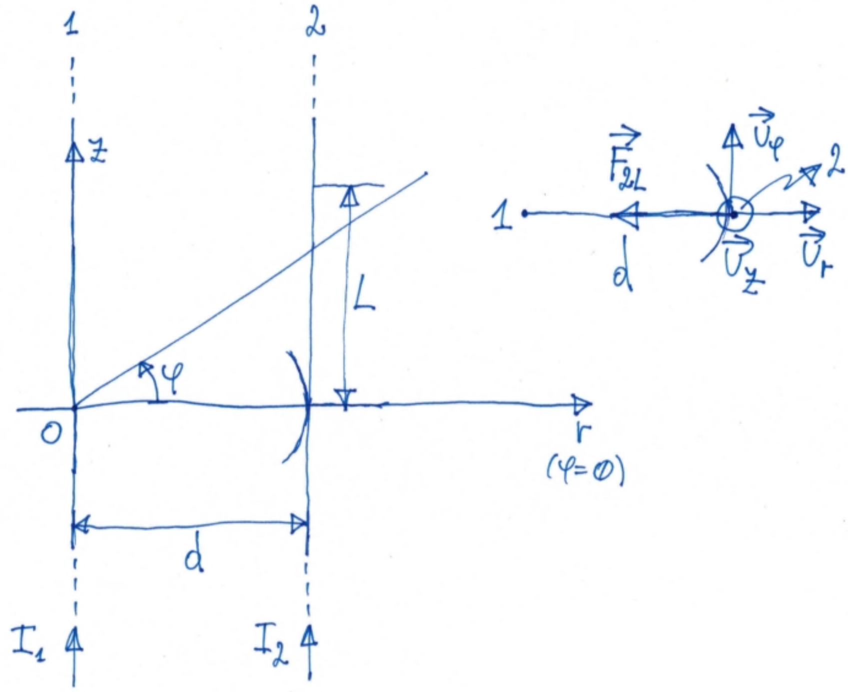

Figure 11.2

We intend to calculate the force acting on each of the two conductors due to the current carried by the other conductor.

We will focus, for simplicity, on the force acting on the second conductor (similar arguments would apply to the force on the first conductor).

First, we choose a cylindrical coordinate system $Or\varphi z$, as shown in Fig. 11.2.

Second, we calculate the magnetic field due to I_1 on conductor 1 at a distance d form it. This is

$$\vec{B}_1 = \frac{\mu_0}{2\pi}\,\frac{I_1}{d}\,\vec{u}_\varphi \tag{11.15}$$

Then, we consider a finite segment of length L of conductor 2 and calculate the force acting on it from (11.9)

$$\begin{aligned}\vec{F}_{2L} &= I_2 \int_0^L \vec{u}_z \times \vec{u}_\varphi\,\frac{\mu_0}{2\pi}\,\frac{I_1}{d}\,d\ell \\ &= -\frac{\mu_0}{2\pi}\,\frac{I_1 I_2}{d}\,L\,\vec{u}_r \end{aligned} \tag{11.16}$$

where $\vec{t} = \vec{u}_z$. This force is normal to both conductors and attractive when the two currents have the same sign or repulsive when they have opposite sign.

Equation (11.16) is used to define the SI current unit, ampère (A): This is the current intensity carried by two straight, indefinite (i.e., $L >> d$), and parallel conductors at $d = 1$m generating a force $\widetilde{F}_{2L} = \widetilde{F}_{1L} = 2 \times 10^{-7}\text{Nm}^{-1}$.

11.3 Work to Move a Current-Carrying Circuit

Consider a filiform circuit γ carrying a steady current I and placed in a region with a magnetic field $\vec{B}_0$ due to other currents, as shown in Fig. 11.3. As shown in the figure, as infinitesimal element $d\ell$ of γ is free to move along the conducting tracks p and q. Suppose to apply a force on $d\ell$ that moves it from the position PQ to the position $P'Q'$ in a quasi-static fashion. This means the force applied to move the element is counter-balanced at each time by the force $d\vec{F}$ due to the magnetic field.

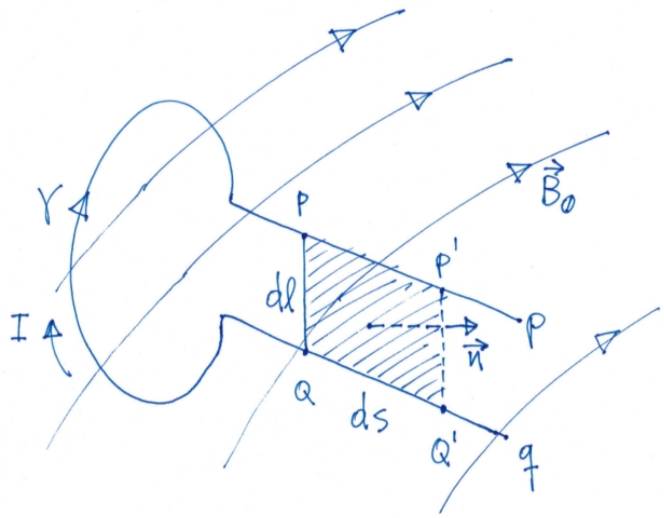

Figure 11.3

Under these conditions, we want to calculate the work of the force $d\vec{F}$ during the movement. We will assume that during a circuit movement the current on it remains the same, i.e., all circuit displacements take place under steady current conditions.

Following the notation in Fig. 11.3, if the displacement length is ds, we have

$$dW = d\vec{F} \cdot \vec{t}_{PP'}\, ds \tag{11.17}$$

where $\vec{t}_{PP'}$ is the tangent unit vector along p. If $\vec{B}$ is the net magnetic field on $d\ell$ due to both the external currents ($\vec{B}_0$) and I, from (11.8) we have

$$dW = (Id\ell\vec{t}_{PQ} \times \vec{B}) \cdot \vec{t}_{PP'}\, ds \tag{11.18}$$

where $\vec{t}_{PQ}$ is the tangent unit vector to γ at $d\ell$. Since

$$\vec{t}_{PP'}\, ds \times \vec{t}_{PQ}\, d\ell = (\vec{t}_{PP'} \times \vec{t}_{PQ}) ds\, d\ell = \vec{n} dA \tag{11.19}$$

where $dA = ds\, d\ell$ is the area of the rectangle $PQQ'P'$ and $\vec{n}$ the normal unit vector for this rectangle, we find

$$dW = I(\vec{t}_{PP'} \times \vec{t}_{PQ} \cdot \vec{B}) ds\, d\ell = I(\vec{n} \cdot \vec{B}) dA \tag{11.20}$$

where we used the cyclic properties of the triple product. The quantity $\vec{n} \cdot \vec{B} dA$ is the infinitesimal flux $d\Phi_t$ of $\vec{B}$ through dA, thus

$$dW = Id\Phi_t \tag{11.21}$$

Consider now a rigid (filiform) circuit that moves with respect to the currents generating $\vec{B}_0$. The discussion that led to Eq. (11.21) can be extended to each infinitesimal element in which the rigid circuit can be decomposed. Figure 11.4 shows a rigid circuit that moved from an initial position A to a final position B.

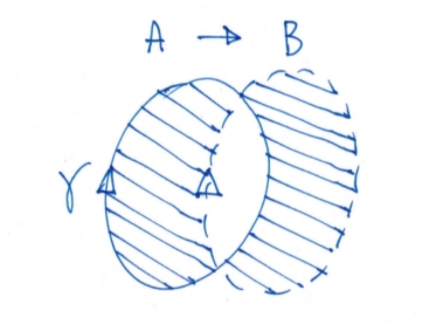

Figure 11.4

In this case, the total work of the field forces is

$$W = I\Delta\Phi_t \tag{11.22}$$

where I is the steady current carried by γ and $\Delta\Phi_t$ the flux through shaded area in Fig. 11.4,

$$\Delta\Phi_t = \Phi_B - \Phi_A \tag{11.23}$$

In this expression, Φ_A and Φ_B are the fluxes linked with γ for position A and B, respectively. Note that, when calculating Φ_A and Φ_B it is not required to take into account the self-induction flux generated by I because this flux remains constant when moving form A to B. The only fluxes to be considered are those due to $\vec{B}_0$, i.e., the mutual-induction fluxes between γ and the external currents. In fact, a rigid circuit cannot generate work on itself.

We can finally extend all our results to the general case of a bulk circuit, the lateral dimensions of which are non-negligible compared to its length. Suppose to divide the entire bulk conductor carrying a total current I (steady) into N arbitrary flux tubes of the current field, each tube with current intensity I_1, I_2, ..., I_N. Assuming that each of these tubes is a quasi-filiform conductor, we can write

$$\begin{aligned} W_1 = I_1(\Phi_1' - \Phi_1), \qquad W_2 = I_2(\Phi_2' - \Phi_2)\ ,\ \ldots\ , \\ W_N = I_N(\Phi_N' - \Phi_N) \end{aligned} \tag{11.24}$$

where W_i is the work of the field forces on the i-th flux tube during the displacement and Φ_i and Φ_i' are, respectively, the linked fluxes for the i-th tube at the initial and final position.

The total work W on the entire circuit is

$$\begin{aligned} W &= W_1 + W_2 + \cdots + W_N \\ &= I_1(\Phi_1' - \Phi_1) + I_2(\Phi_2' - \Phi_2) + \cdots + I_N(\Phi_N' - \Phi_N) \\ &= I_1\Phi_1' + I_2\Phi_2' + \cdots + I_N\Phi_N' \\ &\quad - (I_1\Phi_1 + I_2\Phi_2 + \cdots + I_N\Phi_N) \end{aligned} \tag{11.25}$$

We can thus define a flux Φ linked with the entire bulk circuit such that

$$I\Phi = I_1\Phi_1 + I_2\Phi_2 + \ldots I_N\Phi_N \tag{11.26}$$

By definition,

$$\Phi = \frac{I_1}{I}\,\Phi_1 + \frac{I_2}{I}\,\Phi_2 + \cdots + \frac{I_N}{I}\,\Phi_N \tag{11.27}$$

from which

$$W = I(\Phi' - \Phi) \tag{11.28}$$

where Φ and Φ' are the fluxes linked with the entire bulk circuit at the initial and final position.

Equation (11.27) extends the concept of linked flux to the case of a non-filiform circuit and, thus, makes it possible to define the self- and mutual-inductance coefficients to the case of bulk circuits.

In addition, the results in this section allows us to calculate readily the net magnetic force acting on the parts of a generic current-carrying circuit as well as the net torque acting on it. In fact, if the (rigid) circuit undergoes a translation $d\vec{s}$, the corresponding total work of the magnetic forces must be equal to the work of the net force $\vec{F}$. Hence,

$$dW = \vec{F} \cdot d\vec{s} = I d\Phi \tag{11.29}$$

When $d\vec{s} = dx\vec{u}_x$ in a Cartesian coordinate system $Oxyz$, we have

$$F_x dx = I d\Phi \tag{11.30}$$

where $d\Phi$ is the variation of the flux linked with the circuit carrying a steady current I, during the displacement. Therefore,

$$F_x = I\,\frac{\partial}{\partial x}\,\Phi \tag{11.31a}$$

and, similarly,

$$F_y = I\,\frac{\partial}{\partial y}\,\Phi \tag{11.31b}$$

$$F_z = I\,\frac{\partial}{\partial z}\,\Phi \tag{11.31c}$$

Similarly, the net torque acting on the circuit is

$$\tau_x = I\,\frac{\partial}{\partial \alpha}\,\Phi \tag{11.32a}$$

$$\tau_y = I\,\frac{\partial}{\partial \beta}\,\Phi \tag{11.32b}$$

$$\tau_z = I\,\frac{\partial}{\partial \gamma}\,\Phi \tag{11.32c}$$

where α, β, and γ are the rotation angles about the x-, y-, and z-axis, respectively (cf. Fig. 11.5).

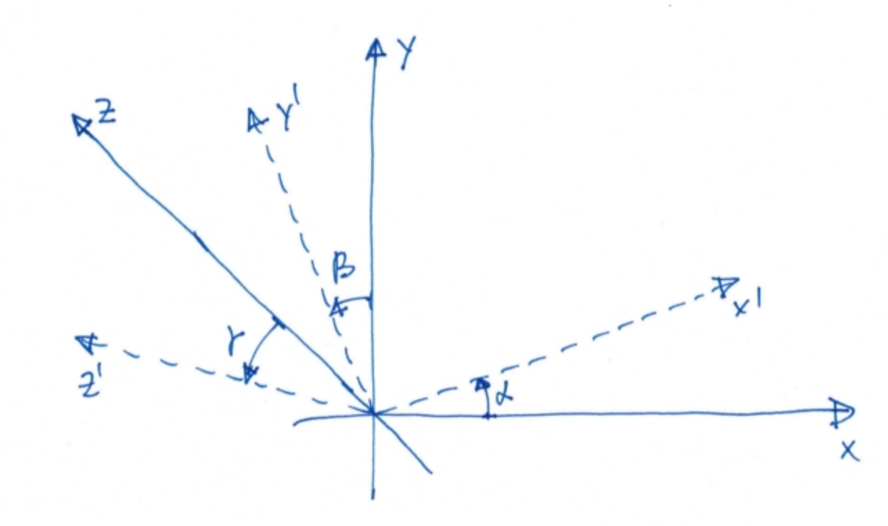

Figure 11.5