PHYS 342 - EM 02

Lecture 8

Consider a quasi-filiform (closed) circuit τ with cross-section dA and carrying a steady current I, as shown in Fig. 8.1. The current is generated by an emf, which is not shown in the figure.

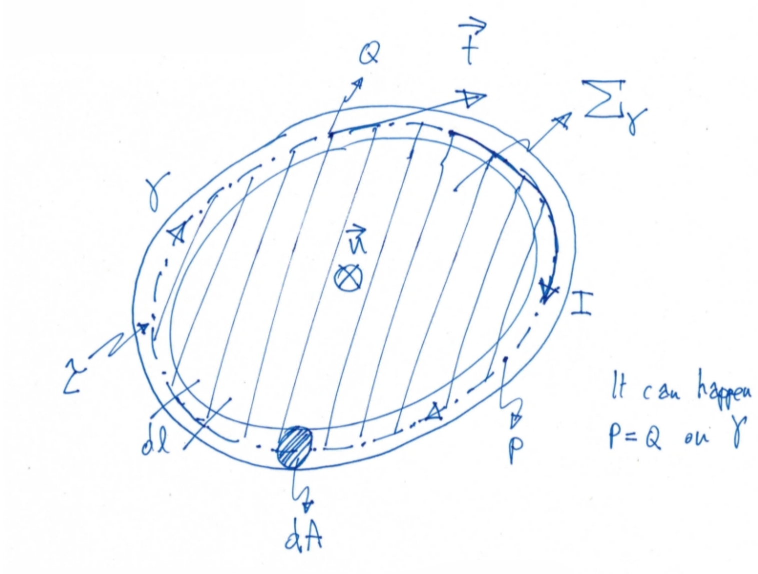

Figure 8.1

The surface Σ_γ is any open surface having γ as a border. The normal unit vector $\vec{n}$ to Σ_γ and the orientation of γ, which is used to evaluate the sign of I, are chosen consistently with the right-hand rule. The line γ is one of the infinite possible longitudinal axes associated with τ. The quasi-filiform nature of the latter, in fact, makes it possible to define longitudinal axes that differ by an infinitesimal distance from each other. The axis γ could be the central longitudinal axis of τ, i.e., the axis passing through the center of dA at each cross-section of τ.

In PHYS 242, we defined a constant called the inductance coefficient of τ with respect to γ as

$$L \equiv \frac{\Phi_\gamma}{I} \tag{8.1}$$

For a single circuit, as τ, this is also called the self-inductance coefficient of the circuit.

In the definition (8.1), the flux Φ_γ is calculated with respect to the special line γ chosen as longitudinal axis of τ. Thus, a different inductance L' will result from a different choice of axis, say γ'. The definition (8.1) leads to an infinite set of inductances, one for each possible axis.

Under quasi-filiform conditions, however, the length of τ is much bigger than its diameter. Hence, the fluxes Φ_γ and $\Phi_{\gamma'}$ linked with γ and γ' (and those for any other longitudinal axis) and generated by the same current I are almost identical. As a consequence, the infinite inductance associated with all possible axes of τ are approximately the same.

It is worth noting that if we were to assume a rigorously filiform conductor, i.e., a conductor with zero diameter, it would lead to a meaningless definition of L. In this case, in fact, even in presence of a finite current, the field $\vec{B}$ would diverge to infinite when approaching the conductor. This can be formally seen from Eq. (3.18), where $r_{QP} \to 0$ when considering field points P closer and closer to source points Q on τ. Since τ is considered to be filiform, Q and P eventually tend to coincide. Therefore, the flux of $\vec{B}$ linked with the circuit would also diverge so would L. As an example, we can consider an infinite straight wire with steady current I. If the wire is strictly filiform the magnitude of the field is given by

$$B = \frac{\mu_0 I}{2\pi r}\,, \tag{8.2}$$

where r is the distance from the wire. It is evident that $B \to +\infty$ for $r \to 0^+$. If the wire is quasi-filiform with cross-section with diameter R, the field is then

$$B = \frac{\mu_0 I r}{2\pi R^2}\,, \quad \text{for} \quad r \in [0, R] \tag{8.3}$$

and

$$B = \frac{\mu_0 I}{2\pi r}\,, \quad \text{for} \quad r \in (R, = \infty] \tag{8.4}$$

This field clearly does not diverge at any point.

Consider now two quasi-filiform circuits τ_1 and τ_2, as shown in Fig. 8.2.

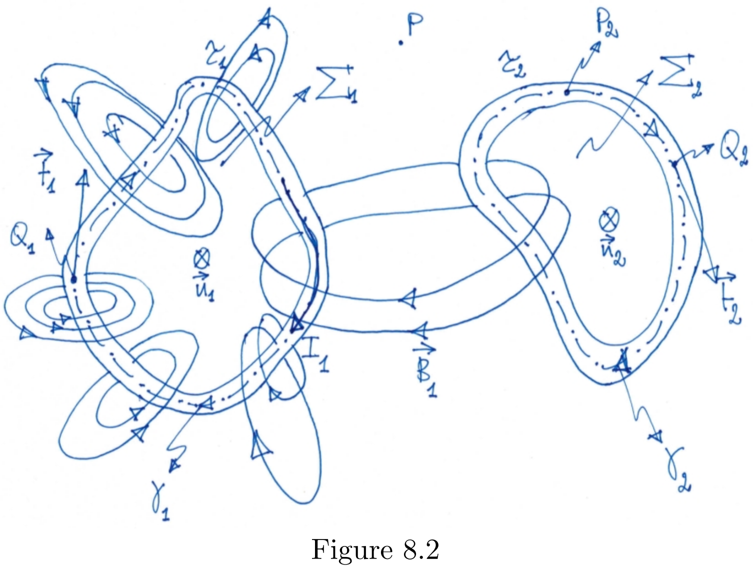

Figure 8.2

The two circuits, which are close to each other, but do not touch, form a circuit system. Assume a steady current I_1 (the sign of which is evaluated with respect to an arbitrary reference direction) flows in τ_1 and no current flows in τ_2. We choose arbitrarily a positive direction on γ_2, which is a longitudinal axis of τ_2. The generic surface Σ_2 that has γ_2 as a border is oriented such that its normal unit vector $\vec{n}_2$ follows the right-hand rule. The flux due to the magnetic field $\vec{B}_1$ generated by I_1 and linked with γ_2 can be written as

$$\Phi_{21} = M_{21} I_1 \tag{8.5}$$

where M_{21} is called the mutual-inductance coefficient of τ_1 on τ_2.

Similarly, when a current I_2 flows in τ_2 and no current in τ_1, we find

$$\Phi_{12} = M_{12} I_2 \tag{8.2'}$$

where Φ_{12} is the flux due to I_2 and linked with τ_1 and M_{12} is the mutual inductance of τ_2 on τ_1.

For a formal calculation of M_{21} and M_{12} we can proceed as follows. The vector potential due to I_1, $\vec{A}_1$, can be calculated at any point in space P from the integral

$$\vec{A}_1(P) = \frac{\mu_0}{4\pi} \iiint\limits_{\tau_1} \frac{\vec{J}_1}{r}\, dV \tag{8.6}$$

where $\vec{J}_1$ is the electric current volume density associated with I_1 and evaluated at each source point Q_1 on γ_1 (we remind that τ_1 is quasi filiform; we can thus assume a point Q_1 inside τ_1 to be a point on γ_1 up to an infinitesimal distance). In (8.6), $r = ||\vec{r}_{Q_1P}||$ is the Euclidean norm of the distance between the generic field point P and Q_1. Under stationary conditions, the quasi-filiform conductor is a flux tube for $\vec{J}_1$.

Therefore, $I_1 = J_1 dA$ and (8.6) can be rewritten as

$$\vec{A}_1(P) = \frac{\mu_0 I_1}{4\pi} \oint\limits_{\gamma_1} \frac{\vec{t}_1}{r}\, d\ell_1 \tag{8.7}$$

where $\vec{t}_1$ is the tangent unit vector at each point Q_1 on γ_1 (cf. Fig. 8.2) and also the direction of J_1, $\vec{J}_1 = J_1\vec{t}_1$, and $d\ell_1$ is an infinitesimal element on γ_1.

From Eq. (4.12), we know that

$$\Phi_{21} = \oint\limits_{\gamma_2} \vec{A}_1 \bullet \vec{t}_2 d\ell_2 \tag{8.8}$$

where $\vec{t}_2$ is indicated in Fig. 8.2 and $d\ell_2$ is an infinitesimal element on γ_2. By combining (8.7) and (8.8), we find

$$\begin{aligned} M_{21} = \frac{\Phi_{21}}{I_1} &= \frac{\mu_0}{4\pi} \oint\limits_{\gamma_2} \Big(\oint\limits_{\gamma_1} \frac{\vec{t}_1}{r}\, d\ell_1 \Big) \bullet \vec{t}_2 d\ell_2 \\ &= \frac{\mu_0}{4\pi} \oint\limits_{\gamma_2} \oint\limits_{\gamma_1} \frac{\vec{t}_1 \bullet \vec{t}_2}{r}\, d\ell_1 d\ell_2 \end{aligned} \tag{8.9}$$

The quantity r in the integrand of (8.9) is always different from zero. In fact, as clear from Fig. 8.2, any source point Q_1 on γ_1 is always different from any field point P_2 on γ_2. Since γ_1 and γ_2 do not share any point, the integrand in (8.9) is well defined. Thus, the integral, which is M_{21}, is also well defined.

Following a similar path, it is possible to calculate M_{12}:

$$M_{12} = \frac{\mu_0}{4\pi} \oint_{\gamma_1} \oint_{\gamma_2} \frac{\vec{t_2} \bullet \vec{t_1}}{r} \, d\ell_2 d\ell_1 \tag{8.9'}$$

From the linearity of integration and for the commutative property of the scalar product, it follows that $M_{21} = M_{12}$.

The expressions (8.9) and (8.9') are called Neumann integrals. These integrals are only valid under the assumption that the flux is proportional to the current that generates it. In the case of ferromagnetic materials, for example, a Neumann-type integral cannot be used.

Can we use a Neumann-type integral for the calculation of the self inductance of a single, closed circuit with steady current I?

Consider again the circuit τ of Fig. 8.1. In order to calculate L by means of Eq. (8.9), we could assume $\gamma_1 = \gamma_2 = \gamma$, $\vec{t_1} = \vec{t_2} = \vec{t}$, and $d\ell_1 = d\ell_2 = d\ell$. We would thus find

$$\begin{aligned} L &= \frac{\mu_0}{4\pi} \oint_{\gamma} \oint_{\gamma} \frac{\vec{t} \bullet \vec{t}}{r} \, d\ell d\ell \\ &= \frac{\mu_0}{4\pi} \oint_{\gamma} \oint_{\gamma} \frac{1}{r} \, d\ell d\ell \end{aligned} \tag{8.10}$$

For each source point Q on γ, the integrand of (8.10) will diverge every time a field point P, also γ, coincides with Q. The inductance given by (8.10) is not well defined.

We must find a trick to circumvent this problem. We will so do by means of an example. We will try to calculate the inductance of a circular ring.