

1.1 Electric interactions: Simple experiments

Simple experiments can reveal "electric interactions"

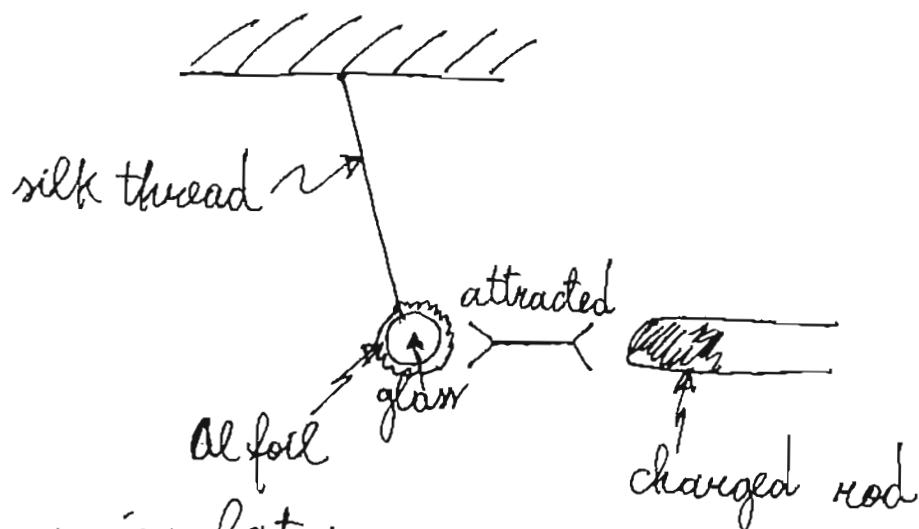
→ attractions

→ repulsions

- Plastic rod rubbed with wool cloth can lift light materials
→ attraction stronger than gravitational attraction from entire planet

we say rod is electrically charged

- Electric pendulum



- Conductors vs. insulators

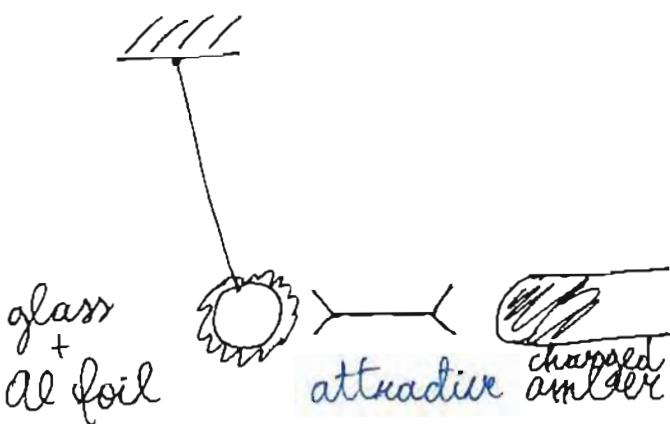
metals and other similar materials even if rubbed with a wool cloth do not act on the pendulum.

If metal has a plastic handle, it can be charged. However, as soon as touched by another metal, or even by a human hand, it loses its charge

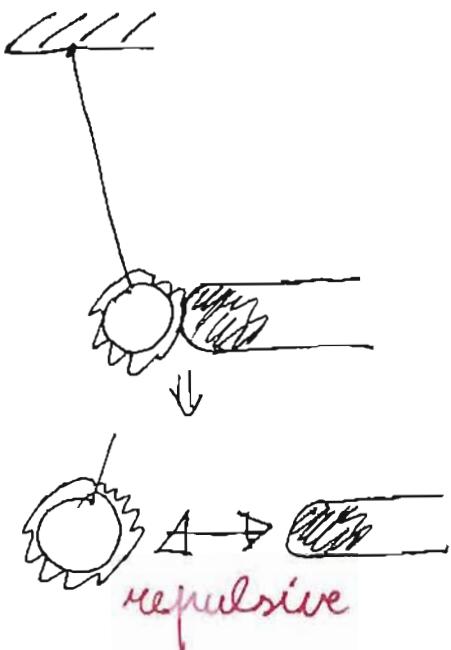
electricity

- positive and negative electric charge

1)



2)



we can thus define

Positive "+" and negative "-" charges:

$$+) \rightarrow \leftarrow (-$$

$$-) \rightarrow \leftarrow (+$$

$$+) \leftrightarrow \leftrightarrow (-$$

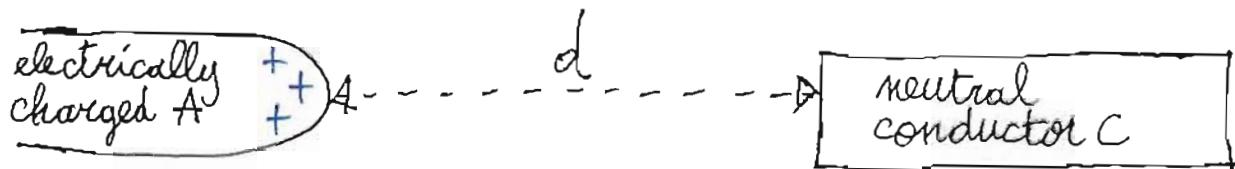
$$-) \leftrightarrow \leftrightarrow (+$$

note that, so far, + and - charges are only a convention detached from any physical quantity. For example, if we blow a minium-sulphur powder on an electrically charged body, sulphur remains attached to + charges and minium to - charges.

since sulphur is yellow and minium red,
we could have called the two type of charges
"yellow" and "red," respectively

3/13

- Electrostatic induction



this can be unveiled with a minium-sulphur powder

if d becomes much larger again $\Rightarrow C$ becomes again neutral

this phenomenon is called electrostatic induction from A to C

- neutral state: "+" and "-" charges destroy each other

- induction: "+" gets separated from "-"

for insulators this phenomenon is called polarization

1.2 Electric charge

4/13

Definition of electric charge:

"Two electrically charged bodies A and B have equal electric charges, with equal sign ("+" or "-"), when they generate equal forces (in magnitude, direction, and sign) on a third body C fixed in the reference frame of A and B"

experimentally, one finds that if C is substituted by C' , the forces generated by A and B remain the same (independence from test body).

Definition of point-like bodies:

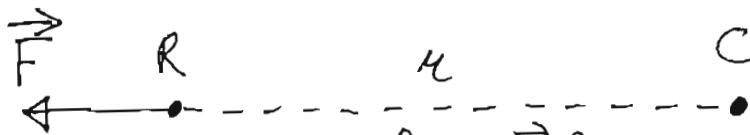
Bodies A and C are point-like bodies if the distance between them is sufficiently larger than their dimensions (same for B and C or C').

In the definition of electric charge, the bodies A, B, C, and C' are considered to be point-like bodies

- Summation of charges

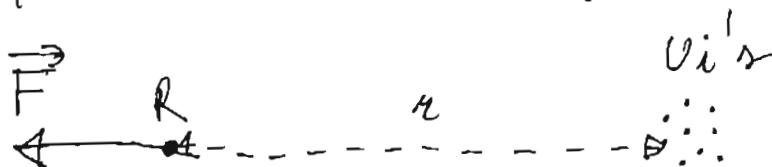
- 1) Consider a body V with "+" or "-" electric charge.
We define this charge to be unity

- 2) Consider a body C charged with the same sign as V and placed at distance r from body R.



We measure a force \vec{F} from C to R

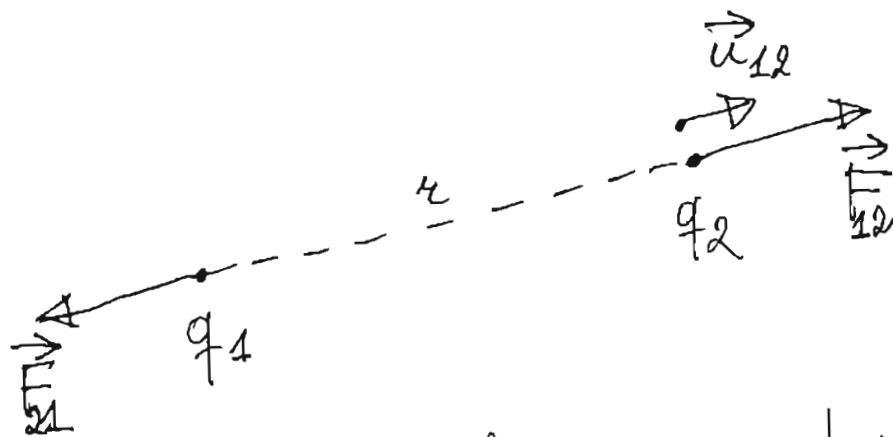
- 3) Finally, consider many bodies v_i with unitary charge. Consider to place enough such bodies next to each other such that we measure the same force \vec{F} as in 2) on a body R placed at distance r from the v_i 's



the number of unitary charges v_i defines the electric charge of C

positive and negative charges are assumed to act with equal forces, but opposite sign on the same R

1.3 Coulomb's law



the force between two charges q_1 and q_2 is:

1) directed along a straight line between q_1 and q_2

2)

$$\sim \frac{|q_1||q_2|}{r^2}$$

3) positive for charges with equal sign and negative for charges with opposite sign

$$\vec{F}_2 = k \frac{q_1 q_2}{r^2} \vec{u}_{12} = -\vec{F}_1$$

↑
action/reaction
principle

- If $\frac{q_1}{|q_1|} = \frac{q_2}{|q_2|} \Rightarrow$ repulsive force

- If $\frac{q_1}{|q_1|} \neq \frac{q_2}{|q_2|} \Rightarrow$ attractive force

in the SI system

[charge] = C (Coulomb)

$$k \underset{\sim}{=} \frac{8.988 \times 10^9 \text{ kg m}^3 \text{s}^{-2} \text{C}^{-2}}{4\pi\epsilon_0}$$

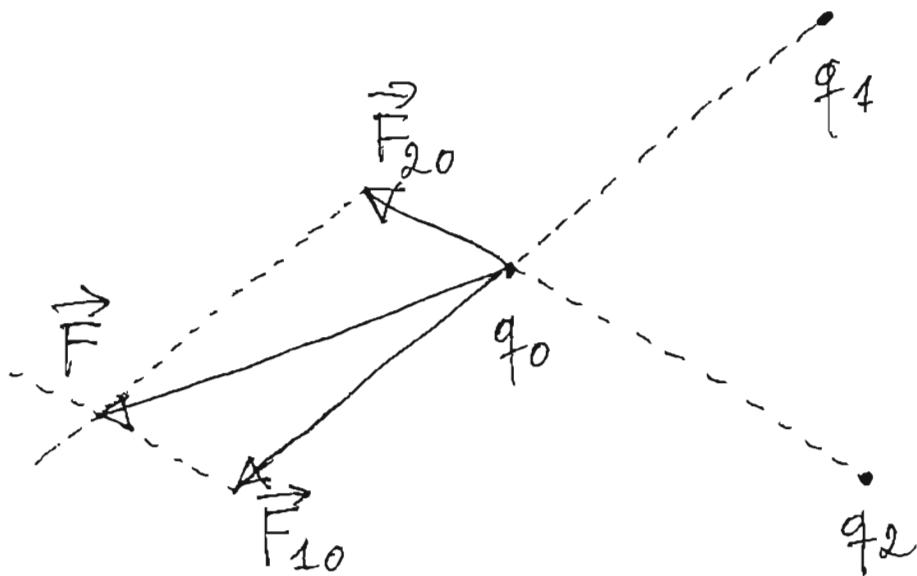
$$\epsilon_0 \approx 8.854 \times 10^{-12} \text{ kg}^{-1} \text{m}^{-3} \text{s}^2 \text{C}^{-2}$$

(vacuum dielectric constant)

1 C is huge; typical values are $\sim 1 \mu\text{C}$ or less

1.4 Superposition principle

4/13



\vec{F}_{10} and \vec{F}_{20} obey Coulomb's law

this principle is different from charge summation
(given by definition) and Coulomb's law (valid only
between two charges)

- Example #1.

$$+|q_1| - |q_2|$$

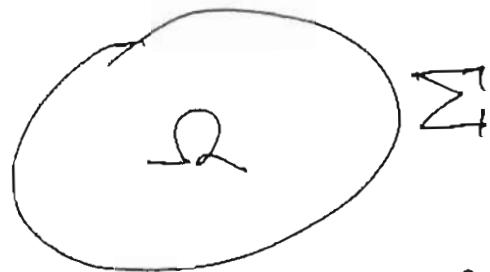
when $|q_1| = |q_2| \Rightarrow \vec{F}_o = \vec{0}$

when $|q_1| \neq |q_2| \Rightarrow \vec{F}_o$ as if

generated by a single $|q_0| = |q_1| - |q_2|$

1.5 Charge conservation principle

8/13



the total charge in an isolated system
(given by the sum of all positive and negative charges) is constant in time
following from experience, if

$$\Delta Q = Q(t_2) - Q(t_1) \neq 0$$

\Rightarrow charged bodies passed through Σ between t_1 and t_2 $\Rightarrow \Sigma$ is not isolated

1.6 Charge quantization

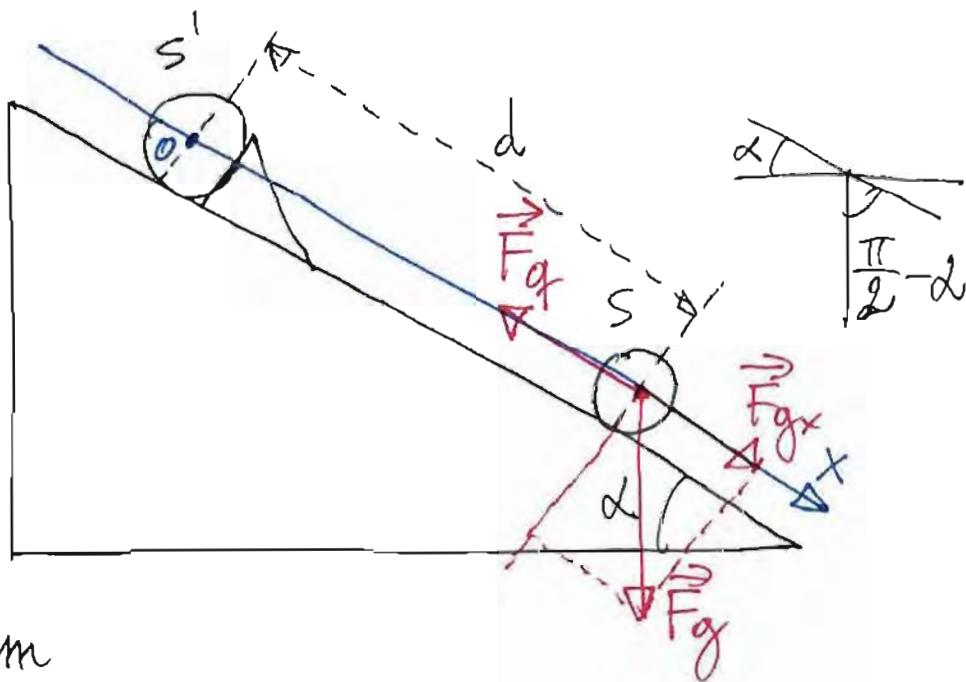
the values assumed by the electric charge
are integer multiples of the electron charge e

e is negative and $|e| \approx 1.602 \times 10^{-19} \text{ C}$

- Example #2:

① A sphere S of mass m is placed on an inclined plane. A second sphere S' is hinged at distance d on the plane, above S .

Assuming S and S' to be electrically charged with equal and opposite charges $+|q|$ and $-|q|$, respectively, calculate the value of q required to prevent S from rolling down the plane.



$$S \rightarrow m$$

$$S \rightarrow +|q|; S' \rightarrow -|q|$$

$$x \propto d$$

② Reference frame

③ Indicate all forces

④ Degrees of freedom (DOF) : \times

⑤ Equations of motion (static)

$$\vec{F}_q + \vec{F}_{gx} = \vec{0}$$

$$F_{gx} = +mg \sin \alpha$$

$$F_q = -k \frac{|q||q|}{d^2}$$

$$mg \sin \alpha - k \frac{|q||q|}{d^2} = 0 \Rightarrow |q| = \sqrt{\frac{mg \sin \alpha}{k}}$$

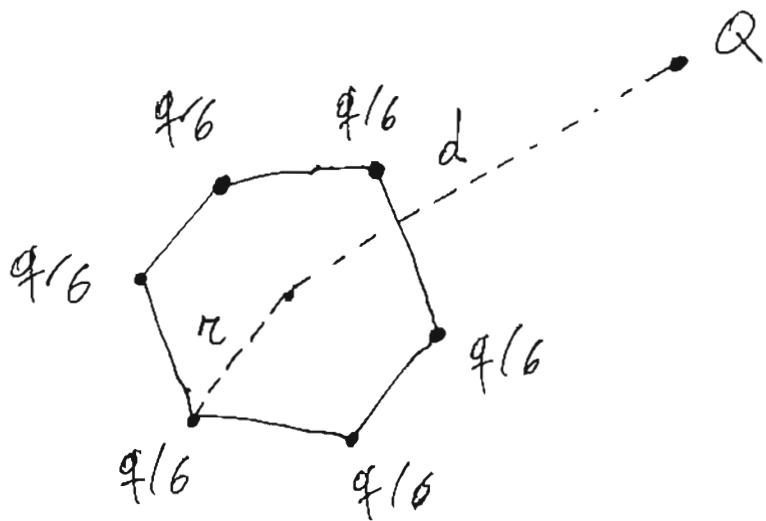
the further the bodies, the larger the charge needed

- Example #3:

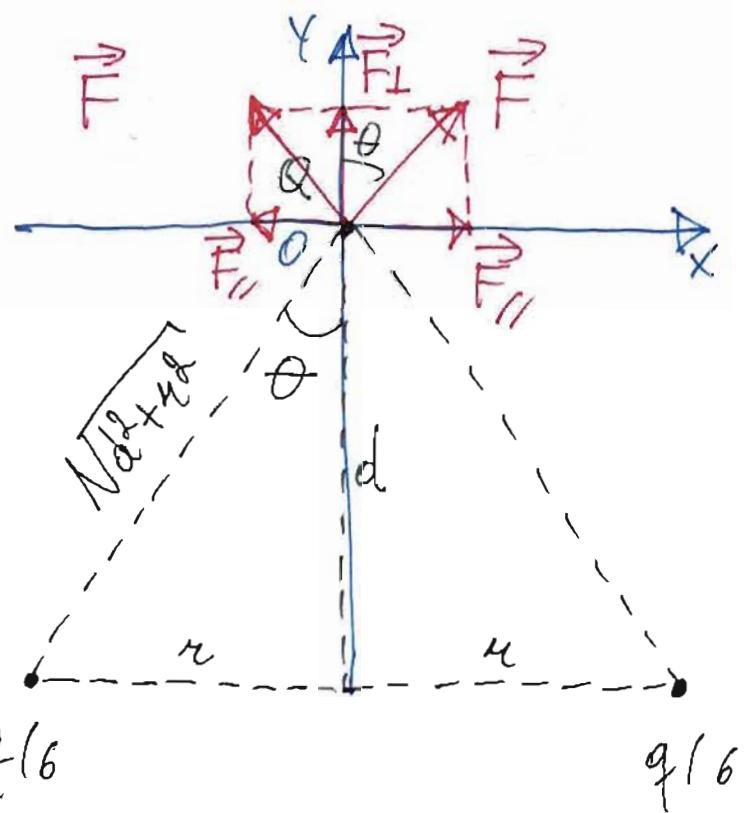
① Consider an hexagon with charges $q/6$ on each vertex. The hexagon is enclosed in a circle of radius r . A charge Q is located at distance d from the center of the circle on the perpendicular to the circle. Both $q > 0$ and $Q > 0$.

1) Calculate \vec{F} on Q .

2) Study the case $d \gg r$.



② Reference frame



③ Indicate all forces

④ Forces

12/13

1) The forces on Q due to each $q/6$ are

$$\vec{F} = \vec{F}_{\parallel} + \vec{F}_{\perp}$$

$$= F K \frac{q Q}{6(d^2 + r^2)} \sin \theta \vec{u}_x + K \frac{q Q}{6(d^2 + r^2)} \cos \theta \vec{u}_y$$

for each opposite pair of charges $q/6$
 \vec{F}_{\parallel} have opposite sign

$$\Rightarrow \sum \vec{F}_{\parallel} = \vec{0}$$

\vec{F}_{tot} must be perpendicular

$$\vec{F}_{\text{tot}} = \sum_{\ell=1}^6 \vec{F}_{\perp}^{\ell} = 6 K \frac{q Q}{6(d^2 + r^2)} \cos \theta \vec{u}_y$$

$$\cos \theta = \frac{d}{\sqrt{d^2 + r^2}}$$

$$\boxed{\vec{F}_{\text{tot}} = K \frac{q Q d}{(d^2 + r^2)^{3/2}} \vec{u}_y}$$

2) When $d \gg r$

$$\vec{F}_{\text{tot}} \approx k \frac{qQd}{d^3} = k \frac{qQ}{d^2}$$

as the Coulomb force between two point-like charges q and Q at distance d

Summary lecture 1

- Positive and negative charge.
- Conductors vs. insulators.
- Definition of electric charge : A and B have equal charge when they generate equal forces on a test body C.
 - point-like bodies.
 - summation of charges.
- Coulomb's law

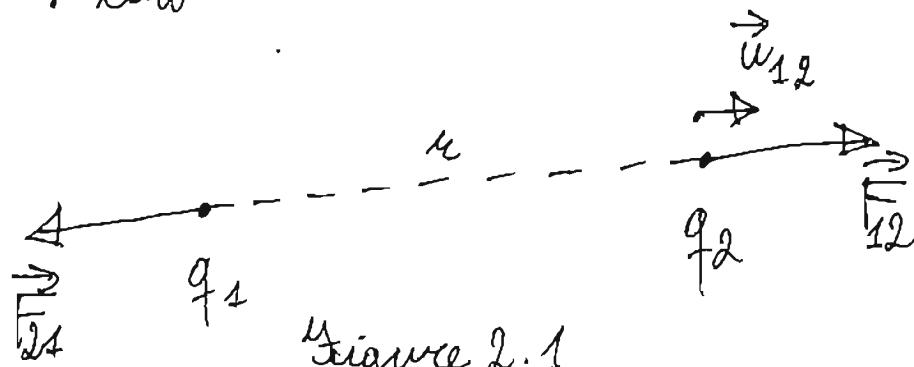


Figure 2.1

$$\vec{F}_{12} = k \frac{q_1 q_2}{r^2} \vec{u}_{12} = -\vec{F}_{21} \quad [C] \quad (2.1)$$

- Superposition principle.
- Charge conservation principle (sum of forces vectors).
- Charge quantization.

2.1 Electric field

We consider two point-like charges q_1 and q_0 fixed in vacuum, and we imagine to measure the force acting on q_0 keeping q_1 fixed in point Q and moving q_0 . This procedure allows us to determine the vector field $\vec{F}_0(P)$ due to the force \vec{F}_0 acting on q_0 , where q_0 is in a generic point P , due to q_1 in point Q . The field $\vec{F}_0(P)$ is defined in the entire space.

Since \vec{F}_0 is given by Coulomb's law, the field is said to be a central field.

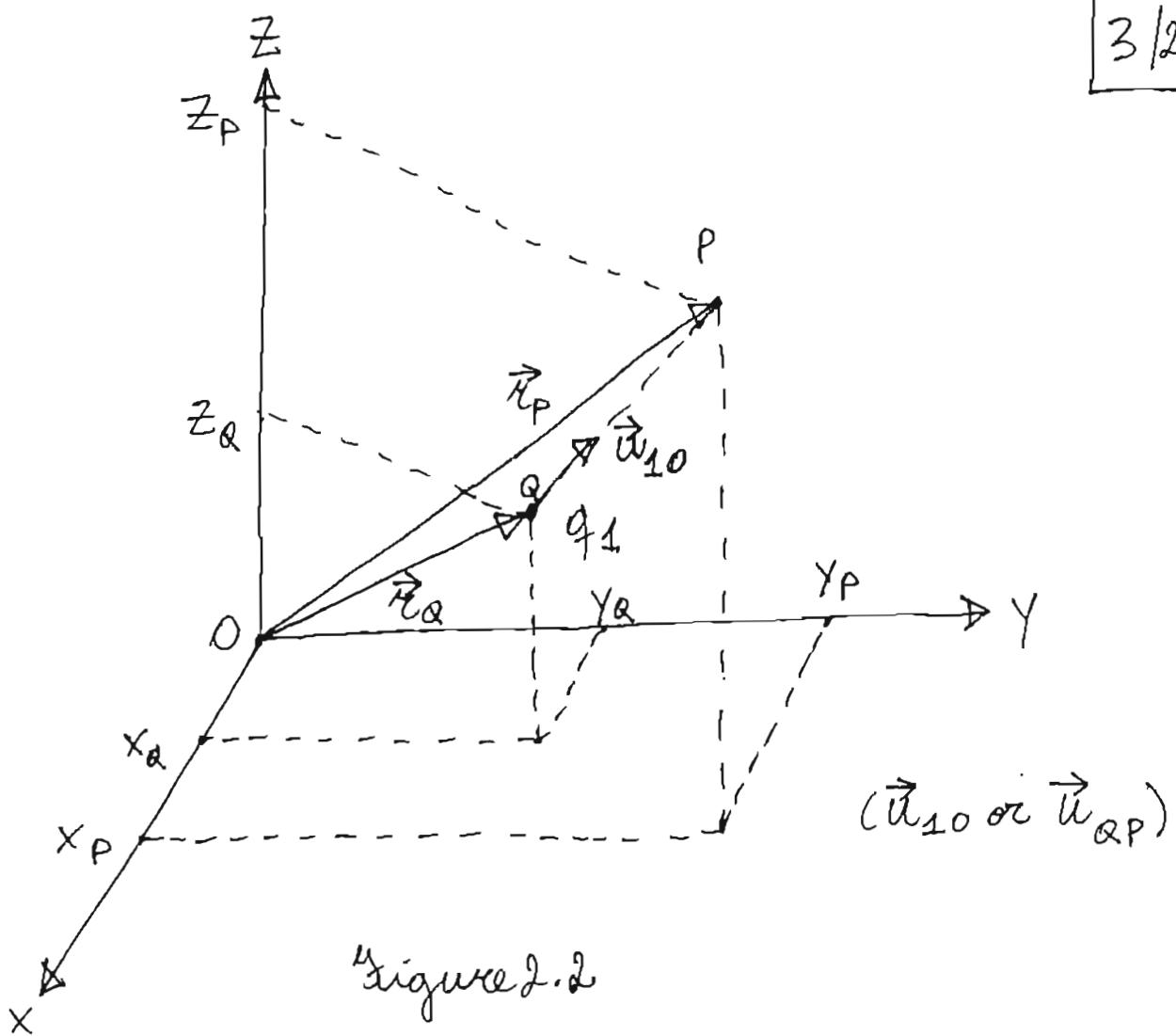
- By substituting q_0 with q'_0 , according to Coulomb's law we obtain a new vector field

$$\vec{F}'_0(P) = \frac{q'_0}{q_0} \vec{F}_0(P). \quad (2.2)$$

- $\vec{E}(P) \rightarrow$ vector field acting on the unitary positive charge due to q_1

$$\Rightarrow \vec{F}_0(P) = q_0 \vec{E}(P), \quad (2.3)$$

where $\vec{E}(P) = \frac{1}{4\pi\epsilon_0} \frac{q_1}{r_{10}^2} \vec{u}_{10}. \quad (2.4)$



$$\vec{u}_{10} = \frac{\vec{r}_P - \vec{r}_Q}{|\vec{r}_P - \vec{r}_Q|} = \frac{(x_p - x_Q)\vec{u}_x + (y_p - y_Q)\vec{u}_y + (z_p - z_Q)\vec{u}_z}{[(x_p - x_Q)^2 + (y_p - y_Q)^2 + (z_p - z_Q)^2]^{1/2}},$$

thus

$$E_x = \frac{q_1}{4\pi\epsilon_0} \frac{x_p - x_Q}{[(x_p - x_Q)^2 + (y_p - y_Q)^2 + (z_p - z_Q)^2]^{3/2}}, \quad (2.5a)$$

$$E_y = \frac{q_1}{4\pi\epsilon_0} \frac{y_p - y_Q}{[(x_p - x_Q)^2 + (y_p - y_Q)^2 + (z_p - z_Q)^2]^{3/2}}, \quad (2.5b)$$

$$E_z = \frac{q_1}{4\pi\epsilon_0} \frac{z_p - z_q}{[(x_p - x_q)^2 + (y_p - y_q)^2 + (z_p - z_q)^2]^{3/2}} \quad (2.5c)$$

- geometric representation of \vec{E} by means of field lines

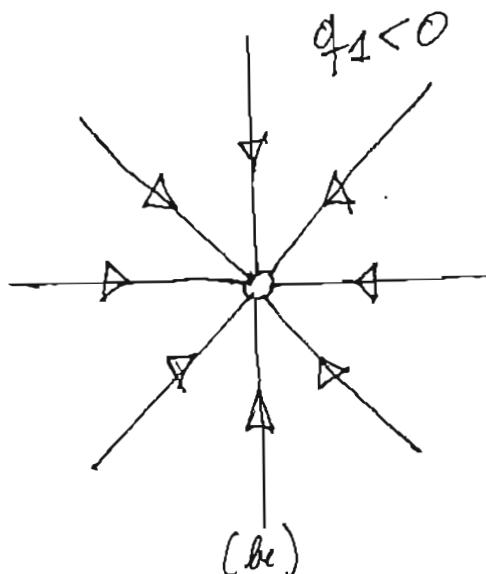
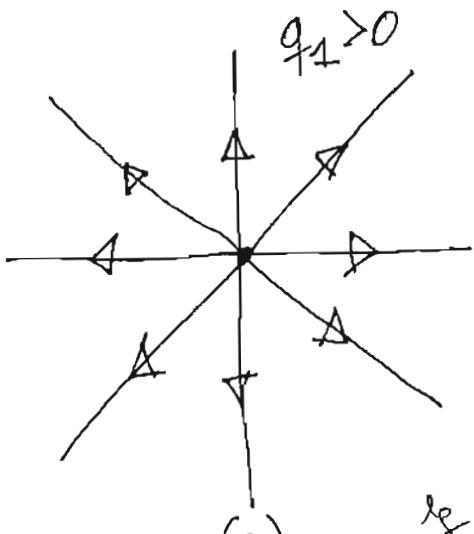


Figure 2.3.

In 3D, the field has a central symmetry, which is natural due to the homogeneity and isotropy of space in the inertial reference frame where the source charge q_1 is fixed.

- Extension from one source-charge q_1 to N source charges

Given N charges in vacuum, q_1, q_2, \dots, q_N , fixed at points Q_1, Q_2, \dots, Q_N in an inertial reference frame, the force due to the N charges on a test charge q_0 at point P is

$$\vec{F}_o(P) = \sum_{k=1}^N \vec{F}_{ko}(P) =$$

$$= \frac{q_0}{4\pi\epsilon_0} \sum_{k=1}^N \frac{q_k}{r_{ko}^2} \vec{u}_{ko}, \quad (2.6)$$

$$= q_0 \vec{E}(P),$$

where $\vec{E}(P) = \frac{1}{4\pi\epsilon_0} \sum_{k=1}^N \frac{q_k}{r_{ko}^2} \vec{u}_{ko}$

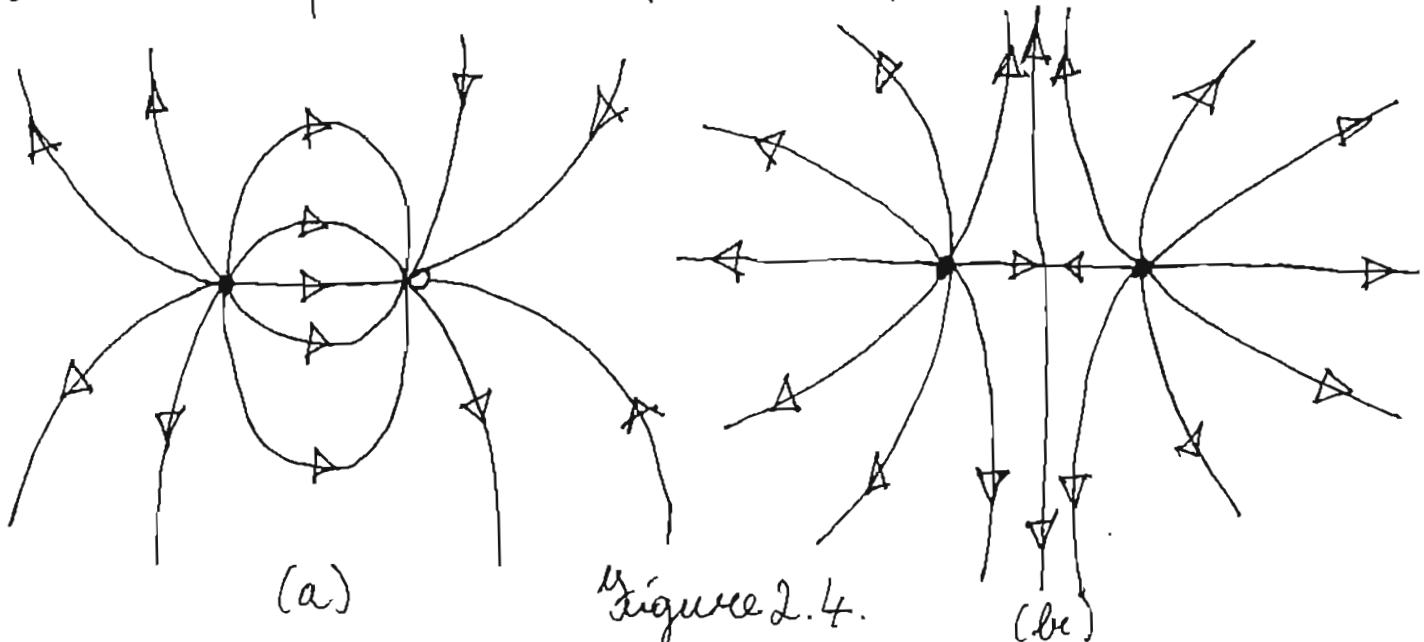
$$= \sum_{k=1}^N E_k(P). \quad (2.7)$$

electric field due
to one source-charge q_k
fixed at point Q_k

Equation (2.7) results from:

- 1) Coulomb's law.
- 2) Superposition principle.

- Geometric representation for multiple charges



- If the charge distribution is unknown, Eq. (2.7) cannot be used. In this case

$$\vec{E}(P) = \frac{\vec{F}_0(P)}{q_0} . \quad (2.8)$$

- 1) If all source charges remained fixed, Eq. (2.8) would be independent from the test charge q_0 .
- 2) However, in reality the presence of q_0 disturbs the distribution of the source charges. Eq. (2.8) would thus give a different result compared to Eq. (2.7), where the distribution of all source charges is known.

3) The disturbance due to q_0 on the source charges increases with the value of q_0 .

Hence, the operative definition of \vec{E} is

$$\vec{E}(P) = \lim_{q_0 \rightarrow 0} \frac{\vec{F}_0(P)}{q_0}. \quad (2.9)$$

Note that $\lim q_0 \rightarrow 0$ is a macroscopic limit.

It is enough that the value of the test charge q_0 is much smaller than the value of each source charge. Also note that, due to charge quantization, $q_0 \neq 0$.

2.2 Continuous charge distribution

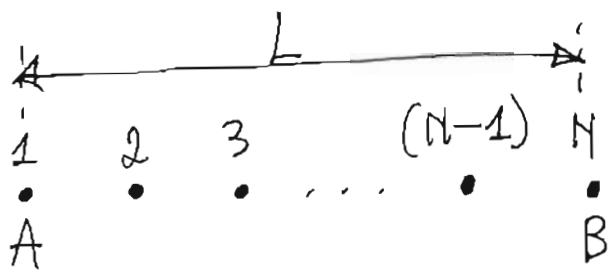


Figure 2.5.

Figure 2.5 shows a distribution of N point-like charges q along a segment AB. The charges are considered to be equal and equally spaced.

- Suppose to increase the number of charges N , while reducing their value q such that $Q = qN = \text{const.}$

This means the inter-charge distance $s = L/(N-1)$ diminishes.

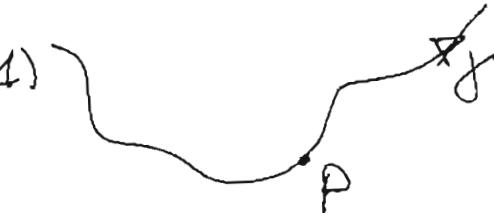
- $\lim N \rightarrow \infty$ (caveat: Charge quantization) \Rightarrow the charges are distributed continuously on AB. The charge on an infinitesimal element dx on AB is

$$dq = \frac{Q}{L} dx = \lambda dx, \quad (2.10)$$

λ : Linear charge density

- Charges distributed on a line γ :

$$\lambda(P) = \left. \frac{dq}{dl} \right|_P, \quad (2.11)$$



$$[\lambda] = C m^{-1}.$$

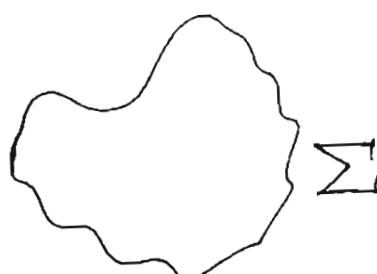
The total charge Q distributed on γ is

$$Q = \int_{\gamma} dl \lambda(P), \quad (2.12)$$

- Surface charge density:

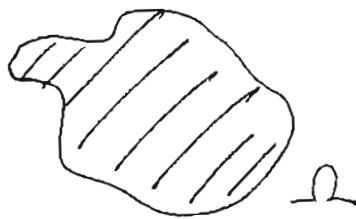
$$\sigma = \frac{dq}{dS}, \quad (2.13)$$

$$[\sigma] = C m^{-2}.$$



- Volume charge density:

$$\sigma = \frac{dq}{dV}, \quad (2.14)$$



$$[\sigma] = C/m^3.$$

- In summary:

$$1) \text{Line } \gamma \Rightarrow Q = \oint_{\gamma} dl \chi(p).$$

$$2) \text{Surface } \Sigma \Rightarrow Q = \iint_{\Sigma} dS \sigma(p), \quad (2.15)$$

$$3) \text{Volume } \Omega \Rightarrow Q = \iiint_{\Omega} dV \sigma(p), \quad (2.16)$$

2.2.1 Physical meaning of a continuous charge distribution

Charge is quantized. A body of volume Ω is made by a caos of electrons and protons acting as point-like charges. These charges are separated by very large distances compared to their dimensions and are characterized by incessant motion. If we were to use Eq. (2.14),

we would obtain a function changing abruptly from point to point, both in space and time. This function would be almost always zero, other than the points where there is an actual charge, where it would assume very large values.

- Eq. (2.16) is defined for

$$S = \lim_{\Delta z \rightarrow 0} \frac{\Delta q}{\Delta z}.$$

- Assume a $\Delta z \approx 1 \text{ cm}^3$ and reduce Δz . The value of S changes until Δz becomes small enough ($\approx 10^{-3} \text{ mm}^3$), when S stabilizes,
- If Δz becomes too small, quantization of charge starts to play a role.
- Δz must be chosen small enough for S to stabilize, but not too small for charge quantization to be important
(note that S should be considered a time average \bar{S} over a short enough, but not too short time interval)

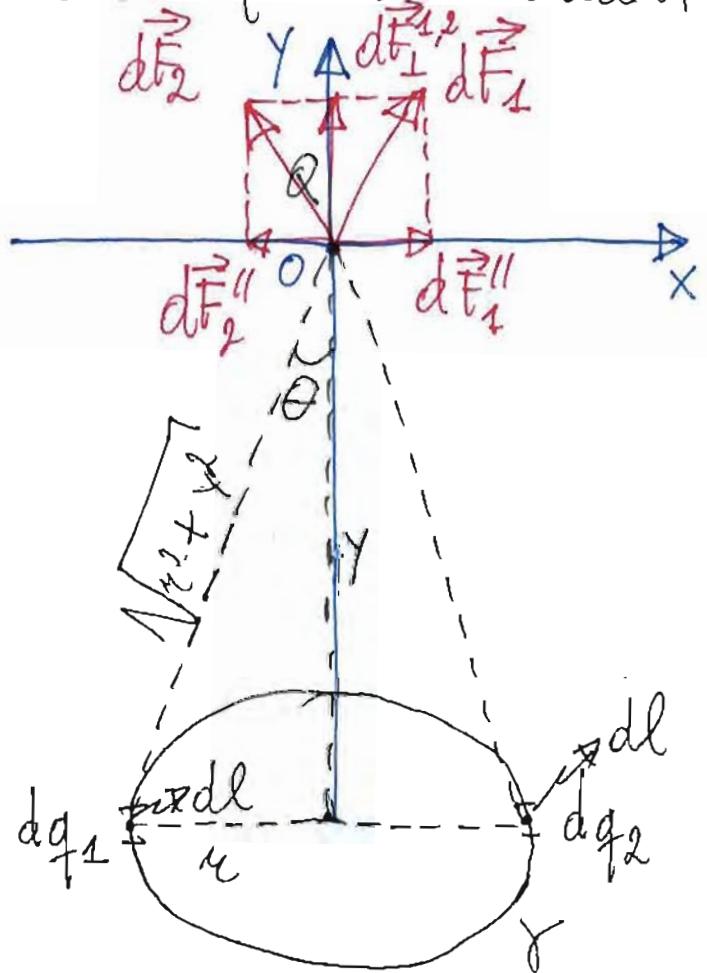
\Rightarrow physics infinitesimal

• Example #1.

① A charge q is distributed along a ring of radius r , with linear density $\lambda = q/2\pi r$. A point-like charge Q is located at distance y from the center of the ring, on the ring axis.

1) Calculate the force \vec{F} generated by q on Q .

2) Calculate for which value of y , \vec{F} is maximum.



② Reference frame

③ Indicate all forces

④ $\text{DOF} = \gamma$.

12/22

⑤ Results:

1) The force $d\vec{F}$ generated by an infinitesimal charge dq is

$$dq = \lambda dl$$

has magnitude

$$dF = \frac{1}{4\pi\epsilon_0} \frac{\lambda dl Q}{r^2 + y^2}$$

The force $d\vec{F}$ can be decomposed in two components, $d\vec{F}''$ and $d\vec{F}^\perp$,

$$d\vec{F} = d\vec{F}'' + d\vec{F}^\perp.$$

For each dq and the dq symmetrically opposed to dq on the ring, the $d\vec{F}''$ cancel each other. Thus, each dq generates only an effective $d\vec{F}^\perp$

$$dF^\perp = dF \cos\theta.$$

By integrating over the ring, the total force \vec{F} is given by

$$\vec{F} = \int_{\gamma} dF + \vec{u}_y = \int_{\gamma} dF \cos \theta \vec{u}_y,$$

where

$$\cos \theta = \frac{y}{\sqrt{x^2 + y^2}}.$$

$$\begin{aligned} \vec{F} &= \int_{\gamma} \frac{1}{4\pi\epsilon_0} \frac{\lambda dl Q}{r^2 + y^2} \frac{y}{\sqrt{x^2 + y^2}} \\ &= \frac{1}{4\pi\epsilon_0} \frac{\lambda Q y}{(x^2 + y^2)^{3/2}} \int_{\gamma} dl \\ &= \boxed{\frac{1}{4\pi\epsilon_0} \frac{q Q y}{(x^2 + y^2)^{3/2}}}. \end{aligned}$$

\vec{F} is repulsive.

- 2) Due to the symmetry of the problem with respect to the plane of ring at $y=0$, we can solve the problem for $y \geq 0$.

We can first observe that for $y=0$, $F=0$ and for $y \rightarrow +\infty$, $F \rightarrow 0$. Since $F \geq 0$ and non zero for some values of y , F should have at least one

maximum. The maximum can be found by 14/23

$$\frac{d}{dy} F = \frac{qQ}{4\pi\epsilon_0} \left[\frac{1}{(x^2+y^2)^{3/2}} - \frac{3}{2} y \frac{2y}{(x^2+y^2)^{5/2}} \right] = 0,$$
$$\Rightarrow \frac{1}{(x^2+y^2)^{5/2}} (y^2 + 4y^2 - 3y^2) = 0$$
$$\Rightarrow \boxed{y = \frac{x}{\sqrt{2}}}.$$

2.3 Electric field generated by a generic charge distribution

15/23

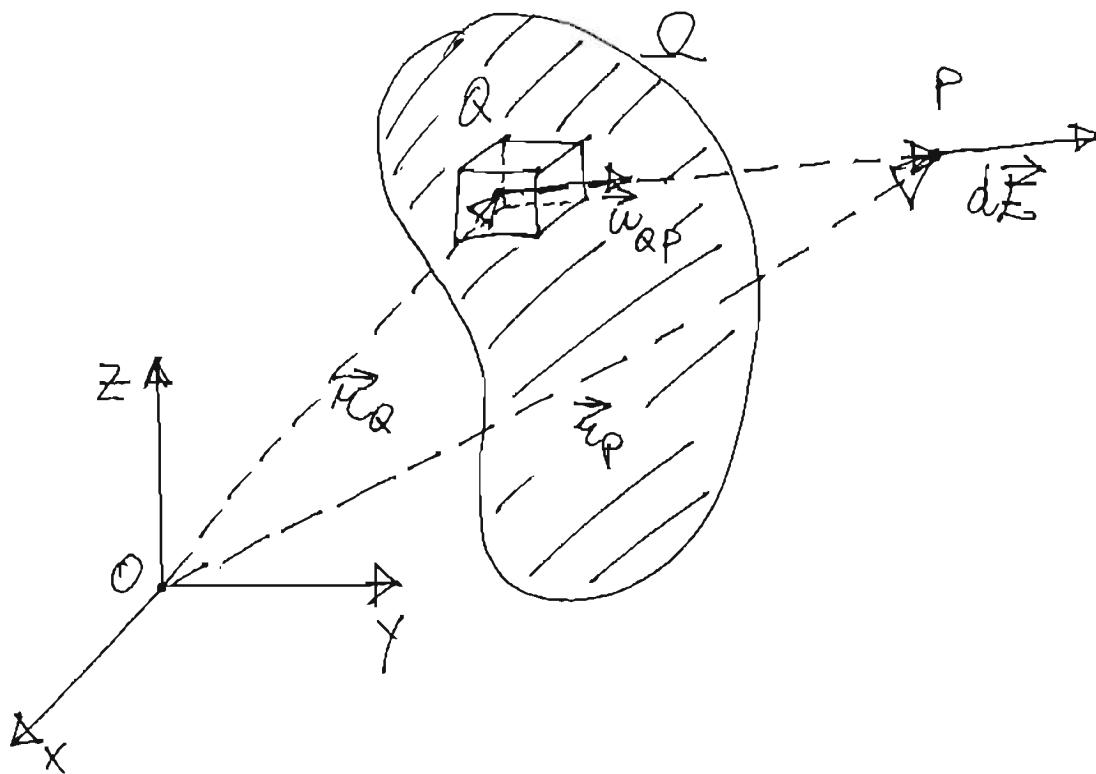


Figure 2.4.

The charge dq in dV is

$$dq = \rho(Q) dV. \quad (2.22)$$

dq can be considered to be point-like \Rightarrow it generates a field $d\vec{E}$ in a field-point P :

$$d\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{dq}{r_{QP}^2} \cdot \vec{u}_{QP} = \frac{1}{4\pi\epsilon_0} \frac{\rho(Q)}{r_{QP}^2} dV \vec{u}_{QP}, \quad (2.23)$$

16/23

where $r_{QP} = |\vec{r}_P - \vec{r}_Q|$. By means of the superposition principle:

$$\vec{E}(P) = \frac{1}{4\pi\epsilon_0} \iiint_Q d\tau \frac{\rho(Q)}{r_{PQ}^2} \vec{u}_{QP}. \quad (2.24)$$

The integral in Eq. (2.24) is a vector integral, extended on the entire region Q where a volume charge density $\rho(Q)$ is defined.

In cartesian coordinates,

$$\begin{aligned} \vec{E}(x_P, y_P, z_P) &= \frac{1}{4\pi\epsilon_0} \iiint_Q \rho(x_Q, y_Q, z_Q) \\ &\times \frac{(x_P - x_Q)\vec{u}_x + (y_P - y_Q)\vec{u}_y + (z_P - z_Q)\vec{u}_z}{[(x_P - x_Q)^2 + (y_P - y_Q)^2 + (z_P - z_Q)^2]^{3/2}} \\ &\times dx_Q dy_Q dz_Q. \quad (2.25) \end{aligned}$$

For linear and surface charge densities we obtain:

$$\vec{E}(P) = \frac{1}{4\pi\epsilon_0} \int dl \frac{\lambda(Q)}{r_{QP}^2} \vec{w}_{QP}, \quad (2.26)$$

$$\vec{E}(P) = \frac{1}{4\pi\epsilon_0} \sum dS \frac{\sigma(Q)}{r_{QP}^2} \vec{w}_{QP}. \quad (2.27)$$

- It's easy to verify that the electric field \vec{E} generated by a finite distribution of charges tends to 0 for $r_{QP} \rightarrow \infty$, at least as $1/r_{QP}^2$.
- Example #2:

- Consider a charge distribution with uniform linear distribution λ on a straight line (cf. Fig.28).

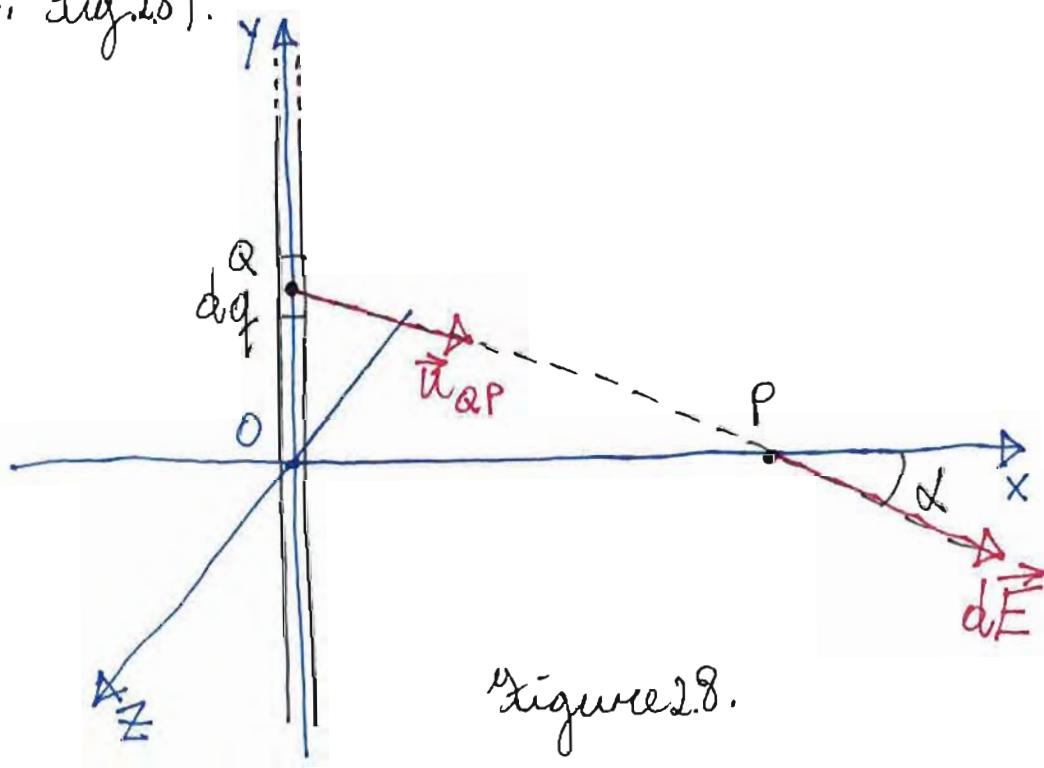


Figure 2.8.

Calculate the electric field generated by λ or a generic point P that does not belong to λ . 18/23

② Reference frame:

- 1) Symmetry argument. The charge distribution λ on ℓ is indistinguishable from any half plane originating from ℓ . This is because any observer fixed in space would see the same charge distribution λ upon rotating ℓ (rotational symmetry). We can thus consider the half plane containing P.
- 2) The reference frame for the half plane containing P is shown in Fig. 2.8.

③ Indicate all fields
Confer Fig. 2.8.

④ DOF

- 1) Symmetry argument. From Fig. 8, for each infinitesimal charge dq at point $+|y|$ there is a charge dq at point $-|y|$. Hence, the total electric field at point P will only have an E_x component. This is because the line is infinite and thus translational invariant.
- 2) DOF ΣX .

⑤ Results

Using Eq. (2.26) with $dl = dy$, we obtain

$$\vec{E}(P) = \frac{1}{4\pi\epsilon_0} \int_{-\infty}^{+\infty} dy \frac{\lambda}{x^2 + y^2} \vec{u}_{QP}.$$

Due to the symmetry argument in point ④, the only component of interest of \vec{E} is E_x :

$$\begin{aligned} E_x &= \frac{\lambda}{4\pi\epsilon_0} \int_{-\infty}^{+\infty} dy \frac{1}{x^2 + y^2} \cos\theta \\ &= -\frac{\lambda}{4\pi\epsilon_0} \int_{-\infty}^{+\infty} dy \frac{x}{(x^2 + y^2)^{3/2}} \\ &= -\frac{\lambda}{2\pi\epsilon_0 x}. \end{aligned}$$

Thus

$$\boxed{\vec{E}(P) = \frac{\lambda}{2\pi\epsilon_0 x} \vec{u}_x}$$

and the field diverges in correspondence of l .

• Example #3:

- ① Consider a charge distribution with constant surface distribution σ on an infinite plane Π .

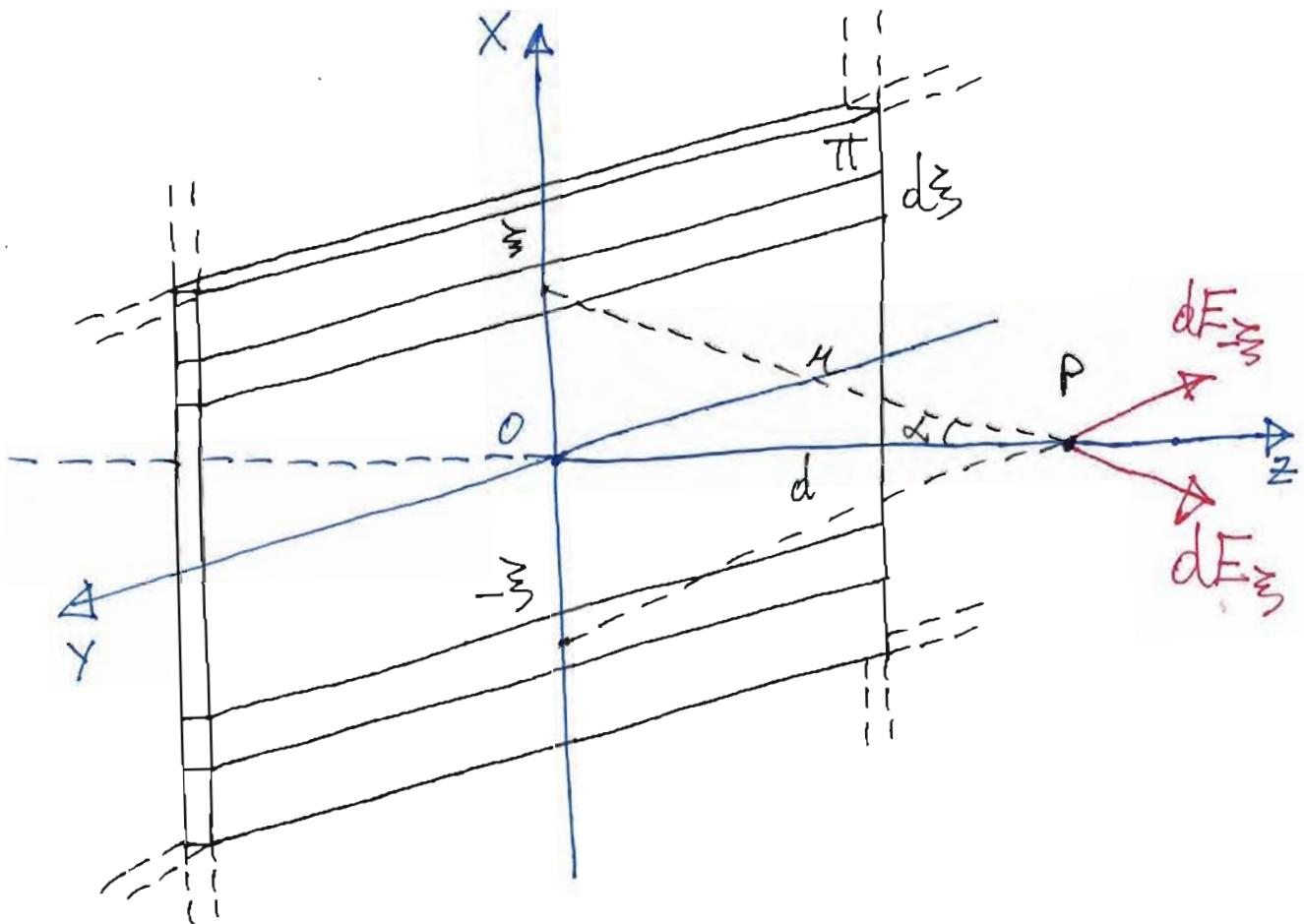


Figure 2.9.

Calculate the electric field generated by σ on a generic point P at distance d from Π .

- ② Preference frame

Confer Fig 2.9.

③ Indicate all fields

From the results in example #2, it follows that each line ξ generates an electric field directed as the line between ξ and P and perpendicular to ξ in the half plane containing ξ and P (cf. Fig 2.9).

④ DOF

1) Symmetry argument. The charge distribution σ on Π is indistinguishable from any point P on the left or right of Π . This is because any observer fixed in space would see the same charge distribution σ upon translating Π vertically and/or horizontally (translation symmetry). We thus expect the total electric field \vec{E} at P to only have a component along the \vec{z} -axis.

2) $\text{DOF} = 2$.

⑤ Results

$$\begin{aligned} d\vec{E}_\xi &= -\frac{\sigma d\xi}{2\pi\epsilon_0 r} \sin\theta \vec{u}_y \\ &\quad + \frac{\sigma d\xi}{2\pi\epsilon_0 r} \cos\theta \vec{u}_z, \end{aligned}$$

$$\vec{dE}_{\xi} = \frac{\sigma d\xi}{2\pi\epsilon_0 r} \sin \theta \vec{u}_y + \frac{\sigma d\xi}{2\pi\epsilon_0 r} \cos \theta \vec{u}_z .$$

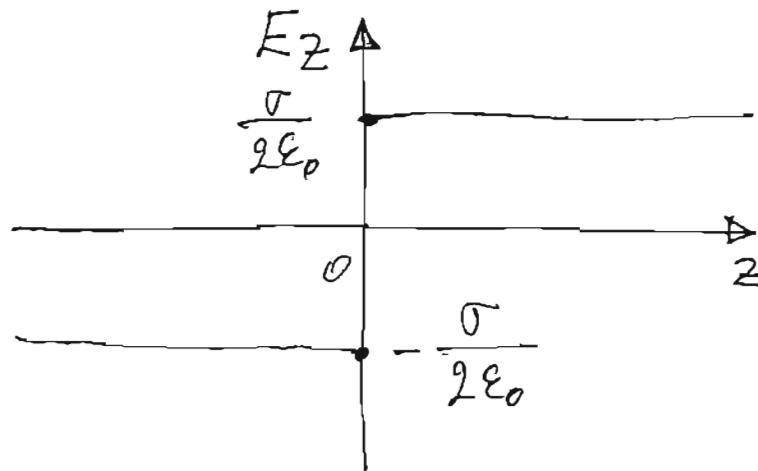
The y components cancel each other (as expected).
Thus,

$$d\vec{E} = 2 \frac{\sigma d\xi}{2\pi\epsilon_0 r} \cos \theta \vec{u}_z .$$

Integrating from 0 to $+\infty$:

$$\begin{aligned} \vec{E} &= \frac{\sigma d}{\pi\epsilon_0} \int_0^{+\infty} d\xi \frac{1}{d^2 + \xi^2} \vec{u}_z = \frac{\sigma}{\pi\epsilon_0} \int_0^{+\infty} dx \frac{1}{4+x^2} \vec{u}_z \\ &= \boxed{\frac{\sigma}{2\epsilon_0} \vec{u}_z} . \end{aligned}$$

The total electric field is independent from the distance between P and T.



Exercise 10.

Figure 10 shows that E_z has a first kind discontinuity for $y=0$: 23/23

$$[E_z] = \frac{\sigma}{\epsilon_0}$$

Summary lecture 2

- The electric field acting on the unitary positive charge due to a source-charge q_1 is defined as

$$\vec{F}_o(p) = q_o \vec{E}(p),$$

where

$$\vec{E}(p) = \frac{1}{4\pi\epsilon_0} \frac{q_1}{r_{10}^2} \vec{u}_{10}.$$

- Electric field due to N charges q_k in vacuum

$$\vec{E}(p) = \frac{1}{4\pi\epsilon_0} \sum_{k=1}^N \frac{q_k}{r_{ko}^2} \vec{u}_{ko} = \sum_{k=1}^N \vec{E}_k(p).$$

- Disturbance due to q_o

$$\vec{E}(p) = \lim_{q_o \rightarrow 0} \frac{\vec{F}_o(p)}{q_o}.$$

- Continuous charge distribution

- On a line γ :

$$\lambda(p) = \left. \frac{dq}{dl} \right|_p, Q = \int_{\gamma} dl \lambda(p),$$

- On a surface Σ :

$$\sigma(p) = \left. \frac{dq}{dS} \right|_p, Q = \iint_{\Sigma} dS \sigma(p).$$

3) On a volume Ω :

2/19

$$\sigma = \frac{dq}{dV}, Q = \iiint_{\Omega} dV \sigma(P).$$

• Electric field generated by a generic charge distribution

1) By a volume Ω :

$$\vec{E}(P) = \frac{1}{4\pi\epsilon_0} \iiint_{\Omega} dV \frac{\sigma(Q)}{r_{PQ}^2} \hat{u}_{QP}.$$

2) By a surface Σ :

$$\vec{E}(P) = \frac{1}{4\pi\epsilon_0} \iint_{\Sigma} dS \frac{\sigma(Q)}{r_{QP}^2} \hat{u}_{QP}.$$

3) By a line γ :

$$\vec{E}(P) = \frac{1}{4\pi\epsilon_0} \int_{\gamma} dl \frac{\lambda(Q)}{r_{QP}^2} \hat{u}_{QP}.$$

3.1 Gauss' theorem

3/19

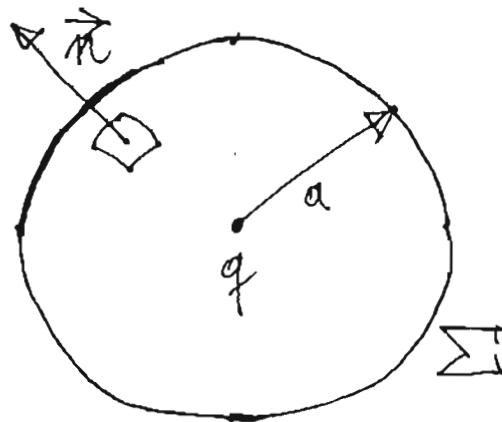


Figure 3.1.

Consider a charge q in the center of a fictitious sphere with surface Σ . \vec{n} is the normal to Σ , directed outward. The flux Φ_{Σ} of the electric field through Σ is:

$$\Phi_{\Sigma} = \iint_{\Sigma} \vec{E} \cdot \vec{n} dS.$$

Since the electric field \vec{E} generated by q is constant on Σ and is directed along \vec{n} , we obtain

$$\Phi_{\Sigma} = E 4\pi a^2.$$

Using Eq. (2.4) for the parameters in Fig. 3.1, we find

$$E = \frac{1}{4\pi\epsilon_0} \frac{q}{a^2}$$

and so

$$\oint \vec{E} \cdot d\vec{s} = \frac{q}{\epsilon_0} . \quad (3.1)$$

- Gauss' theorem for a point-like charge - The flux of the electric field through any closed surface containing a charge is proportional to the value q of the charge, independently from the shape and area of the surface.

Proof.

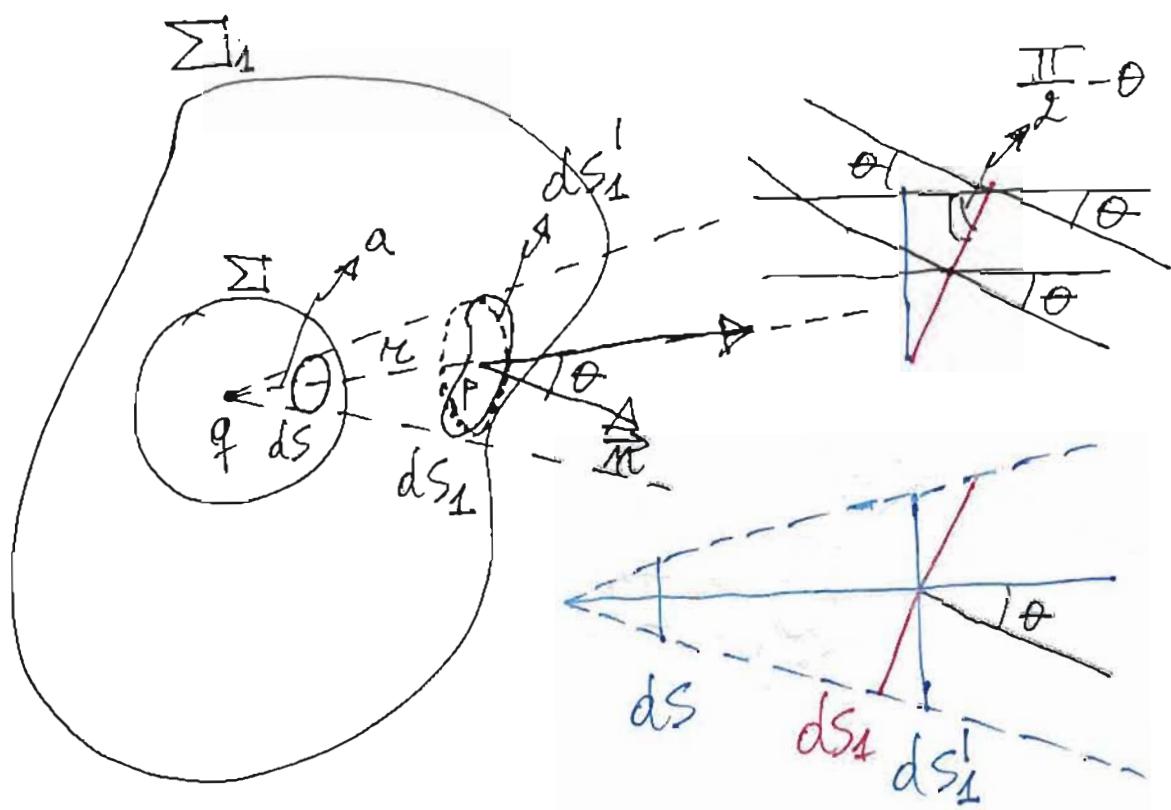


Figure 3.2.

We need to show that the flux through Σ_1 is the same as the flux through Σ . The infinitesimal cone with vertex in q defines an infinitesimal surface ds on Σ and ds_1 on Σ_1 .

The infinitesimal surface dS'_1 is defined by the cone with vertex in q , on the surface of the sphere centered in q and containing the center P of dS_1 . Hence,

$$\frac{dS'_1}{dS} = \frac{r^2}{a^2}, \quad (3.2)$$

where r is the distance between q and P and a the radius of Σ . From Fig. 3.2, it appears clear that

$$dS'_1 = dS_1 \sin\left(\frac{\pi}{2} - \theta\right) = dS_1 \cos\theta,$$

which, using Eq.(3.2), gives:

$$dS_1 = dS \frac{r^2}{a^2} \frac{1}{\cos\theta}.$$

We can now calculate the infinitesimal flux of \vec{E} through dS_1 :

$$d\Phi_1 = \vec{E}(P) \cdot \vec{n} dS_1 = E dS_1 \cos\theta.$$

By using Eq. (2.4), we finally obtain

$$d\Phi_1 = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} dS \frac{r^2}{a^2} \frac{1}{\cos\theta} \cos\theta$$

$$= \frac{1}{4\pi\epsilon_0} \frac{q}{a^2} dS = d\Phi,$$

By repeating this argument for each infinitesimal cone with vertex in q and summing, one obtains that the flux of \vec{E} through the closed surface Σ_1 is the same as that through Σ .

- Corollary - The flux of \vec{E} through any closed surface Σ that does not contain any charge q is zero.
- Extension to the case of a generic charge distribution
If a closed surface Σ contains N point-like charges q_1, q_2, \dots, q_N ,

$$\vec{E} = \sum_{k=1}^N \vec{E}_k .$$

From Gauss' theorem:

$$\iint_{\Sigma} \vec{E} \cdot \vec{n} dS = \sum_{k=1}^N \iint_{\Sigma} \vec{E}_k \cdot \vec{n} dS = \frac{1}{\epsilon_0} \sum_{k=1}^N q_k . \quad (3.3)$$

Similarly, for a continuous charge distribution inside a closed surface Σ :

$$\iint_{\Sigma} \vec{E} \cdot \vec{n} dS = \frac{1}{\epsilon_0} Q , \quad (3.4)$$

where Q is the total charge inside Σ .

3.3 Line integral of a vector field on an oriented curve

Consider a vector field \vec{A} defined in a domain Ω , and a curve γ contained in Ω with extremes M and N (cf. Fig. 3.3). The curve γ is oriented from M to N .

Consider the points P_0, P_1, \dots, P_n on γ . Each segment $P_k P_{k+1}$ is associated with a vector \vec{l}_k . The direction of \vec{l}_k is along the line $P_k P_{k+1}$ with same orientation as γ ; the magnitude of \vec{l}_k is the length l_k of segment $P_k P_{k+1}$. Consider the values $\vec{A}_0, \vec{A}_1, \dots, \vec{A}_n$ of \vec{A} at each point P_0, P_1, \dots, P_n , respectively, given

$$\vec{A}_k \cdot \vec{l}_k, \quad k = 0, 1, \dots, n,$$

we define

$$T_n = \sum_{k=0}^{n-1} \vec{A}_k \cdot \vec{l}_k.$$

If $\lim_{n \rightarrow \infty} T_n$ (where $\vec{l}_k \rightarrow 0$) converges to a finite value, this value is called the line integral of \vec{A} on γ :

$$T_\gamma = \lim_{n \rightarrow \infty} T_n = \int_{\gamma} \vec{A} \cdot d\vec{l}. \quad (3.9)$$

In the case of a closed curve γ , Eq. (3.9) becomes:

$$C_\gamma = \oint_{\gamma} \vec{A} \cdot d\vec{l}. \quad (3.10)$$

Defining $d\vec{l} = \vec{T} dl$, where \vec{T} is the unit vector of the

tangent to γ in the generic point where \vec{A} is considered 11/19
 and dl is the magnitude of the infinitesimal vector $d\vec{l}$,
 we can write:

$$T_\gamma = \int_{\gamma} \vec{A} \cdot \vec{t} dl \quad \text{and} \quad C_\gamma = \oint_{\gamma} \vec{A} \cdot \vec{t} dl. \quad (3.11)$$

In a cartesian frame:

$$\vec{A} \cdot d\vec{l} = A_x(x, y, z) dx + A_y(x, y, z) dy + A_z(x, y, z) dz, \quad (3.12)$$

where dx , dy , and dz are the components of $d\vec{l}$. Thus,

$$T_\gamma = \int_{\gamma} A_x dx + A_y dy + A_z dz, \quad (3.12)$$

$$C_\gamma = \oint_{\gamma} A_x dx + A_y dy + A_z dz.$$

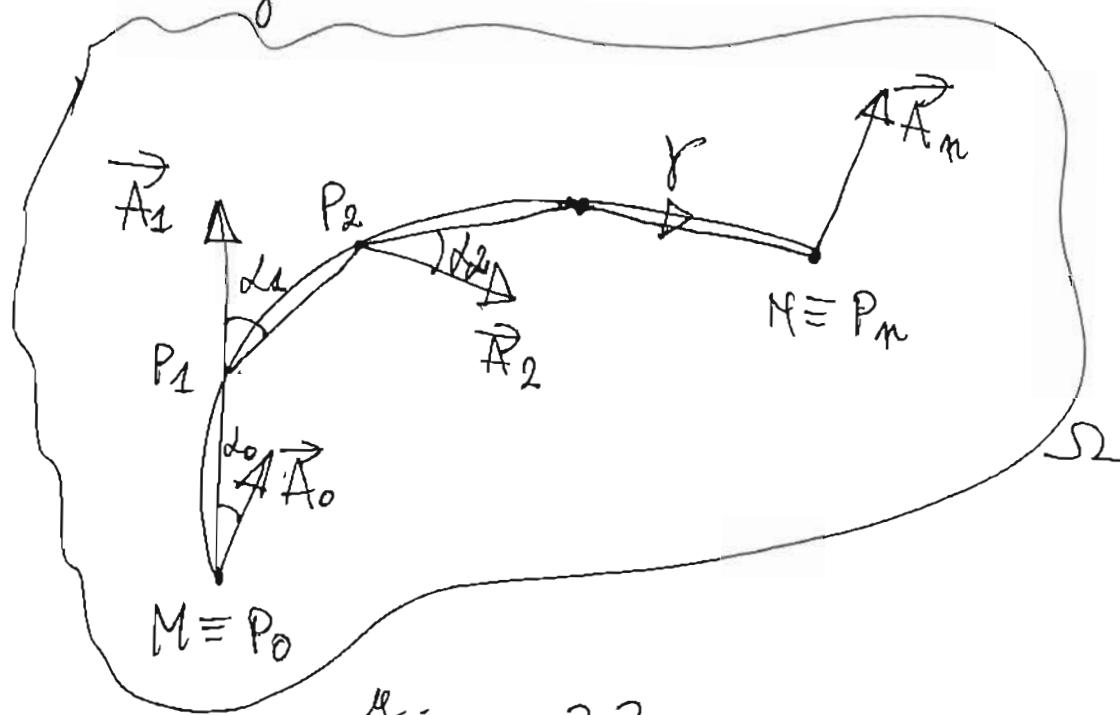


Figure 3.3.

Consider two points M and N on γ , dividing γ into two arcs (1) and (2), both oriented from M to N. 14/19

From Eq. (3.14), it follows that

$$\int_{(1)}^{\vec{A}} \cdot d\vec{l} = \int_{(2)}^{\vec{A}} \cdot d\vec{l} . \quad (3.16)$$

It is also true that

$$\oint_{\gamma}^{\vec{A}} \cdot d\vec{l} = \int_{(1)}^{\vec{A}} \cdot d\vec{l} - \int_{(2)}^{\vec{A}} \cdot d\vec{l} \quad (3.17)$$

and, thus,

$$\oint_{\gamma}^{\vec{A}} \cdot d\vec{l} = 0.$$

Similarly, it is possible to show that if \vec{A} satisfies Eq. (3.15), it must also satisfy Eq. (3.14). Therefore, Eqs. (3.14) and (3.15) are equivalent.

• Example #1

Consider a vector field \vec{A} defined in a domain Ω . Assume the field is directed along a straight line from a center O to each point P in Ω (central field; cf. Fig. 3.6). Further, assume \vec{A} has a spherical symmetry, i.e., it depends only on the distance r between P and O,

$$\vec{A}(P) = f(r) \vec{u}_r ,$$

where \vec{u}_r is the unit vector of the vector line through P and $f(r)$ a function describing how \vec{A} varies with r.

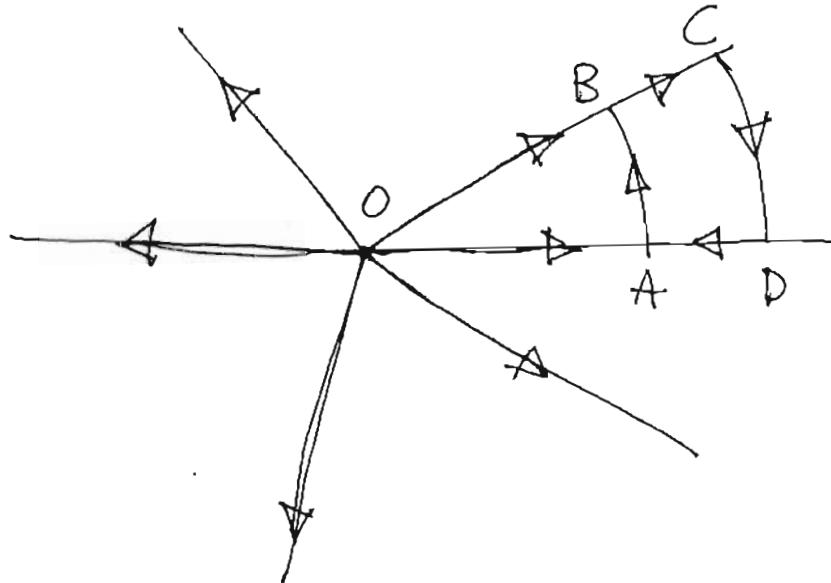


Figure 3.6.

Consider the closed line ABCD in Fig. 3.6. The segments AB and CD are arcs of circles with radius r_1 and r_2 , respectively. Hence,

$$\oint \vec{A} \cdot d\vec{l} = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA} .$$

The terms relative to AB and CD are zero because on those arcs \vec{A} and \vec{t} are orthogonal. The term relative to BC is:

3.4 Irrotational property of the electrostatic field
for a single point-like charge q_1 ,

$$\vec{E}(P) = \frac{1}{4\pi\epsilon_0} \frac{q_1}{r_{10}^2} \hat{u}_{10} . \quad (3.18)$$

According to Eq. (3.18), the vector field \vec{E} is of central type with spherical symmetry. As shown in example #1 in Lec. 3.3, this means that \vec{E} is an irrotational field: For any closed line γ in the domain of definition of \vec{E}

$$\oint_{\gamma} \vec{E} \cdot \vec{t} dl = 0 . \quad (3.19)$$

Note that this result also applies for lines passing through the source charge, as long as the integral is intended as the principal value of the Cauchy integral. The physical meaning of Eq. (3.19) is clear when thinking, based on the definition of \vec{E} , that its line integral is equal to the work generated by the field forces to move a unitary positive charge along the line. Equation (3.19) states that such a work along any closed line γ is zero. In other words, if the field forces generate a positive work along one segment of the line (i.e., the charge "falls" in the field along this segment), in the remaining part of the line they generate

From Eq. (3.49) it follows that each of the integrals in Eq. (3.21) is zero, hence 19/19

$$\oint_{\gamma} \vec{E} \cdot \vec{dl} = 0.$$

This result can readily be extended to the case of a continuous charge distribution. We can thus state the irrotational property of the electrostatic field:

The line integral of the electrostatic field generated by a generic distribution of source charges along any closed line is equal to zero.

It is worth mentioning that Gauss' theorem and the irrotational property of the electrostatic field are perfectly equivalent to Coulomb's law and the superposition principle (the only caveat being that Gauss' theorem and the irrotational property must be defined specifying the field behavior at infinity). In the next lecture, we will examine a few examples where it is more convenient to use Gauss' theorem and the irrotational property to solve problems instead of using directly Coulomb's law and the superposition principle.

Summary lecture 3

- **Gauss' theorem**

The flux of the electric field \vec{E} through any closed surface containing a charge is proportional to the value q of the charge, independently from the shape and area of the surface:

$$\oint_{\sum} \vec{E} \cdot \vec{n} dS = \frac{q}{\epsilon_0} .$$

- The flux of \vec{E} through any closed surface \sum that does not contain any charge q is zero.
- Gauss' theorem for N point-like charges q_1, q_2, \dots, q_N in a close surface \sum :

$$\sum_{k=1}^N \oint_{\sum} \vec{E}_k \cdot \vec{n} dS = \frac{1}{\epsilon_0} \sum_{k=1}^N q_k .$$

$$\oint_{\gamma} \vec{A} \cdot d\vec{l} = 0.$$

- Every vector field of central type with spherical symmetry has a potential.
- Irrotational property of the electrostatic field
The line integral of the electrostatic field generated by a generic distribution of source charges along any closed line is equal to zero;

$$\oint_{\gamma} \vec{E} \cdot d\vec{l} = 0.$$

- Gauss' theorem and the irrotational property field \vec{E} are equivalent to the Coulomb law and the superposition principle.

• Example #1:

① Consider again an infinite straight line γ with a linear charge distribution λ uniformly distributed along the line (λ can be positive or negative).

Calculate the electric field \vec{E} generated by λ at any generic point in space, P (cf. example #2, lecture, page 14; cf. Fig. 4.1). Do not use Coulomb's law and the superposition principle.

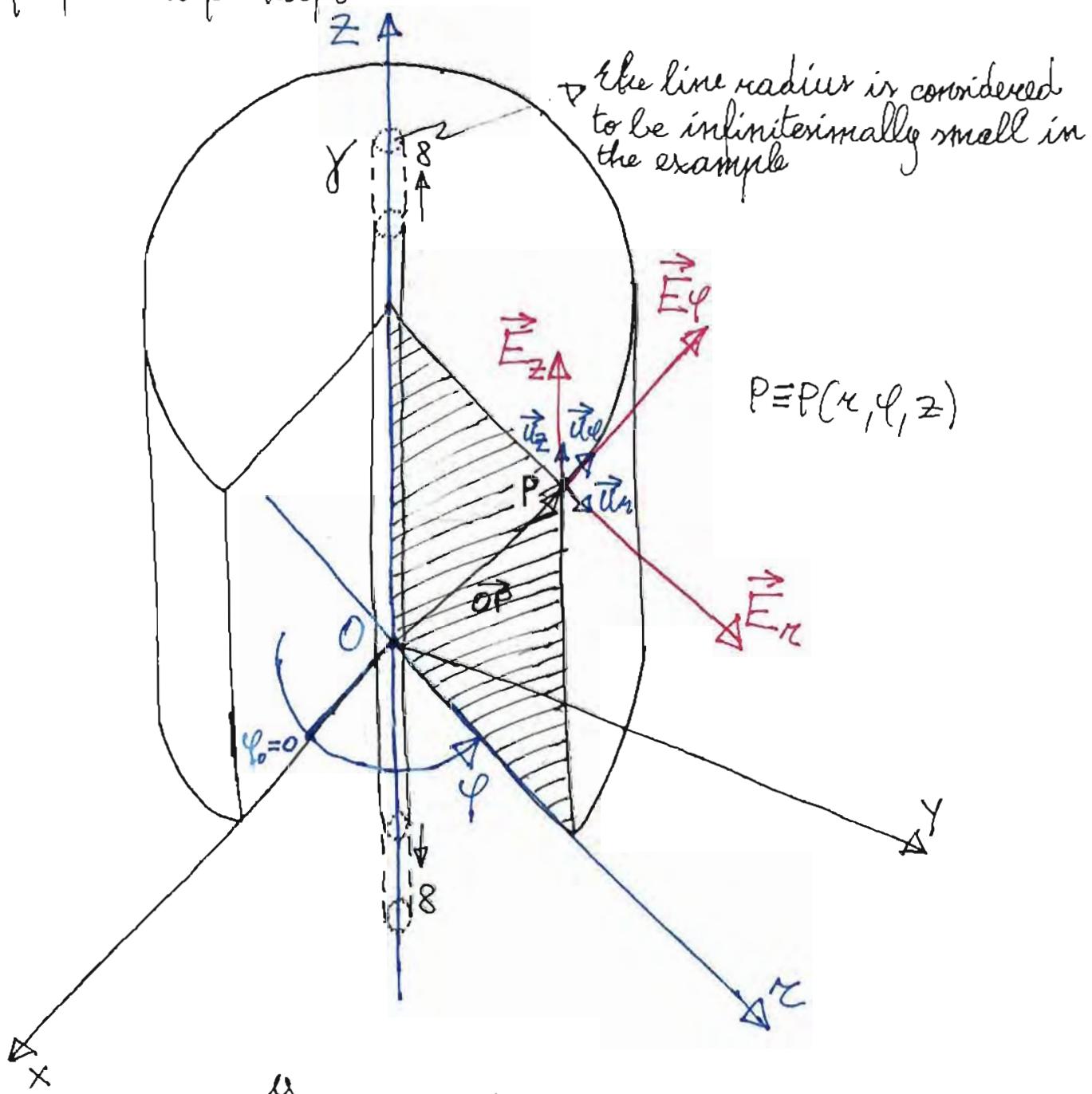


Figure 4.1.

② Reference frame:

The line γ can be thought as a cylinder of infinitesimal radius. As a consequence, the natural choice for the reference frame is a cylindrical reference frame with cylindrical coordinates (cf. next lecture for more details on various types of reference frames).

A cylindrical reference frame $Ox\varphi z$ is shown in Fig. 4.1 together with a condition $\varphi_0 = 0$ defining the zero of the φ coordinate. In this frame, a generic point in space, P , is defined as

$$P \equiv P(r, \varphi, z) \text{ and } \vec{OP} = r \vec{u}_r + \varphi \vec{u}_\varphi + z \vec{u}_z,$$

where \vec{u}_r , \vec{u}_φ , and \vec{u}_z are the unit vectors for the $Ox\varphi z$ frame and r , φ , and z the magnitude of each spatial component of vector \vec{OP} .

③ Indicate all components of the electric field \vec{E} :

The three components of the electric field \vec{E} with respect to the $Ox\varphi z$ reference frame at a generic point in space, P , are shown in Fig. 4.1:

$$\begin{aligned} \vec{E}(P) &= \vec{E} = \vec{E}_r + \vec{E}_\varphi + \vec{E}_z \\ &\quad | \\ &= E_r \vec{u}_r + E_\varphi \vec{u}_\varphi + E_z \vec{u}_z, \quad (4.1) \end{aligned}$$

where E_r , E_φ , and E_z are the magnitudes of the components \vec{E}_r , \vec{E}_φ , and \vec{E}_z , respectively. A magnitude can be a positive or negative number (it is not an absolute value). Note that \vec{E}_r is called the radial component, \vec{E}_φ the tangent component, and \vec{E}_z the vertical component of \vec{E} .

④ DOF:

- Symmetry arguments.

We will use three symmetry arguments to gain as much information as possible on the components of \vec{E} . The symmetry arguments refer to the spatial properties of γ , regardless of any field or reference frame.

(a) Rotation symmetry.

γ remains unchanged after a rotation of any arbitrary angle about its own axis (the rotation can be clockwise or counterclockwise; cf. Fig. 4.2a).

(b) Translation symmetry.

γ remains unchanged after a translation of any arbitrary displacement (upward or downward) along its own axis (cf. Fig. 4.2b).

(c) Reflection symmetry.

γ remains unchanged after a π rotation about any point o' on γ (the rotation can be clockwise or counterclockwise; cf. Fig. 4.2c).

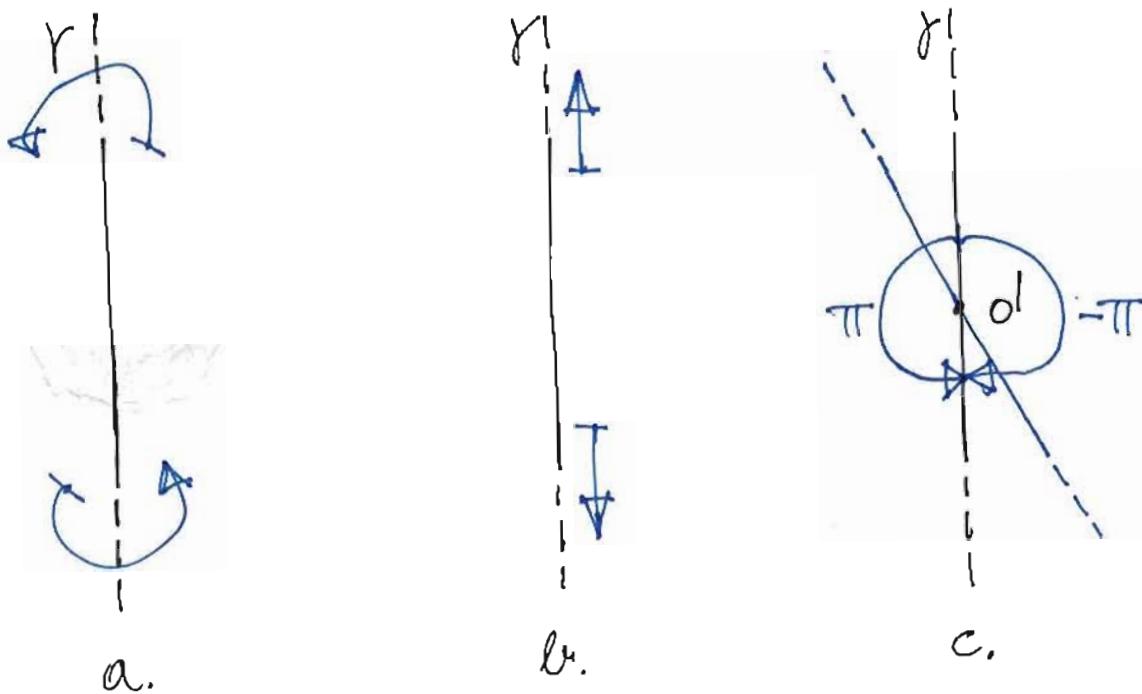


Fig. 4.2.

In the remainder of this section, we will use combinations of these three symmetry arguments.

1) Radial component E_r of the electric field \vec{E} .

In general, E_r can be directed inward or outward at different points in space. The magnitude E_r can be different at any point.

Consider a circle γ_1 lying on a plane perpendicular to γ passing through P and with center on γ , O (cf. Fig. 4.3 a). We assume E_r to be directed outward in P (this will depend on the sign of λ). Hereafter, when performing a transformation on γ (e.g., a rotation about its axis, a translation along its axis, or a reflection about one of its

81

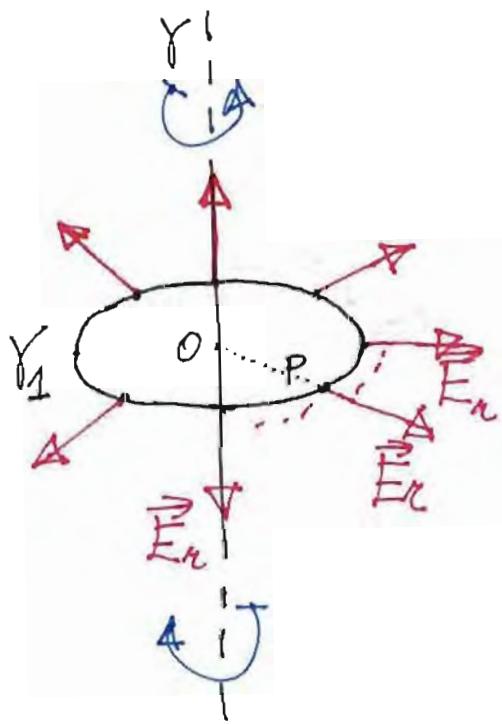
points), the line γ , the component \vec{E}_x , and its application point P are considered to be a single rigid body. Consequently, any transformation on γ applies directly to \vec{E}_x and P . The same assumption will be used for any charge distribution, transformation, electric field \vec{E} , and its application point P in the following.

Therefore, due to the rotation symmetry of γ , the component \vec{E}_x must be the same (direction, magnitude, and sign) at each point of γ_1 (cf. Fig. 4.3a).

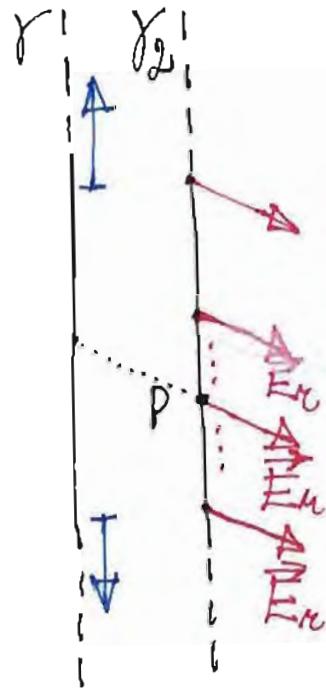
Consider an infinite straight line γ_2 passing through P and parallel to γ (cf. Fig. 4.3b). Due to the translation symmetry of γ , \vec{E}_x must be the same at each point on γ_2 .

We now must perform a consistency check with the reflection symmetry of γ . Consider \vec{E}_x at P and perform a π rotation of γ , \vec{E}_x , and P about a point O on γ different from P . After the transformation, P becomes P' and \vec{E}_x remains the same: γ, P , and \vec{E}_x become γ, P' , and \vec{E}_x . Point P' is on a different plane perpendicular to γ compared to P . However, due to the translation symmetry of γ , P' and \vec{E}_x at P' can be shifted upward to the same plane of P , thus obtaining a new point P'' with \vec{E}_x applied to it. By rotating \vec{E}_x at P'' by an angle π

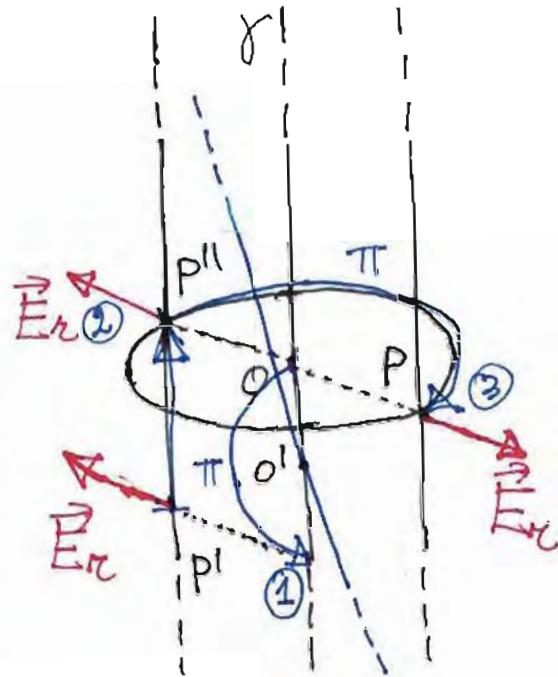
about the axis of γ , we consistently obtain again [9] \vec{E}_r at P. A component \vec{E}_r with $E_r \neq 0$ can thus exist. We also expect E_r to go to zero at infinity.



a.

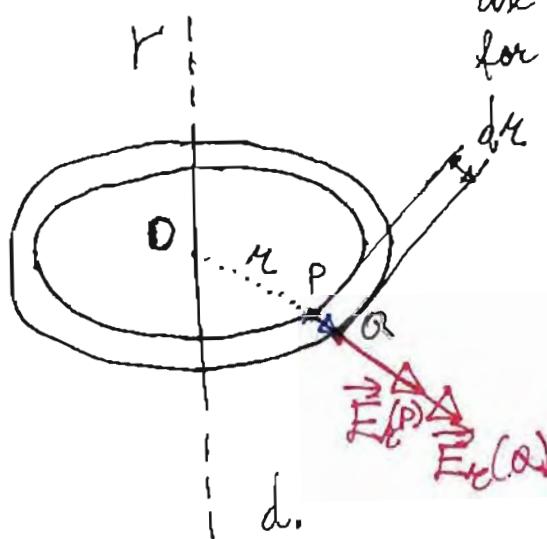


b.



c.

①: Rotation of Y , P, and \vec{E}_r by π about O' (top). ②: Upward translation from P' to P'' . ③: Rotation by π about γ from P'' back to P. Note that we could use $O' = O$ for ①. We used O' for generality.



d.

Fig. 4.3.

We expect \vec{E}_x to be a continuous vector function away from γ . As a consequence, if \vec{E}_x at a point P on a plane perpendicular to γ with distance r from the intersection point O between the plane and γ is directed outward, \vec{E}_x at a point Q on the same plane, but with distance $r+dr$ from O should also be directed outward. By extension, \vec{E}_x should be directed outward everywhere in space. This consideration is consistent with the uniformity of λ : If $\lambda > 0$, \vec{E}_x will be directed outward (remember the electric field is the vector field associated with the force acting on a positive test charge) and if $\lambda < 0$, \vec{E}_x will be directed inward. E_x will go to zero at infinity.

In summary, the symmetries of the line γ with λ allow for a radial component $\vec{E}_r = E_r \hat{r}_r$ of \vec{E} . The component must be directed everywhere in space outward for $\lambda > 0$ and inward for $\lambda < 0$. E_r must go to zero at infinity.

2) Tangent component \vec{E}_φ of the electric field \vec{E} .

In general, \vec{E}_φ can be directed clockwise or counterclockwise at different points in space. The magnitude E_φ can be different at any point.

Consider a circle γ_1 lying on a plane perpendicular to γ passing through P and with center on γ (O) (cf. Fig. 4.4 a). We assume \vec{E}_φ to be directed counterclockwise in P . Due to the rotation symmetry

of γ , the component \vec{E}_φ must be the same (direction, magnitude, and sign) at each point of γ_1 (cf. Fig. 4.4a). [11]

Consider an infinite straight line γ_2 passing through P and parallel to γ (cf. Fig. 4.4b). Due to the translation symmetry of γ , \vec{E}_φ must be the same at each point on γ_2 .

Therefore, \vec{E}_φ must be the same at each point on the lateral surface of any cylinder with central axis on γ . In general, \vec{E}_φ can be different on different cylinders.

We now must perform a consistency check with the reflection symmetry of γ . Consider \vec{E}_φ at P and perform a counterclockwise rotation of γ , P , and \vec{E}_φ at P about a point O' on γ different from O . After the transformation, γ remains the same, P becomes P' , and \vec{E}_φ becomes a vector \vec{E}'_φ with same magnitude as \vec{E}_φ , E_φ , but directed clockwise (i.e., opposite of \vec{E}_φ) and applied to P' (cf. Fig. 4.4c). Point P' is on a different plane perpendicular to γ compared to P . However, due to the translation symmetry of γ , P' and \vec{E}'_φ can be shifted upward to the same place of P , thus obtaining a new point P'' with \vec{E}'_φ applied to it. By rotating γ , P'' , and \vec{E}'_φ at P'' by an angle π clockwise about the axis of γ , we must obtain the original component \vec{E}_φ at P . However, we obtain \vec{E}'_φ at P , which is inconsistent with \vec{E}_φ at P since \vec{E}_φ and \vec{E}'_φ have opposite directions. In order to reconcile this inconsistency with the

3) Vertical component \vec{E}_z of the electric field \vec{E} . 131

In general, \vec{E}_z can be directed upward or downward at different points in space. The magnitude E_z can be different at any point.

Consider a circle γ_1 lying on a plane perpendicular to γ passing through P and with center on γ , o (cf. Fig. 4.5a). We assume \vec{E}_z to be directed upward in P . Due to the rotation symmetry of γ , the component \vec{E}_z must be the same (direction, magnitude, and sign) at each point of γ_1 (cf. Fig. 4.5a).

Consider an infinite straight line γ_2 passing through P and parallel to γ (cf. Fig. 4.5b). Due to the translation symmetry of γ , \vec{E}_z must be the same at each point on γ_2 .

Therefore, as in the case of \vec{E}_ϕ , \vec{E}_z must be the same at each point on the lateral surface of any cylinder with central axis on γ . In general, \vec{E}_z can be different on different cylinders.

We now must perform a consistency check with the reflection symmetry of γ . Consider \vec{E}_z at P and perform a counterclockwise rotation of γ , P , and \vec{E}_z at P about a point o' on γ different from o . After the transformation, γ remains the same, P becomes P' , and \vec{E}_z becomes a vector \vec{E}'_z with same magnitude as \vec{E}_z , E_z , but directed downward (i.e., opposite of \vec{E}_z) and applied to P' (cf. Fig. 4.5c). Point P' is on a different

plane perpendicular to γ compared to P. However, due to the translation symmetry of γ , P' and \vec{E}_z' at P' can be shifted upward to the same plane of P, thus obtaining a new point P'' with \vec{E}_z' applied to it. By rotating γ , P'' , and \vec{E}_z' at P'' by an angle π clockwise about the axis of γ , we must obtain the original component \vec{E}_z at P. However, we obtain \vec{E}_z' at P, which is inconsistent with \vec{E}_z at P since \vec{E}_z and \vec{E}_z' have opposite directions. In order to reconcile this inconsistency with the symmetry properties of γ , it must be $E_z = E_z' = 0$.

In summary, the symmetries of the line γ with λ do not allow for a vertical component $\vec{E}_z = E_z \hat{u}_z$ of \vec{E} . Note that, also in this case, we do not need to consider any further argument on the radial dependence of \vec{E}_z (i.e., an argument similar to that of Fig. 4.3d, but for \vec{E}_z). This is because $\vec{E}_z = \vec{0}$ everywhere. Also note that this finding is consistent with the zero-field condition at infinity.

4) $DOF = \epsilon$.

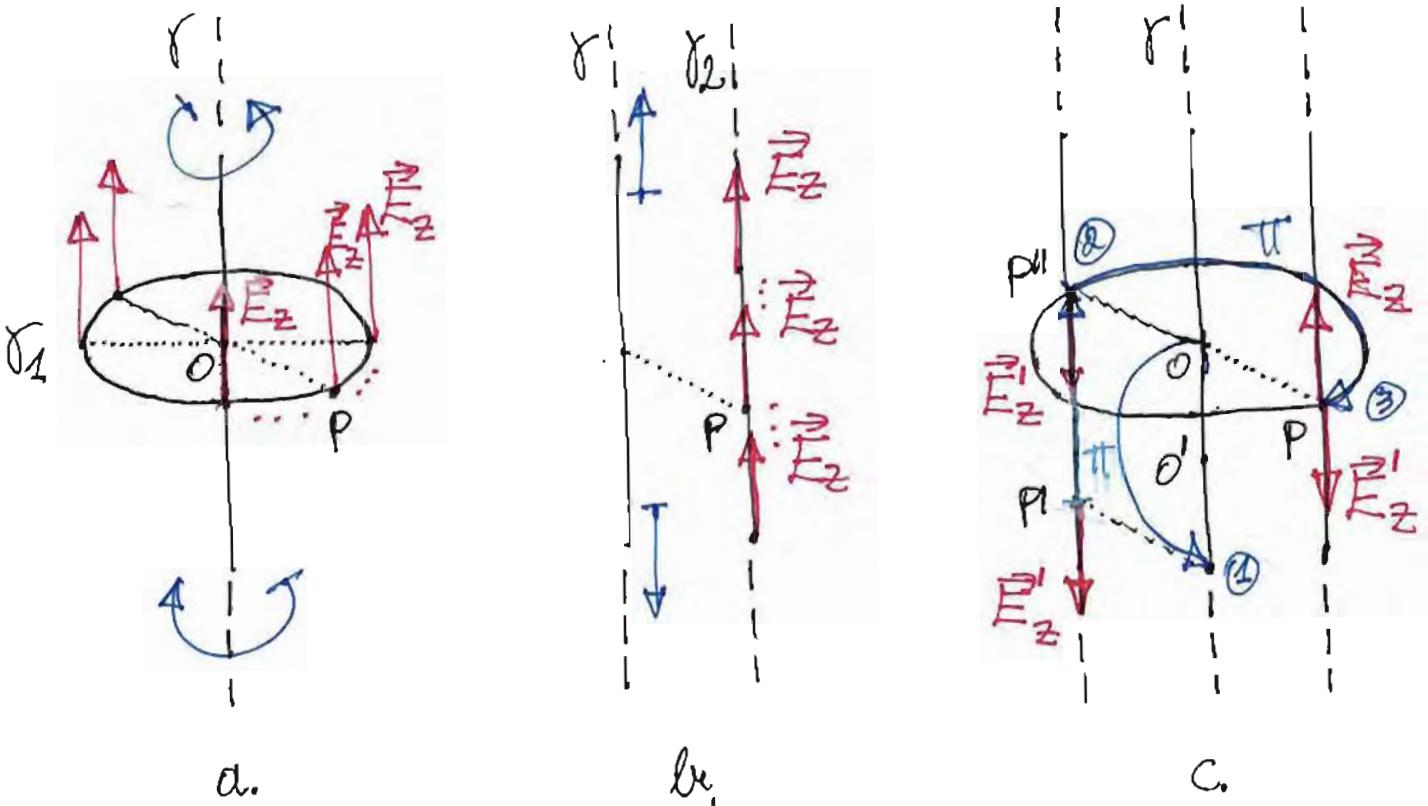


Figure 4.5.

- ①: Rotation of γ , P , and \vec{E}_z by π about O to γ , P' , and \vec{E}'_z .
- ②: Upward translation from γ , P' , and \vec{E}'_z to γ , P'' , and \vec{E}''_z .
- ③: Rotation by π about γ from γ , P'' , and \vec{E}''_z to γ , P , and $\vec{E}'_z \neq \vec{E}_z$.

⑤ Results:

Considering that $\vec{E}_\varphi = \vec{E}_z = \vec{0}$ and \vec{E}_r must be directed either inward or outward everywhere in space, we can resort to Gauss' theorem to calculate E_r .

Due to the rotation and translation symmetry of γ , \vec{E}_r must be the same at each point of the lateral surface of any infinite cylinder with axis on γ . In order to calculate E_r everywhere in space, we thus consider an arbitrary closed cylindric surface Σ of axis γ and

passing through P. The lateral surface of Σ is S [16]
 with height h and its bases are S_1 and S_2 (cf. Fig.
 4.6). Gauss' theorem for surface Σ reads:

$$\begin{aligned} \sum \iint_{\Sigma} \vec{E} \cdot \vec{n} dS &= \iint_S \vec{E} \cdot \vec{n} dS \\ &\quad + \iint_{S_1} \vec{E} \cdot \vec{n} dS + \iint_{S_2} \vec{E} \cdot \vec{n} dS \\ &= \frac{Q_h}{\epsilon_0}, \quad (4.2) \end{aligned}$$

where Q_h is the charge associated with the portion of γ of length h contained in Σ . The only nonzero component of \vec{E} , E_x , is tangent to both surface S_1 and S_2 and normal to surface S. As a consequence,

$$\iint_{S_1} \vec{E} \cdot \vec{n} dS = \iint_{S_2} \vec{E} \cdot \vec{n} dS = 0$$

and

$$\sum \iint_{\Sigma} \vec{E} \cdot \vec{n} dS = \iint_S \vec{E}_x \cdot \vec{n} dS = \iint_S E_x \vec{u}_x \cdot \vec{n} dS$$

$$\int_S \vec{E}_r dS = E_r \iint_S dS$$

$$= E_r 2\pi r h = \frac{Q_h}{\epsilon_0}.$$

In the above equation, we have expressed both \vec{E}_r and \vec{n} according to the reference frame chosen in point ②, $\vec{E}_r = E_r \vec{u}_r$ and $\vec{n} = \vec{u}_r$. In addition, we have assumed the radius of S_1 and S_2 to be r (cf. Fig. 4.6).

Due to the arbitrary choice of Σ , we can thus conclude that

$$E_r = \frac{Q_h}{2\pi\epsilon_0 rh} = \frac{\lambda}{2\pi\epsilon_0 r}$$

and

$$\boxed{\vec{E}_r = \frac{\lambda}{2\pi\epsilon_0 r} \vec{u}_r . \quad (4.3)}$$

The result of Eq. (4.3) is the same as that obtained in example #2, lecture 2, page

In summary, the electric field \vec{E} generated by a linear charge distribution λ on an infinite straight line γ at any generic point P in space is

$$\vec{E}(P) = \vec{E}_x + \vec{E}_\varphi + \vec{E}_z ,$$

with

$$\left\{ \begin{array}{l} \vec{E}_x = \frac{\lambda}{2\pi\epsilon_0 r} \hat{u}_x \\ \vec{E}_\varphi = \vec{0} \\ \vec{E}_z = \vec{0} \end{array} \right. ,$$

where r is the minimum distance between P and γ . As expected, each component of \vec{E} goes to zero at any point P at infinite distance from γ ;

$$\lim_{P \rightarrow \infty} \vec{E}(P) = \lim_{r \rightarrow P+\infty} \vec{E}_x(r) = \vec{0} .$$

- Discussion:

Without resorting to any symmetry argument, Gauss' theorem for an arbitrary closed cylindrical surface Σ would not suffice to calculate any of the components of the electric field \vec{E} (\vec{E}_x , \vec{E}_φ , and \vec{E}_z) generated by a linear charge distribution λ on an infinite straight line γ at any generic point P in space.

In fact, without the knowledge that \vec{E}_x must be the same at each point of the lateral surface of any infinite cylinder with axis on γ , it would be impossible to calculate $\iint_S \vec{E}_x \cdot \vec{n} dS$. The fact that \vec{E}_y

does not contribute to the total flux of \vec{E} through S because \vec{E}_y is perpendicular to \vec{n} on S , S_1 , and S_2 , gives no information on \vec{E}_y . According to Gauss' theorem \vec{E}_y can be nonzero everywhere in space without adding any contribution to the flux through S . At last, without any knowledge on \vec{E}_z , it would be impossible to calculate $\iint_{S_1, S_2} \vec{E}_z \cdot \vec{n} dS$. Hence, the symmetry

arguments are a necessary condition to apply Gauss' theorem to S .

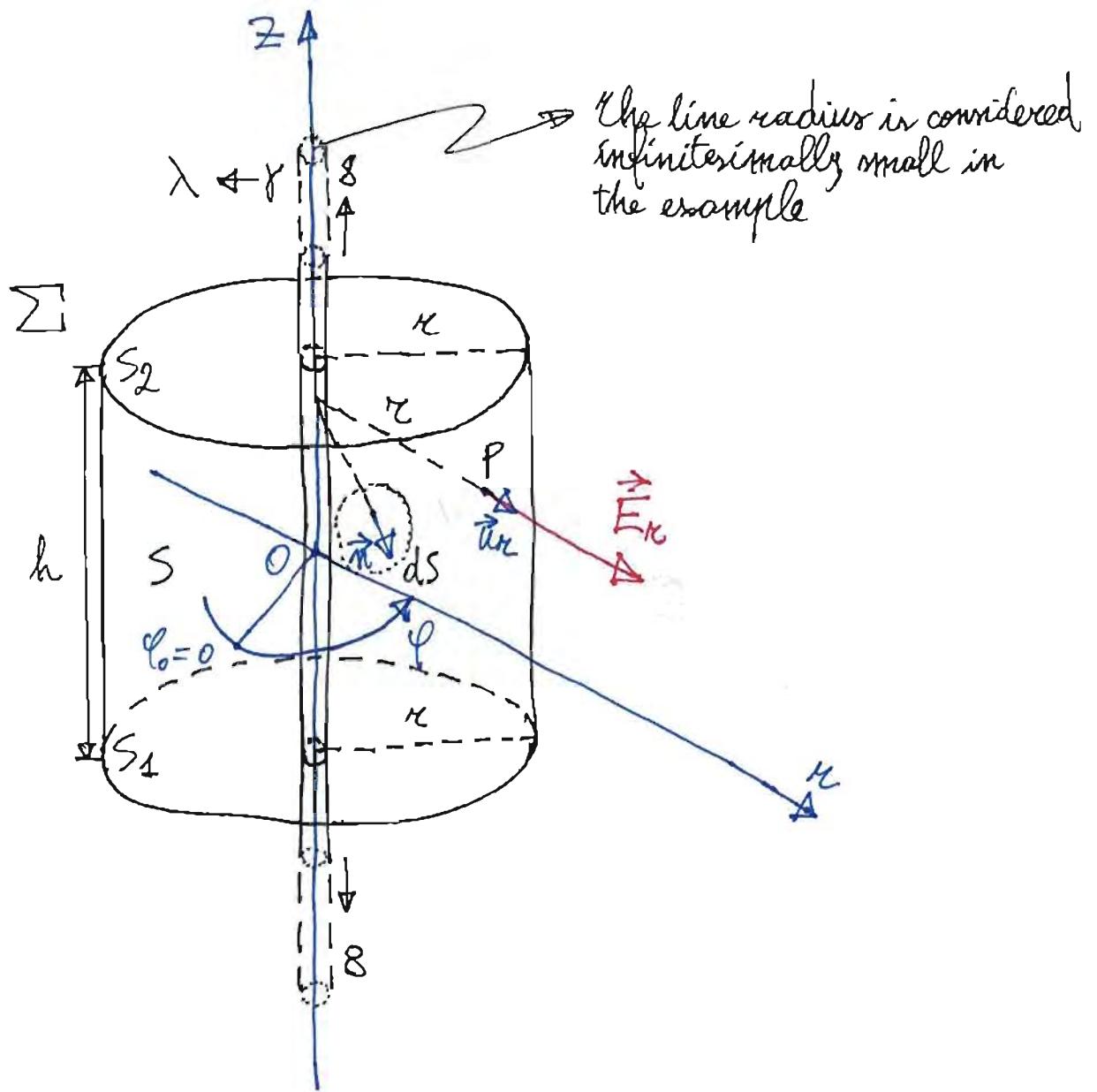


Figure 4.6.

Assume now we do not trust any symmetry argument to find properties of the components of \vec{E} . Instead, we intend to solely resort to, (a), the irrotational property of the electric field \vec{E} , (b), the zero-field condition at infinity, and, (c), Gauss' theorem to calculate all components of \vec{E} . From lecture 3, page 19, we know that properties (a), (b), and (c) are equivalent to Coulomb's law and the superposition principle. Hence, it must be possible to solve the present example without using any symmetry argument. In the remainder of this lecture, we will show that this is indeed true. Using properties (a), (b), and (c) is actually the most general way to solve an electrostatic problem, equally general as using Coulomb's law and the superposition principle. More general than using property (c) and symmetry arguments. Obviously, all three approaches should give the same result.

- Without symmetry arguments.

We will now use three fundamental properties of \vec{E} to gain as much information as possible on the components of \vec{E} .

(a) Irrotational property of \vec{E} ,

For any given closed, oriented curve γ ,

$$\oint \vec{E} \cdot \vec{t} d\ell = 0,$$

where \vec{t} is the unit vector tangent to γ and directed as γ . $d\ell$ is an

(b) Infinitesimal fine element along γ .
Zero-field condition at infinity:

$$\lim_{P \rightarrow \infty} \vec{E}(P) = \vec{0}.$$

22/

(c) Gauss' theorem.

For any given closed surface Σ ,

$$\iint_{\Sigma} \vec{E} \cdot \vec{n} dS = \frac{Q}{\epsilon_0},$$

where \vec{n} is the unit vector normal to dS and directed outward with respect to Σ , Q is the charge contained in Σ , and ϵ_0 the dielectric constant of vacuum. dS is an infinitesimal surface element on Σ .

In the remainder of this section, we will use combinations of these three properties.

1) Radial component E_r of the electric field \vec{E} .

In general, E_r can be directed inward or outward at different points in space. The magnitude E_r can be different at any point.

- ~~Plane or perpendicular to Σ~~

Consider a closed curve γ_3 oriented counterclockwise and lying on a plane perpendicular to γ , the curve γ_3 is a continuous piecewise curve composed by four open curves: Line segment AB, arc BC, line segment CD, and arc DA. Line segments AB and CD are assumed to be directed radially with respect

to γ , and arcs BC and DA , respectively, to be portions of two circles with different radii lying on the plane perpendicular to γ and with center on γ (cf. Fig. 4.4). As a consequence, $\overline{AB} = \overline{CD}$ (where the bar symbol indicates a straight length). It is further assumed that AB and CD are of infinitesimal length, $\overline{AB} = \overline{CD} = |dl|$. In the cylindrical reference frame shown in Fig. 4.4a, point A is at distance $r_A = r + dr$ from γ , point B is at distance $r_B = r$ from γ , point C at distance $r_C = r$ from γ and point D at distance $r_D = r + dr$ from γ . Because of the infinitesimal nature of line segments AB and CD , arcs BC and DA can be assumed to have the same length. Note that the length of a line segment is considered to be a positive number. This is why, e.g., $\overline{AB} = \overline{CD} = |dl|$ (absolute value).

We now want to use the irrotational property of \vec{E} for curve γ_3 . Due to the construction of γ_3 (AB and CD are infinitesimals, \vec{E}_r can be considered to have a constant magnitude on line segments AB and CD). For the sake of generality, here we assume the radial component of \vec{E} on AB , \vec{E}_{r1} , to be directed outward with respect to γ and with magnitude E_{rs} . In the cylindrical reference frame, \vec{E}_{r1} can be written as

$$\vec{E}_{r1} = E_{rs} \hat{u}_r , \quad (4.4)$$

for $r \in [r_A, r_B]$. On AB , the infinitesimal line

vector element $d\vec{l}_1$, being directed along γ , has opposite direction of \vec{E}_{41} . In the cylindrical reference frame, $d\vec{l}_1$ can be written as

$$d\vec{l}_1 = \vec{E}_1 d\vec{l}_1 = -\vec{u}_r d\vec{l}_1 . \quad (4.5)$$

In order to write the magnitude of $d\vec{l}_1$, $d\vec{l}_1$, in the cylindrical reference frame, line segment AB must be parametrized opportunely in such a reference frame. From line integral theory, $d\vec{l}_1$ can generally be written as

$$d\vec{l}_1 = f_1'(r) dr , \quad (4.6)$$

where $f_1(r)$ is a function that parametrizes AB as a function of the coordinate r (we assumed AB to be directed radially on a plane perpendicular to γ ; hence, it is reasonable to parametrize AB as a function of the radial coordinate only). $d\vec{l}_1$ represents a positive infinitesimal distance between two consecutive points on AB, going from A to B, i.e., consistently with the counterclockwise orientation of γ . Since we assumed AB to be infinitesimally small, we can use A and B as the two consecutive points. This argument is independent from the choice of a specific reference frame. However, a positive difference from A to B, becomes a negative difference with respect to the radial coordinate of

the cylindrical reference frame. Thus,

$$f_1(r) = -k$$

and

$$dl_1 = -dr, \quad (4.7)$$

as expected. Equation (4.5) then reads:

$$d\vec{l}_1 = dr \vec{t}_r. \quad (4.8)$$

The length (to be a positive quantity) of an oriented curve is the line integral of a unitary vector field along the curve orientation. For example,

$$\overline{AB} = \int_{AB} \vec{t}_1 \cdot d\vec{l}_1 = \int_{AB} \vec{t}_1 \cdot \vec{t}_1 dl_1 = \int_{AB} dl_1 > 0.$$

In the cylindrical reference frame,

$$\begin{aligned} \overline{AB} &= \int_{r_A}^{r_B} -u_r \cdot \underbrace{\vec{t}_r dr}_{dl_1} = \int_{r_B}^{r_A} dr = [r]_{r_B}^{r_A} \\ &= r_A - r_B = r + dr - r = dr > 0. \quad (4.9) \end{aligned}$$

This is a consistency check with the result of Eq. (4.8). In fact, it is a common mistake to write

$d\vec{l}_1 = -dr \vec{u}_r$ under the wrong assumption
 $d\vec{l}_1 = dr$. If we were to use $-dr \vec{u}_r$ for $d\vec{l}_1$ in
the integral of Eq.(4.9), we would obtain

$$\overline{AB} = \int_{r_A}^{r_B} -\vec{u}_r \cdot (-\vec{u}_r) dr = [r]_{r_A}^{r_B}$$

$$= r_B - r_A = r - (r + dr) = r - r - dr = -dr < 0.$$

Note that $d\vec{l}_1 = d\vec{l}|_{AB}$ by definition.

Due to the construction of γ_3 (AB and CD are infinitesimals), \vec{E}_φ can be considered to be the same at each pair of points $(P_B^{\gamma_2}, P_D^{\gamma_2})$ at the intersection between a semi-infinite radial line from γ and arcs BC and DA (cf. Fig. 4.4). In fact, the two arcs are infinitesimally close to each other. In general, \vec{E}_φ can be different at any pair of such points.

A similar argument as for E_{r1} can be used for the radial component of \vec{E} on CD , E_{r2} . For the sake of generality, we assume E_{r2} to be directed inward with respect to γ and with magnitude E_{r2} . In the cylindrical reference frame, \vec{E}_{r2} can be written as

$$\vec{E}_{r2} = -E_{r2} \vec{u}_r . \quad (4.10)$$

On CD , the infinitesimal line vector element $d\vec{l}_2$, being directed along γ , has opposite direction of \vec{E}_{r2} .

In the cylindrical reference frame, $d\vec{l}_2$ can be written as [24]

$$d\vec{l}_2 = \vec{t}_2 dl_2 = \vec{u}_x dl_2$$

with

$$dl_2 = f'_2(u) du.$$

In this case, since an infinitesimal difference between two consecutive points on CD is the same as along r in the cylindrical reference frame,

$$f'_2(u) = r.$$

Hence,

$$d\vec{l}_2 = dr \vec{u}_x. \quad (4.11)$$

Note that $d\vec{l}_2 = d\vec{l}|_{CD}$ by definition.

We now have in hand a representation of all components necessary to calculate $\oint_3 \vec{E} \cdot d\vec{l}$ both

independently from any reference frame and in the cylindrical reference frame:

on AB, \vec{E}_{r1} is directed outward from r

$$= E_{r1} \vec{u}_r$$

$d\vec{l}_1$ is directed inward towards r

$$= dr \vec{u}_x ,$$

on BC, $\vec{E}_\varphi =$ can be directed clockwise or counterclockwise
 $= \vec{E}_\varphi(P_{BC})$

$d\vec{l}|_{BC} =$ is directed counterclockwise,
 on CD, $\vec{E}_{r_2} =$ is directed inward towards γ
 $= -E_{r_2} \vec{u}_x$

$d\vec{l}_2 =$ is directed outward from γ
 $= dr \vec{u}_x ,$

on DA, $\vec{E}_\varphi =$ can be directed clockwise or counterclockwise
 $= \vec{E}_\varphi(P_{DA})$

$d\vec{l}|_{DA} =$ is directed counterclockwise.

In addition, $\vec{E}_\varphi(P_{BC}^r) = \vec{E}_\varphi(P_{DA}^r)$ for each pair of points (P_{BC}^r, P_{DA}^r) on arcs BC and DA and $d\vec{l}|_{DA} = -d\vec{l}|_{BC}$. However, we do not know the functional dependence of \vec{E}_φ on arcs BC and DA.

The components of \vec{E} and the corresponding

[20]

elements $d\vec{l}$ on AB, BC, CD, and DA of γ_3 are indicated in Fig. 4.4a independently from any reference frame. The components \vec{E}_{r1} and \vec{E}_{r2} and the corresponding elements $d\vec{l}_1$ and $d\vec{l}_2$ on AB and CD of γ_3 , respectively, are indicated in Fig. 4.4b for the cylindrical reference frame.

Due to the irrotational property of \vec{E} :

$$\oint_{\gamma_3} \vec{E} \cdot d\vec{l} = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA} = 0 \quad . \quad (4.12)$$

We will calculate the line integral of Eq.(4.12), (i), independently from any reference frame and, (ii), in the cylindrical reference frame.

(i) In the light of all our assumptions and referring to Fig. 4.4a:

$$\int_{AB} \vec{E} \cdot d\vec{l} = E_{r1} \cos \pi \int_{AB} dl_1 + \cos \frac{\pi}{2} \int_{AB} E_\varphi (P_{AB}) dl_1$$

$$\begin{aligned}
 & + \cos \frac{\pi}{2} \int_{AB} E_z(P_{AB}) dl_1 \\
 = -E_{x1} \overline{AB} , \quad (4.13a)
 \end{aligned}$$

where P_{AB} is a point on AB and $\overline{AB} \equiv \int_{AB} dl_1$.

$$\begin{aligned}
 \int_{BC} \vec{E} \cdot d\vec{l} &= \cos \frac{\pi}{2} \int_{BC} E_z(P_{BC}) dl_1|_{BC} \\
 &+ \int_{BC} \vec{E}_q(P_{BC}) \cdot d\vec{l}|_{BC} \\
 &+ \cos \frac{\pi}{2} \int_{BC} E_z(P_{BC}) dl_1|_{BC} \\
 = \int_{BC} \vec{E}_q(P_{BC}) \cdot d\vec{l}|_{BC} , \quad (4.13b)
 \end{aligned}$$

where P_{BC} is a point on BC .

$$\int_{CD} \vec{E} \cdot d\vec{l} = E_{x2} \cos \pi \int_{CD} dl_2$$

$$\begin{aligned}
 & + \cos \frac{\pi}{2} \int_{CD} E_\varphi(P_{CD}) dl_2 \\
 & + \cos \frac{\pi}{2} \int_{CD} E_z(P_{CD}) dl_2 \\
 = -E_{\varphi 2} \overline{CD}, \quad (4.13c)
 \end{aligned}$$

where P_{CD} is a point on CD and $\overline{CD} \equiv \int_{CD} dl_2$.

$$\begin{aligned}
 \int_{DA} \vec{E} \cdot d\vec{l} &= \cos \frac{\pi}{2} \int_{DA} E_\varphi(P_{DA}) dl|_{DA} \\
 &+ \int_{DA} \vec{E}_\varphi(P_{DA}) \cdot d\vec{l}|_{DA} \\
 &+ \cos \frac{\pi}{2} \int_{DA} E_z(P_{DA}) dl|_{DA} \\
 = \int_{DA} \vec{E}_\varphi(P_{DA}) \cdot d\vec{l}|_{DA}, \quad (4.13d)
 \end{aligned}$$

where P_{DA} is a point on DA .

The line integral of Eq. (4.12) thus reads: [32]

$$\begin{aligned}
 \oint_{\gamma_3} \vec{E} \cdot d\vec{l} &= -E_{r1} \overline{AB} + \int_{BC} \vec{E}_q(P_{BC}) \cdot d\vec{l} \Big|_{BC} \\
 &\quad - E_{r2} \overline{CD} + \int_{DA} \vec{E}_q(P_{DA}) \cdot d\vec{l} \Big|_{DA}. \tag{4.14}
 \end{aligned}$$

Due to the construction of γ_3 , the length of arcs BC and DA is the same. Therefore, we can consider a single arc BC spanned twice by pair of points (P_{BC}^q, P_{DA}^q) , which are in a one-to-one correspondence due to the equal length of arcs BC and DA. Readily,

$$\begin{aligned}
 &\int_{BC} \vec{E}_q(P_{BC}) \cdot d\vec{l} \Big|_{BC} + \int_{DA} \vec{E}_q(P_{DA}) \cdot d\vec{l} \Big|_{DA} \\
 &= \int_{BC} \vec{E}_q(P_{BC}^q) \cdot d\vec{l} \Big|_{BC} + \vec{E}_q(P_{DA}^q) \cdot (-d\vec{l} \Big|_{BC}) \\
 &= \int_{BC} [\vec{E}_q(P_{BC}^q) - \vec{E}_q(P_{DA}^q)] \cdot d\vec{l} \Big|_{BC}
 \end{aligned}$$

$$\int_{BC} \left[\vec{E}_e(P_B^r) - \vec{E}_e(P_C^r) \right] \cdot d\vec{l} \Big|_{BC} = 0. \quad (4.15)$$

Finally, the irrotational property of \vec{E} gives:

$$\oint_{\gamma_3} \vec{E} \cdot d\vec{l} = -E_{x1} \cancel{AB} - E_{x2} \cancel{CD} = 0,$$

which corresponds to the condition

$$E_{x2} = -E_{x1}. \quad (4.16)$$

Note that no assumption on the components \vec{E}_y and \vec{E}_z on AB and CD and on the components \vec{E}_x and \vec{E}_z on BC and DA was made when calculating the line integral. In fact, all these components are perpendicular to the corresponding elements $d\vec{l}$ and, thus, they drop from the line integral.

The condition of Eq. (4.16) means that the component E_{x2} on CD must have the same magnitude of E_{x1} on AB and be directed as E_{x1} , in this case outward from γ . Due to the arbitrary choice of

curve γ_3 (we can arbitrarily choose any radius $r_B = r_C$ and any arc length $BC = DA$; cf. Fig. 4.4c), the condition of Eq. (4.16) means that \vec{E}_r must be the same on any circle on a plane perpendicular to γ and with center in γ . Remarkably, this result recovers the knowledge acquired by means of the rotation symmetry argument for \vec{E}_r .

(ii) In the light of all our considerations and referring to Fig. 4.4b:

$$\begin{aligned} \int_{AB} \vec{E} \cdot d\vec{l} &= \int_A^B \vec{E}_{r1} \cdot d\vec{l}_1 = \int_{rA}^{rB} E_{r1} dr \vec{u}_r \cdot \vec{u}_r \\ &= E_{r1} \int_{rA}^{rB} dr = E_{r1} [r] \Big|_{rA}^{rB} \\ &= E_{r1} (r - r - dr) = -E_{r1} dr. \quad (4.16a) \end{aligned}$$

Since we do not have any knowledge on the functional dependence of \vec{E}_r on BC and DA , we cannot perform line integrals \int_{BC} and \int_{DA} in the cylindrical reference frame. However, we already know

from point (i) that those integrals cancel each other! [35]

$$\int_{BC}^{\vec{E}} \cdot d\vec{l} + \int_{DA}^{\vec{E}} \cdot d\vec{l} = 0 \quad . \quad (4.14 \text{ be})$$

$$\begin{aligned} \int_{CD}^{\vec{E}} \cdot d\vec{l} &= \int_C^D \vec{E}_{r2} \cdot d\vec{l}_2 = - \int_{r_c}^{r_d} E_{r2} dr \vec{u}_r \cdot \vec{u}_r \\ &= - E_{r2} \int_{r_c}^{r_d} dr = - E_{r2} [r] \Big|_{r_c}^{r_d} \\ &= - E_{r2} (r_d - r_c) = - E_{r2} dr. \quad (4.14 \text{ c}) \end{aligned}$$

The line integral of Eq. (4.12) thus reads:

$$\oint_{\gamma_3}^{\vec{E}} \cdot d\vec{l} = - E_{r1} dr - E_{r2} dr = 0,$$

which confirms the condition of Eq. (4.16);

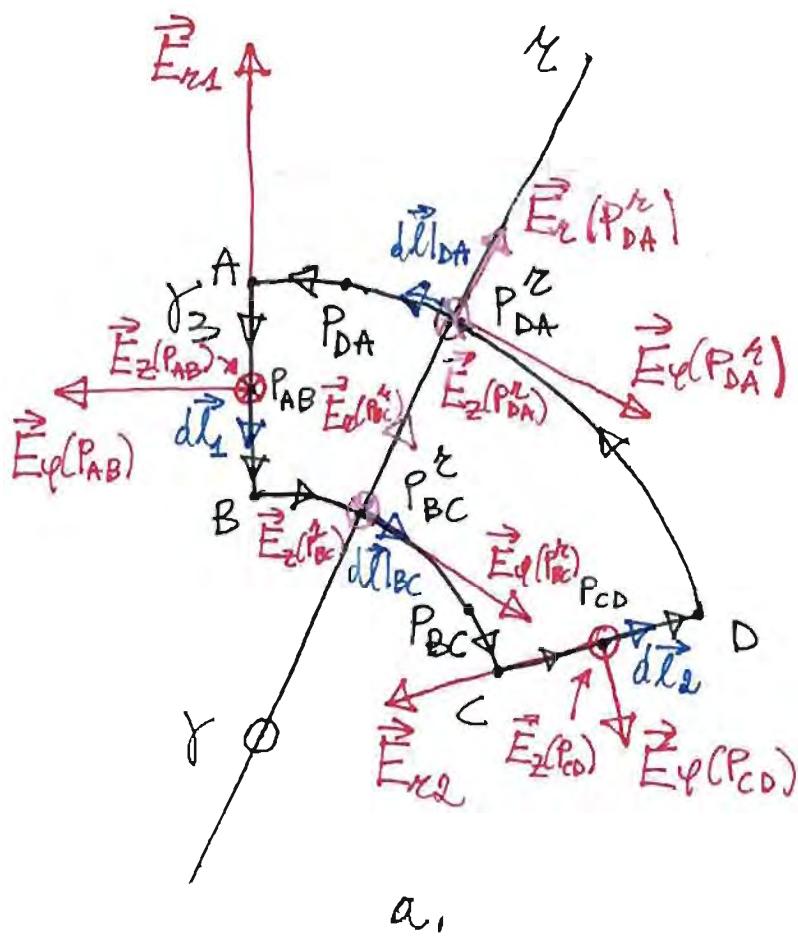
$$E_{r2} = - E_{r1}.$$

Thus, in the cylindrical reference frame

$$\vec{E}_{r1} = E_{r1} \vec{u}_x \quad \text{and} \quad \vec{E}_{r2} = -(-E_{r1}) \vec{u}_x \\ = E_{r1} \vec{u}_x = \vec{E}_{r1},$$

Note that, when calculating the line integrals of Eqs. (4.14a) and (4.14c) we assumed no contribution from the components E_y and E_z . In fact, since we do not know the functional dependence of these components on AB and CD, we must resort to the calculation without reference frame for those components. The contribution from these components is zero.

The calculation of a line integral with respect to a specific reference frame is not only useful as a crosscheck (as in the example above), it is also an important tool when performing final consistency checks of the entire mathematical derivation. This will appear clear towards the very end of this lecture.



a.

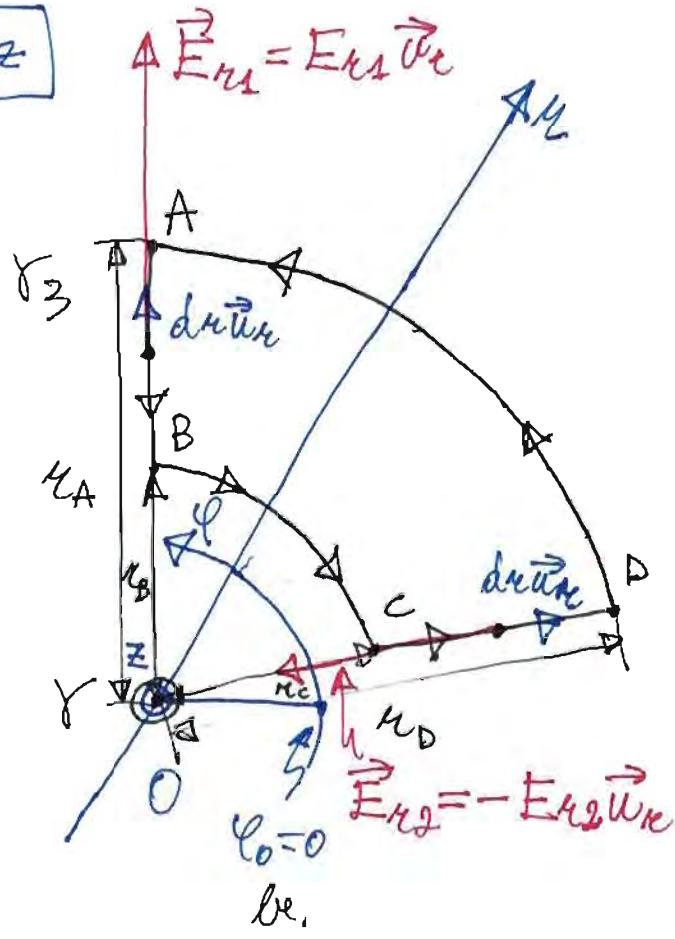
$\vec{E}(P_{BC})$ and $\vec{E}(P_{DA})$ not indicated

$\overline{AB} = \overline{CD}$ is infinitesimally small $\Rightarrow BC \approx DA$

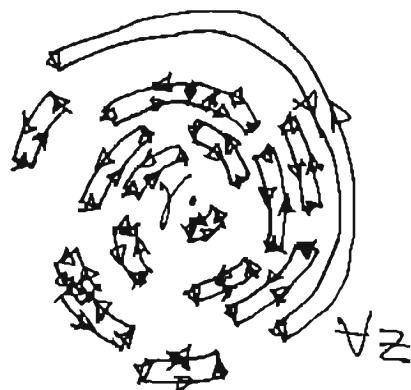
○: Vector pointing outward from the plane

✖: Vector pointing inward into the plane

plane $Ox\& y \perp z$



b.



c.

Figure 4.4.

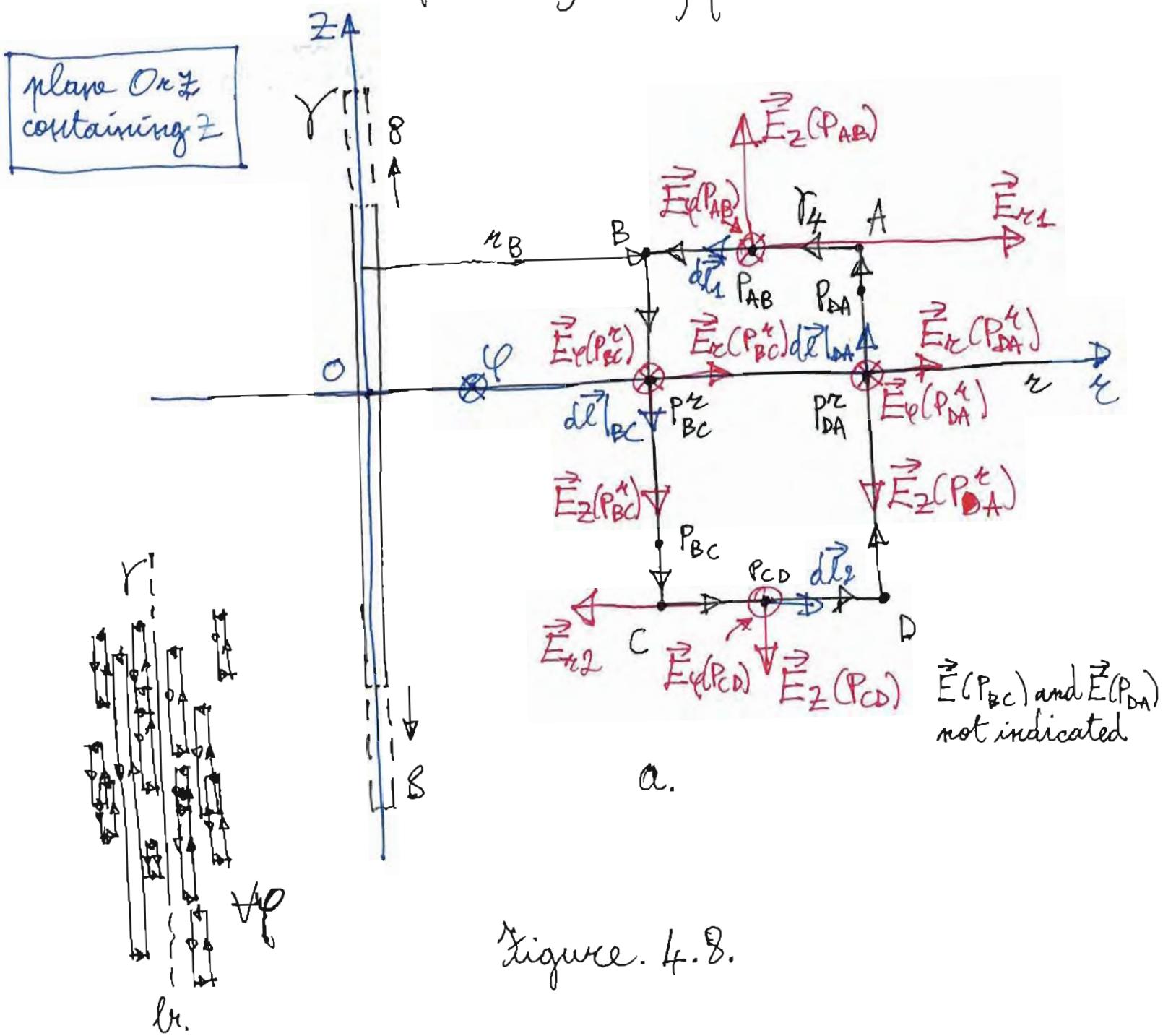
- Plane Orz containing \mathbf{z} .

Consider a closed curve γ_4 oriented counterclockwise and lying on a plane containing \mathbf{z} . The curve γ_4 is a continuous piecewise curve composed by four open curves: line segments AB, BC, CD, and DA. Line segments AB and CD are assumed to be directed radially with respect to \mathbf{z} and line segments BC and DA to be parallel to \mathbf{z} (cf. Fig. 4.8a). As a consequence, $\overline{AB} = \overline{CD}$ and $\overline{BC} = \overline{DA}$. It is further assumed that \overline{AB} and \overline{CD} are of infinitesimal length.

The problem is analogous to that for plane Or ℓ perpendicular to \mathbf{z} . Arcs BC and DA are substituted by line segments BC and DA and \vec{E}_φ on arcs BC and DA by \vec{E}_z on line segments BC and DA. The rest of the components of \vec{E} and elements $d\ell$ are assumed to be the same as in the previous problem. Consequently, due to the irrotational property of \vec{E} we find the same condition of Eq. (4.16), $E_{r2} = -E_{r1}$. This time, E_{r1} and E_{r2} are the magnitudes of vertically separated components \vec{E}_{r1} and \vec{E}_{r2} on line segments AB and CD. Due to the arbitrary choice of γ_4 (we can arbitrarily choose \overline{AB} and \overline{BC} ; cf. Fig. 4.8b), it follows that E_r must be the same on any line parallel to \mathbf{z} . Remarkably, this result recovers the knowledge acquired by means of the translation symmetry argument for \vec{E}_r .

By combining the results for the plane Oxy perpendicular to z and Oxz containing z , we obtain again the result that \vec{E}_x must be the same at each point on the lateral surface of any infinite cylinder with axis y . In addition, due to the zero-field condition at infinity, we expect $\lim_{P \rightarrow \infty} \vec{E}_x(P) = \vec{0}$. This overall finding recovers the knowledge acquired with the rotation, translation, and reflection symmetry of y .

39



3) Vertical component \vec{E}_z of the electric field \vec{E} . [50]

In general E_z can be directed upward or downward at different points in space. The magnitude E_z can be different at any point.

- Plane Oxz containing z .

Consider a closed curve γ_2 oriented counterclockwise and lying on a plane generating from γ . The curve is assumed to contain a portion of γ and to be a continuous piecewise curve composed by four open curves. Line segments AB , BC , CD , and DA . It is assumed that $\overline{AB} = \overline{CD}$ and $\overline{BC} = \overline{DA}$, with $\overline{BB'} = \overline{AA'} \neq \overline{CC'} = \overline{DD'}$ so that γ_2 is symmetric about γ . Point $B' \equiv C'$ and $D' \equiv A'$. It is finally assumed that AB and CD are of infinitesimal length, $\overline{AB} = \overline{CD} = |dl|$ (cf. Fig. 4.11).

The problem is analogous to that for \vec{E}_y on plane Oxy perpendicular to z , where \vec{E}_y on AB and CD is substituted by \vec{E}_z on AB and CD . Here, \vec{E}_z on AB is defined as $\vec{E}_{z1} = \vec{E}_z(P_1)$ and directed upward and \vec{E}_z on CD as $\vec{E}_{z3} = \vec{E}_z(P_3)$ and directed downward consistently with the conventions in Fig. 4.10a. Points $P_{B'}$ and $P_{A'}$ are substituted by points $P_{B'}$ and $P_{A'}$ and all other components of E and elements dl are assumed to be the same as in Fig. 4.10a (cf. Fig. 4.11a).

We thus obtain a condition for \vec{E}_z similar to that of Eq. (4.24): [51]

$$E_z(P_3) = -E_z(P_1) \quad , \quad (4.26)$$

which means

$$E_z(P) = C_p \quad , \quad (4.27)$$

at any point P along the straight infinite line containing points B and C (and, so, A and D), with C_p a generic constant.

Due to the zero-field condition at infinity, it must also be

$$\lim_{P \rightarrow \infty} E_z(P) = 0 \quad . \quad (4.28)$$

By combining the results of Eqs. (4.27) and (4.28), it follows that

$$\lim_{P \rightarrow \infty} E_z(P) = 0 = C_p = E_z(P) \quad , \quad (4.29)$$

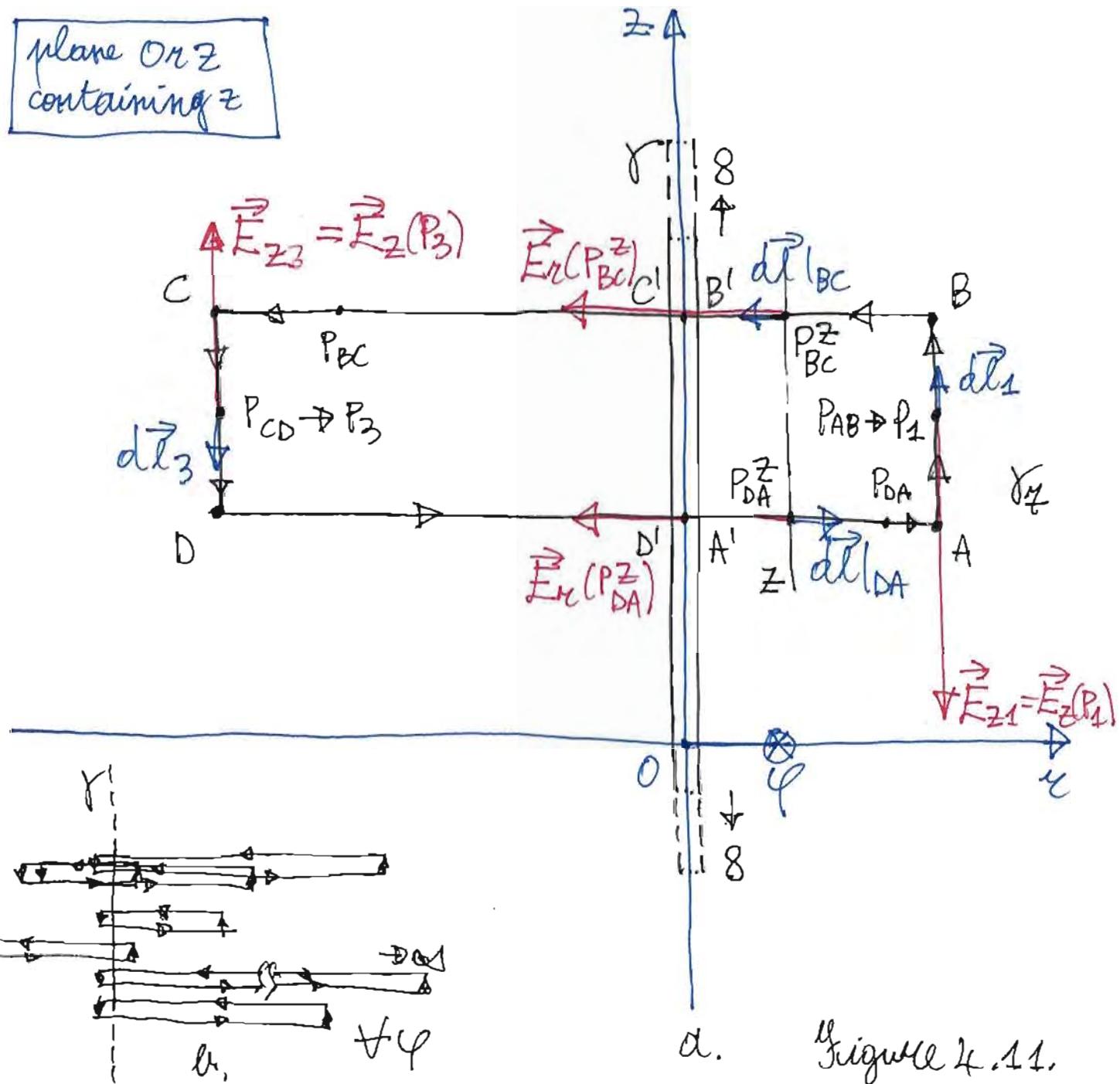
clearly showing that if $\vec{E}_z(P)$ must be zero at infinity, it must be zero at each point P.

This argument can readily be extended to any infinite straight line parallel to that containing points B and C or any plane containing δ (cf. Fig. 4.11).

hence spanning the entire space. We can thus safely conclude that [52]

$$\vec{E}_z = \vec{0} \quad (4.30)$$

everywhere in space. Remarkably, this result recovers the knowledge acquired by means of the rotation, translation, and reflection symmetry arguments.



Soley by means of the irrotational property of \vec{E} and the zero-field condition at infinity, we were able to recover all the knowledge acquired by way of all symmetry arguments on the components of \vec{E} . That is, E_x must be the same at each point on the lateral surface of any infinite cylinder with axis on γ and must go to zero at infinity, $\lim_{r \rightarrow \infty} E_x(r) = 0$, $E_y = 0$ everywhere in space, and $E_z = \vec{0}$ everywhere in space. We can thus use Gauss' theorem for surface Σ , as detailed in point ⑤, pages 15 to 18, in this lecture. This procedure allows us to obtain the magnitude E_r of \vec{E}_r , which is given within Eq. (4.3).

In summary, by means of the irrotational property of \vec{E} , the zero-field condition at infinity, and Gauss' theorem we obtain again

$$\vec{E} = \vec{E}_x + \vec{E}_y + \vec{E}_z ,$$

with

$$\left\{ \vec{E}_x = \frac{\lambda}{2\pi\epsilon_0 r^2} \vec{R}_x \right. , \quad (4.31a)$$

$$\left\{ \vec{E}_y = \vec{0} \right. , \quad (4.31b)$$

$$\left\{ \vec{E}_z = \vec{0} \right. , \quad (4.31c)$$

In particular, Eq. (4.31a) shows that E_x must go to zero at infinity and diverge on γ :

$$\lim_{r \rightarrow \infty} \vec{E}_r(r) = \vec{0} \quad (4.32)$$

and

$$\lim_{r \rightarrow 0^+} \vec{E}_r(r) = +\infty \quad . \quad (4.33)$$

- Integration consistency checks.

In order to obtain the conditions of Eqs.

(4.21) and (4.26), we made use of the fact that all integrals involved in the irrotational property of \vec{E} were well defined. In particular, we assumed

$$\int_{BC} \vec{E}_r(P_{BC}) \cdot d\vec{l} \Big|_{BC} + \int_{DA} \vec{E}_r(P_{DA}) \cdot d\vec{l} \Big|_{DA} = 0 \quad . \quad (4.34)$$

Note that, in both integrals of Eq. (4.34) the integrand function E_r must be evaluated at $r=0$. According to Eq. (4.33), the integrand is ill-defined at $r=0$. This means such integrals cannot be calculated as standard Riemann integrals.

Instead, they have to be calculated as improper integrals. In order to show that Eq. (4.34) remains valid, we must now express \vec{E}_r and $d\vec{l}$ on BC and DA according to the reference frame of choice.

Figure 4.12, which is tailored to the case of Fig. 4.11 (the very same argument would apply to the case of Fig. 4.10), shows line segments BC and DA that belong to curve γ_7 . The two line segments are decomposed into two sets of line sub-segments BB' and $C'C$, with $B' \equiv C'$, and DD' and $A'A$, with $D' \equiv A'$. The component \vec{E}_r and corresponding element $d\vec{l}$ for each line sub-segments are indicated. Finally, a cylindrical reference frame Ox_1x_2 is shown. Referring to this reference frame, we have:

$$\text{on } BB': \vec{E}_r(P_{BB'}) = E_r(r) \vec{u}_r, r \in [r_B, r_{B'}]$$

$$d\vec{l}|_{BB'} = \vec{u}_r dr \quad (\text{d. pages 24 to 27 in this lecture})$$

$$\text{on } C'C: \vec{E}_r(P_{C'C}) = E_r(r) \vec{u}_r, r \in [r_C, r_{C'}]$$

$$d\vec{l}|_{C'C} = \vec{u}_r dr$$

$$\text{on } DD': \vec{E}_r(P_{DD'}) = E_r(r) \vec{u}_r, r \in [r_D, r_{D'}]$$

$$d\vec{l}|_{DD'} = \vec{u}_r dr$$

$$\text{on } A'A: \vec{E}_r(P_{A'A}) = E_r(r) \vec{u}_r, r \in [r_{A'}, r_A]$$

$$d\vec{l}|_{A'A} = \vec{u}_r dr,$$

where $r_B = r_A$, $r_{B'} = r_D = r_{C'} = r_{A'} = r_A + \delta$, with δ an arbitrary small positive quantity, $r_C = r_D$, and $E_r(r)$ given by

Eq. (4.34a). The integral of Eq. (4.34) can thus be written as [56]

$$\begin{aligned}
 & \int_{BB'} \vec{E}_r(P_{BB'}) \cdot d\vec{l} \Big|_{BB'} + \int_{CC'} \vec{E}_r(P_{CC'}) \cdot d\vec{l} \Big|_{CC'} \\
 & + \int_{DD'} \vec{E}_r(P_{DD'}) \cdot d\vec{l} \Big|_{DD'} + \int_{AA'} \vec{E}_r(P_{AA'}) \cdot d\vec{l} \Big|_{AA'} \\
 = & \lim_{\delta \rightarrow 0^+} \left[\frac{\lambda}{2\pi\epsilon_0} \left(\int_{r_B}^{\delta} \frac{1}{r} \vec{u}_r \cdot \vec{u}_r dr + \int_{r_C}^{r_C} \frac{1}{r} \vec{u}_r \cdot \vec{u}_r dr \right. \right. \\
 & \quad \left. \left. + \int_{r_C}^{\delta} \frac{1}{r} \vec{u}_r \cdot \vec{u}_r dr + \int_{r_B}^{r_B} \frac{1}{r} \vec{u}_r \cdot \vec{u}_r dr \right) \right] \\
 = & \lim_{\delta \rightarrow 0^+} \left[\frac{\lambda}{2\pi\epsilon_0} \left(\ln \frac{\delta}{r_B} - \ln \frac{\delta}{r_C} \right. \right. \\
 & \quad \left. \left. + \ln \frac{r_B}{r_C} - \ln \frac{r_B}{r_B} \right) \right] = 0. \quad (4.35)
 \end{aligned}$$

This result is the Cauchy principle value for the integral of Eq. (4.34). The improper integral is thus well defined and equal to zero. This means that

all integrations leading to the conditions of Eqs. [54]
 (4.21) and (4.26) are well defined. As a consequence,
 all information acquired on the components of \vec{E} by means
 of the irrotational property of \vec{E} and the zero-field
 condition at infinity are consistent with the expression of \vec{E}_r
 obtained a posteriori by way of Gauss' theorem.

- Summary.

The results of Eqs. (4.31), (4.32), and (4.33) can be
 equivalently obtained through : (1) Coulomb's law +
 superposition principle ; (2) symmetry arguments +
 Gauss' theorem ; (3) irrotational property of \vec{E} + zero
 field condition at infinity + Gauss' theorem.

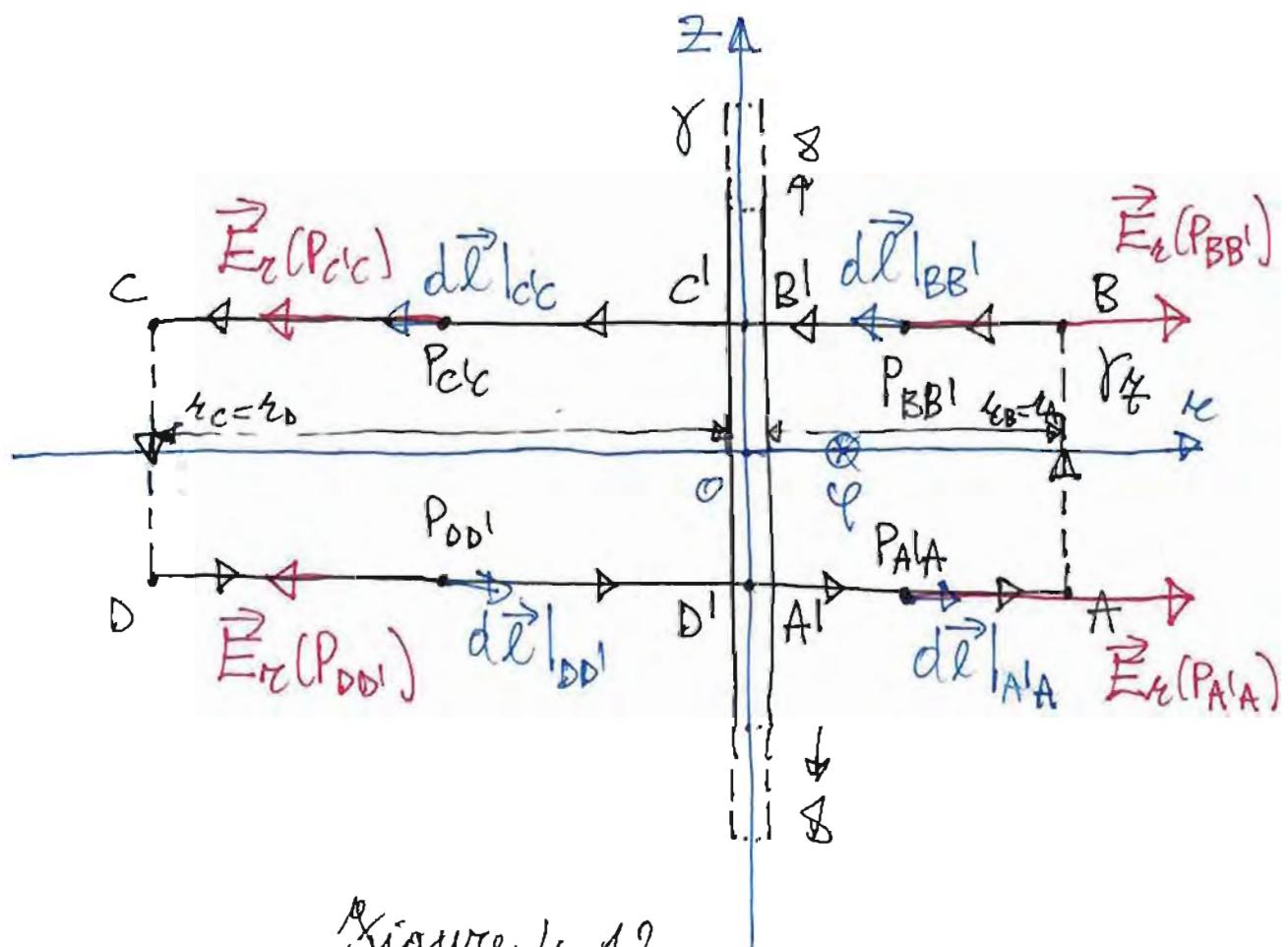


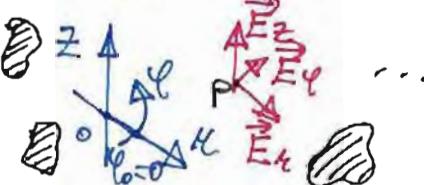
Figure 4.12.

Summary lecture 4

- Electric field \vec{E} generated by an infinite straight line γ with a linear charge distribution λ uniformly distributed along the line.

- given an inertial reference frame and a cylindrical coordinate system associated with it, $Ox\varphi z$:

$$\vec{E} = \vec{E}_z + \vec{E}_\varphi + \vec{E}_x .$$



- The field \vec{E} can be calculated by means of:

- 1) Coulomb's law and the superposition principle (cf. example #2, lecture 2, page 14 and thereafter).
- 2) Symmetry arguments and Gauss' theorem.
- 3) The irrotational property of \vec{E} , the zero-field condition at infinity, and Gauss' theorem.

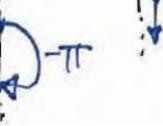
- The symmetry arguments for line γ are:

(a) Rotation symmetry,



(b) Translation symmetry,

(c) Reflection symmetry,

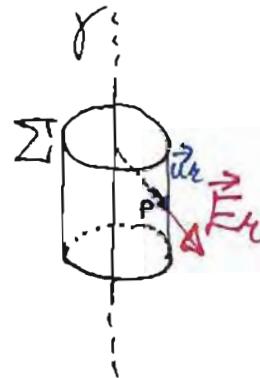


- Due to these arguments, \vec{E} can have a radial component

\vec{E}_r must be the same at each point on the lateral surface of any infinite cylinder with axis δ . However, \vec{E} cannot have a tangent and vertical components, $\vec{E}_\varphi = \vec{0}$ and $\vec{E}_z = \vec{0}$.

- The magnitude E_r can thus be calculated by way of Gauss' theorem for an arbitrary cylindrical surface Σ with axis γ . Hence,

$$\left\{ \begin{array}{l} \vec{E}_r = \frac{\lambda}{2\pi\epsilon_0 r} \vec{u}_r \\ \vec{E}_\varphi = \vec{0} \\ \vec{E}_z = \vec{0} \end{array} \right.$$



and

$$\lim_{P \rightarrow \infty} \vec{E}(P) = \lim_{r \rightarrow \infty} \vec{E}_r(r) = \vec{0} ,$$

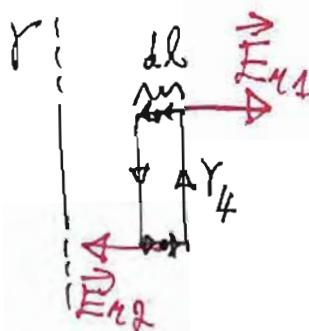
$$\lim_{P \rightarrow 0} \vec{E}(P) = \lim_{r \rightarrow 0^+} \vec{E}_r(r) = +\infty .$$

- Gauss' theorem alone does not allow us to acquire any knowledge on \vec{E} generated by δ .
- Without resorting to the symmetry arguments, it is possible to obtain the same information on the components of \vec{E} by means of the irrotational property of \vec{E} for a set of

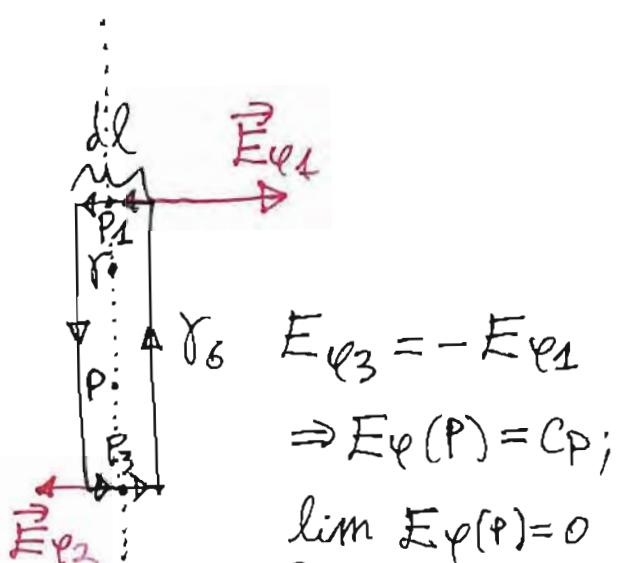
compatible with the line geometry and the zero-field condition at infinity. [3]



$$E_{r2} = -E_{r1}$$



$$E_{r2} = -E_{r1}$$

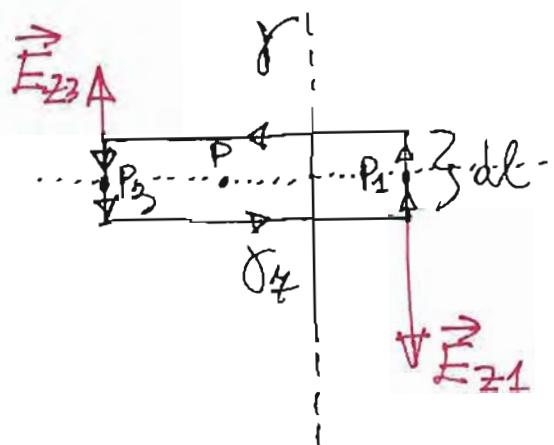


$$E_{r3} = -E_{r1}$$

$$\Rightarrow E_r(p) = C_p;$$

$$\lim_{P \rightarrow \infty} E_r(p) = 0$$

$$\Rightarrow E_r(p) = 0, \forall p$$



$$E_{z3} = -E_{z1} \Rightarrow E_z(p) = C_p;$$

$$\lim_{P \rightarrow \infty} E_z(p) = 0 \Rightarrow E_z(p) = 0, \forall p$$

- Gauss' theorem for Σ allows again to obtain E_r .

- When a line integral intersects γ , the integral must be calculated as an improper integral and its value as the Cauchy principle value.

5.1 Electrostatic field properties in local form.

41

thus far, Gauss' theorem and the irrotational property of the electrostatic field \vec{E} have been expressed in so-called integral form:

$$\oint_{\Sigma} \vec{E} \cdot \vec{n} dS = \frac{Q}{\epsilon_0}, \quad (5.1)$$

where Q is the total charge within a volume bounded by a closed surface Σ , \vec{E} the electrostatic field at each point on Σ , \vec{n} the normal unit vector to an infinitesimal surface element on Σ , dS , and ϵ_0 the dielectric constant of vacuum,

$$\oint_{\gamma} \vec{E} \cdot \vec{T} dl = 0, \quad (5.2)$$

where γ is an arbitrarily oriented closed curve

, \vec{E} the electrostatic field at each point on γ , and \vec{T} the tangent unit vector to an infinitesimal curve element on γ , dl . All curves and surfaces are in the 3D Euclidean space.

Equations (5.1) and (5.2) apply to any arbitrarily small surface Σ and curve γ . Hence, they can be written in local form.

5.1.1 Gauss' theorem in local form.

Case 1.

Consider a charge distribution $\rho(\mathbf{P})$ in any point [5]
 of a region Ω in the 3D Euclidian space. We assume
 the region Ω to be compact in space (i.e., to be closed
 and bounded) with a piecewise smooth boundary Σ .
 We can write Eq. (5.1) as

$$\sum \oint \vec{E} \cdot \vec{n} dS = \frac{1}{\epsilon_0} \iiint_{\Omega} \rho d\tau , \quad (5.3)$$

where $d\tau$ is an infinitesimal volume element in Ω .
 Assuming \vec{E} to be continuously differentiable in a vector
 sense (i.e., each component of \vec{E} in a given inertial reference
 frame, with respect to a specific coordinate system,
 $E_i \in \mathbb{C}^1$) on a neighborhood of Ω , we obtain

$$\sum \oint \vec{E} \cdot \vec{n} dS = \iiint_{\Omega} \operatorname{div} \vec{E} d\tau$$

and, from Eq. (5.3),

$$\iiint_{\Omega} \operatorname{div} \vec{E} d\tau = \frac{1}{\epsilon_0} \iiint_{\Omega} \rho d\tau .$$

Due to the arbitrary choice of Ω , it finally follows

$$\operatorname{div} \vec{E} = \frac{\beta}{\epsilon_0} \quad \dots (5.4)$$

It is obvious that in the points with no charge, $\beta=0$,

$$\operatorname{div} \vec{E} = 0.$$

In this sense, the volume charge density β appears as a source for the vector field \vec{E} .

In order to find the result of Eq. (5.4), we made use of the divergence (Gauss-Ostrogradskij's) theorem (of tutorial). We now intend to set out to find the necessary and sufficient conditions on any given charge distribution for the result of Eq. (5.4) to hold. In fact, when trying to solve an electrostatic problem we are given the charge distribution and we are asked to find \vec{E} . However, in order to apply the divergence theorem we made use of the fact that \vec{E} is continuously differentiable in a vector sense even though, *a priori*, we did not know anything about \vec{E} .

- Direct statement. It is a necessary condition that β be a continuous and limited function on Ω for $\vec{E} \in C^1(\Omega)$ and, thus, Eq. (5.4) to hold.

We assume that $\vec{E} \in C^1$ on a neighborhood of Ω . With respect to a cartesian coordinate system this

means that

$$\vec{E} = E_x(x, y, z) \vec{u}_x + E_y(x, y, z) \vec{u}_y + E_z(x, y, z) \vec{u}_z ,$$

with

$$E_x \in C^0(\Omega)$$

$$E_y \in C^0(\Omega)$$

$$E_z \in C^0(\Omega)$$

and

$$\left\{ \begin{array}{l} \frac{\partial}{\partial x} E_x, \frac{\partial}{\partial y} E_x, \frac{\partial}{\partial z} E_x \in C^0(\Omega) \\ \frac{\partial}{\partial x} E_y, \frac{\partial}{\partial y} E_y, \frac{\partial}{\partial z} E_y \in C^0(\Omega) . \quad (55) \\ \frac{\partial}{\partial x} E_z, \frac{\partial}{\partial y} E_z, \frac{\partial}{\partial z} E_z \in C^0(\Omega) \end{array} \right.$$

We can thus write

$$\operatorname{div} \vec{E} = \vec{\nabla} \cdot \vec{E} = \frac{\partial}{\partial x} E_x + \frac{\partial}{\partial y} E_y + \frac{\partial}{\partial z} E_z .$$

Note that, for the divergence to be well defined, only a subset of the conditions (5.5) must be satisfied. Under those conditions, it follows that $\operatorname{div} \vec{E} \in C^0(\Omega)$.

$$\stackrel{!}{=} \sigma = \lim_{P \rightarrow P_0^+} \operatorname{div} \vec{E}, \quad (5.10)$$

which contradicts the imposed condition of Eq. (5.8), thus proving the counter direct statement. At least one of the derivatives of Eqs. (5.9) must be discontinuous. Note that this last statement does not allow us to draw any conclusions on the cross-derivatives

$$\frac{\partial}{\partial y} E_x, \frac{\partial}{\partial z} E_x, \frac{\partial}{\partial x} E_y, \frac{\partial}{\partial z} E_y,$$

$$\frac{\partial}{\partial x} E_z, \frac{\partial}{\partial y} E_z,$$

which can be continuous even if σ is discontinuous. Here, we assumed that Eq. (5.4) is the entire knowledge we have connecting an arbitrary charge distribution and the electric field \vec{E} . We will soon come back to this issue.

- Inverse statement. It is a sufficient condition that σ be a continuous and limited function in Ω for $\vec{E} \in C^1(\Omega)$ and, thus, Eq. (5.4) to hold.

We now assume $\sigma \in C^0(\Omega)$. This would be consistent with the divergence to be well defined and, hence, continuous, if Eq. (5.4) was to hold [remember that Eq. (5.4) was derived assuming $\vec{E} \in C^1(\Omega)$].

This should not surprise considering the definition of divergence and its physical meaning. 121

$$\operatorname{div}_{P_0} \vec{E} = \lim_{V(z) \rightarrow 0} \frac{\oint_{\Sigma'} \vec{E} \cdot \vec{n} dS}{V(z)},$$

where z is a region in Ω with volume $V(z)$ and bounded by a surface Σ' . \vec{n} is the normal unit vector at each point on Σ' and dS an infinitesimal surface element on Σ' . This definition shows that the divergence quantifies locally the flux of the electric field \vec{E} through an arbitrarily small surface. The divergence can be continuous even if some of its components in cartesian coordinates are not, so long the overall "flux" is continuous.

Our counterexample shows that, in general, it is not sufficient \vec{E} be continuous in Ω for \vec{E} be continuously derivable in a vector sense in Ω . It is not even sufficient for the derivatives of Eq. (5.4) to be continuous! However, it is sufficient for Eq. (5.4) to hold. This naturally leads to the conclusion that the hypothesis on \vec{E} that $\vec{E} \in C^1(\Omega)$ or even that the derivatives (5.4) be continuous can be relaxed, so long $\operatorname{div} \vec{E}$ remains well defined in the sense of

Eqs. (5.11) and (5.12). The continuity of \mathfrak{S} imposes [13] some level of regularity on the derivatives (5.4). We can thus conjecture that if $\mathfrak{S} \in C^0(\Omega)$, then \vec{E} will be such that either the derivatives (5.4) are continuous in Ω or, at least, they are discontinuous under the condition $\operatorname{div} \vec{E}$ be continuous in Ω . Once again, the statement does not allow us to draw any conclusions on the cross-derivatives.

Our conjecture does not prove the inverse statement and only experience or knowledge deriving from different equations can confirm it or not. In our case experience means solving problems.

Before summarizing the role of the continuity of \mathfrak{S} and its property to be limited on the continuity of \vec{E} and its property to be limited, it is worth examining more closely Eq. (2.24) in lecture 2, page 16:

$$\vec{E}(\vec{r}_P) = \frac{1}{4\pi\epsilon_0} \iiint_{\Omega} d\mathfrak{z} \frac{\mathfrak{S}(\vec{r}_{\alpha})}{|\vec{r}_P - \vec{r}_{\alpha}|^2} \vec{u}_{\alpha P}. \quad (5.13)$$

It is worth stressing the fact that in Eq. (5.13) the electric field depends on the free variable \vec{r}_P , while the integral is performed over the variable \vec{r}_{α} that can only assume values in Ω . In general, \vec{r}_P can be anywhere in the 3D Euclidean space, inside or outside Ω .

As in the case of Eq. (5.4), also Eq. (5.13) relates the charge distribution S to \vec{E} . It is important to notice that the integrand function in (5.13),

$$\frac{S(\vec{r}_a)}{|\vec{r}_p - \vec{r}_a|^2},$$

diverges to infinity every time $\vec{r}_p = \vec{r}_a$. In general, this happens at least one time for each \vec{r}_p in Ω . This would lead to the inconvenient result \vec{E} be diverging to infinity, if it was not for the fact that we are actually integrating a divergent function and, as we know from calculus, the operation of integration smoothens discontinuities. We now shall prove this result in general.

Consider a function $f(x, y, z)$ in cartesian coordinates, defined in a compact region Ω of the 3D Euclidean space. Suppose f be limited in each neighborhood of a point $P_0(x_0, y_0, z_0)$ in Ω (eventually including its boundary). Further assume f be limited and integrable in each region $\Omega - w_S$, where w_S indicates any region in Ω containing P_0 and $\Omega - w_S$ indicates the remaining part of Ω . Note that the subscript S refers to the effective diameter of w_S . By definition, the improper integral of f in Ω is

$$\iiint_{\Omega} f(x, y, z) dx dy dz = \lim_{\delta \rightarrow 0} \iiint_{\Omega - \omega_\delta} f(x, y, z) dx dy dz. \quad (5.14)$$

If this limit exists, is finite, and is independent from the way the region ω_δ contracts around point P_0 , the integral (5.14) is said to be convergent, otherwise is divergent.

There are some cases where the integral (5.14) does not exist in general, but it does exist and is finite when considering regions ω_δ that are spheres centered in P_0 . In these cases, the limit is called principle value (according to Cauchy) of the integral (5.14).

When the function $f \geq 0$ in Ω , the following theorem is valid. Considered an arbitrary series \sum_n of spheres (with decreasing diameters δ_n) centered in P_0 , necessary and sufficient condition for the improper integral (5.14) to converge is that the series

$$\iiint_{\Omega - \sum_1} f(x, y, z) dx dy dz, \iiint_{\Omega - \sum_2} f(x, y, z) dx dy dz, \dots,$$

$$\iiint_{\Omega - \sum_n} f(x, y, z) dx dy dz$$

be limited.

Consider now a function

$$f(x, y, z) = \frac{c}{r^\alpha},$$

with $c > 0$, $r = [(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2]^{1/2}$, and arbitrary $\alpha \in \mathbb{N}$. It is easy to prove that the improper integral

$$\iiint_{\Omega} \frac{c}{r^\alpha} dx dy dz,$$

on a sphere Σ with radius R and centered in $P_0(x_0, y_0, z_0)$ converges for $\alpha < 3$ and diverges for $\alpha \geq 3$. Consider a series of spheres Σ_n in Ω , each centered in P_0 and with diameter δ_n decreasing with n . Consider the integral

$$\iiint_{\Omega - \Sigma_n} \frac{c}{r^\alpha} dx dy dz, \quad (5.15)$$

By choosing a spherical coordinate system $O r \theta \varphi$ with center O in P_0 , we obtain

$$\iiint_{\Omega - \Sigma_n} \frac{c}{r^\alpha} dx dy dz = \iiint_{\Omega - \Sigma_n} \frac{c}{r^\alpha} \cdot r^2 \sin \theta dr d\theta d\varphi$$

$$\begin{aligned}
 &= \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta d\theta \int_{\delta_m}^R \frac{C}{r^\lambda} r^2 dr \\
 &= \begin{cases} 4\pi C \int_{\delta_m}^R r^{(2-\lambda)} dr, & 0 < \lambda < 3 \\ 4\pi C \int_{\delta_m}^R \frac{1}{r} dr, & \lambda = 3 \end{cases}
 \end{aligned}$$

By simple integration we thus find

$$\iiint_{\Omega - \sum_m} \frac{C}{r^\lambda} dx dy dz = \begin{cases} \frac{4\pi C}{3-\lambda} [R^{(3-\lambda)} - \delta_m^{(3-\lambda)}], & 0 < \lambda < 3 \\ 4\pi C \ln \frac{R}{\delta_m}, & \lambda = 3 \end{cases} \quad (5.16)$$

In the limit for $n \rightarrow \infty$, it is evident that the series of integrals (5.16) is limited for $0 < \lambda < 3$ and illimited already for $\lambda = 3$. This means the series will be illimited even more so for $\lambda > 3$. This proves our statement:

$$\lim_{n \rightarrow \infty} \iiint_{\Omega - \sum_m} \frac{C}{r^\lambda} dx dy dz = \begin{cases} \frac{4\pi C}{3-\lambda} R^{(3-\lambda)}, & 0 < \lambda < 3 \\ \infty, & \lambda \geq 3 \end{cases}$$

With this general result in hand, we can consider again integral (5.13). [19]

- If $S(\vec{r}_\alpha) \in C^0(\Omega)$ and limited in Ω , the integrand function in (5.13),

$$\frac{S(\vec{r}_\alpha)}{|\vec{r}_p - \vec{r}_\alpha|^2},$$

is $C^0(\Omega)$ and limited in Ω apart from $\vec{r}_p = \vec{r}_\alpha$ because composed by functions that are C^0 and limited in Ω apart from $\vec{r}_p = \vec{r}_\alpha$,

$$\left\{ \begin{array}{l} S(\vec{r}_\alpha) \in C^0(\Omega) \text{ and limited in } \Omega \\ |\vec{r}_p - \vec{r}_\alpha|^2 \in C^0(\Omega) \text{ and limited in } \Omega, \\ \text{apart from } \vec{r}_p = \vec{r}_\alpha \end{array} \right.$$

The special case when $\vec{r}_p = \vec{r}_\alpha$ that could lead to an illimitated integral (5.13) is cured in the sense of the principal value according to Cauchy. In fact, if $S(\vec{r})$ is limited in Ω , assuming S^{\max} be the maximum value assumed by $S(\vec{r}_\alpha)$ in Ω ,

$$S^{\max} = \max \{ S(\vec{r}_\alpha) \}_{\vec{r} \in \Omega},$$

we can certainly write the inequality

$$\frac{s(\vec{r}_Q)}{|\vec{r}_P - \vec{r}_Q|^2} = \frac{s(\vec{r}_Q)}{r^2} \leq \frac{s_{\max}}{r^2},$$

where $r \equiv |\vec{r}_P - \vec{r}_Q|^2$ is the absolute value of the distance between points Q and P . As a consequence, the magnitude $E(\vec{r}_P)$ of $\vec{E}(\vec{r}_P)$ must be such that

$$\frac{1}{4\pi\epsilon_0} \iint_{\Omega} dz \frac{s(\vec{r}_Q)}{r^2} \leq \frac{1}{4\pi\epsilon_0} \iint_{\Omega} dz \frac{s_{\max}}{r^2}.$$

This last improper integral is of the type (5.16) for $\lambda=2$. Hence, it must converge and, as a consequence, also the integral on the left side of the inequality must converge for each point in Ω including when $r=0$ (i.e., $\vec{r}_P=\vec{r}_Q$).

We can thus safely state that if $s(\vec{r}) \in C^0(\Omega)$ and limited in Ω , $\vec{E}(\vec{r}_P)$ calculated with integral (5.13), which is the most general way to calculate $\vec{E}(\vec{r}_P)$ for a given s , is $C^0(\Omega)$ and limited in Ω .

- If $s(\vec{r}_Q)$ is limited in Ω , but discontinuous, the previous statements holds so long $s(\vec{r}_Q)$ is continuous in sub-regions of Ω , e.g., it can be decomposed in $s_1(\vec{r}_Q)$ in sub-region Ω_1 and $s_2(\vec{r}_Q)$ in sub-region Ω_2 , where $\Omega = \Omega_1 \cup \Omega_2$ and $s_1(\vec{r}_Q) \in C^0(\Omega_1)$ and

limited in Ω_1 and $S_2(\vec{r}_\alpha) \in C^0(\Omega_2)$ and limited 29 in Ω_2 . In general, $\Omega = \Omega_1 \cup \Omega_2 \cup \dots \cup \Omega_n$ and $S_i(\vec{r}_\alpha) \in C^0(\Omega_i)$ and limited in Ω_i for $i=1, \dots, n$ and $n \in \mathbb{N}_0$.

The case for $n=2$ is one of the most common ones, e.g., when a charge is uniformly distributed on a sphere of radius R and there is no charge anywhere else in space.

Consider the case $n=2$. The electric field at any point \vec{r}_p in Ω_1 or Ω_2 is given by

$$\vec{E}(\vec{r}_p) = \frac{1}{4\pi\epsilon_0} \left[\iiint_{\Omega_1} d\vec{r} \frac{S_1(\vec{r}_\alpha)}{r_\alpha^2} \vec{u}_{\alpha p} + \iiint_{\Omega_2} d\vec{r} \frac{S_2(\vec{r}_\alpha)}{r_\alpha^2} \vec{u}_{\alpha p} \right], \quad (5.14)$$

because of the superposition principle. In the integral (5.14), $\vec{r}_\alpha \in \Omega_1$ in the left integral, $\vec{r}_\alpha \in \Omega_2$ in the right integral, and \vec{r}_p can be anywhere in Ω in both integrals. In region Ω_1 , the integrand $S_1(\vec{r}_\alpha)/r_\alpha^2$ is continuous and limited for each point \vec{r}_p outside Ω_1 (where it will never happen $\vec{r}_p = \vec{r}_\alpha$) and for each point \vec{r}_p inside Ω_1 apart from $\vec{r}_p = \vec{r}_\alpha$. A similar argument applies to $S_2(\vec{r}_\alpha)/r_\alpha^2$ in Ω_2 . The case $\vec{r}_p = \vec{r}_\alpha$ where either of the two integrands become illimited within the respective region of integration

can, however, be cured again in the sense of principal value according to Cauchy. In fact, assume

$$\left\{ \begin{array}{l} S_1^{\text{Max}} = \max \{ S_1(\vec{r}_0) \}_{\Omega_1} \\ S_2^{\text{Max}} = \max \{ S_2(\vec{r}_0) \}_{\Omega_2} \end{array} \right.$$

we can certainly write the two inequalities

$$\left\{ \begin{array}{l} \frac{S_1(\vec{r}_0)}{r^2} \leq \frac{S_1^{\text{Max}}}{r^2} \\ \frac{S_2(\vec{r}_0)}{r^2} \leq \frac{S_2^{\text{Max}}}{r^2} \end{array} \right.$$

and, thus,

$$\left\{ \begin{array}{l} \frac{1}{4\pi\epsilon_0} \iiint_{\Omega_1} dz \frac{S_1(\vec{r}_0)}{r^2} \leq \frac{1}{4\pi\epsilon_0} \iiint_{\Omega_1} dz \frac{S_1^{\text{Max}}}{r^2} \\ \frac{1}{4\pi\epsilon_0} \iiint_{\Omega_2} dz \frac{S_2(\vec{r}_0)}{r^2} \leq \frac{1}{4\pi\epsilon_0} \iiint_{\Omega_2} dz \frac{S_2^{\text{Max}}}{r^2} \end{array} \right.$$

The two improper integrals on the right side of each inequality are both of the type (5.16) for $\lambda=2$. Hence, they both must converge and, as a consequence, also the two integrals on the left side of each inequality must converge for each point in Ω_1 and Ω_2 , respectively, including when $\chi=0$ (i.e., $\vec{r}_P=\vec{r}_Q$).

We can thus safely state that if $S_1(\vec{r}_Q) \in C^0(\Omega_1)$ and limited in Ω_2 and $S_2(\vec{r}_Q) \in C^0(\Omega_2)$ and limited in Ω_1 , $\vec{E}(\vec{r}_P)$ calculated with integral (5.14), is the sum of two continuous and limited functions and, thus, is $C^0(\Omega)$ and limited in Ω . This can be generalized to any n .

One has to be careful when the discontinuity is a surface charge distribution in the 3D Euclidean space. In that case, \vec{E} is in general discontinuous at the surface (we will come back to this case soon).

- If $S(\vec{r}_Q) \in C^0(\Omega)$ and limited in Ω , we cannot tell anything on the derivatives of the various components of \vec{E} at a generic point $\vec{r}_P \in \Omega$, not even on the first derivatives. Even more so if S is a discontinuous function. In fact, with respect to a cartesian coordinate system, the absolute value of the x component of $\vec{E}(\vec{r}_P)$ can be written as

$$|E_x(x_p, y_p, z_p)| = \frac{1}{4\pi\epsilon_0} \iiint_{\Omega} dx_q dy_q dz_q \frac{\delta(x_q, y_q, z_q)}{(x_p - x_q)^2},$$

where, without loosing generality, we have assumed $y_p = y_q$ and $z_p = z_q$. The integrand $\delta(x_q, y_q, z_q) / (x_p - x_q)^2$ is illimitated for $x_p = x_q$. Hence,

$$\begin{aligned} \frac{\partial}{\partial x_p} |E_x(x_p, y_p, z_p)| &= \frac{1}{4\pi\epsilon_0} \frac{\partial}{\partial x_p} \left[\iiint_{\Omega} dx_q dy_q dz_q \frac{\delta(x_q, y_q, z_q)}{(x_p - x_q)^2} \right] \\ &\neq \frac{1}{4\pi\epsilon_0} \iiint_{\Omega} dx_q dy_q dz_q \frac{\partial}{\partial x_p} \left[\frac{\delta(x_q, y_q, z_q)}{(x_p - x_q)^2} \right]. \end{aligned}$$

In fact, we are not allowed to bring derivatives inside the integral sign for a discontinuous integrand in this case discontinuous in an asymptotic sense (the integrand is illimitated for $x_p = x_q$). We could think of bringing the derivative inside the integral sign under the assumption $x_p \neq x_q$. This would give

$$\frac{1}{4\pi\epsilon_0} \iiint_{\Omega} dx_q dy_q dz_q \frac{\partial \delta(x_q, y_q, z_q)}{(x_p - x_q)^3}, \quad x_p \neq x_q.$$

If we were now to recover the case $x_p = x_\alpha$, which [24]
 will happen when \vec{E} is evaluated at any point in
 Ω , we would obtain a diverging integrand
 $S(x_\alpha, Y_\alpha, Z_\alpha)/(x_p - x_\alpha)^3$. This integrand is of the
 type c/r^λ with $\lambda = 3$. We know that this type of
 function makes the improper integral

$$\iiint_{\Omega} dz \frac{c}{r^3}$$

diverge. Hence, the integral (5.18) diverges when
 recovering the case $x_p = x_\alpha$. There is no reason to
 believe that this should always be the case (in fact,
 the example of a uniform charge distribution S in
 the volume of a sphere provides clear evidence that
 the first derivative of \vec{E}_q along the radial axis
 with respect to r can be limited and continuous inside
 the volume). As a consequence, it does not make any
 sense to bring the derivative inside the integral
 sign and guess on the properties of the derivatives
 of a component of \vec{E} to be continuous and limited.
 [note that the argument for integral (5.18) can be
 generalized to the case where $y_p \neq Y_\alpha$ and $z_p \neq Z_\alpha$
 as well as to any component of \vec{E} and any first derivative
 of each component with respect to x_p , y_p , and z_p].

The only way to determine whether $\vec{E} \in C^1(\Omega)$ and is limited in Ω is to calculate \vec{E} for a given ρ through integral (5.13). [25]

- In summary. Assume we are given a generic charge distribution ρ in Ω . From Eqs.(5.4) and (5.13), it is a necessary condition that $\rho \in C^0(\Omega)$ and be limited in Ω for $\vec{E} \in C^1(\Omega)$ and be limited in Ω . It is a sufficient condition that ρ be discontinuous at some point (or on a line or surface) in Ω for at least one of the derivatives (5.4) be discontinuous at that point (or line or surface). It is not a sufficient condition that $\rho \in C^0(\Omega)$ and be limited in Ω for $\vec{E} \in C^1(\Omega)$ and be limited in Ω . It is, however, sufficient to conjecture that either the derivatives (5.4) are continuous and limited in Ω or they are discontinuous in the sense of conditions (5.11), which still allows to have a well-defined $\text{div } \vec{E}$. Nothing can be said on the cross-derivatives. It is necessary and sufficient condition that $\rho \in C^0(\Omega)$ and be limited in Ω for $\vec{E} \in C^0(\Omega)$ and be limited in Ω . In particular, if ρ is discontinuous in Ω , but it can be decomposed in continuous and limited functions in sub-regions of Ω , it is sufficient for $\vec{E} \in C^0(\Omega)$ and be limited in Ω . The case $\rho = 0$ in Ω is a special case $\rho \in C^0(\Omega)$ and be limited in Ω . We can finally conjecture (examples will

$$\oint \operatorname{curl} \vec{E} \cdot \vec{n} dS = 0 .$$

\sum

Due to the arbitrary choice of γ , it finally follows

$$\operatorname{curl} \vec{E} = \vec{0} . \quad (5.20)$$

Equations (5.4) and (5.20), which are valid in case 1 [$\vec{E} \in C^0(\Omega)$ and limited in Ω], are Gauss' theorem and the irrotational property of \vec{E} in local form. Since $\vec{E} \in C^1(\Omega)$ and is limited in Ω , the local form is actually a differential form in this case.

5.1.3 Gauss' theorem and the irrotational property of \vec{E} .

Case 2.

Consider a continuous and limited charge distribution τ at any point of a surface Σ in the 3D Euclidian space. We assume the region Σ to be smooth. The total charge Q in a volume Ω entirely containing Σ is

$$Q = \iiint_{\Omega} dx dy dz \tau(\vec{r}(x, y, z)) \delta(\vec{r} - \vec{r}_0(x, y, z)) , \quad (5.21)$$

where the integral has been expressed in cartesian coordinates (note that τ has units C/m^2), δ is the delta-Dirac function, $\vec{r} = \vec{r}(x, y, z)$ is any point in Ω , and $\vec{r}_0(x, y, z)$ is a point

on the surface Σ . Due to the presence of the S-Dirac function, Eq. (5.21) clearly shows that the charge distribution T represents a discontinuity in the 3D space. Following our guidelines in the top paragraph of page 26 of this lecture (just before Sec. 5.1.2), in this case we should not resort to Eqs. (5.4) and (5.20) to express Gauss' theorem and the irrotational property of \vec{E} in local form. Instead, we should use Eqs. (5.1) and (5.2) for a special "infinitesimal" surface and line, respectively. In fact, those equations are valid even for discontinuous charge distributions.

- Gauss' theorem.

Consider an infinitesimal closed surface of cylindrical type, Σ_c . Since we want to study the properties of \vec{E} infinitesimally close to Σ , both above and below Σ , the bases S_1 and S_2 of cylinder Σ_c are assumed to be above and below Σ , respectively. The two bases are assumed to be circles with equal radius $d\tau$, from which the infinitesimal nature of the bases. The area of the two bases is thus

$$S_1 = S_2 = \pi d\tau^2. \quad (5.22)$$

The lateral surface S_L of cylinder Σ_c is assumed to have area

$$S_L = 2\pi d\tau \cdot h, \quad (5.23)$$

where h is the height of Σ_c . Curve Σ_c is typically referred to as a "coin-type" curve. In order to study \vec{E}

infinitesimally close to Σ , we will need to consider [29] the limit for h going to zero. Figure 5.1 shows a coin-type curve Σ_c in the vicinity of Σ . The center of Σ_c is a generic point P on Σ . The surface Σ is oriented in such a way that the normal unity vector \vec{n} at P is directed upward with respect to Σ .

We can thus write Eq. (5.3) for Σ_c as

$$\begin{aligned} \oint_{\Sigma_c} \vec{E} \cdot \vec{n} dS &= \iint_{S_1} \vec{E}_1 \cdot \vec{n}_1 dS + \iint_{S_2} \vec{E}_2 \cdot \vec{n}_2 dS \\ &+ \iint_{S_e} \vec{E}_e \cdot \vec{n}_e dS = \iiint_{V_c} d\tau \delta(\vec{r}_0) \delta(\vec{r} - \vec{r}_0) = \iint_S d\sigma, \end{aligned} \quad (5.24)$$

where \vec{E}_1 , \vec{E}_2 , and \vec{E}_e are the electric fields on surfaces S_1 , S_2 , and S_e , respectively, V_c the volume of Σ_c , and S the portion of Σ included in Σ_c . The last two integrals in Eq. (5.24) are different ways to calculate the total charge in Σ_c . Due to the infinitesimal nature of S_1 and S_2 , E_1 and E_2 can be assumed to be constant on S_1 and S_2 . No assumption is made on \vec{E}_e which, in general, can vary arbitrarily on S_e . The last integral of Eq. (5.24) can be written as

$$\iint_{S_e} \vec{E}_e \cdot \vec{n}_e dS = \iint_{S_e} \vec{E}_e \cdot \vec{n}_e \cdot 2\pi dr \cdot h.$$

Note that \vec{n}_e is here the normal unity vector at each point

on S_e . It is easy to convince oneself that

301

$$\lim_{h \rightarrow 0} \iint_{S_e} \vec{E}_e \cdot \vec{n}_e \cdot \Delta T dA \cdot h = 0.$$

In this limit we can thus disregard the contribution of \vec{E}_e to the integral (5.24). In the same limit, it follows that $\vec{n}_1 = \vec{n}$ and $\vec{n}_2 = -\vec{n}$. Inserting Eq. (5.22) in the integral (5.24), the latter reads

$$\lim_{h \rightarrow 0} \sum_{c_i} \iint_{\Sigma_c} \vec{E} \cdot \vec{n} dS = \vec{n} \cdot \vec{E}_1 \cdot \Delta r^2 - \vec{n} \cdot \vec{E}_2 \cdot \Delta r^2 \\ = \frac{1}{\epsilon_0} \sigma \cdot \Delta r^2, \quad (5.25)$$

where we have assumed areas S_1 and S_2 to be equal to the area S of the portion of Σ included in \sum_c . This is a reasonable assumption considering that S_1 and S_2 are infinitesimally small and arbitrarily close to Σ due to the limit on h . Therefore, we can write the integral (5.24) as

$$\vec{n} \cdot (\vec{E}_1 - \vec{E}_2) = \frac{1}{\epsilon_0} \sigma. \quad (5.26)$$

Condition (5.26) shows that the normal components of the electric field \vec{E} infinitesimally above and below a continuous and limited

charge distribution σ on a smooth surface Σ present a discontinuity of the first kind. The electric field components remain limited. The difference between the normal components E_{n1} and E_{n2} at a generic point P on Σ , on each side of Σ , is proportional to the surface charge distribution evaluated at P :

$$E_{n1}(P) - E_{n2}(P) = \frac{1}{\epsilon_0} \sigma(P).$$

It is worth noting that if the charge was distributed with a volume density ρ inside the coin-type surface Σ_c , instead of a surface density σ , the total charge in Σ_c would reduce to zero in the limit for ρ going to zero. It is the surface nature of the charge distribution to assure the charge inside Σ_c is not affected by the limit.

Condition (5.26) implies that the $\text{div } \vec{E}$ at any point on Σ is not well defined, at least in the usual sense. This confirms it is not possible to use Eq. (5.4) in this case. Due to the discontinuity of \vec{E} at any point on Σ , also the $\text{curl } \vec{E}$ at any point on Σ is not well defined, at least in the usual sense. Hence, it is impossible to use Eq. (5.20) in this case. \vec{E} must be at least continuous for $\text{div } \vec{E}$ and $\text{curl } \vec{E}$ to be well defined.

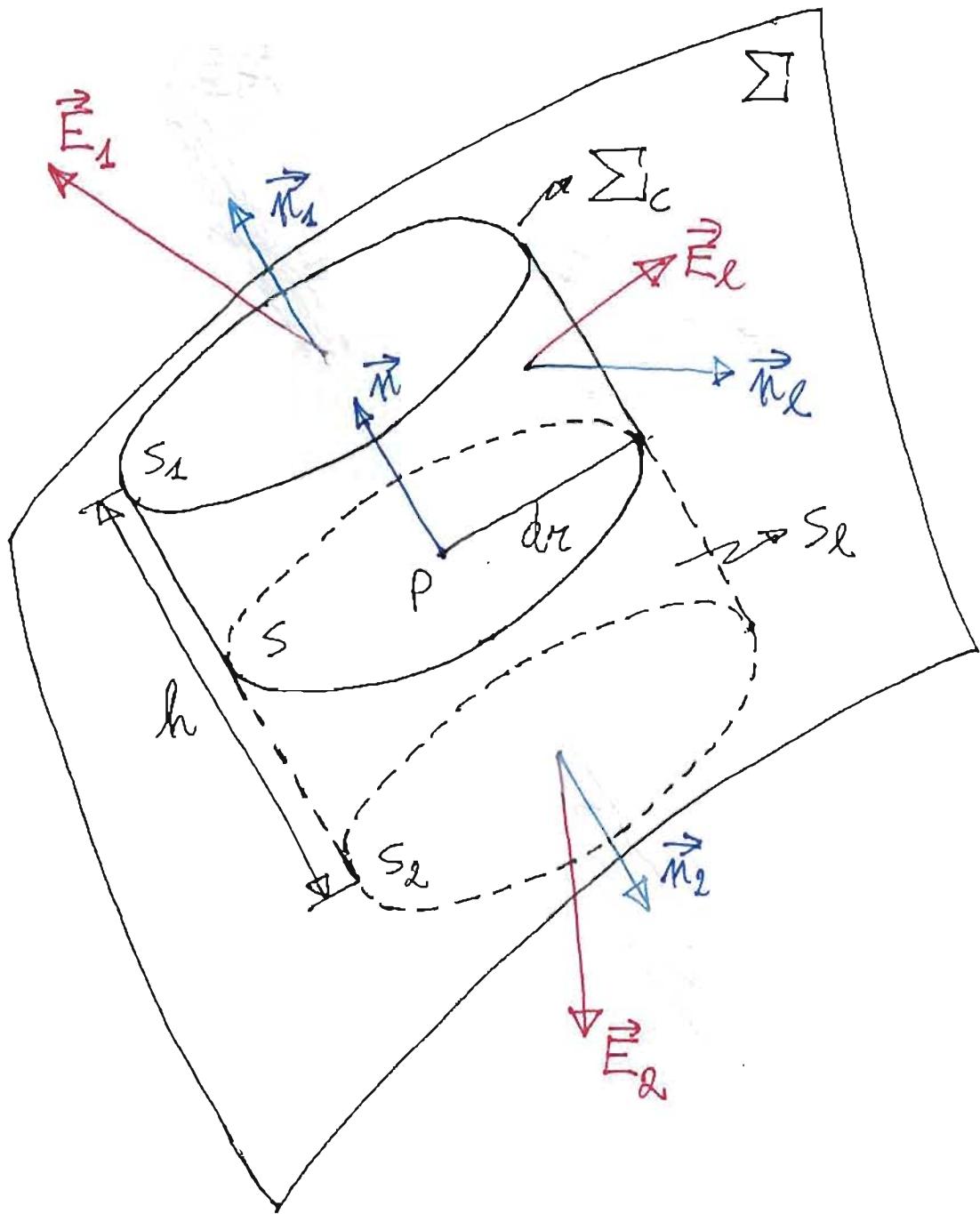


Figure 5.1.
Limit of small h .

- Gerotational property of \vec{E} .

Consider an infinitesimal closed line of rectangular type, $\gamma_c = ABCD$, oriented clockwise. Since we want to study the properties of \vec{E} infinitesimally close to Σ , both above and below Σ , the sides AB and CD are assumed to be above and below Σ , respectively. These two sides are assumed to be line segments with equal length $\overline{AB} = \overline{CD} = dl$, from which the infinitesimal nature of the sides. The sides BC and DA that intersect Σ are assumed to be line segments with equal length $\overline{BC} = \overline{DA} = h$. Curve γ_c is typically referred to as a "cut-type" curve. As for the coin-type curve, in order to study \vec{E} infinitesimally close to Σ , we will need to consider the limit for h going to zero. Figure 5.2 shows a cut-type curve γ_c in the vicinity of Σ . The center of γ_c is a generic point P on Σ . The tangent unit vector \vec{T} at P on Σ is oriented as shown in Fig. 5.2.

We can thus write Eq. (5.2) for γ_c as

$$\begin{aligned} \oint_{\gamma_c} \vec{E} \cdot \vec{T} dl &= \int_{AB} \vec{E}_1 \cdot \vec{E}_1 dl + \int_{BC} \vec{E}_{BC} \cdot \vec{E}_{BC} dl \\ &\quad + \int_{CD} \vec{E}_2 \cdot \vec{E}_2 dl + \int_{DA} \vec{E}_{DA} \cdot \vec{E}_{DA} dl = 0, \end{aligned} \quad (5.24)$$

where \vec{E}_1 , \vec{E}_{BC} , \vec{E}_2 , and \vec{E}_{DA} are the electric fields on

lines AB, BC, CD, and DA, respectively, and $\vec{t}_1, \vec{t}_{BC}, \vec{t}_2$, and \vec{t}_{DA} the corresponding tangent unit vectors. Due to the infinitesimal nature of AB and CD, \vec{E}_1 and \vec{E}_2 can be assumed to be constant on AB and CD, respectively. No assumptions are made on \vec{E}_{BC} and \vec{E}_{DA} which, in general, can vary arbitrarily on BC and DA, respectively. In the limit for h going to zero, it follows that $\vec{t}_1 = \vec{t}$ and $\vec{t}_2 = -\vec{t}$ and the integral (5.24) can be rewritten as

$$\begin{aligned}
 \lim_{h \rightarrow 0} \oint_C \vec{E} \cdot \vec{t} dl &= \vec{E}_1 \cdot \vec{t} dl + \lim_{h \rightarrow 0} \int_{BC} \vec{E}_{BC} \cdot \vec{t}_{BC} dl \\
 &\quad - \vec{E}_2 \cdot \vec{t} dl + \lim_{h \rightarrow 0} \int_{DA} \vec{E}_{DA} \cdot \vec{t}_{DA} dl \\
 &= \vec{E} \cdot (\vec{E}_1 - \vec{E}_2) = \vec{n} \times (\vec{E}_1 - \vec{E}_2) = 0,
 \end{aligned} \tag{5.28}$$

where we have used the commutative property of the scalar product, \vec{n} is the normal unit vector at P, and we have assumed the contributions from the line integrals on BC and DA to be negligible as the line lengths go to zero. Note that the " $\vec{E} \cdot$ " and " $\vec{n} \times$ " forms of condition (5.28) are a scalar and a vector, respectively. Hence, it is slightly improper to assume

them to be equal. However, they are both equal to zero 35 and, thus, lead to the same condition that the tangent components of the electric field \vec{E} infinitesimally above and below a continuous and limited charge distribution τ on a smooth surface Σ are continuous and limited:

$$E_{t_1}(P) = E_{t_2}(P).$$

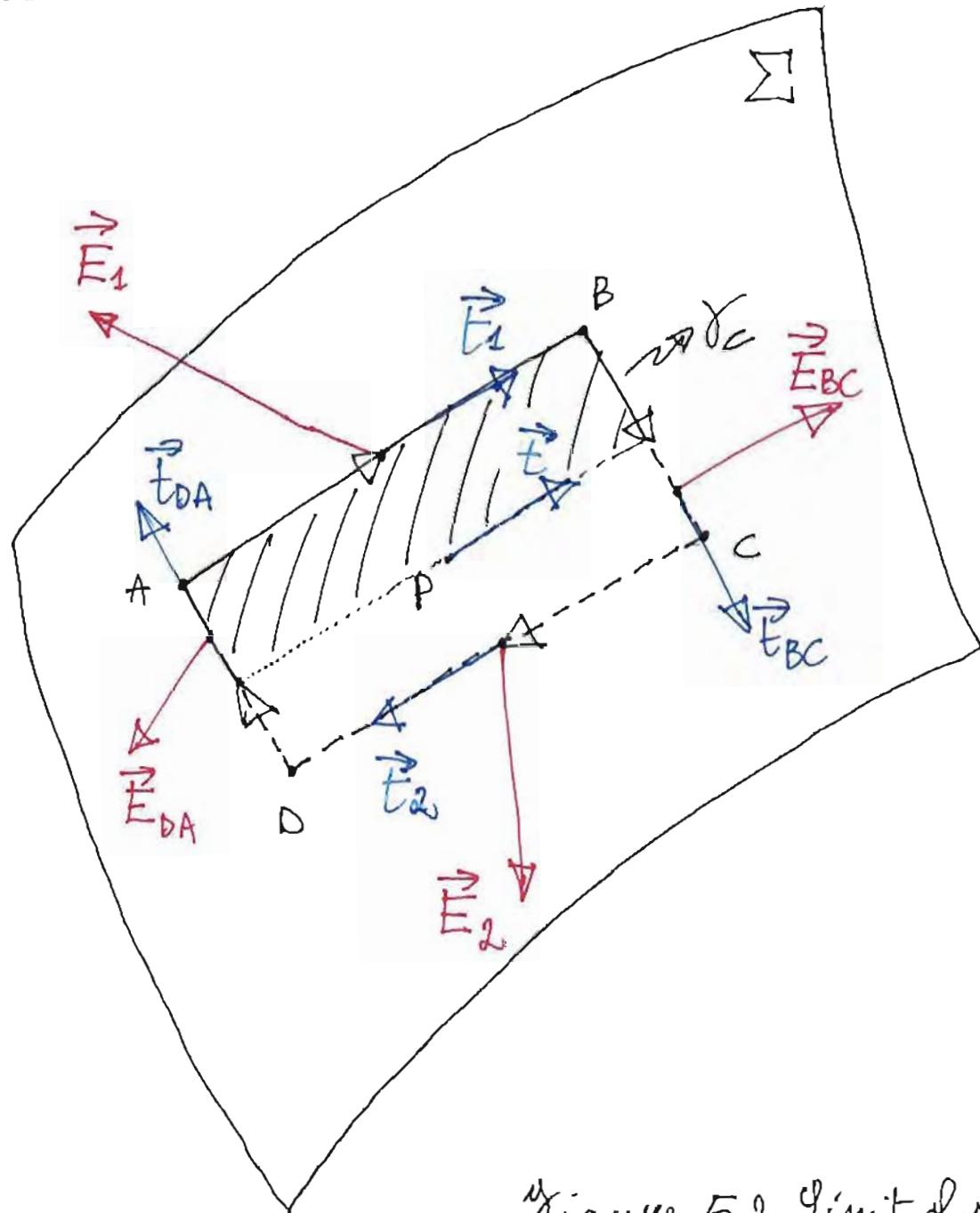


Figure 5.2. Limit of small h .

Case 3.

Consider a compact region Ω in the 3D Euclidian space. Assume a continuous and limited charge distribution s in a sub-region Ω_1 (also compact) of Ω and zero charge in the remaining part of Ω , $\Omega_2 = \Omega - \Omega_1$. The charge distribution presents a discontinuity at the boundary between Ω_1 and Ω_2 . Following again our guidelines on the use of the differential forms in Eqs. (5.4) and (5.20), we shall resort again on coin- and cut-type curve to obtain Gauss' theorem and the irrotational property of \vec{E} in local form (this case can be generalized to two arbitrary densities s_1 and s_2 in Ω_1 and Ω_2 , continuous and limited in each sub-region and discontinuous on the boundary).

Figure 5.3 shows regions Ω , Ω_1 , and Ω_2 , where, in this case, Σ represents the boundary between Ω_1 and Ω_2 . We can thus use the same coin-type curve as in Fig. 5.1 to determine the integral (5.3). This reads:

$$\oint_{\Sigma} \vec{E} \cdot \vec{n} dS = \iint_{S_1} \vec{E}_1 \cdot \vec{n}_1 dS + \iint_{S_2} \vec{E}_2 \cdot \vec{n} dS$$

$$+ \iint_{S_e} \vec{E}_e \cdot \vec{n}_e dS = \iiint_{\Omega_c} d\tau S . \quad (5.20)$$

In the limit of small b , S can be assumed to be constant in

the region of Σ_1 contained in Σ_C . Hence,

341

$$\iiint_{\Sigma_C} dz \, S = S \cdot \pi dr^2 \cdot \frac{1}{2} h .$$

Under these conditions

$$\lim_{h \rightarrow 0} \iiint_{\Sigma_C} dz \, S = \lim_{h \rightarrow 0} \frac{1}{2} S \pi dr^2 \cdot h = 0. \quad (5.30)$$

Note that, as for the case of components \vec{E}_{BC} and \vec{E}_{DA} in Fig. 5.2, we would have obtained the same result as in even without assuming S to be constant. In fact, as soon as the volume of Σ_C goes to zero, there should be zero charge due to a volume charge distribution S . This was not the case for a surface charge distribution because of the presence of the δ -Dirac in the integral (5.24). Due to condition (5.30), in the limit for h going to zero, integral (5.29) reads:

$$\vec{n} \cdot (\vec{E}_1 - \vec{E}_2) = 0 . \quad (5.31)$$

In case 3, the normal components of \vec{E} on each side of Σ (note that Σ is smooth because Σ_1 and Σ_2 are both compact) are continuous:

$$E_{n1}(P) = E_{n2}(P).$$

This result was to be expected in the light of our demonstration in pages 18-22 of this lecture, which is thus confirmed. [381]

The condition of Eq. (5.31), however, does not provide any knowledge on the behavior of the derivatives of \vec{E} on each side of Σ , infinitesimally close to it. We therefore recommend to use a cut-type curve to obtain the local form for the irrotational property of \vec{E} .

- Irrotational property of \vec{E} .

By resorting to the cut-type curve of Fig. 5.2 for case 3, we obtain the very same result of condition (5.28):

$$\vec{t} \cdot (\vec{E}_1 - \vec{E}_2) = 0 . \quad (5.32)$$

Case 4.

Consider a continuous and limited linear charge distribution λ on a smooth line γ in the 3D Euclidian space. Consider a point P_0 on γ and the distance d between P_0 and any point P in the 3D space. When P is very close to P_0 , the line γ appears as a line. Hence, the distance d can be approximated by the radial distance r between P and P_0 (cf. Fig. 5.4). Without entering in details.

- Gauss' theorem.

It can be shown that

$$\lim_{r \rightarrow 0} E_n = \frac{\lambda}{2\pi\epsilon_0 r}, \quad (5.33)$$

where E_n is the magnitude of the normal component of \vec{E} at P_0 on γ , $\vec{E}_n = E_n \vec{u}_z = E_n \vec{n}$. This result should not surprise as for P very close to P_0 (eventually infinitesimally close), γ appears as an infinite field for an infinite line with λ is of the type (5.33). Condition (5.33) implies that the normal component of \vec{E} at any point on γ diverges to infinity asymptotically as $1/r$.

- Circumferential property of \vec{E} .

It can be shown that

$$\lim_{r \rightarrow 0} E_t = \frac{\lambda'}{2\pi\epsilon_0} \ln r, \quad (5.34)$$

where λ' is the derivative of λ along γ and E_t is the magnitude of the tangent component of \vec{E} at P_0 on γ , $\vec{E}_t = E_t \vec{t}$. This result does not contradict the fact that for an infinite straight line the component $\vec{E}_z = \vec{E}_t = \vec{0}$ (cf. lecture 4). In fact, in the case of an infinite straight line $\lambda' = 0$. More in general, from conditions (5.33) and (5.34) it follows that

$$\lim_{P \rightarrow P_0} \frac{E_t}{E_n} = 0 \quad (5.35)$$

because E_n goes to infinity faster ($\sim 1/r$) than E_t ($\sim \ln r$).

This means that in very close proximity (in principle, infinitesimally close) to any line \mathcal{S} with λ , the electric field \vec{E} is perpendicular to \mathcal{S} .

Case 5.

Consider an isolated point-like charge. In this case the electric field diverges as $1/r^2$ in very close proximity of the charge, as expected from Coulomb's law.

- All other cases must be considered individually.

In summary, for cases 1 - 5 the local form of Gauss' theorem and of the irrotational property of \vec{E} are not differential forms. These special local forms represents boundary conditions for \vec{E} . For cases 2 and 3 the boundary conditions are continuity or discontinuity conditions of the first kind. For cases 4 and 5 they are asymptotic conditions.

6.1 Electrostatic field \vec{E} generated by point-like charges. 21

① Consider two point-like charges q_1 and q_2 . The first charge is located at point $Q = Q_1$ and the second at point $Q = Q_2$. The distance $\overline{Q_1 Q_2} = d$.

Calculate \vec{E} at point $P = O$ at the center of line segments $Q_1 Q_2$ and at any point $P = O'$ on the axis of $Q_1 Q_2$ passing through O ! (cf. Fig. 6.1).

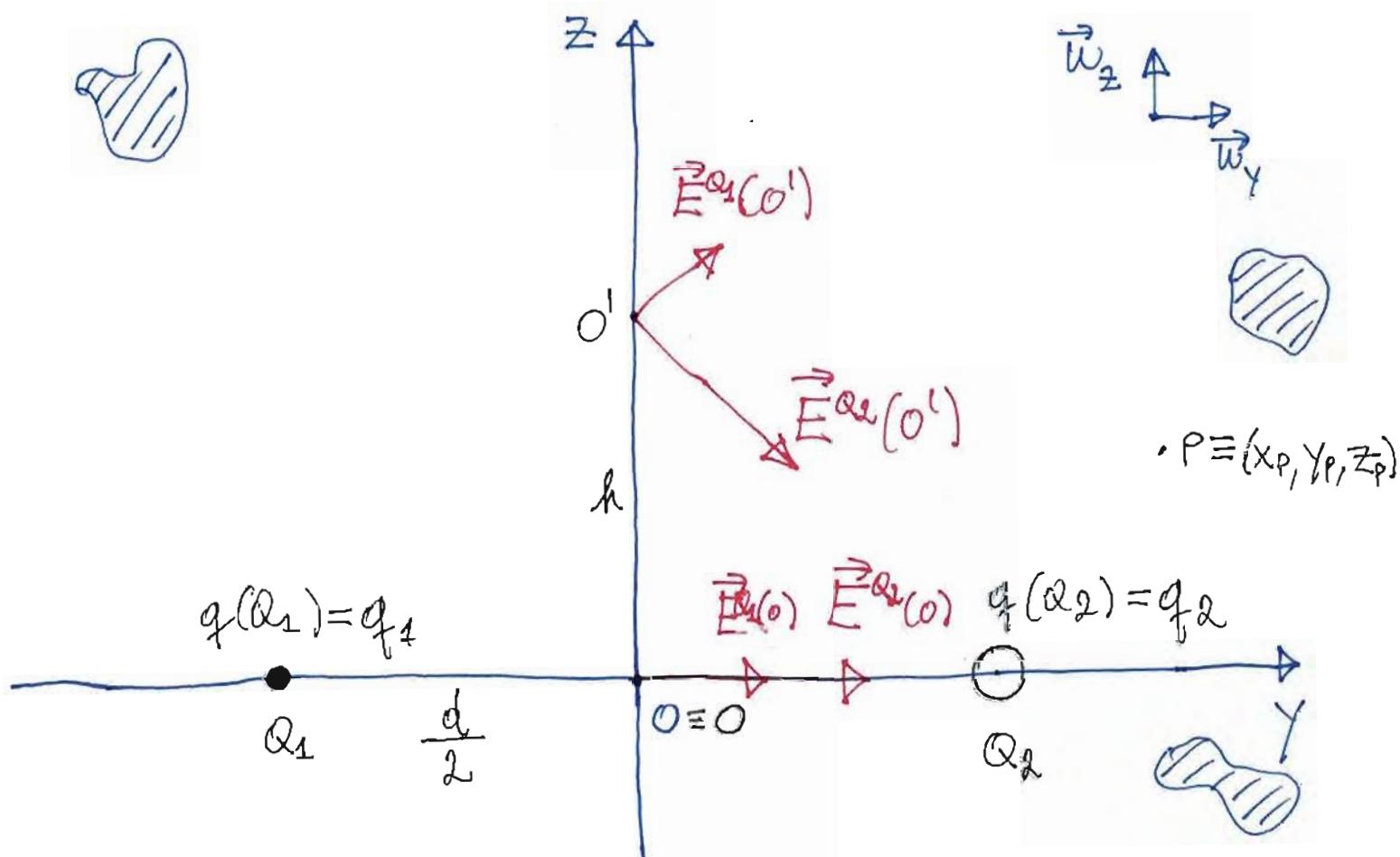


Figure 6.1.

② Preference frame and coordinate system:

The two charges are considered with respect to the inertial reference frame (shaded blue regions) shown in Fig. 6.1. The two charges

belong to a single plane, where their position can be represented with respect to the cartesian coordinate system Oyz shown in Fig. 6.1. The x axis, perpendicular to the plane of the two charges, is not indicated. The origin O has been chosen to coincide with the middle point of $Q_1 Q_2$. Hence, O' is on the z axis. The coordinates of the various points are:

$$\left\{ \begin{array}{l} Q_1 \equiv \left(0, -\frac{d}{2}, 0\right) \\ O \equiv (0, 0, 0) \\ Q_2 \equiv \left(0, +\frac{d}{2}, 0\right) \\ O' \equiv (0, 0, +h) \end{array} \right. . \quad (6.1)$$

The field \vec{E} at any point P can be written in components as

$$\vec{E}(P) = E_x(P) \vec{u}_x + E_y(P) \vec{u}_y + E_z(P) \vec{u}_z .$$

③ Indicate all fields \vec{E} :

A point-like charge q at point Q, $q(Q)$, generates at any point P a field according to Coulomb's law:

$$\vec{E}_{QP} = \frac{1}{4\pi\epsilon_0} \frac{q(Q)}{r_{QP}^2} \vec{u}_{QP} , \quad (6.2)$$

where r_{QP} is the distance between Q and P and \vec{u}_{QP} the corresponding unit vector oriented from Q to P.

The fields generated according to Eq. (6.2) by the charges q_1 and q_2 have been indicated in Fig. 6.1, both at o and o' . 41

④ DOF:

Due to Eq. (6.2), the total field at o and o' must lie on the yz plane and, thus, cannot have an x component. In this, the DOF are the y and z components of \vec{E} .

⑤ Results:

- Point o .

The field generated by $q(Q_1)$ at o , $\vec{E}^{Q_1}(o)$, can be calculated by writing Eq. (6.2) for Q_1 and o in cartesian coordinates:

$$\begin{aligned}\vec{E}^{Q_1}(o) &= E_x^{Q_1}(o) \vec{u}_x + E_y^{Q_1}(o) \vec{u}_y + E_z^{Q_1}(o) \vec{u}_z \\ &= \frac{q_1}{4\pi\epsilon_0} \left(\frac{x_o - x_{Q_1}}{[(x_o - x_{Q_1})^2 + (y_o - y_{Q_1})^2 + (z_o - z_{Q_1})^2]^{3/2}} \vec{u}_x \right. \\ &\quad + \frac{y_o - y_{Q_1}}{[(x_o - x_{Q_1})^2 + (y_o - y_{Q_1})^2 + (z_o - z_{Q_1})^2]^{3/2}} \vec{u}_y \\ &\quad \left. + \frac{z_o - z_{Q_1}}{[(x_o - x_{Q_1})^2 + (y_o - y_{Q_1})^2 + (z_o - z_{Q_1})^2]^{3/2}} \vec{u}_z \right),\end{aligned}\quad (6.3)$$

Inserting conditions (6.1) in Eq. (6.3) we obtain

$$\vec{E}^{Q_1}(O) = E_y^{Q_1}(O) \vec{u}_y = \frac{q_1}{4\pi\epsilon_0} \frac{\frac{d}{2}}{\left(\frac{d}{2}\right)^3} \vec{u}_y$$

$$= \frac{q_1}{\pi\epsilon_0} \frac{1}{d^2} \vec{u}_y \quad . \quad (6.4)$$

Similarly,

$$\vec{E}^{Q_2}(O) = E_y^{Q_2}(O) \vec{u}_y = \frac{q_2}{4\pi\epsilon_0} \frac{Y_0 - Y_{Q_2}}{[(Y_0 - Y_{Q_2})^2]^{3/2}} \vec{u}_y$$

$$= -\frac{q_2}{\pi\epsilon_0} \frac{1}{d^2} \vec{u}_y \quad . \quad (6.5)$$

By means of the superposition principle we can thus calculate the total field at O due to both q_1 and q_2 :

$$\vec{E}(O) = \vec{E}^{Q_1}(O) + \vec{E}^{Q_2}(O)$$

$$= \frac{q_1 - q_2}{\pi\epsilon_0} \frac{1}{d^2} \vec{u}_y \quad . \quad (6.6)$$

As expected from Coulomb's law, the field in O due to q_1 and q_2 has only a y component.

81

radial direction at Q_1 , and the distance between Q_2 and Q_1 has been approximated to be d (cf. Fig. 6.2).

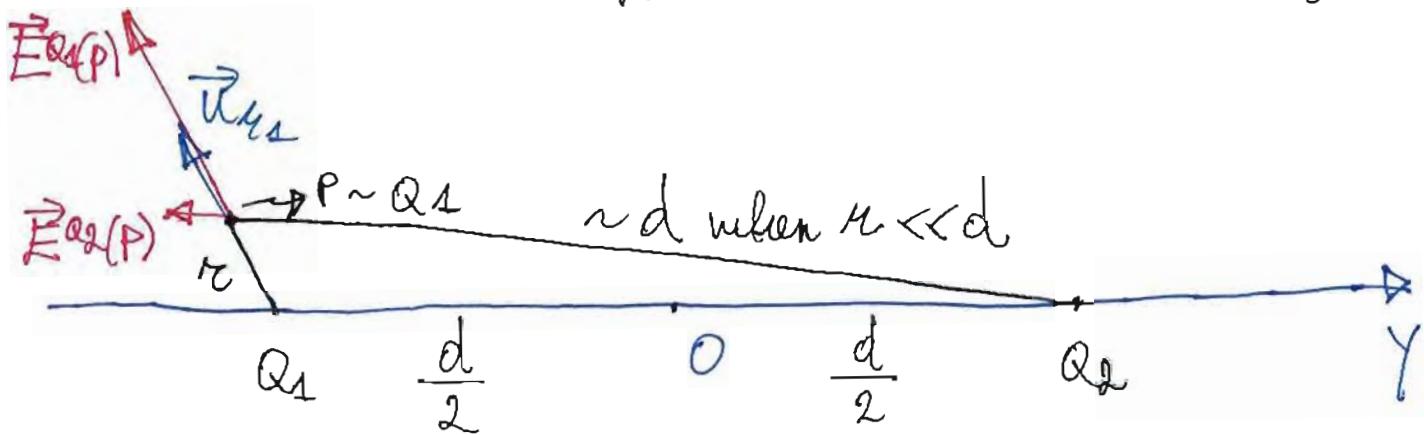


Figure 6.2.

When $r \ll d$, the contribution to the field from Q_2 can be neglected:

$$\vec{E}(P \sim Q_1) \approx \frac{1}{4\pi\epsilon_0} \frac{q_1}{r^2} \vec{u}_{R1} . \quad (6.10)$$

Similarly,

$$\vec{E}(P \sim Q_2) \approx \frac{1}{4\pi\epsilon_0} \frac{q_2}{r^2} \vec{u}_{R2} , \quad (6.11)$$

where \vec{u}_{R2} is the unit vector in the radial direction at Q_2 .

- Any point P on the y axis, $P \equiv (0, y_P, 0)$.

$$\vec{E}(O, Y_p, 0) = \frac{1}{4\pi\epsilon_0} \left(q_1 \frac{(Y_p - Y_{Q1})}{|Y_p - Y_{Q1}|^3} + q_2 \frac{(Y_p - Y_{Q2})}{|Y_p - Y_{Q2}|^3} \right) \vec{u}_y . \quad (6.12)$$

• Case $q_1 = +q$ and $q_2 = -q$ (balanced charges).

From Eqs. (6.6), (6.9), (6.10), (6.11), and (6.12):

$$\left\{ \begin{array}{l} \vec{E}(O) = \frac{2q}{\pi\epsilon_0} \frac{1}{d^2} \vec{u}_y \\ \vec{E}(O') = \frac{q}{4\pi\epsilon_0} \frac{d}{\left[\left(\frac{d}{2}\right)^2 + h^2\right]^{3/2}} \vec{u}_y \\ \vec{E}(\sim Q_1) \approx \frac{q}{4\pi\epsilon_0} \frac{1}{r_1^2} \vec{u}_{u_1} \\ \vec{E}(\sim Q_2) \approx -\frac{q}{4\pi\epsilon_0} \frac{1}{r_2^2} \vec{u}_{u_2} \\ \vec{E}(O, Y_p, 0) = \frac{q}{4\pi\epsilon_0} \left(\frac{(Y_p - Y_{Q1})}{|Y_p - Y_{Q1}|^3} + \frac{(Y_{Q2} - Y_p)}{|Y_p - Y_{Q2}|^3} \right) \vec{u}_y \end{array} \right. . \quad (6.13)$$

The knowledge of Eqs. (6.13) allow us to draw the 101
 \vec{E} field lines for case $q_1 = +q$ and $q_2 = -q$ (cf. Fig. 6.3).

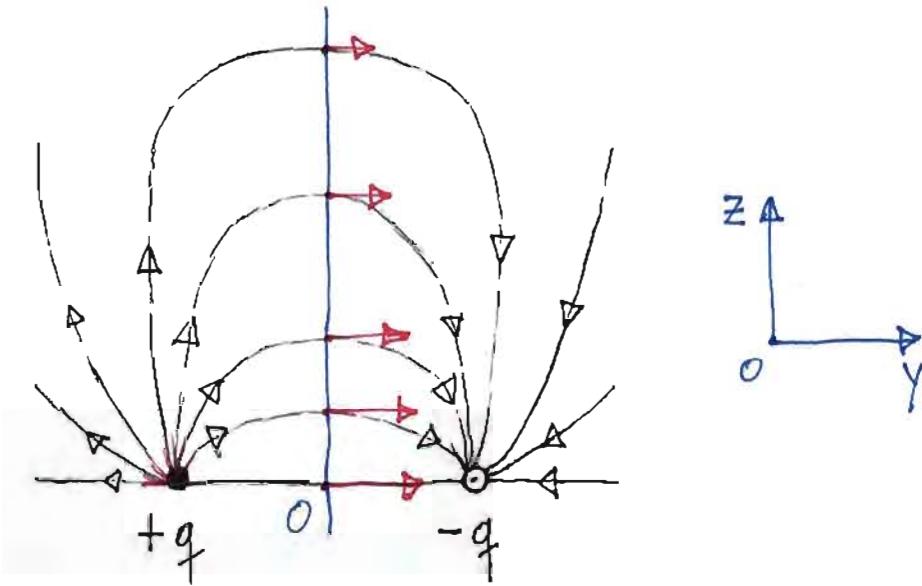


Figure 6.3.

The \vec{E} field lines in Fig. 6.3 are symmetric with respect to the z axis. They are also symmetric with respect to the y axis. This is why we decided not to draw them on the negative side of z .

The knowledge of \vec{E} at o , any point o' , and in the close vicinity of Q_1 and Q_2 (indicated in red in the figure) gives information on the tangent behavior of the \vec{E} field lines at those points. It thus help draw the \vec{E} field lines everywhere on the yz plane, under the reasonable assumption \vec{E} to be as smooth as possible in the regions connecting the various points where we have calculated \vec{E} .

It is useful to consider $\vec{E}(0, Y_p, 0)$ when both $|Y_p - Y_{q1}|, |Y_p - Y_{q2}| \gg d$. In this case, we can assume $|Y_p - Y_{q1}| \approx |Y_p - Y_{q2}|$ and, similarly, $Y_p - Y_{q1} \approx Y_p$ and $Y_{q2} - Y_p \approx -Y_p$. Hence, Eq. (6.12) for $q_1 = +q$ and $q_2 = -q$ reads:

$$\vec{E}(0, Y_p, 0) = \frac{q}{4\pi\epsilon_0} \left(\frac{Y_p}{|Y_p|^3} - \frac{Y_p}{|Y_p|^3} \right) \vec{u}_y$$

\downarrow

$$= \vec{0}.$$

On the y axis, the far field on the positive and negative side of 0 is zero. The two charges appear as a single charge $Q = q - q = 0$. While it is in general true that the field at infinity for any charge distribution has to go to zero, it is not true that the far field (far, but not infinity) even for a charge distribution of total zero charge (neutral charge distribution, as in our case) has to be zero everywhere in space. In our case, it is true that the far field goes to zero on the y axis.

- Case $q_1 = +q$ and $q_2 = -2q$ (unbalanced charges). From Eqs. (6.6), (6.9), (6.10), (6.11), and (6.12):

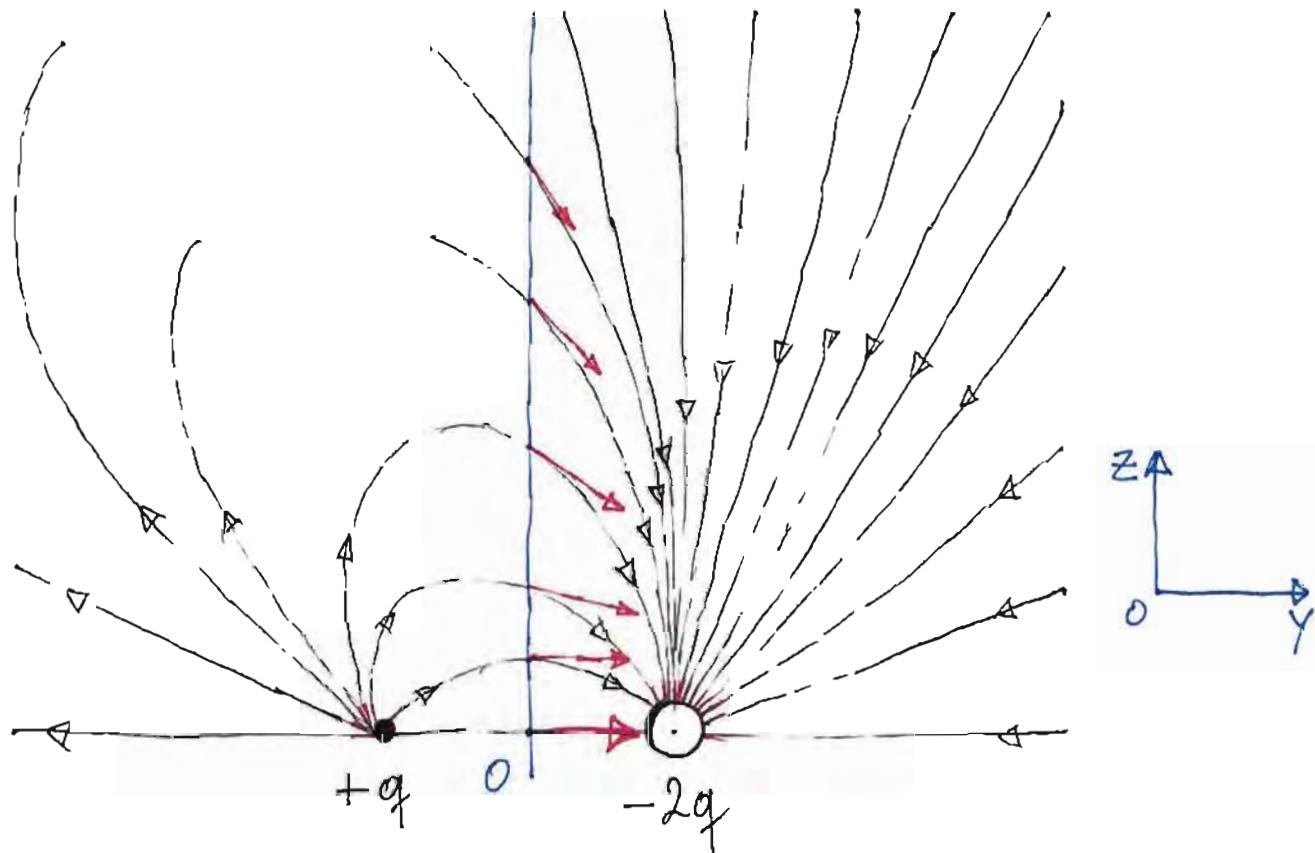


Figure 6.4.

While again the \vec{E} field lines in Fig. 6.4 are symmetric with respect to the y axis, they are asymmetric with respect to the z axis. This is expected from the univalence of the charges.

We note that the y and z components of $\vec{E}(0')$ are proportional, by the same scaling factor, to

$$\left\{ \begin{array}{l} \vec{E}_y(0') \sim \frac{3}{2} d \vec{u}_y \\ \vec{E}_z(0') \sim -h \vec{u}_z \end{array} \right.$$

$$\left\{ \begin{array}{l} \vec{E}_y(0') \sim \frac{3}{2} d \vec{u}_y \\ \vec{E}_z(0') \sim -h \vec{u}_z \end{array} \right.$$

The magnitude of these two components is the same

when $h = \frac{3}{2}d$. This is the fifth red vector from the bottom on the z axis in Fig. 6.4. In this case, the vector displays an angle of 45° with respect to z . The angle is smaller further from o and larger closer to o . In particular, the far field on z for $h \gg d$ is given by

$$\vec{E}_{\text{far}}(0) \approx \frac{q}{4\pi\epsilon_0} \frac{3}{2} \frac{d}{h^3} \vec{u}_y - \frac{q}{4\pi\epsilon_0} \frac{1}{h^2} \vec{u}_z$$

This means the y component of the far field is much smaller than the z component. In other words, the far field tends to point downward to the two charges, along z . This should not surprise because far enough from the two charges, they appear as a single charge $Q = q - 2q = -q$, which attracts field lines to it. Note that the y component of the far field ($\approx 1/h^3$) does not go to zero, it is just smaller than the z component. Of course the field at infinity ($h \rightarrow \infty$) goes to zero.

In this case, the far field on y is given by

$$\vec{E}(0, Y_p, 0) \simeq \frac{q}{4\pi\epsilon_0} \left(\frac{Y_p}{|Y_p|^3} - 2 \frac{Y_p}{|Y_p|^3} \right) \vec{u}_y$$

$$= - \frac{q}{4\pi\epsilon_0} \frac{Y_p}{|Y_p|^3} \vec{u}_y .$$

As expected the two charges behave as a single charge
 $Q = +q - 2q = -q$.

- Case $q_1 = +q$ and $q_2 = +q$ (equal charges).

From Eqs. (6.6), (6.9), (6.10), (6.11), and (6.12):

$$\left\{ \begin{array}{l} \vec{E}(0) = \vec{0} \\ \vec{E}(0^+) = \frac{2q}{4\pi\epsilon_0} \frac{h}{\left[\left(\frac{d}{2} \right)^2 + h^2 \right]^{3/2}} \vec{u}_z \\ \vec{E}(\sim Q_1) \simeq \frac{q}{4\pi\epsilon_0} \frac{1}{q_2} \vec{u}_{\epsilon_1} \quad . \quad (6.15) \\ \vec{E}(\sim Q_2) \simeq \frac{q}{4\pi\epsilon_0} \frac{1}{q_2} \vec{u}_{\epsilon_2} \\ \vec{E}(0, Y_p, 0) = \frac{q}{4\pi\epsilon_0} \left(\frac{(Y_p - Y_{Q_1})}{|Y_p - Y_{Q_1}|^3} + \frac{(Y_p - Y_{Q_2})}{|Y_p - Y_{Q_2}|^3} \right) \vec{u}_y \end{array} \right.$$

Figure 6.5 shows the \vec{E} field lines in this case. [16]

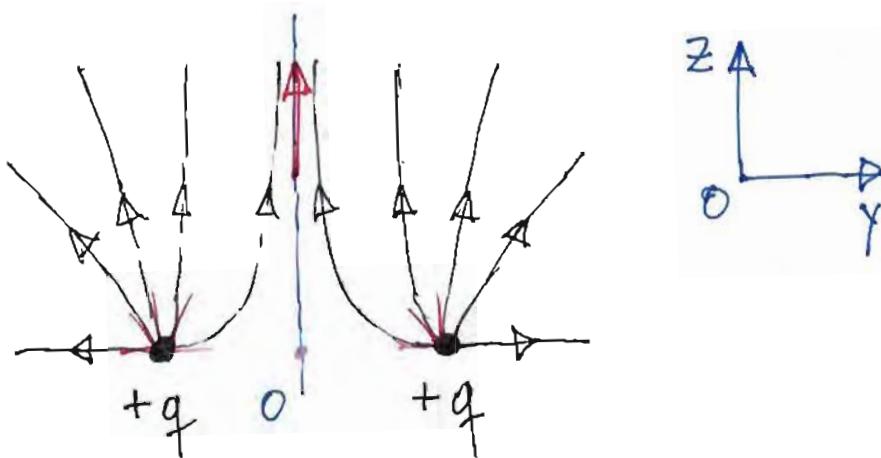


Figure 6.5.

In this case, $\vec{E}(0')$ points always upward along z and $\vec{E}(0, y_p, 0)$ far from the charges is

$$\vec{E}_{\text{far}}(0, y_p) \approx \frac{2q}{4\pi\epsilon_0} \frac{y_p}{|y_p|^3} \hat{y},$$

which is the field of a single charge $Q = 2q$.

The case for $q_1 = -q$ and $q_2 = -q$ is analogous to the one just analyzed.

Note that the case of two charges q_1 and q_2 at a distance d is referred to as electric dipole. We will come back to the electric dipole extensively in the next few lectures.

6.2 Surface and point-like charge densities : The 141 delta-Dirac.

In this section, we study by means of useful examples a formal method to calculate charges from charge densities. The method is based on the delta-Dirac.

- Consider a surface charge density σ . With respect to a cartesian reference frame, assume σ to be constant on the xy plane defined by $z = z_0$. Calculate the total charge contained in a parallelepiped Ω with a rectangular base of sides a and b , and height $2h$. The base is assumed to be parallel to the xy plane and the height half above and half below the plane at $z = z_0$ (cf. Fig. 6.6).

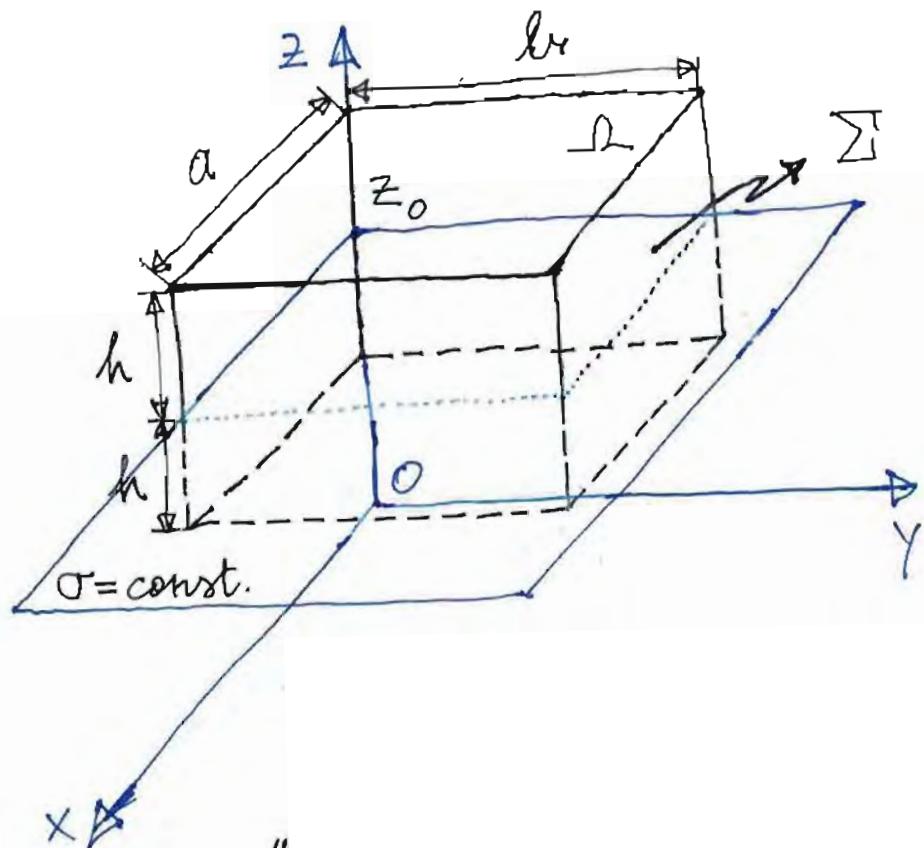


Figure 6.6.

18/

One way to calculate the total charge Q_f contained in the parallelepiped is to resort to the surface integral

$$Q_f = \sum \iint dS \sigma$$

$$= \sigma \int_0^a dx \int_0^{b_x} dy = ab_x \sigma , \quad (6.16)$$

where \sum is the surface intersecting the parallelepiped and the xy plane at z_0 .

A more general way to calculate Q_f is by means of the volume integral

$$Q_f = \iiint_V dV \sigma . \quad (6.17)$$

This integral, however, seems to be not well defined. In fact, σ has by definition units $C m^{-2}$, while dV units m^3 . We can circumvent this problem by using the delta - Dirac. In cartesian coordinate, we can thus rewrite the integral (6.14) as

$$Q_f = \int_0^a dx \int_0^{b_r} dy \int_{z_0-h}^{z_0+h} dz \cdot \sigma(x, y, z) \delta(z - z_0)$$

$$= \int_0^a dx \int_0^{b_r} dy \cdot \sigma(x, y, z_0) = a b_r \cdot \sigma, \quad (6.18)$$

Where we have used the fundamental property of the delta-Dirac:

$$\int_{\xi_0-\varepsilon}^{\xi_0+\varepsilon} d\xi \cdot f(\xi) \delta(\xi - \xi_0) = f(\xi_0) \quad (6.19)$$

and the fact that $\sigma(x, y, z_0) = \sigma = \text{const.}$ The results of Eqs. (6.18) and (6.16) are equal, as expected.

Consider now the case for $h \rightarrow 0$. Physically, this means that h is an arbitrarily small quantity, exactly as ε in integral (6.19). The integral between $(z_0 - h)$ and $(z_0 + h)$ in (6.18), thus, still includes point $z = z_0$:

$$Q_f(h \rightarrow 0) = \lim_{h \rightarrow 0} \int_0^a dx \int_0^{b_r} dy \int_{z_0-h}^{z_0+h} dz \cdot \sigma(x, y, z) \delta(z - z_0)$$

$$\begin{aligned}
 &= \int_0^a dx \int_0^b dy \int_{z_0-\varepsilon}^{z_0+\varepsilon} dz \cdot \Gamma(x, y, z) \delta(z - z_0) \\
 &= \int_0^a dx \int_0^b dy \cdot \Gamma(x, y, z_0) = ab \Gamma.
 \end{aligned}$$

20/-

The result of Eq. (6.18) is valid even in the case of an infinitesimally small volume.

If instead of having a surface charge density σ we had a constant volume charge density δ in \mathcal{L} , the total charge in \mathcal{L} would be:

$$\begin{aligned}
 Q_\delta &= \iiint d\tau \delta \\
 &= \delta \int_0^a dx \int_0^b dy \int_{z_0-h}^{z_0+h} dz = 2abh\delta. \quad (6.20)
 \end{aligned}$$

In this case,

$$Q_\delta(h \rightarrow 0) = \lim_{h \rightarrow 0} (2abh\delta) = 0.$$

Is it possible to describe a point-like charge Q by means of a suitable charge density? For this purpose consider a constant volume charge density δ in a sphere Σ with radius a . In order to emulate a point-like charge, we can continuously reduce a until Q is entirely localized in a single point. We thus obtain a function

$$\left\{ \begin{array}{ll} \delta = 0 & \text{outside } \Sigma \\ \delta = \frac{Q}{\frac{4}{3}\pi a^3} & \text{inside } \Sigma \end{array} \right. . \quad (6.21)$$

The function δ changes during the compression as Q must remain the same. For $a \rightarrow 0$ the function (6.21) becomes ill defined. However, distribution theory makes it possible to define such function, called delta-Dirac, where

$$\left\{ \begin{array}{ll} \delta(P) = 0 & \forall P \neq 0 \\ \iiint_{\Omega} d^3r \delta(P) = 1 & \forall \Omega \text{ including } 0 \end{array} \right. , \quad (6.22)$$

with 0 being the origin of a coordinate system.

It is possible to prove that

[21]

$$\iiint_{\Omega} dz \cdot f(P) \delta(P) = f(0) ,$$

where $f(P)$ is a generic function in Ω . It is also possible to define a delta-Dirac centered at a point P_0 different from the origin, $\delta(P - P_0)$.

We can thus define the charge density s of a point-like charge Q at point P_0 as

$$s(P) = Q \delta(P - P_0) . \quad (6.23)$$

Hence,

$$Q = \iiint_{\Omega} dz \cdot s(P) = \iiint_{\Omega} dz \cdot Q \delta(P - P_0) ,$$

as long as the region Ω includes the point P_0 where Q is located.

6.3 Electric flux in spherical coordinates.

Calculate the flux of a uniform electrostatic field \vec{E} through a surface Σ with the shape of an emisphere of radius r . The emisphere axis is assumed to be parallel to the field \vec{E} (fig. 6.4).

② Reference frame and coordinate system:

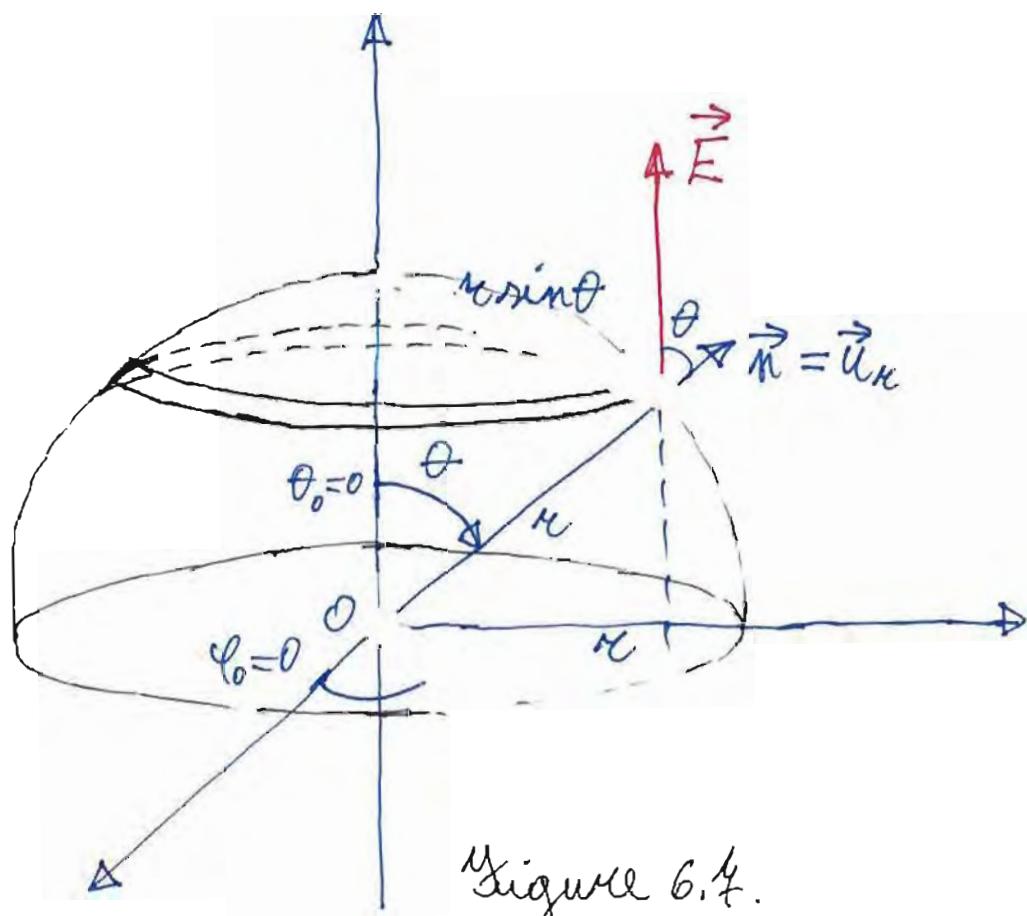
A spherical coordinate system $Oxyz$ with origin O in the center of the emisphere and with respect to an inertial reference frame is indicated in Fig. 6.4.

③ Indicate all fields \vec{E} :

Field \vec{E} , uniform and parallel to the emispher axis, is shown in Fig. 6.4.

④ DOF:

Since the radius r of the emisphere is fixed, the DOF are the angles θ and φ .



⑤ Results:

The infinitesimal surface dS indicated in Fig. 6.4 is given by:

$$dS = r d\theta \int_0^{2\pi} r \sin\theta \cdot d\varphi \\ = 2\pi r^2 \sin\theta d\theta .$$

This is the surface to be used when calculating the flux

$$\Phi = \sum \vec{E} \cdot \vec{n} dS .$$

Due to the conditions on \vec{E} :

$$\vec{E} \cdot \vec{n} = E \cos\theta .$$

Hence,

$$\Phi = \int_0^{\pi/2} E \cos\theta \cdot 2\pi r^2 \sin\theta d\theta \\ = 2\pi r^2 E \int_0^{\pi/2} d\theta \frac{e^{i\theta} + e^{-i\theta}}{2} - \frac{e^{i\theta} - e^{-i\theta}}{2i} .$$

$$\begin{aligned}
 &= 2\pi r^2 E \int_0^{\pi/2} d\theta \frac{1}{2} \sin(2\theta) \\
 &= \pi r^2 E \int_0^{\pi} d(\theta) \frac{1}{2} \sin(2\theta) \\
 &= \frac{\pi r^2 E}{2} \left[-\cos(2\theta) \right]_0^{\pi} = \pi r^2 E .
 \end{aligned}$$

This result shows the interesting property that the flux through Σ due to a uniform field \vec{E} is equal to the magnitude E times the area of the projection of Σ on a plane perpendicular to \vec{E} . This is of course not true in general.

6.4 The $\text{div } \vec{E}$ and $\text{curl } \vec{E}$ for the case of the sphere with different charge densities.

In homework 1 we studied the problem of a sphere of radius R in, case 1, a constant volume charge density $\rho = \rho_0$ inside the sphere and, case 2, a volume charge density $\rho = \rho_0 r/R_0$ from the center of the sphere to an inner radius $R_0 < R$ and $\rho = \rho_0$ between R_0 and R . The electric field \vec{E} everywhere in space was found in both cases.

Case 1.

From homework 1, the field \vec{E} in the entire space is given by:

$$\left\{ \begin{array}{l} \vec{E} = \frac{\sigma_0}{3\epsilon_0} r \vec{u}_r , \quad r < R \\ \vec{E} = \frac{R^3 \sigma_0}{3\epsilon_0} \frac{1}{r^2} \vec{u}_r . \end{array} \right. \quad (6.23a)$$

$$\left\{ \begin{array}{l} \vec{E} = \frac{R^3 \sigma_0}{3\epsilon_0} \frac{1}{r^2} \vec{u}_r . \end{array} \right. \quad r > R \quad (6.23b)$$

In this case, σ is continuous and limited both in the sub-region inside the sphere (where $\sigma = \sigma_0$) and in that outside the sphere (where $\sigma = 0$), but is discontinuous at the boundary $r = R$. From the discussion on continuity in lecture 5 it follows that, in this case, \vec{E} must be continuous and limited in the entire space. The fields (6.23a) and (6.23b) confirm this expectation. In fact, inside the sphere ($r < R$) $\vec{E} = \vec{0}$ at $r=0$ and \vec{E} is continuous and limited. Outside the sphere ($r > R$) \vec{E} is also continuous and limited and $\vec{E} = \vec{0}$ for $r \rightarrow +\infty$. In the neighborhood of the boundary $r = R$, from field (6.23a) it follows that

$$\lim_{r \rightarrow R^-} \vec{E} = \frac{\sigma_0 R}{3\epsilon_0} \vec{u}_r \quad (6.24a)$$

and from field (6.23b) it follows that

$$\lim_{r \rightarrow R^+} \vec{E} = \frac{R^3 S_0}{3 \epsilon_0} \frac{1}{R^2} \vec{u}_R = \frac{S_0 R}{3 \epsilon_0} \vec{u}_R . \quad (6.24b)$$

The limits (6.24a) and (6.24b) are equal, thus, confirming \vec{E} is continuous and limited everywhere in space. However, this does not imply that \vec{E} is continuously derivable everywhere in space.

According to our general guidelines in lecture 5 on the use of Gauss' theorem and the irrotational property of \vec{E} , we should be able to safely use them in differential form inside and outside the sphere, but we should pay attention at $r=R$.

- $\text{div } \vec{E}$.

In the spherical coordinate system $O r \theta \varphi$ chosen to express the fields (6.23a) and (6.23bu), in general

$$\begin{aligned} \text{div } \vec{E} &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 E_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (E_\theta \sin \theta) \\ &\quad + \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} E_\varphi , \end{aligned}$$

where E_r , E_θ , and E_φ are the components of \vec{E} with respect to the coordinate system.

According to the fields (6.23a) and (6.23bu), the components E_θ and E_φ are both zero everywhere in

in lecture 5, case 3, pages 36-38. In particular,
condition (5.31)

[29]

$$\vec{n} \cdot (\vec{E}_1 - \vec{E}_2) = 0$$

in the case of the sphere, case 1, at $r=R$ corresponds to

$$\vec{u}_r \cdot \left(\frac{\sigma_0 R}{3\epsilon_0} \vec{u}_r - \frac{\sigma_0 R}{3\epsilon_0} \vec{u}_\theta \right) = 0 .$$

In other words, the limits (6.24a) and (6.24b) confirm the validity of condition (5.31).

- $\operatorname{curl} \vec{E}$.

In the spherical coordinate system $\Omega r\theta\varphi$, in general

$$\begin{aligned} \operatorname{curl} \vec{E} = & \frac{1}{r \sin\theta} \left[\frac{\partial}{\partial\theta} (E_\varphi \sin\theta) - \frac{\partial}{\partial\varphi} E_\theta \right] \vec{u}_r \\ & + \frac{1}{r \sin\theta} \left[\frac{\partial}{\partial\varphi} E_r - \frac{\partial}{\partial r} (r E_\varphi \sin\theta) \right] \vec{u}_\theta \\ & + \frac{1}{r} \left[\frac{\partial}{\partial r} (r E_\theta) - \frac{\partial}{\partial\theta} E_r \right] \vec{u}_\varphi . \end{aligned}$$

In the case of the sphere, case 1, Eqs. (6.23a) and (6.23b) make it possible to write the $\operatorname{curl} \vec{E}$ as

$$\left\{ \begin{array}{l} \text{curl } \vec{E} = \vec{0} \quad , \quad r < R \quad (6.26a) \\ \lim_{r \rightarrow R^+} \text{curl } \vec{E} = \vec{0} \quad , \quad r = R \quad (6.26b) \\ \text{curl } \vec{E} = \vec{0} \quad , \quad r > R \quad (6.26c) \end{array} \right.$$

The $\text{curl } \vec{E}$ is well defined both inside and outside the sphere, and at the boundary $r=R$. As a consequence, the irrotational property of \vec{E} in differential form can be used everywhere in space, Eqs. (6.26a) - (6.26c).

However, it is desirable to either use both Gauss' theorem and the irrotational property of \vec{E} in differential form or none. So, at the boundary $r=R$ we prefer to resort to the procedure outlined in lecture 5, case 3, pages 36-38, also for the $\text{curl } \vec{E}$. In particular, condition (5.32)

$$\vec{E} \cdot (\vec{E}_1 - \vec{E}_2) = 0$$

in the case of the sphere, case 1, at $r=R$ corresponds to

$$\left\{ \begin{array}{l} \vec{u}_\theta \cdot (\vec{E}_{\theta 1} - \vec{E}_{\theta 2}) = 0 \\ \vec{u}_\varphi \cdot (\vec{E}_{\varphi 1} - \vec{E}_{\varphi 2}) = 0 \end{array} \right.$$

where the subscripts 1 and 2 indicate a component [31] of \vec{E} at the boundary $r=R$, above and below the boundary respectively. Because of Eqs. (6.23a) and (6.23be),
 $\vec{E}_{\theta 1} = \vec{E}_{\theta 2} = \vec{E}_{\varphi 1} = \vec{E}_{\varphi 2} = \vec{0}$. In other words the fields (6.23a) and (6.23be) confirm the validity of condition (5.32).

In summary, in the case of the sphere, case 1, S is continuous and limited in two sub-regions of the entire space (the union of which is the entire space) and is discontinuous and limited at the boundary between the two sub-regions. The field \vec{E} is continuous and limited everywhere in space; its derivatives for the div \vec{E} are continuous and limited in each sub-region, but are discontinuous and limited at the boundary; finally its derivatives for the curl \vec{E} are continuous and limited everywhere in space.

• Case 2. \vec{S}_0

We will focus on the sub-region inside the sphere, $r < R$. In fact, the behavior of S and \vec{E} in the neighborhood of the boundary at $r = R$ is similar to that in case 1.

Inside the sphere, S is continuous and limited with continuous and limited first derivative, except for $r = R_0$,

where the first derivative presents a discontinuity [32] of the first kind. In the entire space inside the sphere:

$$\left\{ \begin{array}{l} \frac{\partial}{\partial r} \rho = \frac{\rho_0}{R_0} \quad , \quad 0 \leq r < R_0 \quad (6.24a) \\ \lim_{r \rightarrow R_0^-} \frac{\partial}{\partial r} \rho = \frac{\rho_0}{R_0} \quad , \quad r \rightarrow R_0^- \quad (6.24b) \\ \lim_{r \rightarrow R_0^+} \frac{\partial}{\partial r} \rho = 0 \quad , \quad r \rightarrow R_0^+ \quad (6.24c) \\ \frac{\partial}{\partial r} \rho = 0 \quad , \quad R_0 < r < R \quad (6.24d) \end{array} \right.$$

In our guidelines in lecture 5 we did not consider the case of charge densities with discontinuous derivatives in the region where they are defined (i.e., of volume densities in a volume or surface densities on a surface). Here, we intend to study the consequences of the discontinuity of Eqs. (6.24b) and (6.24c) on Gauss' theorem and the irrotational property of \vec{E} in local form given the field \vec{E} in the entire space inside the sphere as calculated in homework 1. From that homework,

$$\left\{ \begin{array}{l} \vec{E} = \frac{\sigma_0}{4R_0} r^2 \vec{u}_r , \quad 0 \leq r < R_0 \quad (6.28a) \\ \lim_{r \rightarrow R_0^-} \vec{E} = \frac{\sigma_0 R_0}{4} \vec{u}_r , \quad r \rightarrow R_0^- \quad (6.28b) \\ \lim_{r \rightarrow R_0^+} \vec{E} = \frac{\sigma_0 R_0}{4} \vec{u}_r , \quad r \rightarrow R_0^+ \quad (6.28c) \\ \vec{E} = \left(\frac{\sigma_0}{3} r - \frac{\sigma_0 R_0^3}{12} \frac{1}{4r} \right) \vec{u}_r . \quad R_0 < r < R \quad (6.28d) \end{array} \right.$$

As expected, for a continuous and limited σ , \vec{E} is also continuous and limited.

- $\operatorname{div} \vec{E}$.

From Eqs. (6.28a) – (6.28d) and in a spherical coordinate system $O_r\theta\varphi$, we obtain

$$\left\{ \begin{array}{l} \operatorname{div} \vec{E} = \frac{\sigma_0}{R_0} r , \quad 0 \leq r < R_0 \quad (6.29a) \\ \lim_{r \rightarrow R_0^-} \operatorname{div} \vec{E} = \sigma_0 , \quad r \rightarrow R_0^- \quad (6.29b) \end{array} \right.$$

$$\left. \begin{array}{l} \lim_{r \rightarrow R_0^+} \operatorname{div} \vec{E} = S_0 , \quad r \rightarrow R_0^+ \\ \operatorname{div} \vec{E} = S_0 \end{array} \right\} \begin{array}{l} (6.29c) \\ (6.29d) \end{array}$$

The $\operatorname{div} \vec{E}$ is well defined, continuous and limited, at any point inside the sphere. Eqs. (6.29a) - (6.29d) represent Gauss' theorem in differential form at each of those points.

- $\operatorname{curl} \vec{E}$.

As in case 1, also in case 2 $\operatorname{curl} \vec{E} = \vec{0}$ at any point inside the sphere.

In summary, in the case of the sphere, case 2, 3 is continuous and limited at any point inside the sphere and its first derivative is also continuous and limited except for $r = R_0$, where it presents a discontinuity of the first kind. The field \vec{E} , its derivatives for the $\operatorname{div} \vec{E}$, and its derivatives for the $\operatorname{curl} \vec{E}$ are continuous and limited at any point inside the sphere.

We can thus extend by conjecture our guidelines in lecture 5, stating that a discontinuity (at least of the first kind) in S should not affect the validity of Eqs. (5.4) and (5.20).

6.5 The $\text{div } \vec{E}$ and $\text{curl } \vec{E}$ for the case of an infinite straight line with constant charge density λ .

In lecture 4 we found the field \vec{E} at any point in space:

$$\left\{ \begin{array}{l} \vec{E} = \frac{\lambda}{2\pi\epsilon_0} \frac{1}{r} \vec{u}_r , \quad r > 0 \\ \lim_{r \rightarrow 0^+} \vec{E} = +\infty \end{array} \right. \quad (6.30a)$$

$$\left\{ \begin{array}{l} \lim_{r \rightarrow 0^+} \vec{E} = +\infty \\ \quad \quad \quad r \rightarrow 0^+ \end{array} \right. \quad (6.30b)$$

- $\text{div } \vec{E}$.

In the cylindrical coordinate system $0r\varphi z$ chosen to express the fields (6.30a) and (6.30b), in general

$$\text{div } \vec{E} = \frac{1}{r} \frac{\partial}{\partial r} (r E_r) + \frac{1}{r} \frac{\partial}{\partial \varphi} E_\varphi + \frac{\partial}{\partial z} E_z . \quad (6.31)$$

From Eqs. (6.30a) and (6.30b) it follows that

$$\left\{ \begin{array}{l} \text{div } \vec{E} = 0 \\ \lim_{r \rightarrow 0^+} \text{div } \vec{E} = \text{ill defined} \end{array} \right. , \quad r > 0 \quad (6.32a)$$

$$\left\{ \begin{array}{l} \text{div } \vec{E} = 0 \\ \lim_{r \rightarrow 0^+} \text{div } \vec{E} = \text{ill defined} \end{array} \right. , \quad r \rightarrow 0^+ \quad (6.32b)$$

where Eq. (6.32b) is ill defined because \vec{E} diverges for $r \rightarrow 0^+$ and, thus, it cannot be used in the first term of Eq. (6.31).

For $r > 0$, Eq. (6.32a) correctly represents Gauss' theorem or

differential form. For $r \rightarrow 0^+$, however, we must resort to Eq. (5.33), which appears to be consistent with Eq. (6.30a) upon substituting E_r with E_φ .

$$-\operatorname{curl} \vec{E}$$

In general, in a cylindrical coordinate system

$$\begin{aligned} \operatorname{curl} \vec{E} = & \left(\frac{1}{r} \frac{\partial}{\partial \varphi} E_z - \frac{\partial}{\partial z} E_\varphi \right) \vec{u}_r \\ & + \frac{1}{r} \left[\frac{\partial}{\partial r} (r E_\varphi) - \frac{\partial}{\partial \varphi} E_r \right] \vec{u}_z \\ & + \left(\frac{\partial}{\partial z} E_r - \frac{\partial}{\partial r} E_z \right) \vec{u}_\varphi . \quad (6.33) \end{aligned}$$

From Eqs. (6.30a) and (6.30b) it follows that

$$\left\{ \begin{array}{l} \operatorname{curl} \vec{E} = 0 \quad , \quad r > 0 \\ \lim_{r \rightarrow 0^+} \operatorname{curl} \vec{E} = \text{ill defined} , \quad r \rightarrow 0^+ \end{array} \right. \quad (6.34)$$

where Eq. (6.34b) is ill defined because \vec{E} diverges for $r \rightarrow 0^+$ and, thus, it cannot be used in the second term of the \vec{u}_z component of $\operatorname{curl} \vec{E}$ and in the first term of the \vec{u}_φ component of $\operatorname{curl} \vec{E}$ [cf. Eq. (6.33)].

For $\epsilon > 0$, Eq. (6.34a) correctly represents the irrotational property of \vec{E} in differential form. For $\epsilon \neq 0^+$, however, we must resort to Eq. (5.34). For a straight line $\lambda' = 0$ at any point on the line. Hence, Eq. (5.34) is consistent with the fact that E_y and E_z [and, thus, E_L in Eq. (5.34)] are both zero in Eq. (6.30a).

Note that later in the course we will encounter an example of surface charge density that fulfills all conditions for case 2 in lecture 5.

Phys 242 - §14. Lecture 7 - Field lines, the discontinuity box, and caveats on infinitesimal argument in lecture 4.

11

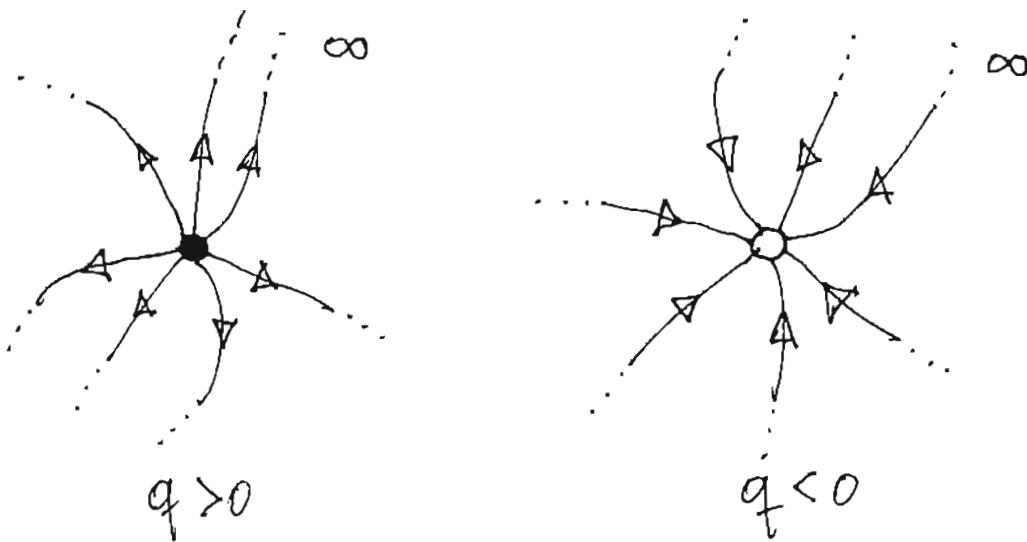
Summary lecture 6.

- Electrostatic dipole of two generic charges q_1 and q_2 .
 - Field \vec{E} at the middle point of the line segment between q_1 and q_2 , $Q_1 Q_2$.
 - Field \vec{E} at any point on the axis of $Q_1 Q_2$.
 - Far field on the axis of $Q_1 Q_2$ and on the line containing q_1 and q_2 .
 - Case $q_1 = +q$ and $q_2 = -q$, $q_1 = +q$ and $q_2 = -2q$, and $q_1 = \pm q$ and $q_2 = \mp q$.
- Rules on how to draw the field lines of the electrostatic field \vec{E} (Michael Faraday revisited).

In general, given a set of field lines in the 3D Euclidean space generated by a generic source charge distribution, the field \vec{E} at any point on a line is directed as the tangent of the line at that point. An arrowhead on the line indicates the sign of \vec{E} as if it was applied, by convention, to a positive unitary test charge. Even though \vec{E} exists at any point in space, for clarity of representation

only a sparse enough number of lines is drawn.
The density of field lines at a given point is proportional to the magnitude (more precisely, to the absolute value of the magnitude) of \vec{E} at that point.

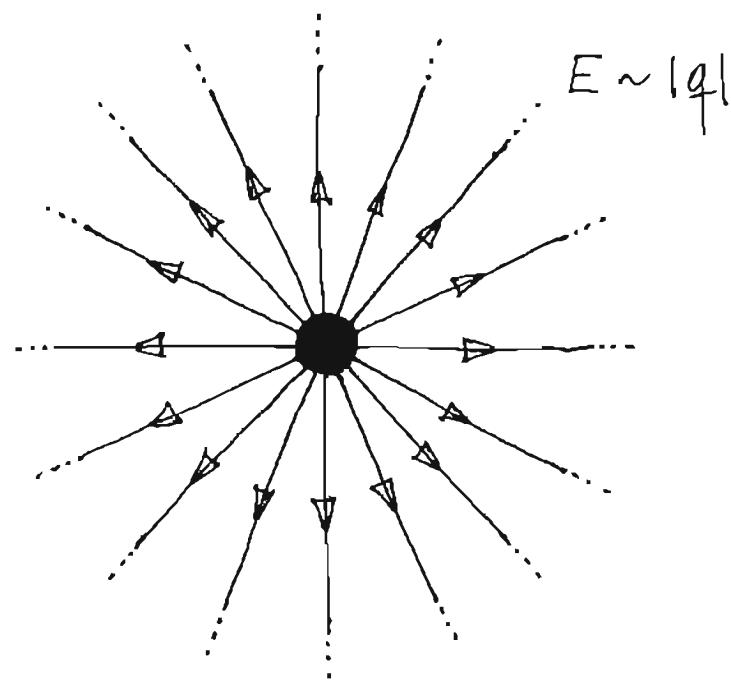
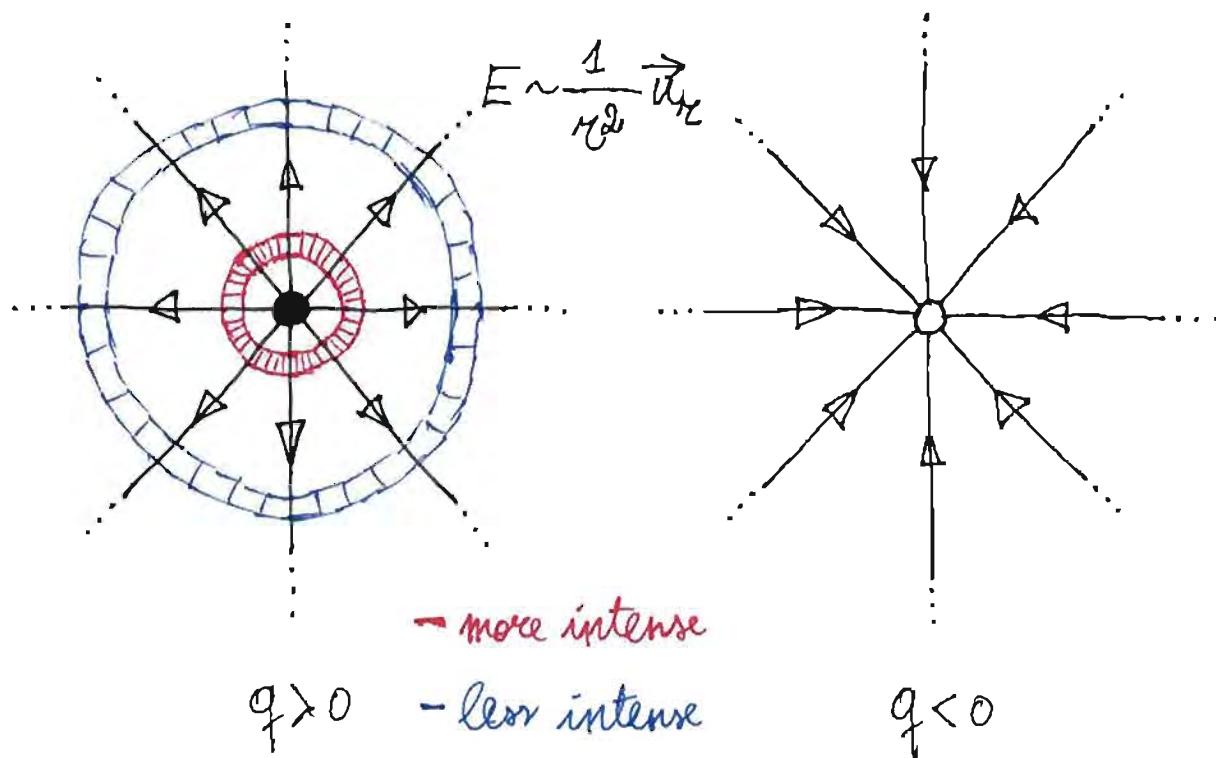
- ① The lines are drawn as black curves with an arrowhead in the middle indicating their orientation.
The field \vec{E} at a point on a line is drawn as a red arrow.
- ② Given a point-like charge distribution $q\delta(P-P_0)$, the field lines are directed from the charge to infinity when $q > 0$. They are directed from infinity to the charge when $q < 0$. This is consistent with Coulomb's Law ($\sim q$ dependence).



- ③ When a point-like charge is isolated in space, due to Coulomb's law the field lines are directed radially from the charge ($\sim r$ dependence), they are more intense close to the charge ($\sim 1/r^2$ dependence), and they are more intense

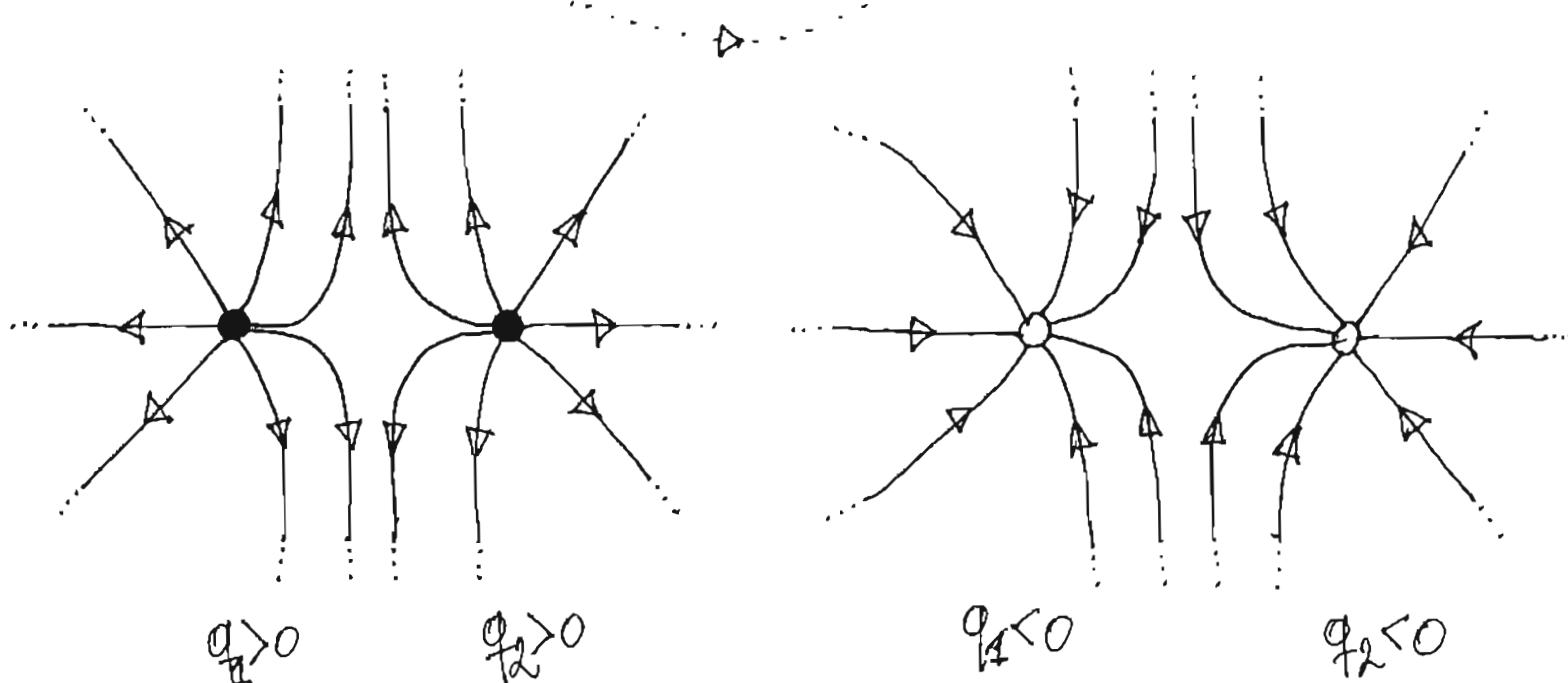
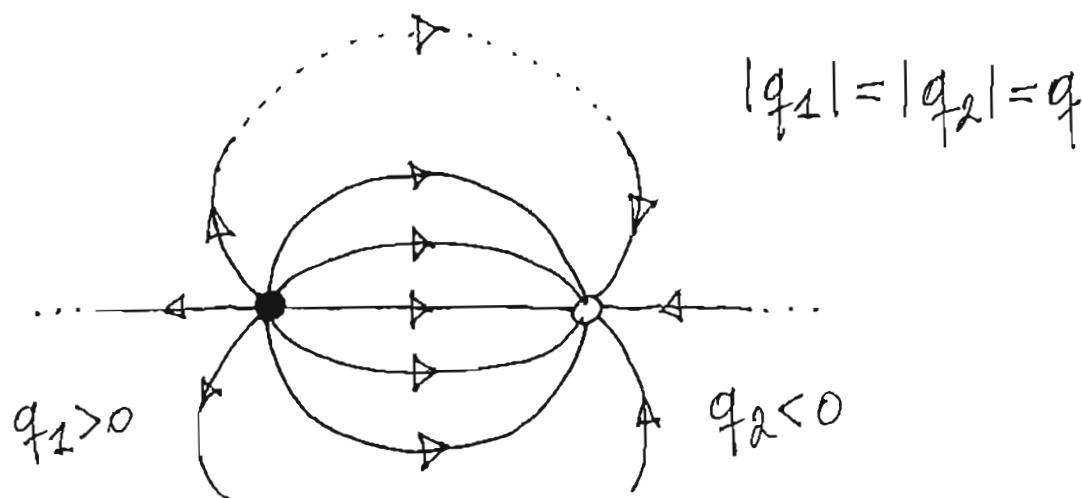
for a trigger charge ($\sim |q|$ dependent).

31



$q' > q > 0$
(similar for a negative
charge)

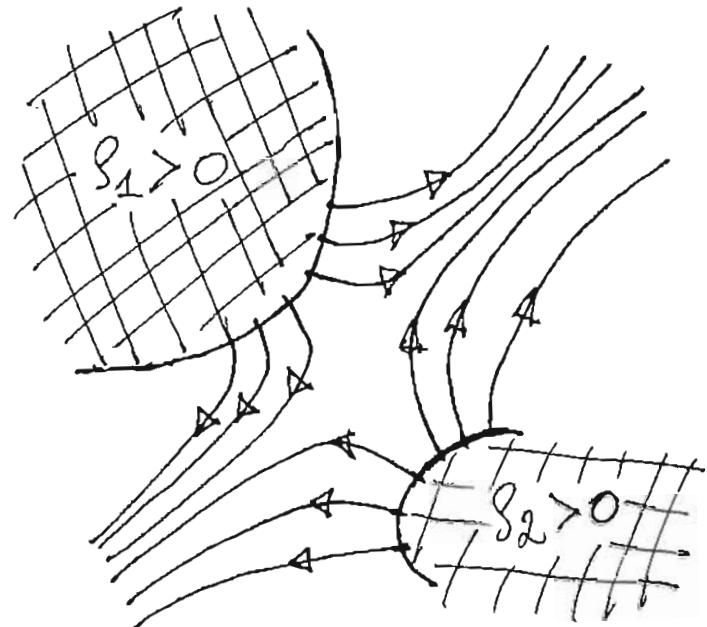
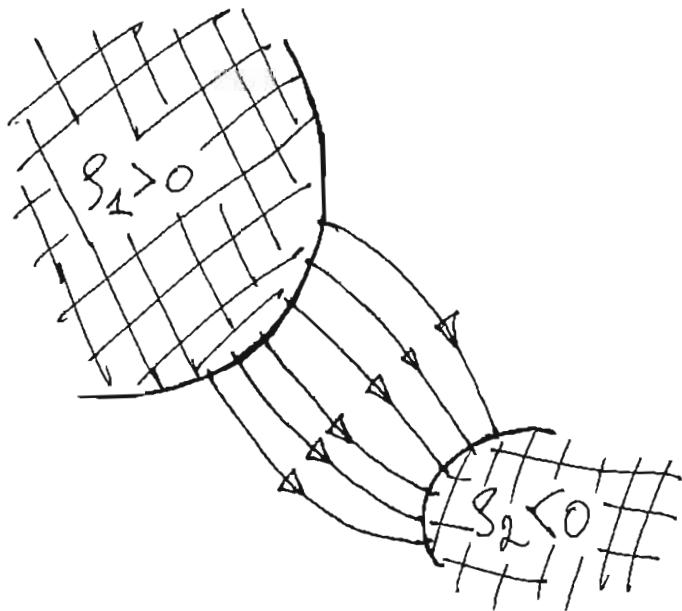
④ Given two isolated point-like charge distributions 41
 $q_1 \delta(P-P_1)$ and $q_2 \delta(P-P_2)$, the field lines are directed from the positive to the negative charge when the charges have opposite sign (e.g., cf. Figs. 6.3 and 6.4). Otherwise the lines are directed from each charge to infinity with orientation depending on the sign of the charges (e.g., cf. Fig. 6.5). Once again, this case is called electrostatic dipole.



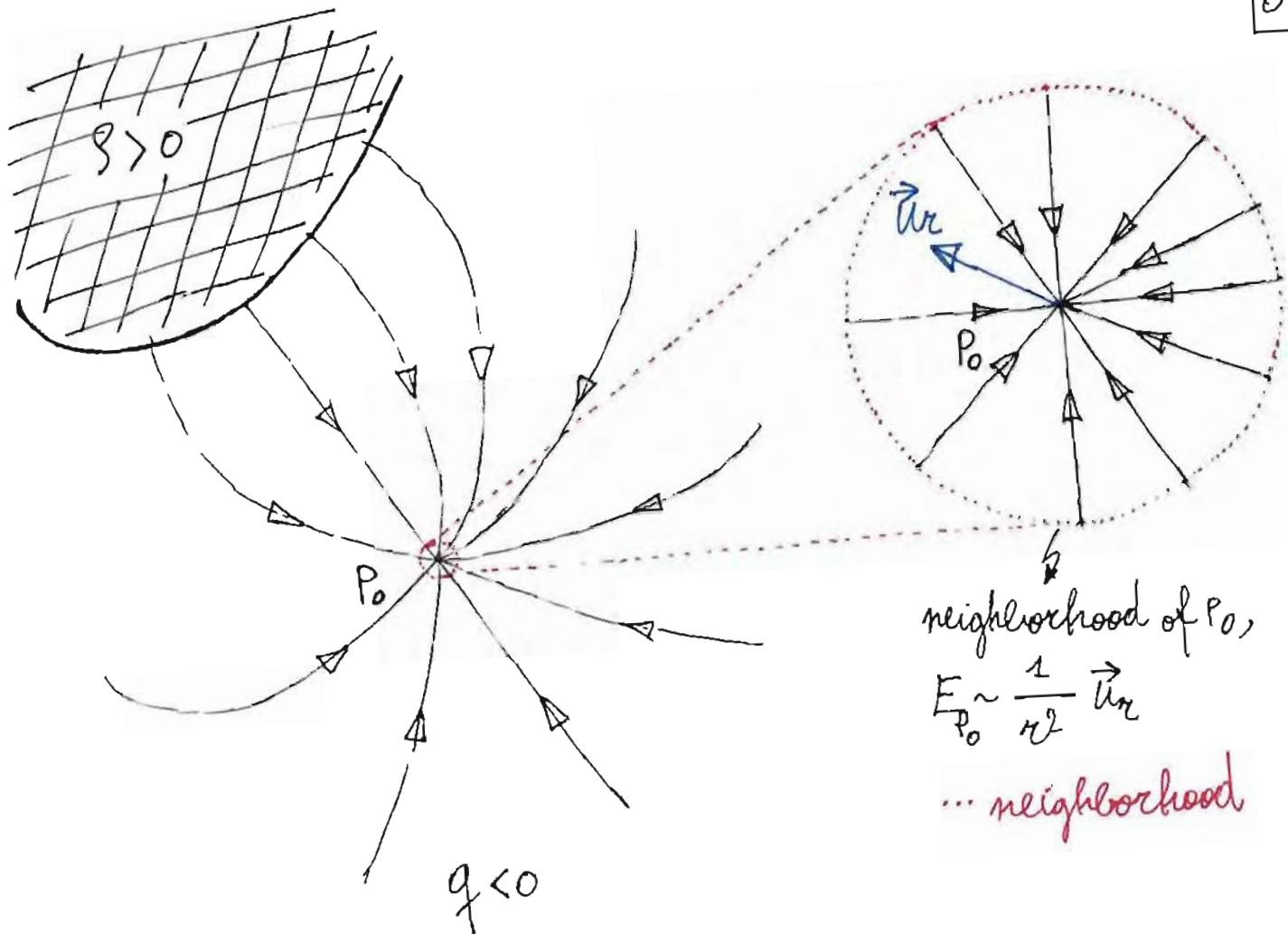
Note that when $|q_1| \neq |q_2|$, the field lines are asymmetric (cf., e.g., Fig. 6.4).

- ⑤ Given two isolated volume charge distributions with densities S_1 and S_2 , in the space between S_1 and S_2 the field lines are directed from the positive to the negative density when the densities have opposite sign. Otherwise the lines are directed from each density to infinity with orientation depending on the sign of the densities.

This should only be considered a rule of thumb. In general, the field lines largely depend on the spatial distribution of S_1 and S_2 and have to be calculated numerically.



- ⑥ When a point-like charge distribution is not isolated in space, i.e., a set of arbitrary charge distributions is present in the vicinity of the point-like charge, for consistency with Coulomb's law, rule ③ still applies in the neighbourhood of the point-like charge.

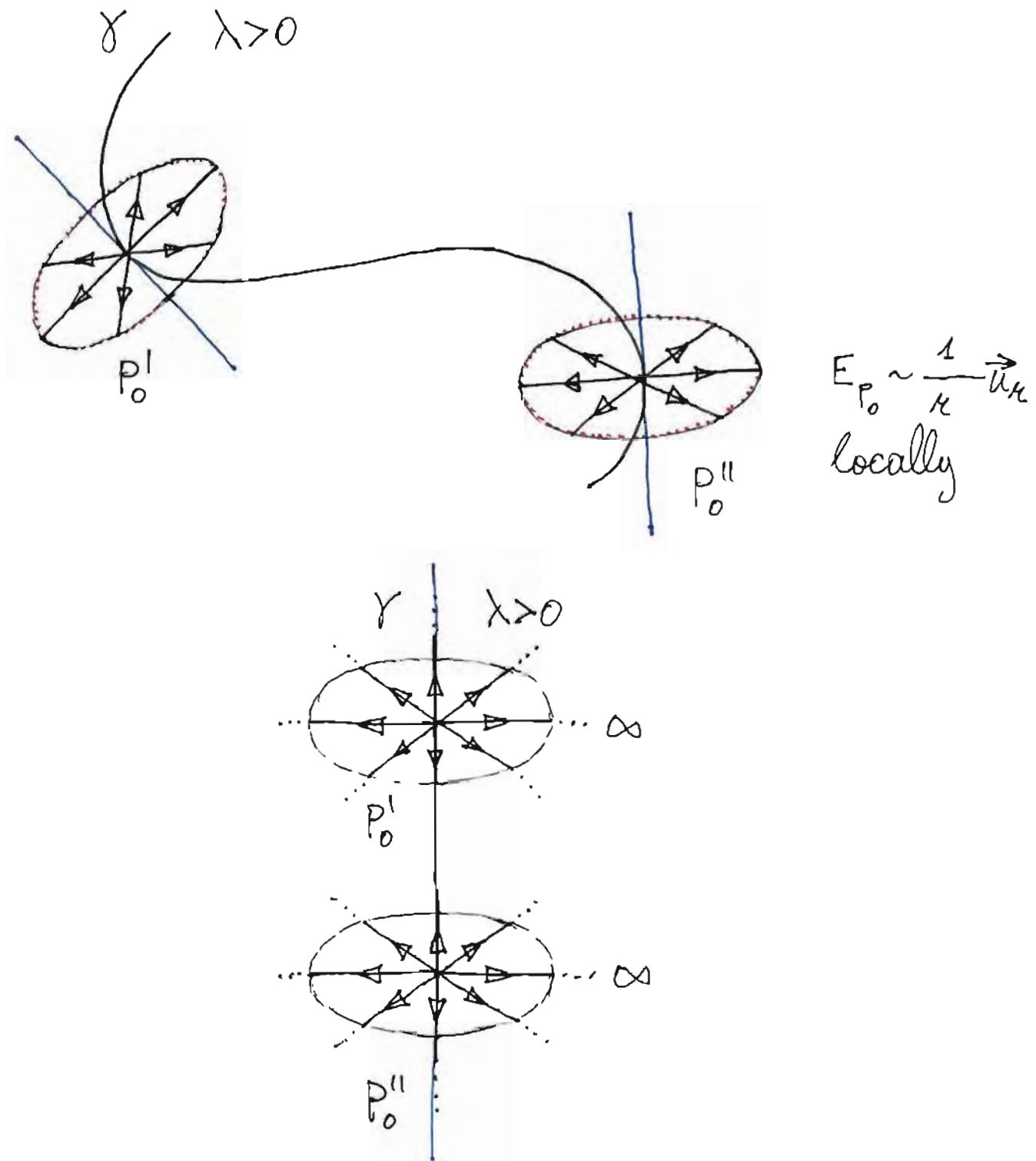


- ⑦ Given a continuous and limited linear charge distribution with density λ on a smooth line γ in the 3D Euclidean space, because of condition (5.35) the field lines in the neighborhood of any point P_0 on γ must be perpendicular to γ . In addition, because of condition (5.33) the field lines must be more intense close to P_0 ($\sim 1/r$ dependence).

A special case is that of an infinite straight line γ (cf. lecture 4). In this case, the field lines are directed radially and with the same intensity from each point on the line to infinity.

As always, the field lines are directed from γ to

infinity or vice versa, depending on whether λ is positive or negative, respectively. [71]



- ⑩ Field lines cannot cross each other. This is due to the unique value of \vec{E} at each point in space.
- ⑪ The field lines generated by a continuous and limited charge distribution in space tend to become less intense away from the distribution. This is because \vec{E} tends to zero at infinity.
- ⑫ In general, given an arbitrary set of charge distributions in space the field lines can be drawn by calculating \vec{E} at a few opportune points. These are typically chosen by symmetry arguments (e.g., the axis of $Q_1 Q_2$ in the case of the dipole)

Finally, 11/

the field lines are assumed to be maximally smooth in all the regions with zero charge or with charge distributions with continuous and limited densities.

The examples of the unbalanced dipole in Fig. 6.4, of the infinite straight line in lecture 4, of the sphere in homework 1 and lecture 6, and that of the infinite plane (to come) summarize well all the most common cases on how to draw field lines.

- The density δ of a point-like charge distribution Q at point P_0 is given by

$$\delta(P) = Q \delta(P - P_0)$$

- $\operatorname{div} \vec{E}$ and $\operatorname{curl} \vec{E}$ for a sphere and for an infinite straight line.

7.1 The discontinuity box.

The physical origin of the result of Eq. (5.4), i.e., Gauss' theorem in differential form, can be better appreciated by resorting to infinitesimal boxes.

Consider a cube (or box) Ω_{box} with infinitesimal dimensions in the 3D Euclidean space. With respect to a Cartesian coordinate system $Oxyz$, the corner of Ω_{box} closest to O is a generic point $P(x, y, z)$ and the edges are parallel to the axes of the coordinate system. The length of the box is dx along the x direction, dy along the y direction, and dz along the z direction, where all lengths are assumed to be equal. Such an infinitesimal cube is shown in Fig. 7.1 (note that any infinitesimal rectangular parallelepiped would work as well). The infinitesimal volume of Ω_{box} is thus

$$dV = dx dy dz . \quad (4.1)$$

The face of Ω_{box} parallel to the yz plane and containing $P(x, y, z)$ is called A_1 in the figure. Similarly, the face parallel to the yz plane and containing $P(x+dx, y, z)$ is called A_2 .

We assume Ω_{box} to be part of a larger region Ω characterized by a volume charge distribution with continuous and limited density ρ .

The electric field at P is

$$\vec{E} = E_x(x, y, z) \vec{u}_x + E_y(x, y, z) \vec{u}_y + E_z(x, y, z) \vec{u}_z , \quad (4.2)$$

where E_x , E_y , and E_z are three generic functions of the coordinates x , y , and z .

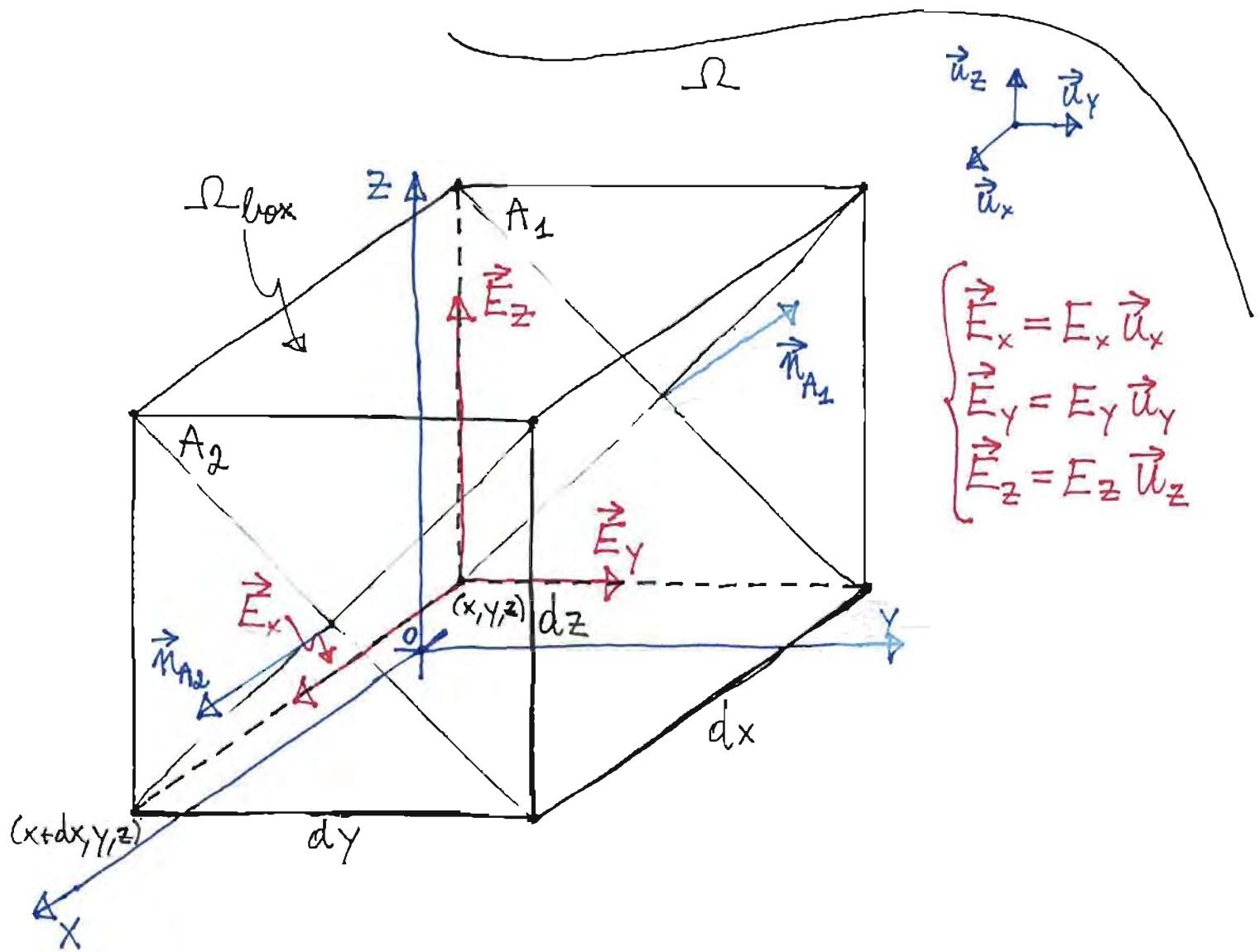


Figure 4.1.

The flux of \vec{E} through A_1 is given by

$$\Phi_{A_1} = \iint_{A_1} \vec{E} \cdot \vec{n}_{A_1} dS , \quad (4.3)$$

where $\vec{n}_{A_1} = -\vec{u}_x$ and $dS = dy dz$. Due to the infinitesimal dimensions of A_1 , we can assume \vec{E} to be constant on A_1 and equal to \vec{E} at $P(x, y, z)$. Hence, the integral (4.3) can be written as

$$\begin{aligned} \Phi_{A_1} &= (E_x^{A_1} \vec{u}_x + E_y^{A_1} \vec{u}_y + E_z^{A_1} \vec{u}_z) \cdot (-\vec{u}_x) \\ &\quad \cdot dy dz \\ &= -E_x^{A_1}(x, y, z) \cdot dy dz . \quad (4.4) \end{aligned}$$

Similarly, the flux of \vec{E} through A_2 is given by

$$\begin{aligned} \Phi_{A_2} &= \iint_{A_2} \vec{E} \cdot \vec{n}_{A_2} dS \\ &= (E_x^{A_2} \vec{u}_x + E_y^{A_2} \vec{u}_y + E_z^{A_2} \vec{u}_z) \cdot \vec{u}_x \\ &\quad \cdot dy dz \end{aligned}$$

$$\vec{E}_Y^{A_3} = E_Y(x, y, z) \vec{u}_Y = E_Y^{A_3} \vec{u}_Y.$$

On A_4 , we expand $E_Y^{A_3}$ in Taylor series and obtain

$$\vec{E}_Y^{A_4} = E_Y^{A_3} \vec{u}_Y + \frac{\partial}{\partial y} E_Y^{A_3} \cdot dy \vec{u}_Y, \quad (4.8)$$

which we assume to be constant on A_4 .

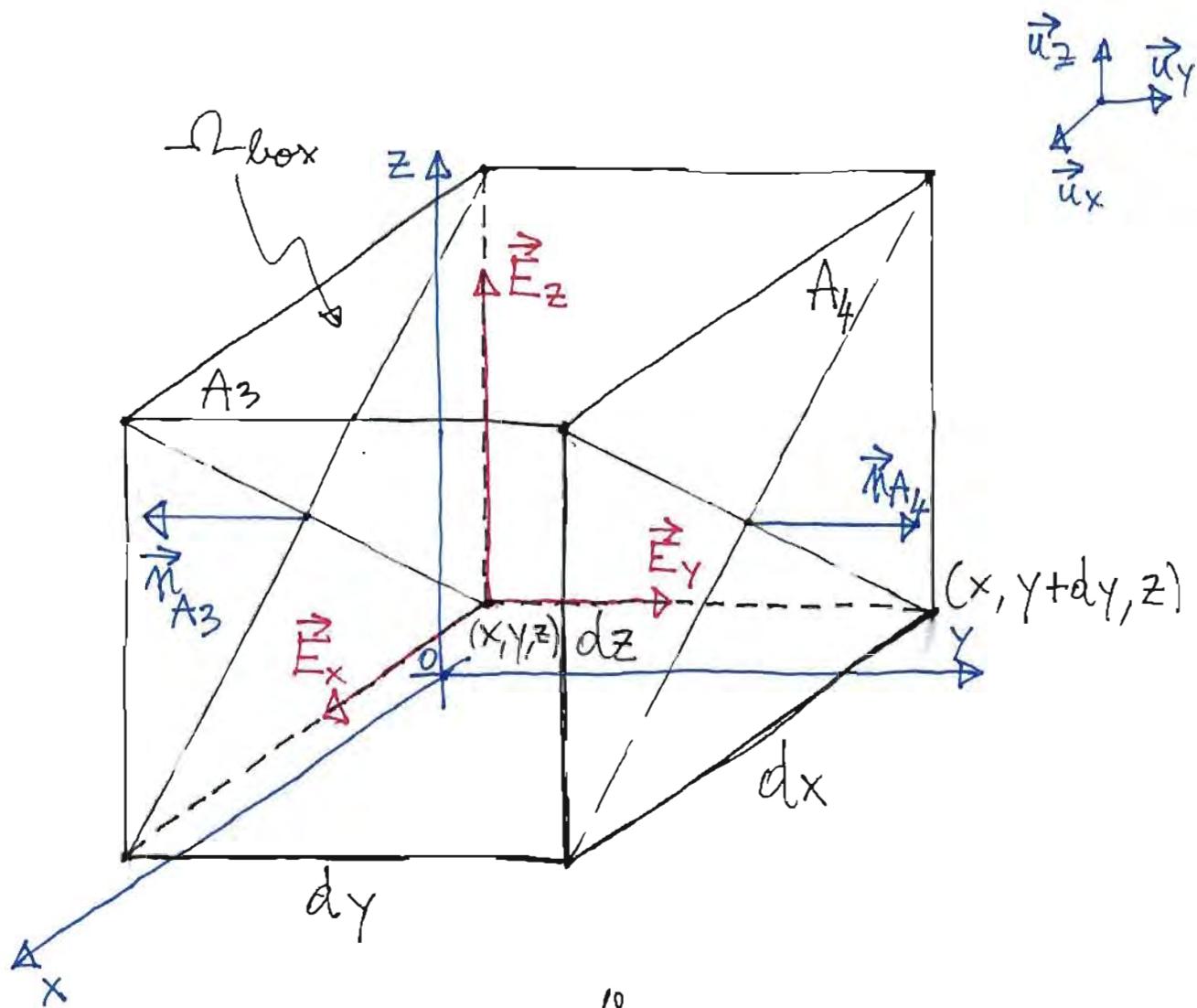


Figure 4.2.

Following the notation in Fig. 4.2, $\vec{n}_{A_3} = -\vec{u}_y$ and [18]
 $\vec{n}_{A_4} = \vec{u}_y$. The total flux through A_3 and A_4 is thus

$$\begin{aligned}
 \oint^{A_3} + \oint^{A_4} &= \iint_{A_3} \vec{E} \cdot \vec{n}_{A_3} dS + \iint_{A_4} \vec{E} \cdot \vec{n}_{A_4} dS \\
 &= - \cancel{E_y^{A_3} \cdot dx dz} + \cancel{E_y^{A_4} \cdot dx dz} \\
 &\quad + \frac{\partial}{\partial y} E_y^{A_3} \cdot dy \cdot dx dz \\
 &= \frac{\partial}{\partial y} E_y^{A_3} \cdot dz = \frac{\partial}{\partial y} E_y(x, y, z) dz. \tag{4.9}
 \end{aligned}$$

Without entering in details, the total flux of \vec{E} through the remaining two faces of Ω_{box} , A_5 and A_6 (parallel to the xy plane) is

$$\oint^{A_5} + \oint^{A_6} = \frac{\partial}{\partial z} E_z(x, y, z) dz. \tag{4.10}$$

According to Gauss' theorem in integral form, calling Σ the outer surface of Ω_{box} , we can conclude that the total flux of \vec{E} through Σ is

$$\begin{aligned}
 \oint_{\Sigma} \vec{E} \cdot \vec{n} dS &= \\
 &= (\oint^{A_1} + \oint^{A_2}) + (\oint^{A_3} + \oint^{A_4}) + (\oint^{A_5} + \oint^{A_6}) \\
 &= \left[\frac{\partial}{\partial x} E_x(x, y, z) + \frac{\partial}{\partial y} E_y(x, y, z) + \frac{\partial}{\partial z} E_z(x, y, z) \right] dz \\
 &= \operatorname{div} \vec{E} \cdot dz = \vec{\nabla} \cdot \vec{E} \cdot dz \\
 &= \frac{1}{\epsilon_0} \iiint_{\Omega_{\text{box}}} S \cdot dz = \frac{1}{\epsilon_0} S \cdot dz,
 \end{aligned}$$

where we have assumed S to be constant in Ω_{box} because of the infinitesimal dimensions of Ω_{box} . Hence,

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} S, \quad (4.11)$$

which is exactly the same result as Eq. (5.4). By means of the infinitesimal box Ω_{box} we have thus rigorously proven Gauss' theorem in differential form.

This procedure allows us to understand the meaning [20] of the divergence of a vector, e.g., of \vec{E} . The divergence of a vector at the point P is the flux (the outgoing flow of the vector) per unit volume, in the neighborhood of P.

Consider now two identical infinitesimal boxes Ω_{box}^A and Ω_{box}^B adjacent to each other, as shown in Fig. 4.3.

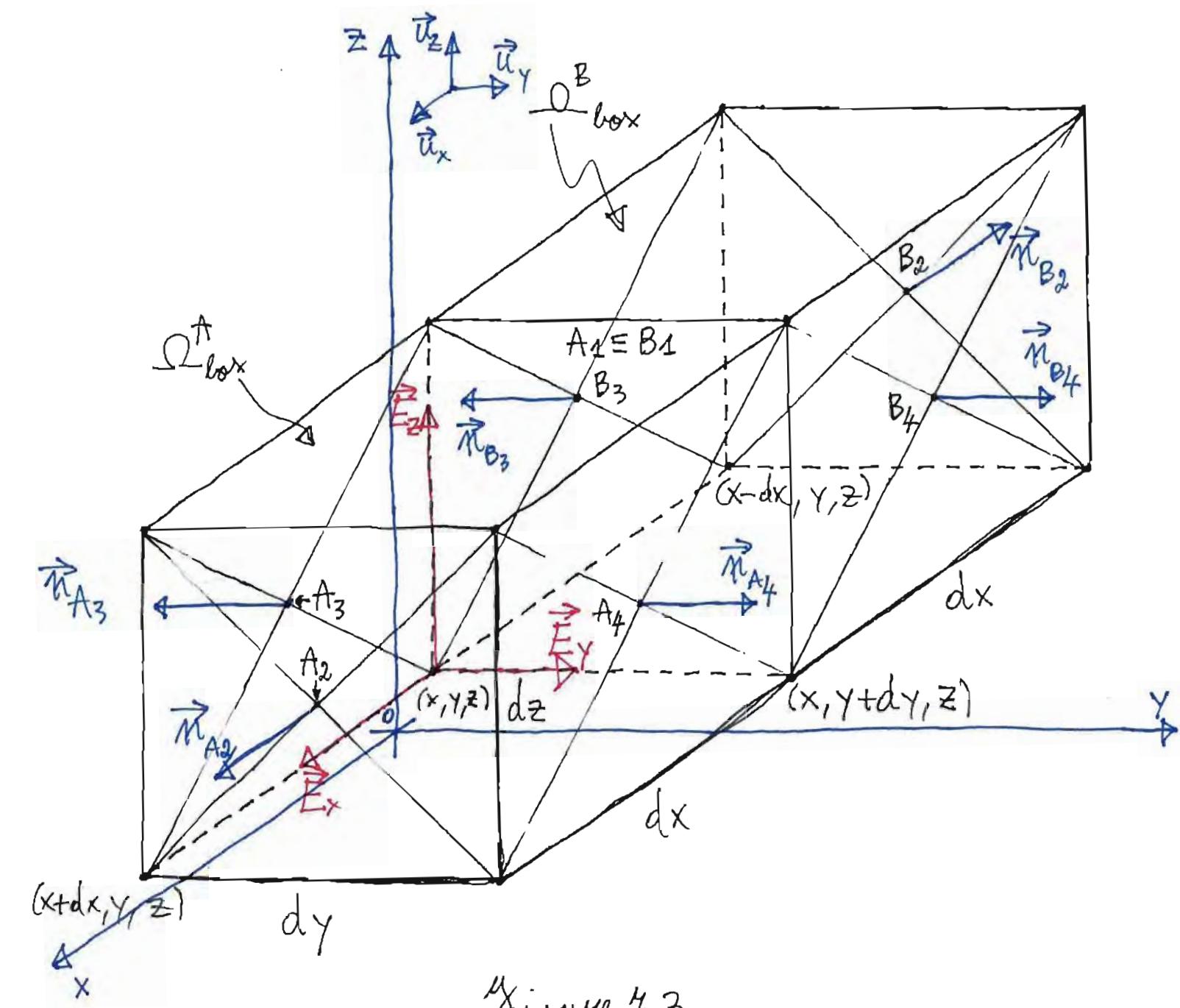


Figure 4.3.

The outer surface of the union of the two boxes, $\Omega_{\text{box}}^A \cup \Omega_{\text{box}}^B$, is given by the faces $A_2, A_3, \text{ and } A_4$ (indicated in the figure) and the faces A_5 and A_6 parallel to the xy plane (not indicated for simplicity) for box Ω_{box}^A , and by the faces $B_2, B_3, \text{ and } B_4$ (indicated) and B_5 and B_6 (not indicated) for Ω_{box}^B . The outer surface is called Σ^{AB} .

The union of the two boxes is assumed to be part of a region of space with a continuous and limited charge density ρ , which is assumed to be constant in both boxes.

Following the notation in the figure, $\vec{n}_{A_2} = \vec{u}_x, \vec{n}_{A_3} = -\vec{u}_y, \vec{n}_{A_4} = \vec{u}_y, \vec{n}_{A_5} = -\vec{u}_z, \text{ and } \vec{n}_{A_6} = \vec{u}_z$ for Ω_{box}^A , and $\vec{n}_{B_2} = -\vec{u}_x, \vec{n}_{B_3} = -\vec{u}_y, \vec{n}_{B_4} = \vec{u}_y, \vec{n}_{B_5} = -\vec{u}_z, \text{ and } \vec{n}_{B_6} = \vec{u}_z$ for Ω_{box}^B . Each component of \vec{E} at $P(x, y, z)$ is assumed to be constant on each corresponding face: $E_x(x, y, z)$ on the internal face $A_1 \equiv B_1; E_y(x, y, z)$ on A_3 and $B_3; E_z(x, y, z)$ on A_5 and B_5 . All derivatives are calculated with respect to $P(x, y, z)$. The total flux of \vec{E} through Σ^{AB} is given by:

$$\oint_{\Sigma^{AB}} \vec{E} \cdot \vec{n} dS = \iint_{A_2} \left\{ \left[E_x(x, y, z) + \frac{\partial}{\partial x} E_x(x, y, z) dx \right] \vec{u}_x \right. \\ \left. + \left[E_y(x, y, z) + \frac{\partial}{\partial x} E_y(x, y, z) dx \right] \vec{u}_y \right\}$$

$$+ \iiint_{B_2} \left\{ \left[E_x(x, y, z) - \frac{\partial}{\partial x} E_x(x, y, z) \cdot dx \right] \vec{u}_x + \left[E_y(x, y, z) \right. \right.$$

$$\left. \left. - \frac{\partial}{\partial x} E_y(x, y, z) \cdot dx \right] \vec{u}_y + \left[E_z(x, y, z) - \frac{\partial}{\partial x} E_z(x, y, z) \cdot dx \right] \vec{u}_z \right\}$$

$$\} \cdot (-\vec{u}_x) \cdot dy dz$$

$$+ \iint_{B_3} \left[E_x(x, y, z) \vec{u}_x + E_y(x, y, z) \vec{u}_y + E_z(x, y, z) \vec{u}_z \right] \cdot (-\vec{u}_y) \cdot dx dz$$

$$+ \iint_{B_4} \left\{ \left[E_x(x, y, z) + \frac{\partial}{\partial y} E_x(x, y, z) \cdot dy \right] \vec{u}_x + \left[E_y(x, y, z) \right. \right.$$

$$\left. \left. + \frac{\partial}{\partial y} E_y(x, y, z) \cdot dy \right] \vec{u}_y + \left[E_z(x, y, z) + \frac{\partial}{\partial y} E_z(x, y, z) \cdot dy \right] \vec{u}_z \right\}$$

$$\} \cdot \vec{u}_y \cdot dx dz$$

$$+ \iint_{B_5} \left[E_x(x, y, z) \vec{u}_x + E_y(x, y, z) \vec{u}_y + E_z(x, y, z) \vec{u}_z \right] \cdot (-\vec{u}_z) \cdot dx dy$$

$$+ \iint_{B_6} \left\{ \left[E_x(x, y, z) + \frac{\partial}{\partial z} E_x(x, y, z) \cdot dz \right] \vec{u}_x + \left[E_y(x, y, z) \right. \right.$$

$$+ \frac{\partial}{\partial z} E_y(x, y, z) \cdot dz \Big] \vec{u}_y + \left[E_z(x, y, z) + \frac{\partial}{\partial z} E_z(x, y, z) \cdot dz \right] \vec{u}_z \quad | \frac{24}{\square}$$

$$\} \cdot \vec{u}_z \cdot dx dy$$

$$\begin{aligned}
 &= E_x(x, y, z) \cdot dy dz + \frac{\partial}{\partial x} E_x(x, y, z) \cdot dz \\
 &\quad - \cancel{E_y(x, y, z)} \cdot dx dz + \cancel{E_y(x, y, z)} \cdot dx dz \\
 &\quad + \frac{\partial}{\partial y} E_y(x, y, z) \cdot dz \\
 &\quad - \cancel{E_z(x, y, z)} \cdot dx dy + \cancel{E_z(x, y, z)} \cdot dx dy \\
 &\quad + \frac{\partial}{\partial z} E_z(x, y, z) \cdot dz \\
 &\quad - \cancel{E_x(x, y, z)} \cdot dy dz + \frac{\partial}{\partial x} E_x(x, y, z) \cdot dz \\
 &\quad - \cancel{E_y(x, y, z)} \cdot dx dz + \cancel{E_y(x, y, z)} \cdot dx dz \\
 &\quad + \frac{\partial}{\partial y} E_y(x, y, z) \cdot dz
 \end{aligned}$$

$$\begin{aligned}
 & -E_z(x, y, z) \cdot dx dy + E_z(x, y, z) \cdot dx dy \\
 & + \frac{\partial}{\partial z} E_z(x, y, z) \cdot dz \\
 = & 2 \left[\frac{\partial}{\partial x} E_x(x, y, z) + \frac{\partial}{\partial y} E_y(x, y, z) + \frac{\partial}{\partial z} E_z(x, y, z) \right] \cdot dz \\
 = & \frac{1}{\epsilon_0} \left(\iiint_{\Omega_{\text{box}}^A} S \cdot dz + \iiint_{\Omega_{\text{box}}^B} S \cdot dz \right) = \frac{1}{\epsilon_0} 2S \cdot dz, \tag{4.12}
 \end{aligned}$$

25/

from which, again, follows Gauss' theorem in differential form:

$$\vec{\nabla} \cdot \vec{E}(x, y, z) = \frac{1}{\epsilon_0} S.$$

This result is obvious when all derivatives involved in the divergence of \vec{E} are continuous. Referring to boxes Ω_{box}^A and Ω_{box}^B , for example, it is possible to study the role played by the continuity of derivatives in the two sectors of space adjacent to $P(x, y, z)$, where P can be approached in the limit with respect to x from the left and right. Choosing a specific point $P = P_0(x_0, y_0, z_0)$, approaching the point along x from the right ("+") means approaching it within the sector delimited by Ω_{box}^A and from the left ("−") within the sector delimited by Ω_{box}^B . In this sectors, the derivatives in $\text{div } \vec{E}$ are continuous when

$$\lim_{x \rightarrow x_0^+} \frac{\partial}{\partial x} E_x(x, y, z) \Big|_{P(x, y_0, z_0)}$$

$$= \lim_{x \rightarrow x_0^-} \frac{\partial}{\partial x} E_x(x, y, z) \Big|_{P(x, y_0, z_0)} = \alpha \quad (4.13a)$$

and, similarly,

$$\left\{ \begin{array}{l} \lim_{x \rightarrow x_0^+} \frac{\partial}{\partial y} E_y \Big|_{P(x, y_0, z_0)} = \lim_{x \rightarrow x_0^-} \frac{\partial}{\partial y} E_y \Big|_{P(x, y_0, z_0)} = \beta \\ \lim_{x \rightarrow x_0^+} \frac{\partial}{\partial z} E_z \Big|_{P(x, y_0, z_0)} = \lim_{x \rightarrow x_0^-} \frac{\partial}{\partial z} E_z \Big|_{P(x, y_0, z_0)} = \gamma \end{array} \right. \quad (4.13b)$$

$$\left\{ \begin{array}{l} \lim_{x \rightarrow x_0^+} \frac{\partial}{\partial y} E_y \Big|_{P(x, y_0, z_0)} = \lim_{x \rightarrow x_0^-} \frac{\partial}{\partial y} E_y \Big|_{P(x, y_0, z_0)} = \beta \\ \lim_{x \rightarrow x_0^+} \frac{\partial}{\partial z} E_z \Big|_{P(x, y_0, z_0)} = \lim_{x \rightarrow x_0^-} \frac{\partial}{\partial z} E_z \Big|_{P(x, y_0, z_0)} = \gamma \end{array} \right. \quad (4.13c)$$

Note that within sectors Ω_{box}^A and Ω_{box}^B point P_0 can be approached along y and z only from the right. This is why all derivatives in the limits above were calculated at fixed y_0 and z_0 .

Under the conditions (4.13a) - (4.13c), the integrals in Eq. (4.12) can be written as

$$\begin{aligned}
 \sum_{AB} \oint \vec{E} \cdot \vec{n} dS &= \iint_{A_2} + \iint_{A_3} + \iint_{A_4} + \iint_{A_5} + \iint_{A_6} \\
 &\quad + \iint_{B_2} + \iint_{B_3} + \iint_{B_4} + \iint_{B_5} + \iint_{B_6} \\
 &= \left[\lim_{x \rightarrow x_0^+} \frac{\partial}{\partial x} E_x(x, y, z) \Big|_{P(x, y_0, z_0)} + \lim_{x \rightarrow x_0^+} \frac{\partial}{\partial y} E_y \Big|_{P(x, y_0, z_0)} \right. \\
 &\quad \left. + \lim_{x \rightarrow x_0^+} \frac{\partial}{\partial z} E_z \Big|_{P(x, y_0, z_0)} \right] \cdot dz \\
 &\quad + \left[\lim_{x \rightarrow x_0^-} \frac{\partial}{\partial x} E_x \Big|_{P(x, y_0, z_0)} + \lim_{x \rightarrow x_0^-} \frac{\partial}{\partial y} E_y \Big|_{P(x, y_0, z_0)} \right. \\
 &\quad \left. + \lim_{x \rightarrow x_0^-} \frac{\partial}{\partial z} E_z \Big|_{P(x, y_0, z_0)} \right] \cdot dz \\
 &= (\alpha + \beta + \gamma) \cdot dz + (\alpha + \beta + \gamma) \cdot dz \\
 &= \frac{1}{\epsilon_0} 2S \cdot dz, \quad (7.14)
 \end{aligned}$$

from which follows that

$$\iint_{\sum^{\text{AB}}} \vec{E} \cdot \vec{n} dS = (\alpha + \beta + \gamma) \cdot dZ + (\beta + \beta + \gamma) \cdot dZ \\ = \frac{1}{\epsilon_0} \mathcal{S} \cdot dZ ,$$

from which it follows that

$$\frac{1}{\epsilon_0} \mathcal{S} = \alpha + \beta + \gamma , \quad (4.16)$$

exactly as in the case of continuous derivatives. This means that for the example of discontinuous derivatives (4.15a) - (4.15c) the total flux of \vec{E} through \sum^{AB} is equal to the example of continuous derivatives (4.13a) - (4.13c). Note that these fluxes were calculated by means of Gauss' theorem in integral form, which is valid even for discontinuous field derivatives.

However, if we now consider the flux of \vec{E} through the outer surface of $\mathcal{D}_{\text{box}}^A$ and $\mathcal{D}_{\text{box}}^B$ separately, under the discontinuity conditions (4.15a) - (4.15c), we obtain

$$\iint_{\sum^A} \vec{E} \cdot \vec{n} dS = \left| \begin{array}{c} \iint \\ A_1 \end{array} \right| + \left| \begin{array}{c} \iint \\ A_2 \end{array} \right| + \left| \begin{array}{c} \iint \\ A_3 \end{array} \right| + \left| \begin{array}{c} \iint \\ A_4 \end{array} \right| + \left| \begin{array}{c} \iint \\ A_5 \end{array} \right| + \left| \begin{array}{c} \iint \\ A_6 \end{array} \right|$$

$$\begin{aligned}
 &= -E_x dy dz + E_x dy dz + \gamma dz \\
 &\quad + \lambda dz + \gamma dz \\
 &= (2\lambda + \gamma) \cdot dz \\
 &= \frac{1}{\epsilon_0} \iiint_{\Omega_{\text{box}}^A} \gamma \cdot dz = \frac{1}{\epsilon_0} \gamma \cdot dz,
 \end{aligned}$$

from which it follows that

$$\frac{1}{\epsilon_0} \gamma = 2\lambda + \gamma \quad (4.14)$$

This result is clearly different from (4.16), indicating that γ cannot be continuous in Ω_{box}^A and Ω_{box}^B . In fact,

$$\begin{aligned}
 \sum_B \oint \vec{E} \cdot \vec{n} dS &= \underbrace{\iint}_{B_1} + \underbrace{\iint}_{B_2} + \underbrace{\iint}_{B_3} + \underbrace{\iint}_{B_4} + \underbrace{\iint}_{B_5} + \underbrace{\iint}_{B_6} \\
 &= (2B + \gamma) \cdot dz \\
 &= \frac{1}{\epsilon_0} \iiint_{\Omega_{\text{box}}^B} \gamma \cdot dz = \frac{1}{\epsilon_0} \gamma \cdot dz,
 \end{aligned}$$

from which it follows that

$$\frac{1}{\epsilon_0} \gamma = 2\beta + \gamma . \quad (4.18)$$

The conditions on γ of Eqs. (4.14) and (4.18) clearly show that γ must present a discontinuity between the boxes Ω_{box}^A and Ω_{box}^B under the discontinuity conditions (4.15a)-(4.15c). In particular, $\gamma = (2\alpha + \gamma)$ in Ω_{box}^A and $\gamma = (2\beta + \gamma)$ in Ω_{box}^B . The hypothesis that γ is continuous is, thus, not valid in this case.

In order to study all possible discontinuities in the derivatives in $\text{div } \vec{E}$, i.e., to consider all approaches along x , y , and z with respect to any point P_0 , more sectors must be considered. It can be shown that 8 sectors corresponding to adjacent identical infinitesimal cubes of the type Ω_{box} , sharing P_0 as a common corner, suffice to study all discontinuities in $\text{div } \vec{E}$. The combination of these 8 cubes is called the discontinuity box.

It is worth mentioning that a procedure similar to the one that led to Eq. (4.11) in this lecture can be used to derive the irrotational property of \vec{E} in differential form. In this case, one has to resort to infinitesimal squares instead of cubes. We leave the proof to the reader.

321

7.2 Caveats on the use of infinitesimal structures in lecture 4.

Consider the infinite straight line γ studied in lecture 4.

Figure 4.8 refers to the argument we used to show that the component \vec{E}_z of \vec{E} does not depend on z . The argument is based on the assumption that \vec{E}_z at point P_{BC}^r is exactly the same as \vec{E}_z at P_{DA}^r because $\overline{AB} = \overline{CD} = dr$. This is extended to any point of the type P_{BC}^r and P_{DA}^r on BC and DA, respectively.

However, in the light of the results in Sec. 4.1, we can now state that the argument above is only true in zeroth order approximation. In first order approximation, in fact, given $\vec{E}_z(P_{BC}^r)$ we can write

$$\vec{E}_z(P_{DA}^r) = E_z(P_{BC}^r) \vec{u}_z + \frac{\partial}{\partial r} E_z(P_{BC}^r) \cdot dr \vec{u}_z . \quad (7.19)$$

As a consequence, $\vec{E}_z(P_{BC}^r) = \vec{E}_z(P_{DA}^r)$ iff $\partial E_z(P_{BC}^r) / \partial r = 0$ or, in other words, E_z does not depend on r . At this point, without resorting to any symmetry argument it is impossible to guarantee that this is true. In fact, we need to wait until the argument represented by Fig. 4.11 to discover not only that \vec{E}_z does not depend on r , but that

it is actually zero everywhere. With this knowledge [33] in hand, we can conclude that \vec{E}_r does not depend on r .

Consider now the argument associated with Fig. 4.11. This is based on the first order assumption that $\vec{E}_r(P_{BC}^z) = \vec{E}_r(P_{DA}^z)$ at all points of type P_{BC}^z and P_{DA}^z on BC and DA, respectively. For the argument to be valid in first order approximation, it is required that $\partial E_r / \partial z = 0$ or, in other words, that E_r does not depend on z . Such an assumption, however, is the result of the first argument (Fig. 4.8), which is based on the second argument (Fig. 4.11). This contradiction clearly points out that no safe conclusion can be drawn in first order approximation.

The only way to circumvent this issue is to resort to symmetry arguments in the case of Fig. 4.11. In fact, because of the translation symmetry of γ we can argue that, to all orders, $\vec{E}_r(P_{BC}^z) = \vec{E}_r(P_{DA}^z)$ and, in general, \vec{E}_r does not depend on z . This allows us to conclude that \vec{E}_z does not depend on r and that $\vec{E}_z = \vec{0}$ everywhere (argument of Fig. 4.11). It also allows us to save the argument of Fig. 4.8 to all orders and, thus, to confirm a posteriori that E_r

does not depend on \vec{z} .

341

We can summarize stating that the use of the irrotational property of \vec{E} for lines including infinitesimal structures and symmetry arguments should opportunely be combined together to solve electrostatic problem. We recommend to identify as many symmetries as possible and to confirm a posteriori all results derived from symmetries by means of the irrotational property of \vec{E} in integral form for suitable lines.

It is worth stressing with another example that extra care must be taken when applying Gauss' theorem and the irrotational property of \vec{E} in integral form to curves including infinitesimal structures.

Consider again the infinite straight line γ . Fig. 4.4 a shows a surface Σ with six faces S_1, S_2, \dots, S_6 . S_1 and S_6 are assumed to be two identical surfaces with infinitesimal dimensions and, thus, equal area dS . Because of the infinitesimal dimensions of S_1 and S_6 , all the other faces are infinitesimally close to each other. Hence, in zeroth order approximation the field \vec{E} generated by γ can be assumed to be the same at each point on S_2 and S_3 and on S_4 and S_5 with equal radial distance from γ . At these points, \vec{E} can be approximated with its value on the radial axis through γ, S_1 and S_6 , $E_{S_1, 6}$. In addition, \vec{E} can assumed to be constant on S_1 and S_6 with values

\vec{E}_1 and \vec{E}_6 , respectively. Under these assumptions, the total flux of \vec{E} through Σ is given by

$$\begin{aligned}
 \sum \oint_{\Sigma} \vec{E} \cdot \vec{n} dS &= \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6} \\
 &= -E_{r_1} dS - \iint_{S_2} E_{\varphi_2(r)} dS + \iint_{S_2} \cancel{E_{\varphi_2(r)} dS} \\
 &\quad - \iint_{S_4} \cancel{E_{z_4}(r) dS} + \iint_{S_4} E_{z_4}(r) dS \\
 &\quad + E_{r_6} dS = 0 , \quad (4.20)
 \end{aligned}$$

where we used the fact that $S_2 = S_3$ and $S_4 = S_5$ due to construction and, similarly, $E_{\varphi_2(r)} = E_{\varphi_3(r)}$ and $E_{z_4(r)} = E_{z_5(r)}$ at each point on $r_{\varphi_{16}}$ between S_1 and S_6 .

Integral (4.20) result in condition

$$E_{r_1} = E_{r_6} , \quad (4.21)$$

which would seem to state that E_r does not depend on r . This contradicts the result of Eq. (4.3).

Note that condition (4.21) was obtained assuming no knowledge on \vec{E}_φ and \vec{E}_z .

It is easy to convince oneself that the wrong result of condition (4.21) was obtained by using a surface Σ which is inconsistent with the rotation symmetry of Υ . A consistent surface Σ' is shown in Fig. 4.4 b. In this case, face S'_1 has an infinitesimal base $r_1 d\varphi$ and an infinitesimal height dz . Similarly, face S'_6 has base $r_6 d\varphi$ and height dz . By construction, faces S_2 and S_3 and S'_4 and S'_5 are the same: $S'_2 = S'_3$ and $S'_4 = S'_5$. As in the case of Σ , in zeroth order approximation \vec{E} can be assumed to be the same at each point on S_2 and S_3 and on S'_4 and S'_5 with equal radial distance from Υ . At these points, \vec{E} can be approximated with its value on the radial axis through Υ , S'_1 , and S'_6 , r_{S_16} . In addition, \vec{E} can be assumed again to be constant on S'_1 and S'_6 with values \vec{E}_1 and \vec{E}_6 , respectively. Under these assumptions, the integral (4.20) can be written as

$$\oint_{\Sigma'} \vec{E} \cdot \vec{n} dS = - E_{r_1} \cdot r_1 d\varphi dz + E_{r_6} \cdot r_6 d\varphi dz = 0 ,$$

from which it follows that

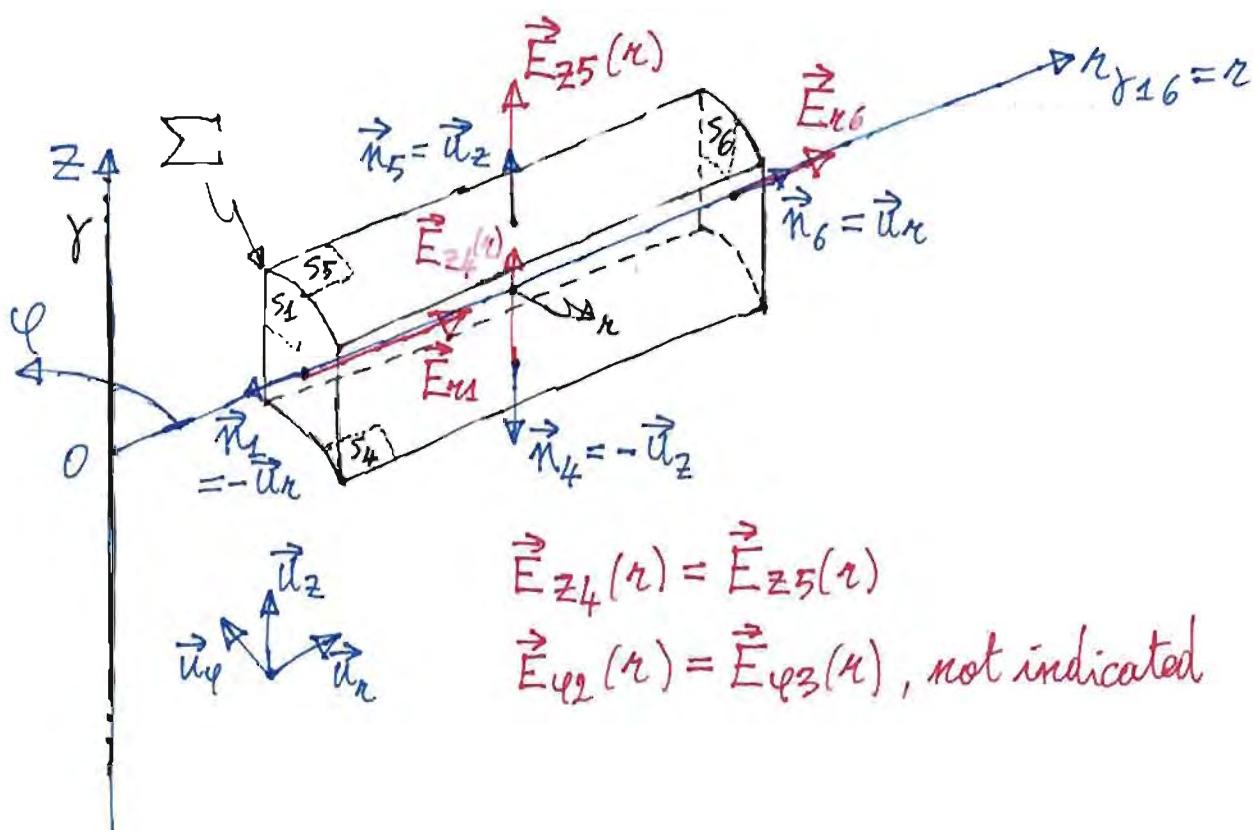
371

$$\frac{E_{r_1}}{E_{r_6}} = \frac{\epsilon_6}{\epsilon_1} \quad . \quad (4.23)$$

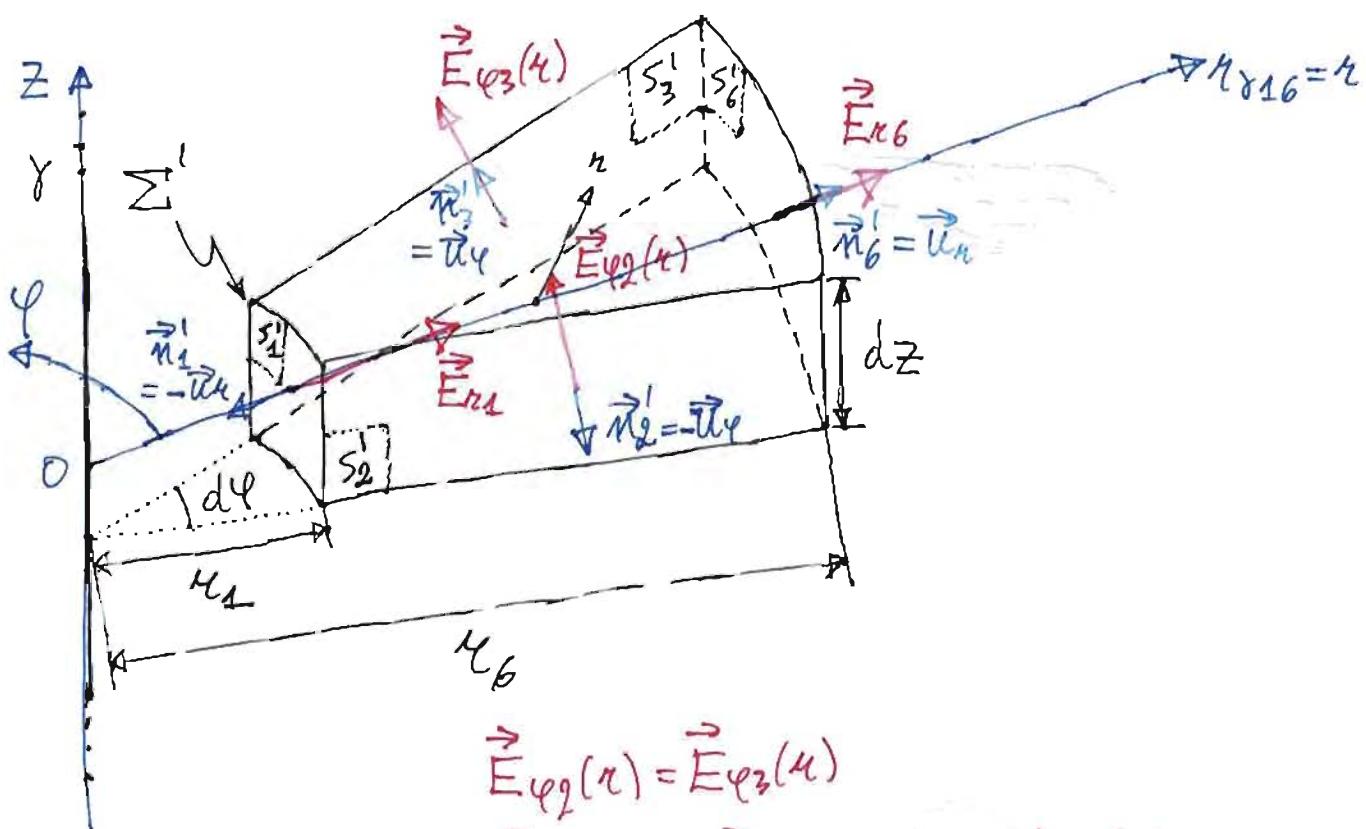
Condition (4.23) is consistent with Eq. (4.3) according to which

$$\left\{ \begin{array}{l} E_{r_1} = \frac{\lambda}{2\pi\epsilon_0\epsilon_1} \\ E_{r_6} = \frac{\lambda}{2\pi\epsilon_0\epsilon_6} \end{array} \right.$$

We can conclude this example stating that, given a charge distribution, the curves used in the integral form of Gauss' theorem and the irrotational property of \vec{E} must be chosen consistently with the geometrical configuration (and, thus, symmetry) of the distribution. That is, those curves must "contain" information on the given charge distribution.



a.



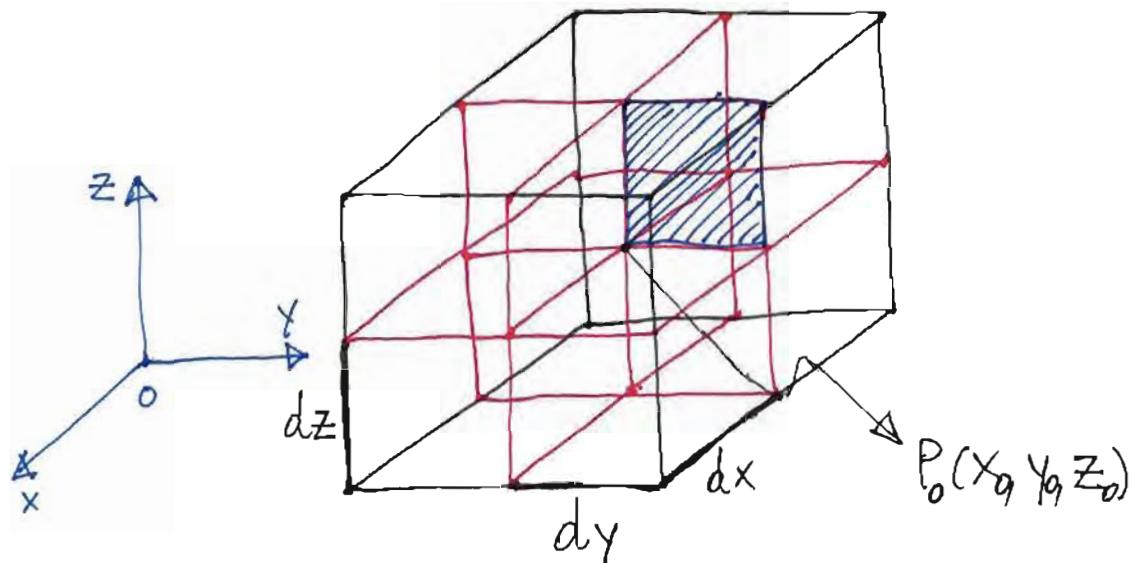
b.

Figure 4.4.

The electrostatic potential / part 1.

Summary lecture 4.

- Field lines rules.
 - Field lines go from positive to negative charges, from positive charges to infinity, and from infinity to negative charges.
 - Field lines are more intense in the neighborhood of larger charges.
 - Field lines never cross each other.
 - Field lines are perpendicular to continuous and limited surface charge distributions and to the discontinuity boundary between continuous and limited volume charge distributions.
 - The field \vec{E} at a point P on a field line is tangent to the line at that point.
- The discontinuity box.
 - The discontinuity box is made by eight infinitesimal cubes, each with edges dx , dy , and dz , all adjacent to each other, and sharing a generic point $P_0(x_0, y_0, z_0)$ in space as a common corner.



- The box makes it possible to study discontinuities in the limit approaching P_0 from the left and right along x, y , or z for all derivatives required to calculate $\vec{\nabla} \cdot \vec{E}$.
- Assuming the component $E_x(P_0)$ to be constant on the face shaded in blue in the cube, the components E_x on the faces in front and behind it in first order approximation are:

$$\left\{ \begin{array}{l} E_x^{\text{front}} = E_x(P_0) + \lim_{x \rightarrow x_0^+} \frac{\partial}{\partial x} E_x(x, y_0, z_0) \cdot dx \\ E_x^{\text{back}} = E_x(P_0) - \lim_{x \rightarrow x_0^-} \frac{\partial}{\partial x} E_x(x, y_0, z_0) \cdot dx \end{array} \right.$$

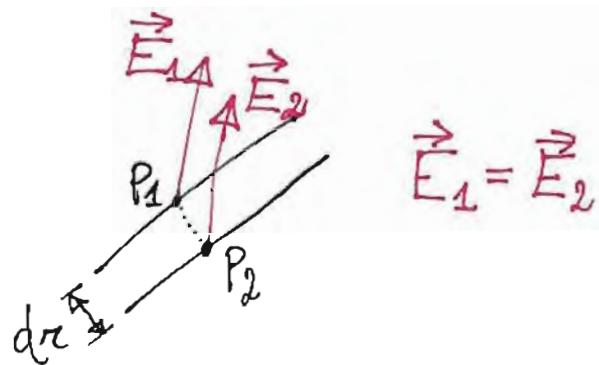
respectively. Similar equations apply to other components and other sectors (cubes).

- The discontinuity box can also be used to demonstrate Gauss' theorem in differential form: 31

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \rho .$$

Caveats on zeroth order approximations.

- Given an arbitrary charge distribution, when applying the irrotational property of \vec{E} or Gauss' theorem to two infinitesimally close lines or surfaces, respectively, in zeroth order approximation \vec{E} can be assumed to be equal on two adjacent points on different lines or surfaces.



Depending on the geometrical configuration of the charge distribution, however, in first order approximation \vec{E}_1 can be different from \vec{E}_2 .

- In general, the geometry of the lines and surfaces used for the irrotational property of \vec{E} and Gauss' theorem must be consistent with the geometry of the given charge distribution.

8.1 Physics history inverted: Coulomb's theorem and the superposition property of \vec{E} derived from Gauss' law and the irrotational principle of \vec{E} . 41

In this section, we will pretend to change the course of history by assuming that Johann Carl Friedrich Gauss, along with helper Michael Faraday, take the place of Charles-Augustin de Coulomb and vice versa. This requires the first two gentlemen to perform a jump back in the past by approximately five decades and, correspondingly, Mr. Coulomb to go back to the future by the same time (cf. Fig. 8.1).

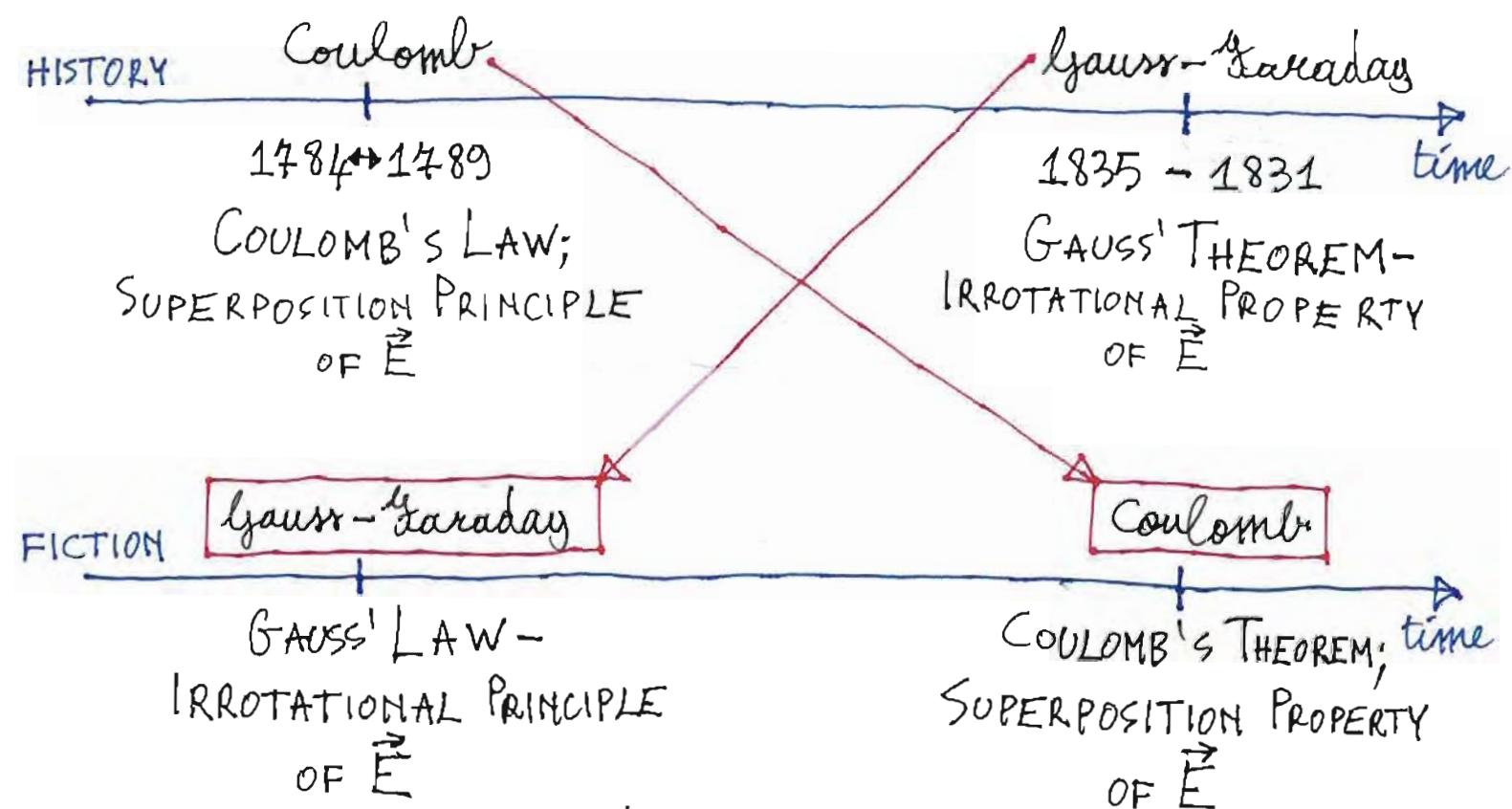


Figure 8.1.

Before proceeding further with our fiction and its consequences on the way electrostatic would have been explained, it is worth giving a few definitions. 5/

8.1.1 Laws vs. theorems in physics.

- **Laws.** A physics' law is a (fundamental) principle that is demonstrated by means of empirical evidence.
 - **Caveat 1.** Note that in theoretical physics a principle can be given without any experimental evidence. However, at a later time, such principles must be confirmed by experiments to be considered legitimate.
 - **Caveat 2.** In the literature, the words law and principle are typically assumed to have the same meaning (e.g., the superposition "principle").

The concept of law in physics is analogous to that of axiom or postulate in mathematics, the main difference being the former is corroborated by experiments.

- **Theorems.** A physics' theorem is a statement (or property) that can be mathematically demonstrated by means solely of physics' laws. Corollaries can be demonstrated using other physics' theorems.
 - **Caveat 1.** In the literature, the words theorem and property are typically assumed to have the same meaning.

(e.g., Gauss' "theorem" or the irrotational "property" of \vec{E}). 61

8.1.2 Gauss and Faraday.

Assume that Gauss went back in time, replacing Coulomb. However, instead of performing the torsion balance experiment (i.e., Coulomb's historical experiment), Gauss decided to perform an experiment based on charged spheres (i.e., our fictional experiment), the apparatus of which is shown in Fig. 8.2.

The apparatus comprises an external metallic sphere Σ_{ext} , with radius $R_{\text{ext}} \approx 1.5 \text{ m}$, containing a smaller metallic sphere Σ_{int} . Both spheres are centered in the same point O . Σ_{int} contains a device E able to reveal the passage of an extremely small charge that goes through it. An image of the screen of E is deflected by means of a mirror B into the axis of a microscope M that, finally, allows an observer to monitor E . A little hole through Σ_{int} and Σ_{ext} makes it possible to monitor E without significantly perturbing the experiment. Two thin conducting wires f_1 and f_2 connect Σ_{ext} to Σ_{int} through E (f_1 does not touch Σ_{int}). If the observer applies a charge Q to Σ_{ext} while constantly monitoring E . If a field $\vec{E} \neq \vec{0}$ was present inside Σ_{ext} , a fraction of

the charge Q, q , would pass through Σ and, thus, would be detected by the observer. This is because a nonzero electrostatic field \vec{E} would generate a force $\vec{F} = q \vec{E} = m_q \vec{a}$ (where m_q is the mass of q) that would move the charge from Σ_{ext} to Σ_{int} . However, repeating the experiment multiple times shows no charge passage through Σ . As a consequence, the observer (in this case Gauss) can safely conclude that $\vec{E} = \vec{0}$ inside Σ_{ext} .

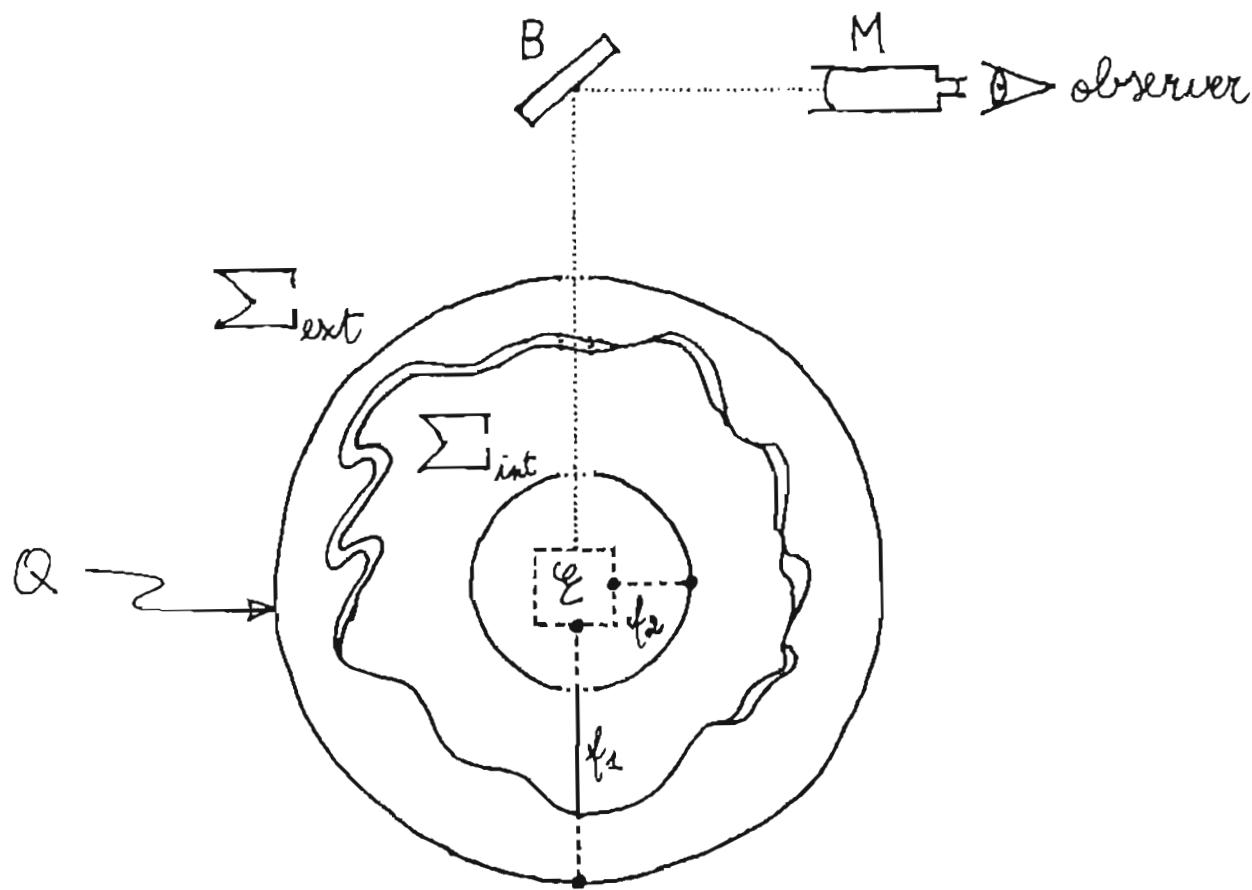


Figure 8.2.

The apparatus of Fig. 8.2 can be simplified as shown in Fig. 8.3a. A metallic hollow sphere is separated from the ground by means of an insulating support. A charge Q , e.g.,

a positive charge, is applied to the sphere. A little metallic ball attached to an insulating handle is lowered through a small aperture into the sphere until it touches the interior wall of the sphere. If the little ball was initially discharged, after extracting it from the sphere it will remain discharged. This can easily be shown by connecting the ball to an electrometer: the pointer of the electrometer will indicate zero before and after lowering the ball into the sphere. Once again, this shows that $\vec{E} = \vec{0}$ inside a charged sphere.

If, however, the ball touches the exterior of the sphere, connecting the ball afterwards to the electrometer will now show that the ball has been charged: the pointer of the electrometer will move from zero (cf. Fig. 8.3 b). This new experiment shows that outside a charged sphere there is a nonzero electrostatic field \vec{E} , which moves a fraction of Q from the sphere to the ball. Note that we will study more in detail the behavior of metallic objects (e.g., spheres) in presence of an electrostatic field \vec{E} in the lectures on conductors.

By performing experiments analogous to those described above, Gauss would have been eventually able to state that "the flux of the electrostatic field \vec{E} through any arbitrary surface is proportional by $1/\epsilon_0$ to the total charge contained within the surface." In this form, Gauss' statement would have appeared to be a fundamental principle demonstrated by empirical evidence rather than a theorem, and, thus, we would have referred to it as "Gauss' law" instead of Gauss' theorem.

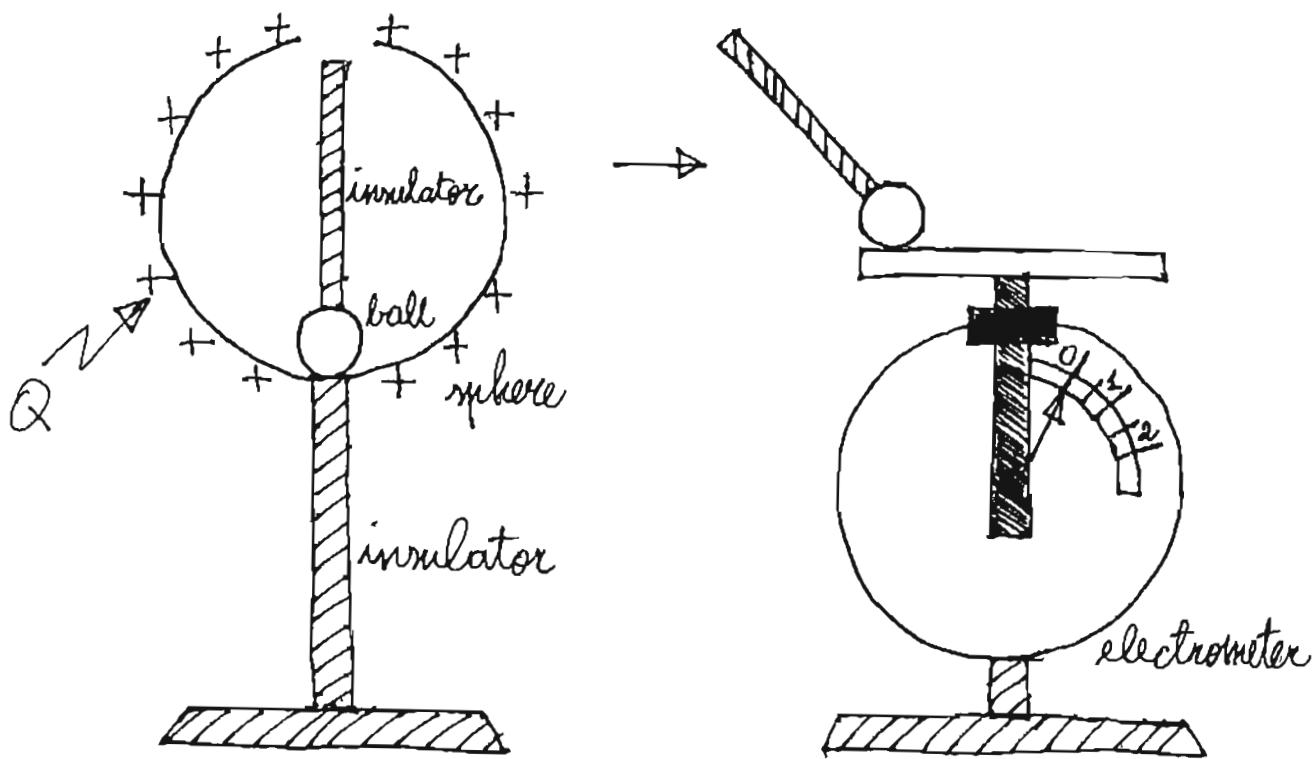
Note that the simple experiments described in this section (cf. Fig. 8.2 and Figs. 8.3a and 8.3b) confirm Gauss' statement. However, in order to be able to deduce such a statement only from empirical evidence Gauss would have needed more advanced experiments. The reader should try to devise a set of such experiments. Here, we assume Gauss was able to do so and, thus, to summarize his findings in mathematical form as:

$$\sum \oint \vec{E} \cdot \vec{n} dS = \frac{1}{\epsilon_0} Q , \quad (8.1)$$

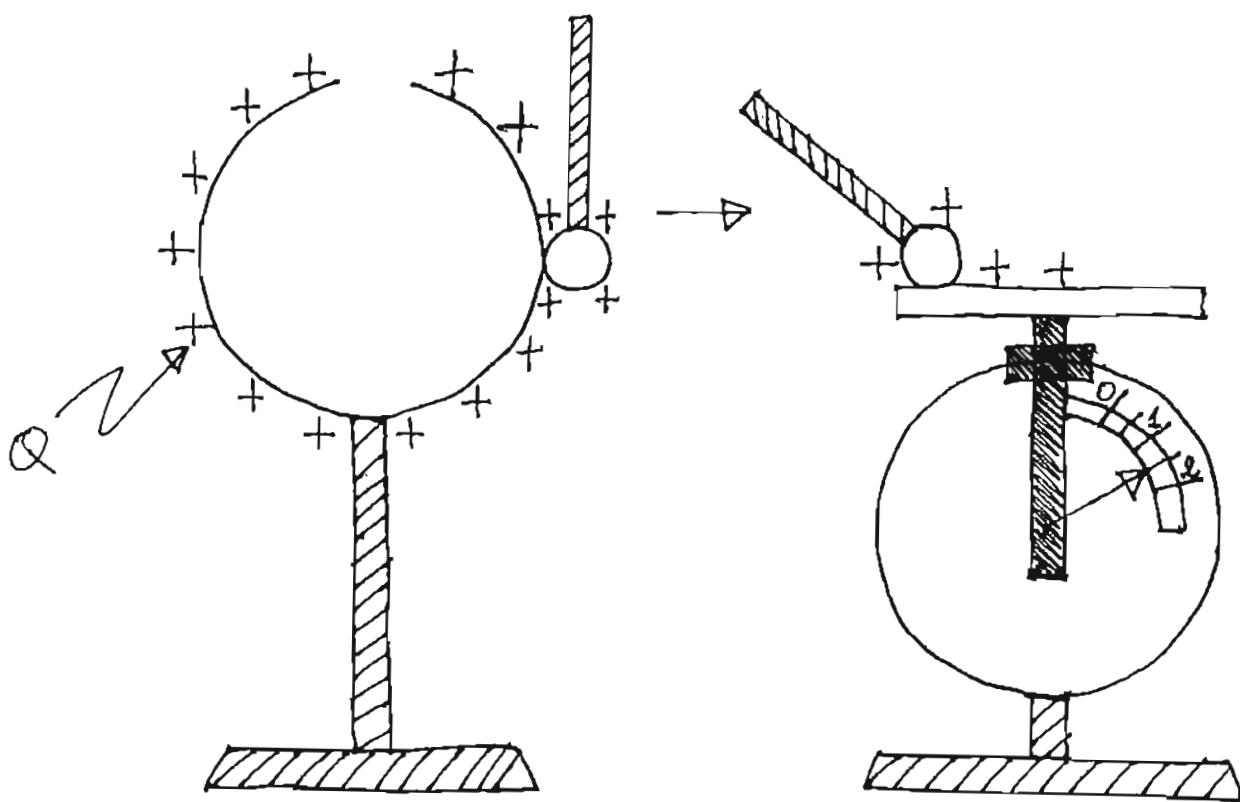
where, as always, \sum is an arbitrary surface containing a total charge Q and $\vec{n} dS$ an infinitesimal surface element on \sum oriented according to the unit vector \vec{n} , which is normal to dS and points outward with respect to \sum .

- Faraday's help.

Assume that also Faraday went back in time, at the same time as Gauss and, at that time, performed a set of field-line experiments. In one such experiments, Faraday considered a charged sphere similar to that depicted in Fig. 8.3a. He then sprayed all around the sphere a charged powder that allowed him to reveal the field lines of the electrostatic field \vec{E} generated by the charged sphere. He found the field lines to be directed



a.



b.

Figure 8.3.

radially with respect to the center of the sphere (cf. Fig. 8.4 a). Faraday repeated many field-line experiments for different charge distributions, never obtaining a field-line configuration as that shown in Fig. 8.4 b.

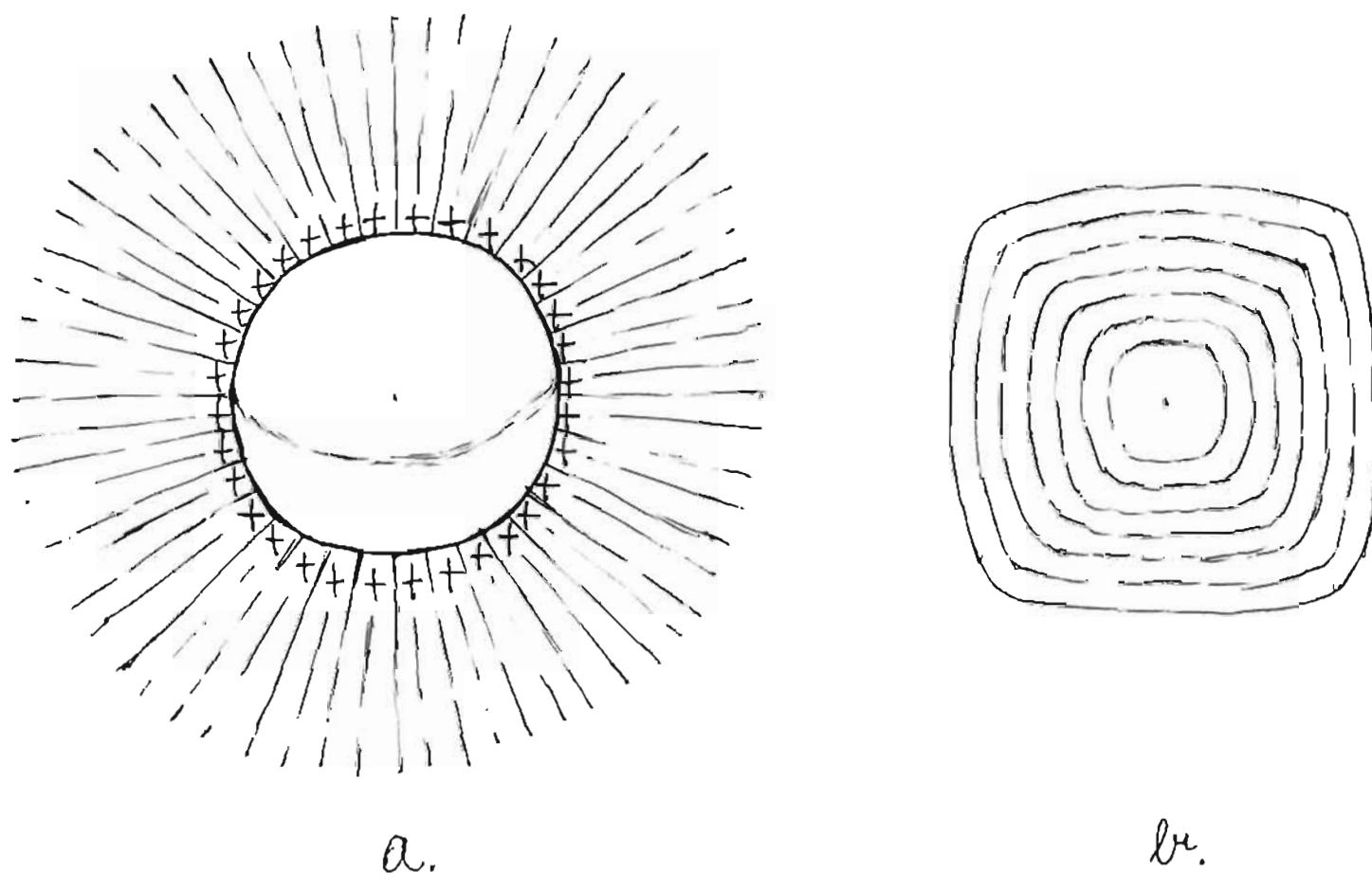


Figure 8.4.

From these findings, Faraday would have been able to deduce an important law of electrostatics, i.e., the irrotational principle of \mathbf{E} . With the help of a mathematician (it is well known that Faraday was a brilliant physicist, but not a strong mathematician), Faraday would have summarized

his experimental results as:

$$\oint_{\gamma} \vec{E} \cdot \vec{dl} = 0 \quad , \quad (8.2)$$

where γ is any arbitrary closed line oriented according to the unit vector \vec{E} tangent to γ at each point on γ and dl is the magnitude of an infinitesimal line element on γ . What in history happens to be a property (or theorem) that can be demonstrated from the characteristic of \vec{E} to be a central field in our fiction becomes a fundamental law demonstrated by empirical evidence.

By means of field-line--type experiments, Faraday would have also been able to show that, given a limited charge distribution confined in a finite region of space, the field lines of \vec{E} become fainter and fainter the further from the charge. In other words, Faraday would have shown the zero-field principle of \vec{E} at infinity.

8.1.3 Coulomb.

Following our fictional time line, a few decades after the experimental findings of Gauss and Faraday, Coulomb came along. Instead of performing experiments and deducing laws, in our fiction Coulomb demonstrated theorems.

With empirical laws (8.1) and (8.2) in hand, Coulomb set out to find the electrostatic field \vec{E} generated by a

(point-like) charge q in the 3D euclidean space.

Figure 8.5 shows the reference frame and associated coordinate system chosen by Coulomb to solve this problem. The coordinate system is a spherical system $Oxyz$ and the reference frame (not shown) is assumed to be inertial.

The figure also shows the vector components of \vec{E} at a generic point $P(r, \theta, \varphi)$, \vec{E}_r , \vec{E}_θ , and \vec{E}_φ .

Since Coulomb a priori has no knowledge on \vec{E} , he assumes three DOF: r , θ , and φ .

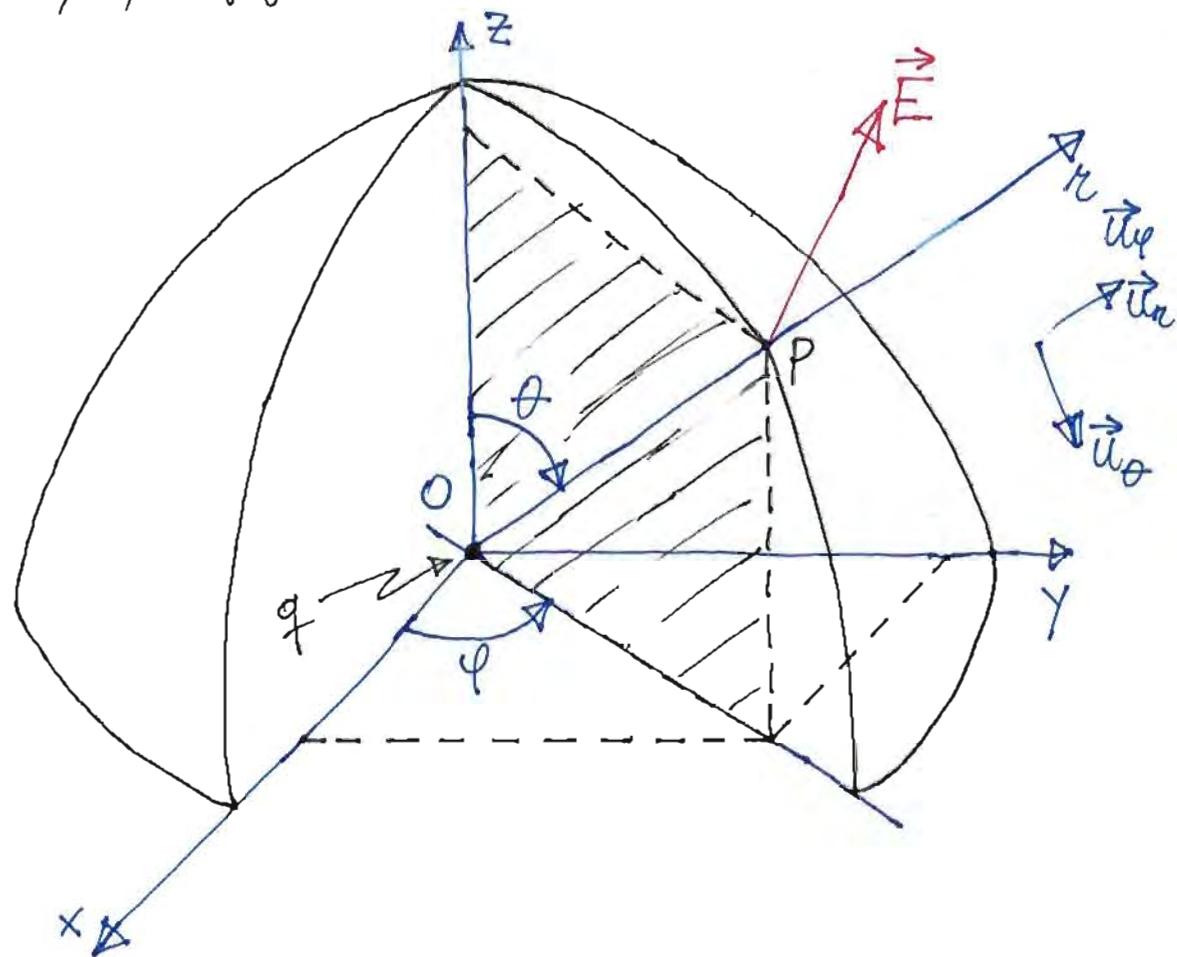


Figure 8.5.

- Coulomb decides to begin his derivation of \vec{E} with \vec{E}_θ . He decides to resort to both Eqs. (8.1) and (8.2) and

symmetry arguments to find \vec{E}_θ (we remind here the reader to the caveats in lecture 4, Sec. 4.2, page 32 and thereafter). Coulomb's strategy is to take advantage of the problem symmetries as much as possible and then, *a posteriori* confirm those arguments by means of Eqs. (8.1) and (8.2).

A point-like charge distribution q is characterized by a rotation symmetry with respect to both θ and φ in the $x\theta\varphi$ system: the charge appears the same to an observer at P for any value of θ and φ (cf. Fig. 8.6 a).

Assume a nonzero \vec{E}_θ at point P (cf. Fig. 8.6 b). Because of the rotation symmetry with respect to θ , \vec{E}_θ must be the same at each point on the closed line (meridian circle) γ_θ in Fig. 8.6 b. This is obviously true also for point P' , which belongs to both γ_θ and the closed line (parallel circle) γ_φ . By rotating \vec{E}_θ at P' by an angle π clockwise (or counter-clockwise) along γ_φ , because of the rotation symmetry with respect to φ we obtain a component \vec{E}_θ at P with the same magnitude of the initial \vec{E}_θ at P , but opposite direction. The only case when this is possible is if $E_\theta = 0$ at P . Due to the arbitrary choice of P , Coulomb can conclude $\vec{E}_\theta = \vec{0}$ everywhere in space.

Coulomb can confirm this finding by assuming a nonzero \vec{E}_θ at each point on γ_θ . Because of the rotation symmetry with respect to θ we keep assuming \vec{E}_θ to be equal at each point on γ_θ (manifestly, this argument is relaxed

compared to the argument based on the Π rotation along \vec{Y}_θ that led us to conclude $\vec{E}_\theta = \vec{0}$ everywhere in space). Under these assumptions, Coulomb can invoke the law of Eq. (8.2) for the case of closed line γ_θ oriented as indicated in Fig. 8.6 b. The only component of \vec{E} contributing to the integral (8.2) is E_θ because the only one directed as $\vec{E} = \vec{U}_\theta$. Hence, Coulomb can write

$$\oint_{\gamma_\theta} \vec{E} \cdot \vec{t} dl = \int_0^{2\pi} E_\theta \vec{U}_\theta \cdot \vec{U}_\theta \cdot r d\theta \\ = 2\pi r \cdot E_\theta = 0 \quad . \quad (8.3)$$

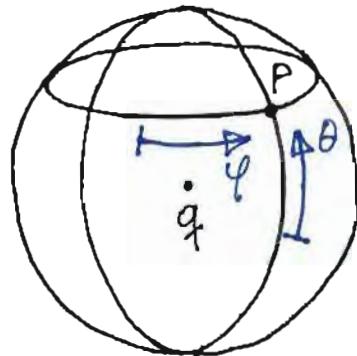
Assuming $r \neq 0$ (i.e., for any P different from the point where q is located), the result of Eq. (8.3) is valid iff $E_\theta = 0$.

Coulomb can now safely conclude that $\vec{E}_\theta = \vec{0}$ everywhere in space.

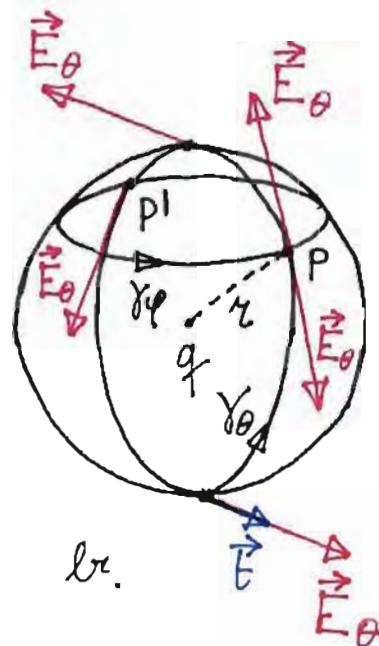
It is worth noting that the use of symmetry arguments combined with Eqs. (8.1) and (8.2) in the present problem is similar to that of a charged sphere (cf. homework 1, problem 1). In fact, as we know from lecture 6, Sec. 6.2 (page 14 and thereafter), a point-like charge distribution can be viewed as the limit $R \rightarrow 0^+$ of a volume charge distribution with constant

density on a sphere of radius R .

- By means of arguments analogous to those used for \vec{E}_θ , Coulomb would have also concluded that $\vec{E}_\varphi = \vec{0}$ everywhere in space (the reader can try to show it).



a.



b.

Figure 8.6.

- The only component of \vec{E} Coulomb must consider is, thus, \vec{E}_r .

Because of the rotation symmetry with respect to θ and φ , \vec{E}_r must be the same at each point on the meridian and parallel circles passing through a generic point $P(r, \theta, \varphi)$, as shown in Fig. 8.4a. In particular, given a nonzero \vec{E}_r at P , a rotation of $\{\vec{P}, \vec{E}_r\}$ by an angle θ along γ_θ leads to vector \vec{E}_r at P' . A rotation of $\{\vec{P}', \vec{E}_r\}$ by an angle π clockwise (or counter-clockwise) along

γ_q leads to vector \vec{E}_r at P, consistently with the initial choice of \vec{E}_r (of Fig. 8.4a). Due to the arbitrary choice of P, Coulomb can conclude that there can be a nonzero component E_r and that such a component must be the same at each point of a generic sphere with radius r and centered in q .

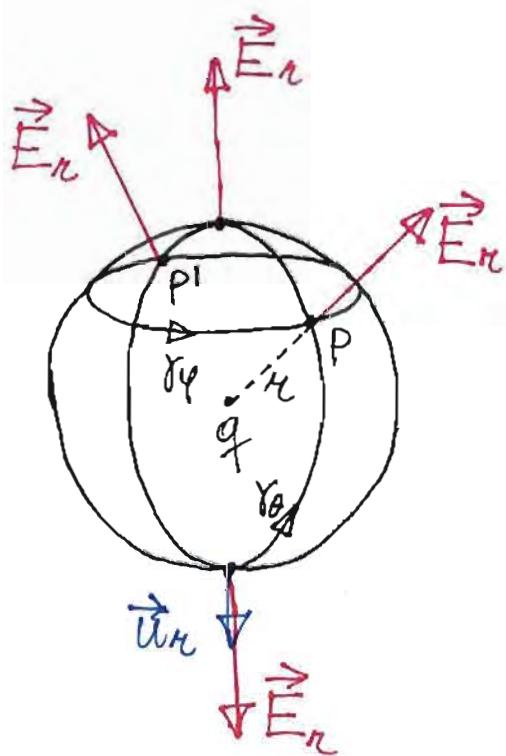
Coulomb can confirm this argument by considering a closed line γ_r as shown in Fig. 8.4b. The line is oriented counter-clockwise and comprises four parts, line segments AB and CD and arcs BC and DA. Segments AB and CD are directed radially with respect to q and arcs BC and DA belong to two circles both centered in q . The radial distance between the arcs is assumed to be infinitesimally small, i.e., $\overline{AB} = \overline{CD} = dr$. Note that, following our caveats in lecture 7 on lines comprising infinitesimal distances, the geometry of γ_r has been chosen consistently with the rotation symmetry of the given charge distribution q . Coulomb can now invoke Eq. (8.2) and calculate the line integral of \vec{E} on γ_r . The only component of \vec{E} contributing to the integral on AB and CD is \vec{E}_r . In addition, \vec{E}_r is assumed to be constant on AB and CD because of segments of infinitesimal length. We define \vec{E}_r on AB to be \vec{E}_{r1} and on CD to be \vec{E}_{r2} . The only component contributing on BC and DA is \vec{E}_θ , which, as we know, is zero everywhere. Hence the integral (8.2) reads

$$\begin{aligned}
 \oint_{\Gamma_r} \vec{E} \cdot \vec{t} dl &= \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA} \\
 &= \int_r^{r+dr} E_{r1} \vec{u}_r \cdot \vec{u}_r \cdot dr \\
 &\quad + \int_r^{r+dr} -E_{r2} \vec{u}_r \cdot \vec{u}_r \cdot dr \\
 &= E_{r1}(r - r - dr) - E_{r2}(r + dr - r) \\
 &= -E_{r1} dr - E_{r2} dr = 0,
 \end{aligned}$$

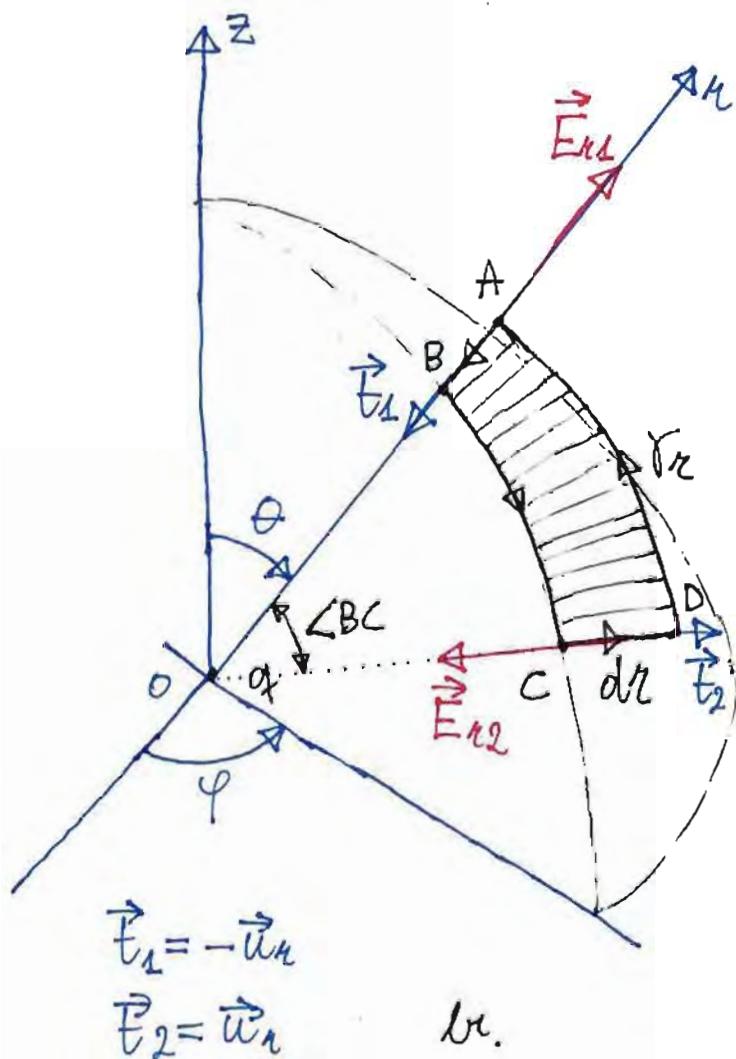
from which it follows that

$$E_{r2} = -E_{r1} \quad . \quad (8.4)$$

Because of condition (8.4) and due to the arbitrary choice of angles θ and ϕ and of the aperture $\angle BC (= \angle DA)$, Coulomb can safely conclude that \vec{E}_r must be the same at each point on a sphere with radius r and centered in q .



a.



b.

Figure 8.7.

By extensive use of symmetry arguments and, most importantly, of Eq. (8.2), i.e., the irrotational principle of \vec{E} , Coulomb has already found that \vec{E} must be directed radially (central field) and E_r must be constant on each sphere centered in q . In order to find any possible functional dependence of E_r on r , θ , and φ , Coulomb must now invoke Gauss' law [Eq. (8.1)]. The first closed surface considered by Coulomb is surface Σ'_r in Fig. 8.8 a. Since Σ'_r contains elements with

infinitesimal dimensions, Coulomb must assure that its geometry is consistent with the rotation symmetry of the charge distribution. For this reason, \sum_i^1 is constructed as the semi-cone obtained from the intersection of a cone with apex q and infinitesimal aperture $d\theta$, with two concentric spheres centered in q and radius r and r' , respectively. Thus, the semi-cone comprises infinitesimal surfaces dS and dS' as bases, and a lateral surface S_l . Because of the infinitesimal dimensions of the two bases, we assume \vec{E} to be constant on each of them and we define it to be \vec{E}_r on dS and \vec{E}'_r on dS' . Since $\vec{E}_\theta = \vec{E}_\varphi = \vec{0}$ everywhere, \vec{E}_r and \vec{E}'_r are the only components of \vec{E} contributing to the integral (8.1) on dS and dS' , respectively. In particular, consistently with our choice in Fig. 8.8a, \vec{E}_r is directed opposite with respect to the unit normal vector \vec{n} on dS , and \vec{E}'_r is directed as the unit normal vector \vec{n}' on dS' . The only components of \vec{E} contributing to the integral (8.1) on S_l would be \vec{E}_θ and \vec{E}_φ (\vec{E}_r is always orthogonal to the normal vector at each point on S_l), which, however, are both zero everywhere. Lastly, the area of the infinitesimal spherical elements dS and dS' can be approximated with the area of the disks obtained by intersecting two planes, both perpendicular to r , with the spheres with radius r and r' centered in q , respectively. Each intersection is a circumference corresponding to the outer border of dS and

dS' , respectively (cf. Fig. 8.8a, inset). Making a negligible projection error, we can thus write:

$$\begin{aligned} dS &= \pi \left(r \sin \frac{d\theta}{2} \right)^2 \simeq \pi \left(r \frac{d\theta}{2} \right)^2 \\ &= \frac{\pi}{4} r^2 \cdot (d\theta)^2 \quad (8.5) \end{aligned}$$

and

$$\begin{aligned} dS' &= \pi \left(r' \sin \frac{d\theta}{2} \right)^2 \simeq \pi \left(r' \frac{d\theta}{2} \right)^2 \\ &= \frac{\pi}{4} r'^2 \cdot (d\theta)^2 \quad (8.6) \end{aligned}$$

Under these assumptions and due to the absence of charge within \sum'_R , Coulomb (me) can calculate the integral (8.1) as

$$\begin{aligned} \oint_{\sum'_R} \vec{E} \cdot \vec{n} dS &= \int_{dS} + \int_{S_e} + \int_{dS'} \\ &= \int_{dS} E_x \vec{u}_x \cdot (-\vec{u}_x) \cdot dS + \int_{dS'} E'_x \vec{u}_x \cdot \vec{u}_x \cdot dS' \end{aligned}$$

$$= E_r \cdot 4\pi r^2 = \frac{1}{\epsilon_0} q, \quad (8.9)$$

from which it follows that

$$E_r = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2}$$

or, more in general,

$$\vec{E}_r = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \vec{u}_r. \quad (8.10)$$

From Eq. (8.10), given a charge q' at point P_2 in space, the force acting on q' due to q at point P_1 is

$$\vec{F}_{12} = \frac{1}{4\pi\epsilon_0} \frac{qq'}{r_{12}^2} \vec{u}_{12}, \quad (8.11)$$

where r_{12} is the absolute value of the distance between points P_1 and P_2 and \vec{u}_{12} the unit vector indicating the direction between P_1 and P_2 . Starting from Gauss' law and the irrotational principle of \vec{E} [Eqs. (8.1) and (8.2)], Coulomb was able to demonstrate Eq. (8.11): Coulomb's

theorem.

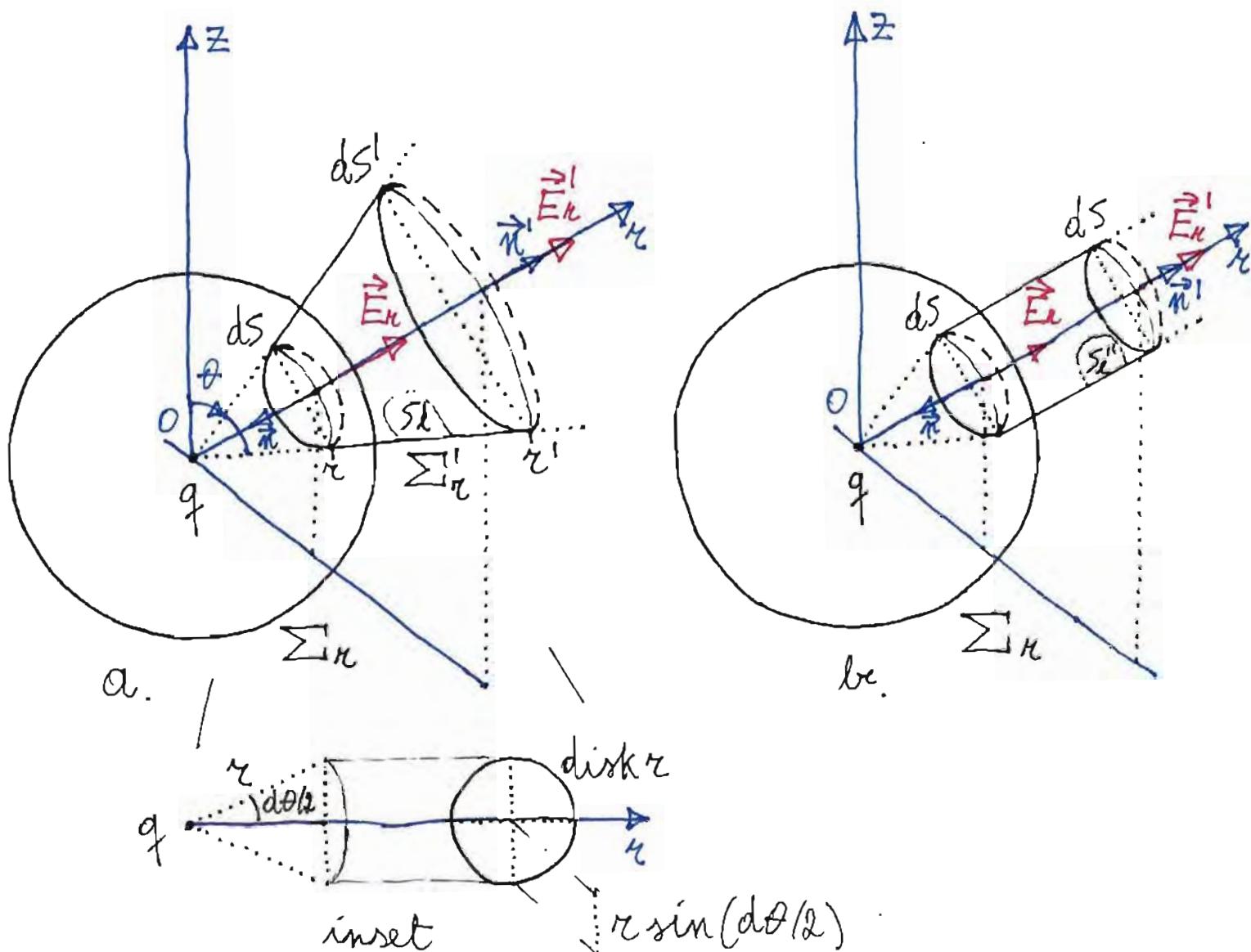


Figure 8.8.

Similar arguments apply to \vec{E}_q . The demonstration [27] of Coulomb's theorem is now complete.

- Coulomb decided then to study two point-like charges q_1 and q_2 at a very small (negligible) distance from each other, but not on top of one another. Coulomb assumed the two charges to be in the neighborhood of a point P in space.

By means of Gauss' law, Coulomb could readily calculate the total flux due to the electrostatic field of q_1 and q_2 through a sphere Σ_r with radius r and centered in P:

$$\Phi_{\text{tot}}^{\Sigma_r} = \frac{1}{\epsilon_0} (q_1 + q_2) \quad . \quad (8.12)$$

By means of his own theorem [e.g., in the form of eq. (8.10)], Coulomb was also able to calculate directly (i.e., using the general definition of flux) the flux due to the electrostatic field \vec{E}_1 of q_1 and, separately, due to \vec{E}_2 of q_2 through Σ_r :

$$\begin{aligned} \Phi_1^{\Sigma_r} &= \iint_{\Sigma_r} \vec{E}_1 \cdot \vec{n} dS = -\frac{1}{4\pi\epsilon_0} q_1 \frac{1}{r^2} \vec{u}_r \cdot \vec{u}_r \\ &\quad \cdot \left[\int_0^{2\pi} d\varphi \int_0^\pi \sin\theta \cdot d\theta \right] \end{aligned}$$

$$= \frac{1}{\epsilon_0} q_1 \quad (8.13a)$$

and

$$\Phi_2^{\Sigma_r} = \iint_{\Sigma_r} \vec{E}_2 \cdot \vec{n} dS = \frac{1}{\epsilon_0} q_2, \quad (8.13b)$$

where both \vec{E}_1 and \vec{E}_2 and \vec{n} were assumed to be directed outward on Σ_r . While each of the two fluxes (8.13a) and (8.13b) is different from the total flux (8.12), their sum is equal to it:

$$\Phi_1^{\Sigma_r} + \Phi_2^{\Sigma_r} = \frac{1}{\epsilon_0} (q_1 + q_2) = \Phi_{\text{tot}}^{\Sigma_r}.$$

In order for this result to be valid, a superposition property of \vec{E} must exist, so that

$$\Phi_{\text{tot}}^{\Sigma_r} = \frac{1}{\epsilon_0} (q_1 + q_2) = \frac{1}{4\pi\epsilon_0} (q_1 + q_2) \frac{1}{r^2} \vec{r}^2 \vec{u} \cdot \vec{u}_r$$

$$\cdot \int_0^{2\pi} d\phi \int_0^\pi \sin\theta d\theta$$

$$= \oint_{\text{c}} (\vec{E}_1 + \vec{E}_2) \cdot \vec{n} dS ,$$

where the total field generated by q_1 and q_2 is $\vec{E}_{\text{tot}} = \vec{E}_1 + \vec{E}_2$. It is easy to prove that the superposition property of \vec{E} is valid for any arbitrary distribution of N point-like charges and, by extension, can be applied to arbitrary continuous charge distributions.

In this way, Coulomb would have been able to demonstrate the superposition property of \vec{E} , which happens to be "built-in" in Gauss' law.

- Our historical and fictional journey demonstrates the equivalence between Coulomb's law and the superposition principle of \vec{E} on one hand and Gauss' theorem and the irrotational property of \vec{E} (with the addition of the zero-field condition at infinity) on the other hand.

8.1.4 Back to history.

According to Feynman, it was Benjamin Franklin to first notice that the electrostatic field inside a charged metallic sphere is zero. He probably used an apparatus similar to that

sketched in Fig. 8.3. When he mentioned his finding to Priestley, the latter conjectured that it might had to do with an inverse square law. In fact, at the time it was known that a spherical shell of matter produces a zero gravitational field inside the shell (and, according to Newton's law, the gravitational field depends on the inverse distance squared). This happened 18 years before Coulomb's torsion balance experiment.

Six years after Franklin's discovery, Cavendish performed a more refined experiment, conceptually very similar to that sketched in Fig. 8.2. He found that upon charging an outer metallic sphere, $1/60$ of the applied charge transferred to an inner sphere connected to the outer one by means of a conducting wire. As it turns out, Cavendish did not publish his null experiment, which, approximately 100 years later, was resumed and further refined by Maxwell. Maxwell found that only $1/21.600$ of the applied charge was transferred to the inner sphere. It was not until 1936 that Plimpton and Lawton, by means of the apparatus sketched in Fig. 8.2, were able to measure a charge transfer of $1/10^9$.

In homework 1, problem 1, the reader was asked to calculate the electrostatic field inside a charged shell both

using the actual Coulomb's law for the electrostatic field [i.e., as in Eq. (8.10)] and by means of a fictitious inverse cube law,

$$\vec{E}_{\text{fict}} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^3} \hat{u}_r . \quad (8.14)$$

Using such a fictitious law resulted in a nonzero field inside the shell, contradicting all experiments. Note that any other exponent $\lambda = 2 + \epsilon$, with ϵ an arbitrary small quantity (positive or negative), would have resulted in a wrong, nonzero electrostatic field inside the shell. The fact that the field is zero inside the shell is equivalent to the field being dependent on $1/r^2$, as in Coulomb's law, and, thus, to Gauss' theorem as well.

Remarkably, it is simple to show that if the field is zero inside a charged shell, the exponent in Coulomb's law must be exactly 2. For this purpose, consider the shell in Fig. 8.9. In the figure, a generic point P inside the shell is the apex of a pair of cones intersecting the shell in two infinitesimal spherical sectors dS_1 and dS_2 . The distance between P and each point on the outer border of dS_1 is r_1 and of dS_2 is r_2 . By way of basic

geometrical arguments (we have seen this earlier in this lecture), it follows that

$$\frac{dS_2}{dS_1} = \frac{\frac{\pi}{4} r_2^2 (d\theta)^2}{\frac{\pi}{4} r_1^2 (d\theta)^2} = \frac{r_2^2}{r_1^2}, \quad (8.15)$$

where we assumed the two cones to have the same aperture $d\theta$. If the surface of the shell is uniformly charged, the charge dq on an infinitesimal surface element is proportional to the area of the element, $dq = \sigma dS$, where σ is a constant surface charge density. Hence,

$$\frac{dq_2}{dq_1} = \frac{\sigma dS_2}{\sigma dS_1} = \frac{dS_2}{dS_1}. \quad (8.16)$$

Assume the exponent in Coulomb's law to be $\lambda (= 2 + \epsilon)$, the ratio between the magnitudes of the fields generated by dq_1 and dq_2 at P is

$$\frac{E_2}{E_1} = \frac{dq_2/r_2^\lambda}{dq_1/r_1^\lambda}. \quad (8.17)$$

By inserting Eqs. (8.15) and (8.16) in Eq. (7.14) we obtain:

$$\frac{E_2}{E_1} = \frac{\frac{dS_2}{dS_1} \frac{r_2^{-\lambda}}{r_1^{-\lambda}}}{\frac{dS_1}{dS_2} \frac{r_1^{-\lambda}}{r_2^{-\lambda}}} = \frac{r_2^2}{r_1^2} \frac{r_2^{-\lambda}}{r_1^{-\lambda}}$$

$$= \frac{r_2^{(2-\lambda)}}{r_1^{(2-\lambda)}} \quad . \quad (8.18)$$

The field at P is zero (as it must be according to experiments), iff $\lambda = 2$ in Eq. (8.18). In fact, in this case

$$\frac{E_2}{E_1} = 1 \quad (8.19)$$

and the two fields cancel each other exactly. This means $\epsilon = 0$. If this was not true, e.g., $\lambda = 3$, we would obtain

$$\frac{E_2}{E_1} = \frac{r_1}{r_2},$$

which means the two fields do not cancel each other since, in general, $r_1 \neq r_2$. [Note that, because of Coulomb's law, \vec{E}_1 and \vec{E}_2 in Eq. (8.19) are directed opposite of each other]

Obviously, the argument can be extended to any dS (and dq) on the shell and for any point P inside the shell (however, note that the center of the shell would be a bad choice to prove the argument as, there, $r_1=r_2$).

As a consequence, the experiment of Plimpton and Lawton confirms Gauss' theorem and shows that the exponent in Coulomb's law must be 2 within one part in a billion. Further refinements of this result were obtained using different experimental methods in the early 40's (cf., e.g., Williams, Teller, and Hill, 1941).

We finally note that if, e.g., $\alpha=3$, the integral in the sense of Cauchy's principle value of Eq. (5.16) would diverge. As a consequence, the electrostatic field generated by any charge distribution with continuous and limited density would also diverge, resulting in catastrophic consequences on our way of living! It appears the exponent in Coulomb's law is just the maximum allowed before such a catastrophe.

One last question remains to be answered, in all experiments spherical structures were used. Is such a special shape required to find $\alpha=2$? The answer is no.

We will come back to this topic in the lectures on conductors.

case $\sigma > 0$

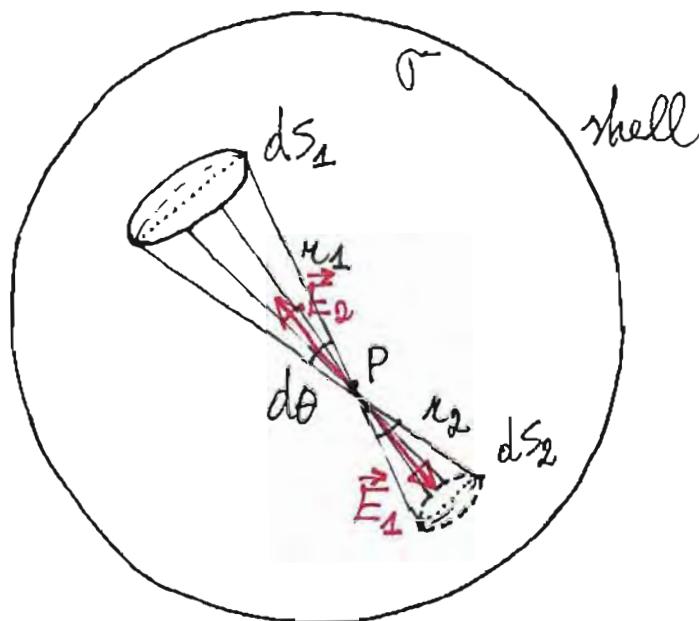


Figure 8.9.

8.1.5 More caveats.

One more comment is in order. Consider Fig. 8.8 b. In our fiction, if Coulomb would have chosen the closed surface \sum_r'' to apply Gauss' law, he would have concluded that \vec{E}_r and \vec{E}_r' were the same. In fact, because of the infinitesimal dimensions of dS , the field \vec{E} can be assumed to be the same at each point on the lateral surface of \sum_r'', S_r'' , with equal radial distance from q . We shall discuss this more in detail in the following.

Surface Σ''_r is a cylindrical surface comprising two equal infinitesimal bases dS and a lateral surface S''_e . First, on the basis closer to q , the field \vec{E}_r can be assumed to be constant on the entire basis and directed outward, thus opposite compared to the unit normal vector \vec{n} on that basis. Hence, on this basis the electrostatic flux is given by

$$\begin{aligned}\oint \vec{E}_r dS &= \vec{E}_r \cdot \vec{n} \cdot dS = \vec{E}_r \vec{a}_r \cdot (-\vec{a}_r) \cdot dS \\ &= -E_r dS . \quad (8.20)\end{aligned}$$

Second, on S''_e to each scalar product $\vec{E} \cdot \vec{n}_e = E n_e$ there exists one equal and opposite, $-E n_e$. Hence, on this surface the overall flux is zero:

$$\oint_{S''_e} \vec{E} \cdot \vec{n} dS = 0 . \quad (8.21)$$

Last, on the basis further from q , the field \vec{E}'_r can be assumed to be constant on the basis and directed outward, thus as the unit normal vector \vec{n}' on that basis. Hence, on this basis the flux is given by

$$\begin{aligned}\Phi_{r_1}^{\text{dS}} &= \vec{E}_r^1 \cdot \vec{n}^1 \cdot dS = E_r^1 \vec{u}_r \cdot \vec{u}_r \cdot dS \\ &= E_r^1 dS \quad . \quad (8.22)\end{aligned}$$

From Gauss' law, the total flux through \sum''_{r_1} is given by the sum of fluxes (8.20), (8.21), and (8.22), and must be equal to zero since there is no charge within \sum''_{r_1} :

$$\begin{aligned}\Phi^{\sum''_{r_1}} &= -E_r dS + E_r^1 dS \\ &= 0 \quad . \quad (8.23)\end{aligned}$$

From this result it would follow that $E_r^1 = E_r$, thus contradicting our previous result (8.8). This is because \sum''_{r_1} is similar to \sum'_{r_1} , but ultimately inconsistent with the rotation symmetry of the given charge distribution. The condition (8.23) is thus wrong, showing one more time that particular care must be taken when considering structures with infinitesimal elements.

We can conclude this section observing that the use of Gauss' law, Eq. (8.1), and the irrotational principle

of \vec{E} , eq. (8.2), and eventually the zero-field condition at infinity can be quite tricky. This integral form approach should be followed when the problem is characterized by specific, clear symmetries. In those cases, the integral approach typically results in "back of the envelope" results and, thus, is very useful. More in general, the safest approach is to invoke the differential form of eqs. (8.1) and (8.2) and set up a system of partial differential equations (PDEs) with boundary conditions similar to those examined in lecture 5. These equations can sometimes be solved analytically, often must be solved numerically. We will discuss more about this method in the upcoming lectures.

As a last note, it is worth mentioning that all problems in electrostatics could actually be solved by computing integrals of the type (2.24). In fact, we now know the equivalence between Coulomb's law and the superposition principle of \vec{E} and Gauss' theorem and the irrotational property of \vec{E} . Those integrals, however, are often though to compute, even numerically.

8.2 The electrostatic potential / part 1.

In the next series of lectures, we move from a vector quantity, the electrostatic field \vec{E} , to a scalar quantity, the electrostatic potential V . The field and potential, which are mathematically related to each other, make it possible to study the electrostatic interaction from different, ultimately equivalent perspectives.

The electrostatic potential V makes it possible to simplify calculations and is very useful solving problems where, a priori, the charge distribution is unknown.

8.2.1 The potential of a point-like charge.

Consider a point-like charge distribution q located at point Q in space. In a spherical coordinate system, a generic point P in space is represented by 3 numbers: r , θ , and φ . Because of its central and symmetric structure, the field \vec{E} generated by q can be represented by the gradient of a scalar function (cf. tutorial 2). In particular, given r the distance between P and Q (r is an absolute value) and C an arbitrary constant, by defining a scalar function $V(P)$ as

$$V(P) = \frac{1}{4\pi\epsilon_0} \frac{q}{r} + C, \quad (8.24)$$

it follows that

$$\vec{E}(P) = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r}_{QP}$$

$$= -\text{grad } V(P) = -\vec{\nabla} V(P) \quad . \quad (8.25)$$

Hence, each function of the type (8.24) represents a scalar potential of the field generated by a point-like charge (cf. also tutorial 2 on the scalar potential).

Because of the arbitrary choice of C in definition (8.24), C can be chosen in such a way that $V=0$ at an arbitrary point P of the field, so long $P \neq Q$. For example, if we want $V(P_0)=0$ for a generic $P_0 \neq Q$, it is sufficient to impose

$$C = -\frac{1}{4\pi\epsilon_0} \frac{q}{r_0},$$

where r_0 is the distance between Q and P_0 .

In particular, among all potentials given by (8.24), only one verifies the zero-potential condition at infinity

$$\lim_{r \rightarrow \infty} V(r) = 0 \quad . \quad (8.26)$$

This is the potential for $C=0$. Unless specific reasons occur, it is common to define precisely this function as the

potential of the field. This potential is called Green function corresponding to normal conditions at infinity.
It is simple to verify that V has the same value at each point on a generic sphere centered in Q . The sphere is thus called equipotential surface of the field (cf. tutorial 2)

Figure 8.10 shows the equipotential surfaces for a point-like charge q in 2D. Note that the equipotentials are perpendicular to the field lines. We can thus add another rule on how to sketch field lines!

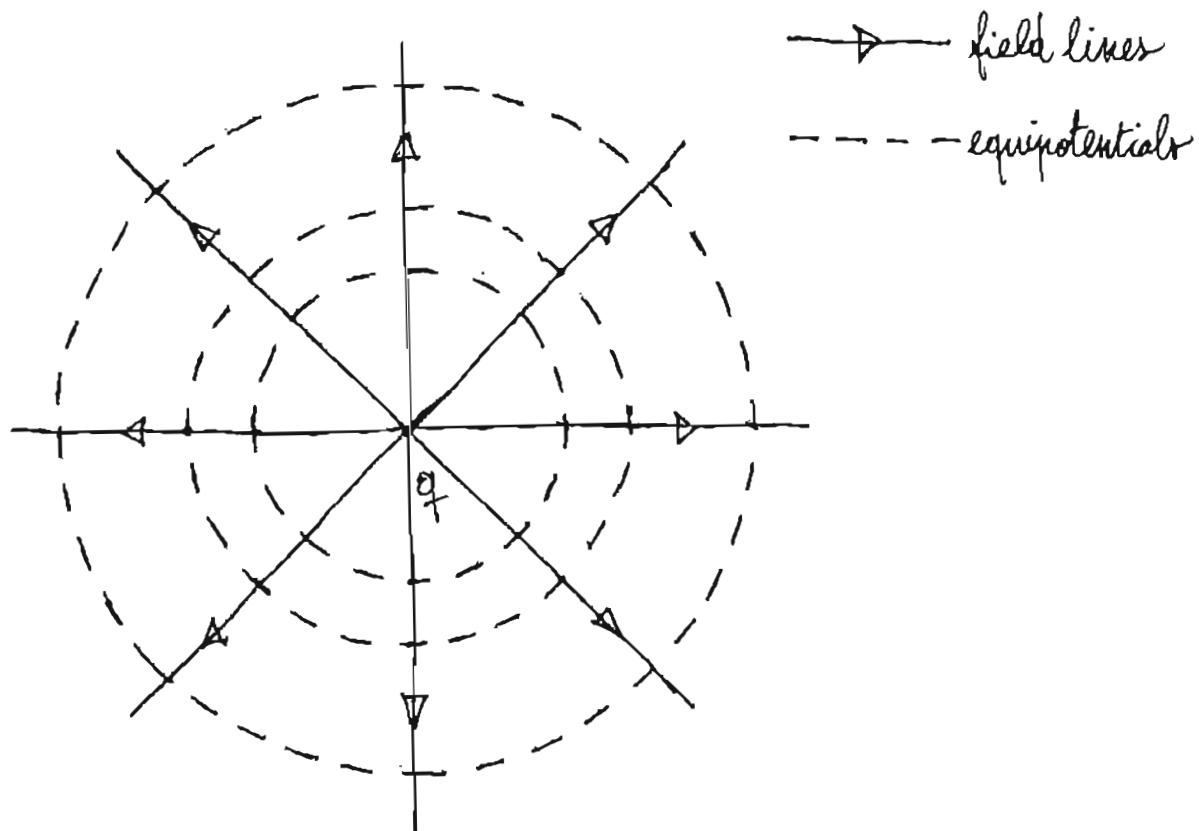


Figure 8.10.

8.2.2 Work of the field forces.

We will now attempt to attach a deeper physical meaning to the concept of potential. 42

Referring again to the simple case of a point-like charge distribution q at point Q , we imagine to introduce a charge q_0 in the field and, by applying a suitable external force to it, to move it from point P_1 to point P_2 along a given line γ . We assume the external force on q_0 to be at each point on γ equal and opposite to the force on q_0 due to q . To ease the intuition, we can assume the external force to be slightly larger than the field force, so that it can overcome it without, however, giving a noticeable acceleration to q_0 . Under these conditions, the charge q_0 moves extremely slowly (in the limit, with zero velocity and acceleration) through a continuous series of equilibrium steps with zero kinetic energy. In this sense, we perform a quasi-static transformation, so that the work of the external force is equal and contrary to that of the field force.

The work W_γ of the field force to move q_0 from P_1 to P_2 on γ is

$$W_\gamma = \int_{\gamma} \vec{F} \cdot \vec{E} dl = q_0 \int_{\gamma} \vec{E} \cdot \vec{E} dl , \quad (8.24)$$

where \vec{E} is the unit tangent vector to the oriented curve γ and dl the infinitesimal arc element on γ .

By substituting (8.25) in (8.24), we obtain

431

$$W_f = -q_0 \int_Y \vec{\nabla} V \cdot \vec{E} dl \quad (8.28)$$

and finally, using the properties of the gradient (cf. tutorial 2),

$$W_f = q_0 [V(P_1) - V(P_2)] , \quad (8.29)$$

where V is any of the infinite potential functions of the field. It is worth noting that (8.29) is independent from the choice of C .

In summary, chosen any of the field potentials with arbitrary C , the difference $V(P_1) - V(P_2)$ for two generic points P_1 and P_2 (different from ∞) is equal to the work of the field forces when moving a unitary positive charge from P_1 to P_2 along any line connecting them. In fact, due to the properties of the gradient that express the irrotational property of \vec{E} , the integral (8.28) depends only on the limiting points of Y , P_1 and P_2 . The work W_f can be positive or negative (or, of course, zero), depending whether the charge displacement is along the field force or against it.

From (8.29), when $q_0 = 1C$ is possible to define the

physical unit for the potential difference. In the SI this is called "volt" (V): Between two points there exists a potential difference (or drop) of 1 V, when the work of the field force to move a charge of 1 C between the two points is 1 J. From this definition, it is also possible to obtain the most commonly used unit for \vec{E} , which is V/m (and its multiples, e.g., kV/cm).

Summary lecture 8.

- Equivalence between Coulomb's law and the superposition principle of \vec{E} and Gauss' theorem and the irrotational property of \vec{E} (with the addition of the zero-field condition at infinity).
 - In physics, laws are principles demonstrated by experimental evidence. Theorems, instead, are demonstrated from laws.
 - The electrostatic field inside a charged metallic sphere is zero, $\vec{E} = \emptyset$. This experimental evidence confirms Gauss' theorem.
 - The field lines of \vec{E} never form closed loops around a charge distribution. This experimental evidence confirms the irrotational property of \vec{E} .
 - The radial nature of Coulomb's law (\vec{E} is a central field) can be demonstrated from symmetry argument and the irrotational property of \vec{E} . This means that, given a point-like charge q , in spherical coordinates

$\vec{E}_\theta = \vec{E}_\varphi = \vec{0}$ everywhere in space (rotation symmetry). 2/25

- Because of symmetry arguments and the irrotational property of \vec{E} , the field of q is the same on each sphere centered in q (again, rotation symmetry). From Gauss' theorem it then follows that $E \sim 1/r^2$, where r is the distance between q and any point in space.
 - The superposition principle of \vec{E} is built-in in Gauss' theorem.
 - The field inside a charged metallic sphere is zero only if the exponent in Coulomb's law is $d=2$ (inverse square law). This is also consistent with Gauss' theorem.
- Basic definitions of electrostatic potential.

- At point P ,

$$\vec{E}(P) = -\vec{\nabla} V(P) ,$$

where, for a point-like charge, the electrostatic potential

$$V(P) = \frac{1}{4\pi\epsilon_0} \frac{q}{r} + C ,$$

with C an arbitrary constant.

- The condition

$$\lim_{r \rightarrow \infty} V(r) = 0$$

corresponds to $C=0$. The associated V is called green function for normal conditions at infinite. 3/25

- For a point-like charge, V is constant on each sphere centered in the charge. These spheres are called equipotential surfaces. The field lines of \vec{E} are perpendicular to the equipotentials.
- The work of the field forces generated by a point-like charge q to move a test charge q_0 from point P_1 to P_2 along an arbitrary line γ is:

$$W_\gamma = -q_0 \int_{\gamma} \vec{\nabla} V \cdot \vec{E} dl$$
$$= q_0 [V(P_1) - V(P_2)] .$$

- The unit of the voltage (difference) is the volt (V).

9.1 Potentials of a generic charge distribution. 4/25

Consider a generic distribution of point-like charges q_1, q_2, \dots, q_n . Each charge generates a field $\vec{E}_1, \vec{E}_2, \dots, \vec{E}_n$. Because of the superposition principle of \vec{E} , it follows that

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \dots + \vec{E}_n ,$$

where \vec{E} is the total field generated by the distribution.

Following Sec. 8.2 in lecture 8, each field $\vec{E}_1, \vec{E}_2, \dots, \vec{E}_n$ can be associated with a potential V_1, V_2, \dots, V_n , respectively. As a consequence,

$$\vec{E} = -\vec{\nabla}V_1 - \vec{\nabla}V_2 + \dots - \vec{\nabla}V_n .$$

Because of the linearity of the gradient, it follows that

$$\begin{aligned} \vec{E} &= -\vec{\nabla}(V_1 + V_2 + \dots + V_n) \\ &\stackrel{!}{=} -\vec{\nabla}V , \quad (9.1) \end{aligned}$$

where

$$V = V_1 + V_2 + \dots + V_n . \quad (9.2)$$

In other words, the field generated by a generic distribution of point-like charges (in limited number) is characterized by a potential, which is equal to the sum of the potentials associated with each point-like charge. This can be

interpreted as the superposition principle of the electrostatic potential V . 5/25

In the case of n point-like charges, the potential of Eq. (8.24) at a generic point P becomes

$$V(CP) = \frac{1}{4\pi\epsilon_0} \sum_{k=1}^n \left(\frac{q_k}{r_k} + c_k \right), \quad (9.3)$$

where r_k is the distance (absolute value) between the point Q_k at which the charge q_k is localized and the generic point P where the potential is evaluated and c_k is the arbitrary constant associated with the generic partial potential generated by q_k . For a limited number of charge the constants c_k can be chosen arbitrarily. Thus,

$$V(P) = \frac{1}{4\pi\epsilon_0} \sum_{k=1}^n \frac{q_k}{r_k} + C, \quad (9.4)$$

with

$$C = \sum_{k=1}^n c_k.$$

In particular, we can choose $C=0$ from which it follows that $V(P)$ goes to zero at infinity. The zero-potential at infinity condition may become ambiguous when the field is not centrally symmetric. Hereafter, we will intend

that $V(P)$ goes to zero at infinity when for any 6/25
arbitrary $\varepsilon > 0$ is possible to determine a spherical
surface Σ such that on the points of the surface and
on those outside the surface

$$|V(P)|_{\Sigma} < \varepsilon .$$

It is easy to show that for $C=0$, the potential $V(P)$
satisfy this condition.

If, instead, we want $V(P)$ to become zero at a point
 P_0 (different from any of the points Q_k where the charges
 q_k are localized), it is enough to choose C such that

$$V(P_0) = \frac{1}{4\pi\epsilon_0} \sum_{k=1}^n \frac{q_k}{r_{k0}} + C = 0 , \quad (9.5)$$

where, in this case, r_{k0} indicates the distance between
any point Q_k and P_0 . Obviously, the same result is
obtained by choosing the partial potential contributions
from each charge such that they each are zero at P_0 :

$$\frac{1}{4\pi\epsilon_0} \frac{q_k}{r_{k0}} + C_k = 0 .$$

We now consider a charge distribution with volume

density δ (in general variable from point to point) 4/25
 in a limited region of space Ω . We can divide this
 charge distribution in infinitesimal regions $d\mathcal{E}$, each
 occupied by an infinitesimal charge

$$dq = \delta d\mathcal{E}.$$

The potential generated by the infinitesimal charge at
 point Q , $\delta(Q)d\mathcal{E}$, at the generic point P can be written as

$$dV_{CP} = \frac{1}{4\pi\epsilon_0} \frac{\delta(Q)}{r_{QP}} d\mathcal{E}, \quad (9.6)$$

where r_{QP} is the distance between Q and P and the arbitrary
 constant in the definition of potential was assumed to be
 zero.

The superposition principle allows us to obtain the
 potential of a continuous charge distribution by integrating
 (9.6) over the entire region Ω where the charges are
 distributed:

$$V(P) = -\frac{1}{4\pi\epsilon_0} \iiint_{\Omega} \frac{\delta(Q)}{r_{QP}} d\mathcal{E}. \quad (9.7)$$

In a cartesian coordinate system, the integral (9.7)

becomes

$$V(x_p, y_p, z_p) = \frac{1}{4\pi\epsilon_0} \iiint_{\Omega} \frac{\delta(x_\alpha, y_\alpha, z_\alpha)}{[(x_p - x_\alpha)^2 + (y_p - y_\alpha)^2 + (z_p - z_\alpha)^2]^{1/2}} \cdot d x_\alpha d y_\alpha d z_\alpha . \quad (9.8)$$

The scalar function $V(P)$ is characterized by interesting analytical and physical properties. First, given a charge distribution with continuous and limited volume density δ , $V(P)$ is continuous and limited everywhere in space (including where the charge is distributed). Consider a point P outside the region Ω where the charge is distributed. In this case, the integrand function in (9.4) is well defined, continuous and limited, at each point in Σ because δ is by assumption continuous and limited in Ω and $r > 0$ for each P outside Ω . Hence, we can safely conclude that the integral in (9.4) is continuous and limited for each P outside Ω . The reader should prove that the integral in (9.4) is well defined, continuous and limited, even for each P inside Ω (hint: use arguments analogous to those that led to the Cauchy's principle value of Eq. (5.16) in lecture 5).

Second, it is easy to convince oneself that if the charge distribution is contained in a limited region of space, the potential V goes to zero at infinity (we will come back

to this point when deriving the multipole expansion | 9/25
of the electrostatic potential). From this property of the potential it follows the analogous property for the field.

Last, also in the case of continuous distributions the work of the field forces to move a unitary positive charge from point P_1 to point P_2 is equal to the potential difference $V(P_1) - V(P_2)$ and, again, the equipotential surfaces are orthogonal to the field lines.

In the case of surface and linear charge distributions we obtain

$$V(P) = \frac{1}{4\pi\epsilon_0} \iint \frac{\sigma(Q)}{r_{QP}} dS + \sum \quad (9.9)$$

and

$$V(P) = \frac{1}{4\pi\epsilon_0} \int \frac{\lambda(Q)}{r_{QP}} dl \quad , \quad (9.10)$$

respectively.

Without demonstration, in the case of surface charge distributions the potential is continuous and limited even at the points where the charge is defined. In the case of linear charge distributions, however, the potential diverges logarithmically at the points where the charge is defined

It is worth mentioning that calculating the potential generated by a known charge distribution reduces to a single integral of a scalar function. Thus, it is much easier to calculate the potential than the field. In addition, once V is known it is straightforward to calculate \vec{E} from (9.1), i.e., by means of simple derivatives.

9.2 The electrostatic potential of charged rings, shells, and spheres.

In this section, we will review a few relevant examples of electrostatic potential. In the first two examples, we will calculate the potential directly from its definition. In the last example, we will calculate it from the knowledge of the electrostatic field.

9.2.1 The electrostatic potential of a charged ring on the ring's axis.

Consider a linear, uniform charge distribution with constant density λ on a circle γ with radius a (cf. Fig. 9.1). We want to calculate the electrostatic potential V on the axis of γ .

9.1.2. The electrostatic potential of a charged shell at any point P in space. 14/25

Consider a surface, uniform charge distribution with constant density σ on a shell Σ with radius R (cf. Fig. 9.2). Calculate V at any point P in space.

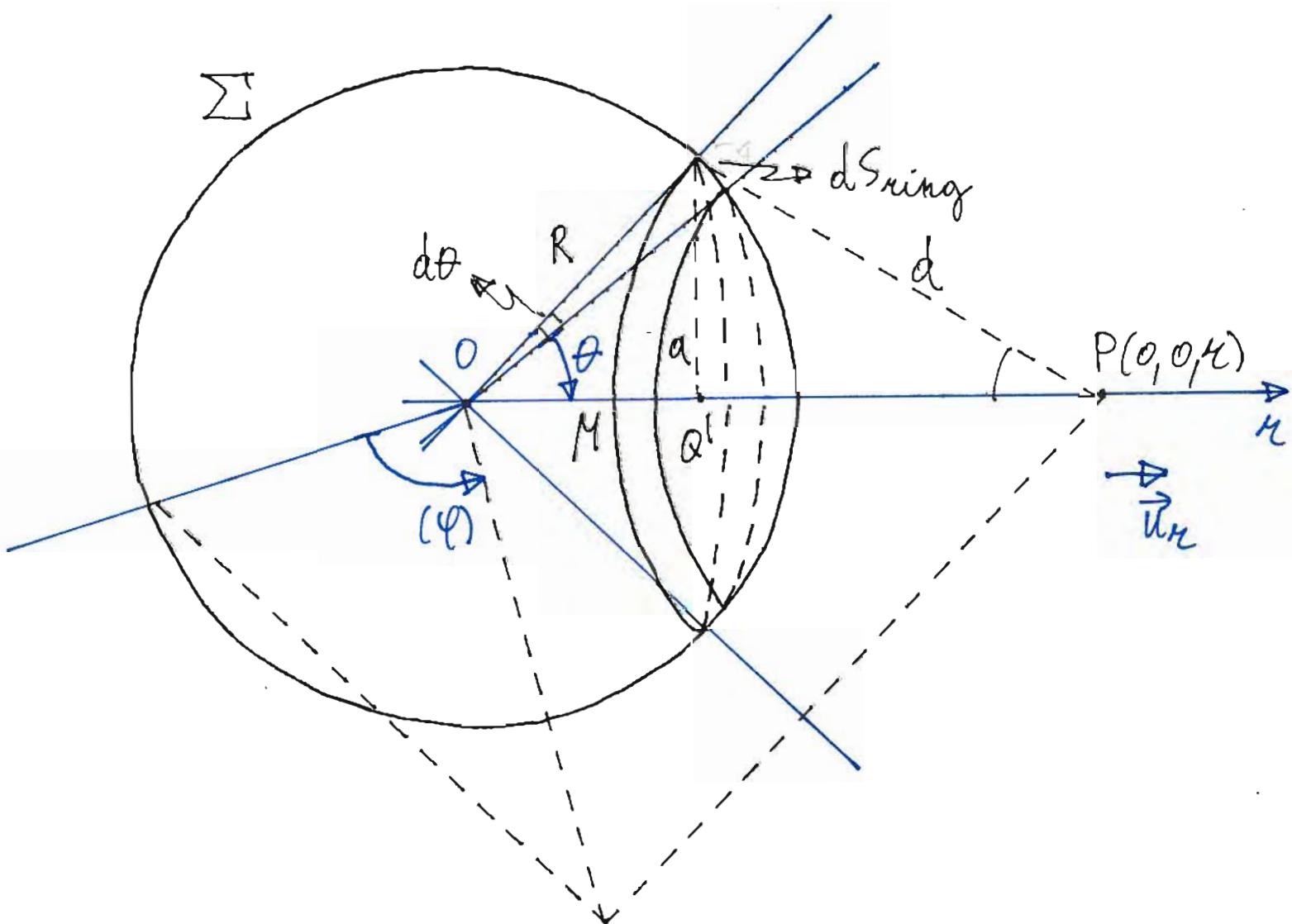


Figure 9.2.

The spherical coordinate system used to solve this problem is shown in Fig. 9.2. 15/25

We proceed as in HA 1.1.3/1). The potential at P due to the ring structure dS_{ring} indicated in the figure is given by:

$$dV = \frac{1}{4\pi\epsilon_0} \frac{\sigma dS_{\text{ring}}}{d},$$

where

$$\begin{aligned} dS_{\text{ring}} &= 2\pi a R \cdot d\theta \\ &= 2\pi R^2 \sin\theta \cdot d\theta \end{aligned}$$

and

$$d = (R^2 + r^2 - 2Rr \cos\theta)^{1/2},$$

where we used Carnot's theorem for triangle QQ'P. Thus,

$$dV = \frac{1}{4\pi\epsilon_0} \frac{\sigma 2\pi R^2 \sin\theta \cdot d\theta}{(R^2 + r^2 - 2Rr \cos\theta)^{1/2}}. \quad (9.13)$$

By integrating on the entire shell Σ , i.e., for $\theta \in [0, \pi]$, we obtain:

$$\begin{aligned}
 V(0,0,\mu) &= \frac{1}{4\pi\epsilon_0} \sum_{n=1}^{\infty} \frac{\sigma 2\pi R^2 \sin \theta}{(R^2 + r^2 - 2Rr \cos \theta)^{1/2}} \cdot d\theta \\
 &= \frac{1}{4\pi\epsilon_0} \int_0^{\pi} \frac{\sigma 2\pi R^2 \sin \theta}{(R^2 + r^2 - 2Rr \cos \theta)^{1/2}} \cdot d\theta \\
 &= \frac{1}{4\pi\epsilon_0} \frac{R}{r} \sigma 2\pi \left[(R^2 + r^2 - 2Rr \cos \theta)^{1/2} \right]_0^{\pi} \\
 &= \frac{\sigma R}{2\epsilon_0 r} \left[(R^2 + r^2 - 2Rr \cos \theta)^{1/2} \right]_0^{\pi} \\
 &= \begin{cases} \frac{\sigma R}{2\epsilon_0 r} [(R+r) - (R-r)], & r \in [0, R] \\ \frac{\sigma R}{2\epsilon_0 r} [(R+r) - (r-R)], & r \in (R, +\infty) \end{cases}
 \end{aligned}$$

(9.14)

Noting that the total charge on Σ is given by:

$$q_{\Sigma} = 4\pi R^2 \sigma ,$$

we finally obtain:

$$V(0,0,r) = \begin{cases} \frac{q_{\Sigma}}{4\pi\epsilon_0 r} & , r \in [0, R] \\ \frac{q_{\Sigma}}{4\pi\epsilon_0 r} & . r \in (R, +\infty) \end{cases}$$
(9.15)

Note that by rotating the axis r by arbitrary angles θ and φ around 0 , Eq. (9.15) makes it possible to calculate V at any point in space P . Equation (9.15) will always be the solution.

Also note that V is continuous at $r=R$:

$$\lim_{r \rightarrow R^-} V(r) = \frac{q \Sigma}{4 \pi \epsilon_0 R}$$

|

$$= \lim_{r \rightarrow R^+} V(r)$$

In addition,

$$\lim_{r \rightarrow +\infty} V(r) = 0$$

Finally, we note that the potential outside the shell is equivalent to that of a point-like charge $q\Sigma$ centered in Σ . Figure 9.3 shows a plot of $V(r)$.

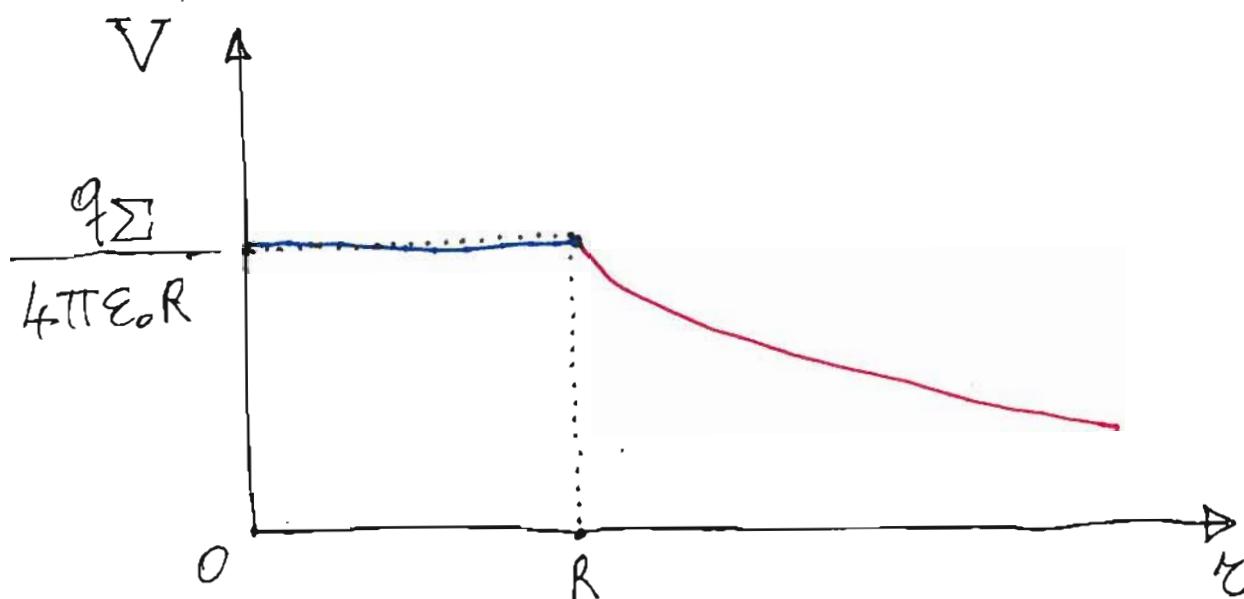


Figure 9.3.

Again by means of definition (8.25), we can find $\vec{E}(P)$:

$$\begin{aligned}
 \vec{E} &= -\vec{\nabla} V \\
 &= -\frac{\partial}{\partial r} V(r) \cdot \vec{u}_r - \frac{1}{r} \frac{\partial}{\partial \theta} V(r) \cdot \vec{u}_\theta \\
 &\quad - \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} V(r) \cdot \vec{u}_\varphi \\
 &= -\frac{\partial}{\partial r} V \cdot \vec{u}_r \\
 &= \begin{cases} 0 & , r \in [0, R] \\ \frac{1}{4\pi\epsilon_0} \frac{q \sum}{r^2} \vec{u}_r & , r \in (R, +\infty) \end{cases}
 \end{aligned}$$

as expected from HA 1.1.3 and HA 1.1.4/1-2).

9.2.3 The electrostatic potential of a charged sphere with known field \vec{E} at any point in space.

From HA 1.1.5 / 1) we know that the field at any point in space for a charged full sphere with radius R is given by:

$$\left\{ \begin{array}{l} \vec{E} = \frac{\sigma_0}{3\epsilon_0} r \cdot \vec{u}_r , \quad r \in [0, R] \\ \vec{E} = \frac{q \sum'_R}{4\pi \epsilon_0 r^2} \cdot \vec{u}_r , \quad r \in (R, +\infty) \end{array} \right.$$

where r is the distance from the center of the sphere on an arbitrary axis passing through the center and on the positive side of the axis,

$$q \sum'_R = \frac{4}{3} \pi R^3 \sigma_0$$

is the total charge contained within Σ' , and σ_0 is a constant charge density. As always the arbitrary choice of the axis r allows us to study only the case $r > 0$, without losing generality.

We can now use definition (8.25) and obtain V

$$\underline{=} -\frac{q \sum_r^1}{4\pi \epsilon_0 r^2} \cdot \vec{u}_r ,$$

from which it follows that

$$dV = -\frac{q \sum_r^1}{4\pi \epsilon_0} \int_R^r \frac{1}{r^2} \cdot dr$$

and so

$$V = -\frac{q \sum_r^1}{4\pi \epsilon_0} \left(\frac{1}{r} - \frac{1}{R} \right) , \quad r \in (R, +\infty) . \quad (9.14)$$

We remind that V is defined up to an arbitrary constant. In the case of (9.14), we want to choose a constant C such that

$$\lim_{r \rightarrow +\infty} \left[\frac{q \sum_r^1}{4\pi \epsilon_0} \left(\frac{1}{r} - \frac{1}{R} \right) + C \right] = 0 .$$

This normal condition at infinity corresponds to

$$C = \frac{1}{4\pi \epsilon_0} \frac{q \sum_r^1}{R} . \quad (9.18)$$

We can thus offset the potential of (9.17) by 23/25
 C and find

$$\begin{aligned} V &= \frac{1}{4\pi\epsilon_0} q \sum'_R \left(\frac{1}{r} - \frac{1}{R} \right) + C \\ &= \frac{1}{4\pi\epsilon_0} \frac{q \sum'_R}{r}, \quad r \in (R, +\infty) . \quad (9.19) \end{aligned}$$

This potential is equal to that of a point-like charge $q \sum'_R$ at the center of the sphere. In particular,

$$\lim_{r \rightarrow R^+} V = \frac{1}{4\pi\epsilon_0} \frac{q \sum'_R}{R} .$$

In the case of (9.16), we can choose an arbitrary constant C' such that

$$\lim_{r \rightarrow R^-} \left(-\frac{\rho_0}{6\epsilon_0} r^2 + C' \right) = \frac{1}{4\pi\epsilon_0} \frac{q \sum'_R}{R} ,$$

in other words, a constant that allows us to connect continuously the potential inside and outside the sphere. This limit gives

$$-\frac{\sigma_0}{6\epsilon_0} R^2 + C^1 = \frac{1}{4\pi\epsilon_0} \frac{q \sum'_R}{R}$$

from which

$$\begin{aligned} C^1 &= \frac{1}{4\pi\epsilon_0} \frac{q \sum'_R}{R} + \frac{1}{2\epsilon_0} \frac{3}{4\pi} \frac{1}{R^3} R^2 q \sum'_R \\ &= \frac{1}{4\pi\epsilon_0} \left(\frac{q \sum'_R}{R} + \frac{q \sum'_R}{2R} \right) \\ &= \frac{1}{4\pi\epsilon_0} \frac{3q \sum'_R}{2R} \quad . \quad (\text{eq. 20}) \end{aligned}$$

We thus find

$$\begin{aligned} V &= -\frac{\sigma_0}{6\epsilon_0} r^2 + \frac{1}{4\pi\epsilon_0} \frac{3q \sum'_R}{2R} \\ &= -\frac{1}{2\epsilon_0} \frac{3}{4\pi} \frac{1}{R^3} q \sum'_R r^2 + \frac{1}{4\pi\epsilon_0} \frac{3q \sum'_R}{2R} \\ &= \frac{1}{4\pi\epsilon_0} \frac{q \sum'_R}{2R} \left(3 - \frac{r^2}{R^2} \right), \quad r \in [0, R] \quad . \quad (\text{eq. 21}) \end{aligned}$$

In particular,

$$V(0) = \frac{1}{4\pi\epsilon_0} \cdot \frac{3}{2} \cdot \frac{q \sum k}{R}$$

Figure 9.4 shows a plot of $V(r)$ for the charged sphere.

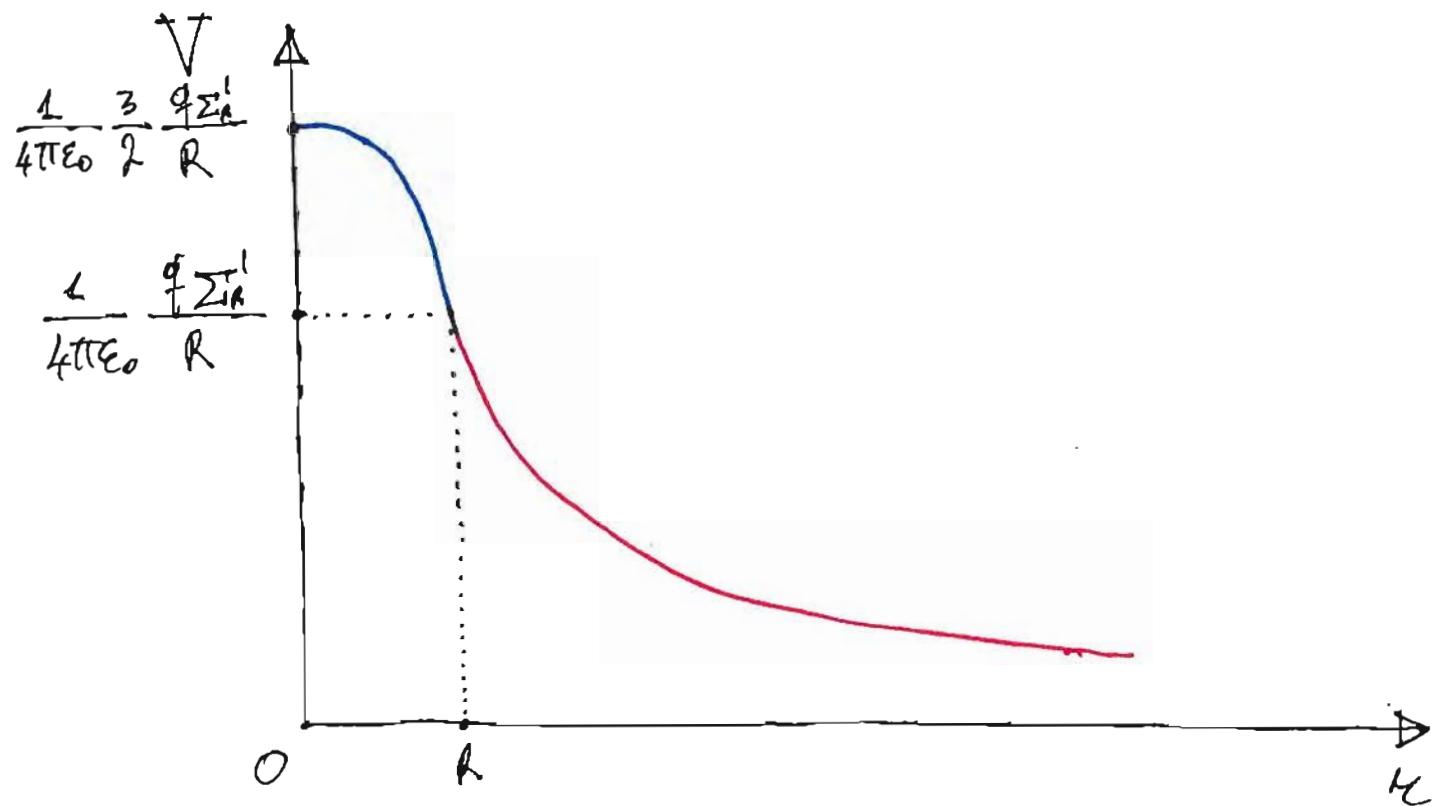


Figure 9.4.

Phys 242 - §14. Lecture 10 - The electrostatic potential / part 3. | 1/34

Summary lecture 9.

- Potential of a generic charge distribution.
 - given n point-like charges q_1, q_2, \dots, q_n ,

$$\begin{aligned}\vec{E} &= \vec{E}_1 + \vec{E}_2 + \dots + \vec{E}_n \\ &\quad | \\ &= -\vec{\nabla} V_1 - \vec{\nabla} V_2 + \dots - \vec{\nabla} V_n \\ &\quad | \\ &= -\vec{\nabla}(V_1 + V_2 + \dots + V_n)\end{aligned}$$

where

$$V = V_1 + V_2 + \dots + V_n .$$

- For n point-like charges

$$V(P) = \frac{1}{4\pi\epsilon_0} \sum_{k=1}^n \left(\frac{q_k}{r_k} + C_k \right) ,$$

where r_k is the distance between q_k and P , and C_k is the arbitrary constant for the partial

potential due to q_K .

- The potential for a continuous volume charge distribution δ in a region Ω is:

$$V(P) = \frac{1}{4\pi\epsilon_0} \iiint_{\Omega} \frac{\delta(Q)}{r_{QP}} d\tau .$$

- For a surface distribution σ on Σ :

$$V(P) = \frac{1}{4\pi\epsilon_0} \sum \iint \frac{\sigma(Q)}{r_{QP}} dS .$$

- For a linear distribution λ on γ :

$$V(P) = \frac{1}{4\pi\epsilon_0} \int \gamma \frac{\lambda(Q)}{r_{QP}} dl .$$

- Given a continuous and limited charge volume density δ in a region Ω , V is continuous and limited everywhere in space.

10.1 Poisson and Laplace equations.

Given a charge distribution ρ in a region Ω of space, at each point where \vec{E} is continuously differentiable

$$\operatorname{div} \vec{E} = \vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \rho \quad , \quad (10.1)$$

$$\operatorname{curl} \vec{E} = \vec{\nabla} \times \vec{E} = \vec{0} \quad . \quad (10.2)$$

By inserting

$$\vec{E} = -\operatorname{grad} V = -\vec{\nabla} V$$

into Eq. (10.2), we find

$$\begin{aligned} \operatorname{curl} (-\operatorname{grad} V) &= \vec{\nabla} \times (-\vec{\nabla} V) \\ &\downarrow \\ &= -\vec{\nabla} \times (\vec{\nabla} V) \\ &\downarrow \\ &= -\vec{\nabla} \times \vec{\nabla} V = 0 \end{aligned}$$

because the vector $\vec{\nabla} V$, which is the formal product of V times the scalar \vec{V} , is "parallel" to $\vec{\nabla}$. Another more physical way to see this is by noting that the vector field $\vec{\nabla} V$, having a potential, must be irrotational (cf. tutorial 3). In other words, (10.2) is identically

verified.

Inserting $\vec{E} = -\vec{\nabla}V$ into (10.1), however, gives

$$\begin{aligned}\operatorname{div}(-\operatorname{grad} V) &= \vec{\nabla} \cdot (-\vec{\nabla}V) \\ &= -\vec{\nabla} \cdot (\vec{\nabla}V) \\ &= -\vec{\nabla} \cdot \vec{\nabla}V \\ &= -\vec{\nabla}^2 V = \frac{1}{\epsilon_0} \delta\end{aligned}$$

and, thus,

$$\vec{\nabla}^2 V = -\frac{1}{\epsilon_0} \delta \quad , \quad (10.3)$$

where $\vec{\nabla}^2$ is called the laplacian (cf. tutorial 3). The (10.3) is named Poisson equation.

In a region of space without any charge distribution (10.3) becomes

$$\vec{\nabla}^2 V = 0 \quad , \quad (10.4)$$

which is called Laplace equation.

Both Poisson and Laplace equations are partial

differential equations. Their analytical form depends on the coordinate system chosen to represent the laplacian operator. Note that Poisson (and Laplace) equation includes both fundamental laws of electrostatic, i.e., Eqs. (10.1) and (10.2). In other words, given a charge distribution $\delta(Q)$, if we were able to find a function $V(P)$ that satisfies (10.3) at each point P , this would certainly be a potential of the field. From $\vec{E} = -\vec{\nabla}V$ we would then immediately obtain $\vec{E}(P)$. Also note that given a function $V(P)$ that satisfies (10.3), any other function

$$V^* = V + C,$$

where C is an arbitrary constant, is also a solution of (10.3). In fact,

$$\vec{\nabla}^2 V^* = \vec{\nabla}^2 V$$

at each point.

Remarkably, from all discussed so far, assume a charge density $\delta(Q)$ is known for each point Q . Further assume that δ is different from zero only in a limited region Σ , i.e., the charge does not extend to infinity.

Under these assumptions, it is possible to show 4/34
that the function

$$V(P) = \frac{1}{4\pi\epsilon_0} \iiint_{\Sigma} \frac{\delta(Q)}{r_{QP}} d\Sigma \quad (10.5)$$

satisfies Poisson equation and, thus, is a potential of the field. The demonstration is rather easy when P is entirely outside the region Σ , i.e., when $r_{QP} \neq 0$ and, thus, the integrand in (10.5) does not diverge to infinity (i.e. is nonsingular). In this case, we can easily compute the laplacian of (10.5) by bringing the laplacian second derivatives inside the integral sign. In cartesian coordinates:

$$\begin{aligned} \vec{\nabla}_P^2 V &= \frac{1}{4\pi\epsilon_0} \iiint_{\Sigma} \delta(Q) \left\{ \frac{\partial^2}{\partial x_P^2} \frac{1}{[(x_P - x_Q)^2 + (y_P - y_Q)^2 + (z_P - z_Q)^2]^{1/2}} \right. \\ &\quad + \frac{\partial^2}{\partial y_P^2} \frac{1}{[(x_P - x_Q)^2 + (y_P - y_Q)^2 + (z_P - z_Q)^2]^{1/2}} \\ &\quad \left. + \frac{\partial^2}{\partial z_P^2} \frac{1}{[(x_P - x_Q)^2 + (y_P - y_Q)^2 + (z_P - z_Q)^2]^{1/2}} \right\} \end{aligned}$$

$$\} \cdot dx_Q dy_Q dz_Q , \quad (10.6)$$

where $\delta(Q) = \delta(x_Q, y_Q, z_Q)$. It is easy to show that

$$\frac{\partial^2}{\partial x_P^2} \frac{1}{[(x_P - x_Q)^2 + (y_P - y_Q)^2 + (z_P - z_Q)^2]^{1/2}} =$$

$$= \frac{1}{[(x_P - x_Q)^2 + (y_P - y_Q)^2 + (z_P - z_Q)^2]^{5/2}}$$

$$= \frac{1}{[(x_P - x_Q)^2 + (y_P - y_Q)^2 + (z_P - z_Q)^2]^{3/2}}$$

$$= 3 \frac{(x_P - x_Q)^2}{r_{QP}^5} - \frac{1}{r_{QP}^3} \quad . \quad (10.4a)$$

Similarly,

$$\frac{\partial^2}{\partial y_P^2} \frac{1}{r_{QP}} = 3 \frac{(y_P - y_Q)^2}{r_{QP}^5} - \frac{1}{r_{QP}^3} \quad (10.4b)$$

and

$$\frac{\partial^2}{\partial z_p^2} \frac{1}{r_{QP}} = 3 \frac{(z_p - z_Q)^2}{r_{QP}^5} - \frac{1}{r_{QP}^3}. \quad (10.4c)$$

9/34

By inserting (10.4a) - (10.4c) into (10.6), we obtain

$$\begin{aligned} \vec{\nabla}^2 V &= \frac{1}{4\pi\epsilon_0} \iiint_{\Sigma} \rho(Q) \left[3 \frac{(x_p - x_Q)^2 + (y_p - y_Q)^2 + (z_p - z_Q)^2}{r_{QP}^5} \right. \\ &\quad \left. - 3 \frac{1}{r_{QP}^3} \right] \cdot dx_Q dy_Q dz_Q \\ &= \frac{1}{4\pi\epsilon_0} \iiint_{\Sigma} \rho(Q) \left(3 \frac{r_{QP}^2}{r_{QP}^5} - 3 \frac{1}{r_{QP}^3} \right) \cdot dx_Q dy_Q dz_Q \\ &= \frac{1}{4\pi\epsilon_0} \iiint_{\Sigma} \rho(Q) \left(3 \cancel{\frac{1}{r_{QP}^3}} - 3 \cancel{\frac{1}{r_{QP}^3}} \right) \cdot dx_Q dy_Q dz_Q \\ &= 0, \quad r_{QP} \neq 0 \text{ and } P \notin \Sigma. \quad (10.8) \end{aligned}$$

This result is consistent with the homogeneous case of Poisson equation, i.e., Laplace equation. In fact, the absence of charges outside the region Σ necessarily implies that $\vec{\nabla}_p^2 V = 0$. The more complicated case inside the

region 2 is left to a more advanced course on electromagnetism. The interested reader can refer to two PDF files uploaded on LEARN, Poisson_Coulomb_Verification_01 and Poisson_Coulomb_Verification_02.

Note that the result of (10.8) can more easily be shown resorting to a spherical coordinate system. In such a system, defining a new variable $(\epsilon_p - \epsilon_\alpha) = \epsilon$ allows us to write (10.6) as

$$\begin{aligned} \vec{\nabla}_r^2 V &= \frac{1}{4\pi\epsilon_0} \iiint_{\Sigma} \delta(\epsilon_\alpha, \theta_\alpha, \varphi_\alpha) \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \frac{1}{r} \right) \\ &\quad \cdot r_\alpha^2 \sin\theta_\alpha \, d\epsilon_\alpha \, d\theta_\alpha \, d\varphi_\alpha \\ &= \frac{1}{4\pi\epsilon_0} \iiint_{\Sigma} \delta(\epsilon_\alpha, \theta_\alpha, \varphi_\alpha) \frac{1}{r^2} \frac{\partial}{\partial r} (-1) \\ &\quad \cdot r_\alpha^2 \sin\theta_\alpha \, d\epsilon_\alpha \, d\theta_\alpha \, d\varphi_\alpha \\ &= 0. \end{aligned}$$

This result obviously confirms (10.8).

Note also that, among all possible field potentials, that given by (10.5) is the only one to become zero at infinity.

We can summarize this section stressing the fact that given the complete knowledge of all source charges in the region of space of interest, the potential and, thus, the field can be determined directly by means of (10.5). However, it often happens that not all source charges are known, which means $S(Q)$ is not completely known in the region of interest. In these more complex cases, having in hand a differential form for the potential (and field) equations, Eqs. (10.3) and (10.4), makes it easier to find a solution. We will soon come back to this problem.

10.2 Energy of a system of point-like charges.

In this section, we intend to define and evaluate the electrostatic energy associated with a system of point-like charges in vacuum.

We first consider the simple case of two charges q_1 and q_2 in vacuum. The charges are located at points Q_1 and Q_2 , respectively, at a distance r_{12} from each other (cf. Fig. 10.1). From the previous sections, we know that if q_1 is maintained fixed at Q_1 while q_2 is moved from a point Q'_2 to Q_2 by means of a quasi-static

transformation, the work generated by the field forces to perform the displacement is independent from the path followed and is given by

$$W = q_2 [V_1(Q_2) - V_1(Q'_2)].$$

This means that when modifying a given initial configuration of two charges by displacing one or both of the charges, the work generated by the field forces depends only on the final charge configuration and not on the specific way the displacement brought from one to the other configuration. This is conceptually similar to the definition in thermodynamics of the internal energy of a physical system as a state function. In fact, also in the electrostatic case we can define an energy that depends only on the "state" of the system (i.e., on the configuration of the two charges) and that, with its variations, gives the work generated by the system when going from one to another configuration.

To uniquely define the energy associated with the state of a system we must choose a reference state that, by convention, has zero energy. In the electrostatic case it seems natural to choose such a state as the state

from infinity to the configuration being studied. 14/34
 Indicating with U_e the electrostatic energy associated with the system, we find

$$\begin{aligned} U_e &= q_2 [V_1(Q_2) - V_1(\infty)] \\ &= q_2 V_1(Q_2) \\ &= \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r_{12}}, \quad (10.9) \end{aligned}$$

where we assumed normal conditions at infinity. Note that when the two charges have opposite sign, they tend to bond to each other and, thus, their energy is negative.
 The energy (10.9) can readily be generalized to the case of any system of point-like charges. Given, for example, three charges q_1 , q_2 , and q_3 initially located at infinity. We move q_1 to point Q_1 . Since at this stage the charges do not interact, this movement requires zero work.
 We then move q_2 to Q_2 , while keeping q_1 fixed at Q_1 . At this stage, q_2 "feels" the interaction with q_1 and, thus, the work done against the interaction forces is given by:

$$W_2 = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r_{12}} \quad . \quad (10.10a)$$

At last, we move q_3 to Q_3 , while keeping both q_1 and q_2 fixed at Q_1 and Q_2 , respectively. At this stage, q_3 feels the interaction with both q_1 and q_2 and, thus, we must perform a work

$$W_3 = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_3}{r_{13}} + \frac{1}{4\pi\epsilon_0} \frac{q_2 q_3}{r_{23}} , \quad (10.10b)$$

where r_{13} and r_{23} are the distances between q_1 and q_3 and q_2 and q_3 , respectively. We can thus conclude that the energy associated with the given configuration of three charges q_1 , q_2 , and q_3 at points Q_1 , Q_2 , and Q_3 is given by

$$U_e = \frac{1}{4\pi\epsilon_0} \left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} \right) , \quad (10.11)$$

i.e., the sum of works (10.10a) and (10.10b). It is worth noting that the energy U_e is independent from the specific way in which the charges have been moved from infinity to the final configuration.

The energy (10.11) can be rewritten in a form easier to generalize noting that, for each value of the indexes, $r_{ij} = r_{ji}$. Hence,

$$\begin{aligned} U_e &= \frac{1}{4\pi\epsilon_0} \left[\frac{1}{2} \left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} \right. \right. \\ &\quad \left. \left. + \frac{q_2 q_1}{r_{21}} + \frac{q_3 q_1}{r_{31}} + \frac{q_3 q_2}{r_{32}} \right) \right] \\ &= \frac{1}{2} \frac{1}{4\pi\epsilon_0} \sum_{i=1}^3 \sum_{\substack{j=1 \\ (j \neq i)}}^3 \frac{q_i q_j}{r_{ij}}, \end{aligned} \quad (10.12)$$

where, as always, r_{ij} is the distance between q_i and q_j . In summary, the energy associated with a system of N point-like charges q_1, q_2, \dots, q_N located at points Q_1, Q_2, \dots, Q_N at distances r_{ij} from each other is given by:

$$U_e = \frac{1}{2} \frac{1}{4\pi\epsilon_0} \sum_{i=1}^N \sum_{\substack{j=1 \\ (j \neq i)}}^N \frac{q_i q_j}{r_{ij}}. \quad (10.13)$$

This result can be written in a different form using the concept of potential. In fact, in (10.13) appears a term

$$V_i = \frac{1}{4\pi\epsilon_0} \sum_{\substack{j=1 \\ (j \neq i)}}^N \frac{q_j}{r_{ij}},$$

which corresponds to the value of the potential generated at point P_i from all charges, excluded q_i . We can thus rewrite (10.13) as

$$U_e = \frac{1}{2} \sum_{i=1}^N q_i V_i . \quad (10.14)$$

The potential V_i to be used in each term of the sum (10.14) is that produced by all charges excluded the i -th. In fact, if we attempted to calculate V_i taking into account also for the charge q_i , we would find meaningless results because the contribution to the potential of a point-like charge diverges at the point occupied by the charge itself.

We finally observe that, if we wanted to choose by convention a nonzero value of the potential at infinity, the work W would be given by $V(P) - V_\infty$ (still having the same value as before) and (10.14) would assume the equivalent form

$$U_e = \frac{1}{2} \sum_{i=1}^N q_i (V_i - V_\infty)$$

$$= \frac{1}{2} \sum_{i=1}^N q_i V_i - \frac{1}{2} V_\infty \sum_{i=1}^N q_i .$$

We now seek an expression for the electrostatic energy associated with a continuous charge distribution with volume density ρ (in general, variable from point to point). We assume to reach the desired charge configuration starting from a situation where the entire charge is divided in many small point-like charges dq , which can be treated as infinitesimal, located at a very large distance from each other so that we can neglect their interactions. We then assume to bring one by one the charges dq to the various points, distributing them in the volume elements $d\tau$, until reaching the desired value of ρ at each point. As always, we assume that the charges have moved very slowly, so that we can neglect their kinetic energy.

Indicating with dU_e the infinitesimal work required to move the infinitesimal increment of charge

dq' to the generic point P , when the potential at 19/34
 P has the value $V'(P)$, we obtain

$$\begin{aligned} dU_e &= dq' \cdot V'(P) \\ &= d\delta' \cdot dz \cdot V'(P), \quad (10.15) \end{aligned}$$

where $d\delta'$ is the infinitesimal increment of density at P and
 dz the infinitesimal volume element centered at P .

Since the work required to build the charge distribution
is independent from the order and mode in which the
various contributions are brought together, we can assume
to move the various parts of the system to the final value
of their electrical charge simultaneously. In other words,
we assume that at each instant of time during the charging
process, the charge density δ' at each point is the same
fraction α of the final density δ :

$$\delta' = \alpha \delta.$$

The infinitesimal increment of charge density is, thus,

$$d\delta' = \delta \cdot d\alpha$$

and (10.15) can be rewritten as

$$dU_e = S(P) V'(P) \cdot d\lambda d\sigma . \quad (10.16)$$

Being all charges equal to the same fraction λ of their final value, also the potential $V'(P)$ will be a fraction λ of the final value $V(P)$:

$$V'(P) = \lambda V(P) ,$$

and, thus, (10.16) becomes

$$dU_e = \lambda S(P) V(P) \cdot d\lambda d\sigma . \quad (10.17)$$

The total electrostatic energy is obtained by summing all terms of the type (10.17) and by varying λ from 0 to 1:

$$\begin{aligned} U_e &= \int_0^1 \lambda \left(\iiint_S V \cdot d\sigma \right) \cdot d\lambda \\ &= \frac{1}{2} \iiint_S V \cdot d\sigma . \quad (10.18) \end{aligned}$$

This result is of the same form as (10.14), with the difference that the operation of integration on a continuum

has substituted the operation of sum. Note that the 21/34
 integral in (10.18) can be extended to any arbitrarily
 large region that contains Σ , so long $\delta = 0$ outside Σ .

Note that in (10.18) there is no need to specify how to calculate the potential. In fact, it is not necessary to exclude the contribution of the charge localized in the point where the potential is considered, because such a contribution is infinitesimally small.

Similar to (10.18), we can write the electrostatic energy associated with a surface or linear charge distributions:

$$U_e = \frac{1}{2} \sum \iint \sigma V \cdot dS \quad (10.19)$$

and

$$U_e = \frac{1}{2} \int \lambda V \cdot dl, \quad (10.20)$$

respectively. Note that (10.20) has physical meaning only when the "wire" on which the charge is distributed has a small diameter, but not zero.

10.3 Energy density of an electrostatic field.

22/34

Consider a volume charge distribution with density ρ in a domain Σ . The electrostatic energy associated with such a system is given by

$$U_e = \frac{1}{2} \iiint_{\Sigma} \rho V \cdot d\tau .$$

Remembering that

$$\operatorname{div} \vec{E} = \frac{1}{\epsilon_0} \rho ,$$

we have

$$\rho V = \epsilon_0 V \operatorname{div} \vec{E} .$$

By means of the vector identity

$$\operatorname{div}(f \vec{A}) = f \operatorname{div} \vec{A} + \vec{A} \cdot \operatorname{grad} f ,$$

where f and \vec{A} are a generic scalar and vector field, respectively, we have

$$\rho V = \epsilon_0 V \operatorname{div} \vec{E}$$

$$= \epsilon_0 \operatorname{div}(\nabla \vec{E}) - \epsilon_0 \vec{E} \cdot \operatorname{grad} V.$$

It is also

$$\vec{E} = -\operatorname{grad} V$$

and, thus,

$$\begin{aligned} \nabla V &= \epsilon_0 \operatorname{div}(\nabla \vec{E}) + \epsilon_0 \vec{E} \cdot \vec{E} \\ &= \epsilon_0 \operatorname{div}(\nabla \vec{E}) + \epsilon_0 E^2 \end{aligned}$$

Equation (10.18) then becomes

$$V_e = \frac{1}{2} \epsilon_0 \iiint_{\Sigma} \operatorname{div}(\nabla \vec{E}) \cdot d\tau + \frac{1}{2} \iiint_{\Sigma} \epsilon_0 E^2 \cdot d\tau.$$

From the divergence theorem it follows that

$$\iiint_{\Sigma} \operatorname{div}(\nabla \vec{E}) \cdot d\tau = \sum \oint \vec{E} \cdot \vec{n} dS,$$

where Σ is the closed surface enclosing the domain Σ where the charges are located. Furthermore, if the charge distribution producing the field is localized

entirely in a finite region of space (all at a finite distance from a given point of reference), the domain Σ and the surface Σ , which we assume to be, e.g., spherical, can be chosen so large to result in a zero surface integral [remember that the integral (10.18) can be extended to any region containing Σ , so long $\delta=0$ outside Σ]. In fact, in the limit the radius r of Σ goes to infinity, the potential V goes to zero at least as r^{-1} and the field E as r^{-2} . Hence, the product EV goes to zero at least as r^{-3} , while the area of Σ increases as r^2 . As a consequence, the surface integral goes to zero at least as r^{-1} and the electrostatic energy V_e can be written as

$$V_e = \frac{1}{2} \iiint_{\Sigma_\infty} \epsilon_0 E^2 \cdot d\Sigma , \quad (10.21)$$

so long the volume integral is extended to the entire space (indicated by Σ_∞). This result shows that the energy V_e can be calculated even without knowing the charge distribution, as long as the field distribution is known everywhere. Moreover, (10.21) can be formally

interpreted stating that the energy V_e is distributed 25/34 in the entire space with a volume energy density (or a specific energy, energy per unit volume)

$$u_e = \frac{dV_e}{dz} \\ = \frac{1}{2} \epsilon_0 E^2 \quad . \quad (10.22)$$

In this way, we can attach another physical meaning to the field, which appears to be the entity where the electrostatic energy of the system of charges is "stored."

As a final remark, it is worth stressing the limitations of applicability of (10.21). Because of the way it has been derived (using, among others, the relation $\operatorname{div} \vec{E} = \rho/\epsilon_0$), (10.21) cannot be used in the case of point-like and linearly distributed charges. For example, using (10.21) in the case of a point-like charge, we have

$$E = \frac{1}{4\pi\epsilon_0} \frac{q}{r},$$

which, integrating over spherical shells gives

$$V_e = \lim_{R \rightarrow \infty} \frac{1}{2} \int_0^R \epsilon_0 \left(\frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \right)^2 4\pi r^2 dr \rightarrow \infty.$$

This result is obviously in contrast with the fact 26/34
that the energy U_e , being defined as energy of interaction,
must be zero in the case of a single point-like charge
at finite.

10.4 Multipole series expansion of the electrostatic potential.

We now intend to show a powerful method that makes it possible to calculate the approximate potential generated by a given charge distribution. This method, known as multipole series expansion, makes it possible to introduce in a transparent fashion specific type of charge distributions, such as dipoles and multipoles, which are of fundamental interest in many sectors of atomic physics, quantum optics, and in the study of insulating materials. Even circuit quantum electrodynamics, which studies the interaction between superconducting quantum bits (qubits) with a microwave electric field, can be reduced to electric dipole in a field.

Consider a generic charge distribution in a limited region of space, so that it can be entirely contained within a sphere of radius a (cf. Fig. 10.2). We intend to study the effects of such a charge distribution at a large distance from the charge (i.e., at points P the distance of

which from the origin O in Fig. 10.2 is much larger [27/34]
than a).

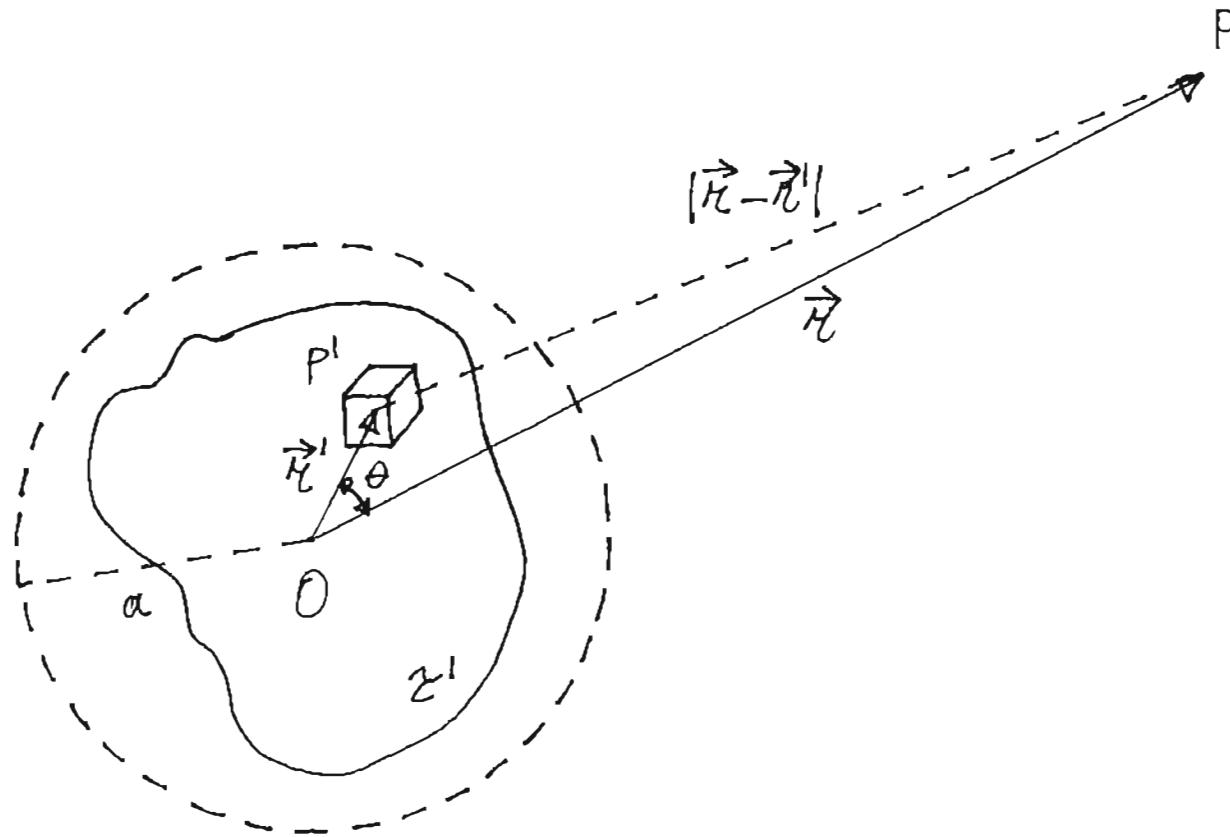


Figure 10.2.

We indicate with \vec{r}' the vector from the origin O to the point P' within the region Σ' occupied by the charge, with $S(\vec{r}')$ the volume charge density at P' , and with \vec{r} the vector from O to the generic observation point P .

The potential at P is given by

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \iiint_{\Sigma'} \frac{S(\vec{r}')}{|\vec{r} - \vec{r}'|} d\Sigma' , \quad (10.23)$$

where $d\mathcal{V}^1$ is an infinitesimal volume element 28/34
 in Σ^1 and $|\vec{r} - \vec{r}'|$ indicates the distance between P and
 P' , which, by means of Carnot's theorem, is given by

$$\begin{aligned} r_{pp'} &= |\vec{r} - \vec{r}'| \\ &= (r^2 + r'^2 - 2rr' \cos\theta)^{1/2}, \quad (10.24) \end{aligned}$$

where r and r' represent the absolute values of vector \vec{r}
 and \vec{r}' , respectively, and θ the angle between them. Since
 we are interested in the values of the potential at points
 very far from the origin, we can assume $r'/r \ll 1$ and,
 thus, we can expand the expression $|\vec{r} - \vec{r}'|$ in power series
 of r'/r limited to the first terms. Noting that

$$\begin{aligned} |\vec{r} - \vec{r}'|^{-1} &= (r^2 + r'^2 - 2rr' \cos\theta)^{-1/2} \\ &= (r^2)^{-1/2} \left(\frac{r^2 + r'^2 - 2rr' \cos\theta}{r^2} \right)^{-1/2} \\ &= \frac{1}{r} \left[1 + \left(\frac{r'^2}{r^2} - 2 \frac{r'}{r} \cos\theta \right) \right]^{-1/2} \end{aligned}$$

and defining

$$\xi = \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\theta ,$$

We can write

$$|\vec{r} - \vec{r}'|^{-1} = \frac{1}{r} (1 + \xi)^{-1/2} .$$

By expanding in MacLaurin series, we then obtain

$$(1 + \xi)^{-1/2} = 1 - \frac{1}{2}\xi + \frac{3}{8}\xi^2 + \dots$$

and, thus,

$$|\vec{r} - \vec{r}'|^{-1} = \frac{1}{r} \left[1 + \left(\frac{r'}{r}\right) \cos\theta - \frac{1}{2} \left(\frac{r'}{r}\right)^2 \right. \\ \left. + \frac{3}{8} \cdot 4 \left(\frac{r'}{r}\right)^2 \cos^2\theta \right. \\ \left. - \frac{3}{8} \cdot 2 \cdot 2 \left(\frac{r'}{r}\right)^2 \left(\frac{r'}{r}\right) \cos\theta \right. \\ \left. + \frac{3}{8} \left(\frac{r'}{r}\right)^4 + \dots \right]$$

$$\begin{aligned}
 &= -\frac{1}{r} \left[1 + \left(\frac{r^1}{r} \right) \cos \theta \right. \\
 &\quad \left. + \left(\frac{r^1}{r} \right)^2 \frac{3 \cos^2 \theta - 1}{2} + \dots \right] \\
 &= -\frac{1}{r} + \frac{r^1}{r^2} \cos \theta + \frac{r^{12}}{r^3} \frac{3 \cos^2 \theta - 1}{2} \\
 &\quad + \dots \\
 &= \frac{1}{r} + \frac{\vec{r}^1 \cdot \vec{r}}{r^3} + \frac{3(\vec{r}^1 \cdot \vec{r})^2}{245} - \frac{r^{12}}{243} + \dots
 \end{aligned}$$

By substituting this expansion into (10.23) and, for example, stopping at the first two terms, we obtain

$$V(\vec{r}) \approx \frac{1}{4\pi\epsilon_0} \iiint_{\vec{r}^1} S(\vec{r}^1) \left(\frac{1}{r} + \frac{\vec{r}^1 \cdot \vec{r}}{r^3} \right) \cdot d\vec{r}^1. \quad (10.25)$$

Since the vector \vec{r} is independent from the variable of integration \vec{r}^1 , we obtain

$$V(\vec{r}) \approx \frac{1}{4\pi\epsilon_0} \left[\frac{1}{r} \iiint_{\vec{r}^1} S(\vec{r}^1) \cdot d\vec{r}^1 + \frac{\vec{r}}{r^3} \cdot \iiint_{\vec{r}^1} S(\vec{r}^1) \vec{r}^1 \cdot d\vec{r}^1 \right]. \quad (10.26)$$

The first term in this expression corresponds to the potential that would be generated at P if the entire charge in region \mathcal{V}^1 ,

$$Q = \iiint_{\mathcal{V}^1} S(\vec{r}^1) \cdot d\mathcal{V}^1,$$

were concentrated at O. It is evident that this should be the first approximation term of the potential "seen" by a distant observer from the charge.

The second term in (10.26) gives a contribution that varies with the inverse distance squared. In this term appears a vector quantity

$$\vec{p} = \iiint_{\mathcal{V}^1} S(\vec{r}^1) \vec{r}^1 \cdot d\mathcal{V}^1, \quad (10.24)$$

which is called dipole moment of the charge distribution. The reason to call this term a "moment" is because the function S is multiplied times vector \vec{r}^1 . The dipole moment accounts for the specific distribution of the charge in \mathcal{V}^1 . This goes beyond what the only charge Q can account for. We can thus rewrite (10.26) as

$$V(\vec{r}) \approx \frac{1}{4\pi\epsilon_0} \left(\frac{Q}{r} + \frac{\vec{r} \cdot \vec{p}}{r^3} \right) ; \quad (10.28)$$

the two terms in (10.28) are called the term of monopole and dipole, respectively.

Note that the monopole term can be zero. This happens when the charge distribution is made up by the same amount of positive and negative charge. However, even if $Q = 0$, the simple approximation (10.28) shows that the electrostatic effects are nonzero. This is due to the presence of the dipole term, which can be different than zero.

We now focus on the dipole term \vec{p} defined by (10.24). When considering systems with zero total charge Q , the dipole moment can be calculated assuming the entire charge concentrated in the system center of charge. Similar to the center of mass, the center of charge is defined by

$$\vec{r}_c = \frac{\iiint_{z'} s(\vec{r}') \vec{r}' \cdot dz'}{\iiint_{z'} s(\vec{r}') \cdot dz'},$$

where \vec{r}_c^1 is the vector distance of the center of charge from the origin. As a consequence,

$$\vec{r}_c^1 Q = \iiint_{z^1} S(\vec{r}') \vec{r}' \cdot d\vec{z}' .$$

Hence, the dipole moment can be calculated by concentrating the charge Q in the center of charge and multiplying it by \vec{r}_c^1 .

In general, the dipole moment depends on the point with respect to which is calculated (i.e., the origin of the vectors \vec{r}'). In particular, it is zero the dipole moment with respect to the center of charge \vec{r}_c^1 . In the next lecture, we will see that, for systems of zero total charge, the dipole moment is independent from the point with respect to which is calculated: In those cases, the dipole moment is an intrinsic property of the system itself.

We note again that (10.26) is an approximation of (10.23) limited to the first two terms only of the series expansion of (10.23). Retaining higher order terms, called multipole terms, we would have

$$\nabla(\vec{V}) = \frac{1}{4\pi\epsilon_0} \left(\frac{k_0}{r} + \frac{k_1}{r^2} + \frac{k_2}{r^3} + \dots \right) , \quad (10.20)$$

where

$$k_0 = \iiint_{\Sigma^1} g(\vec{r}^1) \cdot d\vec{z}^1 ,$$

$$k_1 = \left[\iiint_{\Sigma^1} g(\vec{r}^1) \vec{r}^1 \cdot d\vec{z}^1 \right] \cdot \frac{\vec{r}}{r} ,$$

$$k_2 = \iiint_{\Sigma^1} g(\vec{r}^1) \left[\frac{3}{2} \left(\frac{\vec{r}^1 \cdot \vec{r}}{r} \right)^2 - \frac{r^{12}}{2} \right] \cdot d\vec{z}^1 , \dots$$

The third term contains the quadrupole moment of the distribution, the fourth term the octupole moment, and so forth. In the following, we will only use the first two moments Q and \vec{p} . Note that higher moments play an important role, e.g., in nuclear physics. However, it is important to remember that when the first two terms are zero, the potential at large distances is mostly given by the quadrupole moment. When the quadrupole moment is also zero, by the octupole moment, and so on. The series (10.29) shows that, for a limited charge distribution in space, the potential at infinity goes to zero at least as the inverse distance.

Phys 242 - 914. Lecture 11 - The electrostatic potential / part 4. 1/28

Summary lecture 10.

- Poisson and Laplace equations.
 - In a region of space with a charge distribution ρ ,

$$\vec{\nabla}^2 V = -\frac{1}{\epsilon_0} \rho .$$

This is called Poisson equation.

- In absence of any charge distribution,

$$\vec{\nabla}^2 V = 0 .$$

This equation is the homogeneous case of Poisson equation and is called Laplace equation.

- Energy of a system of point-like charges.
 - For two charges q_1 and q_2 :

$$U_e = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r_{12}}$$

- For N charges:

$$U_e = \frac{1}{2} \frac{1}{4\pi\epsilon_0} \sum_{i=1}^N \sum_{\substack{j=1 \\ (j \neq i)}}^N \frac{q_i q_j}{r_{ij}}$$

$$= \frac{1}{2} \sum_{i=1}^N q_i V_i ,$$

where

$$V_i = \frac{1}{4\pi\epsilon_0} \sum_{\substack{j=1 \\ (j \neq i)}}^N \frac{q_j}{r_{ij}}$$

- For a continuous charge distribution S in a region contained in a domain Σ ,

$$U_e = \frac{1}{2} \iiint_S V \cdot d\Sigma ,$$

where V is the potential at any point in the domain Σ , generated by S . U_e , in general, varies from point to point in Σ .

An important example is that of a 3/28
 single point-like charge q at a point P_0 in
 space. Assuming P_0 to be the origin of a given
 coordinate system,

$$\delta(P) = \delta(\vec{R}) = q \delta(P) = q \delta(\vec{R}).$$

The energy V_e of a spheric domain with radius R and center in q is given by

$$V_e = \frac{1}{2} \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta \cdot d\theta \int_0^R r^2 q \delta(r)$$

$$= \frac{q}{R} \cdot dr$$

$$= 2\pi q^2 \int_0^R r \delta(r) \cdot dr = 2\pi q^2 [r]_0^R = 0.$$

where we used the delta-Dirac property
 $\int_S f(P) \delta(P) \cdot d\mathcal{L} = f(0)$, if 0 is a point in S . In our
 Σ

case the origin is part of the spheric domain of
 integration where V_e is evaluated. The result
 above has to be expected since V_e is defined as
 an interaction energy and, thus, must be zero in

the case of a single charge.

- Energy density of \vec{E} .

The energy U_e in the entire space can be calculated as

$$U_e = \frac{1}{2} \iiint_{\mathbb{R}^3} \epsilon_0 E^2 \cdot d\tau$$

and, thus,

$$u_e = \frac{1}{2} \epsilon_0 E^2$$

is the energy density of E .

Note that calculating U_e from E^2 instead of SV , e.g., in the case of a single point-like charge would result in an infinite energy, in stark contrast with the energy we found earlier, $U_e = 0$. In fact, the $\sim E^2$ form of U_e cannot be used in the case of point-like charges. This is because of the many vector calculus assumptions made to find the $\sim E^2$ form.

Multipoles.

The potential V due to a charge distribution δ in a domain Σ' can be approximated by

$$V(\vec{r}) \approx \frac{1}{4\pi\epsilon_0} \left(\frac{Q}{r} + \frac{\vec{r} \cdot \vec{p}}{r^3} \right) ,$$

where

$$Q = \iiint_{\Sigma'} \delta(\vec{r}') \cdot d\vec{r}'$$

is associated with the monopole term in V , and

$$\vec{p} = \iiint_{\Sigma'} \delta(\vec{r}') \vec{r}' \cdot d\vec{r}'$$

is associated with the dipole term in V .

11.1 The electrostatic dipole.

6/28

The dipole moment of a given charge distribution,

$$\vec{p} = \iiint_{z^1} s(\vec{r}') \vec{r}' \cdot d\vec{z}' ,$$

depends, in general, from the origin chosen to measure distances. It does not depend on the origin when the total charge of the distribution is zero. In this case \vec{p} is an intrinsic characteristic of the system. This can be demonstrated by displacing the origin O by a vector \vec{R} . The new dipole moment is given by

$$\begin{aligned} \vec{p}' &= \iiint_{z^1} (\vec{r}' - \vec{R}) s(\vec{r}') \cdot d\vec{z}' \\ &= \iiint_{z^1} \vec{r}' s(\vec{r}') \cdot d\vec{z}' - Q \vec{R} = \vec{p} - Q \vec{R}, \quad (11.1) \end{aligned}$$

where, as always,

$$Q = \iiint_{z^1} s(\vec{r}') \cdot d\vec{z}' .$$

It is obvious that when $Q=0$, from (11.1) follows that

7/28

$$\vec{p}' = \vec{p} \quad . \quad (11.2)$$

A particularly simple charge distribution that satisfies the condition $Q=0$ is that of two equal point-like charges of opposite sign, and located at a distance d from each other.

The dipole moment for the case of the two charges can be calculated choosing any origin. For example, we can choose the origin to coincide with the point occupied by the negative charge. The dipole moment can be thus calculated straightforwardly and is given by

$$\begin{aligned}\vec{p} &= \iiint_{\mathcal{V}'} [-q \delta(\vec{r}') \vec{r}' + q \delta(\vec{r}' - \vec{d}) \vec{r}'] \cdot d\vec{r}' \\ &= -q [\vec{r}']_o + q [\vec{r}']_{\vec{d}} = q \vec{d}, \quad (11.3)\end{aligned}$$

where \vec{d} is the vector distance between the negative and positive charge. In other words, the dipole moment for the two charges is a vector having : a) the direction of

the line connecting the two charges; b) orientation [8/28] from the negative to the positive charge; c) absolute value given by the product of the positive charge times the distance between the charges. In first approximation, the potential generated at a generic point in space, at a distance much larger than the separation between the two charges, d , is given by (10.28) for $Q=0$:

$$V(\vec{r}) \approx \frac{1}{4\pi\epsilon_0} \frac{\vec{r} \cdot \vec{p}}{r^3} . \quad (11.4)$$

The electric field can be obtained from (11.4) knowing that

$$\vec{E} = -\text{grad } V$$

Choosing as the origin of the coordinates the position occupied by the negative charge and indicating with \vec{r} the vector from the origin to a point P where the field is evaluated, we obtain

$$\vec{E}(\vec{r}) \approx \frac{1}{4\pi\epsilon_0} \left[\frac{3(\vec{r} \cdot \vec{p})\vec{r}}{r^5} - \frac{\vec{p}}{r^3} \right] . \quad (11.5)$$

The so obtained system is called ideal electrostatic dipole. The potential and field of an ideal dipole are given by (11.4) and (11.5), respectively, exactly:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{\vec{r} \cdot \vec{\mu}}{r^3}$$

and

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \left[\frac{3(\vec{r} \cdot \vec{\mu})\vec{r}}{r^5} - \frac{\vec{\mu}}{r^3} \right].$$

The relevance of the ideal dipole resides in the fact that often we are interested in systems of charges with zero global charge and for which only the effects at large distances compared to their dimensions are needed. Examples of such systems are atoms and molecules. These are globally neutral, however, they often have nonzero dipole moment or, under specific conditions, they acquire a dipole moment due to the action of external forces.

11.2 Interactions between an electrostatic dipole and field

When an ideal electrostatic dipole is in an electrostatic field, mechanical actions take place on the dipole. If we assume the two point-like charges in the dipole to be rigidly attached to each other, from a mechanical point of view, the system can be regarded as a rigid body. Hence, with respect to a reduction pole, the mechanical actions on the dipole can be completely described by means of a resulting force and a resulting moment (or effective force and moment). We intend to calculate both the effective force and moment due to the electrostatic field and the interaction energy between the dipole and the field.

- Effective force.

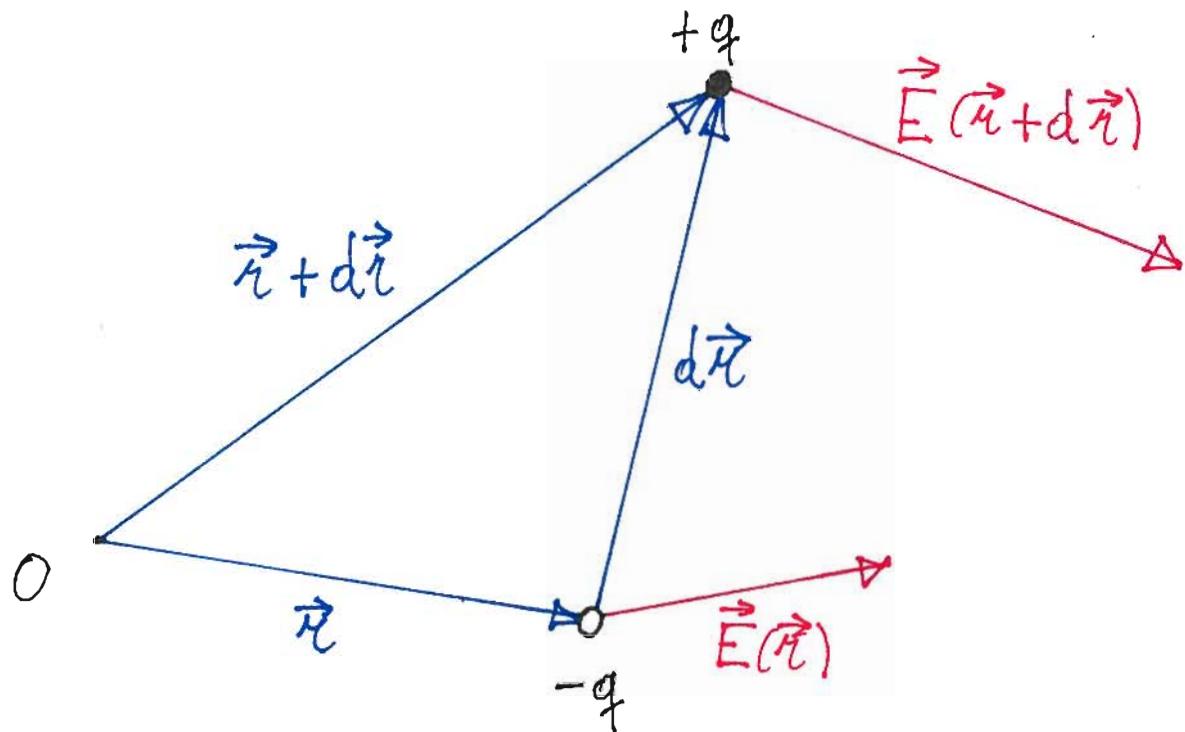


Figure 11.1.

In Fig. 11.1, \vec{r} indicates the vector distance between O and the point occupied by the negative charge $-q$. The value of the electrostatic field at that point is $\vec{E}(\vec{r})$. At the point occupied by the positive charge $+q$, if the distance d between the two charges is very small, i.e., the system tends to an ideal dipole, $d \rightarrow d\vec{r}$, the field is given by

$$\vec{E}' = \vec{E}(\vec{r}') = \vec{E}(\vec{r} + d\vec{r}) = \vec{E}(\vec{r}) + d\vec{E}, \quad (11.6)$$

where $d\vec{E}$ is the field variation corresponding to the infinitesimal displacement $d\vec{r}$ between the two charges.

In orthogonal cartesian coordinates, the vector $d\vec{r}$ is given by its components dx , dy , and dz ,

$$d\vec{r} = dx \cdot \vec{u}_x + dy \cdot \vec{u}_y + dz \cdot \vec{u}_z,$$

while $d\vec{E}$, differential of a vector, is given by

$$d\vec{E} = dE_x \cdot \vec{u}_x + dE_y \cdot \vec{u}_y + dE_z \cdot \vec{u}_z,$$

where dE_x , dE_y , and dE_z are the infinitesimal variation of the components of \vec{E} associated with the infinitesimal displacement $d\vec{r}$. We obtain

$$dE_x = \frac{\partial}{\partial x} E_x \cdot dx + \frac{\partial}{\partial y} E_x \cdot dy + \frac{\partial}{\partial z} E_x \cdot dz$$

$$dE_y = \frac{\partial}{\partial x} E_y \cdot dx + \frac{\partial}{\partial y} E_y \cdot dy + \frac{\partial}{\partial z} E_y \cdot dz, \quad (11.7)$$

$$dE_z = \frac{\partial}{\partial x} E_z \cdot dx + \frac{\partial}{\partial y} E_z \cdot dy + \frac{\partial}{\partial z} E_z \cdot dz$$

which, in matrix form, is

$$\begin{aligned} \begin{bmatrix} dE_x \\ dE_y \\ dE_z \end{bmatrix} &= \begin{bmatrix} \frac{\partial}{\partial x} E_x & \frac{\partial}{\partial y} E_x & \frac{\partial}{\partial z} E_x \\ \frac{\partial}{\partial x} E_y & \frac{\partial}{\partial y} E_y & \frac{\partial}{\partial z} E_y \\ \frac{\partial}{\partial x} E_z & \frac{\partial}{\partial y} E_z & \frac{\partial}{\partial z} E_z \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} \\ &\quad . \end{aligned} \quad (11.8)$$

Before proceeding, it is worth noting the field here discussed is that in absence of the dipole. In fact, since we assumed the dipole to be a rigid body, the reciprocal

actions of the two charges in the dipole are ineffective from the mechanical point of view.

We will later show that the square matrix made with the partial derivatives of the components of \vec{E} is the set of the nine cartesian components of a second order tensor. Such a tensor is called the "gradient of the vector field \vec{E} ." We can thus write (11.8) in compact form as

$$d\vec{E} = (\text{grad } \vec{E}) \cdot d\vec{r}, \quad (11.9)$$

where the symbol " \cdot " indicates the scalar product between the tensor $\text{grad } \vec{E}$ and the vector $d\vec{r}$.

We can now find the effective force \vec{F} acting on the dipole. Indicating the force acting on the negative and positive charge as \vec{F}_- and \vec{F}_+ , respectively, we have

$$\vec{F} = \vec{F}_- + \vec{F}_+ = -q\vec{E} + q(\vec{E} + d\vec{E}) = qd\vec{E}, \quad (11.10)$$

which, in tensor notation, becomes

$$\vec{F} = q \cdot d\vec{r} \cdot \text{grad } \vec{E} = \vec{\mu} \cdot \text{grad } \vec{E},$$

where $\vec{p} = q \cdot d\vec{r}$ is the dipole moment.

- Effective moment.

We now calculate the effective moment due to all forces acting on the dipole. We can choose any arbitrary point as a pole, for example O in Fig. 11.1. The moment with respect to O is

$$\begin{aligned}
 \vec{C}_O &= \vec{r} \times \vec{F}_- + (\vec{r} + d\vec{r}) \times \vec{F}_+ \\
 &= -\vec{r} \times q \vec{E} + (\vec{r} + d\vec{r}) \times q(\vec{E} + d\vec{E}) \\
 &= q d\vec{r} \times \vec{E} + q \vec{r} \times d\vec{E} + q d\vec{r} \times d\vec{E} \\
 &= \vec{p} \times \vec{E} + \vec{r} \times \vec{F} + \vec{p} \times d\vec{E}, \quad (11.11)
 \end{aligned}$$

where $\vec{F} = \vec{F}_- + \vec{F}_+$. The last term in (11.11) is negligible with respect to the other terms, hence

$$\vec{C}_O = \vec{r} \times \vec{F} + \vec{p} \times \vec{E} \quad . \quad (11.12)$$

The first term in (11.12) is the moment of the effective force \vec{F} with respect to the pole O , while the second term depends uniquely on the dipole moment \vec{p} and the

value of \vec{E} at the point occupied by the dipole itself. If the reduction pole O is chosen to be at the position of the negative charge $-q$ (which coincides with the position of the ideal dipole when the distance between the charges goes to zero), the effective moment becomes

$$\vec{C} = \vec{p} \times \vec{E} \quad . \quad (11.13)$$

- In summary, the effective force acting on the dipole is different from zero only for a nonuniform field in the neighborhood occupied by the dipole. In fact, for a uniform field $d\vec{E} = \vec{0}$ and, from (11.10), $\vec{F} = \vec{0}$. For a uniform field, the system is described by \vec{C} of (11.13) only. In this case, if the dipole is free from any other mechanical action the dipole rotates until it aligns with the field, at which point $\vec{C} = \vec{0}$. There are two possible equilibrium positions for which $\vec{C} = \vec{0}$: one when \vec{p} and \vec{E} are parallel (same direction) and one when they are antiparallel (opposite directions). It can be shown that only the first position is stable (i.e., displacing the dipole by means of a small rotation, the dipole tends to

go back to its original position in the first case
and to go away from it in the second case). 18/18

- Field-dipole interaction energy.

We intend to calculate the electrostatic interaction energy between a dipole and a field \vec{E} . This energy is not the energy of the system of the two-point-like charges that form the dipole (which corresponds to the work required to build the dipole starting from two far away charges). In fact, we want to calculate the energy relative to the interaction between an already built dipole and all the other charges that generate the field acting on it. In other words, the energy corresponding to the work required to bring the dipole in a given configuration in \vec{E} from a configuration where the already built dipole did not feel the action of the field because it was far enough from the charges generating \vec{E} (also in this case, the reciprocal actions between the charges building the dipole are mechanically ineffective because the dipole is assumed to be a rigid body).

Assume V is the potential at the point occupied by the negative charge $-q$. Thus (cf. Fig. 11.1), the potential at the point occupied by the positive charge $+q$ is

$$V' = V + dV,$$

where dV is the variation of the potential due to the displacement $d\vec{r}$ (in the limit, zero). It follows that 19/28

$$V + dV = V + \text{grad } V \cdot d\vec{r}$$

and, thus, the electrostatic energy of the dipole in the field is

$$\begin{aligned} U_e &= U_{e^-} + U_{e^+} \\ &\stackrel{|}{=} -qV + q(V + dV) \\ &\stackrel{|}{=} q \cdot d\vec{r} \cdot \text{grad } V \\ &\stackrel{|}{=} -\vec{\mu} \cdot \vec{E} \quad . \quad (11.14) \end{aligned}$$

Note that U_e is not the energy of a system of two charges. In fact, U_{e^-} and U_{e^+} are the electrostatic potential energies of a single (negative or positive, respectively) charge in the potential generated by some other charge (no factor of $1/2$ anywhere!).

The result of Eq. (11.14), electric dipole interaction is of fundamental importance in atomic physics and quantum optics as it is at the basis of the Jaynes-Cummings hamiltonian, which governs the interaction between a two-level atom and a field.

11.3 Distributions of dipoles.

20/28

When considering a system of N dipoles, because of the superposition principle the potential and the field are given by the sum of the contribution from each single dipole.

As for the case of the charges, it is also possible to consider continuous distributions of dipoles. To this end, consider a region of space with volume Δz (very small with respect to the typical scale of experiments, but still finite) and imagine that the region contains a very large number N of quasi-ideal dipoles (i.e., dipole such that the distance a between charges of opposite sign is very small compared to the linear dimensions of Δz). The total dipole moment $\Delta \vec{p}$ in Δz is given by the vector sum of the dipoles \vec{p}_i of each dipole

$$\Delta \vec{p} = \sum_{i=1}^N \vec{p}_i .$$

To better understand the problem, assume all the dipoles are identical and oriented in the same way ($\vec{p}_i = \vec{p}$). Thus,

$$\Delta \vec{T} = N \vec{p} .$$

Introducing the dipole moment per unit volume,

$$\vec{P} = \frac{\Delta \vec{T}}{\Delta V} , \quad (11.15)$$

we readily obtain

$$\vec{P} = n \vec{p} , \quad (11.16)$$

where n is the number of dipoles per unit volume.

The definition of \vec{P} can be easily extended to the case where the dipoles are not identical or are differently oriented,

$$\vec{P} = \frac{1}{\Delta V} \sum_{i=1}^N \vec{p}_i . \quad (11.17)$$

In the case of the charges, a distribution of charges is characterized by a scalar function $\delta(Q)$, which represents the charge per unit volume. Similarly, a distribution of dipoles is characterized by a vector function $\vec{P}(Q)$ giving the dipole moment per unit volume.

22/28

at a generic point \mathbf{Q} . We can imagine the distribution of dipoles as a continuum characterized by the vector function $\vec{P}(\mathbf{Q})$.

We now want to calculate the potential and field generated by a given distribution of dipoles. For example, consider a region Ω where exists a continuous dipole distribution characterized by the function $\vec{P}(\mathbf{Q})$. We then consider a small volume $\Delta \mathcal{V}$ centered at a generic point \mathbf{Q}' and substitute the set of dipoles in $\Delta \mathcal{V}$ with a single dipole with moment

$$\Delta \vec{\Pi} = \vec{P}(\mathbf{Q}') \cdot \Delta \mathcal{V} .$$

From (11.4), we can find the potential at point \mathbf{Q} due this dipole,

$$\Delta V = \frac{1}{4\pi\epsilon_0} \frac{\Delta \vec{\Pi} \cdot (\vec{r} - \vec{r}')}{| \vec{r} - \vec{r}' |^3} .$$

Repeating similar arguments for each volume $\Delta \mathcal{V}$ in which the region Ω can be divided and superimposing the effects, we obtain the potential V at \mathbf{Q} for the entire dipole distribution.

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \iiint_{\Omega} \frac{\vec{P}(\vec{r}') \cdot (\vec{r} - \vec{r}')}{|r - r'|^3} d\vec{r}' . \quad (11.18)$$

11.3.1 Charge distributions equivalent to dipole distributions.

We want to show that a continuous distribution of dipoles is equivalent to a suitable charge distribution in the same region of the dipoles, as far as voltage and field are concerned.

It is easy to show that

$$\text{grad} \frac{1}{|\vec{r} - \vec{r}'|} = \text{grad} \frac{1}{|\vec{s}|} = \text{grad} \frac{1}{s} ,$$

where $|\vec{s}| = s > 0$. In spherical coordinates

$$\vec{s} = \vec{r} - \vec{r}' = s \cdot \vec{u}_r$$

and

$$\text{grad} \frac{1}{s} = -\frac{1}{s^2} \cdot \vec{u}_r = -\frac{1}{|\vec{s}|^2} \frac{\vec{s}}{|\vec{s}|}$$

$$= - \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}$$

and, thus,

$$\frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} = - \operatorname{grad} \frac{1}{|\vec{r} - \vec{r}'|}.$$

Note that the operator "grad" acts on the variable \vec{r} . Indicating with "grad'" the operator acting on the variable \vec{r}' , we obtain

$$\frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} = + \operatorname{grad}' \frac{1}{|\vec{r} - \vec{r}'|}. \quad (11.19)$$

By substituting (11.19) into (11.18), we obtain

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \iiint_{\Omega} \vec{P}(\vec{r}') \cdot \operatorname{grad}' \frac{1}{|\vec{r} - \vec{r}'|} \cdot d\vec{r}'.$$

From vector calculus

$$\vec{P}(\vec{r}') \cdot \operatorname{grad}' \frac{1}{|\vec{r} - \vec{r}'|} = \operatorname{div}' \left(\frac{\vec{P}(\vec{r}')}{|\vec{r} - \vec{r}'|} \right) d\vec{r}'$$

$$-\frac{1}{|\vec{r} - \vec{r}'|} \operatorname{div}' \vec{P}(\vec{r}')$$

and finally

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \iiint_{\Omega} \operatorname{div}' \left(\frac{\vec{P}(\vec{r}')}{|\vec{r} - \vec{r}'|} \right)$$

$$-\frac{1}{4\pi\epsilon_0} \iiint_{\Omega} \frac{1}{|\vec{r} - \vec{r}'|} \operatorname{div}' \vec{P}(\vec{r}') \cdot d\vec{r}'.$$

Using the divergence theorem for the first integral gives

$$V(\vec{r}) = -\frac{1}{4\pi\epsilon_0} \iint_{\Sigma} \frac{\vec{P}(\vec{r}') \cdot \vec{n}}{|\vec{r} - \vec{r}'|} dS$$

$$-\frac{1}{4\pi\epsilon_0} \iiint_{\Omega} \frac{\operatorname{div}'(\vec{P}(\vec{r}'))}{|\vec{r} - \vec{r}''|} \cdot d\vec{r}'' , \quad (11.20)$$

where Σ is the surface that delimits Ω . The two integrals in (11.20) correspond exactly to the

potentials produced by a surface charge distribution on Σ with density

$$\sigma_{eq} = \vec{P} \cdot \vec{n} = P_n \quad (11.21)$$

and by a volume charge distribution in Ω with density

$$\rho_{eq} = -\operatorname{div}^1 \vec{P} \quad (11.22)$$

We can thus state that given a closed surface Σ that contains the entire distribution of dipoles, the flux of \vec{E} through Σ is zero. This can be directly deduced from Gauss' theorem because the sum of the charges in each dipole is zero and, so, also the sum of all the charges of the dipoles in the entire distribution is zero. This conclusion is consistent with (11.21) and (11.22). In fact, evaluating the total charge equivalent to the distribution of dipoles in Ω , we find

$$Q_{tot} = \iint_{\Omega} \rho_{eq} \cdot d\tau + \iint_{\Sigma} \sigma_{eq} \cdot dS$$

$$= \iiint_{\Omega} (-\operatorname{div}^! \vec{P}) \cdot d\tau + \sum \iint \vec{P} \cdot \vec{n} dS,$$

where Ω is the region occupied by the distribution and \sum the surface that delimits it. Using the divergence theorem for the volume integral gives

$$Q_{\text{tot}} = - \sum \iint \vec{P} \cdot \vec{n} dS + \sum \iint \vec{P} \cdot \vec{n} dS = 0.$$

It is quite different the case of a surface \sum' that cuts some of the dipoles (cf. Fig. 11.2). In this case, the sum of the charges contained in \sum' is nonzero and, thus, the flux of \vec{E} through it is different from zero. This also explains how it is possible, given a set of dipoles (each of which has a zero total charge), to obtain an equivalent charge distribution with nonzero local density.

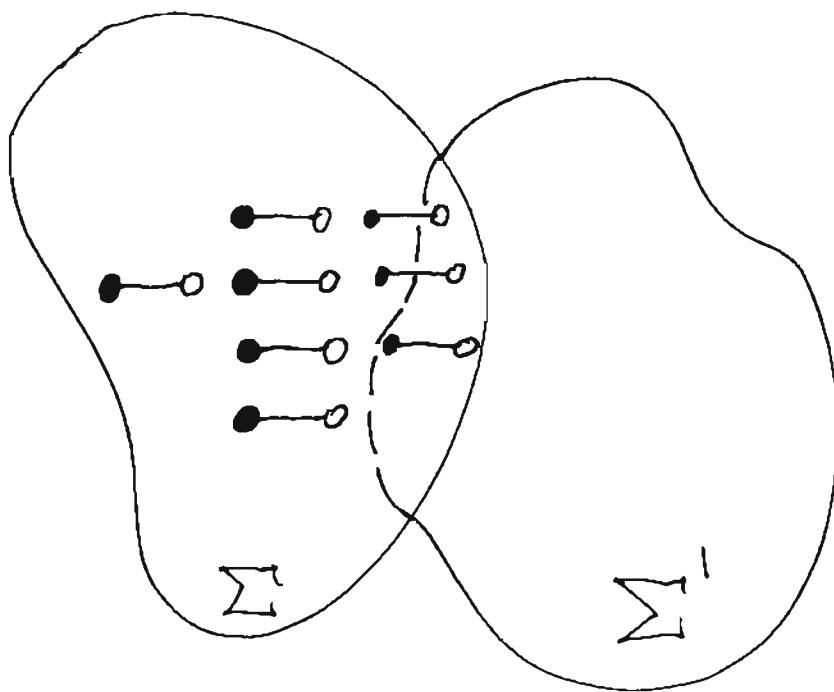


Figure 11.2.

Summary lecture 11.

- Electrostatic dipole.

- An electrostatic dipole is a system of two equal point-like charges of opposite sign and located at a distance d from each other. The total charge is $Q=0$.
- Potential at a distance much larger than d :

$$V(\vec{r}) \approx \frac{1}{4\pi\epsilon_0} \frac{\vec{e} \cdot \vec{p}}{r^3}$$

- Field at a large distance:

$$\vec{E}(\vec{r}) \approx \frac{1}{4\pi\epsilon_0} \left[\frac{3(\vec{r} \cdot \vec{p})\vec{r}}{r^5} - \frac{\vec{p}}{r^3} \right]$$

- Ideal dipole. The distance d is infinitesimally small and the charge q infinitesimally large, so that $qd = \text{const. (finite)}$. In this case, the potential and

field above are exact solutions. The only term [2] surviving in the multipole expansion is the term containing the dipole moment.

- Dipole - field interactions.

- Regarding an ideal electrostatic dipole as a rigid body, the effective force on the dipole in a field is

$$\vec{F} = \vec{\mu} \cdot \text{grad } \vec{E}$$

- Under the same rigid body assumption, the effective moment with respect to a pole O at distance \vec{r} from $(-q)$ and $(\vec{r} + d\vec{r})$ from $(+q)$ is

$$\vec{c}_o = \vec{r} \times \vec{F} + \vec{\mu} \times \vec{E}$$

- The field - dipole interaction energy is given by

$$U_e = -\vec{\mu} \cdot \vec{E}$$

- Distributions of dipoles.

A distribution of dipoles is characterized by a vector function

$$\vec{P}(Q) = \frac{1}{\Delta Z} \sum_{i=1}^N \vec{p}_i .$$

3/

- given a continuous dipole distribution with density \vec{P} and assuming \vec{r}' a dipole point (Q) and \vec{r} a potential point (Q'),

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \iiint_Q -\frac{\vec{P}(\vec{r}') \cdot (\vec{r} - \vec{r}')}{| \vec{r} - \vec{r}' |^3} dZ'.$$

- The potential of a continuous dipole distribution is equivalent to the potentials produced by a surface and a volume charge distribution

$$\sigma_{eq} = \vec{P} \cdot \vec{n}$$

and

$$\beta_{eq} = -\operatorname{div}' \vec{P} ,$$

respectively.

12.1 Electrostatic potential of a double infinite layer. [4]

In the mid-term exam, we studied two parallel infinite planes at distance d from each other. Assuming the left plane to be uniformly charged with a positive surface charge density $+\sigma$ and the plane on the right with a negative charge density $-\sigma$, we found that

$$\begin{cases} \vec{E} = \vec{0} , z \in (-\infty, 0) \text{ et } z \in (d, +\infty) \\ \vec{E} = \frac{\sigma}{\epsilon_0} \cdot \vec{u}_z , z \in [0, d] \end{cases}, \quad (12.1)$$

where z is an axis normal to both planes.

It is then easy to find the potential from

$$\begin{cases} \text{grad } V = 0 , z \in (-\infty, 0) \text{ et } z \in (d, +\infty) \\ \text{grad } V = -\frac{\sigma}{\epsilon_0} \cdot \vec{u}_z , z \in [0, d] \end{cases}. \quad (12.2)$$

which results in

$$\begin{cases} V = C_1 , z \in (-\infty, 0) \\ V = \int_0^z -\frac{\sigma}{\epsilon_0} \cdot dz = -\frac{\sigma}{\epsilon_0} z + C_2 , z \in [0, d] \end{cases}. \quad (12.3)$$

$$\left\{ \begin{array}{l} V = C_3 \\ z \in (d, +\infty) \end{array} \right.$$

5/

We can choose arbitrarily the point between 0 and d at which the potential is zero by choosing opportunely the arbitrary constant C_2 . Then, a suitable choice of the two arbitrary constants C_1 and C_3 makes it possible to plot the potential shown in Fig. 12.1 (the constants C_1 and C_3 can be chosen so that the potential is continuous from $-\infty$ to $+\infty$).

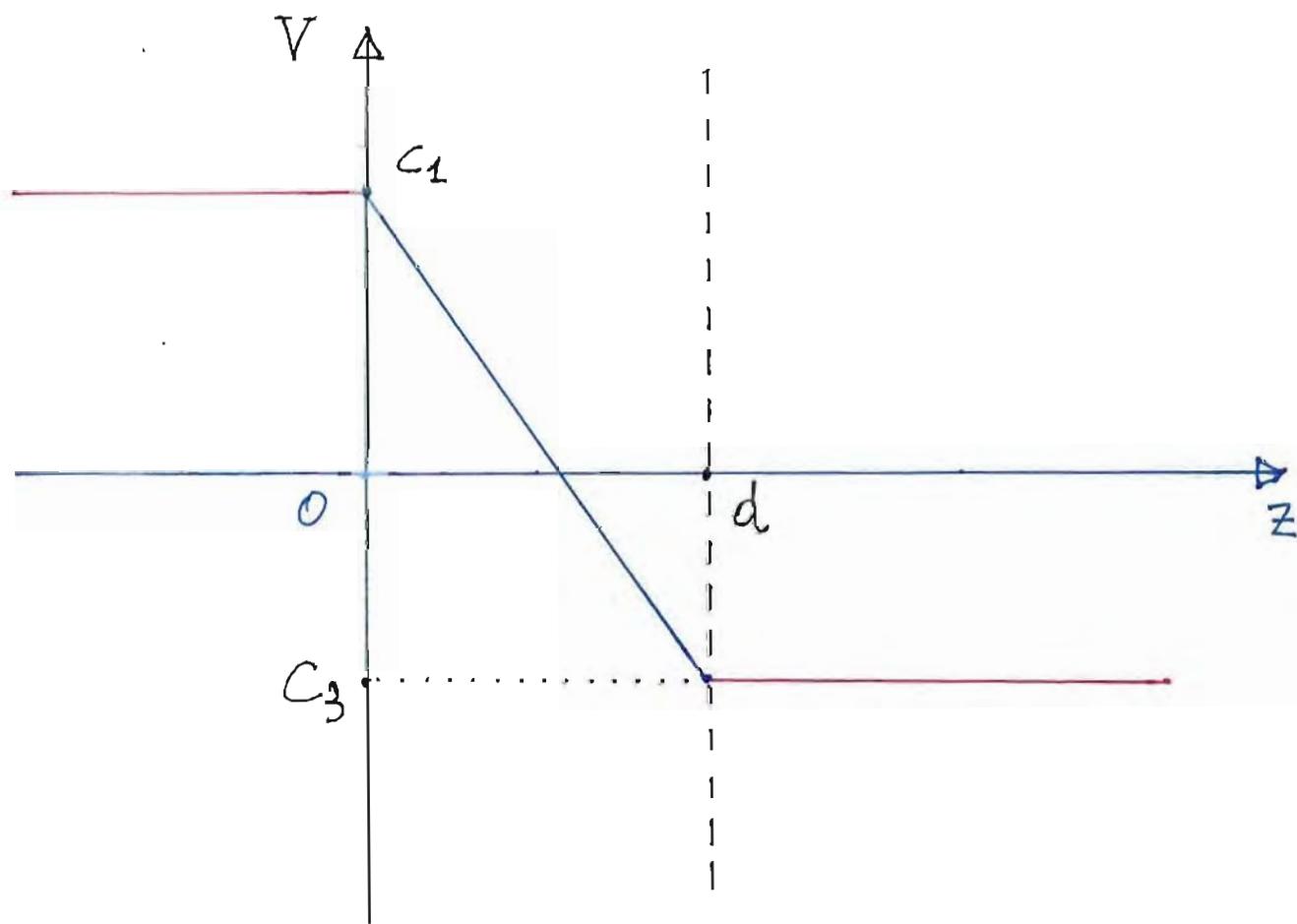


Figure 12.1.

Independently from the choice of the arbitrary constants, however, the potential difference between the two planes is given by [mid equation in (12.3)]

$$V(d) - V(0) = - \frac{\sigma}{\epsilon_0} d \quad . \quad (12.4)$$

This result shows an interesting property of the double layer. Imagine to decrease the distance d between the two layers, while increasing the value of the surface charge density σ so that $\sigma d = \text{const}$. Under these conditions, the potential difference between the two layers remains constant. It is then possible to conceive a scenario where the distance $d \rightarrow 0$ and the potential V has a true discontinuity in correspondence of the distribution of charges. This property is typical for double layers and it is true even if the charge distribution is nonplanar.

Another way to calculate the potential difference between two generic points P_2 and P_1 on the left and right plane, respectively, is by means of the definition of work,

$$W_Y = q_0 \int_Y \vec{E} \cdot \vec{dl} = -q_0 [V(P_2) - V(P_1)] \quad . \quad (12.5)$$

In the case of the double layer, from (12.1) it is clear that each of the two infinite planes is an equipotential surface. We can thus calculate the integral (12.5) on any line γ between any generic point P_2 on the left plane and any generic point P_1 on the right plane. For simplicity of calculation, we can choose the oriented line segment normal to both planes and directed from the negative (right) to the positive (left) planes (cf. Fig. 12.2). It results that \vec{E} and \vec{T} are parallel. Note that the relative sign between \vec{E} and \vec{T} is not necessarily negative [cf. text after

Eq.(12.6)]. Hence, for the field (12.1),

$$\begin{aligned}
 -[V(P_2) - V(P_1)] &= \int_{P_1}^{P_2} \vec{E} \cdot \vec{dl} \\
 &= \int_{P_1}^{P_2} \frac{\sigma}{\epsilon_0} \cdot \vec{u}_z \cdot \vec{E} \cdot \vec{dl} \\
 &= \frac{\sigma}{\epsilon_0} \int_{P_1}^{P_2} \vec{u}_z \cdot \vec{E} \cdot \vec{dl}. \quad (12.6)
 \end{aligned}$$

In (12.6), we are integrating from a point on the right plane, P_1 at $z=d$, to a point on the left plane, P_2 at $z=0$. Since $d > 0$, this means the line integral is defined from a point at "larger" distance to one at "smaller" distance. Therefore, particular care must be taken in choosing $\vec{E} \cdot \vec{dl}$. The length of the straight line segment between P_1 and P_2 along which we are integrating must be positive, $d > 0$. Assume we choose $\vec{E} \cdot \vec{dl} = -\vec{u}_z \cdot dz$. The length $\overline{P_1 P_2}$ can be calculated as

$$\overline{P_1 P_2} = \left| \int_0^d -\vec{u}_z \cdot (-\vec{u}_z \cdot dz) \right|$$

$$= - \int_0^d dz = -d < 0.$$

thus, the correct choice is $\vec{E} \cdot dl = \vec{u}_z \cdot dz$. The integral (12.6) then becomes

$$V(P_1) - V(P_2) = \frac{\sigma}{\epsilon_0} \int_d^0 \vec{u}_z \cdot \vec{u}_z \cdot dz$$

$$= - \frac{\sigma}{\epsilon_0} d$$

or

$$V(d) - V(0) = - \frac{\sigma}{\epsilon_0} d ,$$

which is equal to (12.4).

Note that this result applies also to the case of a

parallel plate capacitor under the condition 10/
 $d \ll \sqrt{A}$, where d is the distance between the two plates and A the area of each plate (each plate is considered to be of finite size).

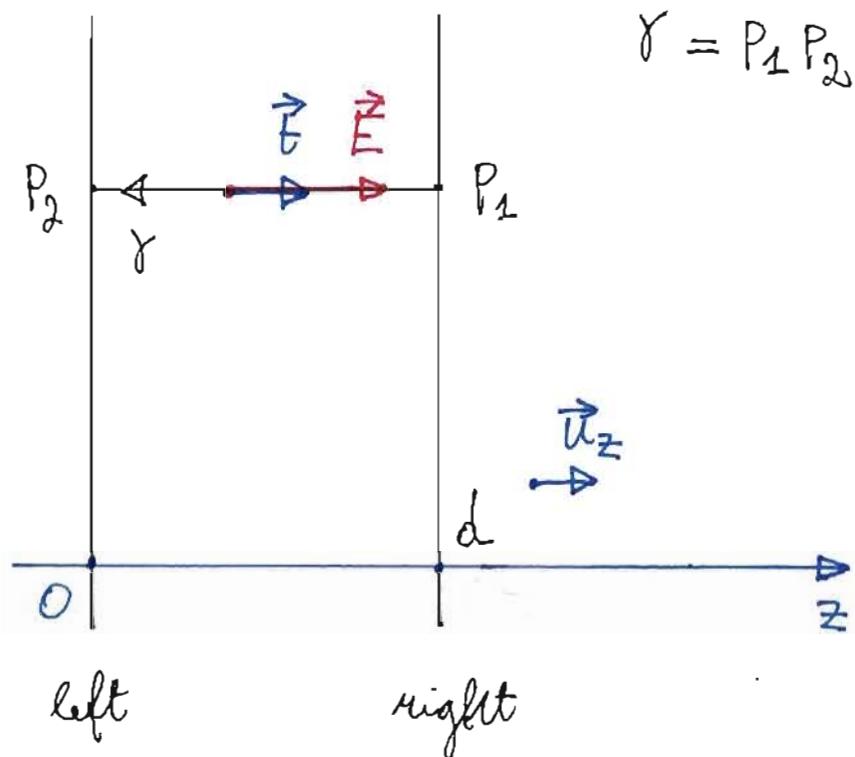


Figure 12.2.

Figure 12.2 shows how to calculate the line integral (12.6). Note again that when integrating from P_1 to P_2 , \vec{E} and \vec{E} are parallel and with the same sign.

12.2 The potential generated by unlimited charge distributions.

We know from (9.3) and (9.4) that the potential generated by a system of N point-like charges in vacuum can be obtained by superimposing the potentials generated by each charge,

$$V(P) = \frac{1}{4\pi\epsilon_0} \sum_{k=1}^N \left(\frac{q_k}{r_k} + c_k \right), \quad (12.7)$$

where c_k s are N arbitrary constants. After choosing the constants, in the case of a limited number N of charges, we have

$$c = \sum_{k=1}^N c_k$$

and, thus,

$$V(P) = \frac{1}{4\pi\epsilon_0} \sum_{k=1}^N \frac{q_k}{r_k} + C. \quad (12.8)$$

It is always possible to choose a value of C such that V becomes zero at a given point (as long as the point

is different from the points occupied by the charges). In particular, for $C=0$ the potential becomes zero at infinity.

The scenario is quite different in the case of an unlimited number N of charges. In this case, in fact, it is not possible to choose the C_k 's in a completely arbitrary manner. Before delving into the details why this is the case, it is worth considering a few examples.

12.2.1 The potential of a single infinite layer.

From lecture two (page 22), we know that the field generated by an infinite single layer (or infinite plane) with respect to a cartesian coordinate system, where the axis Z is normal to the plane, is given by

$$\left\{ \vec{E} = -\frac{\sigma}{2\epsilon_0} \cdot \vec{u}_z \quad , \quad z \in [-\infty, 0] \right. \quad (12.9a)$$

$$\left. \vec{E} = \frac{\sigma}{2\epsilon_0} \cdot \vec{u}_z \quad . \quad z \in [0, +\infty) \right. \quad (12.9b)$$

It is then straightforward to obtain V at any point in space by integration, from $\vec{E} = -\text{grad } V$. For

$z \in (-\infty, 0]$, we use the field (12.9a) and write

[13]

$$\vec{E} = -\frac{\sigma}{2\epsilon_0} \cdot \vec{u}_z = -\frac{d}{dz} V(z) \cdot \vec{u}_z$$

from which

$$dV(z) = \frac{\sigma}{2\epsilon_0} \cdot dz$$

and finally

$$\begin{aligned} \int_0^z dV(z) &= V(z) - V(0) \\ &= \frac{\sigma}{2\epsilon_0} \int_0^z dz = \frac{\sigma}{2\epsilon_0} z . \end{aligned}$$

Calling $V(0) = C_0$, with C_0 an arbitrary constant, we can write

$$V(z) = \frac{\sigma}{2\epsilon_0} z + C_0 . \quad (12.10a)$$

Similarly, for $z \in [0, +\infty)$, we use the field (12.9b) and write

$$\vec{E} = \frac{\sigma}{2\epsilon_0} \cdot \vec{u}_z = -\frac{d}{dz} V(z) \cdot \vec{u}_z$$

from which

$$dV(z) = -\frac{\sigma}{2\epsilon_0} \cdot dz$$

and finally

$$\begin{aligned} \int_0^z dV(z) &= V(z) - V(0) \\ &= -\frac{\sigma}{2\epsilon_0} \int_0^z dz = -\frac{\sigma}{2\epsilon_0} z . \end{aligned}$$

Since $V(0) = C_0$, we can write

$$V(z) = -\frac{\sigma}{2\epsilon_0} z + C_0 \quad (12.10b)$$

Figure 12.3 shows a plot of the potential for $z \in (-\infty, +\infty)$.

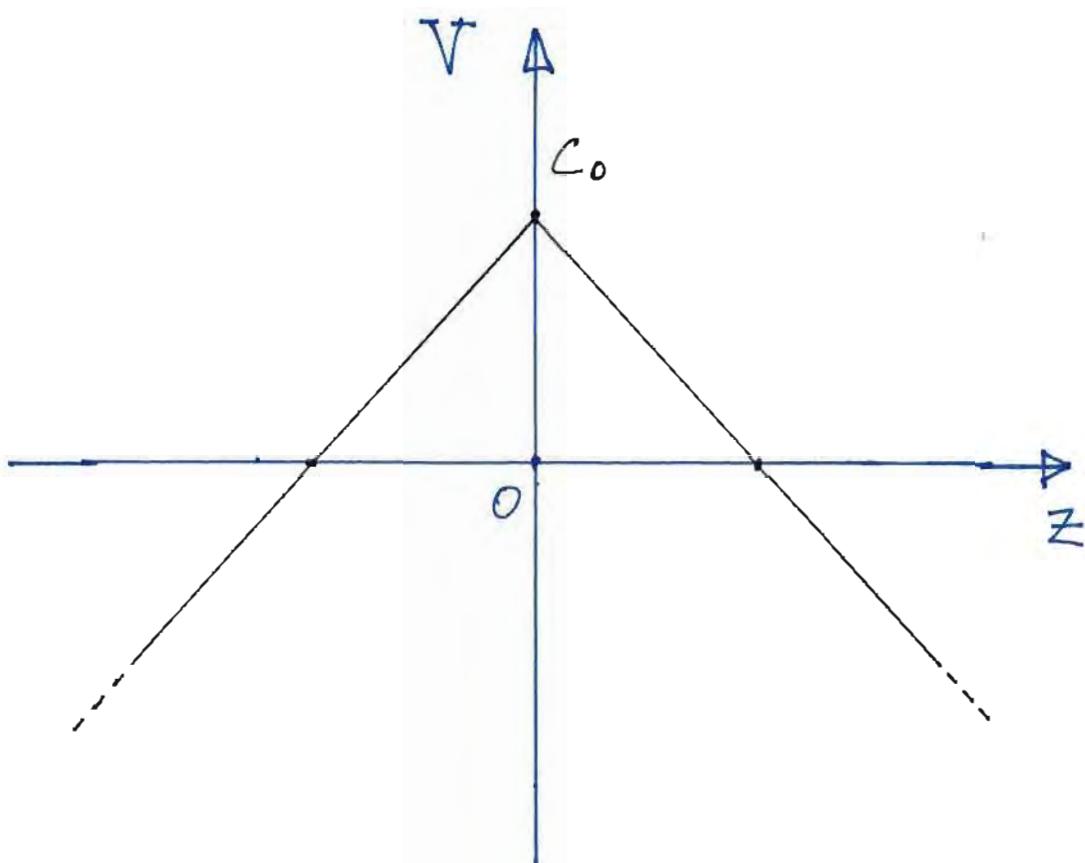


Figure 12.3.

12.2.2 The potential of a charged disk on the disk's axis.

In lecture 9, section 9.2.1, we studied the potential of a charged ring on the ring's axis. The potential is given by Eq. (9.12). By generalizing the notation used in that equation and calling the radius of the ring r instead of a , we have

$$V(z) = \frac{\lambda}{2\epsilon_0} \frac{r}{\sqrt{r^2 + z^2}}$$

From the definition of linear charge density,

$$\lambda = \frac{q}{2\pi r} ,$$

where q is the total charge on the ring, it follows that

$$V(z) = \frac{1}{4\pi\epsilon_0} \frac{q}{\sqrt{r^2 + z^2}} . \quad (12.11)$$

With the result (12.11) in hand, we can easily calculate the potential of a disk uniformly charged with charge q on the disk's axis.

The surface charge density associated with the disk is constant and given by

$$\sigma = \frac{q}{\pi R^2} , \quad (12.12)$$

where R is the radius of the disk.

① Coordinate system.

The coordinate system chosen to solve the problem is a cylindrical $Oxyz$ system, where O coincides with the center of the disk and z with the disk's axis (cf. Fig. 12.4).

② Results.

In order to solve the problem and calculate the potential, we can divide the disk in circular crowns with radii r and $(r+dr)$, respectively, as shown in Fig. 12.4.

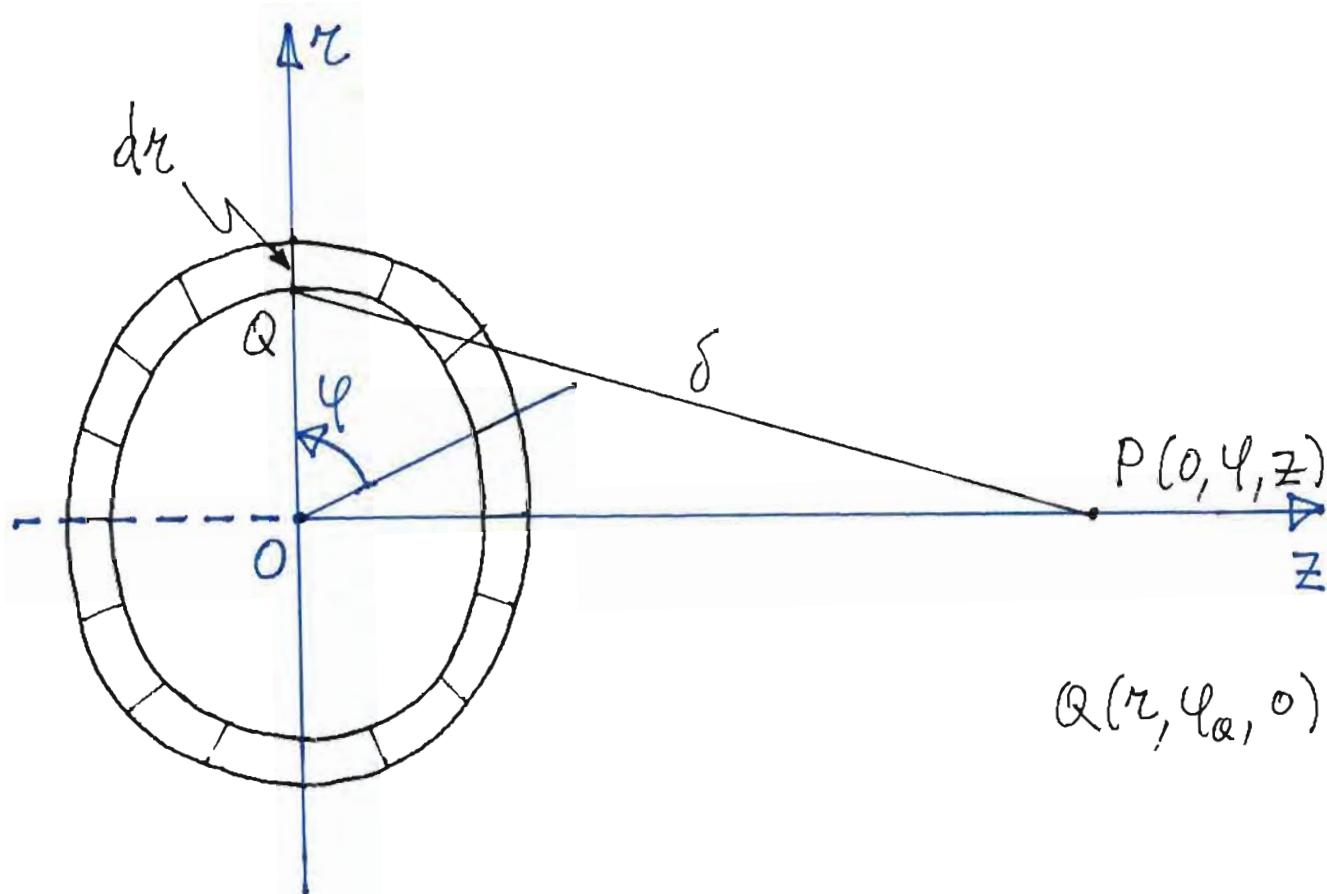


Figure 12.4.

Each circular crown represents an infinitesimal ring-type structure with potential given by Eq. (12.11),

$$dV(z) = \frac{1}{4\pi\epsilon_0} \frac{dq}{\sqrt{r^2 + z^2}},$$

where the infinitesimal charge dq is given by the surface charge density (12.12) times the area of the circular crown,

$$dq = \sigma \cdot 2\pi r \cdot dr.$$

Hence,

$$\begin{aligned} dV(z) &= \frac{1}{4\pi\epsilon_0} \frac{\sigma \cdot 2\pi r \cdot dr}{\sqrt{r^2 + z^2}} \\ &= \frac{\sigma}{2\epsilon_0} \frac{r \cdot dr}{\sqrt{r^2 + z^2}}. \quad (12.13) \end{aligned}$$

By integration for $r \in [0, R]$,

$$V(z) = \frac{\sigma}{2\epsilon_0} \int_0^R \frac{r \cdot dr}{\sqrt{r^2 + z^2}}. \quad (12.14)$$

Noting the triangle OQP in Fig. 12.4, we immediately find the substitution that simplifies the integral (12.14),

19/

$$\delta^2 = r^2 + z^2 . \quad (12.15a)$$

By differentiating δ^2 we then obtain

$$2\delta \cdot d\delta = 2r \cdot dr . \quad (12.15b)$$

Since the disk is characterized by a rotation symmetry with respect to φ and by a reflection symmetry (π rotation) with respect to the r axis (i.e., the disk is symmetric with respect to a change of sign in the Z coordinate), without loss of generality we can consider the case $Z \geq 0$ only. The limits of integration can then be found geometrically by inspecting the triangle OQP . When $r=0$, $\delta=z$ and when $r=R$, $\delta=\sqrt{R^2+z^2}$. Hence,

$$\delta^- = z , \quad (12.15c)$$

$$\delta^+ = \sqrt{R^2 + z^2} \quad . \quad (12.15d)$$

By substituting (12.15a)-(12.15d) into (12.14)
we find

$$\begin{aligned} V(z) &= \frac{\sigma}{2\epsilon_0} \int_{\delta^-}^{\delta^+} \frac{\delta \cdot d\delta}{\delta} \\ &= \frac{\sigma}{2\epsilon_0} (\sqrt{R^2 + z^2} - z) \\ &= \frac{q}{2\pi\epsilon_0} \frac{\sqrt{R^2 + z^2} - z}{R^2}, \quad (12.16) \end{aligned}$$

where we used (12.12). This solution is valid for $z \geq 0$. The solution for $z \in (-\infty, +\infty)$ must be written as

$$V(z) = \frac{q}{2\pi\epsilon_0} \frac{\sqrt{R^2 + z^2} - |z|}{R^2}. \quad (12.14)$$

The absolute value on z can be better understood

resorting to the substitution (12.15a). 21

In fact, for $r=0$

$$\sigma^2 = z^2$$

and so

$$\sigma = \sqrt{z^2} = z > 0.$$

For $z < 0$, this is possible only taking $\sigma = |z|$,
from which (12.14).

12.2.3 The potential of a single infinite layer revisited.
A uniformly charged infinite plane can be divided in circular crowns with radii r and $(r+dr)$, respectively,
exactly as in the case of the disk. In this case, however,
the radius r instead of being limited to R can extend
all the way to infinity (cf. Fig. 12.5).

With respect to an Orz cylindrical system, we
consider the origin O to coincide with the center of the
crowns making up the infinite plane. The potential at
a generic point P is thus given by (12.13), this

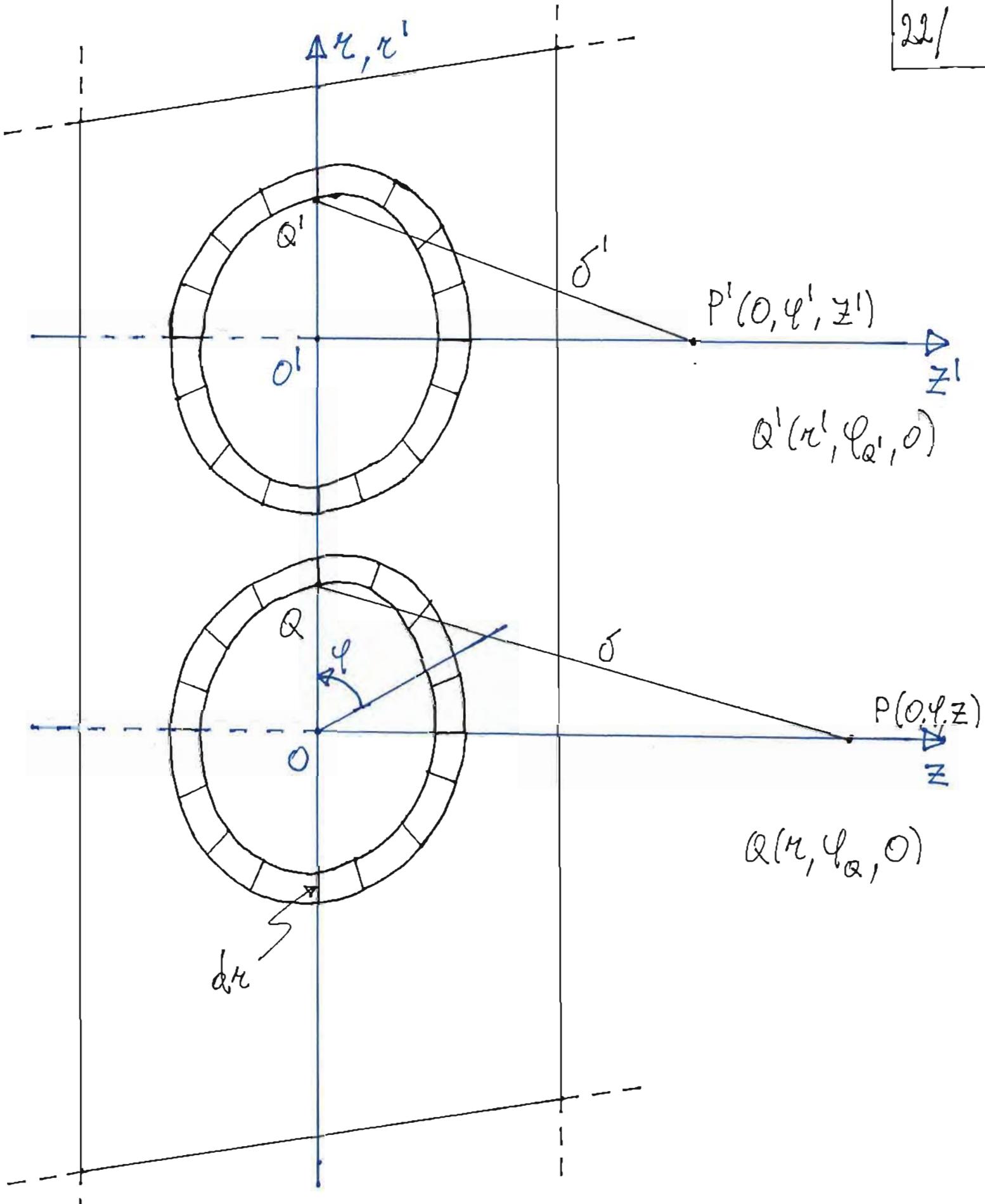


Figure 12.5.

time integrated for $r \in [0, +\infty)$, i.e., $R \rightarrow +\infty$. | 23/
 For $z \in (-\infty, +\infty)$ and using the substitution (12.15a)
 - (12.15d), we obtain

$$\begin{aligned} V(z) &= \frac{\sigma}{2\epsilon_0} \int_0^{+\infty} \frac{r \cdot dr}{\sqrt{r^2 + z^2}} \\ &= \frac{\sigma}{2\epsilon_0} \int_{|z|}^{+\infty} d\delta \\ &= +\frac{\sigma}{|\sigma|} \infty . \quad (12.18) \end{aligned}$$

The potential (12.18) diverges and can be either $-\infty$ or $+\infty$ depending on the sign of σ . The potential of Eq. (12.18) is in stark contrast with that of Eqs. (12.10a) and (12.10b). However, they both seem to be legitimate potentials for a uniformly charged infinite plane.

In order to understand which one is the correct potential, we need to complete our theory on

unlimited charge distributions.

12.2.4 Diverging potentials.

If we arbitrarily choose the C_k 's in (12.4) so that they are all equal to a single constant $C_0 \neq 0$, going the sum in (12.4) to infinity, the entire expression (12.7) would diverge for each value of r ,

$$C_0 = \sum_{k=1}^{\infty} C_k = \sum_{k=1}^{\infty} C_0 = \infty$$

and, thus,

$$\begin{aligned} V(p) &= \frac{1}{4\pi\epsilon_0} \sum_{k=1}^{\infty} \frac{q_k}{r_k} + C_0 \\ &= \infty, \text{ for each } r_k. \end{aligned}$$

In general, it is also impossible to choose the C_k 's so that they are all equal to zero. This last statement can be demonstrated considering a system of equal point-like charges q , located at a unitary distance one from another on an x axis (cf. Fig. 12.6). We now attempt to calculate the potential V

at a generic point P_0 on the x axis. We can [25]
use (12.4), making sure P_0 does not coincide with
any of the points where the charges are located and
imposing $C_k = 0$ for each k . Assuming the origin
 O of the x axis to coincide with the point occupied by
the charge immediately on the left of P_0 and indicating
the position of P_0 on the axis as x_0 , by means of
the superposition principle we have

$$V = V_1 + V_2 , \quad (12.19)$$

where V_1 is the potential due to all charges on the
left of O (including the charge at the origin) and V_2
the potential due to all charges on the right of O . For
 $C_k = 0$, (12.4) gives

$$\left\{ \begin{array}{l} V_1 = \frac{q}{4\pi\epsilon_0} \left(\frac{1}{x_0} + \frac{1}{x_0+1} + \frac{1}{x_0+2} + \dots \right) , \\ V_2 = \frac{q}{4\pi\epsilon_0} \left(\frac{1}{1-x_0} + \frac{1}{1-x_0+1} + \frac{1}{1-x_0+2} + \dots \right) . \end{array} \right. \quad (12.20a)$$

$$\left\{ \begin{array}{l} V_1 = \frac{q}{4\pi\epsilon_0} \left(\frac{1}{x_0} + \frac{1}{x_0+1} + \frac{1}{x_0+2} + \dots \right) , \\ V_2 = \frac{q}{4\pi\epsilon_0} \left(\frac{1}{1-x_0} + \frac{1}{1-x_0+1} + \frac{1}{1-x_0+2} + \dots \right) . \end{array} \right. \quad (12.20b)$$

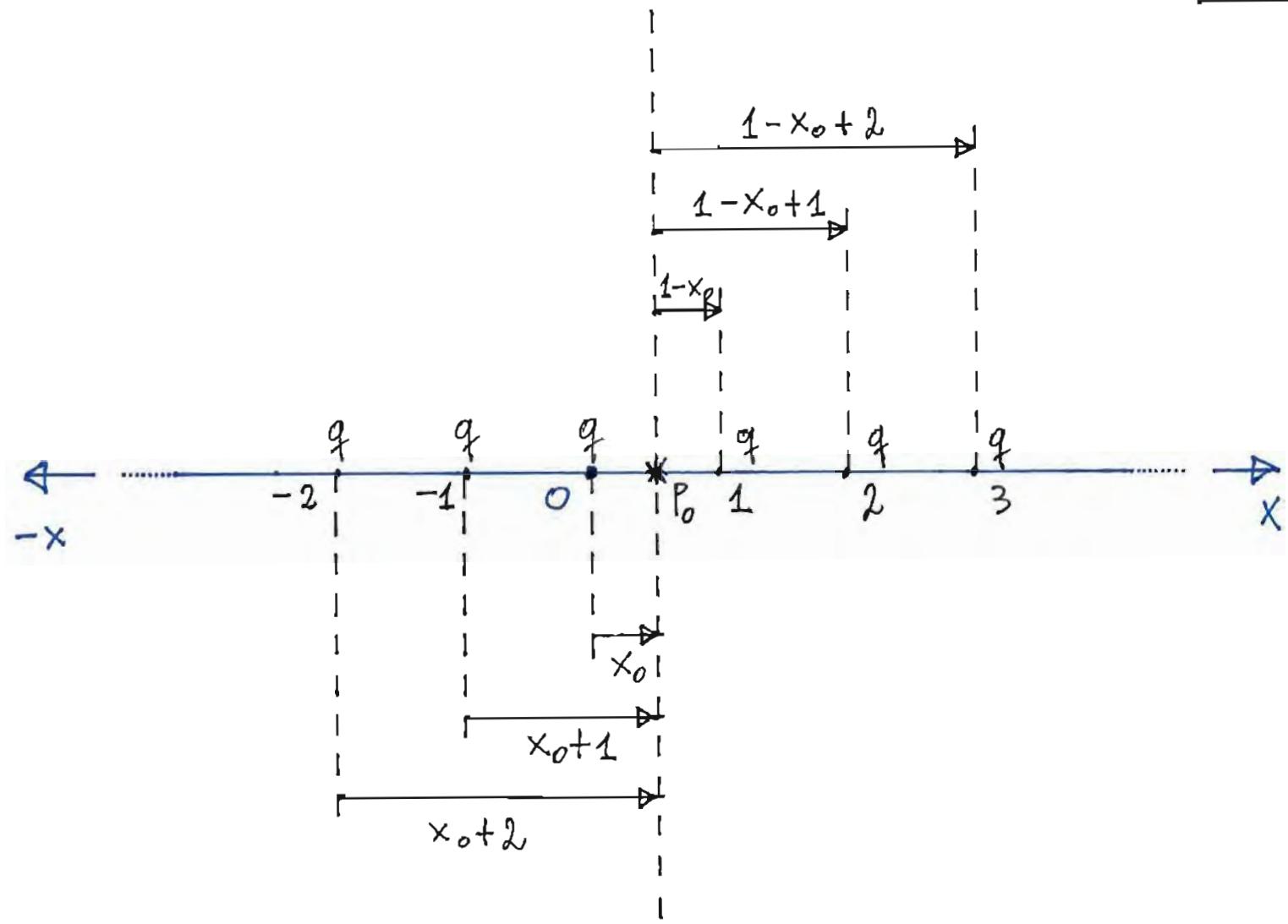


Figure 12.6.

By substituting (12.20a) and (12.20b) into (12.19), we find

$$V(x) = \frac{q}{4\pi\epsilon_0} \left(\sum_{k=0}^{\infty} \frac{1}{x_0+k} + \sum_{k=0}^{\infty} \frac{1}{1-x_0+k} \right). \quad (12.21)$$

given a and b, the integers immediately larger and

smaller than x_0 , we have

$$\left\{ \begin{array}{l} \frac{1}{x_0+k} > \frac{1}{a+k} \end{array} \right. , \quad (12.22a)$$

$$\left\{ \begin{array}{l} \frac{1}{1-x_0+k} > \frac{1}{1-a+k} \end{array} \right. , \quad (12.22b)$$

The series in (12.21) are both upper bounds of the harmonic series and, as such, they both diverge. Thus, by imposing $C_k=0$ for each k in (12.4), the expression of the potential for the case of Fig. 12.6 diverges even at points different from a charge point (e.g., P_0). However, it would be easy to show that the field at those points generated by the unlimited system of charges remains limited and, thus, from integration it would be possible to find a finite potential (the reader is recommended to show that this is true).

The trick to obtain directly (without integrating a field) a finite potential in the case of an unlimited charge distribution is to choose each of the C_k 's so that the potential generated by the k -th charge (to which C_k corresponds to) is zero at a given point P' . The

total potential, sum of all potentials that are zero at P' , will also be zero at P' . 281

Before applying this trick to the case of a charged infinite plane, it is worth summarizing our findings by stating that, since the superposition principle must apply to potentials, it is illegitimate to impose $C_k = 0$ for each k . In fact, e.g., for the case of Fig. 12.6, this would correspond to impose that the potential is zero at a point (the infinite point on the x axis), where a nonzero charge exists (this point is indeed called point of accumulation of the charges): this is impossible.

12.2.5 The potential of a single infinite layer revisited.
With reference to Fig. 12.5, we want to calculate the potential at a generic point P under the condition that the potential is zero at a given point P' . It is enough to impose that the infinitesimal potential generated by the circular crown centered in O is subtracted by the infinitesimal potential due to the crown centred in O' , which is the projection of P' onto the infinite plane, along the Z' axis. Note that any crown centered at a generic point O' can be used to span the entire plane by varying r (or r') from zero to infinity. This is due to the

translation symmetry of the infinite plane along the r (or r') axis, for any choice of ℓ . Hence,

$$dV(z) = \frac{\sigma}{2\epsilon_0} \left[\frac{r \cdot dr}{\sqrt{r^2 + z^2}} - \frac{r' \cdot dr'}{\sqrt{(r')^2 + (z')^2}} \right]. \quad (12.23)$$

By integrating for $r \in [0, +\infty)$ and $r' \in [0, +\infty)$, for $z \in (-\infty, +\infty)$, and by using the substitution (12.15a)–(12.15d), we obtain

$$V(z) = \frac{\sigma}{2\epsilon_0} \left(\int_{|z|}^{+\infty} d\delta - \int_{|z'|}^{+\infty} d\delta' \right), \quad (12.24)$$

where $(\delta')^2 = (r')^2 + (z')^2$. Each of the two integrals in (12.24) is an improper integral that can be rewritten as (each integral clearly diverges for $|\delta|, |\delta'| \rightarrow +\infty$)

$$\lim_{d \rightarrow +\infty} \frac{\sigma}{2\epsilon_0} \left(\int_{|z|}^d d\delta - \int_{|z'|}^d d\delta' \right) =$$

$$\begin{aligned}
 &= \lim_{d \rightarrow +\infty} \frac{\sigma}{2\epsilon_0} (\phi - |z| - \phi + |z'|) \\
 &= \frac{\sigma}{2\epsilon_0} (|z'| - |z|) = -\frac{\sigma}{2\epsilon_0} |z| + \frac{\sigma}{2\epsilon_0} |z'|. \tag{12.25}
 \end{aligned}$$

For $z \in (-\infty, 0]$,

$$V(z) = \frac{\sigma}{2\epsilon_0} z + \frac{\sigma}{2\epsilon_0} |z'| \tag{12.26a}$$

and for $z \in [0, +\infty)$,

$$V(z) = -\frac{\sigma}{2\epsilon_0} z + \frac{\sigma}{2\epsilon_0} |z'| \tag{12.26b}$$

By defining $C_0 = (\sigma/2\epsilon_0)|z'|$, the potential (12.26a) and (12.26b) is exactly the same as (12.10a) and (12.10b). The potential is not infinite everywhere. It becomes infinity only for $|z| \rightarrow \infty$.

At last, it is worth calculating the energy U_e at a generic point P . Remembering that U_e is the

work against the field forces to move a charge q_0 from infinity to P (for example, we can consider q_0 to be a positive test charge), $V_e(P)$ can be calculated either from (12.26a) and (12.26b) or by integrating the field, which we know is given by

$$\vec{E} = \frac{\sigma}{2\epsilon_0} \frac{z}{|z|} \cdot \hat{u}_z, \quad z \in \mathbb{R} \quad (12.24)$$

where the sign is dictated by z .

For $z \in [0, +\infty)$, the integration approach gives

$$V_e(P) = -q_0 \int_{\gamma^+}^z \vec{E} \cdot \vec{dl}$$

Because the field (12.24) is directed along the z axis, we can choose γ^+ to be directed also along the z axis from $+\infty$ to z . Hence,

$$V_e(P) = -q_0 \frac{\sigma}{2\epsilon_0} \int_{+\infty}^z \hat{u}_z \cdot \hat{u}_z \cdot dz$$

$$= q_0 \frac{\sigma}{2\epsilon_0} \int_z^{+\infty} dz$$

$$= q_0 \frac{\sigma}{2\epsilon_0} [z]_z^{+\infty} = +\infty . \quad (12.28a)$$

The approach based on the knowledge of the potential gives

$$\begin{aligned} U_e(P) &= q_0 [V(P) - V(+\infty)] \\ &= q_0 \left[-\frac{\sigma}{2\epsilon_0} z + \phi_0 - (-\infty) - \phi_0 \right] \\ &= +\infty , \quad (12.28a') \end{aligned}$$

which, as expected, is the same result as (12.28a). As always, it is worth checking whether the choice $\vec{E} = +\vec{u}_z dz$ in (12.28a) makes sense. On γ^+ , we are integrating from $+\infty$ to $z \geq 0$. The length of γ^+ has to be positive. Measuring the length by means of a $(-\vec{u}_z)$, we find

$$\int_{+\infty}^z (-\vec{u}_z) \cdot \vec{E} = \int_{+\infty}^z (-\vec{u}_z) \cdot (+\vec{u}_z dz)$$

$$= \int_z^{+\infty} dz = +\infty = \bar{\gamma}^+ > 0 . \quad]$$

33/

Similarly, for $z \in (-\infty, 0]$, the integration approach gives

$$U_e(P) = -q_0 \int_{\gamma^-} \vec{E} \cdot \vec{E} dl ,$$

where γ^- is now a line along the z axis from $-\infty$ to z . Hence,

$$U_e(P) = -q_0 \left(-\frac{\sigma}{2\epsilon_0} \right) \int_{-\infty}^z \vec{u}_z \cdot \vec{u}_z \cdot dz$$

$$= q_0 \frac{\sigma}{2\epsilon_0} \int_{-\infty}^z dz$$

$$= q_0 \frac{\sigma}{2\epsilon_0} [z]_{-\infty}^z = z - (-\infty) = +\infty.$$

(12.28b')

From the potential

$$\begin{aligned} U_e(P) &= q_0 [V(P) - V(-\infty)] \\ &= q_0 \left[\frac{\sigma}{2\epsilon_0} z + \phi_0 - (-\infty) - \phi_0 \right] \\ &= +\infty, \quad (12.28b') \end{aligned}$$

which is the same as (12.28b).

[In this case, on γ^- , we are integrating from $-\infty$ to $z \leq 0$. Also the length of γ^- has to be positive. Measuring the length, as always, by means of a $(+\vec{u}_z)$, in this case, we find

$$\begin{aligned} \int_{-\infty}^z (+\vec{u}_z) \cdot \vec{t} &= \int_{-\infty}^z (+\vec{u}_z) \cdot (+\vec{u}_z \cdot d\gamma) \\ &= \int_{-\infty}^z d\gamma = z - (-\infty) = +\infty = \bar{\gamma} > 0. \end{aligned}$$

The fact that $V_e = +\infty$ should not surprise, simply because the plane here considered is an infinite plane. Even though an infinite charged plane is an unphysical structure (from which the infinite energy), it represents an easy structure that allows to evaluate simply both \vec{E} and V . When considering a physical plane with finite lateral dimensions, the simple results obtained for an infinite plane are valid so long a test charge explores the field in close proximity of the finite plane (i.e., away from the borders and for a distance much smaller than the lateral dimensions of the plane). We remind the reader to problem HA 2.2 and HA 3.³ for a study of the potential of limited and unlimited linear charge distributions.

12.3 Multipole expansion for a continuous volume charge distribution.

Consider a continuous and limited volume charge density δ in a region Ω of the 3D euclidian space. The potential at a point P outside Ω is given by

$$V(P) = \frac{1}{4\pi\epsilon_0} \iiint_{\Omega} \frac{\rho(Q)}{r_{QP}} \cdot d\tau , \quad (12.29)$$

where r_{QP} is the absolute value of the distance between P and the generic infinitesimal volume element $d\tau$ centered at Q (cf. Fig. 12.4). For simplicity, we can rename $r_{QP} = \delta$ and define r to be the magnitude of the distance between P and a point O inside Ω , with $O \neq Q$, and R the magnitude of the distance between O and Q. By indicating as θ the angle \hat{POQ} , from Cosine's theorem

$$\delta^2 = r^2 + R^2 - 2rR \cos\theta , \quad (12.30)$$

from which

$$\delta = \sqrt{r^2 + R^2 - 2rR \cos\theta} . \quad (12.31)$$

Since $r \neq 0$ because P is outside Ω , (12.31) can be rewritten as

$$\delta = r \sqrt{1 + \left(\frac{R}{r}\right)^2 - 2\left(\frac{R}{r}\right) \cos\theta}$$

and by substituting

$$\left\{ \begin{array}{l} \frac{R}{r} = u , \quad (12.32a) \\ \cos \theta = v^*, \quad (12.32b) \end{array} \right.$$

finally as

$$\delta = \epsilon \sqrt{1 + u^2 - 2uv^*} . \quad (12.33)$$

By substituting (12.33) into (12.29), we find

$$V(P) = \frac{1}{4\pi\epsilon_0} \frac{1}{r} \iiint_{-2}^2 \frac{s}{\sqrt{1+u^2-2uv^*}} dz . \quad (12.34)$$

The integrand in (12.34) can be expanded by means of a power series

$$\frac{1}{\sqrt{1+u^2-2uv^*}} = \sum_{n=0}^{\infty} P_n(v^*) \cdot u^n , \quad (12.35)$$

where $P_n(v^*)$ is the Legendre polynomial of order n .

Note that Eq. (12.35) is one way to define the [38]
 Legendre's polynomials. These can also be defined as the
 solutions of the ordinary differential equations with
 variable coefficients

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 .$$

The Legendre's polynomials can be calculated by means
 of the Rodriguez's identity

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n .$$

By means of this identity, the first few polynomials are

$$P_0(x) = 1 ,$$

$$P_1(x) = x ,$$

$$P_2(x) = \frac{1}{2} (3x^2 - 1) ,$$

$$P_3(x) = \frac{1}{2} (5x^3 - 3x) , \dots$$

The series (12.35) converges uniformly in the range
 $x \in (-1, +1)$. By substituting (12.35) into (12.34), we

obtain

39/

$$V(P) = \frac{1}{4\pi\epsilon_0} \frac{1}{r} \iiint_{\Omega} \sum_{n=0}^{\infty} S P_n(v) \cdot u^n \cdot d\tau.$$

The series in the integrand is uniformly convergent and the terms of the series are continuous functions. Thus, the sign of integral and that of series can be safely exchanged,

$$V(P) = \frac{1}{4\pi\epsilon_0} \frac{1}{r} \sum_{n=0}^{\infty} \iiint_{\Omega} S P_n(v) \cdot u^n \cdot d\tau. \quad (12.36)$$

The first terms of the series are

$$V_0 = \frac{1}{4\pi\epsilon_0} \frac{1}{r} \iiint_{\Omega} S \cdot d\tau = \frac{1}{4\pi\epsilon_0} \frac{q}{r}, \quad (12.34a)$$

$$V_1 = \frac{1}{4\pi\epsilon_0} \frac{1}{r} \iiint_{\Omega} \frac{R S \cos\theta}{r} \cdot d\tau, \quad (12.34b)$$

$$V_2 = \frac{1}{4\pi\epsilon_0} \frac{1}{r} \iiint_{\Omega} \frac{R^2(3\cos^2\theta - 1)\beta}{2r^2} d\gamma, \quad (12.37c)$$

where we used the Legendre's polynomials P_0 , P_1 , and P_2 . To summarize the procedure, we first found $V(P)$, we then showed how V depends on r , and, at last, we expanded in power series the part of the potential that does not depend on r (note that this is legitimate even if r appears as a scaling factor in the definition of u).

The term V_0 is the potential at P of the total charge q concentrated at point O .

The term,

$$V_1 = \frac{1}{4\pi\epsilon_0} \frac{\mu}{r^2}, \quad (12.38)$$

is the term of dipole, where

$$\mu = \iiint_{\Omega} \beta R \cos\theta \cdot d\gamma \quad (12.39)$$

is the electrostatic dipole moment along the OP direction.

The next term,

$$V_2 = \frac{1}{4\pi\epsilon_0} \frac{m}{R^3}, \quad (12.40)$$

is the term of quadrupole, where

$$m = \iiint_{\Omega} \frac{1}{2} R^2 (3\cos^2\theta - 1) \rho \cdot d\tau \quad (12.41)$$

is the electrostatic quadrupole moment.

In conclusion,

$$V = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{r} + \frac{p}{r^2} + \frac{m}{r^3} + \dots \right). \quad (12.42)$$

Note that for a discrete system of point-like charges q_i , with respect to a cartesian coordinate system the electrostatic dipole moment is a vector with components

$$p_x = \sum_i q_i x_i; \quad p_y = \sum_i q_i y_i, \quad p_z = \sum_i q_i z_i,$$

where x_i , y_i , and z_i are the coordinates of charge q_i . If the charge distribution is continuous on a volume, [42]

$$p_x = \iiint_{\Omega} x s \cdot d\tau, \quad p_y = \iiint_{\Omega} y s \cdot d\tau, \quad p_z = \iiint_{\Omega} z s \cdot d\tau.$$

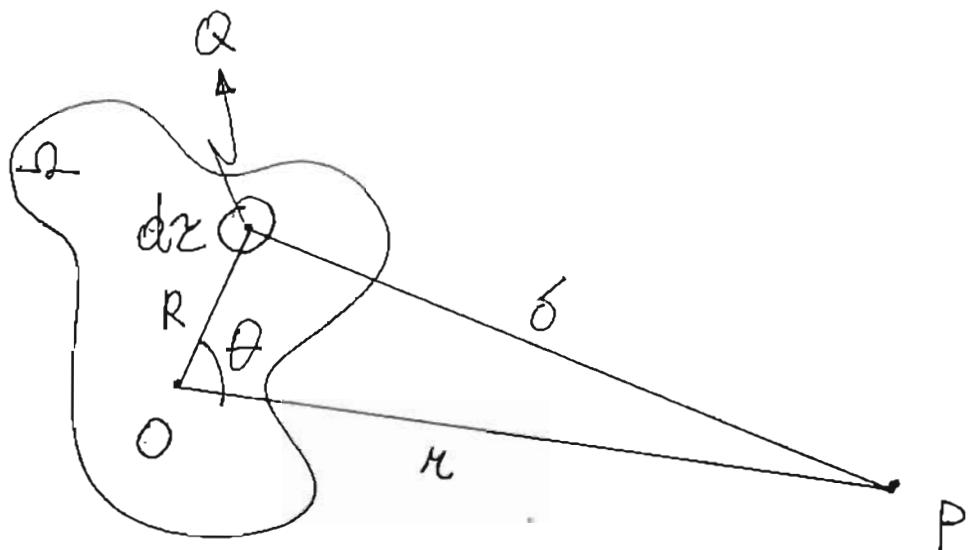


Figure 12.4.

Summary lecture 12.

- Potential difference ΔV between two points in space, P_1 and P_2 .

- After calculating directly a voltage $V(P)$ by means of integration,

$$\Delta V = V(P_2) - V(P_1) .$$

- Alternatively, ΔV can be calculated (obtained) from the work produced by the field forces to move a positive test charge q_0 from P_1 to P_2 along any line γ oriented from P_1 to P_2 ,

$$\Delta V = - \int_{\gamma} \vec{E} \cdot \vec{E} \cdot d\ell .$$

Due to the arbitrariness of γ , this can be chosen as the "easiest" path for the line integral calculation.

• Diverging potentials.

For a system of N point-like charges

$$V(P) = \frac{1}{4\pi\epsilon_0} \sum_{k=1}^N \left(\frac{q_k}{r_k} + c_k \right).$$

Assume $N \rightarrow \infty$.

- If we (arbitrarily) choose all c_k 's to be equal to $c_0 \neq 0$, $V(P)$ clearly diverges.
- By comparison with the harmonic series, choosing all c_k 's to be zero also corresponds to a diverging potential.
- The trick to circumvent this problem is to choose

$$V_k(P) = \frac{1}{4\pi\epsilon_0} \left(\frac{q_k}{r_k} - \frac{q_k}{r_k^1} \right).$$

Energy.

Consider a charge q_0 at a point P in space in a field \vec{E} .

- The energy $U_e(P)$ is the work against the field forces to move q_0 from ∞ to P ,

$$U_e(P) = -q_0 \int_{\gamma} \vec{E} \cdot \vec{dl},$$

where γ is an arbitrary line oriented from ∞ to P .

- If $V(P)$ is a known function,

$$U_e(P) = q_0 [V(P) - V(\infty)].$$

- Multipoles.

Given a point P at distance r from the center O of a charge distribution in a volume Ω with density S (P outside Ω). Named R the distance between O and a point Q inside Ω and $\theta = \hat{P}OQ$,

$$V(P) = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{r} + \frac{p}{r^2} + \frac{m}{r^3} + \dots \right),$$

4/34

where

$$q = \iiint_{\Omega} s \cdot d\tau$$

$$p = \iiint_{\Omega} sr \cos\theta \cdot d\tau$$

$$m = \iiint_{\Omega} \frac{1}{2} R^2 (3 \cos^2\theta - 1) s \cdot d\tau.$$

The total charge is q , p is magnitude of the dipole moment, and m the magnitude of the quadrupole moment.

13.1 Conductors and insulators.

5/37

So far, we considered physical scenarios where we calculated the field given a charge distribution. It often happens, however, that the charge distribution associated with a field is *a priori* unknown and must be determined simultaneously with the field.

Material objects are made by an ensemble of positive and negative charges blended so that, in conditions of electrical neutrality, the total charge in a "physics infinitesimal" of an object is zero.

However, depending on the physical nature of the object, some of the object's charges are free to move within the object. As a consequence, when a neutral body (object) is immersed in an electrostatic field, charges with opposite sign, being acted upon by forces with opposite direction, they move away from each other. Such a separation destroys the local neutrality of the body and, point-by-point, allows for charge distributions. Note that, if no extra charges are externally added to the body, the body remains globally neutral. The so induced charge distribution contributes to the overall

electrostatic field, which, on its own, modifies the local charge distribution in the body. Hence, the main difficulty of this problem: The electrostatic field is generated by both the external charge distributions and by the induced charges, which, on their own, depend on the resulting field.

It is clear that if the distribution of the induced charges was a priori known, the resulting field could be calculated as in the previous lectures. We could simply substitute the body with the distribution of the induced charges and calculate the field generated by all charges (internal and external to the body) as if they were acting in vacuum. (Note that, if it was possible to fix all charges in a body to their respective position, nothing would happen by placing the body itself in an electrostatic field. In fact, due to the immobility of the charges the local electrostatic neutrality of the body would not be perturbed. Independently from the physical nature of the body, the body would be completely transparent for the electrostatic field. In other words, the field would be generated solely by the external charges, as if they were acting in vacuum and the body did not exist.)

However, it is impossible to do so because the charge distribution in a body is the result of a situation of equilibrium between the electrostatic forces acting on the charges and the reaction of the constraints to which the charges are subject to in the body. We can conclude saying that, a priori, it is impossible to give nor the induced charge distribution in the body, neither the resulting field. It is necessary to determine, instead, both the induced charge and the resulting field simultaneously.

The next sections will describe the phenomena that take place in conducting bodies in vacuum (or in air at normal conditions) in presence of an electrostatic field. A few methods to solve some of the problems outlined above will also be introduced.

In the first lecture, we qualitatively defined insulators (also called dielectrics) as the bodies capable to store electric charge for long times and conductors as the bodies that allow a quick charge dispersion. This is only a qualitative distinction. A quantitative study needs the introduction of the concepts of conduction and polarization. In this course, we will limit the discussion to conduction only.

Conductors, in particular metals, are objects with an Avogadro number of charged particles (on the order of one per atom) free to move within the object. Those free charges (in conducting metals these are electrons, in electrolytic solutions positive and negative ions, in ionized gases positive ions and electrons, ...), within the entire region occupied by the body continuously hit each other or the "fixed" particles. In normal conditions, however, they will hardly exit the body. Those free particles can be thought as trapped in a potential well the walls of which coincide with the body's surface (as a liquid in a gravitational field is contained by a glass' walls). In other words, the external surface of the conductor behaves as if it were a charged double layer, which determines a sharp discontinuity of potential when going from inside to outside the body [cf. lecture 12, Eq. (12.4) and the comment thereafter]. It is exactly this discontinuity to determine the value of the potential barrier that prevents electrons to escape the body under normal conditions.

Hereafter, when we will talk about the potential of a conductor, we will intend the potential difference

between the interior of the body and its external surface (i.e., without considering the aforementioned potential well).

Dielectrics, instead, are objects where charged particles are strongly attached to the atoms and molecules to which they belong. Under the action of an external electrostatic field, those particles can only move by a very small distance compared to their equilibrium position and remain attached to the corresponding atoms and molecules.

It is clear that this distinction between conductors and dielectrics refers to an ideal scenario. In reality, also in an insulator there are unbounded charged particles that are free to move under the action of an electrostatic field. Those particles, however, are in a much smaller number than in conductors and their "mobility" is much smaller. As we will see in a few lectures when studying electrical conduction, the attitude to conduct electricity, which distinguishes conductors from insulators, can be measured by means of a suitable physical quantity called electrical

conductivity. For a typical insulator (such as glass, rubber or plastic), the conductivity is approximately 10^{20} times smaller than in a good conductor (such as copper, silver or aluminum).

13.2 Electrostatic equilibrium in homogeneous conductors.

Consider a homogeneous and isothermal conductor. Homogeneous means that it is all made of the same material and isothermal that it is all at the same temperature (i.e., the conductor is assumed to be at the thermodynamic equilibrium at each point). These two conditions are required to exclude the presence of electromotive fields (topic of lectures 16-18).

The conductor is said to be in electrostatic equilibrium when no macroscopic motion of charges takes place within the conductor. This is possible if there are no forces acting on the free charges and, thus, the total macroscopic electrostatic field is zero at each point inside the conductor. To illustrate this statement, consider an initially neutral body that is then located in an external electrostatic field \vec{E}_{ex} generated by a set of charge distributions outside the conductor. Under the effect of \vec{E}_{ex} , the free charges in the conductor move,

thus, perturbing the conductor local neutrality. Even though the conductor remains overall neutral (because no charge has been added to or removed from the body), a distribution of charge with both negative and positive sign is found in it. Such a distribution generates a reaction field \vec{E}_r that is superimposed to \vec{E}_{ex} , generating a resulting field

$$\vec{E} = \vec{E}_{ex} + \vec{E}_r .$$

The dynamics evolves until \vec{E}_r counter-balances the action of \vec{E}_{ex} at each point inside the conductor. Under these conditions,

$$\vec{E} = \vec{E}_{ex} + \vec{E}_r = \vec{0}$$

and the free charges, not being acted upon by any acceleration, stop at a final configuration of electrostatic equilibrium. We can now ask the question whether the conductor always reaches a situation of electrostatic equilibrium or it is possible that this does not happen. It is easy to be convinced that, at least in normal conditions, a state of equilibrium must always be reached. In fact, if this was not the

case, the charges would continue to move indefinitely within the conductor, continuously bouncing against the fixed particles and, thus, transferring to them part of the kinetic energy gained under the action of the total field. This process cannot continue indefinitely if the conductor is acted upon by an electrostatic field because the field is irrotational. This problem is similar to that of a little ball bouncing off a floor under the action of the gravitational field. The ball cannot continue bouncing indefinitely because of the energy transferred to the floor (or any other medium in the proximity of the ball). An exception is that of special type of materials that, when cooled to very low temperatures (on the order of a few K) become superconducting. For these materials, even if the free electrons continually hit against the fixed lattice ions, they do not transfer energy to them because of perfect elastic scattering. Once a macroscopic movement of electrons is initiated, it will continue for even decades. In a normal conductor at room temperature, we can be sure that this does not happen and a state of

equilibrium is rapidly reached (in a good conductor the time required to reach equilibrium, called the relaxation time, is on the order of 10^{-18} - 10^{-19} s). Similar conclusions are reached also when an external charge is added to the conductor. The charge will distribute on the conductor such that, at equilibrium, the electrostatic field inside the conductor is zero everywhere.

13/34

In summary, from the macroscopic point of view a conductor in macroscopic equilibrium is a region inside which $\vec{E} = \vec{0}$ at each point. As a consequence, because of Gauss' theorem

$$\oint S = 0$$

everywhere inside the conductor.

Thus, it is impossible for charges to localize at any point inside a conductor in electrostatic equilibrium.

The scenario is quite different at the external surface of a conductor. In fact, at the interface between the conductor and the space outside, forces due

to the potential barrier previously discussed can balance the forces produced by the electrostatic field, thus preventing charges from leaving the conductor. In equilibrium, a charge with density σ (in general, variable from point to point) distributes on the surface of the conductor so that $\vec{E} = \vec{0}$ at each point inside the conductor. Outside the conductor, the action of σ superimposes to that of all other external charge distributions, resulting in an electrostatic field which is the same as if all surface charges on the conductor were "frozen" (in equilibrium) and the conductor was removed altogether. The presence of a surface charge distribution with density σ on the surface Σ of the conductor implies a discontinuity of the first kind of the normal component of \vec{E} at each point on Σ (note that this is true even if the double layer that determines the characteristic potential barrier of the conductor is superimposed to σ).

Referring, for example, to Fig. 13.1, we find

$$[E_n] = -\frac{\sigma}{\epsilon_0}, \quad (13.1)$$

where $|E_n|$ indicates the difference between the values of the component of \vec{E} normal to Σ for each pair of points with one point slightly above and one slightly below the surface. We remind that the condition (13.1) is the same as condition (5.26) in lecture 5, in other words, case 2 of our Maxwell equations for electrostatics' (cf. lecture 5). At first glance, the reader might be tempted to invoke case 3 of lecture 5 because we are dealing with an object (the conductor) defined as a "volume" in 3D space. The conductor, however, is characterized not only by its volume, but also by a surface that bounds the volume. The volume of the conductor is associated with a charge density ρ , which, by means of Gauss' theorem, we have shown to be zero at each point in the conductor. The surface of the conductor is, instead, associated with a nonzero charge density σ . It is thus clear that the conductor behaves as if it was a surface charge distribution with density σ , leading directly to case 2,

$$\vec{n} \cdot (\vec{E}_1 - \vec{E}_2) = \frac{\sigma}{\epsilon_0} \quad . \quad (13.2)$$

Following the notation in Fig. 13.1, \vec{E}_2 is the field inside the conductor that we know must be zero, 16/34

$$\vec{E}_2 = \vec{0}$$

at each point in the conductor. The unit vector \vec{n} is normal to Σ at a generic point P_0 on Σ and points outside the conductor. The field $\vec{E}_1 = \vec{E}$ is the field outside the conductor. This field is initially evaluated at a point $P \neq P_0$ outside the conductor. In the limit that P approaches P_0 from outside the conductor, \vec{E} and \vec{n} are parallel and (13.2) becomes

$$\lim_{P \rightarrow P_0^+} E_n(P) = \frac{\sigma(P_0)}{\epsilon_0}, \quad (13.3)$$

where $\sigma(P_0)$ is the value of σ at P_0 . Since, in general, the surface density σ varies from point to point on the surface of the conductor, also E_n varies from point to point on Σ .

Similarly, from condition (5.28) we find

$$\lim_{P \rightarrow P_0^+} E_t(P) = 0,$$

which means the field E does not have any tangent component at each point on Σ .

In summary, given a conductor in electrostatic equilibrium in vacuum, the electrostatic field at each point in proximity of the conductor surface (outside the conductor) is normal to the surface. The statement just proven goes under the name of Coulomb's theorem (not to be confused with the "Coulomb's theorem" in the fiction of our lecture 8).

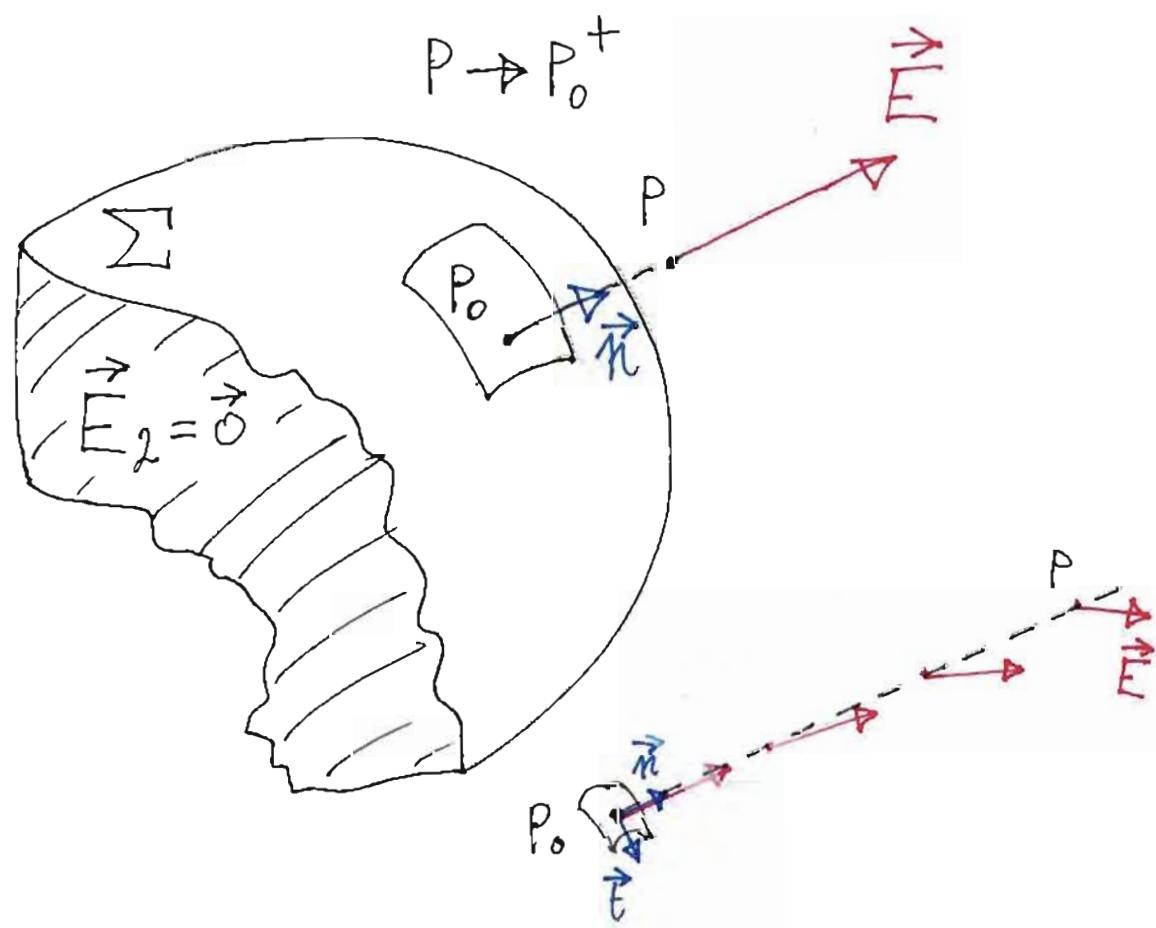


Figure 13.1.

Coulomb's theorem can also be demonstrated [18/34]

from the concept of potential. In fact, since $\vec{E} = \vec{\sigma}$ inside the conductor, it follows that $\vec{\nabla} V = \vec{\sigma}$ and, thus, $V = \text{const.}$ inside the conductor. An argument similar to the Cauchy principle value for \vec{E} in a volume Ω [f. lecture 5, Eq. (5.16)] allows us to show that V is continuous and limited even in correspondence of a surface charge distribution, so long the associated charge density σ is continuous and limited at each point on the surface (we remind that \vec{E} is not continuous in correspondence of surface charge distributions; case 2, lecture 5).

Consider a surface Σ in the 3D Euclidean space, charged with variable density σ . At any point P on Σ ,

$$V(P) = \frac{1}{4\pi\epsilon_0} \iint_{\Sigma} \frac{\sigma(Q)}{r_{QP}} \cdot dS , \quad (13.4)$$

where Q is a generic point on Σ . When $P = Q$, $r_{QP} = 0$ and the integrand in (13.4) is singular (diverges to infinity). We can thus define an improper integral, where we remove by surgery a circular neighborhood Σ_m , with radius δ_m , of point P and assume Σ to be a

circle with radius $R (> \delta_n)$,

19/34

$$\sum \iint \frac{\sigma}{r} \cdot dS \xrightarrow[\delta_n \rightarrow 0]{} \lim \iint \frac{\sigma}{r} \cdot dS , \quad (13.5)$$

$$\sum - \sum_n$$

where we defined $r_{QP} = r$. Assuming $\sigma \in C^0(\Sigma)$ and therein limited and \sum_n to be a series of circles with radii δ_n decreasing with larger n , we can integrate (13.5) in a polar coordinate system $Ox\ell$ centered in P ,

$$\sum \iint \frac{\sigma}{r} \cdot dS = \int_0^{2\pi} d\varphi \int_{\delta_n}^R r \cdot dr \cdot \frac{\sigma}{r}$$

$$= 2\pi \sigma (R - \delta_n) . \quad (13.6)$$

In the limit

$$\lim_{\substack{n \rightarrow \infty \\ (\delta_n \rightarrow 0^+)}} 2\pi \sigma (R - \delta_n) = 2\pi \sigma R , \quad (13.4)$$

which is clearly continuous and limited on Σ , thus proving our thesis.

We are now in a position to safely state that the potential is constant both inside a conductor and on the conductor's surface Σ , $V = V_{\Sigma}$. By definition, the conductor's surface Σ is an equipotential surface. As a consequence, the work of the field forces to move a test charge q_0 between two points P_1 and P_2 on Σ along any oriented line (from P_1 to P_2) also entirely on Σ , γ_{Σ} , is

$$\begin{aligned} W_{\gamma_{\Sigma}} &= q_0 \int_{\gamma_{\Sigma}} \vec{E} \cdot \vec{t} \cdot d\ell \\ &= q_0 [V(P_1) - V(P_2)] \\ &= q_0 [V_{\Sigma} - V_{\Sigma}] = 0 \quad . \quad (13.8) \end{aligned}$$

Equation (13.8) is verified only if $E = 0$ or \vec{E} is normal to \vec{t} at each point on γ_{Σ} . Assuming $E \neq 0$,

it means that \vec{E} must be normal to the conductor's surface Σ at each point on Σ . We can thus write

$$E_n(P) = - \frac{\partial}{\partial n} V(P) , \quad (13.9)$$

where $(\partial/\partial n)$ indicates the derivative outside the conductor at point P along the normal $\vec{n}(P)$ on Σ .

All discussed so far continues to apply even if a charge q is present on the conductor (q has to be considered here as an "extra" charge on the conductor). In this case, the charge q redistributes on the surface of the conductor with a surface density σ , in general variable from point to point. The value of q can readily be calculated as

$$q = \iint_{\Sigma} \sigma \cdot dS = \epsilon_0 \iint_{\Sigma'} E_n \cdot dS , \quad (13.10)$$

where $\Sigma' \supseteq \Sigma$ is a closed surface including the entire conductor (without any other charge except for those localized on the conductor itself).

Consider a charged conductor and assume [22/34]
 to know the charge distribution σ on Σ (the surface
 of the conductor). Furthermore, consider a generic
 point P on Σ and an infinitesimal surface element
 dS centered at P . We intend to calculate the force
 acting on $\sigma \cdot dS$. The field \vec{E} on the outer surface Σ^+ of
 the conductor is normal to the surface with absolute
 value given by

$$E = \frac{\sigma}{\epsilon_0} \quad . \quad (13.11)$$

at first glance, we are tempted to conclude that the
 force on dS is

$$dF = \sigma \cdot dS \cdot E = \frac{\sigma^2}{\epsilon_0} \cdot dS \quad . \quad (13.12)$$

As it turns out, this result is incorrect. In fact, the
 correct value of the force is half the value given by
 (13.12). This is due to the general definition of
 electrostatic field, according to which, in order to calculate
 the force on a charge at P , the charge at P must ideally

be removed while maintaining all the other charges fixed and calculating the field generated by those charges. In our case, the value of \vec{E} that must be multiplied by $\sigma \cdot dS$ is not that given by (13.11), which is generated by all charges on Σ including $\sigma \cdot dS$ itself, but that obtained from (13.11) by subtracting the contribution due to $\sigma \cdot dS$. In order to calculate the contribution of $\sigma \cdot dS$ to the field, we can consider two points P_1 and P_2 in proximity of P , on one and the other side of Σ , respectively. We indicate the field at P_1 and P_2 generated by all charges on Σ except $\sigma \cdot dS$ as \vec{E}_1 and \vec{E}_2 , respectively. Moreover, we indicate the field at P_1 and P_2 generated by $\sigma \cdot dS$ as \vec{E}_1' and \vec{E}_2' , respectively (cf. Fig. 13.2). The field inside the conductor is zero, while outside it must verify Coulomb's theorem, hence

$$\left\{ \begin{array}{l} \vec{E}_1 + \vec{E}_1' = \vec{0} \\ \vec{E}_2 + \vec{E}_2' = \vec{E} \end{array} \right. , \quad (13.13a)$$

$$\left\{ \begin{array}{l} \vec{E}_1' = \frac{\sigma}{\epsilon_0} \hat{n} \\ \vec{E}_2' = \vec{0} \end{array} \right. , \quad (13.13b)$$

where \vec{n} is the unit vector normal to Σ and directed outside the surface. The fields \vec{E}_1 and \vec{E}_2 are the same as the fields on two sides of an infinite layer. Thus, they are both directed along \vec{n} , they have the same absolute value, but opposite sign. The fields \vec{E}_1 and \vec{E}_2 are essentially evaluated at the same point and, thus, they are the same. In summary,

$$\left\{ \begin{array}{l} \vec{E}_2 = -\vec{E}_1 , \quad (13.14a) \\ \vec{E}_2 = \vec{E}_1 = \vec{E} . \quad (13.14b) \end{array} \right.$$

By summing each side of (13.13a) with each side of (13.13b), we obtain

$$2\vec{E} = \vec{E} = \frac{\sigma}{\epsilon_0} \vec{n} . \quad (13.15)$$

In order to calculate the force acting on the charge $\sigma \cdot dS$ is sufficient to multiply such a charge by \vec{E} ,

$$d\vec{F} = \sigma \cdot dS \cdot \vec{\hat{E}} = \sigma \cdot dS \cdot \frac{\vec{E}}{2}$$

$$= -\frac{\sigma^2}{2\epsilon_0} \cdot dS \cdot \vec{n} \quad . \quad (13.16)$$

The force acting on the charge $\sigma \cdot dS$ on the conductor is thus normal to Σ , directed outside the surface regardless from the sign of σ [the density is squared in (13.16)], and proportional to the area dS .

The forces acting on Σ thus behave similarly to those deriving from a pressure

$$p_e = \frac{\sigma^2}{2\epsilon_0} \quad , \quad (13.14)$$

which has units force per unit area and is called electrostatic pressure. We can thus imagine the electrostatic equilibrium on the surface of a conductor as due to the pressure of a fictitious fluid that fills the conductor and tends to expel its charges, in equilibrium with restoring forces on the conductor

surface that prevent the charges from escaping. The restoring forces are due to the potential barrier at the conductor surface.

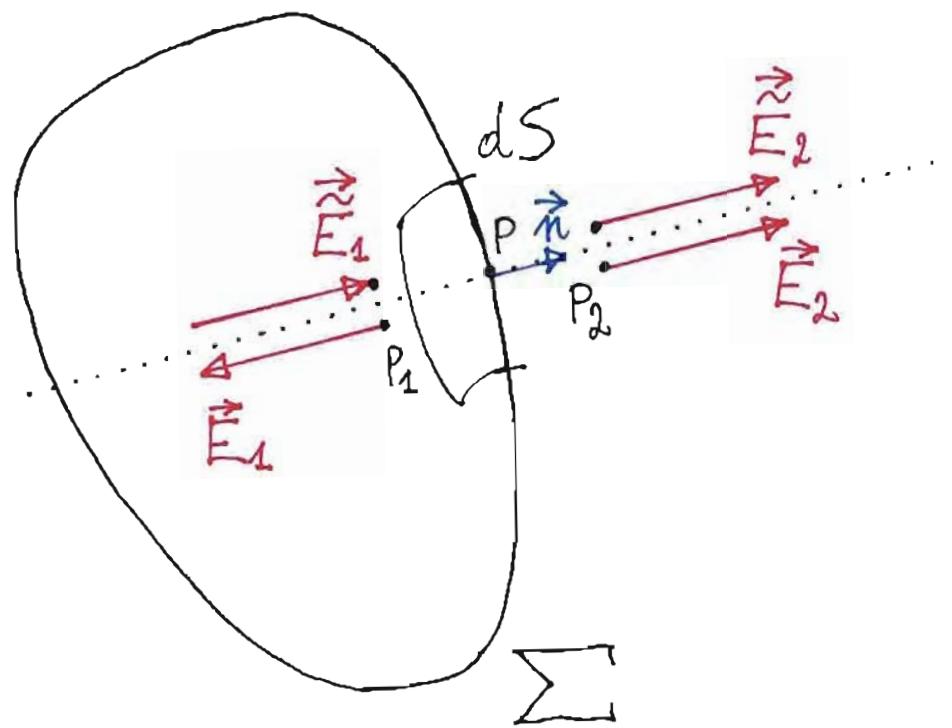


Figure 13.2.

13.3 Field calculation in presence of conductors.

Consider an initially neutral conductor to which a charge q is applied (q is assumed to be finite). This charge distributes on the conductor surface Σ with density σ (variable from point to point) such that

27/34

the field inside the conductor is zero at each point). We assume σ produces a potential V associated with a field \vec{E} . We indicate the internal and external regions of the conductor as Ω^- and Ω^+ , respectively, as shown in Fig. 13.3.

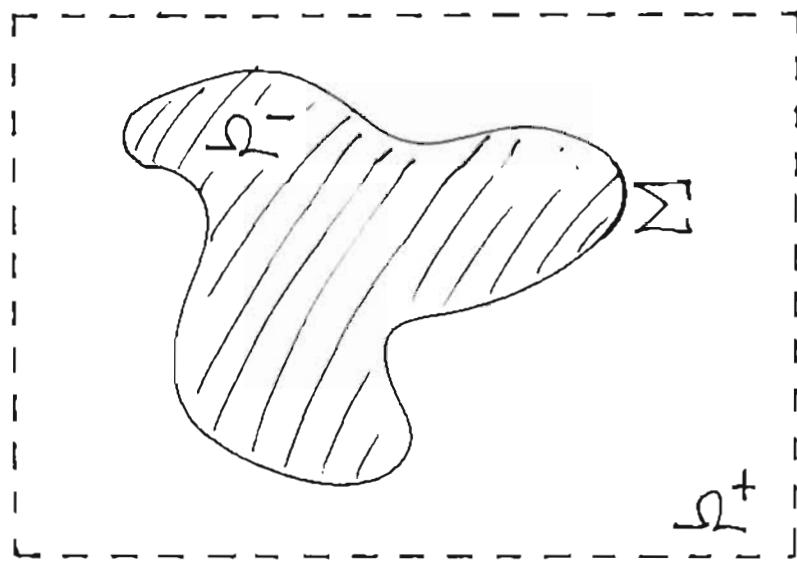


Figure 13.3.

The potential function V must satisfy the following conditions:

- It must be continuous at each point in Ω^+ , including the frontier Σ .
- At each point in Ω^+ , where $\delta=0$, it must satisfy Laplace equation

$$\vec{\nabla}^2 V = 0 .$$

- c) It must have a constant value at each point in Ω , including each point on Σ .
- d) It must verify the condition

$$-\sum \oint_{\Gamma} \frac{\partial}{\partial n} V \cdot dS = \frac{1}{\epsilon_0} q .$$

- e) It must verify the condition

$$\lim_{P \rightarrow \infty} V(P) = 0 .$$

Condition a) is due to the property of the potential to be a continuous function even in correspondence to surface charge distributions. As always, the presence of the potential barrier due to the double layer at the conductor surface is not accounted for. Condition b) accounts for the absence of any charges outside the conductor. About condition c), it is worth mentioning

that, a priori, we only know that V has to be constant on Σ , but its value is unknown. In fact, we are not allowed to arbitrarily choose a value because condition e) must be fulfilled. As for condition d), the integral refers to the outer conductor surface Σ^+ , since the normal derivative of V is discontinuous at each point on Σ^- . At last, condition e) is due to the fact that the charge q distributed with density σ entirely resides at a finite region of space. It is clear that this is true because the considered conductor is assumed to be limited in space. Obviously, we could have chosen any nonzero constant value at infinity by modifying condition e). We remind that condition e) means that for any $\epsilon > 0$, it is possible to find a sphere (enclosing the conductor) outside of which V has values smaller than ϵ (regular condition at infinity).

13.3.1 Conducting sphere.

Consider a conducting sphere in vacuum, with radius R and center O . A charge q is applied to the sphere.

With respect to a spherical coordinate system

Or θ , the sphere is characterized by a rotation symmetry with respect to both θ and ϕ . Hence, the potential V , which we intend to calculate, depends only on the distance r from the center O of the sphere. From condition b), outside the conducting sphere for $r \in (R, +\infty)$, we can write Laplace equation for V in spherical coordinates as

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d}{dr} V \right) = 0 . \quad (13.18)$$

In this case, Laplace equation is a simple ordinary differential equation

$$\begin{aligned} \frac{1}{r^2} 2r \frac{d}{dr} V + \frac{1}{r^2} r^2 \frac{d^2}{dr^2} V &= \\ &= \frac{d^2}{dr^2} V + \frac{2}{r} \frac{d}{dr} V = 0 . \quad (13.19) \end{aligned}$$

It can be simply verified that a general solution to (13.19) is

$$V(r) = \frac{A}{r} + B , \quad (13.20)$$

where A and B are arbitrary constants. Conditions a) and c) can easily be verified. In fact, by arbitrarily setting A and B , $V(r) \in C^0(r \in [R, +\infty))$ and

$$V(R) = \frac{A}{R} + B , \quad (13.21)$$

which means $V(r)$ is constant at each point on Σ (and, thus, at each point inside the sphere). Condition e) is readily verified by imposing $B = 0$,

$$V(r) = \frac{A}{r} . \quad (13.22)$$

At each point on Σ^+

$$\left. \frac{\partial}{\partial n} V \right|_{\Sigma^+} = \left. \frac{\partial}{\partial r} V \right|_{r=R^+} = -\frac{A}{R^2} . \quad (13.23)$$

From condition d) we then obtain

$$-\oint \frac{\partial}{\partial n} V \cdot dS = \frac{A}{R^2} 4\pi R^2 = \frac{1}{\epsilon_0} q , \quad (13.24)$$

from which

$$A = \frac{q}{4\pi\epsilon_0} . \quad (13.25)$$

The solution for V is thus

$$\left\{ \begin{array}{l} V(r) = \frac{1}{4\pi\epsilon_0} \frac{q}{r} , \quad r \in [0, R] \\ \end{array} \right. \quad (13.26a)$$

$$\left\{ \begin{array}{l} V(r) = \frac{1}{4\pi\epsilon_0} \frac{q}{r} . \quad r \in [R, +\infty) \end{array} \right. \quad (13.26b)$$

As expected, for $r > R$ the potential corresponds to that of a point-like charge q located at O .

13.4 Capacitance.

Consider the case of a conducting sphere. We decide to call

$$V_0(r) = +\frac{A}{r} \quad , \quad (13.27)$$

i.e., the potential that assumes a value

$$V_0 = +\frac{A}{R} \quad (13.28)$$

at each point on Σ . It is easy to calculate the charge q_0 as

$$q_0 = -\epsilon_0 \iint_{\Sigma^+} \frac{\partial}{\partial r} V_0(r) \cdot dS \quad (13.29)$$

This charge corresponds to the charge that would be globally present on the conductor when its potential were V_0 as in (13.28).

Imagine to associate a new potential V_1 to the conductor. It can be verified that

$$V_1(r) = \frac{V_1}{V_0} V_0(r) \quad , \quad (13.30)$$

when

34/34

$$V_1(r) = +\frac{\tilde{A}}{r} \quad , \quad (13.31)$$

$$V_1 = +\frac{\tilde{A}}{R} \quad , \quad (13.32)$$

and V_0 and $V_0(r)$ are given by (13.24) and (13.28), respectively. In fact,

$$\begin{aligned} V_1(r) &= +\frac{\tilde{A}}{R} \left(+\frac{R}{A} \right) \left(+\frac{A}{r} \right) \\ &= +\frac{\tilde{A}}{r} . \end{aligned}$$

As it turns out, (13.30) is valid in general for all harmonic functions, i.e., continuously derivable function that satisfy Laplace equations (not only for the simple case of the conducting sphere).

The condition (13.30) means that when the solution to the problem is known for a given value of the potential on the conductor, it is straightforward to obtain the solution for any other value of the potential.

The condition (13.30) also shows that the equipotential surfaces associated with one solution are equipotentials for any other solution (even if, of course, the value of the potential on each surface must be suitably rescaled from case to case). This also means that the field lines are the same for different values of the potential at the conductor. This is a direct consequence of the linearity of Laplace equation.

Note that from the knowledge of $V_1(r)$, we can readily calculate the charge q_1 as

$$\begin{aligned}
 q_1 &= -\epsilon_0 \iint_{\sum^+} \frac{\partial}{\partial r} V_1(r) \cdot dS \\
 &= -\epsilon_0 \frac{V_1}{V_0} \iint_{\sum^+} \frac{\partial}{\partial r} V_0(r) \cdot dS \\
 &= \frac{V_1}{V_0} q_0 \quad . \quad (13.33)
 \end{aligned}$$

thus,

$$\frac{q_1}{V_1} = \frac{q_0}{V_0} \quad . \quad (13.34)$$

From the arbitrariness of q_1 , V_1 , q_0 , and V_0 it follows that the ratio between the charge applied to the conductor and its corresponding potential is a constant, independently from the value of q and V ,

$$\frac{q}{V} = C \quad . \quad (13.35)$$

The constant of proportionality, C , is called the capacitance of the isolated conductor. The capacitance depends only on the conductor geometry. In the case of the conducting sphere (isolated in space), from (13.27) and (13.29)

$$\frac{q}{V} = \frac{q|\Sigma^+}{V|\Sigma^-} = \frac{q_0}{V_0}$$

$$\begin{aligned}
 &= \frac{R}{A} \left[-\epsilon_0 \iint_{\sum^+} \frac{\partial}{\partial r} V(r) \cdot dS \right] \\
 &= \cancel{\frac{R}{A}} \left[+\epsilon_0 \int_0^{2\pi} d\varphi \int_0^\pi \sin\theta \cdot d\theta \right. \\
 &\quad \left. \left(+\frac{1}{r^2} \right) \cancel{R^2} \right] \\
 &= 4\pi \epsilon_0 R . \quad (13.36)
 \end{aligned}$$

For $R = 1\text{m}$, $C \approx 10^{-10}\text{F} = 100\text{nF}$, where the SI units of C are

$$[C] = \frac{[q]}{[V]} = \frac{C}{V} = \text{F (farad)}. \quad (13.37)$$

Summary lecture 13.

- Conductors are objects with an Avogadro number of charged particles free to move within the object.
- A homogeneous and isotherm conductor is in electrostatic equilibrium when no macroscopic motion of charges takes place within the conductor.

- Hence,

$$\vec{E} = 0$$

at each point inside the conductor.

- In addition,

$$\delta = 0$$

at each point inside the conductor.

- In equilibrium, a charge with density σ distributes on the surface of the conductor so that $\vec{E} = \vec{0}$ at each point inside the conductor. The conductor behaves

as a charged shell.

- Coulomb's theorem.

Consider a conductor with surface Σ and a generic point P_0 on Σ . Assuming a point P outside the conductor,

$$\lim_{P \rightarrow P_0^+} E_n(P) = \frac{\sigma(P_0)}{\epsilon_0} .$$

This result is a direct consequence of lecture 5, case 2 for E_n .

- A direct consequence of lecture 5, case 2 for E_t is that

$$\lim_{P \rightarrow P_0^+} E_t(P) = 0 .$$

- The charge q on the conductor surface is given by

$$q = \iint_{\Sigma} \sigma \cdot dS = \epsilon_0 \iint_{\Sigma'} E_n \cdot dS ,$$

where \sum' is a closed surface enclosing the entire conductor and 3/34

$$E_n(P) = - \frac{\partial}{\partial n} V(P)$$

- The force acting on a charge $dq = \sigma \cdot dS$ on \sum is

$$d\vec{F} = \frac{\sigma^2}{2\epsilon_0} \cdot dS \cdot \vec{n}$$

where

$$p_e = \frac{\sigma^2}{2\epsilon_0}$$

is the electrostatic pressure.

- Given a conductor in a region Ω^- bounded by a surface \sum , with Ω^+ a region outside the conductor.
 - The potential V must:
 - $V \in C^0(\Omega^+ \cup \sum)$.
 - $\forall P \in \Omega^+$,

$$\vec{\nabla}^2 V = 0$$

c) $\forall P \in (Q^- \cup \Sigma)$,

$$V(P) = V_0 \text{ (const.)}$$

d)

$$-\oint_{\sum^+} \frac{\partial}{\partial n} V \cdot dS = \frac{1}{\epsilon_0} q .$$

e)

$$\lim_{P \rightarrow \infty} V(P) = 0 .$$

- Capacitance.

$$C = \frac{q}{V} , \quad [C]_{SI} = \frac{C}{V} = F \text{ (farad)} .$$

14.1 The field in hollow conductors: Electrostatic shields.

5/34

Consider a conductor Γ characterized by an internal cavity Ω as shown in Fig. 14.1. A charge q is applied to the conductor. At first glance, we might expect that q distributes both on the external surface Σ_{ext} and on the internal surface Σ_{int} of the conductor. We will prove that, however, the charge only distributes on Σ_{ext} .

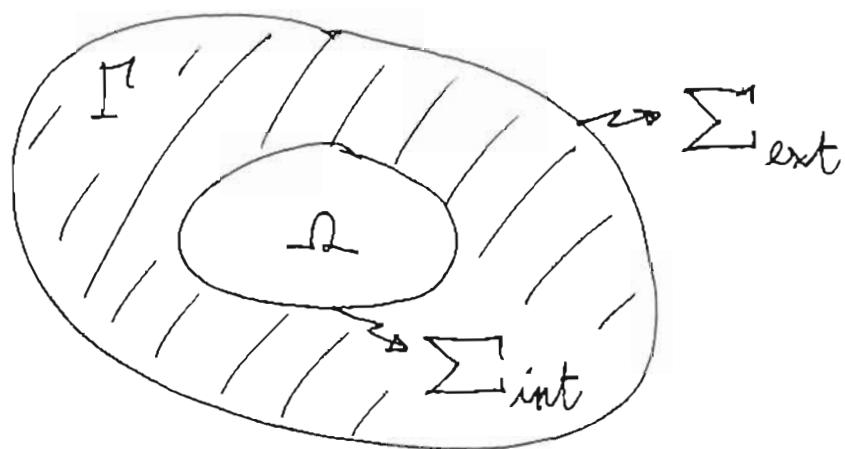


Figure 14.1.

This can be shown by considering the potential function $V(P)$ solution to the problem. Due to the absence of any charges inside the inner cavity Ω , at each point in the region Ω bounded by the surface Σ_{int}

$V(P)$ must satisfy Laplace equation,

$$\vec{\nabla}^2 V = 0 .$$

In addition, $V(P)$ must have a constant value V_0 at each point in the conductor, including each point on \sum_{int} . In summary,

$$V(P) \in C^0(\Omega \cup \sum_{\text{int}}) , \quad (14.1a)$$

must satisfy

$$\vec{\nabla}^2 V = 0 , \quad \forall P \in \Omega \quad (14.1b)$$

and

$$V(P) = V_0 . \quad \forall P \in \sum_{\text{int}} \quad (14.1c)$$

Mathematically, this type of PDE problem is called an internal Dirichlet problem for a constant value of the potential on the frontier (boundary) of the region Ω . In brief, this problem can be summarized as

$$\left\{ \begin{array}{l} \vec{\nabla}^2 V(P) = 0, \quad \forall P \in \Omega \\ V(P)|_{\sum_{\text{int}}} = V_0 \end{array} \right.$$

under the assumption that $V(P) \in C^0(\Omega \cup \sum_{\text{int}})$. By definition (cf. lecture 13), the function $V(P)$ is harmonic in Ω because it satisfies Laplace equation in Ω . From Green's first identity,

$$\begin{aligned} \iint_{\sum} u \frac{\partial}{\partial n} v \cdot dS &= \iint_{\sum} u \cdot (\vec{\nabla} v) \cdot \vec{n} \cdot dS \\ &= \iiint_{\Omega} (\vec{\nabla} u) \cdot (\vec{\nabla} v) \cdot d\varphi \\ &\quad + \iiint_{\Omega} u \cdot (\vec{\nabla}^2 v) \cdot d\varphi, \quad (14.2) \end{aligned}$$

where u and v are harmonic functions defined in a region Ω of the 3D euclidian space with frontier \sum and

$(\partial V / \partial n)$ is the directional derivative of $V(P)$ with respect to a normal unit vector \vec{n} at a generic point P on Σ directed outside the surface (for the external Dirichlet problem) and inside the surface (for the internal Dirichlet problem; cf. tutorial 3 for a definition of directional derivative), in the case $u = v = V$ we find

$$\begin{aligned}
 \sum_{\text{int}} \iint_{\Sigma} V(P) \frac{\partial}{\partial n} V(P) \cdot dS &= \iiint_{\Omega} \|\vec{\nabla} V(P)\|^2 \cdot d\tau \\
 &\quad + \iiint_{\Omega} V(P) \cdot 0 \cdot d\tau \\
 &= \iiint_{\Omega} \|\vec{\nabla} V(P)\|^2 \cdot d\tau,
 \end{aligned} \tag{14.3}$$

where we used (14.1b) to calculate the second volume integral. Under the boundary condition (14.1c), (14.3) becomes

$$\oint_{\sum_{\text{int}}} \mathbf{V}(\mathbf{P}) \cdot \mathbf{d}\mathbf{S} = 0$$

$$\sum_{\text{int}} = \iint_{\Omega} \|\vec{\nabla} V(\mathbf{P})\|^2 \cdot d\mathcal{E},$$

from which it follows that

$$\vec{\nabla} V(\mathbf{P}) \Big|_{\Omega} = 0$$

and, then,

$$V(\mathbf{P}) \Big|_{\Omega \cup \sum_{\text{int}}} = V_0 \quad (14.4)$$

because of the continuity of $V(\mathbf{P})$, condition (14.1a).
Finally, from (14.4)

$$\vec{E}(\mathbf{P}) \Big|_{\Omega \cup \sum_{\text{int}}} = -\vec{\nabla} V(\mathbf{P}) \Big|_{\Omega \cup \sum_{\text{int}}} = \vec{0}. \quad (14.5)$$

In particular, the normal component of \vec{E} at each point on \sum_{int} is zero. A consequence of Coulomb's

theorem is then that the charge density σ on 10/37
 \sum_{int} must be zero. Hence, the entire applied charge
must distribute only on the external surface \sum_{ext} .

A similar argument makes it possible to show that
any charge distribution outside the conductor does not
produce any field inside the cavity. In other words, a
hollow conductor behaves as an electrostatic shield with
respect to the cavity.

Note that, in reality, practical electrostatic shields
are not completely closed hollow conductors (very hard to
implement). Instead, they consist of metallic grids called
"Faraday cages". These are lattices of wires with smaller
or larger unit cells, depending on the applications. In
fact, it is possible to show (it is rather intuitive) that
the fields due to charge distributions external to a cage
"penetrate" through the cage cells only to a distance of a
few cell's linear lengths. For example, a room the
walls, floor, and ceiling of which incorporate a Faraday
cage are electrostatically shielded ambient. Certain
laboratories feature Faraday cages in order not to perturb
experiments due to electromagnetic interference (EMI).

The internal Dirichlet problem (14.1a)-

(14.1c) is based on both Gauss' theorem and the irrotational property of \vec{E} in local form. In fact, Laplace equation (14.1b) was obtained from the differential form of both Maxwell's equations for the electrostatic field. Conclusions similar to (14.4) and (14.5) can be found from the integral form of Maxwell's equations.

Consider Fig. 14.2, representing again a hollow conductor Γ with an inner cavity Ω . A charge q is applied to the external surface Σ_{ext} of Γ . In order to show that $\vec{E} = \vec{0}$ at each point in $\Omega \cup \Sigma_{\text{int}}$, where Σ_{int} is the internal surface of Γ , we can use Gauss' theorem in integral form by defining a gaussian surface Σ that encloses entirely $\Omega \cup \Sigma_{\text{int}}$ and, at the same time, is entirely contained within the conducting region Γ . Under these conditions (cf. Fig. 14.2),

$$\oint_{\Sigma} \vec{E} \cdot \vec{n} \, dS = 0 = \frac{1}{\epsilon_0} q_{\Sigma}, \quad (14.6)$$

where we used the condition $\vec{E} = \vec{0}$ at each point in Γ .

The total flux through Σ is zero and, as a consequence, the total charge inside Σ must be zero, $q_{\Sigma} = 0$. Since the inner cavity Ω is assumed to be empty, this result also implies that the total charge on Σ_{int} must be zero, $q|_{\Sigma_{\text{int}}} = 0$.

12/34

In general, the result of Gauss' theorem of Eq. (14.6) only allows us to conclude that there must be equal amounts of positive and negative charge on Σ_{int} for the total charge $q|_{\Sigma_{\text{int}}} = 0$. In other words, there could be a positive surface charge distribution on one area of Σ_{int} and a negative one on some different area, as shown in Fig. 14.2. Citing Feynman, "Such a thing cannot be ruled out by Gauss' law [theorem]."¹ One more time, Gauss' theorem alone is insufficient to fully characterize the electrostatic field; the irrotational property of \vec{E} must also be invoked.

Before proceeding further, we note that in the simple case of a spherical conducting shell, only by means of (14.6) we could reach the final conclusion that there must be no charge at any point inside the shell, including its internal surface (note that the conducting shell is amongst the simplest types of hollow conductors).

This result can be obtained from symmetry arguments. In fact, because of the rotation symmetry of the shell with respect to both meridian and parallel lines, the asymmetric accumulation of, e.g., a positive surface charge on one part of the shell inner surface and, thus, of a corresponding negative surface charge on another part must be excluded. The only charge distribution on the inner surface that satisfies the symmetries is a zero charge distribution. This, however, could not be true for a hollow conductor of arbitrary geometry, for which symmetry arguments do not apply.

The physical process that takes place on the internal surface Σ_{int} of an arbitrary hollow conductor is that all equal and opposite charges on Σ_{int} move on the surface until they meet and cancel each other completely.

This can be shown by means of the irrotational property of \vec{E} . As shown in Fig. 14.2, assume a positive surface charge is distributed on one area of Σ_{int} . From (14.6) we know that there must be an equal negative surface charge distributed on some other area of Σ_{int} . Since, again, Ω is an empty space (i.e., it does not contain any free charges), in Ω there will be a field \vec{E}

with field lines directed from the positive charges 14/34
 to the negative charges (as in the case of the
 electrostatic dipole; note that, technically, \vec{E} is in
 $\Omega \cup \Sigma_{\text{int}}$). We can now use the irrotational property
 of \vec{E} in integral form by defining a closed line γ
 that crosses $\Omega \cup \Sigma_{\text{int}}$ along the field line between one
 positive charge and one negative charge and that returns
 to the initial point through Γ (cf. Fig. 14.2). Under
 these conditions,

$$\oint_{\gamma} \vec{E} \cdot \vec{t} \cdot d\ell = \int_{\gamma_{\Omega \cup \Sigma_{\text{int}}}} E \cdot d\ell + \int_{\gamma_{\Gamma}} \vec{o} \cdot d\ell \\ \neq 0, \quad (14.4)$$

where $\gamma_{\Omega \cup \Sigma_{\text{int}}}$ is the portion of γ in $\Omega \cup \Sigma_{\text{int}}$ and we
 used the condition $\vec{E} = \vec{o}$ at each point in Γ . The scalar
 product in the first integral simply becomes $E \cdot d\ell$
 because of the definition of field line! The result (14.4)
 contradicts the irrotational property. The only case when
 the property remains valid is for $\vec{E} = \vec{o}$ at each point
 in $\Omega \cup \Sigma_{\text{int}}$. We then obtain the same result as (14.5).

This also means there can be no charges on Σ_{int} . 15/37
 Note that this is true for an empty Ω .

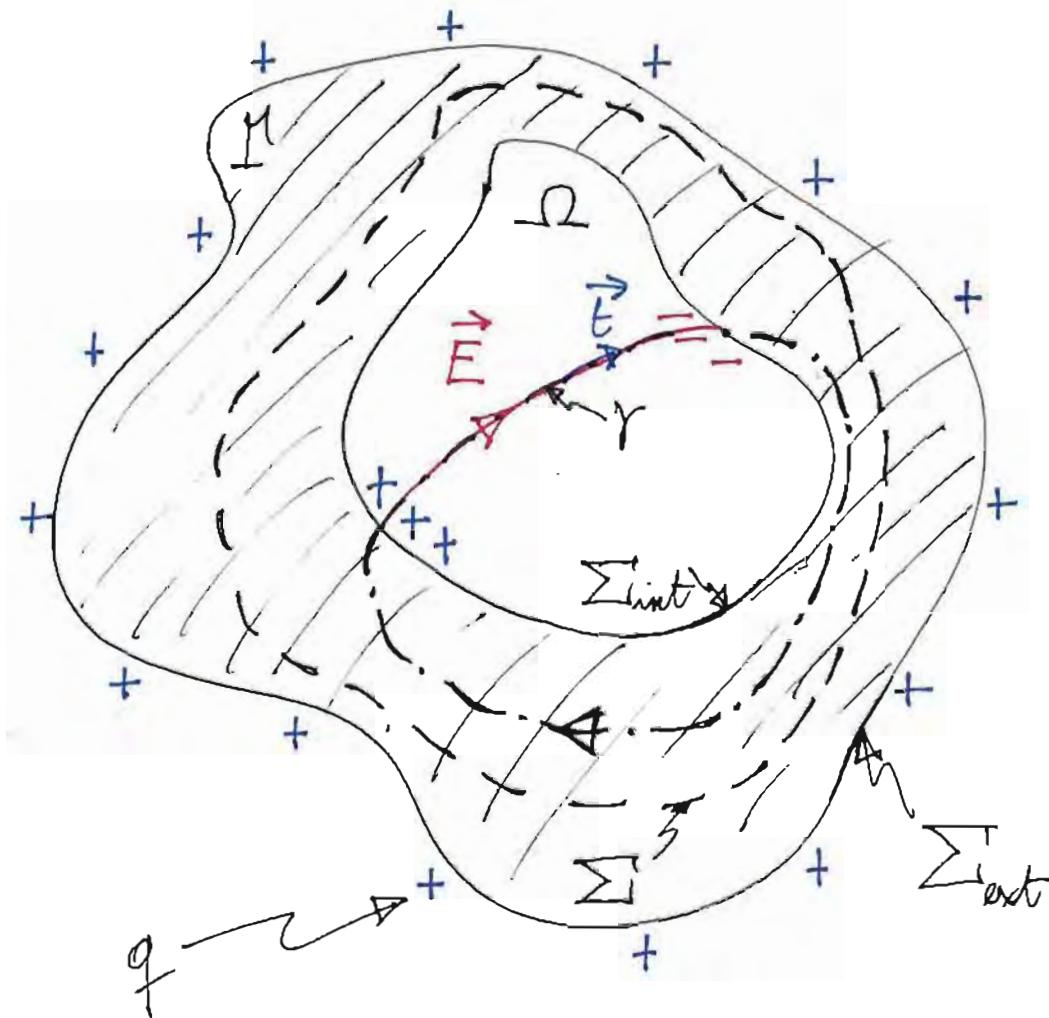


Figure 14.2.

An important property of electrostatic shields is that the behavior of the shield is symmetrical, i.e., any charge distribution inside Ω does not affect the external region to Π . At first glance, this might seem wrong. In fact, if we introduce inside the region Ω of the conductor Π in Fig. 14.2 a charged body C , e.g., a

charged conductor C (cf. Fig. 14.3), it is easy to verify that around C a field \vec{E} exists. Necessarily, the field lines associated with \vec{E} must go from the outer surface of C (assuming C positively charged) to the inner surface of \mathcal{I} , Σ_{int} . We are induced to think that the charge distributed on Σ_{int} must be equal and opposite than that on C . This can easily be demonstrated by using Gauss' theorem for the usual gaussian surface Σ indicated in Fig. 14.3 (dashed blackline). The flux of the electrostatic field through Σ is zero (again because the field is zero at each point in \mathcal{I}). Hence, the charge contained within Σ is zero and, as a consequence, a charge equal and opposite to that on C must distribute on Σ_{int} . In addition, because of the charge conservation principle, the initially neutral conductor \mathcal{I} must remain neutral even after the introduction of C in \mathcal{I} . This is possible only if a charge equal and opposite to that on Σ_{int} distributes on Σ_{ext} (this charge is thus equal to that on C in \mathcal{I}). The charge on Σ_{ext} generates a field outside the cavity. It would, however, be wrong to think that the charged body C inside the cavity affects electrically the region outside. In fact, what actually

alters the electrostatic scenario outside the cavity is the introduction in Ω of a charged body. This event, however, does not take place entirely and solely inside the cavity. The external world must take part to the event by giving away some charge to $C!$ In other words, during this operation the external world is in communication with Ω .

Consider instead what happens when C is already inside the cavity and we only change its position within Ω , or we divide it into two parts, or, in general, we perform any other physical operation that implies the total charge conservation inside the cavity. These operations have no influence on the external world. In other words, the surface charge distribution on Σ_{ext} does not change when the conditions inside the cavity are modified. If, for example, we imagine to change the position of C in Ω , the charge on Σ_{ext} remains unchanged. As a consequence, the calculation of V outside Σ_{ext} can be carried out by means of an external Dirichlet problem regardless from the position of C in Ω .

In summary, the charge on Σ_{ext} and the field

outside Γ , for the same geometry and position of Γ and for the same external conditions (e.g., other nearby charges, other nearby conductors, etc.), are totally independent from what takes place in Ω : The electrostatic shield works both ways, isolating the interior from external influences and viceversa.

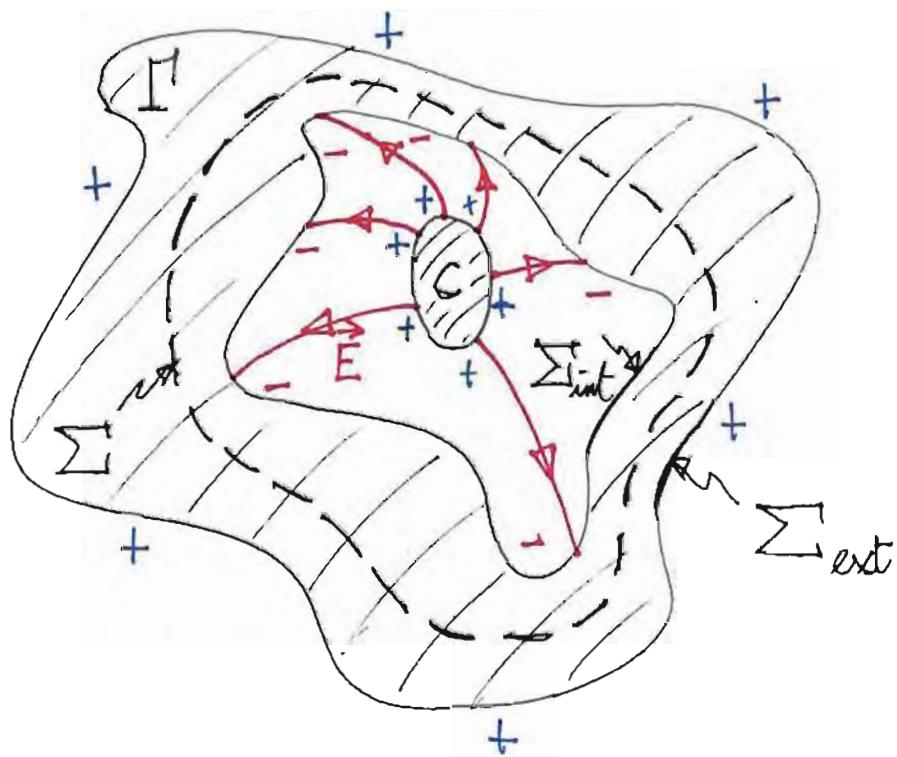


Figure 14.3.

Now that we understand the concept of electrostatic shield, when you will fly next time over the Atlantic ocean on an A380 in the middle of a beautiful thunderstorm you can reassure your shaking neighbor that there is

nothing to worry about: Because of an internal Dirichlet problem (or, if you want, Gauss' theorem and the irrotational property), the aircraft's frame will make a "Faraday cage" that will prevent you to get any electric shock and that will protect the avionics. He might still tell you to get lost, but you can sit back, relax, and enjoy the ride!

Also note that the properties of the electrostatic shields do not depend on the shape of the considered conductor: No matter what its geometry is, $\vec{E} = \vec{0}$ in the inner cavity Ω . This fact makes it possible to perform the Plimpton and Gavilon experiment to an accuracy of better than one part in a billion. Even if the used conducting shell is not perfectly spherical, $\vec{E} = \vec{0}$ inside the shell and the exponent in Coulomb's law is exactly 2 (cf. lecture 8).

14.2 Electrostatic induction (revisited).

We now intend to give a more quantitative description of the concept of electrostatic induction introduced in

lecture 1. We consider to bring an initially neutral conducting body in a region with an electrostatic field generated by a set of fixed charges outside the conductor. Under the action of the field, the charge carriers in the conductor move towards the conductor surface Σ in a very short time and distributes on Σ in a way that counter-balances point by point inside the conductor the external field. As a consequence, the field inside the conductor becomes zero, while outside it gets deformed compare to the configuration without conductor. Figure 14.4 shows this phenomenon in the case of a uniform initial external field.

At this point, we must calculate the induced charge distribution on the conducting body. We proceed assuming a full knowledge on the value and position of the inducing (external) charges q_{ext} . We also assume that their distribution does not change upon introducing the conductive body. This assumption corresponds to consider all charges q_{ext} (external to the conductor) to be fix in the lab reference frame. Consider now the (Known) potential:

$V_{\text{ext}}(P)$ generated at a generic point P by q_{ext} . In general, this potential changes point by point on Σ . In fact, it is the total potential $V(P)$, given by the sum of V_{ext} and the potential generated by the surface charge distribution induced on Σ , V_{ind} , to be constant in the conductor including its surface Σ . The function $V_{\text{ind}}(P)$ must satisfy Laplace equation at each point in the region Ω^+ outside Σ ,

$$\vec{\nabla}^2 V_{\text{ind}}(P) = 0 \quad (14.8)$$

(because the charges generating $V_{\text{ind}}(P)$ are distributed only on the surface Σ). In addition, at each point on Σ must be

$$\begin{aligned} V(P) \Big|_{\Sigma} &= V_{\text{ext}}(P) \Big|_{\Sigma} + V_{\text{ind}}(P) \Big|_{\Sigma} \\ &= \text{const.} \quad (14.9) \end{aligned}$$

If we assume to know the value of V_0 (note that V_0

cannot be arbitrarily chosen because of the overall neutrality of the conductor; we will not show how to determine V_0 here), (14.9) reads

$$\left. V_{\text{ind}}(P) \right|_{\Sigma} = V_0 - \left. V_{\text{ext}}(P) \right|_{\Sigma} . \quad (14.10)$$

We must thus find a function $V_{\text{ind}}(P) \in C^0(\Omega^+ \cup \Sigma)$, that verifies Laplace equation at each point in Ω^+ , assumes on Σ the values given by (14.10), and has a regular behavior at infinity. This is an external Dirichlet problem for the boundary condition (14.10). Suppose we manage to find such a solution $V_{\text{ind}}(P)$. By summing this solution with $V_{\text{ext}}(P)$, we find the potential $V(P)$ outside the conductor. We can then calculate the derivative $(\partial V / \partial n)$ at each point on the exterior of Σ , Σ^+ . Since the derivative at each point on the interior of Σ , Σ^- , must be zero (because V is constant inside the conductor), we can conclude that

$$\frac{\partial}{\partial n} V(P) \Big|_{\Sigma^+} = - \frac{\sigma_{\text{ind}}}{\epsilon_0} \quad . \quad (14.11)$$

This result allows us to find the induced surface charge distribution density σ_{ind} at each point on Σ .

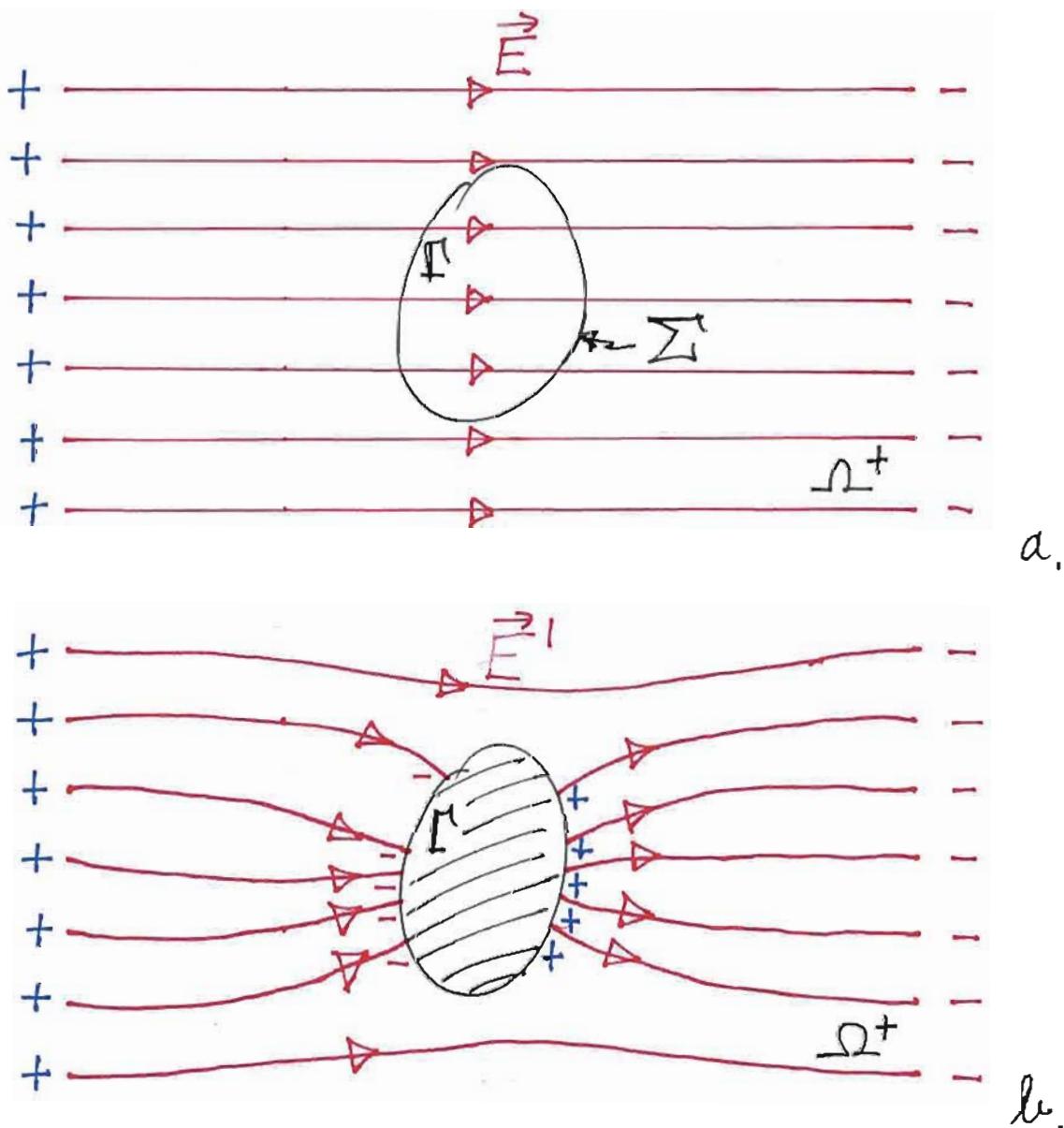


Figure 14.4.

a. Initial condition. b. Start formation of induced charges on Σ .

The concept of electrostatic induction also allows us to revisit the operative definition of electrostatic field.

24/31

If a set of charged conductors is fixed in space by maintaining each of the conductors fixed at its position by suitable forces, the charge distribution on each of them is determined by the position and charge on all the other conductors. It is sufficient to move one of the conductors to modify the charge distribution on the other conductors because of the electrostatic induction. Thus the electrical charges are not fixed and the electrostatic field cannot be measured by means of any arbitrary test charge. A small enough test charge that does not modify the pre-existing charge distribution appreciably must be used. The presence of such a test charge should only produce a negligible perturbation.

For example, consider an infinite plane that separates the space into two regions. On the left side of the plane, the space is filled with a conducting material. On the right side there is vacuum.

The conductor and, thus, the plane that represents its surface are assumed to be neutral. Hence, the field is zero at each point in space. Suppose we want to "measure" such a field by means of a test charge q at a generic point P . After locating q at P , because of electrostatic induction, the surface of the conductor gets charged with sign opposite with respect to that of q . If we choose q to be positive, the plane gets negatively charged and a force due to the induced negative charge acts on q . By measuring the force \vec{F} and by dividing it by q , we obtain a field

$$\vec{E} = \frac{\vec{F}}{q}$$

different from zero! Obviously, this cannot be the null field we originally intended to measure in absence of the test charge q . In order to minimize the error, we should consider \vec{F}/q ratios for values of q as small as possible.

We now intend to calculate the force acting on q . To this end we must find a potential function V such that

a) satisfies Poisson equation

$$\vec{\nabla}^2 V = -\frac{1}{\epsilon_0} q \delta(P)$$

in the region on the right side of the conductor.

Note that $\delta(P)$ is the delta-Dirac centered at P_i ;

- b) is continuous until the conducting plane;
- c) takes a constant value on such a plane;
- d) has a regular behavior at infinity.

As it turns out, finding the solution to this problem is rather complicated.

14.3 The method of images.

An easy way to circumvent the Poisson problem outlined above is to resort to a trick known as the method of images.

In lecture 6, Sec. 6.1, we studied the problem of the balanced electrostatic dipole consisting of two

point-like charges, a positive charge $+q$ and a negative charge $-q$, at a distance d from each other. [27/37]

Figure 14.5 (similar to Fig. 6.3) shows the field lines associated with the dipole together with a few equipotential surfaces. We remind the latter must be normal to the field lines.

Consider the equipotential surface marked Σ_A in the figure. Assume to shape a thin conducting film so that it fits perfectly with Σ_A . The potential on the conducting film must be constant. If we now adjust such a potential to be exactly the constant potential on Σ_A , V_A , and we insert the film right where Σ_A is defined, there would no manner for an observer to measure any effect due to the presence of the film, or its absence (other than by "touching" the film!). Following Segnman, from the simple problem of an electrostatic dipole, we have now solved a new problem: the problem of a complicated curved conductor with a given potential near a point-like charge q . If the bending radius of the conducting film is infinite (or, at least, very large),

We have effectively constructed the case of a closed conducting shell with a point-like charge q nearby. The shell behaves as an electrostatic shield and, thus, the regions inside and outside the shell behave independently. In particular, we could fill the entire shell with a conducting material and obtain the same result. In other words, we have solved the problem of a full conducting sphere close to a charge q (Fig. 14.6). In the region outside the conductor the field is equal to that of two point-like charges, while inside the conductor is zero. In addition, due to Coulomb's theorem the field lines must be normal to the conductor surface. This is obviously the case here because of the definition of equipotential surface.

In summary, we can compute the field in Fig. 14.6 by simply calculating the field due to a charge $+q$ and to a charge $-q$ positioned at a suitable point. The negative charge $-q$ we position behind the conducting surface is called an image charge.

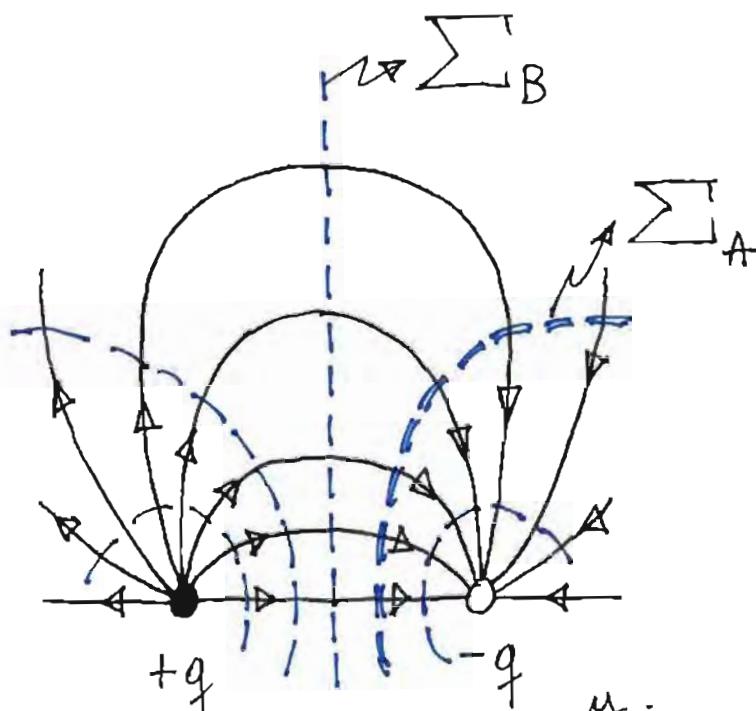


Figure 14.5.

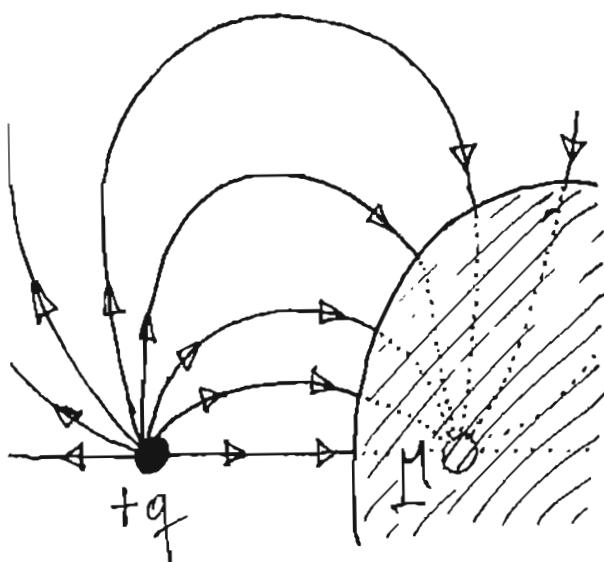


Figure 14.6.

The simplest example on the use of the method of images is for the equipotential surface Σ_B in Fig. 14.5. This makes it possible to solve the problem of

a thin infinite conducting plane facing a point-like positive charge $+q$. The field is obtained from Fig. 14.5 by crossing out the entire region on the right side of Σ_B (cf. Fig. 14.7). Note that the potential due to the charges $+q$ and $-q$ can readily be obtained from the superposition principle,

$$V(P) = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{r_1} - \frac{q}{r_2} \right), \quad (14.13)$$

where r_1 and r_2 are the absolute values of the distances between $+q$ and $-q$ and the generic point P where we want to evaluate the potential. When $P \in \Sigma_B$, $r_1 = r_2$ and, thus,

$$V|_{\Sigma_B} = 0.$$

In other words, we have solved the problem of a positive point-like charge nearby a "grounded" conductor (we will come back to the concept of grounding in the next lecture).

As we know from the theory of electrostatic induction, the positive point-like charge $+q$ induces negative charges on the surface of the conductor. Given a point P_0 on the outer surface (i.e., the surface pointing towards $+q$) of the conductor, the surface density σ associated with the induced negative charges can be calculated from Coulomb's theorem as

$$\lim_{P \rightarrow P_0^+} E_n(P) = \frac{\sigma(P_0)}{\epsilon_0}, \quad (14.14)$$

where E_n is the normal and only possible component of the field at each point on the outer surface of the conductor. In order to calculate $\sigma(P_0)$ we must then calculate the field \vec{E} at P_0 due to both the positive (real) charge $+q$ and the negative (image) charge $-q$. The summation of the two fields, $\vec{E}_e^+(P_0)$ and $\vec{E}_e^-(P_0)$, respectively, coincides with the total field at P_0 due to the conducting plane next to $+q$. Referring to Fig. 14.7, assume a cylindrical coordinate system with origin O at the intersection between the Z axis, which is the

normal to the conductor surface passing through $+q$ and directed from $+q$ to $-q$, and the conductor itself. The generic distance between O and P_0 is r and φ is the angle measured counterclockwise from a reference vertical axis on the conductor surface and passing through O (cf. the figure). Note that the normal unit vector \vec{n} to the conductor surface must be directed from the negative to the positive charge and, thus,

$$\vec{n} = -\vec{t}_z \quad (14.15)$$

with respect to the chosen coordinate system $Ox\varphi z$.

As shown in Fig. 14.4, due to Coulomb's law and the specularity of $+q$ and $-q$, the tangent components $\vec{E}_{te^-}(P_0)$ and $\vec{E}_{te^+}(P_0)$ cancel each other, $\vec{E}_{te^-}(P_0) = -\vec{E}_{te^+}(P_0)$. As expected from Coulomb's theorem, only a normal component $\vec{E}_n(P_0)$ survives: the total field at P_0 is normal to the conductor surface and directed into it (i.e., towards $-q$). At P_0 , the normal component of the field due to $+q$ is given by

$$\vec{E}_{z,e^+}(P_0) = \frac{1}{4\pi\epsilon_0} \frac{q}{\left[\left(\frac{d}{2}\right)^2 + r^2\right]^{1/2}} \cos\delta \cdot \vec{u}_z$$

$$= \frac{1}{4\pi\epsilon_0} \frac{q\left(\frac{d}{2}\right)}{\left[\left(\frac{d}{2}\right)^2 + r^2\right]^{3/2}} \cdot \vec{u}_z, \quad (14.16)$$

where d is the distance between $+q$ and $-q$ and δ is the angle indicated in Fig. 14.4. By projecting this field onto \vec{n} , we obtain the magnitude

$$E_{n,e^+}(P_0) = E_{z,e^+}(P_0) \cdot \vec{u}_z \cdot \vec{n}$$

$$= E_{z,e^+}(P_0) \cdot \vec{u}_z \cdot (-\vec{u}_z)$$

$$= -\frac{1}{4\pi\epsilon_0} \frac{q\left(\frac{d}{2}\right)}{\left[\left(\frac{d}{2}\right)^2 + r^2\right]^{3/2}}. \quad (14.17)$$

Similarly, the normal component at P_0 due to $-q$ is given by

$$\vec{E}_{y,e^-}(P_0) = \vec{E}_{z,e^+}(P_0)$$

and, thus,

$$E_{n,e^-}(P_0) = E_{n,e^+}(P_0) . \quad (14.18)$$

Hence, the magnitude of the total normal component of the field at P_0 is given by

$$E_n(P_0) = -\frac{1}{4\pi\epsilon_0} \frac{qd}{\left[\left(\frac{d}{2}\right)^2 + r^2\right]^{3/2}} \quad (14.19)$$

(the total component is double each component).

Finally, from (14.14) we obtain

$$\begin{aligned} \sigma(P_0) &= \epsilon_0 E_n(P_0) \\ &= -\frac{qd}{4\pi \left[\left(\frac{d}{2}\right)^2 + r^2\right]^{3/2}} . \quad (14.20) \end{aligned}$$

A good check to find if our calculation is correct is to integrate σ over the entire outer surface of the conductor. Proceeding in a similar fashion as for the case of the potential of an infinite charged plane derived from that of a disk (cf. lecture 12), we obtain

$$\begin{aligned}
 q \Big|_{\sum_B^-} &= \oint_{\sum_B^-} \sigma \cdot dS \\
 &= -\frac{qd}{4\pi} \int_0^{2\pi} d\varphi \int_0^{+\infty} \frac{r \cdot dr}{\left[\left(\frac{d}{2}\right)^2 + r^2\right]^{3/2}} \\
 &= \frac{qd}{2} \left[\frac{1}{\sqrt{\left(\frac{d}{2}\right)^2 + r^2}} \right]_0^{+\infty} \\
 &= -q \frac{d}{2} \frac{2}{d} = -q, \quad (14.21)
 \end{aligned}$$

which confirms the presence of an induced negative charge on the conductor (\sum_B^- indicates the outer surface)

of the conductor, facing $+q$).

The last open question is the force \vec{F} acting on $+q$ due to the presence of the charged conducting plane. This can be calculated either by integration from the surface charge density (14.20) or, more simply as the force due to the image charge $-q$. We find

$$\vec{F} = \frac{1}{4\pi\epsilon_0} \frac{q^2}{d^2} \cdot \vec{\nu}_z . \quad (14.22)$$

As a last remark, we notice that the potential function (14.13) fulfills all conditions, a) - d), in page 26. It can be shown that this is the unique solution to our Dirichlet problem.

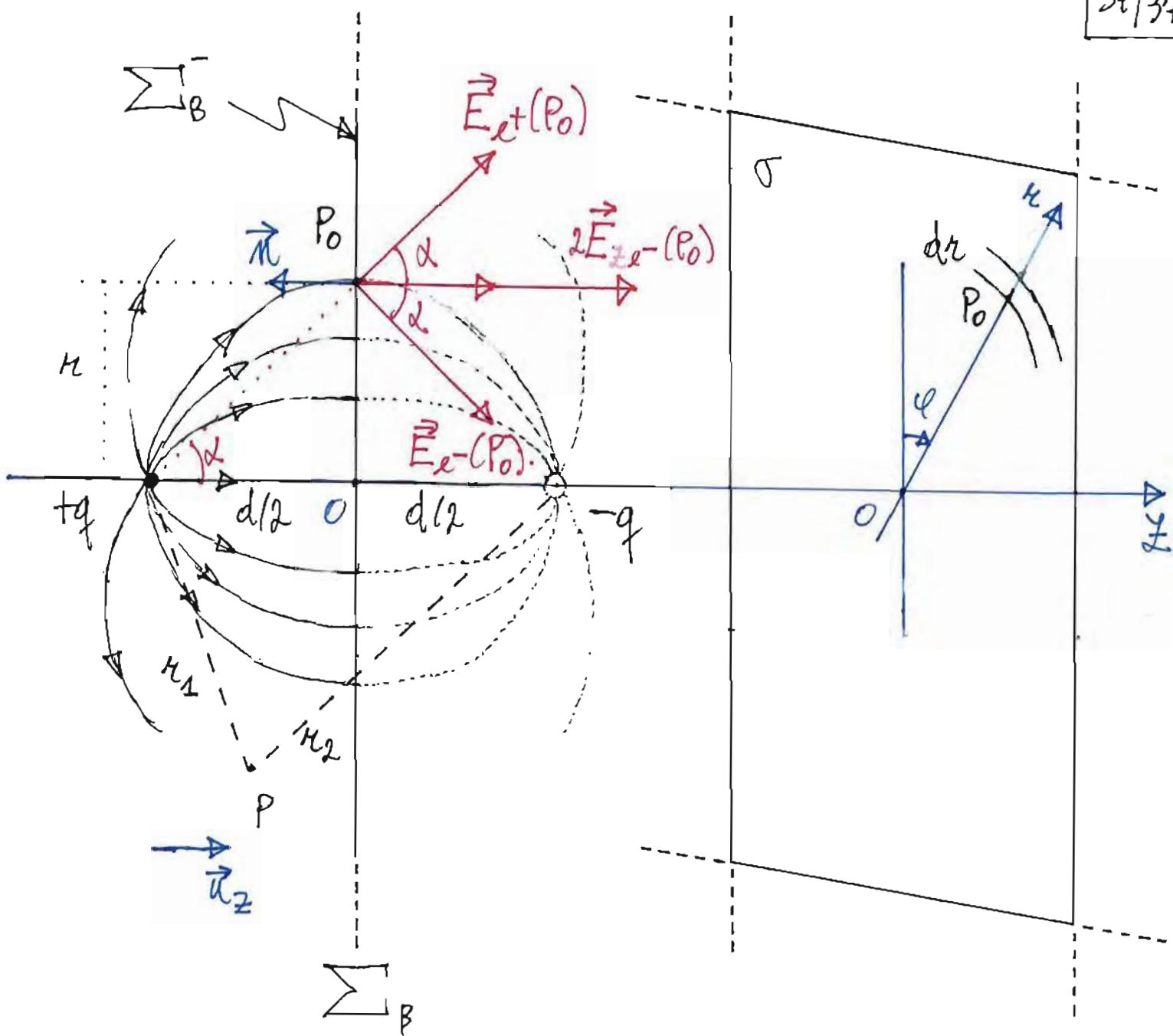


Figure 14.4.

Phys 242 - 914. Lecture 15 - the electrostatic field in presence of conductors in vacuum / part 3. 1/34

Summary lecture 14.

- Electrostatic shields.

Given a conductor Γ with an inner cavity (empty) Ω and internal surface Σ_{int} ,

$$V|_{\Omega \cup \Sigma_{\text{int}}} = V_0 \text{ (const.)}$$

and

$$\vec{E}|_{\Omega \cup \Sigma_{\text{int}}} = \vec{0} .$$

- No charge distributor on Σ_{int} .
- They isolate their interior from external influences and viceversa.

Electrostatic induction.

2/34

Consider a set of inducing charges q_{ext} that generates a potential V_{ext} in a region of space. Upon bringing a conductive body with surface Σ in that region, a set of induced charges q_{ind} distributes on Σ . The induced charges produce a potential V_{ind} such that

$$\vec{\nabla}^2 V_{\text{ind}} = 0$$

in the region outside Σ and

$$V|_{\Sigma} = V_{\text{ext}}|_{\Sigma} + V_{\text{ind}}|_{\Sigma} = V_0 (\text{const.}),$$

which gives the boundary condition

$$V_{\text{ind}}|_{\Sigma} = V_0 - V_{\text{ext}}|_{\Sigma}.$$

- From the total potential V ,

$$\sigma_{\text{ind}} = -\epsilon_0 \frac{\partial}{\partial n} V|_{\Sigma^+},$$

where Σ^+ is the outer surface of the conductor.

3/34

- The method of images.

Consider an infinite conducting plane nearby a single point-like charge $+q$ (the example works even if the plane has finite dimensions, so long the distance from the charge is small compared to the lateral dimensions of the plane). Finding the field lines for this configuration is mathematically hard.

However, the plane is an equipotential surface.

From lecture 6, Sec. 6.1, the plane can be thought as the equipotential associated with a balanced electrostatic dipole with charges $+q$ (the given charge) and $-q$, at equal distance from the two charges.

The field lines of the original problem are, thus, those of the electrostatic dipole crossing out the lines on the region of space from the plane towards the imaginary $-q$ charge (the plane acts as an electrostatic shield). The charge $-q$ is called image charge and this "reverse engineering" technique is called method of images.

Field lines for conductors.

We are now in a position to add two more rules on field lines (cf. list in lecture 4).

- Consider a conductor Ω (interior) $\cup \Sigma$ (surface) in an electrostatic field in vacuum. There is no charge density in Ω , $\rho = 0$, and, thus, $\vec{E} = \vec{0}$ in Ω . The only charge distributes on Σ with density σ . From case 2, lecture 5, \vec{E} must be normal to Σ at each point on Σ (Coulomb's theorem).

The field lines outside a conductor are thus normal to the conductor surface, with zero tangent components, $E_t = 0$. Hence, $F_t = m_q \vec{a}_t = q \vec{E}_t = 0$ and, thus, $\vec{a}_t = \vec{0}$: In equilibrium, the free charges on Σ have zero tangent acceleration and, thus, no current flows in any tangent direction on Σ .

- By means of the method of images, it is possible to reconstruct the field lines of complex charge distributions in the vicinity of conducting bodies.

For example, given a discrete charge distribution of point-like charges nearby a conductor, if the mean distance between the charges and the

conductor is small enough compared to the linear dimensions of the conductor, the conductor can be thought as the equipotential surface of an electrostatic dipole made from one of the charges in the distribution and an image charge associated with the conductor. The total field lines are then obtained by superimposing the lines for each such dipoles, one for each charge in the distribution.

15.1 Capacitors.

6/34

By definition, a capacitor is a physical system characterized by two conductors facing each other and separated by an insulating material (or vacuum). The conductors are called the walls of the capacitor and are charged so that the charge on one wall is equal and opposite than the charge on the other wall. The walls are often made by plates, the linear dimensions of which are much larger than the separation between them.

These simple devices make it possible to create very strong electrostatic fields in limited regions of space and, thus, to store large quantities of electrostatic energy ($U_E \sim E^2$). As it will be shown later, this phenomenon is a consequence of this simple fact: Called A and B the walls of a capacitor with charges $+q$ and $-q$, respectively, the potential difference between the walls is proportional to q and given by

$$V_A - V_B = \frac{q}{C} \quad . \quad (15.1)$$

In (15.1), the coefficient C is an intrinsic characteristic of the capacitor, called capacitance. This coefficient has

the same units [farad (F)] as the analogous coefficient introduced in lecture 13, Sec. 13.4, for an isolated conductor. Note that, the latter can always be regarded as the wall of a capacitor whose second wall is an imaginary equipotential surface at infinity (in reality, e.g., the walls of a lab).

Relation (15.1) can be determined experimentally for each type of capacitor. However, we will try to deduce it from simple theoretical arguments.

We start by considering a general physical system made by two conductor Γ_1 and Γ_2 (cf. Fig. 15.1) with arbitrary shape, isolated from each other, and with respective charges q_1 and q_2 . Assume V_1 and V_2 are the potentials for the two conductors. Defining the surface of each conductor as Σ_1 and Σ_2 , and the space between them Ω , the function V must:

- a) be continuous in $\Sigma_1 \cup \Omega \cup \Sigma_2$;
- b) verify Laplace equation in Ω ;
- c) assume the constant values V_1 and V_2 on Σ_1

and Σ_2 , respectively;

d) be zero at infinity.

The function V is thus solution of an external Dirichlet problem with multiple frontier $\Sigma_1 \cup \Sigma_2$. It can be shown that this problem has a unique solution.

We now indicate with $V_{10}(P)$ and $V_{01}(P)$ the solutions corresponding to the boundary conditions

$$c') \quad \begin{cases} V=1 & \text{on } \Sigma_1, \\ V=0 & \text{on } \Sigma_2; \end{cases}$$

$$c'') \quad \begin{cases} V=0 & \text{on } \Sigma_1, \\ V=1 & \text{on } \Sigma_2, \text{ respectively.} \end{cases}$$

This means that both functions $V_{10}(P)$ and $V_{01}(P)$ satisfy conditions a)-d) and, in addition, V_{10} satisfies c') and V_{01} satisfies c''). Note that for dimensional reasons it is useful to assign the dimension of a potential to the constants V_1 and V_2 and assume the functions $V_{10}(P)$ and $V_{01}(P)$ to be adimensional.

Consider now the function

9/34

$$V(P) = V_1 \cdot V_{10}(P) + V_2 \cdot V_{01}(P), \quad (15.2)$$

where V_1 and V_2 are arbitrary constants. At each point in Ω we have

$$\vec{\nabla}^2 V = V_1 \cdot \vec{\nabla}^2 V_{10} + V_2 \cdot \vec{\nabla}^2 V_{01}. \quad (15.3)$$

Since

$$\vec{\nabla}^2 V_{10} = \vec{\nabla}^2 V_{01} = 0, \quad (15.4)$$

it must be

$$\vec{\nabla}^2 V = 0. \quad (15.5)$$

In addition, as V_{10} and V_{01} go to zero at infinite, so does V . Moreover, the functions V_{10} and V_{01} are both continuous in $\Sigma_1 \cup \Omega \cup \Sigma_2$ and so is V . At last, at each point on Σ_1

$$V \Big|_{\Sigma_1} = V_1 \cdot V_{10} \Big|_{\Sigma_1} + V_2 \cdot V_{01} \Big|_{\Sigma_1}$$

$$= V_1 \cdot 1 + V_2 \cdot 0 = V_1 , \quad (15.6a)$$

and at each point on Σ_2

$$V \Big|_{\Sigma_2} = V_1 \cdot V_{10} \Big|_{\Sigma_2} + V_2 \cdot V_{01} \Big|_{\Sigma_2}$$

$$= V_1 \cdot 0 + V_2 \cdot 1 = V_2 . \quad (15.6b)$$

Since the function V also satisfies to condition c) is the solution of our problem. Obviously, the knowledge of V is based on the knowledge of $V_{10}(P)$ and $V_{01}(P)$, which we assume to be known for now. Those functions depend only on the geometry of the problem, i.e., from the shape of surfaces Σ_1 and Σ_2 . This is the case because the values of these functions on Σ_1 and Σ_2 must satisfy condition c') and c'').

The charges on I_1 and I_2 are

$$q_1 = -\epsilon_0 \iint_{\sum_1} \frac{\partial}{\partial n} V \cdot dS$$

$$= -\epsilon_0 V_1 \iint_{\sum_1} \frac{\partial}{\partial n} V_{10} \cdot dS - \epsilon_0 V_2 \iint_{\sum_1} \frac{\partial}{\partial n} V_{01} \cdot dS,$$

(15.4a)

$$q_2 = -\epsilon_0 \iint_{\sum_2} \frac{\partial}{\partial n} V \cdot dS$$

$$= -\epsilon_0 V_1 \iint_{\sum_2} \frac{\partial}{\partial n} V_{10} \cdot dS - \epsilon_0 V_2 \iint_{\sum_2} \frac{\partial}{\partial n} V_{01} \cdot dS.$$

(15.4b)

By defining

$$C_{11} = -\epsilon_0 \iint_{\sum_1} \frac{\partial}{\partial n} V_{10} dS , \quad (15.8a)$$

$$C_{12} = -\epsilon_0 \iint_{\sum_1} \frac{\partial}{\partial n} V_{01} dS , \quad (15.8b)$$

$$C_{21} = -\epsilon_0 \iint_{\sum_2} \frac{\partial}{\partial n} V_{10} dS , \quad (15.8c)$$

$$C_{22} = -\epsilon_0 \iint_{\sum_2} \frac{\partial}{\partial n} V_{01} dS , \quad (15.8d)$$

we have

$$\begin{cases} q_1 = C_{11} V_1 + C_{12} V_2 \\ q_2 = C_{21} V_1 + C_{22} V_2 \end{cases} , \quad (15.9)$$

or in matrix form

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} . \quad (15.10)$$

The so defined matrix is called the capacitance matrix of the considered physical system. The elements of this matrix depend only on the geometry of the system. It is possible to show (we will not do it here) that the capacitance matrix is symmetric, i. e.,

$$C_{12} = C_{21} . \quad (15.11)$$

At last, from (15.9) and (15.10) we find

$$C_{11} = \left. \frac{q_1}{V_1} \right|_{V_2=0}, \quad C_{12} = \left. \frac{q_1}{V_2} \right|_{V_1=0} = C_{21},$$

$$C_{22} = \left. \frac{q_2}{V_2} \right|_{V_1=0} . \quad (15.12)$$

In order to calculate, for example, the coefficient

C_{11} , we can think to "ground" the conductor Γ_2 , while keeping Γ_1 at the potential V_1 . Once the function $V_{10}(P)$ is calculated, the calculation (or measurement) of the charge q_1 on Γ_1 gives the value of the ratio q_1/V_1 . A similar process applies to the remaining capacitance coefficients.

Note that, in real life, the operation of grounding is realized by connecting a body Γ to the Earth by means of a conducting wire (typically copper) ending in a sharp tip that is pushed into the dirt, deep into the ground. In this way, Γ becomes a small appendix to the huge conducting sphere which is the Earth. As a consequence, Γ must assume the same potential of the Earth. Independently from the charges that can be applied to the Earth (so long they are not huge), the Earth can be assumed to be a conducting sphere isolated in vacuum with a very large capacitance C_T (≈ 400 to $800 \mu F$). The Earth potential V_T , calculated by imposing the potential at infinite to be zero, is still very small,

$$V_T = \frac{q_T}{C_T}$$

Within a good approximation, V_T can be assumed to be zero, even under the assumption that the potential is zero at infinite.

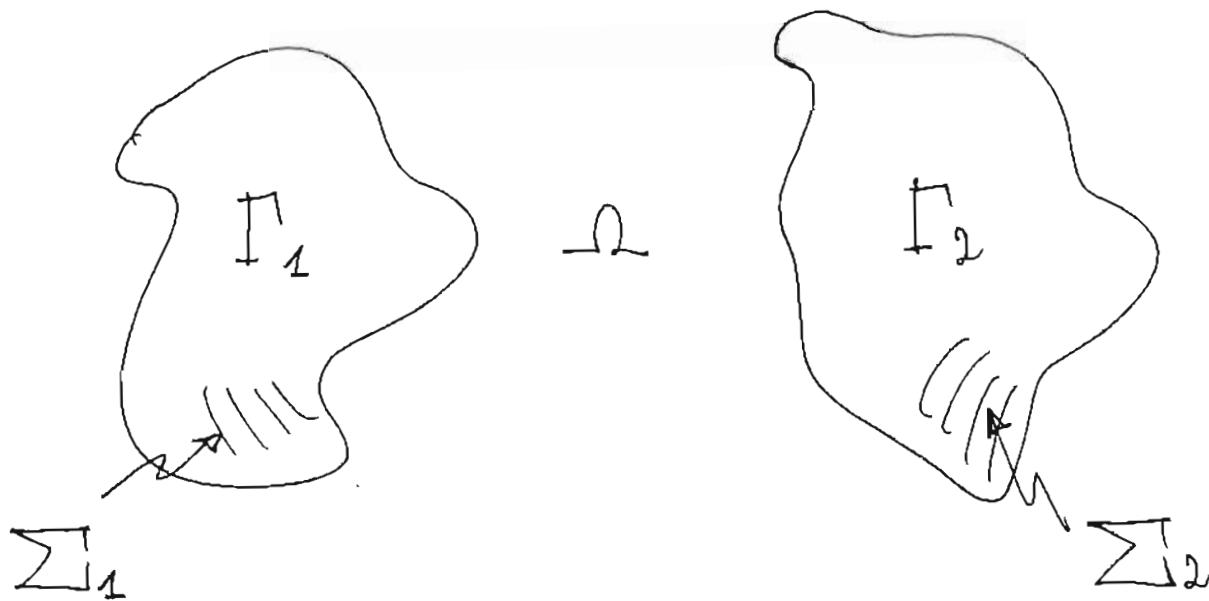


Figure 15.1.

Note that (15.9) and (15.10) can be inverted with respect to V_1 and V_2 . Considering q_1 and q_2 to be known, we find

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^{-1} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}, \quad (15.13)$$

where we indicated with $\|C_{ij}\|^{-1}$ the inverse of matrix $\|C_{ij}\|$. Calling

$$b_{11} = \frac{C_{22}}{|C|}, \quad b_{12} = -\frac{C_{21}}{|C|} = b_{21}, \quad b_{22} = \frac{C_{11}}{|C|},$$

where $|C|$ indicates the determinant of matrix $\|C_{ij}\|$, we have

$$\begin{cases} V_1 = b_{11} q_1 + b_{12} q_2 \\ V_2 = b_{21} q_1 + b_{22} q_2 \end{cases}, \quad (15.14)$$

or

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}. \quad (15.15)$$

We now specialize our discussion to the particular

case (specific to capacitors) where the two conductors are charged so that

$$q_1 = -q_2 = q \quad . \quad (15.16)$$

From (15.9) it follows that

$$q = C_{11}V_1 + C_{12}V_2 = -C_{21}V_1 - C_{22}V_2$$

and then

$$\frac{V_1}{V_2} = -\frac{C_{22} + C_{12}}{C_{11} + C_{21}} \quad . \quad (15.14)$$

So, for the charges on the two conductors to be equal in absolute value and opposite in sign, V_1 and V_2 cannot be arbitrarily chosen: they must satisfy condition (15.17), which means their ratio is fixed when the geometry of the two bodies is given.

By writing V_1 and V_2 as a function of V_1/V_2 and $(V_1 - V_2)$ by means of the identities

$$V_1 = \left(\frac{V_1}{V_2} \right) \frac{V_1 - V_2}{\frac{V_1}{V_2} - 1}$$

and

$$V_2 = \frac{V_1 - V_2}{\frac{V_1}{V_2} - 1}$$

and substituting these expressions in (15.9), we find

$$q = C_{11} \frac{\lambda}{\lambda - 1} \Delta V + C_{12} \frac{1}{\lambda - 1} \Delta V ,$$

where

$$\lambda = - \frac{C_{22} + C_{12}}{C_{11} + C_{21}}$$

and $\Delta V = V_1 - V_2$ [note that we have used only the top relation in (15.9). We could have equally well used the second]. A simple calculation shows that

$$\begin{aligned}
 \lambda - 1 &= - \left(\frac{C_{22} + C_{12}}{C_{11} + C_{21}} + 1 \right) \\
 &= - \frac{C_{22} + C_{12} + C_{11} + C_{21}}{C_{11} + C_{21}} \\
 &= - \frac{\beta}{C_{11} + C_{21}}
 \end{aligned}$$

where $\beta = (C_{11} + C_{12} + C_{21} + C_{22})$. Hence,

$$\begin{aligned}
 q &= \left(C_{11} \frac{C_{22} + C_{12}}{\cancel{C_{11} + C_{21}}} - \frac{\cancel{C_{11} + C_{21}}}{\beta} \right. \\
 &\quad \left. - C_{12} \frac{C_{11} + C_{21}}{\beta} \right) \Delta V \\
 &= \frac{1}{\beta} (C_{11}C_{22} + C_{11}C_{12} - C_{12}C_{11} - C_{12}C_{21}) \Delta V \\
 &= C(V_1 - V_2), \quad (15.18)
 \end{aligned}$$

where

$$C = \frac{C_{11}C_{22} - C_{12}C_{21}}{C_{11} + C_{12} + C_{21} + C_{22}} \quad . \quad (15.19)$$

Thus, the capacity of a capacitor depends only on the system geometry, i.e., on the shape and distance of its walls (note that if the space in between the walls is filled with a dielectric, C also depends on the physical properties of the system).

It is worth noting that if the capacitor's walls are not charged such that $q_1 = -q_2$, the only knowledge of the potential difference between them is insufficient to determine the charge on them. In fact, Eqs. (15.9) show that V_1 and V_2 must be separately given, not only the difference ΔV .

An example of important physical interest is shown in Fig. 15.2. In this case, one of the two walls encloses the other wall completely. When a charge q is applied to the internal wall Π_1 , the field lines in the space in between the two walls is uniquely defined, independently from the value of the charge applied to Π_2 ,

even if this is different from $-q$. This is true because of the properties of electrostatic shields. In the case of Fig. 15.2, the charge q on Γ_1 induces a charge $-q$ on the internal surface Σ_2^- of the wall Γ_2 . In this case, a complete induction is created between Γ_1 and Γ_2 : All field lines generating from Γ_1 end on Σ_2^- . In such a capacitor, the capacity defined as the ratio between q and the potential difference between the walls is independent from the total charge on Γ_2 and is not affected by the presence of external sources (cf. also HA 3.3). A complete induction capacitor is considered an ideal capacitor. In the practical implementation of capacitor one has to try to make a capacitor as close as possible to a complete induction capacitor.

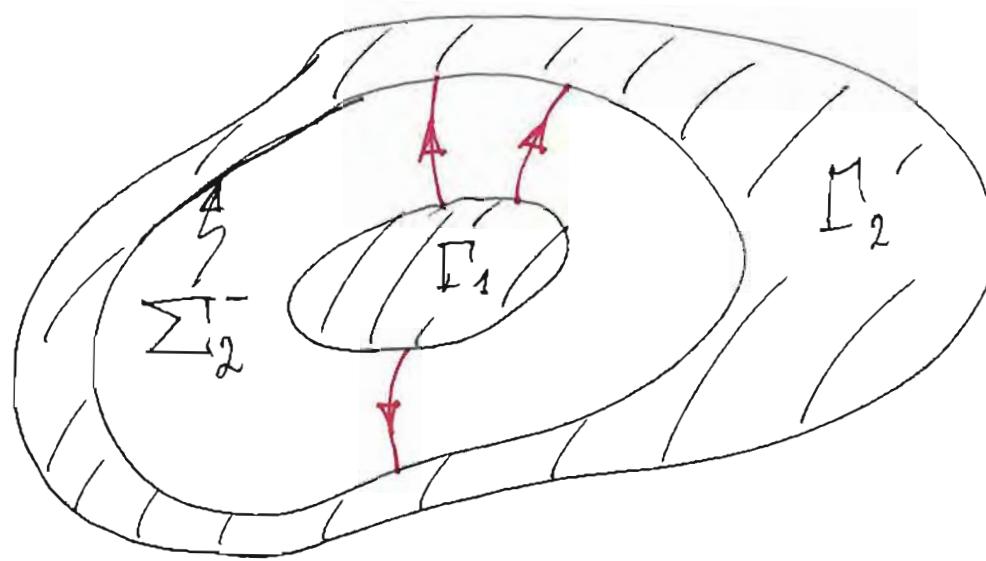


Figure 15.2.

15.1.1 Parallel plate capacitor (cf. also lecture 12). 22/37

Consider a capacitor made by two parallel walls (plates) of area S and separated by a distance d . We want to calculate its capacitance. Note that referring to the geometrical shape of this specific type of capacitor, it is common to indicate graphically any generic capacitor with the symbol



If the distance d between the plates is very small compare to the linear dimensions of the plates, $d \ll \sqrt{S}$, we can reduce the problem to that of two infinite parallel plates at distance d (cf. Fig. 15.3). We already found the potential of two such plates in lecture 12. Here, we want to attack the problem using Laplace equation and opportune boundary conditions.

As always, we choose a cartesian coordinate system where the Z axis is normal to the plates and

the x and y axis are on the left wall. Due to the translation symmetry of both walls with respect to x and y , the potential can only depend on z . It must thus satisfy Laplace equation in the form

$$\vec{\nabla}^2 V(z) = \frac{d^2}{dz^2} V(z) = 0 , \quad (15.20)$$

for $z \in (-\infty, +\infty) - \{0, d\}$.

Note that, while each of the two walls is assumed to be very thin compare to the distance d , we must assume them to have a finite thickness t in order to properly solve the boundary conditions problem associated with (15.20). As shown in the inset of Fig. 15.3, the left wall is characterized by an outer surface \sum_1^- on the left side and \sum_1^+ on the right side. Since the plate is conducting with finite dimensions t and S , no charge can distribute at any point inside the plate. The total charge on the plate is $+q$. Because of charge conservation, there must be a charge $+q/2$ on \sum_1^- and $+q/2$ on \sum_1^+ (we neglect any surface charge that accumulates on the other four lateral surfaces of width t). Similarly, the right

Wall has outer surfaces \sum_2^- and \sum_2^+ at a [24/34]
 distance t from each other (again, these surfaces do
 not include the negligible outer surfaces of width t).

In this case, the total charge on the plate is $-q$.
 Because of charge conservation, a charge $-q/2$ distributes
 on \sum_2^- and an equal charge $-q/2$ distributes on \sum_2^+ .

We can solve the problem by finding the potential
 of each wall alone and, then, obtain the total potential
 by means of the superposition principle.

We begin with the potential of the left wall. In
 this case, the unique solution to Eq. (15.20) is

$$V_1(z) = A_1^- z + B_1^- , \quad (15.21a)$$

for $z \in (-\infty, 0)$ and with A_1^- and B_1^- arbitrary constants.
 The constant A_1^- can be found from Coulomb's theorem
 applied to \sum_1^- ,

$$-\iint_{\sum_1^-} \frac{\partial}{\partial n_1^-} V_1 \cdot dS = +\frac{1}{\epsilon_0} \frac{q}{2} . \quad (15.22a)$$

From Fig. 15.3, we have $\vec{n}_1^- = -\vec{n} = -\vec{u}_z$.
Hence, (15.22a) reads

$$+\sum_{-1} \iint \frac{\partial}{\partial z} V_1(z) \cdot dS = A_1^- S \\ = +\frac{1}{\epsilon_0} \frac{q}{2},$$

from which

$$A_1^- = \frac{q}{2\epsilon_0 S} \quad . \quad (15.22a')$$

Hence,

$$V_1(z) = \frac{q}{2\epsilon_0 S} z + B_1^- \quad . \quad (15.21a')$$

For $z \in (0, +\infty)$, the solution is

$$V_1(z) = A_1^+ z + B_1^+ \quad , \quad (15.21b)$$

with A_1^+ and B_1^+ arbitrary constants. As before, the constant A_1^+ can be found from

$$-\iint_{\sum_1^+} \frac{\partial}{\partial n_1^+} V_1 \cdot dS = +\frac{1}{\epsilon_0} \frac{q}{2} \quad , \quad (15.22b)$$

where, in this case, $\vec{n}_1^+ = +\vec{n} = +\vec{u}_z$ and, thus,

$$-\iint_{\sum_1^+} \frac{\partial}{\partial z} V_1(z) \cdot dS = A_1^+ S \\ = +\frac{1}{\epsilon_0} \frac{q}{2} \quad ,$$

from which

$$A_1^+ = -\frac{q}{2\epsilon_0 S} \quad . \quad (15.22b')$$

Hence,

$$V_1(z) = -\frac{q}{2\epsilon_0 S} z + B_1^+ \quad . \quad (15.21b')$$

For the right wall, the unique solution to eq. (15.20) is

$$\nabla_2(z) = A_2^- z + B_2^- , \quad (15.21c)$$

for $z \in (-\infty, d)$ and with A_2^- and B_2^- arbitrary constants. From Coulomb's theorem

$$-\iint_{\sum_2^-} \frac{\partial}{\partial n_2^-} V_2 \cdot dS = -\frac{1}{\epsilon_0} \frac{q}{2} , \quad (15.22c)$$

where $\vec{n}_2^- = -\vec{n} = -\vec{u}_z$ and, thus,

$$+\iint_{\sum_2^-} \frac{\partial}{\partial z} V_2(z) \cdot dS = A_2^- S \\ = -\frac{1}{\epsilon_0} \frac{q}{2} ,$$

from which

$$A_2^- = -\frac{q}{2\epsilon_0 S} . \quad (15.22c')$$

Hence,

$$V_2(z) = -\frac{q}{2\epsilon_0 S} z + B_2^- \quad . \quad (15.21c')$$

Finally, for $z \in (d, +\infty)$, the solution is

$$V_2(z) = A_2^+ z + B_2^+ \quad , \quad (15.21d)$$

with A_2^+ and B_2^+ arbitrary constants. From Coulomb's theorem

$$-\iint_{\sum_{2^+}} \frac{\partial}{\partial n_2^+} V_2 \cdot dS = -\frac{1}{\epsilon_0} \frac{q}{2} \quad , \quad (15.22d)$$

where $\vec{n}_2^+ = +\vec{n} = +\vec{u}_z$ and, thus,

$$\begin{aligned} -\iint_{\sum_{2^+}} \frac{\partial}{\partial z} V_2(z) \cdot dS &= -A_2^+ S \\ &\stackrel{|}{=} -\frac{1}{\epsilon_0} \frac{q}{2} \end{aligned}$$

from which

$$A_2^+ = -\frac{q}{2\epsilon_0 S} \quad . \quad (15.22d')$$

Hence,

29/34

$$V_2(z) = \frac{q}{2\epsilon_0 S} z + B_2^+ \quad . \quad (15.21d')$$

By means of the superposition principle, the total potential in the region $z \in (-\infty, 0)$ is given by

$$\begin{aligned} V(z) &= V_1(z) + V_2(z) \\ &= \cancel{\frac{q}{2\epsilon_0 S} z + B_1^-} - \cancel{\frac{q}{2\epsilon_0 S} z + B_2^-} \\ &= B_1^- + B_2^- , \quad (15.23a) \end{aligned}$$

where we superimposed (15.21a') with (15.21c').

The potential at each point on the left wall must be equal to a constant V_1 ,

$$V(z) \Big|_{\sum_1} = V(0) = B_1^- + B_2^- = V_1 . \quad (15.24)$$

Hence, for $z \in [-\infty, 0]$,

$$V(z) = V_1 . \quad (15.25a)$$

By superimposing (15.21b') with (15.21c'), 30/34
 we find the total potential for $z \in (0, d)$,

$$\begin{aligned} V(z) &= -\frac{q}{2\epsilon_0 S} z + B_1^+ - \frac{q}{2\epsilon_0 S} z + B_2^- \\ &\stackrel{|}{=} -\frac{q}{\epsilon_0 S} z + (B_1^+ + B_2^-) . \quad (15.23b) \end{aligned}$$

On the left wall,

$$V(z) \Big|_{\sum_1} = V(0) = B_1^+ + B_2^- = V_1 . \quad (15.26)$$

The potential at each point on the right wall must be equal to a constant V_2 ,

$$V(z) \Big|_{\sum_2} = V(d) = -\frac{q}{\epsilon_0 S} d + V_1 = V_2 , \quad (15.27)$$

from which

$$V_2 = V_1 - \frac{q}{\epsilon_0 S} d .$$

From this condition follows that

$$V_2 < V_1 \quad . \quad (15.28)$$

Hence, for $z \in [0, d]$,

$$V(z) = -\frac{q}{\epsilon_0 S} z + V_1 \quad , \quad (15.25b)$$

with a non arbitrary choice of V_2 ,

$$V_2 = V_1 - \frac{q}{\epsilon_0 S} d \quad . \quad (15.25b')$$

By superimposing (15.21 b') with (15.21d'), we finally find the total potential for $z \in (d, +\infty)$,

$$\begin{aligned} V_2(z) &= -\frac{q}{2\epsilon_0 S} z + B_1^+ + \frac{q}{2\epsilon_0 S} z + B_2^+ \\ &= B_1^+ + B_2^+ \quad . \quad (15.23c) \end{aligned}$$

On the right wall,

$$V(z) \Big|_{\sum_{12}} = V(d) = B_1^+ + B_2^+ = V_2 . \quad (15.28)$$

Hence, for $z \in [d, +\infty)$,

$$V(z) = V_2 . \quad (15.25c)$$

In summary, we obtain the same result as in lecture 12, Eq. (12.3),

$$\left\{ \begin{array}{l} V(z) = V_1 , \quad z \in (-\infty, 0] \\ V(z) = -\frac{q}{\epsilon_0 S} z + V_1 , \quad z \in [0, d] \\ V(z) = V_2 = V_1 - \frac{q}{\epsilon_0 S} d , \quad z \in [d, +\infty) \end{array} \right. \quad (15.29)$$

where, in (12.3), $\sigma = q/S$ and $C_1 = V_1$ and $C_2 = V_2$.

The capacitance of the parallel plate capacitor can readily be found from its definition

$$C = \frac{q}{V_1 - V_2} = \frac{q}{\frac{qd}{\epsilon_0 s}} = \epsilon_0 \frac{s}{d} . \quad (15.30)$$

Note that we could have calculated the capacitance without a complete knowledge of all arbitrary constants. In fact, for $z \in [0, d]$ the total potential is given by the superposition of (15.21b) and (15.21c),

$$V(z) = (A_1^+ + A_2^-)z + (B_1^+ + B_2^-) . \quad (15.31)$$

From (15.31),

$$\begin{aligned} V_1 - V_2 &= V(0) - V(d) \\ &= (B_1^+ + B_2^-) - (A_1^+ + A_2^-)d - (B_1^+ + B_2^-) \\ &= -(A_1^+ + A_2^-)d . \quad (15.32) \end{aligned}$$

The charge $+q$ on Σ_1 can be found from (15.22b') and (15.22c'); which, thus, are the only boundary

conditions to be imposed:

[34/37]

$$\frac{q}{2} = -\epsilon_0 S A_1^+,$$

$$\frac{q}{2} = -\epsilon_0 S A_2^-$$

and, thus,

$$q = -(A_1^+ + A_2^-) \epsilon_0 S \quad (15.33)$$

At last,

$$C = \frac{(A_1^+ + A_2^-) \epsilon_0 S}{d} = \epsilon_0 \frac{S}{d}$$

Note that the potential difference in the definition of capacitance is always calculated from the positive to the negative wall.

It is worth performing a dimensional check on the capacitance:

$$[C] = \frac{F}{m} \cdot \frac{m^2}{m} = F.$$

The bottom part of Fig. 15.3 shows the potential of each individual wall as well as the total potential of the capacitor obtained by graphically superimposing the individual potentials. Note that, as expected, the overall potential of each individual wall is continuous at the wall. For simplicity, we have chosen $q = \epsilon_0 = S = d = B_1^- = B_2^- = B_1^+ = V_2 = 1$, $V_1 = 2$, and $B_2^+ = 0$.

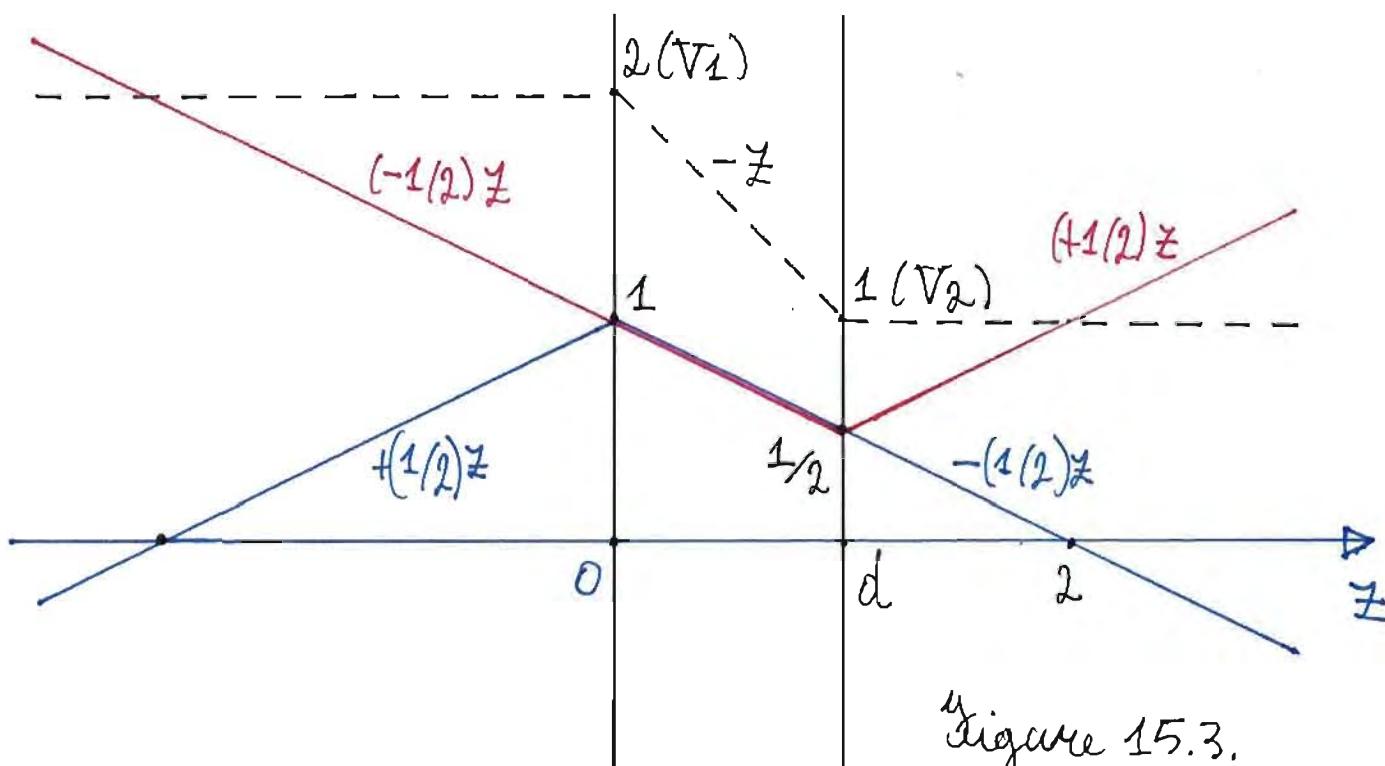
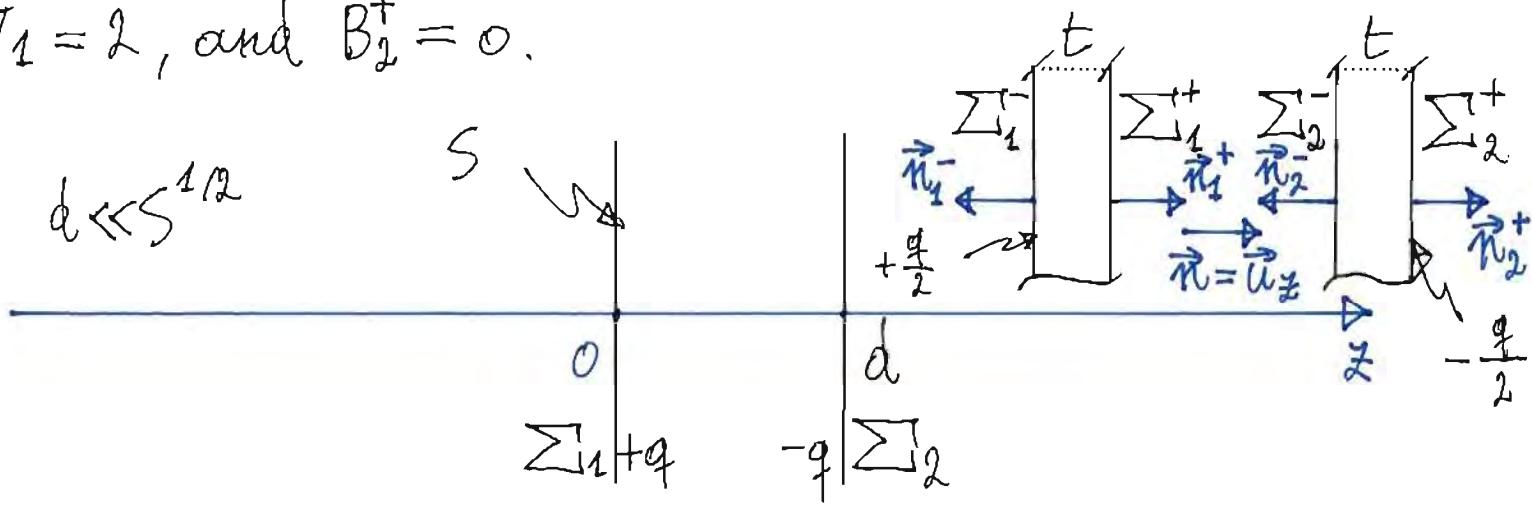


Figure 15.3.

15.2 Electrostatic energy of a system of charged conductors.

From lecture 10, the electrostatic energy associated with a surface charge distribution with density σ is given by

$$U_e = \frac{1}{2} \sum \iint \sigma V \cdot dS , \quad (15.34)$$

where Σ is the region where the charge is distributed.

In a system of n conductors, the electric charge is distributed on the conductors' surface. Naming Σ_i the surface of the i -th conductor Π_i , (15.34) reads

$$U_e = \frac{1}{2} \sum_{i=1}^n \iint_{\Sigma_i} \sigma_i V_i \cdot dS , \quad (15.35)$$

where σ_i is the density associated with the i -th conductor. Note that V_i must be constant on the surface of each conductor. Hence, we can take it outside

the sign of integral. In addition, the total charge [37/34] on Σ_i is given by

$$q_i = \iint_{\Sigma_i} \sigma_i \cdot dS \quad . \quad (15.36)$$

Thus,

$$U_e = \frac{1}{2} \sum_{i=1}^{n_e} q_i V_i \quad . \quad (15.37)$$

This is the wanted expression for the electrostatic energy of a system of n conductors.

In the particular case of a capacitor, the electrostatic energy is given by,

$$\begin{aligned} U_e &= \frac{1}{2} q V_1 - \frac{1}{2} q V_2 = \frac{1}{2} q V \\ &= \frac{1}{2} C V^2 = \frac{1}{2} \frac{q^2}{C} \quad , \quad (15.38) \end{aligned}$$

where q is the capacitor charge, V the potential difference between its walls, and C its capacitance.

Summary lecture 15.

- **Capacitors.**

A capacitor is a system of two conductors (walls) facing each other and separated by an insulator (or vacuum). The walls have a charge $+q$ and $-q$, respectively.

- In general, given two conductors M_1 and M_2 , their charges q_1 and q_2 are related to their voltages V_1 and V_2 by the matrix (voltage = potential)

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix},$$

where

$$C_{11} = \frac{q_1}{V_1} \Big|_{V_2=0}, \quad C_{12} = \frac{q_1}{V_2} \Big|_{V_1=0} = C_{21},$$

$$C_{22} = \frac{q_2}{V_2} \Big|_{V_1=0}$$

- In the case of a capacitor, $q_1 = -q_2 = q$,

$$q = C_{11}V_1 + C_{12}V_2 = -C_{21}V_1 - C_{22}V_2.$$

In particular,

$$q = C(V_1 - V_2) = C \Delta V$$

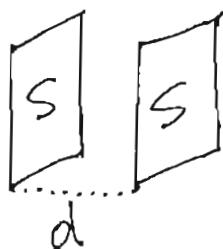
and

$$C = \frac{C_{11}C_{22} - C_{12}C_{21}}{C_{11} + C_{12} + C_{21} + C_{22}}$$

Note that, when calculating C , V_1 should be the voltage of the positively charged wall and V_2 of the negative.

- Parallel plate capacitor.

$$C = \epsilon_0 \frac{s}{d}$$



- Electrostatic energy of a capacitor.

3/29

$$U_e = \frac{1}{2} C V^2$$

$$= \frac{1}{2} \frac{q^2}{C}$$

16.1 Introduction.

So far, we only considered the interaction between charges that are at rest with respect to an inertial reference system. In the next three lectures (and, indeed, the following six), we will study interactions between charges in motion. These are at the basis of the magnetic phenomena known since antiquity.

Before analyzing the relationship between electric and magnetic phenomena and enumerating the corresponding general laws, we must verify whether the definition of electrical charge given in electrostatic (cf. lecture 1) is still valid to describe electrodynamical phenomena.

Consider a point-like charge q_0 , fixed at point $P_0 \equiv (x_0, y_0, z_0)$ in an inertial reference frame K (cf. Fig. 16.1). In the same frame, consider a point-like test body C in a generic motion along a path γ . We measure the force \vec{F} generated by q_0 on C, when C is at a generic point $P \equiv (x, y, z)$ on γ . Note that, in this case, we cannot readily find the force on q_0 as in electrostatic. In fact, the action-reaction principle is not valid here. This is because when C is at P on γ at time t, q_0 "feels" the action due to

C at a previous instant ($t - dt$). The action due to C at t will reach q_0 an instant later, under the assumption that the transmission velocity of the interaction is finite. 5/29

We then substitute C with another charged point-like body C' in motion along a generic path γ' , also going through P. We now measure the force \vec{F}' at P generated by q_0 on C' . By definition, the bodies C and C' have the same charge $q = q'$, when $\vec{F} = \vec{F}'$. It would be possible to verify experimentally that, if $\vec{F} = \vec{F}'$ at P, they are equal at each point in space common to the two paths γ and γ' . Experience shows that two bodies of equal charge at rest (according to the definition given in lecture 1) keep having the same charge when each of them moves in K (according to the new definition just given. The old definition of equal charge is, thus, a special case of the new definition).

Assuming that the additive property of charge continues to be valid also in electrodynamics (cf. lecture 1 for the additive property), we can define the physical quantity electric charge also for moving bodies. Moreover, it is possible to verify experimentally that if a pair of charges

is in a ratio γ , they remain in the same ratio 6/29
 even when they move. Hence, it is natural to associate to
 bodies in motion the same value of charge they had when
 they were at rest. In this sense, electrical charges are
 characterized by a relativistic invariance property.

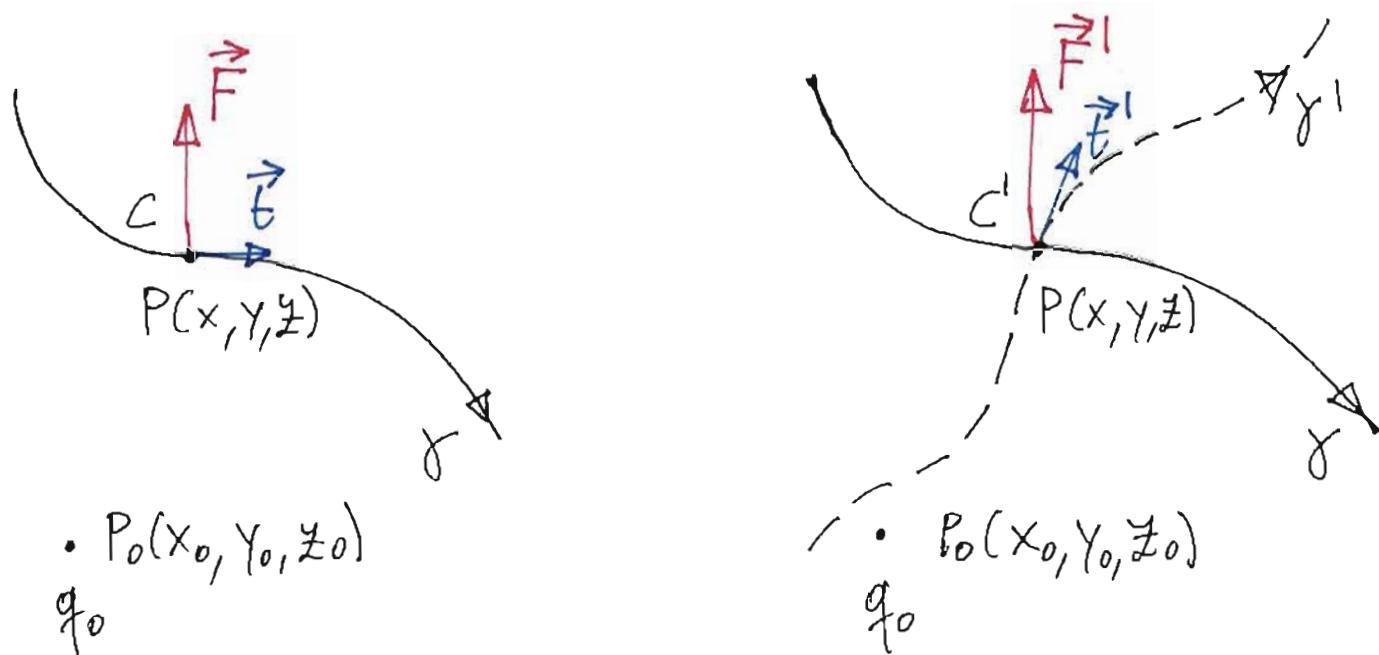


Figure 16.1.

16.2 Interactions between moving charges.

In lecture 2, we have seen that the force generated by a set of source charges, fixed in an inertial reference frame on a test charge q , also fixed in the same frame, can be

written as

7/29

$$\vec{F} = q \vec{E}, \quad (16.1)$$

where \vec{E} is a vector field whose structure depends only on the distribution of the source charges.

Imagine now to perform an experiment as sketched in Fig. 16.1, where the source charge q_0 is fixed in the considered reference frame, while the test charge q moves according to a generic law. Such an experiment would show that the force generated by q_0 on q , when q goes through the generic point P , is the same force we would find if q was fixed at P . In summary, if the source charges are fixed, the motion of the test charge does not affect the forces acting on it due to the source charges.

The last statement is not valid, in general, when the source charges move in space. With this respect, it is worth briefly revising a well-known question in classical mechanics, on how to define and measure the force acting on a moving point-like particle, when the force depends on the velocity of the particle (with given mass m). In this case, since the force cannot be defined from its static

effects (e.g., by means of a dynamometer), by definition we write

$$\vec{F} = \frac{d}{dt} \vec{p}$$

In other words, the force acting on the particle is defined as the time derivative of the linear momentum \vec{p} of the particle.

From experiments, it is found that the force acting on a test charge q going through point P at time t and with velocity \vec{v} in an inertial reference frame is given by

$$\vec{F} = q [\vec{E}(P, t) + \vec{v} \times \vec{B}(P, t)] , \quad (16.2)$$

where $\vec{E}(P, t)$ and $\vec{B}(P, t)$ are vector fields called electric and magnetic field, respectively (note that \vec{B} is also called magnetic induction).

We note that:

- a) the force \vec{F} acting on the test charge is proportional to the value of the charge itself (as in the electrostatic case).

b) If at time t the test charge is fixed at point P , the source charges generate on the test charge a force 9/29

$$\vec{F} = q \vec{E},$$

as it can be shown from (16.2) by imposing $\vec{v} = \vec{0}$. This is the same force obtained in the electrostatic case.

However, the value of $\vec{E}(P, t)$ generated by a distribution of moving source charges (considered at time t in a given reference frame) is, in general, different from the value the same charges would generate if they were fixed at their position at the same instant t . For this reason, the vector \vec{E} , which was called electrostatic field in lectures 1 to 15, is now called electric field. As for the static case, to measure the value of \vec{E} (electric field) at a generic point P , at time t , is enough to measure the force that acts on a positive, unitary test charge, fixed in the given reference frame, at that point and time (cf. lecture 2),

$$\vec{F}(P,t) = q \cdot \vec{E}(P,t) \quad (16.3)$$

As for the influence that the positive test charge q has on the source charges, we can repeat the same arguments given in lecture 2 for the static case: Also in the dynamic case we should define the field by means of a limit process for q going to zero. From (16.3) it appears that the so defined field (for a test charge at rest) is exactly the field that gives the first term in (16.2) of the force acting on the test charge when this moves with velocity \vec{v} .

c) The forces acting on a test charge moving at velocity \vec{v} are the electric force $q \vec{E}$ as well as the magnetic force $q \vec{v} \times \vec{B}$. The latter can be operationally defined as the difference between the total force acting on the test charge q and the electric force,

$$q \vec{v} \times \vec{B} = \vec{F} - q \vec{E} \quad (16.4)$$

Once q , \vec{v} , and \vec{F} are known, (16.4) allows us to

define and measure the magnetic field \vec{B} at point P and time t. More precisely, the operative procedure to define \vec{B} (and \vec{E}) is the following (cf. Fig. 16.2):

i) A first force measurement is realized with q at rest ($\vec{v}_i = \vec{0}$). This measurement makes it possible to determine the field $\vec{E} = \vec{F}_i / q$.

ii) A second force measurement is realized with q moving at an arbitrary velocity \vec{v}_{ii} . This measurement gives the value $\vec{F}_{ii} = \vec{F}_i + q \vec{v}_{ii} \times \vec{B}$. Hence, we find

$$\vec{v}_{ii} \times \vec{B} = \frac{\vec{F}_{ii} - \vec{F}_i}{q} \quad . \quad (16.4')$$

This result shows that \vec{B} lies on a plane normal to vector $(\vec{F}_{ii} - \vec{F}_i)/q$. Thus, (16.4') is insufficient to fully determine the induction vector, of which only gives the component on the aforementioned plane in the direction normal to \vec{v}_{ii} .

iii) A third force measurement is then realized with q moving at velocity \vec{v}_{ii} , where the velocity vector lies on the plane normal to $(\vec{F}_{ii} - \vec{F}_i)/q$ and is normal to \vec{v}_{ii} . This measurement finally gives the component of \vec{B} in the direction of \vec{v}_{ii} .

The dimensions of \vec{B} (or, more correctly, of its magnitude B) can be readily found from (16.2),

$$[w][B] = [E]$$

Hence,

$$[B] = \frac{V}{m} \cdot \frac{s}{m} = \frac{Vs}{m^2} = T$$

This unit is called tesla. One tesla is the value of the field B that generates the force of 1N on a charge of 1C , moving with a velocity of 1m s^{-1} .

The magnetic force is proportional to the velocity of the test charge and is directed normally to \vec{v} . This means that such a force cannot perform any work on the

charge.

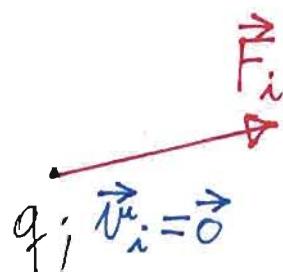
13/29

It is worth noting that \vec{B} depends only on the dynamics (motion) of the source charges. In particular, $\vec{B} = \vec{0}$ when the source charges are fixed. In this case, the only force acting on the test charge is the electric force. While it would be possible to show why and how moving charges generate a magnetic field, at this stage we prefer to accept the existence of \vec{B} as an experimental fact.

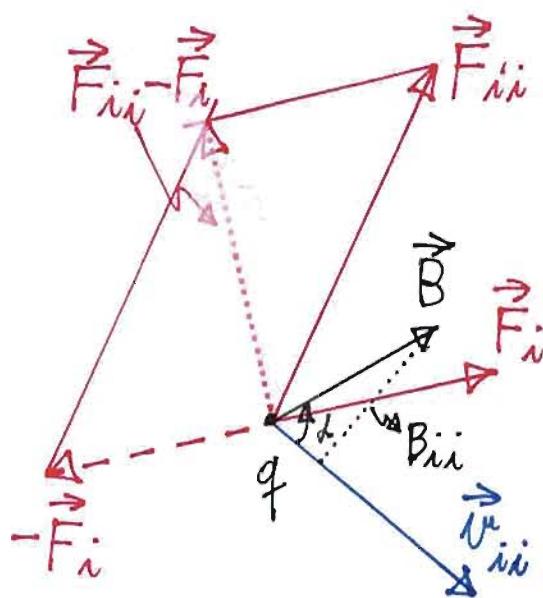
We could now proceed to the calculation of the fields (electric and magnetic) generated by a given distribution of moving source charges in reference frame. Following this path, we would first calculate the fields generated by a single charge and, then, by means of the superposition principle, we could move to the more complex case of fields generated by generic charge distributions. This approach would have the benefit to build on the logical process followed in electrostatics. However, the major downside of this approach is in that, in most applications, we will need to deal with the collective motion of an ensemble of charges (a current), rather than of a single charge. We will thus follow a traditional approach,

starting from the description of the fields associated with currents, to only later arrive at a description of the fields generated by a single charge.

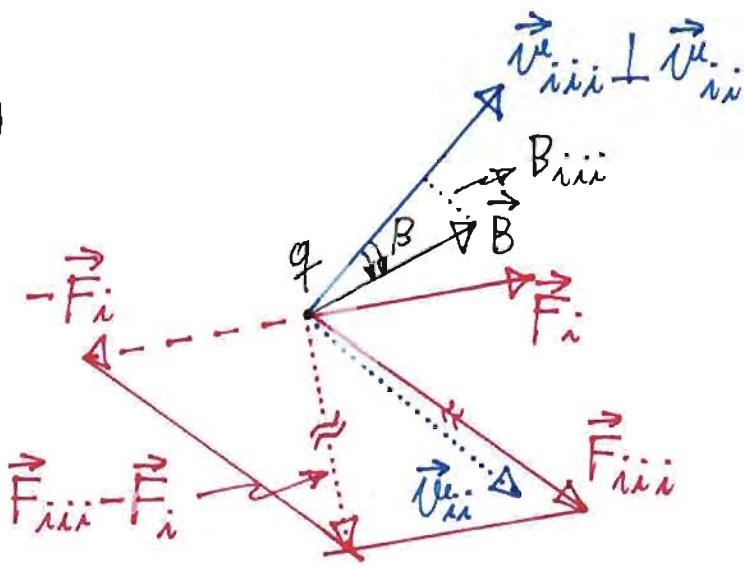
i)



ii)



iii)



$$\beta = \frac{\pi}{2} - \alpha$$

Figure 16.2.

16.3 The electric current.

15/29

Consider two conducting objects A and B, at rest in a given reference frame. Assume that A is charged with a positive charge q_A and that B is neutral. Initially, both objects are in electrostatic equilibrium. Imagine to perturb the equilibrium condition by connecting A and B by means of a conducting wire. In order to be able to observe the resulting physical phenomena for a reasonable time window, it is possible to use a wool thread opportunely soaked in water (the water makes the thread slightly conducting). In fact, a bare wool thread would be, per se, an insulator). By simultaneously measuring the electric charge on A and B, it would make it possible to show that q_A gradually diminishes, while q_B (initially zero) gradually increases. Because of the charge conservation principle, at each time t during the discharging/charging process

$$q_A(t) + q_B(t) = q = \text{const.} \quad (16.5)$$

Qualitatively, it is possible to explain this phenomenon by thinking that the charge flows from A

to B along the wet wool thread. Such a flow, which characterizes each point of the thread at the same time, is called electric current. From (16.5), we obtain by derivation

$$\frac{d}{dt} q_A(t) = - \frac{d}{dt} q_B(t) \quad (16.6)$$

for each time t. Given a cross-section S of the wool thread at any generic point on the thread, we imagine to measure directly the quantity of charge that goes through S in a certain time window. In other words, with a stopwatch at hand, we imagine to "count" all the charges that go through S in a fixed direction (e.g.; from A to B), from the moment the connection between A and B is established. We thus obtain a function $q(t)$ that, for each time t measured by the stopwatch, gives the total quantity of charge that went through S from time $t=0$ to t. Under the conditions assumed for such an experiment, it would result

$$q(t) = q_A(0) - q_A(t) \quad . \quad (16.7)$$

We are now in a position to define a new physical quantity, called the electric current intensity, by means of the relation

$$i(t) = \frac{d}{dt} q(t) \quad . \quad (16.8)$$

This is a scalar quantity that indicates the quantity of charge to go through S in the unit time, for each time t (S , in general, is the cross-section of a conducting wire). The dimensions of i ,

$$[i] = \frac{[q]}{[t]} = \frac{C}{s} \quad ,$$

are called ampere (A) in the SI system. One A, thus, corresponds to a passage through S of $1 C s^{-1}$. In reality, in the SI system it is preferred to define the unit of current independently from the coulomb. In fact, the unit of charge is defined from the unit of current. In any case, a current of 1 A still corresponds to the passage of $1 C s^{-1}$.

In the definition of $q(t)$, the passage of charges

through S is assumed to be counted with respect to a given direction reference. Depending on whether the charges go through S along the given direction or opposite to it, they must be summed or subtracted to the previous value of q . In summary, at each time t when $i > 0$, the charges (positive) go through S along the chosen direction; when, instead, $i < 0$, the charges (positive) go through S in the opposite direction.

So far, we assumed the current is generated by a flow of positive charges. If negative charges also contribute to the flow, the definition of current intensity must be modified by defining the function $q(t)$ as the net charge going through S in the chosen direction. This means that, given a direction reference, when a positive or negative charge goes through S in the chosen direction, its value must be added in an algebraic sense to the previous value of q . If the value of q is positive, the value of i increases, if it is negative, it diminishes. For example, if a positive and a negative charge (with equal absolute value) go through S together, in the same direction, it is equivalent to a neutral particle going through S : Such a particle does not contribute to the

Experiments confirm that, for all electric and magnetic effects, the motion of a positive charge is equivalent to that of a negative charge in the opposite direction.

16.4 Different types of current.

A metallic conductor can be thought as made by a structure of positive charges fixed at the vertices of a lattice and by a "cloud" of electrons free to move in a chaotic manner, continuously colliding against each other and the positive charges.

In conditions of electrostatic equilibrium, the current intensity through a generic section S of the conductor is zero. In fact, due to the chaotic nature of the thermic motion of electrons, the same number of electrons goes through S in one direction and in the opposite direction during the same physics infinitesimal time window Δt . As a consequence, from a macroscopic time scale (which is the scale of interest in the definition of current intensity) the function $q(t)$ is always zero. Hence, also its time derivative is zero.

The scenario is quite different if the free electrons acquire a motion in a specific direction due to suitable forces acting on them. Such a motion, drift, must be superimposed to the background chaotic thermal motion, diffusion. In the electrostatic case, the center of charge of a cloud of n electrons occupies, on average, always the same position (on a macroscopic time scale). In the dynamic case, instead, a component with constant velocity (even if small compare to the thermal component) is superimposed to the thermal motion. This velocity changes the average position of a free electron in time. To get an idea of the order of magnitude of such drift velocities, it is worth mentioning that the electron velocity due to thermal motion in a metallic conductor at room temperature is on the order of 10^5 m s^{-1} , the typical drift velocity is on the order of fractions of millimeter per second. These types of drift current are called conduction currents. Due to the extremely small mass of the free electrons compared to the fixed ions, the current flows without a corresponding mass flow (at least at the macroscopic level). Moreover, a current flow in a conductor does not imply the presence of an electric

net charge on it. In fact, in a physics infinitesimal 21/29 volume of conductor the positive (fixed) charges are, on average, as many as the negative (moving) charges. As a consequence, the total charge can be and keep on being zero, even if the volume is characterized by a continuous flow of negative charges.

Besides conduction currents, there are many other types of currents that depend on the physical properties of the media where they exist. An important example is that of convection currents, which typically take place in liquids and gases. Such currents correspond to macroscopic movements (of fluid-dynamics type) within the liquid or gas, where both negative and positive charges move in different ways. For example, the movement of large masses of air during a thunderstorm gives rise to convection currents. Other types of currents are those associated with the concept of holes in semiconductor devices.

At last, it is worth mentioning that in phenomena that vary with time, besides conduction and convection currents there are also so called displacement currents. These currents can take place in insulators or even in vacuum (in absence of charges of any type). We will study these currents in the last three lectures.

16.5 Current density and continuity equation.

22/29

Consider a region Ω in a conductor where a current i flows. Assume that σ is the volume density of the moving charges only and \vec{v} their velocity at point P and time t (cf. Fig. 16.3). Note that the velocity here considered is the average velocity vector of the microscopic velocities of the free carriers (electrons) in the neighborhood of P . This average coincides, by definition, with the oriented drift velocity superimposed to the chaotic one. Given an infinitesimal surface element dS and fixed arbitrarily the direction (inward or outward) of the normal unit vector \vec{n} to dS , we want to evaluate the charge going through dS in the \vec{n} direction during an infinitesimal time window dt . During this time interval, the charge going through dS is all (and only) the charge that was initially at a distance smaller or equal to $v \cdot dt$ with respect to any point on dS . In other words, the moving free charge carriers going through dS are those contained in an oblique cylinder with base dS and height $\vec{v} \cdot \vec{n} \cdot dt$. Such a charge is given by

$$dq = \sigma \vec{v} \cdot \vec{n} \cdot dt \cdot dS \quad . \quad (16.9)$$

By defining

$$\vec{J} = \mathcal{S} \vec{v} \quad , \quad (16.10)$$

Eq. (16.9) can be rewritten as

$$dq = \vec{J} \cdot \vec{n} \cdot dt \cdot dS = J_n \cdot dt \cdot dS \quad , \quad (16.11)$$

where J_n is the component of vector \vec{J} along the normal \vec{n} at dS .

The vector \vec{J} defined by (16.10) is called current volume density. The SI units of \vec{J} are

$$[J] = \frac{C}{m^3} \cdot \frac{m}{s} = \frac{A}{m^2}$$

and its component along a generic oriented direction \vec{n} gives [as shown by (16.11)] the quantity of charge that goes through a unitary surface (normal to \vec{n}) in the unit time.

Given a generic open surface S , it is possible to calculate (with respect to an arbitrarily chosen direction) the current intensity that goes through S , i.e., the quantity of charge that goes through it in the unit time.

From (16.11), we find

$$i = \iint_S \rho \vec{v} \cdot \vec{n} \cdot dS = \iint_S \vec{j} \cdot \vec{n} \cdot dS \quad . \quad (16.12)$$

Thus, the current intensity i that goes through a generic oriented surface S is given by the flux through S of the current density vector \vec{j} . Note that, in this way, the definition of current intensity (previously only given for filiform conductors) is now extended to the generic case of conductors with any geometry.

Equation (16.10) gives the current density when the free charge carriers are all of the same sign. More in general, (16.10) can be written as

$$\vec{j} = S_+ \vec{v}_+ + S_- \vec{v}_-, \quad (16.10')$$

where S_+ and \vec{v}_+ are the volume charge density and velocity for the positive carriers and S_- and \vec{v}_- for the negative carriers.

In summary, the phenomenon of charge transport associated with a current is described in local form

by a vector quantity (the current density) and in 25/29
 integral form by a scalar quantity (the current
 intensity).

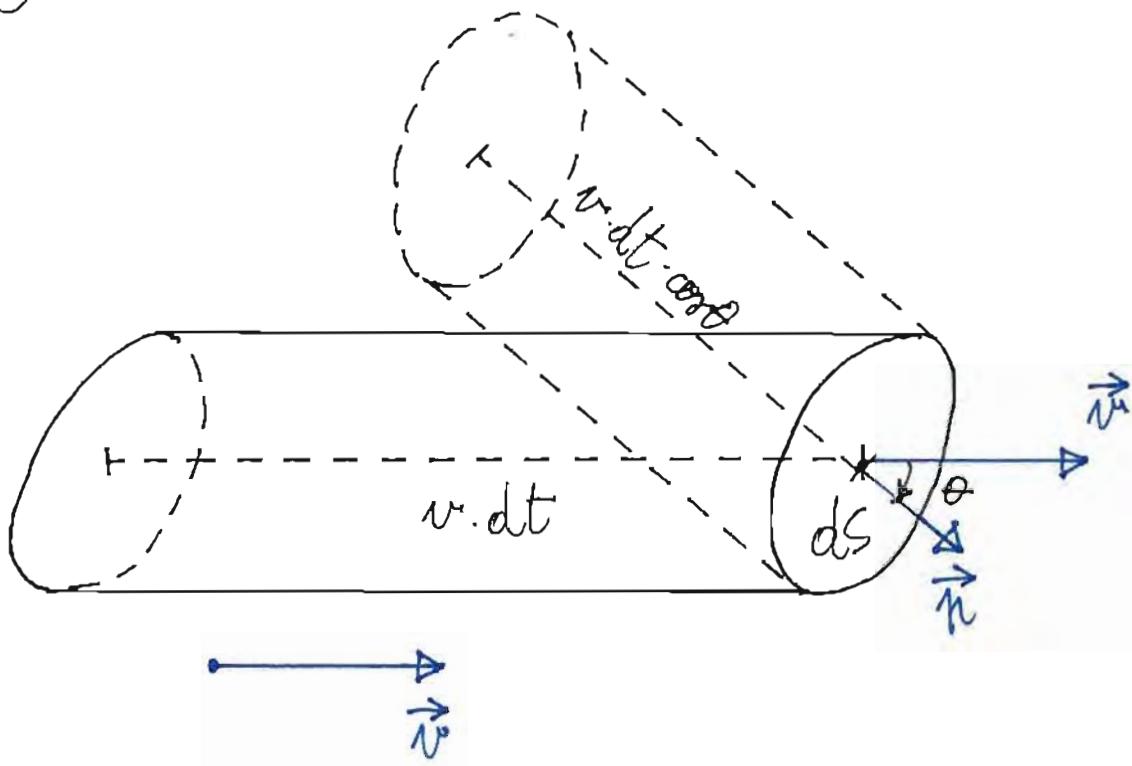


Figure 16.3.

Assume now the conductor where the current flow takes place is a conducting plate S (cf. Fig. 16.4). In this case, we can define at each generic point P on S a vector called surface current density \vec{J}_s ,

$$\vec{J}_s = \sigma \vec{v} \quad , \quad (16.13)$$

where σ is the surface charge density that goes through

26/29

point P with velocity \vec{v} . The dimensions of \vec{J}_S are obviously those of a current per unit length and, thus, the corresponding units are $A\ m^{-1}$. The current intensity that goes through the plate is

$$i = \int_S \vec{J}_S \cdot \vec{n} \cdot d\ell \quad , \quad (16.14)$$

where γ is the input border of S (cf. Fig. 16.4) and \vec{n} is the normal unit vector at a point on γ (\vec{n} is tangent to S).

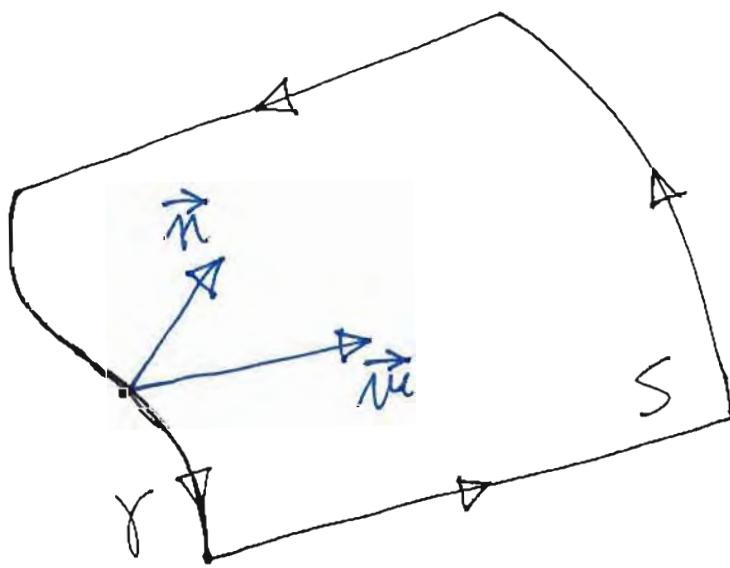


Figure 16.4.

Charge transport fulfills the charge conservation

conservation principle. Given any closed surface Σ , the quantity of charge that goes through it in a generic time interval corresponds to the change of charge contained in the volume enclosed by Σ . The quantity of charge that goes outside Σ in the unit time is

$$i = \iint_{\Sigma} \vec{J} \cdot \vec{n} \cdot dS \quad (16.15)$$

To this must correspond a change of the total charge contained within Σ . Indicating this charge as q , we find

$$\iint_{\Sigma} \vec{J} \cdot \vec{n} \cdot dS = - \frac{d}{dt} q \quad (16.16)$$

The negative sign shows that, in the case of positive charges going out of Σ (i.e., the first term is positive) we find a reduction in q , i.e., $dq/dt < 0$.

Indicating the charge density in the volume Σ

enclosed by Σ as S , we have

28/29

$$q = \iiint_S S \cdot d\mathbf{z}$$

and, thus,

$$\iint_{\Sigma} \vec{J} \cdot \vec{n} \cdot dS = - \frac{d}{dt} \iiint_S S \cdot d\mathbf{z} \quad . \quad (16.14)$$

Since the volume \mathcal{V} is time invariant, the operation of derivative with respect to time and integration in the volume are independent from each other. Hence, the sign of derivative and integral can be exchanged. The charge density S is a function of both the point and time. Thus,

$$\frac{d}{dt} \iiint_S S \cdot d\mathbf{z} = \iiint_S \frac{\partial}{\partial t} S \cdot d\mathbf{z}$$

and, thus,

$$\sum \oint \vec{J} \cdot \vec{n} dS = - \iiint_{\Sigma} \frac{\partial}{\partial t} S \cdot d\vec{x} . \quad (16.18)$$

By means of the divergence theorem for the first term of (16.18), we find

$$\iiint_{\Sigma} \vec{\nabla} \cdot \vec{J} d\vec{x} = - \iiint_{\Sigma} \frac{\partial}{\partial t} S d\vec{x} . \quad (16.19)$$

From the arbitrary choice of Σ and, thus, Σ , (16.19) implies the integrand functions must be equal. Thus, at each point and time, we have

$$\vec{\nabla} \cdot \vec{J} = - \frac{\partial}{\partial t} S(\vec{r}; t) . \quad (16.20)$$

This relation, called the continuity equation, represents the charge conservation principle in local (differential) form. Equations (16.16) and (16.18) represent the principle in integral form.

Summary lecture 16.

- Electric charge in electrodynamics.

From experiments, two objects with equal charge at rest have the same charge when they move with respect to an inertial reference frame.

- Lorentz force.

The force acting on a test charge q going through point P at time t and with velocity \vec{v} in an inertial reference frame is given by

$$\vec{F} = q [\vec{E}(P, t) + \vec{v} \times \vec{B}(P, t)],$$

where \vec{E} is the electric field and \vec{B} the magnetic field generated by moving source charges.

- Electric current.

- Electric current intensity:

$$i(t) = \frac{d}{dt} q(t)$$

- Given the cross-section S of a conducting wire, and a direction reference, when $i > 0$ at each time t , positive charges go through S along the chosen direction; when $i < 0$, the charges go through S in the opposite direction.
- In general, the net charge $q(t)$ must be considered as the algebraic sum of positive and negative charges.

The most common type of current, the conduction current, is the superposition of a thermal (chaotic) motion of electrons (diffusion) and a drift current in a specific direction, generated by a suitable force.

Continuity equation.

- Current volume density:

$$\vec{j} = \rho \vec{v},$$

where ρ is the volume density of charges moving at velocity \vec{v} . 3/33

- Charge conservation principle for moving charges, integral forms:

$$\sum \oint \vec{J} \cdot \vec{n} \cdot dS = - \frac{d}{dt} q(t)$$

or

$$\sum \oint \vec{J} \cdot \vec{n} \cdot dS = - \iiint_{\Sigma} \frac{\partial}{\partial t} \rho S \cdot d\chi .$$

In these equations, χ is a volume enclosed by a surface Σ , $\vec{n} \cdot dS$ is an oriented infinitesimal surface element on Σ , and q is the net charge inside Σ at time t .

- Local form:

$$\vec{\nabla} \cdot \vec{J} = - \frac{\partial}{\partial t} \rho(t) .$$

14.1 Stationary current.

4/33

A local property of a physical system that is independent of time at each point of the system is defined a stationary property. If all the properties of a system are stationary, the system itself is called a stationary system. Consider, for example, a moving liquid and assume to be at a point P fixed in the frame of reference with respect to which the liquid is moving. We then examine the velocities of the liquid masses going through P at following time instants. This is called the Eulerian frame of reference. It can happen that the liquid velocity is constant at point P. When this happens at each point where the liquid flows, the flow is said to be stationary. At different time instants, the velocity measured at the same point P refers to the motion of different liquid particles. If we were to follow one of these particles during its motion, we would find that, even under stationary conditions, the particle velocity varies, in general, along the followed path and, thus, it varies in time. The latter is called Lagrangian frame of reference.

An electric current is stationary if at each point of the medium where it flows both the charge density and current density are stationary. In this case, from Eq. (16.20) it follows that

$$\vec{\nabla} \cdot \vec{J} = 0 \quad (14.1)$$

at each point and time.

Thus, under stationary conditions the field associated with the current density vector is said to be solenoidal. Given any closed surface Σ in the field, it results

$$\iint_{\Sigma} \vec{J} \cdot \vec{n} \cdot dS = 0 \quad (14.2)$$

This means that the quantity of charge that globally goes out of Σ in the unit time is zero. The same amount of charge goes into Σ as, in the same time interval, it goes out. If this was not the case, we would find a time varying quantity of charge within the region enclosed by Σ . This would obviously contradict the stationary condition (which we assumed to be valid).

An important consequence of (14.2) is that the

current flowing in a conductor has the same value through each conductor section. In other words, the conductor represents a flux tube (cf. tutorial 2) for the field associated with vector \vec{J} . This can be demonstrated by considering a piece of conductor with a certain current enclosed by a surface Σ (cf. Fig. 14.1). The considered closed surface comprises two generic conductor sections S_1 and S_2 and a lateral surface S . From (14.2), we find

$$\iint_{S_1} \vec{J} \cdot \vec{n}_1 \cdot dS + \iint_{S_2} \vec{J} \cdot \vec{n}_2 \cdot dS + \iint_S \vec{J} \cdot \vec{n} \cdot dS = 0 , \quad (14.3)$$

where \vec{n}_1 , \vec{n}_2 , and \vec{n} are the normal unit vectors to S_1 , S_2 , and S , respectively. Moreover, given a coin-type surface δS so that the bottom of the surface resides within the conductor and the top outside, in correspondance to a point on S (cf. Fig. 14.2), it results

$$\iint_{\delta S} \vec{J} \cdot \vec{n}' \cdot dS = 0 . \quad (14.4)$$

In the usual limit for the height h of δS (as in

lecture 5, case 2) that goes to zero, we can conclude 7/33
that

$$J_{ni} dS + J_{no} dS = 0 \quad , \quad (14.4')$$

where J_{ni} and J_{no} are the normal components of \vec{J} to the bases of δS immediately inside and outside S , respectively. We hypothesized that a current flows inside the conductor, but not outside. Hence, it must be $J_{no} = 0$ and, from $(14.4')$, also $J_{ni} = 0$. As a consequence, the vector \vec{J} is tangent to the lateral surface of the conductor. Using this result in (14.3) , we obtain

$$\iint_{S_1} \vec{J} \cdot \vec{n}_1 \cdot dS = - \iint_{S_2} \vec{J} \cdot \vec{n}_2 \cdot dS \quad . \quad (14.5)$$

If the direction chosen for \vec{n}_1 is consistent with that chosen for \vec{n}_2 (e.g., both outward with respect to the corresponding bases), (14.5) reads

$$\iint_{S_1} \vec{J} \cdot \vec{n}_2 \cdot dS = \iint_{S_2} \vec{J} \cdot \vec{n}_2 \cdot dS \quad . \quad (14.6)$$

This proves the conductor is a flux tube for \vec{J} . 8/33

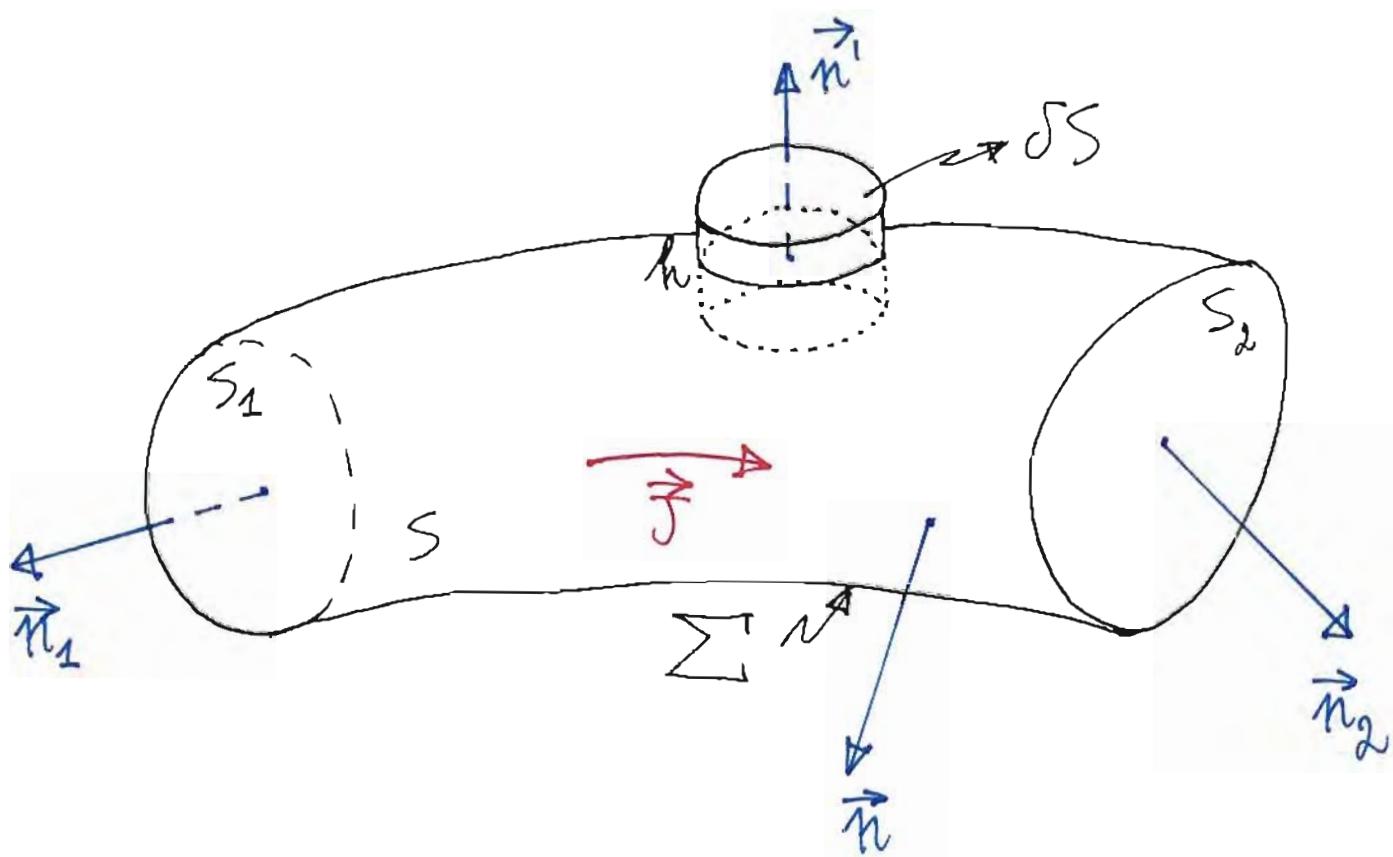


Figure 14.1.

Another important consequence of (14.2) is that, given a current flowing in a medium divided by a surface S into two regions with different physical properties, the components of \vec{J} normal to S have the same value on both "skins" of S , at each point on S :

$$J_{n1} = \lim_{P_1 \rightarrow P_0^-} J_n = \lim_{P_2 \rightarrow P_0^+} J_n = J_{n2} . \quad (14.4)$$

This result can easily be shown by considering again an elementary closed coin-type surface centred at P_0 on S and applying to it (17.2). Note that, however, the tangent components of \vec{J} can show, in this case, a discontinuity.

In summary, the charge conservation principle in the case of stationary currents guarantees the field of vector \vec{J} is solenoidal. In integral form, this can be expressed stating that the flux of \vec{J} out of any closed surface Σ in the field is zero,

$$\iint_{\Sigma} \vec{J} \cdot \vec{n} \cdot dS = 0 .$$

In local form (at each regular point in the medium; differential form), this corresponds to

$$\vec{\nabla} \cdot \vec{J} = 0 .$$

Finally, in correspondence to discontinuity points between media with different physical characteristics,

$$J_{n1} = J_{n2} \quad (\text{local form}) .$$

17.2 The electric field in conductors with a stationary current.

10/33

So far, we treated the phenomenon of electric current from a "cinematic" point of view. That is, we introduced the physical quantities required to describe the current, without, however, showing the causes of the phenomenon. We will now attack this problem, considering for the moment only stationary currents.

It is evident that if in a conductor the charge free carriers move in a specific direction, opportune forces must act on them. These forces can have different origins. For now, we will study the case of electric forces.

In the lectures on conductors, we saw that in a conductor in electrostatic equilibrium, the electrostatic field is zero at each point inside the conductor. If the conductor is not in equilibrium, i. e., a current flows through it, a nonzero electric field is present in the conductor. Such a field, acting on the free charges, makes them move generating the current. The question to ask is then what are the laws describing the behavior of

the electric field in a conductor with stationary current. 11/33

Experiments show that such laws are the same for electrostatic phenomena, both inside and outside the conductor. In fact, even in presence of a stationary current, the electric field obeys the two following fundamental theorems:

$$\iint_{\Sigma} \vec{E} \cdot \vec{n} \cdot dS = \frac{1}{\epsilon_0} q , \quad (\text{Gauss' theorem}) \quad (17.8)$$

$$\oint_{\gamma} \vec{E} \cdot \vec{dl} = 0 , \quad (\text{irrotational property}) \quad (17.9)$$

where Σ and γ are a closed surface and oriented line, respectively.

In summary, a conductor with stationary current is characterized by two vector fields, both stationary: the electric field \vec{E} and the current density field \vec{J} . These fields obey (17.8) and (17.9), and (14.2), respectively. Such laws, however, (or theorems) do not

show the relationship between \vec{E} and \vec{J} . This relationship depends on the physical properties of the conductor and, thus, it cannot be given independently from them.

The relation

$$\vec{J} = f(\vec{E}) , \quad (14.10)$$

characteristic of each conductor, is called constitutive relation. The constitutive relation must be determined case by case experimentally.

For certain conductors at constant temperature, Eq. (14.10) is a linear relation,

$$\vec{J} = g \vec{E} , \quad (14.11)$$

which is called Ohm's law (in local form). If the medium under consideration is isotropic, (14.11) becomes a simple proportionality relation with g a scalar constant (which, in general, depends on the point). In this case, \vec{E} and \vec{J} are parallel and have the same direction. If, instead, the medium is not isotropic, (14.11), written in the components of the two vectors, has typically the form

$$\left\{ \begin{array}{l} J_x = g_{xx} E_x + g_{xy} E_y + g_{xz} E_z , \\ J_y = g_{yx} E_x + g_{yy} E_y + g_{yz} E_z , \\ J_z = g_{zx} E_x + g_{zy} E_y + g_{zz} E_z , \end{array} \right. \quad (14.11)$$

where, in general, g_{ij} depends on the point. Crystalline bodies (which, in general, are anisotropic, at least in the case of single crystals), different directions are not equivalent. The most common conductors are metals, which crystallize in systems with very high symmetry (body-centered cubic, face-centered cubic, ...). Within a good approximation, metals can be assumed to be isotropic, even in the form of single crystals. In most cases, metals are macroscopically isotropic even if the constituent single crystals are not, because the macroscopic properties correspond to measurements on ensembles of many single crystals that are randomly distributed.

The quantity g in (14.11) is called electric conductivity and depends only on the physical nature of the conductor. The inverse quantity $\eta = 1/g$ is called the electric resistivity. The conductivity is often

indicated by the symbol σ (or γ), while the resistivity by ρ . The conductivity is measured in $\text{A V}^{-1} \text{m}^{-1}$ and the resistivity in Vm A^{-1} . We will come back to these units later. 14/33

Relation (14.11) cannot be used for all conductors. With very good approximation, it can be used for metals and metallic alloys and, typically, can be used for electrolytic solutions. However, it cannot be used in the case of ionized gases. For many substances, (14.11) can nevertheless be used at least in some limited range of values of E and J . In fact, even for nonohmic media, such as insulators, it is possible to define an electric resistivity. The materials with lowest resistivity are metals. In the case of copper, for example, $\rho = 1.69 \times 10^{-8} \text{Vm A}^{-1}$ at room temperature. A typical value of ρ for an insulator (e.g., quartz) is on the order of 10^{17}Vm A^{-1} (i.e., approximately 10^{25} times larger than copper).

The conductivity of a medium depends on the physical conditions in which the medium is prepared, i.e., on temperature, pressure, and state of matter (i.e., solid, liquid or gas). A fundamental phenomenon is then the Joule

effect, i. e., the well-known fact that a conductor with current becomes hotter (this is the case of light bulbs or electric heaters, for example).

The heat generated because of the Joule effect is due to the transformation of the work produced by the electric forces. In fact, during their motion the charge free carriers interact with the surrounding atoms and molecules. During such interactions they loose part of the kinetic energy they acquire from the electric field between collisions. The energy lost in the collisions becomes heat. In order to calculate the specific power (i. e., the power per unit volume) dissipated in the conductor, we can consider a flux tube of vector \vec{J} , of length dl and section dS . The charge that goes through dS in the time dt is

$$dq = \vec{J} \cdot \vec{n} \cdot dS \cdot dt \quad , \quad (14.12)$$

where \vec{n} is the unit vector normal to dS . The work produced by the electric field \vec{E} to move dq along dl (note that dl is directed as \vec{J} and $dl/|dl| = \vec{J}/|J|$) is

$$dW = \vec{E} \cdot dl \cdot dq = (\vec{E} \cdot dl) (\vec{J} \cdot \vec{n} \cdot dS \cdot dt) . \quad (14.13)$$

Since \vec{J} and $d\vec{l}$ have the same direction and sign, we can exchange the position of $d\vec{l}$ and \vec{J} . We can also divide by dt and obtain

$$\frac{dW}{dt} = (\vec{E} \cdot \vec{J}) \cdot (d\vec{l} \cdot \vec{n}) \cdot dS = \vec{E} \cdot \vec{J} \cdot dz, \quad (14.14)$$

where dz is the infinitesimal volume of the piece of flux tube considered. Thus, the specific power (power for unit volume) dissipated at point P is

$$P_z = \frac{dW}{dt \cdot dz} = \vec{E} \cdot \vec{J} \quad . \quad (14.15)$$

If Ohm's law can be used for the conductor, we find

$$P_z = E J = \eta J^2 = \rho E^2 \quad . \quad (14.16)$$

17.3 The electromotive force acting in a circuit with current.

In order to maintain a stationary current in a conductor a work must be produced. The work is continuously transformed into heat. This work can be locally produced by an electric field acting on the

charge free carriers. It is thus natural to ask the question whether a stationary current in a conductor can be maintained exclusively by an electric field that obeys Gauss' theorem [(14.8)] and the irrotational property [(14.9)]. The answer is negative. First of all, it is worth noting that a conductor where a current flow takes place (in stationary conditions) must be part of an electric circuit, i.e., a closed path along which current can flow. In fact, under stationary conditions the vector field \vec{J} is solenoidal. In other words, \vec{J} is characterized by closed field lines. Note that this last statement is not generally correct. In fact, the field lines of a solenoidal field can be open, but yet cover a surface in the ergodic sense. In any case, such lines do not have a beginning or end. This is the property used in the present argument. Hence, our hypothesis does not diminish the generality of the argument. Note that the reader might encounter ergodic solenoidal fields in the study of magnetohydrostatic. While it is true that an irrotational field can maintain a current in a part of a conductor, it cannot maintain it in a

closed circuit. In fact, being a conservative field, it produces zero work along any closed line γ ,

$$\oint_{\gamma} \vec{E} \cdot \vec{dl} = 0 \quad . \quad (17.14)$$

Thus, a conservative field cannot produce the required work to maintain a current in the entire circuit.

In the case of ohmic conductors, Eq. (17.14) would read,

$$\eta \oint_{\gamma} \vec{J} \cdot \vec{dl} = 0 \quad , \quad (17.18)$$

which is an impossible result if $\eta \neq 0$. This is because γ is a field line of \vec{J} and, thus, the circulation of \vec{J} can be zero only if $\vec{J} = \vec{0}$ at each point in the circuit. Note that, while in (17.14) γ can be any closed line, in (17.18) we have chosen γ to be a vector field line of \vec{J} : this is a special case of closed line and, thus, (17.14) remains valid.

Thus, in order to maintain a stationary current in a closed circuit it is necessary that forces of different nature (chemical, mechanical, thermic,...), superimposed to the electric conservative forces, act on the charge free carriers. These forces must be able to produce a nonzero work along a closed path. Note that particular care must be taken when distinguishing between electric and nonelectric forces. In fact, at the microscopic level all interactions between atoms, electrons, and nuclei are, indeed, of electric nature. Such a distinction, thus, makes sense only at the macroscopic level. In a macroscopic context, electric forces are only those that are generated by globally non-zero electric charge distributions in macroscopic volumes.

As a consequence, the conducting circuits must be of the type shown in Fig. 14.2. The figure schematically shows two regions with different physical characteristics. Region C is a simple conductor (e.g., metallic), whereas region G (confined between parallel faces) is where the nonelectric phenomena take place. These phenomena

20/33

generate the required nonconservative field. In other words, the nonelectric force field \vec{F}_m acting on the charge free carriers is supposed to exist only in the region G . The electric field \vec{E} , instead, is present in both regions and also in the space outside the conductor. In G takes place the superposition of the forces originated by \vec{E} and the nonelectric forces \vec{F}_m .

Given a closed line γ only partially located in G (cf. Fig. 14.2), it must be

$$\oint_{\gamma} \vec{F}_m \cdot \vec{T} \cdot dl \neq 0 . \quad (14.19)$$

In fact, since $\vec{F}_m = \vec{0}$ outside G , it has to be

$$\oint_{\gamma} \vec{F}_m \cdot \vec{T} \cdot dl = \int_{\gamma_G} \vec{F}_m \cdot \vec{T} \cdot dl \neq 0 , \quad (14.19')$$

where γ_G indicates the part of γ within G . Assuming γ (and, thus, γ_G) to be oriented clockwise and to be

uniform and directed as in Fig. 14.2 inside G , we find

$$\oint \vec{F}_m \cdot \vec{t} \cdot d\ell = F_m l_G , \quad (14.20)$$

γ

where l_G is the distance between the two faces of G .

In order to describe the nonconservative force \vec{F}_m at any point, it is useful to define a vector field \vec{E}_m with the same dimensions of the electric field,

$$\vec{E}_m = \frac{\vec{F}_m}{q} , \quad (14.21)$$

where q is the charge of the carrier that moves in the conductor. The vector \vec{E}_m is called the electromotive field. Despite the fact that has the units of a force divided by a charge, its physical properties are totally different from those of an electric field. In particular, not only \vec{E}_m is nonconservative, but also it depends, in general, on the value of q . In fact, is not necessarily true that \vec{F}_m is proportional to the value of the charge.

Here, we will not delve into the details of the physical origin of \vec{E}_m . Intuitively, we can imagine that the carriers entering G are pushed through G by a microscopic hand with force F_m .

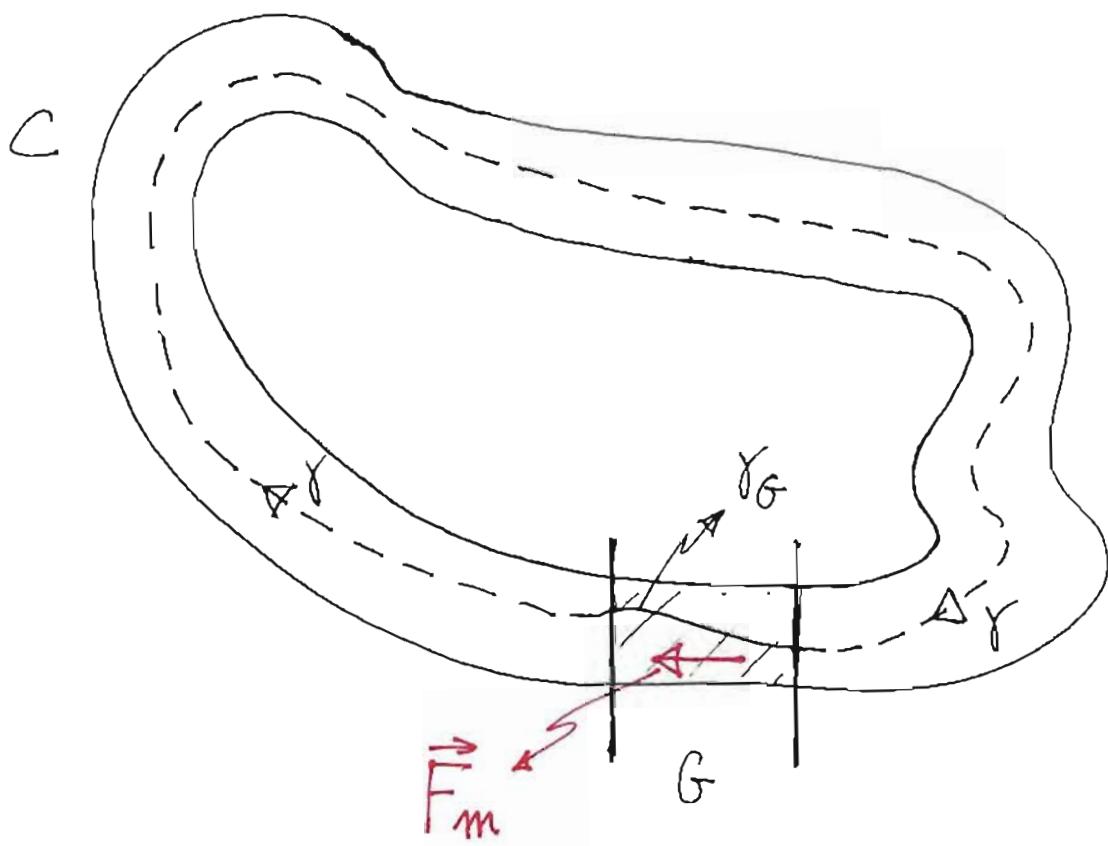


Figure 14.2.

The components of an electric circuit where the forces \vec{F}_m act are called sources. The sources provide the work required to make the current flow.

From (14.19) follows that

$$\mathcal{E} = \oint_{\gamma} \vec{E}_m \cdot \vec{t} \cdot dl \neq 0 , \quad (14.22)$$

where \mathcal{E} is the so called electromotive force (emf) acting along the line γ . The fact that \mathcal{E} is called a force is far from being ideal. In fact, \mathcal{E} has the dimensions of a work divided by a charge and, thus, has the SI units of a volt. \mathcal{E} is a scalar quantity, the sign of which depends on the arbitrary orientation of the line γ . \mathcal{E} represents the ratio W/q between the work of the electromotive field when moving the charge q along the line γ (we remind γ is a closed line) and the charge itself. From (14.19') it obviously follows that

$$\mathcal{E} = \oint_{\gamma} \vec{E}_m \cdot \vec{t} \cdot dl = \int_{\gamma_0} \vec{E}_m \cdot \vec{t} \cdot dl . \quad (14.22')$$

We now intend to study in detail the working

principle of the circuit, considering first the simple case of an open source, i.e., a source disconnected from any external conductor that closes the circuit. The source is characterized by two terminals, A and B, that make it possible to eventually connect the source to an external conductor and, thus, to close the circuit. Assume we know the field distribution of the electromotive field \vec{E}_m at each point in the source G. For example, assume \vec{E}_m is uniformly distributed in G. In the case of an open circuit, there can be no current flowing in the source (as always, we assume a stationary current). Hence, under stationary conditions at each point in G it must be $\vec{F} = \vec{0}$. This implies that the total force acting on the charge carriers (sum of \vec{F}_m and $q\vec{E}$) is zero at each point in G,

$$\vec{E}_m + \vec{E} = \vec{0} \quad . \quad (14.23)$$

In open circuit, the electromotive field \vec{E}_m is balanced point by point by an electrostatic field \vec{E} . In order to explain the origin of \vec{E} it is enough to notice that,

because of the action of \vec{E}_m , the free charges in G start moving and migrate until it is possible, i.e., till the limiting surface of the source (in particular, to the terminals A and B). On such a surface, which is originally neutral, a surface charge distribution takes place. This charge distribution generates a reaction electrostatic field \vec{E}_r . The process continues until the electrostatic field equilibrates the electromotive field point by point. Figure 14.3 shows a qualitative representation of the charge distribution on the surface of the source and of the field lines of the electrostatic field \vec{E} generated by those charges, both inside and outside the source.

Referring to Fig. 14.4, given any line γ_{AB}^{ext} that connects the terminals A and B, outside the source, we have

$$\int_{\gamma_{AB}^{\text{ext}}} \vec{E} \cdot \vec{dl} = V^o(A) - V^o(B) \quad (14.24)$$

because \vec{E} is conservative. This means there is a potential difference $V_{AB}^o = V^o(A) - V^o(B)$ at the terminals (or, better, between them). Furthermore, because of the very definition of \vec{E} ,

$$\int_{\gamma_{AB}^{int}} \vec{E} \cdot \vec{dl} = V^o(A) - V^o(B) , \quad (14.25)$$

where γ_{AB}^{int} is any line connecting A and B, inside the source. In addition, from (14.23) at each point of γ_{AB}^{int} it must be

$$\vec{E}_m = -\vec{E} .$$

Hence,

$$V_{AB}^o = \int_{\gamma_{AB}^{int}} \vec{E} \cdot \vec{dl} = - \int_{\gamma_{AB}^{int}} \vec{E}_m \cdot \vec{dl}$$

$$= \oint_{\gamma} \vec{E}_m \cdot \vec{E} \cdot d\ell = \mathcal{E}, \quad (14.26)$$

where γ is any closed line coinciding with γ_{AB}^{int} at each point inside G , but with opposite direction compared to γ_{AB}^{int} . As a consequence, in the case of an open circuit the potential difference between the source terminals coincides with the emf. Note that, if the electromotive field is uniform, the value of the emf does not depend on the considered line, so long the line crosses the region G only one time. If, instead, the electromotive field is nonuniform in G , the emf can depend on the specific chosen line. In these cases, even when the source is open, there can be currents flowing in its interior.

Referring to Fig. 14.5, assume now to connect the terminals A and B by means of an external conductor C. Due to the action of the electric field \vec{E} , the charge free carriers in C starts moving, generating a

current that modifies the charge distribution at the interface between the entire system (which now also includes C) and the external environment. Once the stationary regime has been reached, the free charges (supposed to be positive, for simplicity), "fall" along C from the point at higher potential (in Fig. 14.5, point A) to that at lower potential (B). The opposite, instead, takes place inside the source, where \vec{E}_m moves the charges against the forces due to the electrostatic field. The electromotive field causes the free charges to rise from the point at a lower potential to that at a higher one. If the conducting material in the source is linear, at each point in the source it must be

$$\vec{J} = \sigma (\vec{E} + \vec{E}_m) , \quad (14.24)$$

where σ is the electric conductivity inside the source. Note that (14.24) represents an extension of Ohm's law (in local form) to the case where both the electric field \vec{E} and the electromotive field \vec{E}_m are simultaneously present.

In the external conductor C, instead, it must be 29/33

$$\vec{J} = g_c \vec{E} \quad , \quad (14.28)$$

where g_c is the conductivity of C.

In stationary conditions, the electromotive field \vec{E}_m is larger (in absolute value) than the electric field \vec{E} inside the source, so that

$$\vec{E}_m + \vec{E} = \eta \vec{J} \quad . \quad (14.29)$$

The fact that \vec{E}_m is larger (in absolute value) than \vec{E} inside G depends on the potential reduction imposed by the external connection between terminals A and B. In the limiting case of infinite conductivity for C, the external connection would result in a short circuit, resulting in a zero potential difference between A and B and, thus, a zero field \vec{E} inside the source.

The potential difference between the terminals A and B is given by

$$\begin{aligned}
 V_{AB} &= V(A) - V(B) \\
 &= \int_{Y_{AB}^{int}} \vec{E} \cdot \vec{dl} , \quad (14.30)
 \end{aligned}$$

where we have now removed the superscript in the potential function, V^0 , that was used to indicate an open circuit situation. In (14.30), Y_{AB}^{int} indicates any line connecting A and B inside G. From (14.29), we have

$$\vec{E} = \eta \vec{J} - \vec{E}_m , \quad (14.31)$$

and, thus,

$$\begin{aligned}
 V_{AB} &= \int_{Y_{AB}^{int}} \eta \vec{J} \cdot \vec{E} \cdot dl - \int_{Y_{AB}^{int}} \vec{E}_m \cdot \vec{E} \cdot dl \\
 &= \mathcal{E} - \int_{Y_{AB}^{int}} \eta (-\vec{J}) \cdot \vec{E} \cdot dl . \quad (14.32)
 \end{aligned}$$

Hence, in the case of a closed circuit, the potential difference between the source terminals does not coincide anymore with the emf. Referring to the directions indicated in Fig. 14.5, we have $\vec{J} \cdot \vec{t} > 0$ i.e., the potential difference is smaller than the emf.

Before concluding this lecture, it is worth mentioning that the point by point balancing between \vec{E}_m and \vec{E} described after (14.23) is only valid at each point inside G . In fact, on the outer skin of G the action of the electromotive field is balanced (at least as far as the normal component to the surface of G is concerned) by the double layer forces that do not allow the charge carriers to exit the body.

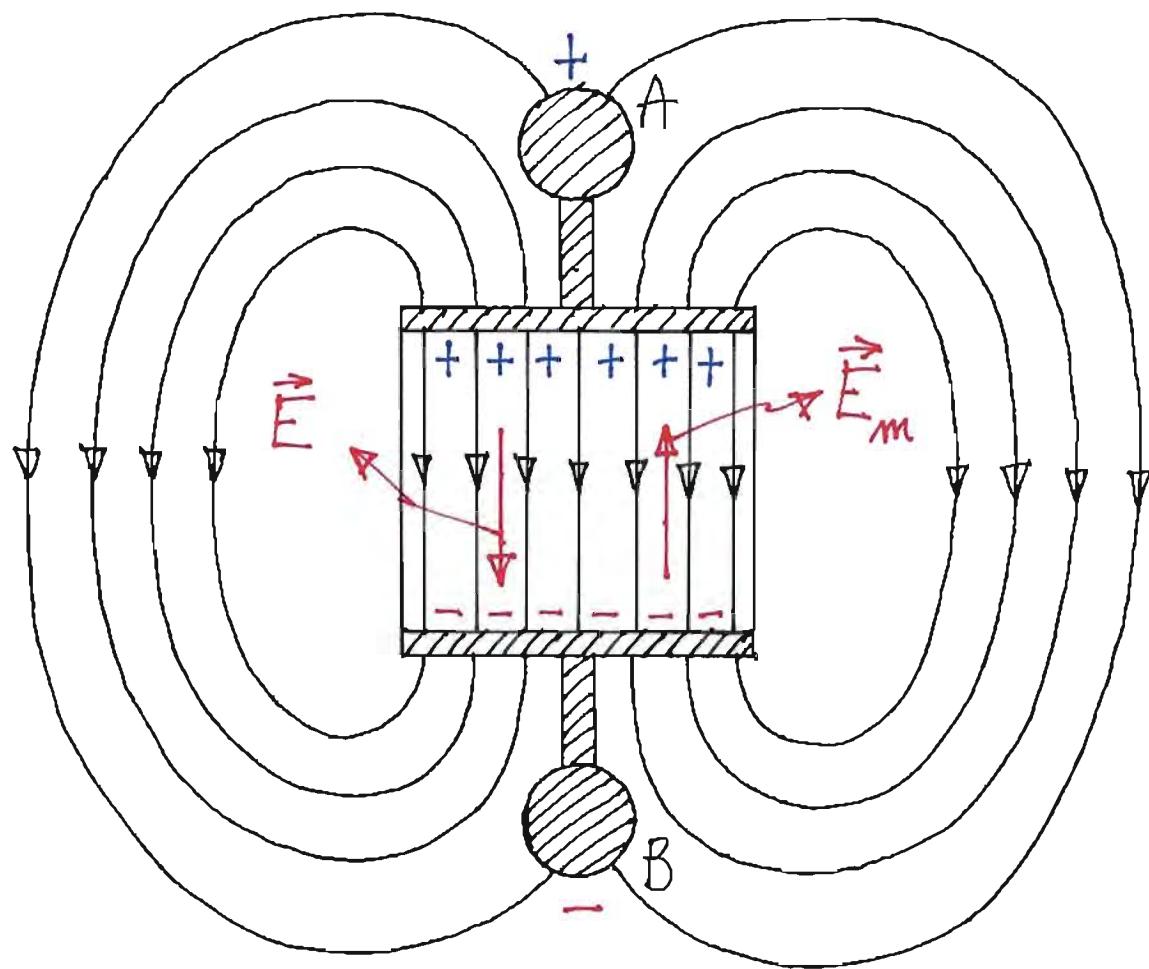


Figure 14.3.

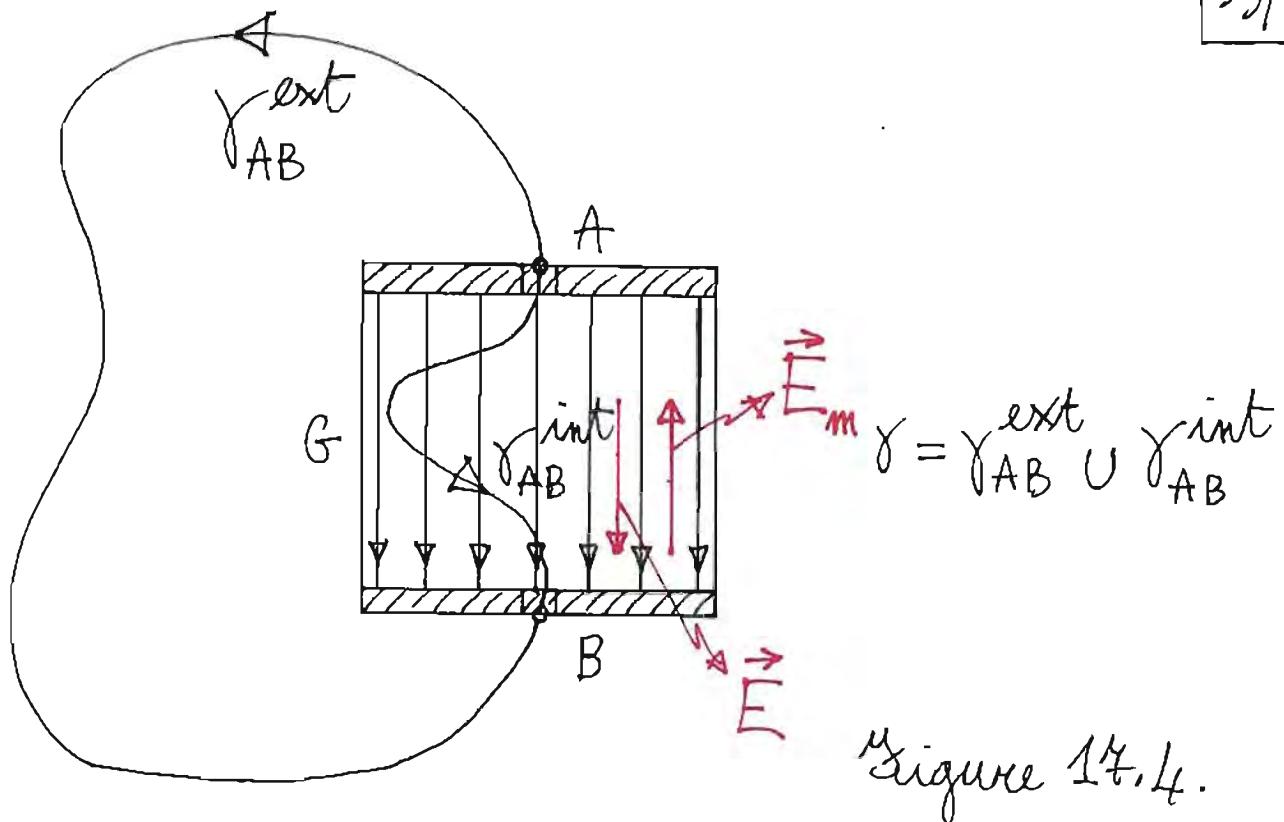


Figure 14.4.

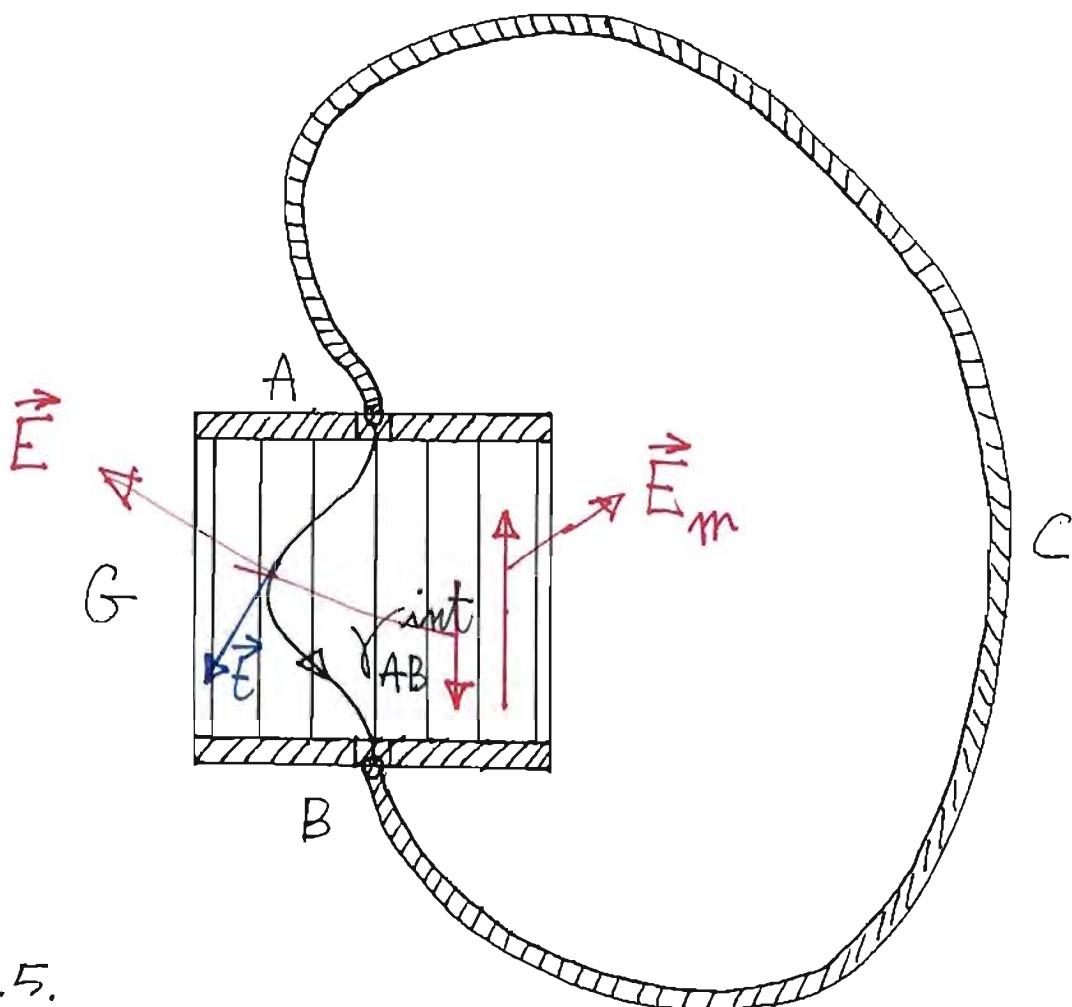


Figure 14.5.

Summary lecture 17.

- Stationary current.

An electric current is stationary if at each point of the medium where it flows both the charge density and current density are stationary:

$$\vec{\nabla} \cdot \vec{J} = 0 \quad (\text{solenoidal vector})$$

at each point and time. The current is independent of time at each point in the medium.

- In integral form, a stationary current is such that

$$\oint_{\Sigma} \vec{J} \cdot \vec{n} \cdot dS = 0 ,$$

i.e., the quantity of charge that goes out of a closed surface Σ within a given medium is zero.

- A conductor is a flux tube for \vec{J} and, [2/23]
thus, \vec{J} is tangent to the lateral surface of the conductor.
- At the discontinuity points between media with different physical properties,

$$J_{n1} = J_{n2}$$

The electric field in conductors with a stationary current.

The field at each point inside a conductor in electrostatic equilibrium is zero, $\vec{E} = \vec{0}$.

If a stationary current flows in the conductor, a nonzero field must exist in the conductor. This field makes the charges to move. The field obeys:

-

$$\oint \vec{E} \cdot \vec{n} \cdot dS = \frac{1}{\epsilon_0} q$$

\sum

and

$$\oint \vec{E} \cdot \vec{dl} = 0 ,$$

γ

as in electrostatics.

- The relation between \vec{J} and \vec{E} for a given conductor,

$$\vec{J} = f(\vec{E}) ,$$

is called the constitutive relation. For certain conductor this relation is linear,

$$\vec{J} = g \vec{E} ,$$

and is called Ohm's law in local form.

The electromotive force acting in a circuit with current.

In stationary conditions, a conductor where a current flow takes place must be part of a (closed) electric circuit due to the solenoidal nature of \vec{J} .

- An irrotational field, such as \vec{E} , can maintain a current in a part of a conductor, but not in a

closed circuit (because of the irrotational property). Thus, a nonconservative field, called the electromotive field, must exist at least in part of the electric circuit. The field is given by 4/23

$$\vec{E}_m = \frac{\vec{F}_m}{q} .$$

- Given an electric circuit γ , the scalar quantity with dimensions of a voltage

$$E = \oint_{\gamma} \vec{E}_m \cdot \vec{t} \cdot dl$$

is called the electromotive force (emf).

- The region where the electromotive force acts is called the source, which is characterized by two terminals A and B where a conductor can be attached (thus creating an electric circuit).

(i) For an open circuit source (i.e., A and B are not connected to any conductor), the potential

difference between A and B is given by

$$V_{AB}^o = \oint \vec{E}_m \cdot \vec{t} \cdot dl = \epsilon ,$$

where γ is a closed line that includes the source

(ii) For a closed circuit with current density \vec{J} ,

$$V_{AB} = \epsilon - \int_{\gamma_{source}} \eta (-\vec{J}) \cdot \vec{t} \cdot dl ,$$

where γ_{source} is a line connecting A and B in the source and $\eta = 1/g$.

18.1 Ohm's and Joule's law in integral form. 6/23

We will show that when two equipotential regions (electrodes) between which a potential difference V exists are connected by means of a linear and isotropic conductor, a current I flows in the conductor such that

$$V = R I \quad , \quad (18.1)$$

where the proportionality constant R is called the electric resistance of the conductor. Equation (18.1) is Ohm's law in integral form. The resistance depends on the conductor material and its geometry, as well as the position of the electrodes. The unit measure of R in the SI is the ohm (Ω), which represents the resistance of a conductor where a current of 1 A flows and that is characterized by a potential difference of 1 V.

Ohm's law can also be written as

$$I = G V \quad , \quad (18.2)$$

where $G = 1/R$ is called the electric conductance of a conductor. The unit of measure of G in the SI is the

siemens (S) or $1/\Omega$. Hence, the units of resistivity 4/23
and conductivity can be written as $\Omega \text{ m}$ and S m^{-1} ,
respectively.

We will now show that (18.1) and (18.2) can be derived from

$$\vec{J} = \sigma \vec{E}$$

and, thus, are valid only for linear conductors. Given a conductor, consider a filiform flux tube for vector \vec{J} (cf. Fig. 18.1). The filiform assumption means that each section S of the tube is so small that both vectors \vec{J} and \vec{E} can be assumed to be constant on each section (in general, \vec{J} and \vec{E} vary between different sections). By indicating with l a curvilinear coordinate (abscissa) along the flux tube axis and considered a cross-section S_k of the tube, with S_k orthogonal to \vec{J} , we have

$$I_k = \vec{J} \cdot \vec{T} \cdot S_k , \quad (18.3)$$

where \vec{T} is the unit vector tangent to the flux tube axis in correspondence of S_k . Since \vec{T} is directed as \vec{J} , (18.3)

reads

$$I_K = JS_K \quad , \quad (18.3')$$

where J is the magnitude of \vec{J} . As a consequence,

$$\begin{aligned} V &= \int_A^B \vec{E} \cdot \vec{E} \cdot dl = \int_A^B \eta \vec{J} \cdot \vec{E} \cdot dl \\ &= \int_A^B \eta \frac{I_K}{S_K} \cdot dl . \quad (18.4) \end{aligned}$$

The area S_K varies, in general, with l . Similarly, if the conductor is assumed to be non homogeneous, the value of the resistivity η also varies with l . The value of I_K , instead, is constant because of the very definition of flux tube. Hence,

$$V = I_K \int_A^B \frac{\eta}{S_K} \cdot dl . \quad (18.5)$$

The quantity

$$R_K = \int_A^B \frac{1}{S_K} \cdot d\ell \quad (18.6)$$

is the resistance of the k -th flux tube. We can thus write

$$V_K = V = R_K I_K \quad . \quad (18.7)$$

The same considerations can be repeated for each of the filiform flux tubes the entire conductor can be thought to be made of. If I is the current through the entire cross-section of the conductor, we have

$$I = \sum_k I_K \quad , \quad (18.8)$$

from which

$$I = \sum_k \frac{V}{R_K} = V \sum_k \frac{1}{R_K} \quad . \quad (18.9)$$

Note that if the cross-section of each filiform flux tube is infinitesimally small, the symbol of sum must be substituted by that of integral. In the following, for

simplicity, we will keep using the sum symbol.
Finally, by indicating

$$\frac{1}{R} = \sum_k \frac{1}{R_k} \quad (18.10)$$

we find

$$I = \frac{V}{R} \quad . \quad (18.11)$$

This result clearly shows that Ohm's law in integral form is a direct result of $\vec{J} = \sigma \vec{E}$.

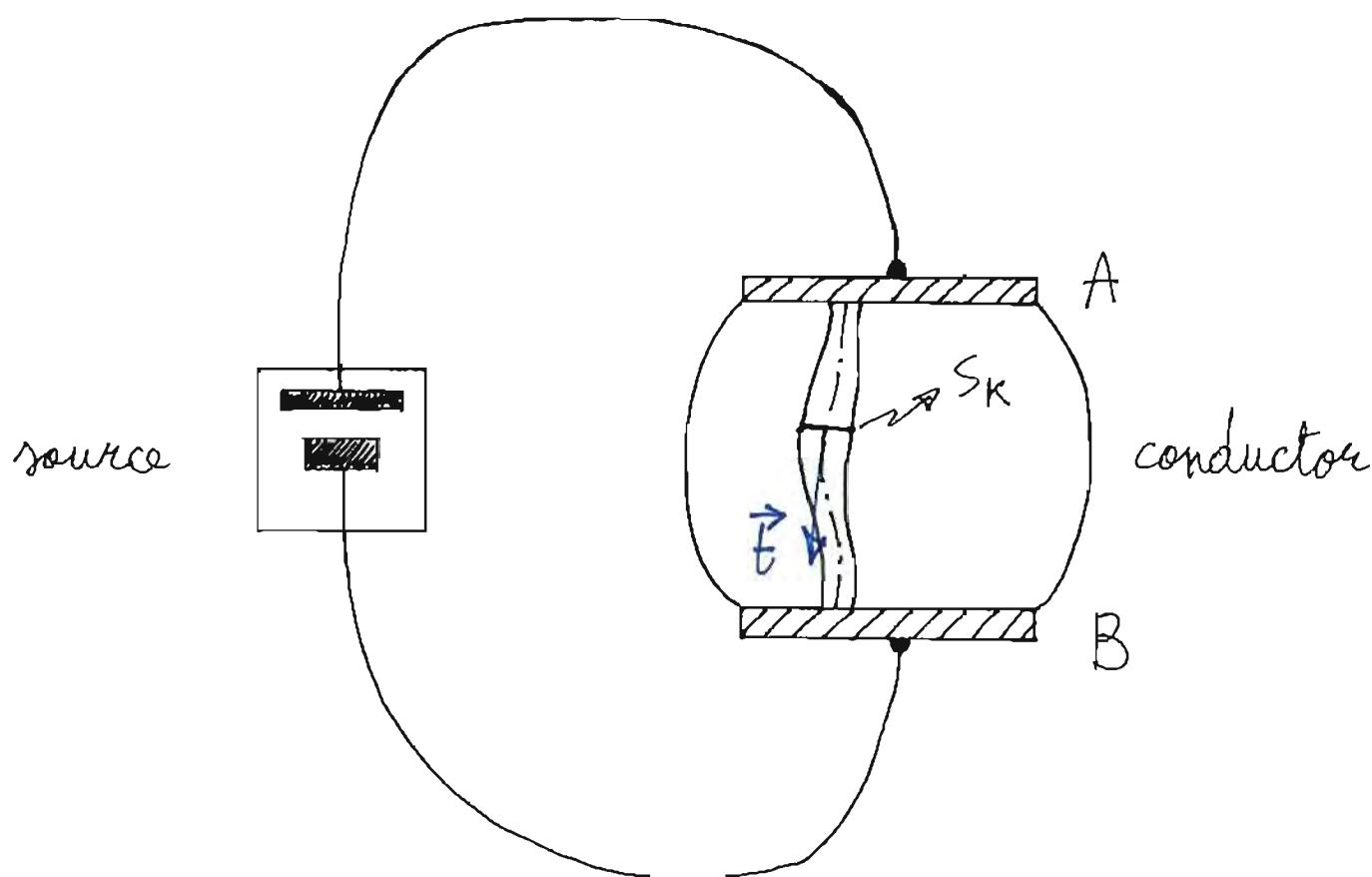


Figure 18.1.

Joule's law in integral form can also be [11/23] demonstrated by resorting to the flux tubes shown in Fig. 18.1. In fact, the power transformed into heat due to the Joule effect on an infinitesimal segment dl of the k -th flux tube (still considered to be filiform) is given by

$$dP_k = \eta J^2 S_k \cdot dl , \quad (18.12)$$

which, remembering that

$$J = \frac{I_k}{S_k} ,$$

can be written as

$$dP_k = \eta \frac{I_k^2}{S_k^2} S_k \cdot dl = \eta \frac{I_k^2}{S_k} \cdot dl . \quad (18.13)$$

Therefore, the total power P_k dissipated on the k -th flux tube is

$$P_k = \int_A^B \eta \frac{I_k^2}{S_k} \cdot dl = R_k I_k^2 , \quad (18.14)$$

where R_k is given by (18.6).

Since $V = R_k I_k$, (18.14) reads

$$P_k = V I_k \quad (18.15)$$

and, thus, the total power dissipated in the entire conductor is given by

$$\begin{aligned} P &= \sum_k P_k = \sum_k V I_k = V \sum_k I_k \\ &= V I \quad . \quad (18.16) \end{aligned}$$

Joule's law follows from (18.16),

$$P = V I = R I^2 = \frac{V^2}{R} \quad , \quad (18.17)$$

where we used Ohm's law (18.1) or (18.11).

Referring to Fig. 14.5, we now consider again the case of an electric circuit comprising an emf source \mathcal{E} closed on a conductor AB. We remind that, in this case, the electromotive field \vec{E}_m prevails on \vec{E} and, thus, a current with density \vec{J} directed as \vec{E}_m is generated. The

vector \vec{J} is thus directed opposite of the tangent unit vector \vec{T} . This means that the scalar quantity in (17.32) [13/23]

$$-\vec{J} \cdot \vec{T}$$

must be actually positive. In particular,

$$-\vec{J} \cdot \vec{T} \cdot S = I \quad , \quad (18.18)$$

where S is the source cross-section. Equation (17.32) can thus be written as

$$\begin{aligned} V_{AB} &= \mathcal{E} - \int_{\gamma_{AB}^{\text{int}}} \eta (-\vec{J}) \cdot \vec{T} \cdot dl \\ &= \mathcal{E} - I \int_{\gamma_{AB}^{\text{int}}} \frac{\eta}{S} \cdot dl \quad , \quad (18.19) \end{aligned}$$

where we assumed \vec{J} to be uniform inside the source and we called I the total current flowing in the source

By defining

$$R_g = \int_{\gamma_{AB}^{\text{int}}} \frac{1}{S} \cdot d\ell , \quad (18.20)$$

we have

$$V_{AB} = \mathcal{E} - R_g I . \quad (18.21)$$

By multiplying both sides of this equation by I , we then obtain

$$\mathcal{E}I = V_{AB}I + R_g I^2 , \quad (18.22)$$

which represents the energetic budget of the entire circuit. In fact, from the very definition of emf, it appears that $\mathcal{E}I$ is the power generated by the source, i.e., the work per unit time produced by the electromotive field acting inside the source. From (18.16), the term $V_{AB}I$ represents the power dissipated in the conductor AB due to the Joule effect and $R_g I^2$ the power dissipated inside the source. In this case, the power generated by the electromotive

field inside the source against the electric field forces is transformed entirely into heat (due to the Joule effect) in the conductor and the source itself.

18.2 Resistance of conductors with current.

We will now study two interesting examples of conductors with currents. In particular, we will calculate the resistance of such conductors starting from the potential equations.

18.2.1 Cylindric conductor with longitudinal current.

Consider a cylindric conductor C , homogeneous, with a generic cross-section, and with resistivity η . The two bases of the cylinder (parallel to each other and normal to the axis of C) are connected to two planar electrodes S_1 and S_2 (with negligible resistivity with respect to η). A potential difference is applied to S_1 and S_2 . Figure 18.2 shows a schematic of the problem.

With respect to a cartesian coordinate system with the Z axis coinciding with the conductor axis, the potential function V is independent from x and y due to the rotation

16/2

symmetry of the conductor about the z axis. at each point in C we thus have

$$\frac{d^2}{dz^2} V(z) = 0 \quad . \quad (18.23)$$

The general solution to this equation is

$$V(z) = Az + B \quad , \quad (18.24)$$

where A and B are arbitrary constants. The electric field \vec{E} is thus uniform and directed along z . Its absolute value

$$|E| = \left| \frac{d}{dz} V(z) \right| = |A| \quad . \quad (18.25)$$

As a consequence, the current density \vec{J} is also uniform and directed along z . The absolute value of \vec{J} is given by

$$|J| = \frac{|E|}{\eta} = \frac{|A|}{\eta} \quad . \quad (18.26)$$

Thus, the current I through the conductor is given by

$$|I| = |J|S = \frac{|A|}{\eta} S , \quad (18.24)$$

where S is the area of the normal cross-section of the conductor (orthogonal to \mathbf{J}).

From (18.24), the potential difference between the conductor electrodes is

$$\Delta V = |V_1 - V_2| = |A|L , \quad (18.28)$$

where V_1 and V_2 are the potentials at S_1 and S_2 , respectively, and L the length of C .

By definition, the resistance is given by

$$R = \frac{\Delta V}{|I|} = \frac{|A|L}{|A|\frac{S}{\eta}} = \eta \frac{L}{S} , \quad (18.29)$$

which is a very useful result to calculate the resistance of filiform conductors (i.e., conductors with longitudinal dimensions much larger of the transversal dimension characteristic of the normal cross-section S , $L \gg \sqrt{S}$).

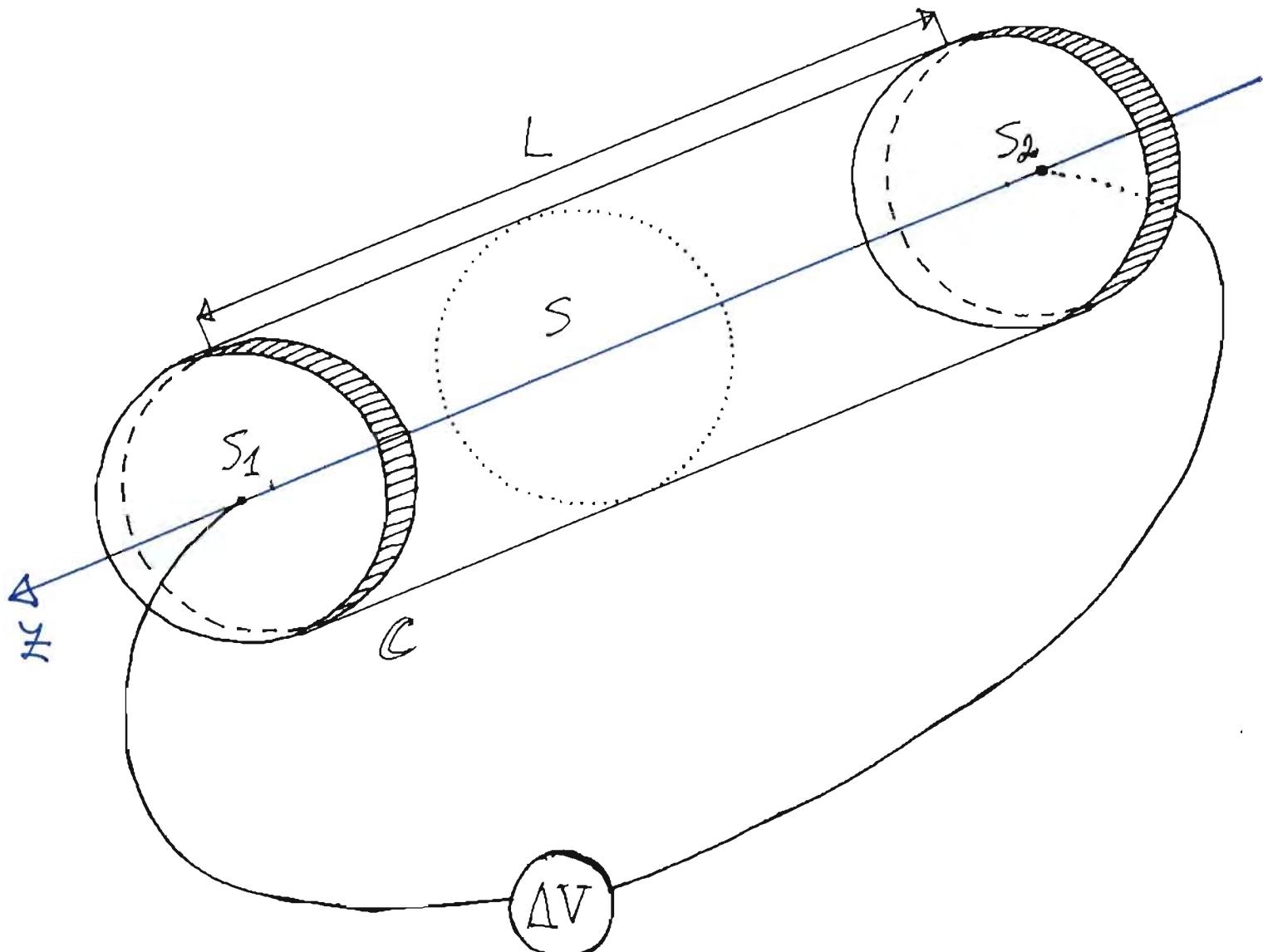


Figure 18. 2.

18.2.2 Hollow cylindric conductor with radial current.

Consider a hollow cylindric conductor \tilde{C} , with a circular cross-section with internal radius R_1 and external radius R_2 , and with resistivity η . Assume the internal and external surfaces, Σ_1 and Σ_2 , respectively, are two electrodes (with resistivity much smaller than η) between which a potential difference V is applied. Figure 18.3 shows a schematic of the problem.

Given a cylindric coordinate system $Ox\varphi z$ with the z axis coinciding with \tilde{C} axis, for symmetry reasons (translation symmetric with respect to z and rotation symmetric with respect to φ), the potential V only depends on r .

Thus, Laplace equation in cylindric coordinates is

$$\frac{d^2}{dr^2} V(r) + \frac{1}{r} \frac{d}{dr} V(r) = 0 \quad . \quad (18.30)$$

By defining

$$\varphi(r) = \frac{d}{dr} V(r) , \quad (18.31)$$

equation (18.30) reads

$$\frac{d}{dr} \varphi(r) + \frac{1}{r} \varphi(r) = 0 , \quad (18.32)$$

the general solution of which is

$$\varphi(r) = \frac{A}{r} , \quad (18.33)$$

where A is an arbitrary constant. By substituting (18.33) into (18.31), we finally find the general solution of (18.30),

$$V(r) = A \ln r + B , \quad (18.34)$$

where A and B are arbitrary constants and r in the natural logarithm should be considered as normalized over 1 m (for obvious dimensions reasons).

The electric field is also directed radially and has absolute value

$$|E| = \left| \frac{d}{dr} V(r) \right| = \frac{|A|}{r} \quad . \quad (18.35)$$

The current density \vec{J} is also directed radially, with absolute value

$$|J| = \frac{1}{\mu} |E| = \frac{1}{\mu} \frac{|A|}{r} \quad . \quad (18.36)$$

The total current that goes through any cylindrical surface with radius r , comprised between Σ_1 and Σ_2 , and coaxial with Σ_1 and Σ_2 is given by

$$\begin{aligned} |I| &= 2\pi r L |J| = 2\pi \mu L \frac{1}{\mu} \frac{|A|}{r} \\ &= \frac{1}{\mu} 2\pi L |A| , \quad (18.37) \end{aligned}$$

where L is the conductor length.

From (18.34), the potential difference between the electrodes is

$$|V| = |A| \ln \left(\frac{R_2}{R_1} \right) \quad (18.38)$$

Under these conditions, the conductor resistance is given by

$$\begin{aligned} R &= \frac{|V|}{|I|} = \frac{\cancel{|A|} \ln \left(\frac{R_2}{R_1} \right)}{\frac{1}{\mu} 2\pi L \cancel{|A|}} \\ &= \mu \frac{\ln \left(\frac{R_2}{R_1} \right)}{2\pi L}, \quad (18.39) \end{aligned}$$

which is a very useful expression when, for example, the so called insulating resistance of a coaxial cable needs to be calculated.

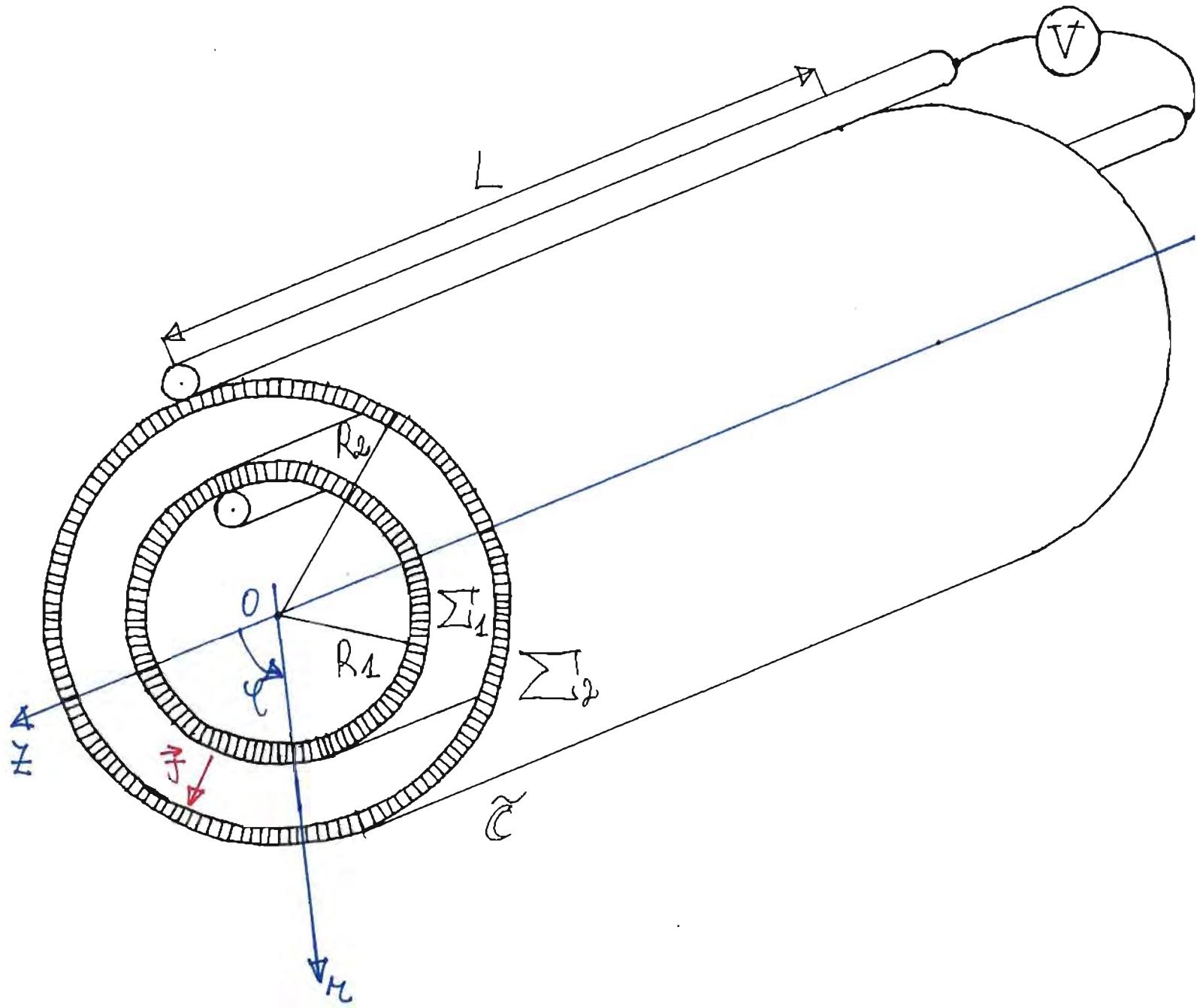


Figure 18.3.

Summary lecture 18.

- Ohm's and Joule's law in integral form.

Consider a linear conductor enclosed between two electrodes A and B. In such a conductor, assume a filiform flux tube for \vec{J} . The generic cross-section of the tube is S_K .

- At S_K , the current is given by

$$I_K = JS_K$$

Hence,

$$\begin{aligned} V_{AB} &= \int_A^B \vec{E} \cdot \vec{dl} \\ &= \int_A^B \eta \frac{I_K}{S_K} \cdot dl \end{aligned}$$

where $\vec{E} = \vec{J}/|\vec{J}|$. Because of the definition of flux tube, I_K is constant along the tube, thus

$$V_{AB} = I_K \int_A^B \frac{\eta}{S_K} \cdot dl .$$

Note that dl is an infinitesimal element on the k -th flux tube. By defining the resistance of such a tube

$$R_K = \int_A^B \frac{\eta}{S_K} \cdot dl ,$$

we find that

$$V_{AB} = R_K I_K .$$

Calling I the current through the cross-section of the entire conductor (i.e., through $S = \sum_k S_k$),

$$I = \sum_k I_k$$

and, thus,

$$I = \sum_k \frac{V_{AB}}{R_K} = V_{AB} \sum_k \frac{1}{R_K} .$$

Finally, by defining

$$R = \sum_k \frac{1}{R_k} ,$$

we find Ohm's law in integral form

$$I = \frac{V_{AB}}{R} .$$

- The power dissipated into heat due to the Joule effect on a piece dl of the k -th flux tube is

$$dP_k = \eta \frac{I_k^2}{S_k} \cdot dl .$$

The total power dissipated on the entire tube is then

$$P_k = \int_A^B \eta \frac{I_k^2}{S_k} \cdot dl = R_k I_k^2 .$$

Since $V_{AB} = R_k I_k$,

$$P_k = V_{AB} I_k$$

and

$$P = \sum_k P_k = V_{AB} I .$$

Typically, V_{AB} is simply called V .

- given a source with emf ϵ connected to the conductor with electrodes A and B, the energetic budget for the closed circuit is

$$\epsilon I = V_{AB} I + R_g I^2 ,$$

where R_g is the source internal resistance.

- Resistance of a cylindric conductor with longitudinal current.

$$R = \eta \frac{L}{S} ,$$

where L is the conductor length and S its cross-section

19.1 Introduction.

5/56

By placing two electric circuits close to each other, when stationary currents flow through both circuits, mutually attractive or repulsive forces act between the circuits. Such forces disappear as soon as (at least) one of the two currents is turned off. The forces between the two circuits are called ponderomotive forces, i.e., forces able to act on the conductor as any other mechanical force, e.g., the weight. The first experiments on those forces were due to H. C. Oersted, J. B. Biot, and F. Savart.

The phenomenon of ponderomotive forces in circuits can be qualitatively explained by thinking that when a conductor is characterized by a current, the motion of the charges generates a magnetic field in the surrounding space. In presence of a second conductor, in proximity of the first conductor, and also characterized by a current, the moving charges in each of the two circuits are acted upon by a force of magnetic origin.

$$\vec{F} = q \vec{v} \times \vec{B} , \quad (19.1)$$

where \vec{v} is the velocity corresponding to the oriented

motion that characterizes the current. Note that, when averaged over a physics infinitesimal time interval, the force acting on a charge is not affected by the term due to the thermic motion.

The moving charge free carriers in the conductor are acted upon both by the forces due to the electric field in the conductor and by those due to the magnetic field. The electric field acts both on the positive ions of the crystalline lattice and on the negative free electrons. However, since the ions are fixed, the electric field acts effectively only on the negative electrons, making them move. In this sense, the electric field cannot have any effect on the entire conductor because, in each macroscopic portion of the conductor, the conductor is neutral.

Given a reference frame, the forces of magnetic origin (if the conductor is at rest) act only on the free charges, normally to the direction of their motion. However, such forces are (typically) insufficient to tear the electrons off the conductor due to the very strong Coulomb-type attraction forces that bind them to the positive fixed ions. The resulting macroscopic effect of the magnetic

force is, thus, a ponderomotive force acting on the entire conductor. 7/56

19.2 The fundamental laws of magnetostatic.

When presenting the laws of the magnetic field generated by stationary currents, in analogy to the electrostatic case, we could proceed starting from the law of the magnetic field generated in vacuum by an elementary current (e.g., by a circular loop). We could then extend this law to the general case of a distribution of currents by means of a superposition principle. As it turns out, this would be a not so easy task. We will thus present the fundamental laws directly (in integral form). From these laws, we will then find the magnetic field generated by a generic distribution of stationary currents. In the electrostatic case, this would correspond to present directly Gauss' theorem and the irrotational property and, then, obtain from them the field for a generic distribution of charges (cf. lecture 8).

The first law of magnetostatic states that the

magnetic field generated by any distribution of currents must be solenoidal. In other words, indicating with \vec{B} the magnetic field and with Σ any closed surface contained within the field, it must be [8/56]

$$\oint \vec{B} \cdot \hat{n} \cdot dS = 0 , \quad (19.2)$$

where \hat{n} is the normal unit vector associated with Σ . The difference to the electrostatic case is clear. In that case, instead of (19.2), Gauss' theorem would apply (according to which the flux of \vec{E} through a closed surface is proportional to the sum of all charges contained within the surface). In the magnetic case, the flux of \vec{B} is always zero: Magnetic charges from where or into which vector line start or end do not exist. The vector lines of \vec{B} do not have any beginning or end (i.e., they are closed) or do extend to infinity.

The law (19.2) is equivalent to

$$\iint_{S_1} \vec{B} \cdot \vec{n}_1 \, dS = \iint_{S_2} \vec{B} \cdot \vec{n}_2 \, dS , \quad (19.3)$$

where S_1 and S_2 are two generic (open) surfaces with common border γ and normal unit vectors \vec{n}_1 and \vec{n}_2 (oriented consistently), respectively. Since the flux of the magnetic field depends only on γ , in the magnetic case the flux is usually referred to as the linked flux with respect to a closed line γ . In this case, it is unnecessary to specify the surface corresponding to γ . The dimensions of the flux of \vec{B} are those of a magnetic field times a surface, $[\Phi] = [B][S]$. The corresponding SI unit measure is, thus, $T\text{m}^2$, which is called the Weber (Wb).

The second law of magnetostatic (also called the circulation law or Ampère's law, or Ampère's circuital law) is related to the circulation of the magnetic field. Differently from the electrostatic field, the magnetostatic field is rotational. Given a generic closed, oriented line γ in vacuum, contained within a magnetic field,

it must be

$$\oint_{\gamma} \vec{B} \cdot \vec{T} \cdot d\ell = \mu_0 I \quad (19.4)$$

where μ_0 is the vacuum permeability constant and \vec{T} a tangent unit vector to γ . By definition, the SI value of the vacuum permeability is $\mu_0 = 4\pi \cdot 10^{-7} \text{ H m}^{-1}$. The term I in (19.4) indicates the sum of all currents linked with γ . It is worth reminding that two closed lines are linked to each other if it is impossible to disconnect them without cutting one of them. A more formal (but less intuitive) definition is that two lines are linked if at least one open surface exists, having one of the two lines as a border and only one intersection point with the other line. The aforementioned sum for I must be intended in an algebraic sense. All currents flowing consistently with the orientation on γ are positive, all the others are negative. Note that the current flow is consistent with the line orientation if the right-hand rule is fulfilled. An example is in order. Consider Fig. 19.1, where two

circuits with current I_1 and I_2 , respectively, are linked with a closed, oriented line γ . The electric circuits form two closed loops γ_1 and γ_2 . According to the figure, the orientation of I_1 on γ_1 has been chosen consistently with the orientation on γ , whereas the orientation of I_2 on γ_2 has been chosen with opposite orientation with respect to γ . In fact, when I_1 crosses the plane surface enclosed by γ , the direction of I_1 is equal to that of \vec{n} for γ . On the contrary, when I_2 crosses the same surface, the direction of I_2 is the opposite of \vec{n} . In summary,

$$\oint_{\gamma} \vec{B} \cdot \vec{T} \, dl = \mu_0 (I_1 - I_2) \quad (19.5)$$

If the closed line γ does not link any current (or it links currents the sum of which is zero), the circulation of the magnetic field is zero. If γ coincides with a vector line, the circulation is certainly nonzero. In fact, since the scalar product $\vec{B} \cdot \vec{T}$ has always the same sign, the circulation is the sum of contributions all with the same sign. Hence, each vector line of the magnetic field

is linked at least with one current.

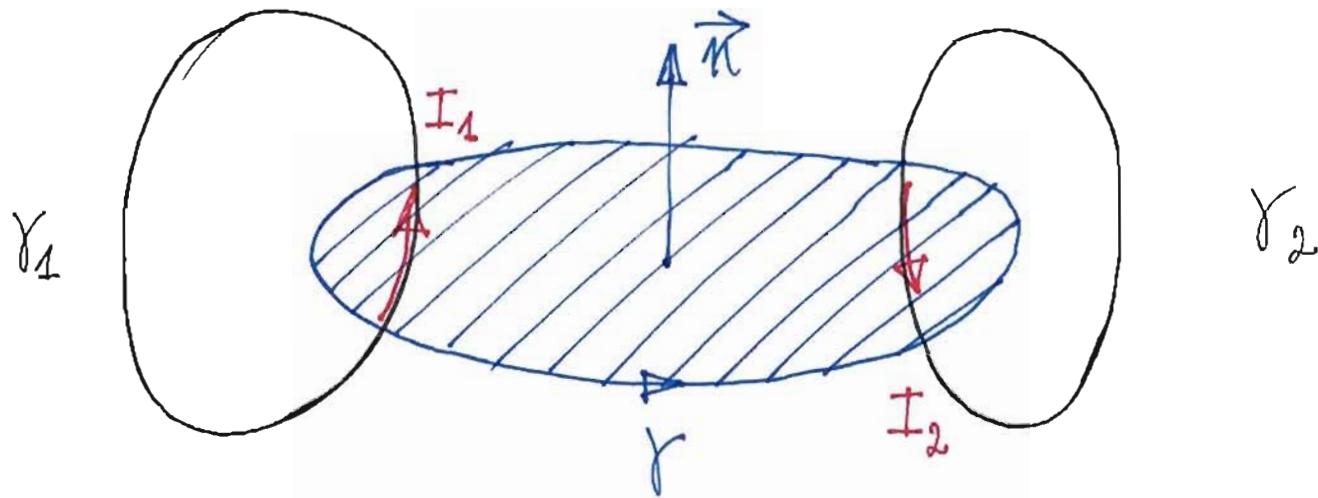


Figure 19.1.

Consider two equally oriented open lines γ_1 and γ_2 with equal limits A and B. It must be

$$\int_{AY_1B} \vec{B} \cdot \vec{E} \cdot d\ell - \int_{AY_2B} \vec{B} \cdot \vec{E} \cdot d\ell = \mu_0 I , \quad (19.6)$$

where I indicates the algebraic sum of all currents linked with the closed line $AY_1B(-\gamma_2)A$. If such a curve does not link any currents (or it links currents the sum of which is zero), we have

$$\int_{A\gamma_1 B} \vec{B} \cdot \vec{F} \cdot d\ell = \int_{A\gamma_2 B} \vec{B} \cdot \vec{F} \cdot d\ell . \quad (19.4)$$

In summary, given internally connected regions of space where there are no currents, it is possible to find there a potential function for the magnetic field. This is because in such regions the circulation of \vec{B} is zero. In presence of currents, this condition is in general not valid.

It is now worth giving an example why the regions must be internally connected. In Fig. 19.2, the point O is the trace of a current I perpendicular to the page. If we imagine that in the plane of the page (and in all planes parallel to it) there is a hole having a closed line γ_1 as a border that surrounds O , the magnetic field is considered in a region that does not contain the current that generates it. Since the domain where the field is considered does not include the region delimited by γ_1 , each closed line entirely contained in the domain does not link any currents contained in the domain. Nevertheless, the field circulation around a closed line that contains point O

is different from zero. In fact, considering the field 14/56
outside γ_1 only cannot change the values of \vec{B} in that region. This
difficulty can easily be resolved by noting that, once the
hole around I is open, the remaining region of the
considered plane is not internally connected. Hence, the
reason for the internal connection assumption.

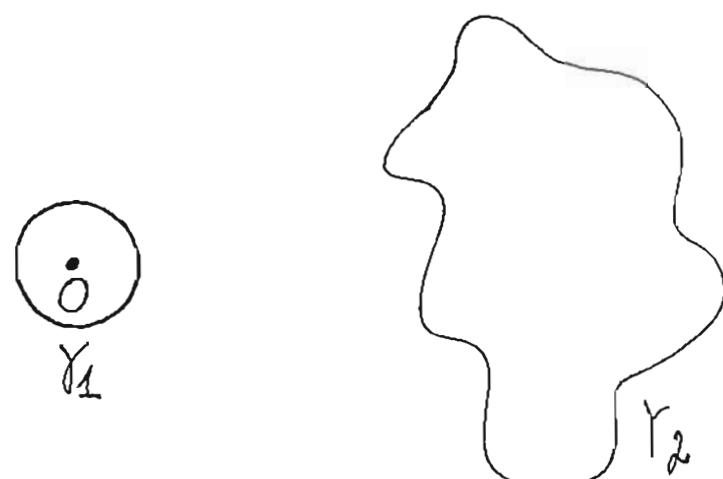


Figure 19.2.

Consider one more example. In Fig. 19.3, point O still
is the trace of a current I perpendicular to the page. A
line of γ_1 tape starts at infinite, goes around point O ,
and goes back to infinite. The region of plane γ_2 , which
is obtained by excluding the region γ_1 (hatched area)
included inside γ_1 , is internally connected. As a consequence,

in this region the magnetic field has a potential (because the current I cannot be linked to any line in Ω_2). Referring to Fig. 19.3, the radius a of the circle that encloses point O can be chosen to be arbitrarily small (so long $a \neq 0$); similarly, the distance δ between the two rays that delimit the rest of the region Ω_1 can also be chosen to be arbitrarily small. Thus, the region Ω_2 can be extended to include almost the entire plane (page). Hence, in contrast to our previous statement, it would seem the magnetic field has a (scalar) potential everywhere on the plane (and on all planes parallel to it). In reality, considering two points facing each other as, e.g., P and P' , and indicating the potential of the field as φ , from the definition of potential we have

$$\varphi(P) - \varphi(P') = \int_{PAA'P'} \vec{B} \cdot \vec{t} \cdot dl \quad . \quad (19.8)$$

When $\delta \rightarrow 0$, the line $PAA'P'$ tends to close around O and, thus, the field line integral tends to the value $\mu_0 I$. As a consequence, when we try to extend the region where \vec{B} has a potential, until the entire domain is included,

16/56

the potential shows an abrupt jump through the infinitesimal distance δ , where the jump is equal to $M_0 I$. Therefore, we must consider a potential the values of which differ by a finite quantity between infinitesimally closed points (such as P and P'). In order to remove such a discontinuity, we should consider multivalued potential functions, i.e., potential functions that assume distinct values at the same point.

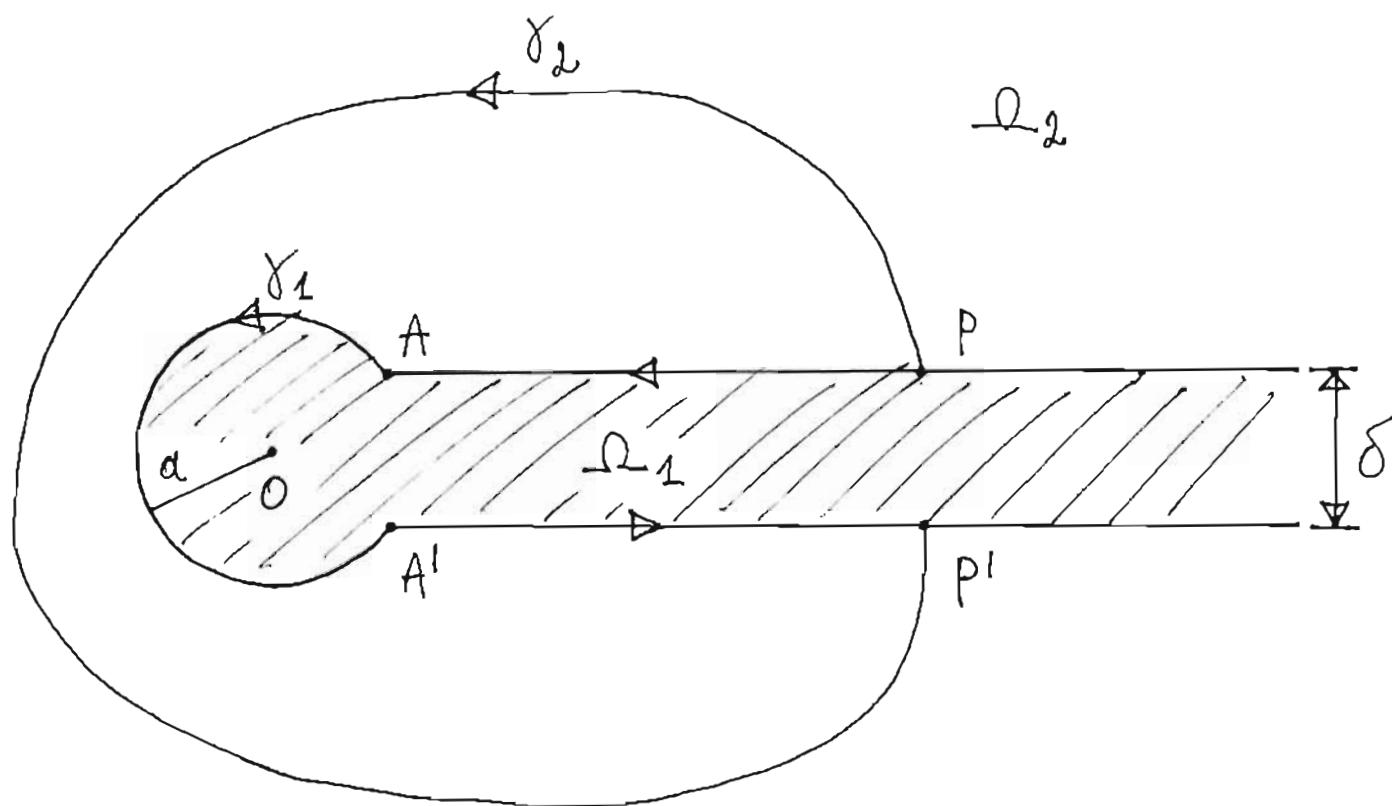


Figure 19.3.

It is worth mentioning that the laws of

magnetostatic (19.2) and (19.4) contain a superposition principle, according to which the magnetic field generated in vacuum by a set of currents is equal to the vector sum of the fields generated by each current (considered alone). In fact, the second term of (19.4) is the sum of the currents generating the field. As shown in pages 24–29 of lecture 8, this is analogous to the superposition principle for the electrostatic fields contained in Gauss' theorem. At last, we observe that also the magnetic field generated by spatially limited current distributions goes to zero at infinite (we will come back to this point when discussing the multipole expansion of the vector field).

It is useful to conclude this section with a table that compares the laws of electrostatic and magnetostatics, showing the dual aspect of these laws (Table 19.1).

	Electrostatics	Magnetostatics
Gauss' theorem vs. solenoidal property of \vec{B}	$\oint \vec{E} \cdot \vec{n} \cdot dS = \frac{1}{\epsilon_0} q$ \sum	$\oint \vec{B} \cdot \vec{n} \cdot dS = 0$ \sum
irrotational property of \vec{E} vs. Ampère's law	$\oint \vec{E} \cdot \vec{t} \cdot dl = 0$ γ	$\oint \vec{B} \cdot \vec{t} \cdot dl = \mu_0 I$ γ

Table 19.1.

19.3 Magnetic fields generated by simple current distributions [19/56]

The first problem to be studied is the magnetostatic dual of the electrostatic field generated by a uniformly charged infinite straight line γ (cf. lecture 4). Arguments similar to those used in lecture 4 will thus be used.

19.3.1 Infinite straight line with stationary current.

Figure 19.4 shows an infinite straight filiform conductor (line) γ with a stationary current I directed upward along the line.

As in lecture 4, the natural choice for the coordinate system is a cylindric system $Oxyz$, where the z axis coincides with γ and has same orientation as I (cf. Fig. 19.4).

The magnetostatic vector \vec{B} at a generic point P is also shown in the figure,

$$\vec{B} = B_x \vec{u}_x + B_\varphi \vec{u}_\varphi + B_z \vec{u}_z . \quad (19.9)$$

We first attempt to solve the problem only by means of symmetry arguments (as in lecture 4, pages 6-14). The symmetries characteristic of this problem are:

(a) Rotation symmetry.

This is exactly as in lecture 4 and is schematically shown in Fig. 19.5 a.

(b) Translation symmetry.

Also as in lecture 4 (shown in Fig. 19.5 b).

(c) Anti-reflection symmetry.

The presence of a current I directed upward breaks the reflection symmetry encountered in lecture 4. This means that, given a vector \vec{B} at a point P , $\{\vec{B}, P\}$, upon rotating γ by a clockwise or counterclockwise π rotation about a pivot O' (O' can be any point on γ), the field obtained at the rotated point P' must be changed in sign, $\{-\vec{B}, P'\}$. Figure 19.5 c shows schematically this type of symmetry.

As in lecture 4, we will now consider each component of \vec{B} and try to find as much information as possible on it by means of the three aforementioned symmetries.

1) Radial component B_r .

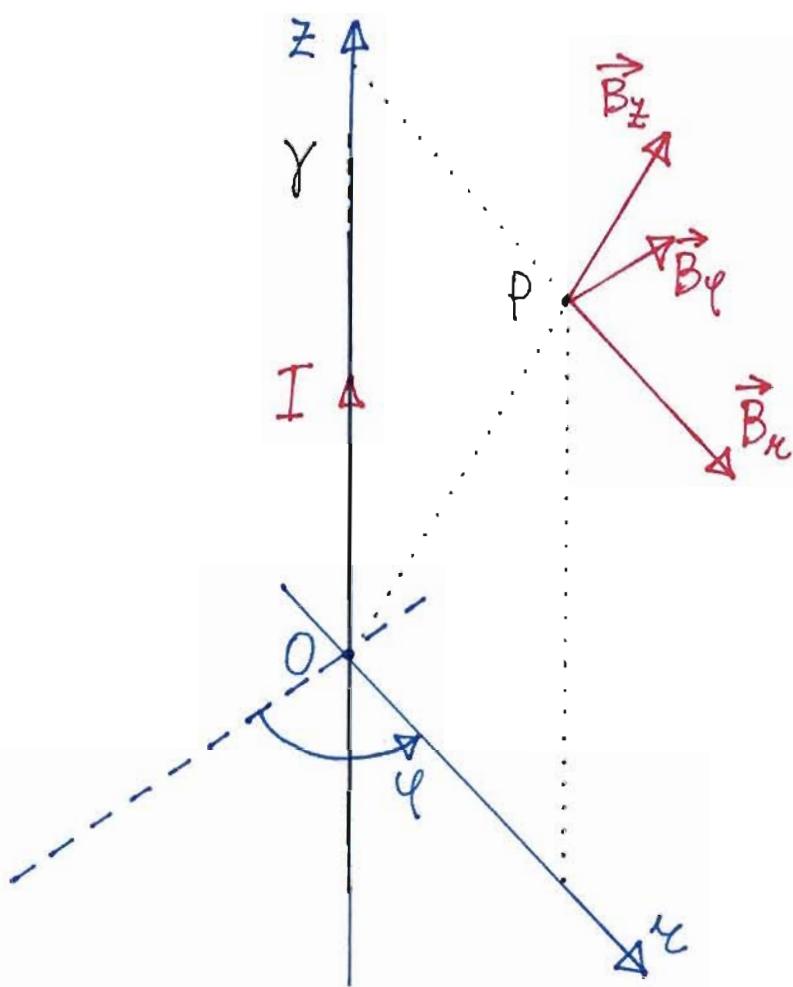


Figure 19.4.

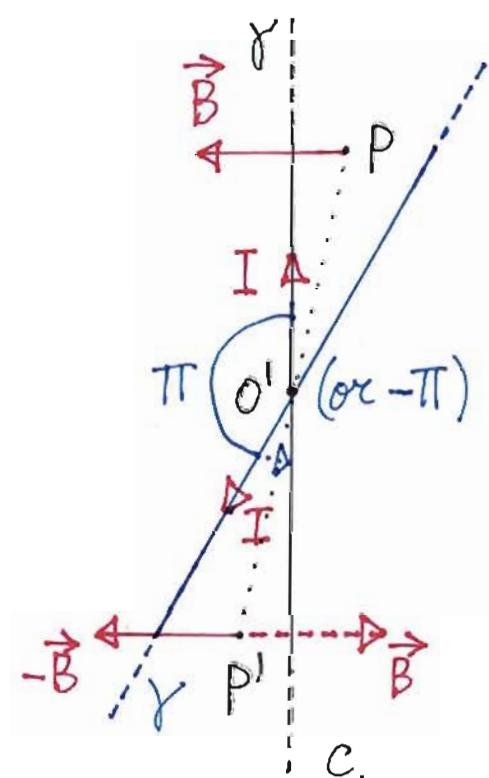
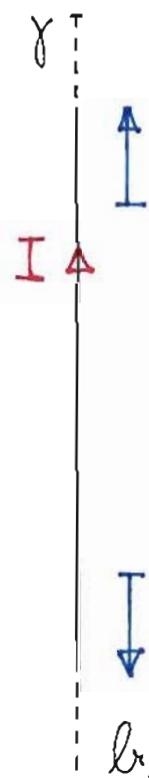
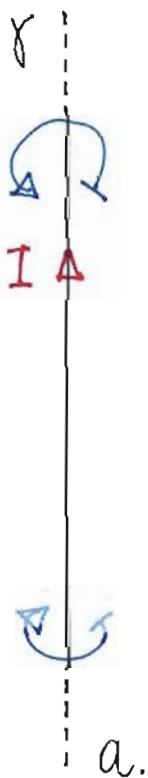


Figure 19.5.

A generic component \vec{B}_r (e.g., pointing outward) at a point P , $\{\vec{B}_r, P\}$, is shown in Fig. 19.6a. Due to the rotation symmetry, if such a component exists, it must be the same at each point on a circle with surface normal to γ and center O_1 at the intersection between such a surface and γ . The circle is called γ_1 in Fig. 19.6a.

Note that, due to the homogeneity and isotropy of space, if \vec{B}_r points outward at P , it must point outward also at $Q = P + dP$. This is the very same argument as for \vec{E}_r , as shown in Fig. 4.3d in lecture 4.

Due to the translation symmetry, \vec{B}_r must also be the same at each point on a line γ_2 parallel to γ , as shown in Fig. 19.6b.

This means that \vec{B}_r must be the same at each point on any cylinder coaxial with γ and must repeat equally on each plane normal to γ .

We can finally make use of the anti-reflection symmetry to perform a consistency check. Consider \vec{B}_r at P , as in Fig. 19.6c, $\{\vec{B}_r, P\}$. Rotate γ as well as $\{\vec{B}_r, P\}$ as a rigid body by an angle π clockwise (or counter-clockwise) about the pivot O' . At the new point P' , we

have the new pair $\{\vec{B}_x, P'\}$ due to the anti-reflection property. Translate $-\vec{B}_x$ at P' upward to P'' , so to obtain $\{\vec{B}_x, P''\}$. At last, rotate $-\vec{B}_x$ at P'' by an angle π clockwise (or counterclockwise) along γ_1 back to P . The new pair $\{\vec{B}_x, P\}$ clearly shows that the only possibility for \vec{B}_x to be compatible with the symmetries of the problem is for $\vec{B}_x = \vec{0}$.

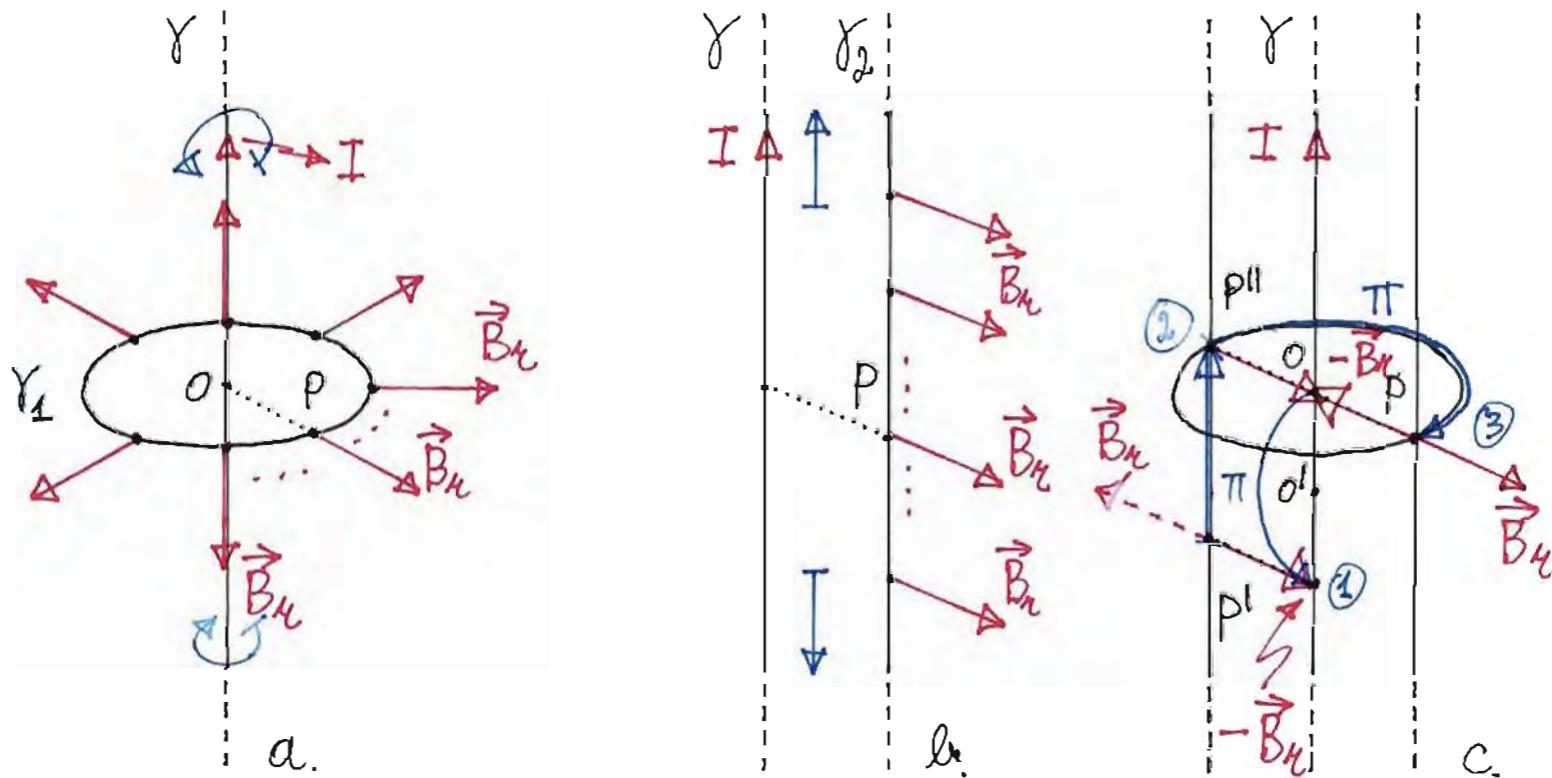


Figure 19.6.

2) Tangent component \vec{B}_φ .

A generic component \vec{B}_φ at a point P , $\{\vec{B}_\varphi, P\}$, is shown

in Fig. 19.4a. Due to the rotation symmetry, \vec{B}_φ [24/56] must be the same at each point on the generic circle γ_1 .

Due to the translation symmetry, \vec{B}_φ must also be the same at each point on the generic line γ_2 , as in Fig. 19.4b.

Hence, \vec{B}_φ must be the same at each point on any cylinder coaxial with γ and must repeat equally on each plane normal to γ .

Finally, the anti-reflection symmetry allows us to perform a consistency check. Consider \vec{B}_φ at P , as in Fig. 19.4c, $\{\vec{B}_\varphi, P\}$. Rotate γ as well as $\{\vec{B}_\varphi, P\}$ as a rigid body by an angle π counterclockwise (or clockwise) about the pivot σ . At the new point P' , we have the new pair $\{-\vec{B}_\varphi, P'\}$ due to the anti-reflection property. Translate $-\vec{B}_\varphi$ upward from P' to P'' , so to obtain $\{-\vec{B}_\varphi, P''\}$. At last, rotate $-\vec{B}_\varphi$ at P'' by an angle π clockwise (or counterclockwise) along γ_1 back to P . The new pair $\{-\vec{B}_\varphi, P\}$ is consistent with the initial pair $\{\vec{B}_\varphi, P\}$, showing that \vec{B}_φ can exist.

3) Vertical component \vec{B}_z .

A component \vec{B}_z at P , $\{\vec{B}_z, P\}$, is shown in Fig. 19.8a. Due to the rotation symmetry, \vec{B}_z must be the same at each point on γ_1 .

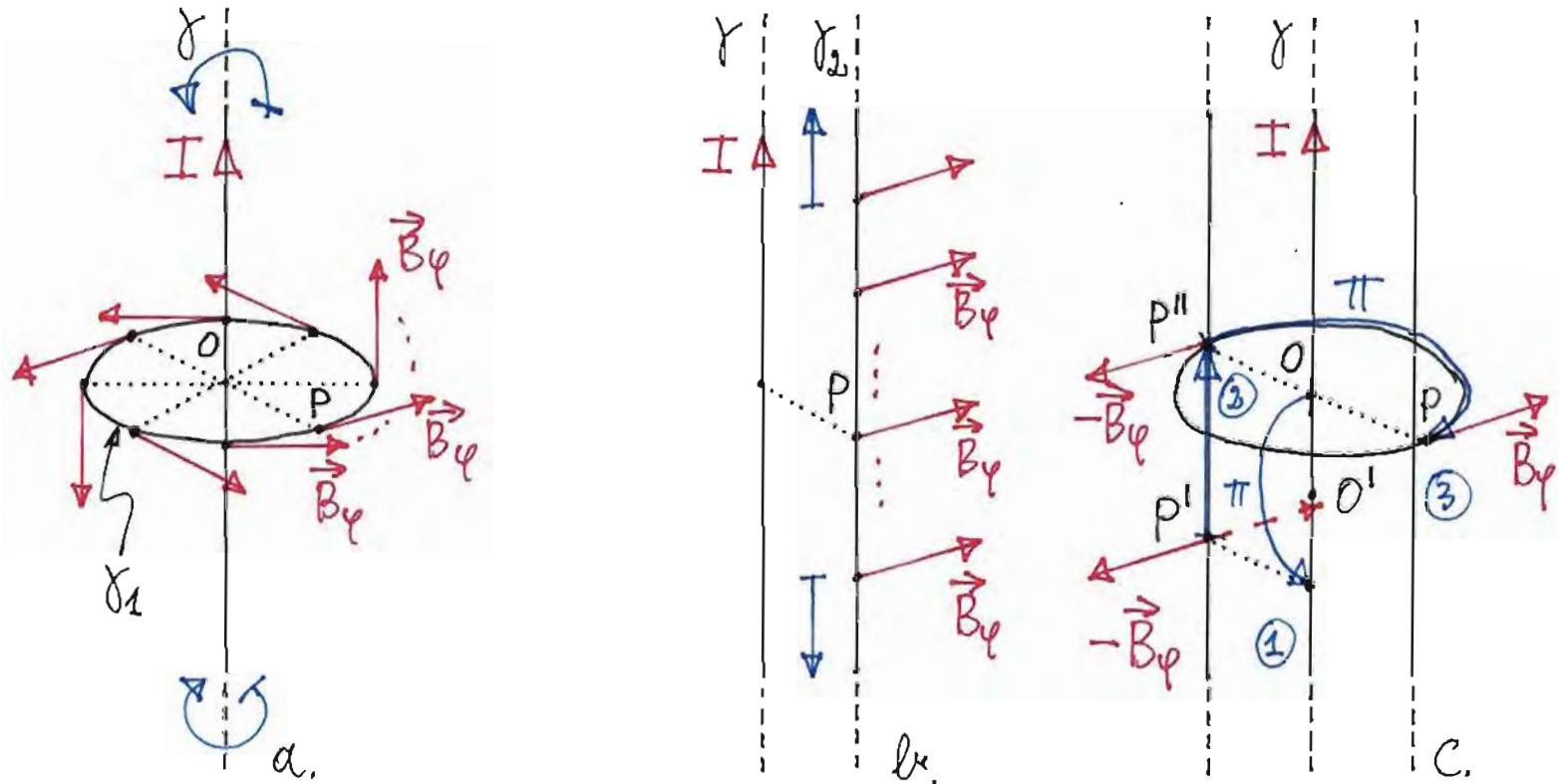


Figure 19.4.

Due to the translation symmetry, \vec{B}_z must be the same at each point on γ_2 , as in Fig. 19.8 b.

The anti-reflection symmetry allows us to perform the usual consistency check. Consider \vec{B}_z at P , as in Fig. 19.8 c, $\{\vec{B}_z, \vec{P}\}$. Rotate γ as well as $\{\vec{B}_z, \vec{P}\}$ as a rigid body by an angle π counterclockwise (or clockwise) about the pivot O' . At P' , we have $\{-\vec{B}_z, \vec{P}'\}$ due to the anti-reflection property. Translate $-\vec{B}_z$ upward from P' to P'' , so to obtain $\{-\vec{B}_z, \vec{P}''\}$. At last, rotate $-\vec{B}_z$ at

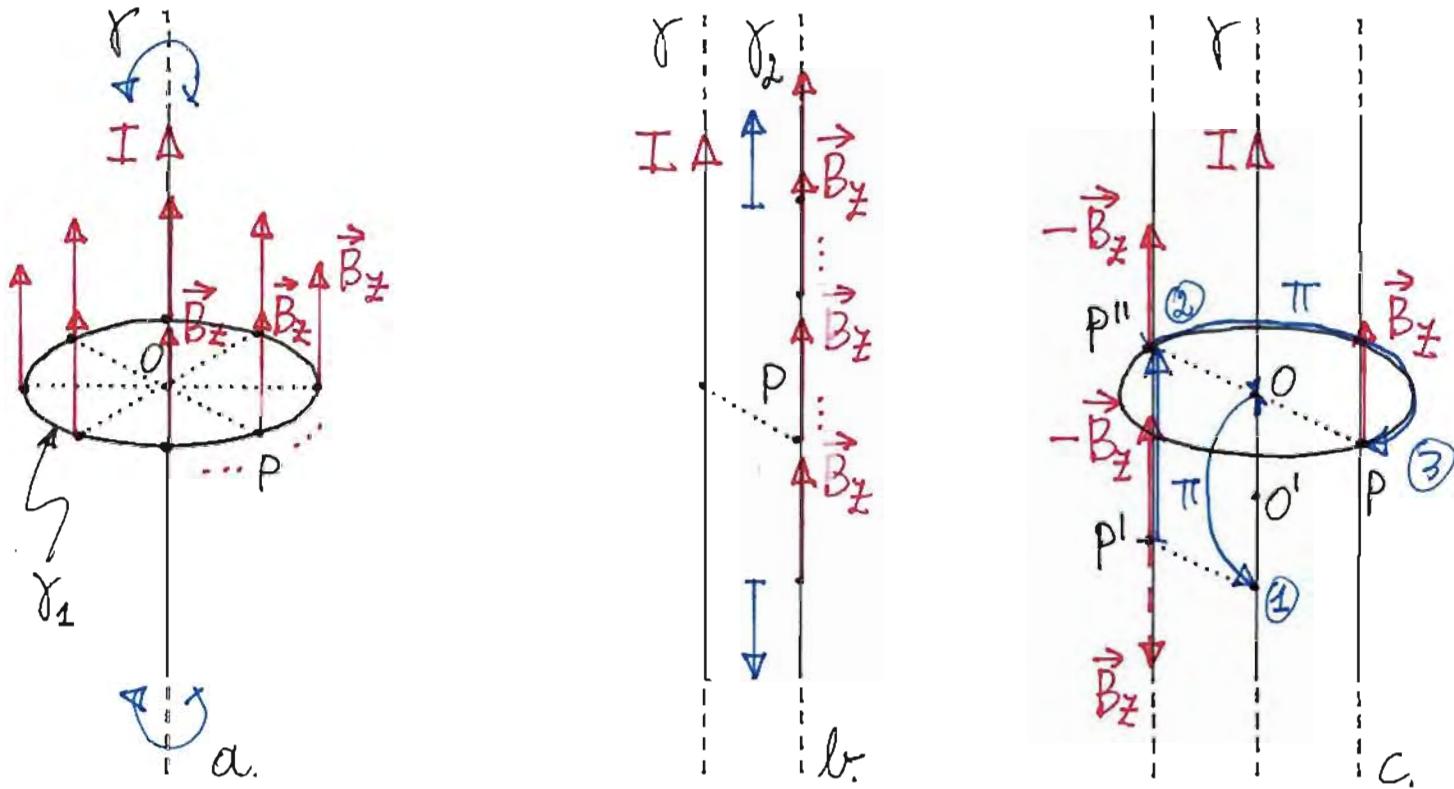


Figure 19.8.

P'' by an angle π counterclockwise (or clockwise) along Y_1 back to P . The pair $\{\vec{B}_z, P\}$ is consistent with the initial pair $\{\vec{B}_z, P\}$. Hence, a component \vec{B}_z could exist.

We can now resort to the fundamental laws of magnetostatic, Eqs. (19.2) and (19.4), to find the magnitude of components \vec{B}_y and \vec{B}_z .

Figure 19.9 shows a closed oriented line Y_1 as well

as a cylindric surface Σ_1 coaxial with γ , completely closed by the bases S_1 and S_2 and the lateral surface S , and with height h . The radius of γ_1 and of a cross-section of Σ_1 normal to γ (e.g., γ_1) is r . The closed line γ_1 and surface Σ_1 will be used to calculate the circulation and flux of \vec{B} , respectively.

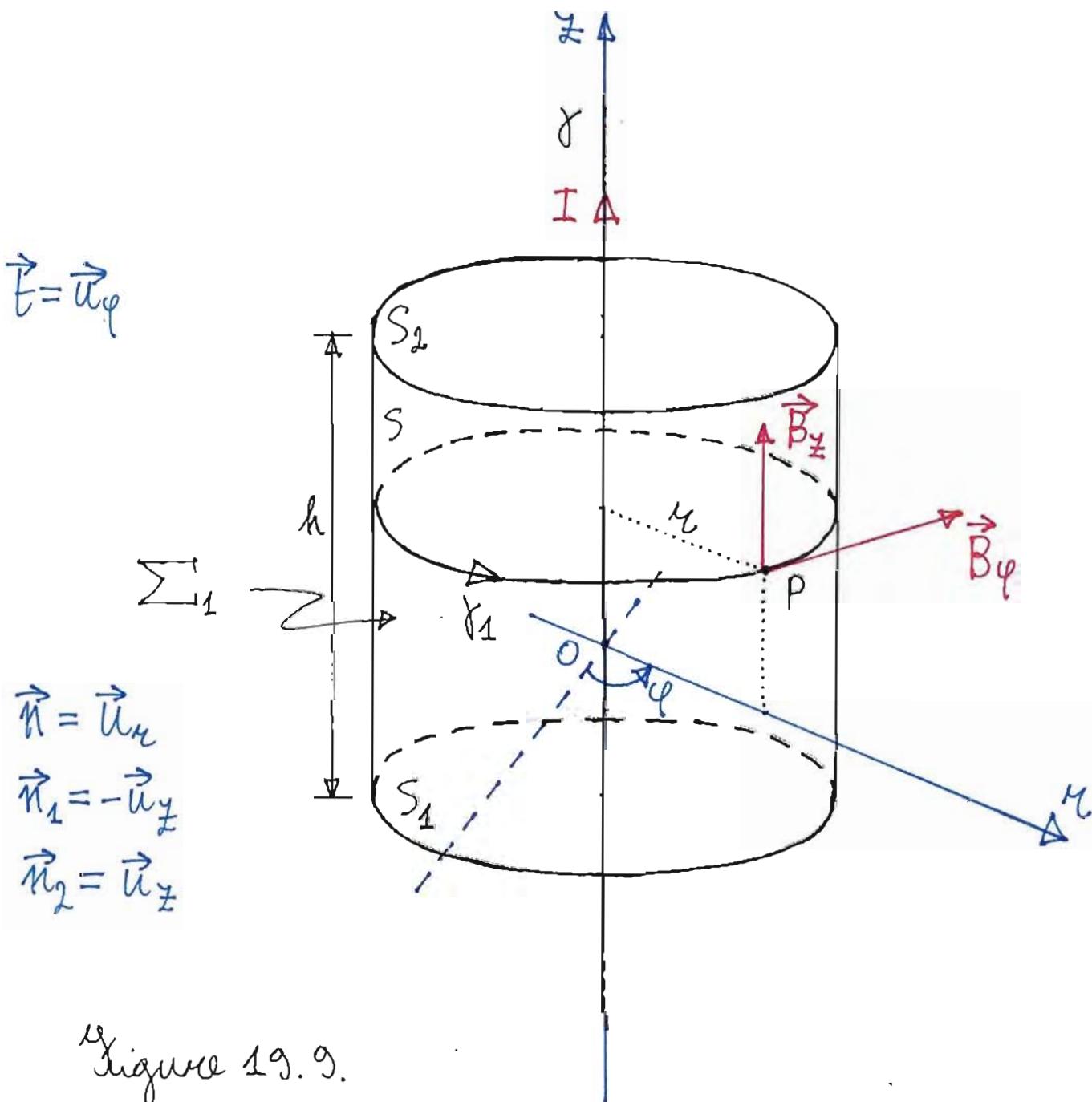


Figure 19.9.

A cylindric coordinate system $O\tau\varphi z$ as well as the [28/56] components \vec{B}_φ and \vec{B}_z are also shown in Fig. 19.9.

From (19.4), and considering that $\vec{E} = \vec{u}_\varphi$, we find

$$\begin{aligned} \oint_{Y_1} \vec{B} \cdot \vec{E} \cdot d\ell &= \oint_{Y_1} B_\varphi \vec{u}_\varphi \cdot \vec{u}_\varphi d\varphi \cdot r \\ &= B_\varphi r \int_0^{2\pi} d\varphi \\ &= 2\pi r B_\varphi = \mu_0 I \quad . \quad (19.10) \end{aligned}$$

Note that the orientation of Y_1 (counterclockwise) has been chosen consistently with the direction (upward) of the linked current I , hence the positive sign in (19.10). From (19.10), it then follows that

$$B_\varphi = \mu_0 \frac{I}{2\pi r} \quad (19.11)$$

and, finally,

$$\vec{B}_\varphi = \mu_0 \frac{I}{2\pi r} \cdot \vec{u}_\varphi \quad . \quad (19.12)$$

Note that, while B_φ diverges in close proximity of the conductor,

$$\lim_{r \rightarrow +\infty} B_\varphi = \lim_{r \rightarrow +\infty} \mu_0 \frac{I}{2\pi r} = 0 \quad . \quad (19.13)$$

What about the component \vec{B}_z ? It is obvious that the circulation of \vec{B}_z along γ_1 is zero, because \vec{B}_z is normal to γ_1 at each point on γ_1 . Choosing a more complicated line, as the one shown in Fig. 19.10, also does not give an further quantitative insight into B_z . The new line, γ_2 , which is also closed and linked with I , comprises two semi-circles, one from point A to B and another from C to D. The line also comprises two straight segments, BC and DA. The line is oriented counterclockwise.

From (19.4), we have

$$\oint_{\gamma_2} \vec{B} \cdot \vec{t} \cdot d\ell = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA} = \mu_0 I \quad . \quad (19.14)$$

The first integral in (19.14) reads

$$\begin{aligned}
 \int_{AB} \vec{B} \cdot \vec{E} \cdot d\ell &= \int_0^{\pi} \vec{B} \cdot \vec{u}_\varphi \cdot d\varphi \cdot r \\
 &= \int_0^{\pi} B_\varphi \cdot \vec{u}_\varphi \cdot \vec{u}_\varphi \cdot d\varphi \cdot r \\
 &= \pi r B_\varphi , \quad (19.15a)
 \end{aligned}$$

where r is the radius of the semicircle AB. The second integral reads

$$\int_{BC} \vec{B} \cdot \vec{E} \cdot d\ell = \int_{z_B}^{z_C} \vec{B} \cdot \vec{u}_z \cdot dz ,$$

where $\vec{E} \cdot d\ell = \vec{u}_z \cdot dz$ because we are integrating in the positive direction of the z axis, from a small value, $z_B = 0$, to a larger value, $z_C > 0$. Thus,

$$\int_{BC} \vec{B} \cdot \vec{E} \cdot d\ell = \int_0^{z_c} B_z \vec{u}_z \cdot \vec{u}_z \cdot dz$$

$$= B_z \int_0^{z_c} dz = B_z z_c . \quad (19.15b)$$

Similarly to the integral on AB, the integral on the semi-circle CD is then

$$\int_{CD} \vec{B} \cdot \vec{E} \cdot d\ell = \int_0^{\pi} \vec{B} \cdot \vec{u}_{\varphi} \cdot \vec{u}_{\varphi} \cdot d\varphi \cdot r$$

$$= \int_0^{\pi} B_{\varphi} \vec{u}_{\varphi} \cdot \vec{u}_{\varphi} \cdot d\varphi \cdot r$$

$$= \pi r B_{\varphi} . \quad (19.15c)$$

At last, the integral on DA reads

$$\int_{DA} \vec{B} \cdot \vec{t} \cdot dl = \int_{z_0}^{z_A} \vec{B} \cdot \vec{u}_z \cdot dz ,$$

where, as for BC, $\vec{t} \cdot dl = \vec{u}_z \cdot dz$ because we are integrating in the negative direction of z , but from a larger value, z_0 , to a smaller value, $z_A = 0$ ($z_0 > z_A$, as in Fig. 19.10).
So,

$$\begin{aligned} \int_{DA} \vec{B} \cdot \vec{t} \cdot dl &= \int_{z_0}^0 B_z \cdot \vec{u}_z \cdot \vec{u}_z \cdot dz \\ &= B_z \int_{z_0}^0 dz = -B_z z_0 . \quad (19.15d) \end{aligned}$$

Assuming $z_c = z_0$, (19.14) becomes

$$\begin{aligned} \oint_{\gamma_2} \vec{B} \cdot \vec{t} \cdot dl &= \pi r B_\varphi + \cancel{z_c B_z} + \pi r B_\varphi - \cancel{z_c B_z} \\ &= 2\pi r B_\varphi = \mu_0 I , \end{aligned}$$

from which follows once again (19.11). The circulation along γ_2 did not give any further information on B_z . [23/56]

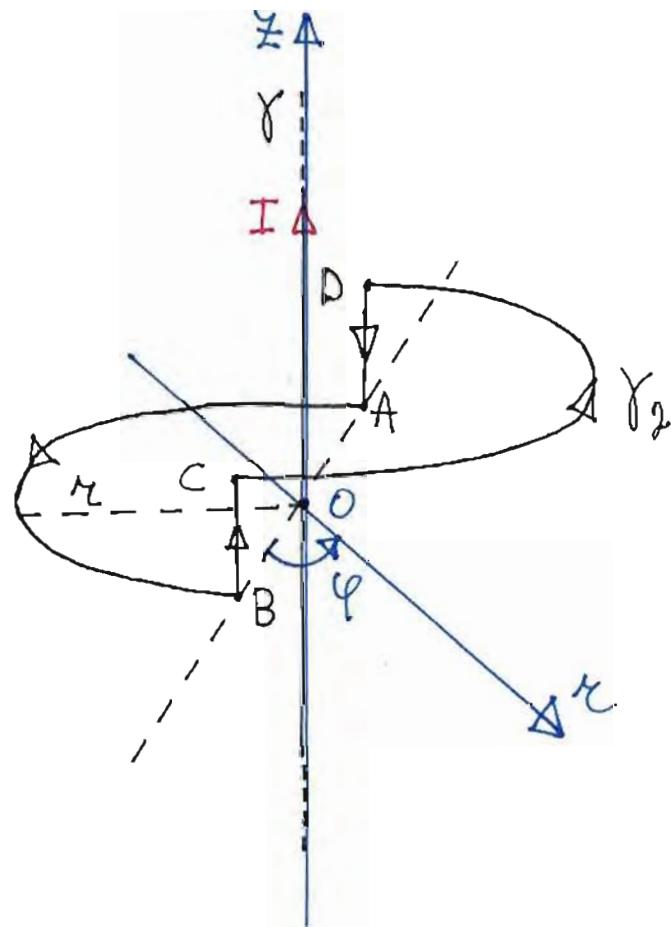


Figure 19.10.

One last possibility to find more information on B_z is by resorting to (19.2) for the closed surface Σ_1 in Fig. 19.9. Since \vec{B}_z is tangent to S , the only contribution to the flux can be through S_1 and S_2 . For $\vec{n}_2 = \vec{u}_z = -\vec{n}_1$, we find

$$\begin{aligned}
 \sum_{i=1}^{\infty} \oint \vec{B} \cdot \vec{n} \cdot dS &= \iint_{S_1} \vec{B} \cdot \vec{n}_1 \cdot dS + \iint_{S_2} \vec{B} \cdot \vec{n}_2 \cdot dS \\
 &= -B_Z \iint_{S_1} \vec{u}_Z \cdot \vec{u}_Z \cdot dS + B_Z \iint_{S_2} \vec{u}_Z \cdot \vec{u}_Z \cdot dS \\
 &= -\pi R^2 B_Z + \pi R^2 B_Z = 0. \quad (19.16)
 \end{aligned}$$

The component B_Z fulfills (19.2) and, thus, it could in principle exist. However, we were unable to find any closed line or surface for which (19.4) or (19.2) could give a quantitative knowledge of B_Z . For a complete solution of the problem, we must thus resort to a set of canonical curves, similar to those used in the electrostatic case of an infinite straight line with a uniform charge distribution. We will now follow a procedure similar to that in lecture 4, using analogous canonical curves.

- Component \vec{B}_r ; plane normal to γ .

Figure 19.11 shows the first canonical curve, γ_3 . The curve comprises two straight line segments AB and CD, and two arcs BC and DA. The line is oriented counter-clockwise. We assume $\overline{AB} = \overline{CD} = dr$. Thus, in zero-order approximation, the field \vec{B} at each point on BC is equal to \vec{B} at each point on DA at a distance dr from the point on BC. Such pair of points are in a bijective correspondence and are called P_{BC}^r and P_{DA}^r , respectively. The only components of \vec{B} contributing to the circulation along γ_3 are B_r on AB and CD and B_ϕ on BC and DA. In particular, B_r can be assumed to be constant on AB and CD; we call the corresponding constant magnitudes B_{r1} and B_{r2} , respectively.

From (19.4),

$$\oint_{\gamma_3} \vec{B} \cdot \vec{t} \cdot dl = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA} = 0 \quad (19.14)$$

since γ_3 does not link any current. The integral on AB is then

$$\int_{AB} \vec{B} \cdot \vec{E} \cdot dl = \int_{r}^{r+dr} B_{r1} \cdot \vec{u}_r \cdot \vec{u}_r \cdot dr$$

$$= B_{r1} (r - r - dr) = -B_{r1} dr, \quad (19.18a)$$

where r is the radius of the circle with center O (where γ is the intersection between γ and the plane where γ_3 lies; cf. Fig. 19.11) that contains the arc BC and $(r+dr)$ the radius of the circle with center O that contains the arc AD . The unit vector \vec{u}_r is according to a cylindrical coordinate system $Oxyz$, as indicated in Fig. 19.11.

The integral on BC is given by

$$\int_{BC} \vec{B} \cdot \vec{E} \cdot dl = \int_{BC} B_\varphi(P_{BC}^r) \cdot \vec{u}_\varphi \cdot \vec{E} \cdot dl \Big|_{BC}, \quad (19.18b)$$

where $dl|_{BC}$ is an infinitesimal element on BC . Following the notation in Fig. 19.11, the integral on CD is

$$\int_{CD} \vec{B} \cdot \vec{t} \cdot dl = \int_{r}^{r+dr} -B_{y2} \cdot \vec{u}_x \cdot \vec{u}_x \cdot dr$$

$$= -B_{y2} (r+dr - r) = -B_{y2} dr, \quad (19.18c)$$

where the negative sign is due to the direction (arbitrarily) chosen for \vec{B}_{y2} in the figure. At last, the integral on DA is given by

$$\int_{DA} \vec{B} \cdot \vec{t} \cdot dl = \int_{DA} B_y(P_{DA}^r) \cdot \vec{u}_y \cdot \vec{t} \cdot dl|_{DA}$$

$$= - \int_{BC} B_y(P_{BC}^r) \cdot \vec{u}_y \cdot \vec{t} \cdot dl|_{BC}, \quad (19.18d)$$

where we used the fact that $B_y(P_{DA}^r) = B_y(P_{BC}^r)$ and that $dl|_{DA} = -dl|_{BC}$.

Summing together integrals (19.18a)–(19.18d), we can rewrite (19.14) as

$$\begin{aligned}
 \oint_{Y_3} \vec{B} \cdot \vec{t} \cdot d\ell &= -B_{r1} \cdot dr + \int_{BC} B_\varphi(P_{BC}^r) \vec{u}_\varphi \cdot \vec{t} \cdot d\ell|_{BC} \\
 &\quad - B_{r2} \cdot dr - \int_{BC} B_\varphi(P_{BC}^r) \vec{u}_\varphi \cdot \vec{t} \cdot d\ell|_{BC} \\
 &= -B_{r1} \cdot dr - B_{r2} \cdot dr = 0 \quad , \quad (19.19)
 \end{aligned}$$

from which we find the condition

$$B_{r2} = -B_{r1} \quad . \quad (19.20)$$

According to (19.20), if a component \vec{B}_r exists, it must be the same on each circle with center O and generic radius r . Note that, the canonical curve Y_3 gives exact solutions in zero-order approximation because associated with the rotation symmetry of the problem (Cf. HA 2.2 and M1.2).

- Component \vec{B}_r ; semiplane generating from Y .

Figure 19.12 shows the second canonical curve, Y_4 . The curve comprises four straight line segments, AB , BC ,

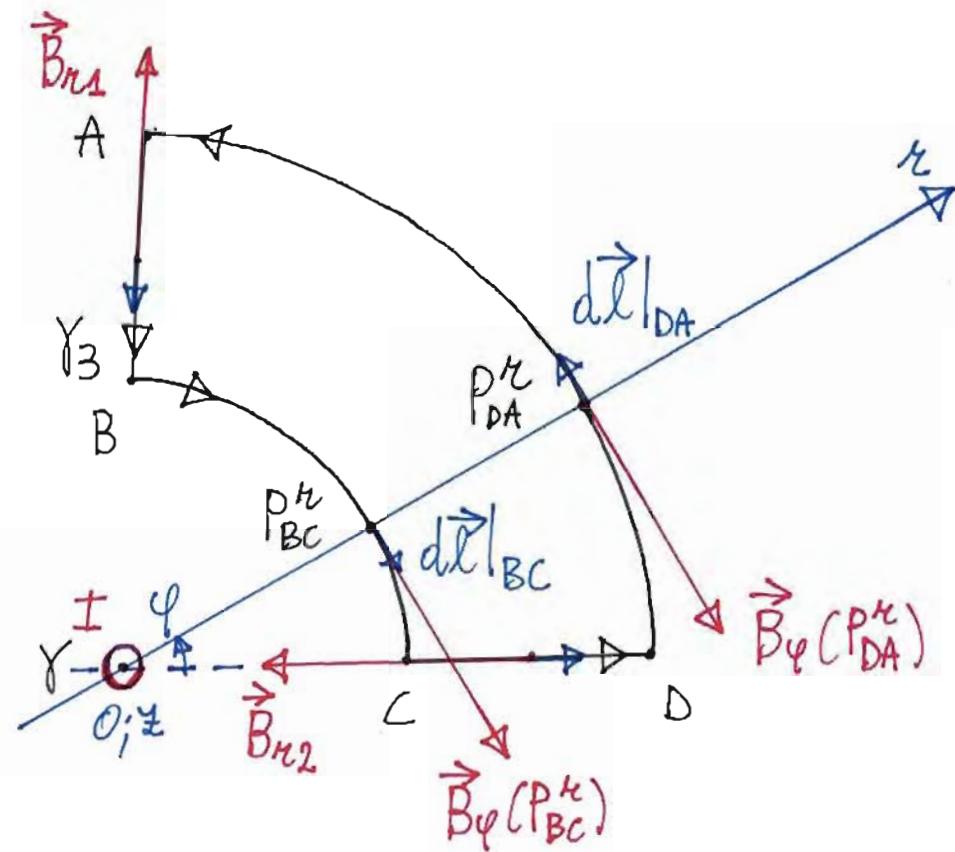


Figure 19.11.

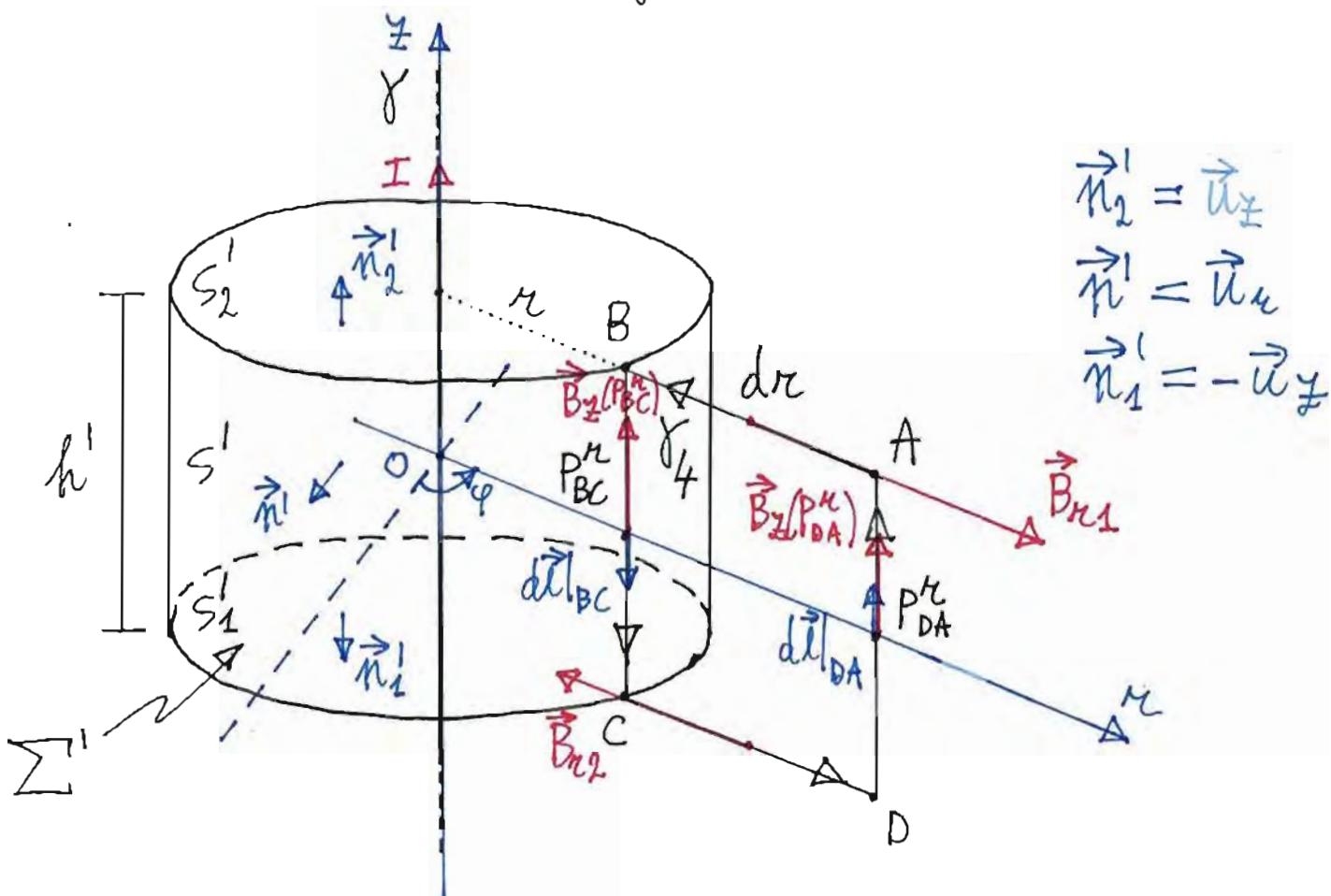


Figure 19.12.

$$\begin{aligned}\vec{n}_2^1 &= \vec{u}_\gamma \\ \vec{n}_1^1 &= \vec{u}_\varphi \\ \vec{n}_1^1 &= -\vec{u}_\gamma\end{aligned}$$

CD , and DA (rectangular curve). The line is oriented counterclockwise. We assume $\overline{AB} = \overline{CD} = dr$. Thus, in zero-order approximation, pairs of points $\{P_{BC}^r, P_{DA}^r\}$ are in bijective correspondance and the field \vec{B} at P_{BC}^r is equal (in zero-order) approximation to \vec{B} at P_{DA}^r . The only components of \vec{B} contributing to the circulation along γ_4 are B_r on AB and CD and B_z on BC and DA . In particular, in zero-order approximation \vec{B}_r can be assumed to be constant and equal to \vec{B}_{r1} on AB and constant and equal to \vec{B}_{r2} on CD .

A cylindric coordinate system $O r \theta z$ is shown in Fig. 19.12. The distance between γ and point B is r .

From (19.4),

$$\oint_{\gamma_4} \vec{B} \cdot \vec{dl} = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA} = 0, \quad (19.21)$$

since γ_4 , again, does not link any current. The integral on AB is

$$\int_{AB} \vec{B} \cdot \vec{E} \cdot d\ell = \int_{r}^{r+dr} B_{x1} \vec{u}_x \cdot \vec{u}_x \cdot dr$$

$$= -B_{x1} dr \quad . \quad (19.22a)$$

The integral on BC is

$$\int_{BC} \vec{B} \cdot \vec{E} \cdot d\ell = \int_{BC} B_z(P_{BC}^r) \vec{u}_z \cdot \vec{E} \cdot d\ell \Big|_{BC} \quad . \quad (19.22b)$$

The integral on CD is

$$\int_{CD} \vec{B} \cdot \vec{E} \cdot d\ell = \int_r^{r+dr} -B_{zz} \vec{u}_z \cdot \vec{u}_z \cdot dr$$

$$= -B_{zz} dr \quad . \quad (19.22c)$$

Finally, the integral on DA is

$$\int_{DA} \vec{B} \cdot \vec{E} \cdot d\ell = \int_{DA} B_z(P_{DA}^r) \cdot \vec{u}_z \cdot \vec{E} \cdot d\ell|_{DA}$$

$$= - \int_{BC} B_z(P_{BC}^r) \cdot \vec{u}_z \cdot \vec{E} \cdot d\ell|_{BC}, \quad (19.22d)$$

where we used the fact that line segment DA can be assumed to be the same as line segment BC due to the bijective correspondence between points P_{BC}^r and P_{DA}^r . In addition, $d\ell|_{DA} = -d\ell|_{BC}$ (regardless of any specific coordinate system).

Integral (19.21) then reads

$$\int_{\gamma_4} \vec{B} \cdot \vec{E} \cdot d\ell = -B_{x1} dr + \int_{BC} B_z(P_{BC}^r) \cdot \vec{u}_z \cdot \vec{E} \cdot d\ell|_{BC}$$

$$-B_{x2} dr - \int_{BC} B_z(P_{BC}^r) \cdot \vec{u}_z \cdot \vec{E} \cdot d\ell|_{BC}$$

$$\frac{1}{r} = -B_{41} \cdot \hat{dr} - B_{42} \cdot \hat{dr} = 0, \quad (19.23)$$

from which we find the condition

$$B_{42} = -B_{41} \quad . \quad (19.24)$$

As a consequence, if \vec{B}_r exists, it must be the same on each line parallel to γ . Note that Figs. 4.4c and 4.8b show the generalization to any generic curve γ_3 and γ_4 , respectively (cf. lecture 4 for more details).

By combining Eqs. (19.20) and (19.24), and the generalizations in Figs. 4.4c and 4.8b, we must conclude that if \vec{B}_r exists, it must be the same on each cylinder coaxial with γ .

We can thus use Eq. (19.2) for the closed surface \sum' indicated in Fig. 19.12. The surface, of height h' , comprises two bases S'_1 and S'_2 and a lateral surface S' . Since \vec{B}_r is tangent to both S'_1 and S'_2 , the only contribution of \vec{B}_r to the flux is through S' , where \vec{B}_r is normal to each point on the surface. The component

\vec{B}_y is tangent to S_1^1 , S_2^1 , and S^1 , whereas \vec{B}_z is tangent to S^1 , but normal to both S_1^1 and S_2^1 . By defining $\vec{n}_1^1 = -\vec{u}_z$, $\vec{n}^1 = \vec{u}_x$, and $\vec{n}_2^1 = \vec{u}_z$, we have

$$\begin{aligned}
 \oint \vec{B} \cdot \vec{n} \cdot dS &= \sum \iint_{S_1^1} B_z \cdot \vec{u}_z \cdot (-\vec{u}_z) \cdot dS \\
 &\quad + \iint_{S^1} B_x \cdot \vec{u}_x \cdot \vec{u}_x \cdot dS \\
 &\quad + \iint_{S_2^1} B_z \cdot \vec{u}_z \cdot \vec{u}_z \cdot dS \\
 &= - \iint_{S_1^1} B_z \cdot dS + 2\pi \epsilon h^1 B_x + \iint_{S_2^1} B_z \cdot dS = 0,
 \end{aligned}$$

(19.25)

where, in general, \vec{B}_y can vary in space. In order to find

a more conclusive relation for B_x , we must thus study \vec{B}_z more closely.

- Component \vec{B}_z ; plane containing γ .

Figure 19.13 shows the third canonical curve, γ_5 . The curve comprises four straight line segments, AB, BC, CD, and DA. Segments BC and DA are split into sub-segments BB' and $C'C$ and DD' and $A'A$, respectively. We assume $\overline{AB} = \overline{CD} = d_z$ and $\overline{BB'} = \overline{A'A} \neq \overline{C'C} = \overline{DD'}$ (asymmetric rectangular curve). In zero-order approximation, pairs of points $\{P_{BC}^z, P_{DA}^z\}$ are in bijective correspondence and the field \vec{B} at P_{BC}^z is equal to \vec{B} at P_{DA}^z . The only components of \vec{B} contributing to the circulation along γ_5 are \vec{B}_z on AB and CD and \vec{B}_x on BC and DA. We can assume \vec{B}_z to be constant on AB and equal to \vec{B}_{z1} and constant and equal to \vec{B}_{z2} on CD.

A cylindric coordinate system $Oxyz$ is shown in Fig. 19.13. The various distances are $\overline{BB'} = \overline{A'A} = r_B$ and $\overline{C'C} = \overline{DD'} = r_C$.

From (19.4),

$$\oint_5 \vec{B} \cdot \vec{E} \cdot d\vec{l} = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA} = 0. \quad (19.26)$$

The integral on AB is

$$\begin{aligned} \int_{AB} \vec{B} \cdot \vec{E} \cdot d\vec{l} &= \int_{z_1}^{z_2} -B_{y1} \vec{u}_y \cdot \vec{u}_z dz \\ &= -B_{y1} dz \quad . \quad (19.27a) \end{aligned}$$

The integral on BC is

$$\int_{BC} \vec{B} \cdot \vec{E} \cdot d\vec{l} = \int_{BC} B_r(P_{BC}) \vec{u}_x \cdot \vec{E} \cdot d\vec{l} \Big|_{BC} \quad . \quad (19.27b)$$

The integral on CD is

$$\int_{CD} \vec{B} \cdot \vec{E} \cdot d\vec{l} = \int_{z_2}^{z_1} B_{yz} \vec{u}_x \cdot \vec{u}_y dy$$

$$= -B_{y2} \cdot dy . \quad (19.27c)$$

Finally, the integral on DA is

$$\begin{aligned} \int_{DA} \vec{B} \cdot \vec{E} \cdot dl &= \int_{DA} B_r(P_{DA}^y) \cdot \vec{u}_r \cdot \vec{E} \cdot dl |_{DA} \\ &= - \int_{BC} B_r(P_{BC}^y) \cdot \vec{u}_y \cdot \vec{E} \cdot dl |_{BC}, \quad (19.27d) \end{aligned}$$

where we used the fact that $\vec{B}_r(P_{BC}^y) = \vec{B}_r(P_{DA}^y)$ and that $dl|_{BC} = -dl|_{DA}$.

Summing (19.27a)-(19.27d), integral (19.26) can be rewritten as

$$\begin{aligned} \int_{\gamma_5} \vec{B} \cdot \vec{E} \cdot dl &= -B_{y1} \cdot dy + \int_{BC} B_r(P_{BC}^y) \cdot \vec{u}_r \cdot \vec{E} \cdot dl |_{BC} \\ &\quad - B_{y2} \cdot dy - \int_{BC} B_r(P_{BC}^y) \cdot \vec{u}_y \cdot \vec{E} \cdot dl |_{BC} \end{aligned}$$

$$\frac{1}{\epsilon} = -B_{Z1} \cdot \cancel{dx^2} - B_{Z2} \cdot \cancel{dx^2} = 0, \quad (19.28)$$

from which we find the condition

$$B_{Z2} = -B_{Z1}. \quad (19.29)$$

Note that curve γ_5 is the same as curve γ_7 in Fig. 4.11 of lecture 4. However, in the magnetostatic case we have chosen it not to cross the line γ in order to guarantee no current is ever linked with it. It is evident that γ_5 can be chosen in a total arbitrary fashion so long it never crosses γ . So, at most, the segments AB or CD can approach γ in the limit their distance from γ remains infinitesimally small. By means of this procedure we can cover almost the entire space, except for a very small neighborhood around γ (a cylindrical infinitesimal neighborhood). Under these assumptions, \vec{B}_Z , if it exists, must be the same at each point in space.

This result is sufficient to simplify (19.25), which can thus be rewritten as

$$\begin{aligned}
 \oint \vec{B} \cdot \vec{n} \cdot dS &= -B_Z \left\{ \int_{S_1'} dS + 2\pi r h' B_R + B_Z \right\} dS \\
 \sum' &\quad | \quad S_1' \quad | \quad S_2' \\
 &= -B_Z \cancel{\pi r^2} + 2\pi r h' B_R + B_Z \cancel{\pi r^2} \\
 &= 2\pi r h' B_R = 0 \quad . \quad (19.30)
 \end{aligned}$$

Assuming $r \neq 0$ (and $h' \neq 0$), (19.30) is fulfilled only if $B_R = 0$. Since \vec{B}_R must be the same at each point on any cylindrical surface coaxial with γ , $\vec{B}_R = \vec{0}$ at each point in space (and in the limit $r \rightarrow 0$ approaching γ).

We can now go back to \vec{B}_Z . Keeping γ_5 entirely on the left or right of γ and keeping either the portion $BB'A'A'$ fixed (γ_5 on the left) or $C'C'D'D'$ fixed (γ_5 on the right), by pushing either $r_c \rightarrow -\infty$ (left) or $r_B \rightarrow +\infty$ (right), we find that

$$\lim_{r_c \rightarrow -\infty} B_Z = \lim_{r_B \rightarrow +\infty} B_Z = 0 \quad , \quad (19.31)$$

because of the zero-field condition at infinite. Note that,

while γ is an infinite line, the fact that is a straight line guarantees that an infinite distance from it is maintained in the limits $r_c \rightarrow -\infty$ and $r_b \rightarrow +\infty$.

If γ was not straight, this would not generally be true (unless it was of finite length, e.g., a loop).

Also note that a current can flow on an infinite line because such a line can be thought as a closed loop with infinite bending radius. A macroscopic current cannot flow on a finite open line.

Conditions (19.29) and (19.31) imply that $\vec{B}_y = \vec{0}$ everywhere in space.

- Component \vec{B}_y ; plane normal to γ .

Figure 19.14 shows the fourth canonical curve, γ_6 . The curve comprises two straight line segments, AB and CD, and two arcs, BC and DA. Consider the cylindrical coordinate system $Oxyz$ shown in the figure. The arcs BC and DA belong to two concentric circles with center O (coinciding with the trace of the infinite line γ on the considered plane) and radii $r_B = r_1$ and $r_A = r_2 >$

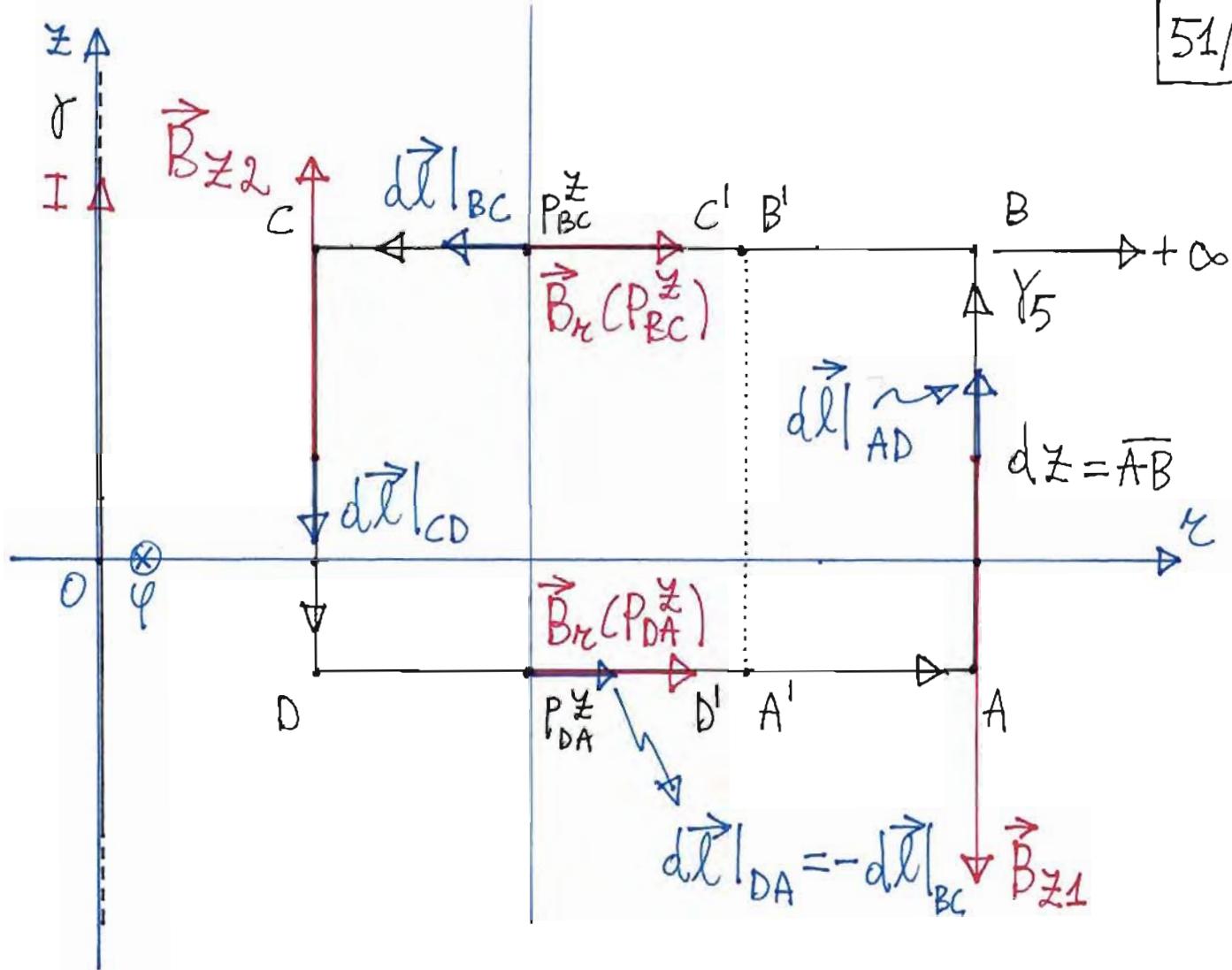


Figure 13.13.

κ_1 , respectively. The angular aperture of both arcs is assumed to be $d\ell$, so that the component \vec{B}_φ can be considered to be constant on BC and DA. Incidentally, \vec{B}_φ is the only component contributing to the line integral (13.4) along γ_6 , on the arcs BC and DA. Since $B_x = B_y = 0$ at each point in space, there are no other contributions to the aforementioned line integral. Thus, integral (13.4) for γ_6 reads (γ_6 oriented counterclockwise)

$$\oint_{\gamma_0} \vec{B} \cdot \vec{t} \cdot dl = \int_{BC} + \int_{DA} = 0 \quad (19.32)$$

(there is no current linked with γ_0). By defining the constant field on BC as $\vec{B}\varphi_1$ and on DA as $\vec{B}\varphi_2$, (19.32) becomes

$$\begin{aligned} \oint_{\gamma_0} \vec{B} \cdot \vec{t} \cdot dl &= \int_{\varphi}^{\varphi} -B\varphi_1 \vec{t}_\varphi \cdot \vec{t}_\varphi d\varphi \cdot r_1 \\ &\quad + \int_{\varphi}^{\varphi+d\varphi} -B\varphi_2 \vec{t}_\varphi \cdot \vec{t}_\varphi d\varphi \cdot r_2 \\ &= -B\varphi_1 (\cancel{\varphi} - \cancel{\varphi} - d\varphi) r_1 - B\varphi_2 (\cancel{\varphi} + d\varphi - \cancel{\varphi}) r_2 \\ &= B\varphi_1 r_1 d\varphi - B\varphi_2 r_2 d\varphi = 0, \end{aligned} \quad (19.33)$$

where the chosen directions for $\vec{B}\varphi_1$ and $\vec{B}\varphi_2$ are

indicated in Fig. 19.14. Hence, we find the condition

53/56

$$\frac{B_{\text{q}_1}}{B_{\text{q}_2}} = \frac{1/\mu_1}{1/\mu_2}, \quad (19.34)$$

according to which, if \vec{B}_{q} exists, it must be constant at each point on any circle with center O and generic radius r . It is worth noting that condition (19.34) is consistent with the previously obtained result for B_{q} , Eq. (19.11). This indicates we are on the right path.

We can now resort to a closed curve, γ_{q} , which concatenates the current I (cf. Fig. 19.14). The curve has been oriented clockwise, consistently with the chosen direction of \vec{B}_{q} . Thus, the normal unit vector \hat{n} to the circle enclosed by γ_{q} is opposite to the direction of I . This means that

$$\oint_{\gamma_7} \vec{B} \cdot \vec{t} \cdot d\ell = -\mu_0 I \quad . \quad (19.35)$$

Curve γ_7 has center O and radius r . Following γ_7 clockwise from the intersection point between γ_7 and the x axis, the angle φ goes from 2π to 0 (opposite direction than the chosen coordinate φ in the given cylindrical coordinate system). For r fixed and larger than zero, (19.35) becomes

$$\begin{aligned} \oint_{\gamma_7} \vec{B} \cdot \vec{t} \cdot d\ell &= -r B_\varphi \int_{2\pi}^0 \vec{u}_\varphi \cdot \vec{u}_\varphi \cdot d\varphi \\ &= -r B_\varphi \cdot [\varphi]_{2\pi}^0 \\ &= 2\pi r B_\varphi = -\mu_0 I \quad . \quad (19.36) \end{aligned}$$

From (19.36) we finally find

$$\vec{B}_\varphi = -\mu_0 \frac{\vec{I}}{2\pi r} \quad . \quad (19.37)$$

As expected from the right-hand rule, from (19.34) follows that \vec{B}_φ must be oriented the opposite of our choice in the figure.

In summary, choosing a counterclockwise direction for \vec{B}_φ , we have

$$\vec{B} = \mu_0 \frac{\vec{I}}{2\pi r} \cdot \hat{u}_\varphi \quad (19.38)$$

at each point in space. Note that the field diverges in proximity of r .

19.3.2 More examples will be shown in tutorial 10.

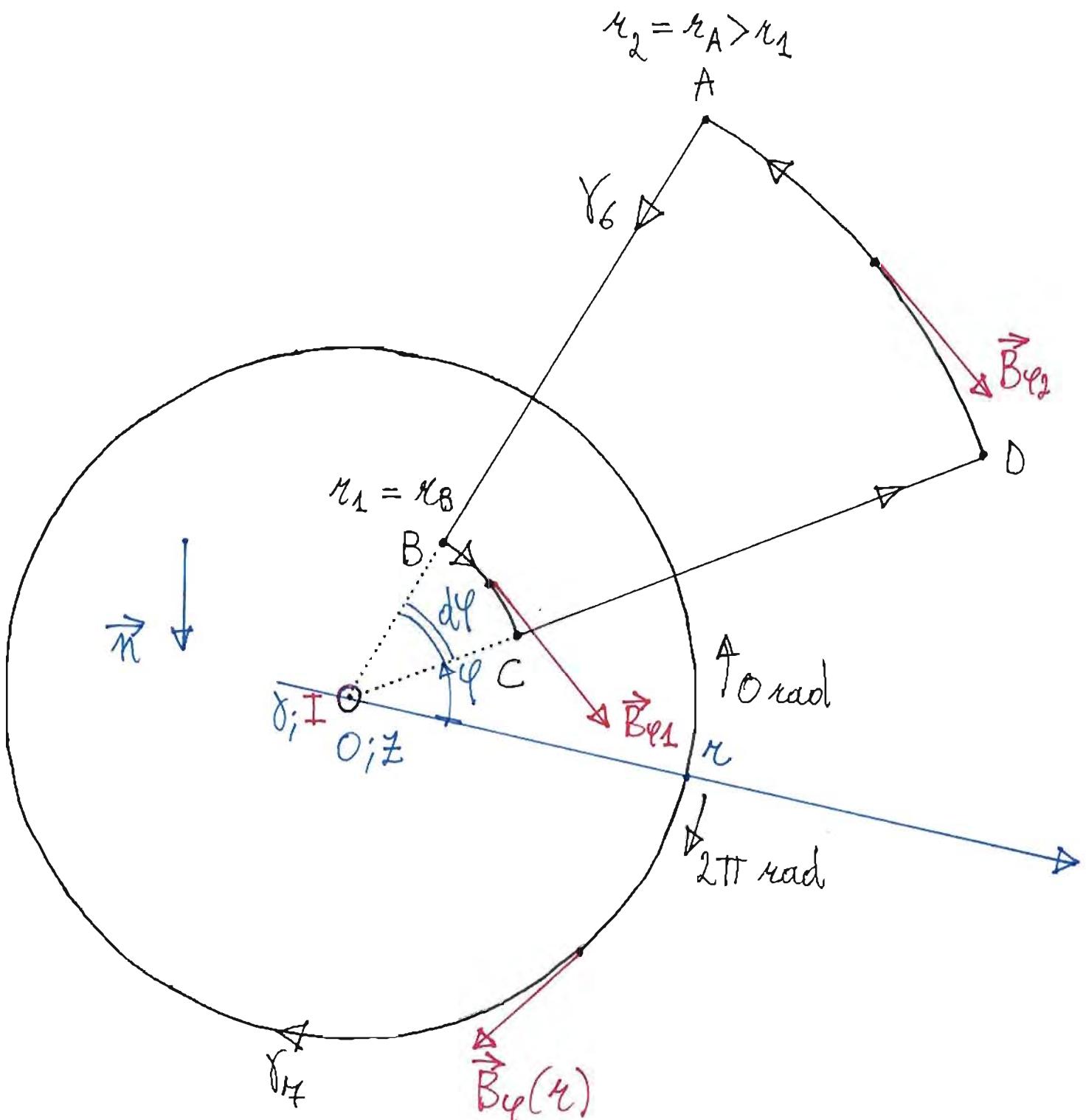


Figure 19.4.

Phys 242 - 914. Lecture 20 - The magnetostatic field in vacuum / part 2. | 1/35

Summary lecture 19.

- Ponderomotive forces.

Two circuits with stationary forces are attracted or repulsed to or from each other due to forces of magnetic origin.

- Such forces originate from

$$\vec{F} = q \vec{v} \times \vec{B}$$

- These forces are typically not strong enough to "remove" the charge free carriers from their conductor. Hence, they manifest themselves as macroscopic (ponderomotive) forces acting on the entire circuit.

- The fundamental laws of magnetostatics.

These laws are similar to Gauss' theorem and the irrotational property, but given as a principle found

from empirical evidence.

- First law.

Given a closed surface Σ in a field \vec{B} ,

$$\oint_{\Sigma} \vec{B} \cdot \vec{n} \cdot dS = 0 .$$

- Second law (Ampère's circuital law).

Given an oriented closed line γ linked with a current I (i.e., the current goes through the area enclosed by the line),

$$\oint_{\gamma} \vec{B} \cdot \vec{T} \cdot dl = \mu_0 I .$$

From these two laws it follows that \vec{B} is solenoidal and rotational.

Note that I is positively linked with γ when its direction coincides with that with the normal unit vector at the surface enclosed by γ (and viceversa).

20.1 The laws of magnetostatics in local form. [3/35]

As in the electrostatic case, we now present the laws of magnetostatics in local form.

We begin considering the case of currents in a stationary regime and distributed with a finite volume density \vec{J} .

- Case 1.

Consider a 3D region Ω in the euclidian space that is enclosed by a closed surface Σ . Assume $\vec{B} \in C^1(\Omega)$, i.e. the magnetostatic field is continuously differentiable in a vector sense in Ω , and therein limited.

• First law of magnetostatics.

The integral form, Eq. (19.2), can be rewritten by means of the divergence theorem as

$$\iiint_{\Omega} \vec{\nabla} \cdot \vec{B} \cdot d\vec{r} = 0 \quad . \quad (20.1)$$

Due to the arbitrariness of Σ and, thus, Ω , at each point where the field \vec{B} is defined, it must then be

$$\vec{\nabla} \cdot \vec{B} = 0 \quad . \quad (20.2)$$

This is the local (differential) form of Eq. (19.2). [4/35]
 Note that Eq. (20.2) is valid also in the special case
 $\vec{f} = \vec{0}$ and, in general, means that \vec{B} (under stationary
 conditions) is a solenoidal field.

- Second law of magnetostatics (Ampère's law).

In Ω , consider an open surface \sum_Y , the border of
 which is an oriented and closed line γ linking a current I .

The integral form, Eq. (19.2), can be rewritten by
 means of Stokes' theorem [we remind we are assuming
 $\vec{B} \in C^1(\Omega)$] as

$$\oint_{\gamma} \vec{B} \cdot \vec{T} \cdot d\ell = \iint_{\sum_Y} \vec{\nabla} \times \vec{B} \cdot \vec{n} \cdot dS , \quad (20.3)$$

where $\vec{T} \cdot d\ell$ is an infinitesimal element of γ (\vec{T} is
 directed consistently with the orientation of γ) and $\vec{n} \cdot dS$
 an infinitesimal element of \sum_Y (\vec{n} is the normal
 unit vector to \sum_Y). As always, $d\ell$ and dS have the
 dimensions of a length and surface, respectively.

Equation (20.3) is only the first term of (19.4). By
 noting that the current I linked with γ in the

second term of (19.4) can be written as the flux 5/35
 through \sum_Y of the current volume density \vec{J} , we have

$$\iint_{\sum_Y} \vec{\nabla} \times \vec{B} \cdot \vec{n} \cdot dS = \mu_0 I$$

$$= \mu_0 \iint_{\sum_Y} \vec{J} \cdot \vec{n} \cdot dS \quad (20.4)$$

Due to the arbitrariness of Y and, thus, \sum_Y , at each point where \vec{B} is defined, it must then be

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \quad (20.5)$$

This is the local (differential) form of Eq. (19.4), which is valid also for $\vec{J} = \vec{0}$ and that, in general, means \vec{B} is a rotational field. The currents associated with \vec{J} that fulfill (20.5) are called the vortices of \vec{B} .

- Case 2.

Consider now a surface \sum in the 3D euclidean space

characterized by a stationary current with a surface density \vec{J}_S . We are in presence of a surface current discontinuity in space. In analogy to Case 2 in lecture 5, we conjecture the impossibility to well-define both divergence and curl in the neighborhood of Σ .

We will thus resort to the integral forms (19.2) and (19.4) for a coin-type surface and cut-type line, respectively. Such curves were already defined in lecture 5, pages 24-35.

• First law of magnetostatics.

Figure 20.1 shows the coin-type surface Σ_c to be used in (19.2). The cross-section between Σ_c and Σ is considered to be an infinitesimal surface dS . Similarly, the two bases of Σ_c , S_1 and S_2 , are assumed to be infinitesimal surfaces dS_1 and dS_2 , respectively. The normal unit vectors to dS_1 , dS , and dS_2 are \vec{n}_1 , \vec{n} , and \vec{n}_2 , respectively. The height of Σ_c is h and its lateral surface S_L , with normal unit vector \vec{n}_L . The center of Σ_c is point P on surface Σ . (P is also the center of dS). Assuming the fields \vec{B}_1 and \vec{B}_2 to be constant at each point on S_1 and S_2 , and assuming a generic field

\vec{B}_e on S_e (cf. Fig. 20.1), Eq. (19.2) can be written [7/35] as

$$\sum_c \oint \vec{B} \cdot \vec{n} \cdot dS = \iint_{S_1} \vec{B}_1 \cdot \vec{n}_1 \cdot dS + \iint_{S_2} \vec{B}_2 \cdot \vec{n}_2 \cdot dS$$

$$+ \iint_{S_e} \vec{B}_e \cdot \vec{n}_e \cdot dS = 0 . \quad (20.6)$$

For a small h , the diameter dr of \sum_c can be assumed to be constant and, thus,

$$\iint_{S_e} \vec{B}_e \cdot \vec{n}_e \cdot dS = \vec{B}_e \cdot \vec{n}_e \cdot 2\pi dr \cdot h . \quad (20.7)$$

Hence,

$$\lim_{h \rightarrow 0} \vec{B}_e \cdot \vec{n}_e \cdot 2\pi dr \cdot h = 0 \quad (20.8)$$

and

$$\lim_{h \rightarrow 0} \vec{n}_1 = \vec{n} = -\lim_{h \rightarrow 0} \vec{n}_2 . \quad (20.9)$$

Thus, (20.6) becomes

$$\begin{aligned}
 \sum_c \oint \vec{B} \cdot \vec{n} \cdot dS &= \iint_{S_1} \vec{B}_1 \cdot \vec{n} \cdot dS - \iint_{S_2} \vec{B}_2 \cdot \vec{n} \cdot dS \\
 &= \vec{B}_1 \cdot \vec{n} \cdot dS_1 - \vec{B}_2 \cdot \vec{n} \cdot dS_2 \\
 &= \vec{B}_1 \cdot \vec{n} \cdot dS - \vec{B}_2 \cdot \vec{n} \cdot dS = 0, \quad (20.10)
 \end{aligned}$$

where we assumed $dS_1 = dS = dS_2$ in the limit for $h \rightarrow 0$. From (20.10) we obtain the condition

$$\vec{n} \cdot (\vec{B}_1 - \vec{B}_2) = 0, \quad (20.11)$$

which is the first law of magnetostatics in local form and states that the normal components of \vec{B} just above and below Σ in the vicinity of P are continuous,

$$B_{n1}(P) - B_{n2}(P) = 0. \quad (20.11')$$

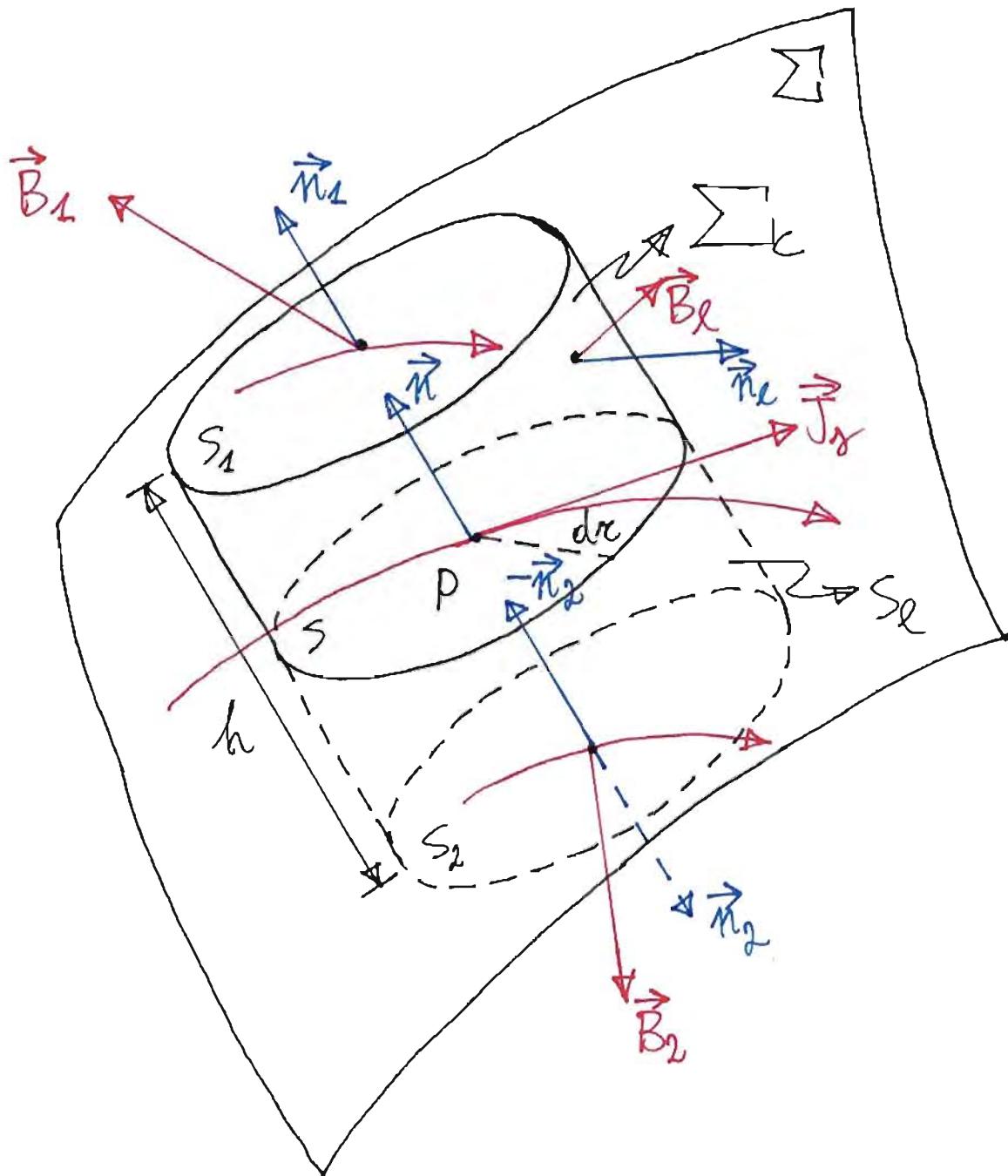


Figure 20.1.

• Second law of magnetostatics (Ampère's law).

Figures 20.2a and 20.2b show the current surface density vector \vec{J}_s at a generic point P on Σ . As indicated in the figures, in general, \vec{J}_s is characterized by two components, one along the tangent unit vector \vec{E} , \vec{J}_s^t , and another along the tangent unit vector \vec{E}'' , $\vec{J}_s^{\prime\prime}$.

As a consequence, in this case we must consider two different cut-type curves, γ_c^t and $\gamma_c^{\prime\prime}$, along which compute the circulation (19.4).

Figure 20.2a shows the first of two such curves, γ_c^t . This is a rectangular curve consisting of four straight line segments AB, BC, CD, and DA. The curve is oriented clockwise with respect to the chosen direction for \vec{J}_s^t . The center of the rectangular surface enclosed by γ_c^t is at a generic point P on Σ . The component \vec{J}_s^t is normal to such a surface at P, $\vec{J}_s^t = J_s^t \cdot \vec{E}^t = J_s^t(P) \cdot \vec{E}^t$. We assume $\overline{AB} = dl_1^t$, $\overline{CD} = dl_2^t$, and $\overline{BC} = \overline{DA} = h$. Because of the infinitesimal dimensions of lines AB and CD, the fields \vec{B}_1 and \vec{B}_2 can be assumed to be constant on AB and CD respectively. No assumptions are made on \vec{B}_{BC} and \vec{B}_{DA} , which can take any value on lines BC and DA, respectively.

The unit vectors tangent to each edge of γ_c^1 are $\vec{E}_1^{||}$, $\vec{E}_{BC}^{||}$, $\vec{E}_2^{||}$, and $\vec{E}_{DA}^{||}$, respectively. Finally, the normal unit vector to \sum at P is \vec{n} . [11/35]

Ampère's law along γ_c^1 reads

$$\oint_{\gamma_c^1} \vec{B} \cdot \vec{E} \cdot d\ell = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA} = \mu_0 I_s^1. \quad (20.12)$$

From the definition of current intensity for a surface current distribution \vec{J}_s , we have

$$\begin{aligned} I_s^1 &= \int_{\gamma_s^1} \vec{J}_s \cdot \vec{E}^1 \cdot d\ell = \int_{\gamma_s^1} (\vec{J}_s^1 + \vec{J}_s^{||}) \cdot \vec{E}^1 \cdot d\ell \\ &= \int_{\gamma_s^1} (J_s^1 \vec{E}^1 + J_s^{||} \vec{E}^{||}) \cdot \vec{E}^1 \cdot d\ell \\ &= J_s^1 \cdot d\ell^1, \quad (20.13) \end{aligned}$$

where γ_s^1 is the cross-section between the rectangular surface enclosed by γ_c^1 and \sum , and $d\ell^1$ its length.

The integral along AB is given by

$$\int_{AB} \vec{B} \cdot \vec{E} \cdot d\ell = \int_{AB} \vec{B}_1 \cdot \vec{E}_1^{\parallel} \cdot d\ell = \vec{B}_1 \cdot \vec{E}_1^{\parallel} \cdot d\ell_1' . \quad (20.14a)$$

The integral along BC is given by

$$\int_{BC} \vec{B} \cdot \vec{E} \cdot d\ell = \int_{BC} \vec{B}_{BC} \cdot \vec{E}_{BC} \cdot d\ell . \quad (20.14b)$$

The integral along CD is given by

$$\int_{CD} \vec{B} \cdot \vec{E} \cdot d\ell = \int_{CD} \vec{B}_2 \cdot \vec{E}_2^{\parallel} \cdot d\ell = \vec{B}_2 \cdot \vec{E}_2^{\parallel} \cdot d\ell_2' . \quad (20.14c)$$

Finally, the integral along DA is given by

$$\int_{DA} \vec{B} \cdot \vec{E} \cdot d\ell = \int_{DA} \vec{B}_{DA} \cdot \vec{E}_{DA} \cdot d\ell . \quad (20.14d)$$

In order to study \vec{B} on the positive page (i.e., just above) and on the negative page (i.e., just below) of Σ at P, we must consider the limit for $h \rightarrow 0$. In this

case,

$$\lim_{h \rightarrow 0} d\ell'_1 = d\ell' = \lim_{h \rightarrow 0} d\ell'_2 ; \quad (20.15)$$

$$\lim_{h \rightarrow 0} BC = 0 = \lim_{h \rightarrow 0} DA ; \quad (20.16)$$

$$\lim_{h \rightarrow 0} \vec{t}'^{\parallel}_1 = \vec{t}'^{\parallel} = - \lim_{h \rightarrow 0} \vec{t}'^{\parallel}_2 . \quad (20.17)$$

Under these conditions, using (20.15) and (20.14) in (20.14a) gives

$$\lim_{h \rightarrow 0} \int_{AB} = \vec{B}_1 \cdot \vec{t}'^{\parallel} d\ell' = B_t^{\parallel} t_1 \cdot d\ell' . \quad (20.14a')$$

Using (20.16) in (20.14b^a) gives

$$\lim_{h \rightarrow 0} \int_{BC} = 0 \quad (20.14b^{'})$$

because the length of integration is zero. Using (20.15) and (20.14) in (20.14c) gives

$$\lim_{h \rightarrow 0} \int_{CD} = - \vec{B}_2 \cdot \vec{T}^{\parallel} \cdot d\ell' = - B_t^{\parallel} d\ell' . \quad (20.14c')$$

Finally, using (20.16) in (20.14d) gives

$$\lim_{h \rightarrow 0} \int_{DA} = 0 , \quad (20.14d')$$

as for (20.14b'). In the limit for $h \rightarrow 0$, (20.12) reads

$$\begin{aligned} \lim_{h \rightarrow 0} \oint_{\gamma'_c} \vec{B} \cdot \vec{T} \cdot d\ell &= B_t^{\parallel} d\ell' - B_t^{\parallel} d\ell' \\ &\quad | \\ &= \mu_0 J_s^{\parallel} d\ell' , \quad (20.18) \end{aligned}$$

where we used (20.13). From (20.18) we find

$$B_t^{\parallel} - B_t^{\parallel} = [B_t^{\parallel}] = \mu_0 J_s^{\parallel} , \quad (20.19)$$

which shows a discontinuity of the first kind for the tangent component of the field along \vec{T}^{\parallel} at each point in close proximity of \sum .

Figure 20.2 b shows the second cut-type curve, γ_c'' . This is exactly the same curve as γ_c' , but rotated by an angle $\pi/2$ about \vec{n} so that the rectangular surface enclosed by γ_c'' is now normal to the direction of \vec{E}'' . For consistency with the signs in (20.17), however, γ_c'' must be oriented counterclockwise with respect to the chosen direction of \vec{J}_S'' (cf. Fig. 20.2 b). For clarity of notation the four vertices of γ_c'' are called E, F, G, and H. The center of the rectangular surface enclosed by γ_c'' is still at P, and the component \vec{J}_S'' is normal to such a surface at P, as indicated in the figure, $\vec{J}_S'' = J_S''(P) \cdot \vec{E}''$. However, due to the orientation of the line, the normal unit vector to the rectangular surface is now $-\vec{E}''$. We further assume $\overline{EF} = dl_1''$, $\overline{GH} = dl_2''$, and $\overline{FG} = \overline{HE} = h$. Moreover, we assume \vec{B}_1 and \vec{B}_2 to be constant on EF and GH, respectively, and \vec{B}_{FG} and \vec{B}_{HE} to be totally generic fields on FG and HE. Finally, the unit vectors tangent to each edge of γ_c'' are \vec{t}_1^I , \vec{t}_{FG} , \vec{t}_2^I , and \vec{t}_{HE} , respectively.

Ampère's law along γ_c'' reads

$$\oint_C \vec{B} \cdot \vec{E} \cdot d\vec{l} = \int_{EF} + \int_{FG} + \int_{GH} + \int_{HE} = -\mu_0 I_s'' , \quad (20.20)$$

where the minus sign is due to the fact that \vec{J}_s'' has opposite direction of $-\vec{E}''$ at P. From the definition of current intensity we have

$$I_s'' = \int_{Y_s''} \vec{J}_s \cdot \vec{E}'' \cdot d\vec{l} = J_s'' \cdot d\vec{l}'' , \quad (20.21)$$

where Y_s'' is the cross-section between the rectangular surface enclosed by γ_c'' and Σ , and $d\vec{l}''$ its length.

The integral along EF is given by

$$\int_{EF} \vec{B} \cdot \vec{E} \cdot d\vec{l} = \int_{EF} \vec{B}_1 \cdot \vec{E}_1 \cdot d\vec{l} = \vec{B}_1 \cdot \vec{E}_1 \cdot d\vec{l}_1'' . \quad (20.22a)$$

The integral along FG is given by

$$\int_{FG} \vec{B} \cdot \vec{E} \cdot d\ell = \int_{FG} \vec{B}_{FG} \cdot \vec{E}_{FG} \cdot d\ell \quad . \quad (20.22b)$$

The integral along GH is given by

$$\int_{GH} \vec{B} \cdot \vec{E} \cdot d\ell = \int_{GH} \vec{B}_2 \cdot \vec{t}_2^I \cdot d\ell = \vec{B}_2 \cdot \vec{t}_2^I \cdot d\ell_2^{II} \quad . \quad (20.22c)$$

Finally, the integral along HE is given by

$$\int_{HE} \vec{B} \cdot \vec{E} \cdot d\ell = \int_{HE} \vec{B}_{HE} \cdot \vec{E}_{HE} \cdot d\ell \quad . \quad (20.22d)$$

In the limit for $h \rightarrow 0$,

$$\lim_{h \rightarrow 0} d\ell_1^{II} = d\ell^{II} = \lim_{h \rightarrow 0} d\ell_2^{II} \quad ; \quad (20.23)$$

$$\lim_{h \rightarrow 0} FG = 0 = \lim_{h \rightarrow 0} HE \quad ; \quad (20.24)$$

$$\lim_{h \rightarrow 0} \vec{t}_1^I = \vec{t}^I = - \lim_{h \rightarrow 0} \vec{t}_2^I \quad . \quad (20.25)$$

Under these conditions,

$$\lim_{h \rightarrow 0} \int_{EF} = B_{t1}^I \cdot dl'' ; \quad (20.22a')$$

$$\lim_{h \rightarrow 0} \int_{FG} = 0 ; \quad (20.22b')$$

$$\lim_{h \rightarrow 0} \int_{GH} = -B_{t2}^I \cdot dl'' ; \quad (20.22c')$$

$$\lim_{h \rightarrow 0} \int_{HE} = 0 . \quad (20.22d')$$

In the limit for $h \rightarrow 0$, (20.20) reads

$$\begin{aligned} \lim_{h \rightarrow 0} \oint_{\gamma_c''} \vec{B} \cdot \vec{dl} &= B_{t1}^I \cancel{dl''} - B_{t2}^I \cancel{dl''} \\ &= -\mu_0 J_s'' \cdot \cancel{dl''} , \quad (20.26) \end{aligned}$$

where we used (20.21). From (20.26) we find

$$B_{t1}^{\parallel} - B_{t2}^{\parallel} = -\mu_0 J_s^{\parallel} \quad , \quad (20.24)$$

which shows a discontinuity of the first kind for the tangent component of the field along \vec{t}^{\parallel} at each point in close proximity of Σ . Maintaining the same notation as in (20.19), (20.24) reads

$$B_{t1}^{\parallel} - B_{t2}^{\parallel} = [B_t^{\parallel}] = -\mu_0 J_s^{\parallel} \quad . \quad (20.28)$$

The discontinuities (20.19) and (20.28) clearly show that neither the divergence nor the curl are well defined in a standard sense at each point on Σ .

Noting that \vec{t}^{\parallel} , \vec{t}^{\perp} , and \vec{n} are the unit vectors of a cartesian coordinate system, (20.19) and (20.28) can be combined together as

$$\vec{n} \times [\vec{B}] = \mu_0 \vec{J}_s \quad , \quad (20.29)$$

where $\vec{n} = (0, 0, 1)$, $[\vec{B}] = [(B_t^{\parallel}, B_t^{\perp}, B_n)] = [(B_{t1}^{\parallel} - B_{t2}^{\parallel}, B_{t1}^{\perp} - B_{t2}^{\perp}, B_{n1} - B_{n2})]$, and $\vec{J}_s = (J_s^{\parallel}, J_s^{\perp}, 0)$.

In fact,

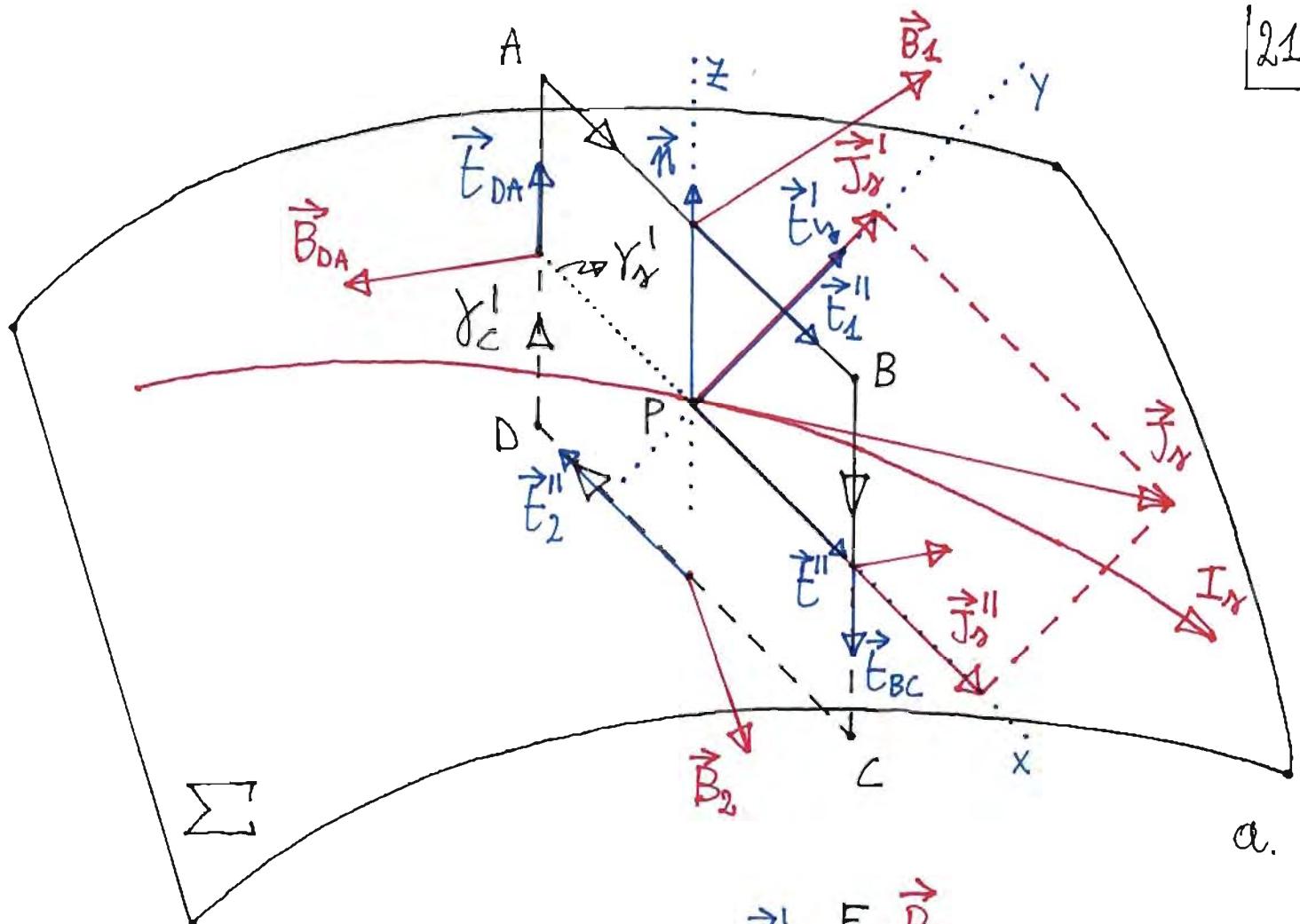
$$\vec{n} \times \vec{B} = \begin{vmatrix} \vec{E}'' & \vec{E}' & \vec{n} \\ 0 & 0 & 1 \\ B_T'' & B_T' & B_n \end{vmatrix}$$

$$= B_T'' \cdot \vec{E}' - B_T' \cdot \vec{E}'' = \mu_0 (J_S' \vec{E}' + J_S'' \vec{E}''),$$

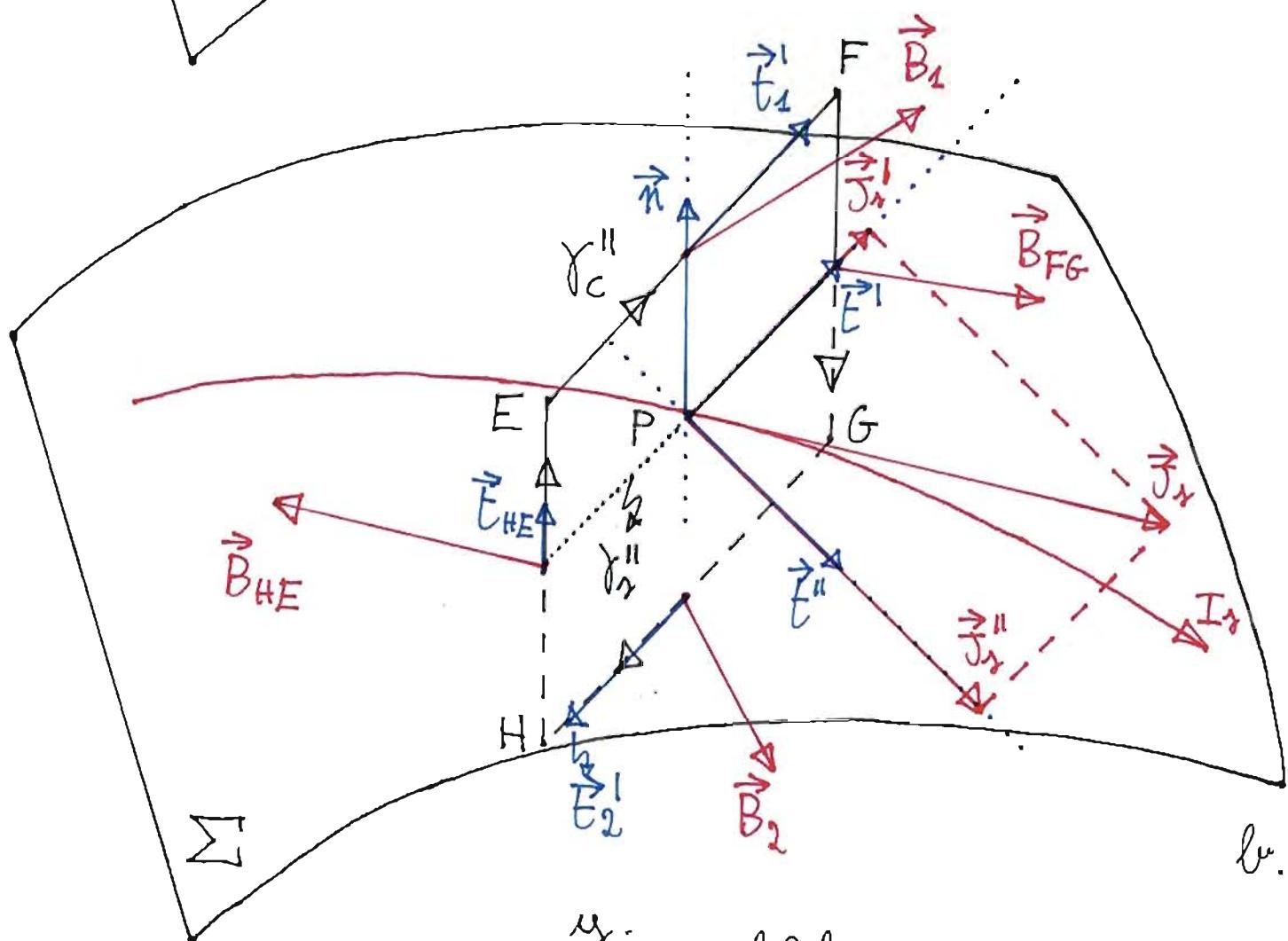
from which

$$\left\{ \begin{array}{l} B_T'' = B_{T1}'' - B_{T2}' = \mu_0 J_S' \\ B_T' = B_{T1}' - B_{T2}' = -\mu_0 J_S'' \end{array} \right.$$

which are equivalent to (20.19) and (20.28), respectively.



a.



b.

Figure 20.2.

- Case 4.

Following the analogy with lecture 5, we finally briefly consider Case 4, the field \vec{B} in proximity of a generic filiform conductor with a stationary current. Note that there is no Case 3 for \vec{B} .

Without giving a demonstration, in close proximity of the aforementioned filiform conductor, \vec{B} tends to have the same behavior as for an infinite straight filiform conductor. In particular,

$$\lim_{r \rightarrow 0^+} B_t = \mu_0 \frac{\pm I}{2\pi r} \quad (20.30)$$

and

$$\lim_{r \rightarrow 0^+} \frac{B_n}{B_t} = 0 \quad , \quad (20.31)$$

where r is the (positive) radial distance from a generic point P in space and a point Q on the conductor, with P in the neighborhood of Q . In addition, B_t and B_n are the tangent and normal components of \vec{B} at P and I is the current flowing in the conductor.

20.2 Fields with vector potential.

23/35

Necessary and sufficient condition for a generic vector field \vec{F} to be expressed as the curl of another vector field \vec{A} is that \vec{F} is solenoidal,

$$\vec{\nabla} \cdot \vec{F} = 0 \quad . \quad (20.32)$$

The condition must be valid at each point where \vec{F} is defined.

It is easy to demonstrate that the condition must be necessary. In fact, if

$$\vec{F} = \vec{\nabla} \times \vec{A} \quad , \quad (20.33)$$

it must also be

$$\vec{\nabla} \cdot \vec{F} = 0 \quad . \quad (20.34)$$

This is because

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0 \quad . \quad (20.35)$$

The last equality can be shown by writing the component of the curl and divergence of a vector field in cartesian coordinates,

$$\begin{aligned}
 \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) &= \frac{\partial}{\partial x} (\vec{\nabla} \times \vec{A})_x + \frac{\partial}{\partial y} (\vec{\nabla} \times \vec{A})_y + \frac{\partial}{\partial z} (\vec{\nabla} \times \vec{A})_z \\
 &= \frac{\partial}{\partial x} \left(\frac{\partial}{\partial y} A_z - \frac{\partial}{\partial z} A_y \right) \\
 &\quad + \frac{\partial}{\partial y} \left(\frac{\partial}{\partial z} A_x - \frac{\partial}{\partial x} A_z \right) \\
 &\quad + \frac{\partial}{\partial z} \left(\frac{\partial}{\partial x} A_y - \frac{\partial}{\partial y} A_x \right) \\
 &= \frac{\partial^2}{\partial x \partial y} A_z - \frac{\partial^2}{\partial x \partial z} A_y \\
 &\quad + \frac{\partial^2}{\partial y \partial z} A_x - \frac{\partial^2}{\partial y \partial x} A_z \\
 &\quad + \frac{\partial^2}{\partial z \partial x} A_y - \frac{\partial^2}{\partial z \partial y} A_x = 0, \quad (20.36)
 \end{aligned}$$

where we assumed all derivatives to be continuous.

To demonstrate that the condition must also be sufficient is a nontrivial task and, thus, we will not

show it in these lectures.

By definition, each vector field \vec{A} that satisfies (20.33) and, thus, (20.35) is called a vector potential of the vector field \vec{F} .

Moreover, given an arbitrary scalar field Φ , we have by definition

$$\vec{\nabla} \times (\vec{\nabla} \cdot \Phi) = 0 , \quad (20.34)$$

because Φ is the scalar potential of an irrotational field. As a consequence, if \vec{A} is a vector potential of \vec{F} , any other vector field \vec{A}' such that

$$\vec{A}' = \vec{A} + \vec{\nabla} \cdot \Phi \quad (20.38)$$

is also a vector potential of \vec{F} (in (20.38), Φ is an arbitrary scalar field). In fact,

$$\begin{aligned} \vec{\nabla} \times (\vec{A}') &= \vec{\nabla} \times (\vec{A} + \vec{\nabla} \cdot \Phi) = \vec{\nabla} \times \vec{A} + \vec{\nabla} \times (\vec{\nabla} \cdot \Phi) \\ &\stackrel{|}{=} \vec{\nabla} \times \vec{A} , \quad (20.39) \end{aligned}$$

where we made use of (20.34).

20.3 Vector potential of \vec{B} and Laplace's elementary theorem 26/35

As shown in Lec. 20.2, every solenoidal vector field has a vector potential. As a consequence, \vec{B} has a vector potential. By calling \vec{A} one arbitrary potential of \vec{B} , we have

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad (20.40)$$

Given an arbitrary scalar field φ , if \vec{A} is a vector potential of \vec{B} , each vector field

$$\vec{A}' = \vec{A} + \vec{\nabla} \cdot \varphi \quad (20.41)$$

is a vector potential of \vec{B} . This means that each magnetic field \vec{B} has infinite vector potentials. From (20.41), the knowledge of one vector potential implies the knowledge of all of them.

By substituting (20.40) into (20.5) we readily obtain

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \mu_0 \vec{j} \quad (20.42)$$

From the definition of vector Laplacian

$$\vec{\nabla}^2 \vec{A} = \vec{\nabla} \cdot (\vec{\nabla} \cdot \vec{A}) - \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) , \quad (20.43)$$

we obtain

$$\vec{\nabla} \cdot (\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A} = \mu_0 \vec{j} . \quad (20.44)$$

Among the infinite vector potentials of \vec{B} it is always possible to choose one which is solenoidal. In fact, given a generic vector potential \vec{A}_0 , which is nonsolenoidal, it is sufficient to consider a scalar function ψ such that at each point of the domain of definition of \vec{A}_0

$$\vec{\nabla}^2 \psi = -\vec{\nabla} \cdot \vec{A}_0 . \quad (20.45)$$

In this case, the vector potential defined by

$$\vec{A} = \vec{A}_0 + \vec{\nabla} \cdot \psi \quad (20.46)$$

is solenoidal by definition.

Choosing \vec{A} such that

$$\vec{\nabla} \cdot \vec{A} = 0 , \quad (20.47)$$

(20.44) can be rewritten as

$$\vec{\nabla}^2 \vec{A} = -\mu_0 \vec{J} \quad (20.48)$$

Each solution of this equation, so long solenoidal, is a vector potential of \vec{B} generated by the given currents \vec{J} . In cartesian coordinates, by projecting (20.48) onto the coordinate axes, we obtain the equivalent system

$$\left\{ \begin{array}{l} \vec{\nabla}^2 A_x = -\mu_0 J_x \\ \vec{\nabla}^2 A_y = -\mu_0 J_y \\ \vec{\nabla}^2 A_z = -\mu_0 J_z \end{array} \right. \quad (20.48')$$

Thus, looking for a solenoidal vector potential associate to a given current distribution is equivalent to look for the solution of a system of three scalar equations, each of which is similar to a Poisson equation. Under the assumption that the three functions A_x , A_y , and A_z go to zero at infinite, the solution of each equation in (20.48') has the form of a coulombian integral (as in the electrostatic case),

$$\left\{ \begin{array}{l} A_x(P) = \frac{\mu_0}{4\pi} \iiint_{\Sigma} \frac{J_x(Q)}{r_{QP}} \cdot d\tau \quad (20.49a) \\ A_y(P) = \frac{\mu_0}{4\pi} \iiint_{\Sigma} \frac{J_y(Q)}{r_{QP}} \cdot d\tau \quad (20.49b) \\ A_z(P) = \frac{\mu_0}{4\pi} \iiint_{\Sigma} \frac{J_z(Q)}{r_{QP}} \cdot d\tau, \quad (20.49c) \end{array} \right.$$

where the components A_x, A_y , and A_z are defined at the generic field-point P , whereas J_x, J_y , and J_z are defined at the generic source-point Q and r_{QP} is the absolute value of the distance between P and Q . The volume integral is defined on the entire region Σ where the currents are distributed.

Equations (20.49a) – (20.49c) can be summarized in a single vector relation that makes it possible to directly calculate the vector potential \vec{A} from the knowledge of \vec{J} ,

$$\vec{A}(P) = \frac{\mu_0}{4\pi} \iiint_{\Sigma} \frac{\vec{J}(Q)}{r_{QP}} \cdot d\tau , \quad (20.4g')$$

so long the distribution of \vec{J} is known at each point in Σ . Once the vector potential \vec{A} is known, \vec{B} can readily be calculated from

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

In this way, at least in principle, the fundamental problem of magnetostatics is solved. This consists in finding the magnetic field generated by a given distribution of currents flowing in conductors in vacuum. The only difference with the electrostatic case is in that the integrand in (20.4g').

By substituting (20.40) into (20.4g'), we obtain

$$\vec{B}(P) = \frac{\mu_0}{4\pi} \vec{\nabla}_P \times \iiint_{\Sigma} \frac{\vec{J}(Q)}{r_{QP}} \cdot d\tau , \quad (20.50)$$

where the sub-script P shows that the operation of

curl must act of the field-point P (and not on the source-point Q). Due to the linearity of the operator curl, we have 31/35

$$\vec{B}(P) = \frac{\mu_0}{4\pi} \iiint_{\Sigma} \vec{\nabla}_P \times \frac{\vec{J}(Q)}{r_{QP}} \cdot d\gamma \quad . \quad (20.51)$$

From the known vector relation

$$\vec{\nabla}_P \times \frac{\vec{J}(Q)}{r_{QP}} = \frac{1}{\mu_{QP}} \vec{\nabla}_P \times \vec{J} - \vec{J} \times \vec{\nabla}_P \cdot \frac{1}{\mu_{QP}}$$

and being $\vec{\nabla}_P \times \vec{J} = \vec{0}$ since \vec{J} is not a function of P, we finally obtain

$$\vec{B}(P) = - \frac{\mu_0}{4\pi} \iiint_{\Sigma} \vec{J}(Q) \times \vec{\nabla}_P \cdot \frac{1}{r_{QP}} \cdot d\gamma \quad . \quad (20.52)$$

Moreover,

$$\vec{\nabla}_P \cdot \frac{1}{r_{QP}} = - \frac{1}{\mu_{QP}^2} \cdot \vec{r}_{QP} = - \frac{\vec{r}_{QP}}{\mu_{QP}^3} \quad . \quad (20.53)$$

and, so,

$$\vec{B}(P) = \frac{\mu_0}{4\pi} \iiint_{\Sigma} \frac{\vec{j}(Q) \times \vec{r}_{QP}}{r_{QP}^3} \cdot d\gamma \quad . \quad (20.54)$$

This expression for \vec{B} is exactly the same as (20.50). However, it makes it possible to more easily obtain a general expression for the magnetic field of a given distribution of filiform currents. For example, consider a filiform circuit with transversal cross-section Σ and characterized by a stationary current I . Due to the filiform nature of the conductor, Σ can be assumed to be infinitesimal dS (cf. Fig. 20.3). Thus, the volume associated with an infinitesimal element dl of the circuit is

$$d\gamma = dS \cdot dl \quad . \quad (20.55)$$

From (20.54) we then have

$$\begin{aligned} \vec{B}(P) &= \frac{\mu_0}{4\pi} \iiint_{\Sigma} \frac{\vec{j}(Q) \times \vec{r}_{QP}}{r_{QP}^3} \cdot d\gamma \\ &= \frac{\mu_0}{4\pi} \iiint_{\Sigma} \frac{\vec{j}(Q) \cdot dS \times \vec{r}_{QP}}{r_{QP}^3} \cdot dl \end{aligned}$$

$$\boxed{\frac{\mu_0}{4\pi} I \oint_{\gamma} \frac{\vec{E} \times \vec{r}_{QP}}{r_{QP}^3} \cdot d\vec{l}} , \quad (20.56)$$

where

$$I = \sum \iint_{\Sigma} J(\vec{r}) \cdot d\vec{S} , \quad (20.54)$$

Σ is oriented so that \vec{E} is its normal unit vector, γ is the conductor's axis (incidentally, \vec{E} is tangent at each point on γ), and Σ is the region where the conductor is defined and the current is distributed [note that

$$\vec{J}(\vec{r}) = J(\vec{r}) \cdot \vec{E} .$$

thus,

$$\begin{aligned} I &= \sum \iint_{\Sigma} \vec{J}(\vec{r}) \cdot \vec{n} \cdot d\vec{S} = \sum \iint_{\Sigma} J(\vec{r}) \vec{E} \cdot \vec{E} \cdot d\vec{S} \\ &= \sum \iint_{\Sigma} J(\vec{r}) \cdot d\vec{S} . \end{aligned}$$

Assuming the entire filiform conductor with axis Y to be divided into infinitesimally small elements dl , from (20.56) the infinitesimal contribution to the field \vec{B} can be calculated as

$$d\vec{B} = \frac{\mu_0}{4\pi} I \cdot dl \frac{\vec{E} \times \vec{r}_{qp}}{r_{qp}^3}, \quad (20.58)$$

which is known as Laplace's elementary equation. It is obvious that this equation does not have independent value from (20.56). In fact, it has no physical meaning to consider an "infinitesimal current element" $I \cdot dl$ regardless from the closed circuit to which it must belong. However, it is remarkable the resemblance with the equation that gives the infinitesimal contribution to the electrostatic field \vec{E} due to an infinitesimal charge dq at point P,

$$d\vec{E} = \frac{1}{4\pi\epsilon_0} dq \frac{\vec{r}_{qp}}{r_{qp}^3}.$$

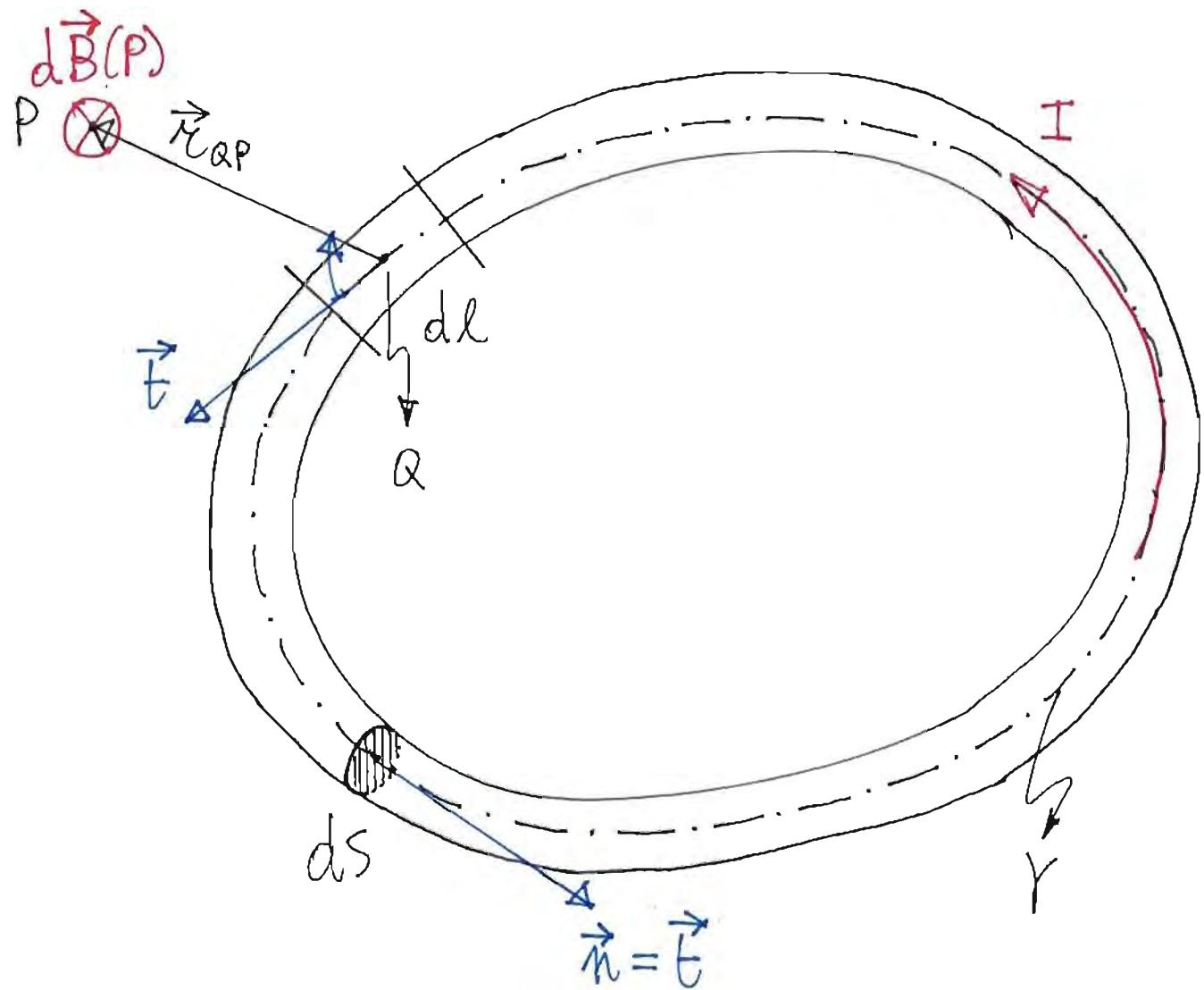


Figure 20.3.

Phys 242 - §14. Lecture 21 - The magnetostatic field in vacuum / part 3. 1/56

Summary lecture 20.

- The laws of magnetostatics in local form.

Case 1. $\vec{J} \in C^0(\Omega)$; limited (stationary)	$\vec{\nabla} \cdot \vec{B} = 0 \quad (1)$ $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \quad (2)$
Case 2. $\vec{J}_s \in C^0(\Sigma)$; limited (stationary)	$\vec{n} \cdot (\vec{B}_1 - \vec{B}_2) = 0 \quad (1)$ $\vec{n} \times (\vec{B}_1 - \vec{B}_2) = \mu_0 \vec{J}_s \quad (2)$
Case 3. Uniform conductor with stationary current I	$\lim_{r \rightarrow 0^+} B_t = \mu_0 \frac{I}{2\pi r} \quad (1)$ $\lim_{r \rightarrow 0^+} \frac{B_n}{B_t} = 0 \quad (2)$

• Vector potential of \vec{B} .

Given a vector field \vec{A} , if

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

\vec{A} is a vector potential of \vec{B} . Given an arbitrary scalar field φ , if \vec{A}' is a vector potential of \vec{B} , also

$$\vec{A}' = \vec{A} + \vec{\nabla} \cdot \varphi$$

is a vector potential of \vec{B} .

- From Ampère's law,

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \mu_0 \vec{J}.$$

By choosing \vec{A} such that $\vec{\nabla} \cdot \vec{A} = 0$ (always possible), we find

$$\vec{\nabla}^2 \vec{A} = -\mu_0 \vec{J}.$$

Assuming normal conditions at infinite for \vec{A} , the solution for \vec{A} is of coulombian type,

$$\vec{A}(P) = \frac{\mu_0}{4\pi} \iiint_{\Sigma} \frac{\vec{J}(Q)}{r_{QP}} \cdot d\chi.$$

- Laplace's theorem.

From $\vec{B} = \vec{\nabla} \times \vec{A}$, we find

$$\vec{B}(P) = \frac{\mu_0}{4\pi} \iiint_{\Sigma} \frac{\vec{j}(Q) \times \vec{r}_{QP}}{r_{QP}^3} \cdot d\tau$$

For a filiform conductor with current I and transversal cross-section dS ,

$$\vec{B}(P) = \frac{\mu_0}{4\pi} I \oint_{\gamma} \frac{\vec{t} \times \vec{r}_{QP}}{r_{QP}^3} \cdot dl .$$

Finally,

$$d\vec{B}(P) = \frac{\mu_0}{4\pi} I \cdot dl \cdot \frac{\vec{t} \times \vec{r}_{QP}}{r_{QP}^3} .$$

21.1 Coefficients of self and mutual inductance of circuits. 4/56

Consider a quasi filiform circuit occupying a region of space Σ and with a stationary current I . The current is generated by an emf not shown in Fig. 21.1, which only shows the conductive part of the circuit. The magnetostatic field \vec{B} can be calculated at each point inside and outside the conductor Σ from

$$\vec{B} = \vec{\nabla} \times \vec{A}, \quad (21.1)$$

where

$$\vec{A} = \frac{\mu_0}{4\pi} \iiint_{\Sigma} \frac{\vec{j}}{r} \cdot d\tau. \quad (21.2)$$

In this equation, \vec{j} is the known volume current density in Σ and r the absolute value of the distance between the point where \vec{A} is evaluated and that where \vec{j} is defined inside Σ (it is clear the latter varies and, thus, the necessity of the volume integral).

By indicating with γ the closed line representing

the longitudinal axis of the conductor and with 5/56
 Φ_γ the flux of the magnetic field generated by I and linked
 with γ , we have

$$\Phi_\gamma = \iint_{\sum_\gamma} \vec{B} \cdot \vec{n} \cdot dS \quad . \quad (21.3)$$

In this equation, \sum_γ is any open surface having γ as border and oriented such that the positive direction of the normal unit vector \vec{n} (to \sum_γ) and the orientation on γ used to evaluate I are consistent with the right-hand rule. From Stokes' theorem and the properties of \vec{A} , we obtain

$$\Phi_\gamma = \iint_{\sum_\gamma} (\vec{\nabla} \times \vec{A}) \cdot \vec{n} \cdot dS = \oint_{\gamma} \vec{A} \cdot \vec{t} \cdot dl \quad . \quad (21.4)$$

Thus, if the distribution of \vec{J} in the conductor is known, (21.2) gives \vec{A} and (21.4) gives Φ_γ .

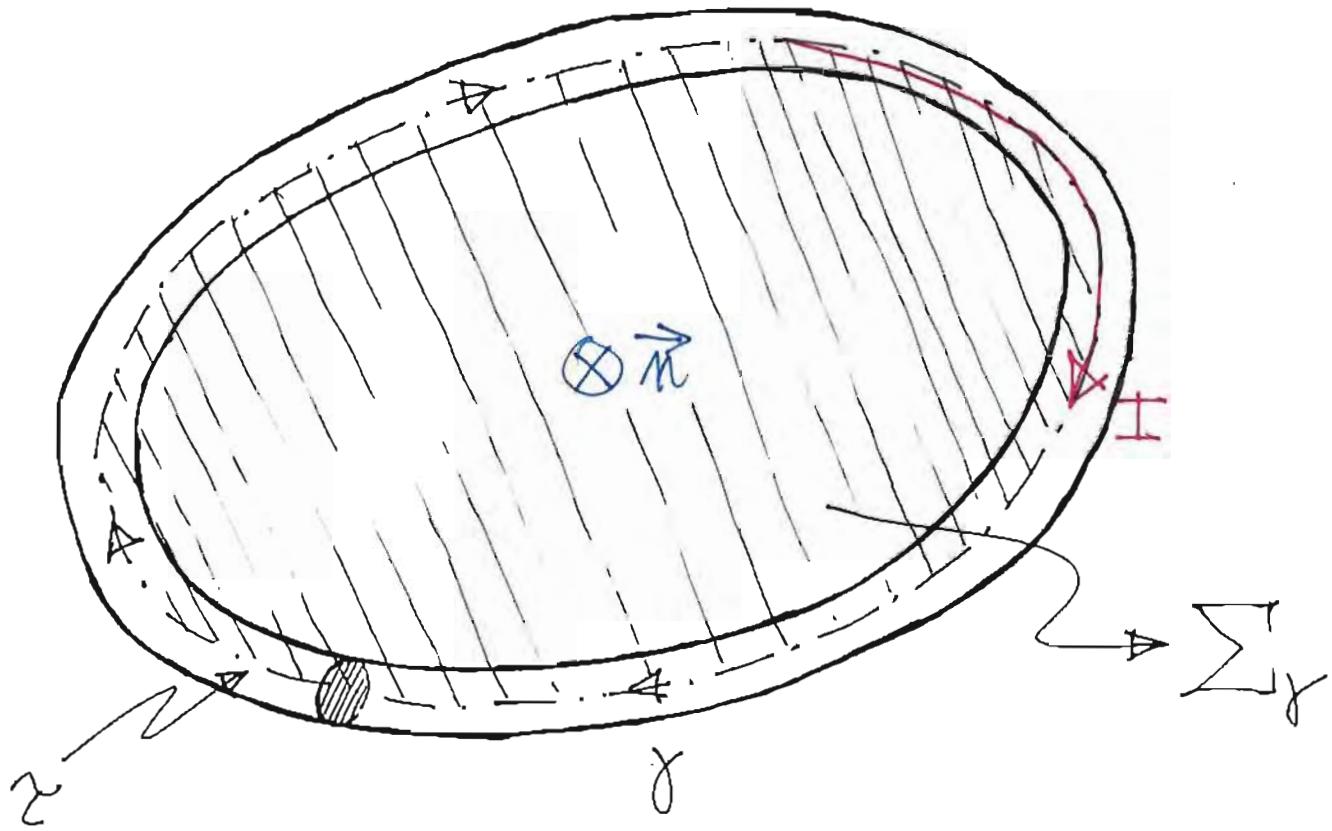


Figure 21.1.

By means of a new emf source, we change the value of \vec{J} at each point inside the circuit until we obtain a new current distribution

$$\vec{J}' = k \vec{J} \quad , \quad (21.5)$$

where k is an arbitrary constant. Under these conditions we have

$$I' = k I \quad . \quad (21.6)$$

From (21.2) and (21.1), also \vec{A} and \vec{B} vary linearly, 7/56

$$\vec{A}' = k \vec{A} \quad (21.4)$$

and

$$\vec{B}' = k \vec{B} \quad (21.8)$$

As a consequence,

$$\dot{\Phi}_Y' = k \dot{\Phi}_Y \quad (21.9)$$

where $\dot{\Phi}_Y'$ is the new value of the flux linked with Y . By defining as $\dot{\Phi}_{1Y}$ the value of $\dot{\Phi}_Y$ corresponding to a unitary current flowing in the circuit, we have

$$\dot{\Phi}_Y = \dot{\Phi}_{1Y} \frac{I}{1} = \frac{\dot{\Phi}_{1Y}}{1} I \quad (21.10)$$

where the "one" in the denominator is the value of the unitary current, 1A. By defining the ratio $\dot{\Phi}_{1Y}/1$ as a coefficient L , we can write

$$\Phi_{\gamma} = LI \quad , \quad (21.11)$$

where L is called the self-inductance coefficient of the circuit with respect to γ . Note that this definition is possible because of (21.9). In fact, $K = I/1$ in (21.10).

The dimensions of L are those of a flux over a current (magnetic flux); hence, its SI unit is Wb/A , which is called "henry" (H).

Because of its very definition, L must be nonnegative. In fact, when I is positive (as, e.g., in the case of Fig. 21.1), the vector lines of the magnetic field point to the same direction of \vec{n} with respect to γ . As a consequence, also Φ_{γ} must be positive. Similarly, when I is negative, also Φ_{γ} is negative. In both cases, the ratio Φ_{γ}/I is positive (or zero).

The flux Φ_{γ} has been defined with respect to a closed line γ representing the longitudinal axis of the circuit. We could repeat all the arguments for a different line γ' , also closed and entirely contained within Σ . If the length of γ is much larger than its diameter, the

19/56

fluxes linked with γ and γ' and generated by the same current I are almost identical. Hence, the value of the inductance L calculated with respect to γ' would be approximately the same as that calculated with respect to γ . Note that, assuming the conductor to be rigorously filiform (instead of quasi filiform), i.e., with zero diameter, would lead to a meaningless definition of L . In this case, in fact, even in presence of a finite current, the field \vec{B} would diverge to infinite when approaching the conductor. As a consequence, the flux of \vec{B} linked with the circuit would also be infinite and, so, the ratio Φ_γ/I as well. As a matter of fact, assuming that a finite current flows in a filiform circuit does not have physical meaning.

Consider now two quasi filiform circuits γ_1 and γ_2 , as shown in Fig. 21.2. The two circuits, which are close to each other, but do not touch each other, form a circuit system. Assume a stationary current I_1 (the sign of which is evaluated with respect to an arbitrary reference direction) flows in γ_1 , whereas no current flows in γ_2 . We choose arbitrarily a positive direction

10/56

on γ_2 , which is the longitudinal axis of x_2 . The generic surface Σ_2 that has γ_2 as a border is oriented such that its normal unit vector \vec{n}_2 follows the usual right-hand rule. We intend to determine the flux due to the magnetic field generated by I_1 and linked with γ_2 . Following an argument similar to that used in the self-inductance case, we find

$$\Phi_{21} = M_{21} I_1 \quad . \quad (21.12)$$

In this equation, Φ_{21} is the flux generated by I_1 and linked with γ_2 and M_{21} a proportionality factor with units of an inductance (i.e., measurable in H as L). This factor is called the mutual-inductance coefficient of the first circuit on the second. Note that sometimes L and M are called coefficients of self (or auto) and mutual (or cross) induction.

Depending on the chosen orientation of the circuits, M_{21} can be larger or smaller than zero. In Fig. 21.2, for example, the vector lines of the field \vec{B} generated

by a positive current I_1 are such that at each point on the surface Σ_2 , \vec{B} and \vec{n}_2 have opposite direction. Hence, $\oint \vec{B} \cdot d\vec{l}_2 < 0$. In this case, the coefficient M_{21} is negative, $M_{21} < 0$. Note that the vector lines linked with both circuits are the only lines to contribute to the flux of mutual induction (e.g., $\oint \vec{B} \cdot d\vec{l}_1$). If, while maintaining the same orientation for the first circuit, we invert the orientation of the second (and, thus, the positive direction of the normal \vec{n}_2 to Σ_2), the flux $\oint \vec{B} \cdot d\vec{l}_1$ generated by the same current I_1 would be positive. In this case, $M_{21} > 0$.

Consider now a case, symmetric of the previous one, where current flows only in Σ_2 . In this case,

$$\oint \vec{B} \cdot d\vec{l}_1 = M_{12} I_2 \quad , \quad (21.13)$$

where $\oint \vec{B} \cdot d\vec{l}_1$ is the flux generated by I_2 and linked with Σ_1 and M_{12} is the mutual inductance of the second circuit on the first.

It is easy to verify that

$$M_{21} = M_{12} \quad (21.14)$$

It is thus possible to define a single mutual inductance $M = M_{21} = M_{12}$ between the two circuits.

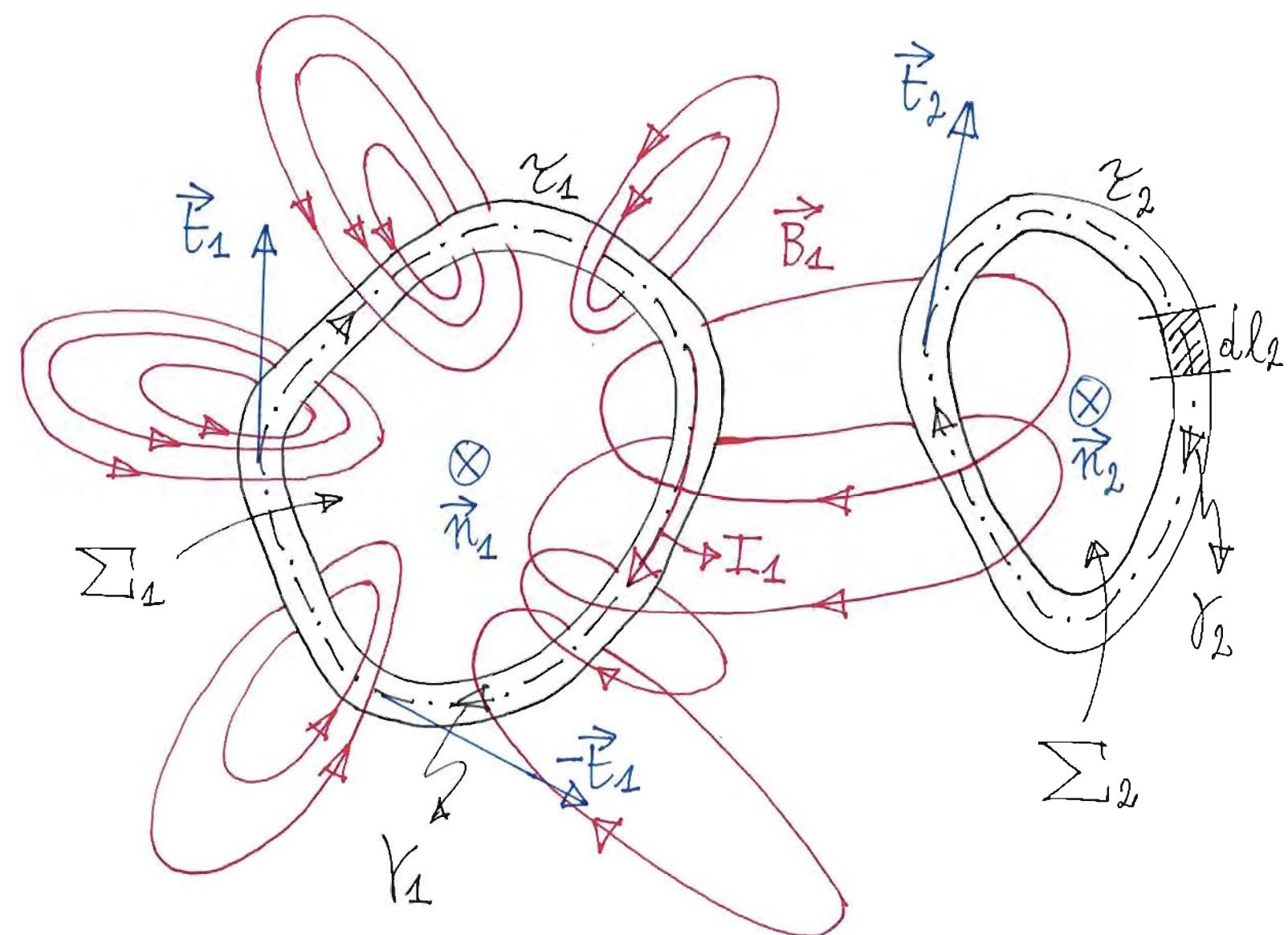


Figure 21.2.

We now move to a more rigorous calculation | 13/56

of M_{21} and M_{12} .

Firstly, we attempt to explain the origin of the vector lines of the magnetostatic field \vec{B}_1 in Fig. 21.2. In order to sketch the lines qualitatively, we will resort to both Laplace's theorem and the solenoidal property of \vec{B} . Figure 21.3 shows the quasi filiform circuit that occupies the region of space Σ_1 . As in Fig. 21.2, γ_1 is one of the possible longitudinal axes of Σ_1 . As shown in the figure, γ_1 is a closed line oriented clockwise. Due to the quasi filiform nature of the circuit, we can assume all longitudinal axes of Σ_1 to coincide with γ_1 (this is correct up to an infinitesimal distance). Under these conditions, the unit tangent vector at each point Q on γ_1 , Q_1 , is $\vec{E}_1(Q_1)$. Among all possible open surfaces having γ_1 as a border, Σ_1 is the surface belonging to the plane on which Σ_1 is assumed to lie on (in Fig. 21.3 such a plane is the page where the circuit is drawn). Due to the chosen orientation of γ_1 , the normal unit vector \vec{n}_1 to Σ_1 must point inside the page (right-hand rule).

The field \vec{B}_1 is generated by a stationary current I_1 , which is assumed to be positive and, thus, oriented

clockwise, consistently with the orientation of γ_1 . [14/56]
As a consequence, each point Q_1 is a source point for
 \vec{B}_1 .

For simplicity, in Fig. 21.3 the circuit γ_2 is not shown. Only the longitudinal axis γ_2 is reported. As for γ_1 , the open surface Σ_2 having γ_2 as a border belongs to the plane on which γ_2 is assumed to lie. Note that, for simplicity, we also assume γ_1 and γ_2 to lie on the same plane. This means that Σ_1 and Σ_2 belong to the same plane.

As shown in Fig. 21.3, we now consider two infinitesimal current element $I_1 \cdot d\ell_1$ and $I_1 \cdot d\ell'_1$ at the source points Q_1 and Q'_1 , respectively. The tangent unit vectors at these source points are \vec{t}_1 and \vec{t}'_1 , respectively. We intend to calculate the infinitesimal fields due to the elements at Q_1 and Q'_1 at two different field point P_1 and P_2 . The point P_1 is one point of Σ_1 and P_2 of Σ_2 . Such fields can be calculated by means of Laplace's theorem [Eq. (20.58)]. Since we are interested in the qualitative behavior of the field lines of \vec{B}_1 , the

only part of (20.58) we need to consider is

15/56

$$\vec{E} \times \vec{r}_{QP}, \quad (21.15)$$

which gives the direction of \vec{B} (or $d\vec{B}$). The distance vectors from Q_1 and Q'_1 to P_1 and P_2 are indicated in Fig. 21.3 along with the tangent unit vectors at Q_1 and Q'_1 . By folding the latter onto the former (right-hand rule), we obtain the directions of the four infinitesimal fields $d\vec{B}_{Q_1}(P_1)$, $d\vec{B}_{Q_1}(P_2)$, $d\vec{B}_{Q'_1}(P_1)$, and $d\vec{B}_{Q'_1}(P_2)$. It is clear that all four fields are normal to the plane of Σ_1 and Σ_2 . However, both $d\vec{B}_{Q_1}(P_1)$ and $d\vec{B}_{Q'_1}(P_1)$ point into the plane, whereas both $d\vec{B}_{Q_1}(P_2)$ and $d\vec{B}_{Q'_1}(P_2)$ point away from the plane. The argument is valid for any source point on Σ_1 . Hence, by means of the superposition principle for \vec{B} , $\vec{B}_{Q_1}(P_1)$, where here Q_1 is any source point, points into the plane and $\vec{B}_{Q_1}(P_2)$ points away from the plane.

Assuming \vec{B}_1 is a continuous function (there is no reason to assume the contrary since we are considering \vec{B}_1 in a region outside two quasi filiform conductors,

and there are no surface current densities in the 16/56
 region), due to the solenoidal property of \vec{B}_1 , the
 field lines must form closed loops. The same lines must
 go into the page at P_1 and out of the page at P_2 . The
 lines sketched in Fig. 21.3 (and 21.2) fulfill both these
 conditions and, thus, represent possible field lines for
 \vec{B}_1 .

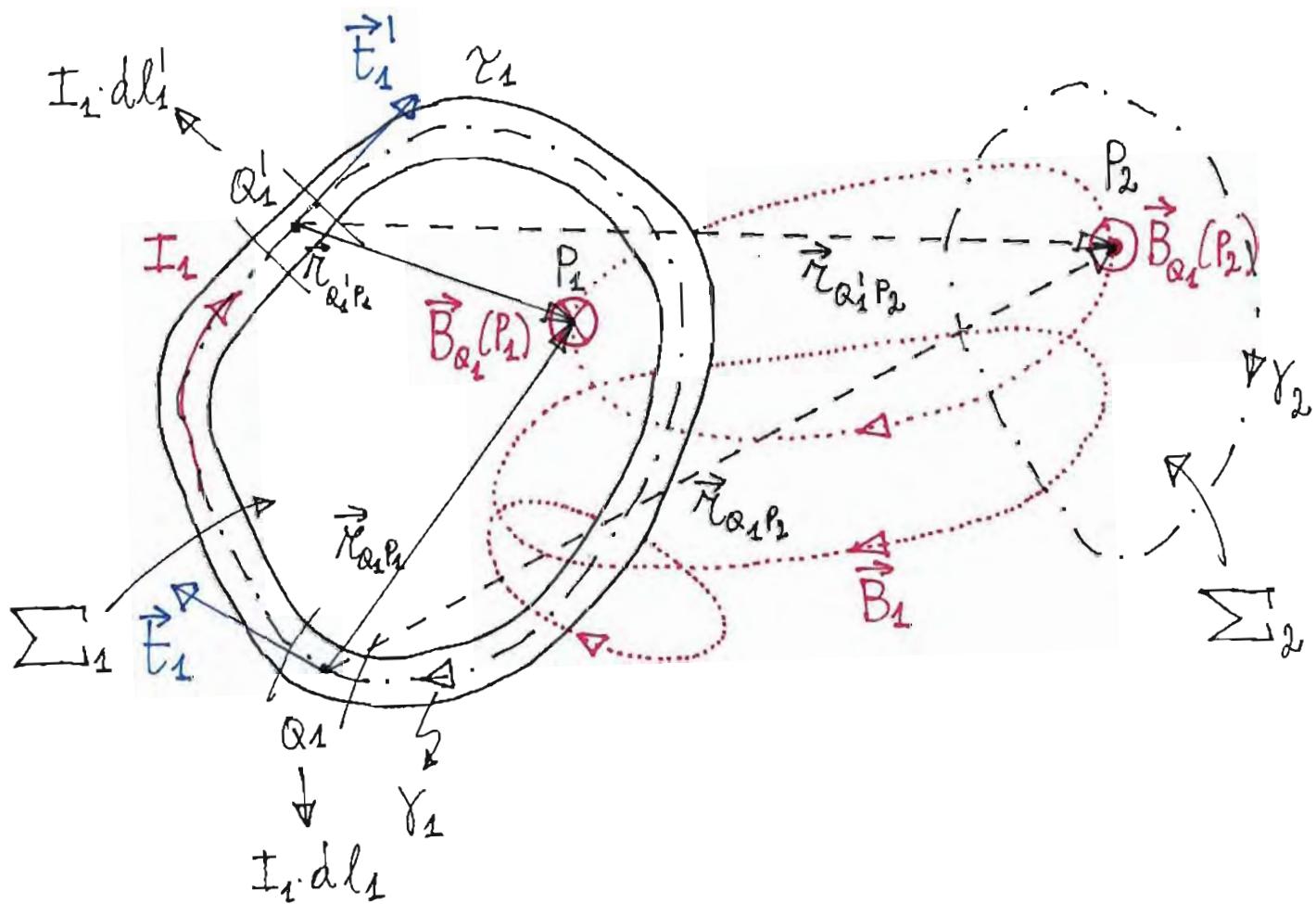


Figure 21.3.

Now that we understand how the lines of \vec{B}_1 are linked to γ_2 (and, thus, γ_2), we could calculate M_{21} from

the knowledge of \vec{B}_1 and the flux that \vec{B}_1 threads through γ_2 . However, we prefer to resort to the distribution of the vector potential \vec{A}_1 generated by I_1 . The vector potential \vec{A}_1 at any point in space P can be calculated from (21.2) as

$$\vec{A}_1(P) = \frac{\mu_0}{4\pi} \iiint_{\Sigma_1} \frac{\vec{J}_1}{r} \cdot d\vec{z} , \quad (21.16)$$

where \vec{J}_1 , which is the electric current volume density associated with I_1 , is evaluated at each source point Q_1 on γ_1 (we remind Σ_1 is quasi filiform; we can thus assume a point Q_1 inside Σ_1 to be a point on γ_1 up to an infinitesimal distance). In (21.16), $r = \|\vec{r}_{Q_1 P}\|$ is the absolute value (i.e., the Euclidian norm) of the distance between the generic field point P and Q_1 . Under stationary conditions, the quasi filiform conductor is a flux tube for \vec{J}_1 . Hence, $I_1 = J_1 \cdot dS$, where dS is the cross-section of Σ_1 . We can thus rewrite (21.16) as

$$\vec{A}_1(P) = \frac{\mu_0 I_1}{4\pi} \oint_{\gamma_1} \frac{\vec{E}_1}{r} \cdot d\vec{l}_1 , \quad (21.17)$$

where \vec{E}_1 is the tangent unit vector at each point 18/56
 Q_1 on γ_1 and also the direction of J_1 , $\vec{J}_1 = J_1 \cdot \vec{E}_1$ and
 $d\ell_1$ an infinitesimal element on γ_1 . Figure 21.4 shows
the longitudinal axes γ_1 and γ_2 of Σ_1 and Σ_2 , respectively.
It also shows the source points Q_1 , a generic field
point P , and the special field points P_2 on γ_2 .

From Stokes theorem, we know that [cf. eq. (21.4)]

$$\Phi_{21} = \oint_{\gamma_2} \vec{A}_1 \cdot \vec{E}_2 \cdot d\ell_2 , \quad (21.18)$$

where \vec{E}_2 is a tangent unit vector at any field point
 P_2 on γ_2 and $d\ell_2$ an infinitesimal element on γ_2 . Note
that, (21.18) confirms the flux threaded by field \vec{B}_1 (or
any other field \vec{B} , in general) depends only on the border
 γ_2 and not on the specific open surface Σ_2 that has
 γ_2 as a border.

We can combine (21.17) and (21.18) to obtain

$$M_{21} = \frac{\Phi_{21}}{I_1} = \frac{\mu_0}{4\pi} \oint_{\gamma_2} \left(\oint_{\gamma_1} \frac{\vec{E}_1}{r} \cdot d\ell_1 \right) \cdot \vec{E}_2 \cdot d\ell_2$$

$$= \frac{\mu_0}{4\pi} \oint_{Y_2} \oint_{Y_1} \frac{\vec{E}_1 \cdot \vec{E}_2}{r} \cdot d\vec{l}_1 \cdot d\vec{l}_2 . \quad (21.19)$$

While (21.19) is an elegant mathematical expression, when calculating M_{21} we recommend to first evaluate \vec{A}_1 at any point in space and, then, calculate its circulation along Y_2 . The compactness of (21.19) usually leads to errors.

It is worth noting that the quantity r in the denominator of the integrand in (21.19) or (21.17) is always different from zero. This is clear from Fig. 21.4, from which it appears that any source point Q_1 on Y_1 will always be different from any field point P_2 on Y_2 . Hence, in the case of the mutual inductance M_{21} , the integrand function is always well defined because Y_1 and Y_2 are different, and the integral (21.19) is always well defined.

A procedure similar to the one that led to integral (21.19) makes it possible to calculate the mutual

inductance M_{12} . In this case, we set $I_1 = 0$ and [20/56] calculate the flux Φ_{12} that a positive test current I_2 (also stationary) generates in \sum_1 . Using the same notation as in (21.19), we obtain

$$M_{12} = \frac{\Phi_{12}}{I_2} = \frac{\mu_0}{4\pi} \oint_{Y_1} \left(\oint_{Y_2} \frac{\vec{E}_2}{r} \cdot d\vec{l}_2 \right) \cdot \vec{E}_1 \cdot d\vec{l}_1$$

$$= \frac{\mu_0}{4\pi} \oint_{Y_1} \oint_{Y_2} \frac{\vec{E}_2 \cdot \vec{E}_1}{r} \cdot d\vec{l}_2 \cdot d\vec{l}_1 . \quad (21.20)$$

Because of the linearity of integration and for the commutative property of the scalar product, it is evident that M_{21} given by (21.19) is equal to M_{12} given by (21.20), $M_{21} = M_{12}$. For this reason, the mutual-inductance coefficient is typically simply called $M (= M_{12} = M_{21})$.

The results (21.19) and (21.20) are called Neumann integrals. The Neumann integrals are valid only under the assumption that the flux (either Φ_{21} or Φ_{12}) is

proportional to the current that generates it.
 In this case, the Neumann integrals are very useful tools (typically used in numerical integration softwares) for the calculation of the mutual inductance between two circuits. In the case of ferromagnetic materials (not treated in these lectures), for example, there is no proportionality between flux and current. Hence, the integrals (21.19) and (21.20) cannot be used.

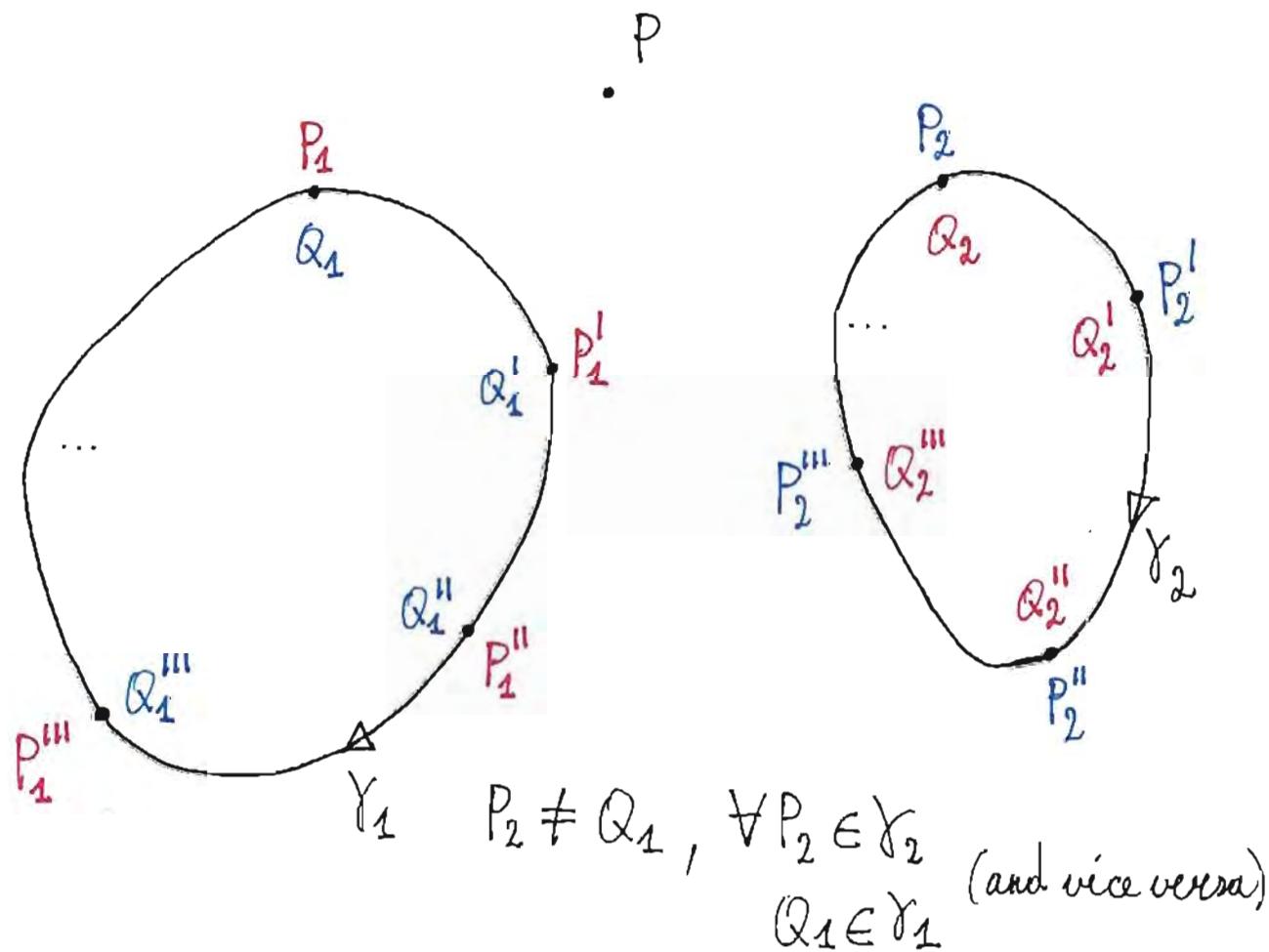


Figure 21.4.

It is obvious that, similarly to (21.19), also the integral (21.20) is always well defined because its integrand is well defined. This is clear from Fig. 21.4: source and field points are always different.

In the light of this observation, the question arises whether it is possible or not to define a Neumann-type integral for the calculation of the self inductance of a given quasi filiform circuit. The answer is more involved than for the simple case of the mutual inductance. Figure 21.5 shows a closed quasi filiform conductor similar to that sketched in Fig. 21.1. The conductor has cross-section dS and is characterized by a stationary (positive) test current I . The circuit is assumed to lie on a plane belonging to the page on which is drawn. One possible longitudinal axis is γ . This is a closed loop oriented, e.g., clockwise and bordering an open surface Σ_γ on the circuit plane. At first glance, we might think of defining a Neumann integral of type

$$\begin{aligned}
 L = L_y &= -\frac{\Phi_y}{I} = \frac{\mu_0}{4\pi} \oint_Y \left(\oint_Y \frac{\vec{E}}{r} \cdot d\vec{l} \right) \cdot \vec{E} \cdot d\vec{l} \\
 &= \frac{\mu_0}{4\pi} \iint_Y \frac{\vec{E} \cdot \vec{E}}{r} \cdot dl \cdot dl \\
 &= \frac{\mu_0}{4\pi} \iint_Y \frac{1}{r} \cdot dl \cdot dl,
 \end{aligned} \tag{21.21}$$

where \vec{E} is a tangent unit vector to γ and dl an infinitesimal element on γ . It is clear that for each source point Q on γ , the integrand in (21.21) will present a singularity (diverge) every time a field point P also on γ coincides with Q . Hence, the integrand function in (21.21), in general, is not well defined. The problem can be circumvented noting that the given conductor is quasi filiform. As a consequence, we can

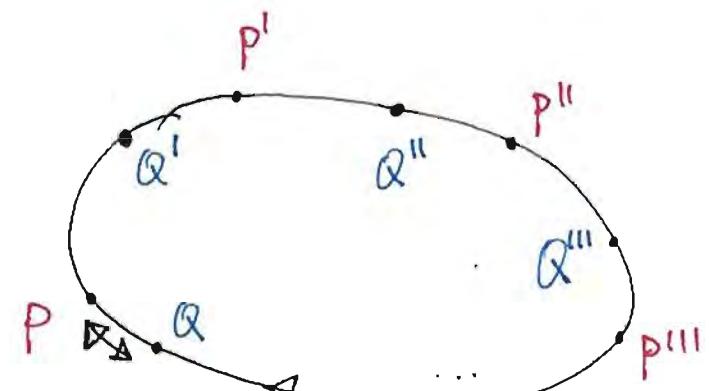
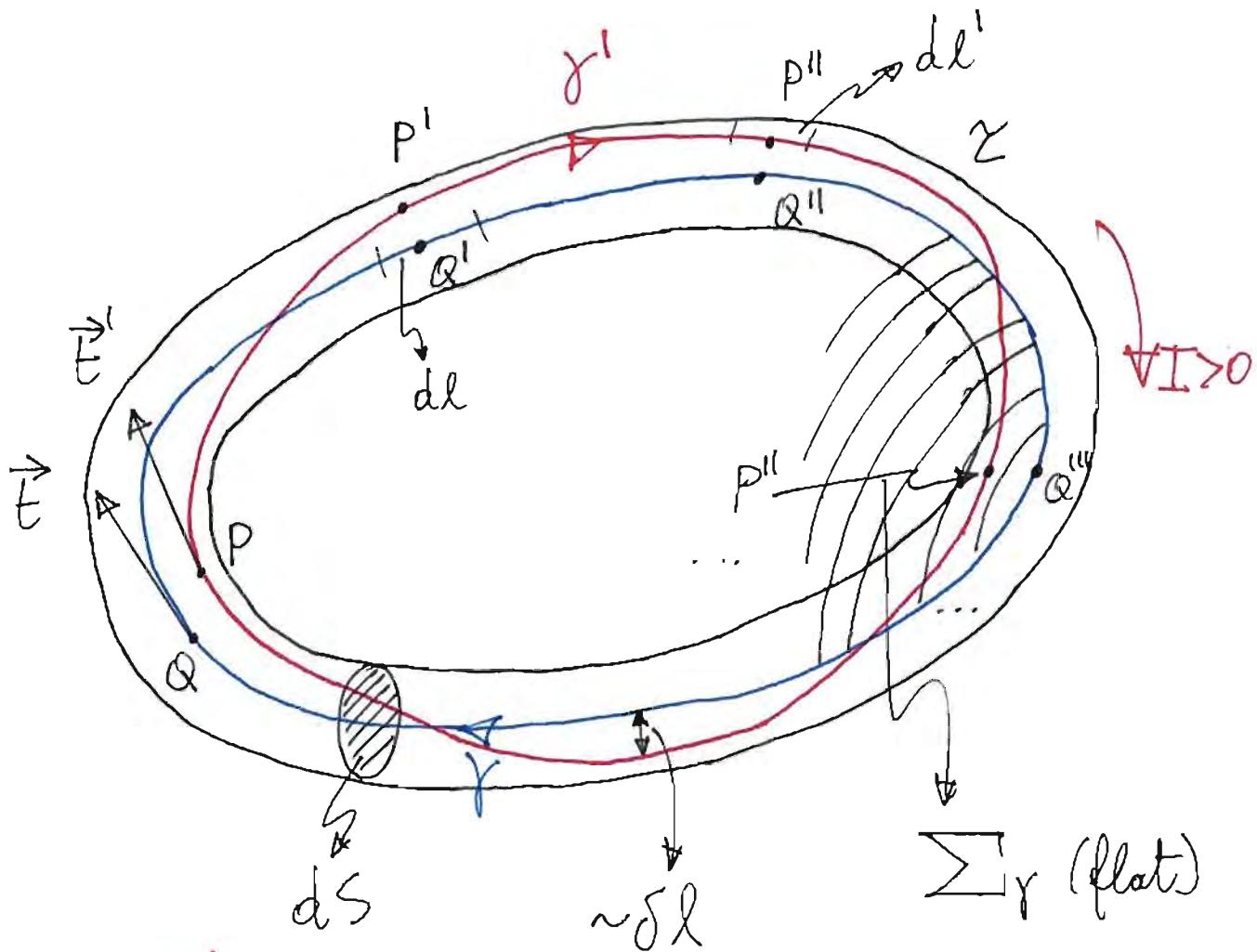
calculate the inner integral in (21.21) on the longitudinal axis γ and the outer integral on a different axis γ' at an average distance δl from γ . We can thus define an improper Neumann-type integral

$$L_\gamma = \lim_{\delta l \rightarrow 0} \frac{\mu_0}{4\pi} \oint_{\gamma'} \left(\oint_{\gamma} \frac{\vec{E}}{r} \cdot d\vec{l} \right) \cdot \vec{E}' \cdot d\vec{l}'$$

$$= \frac{\mu_0}{4\pi} \lim_{\delta l \rightarrow 0} \oint_{\gamma'} \oint_{\gamma} \frac{\vec{E} \cdot \vec{E}'}{r} \cdot d\vec{l} \cdot d\vec{l}', \quad (21.22)$$

where $\vec{E}' \cdot d\vec{l}'$ is an oriented infinitesimal element on γ' , which, consistently with γ , is also a closed loop oriented clockwise (cf. Fig. 21.5). If the limit (21.22) converges, the integral is the Cauchy's value of (21.21). Due to the quasi filiform nature of the conductor, it is clear that any pair of axes of type γ and γ' will give the same result (because all the axes of the conductor are the same up to infinitesimal distances). The reader

can try to calculate the self inductance of a circular loop by means of (21.22). This is not an easy task. 25/56



$$P = Q \Rightarrow \gamma = 0$$

Figure 21.5.

- Extra note on the vector lines of field \vec{B}_1 .

As shown in Fig. 21.3, $\vec{B}_{Q_1}(P_1)$ points into the plane where the circuit γ_1 lies. On the contrary, $\vec{B}_{Q_1}(P_2)$ points away from the same plane. By definition, the vector lines of \vec{B}_{Q_1} for simplicity indicated as \vec{B}_1 in the figure, must be tangent to $\vec{B}_1(P_1)$ and $\vec{B}_1(P_2)$, respectively. Figure 21.6 shows a prospective view of the same circuit. In general, the vector lines of \vec{B}_1 could follow the pattern in Fig. 21.6, while still being tangent to $\vec{B}_1(P_1)$ and $\vec{B}_1(P_2)$ at points P_1 and P_2 .

However, because of Maierza's conjecture, the vector lines of \vec{B}_1 will tend to show the minimum number of inflection points. As a consequence, the actual vector lines of \vec{B}_1 are those shown in Fig. 21.3, which are also confirmed by experiments.

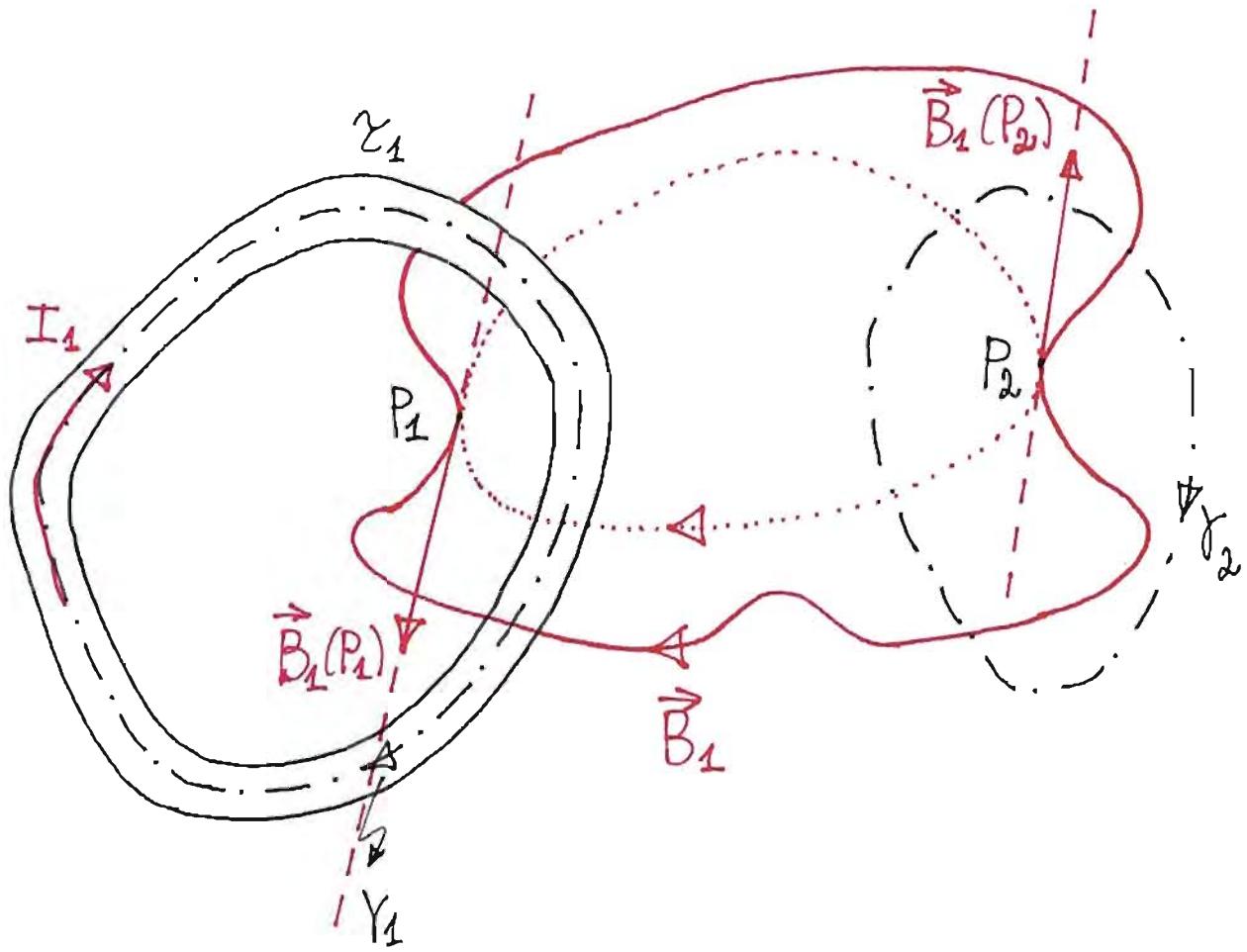


Figure 21, 6.

21.2 Selected examples on self and mutual inductance.

28/56

We will now study two useful circuits and calculate their self and mutual inductance.

21.2.1 Toroidal solenoid.

Consider a conducting hollow toroid with circular cross-section and revolution symmetry axis γ_z , as shown in Fig. 21.7. Given a generic cross-section S of such a conductor, determined by one semiplane originating from γ_z , assume the vector lines of the electric volume current density \vec{J} are all contained within one such meridian semiplane.

The figure also shows a cylindric coordinate system $\gamma_r \gamma_\theta \gamma_z$, with center O in the middle point of the torus, and γ_z axis coinciding with γ_z . The central longitudinal axis of the torus, i.e., the longitudinal axis passing through the center of each cross-section, is named γ . The distance of γ to the vertical axis γ_z is the radius r of a circle with center O . By varying r is possible to select slightly different longitudinal axes.

Assuming the cross-section S has a small thickness compare to the maximum lenght $2\pi r$ of the torus, we

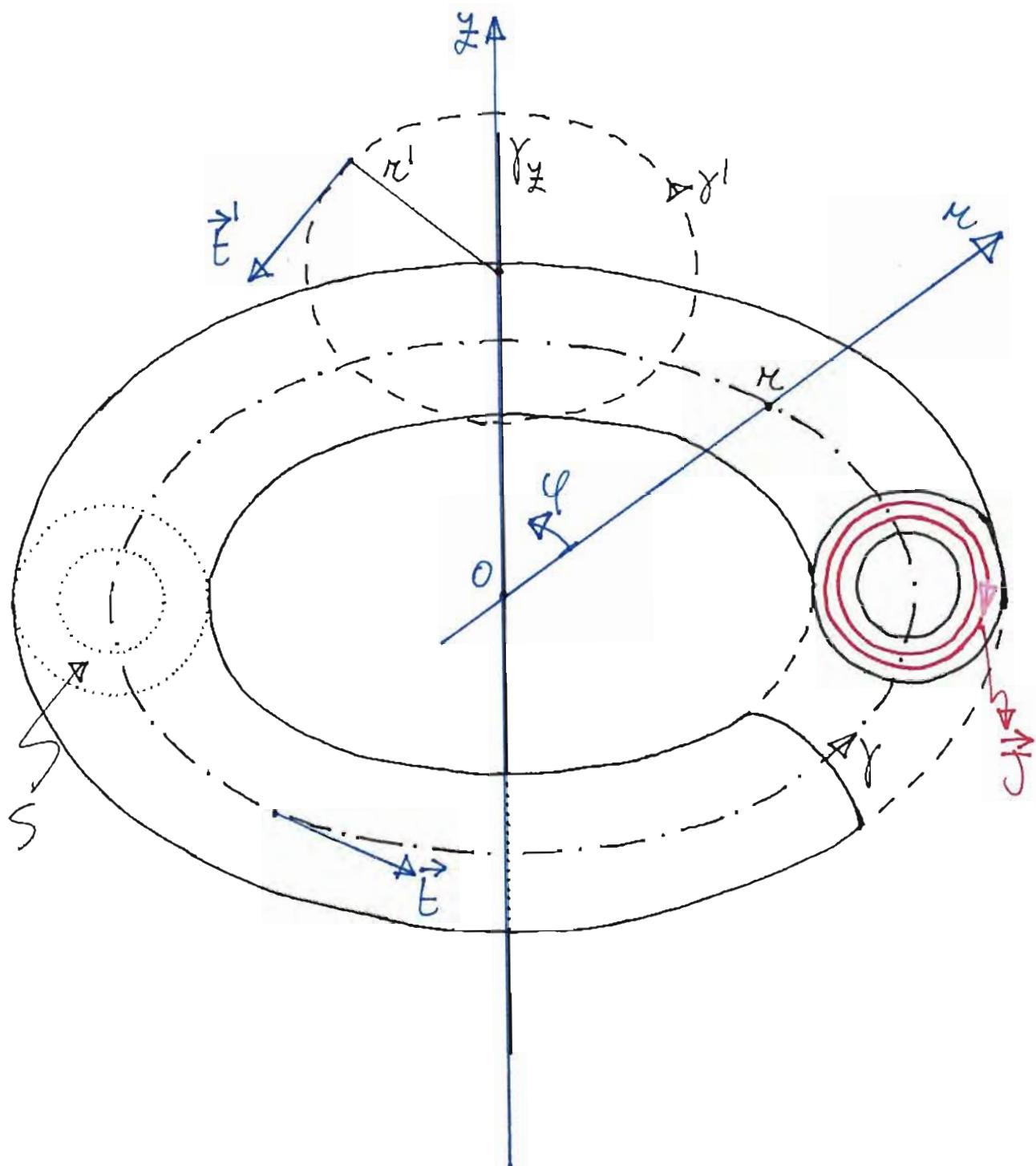


Figure 21.7.

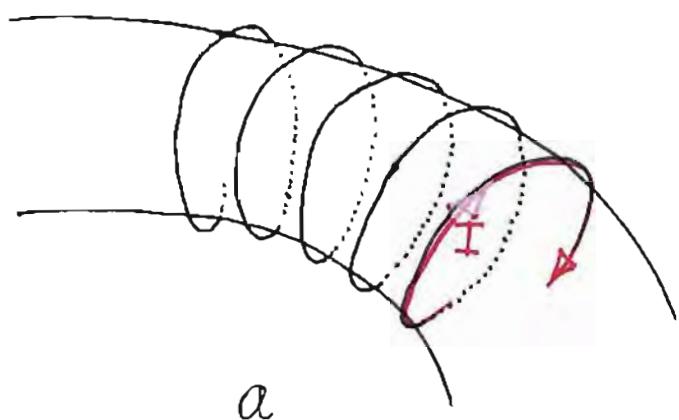
can assume the system under consideration to be made by a single filiform conductor wound around a medium (e.g., plastic) with shape of the given torus. Figure 21.8a

shows an example of such a construction for a portion of the torus. Figures 21.8b and 21.8c show a completely generic cross-section and a simple (and common) rectangular cross-section for the torus. As it turns out, the distribution of \vec{B} inside the torus does not depend on the specific torus' cross-section (see below). In order to better approximate the original (solid) conducting toroid, the total number of loops of the filiform conductor should be as large as possible. In addition, the loop distribution should be as uniform as possible so that each loop is approximately entirely confined within one of the aforementioned meridian semiplanes. Such a system is called toroidal solenoid. Most inductors are realized as toroidal solenoid. It is clear that the filiform wire must be eventually connected to an external emf source in order to force a current I into the coil.

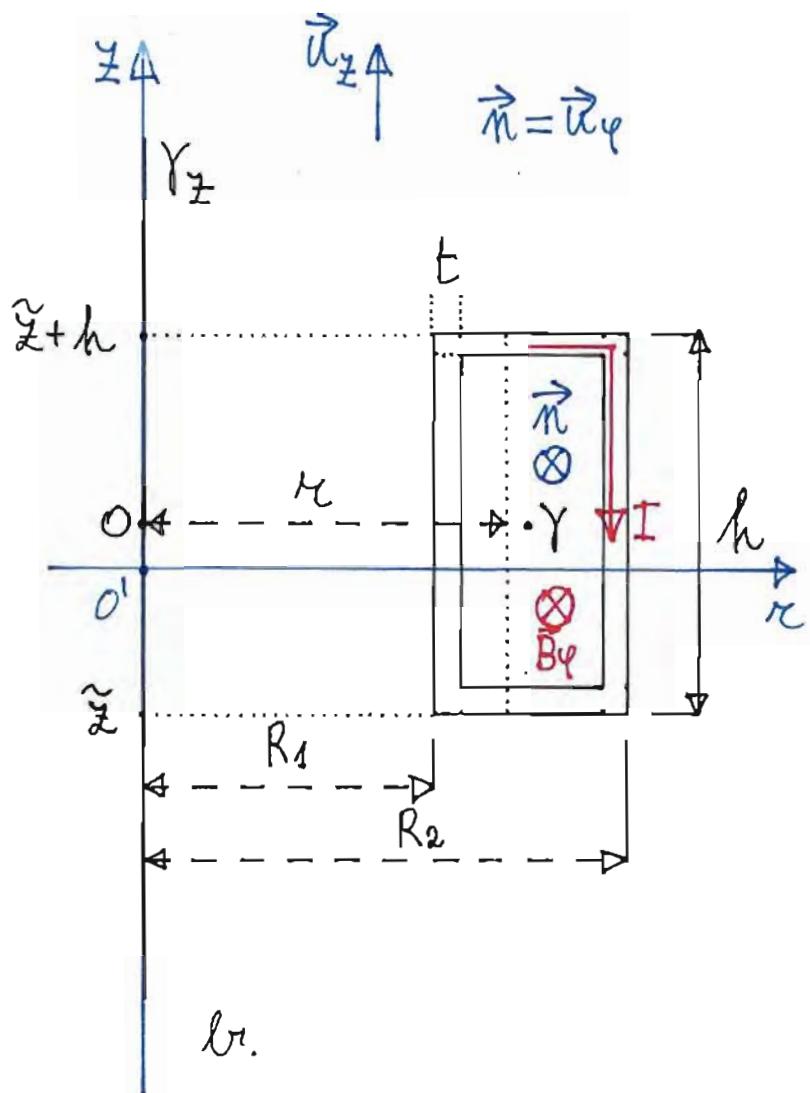
The field \vec{B} generated by the toroidal solenoid at any point in space can be found by means of symmetry arguments and Ampère's law.

A toroidal solenoid with current I is characterized by two symmetries:

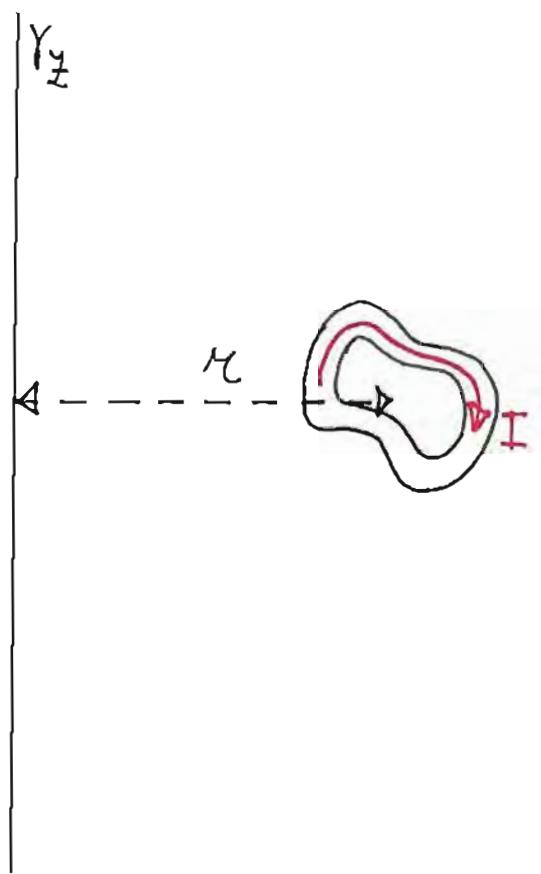
(a) Rotation symmetry with respect to the revolution



a.



b.



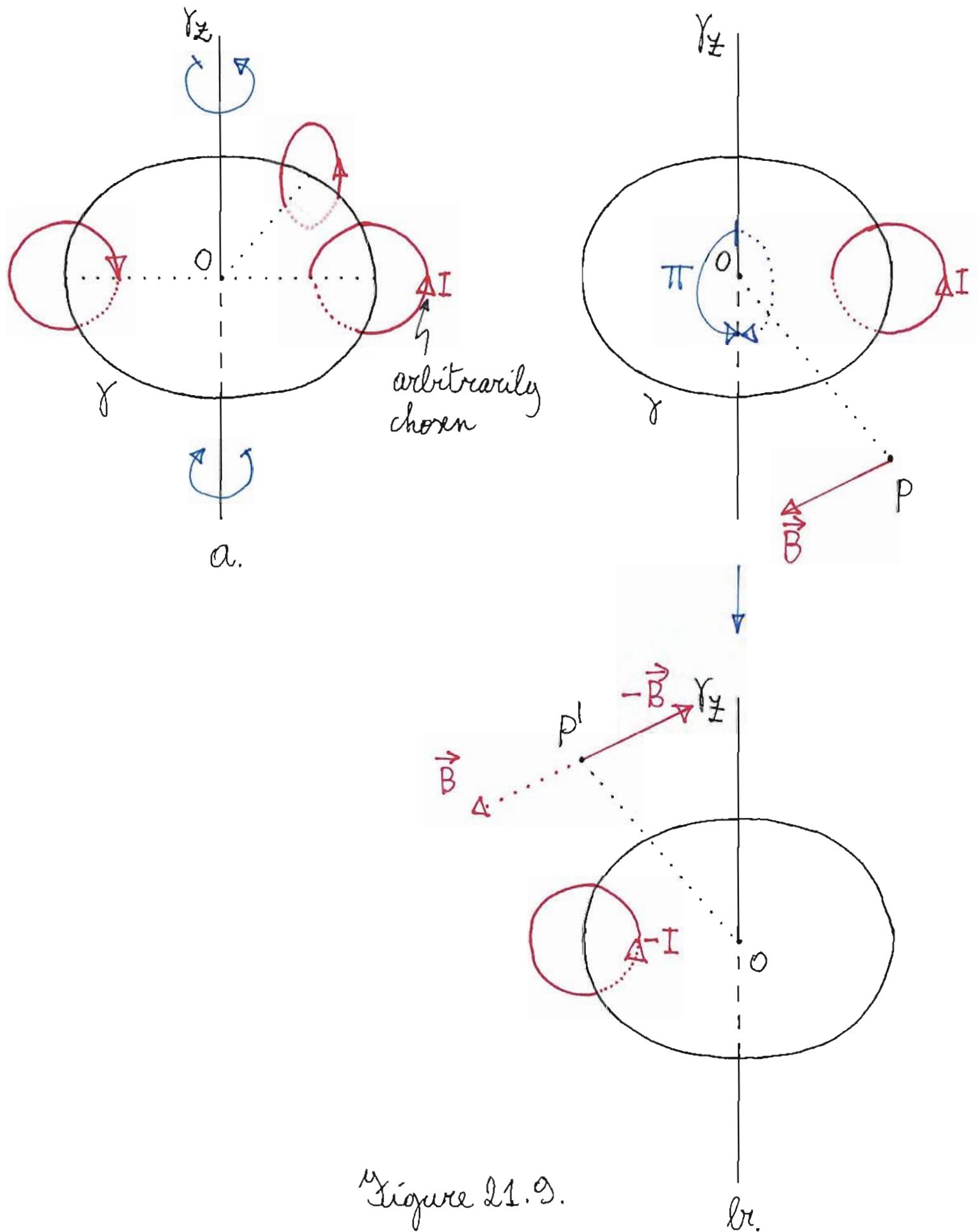
c.

Figure 21.8.

axis \hat{Y}_2 . The solenoid and current direction remain the same upon rotating the system by any angle clockwise or counterclockwise about \hat{Y}_2 . Figure 21.9a shows this type of symmetry. In the figure, the solenoid is represented by its central longitudinal axis \hat{Y} (for simplicity). Three loops with the same current I flowing consistently on them are also shown.

(b) Anti-reflection symmetry with respect to the center O . The top panel in Fig. 21.9b shows the usual solenoid \hat{Y} and one loop with current I flowing counterclockwise. In addition, the figure shows a generic field \vec{B} at any point in space (inside or outside the solenoid).

By considering the entire system as a rigid body (\hat{Y}, I, \vec{B} , and P), a reflection by an angle π clockwise or counterclockwise about O leads to a new system that differs from the initial one for the orientation of I . Hence, for consistency the direction of \vec{B} at P^1 must be inverted. Under these conditions the symmetry can still be used.



We can now study the three components of field \vec{B} at any point P in space. With respect to the $OxOyOz$ coordinate system indicated in Fig. 21.4, the three components are B_x , B_y , and B_z . 34/56

1) Radial component B_r .

Figure 21.10a illustrates the rotation symmetry argument. As always, the toroidal solenoid is represented by its longitudinal axis γ . For simplicity, no current loop is shown and point P is chosen to be on γ (i.e., inside the solenoid). The vertical axis γ_z and center O of the solenoid are also shown.

Because of the rotation symmetry, given B_r at P , B_r must be the same (i.e., radially directed -inward or outward- and with constant magnitude) at each point P on γ , or any other circle centered in O .

Figure 21.10b illustrates the anti-reflection symmetry argument.

In step 1, a pair $\{\vec{B}_x, P\}$ is rotated, e.g., by an angle π counterclockwise about O . Because of the anti-reflection symmetry, the sign of \vec{B}_x in the new pair at P' must be changed, obtaining $\{-\vec{B}_x, P'\}$.

In step 2, a rotation by an angle π clockwise about

γ_2 leads to the final pair $\{\vec{B}_r, P\}$, which is 35/56 obviously inconsistent with the original pair $\{\vec{B}_r, P\}$, as clearly shown by the figure. As a consequence, $B_r = 0$ at each point on any line γ with generic radius r . Therefore, $\vec{B}_r = \vec{0}$ at each point in space.

2) Tangent component \vec{B}_φ .

Figure 21.11a illustrates again the rotation symmetry argument. It is evident that, if a component \vec{B}_φ exists, it must be the same (i.e., tangentially directed and with constant magnitude) at each point P on γ .

Figure 21.11 b illustrates the anti-reflection symmetry argument. As before, step 1 brings a pair $\{\vec{B}_\varphi, P\}$ to a new pair $\{-\vec{B}_\varphi, P'\}$ (anti-reflection). Step 2 brings the pair $\{-\vec{B}_\varphi, P'\}$ to $\{\vec{B}_\varphi, P\}$ (rotation), which is consistent with the initial pair at P (cf. figure). As a consequence, a nonzero component \vec{B}_φ can exist and, if it does, it must be the same (i.e., tangentially directed and with constant magnitude) at each point on any line γ .

3) Vertical component \vec{B}_z .

Figures 21.12a and 21.12b illustrate the usual rotation and anti-reflection symmetry arguments. 36/56

Because of the rotation symmetry, if \vec{B}_z exists, it must be the same (i.e., vertically directed and with constant magnitude) at each point on γ . It is clear that, in general, \vec{B}_z components on lines γ with different radii will be different in magnitude.

Because of the anti-reflection symmetry, in step 1, a pair $\{\vec{B}_z, P\}$ goes to $\{-\vec{B}_z, P'\}$ and then, in step 2, the pair $\{-\vec{B}_z, P'\}$ goes to $\{\vec{B}_z, P\}$, which is consistent with the initial pair (cf. figure). As a consequence, from simple symmetry arguments, a nonzero component \vec{B}_z can exist.

However, experimental evidence shows that $\vec{B}_z = \vec{0}$ at each point in space. While the symmetry arguments are insufficient to prove this fact, it would be possible to show that $\vec{B}_z = \vec{0}$ everywhere by resorting to Laplace's theorem. Due to the complexity of the toroidal solenoid, this is a nontrivial task and, thus we will not show it in these notes.

Note that the absence of translation symmetry implies that equal components (i.e., same direction and constant

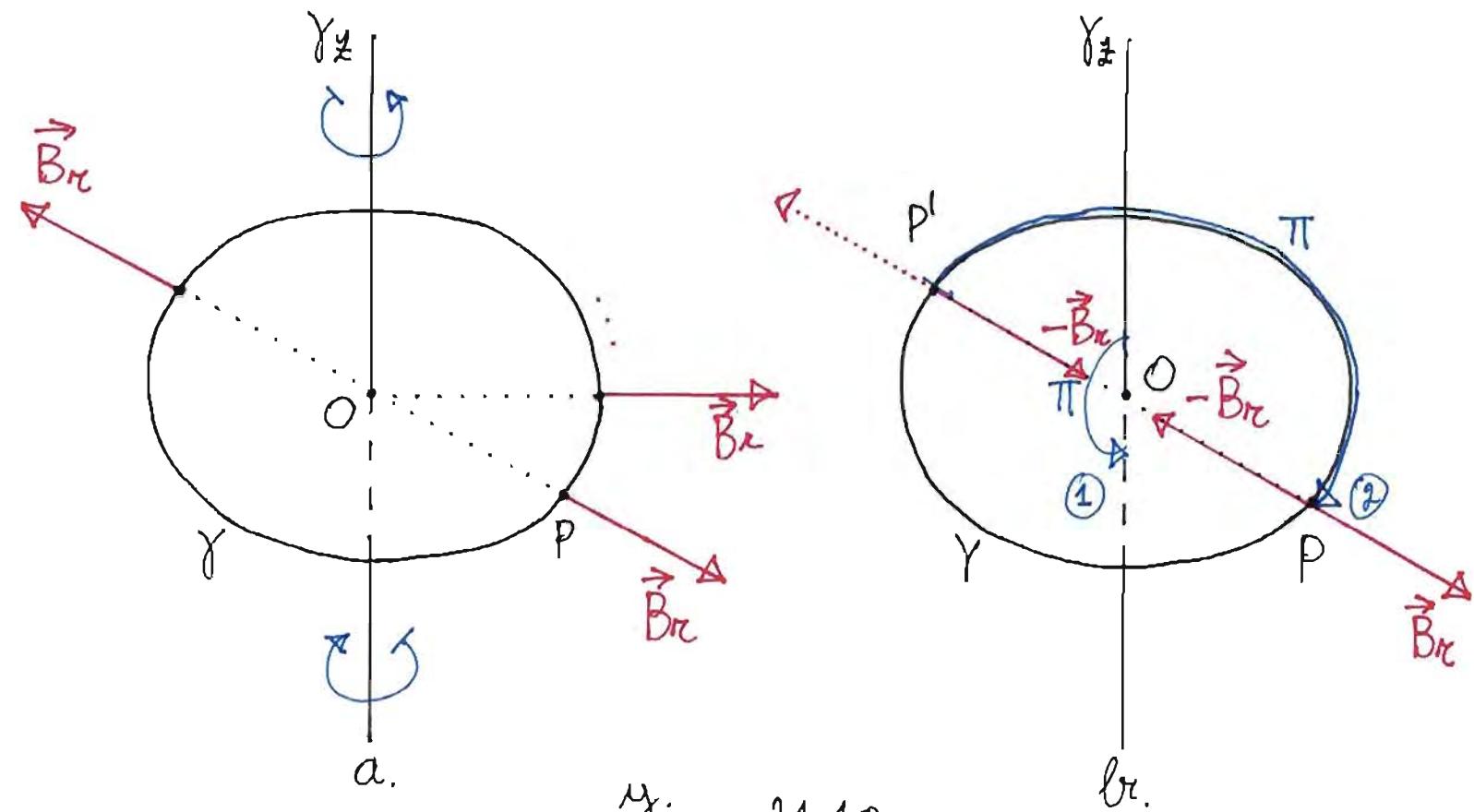


Figure 21.10.

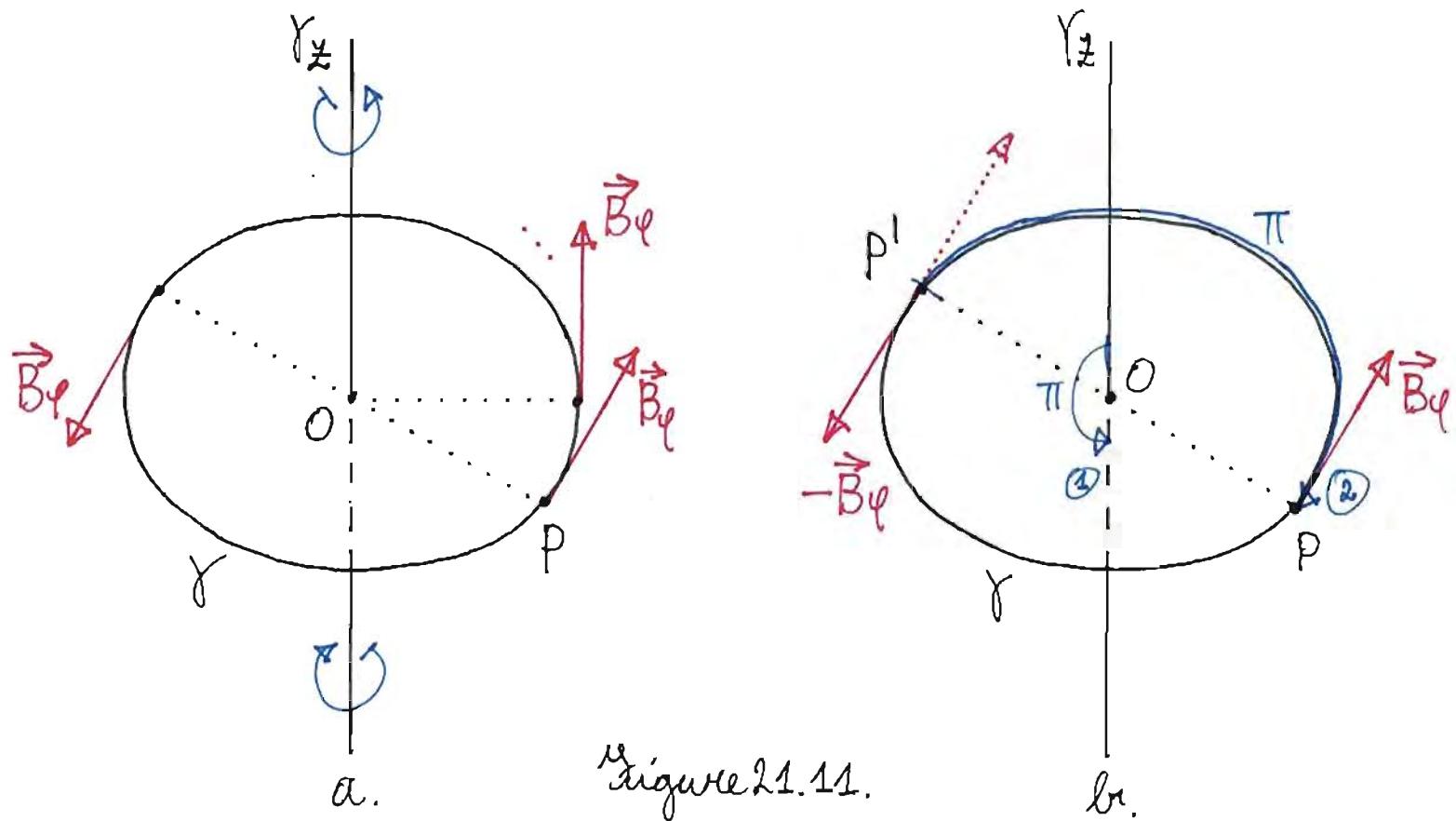


Figure 21.11.

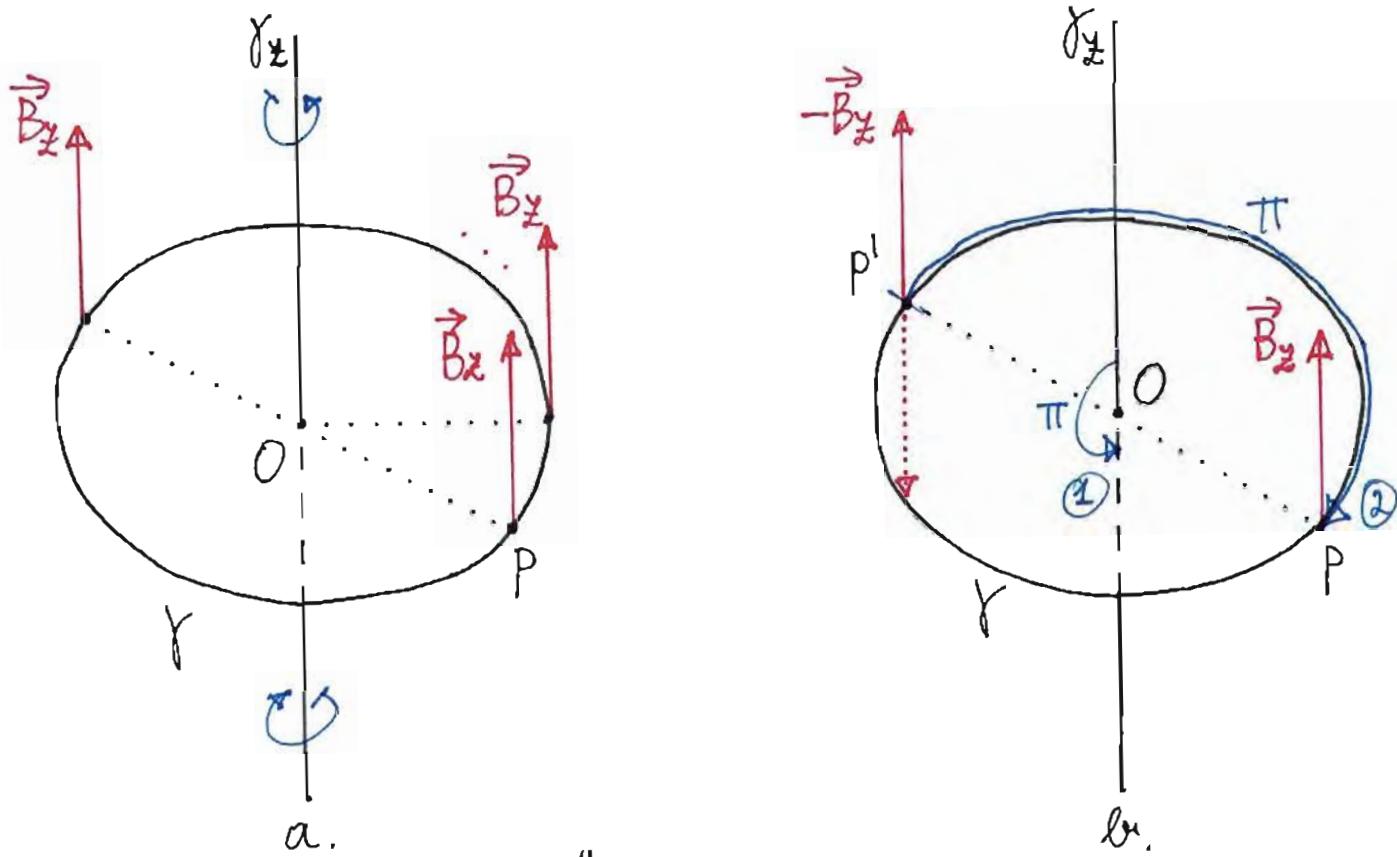


Figure 21.12.

magnitude) on a line γ are different from the corresponding component on a line γ' with center in O , radius r (equal to the radius of γ), but on a plane parallel to the one where γ lies on. In other words, we cannot state that a component can be the same at each point on the lateral surface of a cylinder with axis Y_z .

It is also clear that the magnitude of a given component is, in general, different on circles with different radii.

Finally, because of the homogeneity and isotropy of the

3D Euclidian space (excluding the region where the loops of the solenoid are defined; we remind that the conductor forming the solenoid is assumed to be filiform and, thus, such a region is negligibly small) if one component of \vec{B} has a specific direction at a point P , it will maintain that direction in the neighborhood of $P, P + dP$ (cf. lecture 4, Fig. 4.3d).

39/56

We can now calculate the component \vec{B}_ϕ of the field \vec{B} generated by the toroidal solenoid, at any point P in space, by resorting to Ampère's law,

$$\oint \vec{B} \cdot \vec{dl} = \mu_0 I_e \quad , \quad (21.23)$$

γ_a

where γ_a is a closed oriented line in the region where \vec{B} is defined and I_e the current (stationary) linked with γ_a .

- Region outside the toroidal solenoid.

In this case, $\gamma_a = \gamma'$, where γ' is a closed line lying on a plane parallel to the plane on which lies γ (cf. Fig. 21.4), with center O' on the Y_Z axis, generic radius

r' , and oriented counterclockwise. The line γ' lies in a region of space entirely outside the toroidal solenoid. 40/56

Because of the very construction of the solenoid, it is always possible to find a transformation (involving translations, rotations, and positive or negative contractions) that reduces γ' to a single point, without ever "cutting" the solenoid. Such a transformation is called a zero homotopy. A direct consequence of the zero homotopy is that the current linked with γ' is always zero, $I_e = 0$.

By defining \vec{t}' the unit tangent vector to γ' and dl' an infinitesimal element on γ' (cf. Fig. 21.7), we obtain

$$\begin{aligned} \oint_{\gamma'} \vec{B} \cdot \vec{t}' \cdot dl' &= \oint_{\gamma'} \vec{B}_q \cdot \vec{u}_q \cdot dl' \\ &= \oint_{\gamma'} B_q \vec{u}_q \cdot \vec{u}_q \cdot r' \cdot dr \end{aligned}$$

$$\boxed{B_\varphi r' \int_0^{2\pi} d\varphi}$$

$$= 2\pi r' B_\varphi = 0 \quad , \quad (21.24)$$

where we used the fact that $\vec{r}' = \vec{r}_\varphi$ with respect to the $\vec{r}\varphi\vec{z}$ coordinate system indicated in Fig. 21.7 and $d\varphi$ is an infinitesimal angle. Assuming $r' \neq 0$ (or, at most, a very small, yet nonzero, quantity), Eq. (21.24) is fulfilled iff $B_\varphi = 0$. The only region where B_φ must not necessarily be zero for (21.24) to be fulfilled is on the \vec{y}_z axis, where $r' = 0$. In this case, (21.24) is valid even if $B_\varphi \neq 0$. However, the absence of any source current and of any conducting or insulating material in correspondence of \vec{y}_z implies that if $B_\varphi = 0$ in the neighborhood of \vec{y}_z (i.e. in a cylindrical region with vertical central axis \vec{y}_z and infinitesimal radius), it will be zero also at each point on \vec{y}_z . Hence, $\vec{B}_\varphi = \vec{0}$ at each point outside the solenoid.

- Region inside the toroidal solenoid.

In this case, $\vec{r}_a = \vec{r}$, where the radius r can vary so

long γ remains confined within the inner region 42/56
 of the solenoid, $\vec{E} = \vec{u}_\varphi$ (as before), $dl = r \cdot d\varphi$, and

$$I_e = NI \quad , \quad (21.25)$$

where I is the current flowing in one loop of the solenoid.
 Since I flows clockwise, it is consistent with \vec{u}_z when crossing the
 surface with border γ on the plane of γ . Hence, $I_e > 0$ and

$$\begin{aligned} \oint_{\gamma} \vec{B}_\varphi \cdot \vec{u}_\varphi \cdot r \, d\varphi &= r \int_0^{2\pi} \vec{B}_\varphi \cdot \vec{u}_\varphi \cdot \vec{u}_\varphi \cdot d\varphi \\ &= r B_\varphi \int_0^{2\pi} d\varphi \\ &= 2\pi r B_\varphi = \mu_0 N I . \quad (21.26) \end{aligned}$$

From (21.26) we finally obtain

$$B_\varphi = \frac{\mu_0}{2\pi} \frac{NI}{r} \quad , \quad (21.27)$$

where r is always different from zero.

Note that, inside the toroidal solenoid, the field

does not depend on the specific conductor cross-section, which, e.g., can be of the type shown in Figs. 21.8b or 21.8c. The field depends only on the total current NI and the distance r from γ_z . Furthermore, the field is identical to that of a single infinite straight conductor, directed along γ_z and with current intensity NI .

In summary,

$$\left\{ \begin{array}{l} \vec{B} = \vec{0} \quad , \text{ outside solenoid (21.28a)} \\ \vec{B} = \frac{\mu_0}{2\pi} \frac{NI}{r} \cdot \hat{t}_\varphi \quad . \text{ inside solenoid (21.28b)} \end{array} \right.$$

- Self inductance of a toroidal solenoid.

Consider a toroidal solenoid with a rectangular cross-section, as shown in Fig. 21.8 b. The internal radius of the solenoid is R_1 and the external radius R_2 . The solenoid comprises a total number N of loops (or windings), which are uniformly distributed along its longitudinal central axis γ . The lateral dimensions of one winding are $(R_2 - R_1) h$. The distance between a line parallel to h (the black

dashed line inside the rectangle in Fig. 21.8 b.; note that we here assume the thickness t of the rectangle to be zero) and the vertical central axis γ_z is r .

In order to calculate the self inductance of such a solenoid, we need to calculate the flux linked with the closed line (in reality it must be open at one point for connection to a suitable source) that represents the entire solenoid coil. By definition, we would then need to calculate the surface integral of the field \vec{B} generated by the coil through a generic surface having the coil as a border. This calculation is rather complicated due to the complex geometric shape (helicoidal) of the coil itself. However, we can estimate the total flux Φ linked with the entire coil as the sum of the partial fluxes linked with each single loop, Φ_e , where the loops are considered to be closed lines (in our case, rectangular). This is a good approximation that, as it turns out from comparison with experiments, leads to a negligible error. In addition, there is a bijective correspondence between each point on two different surfaces bordered by a loop and the field \vec{B} is equal at corresponding points due to (21.8 b). As a consequence, all partial fluxes Φ_e are equal to each other.

and, thus,

45/56

$$\Phi = N \Phi_e \quad . \quad (21.29)$$

The partial flux Φ_e can readily be calculated from (21.28 b) for the case of Fig. 21.8 b. We obtain

$$\Phi_e = \iint_{\text{rect.}} \vec{B} \cdot \vec{n} \cdot dS = \iint_{\text{rect.}} B_\varphi \vec{u}_\varphi \cdot \vec{u}_\varphi \cdot dS$$

$$= \iint_{\text{rect.}} B_\varphi \cdot dr \cdot dz$$

$$= \int_{R_1}^{R_2} \int_{\tilde{z}}^{\tilde{z}+h} \frac{M_0}{2\pi} \frac{NI}{r} \cdot dr \cdot dz$$

$$= \frac{M_0}{2\pi} NI \left[\ln r \right]_{R_1}^{R_2} \left[z \right]_{\tilde{z}}^{\tilde{z}+h}$$

$$\boxed{= \frac{\mu_0}{2\pi} NI h \ln \frac{R_2}{R_1}} \quad . \quad (21.30)$$

In this integral, "rect." refers to one rectangular loop of the solenoid, \vec{n} is the normal unit vector to the surface defined by the loop (cf. Fig. 21.8b), I is positive test current oriented clockwise consistently with the clockwise orientation of the loop. Hence, with respect to the $O'xyz$ cylindric coordinate system indicated in the figure (note that we arbitrarily chose O' , which can coincide with the center O of the toroid), $\vec{n} = \hat{u}_\varphi$ pointing into the page. This means the flux given by (21.30) is positive, $\Phi_e > 0$. Finally, we have chosen the vertical coordinate of the bottom edge of the rectangle to be an arbitrary value \tilde{z} . Also note that, from (21.28b) $\vec{B} = \vec{B}_\varphi$ points into the page (hence, $\Phi_e > 0$).

By definition, the self inductance of the entire solenoid is given by

$$L = \frac{\Phi}{I} = \frac{N \Phi_e}{I} = \frac{\mu_0}{2\pi} N^2 h \ln \frac{R_2}{R_1} , \quad (21.31)$$

where we used both (21.29) and (21.30). As expected, the dimension of L are those of a flux over a current, giving

the units of henry,

$$[L] = \frac{H}{m} m = H .$$

- Mutual inductance between a straight infinite line and a toroidal solenoid.

Consider a system of two circuits, comprising a toroidal solenoid with central vertical axis y_2 and rectangular loops and a straight infinite filiform conductor on the y_2 axis (cf. Fig. 21.13). A positive test current I_1 flows in the straight infinite line, pointing upward. Since we intend to calculate the mutual inductance between the two circuits and since $M_{12} = M_{21}$ (where the subscripts refer to the straight line, circuit 1, and the solenoid, circuit 2), we can either calculate M_{12} or M_{21} . We decide to calculate M_{21} . We remind that the mutual inductance can be positive or negative, depending on the orientation of the circuits. As an example, we choose the orientation on each rectangular loop of the solenoid to be counterclockwise and switch off any current on the solenoid, $I_2 = 0$ (we want to calculate M_{21}). With these conventions, the field \vec{B}_1 generated by the infinite straight line is tangentially directed on each

circle with axis \hat{y}_z , points counterclockwise, and is characterized by a constant magnitude at each point of the lateral surface of a cylinder coaxial with \hat{y}_z (cf. Fig. 21.13). The figure also shows a cylindrical coordinate system $Oxyz$, which is consistent with the chosen direction on circuit 1. We remind the reader to lecture 19 for more details on the field \vec{B} generated by an infinite straight line. Note that, by default we are under stationary conditions. The figure shows also two rectangular cross-sections of the solenoid, each of which is characterized by an internal radius R_1 and an external radius R_2 and a height h . The surface defined by each rectangular loop has area $(R_2 - R_1)h$ and its border line is oriented counterclockwise, so that its normal unit vector $\vec{n}_2 = -\hat{u}_y$.

Hence, the flux due to \vec{B}_1 through one rectangular loop is given by

$$\begin{aligned} \Phi_{e21} &= \iint_{\text{rect.2}} \vec{B}_1 \cdot \vec{n}_2 \cdot dS = \iint_{\text{rect.2}} B_{y1} \cdot \hat{u}_y \cdot (-\hat{u}_y) \cdot dS \\ &= -\frac{\mu_0}{2\pi} I_1 h \int_{R_1}^{R_2} \frac{1}{r} \cdot dr \end{aligned}$$

$$\frac{1}{\mu_0} = -\frac{\mu_0}{2\pi} I_1 h \ln \frac{R_2}{R_1}, \quad (21.32)$$

where we used the result of lecture 19 for \vec{B}_1 ,

$$\vec{B}_1 = \vec{B}_{41} = B_{41} \hat{u}_4 = \frac{\mu_0}{2\pi} \frac{I_1}{r}.$$

An argument similar to that used to calculate the self inductance of a toroidal solenoid makes it possible to find the total flux due to \vec{B}_1 and linked with the solenoid,

$$\begin{aligned} \Phi_{21} &= N \Phi_{121} \\ &= -\frac{\mu_0}{2\pi} NI_1 h \ln \frac{R_2}{R_1}, \quad (21.33) \end{aligned}$$

where N is the total number of loops in the solenoid. From the definition of mutual inductance we finally obtain

$$M_{21} = \frac{\Phi_{21}}{I_1} = -\frac{\mu_0}{2\pi} Nh \ln \frac{R_2}{R_1}. \quad (21.34)$$

The coefficient is in this case negative and its units are

The same result could be obtained by means of a Neumann integral, but it would be much more complicated.

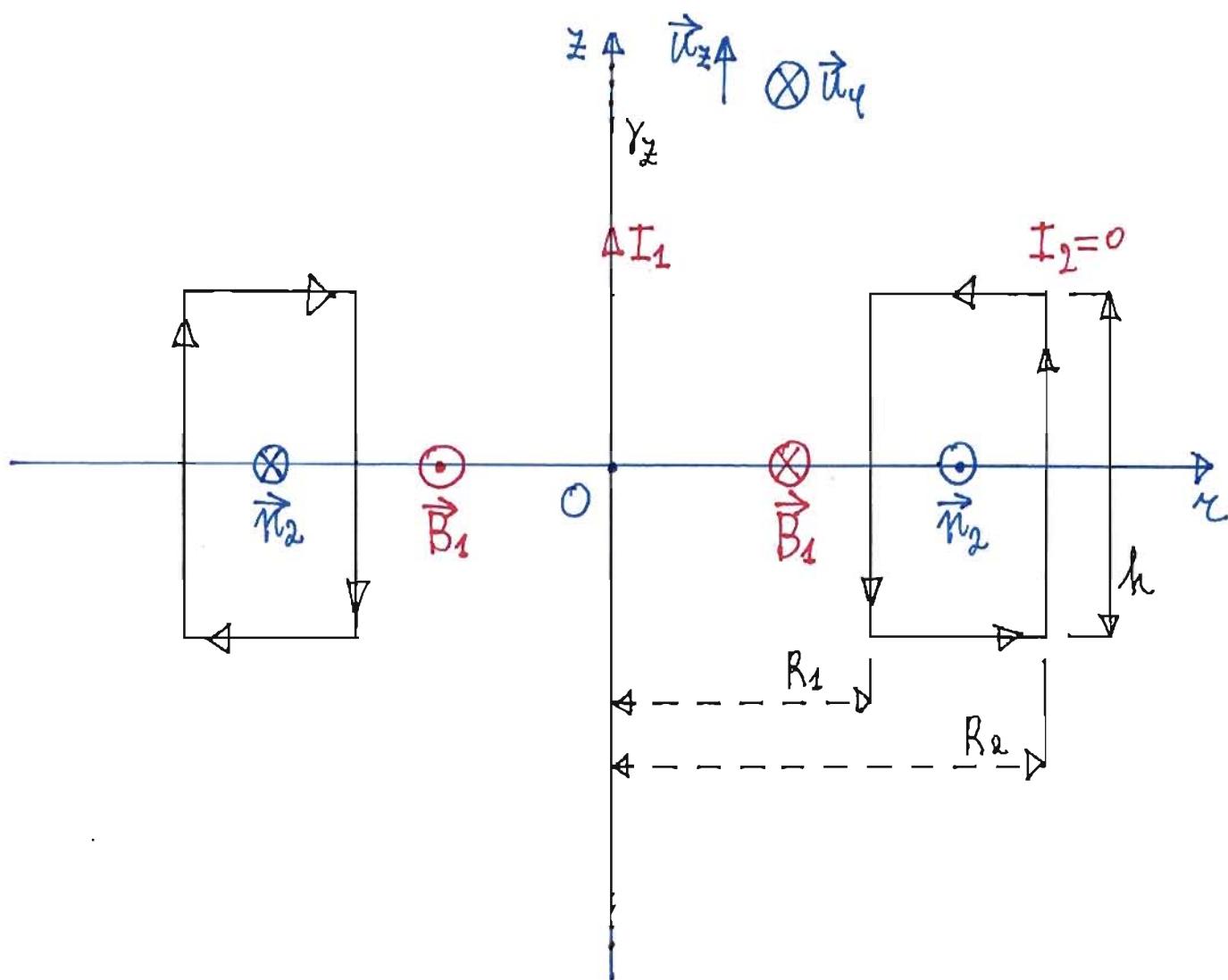


Figure 21.13.

21.2.2 Infinite straight solenoid.

A particularly interesting limiting case of a toroidal solenoid is when the radius r of the central longitudinal axis γ of the toroidal solenoid of Fig. 21.7 tends to infinite. In this case, the toroidal solenoid transforms into an infinite straight solenoid.

As always, when dealing with indefinite (infinite) structures, it is opportune to resort to "densities." Therefore given the total number of loops N of the original (finite) solenoid, we can define the loop density as the number of loops in the coil per unit length,

$$n = \frac{N}{l} = \frac{N}{2\pi r} \quad . \quad (21.35)$$

The magnitude of the field \vec{B} inside the solenoid, which is tangent to any line of γ type, is thus given by [21.8b]

$$\begin{aligned} B &= \frac{\mu_0}{2\pi} N \frac{I}{r} \\ &= \frac{\mu_0}{2\pi} n l \frac{I}{r} = \mu_0 n I \frac{2\pi r}{2\pi r} \quad , \quad (21.36) \end{aligned}$$

where we used (21.35) and, as always, I is positive test

current flowing in one loop of the solenoid.

For an infinite straight solenoid it must then be

$$B = \lim_{r \rightarrow +\infty} \mu_0 n I \frac{\frac{2\pi r}{2\pi r}}{\frac{2\pi r}{2\pi r}} = \mu_0 n I \quad . \quad (21.34)$$

In this case, the field \vec{B} is uniform inside the straight solenoid (i.e., it does not depend on r), directed along the solenoid central longitudinal axis, and with sign dictated by the right-hand rule (the field \vec{B} is the thumb and the current I the rest of the hand). As in the case of the toroidal solenoid, the field \vec{B} is zero outside the solenoid.

Under the sole assumption that the vector lines of \vec{B} are parallel to the longitudinal axis of the solenoid, it is possible to reach the result (21.34) by means of Ampère's theorem. The line $\tilde{\gamma}$ used to calculate the circulation of \vec{B} is shown in Fig. 21.14. The figure shows a lateral cross-section through the central longitudinal axis of the solenoid. Only a portion of the infinite (straight) solenoid is sketched. The dots and crosses indicate the flow direction of the current I in the solenoid. The line $\tilde{\gamma}$ comprises four edges, AB, BC, CD, and DA and is oriented counterclockwise.

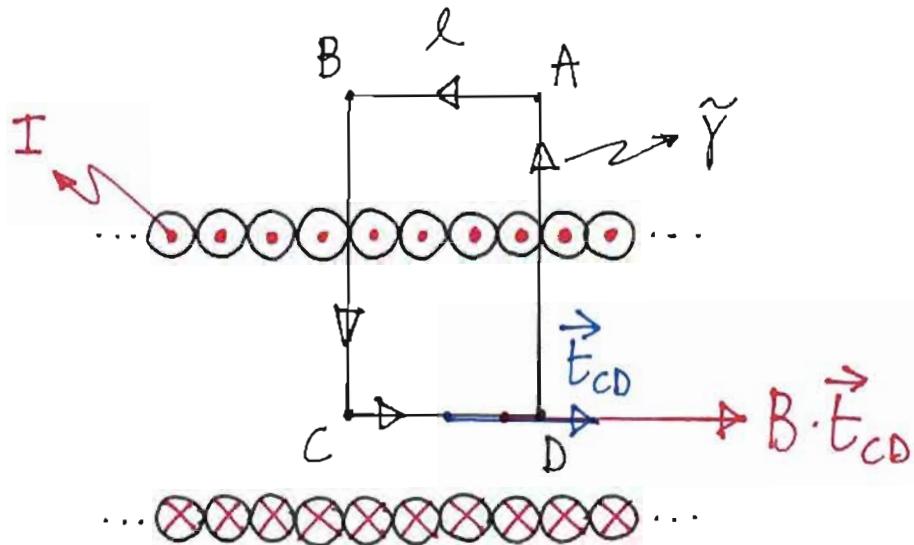
wise. The edges, which are all straight line segments, are positioned so that AB resides outside the solenoid, CD inside, and BC and DA are normal to the solenoid longitudinal axis. Because of the orientation of $\tilde{\gamma}$, the normal unit vector to the surface defined by $\tilde{\gamma}$ points away from the plane on which $\tilde{\gamma}$ lies (right-hand rule). Hence, the current linked with $\tilde{\gamma}$ has a positive sign and is equal to NI , where N is total number of loops of the solenoid linked with $\tilde{\gamma}$. There is zero field outside the solenoid, no field component along BC and DA; assuming $\vec{B} = B \cdot \vec{E}_{CD}$ inside the solenoid along CD, where \vec{E}_{CD} is the unit vector tangent to CD, and assuming $\overline{CD} = l$, the circulation of \vec{B} along $\tilde{\gamma}$ reads

$$\oint_{\tilde{\gamma}} \vec{B} \cdot \vec{dl} = \int_{CD} B \cdot \vec{E}_{CD} \cdot \vec{E}_{CD} \cdot dl$$

$$= B \cdot \overline{CD} = B \cdot l = \mu_0 NI . \quad (21.38)$$

From (21.38) we finally obtain again

$$B = \mu_0 \frac{N}{l} I = \mu_0 N I$$



$$\overline{AB} = \overline{CD}$$

Figure 21.14.

Self inductance of a "long" solenoid.

Consider a straight solenoid the length of which is much longer than its transversal dimension, $l \gg \sqrt{S}$, where S is the area of one loop of the solenoid. If the loop density is big enough and the windings are uniform enough, the field B inside the solenoid can be approximated by (21.34), with direction along the

solenoid longitudinal axis, and sign given by the right-hand rule [depending on the current orientation; in (21.34), we choose $I > 0$].

Proceeding as for the toroidal solenoid, due to the uniformity of \vec{B} inside the solenoid, the partial fluxes through each loop, Φ_e , are all equal and, thus, the total flux through the entire solenoid can be approximated by

$$\Phi = N \Phi_e , \quad (21.39)$$

where N is the total number of loops in the solenoid and

$$\Phi_e = BS = \mu_0 n IS = \mu_0 \frac{N}{l} IS . \quad (21.40)$$

Therefore,

$$\Phi = \mu_0 \frac{N^2}{l} IS = \mu_0 \left(\frac{N}{l} \right)^2 l S = \mu_0 n^2 l S . \quad (21.41)$$

From the definition of self inductance

$$L = \frac{\Phi}{I} = \mu_0 \frac{N^2}{l} S = \mu_0 n^2 l S , \quad (21.42)$$

which has units

$$[L] = \text{H m}^{-1} \text{m}^1 \text{m}^2 = \text{H} .$$

Note that from (21.42) it is also possible to calculate the inductance per unit length as

$$\tilde{L} = \frac{L}{l} = \mu_0 n^2 S \quad , \quad (21.43)$$

which is very useful when the solenoid length is unknown.

Finally note that any long solenoid must be connected to an emf source by means of two leads. In typical applications the length of the leads is negligible compared to that of the solenoid itself. Hence, the self inductance given by (21.42) remains a good approximation.