PHYS 342 - EM 02

Lecture 10

In the next couple of lectures, we will focus on the study of the forces acting on electrically charged conductors and on circuits carrying electrical current.

10.1 Forces on Electrically Charged Conductors

The knowledge of the electrostatic energy of a system of conductors given by Eq. (7.7) makes it possible to calculate the forces acting on the conductors. This task can easily be achieved by using the principle of conservation of energy.

The first case we consider is when the system of conductors is an isolated system. This means the total charge of the system is constant in time. Under the further assumption that each conductor in the system is spatially isolated, i.e., all conductors do not touch each other, we conclude that the charge on each conductor is also a constant in time.

Among all conductors in the system, suppose that one of them is free to translate by a distance $d\vec{r}$ under the action of the electric forces due to the other conductors, which are considered to be fixed at their positions in space. To visualize the process, we can imagine to loosen the constraints on one conductor, thereby enabling small spatial displacements. In this case, the mechanical work produced by the field forces is

$$dW = \vec{F} \cdot d\vec{r} \tag{10.1}$$

where $\vec{F}$ is the net force of all electric forces acting on the conductor. Figure 10.1 shows a schematic of the system under consideration.

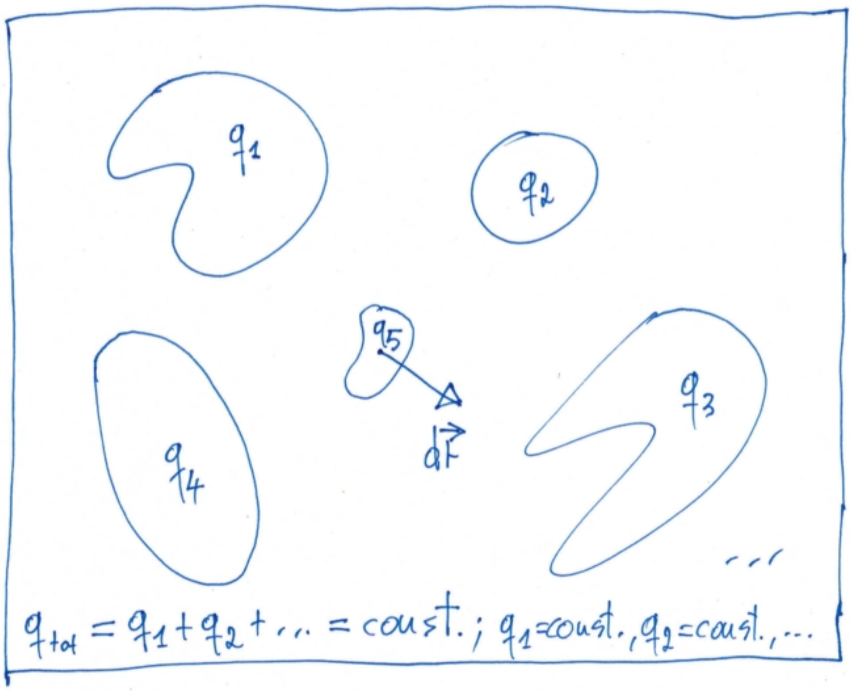

Figure 10.1

Since the system is isolated, the work (10.1) must be against the electrostatic energy U_e of the system

$$dW = -dU_e\Big|_q \tag{10.2}$$

where the differential is calculated assuming constant charges.

By combining (10.1) and (10.2), we obtain

$$\vec{F} \bullet d\vec{r} = -dU_e\Big|_q \tag{10.3}$$

Then, by using the definition of unit vector $\vec{u}_r$ along the generic direction r and that of total differential, we find

$$\begin{aligned} \vec{F} \bullet \left(\frac{\vec{r}}{r}\right) dr &= \vec{F} \bullet \vec{u}_r dr = F_r dr \\ &= -\frac{\partial}{\partial r} U_e\Big|_q dr \end{aligned} \tag{10.4}$$

where F_r is the component of $\vec{F}$ along r. The result (10.4) can finally be simplified as

$$F_r = -\frac{\partial}{\partial r} U_e\Big|_q \tag{10.5}$$

which is the result we were looking for.

The second case is when the conductors form an open system, i.e., a non-isolated system. In such an open system, each conductor is maintained at a constant potential by means of external sources of electric energy, i.e., electric batteries (cf. PHYS 242). Figure 10.2 shows the same system as in Fig. 10.1, but in open configuration: Electric charge can move from the sources onto the conductors, thus keeping them at a constant potential.

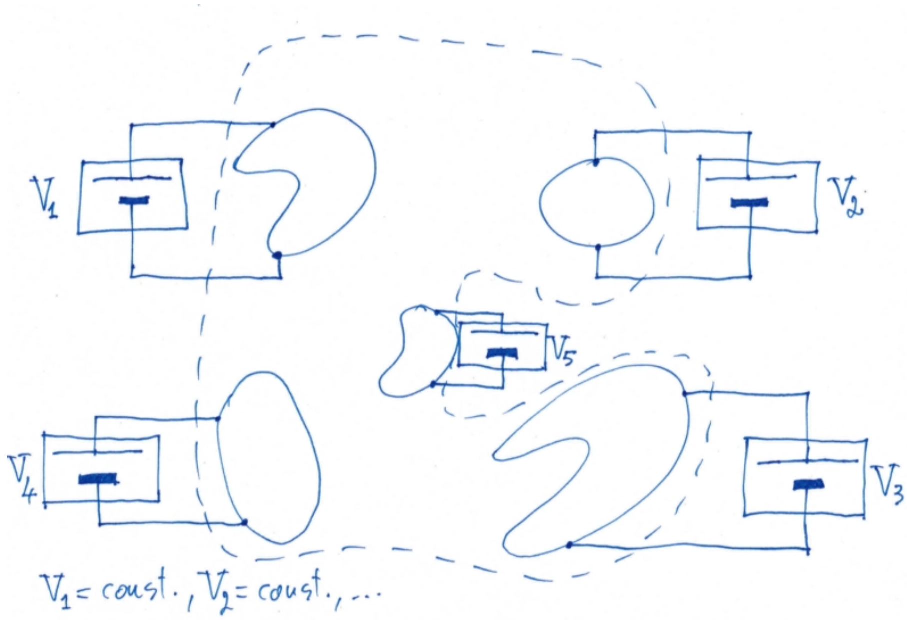

Figure 10.2

In this case, Eq. (10.1) is still valid. This time, however, the mechanical work is against both the electric energy of the system of conductors and that of the external sources. Thus, the equation for the conservation of energy, which is given by Eq. (10.2) in the first case, now becomes

$$dW = -dU_e\Big|_\phi + dU_{ext} \tag{10.6}$$

where $dU_e|_\phi$ is the differential of the electrostatic energy of the system of conductors calculated assuming constant potentials and dU_{ext} indicates the differential of the electric energy due to the external sources. Note that the positive sign of dU_{ext} is due to the fact that the sources are "external" to the system.

In order to write the net force $\vec{F}$ as a function of the electrostatic energy U_e, it is necessary to find a mathematical connection between $dU_e|_\phi$ and dU_{ext}. The energy generated by the external sources to increase by dq_i the charge of the i-th conductor is $\phi_i dq_i$, where ϕ_i is the electrostatic potential for that conductor. Hence, assuming independent sources

$$dU_{ext} = \sum_{i=1}^{N} \phi_i dq_i \tag{10.7}$$

where N is the total number of conductors. The absence of the factor 1/2 is due to the assumption of independence. Assuming constant potentials, by differentiating Eq. (7.7), we find

$$dU_e\Big|_\phi = \frac{1}{2}\sum_{i=1}^{N} \phi_i dq_i \tag{10.8}$$

By comparing Eqs. (10.7) and (10.8), we then obtain

$$dU_{ext} = 2dU_e\Big|_\phi \tag{10.9}$$

Using (10.9) in (10.6), we have

$$dW = -dU_e\Big|_\phi + 2dU_e\Big|_\phi = dU_e\Big|_\phi \tag{10.10}$$

We note that, because of (10.6) and (10.10) the electric work dU_{ext} is equally distributed between mechanical work and electrostatic energy. From the definition of work and Eq. (10.10) it follows that

$$\vec{F} \bullet d\vec{r} = dU_e\Big|_{\phi} \tag{10.11}$$

and, finally,

$$F_r = \frac{\partial}{\partial r} U_e\Big|_{\phi} \tag{10.12}$$

where we followed a derivation similar to that in Eqs. (10.4) and (10.5).

In summary, given a system of electric conductors having a specific geometrical configuration and electric state (i.e., given electric charges and potentials for the conductors), in static conditions the forces acting between conductors depend only on the geometrical configuration and electric state of the system. As a consequence, since Eqs. (10.5) and (10.12) are different expressions of the same forces, they must give the same result.

10.2 Force for Parallel-Plate Condensers

Consider a simple parallel-plate condenser, the capacitance of which was calculated in PHYS 242. We want to calculate the force between the two plates of the condenser.

Assuming the area A of each plate to be much larger than the distance d between the plates, $\sqrt{A} >> d$, from simple symmetry arguments (and our previous knowledge of the field $\vec{E}$ for this type of condenser) it follows that the force must be directed normally to the plates. In addition, from the very definition of condenser it also follows that such a force must be attractive. As shown in Fig. 10.3, we choose a Cartesian coordinate system $Oxyz$, with the yz-plane coinciding with one of the two plates and the x-axis directed towards the other plate.

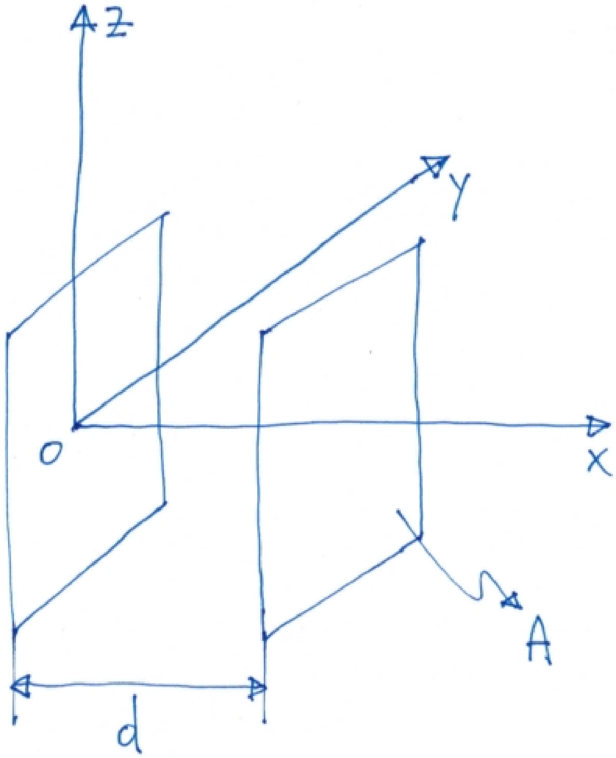

Figure 10.3

We slightly move the two plates with respect to each other along the x direction, which coincides with the direction of the force $\vec{F}$ between the plates.

First, we assume the condenser to be isolated. For example, we charge the condenser by means of a battery. We then disconnect it from the charge source and imagine to perform the displacement under these conditions.

If C is the capacitance of the condenser, from Eqs. (7.8) and (10.5), we have

$$F_x = -\frac{d}{dx}\left(\frac{1}{2}\frac{q^2}{C}\right) \tag{10.13}$$

where q, which is the positive charge on one of the two plates, must be constant for an isolated system. Thus,

$$\begin{aligned} F_x dx &= -d\left(\frac{1}{2}\frac{q^2}{C}\right) = -\frac{1}{2}q^2 d\left(\frac{1}{C}\right) \\ &= \frac{1}{2}\frac{q^2}{C^2}dC \end{aligned} \tag{10.14}$$

For an arbitrary distance x, the capacitance of a parallel-plate condenser is

$$C = \epsilon_0 \frac{A}{x} \tag{10.15}$$

For the condenser in Fig. 10.3, a small displacement dx along the x direction results in (Taylor)

$$\widetilde{C} \simeq C + dC = \epsilon_0 \frac{A}{d} - \epsilon_0 \frac{A}{d^2}dx \tag{10.16}$$

From which

$$dC = -\epsilon_0 \frac{A}{d^2} dx \tag{10.17}$$

Using this result in (10.14), we obtain

$$F_x dx = -\frac{1}{2} \frac{q^2}{C^2} \epsilon_0 \frac{A}{d^2} dx \tag{10.18}$$

or

$$\begin{aligned} F_x &= -\frac{1}{2} \frac{q^2 \cancel{d^2}}{\epsilon_0^{\cancel{2}} A^{\cancel{2}}} \cancel{\epsilon_0} \frac{\cancel{A}}{\cancel{d^2}} = -\frac{1}{2} \frac{q^2}{\epsilon_0 A} \\ &= -\frac{1}{2} \frac{\sigma^2}{\epsilon_0} A \end{aligned} \tag{10.19}$$

where we used the fact that $C = \epsilon_0 A/d$ and $\sigma = q/A$. The latter is the surface charge density on the plates. The force with respect to surface is thus

$$\frac{F_x}{A} = -\frac{1}{2} \frac{\sigma^2}{\epsilon_0} \tag{10.20}$$

The minus sign indicates that the force, when applied to the plate we assumed to displace by a quantity dx, is directed towards the negative side of the x-axis.

Second, we assume the condenser to be non isolated. For example, we assume it is always connected to a battery.

From Eqs. (7.8) and (10.12), we have

$$F_x dx = d\left[\frac{1}{2} C \big(\Delta\phi\big)^2\right] \tag{10.21}$$

where $\Delta\phi$, which is the potential difference between the two plates calculated from the positive to the negative plate, must be constant for a non-isolated system. Thus,

$$F_x dx = \frac{1}{2} \big(\Delta\phi\big)^2 dC \tag{10.22}$$

Using again (10.17) this time in (10.22), we obtain

$$F_x = -\frac{1}{2} \big(\Delta\phi\big)^2 \epsilon_0 \frac{A}{d^2} \tag{10.23}$$

From $C = \epsilon_0 A/d$ and $\sigma = q/A$, we finally find

$$\begin{aligned} F_x &= -\frac{1}{2} \big(\Delta\phi\big)^2 \frac{1}{\epsilon_0 A} \left(\epsilon_0 \frac{A}{d}\right)^2 \\ &= -\frac{1}{2} \big(\Delta\phi\big)^2 \frac{1}{\epsilon_0 A} C^2 \\ &= -\frac{1}{2} \frac{q^2}{\epsilon_0 A} \\ &= -\frac{1}{2} \frac{\sigma^2}{\epsilon_0} A \end{aligned} \tag{10.24}$$

where we used the definition of capacitance $q = C\,\Delta\phi$.

As expected, this result is exactly the same as Eq. (10.19).