PHYS 342 - EM 02

Lecture 9

9.1 Self Inductance of a Quasi-Filiform Circular Ring (Maxwell's Inductance)

Circular rings are simple structures commonly encountered in applications and, yet, the calculation of their self inductance is not an easy problem.

Consider a quasi-filiform circular conductor (i.e., a ring) with radius a and circular cross-section πr^2, where $r << a$ (quasi filiform condition). The ring carries a steady current I.

We want to use a Neumann-type integral to calculate the self-inductance coefficient L of the ring. The simple integral (8.10), however, would diverge. Instead, we will try to emulate a single ring as a pair of coaxial rings with radii a and A, respectively (cf. Fig. 9.1), and calculate the mutual inductance M between them. In the limit $a = A$, M should become the self inductance L of a single ring.

As it turns out, it is easier to solve this problem by assuming a vertical displacement δ between the centers of the two coaxial rings (also shown in Fig. 9.1). It is clear that the two planes containing the rings are assumed to be parallel to each other.

For the calculation of M, we will assume the two rings to be rigorously filiform conductors. The fact that the original single ring has cross-section with radius r will reappear a the very end of the calculation, where we will find a mathematical connection between r and δ.

In summary, we will calculate the self inductance as

$$L = \lim_{\delta \to 0^+} M(\delta; a = A) \tag{9.1}$$

and then express δ as a function of r by simple geometric arguments. The limit (9.1) is key to avoid a diverging Neumann integral. This limit should be intended in a physical sense, i.e., δ is a very small, but finite quantity in the limit.

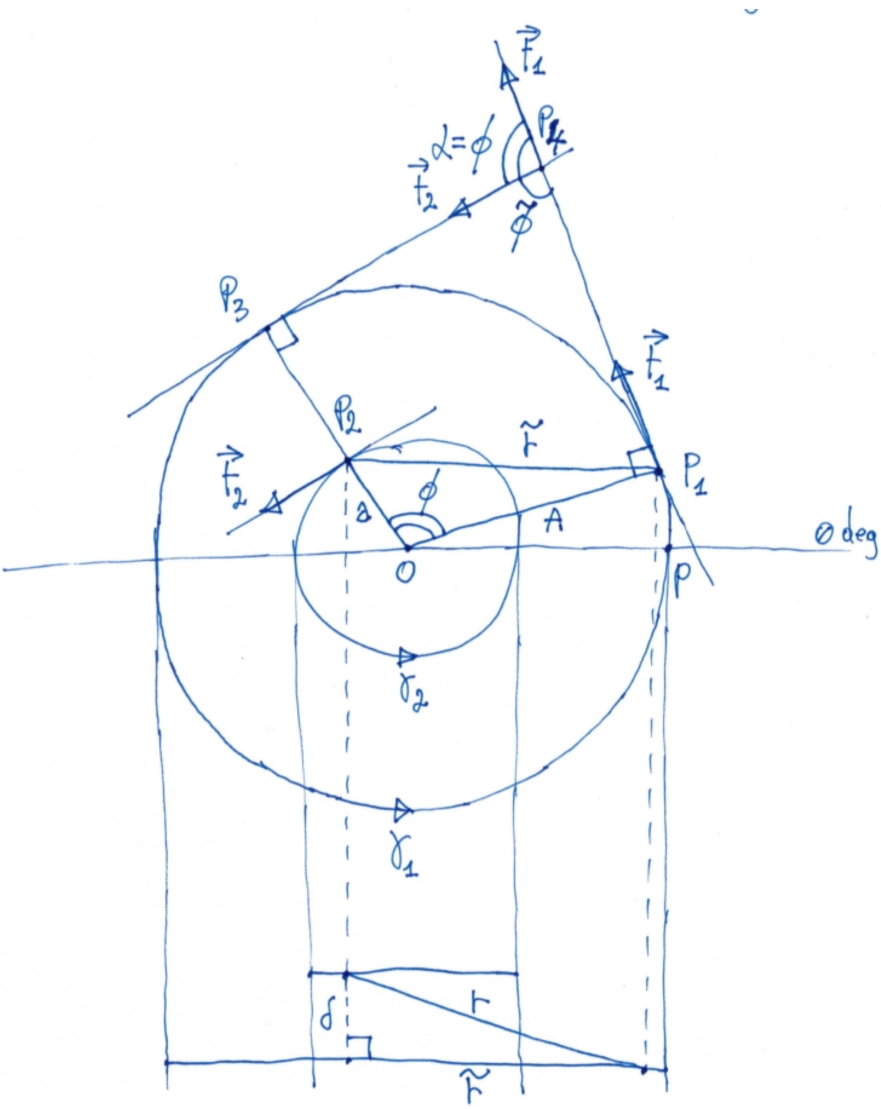

Figure 9.1

Referring to the notation in Fig. 9.1, we can readily obtain each element in the Neumann integral (8.9) for the calculation of the mutual inductance M_{21} between the filiform rings γ_1 and γ_2. We find:

$$r = \sqrt{\tilde{r}^2 + \delta^2} \tag{9.2a}$$

$$\widetilde{r} = \overline{P_1 P_2} = \sqrt{A^2 + a^2 - 2Aa\cos\phi} \tag{9.2b}$$

$$\phi = \widehat{POP_2} - \widehat{POP_1} \equiv \varphi_2 - \varphi_1 \tag{9.2c}$$

for which

$$r = \sqrt{A^2 + a^2 + \delta^2 - 2Aa\cos(\varphi_2 - \varphi_1)} \tag{9.2d}$$

$$d\ell_1 = A\,d\varphi_1, \qquad \varphi_1 \in [0, 2\pi] \tag{9.2e}$$

$$d\ell_2 = A\,d\varphi_2, \qquad \varphi_2 \in [0, 2\pi] \tag{9.2f}$$

From Fig. 9.1, we also find that

$$\vec{t}_1 \bullet \vec{t}_2 = \cos\alpha \tag{9.2g}$$

where α can easily be found considering the quadrilateral $OP_1P_4P_3$. In fact, the interior angles of a simple and planar quadrilateral add up to 2π. Thus,

$$\phi + \widehat{OP_1P_4} + \widetilde{\phi} + \widehat{P_4P_3O} = \phi + \frac{\pi}{2} + \widetilde{\phi} + \frac{\pi}{2} = 2\pi \tag{9.3}$$

for which

$$\widetilde{\phi} = \pi - \phi \tag{9.4}$$

From Fig. 9.1, it also must be

$$\widetilde{\phi} + \alpha = \pi \tag{9.5}$$

and, thus, using (9.4)

$$\alpha = \pi - \widetilde{\phi} = \phi \tag{9.6}$$

We can finally write (9.2g) as

$$\vec{t}_1 \bullet \vec{t}_2 = \cos\phi = \cos(\varphi_2 - \varphi_1) \tag{9.2h}$$

By using (9.2d), (9.2e), (9.2f), and (9.2h), we can write the integral (8.9) as

$$M_{21} = \frac{\mu_0}{4\pi} \int_0^{2\pi}\int_0^{2\pi} \frac{Aa\cos(\varphi_2 - \varphi_1)d\varphi_1 d\varphi_2}{\sqrt{A^2 + a^2 + \delta^2 - 2Aa\cos(\varphi_2 - \varphi_1)}} \tag{9.7}$$

This integral can be exactly solved in the form

$$M_{21} = -\mu_0\sqrt{Aa}\left[\left(k - \frac{2}{k}\right)F(k) + \frac{2}{k}\,E(k)\right] \tag{9.8}$$

where

$$k = \frac{2\sqrt{Aa}}{\sqrt{(A+a)^2 + \delta^2}} \tag{9.9}$$

is the modulus of the complete elliptic integrals of the first and second kind

$$\begin{cases} F(k) = \displaystyle\int_0^{\pi/2} \frac{1}{\Delta}\,d\varphi & \text{(9.10a)} \\ E(k) = \displaystyle\int_0^{\pi/2} \Delta d\varphi & \text{(9.10b)} \end{cases}$$

respectively, with $\Delta = \sqrt{1 - k^2\sin^2\varphi}$.

It can be shown that, in general, $F(k)$ and $E(k)$ can be expanded in series:

$$\begin{cases} F(k) = \dfrac{\pi}{2}\left[1 + \left(\dfrac{1}{2}\right)^2 k^2 + \left(\dfrac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \left(\dfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 k^6 + \cdots\right] & \text{(9.11a)} \\ E(k) = \dfrac{\pi}{2}\left[1 - \left(\dfrac{1}{2}\right)^2 k^2 - \left(\dfrac{1\cdot 3}{2\cdot 4}\right)^2 \dfrac{k^4}{3} - \left(\dfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 \dfrac{k^6}{5} \cdots\right] & \text{(9.11b)} \end{cases}$$

Hence, when k is small $F(k)$ and $E(k)$ can easily be evaluated from (9.11a) and (9.11b).

When $A = a$ and $\delta \to 0^+$, which are the conditions used in the limit (9.1) for the calculation of L, $k \to 1^-$, as it can readily be shown from (9.9). Indeed, the maximum value of k is $k = 1$. When $k \sim 1$, the series (9.11b) converges quite rapidly due to the minus sign in front of all terms with k. On the contrary, the series (9.11a) converges very slowly because all its terms are positive and add all up. As it turns out, this series converges *very* slowly and a huge number of terms must be considered to obtain a reasonable approximation for $F(k)$.

When k is nearly equal to unity, a different type of series expansion must be used for F and E:

$$\begin{cases} F = \ln \dfrac{4}{k_1} + \left(\dfrac{1}{2}\right)^2 k_1^2 \left(\ln \dfrac{4}{k_1} - \dfrac{2}{1 \cdot 2}\right) \\[2ex] \qquad + \left(\dfrac{1 \cdot 3}{2 \cdot 4}\right)^2 k_1^4 \left(\ln \dfrac{4}{k_1} - \dfrac{2}{1 \cdot 2} - \dfrac{2}{3 \cdot 4}\right) + \ldots & \text{(9.12a)} \\[2ex] E = 1 + \dfrac{1}{2} k_1^2 \left(\ln \dfrac{4}{k_1} - \dfrac{1}{1 \cdot 2}\right) \\[2ex] \qquad + \dfrac{1^2 \cdot 3}{2^2 \cdot 4} k_1^4 \left(\ln \dfrac{4}{k_1} - \dfrac{2}{1 \cdot 2} - \dfrac{1}{3 \cdot 4}\right) + \ldots & \text{(9.12b)} \end{cases}$$

where

$$k_1^2 = 1 - k^2 \tag{9.13}$$

which is a small quantity for $k \sim 1$. The expansion (9.12a) and (9.12b) are due to Legendre and play a pivotal role in our calculation of L for the ring.

By approximating

$$\begin{cases} F \sim \ln \dfrac{4}{k_1} + \dfrac{1}{4} k_1^2 \ln \dfrac{4}{k_1} - \dfrac{1}{4} k_1^2 & \text{(9.14a)} \\[2ex] E \sim 1 + \dfrac{1}{2} k_1^2 \ln \dfrac{4}{k_1} - \dfrac{1}{4} k_1^2 & \text{(9.14b)} \end{cases}$$

that is, by considering only the first line in (9.12a) and (9.12b) and with simple algebra, we can give an approximate solution for M_{21}. By inserting (9.14a) and (9.14b) into (9.8), we obtain

$$\begin{aligned} M_{21} &= -\mu_0 \sqrt{Aa} \left(kF - \frac{2}{k} F + \frac{2}{k} E \right) \\ &\sim -\mu_0 \sqrt{Aa} \left(k \ln \frac{4}{k_1} + k \frac{1}{4} k_1^2 \ln \frac{4}{k_1} - k \frac{1}{4} k_1^2 \right. \\ &\qquad - \frac{2}{k} \ln \frac{4}{k_1} - \frac{\cancel{2}}{k} \frac{1}{\cancel{4}_{2}} k_1^2 \ln \frac{4}{k_1} + \frac{2}{\cancel{k}} \frac{\cancel{1}}{4} k_1^2 \\ &\qquad \left. + \frac{2}{k} + \frac{\cancel{2}}{k} \frac{1}{\cancel{2}} k_1^2 \ln \frac{4}{k_1} - \frac{2}{\cancel{k}} \frac{\cancel{1}}{4} k_1^2 \right) \\ &= \mu_0 \sqrt{Aa} \left[-k \frac{1}{4} k_1^2 + \frac{2}{k} \right. \\ &\qquad \left. + \ln \frac{4}{k_1} \cdot \left(k + k \frac{1}{4} k_1^2 - \frac{2}{k} + \frac{1}{k} k_1^2 \right) \right] \\ &= \mu_0 \sqrt{Aa} \left[\frac{k k_1^2}{4} \; \frac{2}{k} \right. \end{aligned}$$

$$- \Big(k + \frac{kk_1^2}{4} - \frac{2}{k} + \frac{k_1^2}{k}\Big) \ln \frac{4}{k_1}\Bigg] \tag{9.15}$$

Assume $A = a$, from (9.9)

$$k = \frac{2a}{\sqrt{4a^2 + \delta^2}} \tag{9.16}$$

and so

$$k^2 = \frac{\cancel{4a^2}}{\cancel{4a^2}\Big(1 + \frac{\delta^2}{4a^2}\Big)} = \frac{1}{1 + \Big(\frac{\delta}{2a}\Big)^2} = \frac{1}{1 + \xi} \tag{9.17}$$

where

$$\xi \equiv \Big(\frac{\delta}{2a}\Big)^2 \tag{9.18}$$

For small values of δ (as in the limit (9.1)), a simple Maclaurin series allows us to approximate k^2 as

$$k^2 \sim 1 - \xi \tag{9.19}$$

From (9.13) it then follows that

$$k_1^2 \sim \cancel{1} - \cancel{1} + \xi = \xi \tag{9.20}$$

Finally,

$$k = \frac{2a}{\sqrt{4a^2\Big(1 + \frac{\delta^2}{4a^2}\Big)}} = \frac{1}{\sqrt{1 + \xi}} \sim 1 - \frac{1}{2}\xi \tag{9.21}$$

By inserting (9.21) and (9.20) into (9.15), we obtain

$$\begin{aligned} M_{21} \sim \mu_0 a \Bigg[& \frac{1}{4}\Big(1 - \frac{1}{2}\xi\Big)\xi - \frac{2}{1 - \xi/2} \\ & - \Big(\Big(1 - \frac{1}{2}\xi\Big) + \frac{1}{4}\Big(1 - \frac{1}{2}\xi\Big)\xi \\ & - \frac{2}{1 - \xi/2} + \frac{\xi}{1 - \xi/2}\Big) \ln \frac{4}{\sqrt{\xi}}\Bigg] \end{aligned} \tag{9.22}$$

In this equation, $\xi \sim \delta^2$ and $\sqrt{\xi} \sim \delta$. As a consequence, in the limit $\delta \to 0^+$, ξ is an infinitesimal of higher order compared to $\sqrt{\xi}$. In other words, in this limit ξ goes to zero faster than $\sqrt{\xi}$. Thus, for the result of Eq. (9.22), the limit (9.1) gives

$$\begin{aligned} \lim_{\delta \to 0^+} M_{21} &= \mu_0 a \Bigg[-2 - (1 - 2) \ln \frac{4}{\sqrt{\xi}}\Bigg] \\ &= \mu_0 a \Big(\ln \frac{4}{\delta/2a} - 2\Big) \\ &= \mu_0 a \Big(\ln \frac{8a}{\delta} - 2\Big) \end{aligned} \tag{9.23}$$

The result of Eq. (9.23) is an approximate solution for the self inductance L of the ring. Note that this result must be used with the grain of salt. In fact, for example, for $\delta = 0$, Eq. (9.23) goes to $+\infty$. However, the approximation is very good so long $\delta \neq 0$, even for very small values of δ ($\ln x$ increases very slowly for increasing values of x).

The last bit we need to work out is the connection between δ and r. This connection is due to Maxwell. Maxwell demonstrated that the most accurate approximation for L is obtained when assuming δ to be the geometrical mean distance of every pair of points in the section of the conductor. For a quasi-filiform conductor with radius r, the geometrical mean distance is (we will not prove it):

$$\delta = e^{-1/4} r = \mathcal{G} r \tag{9.24}$$

By inserting this value into (9.23), we find

$$\begin{aligned} L \sim \mu_0 a \Big(\ln \frac{8a}{\mathcal{G} r} - 2 \Big) &= \mu_0 a \Big(\ln \frac{8a}{r} - \ln \mathcal{G} - 2 \Big) \\ &\sim \mu_0 a \Big(\ln \frac{8a}{r} - 1.75 \Big) \end{aligned} \tag{9.25}$$

This result is known as the Maxwell's inductance for a circular quasi-filiform conductor with radius a and cross-section πr^2.