ECE 106 - Tutorial

Quiz #4

Taught by: Dr. Firas Mansour and Dr. Bajcsy

John Rinehart

June 7, 2015

Contents

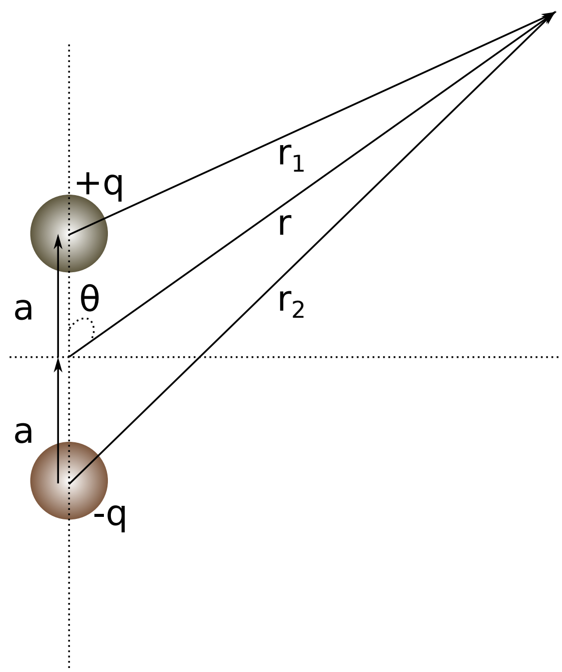

Figure 1: Diagram of Quiz Problem #4

Quiz 4

Two charges are separated by a distance of 2a. One charge, $q_1 = +q$ and the other charge $q_2 = -q$. Find the potential due to these charges a very far distance from the two charges. To solve for this potential, express the potential everywhere in space as the sum of the potentials generated by each point charge. Then, perform a 1st order expansion of the potential.

Okay, the instructions tell us what to do at the very beginning. Our first goal will be to express the potential everywhere in space. We know that the potential $V(\vec{r})$ at a point $\vec{r}$ due to a point charge $V(\vec{r}) = \frac{kq}{r}$. Now, that r in the denominator is the distance that I am away from the point charge. So, for a point charge not located at the origin, I should really express this as $V(\vec{r}) = \frac{kq}{|\vec{r}-\vec{r}'|}$. See reffig:charges.pdf for a diagram that explains this relationship. For this problem, then, the potential at a point $\vec{r}$ is

$$V(\vec{r}) = \frac{kq_1}{r_1} + \frac{kq_2}{r_2}$$

Now, $\vec{r_2} = \vec{a} + \vec{r}$ and $\vec{r_1} = \vec{r} - \vec{a}$. Here a is the vector that points from the negative charge to the origin. It is also the negative of the vector that points from the positive charge to the origin. Thus, $|\vec{r_2}| = |\vec{a} + \vec{r}|$ and $|\vec{r_1}| = |\vec{r} - \vec{a}|$. It would be nice if we could get rid of these absolute value signs. We really want to know how big $\vec{r_2}$ is and how big $\vec{r_1}$ is. For that, we'll employ the following identity: $|\vec{v}| = \sqrt{\vec{v} \cdot \vec{v}}$. You can justify this by realizing that $\vec{a} \cdot \vec{b} = |\vec{a}|.|\vec{b}| \cos\theta$.

$$V(\vec{r}) = \frac{kq_1}{\sqrt{\vec{r} \cdot \vec{r} + \vec{a} \cdot \vec{a} - 2|\vec{a}||\vec{r}| \cos\theta}} + \frac{kq_2}{\sqrt{\vec{r} \cdot \vec{r} + \vec{a} \cdot \vec{a} + 2|\vec{a}||\vec{r}| \cos\theta}}$$

But, $\vec{r} \cdot \vec{r}$ is simply $|\vec{r}|^2 = r^2$. Also,$\vec{a} \cdot \vec{a}$ is simply $|\vec{a}|^2 = a^2$. Thus,

$$V(\vec{r}) = \frac{kq}{\sqrt{r^2 + a^2 - 2ar\cos\theta}} + \frac{kq}{\sqrt{r^2 + a^2 + 2ar\cos\theta}}$$

Quiz 4 continued on next page. . .

Note that I have also inserted the signs of the two charges in the last line. Now, the problem asks us to solve for the potential when $r \gg a$. Thus, I will factor out an r from the previous expression's denominator to express everything in terms of the ratio $\frac{a}{r}$.

$$\begin{aligned} V(\vec{r}) &= \frac{kq}{r}\left(\frac{1}{\sqrt{1+\left(\frac{a}{r}\right)^2 - 2\left(\frac{a}{r}\right)\cos\theta}} - \frac{1}{\sqrt{1+\left(\frac{a}{r}\right)^2 + 2\left(\frac{a}{r}\right)\cos\theta}}\right) \\ &= \frac{kq}{r}\left(\frac{1}{\sqrt{1+\epsilon^2 - 2\,\epsilon\cos\theta}} - \frac{1}{\sqrt{1+\epsilon^2+2\epsilon\cos\theta}}\right) \end{aligned}$$

Here, I have replaced the quantity $\frac{a}{r}$ with the small quantity ϵ. This is nothing more than simple variable substitution. Now, I realize that I can expand this function about small ϵ and I would be done. However, expanding about ϵ would make my life a litle difficult because I would have to take derivatives of $V(\epsilon)$ with respect to ϵ which would involve using the chain rule to evaluate the derivative of $(1+\epsilon^2 \pm 2\epsilon\cos\theta)^{-.5}$. Okay, at this point, I have two ways that I can solve the problem so I will diverge and I will show you the preferred way, first (this is the way that I requested you all do it). I recommend you read both approaches.

Expanding Around f

I'm going to do a little trick that makes my life a lot easier. Note that $\epsilon^2 \pm 2\epsilon\cos\theta$ is just a function of ϵ. I could just wrap this whole expression up inside something else. Let's call $f^+(\epsilon) \equiv \epsilon^2 + 2\epsilon\cos\theta$ and $f^-(\epsilon) \equiv \epsilon^2 - 2\epsilon\cos\theta$ (triple horizontal bars mark a definition).

Note that as $\epsilon \to 0$, $f^{\pm}(\epsilon) \to 0$. Thus, expanding about small ϵ is the same as expanding around small f. So, for any value of r we are considering (assuming, still, that $r \gg a$) we can expand $V(f^+, f^-)$ around small f^+ and f^-. Our expression for $V(f^+, f^-)$, now, is:

$$\begin{aligned} V(\vec{r}) &= \frac{kq}{r}\left(\frac{1}{\sqrt{1+f^-(\epsilon)}} - \frac{1}{\sqrt{1+f^+(\epsilon)}}\right) \\ &= \frac{kq}{r}\left(\left[1+f^-(\epsilon)\right]^{-.5} - \left[(1+f^+(\epsilon)\right]^{-.5}\right) \end{aligned} \tag{1}$$

Now, I will introduce the math necessary to perform a Maclaurin series expansion of $V(\vec{R})$. A Maclaurin series expansion of a function $g(x)$ is the Taylor series expansion of g about $x = 0$ (yeah, that gets a special name). Just remember that a Maclaurin series is a Taylor series of a function centered about zero. The Taylor series of a function $g(x)$ about the point $x = a$, $T(g(x))|_{x=a}$ is defined as:

$$T(g(x))|_{x=a} = \sum_{n=0}^{\infty} \frac{g^n(a)(x-a)^n}{n!}$$

Here, I have used the notation $g^n(a)$ as a succinct form of $\frac{\mathrm{d}^n g(x)}{\mathrm{d}x^n}|_{x=a}$. For a Maclaurin series (one where $a = 0$) this can be reduced to:

$$T(g(x))_{x=a} = \sum_{n=0}^{\infty} \frac{g^n(0)x^n}{n!}$$

The most significant reduction to this expression is in the $(x-a)^n$ term. We have only been asked to expand the potential to first order. But, for demonstrative purposes, let's expand to 2nd order. At this point you should be asking the question: "How do I expand $V(f^+, f^-)$? There are two variables: f^+ and f^-". The way in which we're going to get around this is by expanding two other functions $g^+ = (1+f^+)^{-.5}$

and $g^- = (1+f^-)^{-.5}$ to second order in f^+ and f^-, respectively. Then, we'll combine the expressions to solve for V. An approximate expression for $V(\vec{r})$ then, to second order is:

$$V = \frac{kq}{r}\left(g^-(f^-) - g^+(f^+)\right)$$
$$V \approx \frac{kq}{r}\left(\left(g^{-0}(0) + g^{-1}(0)f^- + \frac{g^{-2}(0)f^{-2}}{2}\right) - \left(g^{+0}(0) + g^{+1}(0)f^+ + \frac{g^{+2}(0)f^{+2}}{2}\right)\right)$$

Okay, that looks pretty complicated, but it's not. The first three terms in parentheses are just the 0 to 2nd order terms of the Maclaurin series expansions of $g^+ = (1+f^+)^{-.5}$ and the second three terms are just the 0 to 2nd order terms of $g^- = (1+f^-)^{-.5}$ - Don't forget that I'm using the shorthand $f^n(a) = \frac{d^n f(x)}{dx^n}|_{x=a}$. One thing that simplifies the previous expression a lot is the fact that $g^+(0) = g^-(0) = 1$. I will substitute this expression in and, in the following lines, I will take the 1st and second derivatives of g^+ and g^- and I'll evaluate $g^{\pm 1}(0)$ and $g^{\pm 2}(0)$.

$$V \approx \frac{kq}{r}\left(\left(1 + g^{-1}(0)f^- + \frac{g^{-2}(0)f^{-2}}{2}\right) - \left(1 + g^{+1}(0)f^+ + \frac{g^{+2}(0)f^{+2}}{2}\right)\right)$$

Now, evauluating $g^{+1}(0)$:

$$\begin{aligned} g^{+1}(0) &= \frac{\mathrm{d}}{\mathrm{d}f^+}(1-f^+)^{-.5}|_{f^+=0} \\ &= -\frac{1}{2}(1-f^+(0))^{-\frac{3}{2}} \\ &= -\frac{1}{2} \end{aligned}$$

Remember that $f^+(0) = 0$. Now, evaluating $g^{+2}(0)$:

$$\begin{aligned} g^{+2}(0) &= \frac{\mathrm{d}^2}{\mathrm{d}f^{+2}}(1-f^+)^{-.5}|_{f^+=0} \\ &= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}f^+}(1-f^+)^{-\frac{3}{2}}|_{f^+=0} \\ &= \frac{3}{4}\left(1-f^+(0)\right) \\ &= \frac{3}{4} \end{aligned}$$

Now, I only found the 2nd order derivative of $g^+(f^+)$. I have to do the whole thing again for $g^-(f^-)$. But, before I do that, realize that the only thing that would change if I did this would be that everywhere I had an f^+ in the previous expressions I would have an f^-, now. But $f^+(0) = f^-(0) = 0$ so my derivatives for g^- would be the same as those for g^+. Thus, I can just recycle my results. Now, I can finally write my expression for $V\left(f^+(\epsilon), f^-(\epsilon)\right)$:

$$V \approx \frac{kq}{r}\left(\left(1 - \frac{1}{2}f^- + \frac{3}{4}\frac{f^{-2}}{2}\right) - \left(1 - \frac{1}{2}f^+ + \frac{3}{4}\frac{f^{+2}}{2}\right)\right)$$

Quiz 4 continued on next page. . .

Okay, notice that the two "1s" cancel. I can substitute my expressions for $f^{+}(\epsilon)$ and $f^{-}(\epsilon)$ into the previous expression:

$$V \approx \frac{kq}{r}\left(\left(-\frac{1}{2}\left(\epsilon^2-2\epsilon\cos\theta\right)+\frac{3}{4}\frac{\left(\epsilon^2-2\epsilon\cos\theta\right)^2}{2}\right)-\left(-\frac{1}{2}(\epsilon^2+2\epsilon\cos\theta)+\frac{3}{4}\frac{\left(\epsilon^2+2\epsilon\cos\theta\right)^2}{2}\right)\right)$$
$$V \approx \frac{kq}{r}\left(-\frac{1}{2}\epsilon^2+\epsilon\cos\theta+\frac{3}{8}\left(\epsilon^2-2\epsilon\cos\theta\right)^2+\frac{1}{2}\epsilon^2+\epsilon\cos\theta-\frac{3}{8}\left(\epsilon^2+2\epsilon\cos\theta\right)^2\right)$$

Let's simplify this expression. Notice that the ϵ^2 and ϵ^4 terms from all the terms will go away (even the term that I have not explicitly squared, yet). Only linear and cubic powers of ϵ will survive. For future reference note that the squared (second order) terms are $-\frac{1}{2}\epsilon^2+\frac{3}{2}\epsilon^2\cos^2\theta$ for both expansions.

$$\begin{aligned} V &\approx \frac{kq}{r}\left(\epsilon\cos\theta-\frac{3}{2}\epsilon^3\cos\theta+\epsilon\cos\theta-\frac{3}{2}\epsilon^3\cos\theta\right) \\ &= \frac{kq}{r}\left(2\epsilon\cos\theta-3\epsilon^3\cos\theta\right) \end{aligned}$$

Now, for sufficiently small $\epsilon \equiv \frac{a}{r}$ we can justify that ϵ^3 is too small to worry about. Thus, to **second order** in ϵ, our expression for V is:

$$V(\epsilon) = \frac{2kq\epsilon\cos\theta}{r}$$

Substituting the expression for ϵ yields the final answer:

$$V(\vec{r} \gg a) = \frac{2kqa\cos\theta}{r^2}$$

Note that I can say that this is the expansion to second order because the second order term is identically zero. Some of you might be wondering why I have that $V(\vec{r})$ (that is, V is a function of a vector). Well, what I mean when I write this is that if you give me any position in 3-d space, I can find you the potential there. Thus, V is a function of $\vec{r}$, the position where you're located. Associated with that $\vec{r}$ there is some distance you are away from the origin r and an angle that you are away from the axis of the line that connects the two charges θ. Thus, V is a function of r. I hope that this makes sense.

Expanding Around ϵ

Okay, I will now expand abount small ϵ and I will assume that you will have read the previous solution ("Expanding Around f"). I will start from the point where I have the potential expressed en terms of ϵ.

$$\begin{aligned} V(\epsilon) &= \frac{kq}{r}\left(\frac{1}{\sqrt{1+\epsilon^2-2\ \epsilon\cos\theta}}-\frac{1}{\sqrt{1+\epsilon^2+2\epsilon\cos\theta}}\right) \\ &= \frac{kq}{r}\left(g^{-}(\epsilon)-g^{+}(\epsilon)\right) \end{aligned}$$

Now, I'm going to assume that ϵ is very small and perform a Maclaurin series about $\epsilon = 0$ for both g^{+} and g^{-}.

Quiz 4 continued on next page. . .

$$
\begin{aligned}
g^{-}(\epsilon) \approx\ & 1 \\
& - \frac{1}{2}\left(\left(1+\epsilon^2-2\epsilon\cos\theta\right)^{-1.5}(2\epsilon-2\cos\theta)\right)\Big|_{\epsilon=0}\epsilon \\
& + \left(\frac{3}{4}(2\epsilon-2\cos\theta)^2\left(\epsilon^2-2\epsilon\cos\theta+1\right)^{-2.5}-\left(\epsilon^2-2\epsilon\cos\theta+1\right)^{-1.5}\right)\Big|_{\epsilon=0}\frac{\epsilon^2}{2} \\
& = 1+\epsilon\cos\theta+\frac{3}{2}\epsilon^2\cos^2\theta-\frac{1}{2}\epsilon^2
\end{aligned}
$$

Notice that $g^{-}(-\epsilon) = g^{+}(\epsilon)$ in that if I substitute ϵ for $-\epsilon$ into $g^{-}(\epsilon)$ I will obtain $g^{+}(\epsilon)$. Thus, I know the expansion, now, for $g^{+}(\epsilon)$:

$$
\begin{aligned}
g^{-}(\epsilon) \approx\ & 1 \\
& - \frac{1}{2}\left(\left(1+\epsilon^2+2\epsilon\cos\theta\right)^{-1.5}(2\epsilon+2\cos\theta)\right)\Big|_{\epsilon=0}\epsilon \\
& + \left(\frac{3}{4}(2\epsilon+2\cos\theta)^2\left(\epsilon^2+2\epsilon\cos\theta+1\right)^{-2.5}-\left(\epsilon^2+2\epsilon\cos\theta+1\right)^{-1.5}\right)\Big|_{\epsilon=0}\frac{\epsilon^2}{2} \\
& = 1-\epsilon\cos\theta+\frac{3}{2}\epsilon^2\cos^2\theta-\frac{1}{2}\epsilon^2
\end{aligned}
$$

Now, if I right my expression for $V(\epsilon)$ in terms of my expansion I obtain:

$$
\begin{aligned}
V(\epsilon) &= \frac{kq}{r}\left(g^{-}(\epsilon)-g^{+}(\epsilon)\right) \\
&= \frac{kq}{r}\left(\left(1+\epsilon\cos\theta+\frac{3}{2}\epsilon^2\cos^2\theta-\frac{1}{2}\epsilon^2\right)-\left(1-\epsilon\cos\theta+\frac{3}{2}\epsilon^2\cos^2\theta-\frac{1}{2}\epsilon^2\right)\right) \\
&= \frac{2kq\epsilon\cos\theta}{r}
\end{aligned}
$$

Substituting $\frac{a}{r} \to \epsilon$:

$$V(\vec{r}) = \frac{2kqa\cos\theta}{r^2}$$

This is the same expression we obtained earlier. Even the second order terms that dropped out of this solution are the same second order terms that dropped out in the earlier solution. See, it doesn't matter what you expand as long as you are Taking derivatives right. The first method was a little easier because you don't have to worry about using the chain rule to evaluate the derivative with respect to ϵ.

Thus concludes the solution for Quiz #4.