Lecture 3

3.1 On the Rotational Character of Ampère's Law

The magnetostatic field $\vec{B}$ is rotational, i.e.,

$$\oint_{\gamma} \vec{B} \cdot \vec{t} d\ell = \mu_0 I \tag{3.1}$$

The current I is the algebraic sum of all currents linked with γ. Two closed lines are linked to each other if it is impossible to disconnect them without cutting one of them.

In Fig. 3.1 below two currents I_1 and I_2 are linked with an oriented curve γ.

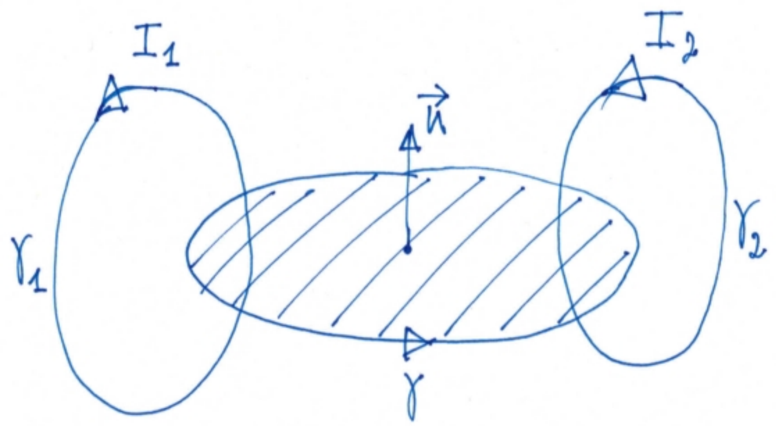

Figure 3.1

In this case, the direction of I_1 is the same as $\vec{n}$ when I_1 crosses the surface associated with $\vec{n}$. Thus, I_1 is assumed to be positive in Ampère's law. On the contrary, I_2 is negative. In summary, for this example

$$\oint_{\gamma} \vec{B} \cdot \vec{t} d\ell = \mu_0 (I_1 - I_2) \tag{3.2}$$

When γ coincides with a vector line of $\vec{B}$, the product $\vec{B} \cdot \vec{t}$ is either always positive or negative, but non-zero. Hence, each vector line of $\vec{B}$ must be linked with at least one current. We remind that, because of the solenoidal property of $\vec{B}$, the vector lines of $\vec{B}$ are closed.

When γ does not link any current, $\vec{B}$ appears to be a rotational field, i.e.,

$$\oint_{\gamma} \vec{B} \cdot \vec{t} d\ell = 0 \tag{3.3}$$

exactly as for the electrostatic field $\vec{E}$. Consider two equally oriented lines γ_1 and γ_2 and assume the lines are open with equal limits A and B. Assuming a total current I is linked with the closed line $A\gamma_1 B(-\gamma_2)A$ (cf. Fig. 3.2),

$$\int_{A\gamma_1 B} \vec{B} \cdot \vec{t} d\ell - \int_{A\gamma_2 B} \vec{B} \cdot \vec{t} d\ell = \mu_0 I \tag{3.4}$$

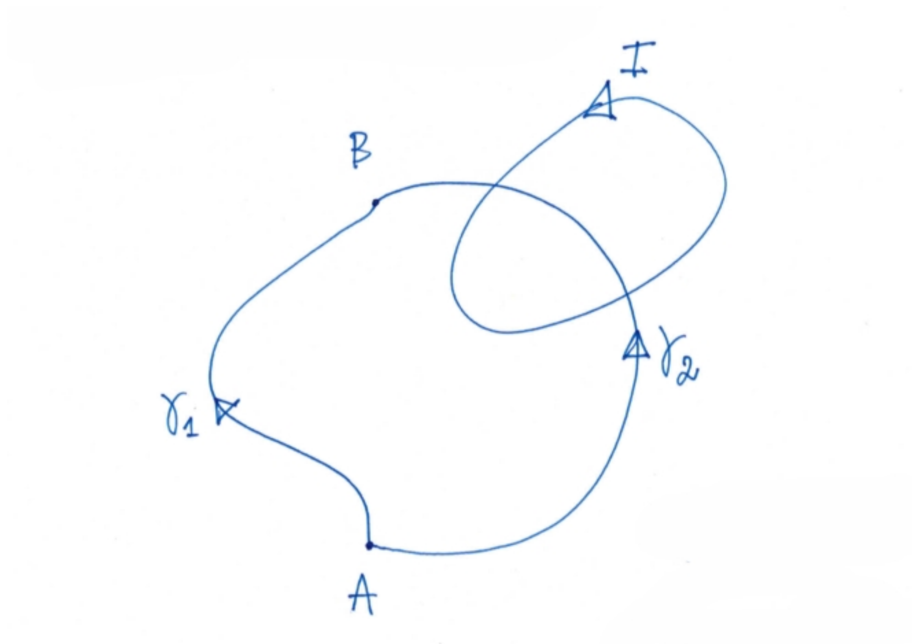

Figure 3.2

When a curve as $A\gamma_1 B(-\gamma_2)A$ does not link any current or a total current $I = 0$, then

$$\int_{A\gamma_1 B} \vec{B} \cdot \vec{t} d\ell = \int_{A\gamma_2 B} \vec{B} \cdot \vec{t} d\ell \tag{3.5}$$

which is equivalent to

$$\oint_{A\gamma_1 B(-\gamma_2)A} \vec{B} \cdot \vec{t} d\ell = 0 \tag{3.6}$$

As a consequence, within internally connected regions of space with no currents, it is possible to find a potential function for $\vec{B}$. Note that an internally connected region of space is such that given a closed loop in the region is always possible to contract it to a point without ever exiting the region (also called simply-connected region).

In Fig. 3.3, point O is the trace of a current I perpendicular to the page. The current I is a uniform current on a straight filiform infinite conductor. This current clearly generates a field $\vec{B}$ in space. In the plane of the page and in all planes parallel to it imagine to cut a hole with closed border $\widetilde{\gamma}$ surrounding O. When now considering $\vec{B}$ in the region of space with the hole around O, the field is considered in a region that does not contain the current which generates it!

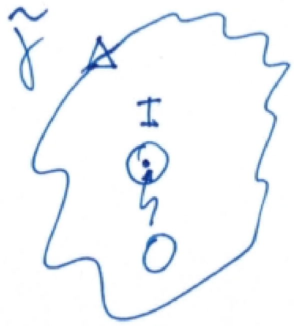

Figure 3.3

This leads to an apparent difficulty. In fact, each closed line entirely contained in the region with the hole does not link any current and, thus, the circulation of $\vec{B}$ must be zero. However, the field $\vec{B}$ generated by I is characterized by circular field lines with center in O (cf. PHYS 242 and tutorial 1) and considering $\vec{B}$ outside $\widetilde{\gamma}$ cannot change this fact. Therefore, the circulation of $\vec{B}$ around a closed, e.g., circular, line with O as center (i.e., a vector line of $\vec{B}$) outside $\widetilde{\gamma}$ must be different than zero! This difficulty is resolved by considering only internally connected regions of space, where cutting hole is not allowed.

Consider again an infinite straight line with current I crossing a plane at point O. This time, however, we cut from the plane a region Ω_1 delimited by a curve γ_1 that starts at infinite, goes around O, and goes back to infinite.

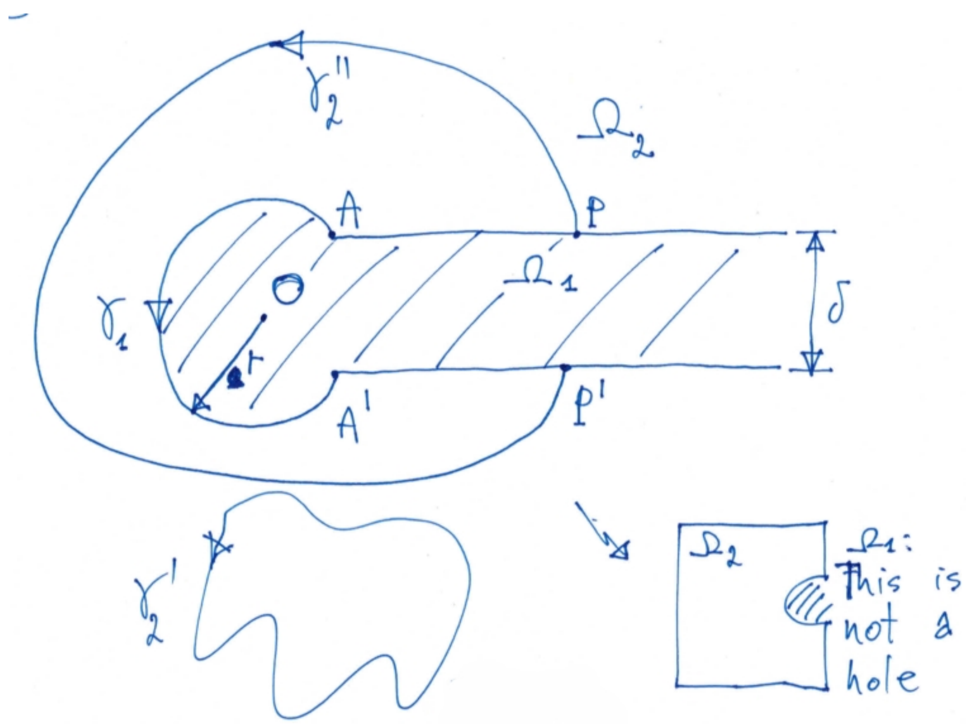

Figure 3.4

Since Ω_1 is not a hole, Ω_2 is an internally connected region. Additionally, any closed line in Ω_2 (e.g., γ_2' or $\gamma_2'' = PP'A'AP$) does not link I. Thus, a scalar potential φ can be associated with $\vec{B}$ in Ω_2.

Referring to Fig. 3.4, the radius r and distance δ can be made arbitrary small (so long r, $\delta \neq 0$). In this way, it would seem that φ can be defined almost everywhere in space. In the limit $r, \delta \to 0^+$, φ would be defined everywhere! This is not entirely correct. Consider P and P' in Fig. 3.4. By definition,

$$\varphi(P) - \varphi(P') = \int\limits_{\substack{PAA'P \\ \text{on } \gamma_1}} \vec{B} \cdot \vec{t} d\ell \tag{3.7}$$

When $\delta \to 0^+$, $PAA'P$ on γ_1 tends to close around O and

$$\varphi(P) - \varphi(P') = \mu_0 I \tag{3.8}$$

The scalar potential φ of $\vec{B}$ is characterized by a discontinuity of the first kind equal to $\mu_0 I$ between two infinitesimally closed points P and P'. This happens every time we attempt to extend the region where φ is defined to include the entire space (including the current generating $\vec{B}$). This means the scalar potential of $\vec{B}$ is a multivalued function. Thus, a scalar potential is not a good representation of $\vec{B}$.

3.2 Laplace's elementary law (or Biot-Savart law)

In PHYS 242 and in the summary in Lecture 2, we introduced a vector potential $\vec{A}$ for $\vec{B}$. We also saw that the vector potential $\vec{A}$ at a generic point P due to a current density $\vec{J}$ at points Q in τ is given by

$$\vec{A}(P) = \frac{\mu_0}{4\pi} \iiint\limits_{\tau} \frac{\vec{J}(Q)}{r_{QP}} dV \tag{3.9}$$

By the definition $\vec{B} = \vec{\nabla} \times \vec{A}$, if follows

$$\vec{B}(P) = \frac{\mu_0}{4\pi} \vec{\nabla}_P \times \iiint\limits_{\tau} \frac{\vec{J}(Q)}{r_{QP}} \, dV \tag{3.10}$$

Due to the linearity of "$\vec{\nabla}_P \times$", we can write

$$\vec{B}(P) = \frac{\mu_0}{4\pi} \iiint\limits_{\tau} \vec{\nabla}_P \times \frac{\vec{J}(Q)}{r_{QP}} \, dV \tag{3.11}$$

From the vector relation

$$\vec{\nabla}_P \times \frac{\vec{J}(Q)}{r_{QP}} = \frac{1}{r_{QP}} \vec{\nabla}_P \times \vec{J} - \vec{J} \times \vec{\nabla}_P \frac{1}{r_{QP}} \tag{3.12}$$

and being $\vec{\nabla}_P \times \vec{J}(Q) = \vec{0}$ we readily obtain

$$\vec{B}(P) = \frac{-\mu_0}{4\pi} \iiint\limits_{\tau} \vec{J}(Q) \times \vec{\nabla}_P \frac{1}{r_{QP}} \, dV \tag{3.13}$$

From the definition of gradient,

$$\vec{\nabla}_P \frac{1}{r_{QP}} = -\frac{1}{r_{QP}^2} \vec{u}_{QP} = -\frac{\vec{r}_{QP}}{r_{QP}^3} \tag{3.14}$$

Finally,

$$\vec{B}(P) = \frac{\mu_0}{4\pi} \iiint\limits_{\tau} \frac{\vec{J}(Q) \times \vec{r}_{QP}}{r_{QP}^3} \, dV \tag{3.15}$$

Consider a filiform conductor with transversal cross-section dA and with a stationary current I (cf. Fig. 3.5). The volume of an infinitesimal element $d\ell$ of the conductor is

$$dV = dA d\ell \tag{3.16}$$

From Eq. (3.15), we find

$$\begin{aligned} \vec{B}(P) &= \frac{\mu_0}{4\pi} \iiint\limits_{\tau} \frac{\vec{J}(Q) dA \times \vec{r}_{QP}}{r_{QP}^3} \, d\ell \\ &= \frac{\mu_0}{4\pi} I \oint\limits_{\gamma} \frac{\vec{t} \times \vec{r}_{QP}}{r_{QP}^3} \, d\ell \end{aligned} \tag{3.17}$$

Where $I = \iint\limits_{dA} J(Q) dA$. Note that $\vec{J}(Q) = J(Q)\vec{t}$ and

$$\begin{aligned} I &= \iint\limits_{dA} \vec{J}(Q) \bullet \vec{n} dA = \iint\limits_{dA} J(Q)\vec{t} \bullet \vec{t} dA \\ &= \iint\limits_{dA} J(Q) dA \end{aligned}$$

The infinitesimal contribution to $\vec{B}$ due to $d\ell$ is thus

$$d\vec{B}(P) = \frac{\mu_0}{4\pi} I d\ell \, \frac{\vec{t} \times \vec{r}_{QP}}{r^3_{QP}} \tag{3.18}$$

This is known as Laplace's elementary law. This equation does not have independent value from Eq. (3.17) because it has no physical meaning to consider an "infinitesimal current element" $Id\ell$ regardless form the closed circuit it is part of.

The resemblance between Eg. (3.18) and

$$d\vec{E}(P) = \frac{1}{4\pi\epsilon_0} dq \, \frac{\vec{r}_{QP}}{r^3_{QP}}$$

valid in electrostatic is remarkable.

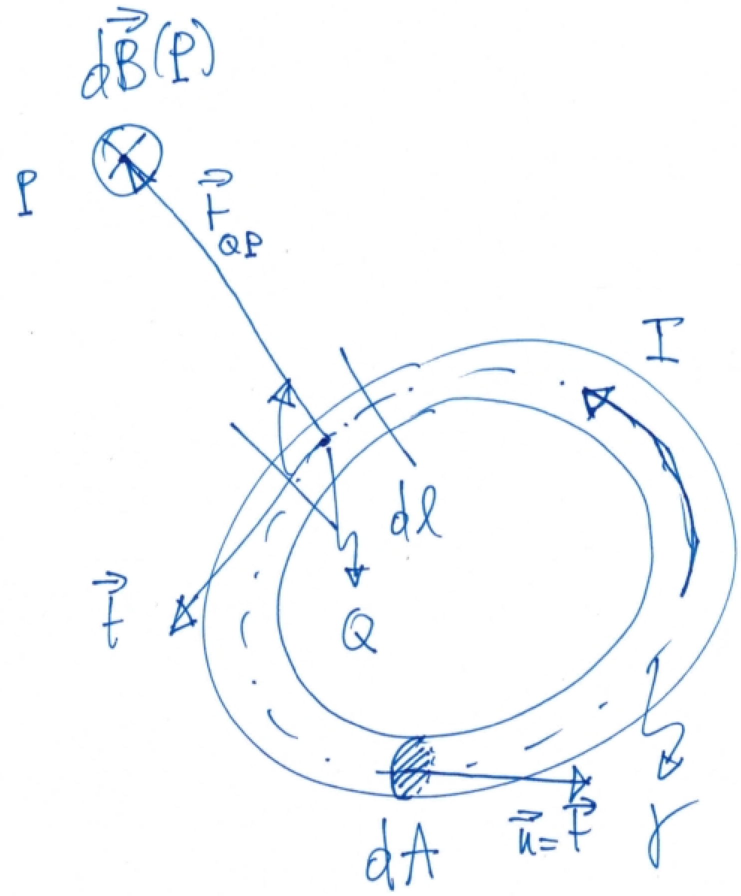

Figure 3.5