This problem set covers the first 15 lectures of the course.

1. Consider a straight quasi-filiform conductor of infinite length, carrying a steady current I (cf. Fig. 1.1).

   1.1. Calculate the magnetostatic field $\vec{B}$ due to the conductor at any point P in space by means of Laplace's elementary law (Eq. (3.18) of the course). [15 points]

   1.2. Show under what conditions the field $\vec{B}(P)$ calculated in 1.1 can be represented by a scalar potential. Is such a potential always a single-valued function? [5 points]

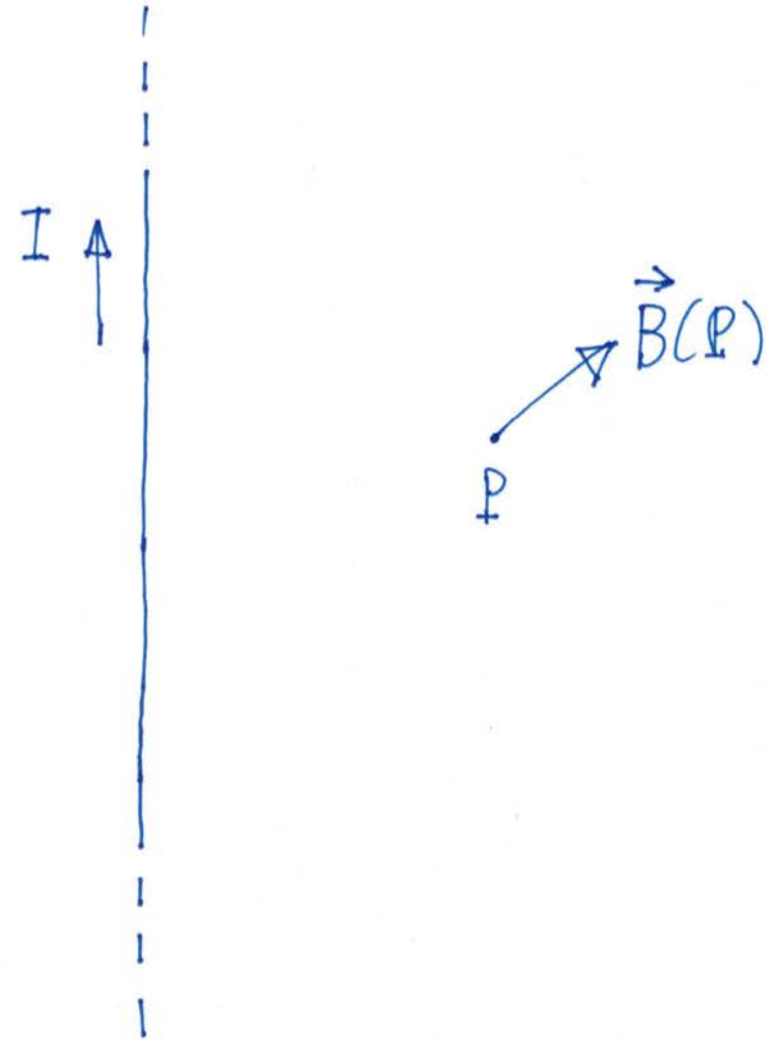


Figure 1.1.

1.1. First, we choose a cylindrical coordinate system $Or\varphi z$, with axis r passing through P and axis z coinciding with the conductor and oriented as the current I (cf. Fig. 1.2).

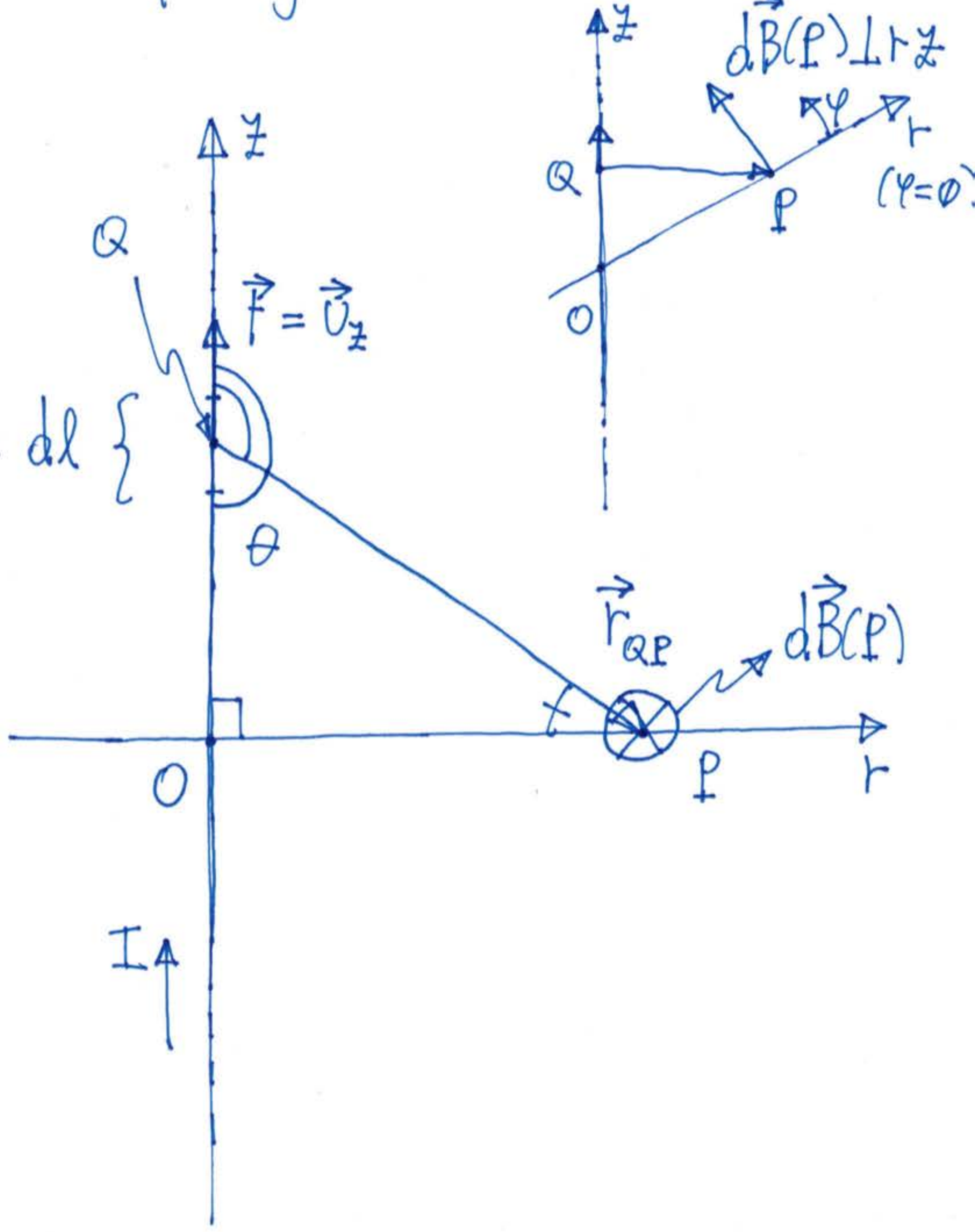


Figure 1.2.

From Laplace's elementary law, we have

$$d\vec{B}(P) = \frac{\mu_0}{4\pi} I\, dl \, \frac{\vec{t} \times \vec{r}_{QP}}{r_{QP}^3} \qquad (1.1P)$$

which is the infinitesimal magnetostatic field at P due to an infinitesimal current element $I dl$ at Q on the wire. From Fig. 1.2,

$$dl = dz \qquad (1.2aP)$$

$$\vec{t} = \vec{u}_z \qquad (1.2bP)$$

$$\frac{\vec{r}_{QP}}{r_{QP}} = \frac{P - Q}{\overline{QP}} \qquad (1.2cP)$$

$$r_{QP} = \left(r^2 + z^2 \right)^{1/2} \qquad (1.2dP)$$

Thus,

$$\vec{t} \times \vec{r}_{QP} = r_{QP} \sin(\pi - \theta) \vec{u}_\varphi$$
$$= r_{QP} \sin\theta \, \vec{u}_\varphi \qquad (1.3P)$$

where

$$\sin\theta = \frac{r}{r_{QP}} \qquad (1.4P)$$

Equation (1.1P) can then be written as

$$d\vec{B}(P) = \frac{\mu_0 I}{4\pi} \frac{r}{r_{QP}^3} dz \, \vec{u}_\varphi$$
$$= \frac{\mu_0 I}{4\pi} \frac{r}{(r^2 + z^2)^{3/2}} dz \, \vec{u}_\varphi \qquad (1.5P)$$

It is convenient to rewrite this result in terms of $\sin\theta$:

$$d\vec{B}(P) = \frac{\mu_0 I}{4\pi}\, r\, \frac{\sin^3\theta}{r^3}\, dz\, \vec{u}_\varphi$$

$$= \frac{\mu_0 I}{4\pi}\, \frac{\sin^3\theta}{r^2}\, dz\, \vec{u}_\varphi \qquad (1.6P)$$

where we used $r_{QP} = r/\sin\theta$ from (1.4P).
From

$$\tan\theta = \frac{r}{z} \qquad (1.7P)$$

we have

$$z = \frac{r}{\tan\theta} \qquad (1.8P)$$

and, thus,

$$dz = -\frac{r}{\sin^2\theta}\, d\theta \qquad (1.9P)$$

Hence, (1.6 P) becomes

$$d\vec{B}(P) = \frac{\mu_0 I}{4\pi} \frac{\sin^3\theta}{r^2} \left(-\frac{r}{\sin^2\theta} d\theta\right) \vec{u}_\varphi$$

$$= -\frac{\mu_0 I}{4\pi} \frac{1}{r} \sin\theta \, d\theta \, \vec{u}_\varphi \qquad (1.10P)$$

We now need to integrate (1.10P) over the entire conductor. As shown in Fig. 1.3, the angle $\theta \to \pi$ when $z \to -\infty$ and $\theta \to 0$ when $z \to +\infty$. As a consequence,

$$\vec{B}(P) = \int_{conductor} d\vec{B}(P) = -\frac{\mu_0 I}{4\pi r} \vec{u}_\varphi \int_{\pi}^{0} \sin\theta \, d\theta$$

$$= -\frac{\mu_0 I}{4\pi r} \vec{u}_\varphi \left[-\cos\theta\right]_{\pi}^{0}$$

$$= \frac{\mu_0 I}{2\pi r} \vec{u}_\varphi \qquad (1.11P)$$

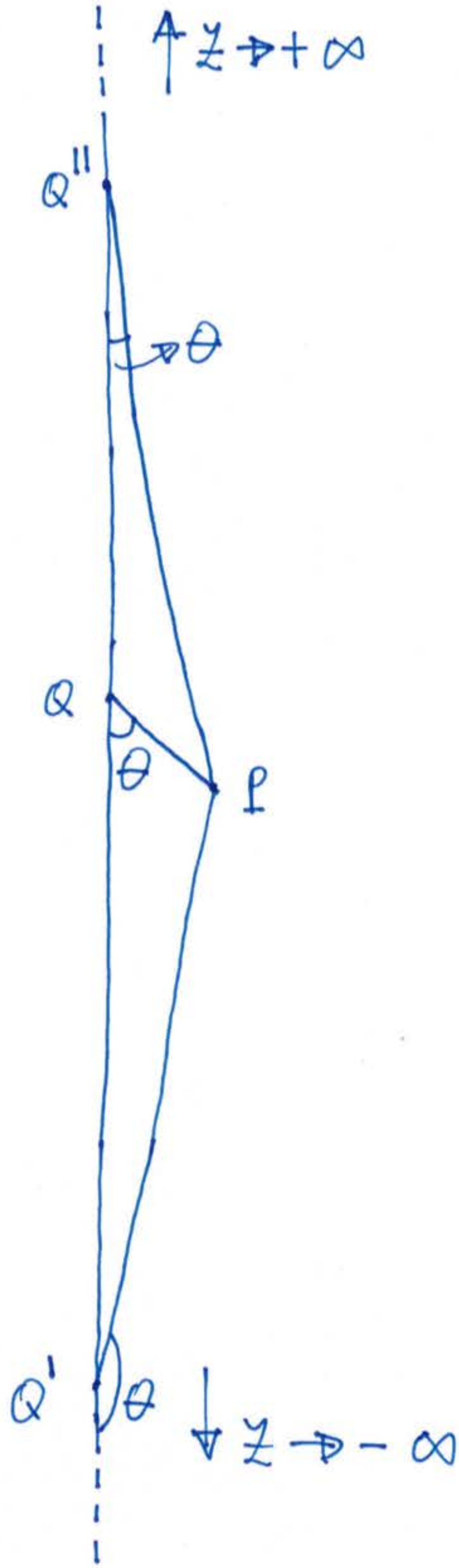


Figure 1.3.

1.2. From (1.11P), the scalar potential (9)
associated with $\vec{B}(P)$ must be a
function of the type

$$\psi(P) = \frac{\mu_0 I}{2\pi}\varphi \qquad (1.12P)$$

In fact, from the components of the
gradient in cylindrical coordinates, we
have

$$(\vec{\nabla}\psi)_r = \frac{\partial}{\partial r}\psi = 0 \qquad (1.13aP)$$

$$(\vec{\nabla}\psi)_\varphi = \frac{1}{r}\frac{\partial}{\partial\varphi}\psi = \frac{\mu_0 I}{2\pi r} \qquad (1.13bP)$$

$$(\vec{\nabla}\psi)_z = \frac{\partial}{\partial z}\psi = 0 \qquad (1.13cP)$$

and, thus,

$$\vec{B}(P) = \vec{\nabla}\Psi(P) \qquad (1.14P)$$

Equation (1.12P) is a multi-valued function when considered on any entire plane parallel to the plane $Or\varphi$. However, when excluding from any such plane a ray originating from the intersection point between the conductor and the plane (e.g., the ray from point O on the plane $Or\varphi$ in Fig. 1.4), Eq. (1.12P) becomes a single-valued function. Thus, considering only open lines, such as γ_1 and γ_2 in Fig. 1.4, we have

$$\int\limits_{P_1\gamma_1 P_2} \vec{B}\cdot\vec{t}\,dl = \Psi(P_2) - \Psi(P_1) = \int\limits_{P_1\gamma_2 P_2} \vec{B}\cdot\vec{t}\,dl$$

$$= \frac{\mu_0 I}{2\pi}(\varphi_2 - \varphi_1) \qquad (1.15P)$$

Considering instead a closed line γ_3 that contains O, we have (11)

$$\oint_{\gamma_3} \vec{B}\cdot\vec{t}\,dl = \psi(P) - \psi(P) = \frac{\mu_0 I}{2\pi}(2\pi - \varphi)$$
$$= \mu_0 I \qquad (1.16P)$$

under the assumption to assign value 0 to φ in P, when P is the starting point of γ_3, and value 2π, when P is the ending point of γ_3. Thus, we must assign two different values to φ for the same point P. Equation (1.16P) is then consistent with Ampère's law, but ψ is a multi-valued function. In order to cure this illness of ψ we must remove a ray originating from the conductor on any of the usual planes, e.g., a ray r from the $Or\varphi$ plane.

In this case, however, it is impossible to consider closed line that contain, e.g., O (r has been removed). The so defined potential is therefore characterized by a discontinuity of the first kind in correspondence of the points of, e.g., r. In fact,

$$\lim_{\varphi_1 \to 0} \psi(P_1) = 0 \,, \quad \lim_{\varphi_2 \to 2\pi} \psi(P_2) = \mu_0 I \quad (1.17)$$

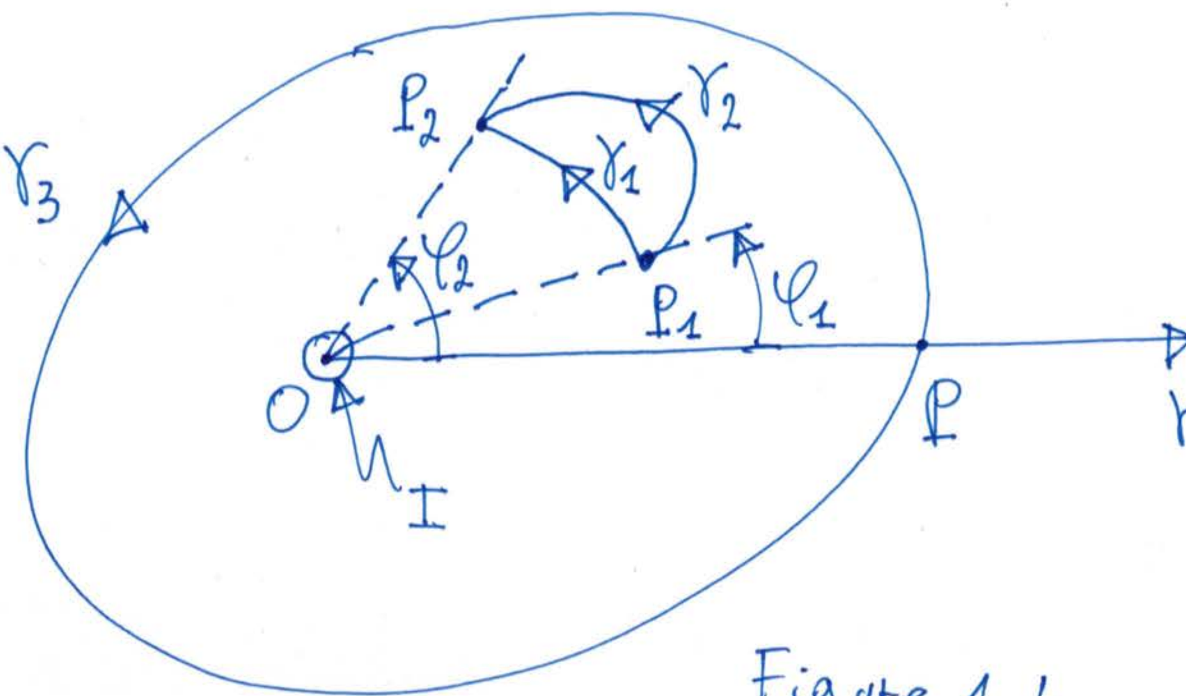


Figure 1.4.

2. Consider a semi-circular quasi-filiform conductor starting at point A and ending at point B connected to a straigth quasi-filiform conductor between points B and A (cf. Fig. 1.5). The structure carries a steady current I that flows counterclockwise on the circular line γ_1 and from B to A on the straigth line γ_2. The radius of the semi-circular conductor is R.

Calculate the magnetostatic field $\vec{B}$ at any point P on the axis of the semi circle by means of Laplace's elementary law. [20 points]

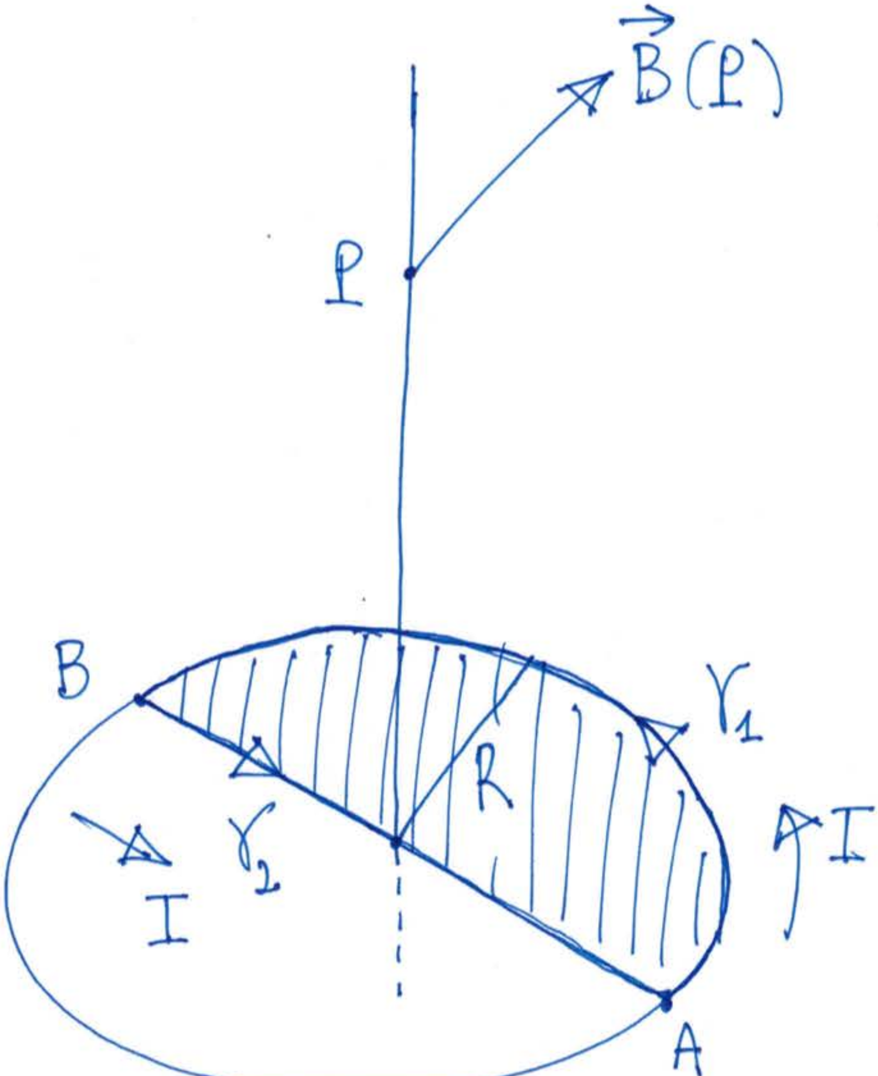


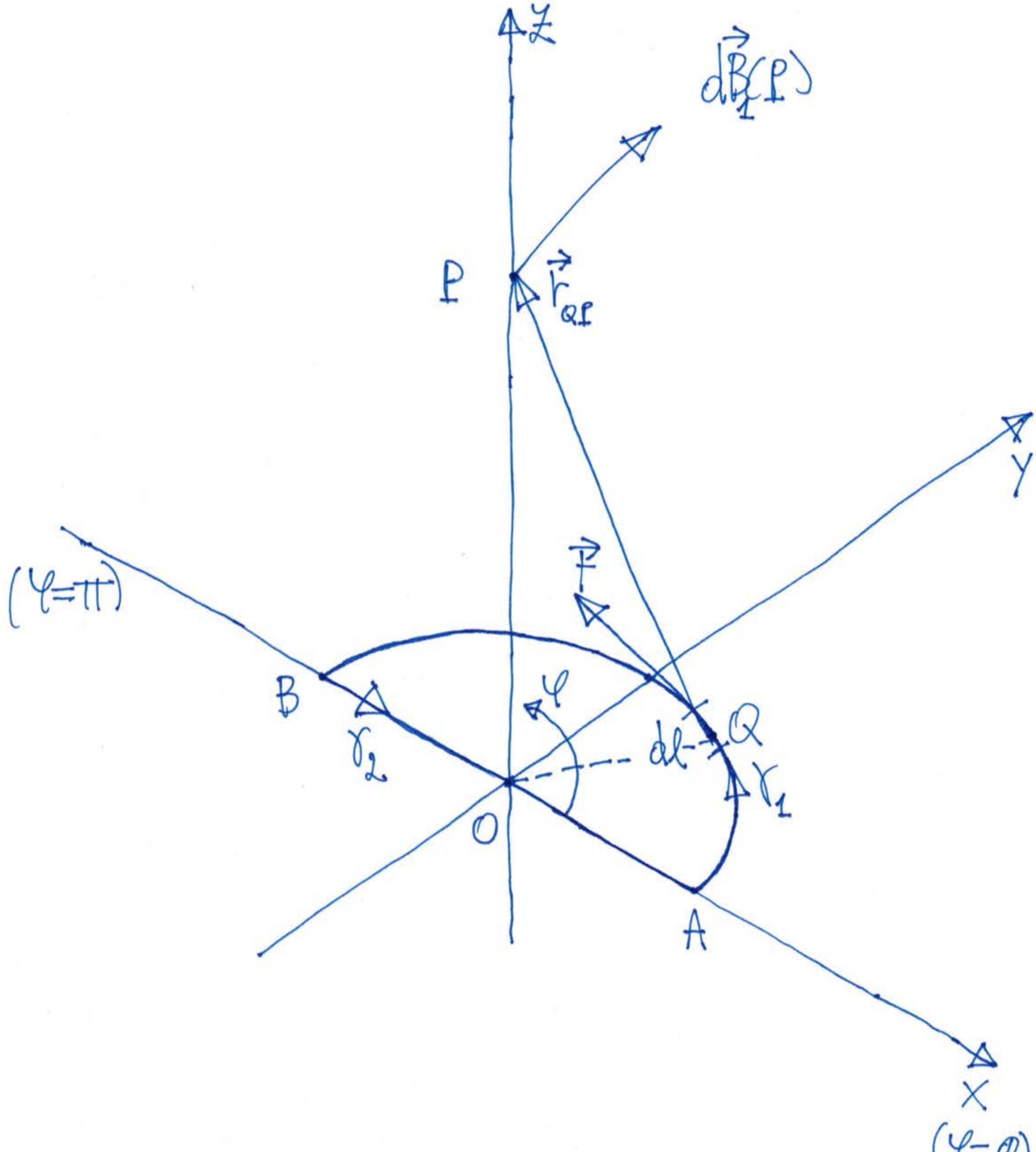
Figure 1.5.
The area of the circuit $\gamma_1 \cup \gamma_2$ to be considered has been hashed.

———— o ————

We first solve for the magnetostatic field $\vec{B}_1(P)$ due to γ_1 and then for $\vec{B}_2(P)$ due to γ_2. Finally, we superimpose the two fields:

$$\vec{B}(P) = \vec{B}_1(P) + \vec{B}_2(P) \tag{1.18P}$$

We choose a Cartesian coordinate system $Oxyz$ with axis z on the semi-circle axis containing P and axis x on the line segment BA, as shown in Fig. 1.6. The angle $\varphi = 0$ on the positive x axis and $\varphi = \pi$ on the negative x axis (i.e. the angle is oriented counterclockwise). In this system, P is at distance z from O.



Z
$d\vec{B}_1(P)$
P
$\vec{r}_{QP}$
Y
$\vec{F}$
$(\varphi=\pi)$
B
γ_2
φ
Q
dl
γ_1
O
A
X
$(\varphi=0)$

Figure 1.6.

- Line γ_1

Consider a generic point Q on γ_1. The infinitesimal line element dl on γ_1 is assumed to be centered in Q and, with respect to the notation in Fig. 1.6, is given by

$$dl = R\, d\varphi \tag{1.19P}$$

The unit vector $\vec{t} = \vec{u}_\varphi$ is tangent to γ_1 at Q. The norm of vector $\vec{r}_{QP}$ is given by

$$r_{QP} = \left(R^2 + z^2 \right)^{1/2} \tag{1.20P}$$

In addition, we note that $\vec{t} \perp \vec{r}_{QP}$ for each point Q on γ_1. Thus, from Laplace's elementary law we find that

$$dB_1(P) = \frac{\mu_0}{4\pi} I R\, d\varphi \frac{1}{(R^2 + z^2)} \tag{1.21P}$$

where we used (1.19P), (1.20P), and the fact that $\vec{r} \perp \vec{r}_{QP}$ in Eq. (1.1P). The direction of $d\vec{B}_1(P)$ is normal to the plane containing $\vec{r}$ and $\vec{r}_{QP}$. Thus, in the plane of the triangle $\widehat{OQP}$, $d\vec{B}_1$ can be decomposed in a normal and a parallel component, as shown in Fig. 1.7. Hence,

$$d\vec{B}_1 = d\vec{B}_1^{\perp} + d\vec{B}_1^{\parallel} \qquad (1.22P)$$

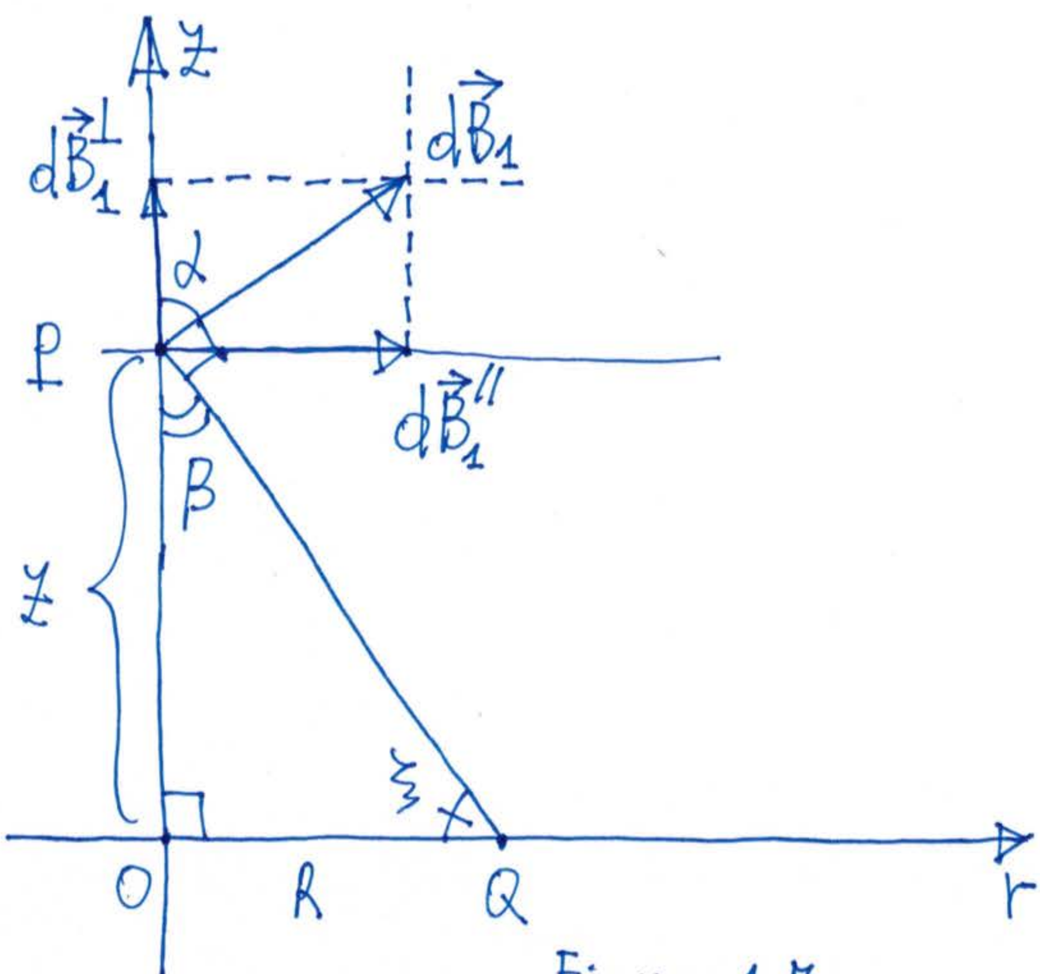


Figure 1.7.

Referring to the notation in Fig. 1.7, we have

$$\alpha + \frac{\pi}{2} + \beta = \pi \tag{1.23P}$$

from which $\beta = (\pi/2) - \alpha$ and

$$\xi + \frac{\pi}{2} + \beta = \pi$$

from which

$$\xi = \frac{\pi}{2} - \beta = \cancel{\frac{\pi}{2}} - \cancel{\frac{\pi}{2}} + \alpha = \alpha \tag{1.24P}$$

In addition,

$$d\vec{B}_1^{\perp} = dB_1 \cos\alpha \, \vec{u}_z = d\vec{B}_1^{z} \tag{1.25P}$$

and

$$d\vec{B}_1^{\parallel} = dB_1 \sin\alpha \, \vec{u}_r \tag{1.26P}$$

where

$$\cos\alpha = \frac{R}{(R^2+z^2)^{1/2}} \tag{1.27P}$$

and

$$\sin\alpha = \frac{z}{(R^2+z^2)^{1/2}} \tag{1.28P}$$

The component $d\vec{B}_1^{\parallel}$ can be further decomposed in the xy plane, as shown in Fig. 1.8.

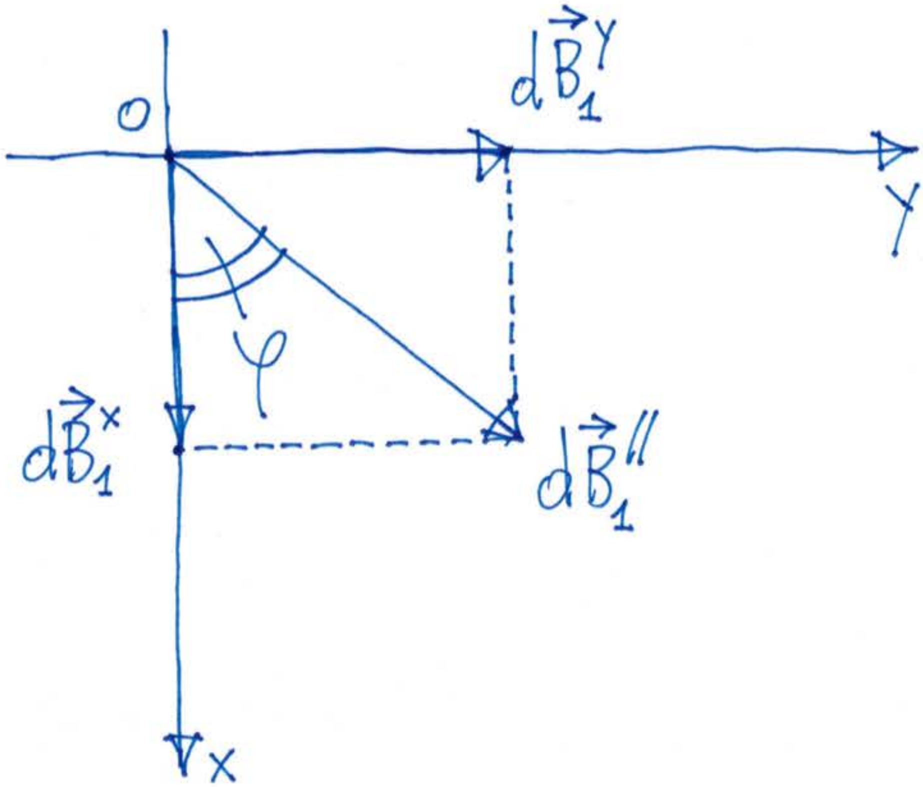


Figure 1.8.

We find

$$d\vec{B}_1^{\parallel} = d\vec{B}_1^{x} + d\vec{B}_1^{y}$$

$$= dB_1 \sin\alpha \cos\varphi \, \vec{U}_x$$

$$+ dB_1 \sin\alpha \sin\varphi \, \vec{U}_y \qquad (1.29P)$$

Using (1.25P), (1.29P), (1.27P), and (1.28P) in (1.21P), we have

$$d\vec{B}_1(P) = \frac{\mu_0}{4\pi} \frac{IR}{(R^2+z^2)} \left[\frac{R}{(R^2+z^2)^{1/2}} \vec{U}_z \, d\varphi \right.$$

$$+ \frac{z}{(R^2+z^2)^{1/2}} \cos\varphi \, \vec{U}_x \, d\varphi$$

$$\left. + \frac{z}{(R^2+z^2)^{1/2}} \sin\varphi \, \vec{U}_y \, d\varphi \right] \qquad (1.30P)$$

This can be written as

$$d\vec{B}_1(P) = \frac{\mu_0}{4\pi} \frac{IR}{(R^2+z^2)^{3/2}} \Big(R\,d\varphi\,\vec{u}_z + z\cos\varphi\,d\varphi\,\vec{u}_x + z\sin\varphi\,d\varphi\,\vec{u}_y \Big) \quad (1.31P)$$

The total field $\vec{B}_1(P)$ can be found by integrating (1.31P) along γ_1 from A to B, i.e., from $\varphi = 0$ to $\varphi = \pi$. Thus,

$$\vec{B}_1(P) = \frac{\mu_0}{4\pi} \frac{IR}{(R^2+z^2)^{3/2}} \left(R\vec{u}_z \int_0^\pi d\varphi + z\vec{u}_x \int_0^\pi \cos\varphi\,d\varphi + z\vec{u}_y \int_0^\pi \sin\varphi\,d\varphi \right)$$

$$= \frac{\mu_0}{4\pi} \frac{IR}{(R^2+z^2)^{3/2}} \Big(\pi R \vec{u}_z + \left[\sin\varphi\right]_0^{\pi} z \vec{u}_x + \left[-\cos\varphi\right]_0^{\pi} z \vec{u}_y \Big)$$

$$= \frac{\mu_0}{4\pi} \frac{IR}{(R^2+z^2)^{3/2}} \left(\pi R \vec{u}_z + 2z \vec{u}_y \right) \tag{1.32P}$$

The fact that the x component goes to zero should not surprise because of the symmetry of γ_1 with respect to the y axis. As expected, when integrating (1.32P) from 0 to 2π we fiend the field of a circular loop:

$$\vec{B}_{loop} = \frac{\mu_0}{2} \frac{IR^2}{(R^2+z^2)^{3/2}} \vec{u}_z \tag{1.32P'}$$

• Line γ_2

The field $\vec{B}_2(P)$ due to the current on the line segment γ_2 can readily be found by integrating (1.10P) between the angles θ_B and θ_A indicated in Fig. 1.9.

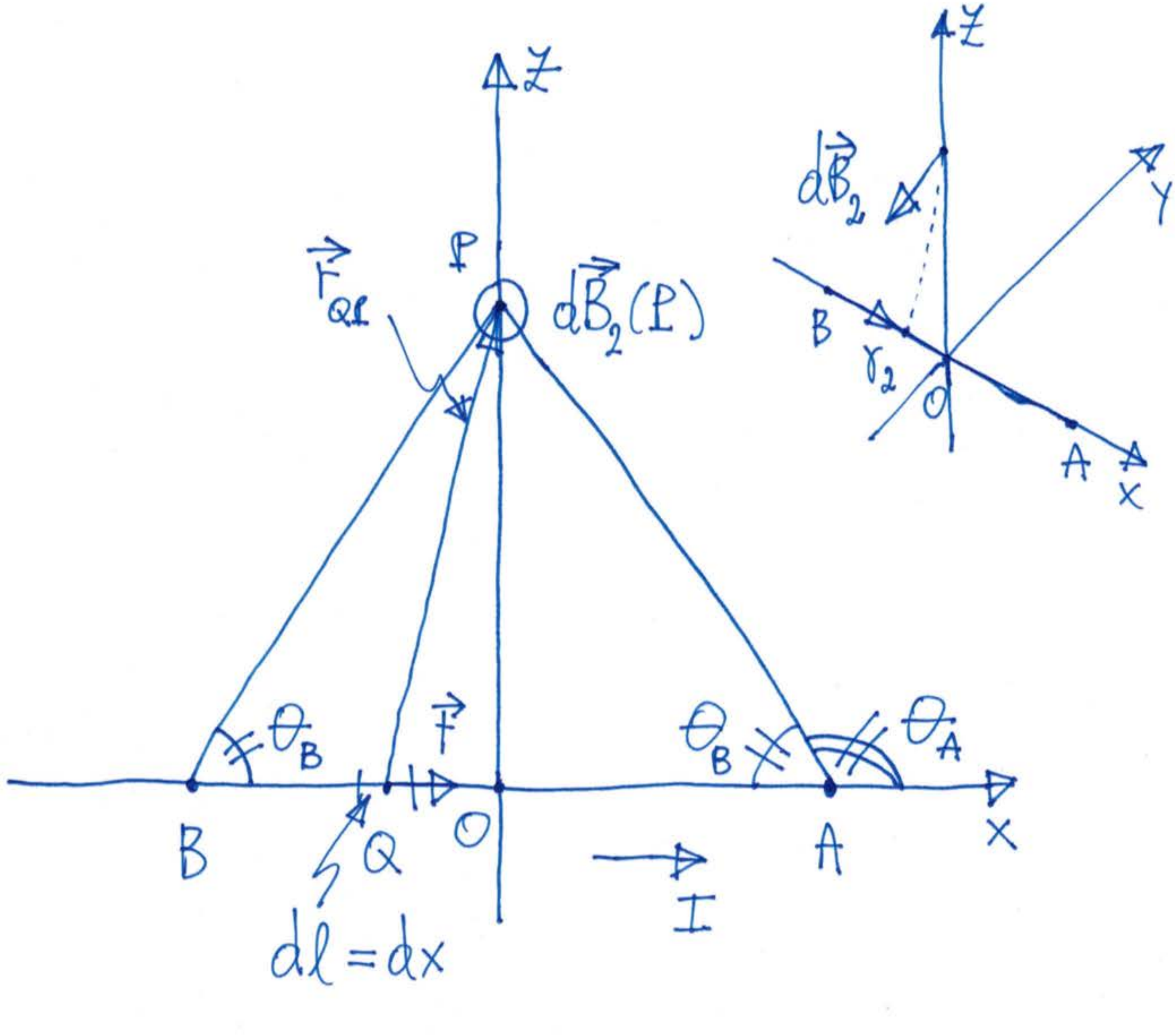


Figure 1.9.

First, we note that

$$\theta_A = \pi - \theta_B \qquad (1.33P)$$

Second,

$$\sin\theta_B = \frac{z}{(R^2+z^2)^{1/2}} \qquad (1.34P)$$

Hence,

$$\theta_B = \sin^{-1}\left[\frac{z}{(R^2+z^2)^{1/2}}\right] \qquad (1.35P)$$

Equation (1.10P) was obtained from (1.1P) for a steady current I directed as the z-axis, with $dl = dz$, $\vec{t} = \vec{u}_z$, and $\overline{OP} = r$. In the case of line γ_2, the steady current I is directed as the x-axis. Both in (1.10P) and for γ_2, I flows consistently with the positive sign of the z-axis and x-axis, respectively. Thus, all signs are consistent in the two cases. For γ_2, $d\ell = dx$, $\vec{F} = \vec{u}_x$, and $\overline{OP} = z$. 26

In addition, $\vec{u}_\varphi$ in (1.10P) is $\vec{u}_y$ for γ_2. As a consequence, from (1.11P) we find

$$\vec{B}_2(P) = -\frac{\mu_0 I}{4\pi z}\vec{u}_y \int_{\theta_B}^{(\pi-\theta_B)} \sin\theta \, d\theta$$

$$= -\frac{\mu_0 I}{4\pi z}\vec{u}_y \left[-\cos\theta\right]_{\theta_B}^{(\pi-\theta_B)}$$

$$= -\frac{\mu_0 I}{4\pi z}\vec{u}_y \, 2\cos\theta_B$$

$$= -\frac{\mu_0 I}{2\pi z}\cos\left\{\sin^{-1}\left[\frac{z}{(R^2+z^2)^{1/2}}\right]\right\}\vec{u}_y$$

(1.36P)

The negative sign is consistent with the righthand rule for I on γ_2.

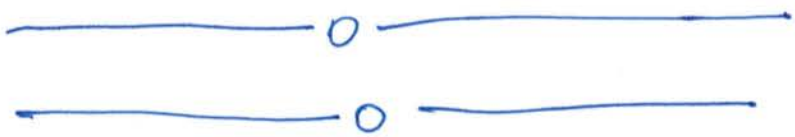
- Line $\gamma_1 \cup \gamma_2$

The total field $\vec{B}(P)$ is finally found from (1.18P), superimposing the fields (1.32P) and (1.36P):

$$\vec{B}(P) = \frac{\mu_0 I}{\pi} \left\{ \left[\frac{zR}{2(R^2+z^2)^{3/2}} - \frac{1}{2z} \cos\theta_B \right] \vec{u}_y + \frac{\pi R^2}{4(R^2+z^2)^{3/2}} \vec{u}_z \right\} \qquad (1.37P)$$

The z component, as expected from the vector product in (1.1P), is positive for a positive or negative z. From the vector product, the sign of the y component due to γ_1 is consistent with the sign of z. Since $\cos\theta_B > 0$, the sign of the y

component due to γ_2 is consistent with the sign of z.



3. Consider an infinite conducting plane of negligible thickness carrying a surface steady current with density $\vec{J}$ uniformly directed on the plane (cf. Fig. 1.10P).

Calculate the magnetostatic field $\vec{B}$ due to $\vec{J}$ at any point P in space by means of the vector potential. [20 points]

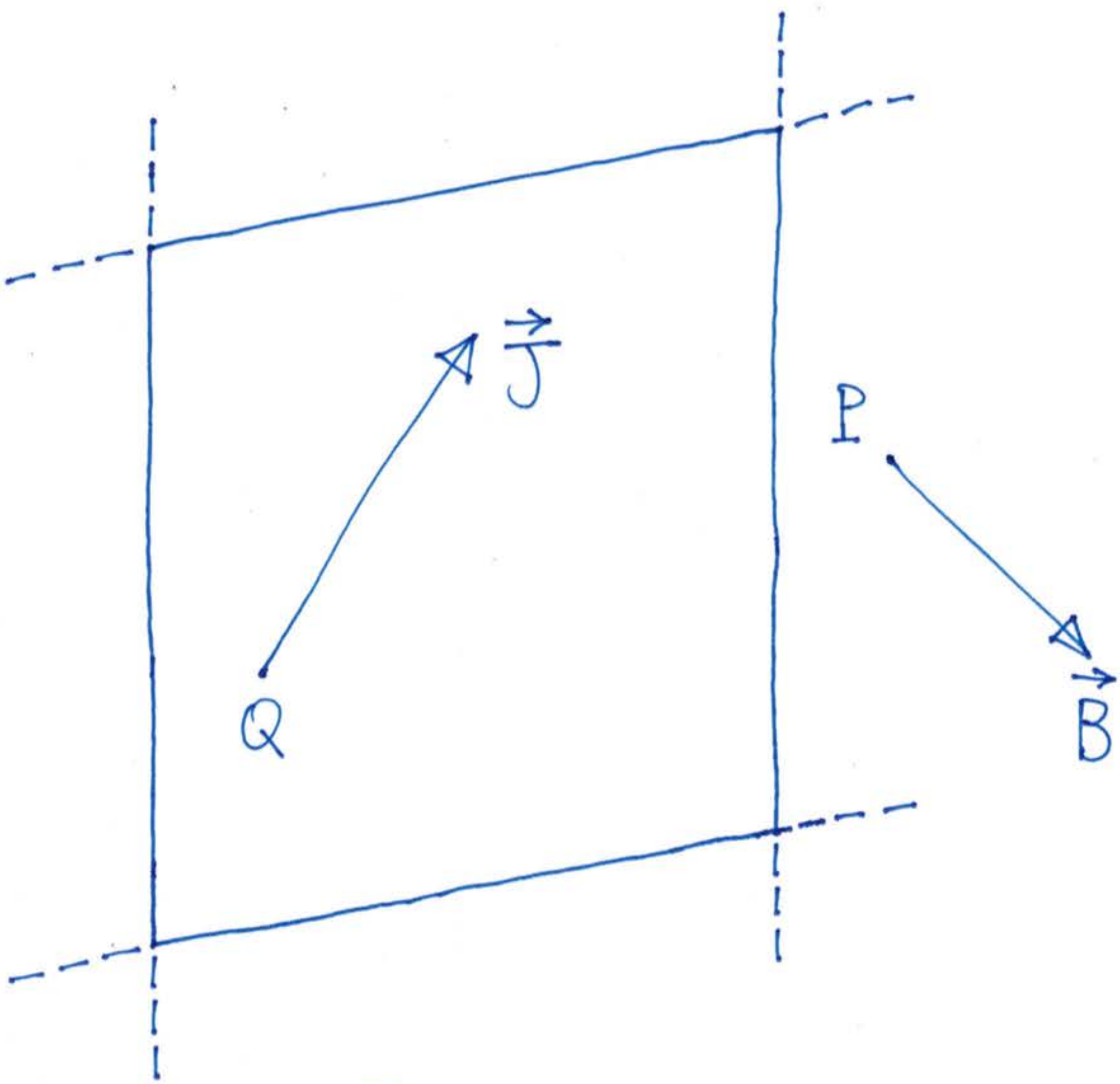


Figure 1.10P.

Since $\vec{J}$ is uniformly directed on the plane, we choose a Cartesian coordinate system with y-axis entirely on the plane and directed and oriented as $\vec{J}$. The z-axis is normal to the plane, as shown in Fig. 1.11P.

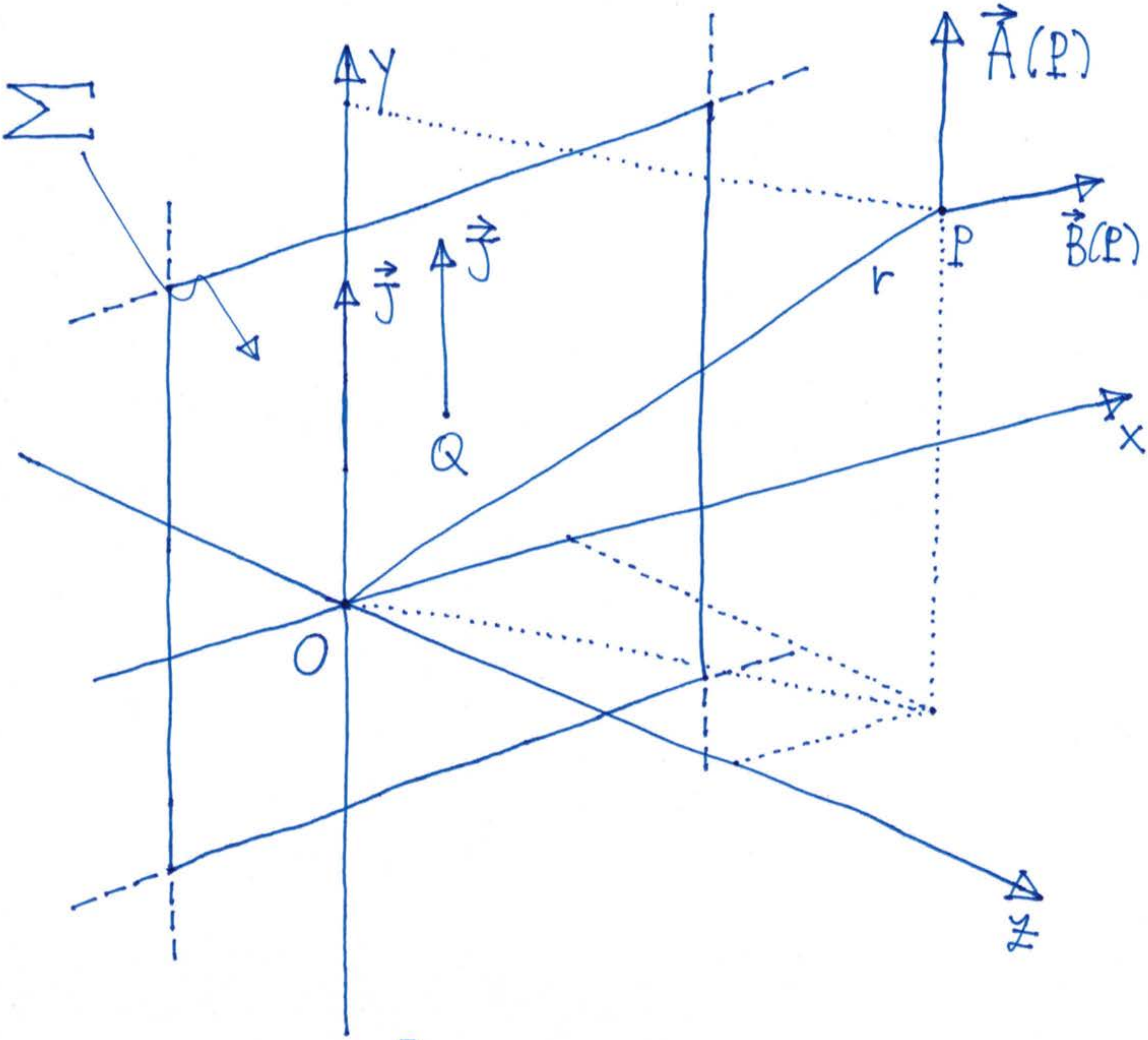


Figure 1.11P.

With respect to the Oxyz coordinate system of Fig. 1.11 P,

$$\vec{J} = J_y \vec{U}_y \qquad (1.38\,P)$$

Equation (4.19) can be written for plane Σ in two dimensions as

$$\vec{A}(P) = \frac{\mu_0}{4\pi} \iint_{\Sigma} \frac{J_y(Q)\vec{U}_y}{r}\, dA \qquad (1.39\,P)$$

The same equations have the same solutions. Thus, by substituting

$$\sigma(Q) = \frac{J_y(Q)}{c^2} = \mu_0 \varepsilon_0 J_y(Q) \qquad (1.40\,P)$$

that is

$$J_y(Q) = \frac{1}{\mu_0 \varepsilon_0} \sigma(Q) \qquad (1.41P)$$

we can write the maguitude of Eq. (1.39P) as

$$\phi(P) = \frac{1}{4\pi\varepsilon_0} \iint_{\Sigma} \frac{\sigma}{r} dA \qquad (1.42P)$$

which represents the electrostatic potential of a uniformly charged infinite plane Σ with constant surface charge density σ given by (1.41P). From PHYS242 we already know the solution of (1.42P). With respect to the coordinate system of Fig. 1.11P and setting the potential ϕ to be zero in correspondance of the surface of the plane

Σ, we have

$$\phi(z) = \begin{cases} +\dfrac{\sigma}{2\varepsilon_0} z \,, \ \forall z \in (-\infty, 0] & (1.43aP) \\[2em] -\dfrac{\sigma}{2\varepsilon_0} z \,, \ \forall z \in [0, +\infty) & (1.43bP) \end{cases}$$

and

$$\vec{A}(P) = \vec{A}(z) = \phi(z)\, \vec{U}_y \qquad (1.44P)$$

Thus, from (1.40P) we finally find

$$\vec{A}(z) = \begin{cases} +\dfrac{\mu_0 J_y}{2} z\, \vec{U}_y \,, \ \forall z \in (-\infty, 0] & (1.45aP) \\[2em] -\dfrac{\mu_0 J_y}{2} z\, \vec{U}_y \,, \ \forall z \in [0, +\infty) & (1.45bP) \end{cases}$$

From the definition of vector potential, we have

$$\vec{B}(P) = \vec{\nabla} \times \vec{A}(P) \tag{1.46P}$$

In Cartesian coordinates the curl is given by

$$\vec{\nabla} \times \vec{A} = (\vec{\nabla} \times A)_x + (\vec{\nabla} \times \vec{A})_y + (\vec{\nabla} \times \vec{A})_z \tag{1.47P}$$

where

$$(\vec{\nabla} \times \vec{A})_x = \frac{\partial}{\partial y} A_z - \frac{\partial}{\partial z} A_y \tag{1.48aP}$$

$$(\vec{\nabla} \times \vec{A})_y = \frac{\partial}{\partial z} A_x - \frac{\partial}{\partial x} A_z \tag{1.48bP}$$

$$(\vec{\nabla} \times A)_z = \frac{\partial}{\partial x} A_y - \frac{\partial}{\partial y} A_x \tag{1.48cP}$$

For the vector potential (1.45P), the only nonzero component is the x component given by (1.48aP). The only nonzero term of this component is the second term. Thus, we find 35

$$\vec{B}(P) = \begin{cases} -\dfrac{\mu_0 J_y}{2}\vec{U}_x \,, & \forall x, y \in (-\infty, +\infty) \\ & \forall z \in (-\infty, 0] \quad (1.49aP) \\ \\ +\dfrac{\mu_0 J_y}{2}\vec{U}_x \,, & \forall x, y \in (-\infty, +\infty) \\ & \forall z \in [0, +\infty) \quad (1.49bP) \end{cases}$$

This field, which is plotted in Fig. 1.12P, presents a discontinuity of the first kind at $z=0$. The direction of $\vec{B}$ along the x-axis should not surprise. In fact, we can decompose Σ into an infinite set of infinitely long filiform conductors adjacent to each other. The conductors are all parallel to each other and parallel to the y-axis. Figure 1.13P shows a top view of the wires, as seen from the y-axis.

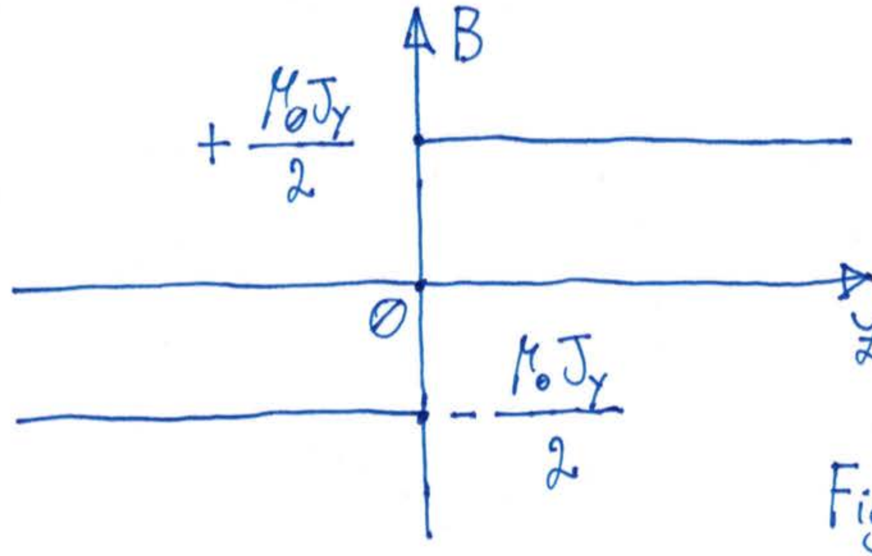


Figure 1.12P.

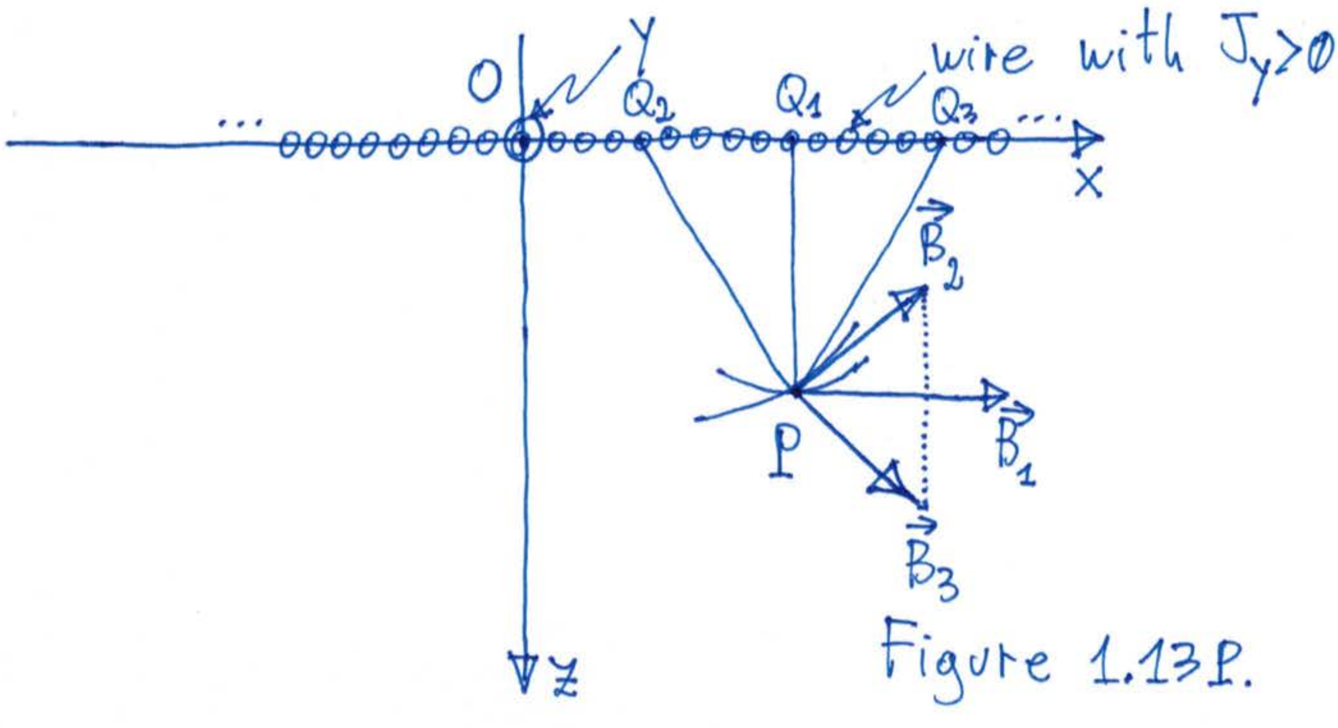


Figure 1.13P.

The figure shows three fields $\vec{B}_1$, $\vec{B}_2$, and $\vec{B}_3$ at P due to three wires Q_1, Q_2, and Q_3. Each wire generates a field given by (1.11P). Because the plane is infinite, it is always possible to find two wires of type Q_2 and Q_3 at equal x distance from Q_1. Thus, the only nonzero component due to all wires is directed as $\vec{u}_x$, in the positive direction for $z \geq 0$ and in the negative direction for $z \leq 0$.

4. Consider the same infinite plane with current $\vec{J}$ of problem 3 in this set. Assume a particle with charge $-|q|$ is free to move in front of the plane at a constant velocity $\vec{v}$; the velocity is parallel to the plane, directed as $\vec{J}$, and oriented opposite of $\vec{J}$. Further assume the velocity of the charge carriers (negative) on the surface of the plane is also $\vec{v}$. The thickness of the plane is negligible and, thus, only surface conduction electrons can exist on it (cf. Fig 1.14ℓ).

Calculate the force acting on the particle $-|q|$ both in the reference frame of the plane and in that of the particle. [20 points]

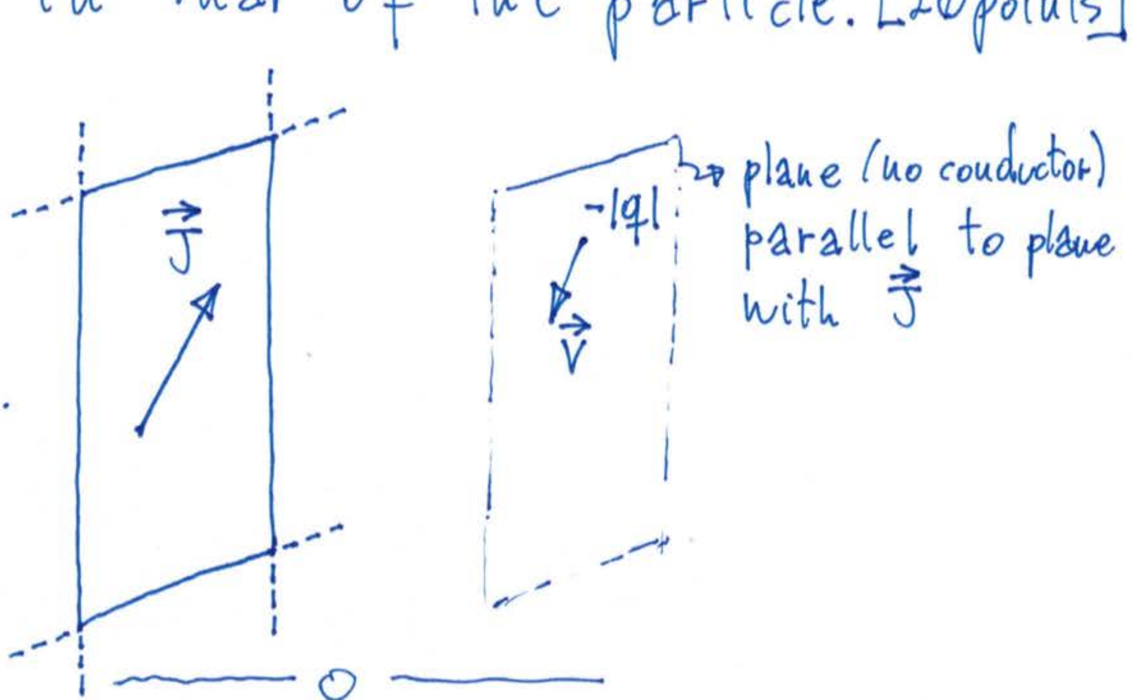


Figure 1.14ℓ.

We consider the same Cartesian coordinate system of Fig. 1.11P, where

$$\vec{J} = J_y \vec{u}_y \, , \quad \text{with } J_y > 0 \qquad (1.50P)$$

and

$$\vec{v} = -|v| \vec{u}_y \, , \quad (v < 0) \qquad (1.51P)$$

We assume the particle $-|q|$ to be in a region of space where $z > 0$ (similar arguments would apply for $z < 0$). In this case, the field $\vec{B}$ acting on the particle is given by (1.49bP),

$$\vec{B} = \frac{\mu_0 J_y}{2} \vec{u}_x \qquad (1.52P)$$

From Eq. (5.5) of the lectures in the case of a surface current characterized by a density of carriers σ_- (σ_+ for the fixed charges), we have

$$J_y = \sigma_- v = \sigma_+ |v| \qquad (1.53P)$$

In the reference frame of the plane, $\vec{E}=\vec{0}$ and, thus, the force acting on $-|q|$ is given by Lorentz force,

$$\vec{F} = -|q|\,\vec{v}\times\vec{B} = -|q|\,(-|v|)\,\vec{u}_y \times \vec{u}_x \frac{\mu_0}{2}\sigma_+ |v|$$

$$= -\frac{|q|\mu_0}{2}\sigma_+ v^2\,\vec{u}_z$$

$$= -\frac{|q|}{2\varepsilon_0}\sigma_+ \frac{v^2}{c^2}\,\vec{u}_z \qquad (1.54P)$$

where we used (1.50P)-(1.53P). Given a rectangular piece of the plane Σ with finite sides L and l, Eq. (5.8) of the lectures becomes

$$q = \sigma L l \qquad (1.55P)$$

Thus, in the reference plane of the

particle we can use Eq. (5.18) of the lectures by simply substituting ρ with σ:

$$\sigma' = \sigma_+ \frac{\frac{v^2}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \qquad (1.56P)$$

In this reference frame, the moving plane (with velocity $-\vec{v}$ along the y-axis) generates a magnetostatic field $\vec{B}'$ that does not generate any effect on $-|q|$, now at rest. The plane also appears to be electrically charged with constant charge density (uniform distribution) given by (1.56P). The field $\vec{E}'$ in front of the plane for $z>0$ is (cf. PHYS242)

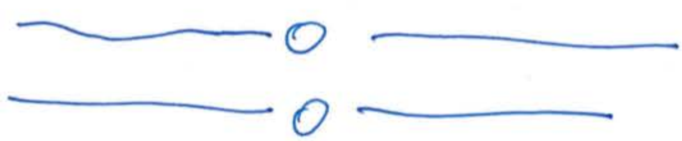
$$\vec{E}' = \frac{\sigma'}{2\varepsilon_0}\vec{u}_z = \frac{1}{2\varepsilon_0}\sigma_+ \frac{\frac{v^2}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}\vec{u}_z$$

(1.57P)

Thus, the force on $-|q|$ is

$$\vec{F}' = -\frac{|q|}{2\varepsilon_0}\sigma_+ \frac{\frac{v^2}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}\vec{u}_z \qquad (1.58P)$$

For small velocities this force is approximately equal to the force found in the frame of reference of the plane, (1.54P).



5. Consider a solid sphere (hard sphere) of radius a and center O. The sphere is made of a perfect conductor in electrostatic equilibrium in vaccum and is at potential V. A conductive shell of radius b and center O surrounds the hard sphere. The shell is at zero potential. Figure 1.15P shows the system under consideration.

By means of the weak form of Poisson's problem calculate the approximate capacitance of the system under consideration, in first and second order (as C_1 and C_2 in lecture 4). Compare the approximate solutions to the exact ones obtained from

$$C = 4\pi\varepsilon_0 \frac{ab}{b-a} \qquad (1.59P)$$

In the calculations, assume $a = 1\,m$. For the first order approximation C_1, consider the ratios

$b/a = 2, 4, 10, 100; 1.5, 1.1$. For C_2 only the ratios $b/a = 10; 1.5$. For C_2, we recommend to use WolframAlpha extensively. [20 points]

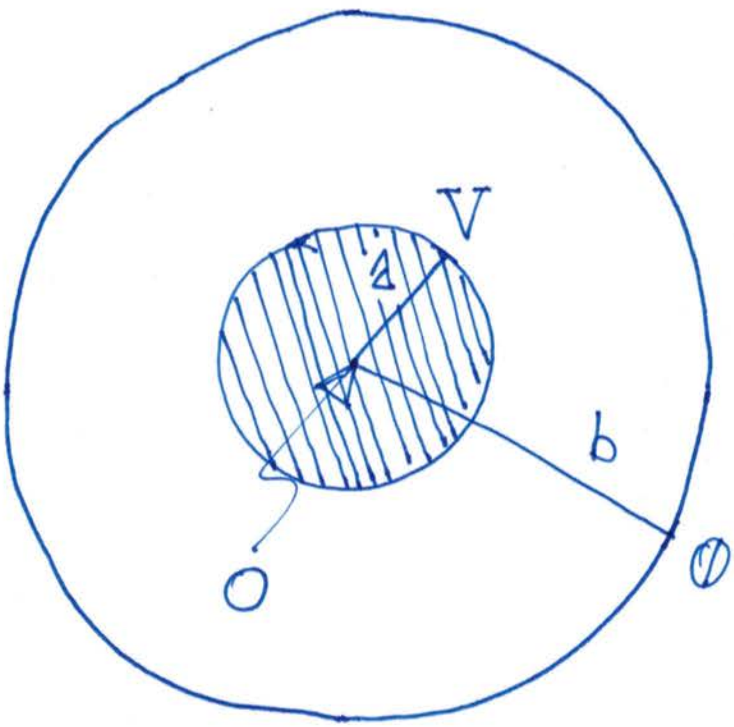


Figure 1.15P.

Consider a ~~[illegible]~~ spherical coordinate system $Or\varphi\theta$, as shown in Fig. 1.16P. 45

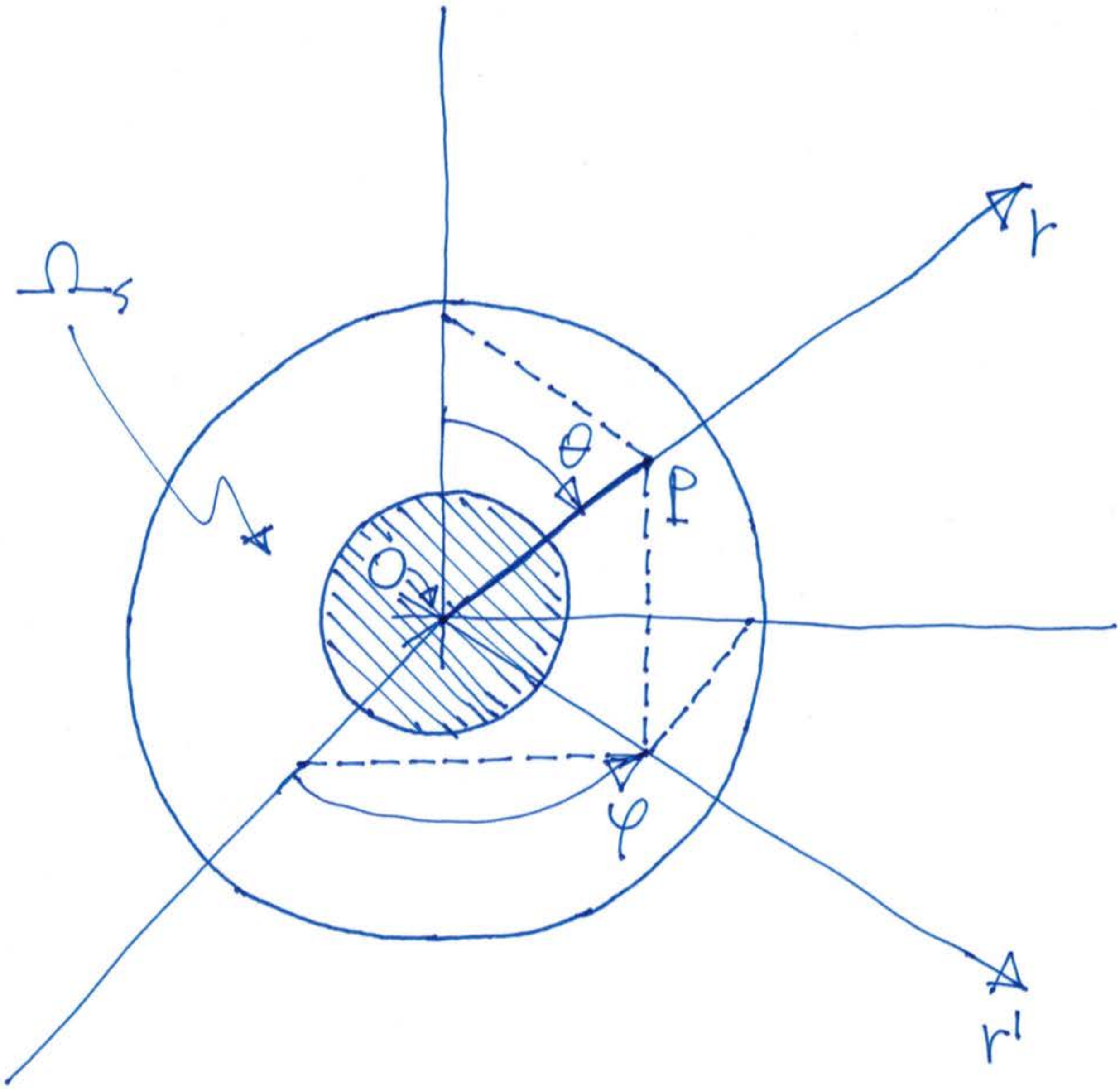


Figure 1.16P.

From Eq. (4.10) of the lectures, the

capacitance of the system can be calculated as

$$\frac{1}{2} C V^2 = \frac{1}{2} \varepsilon_0 \iiint_{\Omega_s} (\vec{\nabla} \phi)^2 \, dV \quad (1.60P)$$

where the system Ω_s is the region of space between the hard sphere and the shell, i.e., for any point P with $r \in [a, b]$, $\varphi \in [0, 2\pi]$, and $\theta \in [0, \pi]$. We now follow lecture 7.

- Case C_1

From the symmetries of the system, we expect the electrostatic field $\vec{E}$ and the potential ϕ to only depend on r (cf. PHYS 242). Suppose that $\vec{E}$ is constant in Ω_s. A possible potential that fulfills the boundary conditions $\phi(r=a) = V$ and

$\phi(r=b)=0$ is

$$\phi(r) = V\left(1 - \frac{r-a}{b-a}\right) \qquad (1.61P)$$

From the definition of gradient in spherical coordinates and the fact that in such coordinates $dV = r^2 dr\, d\varphi \sin\theta\, d\theta$, we can compute the condition that C_1, the first order approximate capacitance, must satisfy:

$$\frac{1}{2} C_1 V^2 \lesssim \frac{1}{2} \varepsilon_0 \int_a^b \left[\frac{\partial}{\partial r}\phi(r)\right]^2 r^2 dr \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\, d\theta \qquad (1.62P)$$

From (1.61 P), we have

$$\frac{\partial}{\partial r}\phi(r) = -\frac{V}{b-a} \qquad (1.63P)$$

Thus, (1.62P) becomes

$$\cancel{\frac{1}{2}} C_1 \cancel{V^2} \lessapprox \cancel{\frac{1}{2}} \varepsilon_0 \int_a^b \frac{\cancel{V^2}}{(b-a)^2} r^2 dr \, 2\pi \left[-\cos\theta\right]_0^\pi$$

$$= \varepsilon_0 \frac{1}{(b-a)^2} \left[\frac{1}{3} r^3\right]_a^b 2\pi \, 2$$

$$= \frac{4}{3}\pi\varepsilon_0 \frac{b^3 - a^3}{(b-a)^2} \qquad (1.64P)$$

Assuming $a = 1\,m$, we need to compare the exact capacitance

$$C = 4\pi\varepsilon_0 \frac{b}{b-1} \qquad (1.65P)$$

to the approximate capacitance

$$C_1 \approx \frac{4}{3}\pi\varepsilon_0 \frac{b^3 - 1}{(b-1)^2} \qquad (1.66P)$$

Note that in (1.66P) we used the approximate sign only as a reminder that C_1 is an approximate solution. Formally, we should use the equal sign because we are actually assigning an exact expression to it. Table 1.1P shows a comparison between the numerical values obtained for (1.65P) and (1.66P) for various ratios of b/a (with a=1m). Note that we renormalized both C and C_1 by $4\pi\varepsilon_0$.

$\frac{b}{a} = b$	$\frac{C}{4\pi\varepsilon_0} = \frac{b}{b-1}$	$\frac{C_1}{4\pi\varepsilon_0} = \frac{b^3-1}{3(b-1)^2}$
2	2	2.33
4	1.33	2.33
10	1.11	4.11
100	1.0101	34.01
1.5	3	3.167
1.1	11	11.03

Table 1.1P.

- Case C_2

We now suppose $\vec{E}$ to linearly depend on r. In this case, a possible potential such that $\phi(r=a) = V$ and $\phi(r=b) = 0$ is

$$\phi(r) = V\left[1+\alpha\left(\frac{r-a}{b-a}\right)-(1+\alpha)\left(\frac{r-a}{b-a}\right)^2\right] \qquad (1.67P)$$

Following the steps after (1.61P), assuming $a=1\,m$, and integrating with respect to r by means of Wolfram Alpha, we directly obtain

$$C_2(\alpha) \approx 4\pi\varepsilon_0 \left\{\frac{1}{(1-b)^4}\left[-(\alpha+1)b^4(\alpha+2+\alpha b) + \frac{1}{3}b^3(\alpha+2+\alpha b)^2+\frac{4}{5}(\alpha+1)^2 b^5 + (\alpha+1)(\alpha+2+\alpha b) - \frac{1}{3}(\alpha+2+\alpha b)^2 - \frac{4}{5}(\alpha+1)^2\right]\right\} \qquad (1.68P)$$

In order to find the minimum of C_2 with respect to α, we must calculate the value of α for which

$$\frac{d}{d\alpha}C_2(\alpha)=0 \tag{1.69P}$$

Using (1.68P) in (1.69P), we have

$$-2\alpha b^4-2b^4-2\alpha b^5-b^4-b^5+\frac{1}{3}b^3 2\alpha(1+b)^2$$
$$+\frac{4}{3}b^3(1+b)+\frac{8}{5}\alpha b^5+\frac{8}{5}b^5+2\alpha+2+2\alpha b$$
$$+1+b-\frac{1}{3}2\alpha(1+b)^2-\frac{4}{3}(1+b)-\frac{8}{5}\alpha-\frac{8}{5}=0 \tag{1.70P}$$

from which

$$\alpha\Big[-2b^4-2b^5+\frac{2}{3}b^3(1+b)^2+\frac{8}{5}b^5+2+2b$$
$$-\frac{2}{3}(1+b)^2-\frac{8}{5}\Big]=2b^4+b^4+b^5-\frac{4}{3}b^3(1+b)$$
$$-\frac{8}{5}b^5-2-1-b+\frac{4}{3}(1+b)+\frac{8}{5} \tag{1.71P}$$

Wolfram Alpha gives

$$\tilde{\alpha} = \frac{-9b^2 - 2b + 1}{4b^2 + 2b + 4} \tag{1.72P}$$

We can now calculate $\tilde{\alpha}$ for $b = 10$ and $b = 1.5$, as requested. We find

$$\tilde{\alpha}(b = 10) \simeq -2.167 \tag{1.73P}$$

and

$$\tilde{\alpha}(b = 1.5) \simeq -1.391 \tag{1.74P}$$

Using these results in (1.68P) for $b = 10$ and $b = 1.5$, respectively, and renormalizing over $4\pi\varepsilon_0$, we find (bar indicates normalization)

$$\bar{C}_2(\tilde{\alpha} = -2.167; b = 10) \simeq 1.971 \tag{1.75P}$$

and

$$\bar{C}_2(\tilde{\alpha} = -1.391; b = 1.5) \simeq 3.004 \tag{1.76P}$$

As expected, these results are much more accurate than those found for C_1 (cf. Table 1.1 P).

6. The Neumann integral (9.7) in the lecture notes represents the mutual inductance M_{21} between two concentric circles of radii A and a, respectively, at a distance δ from each other.

6.1. Find a similar Neumann integral (no need to solve it) for the case of two concentric ellipses with semi-major axis A and a and semi-minor axis B and b, respectively. Assume the two ellipses belong to two parallel planes at a distance δ from each other, as shown in Fig. 1.17P. [15 points]

6.2. Show that the result found in 6.1 reduces to (9.7) when $A=B$ and $a=b$. Consider then the solution (9.8) of (9.7). Assume $A=a=1\,m$ and $\delta=e^{-1/4}\,r$, with $r=1\,mm$. Under these conditions, δ can be assumed to be small. Using these numerical values and following closely all derivations in lecture 9, compare the approximate result for M_{21} when considering the first two terms (i.e., up to k^2) in the series expansions (9.11a) and (9.11b) for

F(K) and E(K) to the approximate result obtained when considering just the first term in the series expansions (9.12a) and (9.12b), i.e., $\ln 4/k_1$ and 1. [5 points]

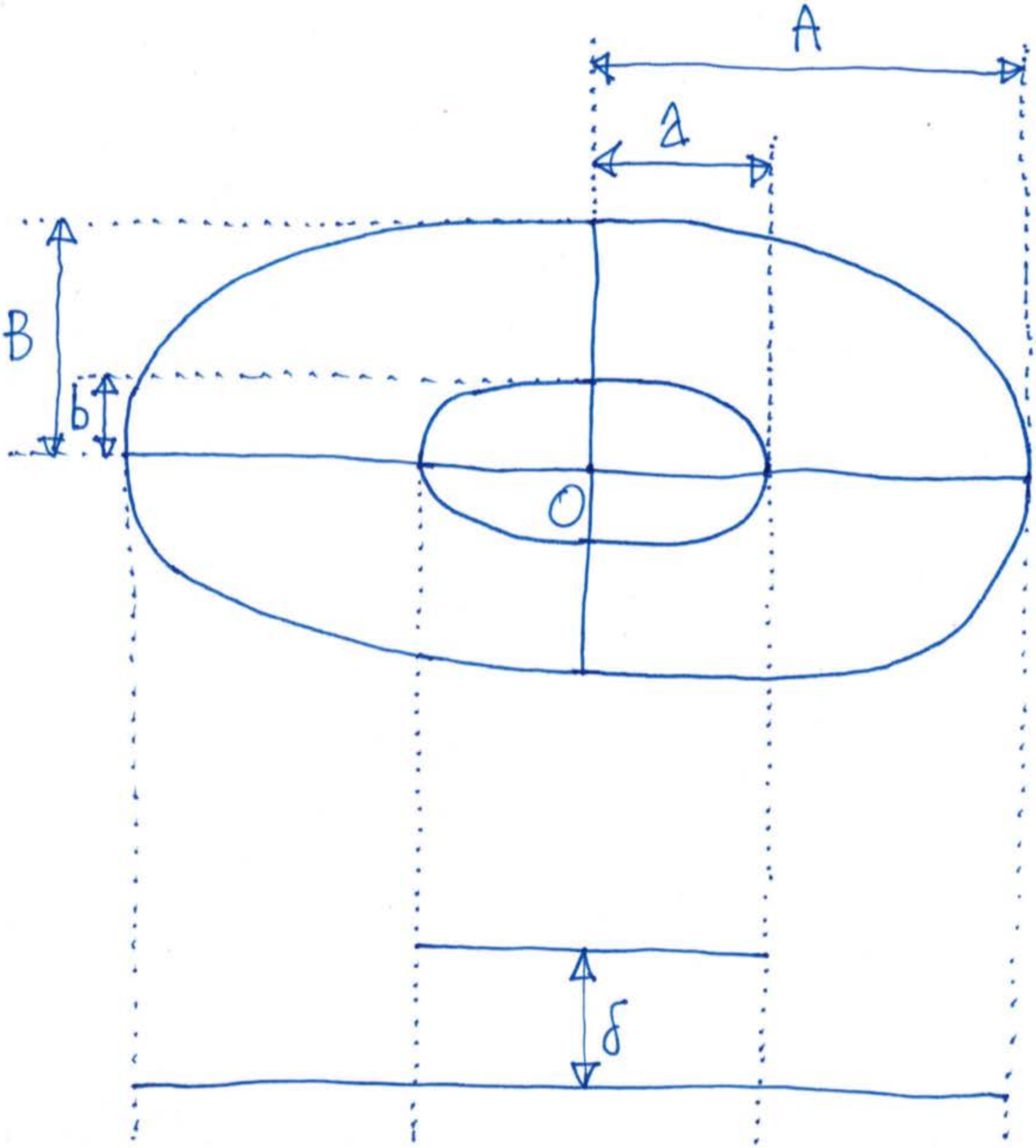


Figure 1.17P.

We follow a similar path as in the first part of lecture 9. 57

6.1. We consider a Cartesian coordinate system Oxy, as shown in Fig. 1.18P.

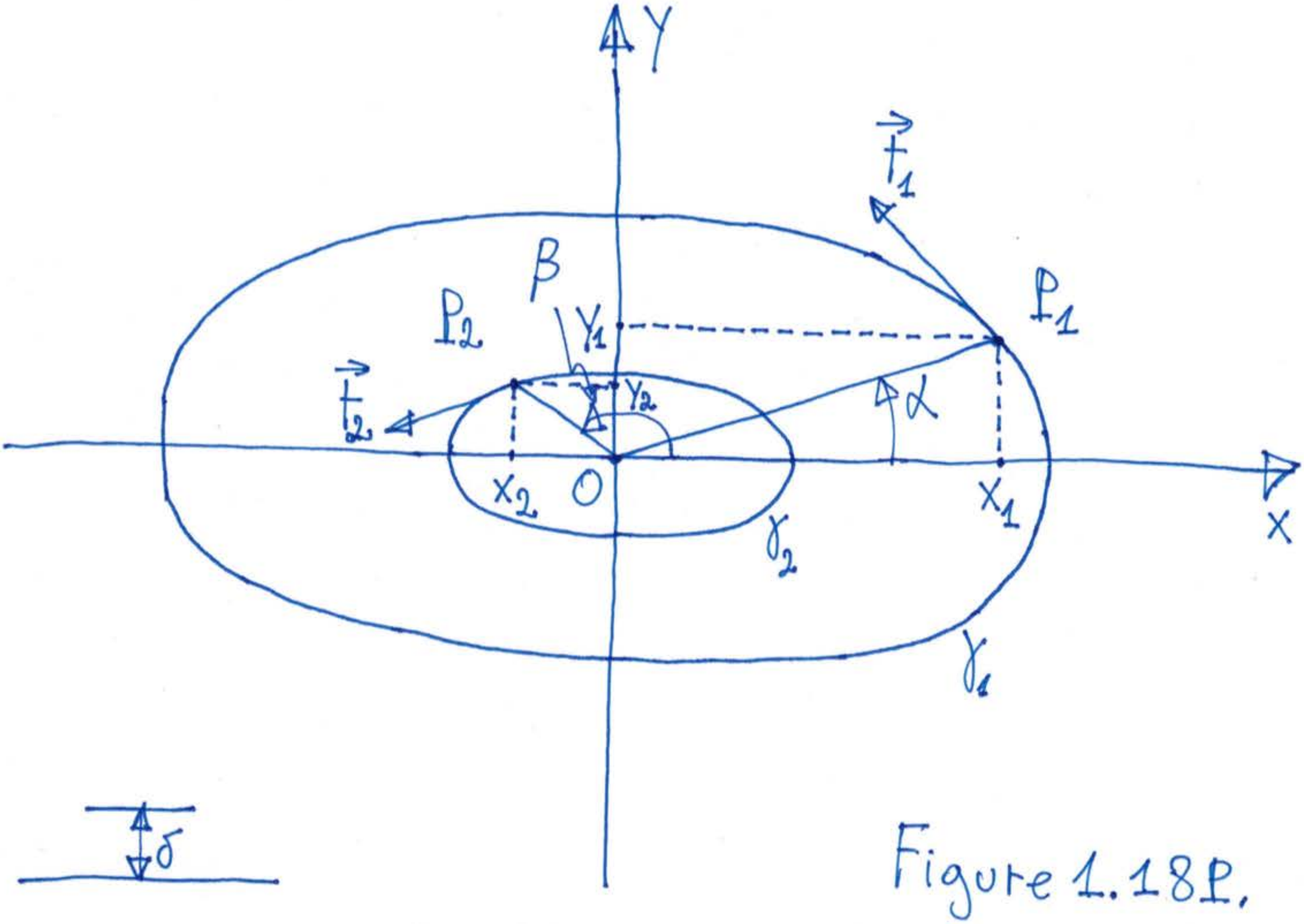


Figure 1.18P.

We want to find the Neumann integral

$$M_{21} = \frac{\mu_0}{4\pi} \oint_{\gamma_2} \oint_{\gamma_1} \frac{\vec{t}_1 \cdot \vec{t}_2}{r} \, dl_1 \, dl_2 \qquad (1.77P)$$

for the case of the two ellipses of Fig. 1.18P. In general,

$$r = \left(\overline{P_1 P_2}\right)_\delta \qquad (1.78P)$$

When $\delta = 0$, this distance becomes the planar distance between P_1 and P_2,

$$\tilde{r} = \left(\overline{P_1 P_2}\right)_0 \qquad (1.79P)$$

where the subscripts δ and 0 in (1.78P) and (1.79P) indicate the general case for $\delta \neq 0$ and the special one for $\delta = 0$, respectively. The distance (1.79P) can be found from the parametric equations of the two ellipses as the magnitude (norm) of the difference between vectors $\vec{OP_2}$ and $\vec{OP_1}$ in Cartesian coordinates. The parametric equations for the two ellipse are:

$$\gamma_1 = \begin{cases} X_1 = A\cos\alpha \\ Y_1 = B\sin\alpha \end{cases} \tag{1.79P}$$

and

$$\gamma_2 = \begin{cases} X_2 = a\cos\beta \\ Y_2 = b\sin\beta \end{cases} \tag{1.80P}$$

with $\alpha, \beta \in [\phi, 2\pi]$. The vectors $\vec{OP}_1$ and $\vec{OP}_2$ can then be written in Cartesian coordinates as

$$\vec{OP}_1 = (X_1, Y_1) = X_1 \vec{U}_x + Y_1 \vec{U}_y \tag{1.81P}$$

and

$$\vec{OP}_2 = (X_2, Y_2) = X_2 \vec{U}_x + Y_2 \vec{U}_y \tag{1.82P}$$

where X_1, Y_1, X_2, and Y_2 are given by (1.79P) and (1.8ϕP). It is evident that these two vectors would belong to two concentric circles if $A=B=R_1$ and $a=b=R_2$. In general, from (1.81P) and (1.82P) we find

$$\tilde{r} = (\overline{P_1P_2})_0 = \|\vec{OP_2} - \vec{OP_1}\|_0 = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$= \sqrt{A^2\cos^2\alpha - 2Aa\cos\alpha\cos\beta + a^2\cos^2\beta + B^2\sin^2\alpha - 2Bb\sin\alpha\sin\beta + b^2\sin^2\beta} \quad (1.83P)$$

When $\delta \neq 0$, the actual distance between P_1 and P_2 can be calculated from the Pythagorean theorem as

$$r = (\overline{P_1P_2})_\delta = \sqrt{\tilde{r}^2 + \delta^2}$$

$$= \sqrt{A^2\cos^2\alpha - 2Aa\cos\alpha\cos\beta + a^2\cos^2\beta + B^2\sin^2\alpha - 2Bb\sin\alpha\sin\beta + b^2\sin^2\beta + \delta^2} \quad (1.84P)$$

In order to compute the scalar product $(\vec{t}_1 \cdot \vec{t}_2)$ in integral (1.77P), we must represent the unit vectors $\vec{t}_1$ and $\vec{t}_2$ tangent to the two ellipses at points P_1 and P_2, respectively, in Cartesian coordinates (it is sufficient on the Oxy plane). In these coordinates, from (1.81P) and (1.82P) we know that the vectors representing P_1 and P_2 with respect to O are given by

$$\vec{r}_1 = \overrightarrow{OP_1} = A\cos\alpha\,\vec{u}_x + B\sin\alpha\,\vec{u}_y \qquad (1.85P)$$

and

$$\vec{r}_2 = \overrightarrow{OP_2} = a\cos\beta\,\vec{u}_x + b\sin\beta\,\vec{u}_y \qquad (1.86P)$$

As a consequence, the vectors $\vec{t}_1$ and $\vec{t}_2$ can be found by deriving (1.85P) and (1.86P) with respect to α and β and by normalizing the result to unity. We find

$$\vec{f}_1 = \left[\frac{\partial}{\partial \alpha}(A\cos\alpha)\vec{U}_x + \frac{\partial}{\partial \beta}(B\sin\alpha)\vec{U}_y \right] \left\{ \left[\frac{\partial}{\partial \alpha}(A\cos\alpha) \right]^2 + \left[\frac{\partial}{\partial \beta}(B\sin\alpha) \right]^2 \right\}^{-1/2}$$

$$= \frac{1}{\sqrt{A^2\sin^2\alpha + B^2\cos^2\alpha}} \left(-A\sin\alpha\,\vec{U}_x + B\cos\alpha\,\vec{U}_y \right) \tag{1.87P}$$

and

$$\vec{f}_2 = \left[\frac{\partial}{\partial \beta}(a\cos\beta)\vec{U}_x + \frac{\partial}{\partial \beta}(b\sin\beta)\vec{U}_y \right] \left\{ \left[\frac{\partial}{\partial \beta}(a\cos\beta) \right]^2 + \left[\frac{\partial}{\partial \beta}(b\sin\beta) \right]^2 \right\}^{-1/2}$$

$$= \frac{1}{\sqrt{a^2\sin^2\beta + b^2\cos^2\beta}} \left(-a\sin\beta\,\vec{U}_x + b\cos\beta\,\vec{U}_y \right) \tag{1.88P}$$

As a sanity check, when $A=B=a=b=1$
and for $\alpha=\beta=0$, from (1.87P) and (1.88P) we have

$$\vec{t}_1=\vec{t}_2=\vec{U}_y \qquad (1.89P)$$

which is the tangent unit vector to a unitary circle in correspondance of the x-axis. From (1.87P) and (1.88P) we readily find

$$\vec{t}_1\cdot\vec{t}_2=\frac{1}{\sqrt{(A^2\sin^2\alpha+B^2\cos^2\alpha)(a^2\sin^2\beta+b^2\cos^2\beta)}}$$

$$(Aa\sin\alpha\sin\beta+Bb\cos\alpha\cos\beta)$$

$$(1.90P)$$

When $A=B=a=b=1$ and for $\alpha=\beta=0$, from (1.90P) we have

$$\vec{t}_1 \cdot \vec{t}_2 = \frac{1}{1} \left(\sin\Phi \sin\Phi + \cos\Phi \cos\Phi \right) = 1 \qquad (1.91P)$$

as expected from (1.89P) (in fact $\vec{U}_y \cdot \vec{U}_y = 1$). At last, the infinitesimal ellipse elements dl_1 and dl_2 can be found from the differentials of the components of vectors (1.85P) and (1.86P) and the Pythagorean theorem. We have

$$\begin{aligned} dl_1 &= \sqrt{[d(A\cos\alpha)]^2 + [d(B\sin\alpha)]^2} \\ &= \sqrt{A^2 \sin^2\alpha \, (d\alpha)^2 + B^2 \cos^2\alpha \, (d\alpha)^2} \\ &= \sqrt{A^2 \sin^2\alpha + B^2 \cos^2\alpha} \; d\alpha \end{aligned} \qquad (1.91P)$$

and

$$dl_2 = \sqrt{[d(a\cos\beta)]^2 + [d(b\sin\beta)]^2}$$

$$\quad =$$

$$= \sqrt{a^2 \sin^2\beta \, (d\beta)^2 + b^2 \cos^2\beta \, (d\beta)^2}$$

$$= \sqrt{a^2 \sin^2\beta + b^2 \cos^2\beta} \; d\beta \qquad (1.92P)$$

When $A = B = a = b = 1$, from (1.91P) and (1.92P) we have

$$dl_1 = 1 \, d\alpha \qquad (1.93P)$$

and

$$dl_2 = 1 \, d\beta \qquad (1.94P)$$

which are the infinitesimal arc elements of two unit circles. Using (1.84P), (1.90P), (1.91P), (1.92P), and for $\alpha, \beta \in [0, 2\pi]$ in (1.77P), we obtain the Neumann integral we were looking for. This is:

$$M_{21} = \frac{\mu_0}{4\pi} \int_0^{2\pi} \int_0^{2\pi}$$

$$\frac{1}{\sqrt{A^2\cos^2\alpha - 2Aa\cos\alpha\cos\beta + a^2\cos^2\beta + B^2\sin^2\alpha - 2Bb\sin\alpha\sin\beta + b^2\sin^2\beta + \delta^2}}$$

$$\frac{Aa\sin\alpha\sin\beta + Bb\cos\alpha\cos\beta}{\cancel{\sqrt{(A^2\sin^2\alpha + B^2\cos^2\alpha)(a^2\sin^2\beta + b^2\cos^2\beta)}}}$$

$$\cancel{\sqrt{(A^2\sin^2\alpha + B^2\cos^2\alpha)(a^2\sin^2\beta + b^2\cos^2\beta)}}\ d\alpha\, d\beta$$

$$= \frac{\mu_0}{4\pi} \int_0^{2\pi} d\alpha \int_0^{2\pi} d\beta$$

$$\frac{Aa\sin\alpha\sin\beta + Bb\cos\alpha\cos\beta}{\sqrt{A^2\cos^2\alpha - 2Aa\cos\alpha\cos\beta + a^2\cos^2\beta + B^2\sin^2\alpha - 2Bb\sin\alpha\sin\beta + b^2\sin^2\beta + \delta^2}}$$

$$(1.95P)$$

which is the Neumann integral for the (67)
mutual inductance for the two ellipses of Fig. 1.17P. This is not an easy integral and, for example, can be solved numerically.

6.2. When $A = B$, $a = b$, and defining $\alpha = \varphi_1$ and $\beta = \varphi_2$, from (1.95P) we have

$$M_{21} = \frac{\mu_0}{4\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{Aa(\cos\varphi_1 \cos\varphi_2 + \sin\varphi_1 \sin\varphi_2)\, d\varphi_1\, d\varphi_2}{\sqrt{A^2 + a^2 + \delta^2 - 2Aa(\cos\varphi_1 \cos\varphi_2 + \sin\varphi_1 \sin\varphi_2)}}$$

$$= \frac{\mu_0}{4\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{Aa\cos(\varphi_2 - \varphi_1)\, d\varphi_1\, d\varphi_2}{\sqrt{A^2 + a^2 + \delta^2 - 2Aa\cos(\varphi_2 - \varphi_1)}}$$

(1.96P)

where we used the property $(\cos\varphi_1 \cos\varphi_2 + \sin\varphi_1 \sin\varphi_2) = \cos(\varphi_1 - \varphi_2) = \cos(\varphi_2 - \varphi_1)$. The integral (1.96P) is exactly the same as integral (9.7).

The solution to this integral is given by Eq. (9.8) in the lecture notes. By truncating the series (9.11a) and (9.11b) to the second term, we obtain an approximate solution of (1.96P),

$$M_{21} \approx -\mu_0\sqrt{Aa}\left[\left(k-\frac{2}{k}\right)\left(\frac{\pi}{2}+\frac{\pi}{8}k^2\right) + \frac{2}{k}\left(\frac{\pi}{2}-\frac{\pi}{8}k^2\right)\right]$$

$$= -\mu_0\sqrt{Aa}\left(\cancel{k\frac{\pi}{2}} + k^3\frac{\pi}{8} - \cancel{\frac{1}{k}\pi} - \cancel{k\frac{\pi}{4}} + \cancel{\frac{1}{k}\pi} - \cancel{k\frac{\pi}{4}}\right)$$

$$= -\mu_0\sqrt{Aa}\,k^3\frac{\pi}{8} \qquad (1.97P)$$

Assuming $A=a$ and small values of δ and using the same definitions as in the notes of lecture 9, we have

$$\xi = \left(\frac{\delta}{2a}\right)^2 \qquad (1.98P)$$

and

$$K \sim 1 - \frac{1}{2}\xi \qquad (1.99P)$$

By inserting (1.99P) into (1.97P), we obtain

$$M_{21} \sim -\mu_0 a \left(1 - \frac{1}{2}\xi\right)^3 \frac{\pi}{8}$$

$$= -\mu_0 a \left(1 - \frac{3}{2}\xi + \frac{3}{4}\xi^2 - \frac{1}{8}\xi^3\right)\frac{\pi}{8} \qquad (1.100P)$$

For small values of δ, which is a good approximation of the limit for $\delta \to 0^+$, the terms proportional to ξ^2 and ξ^3 in (1.100P) are of higher order in δ compare to the term proportional to ξ. Hence,

$$M_{21} \sim \mu_0 a \frac{\pi}{8}\left(\frac{3}{2}\xi - 1\right)$$

$$= \mu_0 a \frac{\pi}{8}\left(\frac{3\delta^2}{8a^2} - 1\right)$$

$$= \mu_0 a \left(\frac{3\delta^2 \pi}{64 a^2} - \frac{\pi}{8}\right) \sim L \qquad (1.101P)$$

Finally, by means of the substitution (9.24), $\delta = e^{-1/4} r$, we find

$$M_{21} \sim L \sim \mu_0 a \left(\frac{3 e^{-1/2} r^2 \pi}{64 a^2} - \frac{\pi}{8}\right) \qquad (1.102P)$$

For example, for $a = 1m$ and $r = 1mm$, we have

$$L \simeq -0.393 \mu_0 \qquad (1.103P)$$

Since the self-inductance must be a positive quantity, this result is unphysical. This means that using the series (9.11a) and (9.11b) up to the second term does not give a reasonable approximation for L (as expected from the discussion in lecture 9). By truncating, instead, the series (9.12a) and (9.12b) to the first term, we obtain a different approximate solution of (1.96P), that is

$$M_{21} \approx -\mu_0 \sqrt{Aa}\left[\left(k - \frac{2}{k}\right)\ln\frac{4}{k_1} + \frac{2}{k}\right] \quad (1.104P)$$

For $A = a$ and small values of δ, from (9.20) we have

$$k_1 \sim \sqrt{\xi} \quad (1.105P)$$

By inserting (1.99P) and (1.105P) into (1.104P), we obtain

$$M_{21} \sim -\mu_0 a\left(k\ln\frac{4}{k_1} - \frac{2}{k}\ln\frac{4}{k_1} + \frac{2}{k}\right)$$

$$= -\mu_0 a \left(\ln\frac{4}{\sqrt{\xi}} - \frac{1}{2}\xi \ln\frac{4}{\sqrt{\xi}} \right.$$

$$\left. - \frac{2}{1-\frac{1}{2}\xi} \ln\frac{4}{\sqrt{\xi}} + \frac{2}{1-\frac{1}{2}\xi} \right)$$

(1.106P)

For small values of δ, ξ is second order and $\sqrt{\xi}$ first order in δ. Hence,

$$M_{21} \sim \mu_0 a \left(\ln\frac{4}{\sqrt{\xi}} - 2 \right)$$

$$= \mu_0 a \left(\ln\frac{8a}{\delta} - 2 \right) \sim L \qquad (1.107P)$$

Using (9.24) we finally obtain

$$M_{21} \sim L \sim \mu_0 a \left(\ln\frac{8a}{r} - 1.75 \right) \qquad (1.108P)$$

which is the same as (9.25). For $a = 1\,m$ and

$r = 1\,mm$, we have

$$L \simeq +7.237\,\mu_0 \qquad (1.109P)$$

This result is positive and is a good approximation for L. Since (1.102P) and (1.108P) were obtained starting from the same Neumann integral with the same sign conventions, we have a confirmation that the approximation (1.102P) is by far not a good approximation for L. In fact, the mutual inductance (1.102P), for example the numerical value (1.103P), can in general be negative. However, in this case also (1.108P), and the numerical value (1.109P) obtained for the same parameters of (1.103P), should be negative. Of course, this is not the case. Note that the first term of (9.12a) and (9.12b) is sufficient to obtain the result (9.25).

7. Consider a parallel-plate condenser with plates of area A and placed at a distance d ($\ll \sqrt{A}$) from each other, in vacuum. The condenser gets charged by applying a potential difference $\Delta\phi$ between the two plates by means of a generator (battery). After disconnecting the condenser from the battery, one of the two plates is displaced in space, while remaining parallel to the other plate, until the distance between the plates becomes $d' > d$.

7.1. Calculate the potential difference $\Delta\phi'$ between the plates at the end of the displacement. [5 points]

7.2. Calculate the work W required to displace the plate, first, by means of the concept of electrostatic energy and, second, by means of Eq. (10.20) of the lecture notes. [15 points]

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We refer to Fig. 10.3 in the lecture notes and the $Oxyz$ Cartesian coordinate system therein.

7.1. During the plate displacement, the condenser charge remains constant. Thus, from the definition of capacitance we have

$$C\,\Delta\phi = C'\,\Delta\phi' \tag{1.110P}$$

where

$$C = \frac{\varepsilon_0 A}{d} \tag{1.111P}$$

and

$$C' = \frac{\varepsilon_0 A}{d'} \tag{1.112P}$$

By substituting (1.111P) and (1.112P) into (1.110P), we obtain

$$\frac{\Delta\phi}{d} = \frac{\Delta\phi'}{d'} \tag{1.113P}$$

Note that this is consistent with the condenser being an isolated system (i.e., disconnected from the battery) after being charged. From (1.113P), we have

$$\Delta\phi' = \frac{d'}{d}\Delta\phi \tag{1.114P}$$

which is the first result we were looking for.

7.2. In order to calculate the work W from the electrostatic energy, we proceed as follows. The initial energy U and the final energy U' are given by (cf. Eq. (7.8) in the lecture notes)

$$U = \frac{1}{2}C(\Delta\phi)^2 \tag{1.115P}$$

and

$$U' = \frac{1}{2}C'(\Delta\phi')^2 = \frac{1}{2}C'\left(\frac{d'}{d}\right)^2(\Delta\phi)^2 \tag{1.116P}$$

respectively, where, in (1.116P), we made use of (1.114P). Dividing (1.112P) by (1.111P), we find

$$\frac{C'}{C} = \frac{d}{d'} \tag{1.117P}$$

from which

$$C' = \frac{d}{d'} C \tag{1.118P}$$

Using (1.118P) in (1.116P), we have

$$\begin{aligned} U' &= \frac{1}{2} \frac{d}{d'} \left(\frac{d'}{d}\right)^2 C (\Delta\phi)^2 \\ &= \frac{1}{2} \frac{d'}{d} C (\Delta\phi)^2 \\ &= \frac{d'}{d} U \end{aligned} \tag{1.119P}$$

Finally,

$$W = U' - U = \frac{d' - d}{d} U \tag{1.120P}$$

Another way to calculate W is by means of Eq.(10.20), which, in vector form with respect to Oxyz, reads

$$\vec{F}_x = -\frac{1}{2}\frac{\sigma^2}{\varepsilon_0}A\,\vec{u}_x = -\frac{1}{2}\frac{(\sigma A)^2}{\varepsilon_0 A}\vec{u}_x = -\frac{1}{2}\frac{q^2}{\varepsilon_0 A}\vec{u}_x \qquad (1.121P)$$

where we defined the charge on the condenser plates as $|q| = \sigma A$. The force (1.121P) is the force acting on the condenser's plates after charging the condenser and must be constant during the plate displacement since during this action the condenser is isolated (the battery is disconnected) and, thus, $q = \text{const}$. The force required to displace the plate must be exactly the opposite of $\vec{F}_x$ of (1.121P) (quasi-static displacement). Hence,

$$W = -\vec{F}_x \cdot \vec{u}_x \,\Delta x$$

$$= \frac{1}{2} \frac{q^2}{\varepsilon_0 A} \vec{U}_x \cdot \vec{U}_x (d' - d)$$

$$= \frac{1}{2} \frac{q^2}{\varepsilon_0 A} (d' - d) \qquad (1.122P)$$

From the definition of capacitance

$$q = C \Delta \phi \qquad (1.123P)$$

Thus, (1.122P) becomes

$$W = \frac{1}{2} \frac{C^2 (\Delta\phi)^2}{\varepsilon_0 A} (d' - d)$$

$$= \frac{1}{2} C (\Delta\phi)^2 (d' - d) \frac{C}{\varepsilon_0 A}$$

$$= \frac{d' - d}{d} U \qquad (1.124P)$$

which is the same as (1.120P).

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8. Consider a straight quasi-filiform conductor of length ℓ attached on one end to a rotating axis, which is normal to the plane containing the conductor (cf. Fig. 1.19P). The conductor ℓ rotates about the axis at a frequency ν. The other end of the conductor moves around a closed conducting quasi-filiform loop of radius ℓ and center on the axis. Suppose the axis to be a qusi-filiform conductor and assume the conductor ℓ to touch the loop at all times. Moreover, suppose there is no mechanical friction between the conductor ℓ and the loop. The entire system is supposed to be in an external uniform magnetostatic field $\vec{B}_0$ directed as the axis and oriented upward. Suppose the electric resistance of the conductor ℓ, the vertical axis, and the loop to be negligible. As shown in the figure, the axis can be connected to the loop by means of a resistance R, which can be opened or

closed by means of a switch.

8.1. Assume the switch is open. Calculate the potential difference between the axis and the loop and show which of these two elements is at larger potential if the conductor ℓ rotates in the counterclockwise direction. In addition, calculate the current I carried by the conductor ℓ when the switch is close.
[10 points]

8.2. Calculate the mechanical torque that an engine must apply to the conductor ℓ in order to keep it rotating both when the switch is close and open. In both cases, calculate the mechanical power due to the engine.
[10 points]

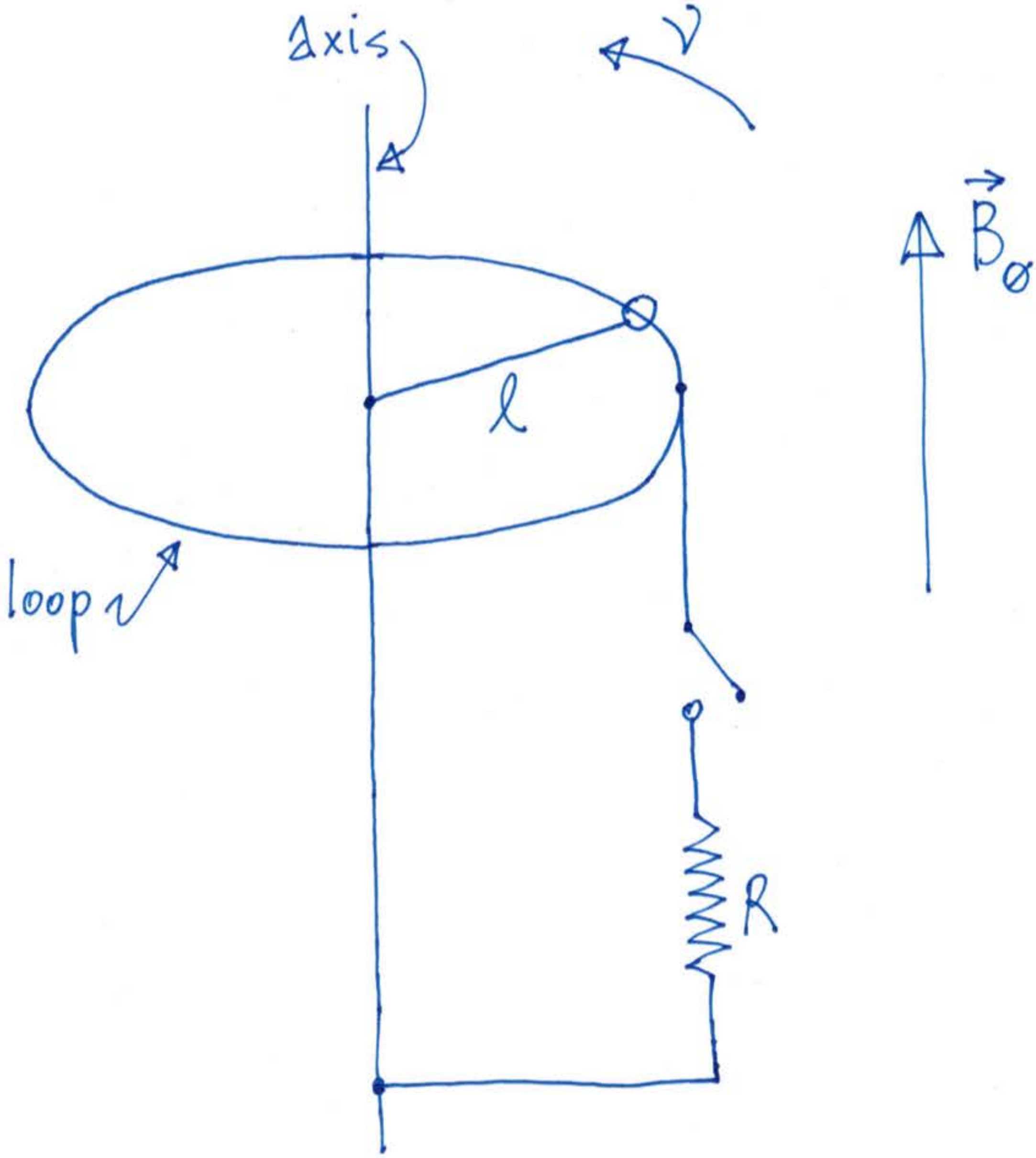


Figure 1.19 P.

In order to solve this problem, we will follow the notes of lecture 10. Consider a cylindrical coordinate system $Or\varphi z$, as shown in Fig. 1.20P.

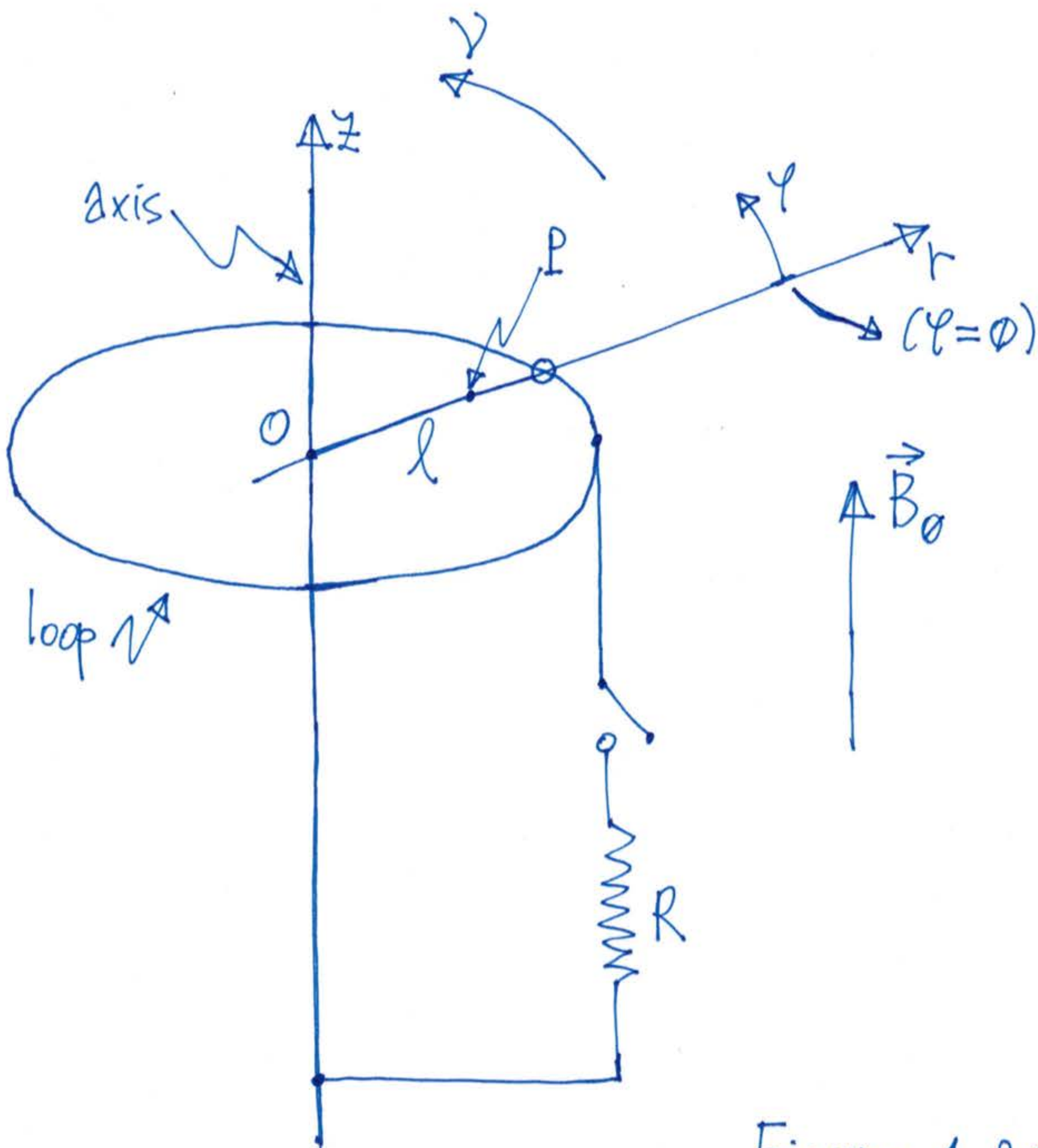


Figure 1.20P.

8.1. The conduction electrons in the quasi-filiform conductor ℓ move together with the conductor in the external field $\vec{B}_0$. Consider a point P on the conductor ℓ such that the distance between P and the axis is $r \in [0, \ell]$, as shown in Fig. 1.20P. The velocity of P is given by

$$\vec{v} = \omega r \, \vec{u}_\varphi = 2\pi\nu r \, \vec{u}_\varphi \tag{1.125P}$$

The Lorentz force acting on the electrons is thus

$$\vec{F}_e = e \, \vec{v} \times \vec{B}_0 \tag{1.126P}$$

where e is the electron charge (negative) and

$$\vec{B}_0 = B_0 \, \vec{u}_z \tag{1.127P}$$

Hence, the electrons are acted upon by a field

$$\vec{E} = \frac{\vec{F}_e}{e} = \vec{v} \times \vec{B}_0$$

$$= 2\pi \nu r \vec{U}_\varphi \times \vec{U}_z B_0$$

$$= 2\pi \nu r B_0 \vec{U}_r \tag{1.128P}$$

where we assumed $\vec{v}$ to be directed counterclockwise, as in Fig. 1.20 P. Under these conditions, the electrons, which are characterized by a negative charge, on the conductor ℓ are pushed from the loop to the axis. As a consequence, negative charges accumulate in the region of the axis, which is thus at a lower potential compare to the region of the loop, where positive charges are accumulated. Under equilibrium conditions, a reaction field $\vec{E}_r$ equal and opposite of (1.128P) appears on the conductor ℓ due to the electrons that moved towards the axis,

$$\vec{E}_r = -2\pi \nu r B_0 \vec{U}_r \tag{1.129P}$$

By definition, the potential difference between

loop and axis in the case of open circuit (86)
is equal to the electromotive force $\mathcal{E}$ due to the field (1.128P), which behaves as the electromotive field $\vec{E}_m$ associated with conductor ℓ that acts as an electric source (cf. PHYS 242 on the electric current). Thus,

$$\Delta\phi^0 = [\phi(r) - \phi(0)]^0 = \mathcal{E}$$

$$= \int_{\text{cond.}\,\ell} \vec{E}_m \cdot \hat{t}\, d\ell = \int_0^\ell \vec{E} \cdot \vec{u}_r\, dr$$

$$= \int_0^\ell 2\pi\nu B_0\, r\, \vec{u}_r \cdot \vec{u}_r\, dr$$

$$= 2\pi\nu B_0 \int_0^\ell r\, dr = \pi\nu B_0 \ell^2 \qquad (1.130\text{P})$$

In summary,

$$\Delta\phi^0 = \pi\nu B_0 \ell^2 \qquad (1.131\text{P})$$

Thus, when the switch is close, the current through the entire circuit (i.e., conductor ℓ, R, and axis) can be found from Ohm's law: (87)

$$I = \frac{\Delta\phi^{o}}{R} = \frac{\pi \nu B_{0} \ell^{2}}{R} \tag{1.132P}$$

This current, due to positive carriers by definition, must flow from the larger to the smaller (higher to lower) potential in the circuit outside conductor ℓ (in fact, conductor ℓ is the effective source of the system). This means that, along conductor ℓ, I must flow from the axis to the loop.

8.2. From Eq. (11.8) in the lecture notes, an infinitesimal element dr of conductor ℓ is acted upon by a force

$$d\vec{F} = I\, dr\, \vec{u}_r \times \vec{B}_0$$

$$= \frac{\pi \nu B_0 \ell^2}{R} \vec{u}_r \times \vec{u}_z B_0\, dr$$

$$= \frac{\pi \nu B_0^2 \ell^2}{R}(-\vec{U}_\varphi)\,dr$$

$$= -\vec{U}_\varphi \frac{\pi \nu (B_0 \ell)^2}{R}\,dr \qquad (1.133P)$$

Note that this force is opposed to the motion of conductor ℓ. The mechanical torque with respect to O (pivot) due to $d\vec{F}$ is given by

$$d\vec{\tau}_0 = \vec{r} \times d\vec{F}$$

$$= -r\vec{U}_r \times \vec{U}_\varphi \frac{\pi \nu (B_0 \ell)^2}{R}\,dr$$

$$= -\frac{\pi \nu (B_0 \ell)^2}{R}\,r\,dr\,\vec{U}_z \qquad (1.134P)$$

By integration, the total torque is given by

$$\vec{\tau}_0 = \int_0^\ell d\vec{\tau}_0$$

$$= -\frac{\pi \nu (B_0 \ell)^2}{R} \vec{u}_z \int_0^\ell r\,dr$$

$$= -\frac{\pi \nu B_0^2 \ell^4}{2R} \vec{u}_z$$

$$= -\frac{1}{2} I B_0 \ell^2 \vec{u}_z \qquad (1.135P)$$

This means that the total torque due to the external forces (i.e., the engine) to maintain the constant rotation must be the opposite of $\vec{\tau}_0$,

$$\vec{\tau}_0^{ex} = \frac{1}{2} I B_0 \ell^2 \vec{u}_z \qquad (1.136P)$$

As expected, this torque is zero when the switch is open,

$$\vec{\tau}_0^{ex} = \vec{0}, \qquad I = 0 \qquad (1.137P)$$

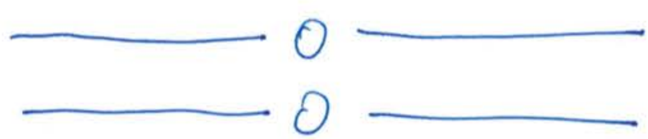
Finally, the mechanical power generated by the engine to maintain the rotation is

$$P = \vec{\tau}_0^{\,ex} \cdot \vec{\omega} = \vec{\tau}_0^{\,ex} \cdot \vec{u}_z \, \omega$$

$$= \frac{1}{2} I B_0 \ell^2 \omega \qquad (1.138P)$$

which, in case of open circuit, is

$$P = 0, \qquad I = 0 \qquad (1.139P)$$



9. Consider a small magnetic needle made of iron. The needle can be modelled as a parallelepiped of square section with sides of length a and height $\ell \gg a$. Assume the magnetic dipole moment of the needle is $\vec{\mu}$ and assume the needle is free to oscillate without mechanical friction about an axis passing through its center of mass and normal to it. When the magnetic needle is placed in an external magnetic field $\vec{B}_0$, uniform and with field lines on the plane of the needle, the needle tends to align parallel to $\vec{B}_0$ (this could be a compass in the Earth's magnetic field). Figure 1.21P shows the system under consideration.

Calculate the period of the small oscillations when the needle is slightly displaced by rotation from its equilibrium condition (equilibrium condition: $\vec{\mu}$ and $\vec{B}_0$ are parallel with equal sign). Assume the density of iron with respect to water is ρ. [Hint: See last part of lecture 15]. [20 points]

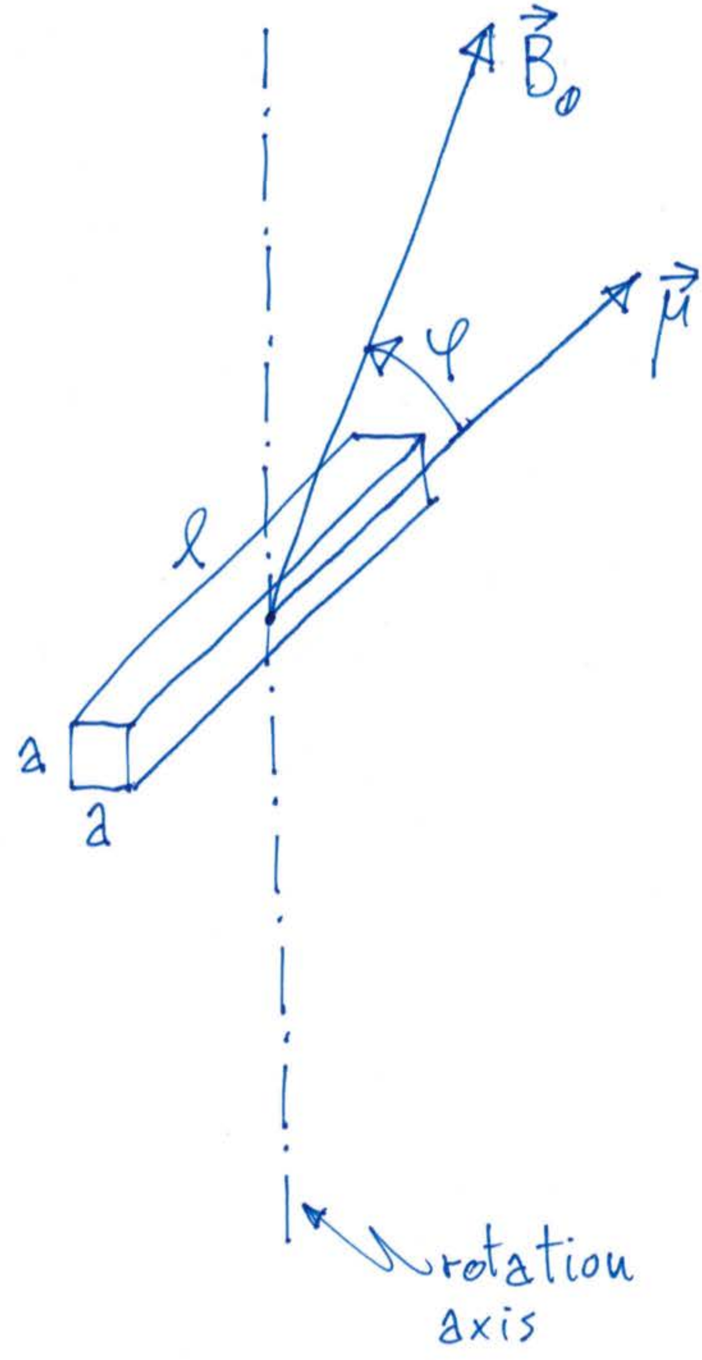


Figure 1.21 P.

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Consider a cylindrical coordinate system Orφz, where O coincides with the center of mass of the magnetic needle, the rφ-plane contains both vector $\vec{\mu}$ and $\vec{B}_0$, and the z-axis coincides with the axis of rotation of the magnetic needle, as shown in Fig. 1.22 P. (93)

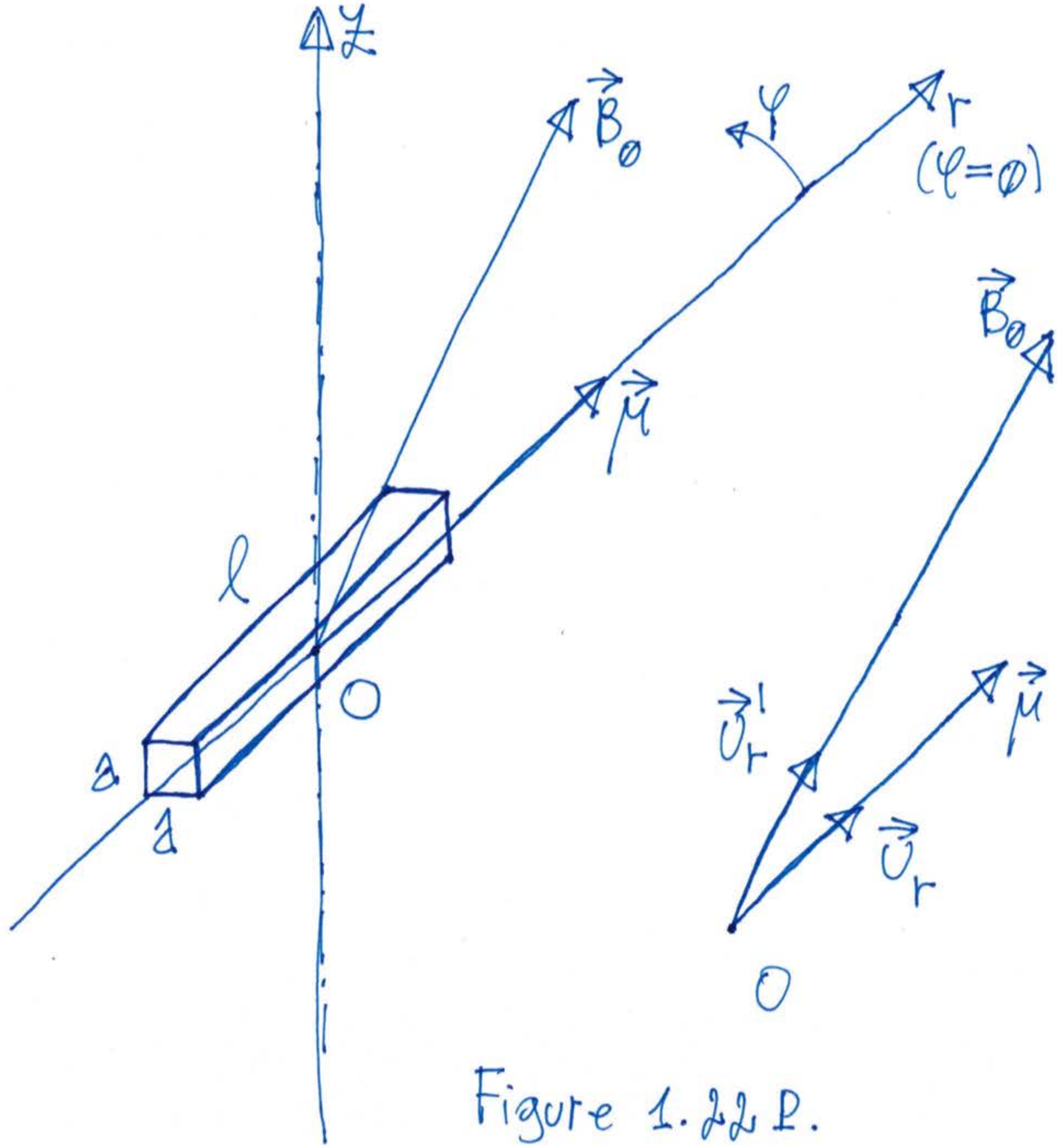

Figure 1.22 P.

A rigid body rotating about a fixed axis obeys Euler's second law: (94)

$$\vec{\tau}_O^{ex} = \frac{d}{dt}\vec{L}_O \qquad (1.140P)$$

where $\vec{\tau}_O^{ex}$ is the torque of the external forces with respect to the pivot O and $\vec{L}_O$ is the system's angular momentum with respect to the same pivot. In our case,

$$\vec{L}_O = \mathcal{I}\vec{\omega} = \mathcal{I}\omega\,\vec{u}_z = \mathcal{I}\frac{d}{dt}\varphi\,\vec{u}_z \qquad (1.141P)$$

where $\mathcal{I}$ is the moment of inertia of the (magnetic) needle and ω its angular velocity. Hence,

$$\vec{\tau}_O^{ex} = \mathcal{I}\frac{d^2}{dt^2}\varphi\,\vec{u}_z \qquad (1.142P)$$

From lecture 15, we know that the

mechanical torque associated with a magnetic dipole moment $\vec{\mu}$ in an external field $\vec{B}_0$ is

$$\vec{\tau} = \vec{\mu} \times \vec{B}_0 \tag{1.143P}$$

With respect to the $Or\varphi z$ system (cf. Fig. 1.22P),

$$\vec{\mu} = \mu \, \vec{u}_r \tag{1.144P}$$

and

$$\vec{B}_0 = B_0 \, \vec{u}_r^{\,\prime} \tag{1.145P}$$

As a consequence,

$$\vec{\tau}_0^{\,ex} = -\vec{\tau} = -\vec{\mu} \times \vec{B}_0$$

$$= -\mu \vec{u}_r \times \vec{u}_r^{\,\prime} B_0 = -\mu B_0 \sin\varphi \, \vec{u}_z \tag{1.146P}$$

By substituting (1.146P) into (1.142P), we have

$$-\mu B_0 \sin\varphi \, \vec{u}_z = \mathcal{J} \frac{d^2}{dt^2} \varphi \, \vec{u}_z \qquad (1.147P)$$

or

$$\mathcal{J} \frac{d^2}{dt^2} \varphi + \mu B_0 \sin\varphi = 0 \qquad (1.148P)$$

which is the exact differential equation of the needle's oscillation. For small oscillations (i.e., slight displacement),

$$\sin\varphi \sim \varphi \qquad (1.149P)$$

and (1.148P) becomes

$$\mathcal{J} \frac{d^2}{dt^2} \varphi + \mu B_0 \varphi = 0 \qquad (1.150P)$$

which is the differential equation of the harmonic motion. A general solution of this equation is

$$\varphi(t) = A \sin(\omega t + \phi_0) \tag{1.151P}$$

where A is the oscillation amplitude and ϕ_0 an offset phase. The angular frequency ω is given by

$$\omega = \sqrt{\frac{\mu B_0}{\mathcal{I}}} \tag{1.152P}$$

Since $\omega = 2\pi\nu = 2\pi/T$, from (1.152P) we have

$$T = 2\pi \sqrt{\frac{\mathcal{I}}{\mu B_0}} \tag{1.153P}$$

We remind that the moment of inertia of a parallelepiped of height $l \gg a$ is given by

$$\mathcal{I} = \frac{1}{3} m l^2 \tag{1.154P}$$

where m is the mass of the needle.
If the needle is made of iron with density ρ, we have

$$\mathcal{J} = \frac{1}{3}\rho a^2 l^3 \tag{1.155P}$$

and finally

$$T = 2\pi a l \sqrt{\frac{\rho l}{3\mu B_0}} \tag{1.156P}$$

10. Consider a thin disk of radius R and uniformly charged with total electric charge q. Assume the thickness of the disk is negligible. Refer to Fig. 1.23 P.

10.1. By means of the multipole series expansion, calculate the electrostatic potential at a generic point P on the disk central axis (which is normal to the disk) at a distance h from the center of the disk, with $h \gg R$. Stop the series to the quadrupole term. [10 points]

10.2. Using the same method, calculate the potential at a generic point P' on a line passing through the center of the disk and belonging to the disk plane. As before, indicate with h ($\gg R$) the distance between P' and the disk center. [10 points]

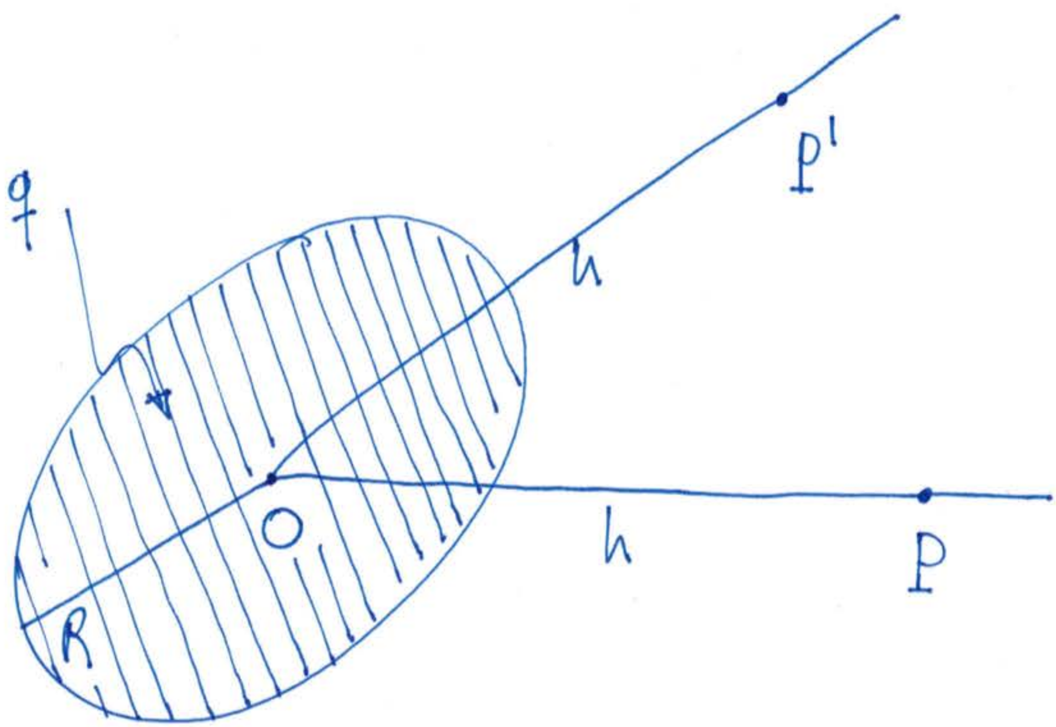

Figure 1.23 P.

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10. Consider a Cartesian coordinate system Oxyz, with origin O in the center of the disk, xy-plane coinciding with the plane of the disk, x-axis containing point P', and z-axis normal to the disk and passing through P, as shown in Fig. 1.24 P. The figure also shows a cylindrical coordinate system $Or\varphi z$.

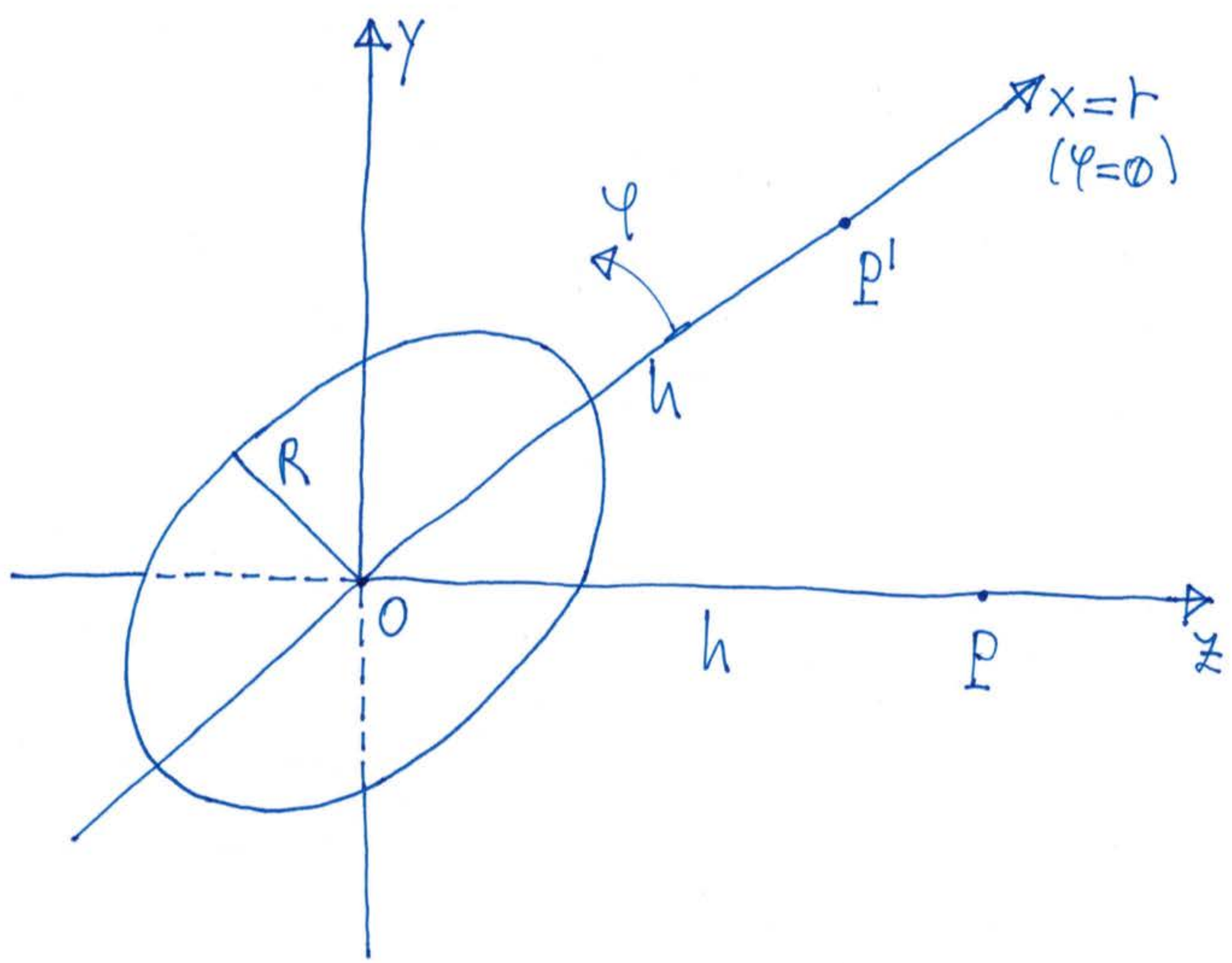

Figure 1.24 P.

To use the same notation of lecture 12, we rename $R=a$. The electrostatic potential at a generic point $\tilde{P}$ at a very large distance from the disk is given by (12.17) of lecture 12,

$$\phi(\tilde{P}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r} + \frac{p}{r^2} + \frac{m}{r^3} + \dots\right) \qquad (1.157P)$$

where q is the total electric charge of the disk as if it were concentrated in O, p is the disk electrostatic dipole moment [Eq. (12.14)], and m its quadrupole moment [Eq. (12.16)]. When $\tilde{P}=P, P'$, $r=h$. We also define the surface charge density of the disk

$$\sigma = \frac{q}{\pi R^2} = \frac{q}{\pi a^2} \qquad (1.158P)$$

which is constant. Hereafter, R is the same as in lecture 12. In the case of the two-dimensional charge density (1.158P), Eq. (12.14) becomes

$$P = \iint_{\text{disk}} \sigma R \cos\theta \, dA = \sigma \int_0^{2\pi} d\varphi \cos\theta \int_0^a dR\, R^2 \qquad (1.159P)$$

Similarly, Eq. (12.16) for σ becomes

$$m = \iint_{\text{disk}} \sigma \frac{1}{2} R^2 (3\cos^2\theta - 1)\, dA = \frac{1}{2}\sigma \int_0^{2\pi} d\varphi\, (3\cos^2\theta - 1) \int_0^a dR\, R^3 = \frac{3}{2}\sigma \int_0^{2\pi} d\varphi \cos^2\theta \int_0^a dR\, R^3$$

$$-\frac{1}{2}\sigma\int_0^{2\pi} d\varphi \int_0^a dR\, R^3 \tag{1.160P}$$

10.1. For point P, $\theta=\pi/2$ and, thus, $\cos\theta=0$. Hence, from (1.159P)

$$p(P)=0 \tag{1.161P}$$

and from (1.160P)

$$\begin{aligned} m(P) &= -\frac{1}{2}\sigma\, 2\pi\left[\frac{1}{4}R^4\right]_0^a \\ &= -\sigma\pi\frac{1}{4}a^4 \\ &= -\frac{q}{\cancel{\pi}\cancel{a^2}}\cancel{\pi}\frac{1}{4}\cancel{a^4}^{\,2} \\ &= -\frac{qa^2}{4} \end{aligned} \tag{1.162P}$$

From (1.157P)

$$\phi(P) \approx \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{h} - \frac{qa^2}{4h^3}\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{h}\left(1 - \frac{1}{4}\frac{a^2}{h^2}\right) \qquad (1.163P)$$

If $a \to 0^+$ (i.e., the disk is a point-like charge) we have

$$\phi(P) \approx \frac{1}{4\pi\varepsilon_0}\frac{q}{h} \qquad (1.164P)$$

as expected.

10.2. For point P', $\theta = \varphi$. Hence, from (1.159P)

$$p(P') = \sigma \int_0^{2\pi} d\varphi \cos\varphi \int_0^a dR\, R^2$$

$$= \sigma \left[\sin\varphi\right]_0^{2\pi} \int_0^a dR\, R^2$$

$$= 0 \qquad (1.165P)$$

and from (1.160P)

$$m(P') = \frac{3}{2}\sigma \int_0^{2\pi} d\varphi \cos^2\varphi \int_0^a dR\, R^3$$

$$- \frac{1}{2}\sigma\, 2\pi \left[\frac{1}{4} R^4\right]_0^a$$

$$= \frac{3}{2}\sigma \left[\frac{\varphi}{2} + \frac{1}{4}\sin(2\varphi)\right]_0^{2\pi} \left[\frac{1}{4} R^4\right]_0^a$$

$$- \frac{q a^2}{4}$$

$$= \frac{3}{8}\sigma \pi a^4 - \frac{q a^2}{4}$$

$$= \frac{3}{8} \frac{q}{\cancel{\pi} \cancel{a^2}} \cancel{\pi}\, a^{\cancel{4}\,2} - \frac{q a^2}{4}$$

$$= \frac{1}{8} q a^2 = \frac{q a^2}{8} \tag{1.166P}$$

From (1.157P)

$$\phi(P') \approx \frac{1}{4\pi\varepsilon_0} \left(\frac{q}{h} + \frac{q a^2}{8 h^3} \right)$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{q}{h} \left(1 + \frac{1}{8} \frac{a^2}{h^2} \right) \tag{1.167P}$$

If $a \to 0^+$, we have again (1.164P).

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