PHYS 342 - EM 02

Lecture 2

2.1 Electric Current

The electric current intensity is defined as

$$i(t) = \frac{d}{dt}\, q(t) \tag{2.1}$$

A metallic conductor can be thought as a lattice of positive fixed charges and a cloud of electrons that are free to move. The electrons can collide with themselves and the positive charges. Under electrostatic equilibrium conditions, the electrons undergo a chaotic thermal motion. Thus, given a cross-section A of a conductor, the same number of electrons crosses A in one direction as in the opposite direction. This means that on a macroscopic time scale, $q(t) = 0$ at all times. This dynamics is called diffusion.

When a force is applied to the electrons, the center of mass of the cloud of electrons can move at a certain velocity. This dynamics, called drift, gives rise to conduction currents.

Consider a region Ω of a conductor characterized by an electric current. ρ_f is the volume density of the moving charges (charge free carriers) and $\vec{v}$ their velocity at point P and time t. Given a surface element with area dA and a normal unit vector $\vec{n}$ to dA, the total charge that crosses dA in a time dt is

$$dq = \rho_f \vec{v} \cdot \vec{n}\; dt\; dA \tag{2.2}$$

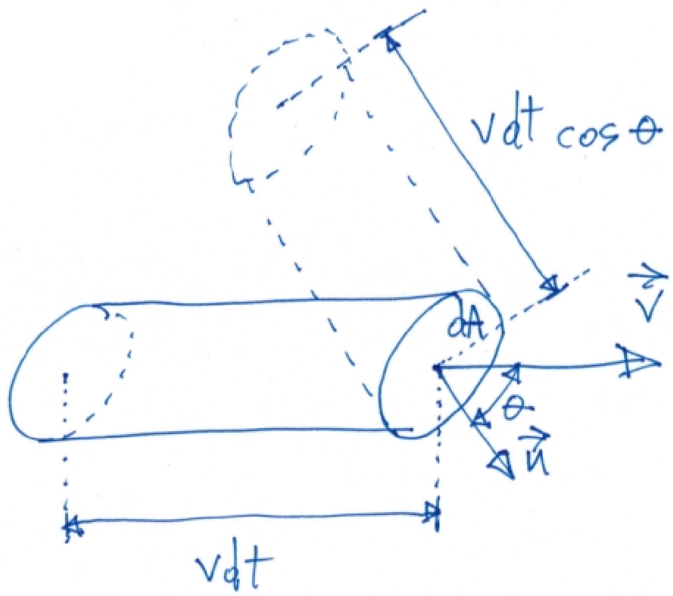

Figure 2.1

The current volume density is defined as

$$\vec{J} = \rho_f \vec{v} \tag{2.3}$$

Given a generic open surface with area A, the current intensity through A with respect to an arbitrary direction $\vec{n}$ is

$$i = \iint_A \rho \vec{v} \cdot \vec{n} \, dA = \iint_A \vec{J} \cdot \vec{n} \, dA \tag{2.4}$$

Charge transport fulfils the charge conservation principle, i.e., given a closed surface Σ, the charge that crosses Σ in a generic time window corresponds to the change of charge within the volume Ω enclosed by Σ. Mathematically,

$$\oint\!\!\oint_\Sigma \vec{J} \cdot \vec{n} \, dA = \iiint_\Omega \vec{\nabla} \cdot \vec{J} \, dV = - \iiint_\Omega \frac{\partial}{\partial t} \rho dV \tag{2.5}$$

where ρ is the volume charge density in Ω.

Given the arbitrariness of Σ and Ω, this means that

$$\vec{\nabla} \cdot \vec{J} = - \frac{\partial}{\partial t} \rho \tag{2.6}$$

This is called continuity equation.

An electric current is said to be stationary when both charge and current densities are time independent at each point in a conductor. In this case,

$$\vec{\nabla} \cdot \vec{J} = 0 \tag{2.7}$$

or

$$\oint\!\!\oint_\Sigma \vec{J} \cdot \vec{n} \, dA = 0 \tag{2.8}$$

The field $\vec{J}$ is said to be solenoidal.

Under stationary conditions, the laws for the electric field inside and outside a conductor with current are

$$\oiint_{\Sigma} \vec{E} \bullet \vec{n} \, dA = \frac{1}{\epsilon_0} Q \tag{2.9}$$

$$\oint_{\gamma} \vec{E} \bullet \vec{t} \, d\ell = 0 \tag{2.10}$$

In general,

$$\vec{J} = f\left(\vec{E}\right) \tag{2.11}$$

where the function f depends on the conductor. This is called constitutive relation.

For certain conductors at constant temperature, the constitutive relation is linear and is called Ohm's law (in local form):

$$\vec{J} = g\vec{E} \tag{2.12}$$

where g is called electric conductivity. In integral form, Ohm's law reads

$$\Delta\phi = RI \tag{2.13}$$

where $\Delta\phi$ is the potential difference (drop) across a conductor of resistance R that carries a current I.

2.2 Magnetostatic Field in Vacuum

The magnetostatic field generated by a generic distribution of stationary current is solenoidal

$$\oiint_{\Sigma} \vec{B} \bullet \vec{n} \, dA = 0 \tag{2.14}$$

This means that magnetic charge (i.e., magnetic monopoles) were never found. The field $\vec{B}$ is also rotational (Ampère's law):

$$\oint_{\gamma} \vec{B} \bullet \vec{t} \, d\ell = \mu_0 I \tag{2.15}$$

where I is a stationary current. In particular, I is the algebraic sum of all currents linked with γ.

In local form, the solenoidal property can be written as

$$\vec{\nabla} \bullet \vec{B} = 0 \tag{2.16}$$

in the case of a stationary current with finite (or zero) volume density $\vec{J}$. For a current with surface density $\vec{J_S}$

$$\vec{n} \bullet \left(\vec{B_1} - \vec{B_2}\right) = 0 \tag{2.17}$$

where $\vec{B_1}$ and $\vec{B_2}$ are the fields slightly above and below the surface where $\vec{J_S}$ is defined.

Similarly, Ampère's law can be written as

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \tag{2.18}$$

and

$$\vec{u} \times \left(\vec{B}_1 - \vec{B}_2\right) = \mu_0 \vec{J}_S \tag{2.19}$$

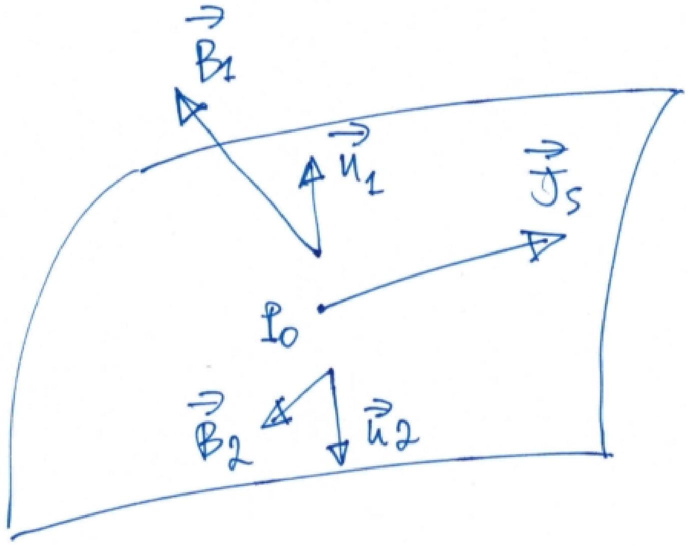

Figure 2.2

We remind that a solenoidal field is characterized by a vector potential. The vector potential $\vec{A}$ of $\vec{B}$ is such that

$$\vec{\nabla} \times \vec{A} = \vec{B} \tag{2.20}$$

It is always possible to find a field $\vec{A}$ such that

$$\vec{\nabla} \cdot \vec{A} = 0 \tag{2.21}$$

Thus, from (2.18) and (2.20)

$$\begin{aligned} \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) &= \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A} \\ &= -\vec{\nabla}^2 \vec{A} = \mu_0 \vec{J} \end{aligned} \tag{2.22}$$

or

$$\vec{\nabla}^2 \vec{A} = -\mu_0 \vec{J} \tag{2.23}$$

where we used (2.21) and a well-known vector calculus identity.

In a Cartesian coordinate system

$$\begin{cases} \vec{\nabla}^2 A_x = -\mu_0 J_x \\ \vec{\nabla}^2 A_y = -\mu_0 J_y \\ \vec{\nabla}^2 A_z = -\mu_0 J_z \end{cases} \tag{2.24}$$

Assuming the functions A_x, A_y, and A_z go to zero at infinity (e.g., for a limited $\vec{J}$), the solutions to the above system are of "Coulomb type:"

$$\begin{cases} A_x(P) & = & \dfrac{\mu_0}{4\pi} \iiint\limits_{\Omega} \dfrac{J_x(Q)}{r_{QP}}\, dV \\ A_y(P) & = & \dfrac{\mu_0}{4\pi} \iiint\limits_{\Omega} \dfrac{J_y(Q)}{r_{QP}}\, dV \\ A_z(P) & = & \dfrac{\mu_0}{4\pi} \iiint\limits_{\Omega} \dfrac{J_z(Q)}{r_{QP}}\, dV \end{cases} \tag{2.25}$$

At last, we remind the definition of inductance. Given a field B generated by a current I on a line γ, a new current

$$I' = kI \tag{2.26}$$

where k is a proportionality constant, also on γ will generate a field

$$B' = kB \tag{2.27}$$

If Φ_γ is the flux through a surface bordered by γ due to B,

$$\Phi'_\gamma = k\Phi_\gamma = \frac{I'}{I}\,\Phi_\gamma \tag{2.28}$$

If $I = 1A$,

$$\Phi'_\gamma = I'\,\frac{\Phi_\gamma}{1A} = I'L \tag{2.29}$$

where L is the inductance associated with γ.