Lecture 13

We continue our study of an ideal electrostatic dipole in an external electrostatic field.

13.1 Field Dipole Interaction Energy

We want to calculate the interaction energy between an ideal electrostatic dipole and an external electrostatic field. The dipole is characterized by a moment $\vec{p}$ and the field by a vector $\vec{E}$.

The interaction energy of interest is not the energy of a system of two point-like charges that form the dipole. That energy would be the work required to build the dipole starting from two charges at a very large distance. Instead, we want to calculate the energy for the interaction between an already built dipole and all the other charges that generate the field acting on it. The energy we are seeking corresponds to the work necessary to move an already built dipole from a configuration where the dipole does not interact with the field (e.g., because the charges generating the field are very far from the dipole) to a configuration where the dipole interacts with the field. We remind that the dipole is supposed to be a rigid body and, thus, the action due to the two charges are mechanically ineffective.

Following the nomenclature in Fig. 12.3, assume ϕ is the electrostatic potential at the point occupied by $(-q)$. The potential at the point occupied by $(+q)$ is, then,

$$\widetilde{\phi} = \phi(\vec{r} + \vec{dr}) = \phi(\vec{r}) + d\phi \tag{13.1}$$

where $d\phi$ is the potential variation corresponding to the infinitesimal displacement $\vec{dr}$.

From (13.1) and the definition of directional derivative, it follows that

$$\phi(\vec{r}) + d\phi = \phi(\vec{r}) + \vec{\nabla}\phi(\vec{r}) \cdot \vec{dr} \tag{13.2}$$

Thus, the electrostatic interaction energy of the dipole in the field is given by

$$\begin{aligned} U_I &= U_- + U_+ \\ &= -q\phi + q(\phi + d\phi) \\ &= q\vec{dr} \cdot \vec{\nabla}\phi \\ &= -\vec{p} \cdot \vec{E} \end{aligned} \tag{13.3}$$

Note that U_{I} is not the energy of a system of two charges. In fact, U_- and U_+ are the electrostatic potential energies of a single (negative or positive, respectively) charge in the potential generated by some other charge distribution (this is why there is no factor 1/2 in (13.3)).

We will now consider the important case of N dipoles in space.

13.2 Distribution of Dipoles

Thanks to the superposition principle, the electrostatic potential and field for a system of N dipoles is given by the sum of the potential and field of each dipole.

As for electric charges, it is also possible to consider continuous distributions of dipoles. To this end, consider a region of space with volume ΔV (assumed to be very small compared to the typical scale of the experiments, but still finite). Suppose the region contains a very large number N of quasi-ideal dipoles. In this case, quasi-ideal means the characteristic distance d of the dipoles is much smaller than $\sqrt[3]{\Delta V}$. The total dipole moment $\Delta\vec{\Pi}$ in ΔV is given by the vector sum of the dipole moments $\vec{p_i}$ of each quasi-ideal dipole,

$$\Delta\vec{\Pi} = \sum_{i=1}^{N} \vec{p_i} \tag{13.4}$$

Suppose all quasi-ideal dipoles are identical and oriented in the same way, i.e., $\vec{p_i} = \vec{p}$. Thus,

$$\Delta\vec{\Pi} = N\vec{p} \tag{13.5}$$

Introducing the dipole moment per unit volume,

$$\vec{P} = \frac{\Delta\vec{\Pi}}{\Delta V} \tag{13.6}$$

we readily obtain

$$\vec{P} = n\vec{p} \tag{13.7}$$

where n is the number of dipoles per unit volume.

The definition of $\vec{P}$ can be easily extended to the case where the dipoles are not identical and have different orientations,

$$\vec{P} = \frac{1}{\Delta V} \sum_{i=1}^{N} \vec{p_i} \tag{13.8}$$

For electric charges, a charge distribution is characterized by a scalar function $\rho(Q)$, which represents the charge per unit volume at a generic point Q. Similarly, a dipole distribution is characterized by a vector function $\vec{P}(Q)$ that gives the dipole moment per unit volume at Q.

We now want to calculate the electrostatic potential and field due to a dipole distribution. The latter can be regarded as a dipole continuum with a vector density $\vec{P}(Q)$. Consider a region of space Ω with a continuous dipole distribution with vector density $\vec{P}(Q)$. We then consider a small volume ΔV centered at a generic point Q' in Ω and substitute the set of dipoles in ΔV with a single effective dipole with moment

$$\Delta\vec{\Pi} = \vec{P}(Q')\Delta V \tag{13.9}$$

From

$$\phi(\vec{r}) \approx \frac{1}{4\pi\epsilon_0}\,\frac{\vec{r}\cdot\vec{p}}{r^3} \tag{13.10}$$

we can find the potential at point Q due to the effective dipole,

$$\Delta\phi \approx \frac{1}{4\pi\epsilon_0}\,\frac{\Delta\vec{\Pi}\bullet(\vec{r}-\vec{r}')}{\|\vec{r}-\vec{r}'\|^3} \tag{13.11}$$

By repeating the same argument for each volume ΔV in which the region Ω can be divided and superimposing the effects, we obtain the potential ϕ at Q for the entire dipole distribution,

$$\phi(\vec{r}) = \phi(Q) \approx \frac{1}{4\pi\epsilon_0}\iiint\limits_{\Omega}\frac{\vec{P}(\vec{r}')\bullet(\vec{r}-\vec{r}')}{\|\vec{r}-\vec{r}'\|^3}\,dV' \tag{13.12}$$

Note that the approximate sign is because the dipoles are supposed to be quasi ideal.

The field can be calculated as $\vec{E}(\vec{r}) = -\vec{\nabla}\phi(\vec{r})$.

13.3 Charge Distributions Equivalent to Dipole Distributions

We want to show that a continuous dipole distribution in a given region of space is equivalent to a suitable charge distribution in the same region, as far as potential and field are concerned.

From simple vector calculus,

$$\vec{\nabla}\frac{1}{\|\vec{r}-\vec{r}'\|} = \vec{\nabla}\frac{1}{\|\vec{\rho}\|} = \vec{\nabla}\frac{1}{\rho} \tag{13.13}$$

where $\|\vec{\rho}\| = \rho > 0$. In spherical coordinates,

$$\vec{\rho} = \vec{r}-\vec{r}' = \rho\vec{u_r} \tag{13.14}$$

and, thus,

$$\begin{aligned}\vec{\nabla}\frac{1}{\rho} &= -\frac{1}{\rho^2}\,\vec{u_r} = -\frac{1}{\|\vec{\rho}\|^2}\,\frac{\vec{\rho}}{\|\vec{\rho}\|}\\ &= -\frac{\vec{r}-\vec{r}'}{\|\vec{r}-\vec{r}'\|^3}\end{aligned} \tag{13.15}$$

it follows that

$$\frac{\vec{r}-\vec{r}'}{\|\vec{r}-\vec{r}'\|^3} = -\vec{\nabla}\frac{1}{\|\vec{r}-\vec{r}'\|} \tag{13.16}$$

The operator "$\vec{\nabla}$" acts on the variable $\vec{r}$. Indicating with "$\vec{\nabla}'$" the operator acting on the variable $\vec{r}'$, we have

$$\frac{\vec{r}-\vec{r}'}{\|\vec{r}-\vec{r}'\|^3} = +\vec{\nabla}\frac{1}{\|\vec{r}-\vec{r}'\|} \tag{13.17}$$

By substituting (13.17) into (13.12), we find

$$\phi(\vec{r}) \approx \frac{1}{4\pi\epsilon_0}\iiint\limits_{\Omega}\vec{P}(\vec{r}')\bullet\vec{\nabla}'\frac{1}{\|\vec{r}-\vec{r}'\|}\,dV' \tag{13.18}$$

From vector calculus,

$$\vec{P}(\vec{r}') \cdot \vec{\nabla}' \frac{1}{\|\vec{r}-\vec{r}'\|} = \vec{\nabla}' \cdot \left[\frac{\vec{P}(\vec{r}')}{\|\vec{r}-\vec{r}'\|} \right] - \frac{1}{\|\vec{r}-\vec{r}'\|} \vec{\nabla}' \cdot \vec{P}(\vec{r}') \tag{13.19}$$

Finally,

$$\begin{aligned} \phi(\vec{r}) &\approx \frac{1}{4\pi\epsilon_0} \iiint\limits_{\Omega} \vec{\nabla}' \cdot \left[\frac{\vec{P}(\vec{r}')}{\|\vec{r}-\vec{r}'\|} \right] dV' \\ &- \frac{1}{4\pi\epsilon_0} \iiint\limits_{\Omega} \frac{1}{\|\vec{r}-\vec{r}'\|} \vec{\nabla}' \cdot \vec{P}(\vec{r}') dV' \end{aligned} \tag{13.20}$$

Using the divergence theorem for the first integral, we have

$$\begin{aligned} \phi(\vec{r}) &\approx \frac{1}{4\pi\epsilon_0} \oint\!\!\!\oint\limits_{\Sigma} \frac{\vec{P}(\vec{r}') \cdot \vec{n}}{\|\vec{r}-\vec{r}'\|} dA \\ &- \frac{1}{4\pi\epsilon_0} \iiint\limits_{\Omega} \frac{\vec{\nabla}' \cdot [\vec{P}(\vec{r}')]}{\|\vec{r}-\vec{r}'\|} dV' \end{aligned} \tag{13.21}$$

where Σ is the closed surface associated with Ω and $\vec{n}$ its normal unit vector.

The two integrals in (13.21) correspond to the potentials due to a surface charge distribution on Σ with density

$$\sigma_{eq} = \vec{P} \cdot \vec{n} = P_n \tag{13.22}$$

and a volume charge distribution in Ω with density

$$\rho_{eq} = -\vec{\nabla}' \cdot \vec{P} \tag{13.23}$$

Given a closed surface Σ that contains the entire distribution of dipoles, the flux of $\vec{E}$ through Σ is zero. This can be directly deduced from Gauss' theorem because the sum of the opposite charges in each dipole is zero and, thus, the sum of all charges for the dipoles in the entire distribution is zero. This conclusion is consistent with (13.22) and (13.23). In fact, the total charge for the distribution of dipoles in Ω is given by

$$\begin{aligned} q_{tot} &= \iiint\limits_{\Omega} \rho_{eq} dV + \oint\!\!\!\oint\limits_{\Sigma} \sigma_{eq} dA \\ &= \iiint\limits_{\Omega} (-\vec{\nabla}' \cdot \vec{P}) dV + \oint\!\!\!\oint\limits_{\Sigma} \vec{P} \cdot \vec{n} dA \end{aligned} \tag{13.24}$$

Using the divergence theorem for the volume integral, we have

$$q_{tot} = -\oint\!\!\!\oint\limits_{\Sigma} \vec{P} \cdot \vec{n} dA + \oint\!\!\!\oint\limits_{\Sigma} \vec{P} \cdot \vec{n} dA = 0 \tag{13.25}$$

In this case, Σ contains Ω completely. If, instead, we were to consider a surface Σ' that cuts some of the dipoles, as shown in Fig. 13.1, the sum of all charges within

Σ' would be different from zero and so would be the flux of $\vec{E}$ through it. This case explains how, given a set of dipoles (each of which has zero total charge), it is possible to find an equivalent charge distribution with a nonzero local density.

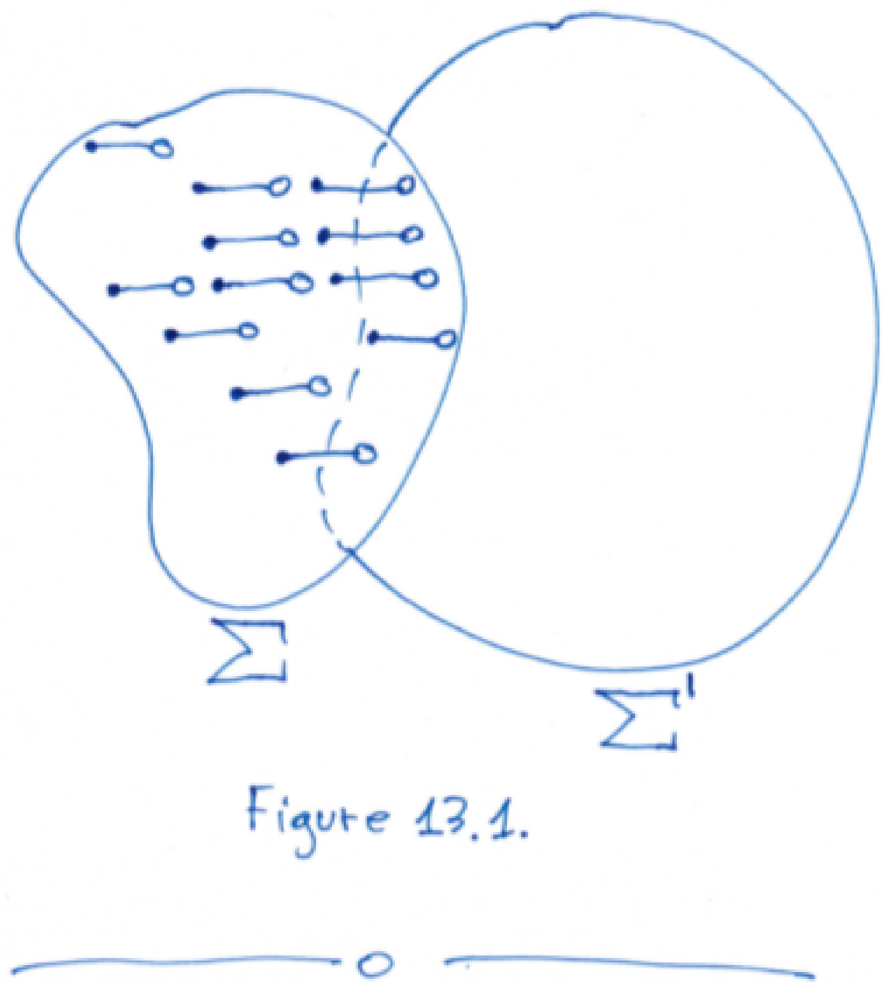

Figure 13.1