PHYS 342 - EM 02

Lecture 14

We will now turn to the case of the multipole expansion for the magnetostatic field.

14.1 Multipole series expansion of the magnetostatic vector potential for a continuous and limited volume electric current distribution

Using the same nomeuclature as in Fig. 12.1, consider a limited region of space Ω where is defined a continuous volume electric current distribution with density $\vec{J}$. We want to study the effects due to the current at a very large distance from it.

The magnetostatic vector potential $\vec{A}$ at point P due to $\vec{J}$ is

$$\vec{A}(P) = \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \iiint\limits_{\Omega} \frac{\vec{J}(\vec{R})}{\|\vec{r} - \vec{R}\|} \, dV \tag{14.1}$$

Considering the points P such that

$$\frac{\vec{R}}{\vec{r}} << 1 \tag{14.2}$$

we can use the same approximation that led to (12.18). By substituting $\phi(\vec{r})$ with $\vec{A}(\vec{r})$, $1/4\pi\epsilon_0$ with $\mu_0/4\pi$, and $\rho(\vec{R})$ with $\vec{J}(\vec{R})$, we find

$$\vec{A}(\vec{r}) \approx \frac{\mu_0}{4\pi} \iiint\limits_{\Omega} \vec{J}(\vec{R}) \left(\frac{1}{r} + \frac{\vec{R} \cdot \vec{r}}{r^3} \right) dV \tag{14.3}$$

By defining

$$\vec{A}_0(\vec{r}) = \frac{\mu_0}{4\pi} \iiint\limits_{\Omega} \vec{J}(\vec{R}) \, \frac{1}{r} \, dV \tag{14.4a}$$

$$\vec{A}_1(\vec{r}) = \frac{\mu_0}{4\pi} \iiint\limits_{\Omega} \vec{J}(\vec{R}) \, \frac{\vec{R} \cdot \vec{r}}{r^3} \, dV \tag{14.4b}$$

we can write

$$\vec{A}(\vec{r}) \approx \vec{A}_0(\vec{r}) + \vec{A}_1(\vec{r}) \tag{14.5}$$

The first term in (14.5) is called the term of magnetostatic monopole and the second term of magnetostatic dipole.

In the magnetostatic case, the term of monopole is always zero. In general, this is clear because of the way the vector potential of Eq. (14.1) was obtained, which requires $\vec{\nabla} \cdot \vec{B} = 0 = \rho_{\rm m}$, i.e., the "magnetic charge" $\rho_{\rm m} = 0$. In analogy to the elctrostatic case, the term of magnetic monopole is the total magnetic charge in Ω, which, thus, must be zero. We can demonstate this also as follows. Since the integral (14.4a) is calculated with respect to $\vec{R}$, we can take r outside the sign of integral,

$$\vec{A}_0(\vec{r}) = \frac{\mu_0}{4\pi} \frac{1}{r} \iiint\limits_{\Omega} \vec{J}(\vec{R}) dV \tag{14.6}$$

In order to calculate this integral, we divide the entire region Ω into many infinitesimal (closed) circuits, each of which is realized as a flux tube of $\vec{J}$. Thus, the integral becomes the vector sum of the contributions from each flux tube. Consider the generic infinitesimal contribution due to one flux tube. Suppose the closed and oriented line γ is the main axis of the flux tube under consideration and dA the area, generally variable along γ, of the normal cross section of the tube. We have

$$\begin{aligned} d\vec{A}_0 &= \frac{\mu_0}{4\pi} \frac{1}{r} \oint\limits_{\gamma} \vec{J}(\vec{R}) dA \, dl \\ &= \frac{\mu_0}{4\pi} \frac{1}{r} \oint\limits_{\gamma} J(\vec{R}) dA \, \vec{t} \, dl \end{aligned} \tag{14.7}$$

where $\vec{t}$ is the tangent unit vector at each point on γ. We assume $\vec{t}$ to have the same direction and sign of $\vec{J}$ at each point on γ. Note that all vectors $\vec{t}dl$ are consecutive along γ. Altogether, they form a closed polygon and, hence,

$$\oint\limits_{\gamma} \vec{t} dl = 0 \tag{14.8}$$

It is easy to be convinced that (14.8) is correct: This is nothing but the vector sum of a continuous series of very short adjacent vectors, from one point to the very same point. Then, because of the solenoidal property of $\vec{J}$, the infinitesimal current $J(\vec{R})dA$ must be constant along γ. Thus,

$$d\vec{A}_0 = \frac{\mu_0}{4\pi} \frac{1}{r} J(\vec{R}) dA \oint\limits_{\gamma} \vec{t} dl = 0 \tag{14.9}$$

where we used the property (14.8). Since the contribution from a single infinitesimal flux tube is zero, it must be

$$\iiint\limits_{\Omega} \vec{J}(\vec{R}) dV = 0 \tag{14.10}$$

from which it follows that

$$\vec{A}_0(\vec{r}) = 0 \tag{14.11}$$

This important result is one incarnation of a fundamental principle of magnetism, that is the absence of so called magnetic monopoles. This statement must be intended as follows: So far, there has been no empirical evidence of the existance of magnetic monopoles (which, theoretically, could exist). Consider a magnet with two poles, North and South (we will come back to this concept when discussing magnetic materials). After breaking the magnet into two pieces, each sub-magnet will still have both a North and a South pole. This means that it is impossible to find a nonzero magnetic charge, i.e., a piece of matter with only a North or a South pole (those to be assumed as net quantities in an algebraic sense). The amount of North and South poles always balances to zero. The term of magnetic monopole is equivalent to the term of electric monopole, which represents the total electric charge of the system [cf. Eq. (12.19(a)]. The term of magnetic monopole is, thus, the total magnetic charge of the system, which must be zero.

Consider now the term of magnetic dipole in (14.5). From simple vector calculus,

$$\begin{aligned} \frac{\vec{R}\bullet\vec{r}}{r^3} &= \vec{R}\bullet\frac{\vec{r}}{r}\frac{1}{r^2} = \vec{R}\bullet\vec{u}_r\frac{1}{r^2} \\ &= -\vec{R}\bullet\vec{\nabla}\frac{1}{r} \end{aligned} \tag{14.12}$$

Thus,

$$\begin{aligned} \vec{A}_1(\vec{r}) &= \frac{\mu_0}{4\pi}\iiint\limits_{\Omega}\vec{J}(\vec{R})\,\frac{\vec{R}\bullet\vec{r}}{r^3}\,dV \\ &= -\frac{\mu_0}{4\pi}\iiint\limits_{\Omega}\vec{J}(\vec{R})\left(\vec{R}\bullet\vec{\nabla}\frac{1}{r}\right)dV \end{aligned} \tag{14.13}$$

By writing

$$\begin{aligned} \vec{J}(\vec{R})\left(\vec{R}\bullet\vec{\nabla}\frac{1}{r}\right) &= \frac{1}{2}\vec{J}(\vec{R})\left(\vec{R}\bullet\vec{\nabla}\frac{1}{r}\right) + \frac{1}{2}\vec{J}(\vec{R})\left(\vec{R}\bullet\vec{\nabla}\frac{1}{r}\right) \\ &\quad -\frac{1}{2}\vec{R}\left[\vec{J}(\vec{R})\bullet\vec{\nabla}\frac{1}{r}\right] + \frac{1}{2}\vec{R}\left[\vec{J}(\vec{R})\bullet\vec{\nabla}\frac{1}{r}\right] \end{aligned} \tag{14.14}$$

we have

$$\begin{aligned} \vec{A}_1(\vec{r}) = &-\frac{\mu_0}{4\pi}\iiint\limits_{\Omega}\frac{1}{2}\left\{\vec{J}(\vec{R})\left(\vec{R}\bullet\vec{\nabla}\frac{1}{r}\right) - \vec{R}\left[\vec{J}(\vec{R})\bullet\vec{\nabla}\frac{1}{r}\right]\right\}dV \\ &-\frac{\mu_0}{4\pi}\iiint\limits_{\Omega}\frac{1}{2}\left\{\vec{J}(\vec{R})\left(\vec{R}\bullet\vec{\nabla}\frac{1}{r}\right) + \vec{R}\left[\vec{J}(\vec{R})\bullet\vec{\nabla}\frac{1}{r}\right]\right\}dV \end{aligned} \tag{14.15}$$

The second integral on the right-hand side of (14.15) is zero. In fact, by dividing once again the entire current distribution into infinitesimal flux tubes (i.e., the region Ω is divided into an infinite set of flux tubes), we have

$$\vec{J}(\vec{R})dV = J(\vec{R})dA\,\vec{t}\,dl = dI\,\vec{t}\,dl \tag{14.16}$$

where dI is the current carried by a generic infinitesimal flux tube (as always, under steady conditions). The contribution from such a flux tube of axis γ to the second integral in (14.15) is

$$d\vec{A}_1^* = -\frac{\mu_0}{4\pi}dI\oint\limits_{\gamma}\frac{1}{2}\left[\vec{t}\left(\vec{R}\bullet\vec{\nabla}\frac{1}{r}\right) + \vec{R}\left(\vec{t}\bullet\vec{\nabla}\frac{1}{r}\right)\right]dl \tag{14.17}$$

Since $\vec{t}dl = d\vec{R}$, we then have

$$d\vec{A}_1^* = -\frac{\mu_0}{4\pi}\, dI \oint_\gamma \frac{1}{2}\left[d\vec{R}\left(\vec{R}\cdot\ \vec{\nabla}\frac{1}{r}\right) + \vec{R}\left(d\vec{R}\cdot\ \vec{\nabla}\frac{1}{r}\right)\right] \tag{14.18}$$

Note that $d\vec{R}$ is an infinitesimal at any point inside Ω, which has been divided into infinite $\vec{t}dl$ elements. Moreover,

$$d\vec{R}\left(\vec{R}\cdot\ \vec{\nabla}\frac{1}{r}\right) + \vec{R}\left(d\vec{R}\cdot\ \vec{\nabla}\frac{1}{r}\right) = d'\left[\vec{R}\left(\vec{R}\cdot\ \vec{\nabla}\frac{1}{r}\right)\right] \tag{14.19}$$

where d' indicates the operation of derivatives with respect to the variable $\vec{R}$. This demonstrates that vector $d\vec{A}_1^*$ is the line integral along γ of an exact differential. Since γ is a closed line $d\vec{A}_1^*$ must be zero. This result can be extended to each flux tube in Ω. Therefore, the second integral in (14.15) must be zero and we have

$$\vec{A}_1(\vec{r}) = -\frac{\mu_0}{4\pi}\iiint_\Omega \frac{1}{2}\left\{\vec{J}(\vec{R})\left(\vec{R}\cdot\ \vec{\nabla}\frac{1}{r}\right) - \vec{R}\left[\vec{J}(\vec{R})\cdot\ \vec{\nabla}\frac{1}{r}\right]\right\}dV \tag{14.20}$$

From the vector identity

$$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\ \vec{b} - (\vec{a}\cdot\vec{b})\ \vec{c} \tag{14.21}$$

which is valid for each tern of vectors $\vec{a}, \vec{b}$, and $\vec{c}$, we have

$$\begin{aligned}\vec{J}(\vec{R})\left(\vec{R}\cdot\ \vec{\nabla}\frac{1}{r}\right) - \vec{R}\left[\vec{J}(\vec{R})\cdot\ \vec{\nabla}\frac{1}{r}\right] &= \left[\vec{\nabla}\frac{1}{r}\times\vec{J}(\vec{R})\right]\times\vec{R}\\ &= -\vec{R}\times\left[\vec{\nabla}\frac{1}{r}\times\vec{J}(\vec{R})\right]\\ &= \left[\vec{R}\times\vec{J}(\vec{R})\right]\times\vec{\nabla}\frac{1}{r}\end{aligned} \tag{14.22}$$

and, thus,

$$\vec{A}_1(\vec{r}) = -\frac{\mu_0}{4\pi}\,\frac{1}{2}\iiint_\Omega\left\{[\vec{R}\times\vec{J}(\vec{R})]\times\ \vec{\nabla}\frac{1}{r}\right\}dV \tag{14.23}$$

Since the vector $\vec{\nabla}(1/r)$ is independent from the variable of integration, we find

$$\vec{A}_1(\vec{r}) = -\frac{\mu_0}{4\pi}\left\{\frac{1}{2}\iiint_\Omega[\vec{R}\times\vec{J}(\vec{R})]dV\right\}\times\ \vec{\nabla}\frac{1}{r} \tag{14.24}$$

The vector

$$\vec{m} = \frac{1}{2}\iiint_\Omega[\vec{R}\times\vec{J}(\vec{R})]dV \tag{14.25}$$

in the integral (14.24) is called magnetostatic (or simply magnetic) dipole moment for $\vec{J}$ with respect to O (in Fig. 12.1). Using (14.25) in (14.24) we have

$$\vec{A}_1(\vec{r}) = -\frac{\mu_0}{4\pi}\vec{m}\times\ \vec{\nabla}\frac{1}{r} = \frac{\mu_0}{4\pi}\,\frac{\vec{m}\times\vec{r}}{r^3} \tag{14.26}$$

The vector $\vec{m}$ defined by (14.25) is independent from the point O from which the vectors $\vec{R}$ originate. In fact, given a new origin O' and calling ρ' the distances from it, it follows that

$$\vec{\rho'} = \vec{R} + \vec{r}_{O'O} \tag{14.27}$$

where $\vec{r}_{O'O}$ is the vector between points O' and O (directed from O' to O). By substituting (14.27) in (14.25), we obtain

$$\begin{aligned} \vec{m}' &= \frac{1}{2} \iiint\limits_{\Omega} \vec{\rho'} \times \vec{J}(\vec{\rho'}) dV = \frac{1}{2} \iiint\limits_{\Omega} (\vec{R} + \vec{r}_{O'O}) \times \vec{J}(\vec{\rho'}) dV \\ &= \frac{1}{2} \iiint\limits_{\Omega} \vec{R} \times \vec{J}(\vec{\rho'}) dV + \frac{1}{2} \iiint\limits_{\Omega} \vec{r}_{O'O} \times \vec{J}(\vec{\rho'}) dV \end{aligned} \tag{14.28}$$

The vector $\vec{r}_{O'O}$ is independent from the variable of integration. Hence,

$$\vec{m'} = \frac{1}{2} \iiint\limits_{\Omega} \vec{R} \times \vec{J}(\vec{\rho'}) dV + \frac{1}{2} \vec{r}_{O'O} \times \iiint\limits_{\Omega} \vec{J}(\vec{\rho'}) dV \tag{14.29}$$

From (14.10), we then find

$$\vec{m'} = \frac{1}{2} \iiint\limits_{\Omega} \vec{R} \times \vec{J}(\vec{\rho'}) dV \tag{14.30}$$

If $\vec{\rho'}$ indicates point Q in Ω with respect to O' and $\vec{R}$ indicates the same point Q with respect to O (cf. Fig. 14.1), from (14.30) it follows that

$$\vec{m'} = \vec{m} \tag{14.31}$$

In summary, in the electostatic case the dipole moment for a given charge distribution is independent from the chosen origin only when the total charge is zero. In the magnetostatic case this is always true because of (14.10).

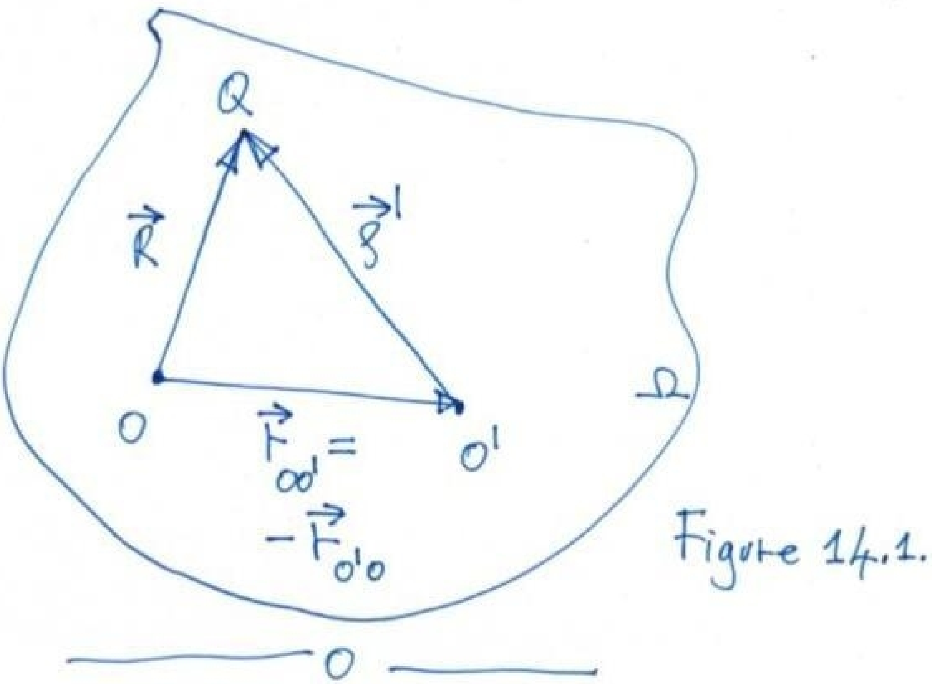

Figure 14.1