PHYS 342 - EM 02

Lecture 5

5.1 Lorentz Force

As we saw in PHYS 242, the force on an electric charge depends on where the charge is in space and on how fast is moving.

The first part of the force is the electric force, which is independent of the motion of the charge q and is described by the electric field $\vec{E}$. The second part is the magnetic force, which depends on the velocity $\vec{v}$ of the charge and on the magnetic field $\vec{B}$. In summary, the total electromagnetic force can be written as

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \tag{5.1}$$

This force is called Lorentz force and is as fundamental as Newton's law of mechanics.

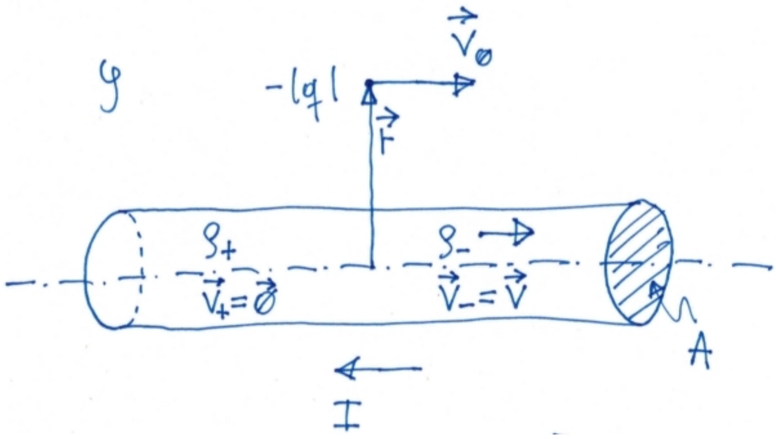
5.1.1 The Relativity of Electric and Magnetic Fields

In Eq. (5.1) we did not specify the reference frame with respect to which the velocity $\vec{v}$ us defined.

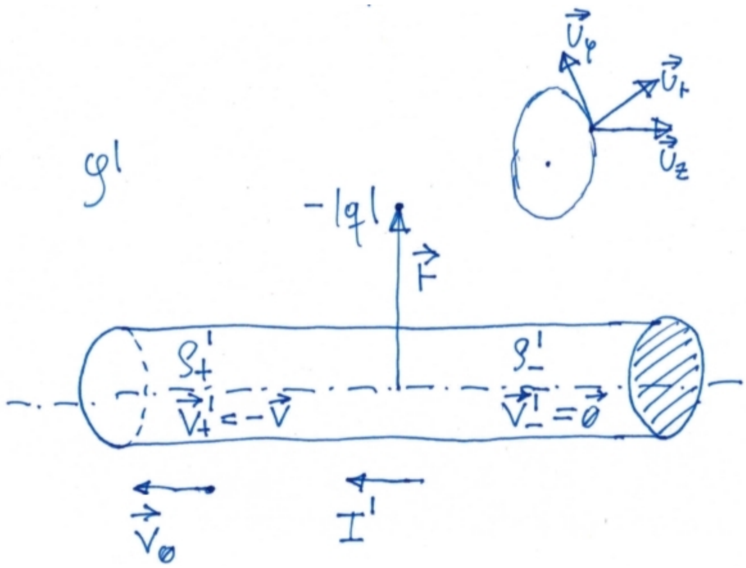
To begin with, we assume the relativity principle is applicable to electromagnetism. We mean that, in the framework of special relativity, Maxwell's equations have the same form in all inertial frames of reference.

Consider a negative charge $-|q|$ moving with velocity $\vec{v}_0$ parallel to a wire carrying a steady, uniform current I, as shown in Fig. 5.1. The wire has cylindrical shape with cross-section A and is assumed to be straight and indefinitely long.

We will study this problem in two different frames of reference, $\mathscr{S}$ and $\mathscr{S}'$. Frame $\mathscr{S}$ is fixed with respect to the wire (cf. Fig. 5.1a) and frame $\mathscr{S}'$ is fixed with respect to the charge (cf. Fig. 5.1b).



(a)



(b)

Figure 5.1

It is clear that in the $\mathscr{S}$-frame, the charged particle is subjected to a magnetic force due to the magnetic field generated by I [cf. Lecture 4, Eq. (4.25)] acting on the moving particle. If the particle were moving freely, it would curve towards the wire (the charge is negative). In the $\mathscr{S}'$-frame, however, there can be no magnetic force on the particle because the particle is at rest in that frame. Does the particle stay where it is in $\mathscr{S}'$? According to the principle of relativity we should see the particle move also in $\mathscr{S}'$.

In a normal conductor (e.g., copper) the electric currents come from the motion of some of the negative electrons (called the conduction electrons), while the positive nuclear charges and the rest of the electrons stay fixed in the body of the material.

We assume the volume density of the conduction electrons to be ρ_- and their velocity in $\mathscr{S}$ to be $\vec{v}_- = \vec{v}$. The density of the charges at rest in $\mathscr{S}$ is $\rho_+(\vec{v}_+ = \vec{0})$. Since we are considering an uncharged wire, it must be $\rho_+ + \rho_- = 0$. As a consequence, there is no electric field outside the wire. The force on the moving particle $-|q|$ is thus

$$\vec{F} = -|q|\,\vec{v}_0 \times \vec{B} \tag{5.2}$$

Using the result (4.25) for $\vec{B}$ and noting that I points in the opposite direction of $\vec{v}$, we obtain

$$\vec{F} = -|q|\, v_0 \vec{u}_z \times \vec{u}_\varphi \, \frac{\mu_0(-|I|)}{2\pi} \, \frac{1}{r}, \qquad r > 0 \tag{5.3}$$

where we represented the various vectors in a cylindrical coordinate system, as indicated in Fig. 5.1. Hence,

$$\vec{F} = -\frac{|q|\, v_0 \mu_0 \,|I|}{2\pi} \, \frac{1}{r} \, \vec{u}_r, \qquad r > 0 \tag{5.4}$$

By means of Eq. (2.3) and Eq. (2.4) in the case of a uniform, steady current, we have

$$I = \rho_- vA \qquad \text{or} \qquad |I| = -\rho_- vA = \rho_+ vA \tag{5.5}$$

Then,

$$\vec{F} = -\frac{|q|\, v_0 \mu_0 \rho_+ vA}{2\pi} \, \frac{1}{r} \, \vec{u}_r, \qquad r > 0 \tag{5.6}$$

Without loosing generality, we further assume $v_0 = v$. Hence,

$$\vec{F} = -\frac{|q|}{2\pi\epsilon_0} \, \frac{\rho_+ A}{r} \, \frac{v^2}{c^2} \, \vec{u}_r, \qquad r > 0 \tag{5.7}$$

In $\mathscr{S}'$, the particle is at rest and the wire moves towards the left with velocity v (cf. Fig. 5.1b). The positive charges moving with the wire will generate a magnetic field $\vec{B}'$. However, the particle is now at rest and, thus, $\vec{B}'$ provides no magnetic force on it. If there is any force on the particle, it must be an electric force. It must be that an uncharged (neutral) wire with a current appears to be charged when set in motion.

We must try to compute the charge density in the wire in $\mathscr{S}'$ from our knowledge about it in $\mathscr{S}$. At first glance, one may think the charge densities are the same in $\mathscr{S}$ and $\mathscr{S}'$. However, from special relativity we know that lengths are changed between $\mathscr{S}$ and $\mathscr{S}'$. Since charge densities depend on the volume occupied by charges, the densities must also change.

Going from $\mathscr{S}$ to $\mathscr{S}'$, the apparent mass of a particle changes by $(1 - v^2/c^2)^{-1/2}$. An important question is whether charge undergoes a similar change between $\mathscr{S}$ and $\mathscr{S}'$. As it turns out, because of charge conservation, charges are always the same, moving or not. This statement is fully confirmed by empirical evidence. The charge q of a particle is an invariant scalar quantity, independent from the frame of reference. In a given frame, the charge density of a distribution of charges is proportional to the number of charges per unit volume. Thus, we only need to worry about the fact that such a volume can change because of the relativistic contraction of distances.

Consider now our moving wire. For the sake of generality, consider a piece of wire of length L, in which there is a generic charge density ρ of stationary charges. The total charge in this piece of wire is

$$q = \rho L A \tag{5.8}$$

If the same charges are observed in a different frame that moves at velocity v, they are found in a piece of the material with the shorter length

$$L' = L\sqrt{1 - \frac{v^2}{c^2}} \tag{5.9}$$

but with the same area A since dimensions transverse to the motion are unchanged (cf. Fig. 5.2).

If we call ρ' the density of charges in the frame in which they are moving, the total charge will be

$$q' = \rho' L' A \tag{5.10}$$

It must be

$$q' = q \tag{5.11}$$

because of the total charge invariance. Thus,

$$\rho' L' = \rho L \tag{5.12}$$

and finally, from (5.9),

$$\rho' = \frac{\rho}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{5.13}$$

The charge density of a moving distribution of charges varies in the same way as the relativistic mass of a particle.

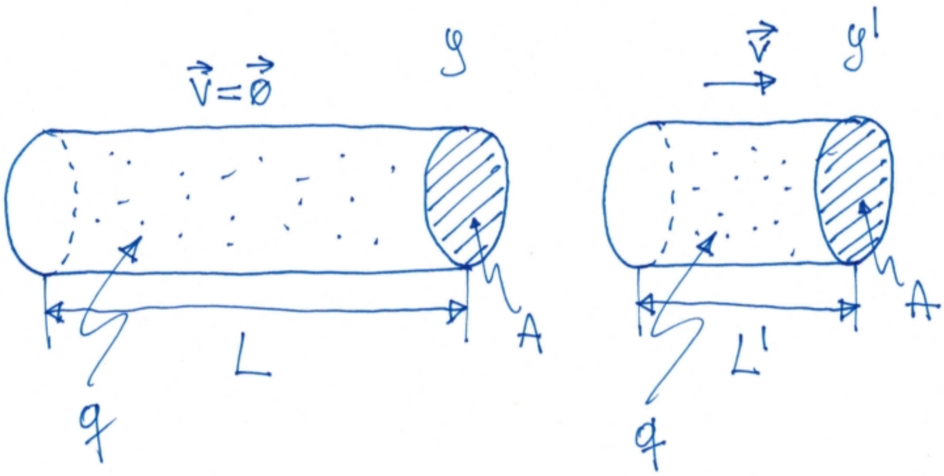


Figure 5.2

We can now use the general result of Eq. (5.13) for the positive charge density ρ_+ of our wire. These charges are at rest in $\mathscr{S}(\vec{v}_+ = \vec{0})$. However, in $\mathscr{S}'$ we find

$$\rho'_+ = \frac{\rho_+}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{5.14}$$

because $\vec{v'}_+ = -\vec{v}$.

The positive charges are at rest in $\mathscr{S}$ and move at speed $-\vec{v}$ in $\mathscr{S}'$. The negative charges, on the contrary, are at rest in $\mathscr{S}'$ and move at speed $\vec{v}$ in $\mathscr{S}$. The two type of charges are characterized by a dual behaviour. The general result of Eq. (5.13) must reflect this fact and, thus, the role of ρ_- and ρ'_- must be swapped compared to that of ρ_+ and ρ'_+:

$$\rho_- = \frac{\rho'_-}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{5.15}$$

or

$$\rho'_- = \rho_- \sqrt{1 - \frac{v^2}{c^2}} \tag{5.15'}$$

Note that the actual sign of $\vec{v}$ does not matter here since the square of the speed only enters in Eq. (5.13).

In $\mathscr{S}'$, the net charge density ρ' is given by

$$\rho' = \rho'_+ + \rho'_- \tag{5.16}$$

Using Eqs. (5.14) and (5.15'), we get

$$\rho' = \frac{\rho_+}{\sqrt{1 - \frac{v^2}{c^2}}} + \rho_- \sqrt{1 - \frac{v^2}{c^2}} \tag{5.17}$$

Because of the neutrality of the wire in $\mathscr{S}$ (stationary wire), $\rho_- = -\rho_+$, we have

$$\rho' = \rho_+ \frac{\frac{v^2}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{5.18}$$

The wire in $\mathscr{S}'$ (moving wire) is positively charged with density ρ' given by (5.18) and, thus, generates an electric field $\vec{E}'$ at the external stationary charged particle. We found the solution to this simple problem in PHYS 242:

$$\begin{aligned} \vec{E}' &= \frac{\lambda}{2\pi\epsilon_0 r}\,\vec{u}_r = \frac{\rho' A}{2\pi\epsilon_0 r}\,\vec{u}_r \\ &= \frac{\rho_+ A \,\frac{v^2}{c^2}}{2\pi\epsilon_0 \sqrt{1 - \frac{v^2}{c^2}}}\,\frac{1}{r}\,\vec{u}_r \end{aligned} \tag{5.19}$$

Thus, the force on $-|q|$ in $\mathscr{S}'$ is

$$\vec{F}' = -\frac{|q|}{2\pi\epsilon_0}\,\frac{\rho_+ A}{r}\,\frac{\frac{v^2}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}\,\vec{u}_r, \qquad r > 0 \tag{5.20}$$

This force is directed towards the wire, precisely as the magnetic force (5.7) in $\mathscr{S}$. By comparing the magnitude of the electric force $\vec{F}'$ in $\mathscr{S}$ given by Eq. (5.7), we find that

$$F' = \frac{F}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{5.21}$$

For small velocities, i.e., $v << c$, the two forces are equal. In this case, we can now state that electricity and magnetism are two different representations of the same phenomenon, the electromagnetic interaction of particles. The separation of this interaction into electric and magnetic parts depends on the reference frame chosen for the description. Note that (5.1) is not to be altered if the source of $\vec{E}$ and $\vec{B}$ is moving. The valves of $\vec{E}$ and $\vec{B}$, however, will be altered by the motion.